Evolution of viscous conducting drops subject to rotation and electric fields

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12th International Conference on Free Boundary Problems Theory and Applications
Germany 11 - 15 June 2012
**Mathematical Model**

Two cases: \( L = I\omega \)
- Electric field \( \mathcal{E}_\infty \) and constant \( L \).
- Charge \( Q \) and constant \( L \).

**Notation**

- \( D_1(t) \) Fluid drop domain
- \( \rho_1 \) Drop density
- \( \mu_1 \) Drop viscosity
- \( p^{(1)} \) Drop pressure
- \( u^{(1)} \) Drop velocity

- \( D_2(t) \) Outer fluid domain
- \( \rho_2 \) Outer fluid density
- \( \mu_2 \) Outer fluid viscosity
- \( p^{(2)} \) Outer fluid pressure
- \( u^{(2)} \) Outer fluid velocity

\( \omega \) Angular velocity
\( \partial D(t) \) Fluid interface
Mathematical Model

Stokes equation

\[
\begin{cases}
\mu_i \Delta u^{(i)} - \nabla \Pi^{(i)} = 0, & \text{in } D_i(t) \\
\nabla \cdot u^{(i)} = 0, & \text{in } D_i(t)
\end{cases}, \quad i \in \{1, 2\}
\]

Boundary condition

\[
(T^{(2)} - T^{(1)}) n = \left(2\gamma H - \rho \frac{L^2}{2T^2} r_{axis}^2 - \frac{\sigma^2}{2\varepsilon_0}\right) n, \quad \text{on } \partial D(t)
\]

Laplace equation

\[
\begin{cases}
\Delta \mathcal{V} = 0, & \text{in } D_2(t) \\
\mathcal{V} = \mathcal{V}_0, & \text{in } \partial D(t) \\
\mathcal{V} \to -\varepsilon_\infty z + O(|r|^{-1}), & \text{as } |r| \to \infty
\end{cases}
\]
Fredholm integral equation of the $2^{nd}$ kind for the velocity:

$$u_j(r') = -\frac{1}{4\pi (\mu_1 + \mu_2)} \int_{\partial \mathcal{D}(t)} f_i(r) G_{ij}(r, r') dS(r)$$

Single layer potential

$$-\frac{\mu_1 - \mu_2}{4\pi (\mu_1 + \mu_2)} \int_{\partial \mathcal{D}(t)} u_i(r) T_{ijk}(r, r') n_k(r) dS(r)$$

Double layer potential

$$G_{ij}(r, r') = \frac{\delta_{ij}}{|r - r'|} + \frac{(r_i - r'_i)(r_j - r'_j)}{|r - r'|^3}$$

Stokeslet

$$T_{ijk}(r, r') = -6\frac{(r_i - r'_i)(r_j - r'_j)(r_k - r'_k)}{|r - r'|^5}$$

Tenselet

$$f_i(r) = \left[ 2\gamma H(r) - \frac{L^2}{2\mathcal{L}^2} r_{axis}^2 - \frac{\sigma^2}{2\varepsilon_0} \right] n_i(r)$$

Traction

where $i, j, k \in \{1, 2, 3\}$ and $r, r' \in \partial \mathcal{D}(t)$. 
Problem
Numerical Method
Results
Conclusions & Future Research Lines

Numerical algorithm

# Algorithm

1. Compute the volume and moment of inertia about the $z$-axis.
2. Calculate mean curvature and charge density of drop.
3. Solve linear system to obtain velocity field at the boundary.
4. Move the boundary with an Euler explicit scheme:
   \[ \mathbf{r}(t_{n+1}) = \mathbf{r}(t_n) + \mathbf{u}(t_n) \Delta t. \]
5. Regularization of the mesh (if necessary):
   - Delaunay remeshing.
   - Mesh relaxation.
   - Mesh refinement.

Repeat above steps until $t_{\text{max}}$ is reached.
Charge and rotation at constant $L$

**Asymptotic expansion:** $L^2 \ll 1$

**Young-Laplace equation**

\[
\delta p = 2\gamma H - \varrho \frac{L^2}{2L^2} r^2_{axis} - \frac{\sigma^2}{2\varepsilon_0}, \quad \text{on } \partial D
\]

**Spheroidal approximation:**

**Energy formulation**

Minimize: $E_{total} = E_{area} + E_{kinetic} + E_{electrostatic}$, $V = 1$
Charge and rotation at constant $L$

![Graph](image)

Fissibility ratio vs. Deformation ($L = 0.1$)

Legend:
- Blue: Numerical Equilibrium Solutions
- Black: Prolate Model Solutions
- Green: Oblate Model Solutions
- Pink: Approx. Equilibrium Solutions (Prolate)
- Red: Approx. Equilibrium Solutions (Oblate)
Charge and rotation at constant $L$

Equilibrium shapes for $L = 0.1$

$\chi = 0.892$
Charge and rotation at constant $L$

Stability analysis
Electric field and rotation at constant $L$

\[ E_{\infty}^2 = h(\alpha) + g(\alpha) L^2 \]
Electric field and rotation at constant $L$

Stability curve for axisymmetric shapes

$y = 0.70458x^2 + 0.089804x + 0.57104$

Numerical results
Quadratic fitting
Taylor cones

Centrifugal force vs. Surface tension \((L = 0.2, E_\infty = 0.9)\)
Self-similarity

\[(z - z_{tip})\kappa_{tip} \]

\[r\kappa_{tip} \]
Taylor cones

Semi-angle convergence ($E_\infty = 0.9$)

$\beta^* \approx 0.535$
Conclusions & Future Research Lines

Conclusions

- Adaptive BEM to simulate droplet evolution.
- Theoretical models to approximate charged rotating drops.
- Stability analysis shows ellipsoidal configurations and singularities (Taylor cones and two-lobed drop breakup).
- Linear relationship for shapes of rotating drops subject to uniform electric fields with same aspect ratio.
- Taylor cone semiangle is not affected by small rotations.

Research Lines

- Evolution of charged rotating drops subject to electric fields.
- Describe the space of parameters \((\chi, E_\infty, L)\).
- Understand the role of rotation on the stability of the system.
Thank you for your attention.

Questions?
Mean curvature

**Algorithm 1** Paraboloid fitting

1: Take $n_p$ initial approximation to the outward unit normal to $\partial D$ at $p$.
2: repeat
3: Choose local coordinates $(x', y', z')$ at $p$ and $z'$-axis along $n_p$.
4: Find $(x'_i, y'_i, z'_i)$ coordinates of the adjacent nodes to $p$.
5: Minimize:

$$F = \sum_{i=1}^{N_p} \left( Ax'_i + By'_i + C(x'_i)^2 + Dx'_i y'_i + E(y'_i)^2 - z'_i \right)^2 .$$

6: $(n_p)_n \leftarrow (-A, -B, 1)/(1 + A + B)^{1/2}$.
7: until $|(n_p)_n - n_p| < \varepsilon$.
8: Mean curvature $k_p = \frac{(1 + B^2)C - ABD + (1 + A^2)E}{(1 + A^2 + B^2)^{3/2}}$. 

Fast convergence iterative method
Surface charge density

Write:

\[ \sigma = \mathcal{V}_0 \sigma_0 + \sigma_{ind} \]

Solve:

\[ \mathcal{E}_\infty z_0 = \frac{1}{4\pi \varepsilon_0} \int_{\partial D(t)} \frac{\sigma_{ind}(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} dS \quad , \quad 1 = \frac{1}{4\pi \varepsilon_0} \int_{\partial D(t)} \frac{\sigma_0(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} dS \]

with:

\[ Q = \mathcal{V}_0 \int_{\partial D(t)} \sigma_0(\mathbf{x}) dS + \int_{\partial D(t)} \sigma_{ind}(\mathbf{x}) dS \]

Linear system at barycenters \( \mathbf{x}_l, \mathbf{x}_j \) of triangles:

\[ \int_{\partial D(t)} \frac{\sigma(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_i\|} dS \approx \sum_{j=1}^{M} \lambda_{ij} \sigma_j \quad , \quad \lambda_{ij} = \int_{T_j} \frac{dS}{\|\mathbf{x} - \mathbf{x}_i\|} \]

\[ \sigma_j = \sigma(\mathbf{x}_j) \]
Surface charge density

- Case $i = j$
  
  $$\lambda_{ii} = \sum_{k=1}^{6} \int_{T_{ik}} \int d\rho \, d\theta = \sum_{k=1}^{6} a_k \ln (\sec (\alpha_k) + \tan (\alpha_k))$$

- Case $i \neq j$
  
  $$\lambda_{ij} = \sum_{k=1}^{N_s} \lambda_{ij,k}, \quad \lambda_{ij,k} = \frac{\text{Area} (T_{jk})}{\| b_{jk} - x_i \|}$$