A symmetry result for the Ornstein-Uhlenbeck operator

Matteo Novaga

University of Padova

Free Boundary Problems, 11–15 june 2012

De Giorgi's conjecture

A well-known conjecture by De Giorgi asks if bounded entire solutions to the equation

$$\Delta u = u^3 - u$$

which are monotone in some direction are in fact one-dimensional, in the sense that the level sets of u are hyperplanes, at least in dimension $n \leq 9$.

The conjecture has been proved by Ghoussoub and Gui in dimension $n = 2$, and by Ambrosio and Cabré in dimension $n = 3$, and a counterexample has been given by del Pino, Kowalczyk and Wei for $n = 9$.

While the conjecture is still open for $4 \leq n \leq 8$, a proof has been presented by Savin under the additional assumption that u connects -1 to 1 along the direction where it increases.

A variant of the conjecture

We are interested in the same question when the Laplacian is replaced by the Ornstein-Uhlenbeck operator, so that the equation becomes

$$\Delta u - \langle x, \nabla u \rangle = f(u), \qquad (OUE)$$

where f is a C^1-function.

A weighted Allen-Cahn Energy

Notice that (OUE) is the Euler-Lagrange equation of the Allen-Cahn type functional

$$E[u] = \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + F(u) \right) e^{-\frac{x^2}{2}}\, dx$$

where $F'(u) = f(u)$.

By elliptic regularity [Lunardi, TAMS 1997], any bounded weak solution u of (OUE) is also a classical solution and satisfies $E[u] < +\infty$ (this is not true in the Euclidean case).

The main result

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded solution to

$$\Delta u - \langle x, \nabla u \rangle = f(u),$$

satisfying

$$\langle \nabla u(x), w \rangle > 0 \qquad x \in \mathbb{R}^n$$

for some $w \in \mathbb{R}^n$. Then u is one-dimensional.

Notice that, in this case, there is no restriction on the dimension.

The linearized equation

The derivative $u_i = \frac{\partial u}{\partial x_i}$ satisfies

$$\int_{\mathbb{R}^n} \big(\langle \nabla u_i, \nabla \varphi \rangle + f'(u) u_i \varphi + u_i \varphi \big) \, d\mu = 0$$

for all $\varphi \in C_c^1(\mathbb{R}^n)$, where we set $d\mu(x) = e^{-\frac{x^2}{2}} \, dx$.

A variational inequality

A monotone solution to (OUE) satisfies the following inequality, which bounds from below the second derivative of the energy functional:

$$\int_{\mathbb{R}^n} \left(|\nabla\varphi|^2 + f'(u)\varphi^2\right)\, d\mu \geq -\int_{\mathbb{R}^n} \varphi^2 \, d\mu.$$

Indeed, assuming that u is monotone along e_1, we apply the previous equality with $i = 1$ and test function φ^2/u_1, and we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n} -f'(u)\varphi^2 - \varphi^2 \, d\mu \\
&= \int_{\mathbb{R}^n} \langle \nabla u_1, \nabla(\varphi^2/u_1) \rangle \, d\mu \\
&= \int_{\mathbb{R}^n} 2(\varphi/u_1)\langle \nabla u_1, \nabla\varphi \rangle - (\varphi/u_1)^2 |\nabla u_1|^2 \, d\mu \\
&= \int_{\mathbb{R}^n} |\nabla\varphi|^2 - \Big|(\varphi/u_1)\nabla u_1 - \nabla\varphi\Big|^2 \, d\mu \\
&\leq \int_{\mathbb{R}^n} |\nabla\varphi|^2 \, d\mu.
\end{aligned}
$$

A geometric Poincaré inequality

By the previous inequality, applied with test function $|\nabla u|\varphi$, we obtain that u satisfies:

$$\int_{\mathbb{R}^n} \left(|\nabla^2 u|^2 - \big|\nabla|\nabla u|\big|^2\right)\varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu$$

where

$$|\nabla^2 u|^2 := \sum_{i,j} u_{ij}^2 \, .$$

The left-hand side can be written as

$$|\nabla^2 u|^2 - \Big|\nabla|\nabla u|\Big|^2 = |\nabla u|^2 \mathcal{K}^2 + \Big|\nabla_T|\nabla u|\Big|^2$$

where

$$\mathcal{K}^2 = \sum_{i=1}^{n-1} \kappa_i^2$$

and

$$\nabla_T g = \nabla g - \langle \nabla g, \frac{\nabla u}{|\nabla u|} \rangle \frac{\nabla u}{|\nabla u|} \, .$$

Conclusion

We have

$$\int_{\mathbb{R}^n} \left(|\nabla u|^2 \mathcal{K}^2 + \left|\nabla_T |\nabla u|\right|^2\right) \varphi^2 \, d\mu \le \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu$$

for all $\varphi \in C_c^1(\mathbb{R}^n)$.

Taking $\varphi(x) = \Phi(|x|)$ with
$\Phi(t) = 1$ if $t \le R$,
$\Phi(t) = 0$ if $t \ge R+1$,
$|\Phi'(t)| \le 2$ for any t,
we get

$$\int_{|x|\le R} \left(|\nabla u|^2 \mathcal{K}^2 + \left|\nabla_T |\nabla u|\right|^2\right) d\mu \le 4 \int_{R\le|x|\le R+1} |\nabla u|^2 \, d\mu.$$

Recalling that

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu < +\infty$$

it follows that

$$|\nabla u|^2 \mathcal{K}^2 + \Big|\nabla_T |\nabla u|\Big|^2 = 0$$

which implies that u is one-dimensional.

Relation with Mean Curvature Flow

This result is related to the Bernstein problem in the Gauss space $(\mathbb{R}^n, \mu)$, which asks for flatness of entire minimal surfaces which are graphs in some direction.

Minimal surfaces in the Gauss space are interesting geometric objects, since they correspond to self-similar shrinkers of the mean curvature flow, and satisfy the equation

$$\kappa = \langle x, \nu \rangle$$

which can be seen as suitable limit of (OUE) .

The Bernstein problem in the Gauss space has been solved in [Ecker and Huisken, Ann. of Math. 1998] under a volume-growth condition, and in [Wang, Geom. Dedicata 2011] in the general case.

Thank you!