Existence of weak solutions for a diffuse interface model for a two-phase flow with degenerate mobility

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June 2012
12. International Conference on Free Boundary Problems

Contents

Basics of the model

Different Navier-Stokes/Cahn-Hilliard models

- "Model H" of Halperin and Hohenberg,
- Lowengrub and Truskinovsky,
- Ding, Spelt and Shu,
- Abels, Garcke and Grün,
- Aki, Dreyer, Giesselmann and Kraus.

Basis of the model in Abels, Garcke and Grün

Mixture of two fluids filling a domain $\Omega \subset \mathbb{R}^3$ with

- mass densities ρ_j, total mass density $\rho = \rho_1 + \rho_2$,
- specific constant mass densites $\tilde{\rho}_j$, volume fraction $\varphi_j = \frac{\rho_j}{\tilde{\rho}_j}$,
- velocities $\mathbf{v}_j$, volume averaged velocity $\mathbf{v} = \varphi_1 \mathbf{v}_1 + \varphi_2 \mathbf{v}_2$.

Conservation laws of mass, species, momentum with stress tensors, then rational continuum mechanics.

The model

Navier-Stokes/Cahn-Hilliard phase field model

$$\begin{aligned} &\partial_t(\rho \mathbf{v}) + \mathrm{div}\,(\rho \mathbf{v} \otimes \mathbf{v}) - \mathrm{div}\,(2\eta(\varphi) D\mathbf{v}) + \nabla p \\ &-\mathrm{div}\left(\mathbf{v} \otimes \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m(\varphi)\nabla\mu\right) = -\varepsilon\hat{\sigma}\mathrm{div}\,(\nabla\varphi \otimes \nabla\varphi)\,, \end{aligned} \tag{1}$$

$$\mathrm{div}\;\mathbf{v} = 0\,, \tag{2}$$

$$(\partial_t \varphi + \mathbf{v}\cdot\nabla\varphi) = \mathrm{div}\,(m(\varphi)\nabla\mu)\,, \tag{3}$$

$$\mu = \frac{\hat{\sigma}}{\varepsilon}\Psi'(\varphi) - \varepsilon\hat{\sigma}\Delta\varphi\,. \tag{4}$$

Remark

Smooth solutions in $Q = \Omega \times (0, \infty)$ with suitable boundary conditions fulfill the energy estimate

$$\frac{d}{dt}E_{\mathrm{tot}}\,(\varphi(t), \mathbf{v}(t)) + \int_Q 2\eta(\varphi)|D\mathbf{v}|^2 + \int_Q m(\varphi)|\nabla\mu|^2 = 0\,.$$

Reformulation of the model

Setting $\hat{\sigma} = \varepsilon = 1$ and $\widetilde{\mathbf{J}} = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m(\varphi)\nabla\mu$ and identifying gradient terms we get we get:

Navier-Stokes/Cahn-Hilliard phase field model (short)

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \operatorname{div}\left(\mathbf{v}\otimes\widetilde{\mathbf{J}}\right) + \nabla g = \mu\nabla\varphi\,, \tag{5}$$

$$\operatorname{div}\mathbf{v} = 0\,, \tag{6}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}\left(m(\varphi)\nabla\mu\right)\,, \tag{7}$$

$$\mu = \Psi'(\varphi) - \Delta\varphi\,. \tag{8}$$

Assumptions on Ψ

Assumptions on the homogenous free energy density

For the free energy density $f(\varphi, \nabla\varphi) = \frac{\varepsilon\hat{\sigma}}{2}|\nabla\varphi|^2 + \frac{\hat{\sigma}}{\varepsilon}\Psi(\varphi)$ we assume that $\Psi \in C([-1,1]) \cap C^2((-1,1))$ fulfills

$$\lim_{s\to\pm1} \Psi'(s) = \pm\infty \quad \text{and} \quad \Psi''(s) \geq -\kappa \quad \text{for some } \kappa \in \mathbb{R}\,.$$

Extend Ψ through $\Psi(s) = \infty$ if $s \in \mathbb{R}\setminus[-1,1]$.

Example

$$\Psi(s) = \frac{\theta}{2}\left((1+s)\ln(1+s) + (1-s)\ln(1-s)\right) - \frac{\theta_c}{2}s^2\,.$$

The model

Navier-Stokes/Cahn-Hilliard phase field model (full)

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \operatorname{div}\left(\mathbf{v}\otimes\widetilde{\mathbf{J}}\right) + \nabla g = \mu\nabla\varphi\,, \tag{9}$$

$$\operatorname{div}\mathbf{v} = 0\,, \tag{10}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu)\,, \tag{11}$$

$$\mu + \kappa\varphi = \Psi_0'(\varphi) - \Delta\varphi\,, \tag{12}$$

in $\Omega\times(0,\infty)$ with boundary and initial conditions

$$\mathbf{v} = 0 \qquad \text{on } \partial\Omega\times(0,\infty)\,, \tag{13}$$

$$\partial_n\varphi = \partial_n\mu = 0 \qquad \text{on } \partial\Omega\times(0,\infty)\,, \tag{14}$$

$$(\mathbf{v},\varphi)|_{t=0} = (\mathbf{v}_0,\varphi_0) \qquad \text{in } \Omega\,. \tag{15}$$

Definition and existence of weak solutions

Weak solution

Let $0 < T < \infty$, $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in H^1(\Omega)$ with $\Psi'(\varphi_0) \in L^2(\Omega)$ and $m \in C^1(\mathbb{R})$, $\eta \in C^0(\mathbb{R})$ with $0 < m_0 \leq m(s), \eta(s) \leq C$ be given. Then a triple $(\mathbf{v}, \varphi, \mu)$ is called a weak solution of (9)-(15) if it has the following regularity

$$\mathbf{v} \in L^2(0, T; H^1_0(\Omega)^d) \cap BC_w([0, T); L^2_\sigma(\Omega)),$$
$$\varphi \in L^2(0, T; H^2(\Omega)) \cap BC_w([0, T); H^1(\Omega)) \text{ with } \Psi'(\varphi) \in L^2(0, T; L^2(\Omega)),$$
$$\mu \in L^2(0, T; H^1(\Omega))$$

and if the following equations are fulfilled.

Definition and existence of weak solutions

$$-\int_{\Omega_T} \rho \mathbf{v} \cdot \partial_t \boldsymbol{\psi} + \int_{\Omega_T} \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) \cdot \boldsymbol{\psi} + \int_{\Omega_T} 2\eta(\varphi) D\mathbf{v} : \nabla \boldsymbol{\psi}$$
$$= \int_{\Omega_T} \mu \nabla \varphi \cdot \boldsymbol{\psi} + \int_{\Omega_T} \left(\mathbf{v} \otimes \widetilde{\mathbf{J}}\right) : \nabla \boldsymbol{\psi} \tag{16}$$

for all $\boldsymbol{\psi} \in [C_0^\infty(\Omega \times (0,T))]^d$ with $\operatorname{div} \boldsymbol{\psi} = 0$ and

$$-\int_{\Omega_T} \varphi \, \partial_t \zeta + \int_{\Omega_T} (\mathbf{v} \cdot \nabla \varphi) \, \zeta = -\int_{\Omega_T} m(\varphi) \nabla \mu \cdot \nabla \zeta \tag{17}$$

for all $\zeta \in C_0^\infty((0,T); C^\infty(\overline{\Omega}))$,

$$\mu + \kappa \varphi = \Psi_0'(\varphi) - \Delta \varphi \quad \text{a.e. in } \Omega \text{ and} \tag{18}$$

Definition and existence of weak solutions

$$(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_0, \varphi_0)\,. \tag{19}$$

Moreover

$$E_{\text{tot}}(\varphi(t), \mathbf{v}(t)) + \int_{\Omega_{(s,t)}} 2\eta(\varphi)|D\mathbf{v}|^2 + \int_{\Omega_{(s,t)}} m(\varphi)|\nabla\mu|^2 \le E_{\text{tot}}(\varphi(s), \mathbf{v}(s)) \tag{20}$$

holds for all $t \in [s, T)$ and almost all $s \in [0, T)$ (including $s = 0$).

Theorem

Under the above assumptions there exists a weak solution $(\mathbf{v}, \varphi, \mu)$ *of* (9)-(15).

Existence

Sketch of Proof

Let $(\mathbf{v}_n, \varphi_n, \mu_n)$ with initial values $(\mathbf{v}_{n,0}, \varphi_{n,0})$ be solutions of (16)-(18), which fulfill the energy estimate (20). This leads to the bounds:

$$\begin{aligned} \mathbf{v}_n &\text{ is bounded in } L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d)\,, \\ \varphi_n &\text{ is bounded in } L^\infty(0,T;H^1(\Omega)) \text{ and} \\ \nabla\mu_n &\text{ is bounded in } L^2(0,T;L^2(\Omega)^d)\,. \end{aligned}$$

Remark

Approximate solutions are constructed by an implicit time discretization.

Existence

There exist $(\mathbf{v}, \varphi, \mu)$ as in the definition of weak solution, such that

Limits

$$
\begin{aligned}
(\mathbf{v}_n, \varphi_n, \mu_n) &\rightharpoonup (\mathbf{v}, \varphi, \mu) && \text{in } L^2(0,T; H^1(\Omega)^d \times H^1(\Omega) \times H^1(\Omega)),\\
(\mathbf{v}_n, \varphi_n) &\rightharpoonup^* (\mathbf{v}, \varphi) && \text{in } L^\infty(0,T; L^2(\Omega)^d \times H^1(\Omega)),\\
(\varphi_n, \rho_n) &\to (\varphi, \rho) && \text{in } L^2(0,T; H^1(\Omega) \times H^1(\Omega)),\\
\mathbf{v}_n &\to \mathbf{v} && \text{in } L^2(0,T; L^2(\Omega)^d).
\end{aligned}
$$

With this limits

- Limit in the equations for $(\mathbf{v}_n, \varphi_n, \mu_n)$ to get (16)-(18),
- Limit in the initial conditions to get $(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_0, \varphi_0)$.

If the approximate solutions arise as solutions of an implicit time discretization, then:

- Limit in a linearly extended version of the discrete energy estimate. ($\Box$)

Degenerate mobility

We consider now the following problem

Navier-Stokes/Cahn-Hilliard phase field model (with deg. mobility)

$$\partial_t(\rho \mathbf{v}) + \mathrm{div}\,(\rho \mathbf{v} \otimes \mathbf{v}) - \mathrm{div}\,(2\eta(\varphi) D\mathbf{v}) + \mathrm{div}\,(\mathbf{v} \otimes \alpha \mathbf{J}) + \nabla g = -\Delta\varphi\nabla\varphi \quad (NSE)\,, \tag{21}$$

$$\mathrm{div}\,\mathbf{v} = 0\,, \tag{22}$$

$$\partial_t \varphi + \mathbf{v} \cdot \nabla\varphi = -\,\mathrm{div}\,\mathbf{J}\,, \tag{23}$$

$$\mathbf{J} = -m(\varphi)\nabla\mu\,, \tag{24}$$

$$\mu = \Psi'(\varphi) - \Delta\varphi\,. \tag{25}$$

Here the homogeneous free energy Ψ is smooth and the mobility m is given through

$$m(\varphi) = \begin{cases} 1 - \varphi^2\,, & \textit{if } |\varphi| \leq 1\,, \\ 0\,, & \textit{else} \end{cases}$$

Degenerate mobility

Weak formulation for unknowns $(\mathbf{v}, \varphi, \mathbf{J})$:

$$(NSE) \text{ weak as before },$$

$$\int_{\Omega_T} \varphi\, \partial_t \zeta + \int_{\Omega_T} (\mathbf{v} \cdot \nabla \varphi)\, \zeta = \int_{\Omega_T} \mathbf{J} \cdot \nabla \zeta$$

for all $\zeta \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and

$$\int_{\Omega_T} \mathbf{J} \cdot \boldsymbol{\eta} = \int_{\Omega_T} \left(\Psi'(\varphi) - \Delta\varphi\right) \operatorname{div}(m(\varphi)\boldsymbol{\eta})$$

for all $\boldsymbol{\eta} \in L^2(0, T; H^1(\Omega)^d) \cap L^\infty(\Omega_T)^d$.

The last line is a weak version of

$$\mathbf{J} = -m(\varphi)\, \nabla \left(\Psi'(\varphi) - \Delta\varphi\right) \, .$$

Degenerate mobility

Approximate the mobility $m(\varphi)$ by a positive mobility $m_\varepsilon(\varphi)$ and set

$$\Psi_\varepsilon(\varphi) = \Psi(\varphi) + \varepsilon\hat{\Psi}(\varphi)\,, \text{ where}$$
$$\hat{\Psi}(\varphi) = \frac{\theta}{2}\left((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi)\right)\,.$$

Then we consider the following approximate problem

$$(NSE)_\varepsilon \text{ and } \operatorname{div}\mathbf{v} = 0\,, \text{ where } \mathbf{J}_\varepsilon = -m_\varepsilon(\varphi)\nabla\mu\,, \tag{26}$$
$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m_\varepsilon(\varphi)\nabla\mu)\,, \tag{27}$$
$$\mu = \Psi'_\varepsilon(\varphi) - \Delta\varphi\,. \tag{28}$$

Let $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ be a weak solution to this problem with $\mathbf{J}_\varepsilon = -m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon$. Note that here already $|\varphi_\varepsilon| < 1$ a.e. .

Case $\mathbf{v} = 0$ (Cahn-Hilliard equation): Elliott and Garcke 1996.

Degenerate mobility

Lemma (Energy estimates for the approximate solutions)

$$(a)\ \operatorname{esssup}_{0\le t\le T}\int_\Omega\left(\rho(\varphi_\varepsilon(t))\frac{|\mathbf{v}_\varepsilon(t)|^2}{2}+\frac{1}{2}|\nabla\varphi_\varepsilon(t)|^2+\Psi_\varepsilon(\varphi_\varepsilon(t))\right)$$
$$+\int_{\Omega_T}2\eta(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2+\int_{\Omega_T}m_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2\le C\,, \tag{29}$$

$$(b)\ \operatorname{esssup}_{0\le t\le T}\int_\Omega M_\varepsilon(\varphi_\varepsilon(t))+\int_{\Omega_T}\left(|\Delta\varphi_\varepsilon|^2+\Psi''_\varepsilon(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2\right)\le C\,, \tag{30}$$

$$(c)\ \varepsilon^3\int_{\Omega_T}|\hat{\Psi}'(\varphi_\varepsilon)|^2\le C\,, \tag{31}$$

$$(d)\ \int_{\Omega_T}|\mathbf{J}_\varepsilon|^2\le C\,. \tag{32}$$

Here we set $M''_\varepsilon(u)=\frac{1}{m_\varepsilon(u)}$ *with* $M'_\varepsilon(0)=M_\varepsilon(0)=0$.

Degenerate mobility

Formal proof of (b): Test (27) with $M'_\varepsilon(\varphi_\varepsilon)$. Crucial point:

$$\int_{\Omega_T} m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon \cdot \nabla M'_\varepsilon(\varphi_\varepsilon) = \int_{\Omega_T} \underbrace{m_\varepsilon(\varphi_\varepsilon)M''_\varepsilon(\varphi_\varepsilon)}_{=1}\nabla\mu_\varepsilon \cdot \nabla\varphi_\varepsilon$$

Formal proof of (c): Let $E_\varepsilon : L^2(\Omega) \to \mathbb{R}$ be given by

$$E_\varepsilon(\varphi) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \varepsilon\int_\Omega \hat{\Psi}(\varphi) & \text{for } \varphi \in \operatorname{dom} E_\varepsilon = \{|\varphi| \le 1\}\,, \\ +\infty & \text{else}\,. \end{cases}$$

Then with $\partial E_\varepsilon(\varphi_\varepsilon) = \varepsilon P_0 \hat{\Psi}'(\varphi_\varepsilon) - \Delta\varphi_\varepsilon = P_0\mu_\varepsilon - P_0\Psi'(\varphi_\varepsilon)$ there holds:

$$\begin{aligned} \|\varphi_\varepsilon\|_{H^2} + \varepsilon\|\hat{\Psi}'(\varphi_\varepsilon)\|_{L^2} &\le C(\|\varphi_\varepsilon\|_{L^2})(\|\partial E_\varepsilon(\varphi_\varepsilon)\|_{L^2} + 1) \\ &\le C(\|\nabla\mu_\varepsilon\|_{L^2} + 1)\,. \end{aligned}$$

Finally use $\varepsilon\int_{\Omega_T} |\nabla\mu_\varepsilon|^2 \le \int_{\Omega_T} m_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2 \le C$.

Degenerate mobility

Passing to the limit for $\varepsilon \to 0$ in

$$\int_{\Omega_T} \boldsymbol{J}_\varepsilon \cdot \boldsymbol{\eta} = \int_{\Omega_T} \left(\Psi'_\varepsilon(\varphi_\varepsilon) - \Delta\varphi_\varepsilon\right) \operatorname{div}(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta})\,.$$

One thing to show is

$$\varepsilon \int_{\Omega_T} \hat{\Psi}'(\varphi_\varepsilon) \operatorname{div}(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}) \to 0\,.$$

To this end

$$\varepsilon \int_{|\varphi_\varepsilon| \le 1-\varepsilon} \underbrace{\hat{\Psi}'(\varphi_\varepsilon)}_{O(ln(\varepsilon))} \operatorname{div}(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}) \quad \text{and}$$

$$\varepsilon \int_{|\varphi_\varepsilon| > 1-\varepsilon} \hat{\Psi}'(\varphi_\varepsilon) \operatorname{div}(\underbrace{m_\varepsilon(\varphi_\varepsilon)}_{\equiv \varepsilon(2-\varepsilon)}\boldsymbol{\eta})\,.$$

Conclusions

Conclusions

- We considered a thermodynamical consistent diffuse interface model for incompressible flows with different densities.
- The model leads to an energy estimate, which is important for the a-priori estimates.
- We show existence of weak solutions by an implicit time discretization.
- We include a degenerate mobility.

Thank you for your attention!