ALGEBRAIC TOPOLOGY I - IV.5

STEFAN FRIEDL

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1. Topological spaces

In this chapter we give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties and constructions. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

1.1. **The definition of a topological space.**

In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are the lemmas from this Section 1.1. Usually we use them without referring to them explicitly.

We start out with a basic definition of set theory.

Definition. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. We write

$$\begin{aligned} \bigcup_{i\in I} U_i &= \{x \in X \mid \text{ there exists an } i \in I \text{ with } x \in X_i\} \\ \bigcap_{i\in I} U_i &= \{x \in X \mid \text{ for every } i \in I \text{ we have } x \in X_i\}. \end{aligned}$$

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.[1]

We continue with two elementary lemmas from set theory. Even though they are elementary it is helpful to have them on record.

Lemma 1.1. *Let $f\colon X \to Y$ be a map between two sets X and Y.*[2] *Then*

$$\begin{array}{lrcl}
(1) & \textit{for every } U \subset X \textit{ we have} & U & \subset & f^{-1}(f(U)) \\
 & \textit{furthermore, if } f \textit{ is injective, then we have} & U & = & f^{-1}(f(U)) \\
(2) & \textit{for every } V \subset Y \textit{ we have} & V & \supset & f(f^{-1}(V)) \\
 & \textit{furthermore, if } f \textit{ is surjective, then we have} & V & = & f(f^{-1}(V)) \\
(3) & \textit{for every family } \{U_i\}_{i\in I} \textit{ of subsets of } X \textit{ we have} & f\Big(\bigcup_{i\in I} U_i\Big) & = & \bigcup_{i\in I} f(U_i) \\
(4) & \textit{for every family } \{U_i\}_{i\in I} \textit{ of subsets of } X \textit{ we have} & f\Big(\bigcap_{i\in I} U_i\Big) & \subset & \bigcap_{i\in I} f(U_i) \\
(5) & \textit{for every family } \{V_i\}_{i\in I} \textit{ of subsets of } Y \textit{ we have} & f^{-1}\Big(\bigcup_{i\in I} V_i\Big) & = & \bigcup_{i\in I} f^{-1}(V_i) \\
(6) & \textit{for every family } \{V_i\}_{i\in I} \textit{ of subsets of } Y \textit{ we have} & f^{-1}\Big(\bigcap_{i\in I} V_i\Big) & = & \bigcap_{i\in I} f^{-1}(V_i) \\
(7) & \textit{for every } V \subset Y \textit{ we have} & f^{-1}(Y \setminus V) & = & X \setminus f^{-1}(V).
\end{array}$$

[1] This simple observation will be useful at times.

[2] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicholas Bourbaki's "Théorie des ensembles" first published in 1954, see [Bou2].

Lemma 1.2. (De Morgan's laws) *Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $Y \subset X$ we denote its complement by Y^c, then*

$$\Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c \quad \textit{and} \quad \Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \Big(\bigcap_{i\in I} A_i\Big) = \bigcup_{i\in I}(X \setminus A_i) \quad \textit{and} \quad X \setminus \Big(\bigcup_{i\in I} A_i\Big) = \bigcap_{i\in I}(X \setminus A_i).$$

We continue with the following notation.

Notation.

(1) We denote by $\varnothing$ the empty set.
(2) Given a set X we denote by $\mathcal{P}(X)$ its *power set*, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

The following definition will keep us occupied for the rest of these notes.

Definition.

(1) Let X be a set. A *topology on X* is a subset $\mathcal{T}$ of $\mathcal{P}(X)$[3] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

   The sets in $\mathcal{T}$ are called *open with respect to $\mathcal{T}$*. If $\mathcal{T}$ is understood from the context, then we just say *open*.
(2) A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a *topology on X*.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".
(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open sets is again open,
   (c) the union of *arbitrarily many* open sets is again open.

We start out with some very basic examples of topological spaces.

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the *trivial topology on X*.
(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the *discrete topology on X*. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.

[3] In more humble words, $\mathcal{T}$ is a set of subsets of X.

(3) Let $X = \mathbb{R}$ and let $\mathcal{T}$ be defined as follows:

$$U \in \mathcal{T} \;:\Longleftrightarrow\; \text{either } U = \varnothing \text{ or } U \text{ is the complement of finitely many points in } \mathbb{R}.$$

For example the sets $\varnothing, \mathbb{R}\setminus\{\pi\}, \mathbb{R}\setminus\{-1, \sqrt{2}\}$ and also $\mathbb{R}$ (since it is the complement of zero points) lie in $\mathcal{T}$. It follows easily from de Morgan's laws that $\mathcal{T}$ is a topology on $X = \mathbb{R}$.

Surely some of the most interesting examples of topological spaces arise from metric spaces. For completeness' sake we recall the definition of metric spaces.

Definition. A *metric space* is a pair (X, d) consisting of a set X and a *metric* d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\}, \\ (x, y) & \mapsto & d(x, y), \end{array}$$

with the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) for all $x, y \in X$ we have

$$d(x, y) \;=\; d(y, x) \qquad\qquad \text{(symmetry)}$$

(3) for all $x, y, z \in X$ we have

$$d(x, z) \;\leq\; d(x, y) + d(y, z) \qquad \text{(triangle inequality)}.$$

Sometimes we refer to $d(x, y)$ as the *distance between x and y*.

Example. In the following we consider $\mathbb{R}^n$ as a metric space with the *Euclidean metric* given by

$$d(x, y) \;:=\; \|x - y\| \quad \text{where} \quad \|z\| \;=\; \|(z_1, \ldots, z_n)\| \;:=\; \sqrt{z_1^2 + \cdots + z_n^2}\,.$$

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 1.3. *Let (X, d) be a metric space. A subset U of X is called* open *if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) := \{y \in X \,|\, d(x, y) < \epsilon\}$ is contained in U. These open sets define a topology on X.*

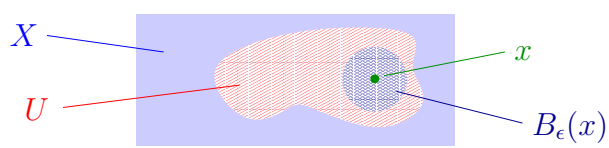

FIGURE 1. Illustration of Lemma 1.3.

Proof.

(1) It is clear that the empty set and X itself are open.

(2) Suppose $U_1, \dots, U_k$ are open subsets of X. We need to show that the intersection $U_1 \cap U_2 \cap \dots \cap U_k$ is also open. Thus let $x \in U_1 \cap U_2 \cap \dots \cap U_k$. For each $i \in \{1, \dots, k\}$ there exists an $\epsilon_i > 0$ with $B_{\epsilon_i}(x) \subset U_i$. We set $\epsilon := \min\{\epsilon_1, \dots, \epsilon_k\}$. Then

$$\begin{array}{ccccc} B_\epsilon(x) & \underset{\substack{\uparrow \\ \text{since } \epsilon \le \epsilon_i}}{\subset} & B_{\epsilon_1}(x) \cap \dots \cap B_{\epsilon_k}(x) & \underset{\substack{\uparrow \\ \text{since } B_{\epsilon_i}(x) \subset U_i}}{\subset} & U_1 \cap U_2 \cap \dots \cap U_k. \end{array}$$

(3) Let $\{U_i\}_{i \in I}$ be a family of open sets. We want to show that $\bigcup_{i \in I} U_i$ is also open. Thus let $x \in \bigcup_{i \in I} U_i$. This means that there exists an $i \in I$ with $x \in U_i$. Since U_i is open there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset U_i$. Thus we have

$$B_\epsilon(x) \ \subset \ U_i \ \subset \ \bigcup_{i \in I} U_i.$$

□

Convention. Given a metric space (X, d) we always view X as a topological space with the topology defined in Lemma 1.3. In particular, unless we say explicitly otherwise, we view $\mathbb{R}^n$ with the topology arising from the Euclidean metric.

We proceed with the next definition.

Definition. Let X be a topological space. We say a subset A of X is *closed* if the complement $X \setminus A$ is open.

The following lemma is an immediate consequence of the definition of a topology and de Morgan's laws, see Lemma 1.2. The lemma mirrors the remark on page 42.

Lemma 1.4. *Let X be a topological space.*

(1) *the empty set and the entire set X are closed,*
(2) *the intersection of* arbitrarily many *closed sets is again closed,*
(3) *the union of* finitely many *closed sets is again closed.*

Remark. Let X be a set. If we specify what we mean by closed sets such that the three properties of Lemma 1.4 are satisfied, then it follows once again from de Morgan's laws that the complements of the closed sets define a topology.

The following proposition gives us important examples of closed and open subsets.

Proposition 1.5. *Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$. Then the following statements hold:*

(1) *the set $B_r(y) = \{x \in X \mid d(x, y) < r\}$ is open in X,*
(2) *the set $\overline{B}_r(y) = \{x \in X \mid d(x, y) \le r\}$ is closed in X,*
(3) *the set $S_r(y) = \{x \in X \mid d(x, y) = r\}$ is closed in X.*

Proof. Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.

(1) Let $x \in B_r(y)$. We set $s := d(x,y)$. By definition of $B_r(y)$ we have $s < r$. We set $t := r - s$. Since $t > 0$ it suffices to show that $B_t(x) \subset B_r(y)$. Thus let $z \in B_t(x)$. We see that

$$d(y,z) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} d(x,y) + d(x,z) \ < \ s + t \ = \ r.$$

Thus we see that $z \in B_r(y)$, which is what we needed to show. We refer to Figure 2 for an illustration.

(2) Almost the same argument as in (1) shows that $X \setminus \overline{B}_r(y)$ is open, which implies that $\overline{B}_r(y)$ is closed.

(3) We have $X \setminus S_r(y) = B_r(y) \cup (X \setminus \overline{B}_r(y))$. By (1) and (2) we see that $X \setminus S_r(y)$ is the union of two open sets, thus by the third property of a topological space we see that $X \setminus S_r(y)$ is open, thus $S_r(y)$ is indeed closed. □

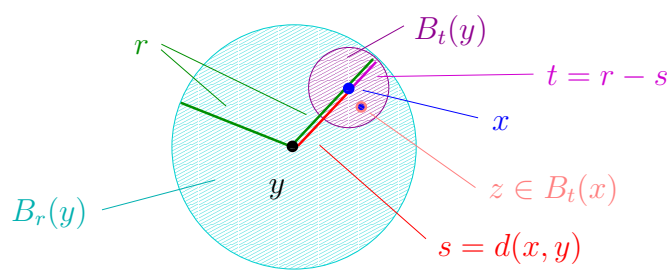

FIGURE 2.

The most frequent application of Proposition 1.5 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation.

Notation. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write

$$\begin{array}{rcll} B_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| < r\} & \text{called } \textit{open } n\textit{-ball} \\ \overline{B}_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| \leq r\} & \text{called } \textit{closed } n\textit{-ball} \\ S_r^{n-1}(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| = r\} & \text{called } (n-1)\textit{-sphere}. \end{array}$$

We adopt the following conventions:

(1) Not surprisingly a 2-ball is often called a *disk* and a 1-sphere is usually referred to as a circle.
(2) If the parameter n is clear from the context we sometimes drop it from the notation.
(3) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(4) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.

We sketch B^1, B^2, S^1 and S^2 in Figure 3. Of course one cannot easily sketch the 3-sphere S^3. But it is worth pointing out that in some form it has been around for a while. In [Lip, Chapter 2] and [Pet] the argument is made that the universe in Dante's Divina Commedia is precisely the 3-sphere.

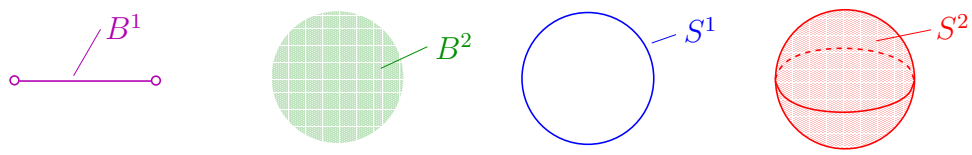

FIGURE 3.

The next definition gives us a rich source of new examples of topological spaces.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset. It follows immediately from the definitions that

$$\mathcal{S} \ := \ \{Y \cap U \,|\, U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the *subspace topology* on Y.

Convention. Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

We continue with the following useful lemma.

Lemma 1.6. *Let X be a topological space and let $Y \subset X$ be a subset. We equip Y with the subspace topology.*

(1) *A set $V \subset Y$ is open in Y if and only if there exists an open subset $U \subset X$ with $V = U \cap Y$.*
(2) *A set $B \subset Y$ is closed in Y if and only if there exists a closed subset $A \subset X$ with $B = A \cap Y$.*

Proof $(*)$. [4] The first statement is just a reformulation of the definition of the subspace topology. We turn to the proof of the second statement. Now let $B \subset Y$. Elementary set theory shows that for any subset U of X we have

$$(*) \qquad U \cap Y \ = \ Y \setminus B \quad \Longleftrightarrow \quad (X \setminus U) \cap Y \ = \ B.$$

We refer to Figure 4 for an illustration. Now the second statement is basically obvious. Indeed, we have

$$\begin{aligned} B \text{ is closed in } Y \ &\Longleftrightarrow \ Y \setminus B \text{ is open in } Y \\ &\Longleftrightarrow \ \text{there exists } U \subset Y \text{ open with } U \cap Y = Y \setminus B \\ &\underset{\text{by } (*)}{\Longleftrightarrow} \ \text{there exists } U \subset Y \text{ open with } (X \setminus U) \cap Y = B \\ &\Longleftrightarrow \ \text{there exists } A \subset X \text{ closed with } A \cap Y = B. \end{aligned}$$

[4] We indicate proofs and statements that can safely be skipped at a first reading of these notes by the symbol $(*)$.

□

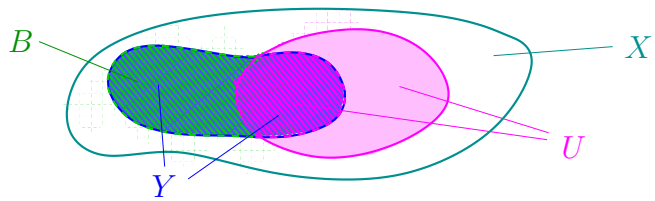

FIGURE 4. Illustration for the proof of Lemma 1.6.

Example. We consider $Y = (-1, 2]$ equipped with the subspace topology coming from $\mathbb{R}$.

(1) It follows from Lemma 1.6 that $V = (0, 2]$ is an open subset of Y, since we can write $V = (0, 4) \cap Y$ and $(0, 4)$ is an open subset of $\mathbb{R}$.
(2) It follows from Lemma 1.6 that $B = (-1, 0]$ is a closed subset of Y, since we can write $B = [-3, 0] \cap Y$ and $[-3, 0]$ is a closed subset of $\mathbb{R}$.

These two examples are illustrated in Figure 5.

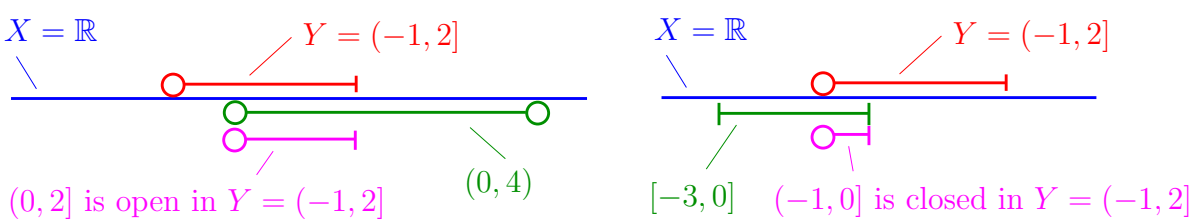

FIGURE 5.

The following lemma is crucial for many arguments.

Lemma 1.7. *Let X be a topological space and let M be a subset of X.*

(1) *Let $\{U_i\}_{i \in I}$ be a family of open subsets X with $\bigcup_{i \in I} U_i = X$. The following two statements hold:*
   (a) *M is open if and only if each intersection $M \cap U_i$ is open in U_i,*
   (b) *M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.*
(2) *Let $A_1, \ldots, A_m$ be a finite collection of closed subsets of X with $A_1 \cup \cdots \cup A_m = X$. The following two statements hold:*
   (a) *M is open if and only if each intersection $M \cap A_i$ is open in A_i,*
   (b) *M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.*

Remark. Note that in Lemma 1.7 we allow an arbitrary family of open subsets but we only allow a finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X = \mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X = \mathbb{R}$ given by all subsets consisting of a single point. Let $M \subset \mathbb{R}$ be a subset. For each $i \in I$ the intersection $M \cap \{A_i\}$ is either the empty set or all of A_i, in particular the intersection $M \cap \{A_i\}$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$.

Proof. Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be a family of open subsets of X with $\bigcup_{i\in I} U_i = X$. The "only if"-directions of (a) and (b) are an immediate consequence of Lemma 1.6. It remains to prove the "if"-directions.

(a) Suppose that for every $i \in I$ the intersection $M \cap U_i$ is an open subset of U_i. By definition of the subspace topology this means that for each $i \in I$ there exists an open subset Z_i of X with $Z_i \cap U_i = M_i \cap U_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } \bigcup_{i\in I} U_i = X}}{=} \bigcup_{i\in I}(M \cap U_i) = \bigcup_{i\in I} \overbrace{(Z_i \cap U_i)}^{\text{intersection of open subsets of } X\text{, hence open in } X} \underset{\substack{\uparrow \\ \text{union of open subsets, hence open}}}{=} \text{open subset of } X.$$

(b) Now suppose that each intersection $M \cap U_i$ is closed in U_i. In other words, each set $U_i \setminus (M \cap U_i)$ is open in U_i. Since $(X \setminus M) \cap U_i = U_i \setminus (M \cap U_i)$ we obtain from (a) that $X \setminus M$ is open in X. In other words, M is closed in X.

(2) Let $A_1, \dots, A_m$ be a finite collection of closed subsets of X with $A_1 \cup \dots \cup A_m = X$. As in (1) it remains to prove the "if"-directions.

(b) We suppose that each for each $i \in \{1, \dots, m\}$ the intersection $M \cap A_i$ is a closed subset of A_i. It follows from Lemma 1.6 that for each $i \in \{1, \dots, m\}$ there exists a closed subset Z_i of X with $Z_i \cap A_i = M_i \cap A_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } A_1 \cup \dots \cup A_m = X}}{=} \bigcup_{i=1}^{m}(M \cap A_i) = \bigcup_{i=1}^{m} \overbrace{(Z_i \cap A_i)}^{\text{intersection of closed subsets of } X\text{, hence closed in } X} \underset{\substack{\uparrow \\ \text{union of finitely many closed subsets,} \\ \text{hence closed}}}{=} \text{closed subset of } X$$

(a) As in (1) we can reduce the proof of (a) to the proof of (b). □

We continue with the next basic definition.

Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say $U \subset X$ is a *neighborhood of* A if there exists an open set V such that $A \subset V \subset U$. We say U is an *open neighborhood of* A, if U is furthermore open. This definition is illustrated in Figure 6.

(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.

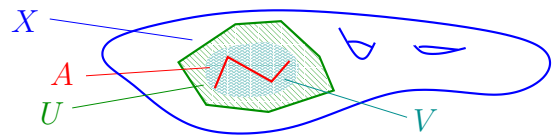

FIGURE 6.

Remark. Some books, see e.g. [Mun3, p. 96], demand that a neighborhood is open. Our definition of a neighborhood agrees with the definition in most books, e.g. [Bre, Hat, Jä1].

Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X.

The following lemma gives a useful criterion for showing that a given subset is open.

Lemma 1.8. *Let X be a topological space and let $U \subset X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.*

Proof $(*)$. Let X be a topological space and let $U \subset X$ be a subset. Suppose that given any $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \underset{\substack{\uparrow \\ \text{since each } W_x \text{ is contained in } U}}{\subset} U.$$

It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. $\square$

We continue with the following definition which will show up every now and then.

Definition. Let X be a topological space. We say a subset A of X is *dense* if every non-empty open subset of X has a non-empty intersection with A.

Examples.

(1) Let $n \in \mathbb{N}$. It follows from basic facts in real analysis that $A = \mathbb{Q}^n$ is a dense subset of $X = \mathbb{R}^n$.
(2) Here is an immediate, perhaps not entirely serious application of topology to the eternal problem for how to capture a lion in the desert [Bar]: We give the desert the *leonine topology*, in which a subset is closed if it is the whole desert or if it contains no lions.[5] The set of lions is now a dense subset.[6] Now put an open cage in the

[5]Why is this a topology?
[6]Why is that?

desert. By density it contains a lion. Once we shut the cage we have caught the lion![7]

After this joke of dubious taste we return to serious mathematics.

Lemma 1.9. *Every open subset of $\mathbb{R}^n$ is the union of countably many open balls.*

Proof $(*)$. Let U be an open subset of $\mathbb{R}^n$. For $x \in U$ we define

$$\epsilon_x \ := \ \sup\{\mu \in [0,1] \,|\, B_\mu(x) \subset U\}.$$

The hypothesis that U is open implies that $\epsilon_x > 0$ and it implies that $B_{\epsilon_x}(x) \subset U$. Since $U \cap \mathbb{Q}^n$ is a countable set it suffices to prove the following claim.

Claim. The subset U is the union of all $B_{\epsilon_x}(x)$ with $x \in U \cap \mathbb{Q}^n$.

Let $y \in U$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can pick $x \in U \cap \mathbb{Q}^n$ with $\|x - y\| < \frac{1}{2}\epsilon_y$. Note that this implies that $y \in B_{\frac{1}{2}\epsilon_y}(x)$. Thus it remains to prove the following claim.

Claim. We have $B_{\frac{1}{2}\epsilon_y}(x) \subset B_{\epsilon_x}(x)$.

Recall that we pointed out above that $B_{\epsilon_y}(y) \subset U$. Since $\|x - y\| < \frac{1}{2}\epsilon_y$ we have by the triangle inequality that $B_{\frac{1}{2}\epsilon_y}(x) \subset B_{\epsilon_y}(y)$. Thus $B_{\frac{1}{2}\epsilon_y}(x) \subset U$. By definition of ϵ_x this implies that $B_{\frac{1}{2}\epsilon_y}(x) \subset B_{\epsilon_x}(x)$. This concludes the proof of the claim. We refer to Figure 7 for an illustration. □

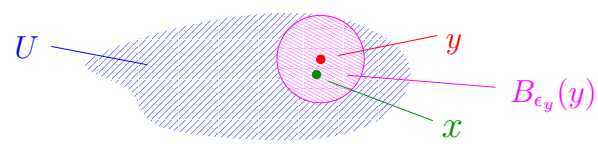

FIGURE 7. Illustration for the proof of Lemma 1.9.

We move on to the next definition.

Definition. Let X be a topological space and let A be a subset of X.

(1) The *interior* $\mathring{A}$ is defined as the union of all open sets of X that are contained in A.
(2) The *closure* $\overline{A}$ of A is defined as the intersection of all closed sets in X that contain A.
(3) The *boundary of A in X* is defined as $\partial A := \overline{A} \setminus \mathring{A}$.

We will give examples in a second, but first we record a few observations that follow almost immediately from the definitions.

Lemma 1.10. *Let X be a topological space and let A be a subset of X.*

(1) *The interior $\mathring{A}$ is an open subset of X.*

[7]This and many other mathematical approaches to hunting lions can be found in [Boa].

(2) *The closure* $\overline{A}$ *of* A *is a closed subset of* X.
(3) *We have* $\mathring{A} \subset A \subset \overline{A}$.
(4) *If* A *is open, then* $\mathring{A} = A$.
(5) *If* A *is closed, then* $\overline{A} = A$.

Proof.
(1) The union of arbitrarily many open sets is, by definition of a topology, again an open set. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In Lemma 1.4 we saw that the intersection of arbitrarily many closed sets is again a closed set. It follows immediately that the closure $\overline{A}$ is a closed subset of X.

The remaining three statements follow immediately from the definitions. □

The above definition is succinct and precise, but using the definition it is perhaps difficult to find the interior and the closure of a given subset of a topological space. In many situations the following lemma takes care of that difficulty.

Lemma 1.11. *Let* X *be a topological space and let* A *be a subset of* X. *We have the following three equalities:*

$$\begin{array}{lrcl}
(1) & \mathring{A} & = & \{x \in A \mid \textit{there exists a neighborhood } U \textit{ of } x \textit{ that is contained in } A\} \\
(2) & \overline{A} & = & \{x \in X \mid \textit{every neighborhood of } x \textit{ contains at least one point in } A\} \\
(3) & \partial A & = & \{x \in X \mid \textit{every neighborhood of } x \textit{ contains at least one point in } A \\
 & & & \qquad\qquad \textit{and one point that does not lie in } A\}
\end{array}$$

Proof. The proof of the lemma is Exercise 1.3. □

Using Lemma 1.11 we can give examples of interiors, closures and boundaries.

Examples.
(1) We consider the topological space $X = \mathbb{R}$ and the subset $A = [-1, 2)$. Using Lemma 1.11 one can easily see that the interior of A is the open interval $(-1, 2)$ and that the closure of A is the closed interval $[-1, 2]$. Furthermore $\partial A = \{-1, 2\}$.
(2) Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We view $B^n_r(x)$, $\overline{B}^n_r(x)$ and $S^{n-1}_r(x)$ as subsets of $\mathbb{R}^n$. Using Lemma 1.11 one can show fairly easily that
$$\overline{B^n_r(x)} \;=\; \overline{B}^n_r(x) \quad \text{and} \quad \partial\overline{B}^n_r(x) \;=\; \partial B^n_r(x) \;=\; \partial S^{n-1}_r(x) \;=\; S^{n-1}_r(x).$$
(3) In Figure 8 we show a subset of $\mathbb{R}^2$, and we also show its interior $\mathring{A}$, its closure $\overline{A}$ and its boundary ∂A.
(4) It is rather unfortunate that the notations $\mathring{A}$, $\overline{A}$ and ∂A hide the role of X. For example let us consider $A = [0, 2]$. If we view it as a subset of $X = \mathbb{R}$, then $\partial A = \partial[0, 2] = \{0, 2\}$. On the other hand, if we view $A = [0, 2]$ as a subset of $X = \mathbb{R}_{\geq 0}$, then $\partial A = \partial[0, 2] = \{2\}$. This ambiguity in the notation can sometimes be a source of confusion. Thus it would be more appropriate to specify the role of X in the above notation. But mathematicians, as all humans, tend to be lazy and thus tend to simplify the notation.

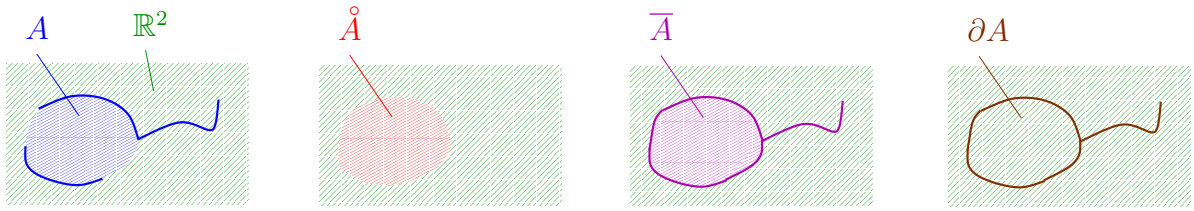

FIGURE 8.

We conclude this section with the following technical lemma.

Lemma 1.12. $(*)$ *Let X be a topological space and let $\{U_i\}_{i\in I}$ be a family of subsets. The following inclusions hold:*

$$(1) \quad \text{interior of } \bigcup_{i\in I} U_i \;\supseteq\; \bigcup_{i\in I} \mathring{U}_i \qquad \text{and} \qquad (2) \quad \text{closure of } \bigcap_{i\in I} U_i \;\subset\; \bigcap_{i\in I} \overline{U}_i.$$

Furthermore, if we are given finitely many subsets $U_1, \dots, U_n$ of X, then the following equalities hold:

$$(3) \quad \text{interior of } \bigcap_{i=1}^{n} U_i \;=\; \bigcap_{i=1}^{n} \mathring{U}_i \qquad \text{and} \qquad (4) \quad \text{closure of } \bigcup_{i=1}^{n} U_i \;=\; \bigcup_{i=1}^{n} \overline{U}_i.$$

Example. It is easy to come up with examples of two subsets U_1, U_2 in $\mathbb{R}^n$ such that inclusions in Lemma 1.12 (1) and (2) are actually proper inclusions. Some examples are illustrated in Figure 9.

FIGURE 9.

Proof. The statements follow fairly easily from the definitions. We leave it to the reader to verify the statements. □

1.2. **The Hausdorff property and convergence of sequences.** Before we introduce the Hausdorff property we introduce two more examples of topological spaces. Both are of the type "$\mathbb{R}^n$ together with an extra point".

Examples.

(1) Let $n \in \mathbb{N}$. We consider the set

$$X \;:=\; \mathbb{R}^n \cup \{\infty\},$$

i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say $U \subset X$ is open, if both of the following two conditions are satisfied:

(a) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,

(b) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.

It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as "$\mathbb{R}^n$ with a point at infinity". For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the "line with a point at infinity".

(2) We consider the set

$$X \;:=\; \mathbb{R} \cup \{*\},$$

i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:

(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,

(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.

One can easily verify that this is indeed a topology on X. We refer to this topological space as the "line with two zeros".

Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity" we will soon see that they are strikingly different topological spaces.

Now we provide the first definition of this section.

Definition. We say a topological space X is *Hausdorff*, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.

The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 1.13. *Let (X, d) be a metric space. The corresponding topological space is Hausdorff.*

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from Proposition 1.5 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. $\square$

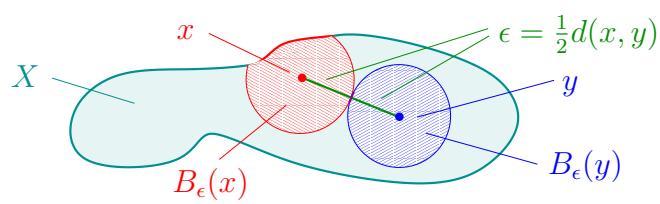

FIGURE 10.

Example. In Exercise 1.8 we will see that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that on the other hand "the line with two zeros" is not Hausdorff.

The following lemma is basically trivial.

Lemma 1.14. *Let X be a topological space and let A be a subset of X. If X is Hausdorff, then A is also Hausdorff.*

Next we introduce the notion of a convergent sequence.

Definition. Let X be a topological space. We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ *converges* if there exists an $a \in X$ such that for any neighborhood U of a there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \in U$. We refer to a as a *limit* of the sequence.

Example. We consider the line with two zeros, i.e. the topological space $X = \mathbb{R} \cup \{*\}$ defined above. The sequence $a_n = \frac{1}{n}$ converges, more or less evidently with limit 0. But strikingly enough, it follows easily from the definitions that the sequence also converges to $*$. Thus we see that the limit is not unique.

The following lemma is most likely already familiar to the reader from real analysis.

Lemma 1.15. *Let X be a topological space. If X is Hausdorff, then every convergent sequence in X has a unique limit.*

Proof. Assume there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X that converges to two different points x and y. Since X is Hausdorff there exist two disjoint neighborhoods U and V of x and y. But by our hypothesis, for large enough n the a_n have to lie in U and V at the same time. But that is not possible and we have thus obtained a contradiction. □

1.3. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition. Let X be a topological space.

(1) An *open covering* of a subset $A \subset X$ is a family $\{U_i\}_{i\in I}$ of open subsets of X with $A \subset \bigcup_{i\in I} U_i$.
(2) We say a topological space X is *compact* if for each open covering $\{U_i\}_{i\in I}$ of X there exist finitely many indices $i_1, \dots, i_k \in I$ such that
$$X = U_{i_1} \cup \cdots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact topological space by Pavel Alexandrov and Pavel Urysohn [AU] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of compactness in topology.

Example. Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.

In the following we will often use the following convention.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact). We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Examples.

(1) The set $\mathbb{Z} \subset \mathbb{R}$ is easily seen to be a discrete subset.
(2) Any finite set of a topological space is, by the above, a compact subset.

Frequently and often subconsciously the following elementary lemma is employed.

Lemma 1.16. *Let X be a topological space, let $A \subset X$ be a subset and let $\{U_i\}_{i \in I}$ be an open covering of A. If A is compact, then there exist finitely many indices $i_1, \ldots, i_k \in I$ such that*

$$A \subset U_{i_1} \cup \cdots \cup U_{i_k}.$$

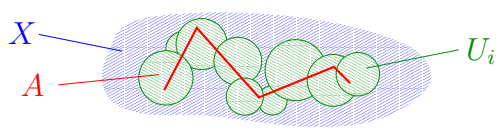

FIGURE 11. Illustration of Lemma 1.16.

Proof $(*)$. Given $i \in I$ we set $V_i := U_i \cap A$. By definition of the subspace topology each V_i is an open subset of A. Thus $\{V_i\}_{i \in I}$ is an open covering of A. Since A is compact there exist $i_1, \ldots, i_k \in I$ with $A = V_{i_1} \cup \cdots \cup V_{i_k}$. But this implies that $A \subset U_{i_1} \cup \cdots \cup U_{i_k}$. □

We continue with the following frequently employed lemma.

Lemma 1.17. *Let X be a topological space. If $K_1, \ldots, K_m$ are finitely many compact subsets of X, then the union $K_1 \cup \cdots \cup K_m$ is also compact.*

Proof. The statement follows easily from the definitions, we leave it to the reader to fill in the details. □

Before we give some more interesting examples of compact topological spaces we want to discuss a few properties of compact topological spaces and compact subsets. The following lemma is frequently used, often subconsciously, in topology.

Lemma 1.18.

(1) *Let X be a topological space. If X is compact, then any closed subset of A is also compact.*
(2) *Let X be a topological space and let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.*

Example. Let X be a topological space. As we pointed out on page 54, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from Lemma 1.18 that any finite subset of X, in particular any point in X, is a closed subset of X.

Proof.

(1) Let X be a compact topological space and let A be a closed subset. Let $\{U_i\}_{i\in I}$ be an open covering of A. By definition of the subset topology there exists for every $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is closed the complement $X \setminus A$ is open. The open subsets $\{V_i\}_{i\in I}$ together with $X \setminus A$ are an open covering of X. Since X is compact we can cover X by finitely many of these open sets. But then we can also cover A by finitely many of the U_i.

(2) Let X be a Hausdorff space and let $A \subset X$ be a compact subset. We want to show that $X \setminus A$ is open. By Lemma 1.8 it suffices to prove the following claim.

Claim. Let $x \in X \setminus A$. Then there exists an open neighborhood V of x that is contained in $X \setminus A$.

We apply the Hausdorff property to x and every $y \in A$. For every $y \in A$ we obtain disjoint open neighborhoods U_y of y and V_y of x. Evidently we have

$$A \;=\; \bigcup_{y\in A} \{y\} \;\subset\; \bigcup_{y\in A} (U_y \cap A) \;\subset\; A.$$

Thus we see that $\{U_y \cap A\}_{y\in A}$ is an open covering of A. Since A is compact there exist $y_1, \dots, y_k \in A$ such that

$$A \;=\; \bigcup_{i=1}^{k} (U_{y_i} \cap A).$$

Now we consider

$$V \;:=\; \bigcap_{i=1}^{k} V_{y_i}.$$

Since V is the intersection of finitely many open sets, it is open itself. Furthermore V does not intersect any of the U_{y_i}, $i = 1, \dots, k$. Hence V is disjoint from $A \subset U_{y_1} \cup \dots \cup U_{y_k}$. This concludes the proof of the claim. □

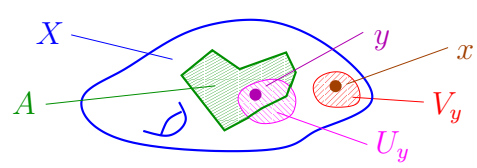

FIGURE 12. Illustration for the proof of Lemma 1.18 (2).

Before we move on to giving more interesting examples of compact topological spaces we state and prove the following elementary lemma.

Lemma 1.19.

(1) *Every discrete topological space that is compact is finite.*

(2) *Let X be a compact topological space and let A be a closed subset. If A is discrete, then A is finite.*

Proof $(*)$.

(1) Let X be a discrete topological space. Each point thus defines an open set. Evidently X is covered by all the open sets given by the points in X. Since X is compact we can cover X by finitely many of those open sets. But this means that X contains only finitely many points.
(2) Let X be a compact topological space and let A be a closed subset. It follows from Lemma 1.18 (1) that A is also compact. So if A is discrete, then it follows from (1) that A is finite. □

We continue with the following self-explanatory definition.

Definition. A subset A of $\mathbb{R}^n$ is called *bounded* if there exists a $C \geq 0$ such that $A \subset \overline{B}^n_C(0)$.

Example. Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows easily from the triangle inequality that the balls $B^n_r(y)$, $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are bounded.

Lemma 1.20. *Let $A \subset \mathbb{R}^n$ be a subset. If A is compact, then A is closed and it is bounded.*

Proof. Let $A \subset \mathbb{R}^n$ be a compact subset.

(1) We want to show that $\mathbb{R}^n \setminus A$ is open. Let $x \in \mathbb{R}^n \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B^n_\epsilon(x) \subset \mathbb{R}^n \setminus A$. Given $j \in \mathbb{N}$ we consider the open subset $U_j = (\mathbb{R}^n \setminus \overline{B}^n_{\frac{1}{j}}(x)) \cap A$. Evidently these sets cover A. Since A is compact we know that there exist $j_1, \ldots, j_k$ with $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $\epsilon = \min\{\frac{1}{j_1}, \ldots, \frac{1}{j_k}\}$. We obtain that $A \subset \mathbb{R}^n \setminus \overline{B}^n_\epsilon(x)$. Thus we have found the desired ϵ.
(2) We consider the open subsets $U_j = B^n_j(0) \cap A$ with $j \in \mathbb{N}$. Evidently these sets cover A. Since A is compact we know that there exist $j_1, \ldots, j_k$ such that $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $C = \max\{j_1, \ldots, j_k\}$. We obtain that $A \subset \overline{B}^n_C(0)$. □

The ever-popular Heine-Borel Theorem says that mercifully the converse to Lemma 1.20 holds.

Theorem 1.21. (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.*

Examples.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 1.21 and Proposition 1.5 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.
(2) It is an amusing exercise, see Exercise 1.8, to show, using the Heine-Borel Theorem 1.21, that "$\mathbb{R}^n$ with a point at infinity" is compact.
(3) It is an equally entertaining exercise, see Exercise 1.8, to show that the "line with two zeros" is not compact.
(4) We set $C_0 = [0, 1]$. We define sets C_i, $i \in \mathbb{N}$ iteratively by setting

$$C_i \ := \ \tfrac{1}{3}C_{i-1} \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C_{i-1}\right)$$

i.e. we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. See Figure 13. The *Cantor set* is defined as

$$\text{Cantor set } C \ := \bigcap_{i\in\mathbb{N}_0} C_i.$$

Each C_i is closed, hence it follows from Lemma 1.4 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 1.21 implies that the Cantor set is compact.

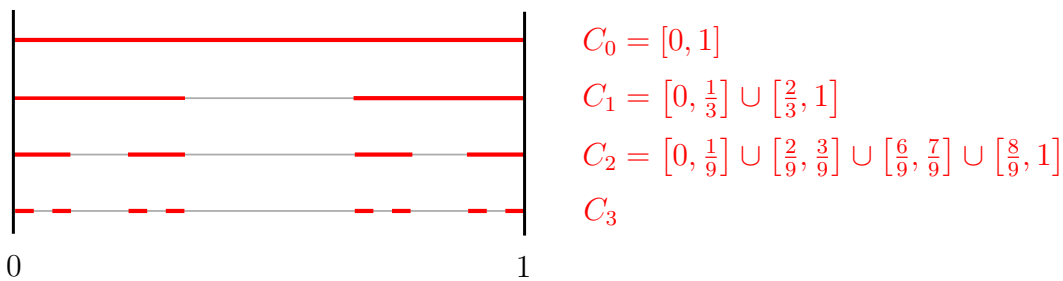

FIGURE 13.

(5) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that people like to replace "compact" by "bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1]\cap\mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$, but as we will see in Exercise 1.17, it is not compact.

The following proposition is the key ingredient in the proof of the Heine-Borel Theorem 1.21.

Proposition 1.22. *Let $n \in \mathbb{N}$ and let $a_i < b_i$, $i = 1,\dots,n$ be real numbers. The hyperrectangle*

$$\underbrace{[a_1,b_1] \times \ldots \times [a_n,b_n]}_{:=\{(x_1,\dots,x_n)\in\mathbb{R}^n \,|\, x_i\in[a_i,b_i] \text{ for } i=1,\dots,n\}} \ \subset \ \mathbb{R}^n$$

is compact.

Proof. To simplify the notation we only treat the case $n = 1$. The general case is treated in a very similar fashion.[8] To simplify the notation we assume that $a = 0$ and $b = 1$. Thus we will now show that $I := [0,1]$ is compact. In an attempt at a proof by contradiction we suppose that I is not compact. This means that there exists an open overing $\{U_j\}_{j\in J}$ such that I cannot be covered by finitely many of the U_j.

[8]The full details are for example provided in [Rudi, Theorem 2.4.1].

Claim. There exists a sequence of closed intervals

$$I \;=\; I_0 \;\supset\; I_1 \;\supset\; I_2 \;\supset\; \dots$$

such that for any $m \in \mathbb{N}_0$ the following two conditions hold:

(1) I_m cannot be covered by finitely many of the U_j,
(2) the length of I_m equals 2^{-m}.

We prove the claim by induction. We set $I_0 := I$. Suppose we have already found $I_0, I_1, \dots, I_m$. We divide the interval I_m into two closed intervals I' and I'' of half the length. Since I_m cannot be covered by finitely many U_j it follows that at least one of the intervals I' and I'' cannot be covered by finitely many U_j either. We declare this interval to be I_{m+1}. This concludes the proof of the claim.

Claim. There exists an $x \in I$ with $x \in \bigcap\limits_{m\in\mathbb{N}} I_m$.

For every $m \in \mathbb{N}$ we pick an $a_m \in I_m$. For any $m \geq n$ we know that a_m and a_n both lie in I_n. In particular

$$|a_m - a_n| \;\leq\; \text{length of } I_n \;=\; \frac{1}{2^n}.$$

It follows that $(a_m)_{m\in\mathbb{N}}$ is a Cauchy sequence. We denote by x the limit of this Cauchy sequence. We want to show that $x \in \bigcap\limits_{m\in\mathbb{N}} I_m$. Put differently, we want to show that for every $m \in \mathbb{N}$ we have $x \in I_m$. Thus let $m \in \mathbb{N}$. For all $n \geq m$ we have $a_n \in I_n \subset I_m$. In particular, $(a_n)_{n\geq m}$ is a sequence that converges in I_m. Since I_m is closed we see[9] that $\lim_{n\to\infty} a_n \in I_m$. This concludes the proof of the claim.

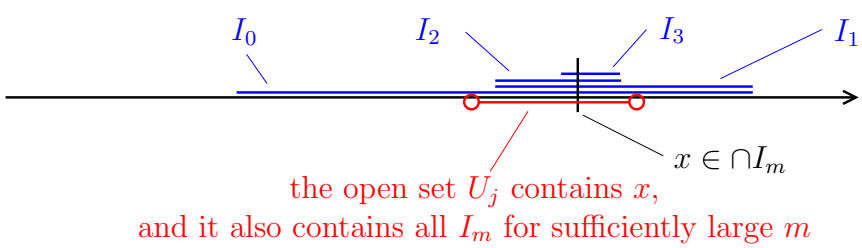

FIGURE 14. Illustration for the proof of Proposition 1.22.

Since $\{U_j\}_{j\in J}$ is an open covering of I there exists a $j \in J$ with $x \in U_j$. Since U_j is open there exists an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \cap I \subset U_j$. We pick $m \in \mathbb{N}$ with $2^{-m} < \epsilon$. Now note that

$$\begin{array}{ccccccc} I_m & \underset{\substack{\uparrow \\ \text{since } x \in I_m \text{ and since} \\ \text{the length of } I_m \text{ equals } 2^{-m}}}{\subset} & [x-2^{-m}, x+2^{-m}] \cap I & \underset{\substack{\uparrow \\ \text{choice of } m}}{\subset} & (x-\epsilon, x+\epsilon) \cap I & \underset{\substack{\uparrow \\ \text{choice of } U_j}}{\subset} & U_j \end{array}$$

[9]Why do we see this?

i.e. I_m is contained in a unique U_j. But this contradicts property (1) of the intervals $I_m, m \in \mathbb{N}$. $\square$

Now we can finally give the proof of the Heine-Borel Theorem 1.21.

Proof of the Heine-Borel Theorem 1.21. In light of Lemma 1.20 we only need to show that if A is a bounded and closed subset of $\mathbb{R}^n$, then it is compact. Since A is bounded there exists a $C \geq 0$ such that $A \subset [-C, C]^n$. By Proposition 1.22 we know that $A \subset [-C, C]^n$ is compact. Since A is closed in $\mathbb{R}^n$ it follows from Lemma 1.6 that is also a closed subset of $[-C, C]^n$. It follows from Lemma 1.18 that A itself is compact. $\square$

1.4. **Bases of a topology.** We start out with the following definition.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say $\mathcal{B}$ has the *basis property* if the following two conditions are satisfied:

(B1) For any $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.
(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. (See Figure 15 for an illustration.)

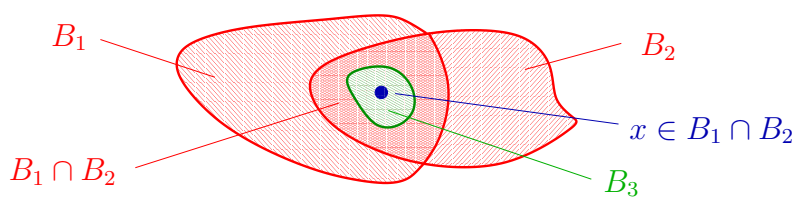

FIGURE 15.

The following lemma justifies our sudden interest in this new definition.

Lemma 1.23. *Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We define*

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \textit{for every } x \in V \textit{ there exists a } B \in \mathcal{B} \textit{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the *topology on X generated by $\mathcal{B}$*.
(2) Conversely, if $\mathcal{T}$ is a given topology on X and if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a *basis of the topology $\mathcal{T}$ of X*.

Examples.

(1) If (X, d) is a metric space, then

$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subset\; \mathcal{P}(X)$$

satisfies (B1) and (B2).[10] The topology generated by this family of subsets is, by definition, precisely the topology we already introduced on page 44.

(2) Let X be a set and let $\mathcal{T}$ be a topology on X. It follows easily from the definitions that $\mathcal{T}$ has the basis property. Furthermore it follows from Lemma 1.8 that the topology generated by $\mathcal{T}$ is in fact $\mathcal{T}$.

For completeness we provide the elementary proof of Lemma 1.23.

Proof $(*)$. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We write $\mathcal{T} := \mathcal{T}(\mathcal{B})$. We need to show that $\mathcal{T}$ satisfies the three properties (1), (2) and (3) in the definition of a topological space, see page 42. Given any $x \in X$ there exists by (B1) a $B \in \mathcal{B}$ with $x \in B$. By definition we see that $X \in \mathcal{T}$. Furthermore it is clear that the empty set $\varnothing$ lies in $\mathcal{T}$.
We need to show that the intersection of finitely many sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

First we consider the case of two sets in $\mathcal{T}$. Thus let $U, V \in \mathcal{T}$. We need to show that $U \cap V \in \mathcal{T}$. Let $x \in U \cap V$. Since U and V lie in $\mathcal{T}$ there exist $B, C \in \mathcal{B}$ with $x \in B \subset U$ and $x \in C \subset V$. By (B2) there exists a $D \in \mathcal{B}$ with $x \in D$ and $D \subset B \cap C$. In particular we have $x \in D \subset U \cap V$. This shows that $U \cap V \in \mathcal{T}$.

Now let $U_1, \ldots, U_k$ be sets in $\mathcal{T}$. Using the above statement and an elementary induction argument we see that $U_1 \cap \cdots \cap U_k$ lies in $\mathcal{T}$ as well.
Let $U_j, j \in J$ be a family of sets in $\mathcal{T}$. We need to show that the union $\bigcup_{j \in J} U_j$ lies in $\mathcal{T}$. Thus let $x \in \bigcup_{j \in J} U_j$. This means that there exists a $k \in J$ with $x \in U_k$. Since $U_k \in \mathcal{T}$ there exists a $B_x \in \mathcal{B}$ with $x \in B_x \subset U_k$. But then we also have

$$x \in B_x \subset U_k \subset \bigcup_{j \in J} U_j. \qquad \square$$

The following lemma can be used to show that two topologies on a given set agree.

Lemma 1.24. *Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies for X. Suppose that $\mathcal{C}$ is a basis for $\mathcal{T}$. (As discussed above we could take $\mathcal{C} = \mathcal{T}$.) If given any $x \in X$ and given any $V \in \mathcal{C}$ there exists a $U \in \mathcal{S}$ with $x \in U \subset V$, then $\mathcal{T} \subset \mathcal{S}$.*

Example. Let $n \in \mathbb{N}$. We consider the two metrics

$$\begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \|v - w\| \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \max\{|v_i - w_i| \,|\, i \in \{1, \ldots, n\}\}. \end{array}$$

The former is the Euclidean metric that we introduced on page 43. We refer to the second metric as the *maximum's metric*. We leave it as an exercise to the dedicated reader to show, using Lemma 1.24, that the two topologies on $\mathbb{R}^n$ agree.

Proof. It follows immediately from Lemma 1.8 that any set in $\mathcal{C}$ is open in $\mathcal{S}$. But then, once again using Lemma 1.8 we see that any set in $\mathcal{T}$ is open in $\mathcal{S}$, i.e. $\mathcal{T} \subset \mathcal{S}$. $\square$

Sometimes the following two lemmas can be useful.

[10] Why is (B2) satisfied?

Lemma 1.25. *Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. If $\mathcal{B}$ has the basis property, then*

$$V \subset X \text{ is open with respect to } \mathcal{T}(\mathcal{B}) \quad \Longleftrightarrow \quad V \text{ is the union of sets in } \mathcal{B}.$$

Proof of Lemma 1.25 $(*)$. We first prove the "$\Leftarrow$"-direction. Thus let V be the union of sets in $\mathcal{B}$. Since sets in $\mathcal{B}$ are open with respect to $\mathcal{T}(\mathcal{B})$ and since $\mathcal{T}(\mathcal{B})$ is a topology we see that $V \in \mathcal{T}(\mathcal{B})$.

Now we prove the "$\Rightarrow$"-direction. Thus we suppose that $V \subset X$ is open with respect to $\mathcal{T}(\mathcal{B})$. For each $x \in V$ there exists, by definition of $\mathcal{T}(\mathcal{B})$, a set $B_x \in \mathcal{B}$ with $x \in B_x \subset V$. We obtain that

$$V \;=\; \bigcup_{x\in V} \{x\} \;\subset\; \bigcup_{x\in V} B_x \;\subset\; V, \qquad \text{thus it follows that } V \;=\; \bigcup_{x\in V} B_x. \qquad \square$$

Lemma 1.26. *Let $\mathcal{B}$ be a basis for the topology of a topological space X. Furthermore let $\{U_j\}_{j\in J}$ be an open covering of X, i.e. $\{U_j\}_{j\in J}$ is a family of open sets with $\bigcup_{j\in J} U_j = X$. Then*

$$\{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

Proof $(*)$. We write

$$\mathcal{C} := \{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}.$$

Since $\mathcal{C} \subset \mathcal{B}$ we have $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}(\mathcal{B})$. It remains to show the reverse inclusion $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{C})$. It follows easily from Lemma 1.8 that it suffices to show that $\mathcal{B} \subset \mathcal{T}(\mathcal{C})$. In other words, it suffices to prove the following claim.

Claim. Given any $x \in X$ and any $B \in \mathcal{B}$ with $x \in B$ there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$.

So let $x \in X$ and let $B \in \mathcal{B}$ with $x \in B$. By hypothesis there exists a j such that $x \in U_j$. Since U_j is open and since $\mathcal{B}$ is a basis for the topology of X there exists a $B' \in \mathcal{B}$ with $x \in B' \subset U_j$. By (B2) there exists a $B'' \in B$ with $x \in B''$ and $B'' \subset B \cap B' \subset U_j$. By definition we have $B'' \in \mathcal{C}$. This concludes the proof of the claim and thus of the lemma. (We refer to Figure 16 for an illustration of the argument.) $\square$

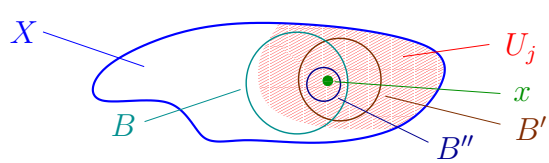

FIGURE 16. Illustration for the proof of Lemma 1.26.

Example. As we pointed out above, $\mathcal{B} = \{B_r^n(x) \,|\, r > 0 \text{ and } x \in \mathbb{R}^n\}$ is a basis for the usual topology of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the family of all open sets that are contained in an open ball of radius 1. Then the open sets in $\mathcal{U}$ cover $\mathbb{R}^n$. It follows from Lemma 1.26 that

$$\{B_r^n(x) \,|\, r \in (0,1) \text{ and } x \in \mathbb{R}^n\}$$

is also a basis for $\mathbb{R}^n$.

The following lemma gives us in particular another source of topologies.

Lemma 1.27. *Let X be a set. For any $\mathcal{C} \subset \mathcal{P}(X)$ the set*

$\mathcal{B}(\mathcal{C})$:= *all subsets of X that can be written as intersections of a finite family of sets in $\mathcal{C}$*

has the basis property.

Proof $(*)$.

(1) On page 41 we pointed out that if we take the empty family of subsets of X, then the intersection is all of X. In other words, we see that $X \in \mathcal{B}(\mathcal{C})$. It follows that $\mathcal{B}(\mathcal{C})$ satisfies (B1).
(2) It is elementary to verify that $\mathcal{B}(\mathcal{C})$ satisfies (B2). □

The previous lemma leads us to the following definition.

Definition. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$ with $\bigcup_{C \in \mathcal{C}} C = X$.

(1) We define $\mathcal{B} = \mathcal{B}(\mathcal{C})$ as in Lemma 1.27 and we refer to $\mathcal{T}(\mathcal{B})$ as the *topology generated by $\mathcal{C}$*.
(2) Conversely, if $\mathcal{T}$ is a topology on X and if the topology generated by $\mathcal{C}$ agrees with $\mathcal{T}$, then we say that $\mathcal{C}$ is a *subbasis for $\mathcal{T}$*.

Example.

(1) Let $\mathcal{C}$ be the set of all intervals in $\mathbb{R}$ of length one. One can easily verify that $\mathcal{C}$ generates the usual topology on $\mathbb{R}$.
(2) It follows easily from the definitions that a basis of a given topology is also a subbasis of the topology.

The following lemma gives us a convenient criterion for showing that a set of open subsets is in fact a (sub-) basis for a given topology.

Lemma 1.28. *Let X be a topological space and let $\mathcal{C}$ be a family of open sets.*

(1) *Suppose $\mathcal{C}$ has the following property:*

   given any open set U and given any $x \in U$ there exists a $C \in \mathcal{C}$ with $x \in C \subset U$.

   Then $\mathcal{C}$ is a basis for the topology of X.
(2) *Suppose $\mathcal{B}$ is a basis for the topology of X. If given any $x \in X$ and given any $B \in \mathcal{B}$ there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $x \in C_1 \cap \cdots \cap C_m \subset B$, then $\mathcal{C}$ is a subbasis for the topology of X.*

Example. Using Lemma 1.28 (1) and using the density of $\mathbb{Q} \subset \mathbb{R}$ one can easily show that the set of *rational* intervals forms a basis for the usual topology on $\mathbb{R}$. In particular $\mathbb{R}$ has a countable basis for its topology.

Proof $(*)$.

(1) We denote by $\mathcal{T}$ the topology of X. First we show that $\mathcal{C}$ has the basis property:
   (B1) If we apply the property of $\mathcal{C}$ to $U = X$ then we obtain immediately that given any $x \in X$ there exists a $C \in \mathcal{C}$ with $x \in C$.
   (B2) Let $x \in C_1 \cap C_2$ with $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open the intersection $U = C_1 \cap C_2$ is also open. It follows from the hypothesis on $\mathcal{C}$, applied to U, that there exists a $C \in \mathcal{C}$ with $x \in C \subset C_1 \cap C_2$.

   We denote by $\mathcal{S}$ the topology on X that is generated by $\mathcal{C}$. We need to show that $\mathcal{S} = \mathcal{T}$.

   First we show the inclusion $\mathcal{S} \subset \mathcal{T}$. Since all sets in $\mathcal{C}$ are open with respect to $\mathcal{T}$ it follows from Lemma 1.25 that $\mathcal{S} \subset \mathcal{T}$.

   Finally we show the reverse inclusion $\mathcal{T} \subset \mathcal{S}$. Thus let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{S}$. Thus let $x \in U$. By our hypothesis there exists a $C \in \mathcal{C}$ with $x \in C \subset U$. Therefore by definition we have $U \in \mathcal{S}$. We have thus shown that $\mathcal{T} \subset \mathcal{S}$.

(2) We leave the verification of this statement as a rather uninspiring exercise to the reader. $\square$.

We conclude this section with the following not terribly inspiring lemma.

Lemma 1.29. $(*)$ *Let X be a topological space and let $A \subset X$ be a subset. If $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \,|\, B \in \mathcal{B}\}$ is a basis for the topology of A, equipped with the subspace topology.*

Proof $(*)$. We prove the lemma using Lemma 1.28 (1). Thus let $U \subset A$ be an open subset in A and let $x \in U$. By definition of the subspace topology there exists an open subset $V \subset X$ of X with $U = V \cap A$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $B \in \mathcal{B}$ with $x \in B \subset V$. This implies that $x \in B \cap A \subset V \cap A = U$. It follows from Lemma 1.28 (1) that $\{B \cap A \,|\, B \in \mathcal{B}\}$ is indeed a basis for the topology of A. $\square$

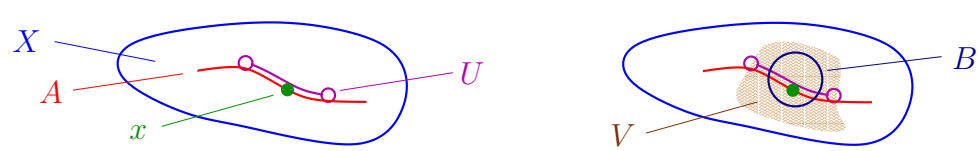

FIGURE 17. Illustration for the proof of Lemma 1.29.

Example. We consider

$$S^1 \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \;\subset\; \mathbb{R}^2.$$

As we saw on page 60, a basis for the topology on $\mathbb{R}^2$ is given by

$$\mathcal{B} \;=\; \{B_r(x,y) \,|\, (x,y) \in \mathbb{R}^2 \text{ and } r > 0\}.$$

The intersection of such an open ball with S^1 is an "open interval on S^1", i.e. it is a subset of the form $\{e^{i\varphi} \,|\, \varphi \in (a,b)\}$. It follows from Lemma 1.29 that these "open intervals" form a basis for the topology on S^1. We refer to Figure 18 for an illustration.

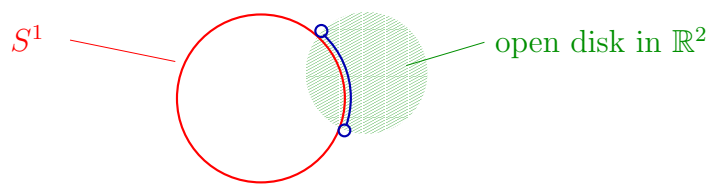

FIGURE 18.

1.5. **Continuous maps.** The following is the key definition in the context of topological spaces.

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is *continuous*, if for each open set U in Y the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then any map $f\colon X \to Y$ is continuous.
(2) Let $n \in \mathbb{N}$. It follows easily from the definitions that for each $i \in \{1,\dots,n\}$ the projection map
$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1,\dots,x_n) & \to & x_i \end{array}$$
is continuous.
(3) Let X be a topological space. On many occasions we will use the following elementary facts
   (a) If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the product and the sum of f and g, i.e. the maps $x \mapsto f(x) + g(x)$ and $x \mapsto f(x) \cdot g(x)$, are also continuous.
   (b) If $f\colon X \to \mathbb{R}$ is a map that is non-zero everywhere, then $x \mapsto \frac{1}{f(x)}$ defines a continuous map on X.
(4) We will also often use the elementary fact that if $f = (f_1,\dots,f_n)\colon X \to \mathbb{R}^n$ is a map, then f is continuous if and only if each $f_i\colon X \to \mathbb{R}$ is continuous.
(5) Let A be a real $(m \times n)$-matrix. It follows from (2), (3) and (4) that the map
$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & Av \end{array}$$
is continuous.

(6) We consider the set $X = \{A, B, C, D\}$ where the topology is given by the set

$$\mathcal{T} := \{\varnothing, \{A\}, \{C\}, \{A, C\}, \{A, B, C\}, \{A, C, D\}, X\}.$$

We refer to this to topological spaces as the *pseudocircle*. It follows easily from the definitions that the map

$$\begin{array}{rcl} f\colon S^1 & \to & X \\ e^{it} & \mapsto & \begin{cases} A, & \text{if } t \in (-\frac{\pi}{4}, \frac{\pi}{4}), \\ B, & \text{if } t = \frac{\pi}{4}, \\ C & \text{if } t \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\ D & \text{if } t = \frac{3\pi}{4}. \end{cases} \end{array}$$

is continuous. This map is illustrated in Figure 19.

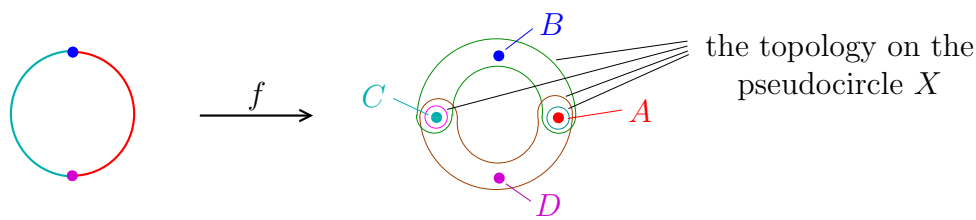

FIGURE 19.

The following lemma will be used on almost every page of these notes.

Lemma 1.30.

(1) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.*
(2) *Let X be a topological space and let A be a subset. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.*
(3) *Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.*
(4) *Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.*

Proof.

(1) Let $U \subset Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } g \text{ is continuous and } U \text{ is open}}}{=} f^{-1}(\text{the open set } g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} \text{open}.$$

(2) This statement follows immediately from the definitions.

(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.
(4) For clarity we denote by $g\colon X \to Z$ the map given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset X$ with $U = Z \cap V$. Thus we see that

$$\begin{array}{ccccccccc} g^{-1}(U) & = & g^{-1}(Z \cap V) & \underset{\substack{\uparrow \\ \text{definition of } g}}{=} & f^{-1}(Z \cap V) & \underset{\substack{\uparrow \\ \text{since } f(X) \subset Z}}{=} & f^{-1}(V) & \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} & \text{open subset of } X. \end{array}$$

□

At times it is also useful to have a characterization of continuous maps in terms of closed sets.

Lemma 1.31. *A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed set A in Y the preimage $f^{-1}(A)$ is closed in X.*

Proof. This lemma is a straightforward consequence of Lemma 1.1 (7) and the definitions. We leave it to the reader to fill in the details. □

By Lemma 1.30 (3) we know that the restriction of a continuous map to any subset is still continuous. The following lemma can be viewed as a converse: if "enough" restrictions of a given map are continuous, then the original map is also continuous.

Lemma 1.32. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If there exists a family of open subsets $\{W_i\}_{i \in I}$ of X with $\bigcup_{i \in I} W_i = X$ such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.*
(2) *If there exists a* finite *collection of closed sets $A_1, \ldots, A_m$ of X with $A_1 \cup \cdots \cup A_m = X$ such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.*

In practice often the following variation on (2) *gets used:*

(2') *If X is Hausdorff and if there exists a* finite *collection of compact sets $A_1, \ldots, A_m$ of X with $A_1 \cup \cdots \cup A_m = X$ such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.*

Remark.

(1) In practice Lemma 1.32 (2) often gets used the following way: We are given two topological spaces X and Y and closed subsets $A_1, \ldots, A_m$ of X. Furthermore we are given continuous maps $f_i\colon A_i \to Y$, $i = 1, \ldots, m$. If the f_i agree on the overlaps, then the map

$$\begin{array}{ccl} X & \to & Y \\ P & \mapsto & f_i(P) \quad \text{if } P \in A_i \end{array}$$

is well-defined and it follows from Lemma 1.32 (2) that it is continuous.

(2) Note that in Lemma 1.32 (1) we are allowed to deal with arbitrarily many open subsets whereas in Lemma 1.32 (2) we are restricted to finitely many closed subsets. It is clear that Lemma 1.32 (2) cannot be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f\colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\}$ is continuous, but clearly not every map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Let $f\colon X \to Y$ be a map between topological spaces.

(1) Let $\{W_i\}_{i\in I}$ be a family of open subsets of X with $\bigcup_{i\in I} W_i = X$ such that each $f_i := f|_{W_i}$ is continuous. Let $U \subset Y$ be an open set. We want to show that $f^{-1}(U)$ is an open subset of X. By Lemma 1.7 (1a) it suffices to show for every $i \in I$ the intersection $f^{-1}(U) \cap W_i$ is an open subset of W_i. But $f^{-1}(U) \cap W_i = f_i^{-1}(U)$ is indeed open in W_i since by hypothesis each f_i is continuous.
(2) The proof of this statement is almost identical to the proof of (1), we just need to replace Lemma 1.7 (1a) by Lemma 1.7 (2a).
(3) This statement follows immediately from (2) and Lemma 1.18 (2). □

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "$\epsilon - \delta$"–definition of continuity agrees with the above definition of continuity.

Proposition 1.33. *Let $f\colon X \to Y$ be a map between metric spaces. Then the following holds:*

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x'\in B_\delta(x)}{\forall}\ d(f(x), f(x')) < \epsilon.$$

Remark. We use Proposition 1.33 to import from real analysis the various continuous functions on $\mathbb{R}$ and $\mathbb{C}$, e.g. polynomials, trigonometric functions and the exponential function. The replacement of the classic "$\epsilon - \delta$"–definition of continuity by the definition in terms of open sets can even be helpful in the context of real analysis. For example, let us consider the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases} \end{aligned}$$

With the "$\epsilon - \delta$"–definition it is slightly annoying to prove continuity at 0. But applying Lemma 1.32 (2) to the closed subsets $A_1 = (-\infty, 0]$ and $A_2 = [0, \infty)$ it is totally evident that this function is continuous. We refer to Figure 20 for an illustration.

The proof of Proposition 1.33 requires some preparations. In particular it builds on the following proposition that frequently makes it much easier to show that a given map is continuous.

Proposition 1.34. *Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of Y (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds*

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

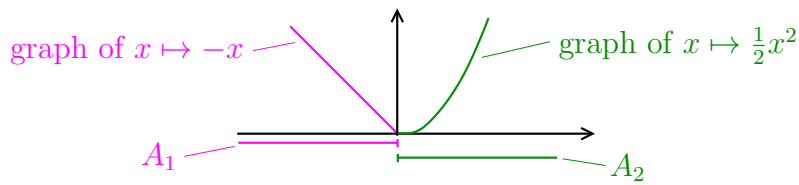

FIGURE 20.

Proof $(*)$. Let $f\colon X \to Y$ be a map between topological spaces and let $\mathcal{C}$ be a subbasis for the topology of Y. The "$\Rightarrow$"-direction of the proposition follows from the observation that elements of a subbasis are in fact open. Thus let us turn to the "$\Leftarrow$"-direction. We now assume that for each $C \in \mathcal{C}$ the preimage $f^{-1}(C)$ is open in X. We start out with a little observation. Suppose we are given $C_1, \ldots, C_k \in \mathcal{C}$, then

$$f^{-1}(C_1 \cap \cdots \cap C_k) \overset{\text{Lemma 1.1}}{\overset{\downarrow}{=}} \underbrace{f^{-1}(C_1)}_{\text{open in } X} \cap \cdots \cap \underbrace{f^{-1}(C_k)}_{\text{open in } X}$$

is also open in X. This shows, by definition, that for each $B \in \mathcal{B}(\mathcal{C})$ the preimage $f^{-1}(B)$ is open in X.

Now we turn to the actual proof that f is continuous. Thus let $U \subset Y$ be an open subset. By Lemma 1.25 there exists a family $\{B_j\}_{j \in J}$ of sets in $\mathcal{B}$ with

$$U = \bigcup_{j \in J} B_j.$$

Then we see that

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{j \in J} B_j\Big) \overset{\text{Lemma 1.1}}{\overset{\downarrow}{=}} \underbrace{\bigcup_{j \in J} \underbrace{f^{-1}(B_j).}_{\text{open in } X}}_{\substack{\text{union of open} \\ \text{sets, thus open in } X}}$$

□

Now that we have Proposition 1.34 under the belt we can easily provide a proof of Proposition 1.33.

Proof of Proposition 1.33$(*)$. Let $f\colon X \to Y$ be a map between metric spaces. We need to show that

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \forall_{x \in X} \ \forall_{\epsilon > 0} \ \exists_{\delta > 0} \ \forall_{y \in B_\delta(x)} \ d(f(x), f(y)) < \epsilon.$$

We prove the two directions separately:

"$\Rightarrow$" Let $x \in X$ and let $\epsilon > 0$. We consider $V := f^{-1}(B_\epsilon(f(x))) \subset X$. Since f is continuous we know that V is open. By the definition of the topology on a metric space there exists a $\delta > 0$ such that $B_\delta(x) \subset V$. This delta has the desired property.

"$\Leftarrow$" On page 60 we saw that the open balls form a basis for the topology of metric spaces. Thus by Proposition 1.34 it suffices to show that if $y \in Y$ and $\mu > 0$, then $f^{-1}(B_\mu(y))$ is an open subset of X. By the definition of the topology on X it suffices to prove the following claim.

Claim. Given any $x \in f^{-1}(B_\mu(y))$ there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$.

We set $\delta = \mu - d(f(x), y)$. Note that $\delta > 0$. By the triangle inequality we have $B_\delta(f(x)) \subset B_\mu(y)$. Furthermore by our hypothesis on f there exists an $\epsilon > 0$ such that $f(B_\epsilon(x)) \subset B_\delta(f(x))$. Thus we see that $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$. We refer to Figure 21 for an illustration of this argument. □

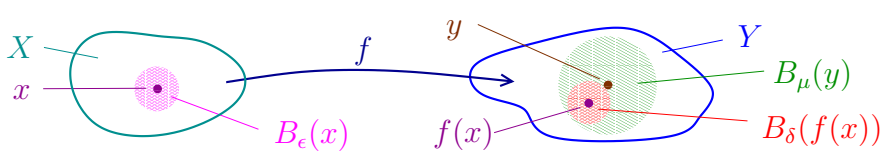

FIGURE 21. Illustration of the proof of Proposition 1.33.

We continue with the following straightforward definition.

Definition. Let $f\colon X \to Y$ be a map between topological spaces.

(1) We say f is *open* if the image of every open subset of X is open in Y.
(2) We say f is *closed* if the image of every closed subset of X is closed in Y.

Example. The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map but it is not a closed map.[11] On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map but it is a closed map.

Lemma 1.35. *Let X be a topological space.*

(1) *If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.*
(2) *If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.*

Proof.

(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1), we just not to appeal to Lemma 1.6 (2) to make the necessary adjustments. □

[11] Why is not a closed map?

We have the following analogue to Proposition 1.34.

Lemma 1.36. *Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{B}$ be a basis for the topology of X. Then the following holds*

$$f \text{ is open} \quad \Longleftrightarrow \quad \text{for each } B \in \mathcal{B} \text{ the image } f(B) \text{ is open in } Y.$$

Proof. This statement follows easily from Lemmas 1.25 and 1.1. □

The following lemma states perhaps the most important feature of compact sets.

Lemma 1.37.

(1) *Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.*
(2) *Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that*

$$f(x_0) \;\le\; f(x) \;\le\; f(x_1) \quad \text{for all } x \in X.$$

(3) *Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.*

Example. Let X be a topological space, let Y be a subset and let $K \subset Y$ be another subset. If K is a compact subset of Y, i.e. then K is also a compact subset of X. This elementary observation follows immediately from Lemma 1.37 and the fact, observed above, that inclusion maps are continuous.

Remark. The statement of Lemma 1.37 (1) is really quite astounding. More precisely, we already saw that the image of an open set under a continuous map is not necessarily open. Similarly, the image of a closed set under a continuous map is not necessarily closed. For example consider the inclusion map $i\colon X = (0,1] \to Y = \mathbb{R}$. Evidently X is a closed subset of itself, but $i(X)$ is not a closed subset of $\mathbb{R}$.

Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open covering of $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

$$X \;\underset{\substack{\uparrow \\ \text{Lemma 1.1}}}{\subset}\; f^{-1}(f(X)) \;\subset\; f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \;\underset{\substack{\uparrow \\ \text{Lemma 1.1}}}{=}\; \bigcup_{i\in I} f^{-1}(U_i).$$

By the continuity of f we obtain that $\{f^{-1}(U_i)\}_{i\in I}$ is an open covering of X. Since X is compact there exist $i_1, \dots, i_k \in I$ such that

$$X \;\subset\; f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k}).$$

We see that

$$\begin{aligned} f(X) \;&\subset\; f\left(f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\substack{\uparrow \\ \text{Lemma 1.1}}}{=}\; f\left(f^{-1}(U_{i_1})\right) \cup \cdots \cup f\left(f^{-1}(U_{i_k})\right) \;\underset{\substack{\uparrow \\ \text{Lemma 1.1}}}{\subset}\; U_{i_1} \cup \cdots \cup U_{i_k}. \end{aligned}$$

(2) It follows from (1) and the Heine-Borel Theorem 1.21 that $f(X) \subset \mathbb{R}$ is bounded and closed. Thus $y_0 := \inf(f(X))$ and $y_1 := \sup(f(x))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.
(3) This statement is an immediate consequence of Lemma 1.18 (1) and the first statement. □

On many occasions we will use the following elementary lemma.

Lemma 1.38. $(*)$ *Let $f\colon K \to X$ be a continuous map from a compact topological space K to a topological space X.*[12] *Suppose there exists a sequence $U_1, U_2, \dots$ of subsets of X with the following properties:*

(1) *each U_i is open,*
(2) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$,*
(3) *we have $X = \bigcup_{i\in\mathbb{N}} U_i$.*

Then there exists a $k \in \mathbb{N}$ with $f(K) \subset U_k$.

Proof $(*)$. For $i \in \mathbb{N}$ we write $V_i = f^{-1}(U_i)$. Since f is continuous it follows from (1) that each V_i is an open subset of K. By (3) the V_i cover all of K. Since K is compact there exist $i_1, \dots, i_n$ such that $K = V_{i_1} \cup \dots \cup V_{i_n}$. We set $k := \max\{i_1, \dots, i_n\}$. Since the U_i are nested the V_i are also nested. It follows that K is already contained in V_k, put differently, $f(K)$ is contained in U_k. □

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is a *homeomorphism* if the following three properties are satisfied:

(1) f is continuous,
(2) f is bijective,
(3) the inverse map $f^{-1}\colon Y \to X$ is also continuous.

If there exists a homeomorphism between X and Y we say that X and Y are *homeomorphic* and sometimes we write $X \cong Y$.

Remark. If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

Convention. At times we refer to a topological space that is homeomorphic to a ball or to a sphere also as a ball or a sphere.

Examples.

[12] A common application is that K is a compact subset of X and f is the inclusion map.

(1) Given $n \in \mathbb{N}$ we denote by
$$S^n_{\geq 0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$$
the upper hemisphere and we define similarly the lower hemisphere $S^n_{\leq 0}$ to be
$$S^n_{\leq 0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \leq 0\}.$$
The maps
$$\begin{array}{rcl} \varphi\colon \overline{B}^n & \mapsto & S^n_{\geq 0} \\ x & \mapsto & \left(x, \sqrt{1 - \|x\|^2}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^n_{\geq 0} & \to & \overline{B}^n \\ (x_1, \ldots, x_n, x_{n+1}) & \mapsto & (x_1, \ldots, x_n) \end{array}$$
are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) Let $A \in \mathrm{GL}(n, \mathbb{R})$ be an invertible real matrix. It follows from the discussion on page 65 that the map $\mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A is a homeomorphism. This simple observation has a useful consequence: Let V be an n-dimensional real vector space. We pick an isomorphism $\varphi\colon \mathbb{R}^n \to V$ and we equip V with the unique topology that turns φ into a homeomorphism. By the above discussion this definition of a topology does not depend on the choice of φ. Unless we say something else any finite-dimensional real vector space will be equipped with this topology. Note that with these topologies any isomorphism between real vector spaces of the same finite dimension is also a homeomorphism.

(3) We consider the map
$$\begin{array}{rcl} f\colon [0,1) & \to & S^1 \\ t & \mapsto & e^{2\pi i t}. \end{array}$$
The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. We illustrate this example in Figure 22. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away under the warm, but mistaken, impression that one now has found a homeomorphism.

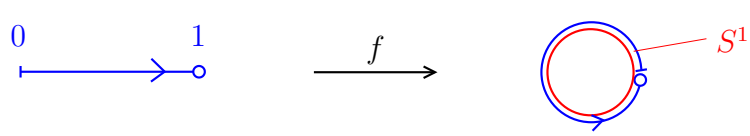

FIGURE 22.

In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$,

let alone to verify that it is continuous. Fortunately we have two results which often help us in circumventing this difficulty.

Lemma 1.39. $(*)$ *Let $f\colon X \to Y$ be a map between topological spaces that is continuous and a bijection. If f is open or if f is closed, then f is a homeomorphism.*

Proof $(*)$. Let $f\colon X \to Y$ be a map between topological spaces that is continuous and that is a bijection. First assume that f is open. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open set. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by Lemma 1.31. $\square$

Example. Let U be an open subset of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$ be a strictly monotonously increasing map, i.e. for any $x < y$ in U we have $f(x) < f(y)$. Note that open intervals form a basis for the topology of U. By the Intermediate Value Theorem the image of an open interval under f is again an open interval. Thus it follows from Lemma 1.36 that $f\colon U \to \mathbb{R}$ is an open map. Lemma 1.39 implies that $f\colon U \to f(U)$ is a homeomorphism.

The next proposition is certainly one of the most frequently used results in topology.

Proposition 1.40. *Let $f\colon X \to Y$ be a bijective continuous map between topological spaces. If X is compact and if Y is Hausdorff, then f is a homeomorphism.*

Remark. In the above example we saw that certainly we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? In Exercise 1.14 we will come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Proof. Let $f\colon X \to Y$ be a bijective continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff. We need to show that the inverse map $g := f^{-1}\colon Y \to X$ is continuous. By Lemma 1.31 it suffices to show that for any closed subset A of X the corresponding preimage $g^{-1}(A)$ is closed in Y. In fact we have

$$\begin{array}{ccccccc} A \subset X \text{ closed} & \Rightarrow & A \text{ is compact} & \Rightarrow & f(A) \text{ is compact} & \Rightarrow & f(A) = g^{-1}(A) \text{ is closed.} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by Lemma 1.18 since } X \text{ is compact} & & \text{Lemma 1.37 (1)} & & \text{by Lemma 1.18 (2) since } Y \text{ is Hausdorff} & \end{array}$$

$\square$

We formulate our first application of Proposition 1.40 as a lemma.

Lemma 1.41. *We consider the map*

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \dots, x_{n+1}) & \mapsto & \begin{cases} \left(\dfrac{x_1}{1 - x_{n+1}}, \dots, \dfrac{x_n}{1 - x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ *with the topology that we introduced on page 53. We refer to* Φ *as the* stereographic projection. *This map has the following properties:*

(1) *The map* Φ *sends the "North Pole"* $N = (0, \ldots, 0, 1) \in S^n$ *to* ∞.
(2) *For any* $P \in S^n$ *that does not equal the North Pole* N *the point* $\Phi(P) \in \mathbb{R}^n$ *is the unique point such that the ray emanating from* N *and that goes through* P *intersects the plane* $\mathbb{R}^n \times \{0\}$ *in* $(\Phi(P), 0)$. *We refer to Figure 23 for an illustration.*
(3) *For any* $(v_1, \ldots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ *we have* $\Phi(v_1, \ldots, v_n, 0) = (v_1, \ldots, v_n)$.
(4) *The map* Φ *is continuous.*
(5) *The map* Φ *is a homeomorphism.*

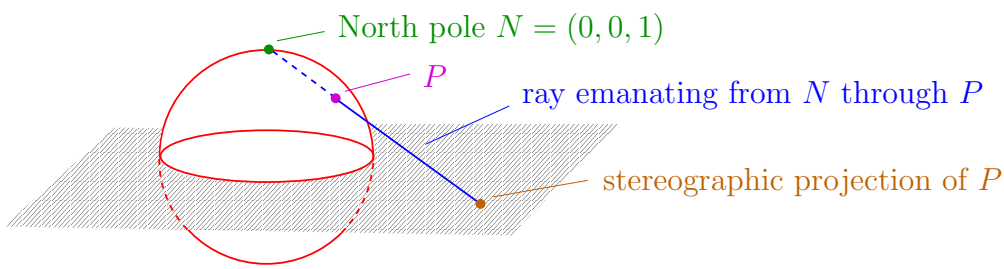

FIGURE 23. Stereographic projection from $S^2 \setminus \{(0, 0, 1)\}$ onto $\mathbb{R}^2$.

Proof.

(1) This statement follows immediately from the definition of Φ.
(2) This statement follows from an elementary calculation which we leave to the reader.
(3) This statement follows immediately from the definition of Φ.
(4) We will prove this statement in Exercise 1.21.
(5) We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 1.21. Furthermore note that on page 53 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from Proposition 1.40 that Φ is a homeomorphism. □

Throughout the remainder of these lecture notes the following convention applies:

Convention. Henceforth any map between two topological spaces is assumed to be continuous, unless we say explicitly otherwise.

1.6. **Normal topological spaces** $(*)$**.** There are many other "separation axioms" that are similar to the Hausdorff condition. We refer to [Mun3, Chapter 4.2] for more details. The only property we will need at some point is the following.

Definition. A topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.

The following lemma nicely combines two key notions that we have introduced so far.

Lemma 1.42. $(*)$ *Let X be a topological space. If X is compact and Hausdorff, then it is also normal.*

Proof $(*)$. Let X be a topological space that is compact and Hausdorff. Furthermore let A and B be two disjoint closed subsets of X. Note that Lemma 1.18 (1) implies that A and B are also compact.

We start out with the following claim.

Claim. Given any $a \in A$ there exists an open neighborhood U_a of a and an open neighborhood V_a of B that are disjoint.

Let $a \in A$. Since X is Hausdorff we can find for any $b \in B$ disjoint open neighborhoods R_b and S_b of a and b. Since B is compact there exist $b_1, \dots, b_n \in B$ such that $B \subset S_{b_1} \cup \dots \cup S_{b_n}$. The open sets

$$U_a := R_{b_1} \cap \dots \cap R_{b_n} \quad \text{and} \quad V_a := S_{b_1} \cup \dots \cup S_{b_n}$$

have the desired properties. This concludes the proof of the claim.

Now we can prove the actual lemma. Since A is compact we can find $a_1, \dots, a_m \in A$ such that $A \subset U_{a_1} \cup \dots \cup U_{a_m}$. The open sets

$$U := U_{a_1} \cup \dots \cup U_{a_m} \quad \text{and} \quad V := V_{a_1} \cap \dots \cap V_{a_m}$$

have the desired properties. □

Remark. Mathematical psychology tells us that Hausdorff does not imply normal. Indeed, an example of a topological space that is Hausdorff but not normal is given in [Mun3, Chapter 4.2].

The following lemma is basically just a convenient reformulation of being normal.

Lemma 1.43. *Let X be a topological space and let $A \subset X$ be a closed subset and $U \subset X$ be an open subset with $A \subset U$. If X is normal, then there exists an open subset V with $A \subset V \subset \overline{V} \subset U$.*

FIGURE 24. Illustration of Lemma 1.43.

Proof. Let X be a topological space and let $A \subset X$ be a closed subset and $U \subset X$ be an open subset with $A \subset U$. We set $B := X \setminus U$. This is a closed subset with $A \cap B = \varnothing$. Since X is normal there exist disjoint neighborhoods V of A and W of B. It remains to show that $\overline{V} \subset U$. In fact we have $V \subset X \setminus W \subset U$. But since W is open we know that $X \setminus W$ is closed. By definition of $\overline{V}$ we have $\overline{V} \subset X \setminus W$. □

Lemma 1.44. *Let X be a topological space and let $U_1, \dots, U_m$ be open subsets such that $X = \bigcup_{i=1}^{m} U_i$. If X is normal, then there exist open subsets $V_i \subset U_i$ such that $\overline{V}_i \subset U_i$ and such that $X = \bigcup_{i=1}^{m} V_i$.*

Remark. A much stronger version of this lemma can be found in [Dug, Theorem 6.1] or alternatively in [Grote, Satz 43].

Proof $(*)$. To simplify the notation we only discuss the case $m = 2$. We leave it to the reader to prove the general case.

Thus let X be a topological space that is normal. Furthermore let U_1 and U_2 be two open subsets with $U_1 \cup U_2 = X$. This data is equivalent to the statement that $X \setminus U_1$ and $X \setminus U_2$ are disjoint closed subsets of X. Since X is normal there exists, by Lemma 1.43, a subset $W_1 \subset X$ with $X \setminus U_1 \subset W_1 \subset \overline{W_1} \subset U_2$. In particular we have $\overline{W_1} \cap (X \setminus U_2) = \varnothing$. Again, because of Lemma 1.43, there exists a subset $W_2 \subset X$ with $X \setminus U_2 \subset W_2 \subset \overline{W_2} \subset X \setminus \overline{W_1}$. In particular we have $\overline{W_2} \cap \overline{W_1} = \varnothing$. We claim that $V_1 = X \setminus \overline{W_1}$ and $V_2 = X \setminus \overline{W_2}$ have the desired properties.

First note that for $i = 1, 2$ we have

$$V_i \;=\; X \setminus \overline{W_i} \;\subset\; X \setminus W_i \;\subset\; U_i.$$

Finally note that

$$V_1 \cup V_2 \;=\; (X \setminus \overline{W_1}) \cup (X \setminus \overline{W_1}) \;=\; X \setminus (\overline{W_1} \cap \overline{W_2}) \;=\; X.$$

We refer to Figure 25 for an illustration. □

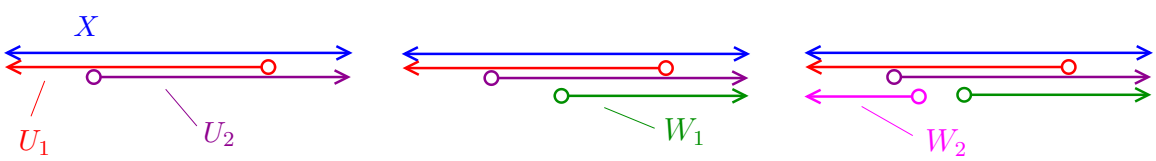

FIGURE 25.

We conclude this discussion of normal topological spaces with two significantly deeper results, namely Urysohn's Lemma and the Tietze Extension Theorem. We will not make use of these results, thus we will not provide the (non-trivial!) proofs.

We start out with what is known as Urysohn's Lemma. Despite its modest name it is actually highly non-trivial

Lemma 1.45. (Urysohn's Lemma) *Let X be a topological space X. If X is normal, then given any two disjoint closed subsets A and B there exists a continuous function $f\colon X \to \mathbb{R}$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. Proofs are provided in many text books on general topology, see e.g. [Kel, Lemma 4.4], [Mun3, Theorem 4.3.1] or [Jä1, p. 109]. □

Theorem 1.46. (Tietze Extension Theorem) *Let X be a topological space, let $A \subset X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a continuous map. If X is normal, then there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$.*

Remark. Let X be a topological space and let A and B be two disjoint closed subsets. We consider the map on $A \cup B$ that is given by $f|_A \equiv 0$ and $f|_B \equiv 1$. If X is normal, then we can extend f by the Tietze Extension Theorem 1.46 to a continuous map on all of X. This shows that Urysohn's Lemma is an immediate consequence of the Tietze Extension Theorem 1.46.

Proof. We refer to [Kel, p. 242], [Mun3, Theorem 4.3.2] or [Jä1, p. 114] for proofs. □

1.7. **Convex subsets.** We start out with the definition of a convex subset.

Definition.

(1) We say that a subset $A \subset \mathbb{R}^n$ is *convex*, if for any two points P and Q in A the segment $\overline{PQ} := \{tP + (1-t)Q \,|\, t \in [0,1]\}$ also lies in A.
(2) Given a subset S of $\mathbb{R}^k$ the *convex hull* of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

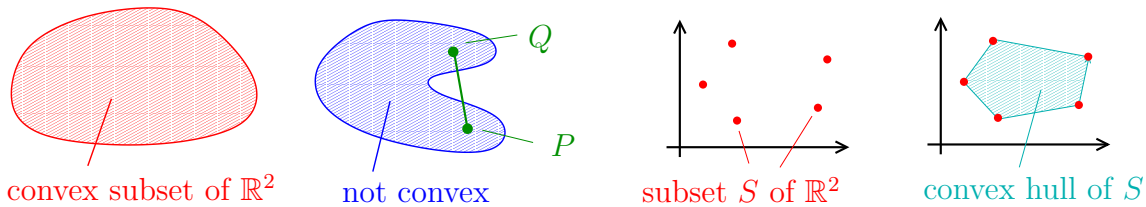

FIGURE 26.

Examples.

(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\substack{\uparrow \\ \text{triangle inequality and homogeneity of } \|-\|}}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\substack{\uparrow \\ \text{since } P,Q \in B^n_\epsilon(x)}}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

We have thus shown that $t \cdot P + (1-t) \cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.

(2) It is elementary to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

(3) It is an invigorating exercise to show that the convex hull of points $P_1, \dots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{ \sum_{i=1}^{n} t_i P_i \,\Big|\, t_1, \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \Big\}.$$

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 1.47.

(1) *If A is a bounded open convex subset of $\mathbb{R}^n$, then given any $Q \in A$ there exists a canonical*[13] *homeomorphism*

$$f_Q \colon A \;\to\; B^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

with $f_Q(Q) = 0$.

(2) *If A is a bounded closed*[14] *convex subset of $\mathbb{R}^n$, then given any point Q in the interior of A*[15] *there exists a canonical homeomorphism*[16]

$$f_Q \colon A \;\to\; \overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

with the following properies:

(a) *the homeomorphism f_Q sends Q to the origin,*

(b) *the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$,*

Examples.

(1) The open square $(0,1) \times (0,1)$ is clearly bounded and convex, thus it follows from Proposition 1.47 (1) that it is homeomorphic to the open disk B^2.

(2) It follows from Proposition 1.47 (2) that any triangle, i.e. any subset of $\mathbb{R}^2$ of the form $A = \{P + sv + tw \mid s, t \in [0,1] \text{ and } s + t \leq 1\}$ where $P \in \mathbb{R}^2$ and v, w are two linearly independent vectors, is homeomorphic to $\overline{B}^2$.

For readers who like gory details we present the more technical sibling of Proposition 1.47.

Proposition 1.48.

(1) *Let A is a bounded open convex subset of $\mathbb{R}^n$ and let $Q \in A$. Given $x \in A \setminus \{Q\}$ we define*[17]

$$\rho(x) \;:=\; \sup\big\{ \|rx\| \,\big|\, r \in \mathbb{R}_{>0} \textit{ and } r(x - Q) + Q \in A \big\}.$$

[13]Throughout the lecture notes we will on occasions write that "some objects exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.

[14]Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 1.21 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to statement (1) of the lemma.

[15]Note that the interior might well be empty, i.e. such a point Q might not exist.

[16]This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.

[17]The statement becomes slightly more readable if one sets $Q = 0$.

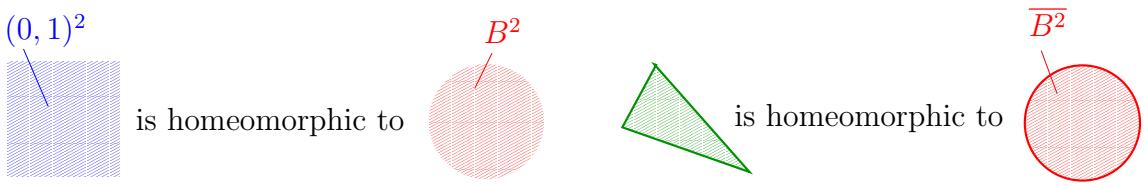

FIGURE 27.

The supremum exists for each $x \in A$ and the map

$$\begin{array}{rcl} f_Q\colon A & \to & B^n \\ x & \mapsto & \begin{cases} (x-Q)\cdot \frac{1}{\rho(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

is a homeomorphism with $f_Q(Q) = 0$.

(2) *Let A be a bounded closed convex subset of $\mathbb{R}^n$. If Q is a point in the interior of A, then the same expression as in (1) defines a homeomorphism $f_Q\colon A \to \overline{B}^n$ with $f_Q(Q) = 0$. Furthermore f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.*

Example. For $i = 1, \dots, n$ let $a_i < b_i$ be real numbers. We consider the "hyperrectangle" $[a_1, b_1] \times \cdots \times [a_n, b_n]$. It follows from the discussion on page 78 and from Proposition 1.22 that the hyperrectangle is convex and compact. Proposition 1.48 (2) gives us an explicit homeomorphism

$$f_Q\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \ \to \ \overline{B}^n$$

that sends the boundary of the hyperrectangle to the sphere S^{n-1} and that sends the "midpoint" $Q := (\frac{a_1+b_1}{2}, \dots, \frac{a_n+b_n}{2})$ to the origin. Throughout this course we will use this homeomorphism to identify the left-hand side with the right-hand side. It follows easily from the explicit description of f_Q that the restriction of f_Q to each open cone on a side is smooth.[18] The homeomorphism from Proposition 1.48 (2) is illustrated in Figure 28.

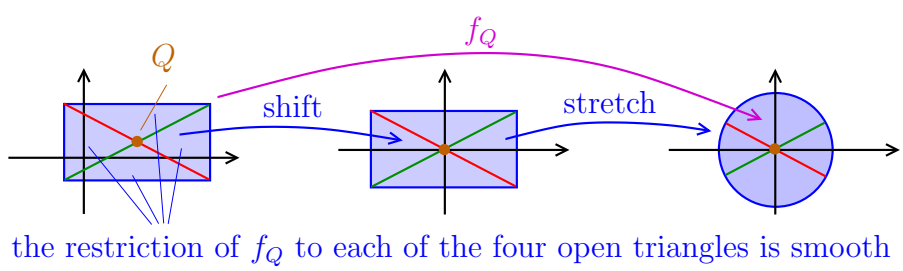

FIGURE 28.

[18]We leave it to the reader to make it precise what "open cone on a side" means. We will make use of this observation only once, and then it should be clear what we mean.

Proof $(*)$.

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that Q is the origin, in particular we can assume that the origin 0 lies in the interior of A. Given $x \in A \setminus \{0\}$ we define

$$\rho(x) \;:=\; \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } rx \in A\big\}$$

and

$$f(x) \;:=\; x \cdot \frac{\rho(x)}{\|x\|}.$$

Since A is closed we have $f(x) \in A$.

Claim. The map $\rho\colon A \setminus \{0\} \to \mathbb{R}$ is positive, continuous and bounded from above and bounded from below by 1.

Since A is bounded it follows that ρ is bounded from above. It follows from the definition of ρ that ρ is bounded from below by 1. Therefore it remains to show that ρ is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

(a) there exists an open neighborhood U of x in $A \setminus \{0\}$ such that $\rho(y) > \rho(x) - \epsilon$ for all $y \in U$,

(b) there exists an open neighborhood V of x in $A \setminus \{0\}$ such that $\rho(y) < \rho(x) + \epsilon$ for all $y \in V$.

We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho(x))$.

Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subset A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f(x)\}$ is still contained[19] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho(x)-\epsilon}$ the sphere of radius $\rho(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho(x)-\epsilon}$ that is contained in C'. We set

$$U \;:=\; \{rz \,|\, z \in U' \text{ and } r \in (0,1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho(x) - \epsilon) \cdot y \in A$, i.e. for any $y \in A$ we have $\rho(y) \geq \rho(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [Be, Chapter 11.3] for full details. This concludes the proof of the claim. Now we consider the map

$$\begin{aligned} f_Q\colon A &\to \overline{B}^n \\ x &\mapsto \begin{cases} x \cdot \frac{1}{\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{aligned}$$

[19] Why is that the case?

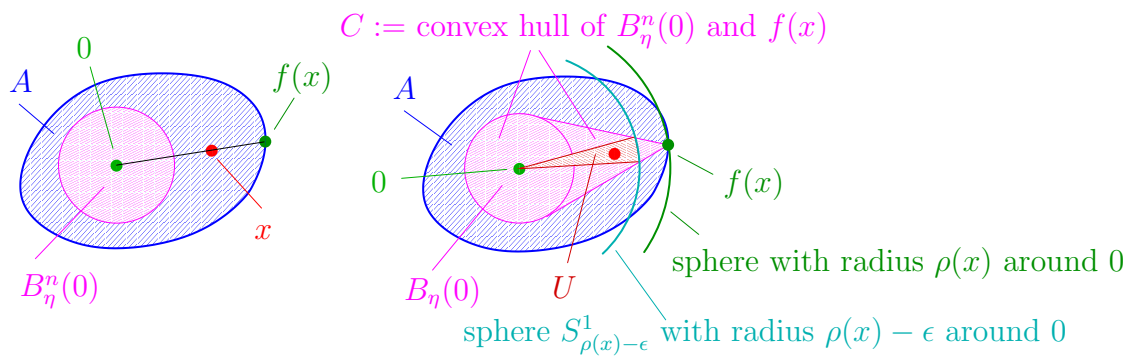

FIGURE 29. Illustration for the proof of Proposition 1.48.

It is straightforward to verify that this map is bijective[20] and it follows from the above claim[21] that f_Q is continuous. Note that our hypothesis on A and Heine-Borel Theorem 1.21 imply that A is compact. Thus we obtain from Proposition 1.40 that f_Q is in fact a homeomorphism. We leave it as a, potentially slightly annoying, exercise to verify that $f_Q(\partial A) = S^{n-1}$.

(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex we know by Exercise 1.4 that $A = \mathring{C}$. One fairly easily verifies that C is convex. It is clear that C is bounded and it follows from Lemma 1.10 that C is closed. One verifies easily that the map f_Q defined for $C = \overline{A}$ restricts to the map f_Q defined for A. Thus it follows from (2) that the map $f_Q \colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_Q \colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. □

Remark. One again let A be a bounded closed convex subset A of $\mathbb{R}^n$ and let Q be a point in the interior of A. Proposition 1.48 provides us with a homeomorphism $f \colon A \to \overline{B}^n$. Since ∂A is not necessarily "smooth" (here we use smooth in a naive sense) one cannot hope to get such a map f that is smooth. But it does make sense to ask whether one can find such a map f such that the restriction to the interior $\mathring{A}$ of A is smooth, or even better, a diffeomorphism. In general the homeomorphism that we provide in Proposition 1.48 will not have this property. But we can modify the above construction to obtain a homeomorphism $f \colon A \to \overline{B}^n$ such that the restriction to $\mathring{A} \to B^n$ is a diffeomorphism. This can be achieved by replacing the linear stretching on the rays by an appropriately chosen different map,

[20]It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_Q that for each $v \in S^{n-1}$ the map f_Q restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

$$A \cap \{rv \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{rv \,|\, r \in \mathbb{R}_{>0}\}.$$

[21]From the continuity of ρ it follows that f_Q is continuous on $A \setminus \{0\}$. From the fact that ρ is bounded it follows easily that f_Q is continuous in 0.

e.g. we should stretch by 1 close to the origin to obtain differentiability at 0. It takes some effort to carry out this program rigorously, details can be found in [Fer, Satz 237].[22] Other approaches to finding homeomorphisms $f\colon A \to \overline{B}^n$ that restrict to a diffeomorphism $\mathring{A} \to B^n$ are sketched in [BJ, p. 86] and [GT, p. 60].

1.8. **The disjoint union of topological spaces** $(*)$**.** we start out with a short set-theoretic discussion. Sometimes, given two sets A and B one would like to consider a new set that is the "disjoint union of A and B". This leads us to the following, arguably slightly awkward definition.

Definition.

(1) Let A and B be two sets. We define the *disjoint union of A and B* as[23]

$$A \sqcup B \;:=\; (A \times \{1\}) \cup (B \times \{2\}).$$

(2) If $\{A_i\}_{i\in I}$ is a family of sets, then we define the *disjoint union* as

$$\bigsqcup_{i\in I} A_i \;:=\; \bigcup_{i\in I} (A_i \times \{i\}).$$

Convention. Let A and B be two sets. In most situations, when there is no danger of confusion, we use the obvious bijections $A = A \times \{1\}$ and $B = B \times \{1\}$ to identify A and B with the corresponding subsets of $A \sqcup B$. In particular, if A and B are already disjoint, then we make the identification $A \sqcup B = A \cup B$. The same convention applies to the disjoint union $\bigsqcup_{i\in I} A_i$. At times the author of these notes will rely on the good will of the reader to correctly interpret a given statement.

Definition. Given a family $\{X_i\}_{i\in I}$ of topological spaces we equip the disjoint union $\bigsqcup_{i\in I} X_i$ with the *disjoint union topology* that is defined by the condition that $U \subset \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $U \cap X_i$ is an open subset of X_i.

Example.

(1) One can easily verify that the topological space $S^1 \sqcup S^1$ is homeomorphic to the topological space $S^1(-2,0) \cup S^1(2,0) \subset \mathbb{R}^2$.
(2) In Exercise 1.22 we will see that the topological space $[-1,0] \sqcup (0,1]$ is homeomorphic to $[-2,-1] \cup (0,1] \subset \mathbb{R}$ and that it is not homeomorphic to $[-1,1]$.

The following lemma summarizes a few very basic properties of the disjoint union of two topological spaces. We leave the dreadfully dull proof as an exercise to the reader.

Lemma 1.49. $(*)$ *Let X, Y and Z be topological spaces.*

[22] It is worth noting that this diffeomorphism is orientation-preserving. (See later for the definition of orientation-preserving.)

[23] In other words, we replace A and B by sets that are disjoint but that admit obvious bijections to A and B.

(1) *The inclusion maps* $X \to X \sqcup Y$ *and* $Y \to X \sqcup Y$ *are homeomorphisms onto their images.*
(2) *A map* $f\colon X \sqcup Y \to Z$ *is continuous if and only if* $f|_X$ *and* $f|_Y$ *are continuous.*
(3) *Let* $f\colon X \to A$ *and* $g\colon Y \to B$ *be two continuous maps to topological spaces* A *and* B*. The map* $f \sqcup g\colon X \sqcup Y \to A \sqcup B$ *that is defined by* f *on* X *and by* g *on* Y *is continuous.*
(4) *The obvious bijection* $X \sqcup Y \to Y \sqcup X$ *is a homeomorphism.*
(5) *The obvious bijection* $(X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z)$ *is a homeomorphism.*
(6) *The topological spaces* X *and* Y *are compact if and only if* $X \sqcup Y$ *is compact.*

For Statements (1)–(4) *the obvious analogues also hold if we consider the disjoint union of arbitrary families of topological spaces.*

1.9. **The product of finitely many topological spaces.** In the following two sections we introduce and discuss the topology on the product of topological spaces. In this section we deal with the product of *finitely* many topological spaces which is what we are mostly interested in. In the following section we will quickly discuss the product of arbitrarily many topological spaces.

Lemma 1.50. *Let* $X_1, \dots, X_k$ *be topological spaces. The set*

$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\; \textit{each } U_i \textit{ is open in } X_i\}$$

has the basis property from page 60.

Proof (∗). We need to verify that $\mathcal{B}$ satisfies the conditions (B1) and (B2) formulated on page 60.

(B1) We have $X_1 \times \cdots \times X_k \in \mathcal{B}$. Thus given any $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$ there exists a set in $\mathcal{B}$, namely $X_1 \times \cdots \times X_k$, which contains $(x_1, \dots, x_k)$. Thus $\mathcal{B}$ satisfies (B1).
(B2) Let $U_1 \times \cdots \times U_k$ and $V_1 \times \cdots \times V_k$ be two sets in $\mathcal{B}$. Then

$$(U_1 \times \cdots \times U_k) \cap (V_1 \times \cdots \times V_k) \;=\; (U_1 \cap V_1) \times \cdots \times (U_k \cap V_k)$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2). □

Definition. Let $X_1, \dots, X_k$ be topological spaces. The *product topology on* $X_1 \times \cdots \times X_k$ is the topology on $X_1 \times \cdots \times X_k$ that is generated by

$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\; \text{each } U_i \text{ is open in } X_i\}.$$

Convention. Given topological spaces $X_1, \dots, X_k$ we always equip $X_1 \times \cdots \times X_k$ with the product topology, unless we explicitly say something else.

Example. Let X be a topological space and let $\{P\}$ be the topological space consisting of a single point (note that $\{X\}$ has a unique topological structure!). The obvious bijection $X \to X \times \{P\}$ is basically by definition a homeomorphism. On many occasions we will use this homeomorphism to make the identification $X = X \times \{P\}$.

The following lemma will get used implicitly on numerous occasions.

Lemma 1.51. $(*)$ *Let $X_1, \dots, X_k$ be topological spaces. If $A_i \subset X_i$, $i = 1, \dots, k$ are closed subsets, then $A_1 \times \cdots \times A_k$ is a closed subset of $X_1 \times \cdots \times X_k$.*

Proof $(*)$. We have

$$(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k) \underset{\substack{\uparrow \\ \text{basic set theory}}}{=} \bigcap_{i=1}^{k} \underbrace{X_1 \times \cdots \times (X_i \setminus A_i) \times \cdots \times X_k}_{\text{open in } X_1 \times \cdots \times X_k \text{ since } A_i \text{ is closed}}.$$

Thus we see that $(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k)$ is the intersection of finitely many open sets, i.e. the set itself is open, which implies by definition that $A_1 \times \cdots \times A_k$ is closed. $\square$

The following lemma seems obvious, but as so many statements in general topology, the actual proof requires a little bit of thought.

Lemma 1.52. $(*)$ *For any $k, l \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \mathbb{R}^k \times \mathbb{R}^l & \to & \mathbb{R}^{k+l} \\ ((x_1, \dots, x_k), (y_1, \dots, y_l)) & \mapsto & (x_1, \dots, x_k, y_1, \dots, y_l) \end{array}$$

is a homeomorphism.

Convention. Henceforth, given $k_1, \dots, k_m \in \mathbb{N}$ we will always identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with $\mathbb{R}^{k_1 + \cdots + k_m}$. In particular, given $X_i \subset \mathbb{R}^{k_i}$, $i = 1, \dots, m$, we view $X_1 \times \cdots \times X_m$ as a subset of $\mathbb{R}^{k_1 + \cdots + k_m}$.

Proof $(*)$. It is clear that the map is a bijection. So it remains to show that the map and its inverse are continuous. The continuity of both maps can be shown using Proposition 1.34 and Lemma 1.8. We leave it to the reader to fill in the details. $\square$

Examples.

(1) Let $s_1, \dots, s_n \in \mathbb{N}_0$. We set $s = s_1 + \cdots + s_n$. By the above convention we can view $B^{s_1} \times \cdots \times B^{s_n}$ as a subset of $\mathbb{R}^s$. By definition this is an open subset of $\mathbb{R}^s$ and one verifies easily that it is convex and bounded. Thus it follows from Proposition 1.47 that there exists a canonical homeomorphism $B^{s_1} \times \cdots \times B^{s_n}$ to the ball B^s that sends $(0, \dots, 0)$ to the origin. The same statement holds for the corresponding closed balls.

(2) Let (X, d) be a metric space. In Exercise 1.26 we will see that the map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

is continuous.

Before we can discuss more examples we introduce the following convention:

Convention.

(1) Given $m \in \mathbb{N}_0$ we make the identification

$$\begin{array}{rcl} \mathbb{C}^m & \xrightarrow{=} & \mathbb{R}^{2m} \\ (x_1 + iy_1, \dots, x_m + iy_m) & \mapsto & (x_1, y_1, x_2, y_2, \dots, x_m, y_m). \end{array}$$

(2) Given $m, n \in \mathbb{N}_0$ we make the identification

$$\begin{array}{rcl} M(m \times n, \mathbb{R}) & \xrightarrow{=} & \mathbb{R}^{m \cdot n} \\ (a_{ij})_{i=1,\dots,m, j=1,\dots,n} & \mapsto & (a_{11}, \dots, a_{1n}, \dots, a_{n1}, \dots, a_{nm}) \end{array}$$

Similarly we identify $M(m \times n, \mathbb{C})$ with $\mathbb{R}^{2mn}$.

(3) We use these maps to view (every subset of) $\mathbb{C}^m$, $M(m \times n, \mathbb{R})$ and $M(m \times n, \mathbb{C})$ as topological spaces.

Examples.

(1) It follows almost immediately from our conventions and the examples on page 65 that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} M(k \times l, \mathbb{R}) \times M(l \times m) & \to & M(k \times m) \\ (A, B) & \mapsto & A \cdot B \end{array}$$

is continuous. The same also applies to matrices over $\mathbb{C}$.

(2) It follows from (1) and Lemma 1.31 that

$$\mathrm{O}(n) = \{A \in M(n \times n; \mathbb{R}) \mid A^T A = \mathrm{id}\} = f^{-1}(\mathrm{id}_n) \quad \text{with} \quad \begin{array}{l} M(n \times n; \mathbb{R}) \xrightarrow{f} M(n \times n; \mathbb{R}) \\ \qquad A \mapsto A \cdot A^T \end{array}$$

is a closed subset of $\mathbb{R}^{n^2}$. Since each column of an orthogonal matrix has norm one we see that $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$ is a bounded subset. Thus it follows from the Heine-Borel Theorem 1.21 that $\mathrm{O}(n)$ is compact. Basically the same argument shows that the set $\mathrm{U}(n)$ of unitary $n \times n$-matrices is a compact topological space.

The following lemma summarizes some key properties of the product topology. We will make use of this lemma on so many occasions that usually we do not mention it explicitly.

Lemma 1.53.

(1) (a) *The product topology is "commutative" in the sense that given two topological spaces A and B the obvious map $A \times B \to B \times A$ is a homeomorphism.*

(b) *The product topology is "associative" in the sense that given three topological spaces A, B and C the identity map is a homeomorphism between $(A \times B) \times C$, $A \times B \times C$ and $A \times (B \times C)$.*

(c) *The product topology commutes with subspace topology. More precisely, let X and Y be two topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. Then the subspace topology of $A \times B \subset X \times Y$ agrees with product topology of A and B.*

(2) *Given any two topological spaces X and Y the obvious bijection between $X \times Y$ and $Y \times X$ is a homeomorphism.*

(3) *Let $X_1, \dots, X_k$ be topological spaces.*
   (a) *For each $i \in \{1, \dots, k\}$ the projection map*
$$\begin{array}{rcl} X_1 \times \cdots \times X_k & \to & X_i \\ (x_1, \dots, x_k) & \mapsto & x_i \end{array}$$
   is continuous. In fact, the product topology on $X_1 \times \cdots \times X_k$ is the smallest topology[24] *on $X_1 \times \cdots \times X_k$ that has the property that all the projection maps are continuous.*
   (b) *Let $i \in \{1, \dots, k\}$ and for $j \neq i$ let $x_j \in X_j$. The inclusion map*
$$\begin{array}{rcl} X_i & \to & X_1 \times \cdots \times X_k \\ x & \mapsto & (x_1, \dots, x, \dots, x_k) \end{array}$$
   is continuous. Furthermore the above inclusion map induces a homeomorphism
$$X_i \xrightarrow{\cong} \{x_1\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \times \{x_k\}.$$
   (c) *If $f_i \colon X_i \to Y_i$, $i = 1, \dots, k$ are continuous maps between topological spaces, then the map*
$$\begin{array}{rcl} X_1 \times \cdots \times X_k & \to & Y_1 \times \cdots \times Y_k \\ (x_1, \dots, x_k) & \mapsto & (f_1(x_1), \dots, f_k(x_k)) \end{array}$$
   is continuous.
   (d) *If $f_i \colon X \to Y_i$, $i = 1, \dots, k$ are continuous maps between topological spaces, then the map*
$$\begin{array}{rcl} X & \to & Y_1 \times \cdots \times Y_k \\ x & \mapsto & (f_1(x), \dots, f_k(x)) \end{array}$$
   is continuous.

Proof. All the statements are easily verified by hand. I have never met anybody who wanted to see the full details. □

The following remark can be viewed as a sibling of the future Remark 1.67.

Remark 1.54. *We consider maps in and out of a product space $Y_1 \times Y_2$:*

(1) *In practice it is usually easy to show that a given map $X \to Y_1 \times Y_2$ to a product space is continuous. More precisely, by Lemma 1.53 (3d) we "just" need to show that the two coordinate maps $X \to Y_1$ and $X \to Y_2$ are continuous.*
(2) *There is usually no easy way to show that a map $Y_1 \times Y_2 \to X$ out of a product space is continuous.*

The following lemma is so essential that usually we will not mention it explicitly.

Lemma 1.55. *Let $X_1, \dots, X_k$ be non-empty topological spaces.*

[24] Here "smallest" means that any topology on $X_1 \times \cdots \times X_k$ that has that property contains the product topology.

(1) *The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.*
(2) *The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.*

Remark. Lemma 1.55 (2) also shows how the case $n = 1$ in Proposition 1.22 implies the general case of Proposition 1.22.

Proof $(*)$. By induction we only have to deal with the case $k = 2$. Thus let X and Y be two topological spaces.

First we deal with the "$\Rightarrow$"-direction of statement (1). Thus we assume that X and Y are Hausdorff. Let (x, y) and (x', y') be two distinct points in $X \times Y$. Without loss of generality we can assume that $x \neq x'$. By our hypothesis there exist open neighborhoods U of x and U' of x' with $U \cap U' = \varnothing$. Note that

$$(U \times Y) \cap (U' \times Y) \;=\; (U \cap U') \times Y \;=\; \varnothing \times Y \;=\; \varnothing.$$

By definition of the product topology the sets $U \times Y$ and $U' \times Y$ are open in $X \times Y$. We have thus found disjoint open neighborhoods of (x, y) and (x', y'). Therefore $X \times Y$ is Hausdorff. (This argument is sketched in Figure 30 to the left.)

Now we deal with the '$\Leftarrow$"-direction of statement (1). Therefore we suppose that $X \times Y$ is Hausdorff. We want to show that X is Hausdorff. Thus let x and x' be two different points in X. Since Y is non-empty there exists a $y \in Y$. Furthermore, since $X \times Y$ is Hausdorff there exist disjoint open neighborhoods W and W' of (x, y) and (x', y). By definition of the product topology there exist open neighborhoods U of x and U' of x' and open neighborhoods V of y and V' of y such that $(U \times V) \cap (U' \times V') = \varnothing$. In particular we see that $(U \times \{y\}) \cap (U' \times \{y\}) = \varnothing$. But this implies that $U \cap U' = \varnothing$. In particular U and U' are disjoint open neighborhoods of x and x'. Thus we have shown that X is Hausdorff. (This argument is sketched in Figure 30 to the right.) The proof that Y is also Hausdorff is evidently basically the same.

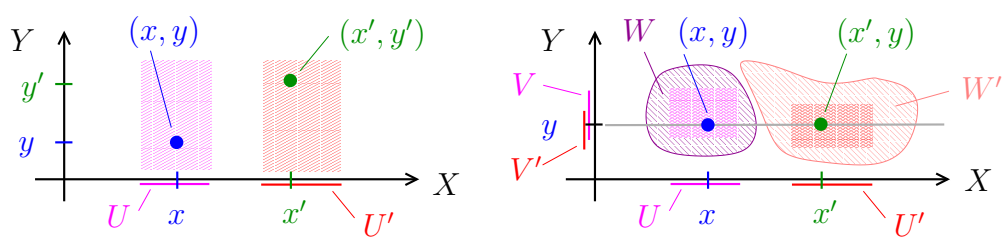

FIGURE 30. Illustration of the proof of Lemma 1.55 (1).

The '$\Rightarrow$"-direction of statement (2) is fortunately elementary, it follows immediately from Lemmas 1.53 and Lemma 1.37 (1).

Finally we deal with '$\Leftarrow$"-direction of statement (2). This is by far the hardest part of the lemma. We start out with the following claim.

Claim. Let $x_0 \in X$. If N is an open set of $X \times Y$ that contains $\{x_0\} \times Y$, then there exists a neighborhood W of $x_0 \in X$ such that $W \times Y \subset N$.

By definition of the product topology there exist sets of the form $U_i \times V_i$, $i \in I$ with the following properties:

(1) each U_i is open in X and each V_i is open in Y,
(2) each U_i contains x_0, i.e. for each i we have $\{x_0\} \times V_i \subset U_i \times V_i$,
(3) each $U_i \times V_i$ is contained in N,
(4) $\{x_0\} \times Y$ is contained in the union of the $U_i \times V_i$.

Since $\{x_0\} \times Y$ is compact we can arrange that I is a finite set, i.e. we can assume that $I = \{1, \ldots, n\}$. We set $W = U_1 \cap \cdots \cap U_n$. This is an open subset of X that contains x_0. It follows immediately from (3) and the definition of W that $W \times Y \subset N$. This concludes the proof of the claim.

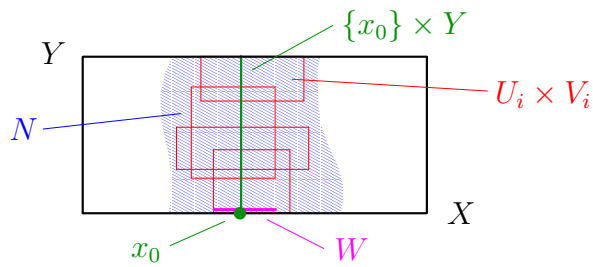

FIGURE 31. Illustration of the proof of Lemma 1.55 (2).

Now we turn to the actual proof that $X \times Y$ is compact. Thus suppose we are given an open covering $\{A_i\}_{i \in I}$ of $X \times Y$. Let $x \in X$. Since Y is compact there exists a finite set $I(x) \subset I$ such that $\{x\} \times Y \subset \bigcup_{i \in I(x)} A_i$. By the above claim there exists an open neighborhood $W(x)$ of x such that $W(x) \times Y \subset \bigcup_{i \in I(x)} A_i$.

All these open sets $\{W(x)\}_{x \in X}$ are an open covering of X. Since X is compact there exist $x_1, \ldots, x_k \in X$ such that $X = W(x_1) \cup \cdots \cup W(x_k)$. But then we see that

$$X \times Y \;=\; (W(x_1) \cup \cdots \cup W(x_k)) \times Y \;=\; \bigcup_{j=1}^{k} W(x_j) \times Y \;\subset\; \bigcup_{j=1}^{k} \bigcup_{i \in I(x_j)} A_i \;\subset\; X \times Y.$$

Thus we have successfully covered $X \times Y$ by finitely many sets in the given open covering. $\square$

Before we can state our next lemma we need to introduce the following definition.

Definition. Let X be a metric space. Given two non-empty subsets A and B we refer to

$$d(A, B) \;=\; \inf\{d(a, b) \,|\, a \in A \text{ and } b \in B\}$$

as the *distance between A and B*. If $A = \{a\}$ consists of a single point, then we write $d(a, B) := d(\{a\}, B)$.

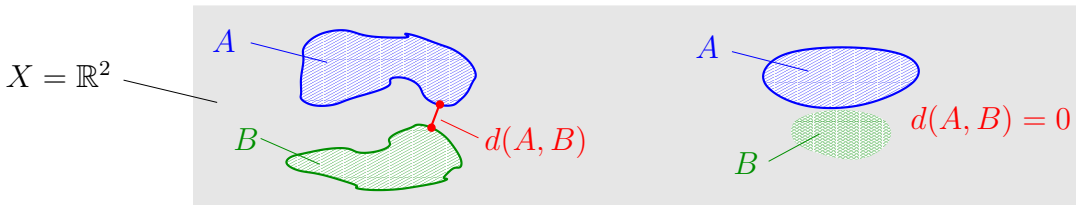

FIGURE 32.

Example. If two subsets A and B intersect, then we evidently have $d(A, B) = 0$. But in general the converse does not hold. For example if we consider $X = \mathbb{R}$, then

$$d\big(\underbrace{\text{open interval } (0,1)}_{=A}, \underbrace{\text{closed interval } [1,2]}_{=B}\big) \;=\; 0,$$

but evidently the intervals do not intersect.

The reader might justly suspect that the phenomenon of the last example is due to the fact that one of the sets is not compact. In several arguments the following corollary will play a crucial role.

Corollary 1.56. *Let X be a metric space and let K and L be two non-empty subsets. If both K and L are compact (for example if one or both of them are finite), then there exist $x \in K$ and $y \in L$ with*

$$d(x, y) \;=\; d(K, L).$$

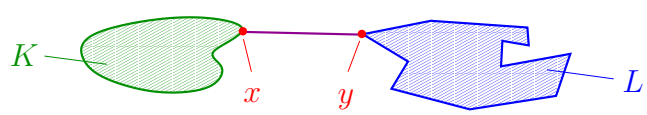

FIGURE 33.

Proof. It follows from our hypothesis and Lemma 1.55 that $K \times L$ is compact. As we pointed out above, the metric $d\colon K \times L \to \mathbb{R}_{\geq 0}$ is continuous. The corollary is now an immediate consequence of Lemma 1.37 (2). □

In Exercise 1.27 we will prove the following lemma.

Lemma 1.57. *Let X be a metric space. If $B \subset X$ is a non-empty subset, then the map*

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, B) := \inf\{d(x, b) \,|\, b \in B\} \end{array}$$

is continuous.

Next we introduce some of the most popular product topological spaces.

Definition.

(1) We refer to $(S^1)^n$ as the *n-dimensional torus*. For $n = 2$ we just say *the torus*.
(2) We refer to $S^1 \times \overline{B}^2$ as the *solid torus* and we refer to $S^1 \times B^2$ as the *open solid torus*.

We deduce from Lemma 1.55 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

The following lemma allows us to view (solid) tori as subsets of $\mathbb{R}^3$.

Lemma 1.58. *We consider the map*[25]

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^2 \times S^1 & \to & \mathbb{R}^3 \\ ((x,y), e^{i\varphi}) & \to & \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\textit{rotation around } z\textit{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{2}x \\ 0 \\ -\frac{1}{2}y \end{pmatrix}}_{\substack{\textit{describes circle} \\ \textit{in } xz\textit{-plane}}} \end{array}$$

The restrictions of Θ to $S^1 \times S^1$, $\overline{B}^2 \times S^1$ and $B^2 \times S^1$ are homeomorphisms onto their images. (The restriction of Θ to $S^1 \times S^1$ is illustrated in Figure 34).

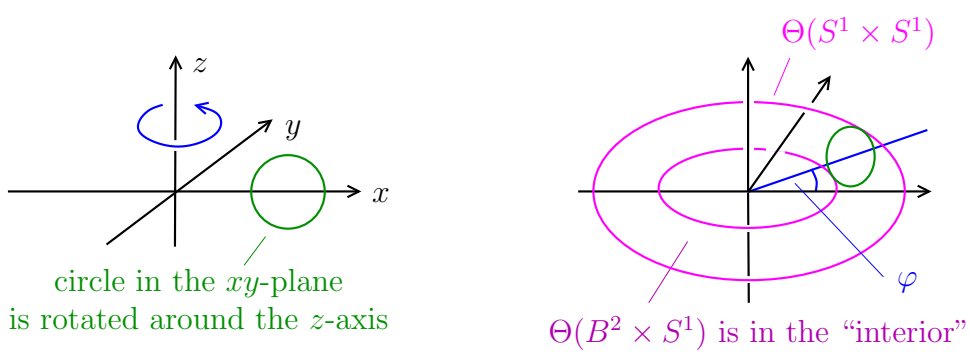

FIGURE 34.

Proof. It follows easily from the example on page 86 that the map $\Theta\colon S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ is continuous. An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. Since the solid torus $\overline{B}^2 \times S^1$ is compact we obtain from Proposition 1.40 that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is a homeomorphism. But then it follows also that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are homeomorphisms onto their images. $\square$

[25]The map is chosen in such a way that it is orientation-preserving (we will introduce the notions of orientation and orientation later on) and such that $\Theta(\{(0,0)\} \times S^1) = S^1 \times \{0\}$.

Lemma 1.58 allows us to view (solid) tori as subsets of $\mathbb{R}^3$. The typical picture is shown in Figure 35. We refer to the images under these maps as the *standard* (*solid*) (*open*) *torus in* $\mathbb{R}^3$. Unfortunately, in a picture, it is hard to distinguish between tori, solid tori and open solid tori.

$S^1 \times S^1$ is homeomorphic to

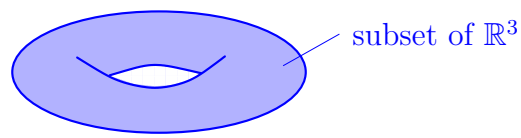

FIGURE 35.

We conclude this discussion of products of topological spaces with the following lemma.

Lemma 1.59. *Let $X_1, \dots, X_k$ be topological spaces and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases for these topological spaces. Then*

$$\mathcal{C} \ := \ \{B_1 \times \cdots \times B_k \mid \text{ for each } i \in \{1, \dots, k\} \text{ we have } B_i \in \mathcal{B}_i\}$$

is a basis for $X_1 \times \cdots \times X_k$.

Example. On page 64 we saw that the *rational* intervals form a basis for the usual topology on $\mathbb{R}$. Together with Lemma 1.59 it follows that the set $\mathcal{B}$ of rational hyperrectangles, i.e. that the set

$$\mathcal{B} \ = \ \{\text{all hyperrectangles } [a_1, b_1] \times \cdots \times [a_k, b_k] \text{ with } a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Q}\} \ \subset \ \mathcal{P}(\mathbb{R}^k)$$

forms a basis for the standard topology on $\mathbb{R}^k$. In particular we see that $\mathbb{R}^k$ admits a basis for the topology that consists of only countably many sets.

Proof $(*)$. The idea is to prove the lemma using Lemma 1.28 (1). Thus let $W \subset X_1 \times \cdots \times X_k$ be an open set and let $(x_1, \dots, x_k) \in W$. By definition of the product topology there exist open sets $U_i \subset X_i$, $i = 1, \dots, k$ with $(x_1, \dots, x_k) \in U_1 \times \cdots \times U_k$ and $U_1 \times \cdots \times U_k \subset W$.

Let $i \in \{1, \dots, k\}$. Since $\mathcal{B}_i$ is a basis for X_i we see that there exists a $B_i \in \mathcal{B}_i$ with $x \in B_i \subset U_i$. It follows that

$$(x_1, \dots, x_k) \ \in \ B_1 \times \cdots \times B_k \ \subset \ U_1 \times \cdots \times U_k \ \subset \ W.$$

We deduce from Lemma 1.28 (1) that the given set $\mathcal{C}$ is indeed a basis for $X_1 \times \cdots \times X_k$. $\square$

1.10. **The product of arbitrarily many topological spaces** $(*)$**.** In this section we discuss the product of arbitrarily many topological spaces. In these notes we will only rarely deal with the case of the product of infinitely many topological spaces. Therefore we decided to deal with this case separately.

Let $\{X_i\}_{i \in I}$ a family of sets. The product of these sets is defined as the set

$$\prod_{i \in I} X_i \ := \ \Big\{ f \colon I \to \bigcup_{i \in I} X_i \,\Big|\, \text{ for every } i \in I \text{ we have } f(i) \in X_i \Big\}.$$

If the X_i are topological spaces, then we intend to equip the product with a suitable topology. Before we specify the topology we should decide what statements we want to hold. It seems reasonable to demand the following:

(A) given any $i \in I$ the projection map

$$\begin{array}{rcl} p_i \colon \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

should be continuous,

(B) given a topological space Y and continuous maps $f_i \colon Y \to X_i$ the resulting map

$$\begin{array}{rcl} F \colon Y & \to & \prod_{i\in I} X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(y) \end{array} \right). \end{array}$$

should be continuous.

The following construction provides us the with the desired topology on $\prod_{i\in I} X_i$.

Lemma 1.60. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We set*

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}.$$

Then the following statements hold:

(1) *The set $\mathcal{B}$ has the basis property that we introduced on page 60.*
(2) *The topology $\mathcal{T}(\mathcal{B})$ is the smallest topology such that the above properties (A) and (B) hold.*

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The proof that $\mathcal{B}$ has the basis property is basically identical to the proof of Lemma 1.50.
(2) (A) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.
(B) Suppose we are given Y and f_i as above. We denote by F the map defined as above. We claim that F is continuous. By Proposition 1.34 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{U_i \neq X_i} \underbrace{f_i^{-1}(U_i)}_{\text{open in } Y} = \begin{array}{c}\text{intersection of finitely} \\ \text{many open subsets of } Y.\end{array}$$

This concludes the proof that F is continuous.

We have thus shown that $\mathcal{T}(\mathcal{B})$ has the desired properties (A) and (B). But it is also clear that if a topology $\mathcal{S}$ on $\prod_{i\in I} X_i$ has the properties (A) and (B), then $\mathcal{B} \subset \mathcal{S}$.

But this implies that $\mathcal{T}(\mathcal{B}) \subset \mathcal{S}$. This shows that $\mathcal{T}(\mathcal{B})$ is the smallest topology with the desired properties. $\square$

The lemma leads us to the following definition.

Definition. Let $\{X_i\}_{i\in I}$ a family of topological spaces. We refer to the topology on $\prod_{i\in I} X_i$ that we defined in Lemma 1.60 as the *product topology* on $\prod_{i\in I} X_i$.

On several occasions we will use, consciously or unconsciously, the following lemma.

Lemma 1.61.

(1) *If $\{X_i\}_{i\in I}$ is a finite family of topological spaces, i.e. if I is finite, then the above product topology agrees with the product topology introduced on page 84.*
(2) *If we are given finitely many families of topological spaces $\{X_i\}_{i\in I_1}, \dots, \{X_i\}_{i\in I_k}$, then the obvious map*

$$\underbrace{\left(\prod_{i\in I_1} X_i\right) \times \cdots \times \left(\prod_{i\in I_k} X_i\right)}_{\text{product of } k \text{ topological spaces}} \to \prod_{i\in I_1 \sqcup \cdots \sqcup I_k} X_i$$

is a homeomorphism.

With basically the same proof as in Lemma 1.55 (1) one can show the following lemma.

Lemma 1.62. *Let $\{X_i\}_{i\in I}$ be a family of non-empty topological spaces. The product $\prod_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.*

The obvious generalization of Lemma 1.55 (2) also holds. But the proof is now considerably harder and thus the lemma gets upgraded to a theorem:

Theorem 1.63. (Tychonoff's Theorem)[26] *The product of arbitrarily many compact topological spaces is again a compact topological space.*

Sketch of proof. We will not really make use of Tychonoff's Theorem thus we limit ourselves to a sketch of the proof for $I = \mathbb{N}$. Thus let $\{X_i\}_{i\in\mathbb{N}}$ be a family of compact topological spaces. Furthermore let $\{U_j\}_{j\in J}$ be a family of open sets of $\prod_{i\in I} X_i$ such that we cannot cover $\prod_{i\in I} X_i$ by finitely many sets in the given open covering $\{U_j\}_{j\in J}$. We need to show that $\bigcup_{j\in J} U_j$ is a proper subset of $\prod_{i\in I} X_i$.

Claim. There exist $x_i \in X_i$, $i \in \mathbb{N}$ with the following property: Given any $k \in \mathbb{N}$ and given any open neighborhoods V_i of x_i, $i = 1, \dots, k$ the product

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots$$

[26] Andrey Tikhonov (1906-1993) was a Russian mathematician. The spelling for the theorem is due to the fact that Tikhonov initially published in German and his name got transcribed from the Cyrillic alphabet to the Latin alphabet as Tychonoff.

cannot be covered by finitely many sets in the given open covering.

The sequence $\{x_i\}_{i\in\mathbb{N}}$ is defined iteratively. The key to the proof is that we already know by Lemma 1.55 (2) that any finite product $X_1 \times \cdots \times X_k$ is compact. We leave it as an exercise to fill in the details. This concludes the proof of the claim.

The following claim concludes the proof of the theorem for $I = \mathbb{N}$.

Claim. The point $(x_1, x_2, \dots) \in \prod_{i\in I} X_i$ does not lie any of the U_j.

Suppose there exists a $j \in J$ with $(x_1, x_2, \dots) \in U_j$. By definition of the topology on $\prod_{i\in I} X_i$ (see Lemma 1.23) there exists a $k \in \mathbb{N}$ and open neighborhoods V_i of x_i, $i = 1, \dots, k$ such that

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots \ \subset \ U_j.$$

But by construction of $(x_1, x_2, \dots)$ this is impossible. This concludes the proof of the claim.

The proof of the general case of the Tychonoff Theorem can be done in a very similar way by equipping the index set I with a "total order". We refer to [Wri] for details. Alternatively proofs of Tychonoff's Theorem are provided in most textbooks on general topology, see e.g. [Mun3, Chapter 5.1] or [Jä1, Chapter X]. A particularly concise proof is also given in [Chf]. □

We continue with a very instructive example.

Example. We equip $\{0, 2\}$ with the discrete topology. We consider the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from page 58} \\ (f\colon \mathbb{N} \to \{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{aligned}$$

It is elementary to see that φ takes values in C. It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 1.63 and the right-hand is evidently Hausdorff. In Exercise 1.29 we verify that φ is continuous. It thus follows from Proposition 1.40 that φ is a homeomorphism.

Initially the definition of the Cantor set is nicely explicit, but the definition looks rather arbitrary. The previous example gives a much more natural definition of the Cantor set. Amazingly there exists a characterization of the Cantor set, up to homeomorphism, in terms of basic topological properties. To state the proposition we need two easy definitions.

Definition. Let X be a topological space. A point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.

Examples.

(1) We consider $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. The only point in X that is not isolated is 0.
(2) No point in the Cantor set is isolated.
(3) No point in $\prod_{i\in\mathbb{N}} \{0, 2\}$ is isolated.

(4) The topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$ do not contain any isolated points.

Definition. A topological space X is called *totally disconnected* if the only connected subsets are the empty set and the subsets consisting of a single point.

Definition. It is straightforward to verify that the following topological spaces are totally disconnected:

(1) the Cantor set,
(2) the infinite product $\prod_{i \in \mathbb{N}} \{0, 2\}$,
(3) discrete topological spaces,
(4) the topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$.

We conclude this section with the following amusing proposition.

Proposition 1.64. *Let X be a topological space that is compact and totally disconnected. If X contains no isolated points, then X is homeomorphic to the Cantor set.*

Proof. We will not make use of the proposition, hence we will not prove it. A proof is for example given in [Val, Theorem 7.8] or [Wil, Theorem 30.3]. □

Example. By the above examples Proposition 1.64 implies not only that the Cantor set is homeomorphic to $\prod_{i \in \mathbb{N}} \{0, 2\}$ but also to $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$. In particular we obtain the weird result that $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are homeomorphic.[27]

1.11. **Quotients of topological spaces.** We recall the notion of an equivalence relation and an equivalence class, even though almost certainly the reader will have encountered these notions already on numerous occasions elsewhere.

Definition.

(1) A *relation* on a set S is a subset E of $S \times S$. Often, if $(s, t) \in E$, then we write $s \sim t$.
(2) An *equivalence relation* on a set S is a subset E of $S \times S$ with the following three properties:
   (a) for any $s \in S$ we have $(s, s) \in E$ (reflexivity),
   (b) if $(s, t) \in E$, then also $(t, s) \in E$ (symmetry),
   (c) if (s, t) and (t, u) lie in E, then (s, u) also lies in E (transitivity).

   As above, if $(s, t) \in E$, then we write $s \sim t$. In this notation the above three properties translate into
   (a) for any $s \in S$ we have $s \sim s$ (reflexivity),
   (b) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry),
   (c) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity).

[27] Can you prove directly that $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are homeomorphic?

An *equivalence class* of $(S, \sim)$ is a *non-empty* subset C of S with the following two properties:
(a) given any two $s, t \in C$ we have $s \sim t$, and
(b) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

We refer to [Halm, Section 7] for a confirmation of the above definitions.

We also introduce the following notation.

Notation. Given an equivalence relation $\sim$ on a set X and given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x. We refer to the map

$$\begin{array}{ccc} X & \to & X/\sim \\ x & \mapsto & [x] \end{array}$$

as the *canonical projection*. Sometimes $X/\sim$ is called a *quotient* of X.

After these preliminaries we return to topological spaces.

Lemma 1.65. *Let $\sim$ be an equivalence relation on a topological space X. If we denote by $p\colon X \to X/\sim$ the canonical projection map from X onto the set of equivalence classes $X/\sim$, then the following statements hold:*

(1) *The set*
$$\mathcal{T} := \{U \subset X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\sim$.[28]
(2) *Let $A \subset X/\sim$ be a subset. The set A is closed in $X/\sim$ if and only if the preimage $p^{-1}(A)$ is closed in X.*
(3) *The projection map $p\colon X \to X/\sim$ is continuous.*
(4) *If X is compact, then $X/\sim$ is also compact.*

Proof.

(1) This statement follows immediately from Lemma 1.1 (5) and (6).
(2) This statement follows easily from Lemma 1.1 (7).
(3) This statement is an immediate consequence of the definitions.
(4) This statement follows immediately from (2) and from Lemma 1.37. □

The following definition will not come as a surprise.

Definition. Let $\sim$ be an equivalence relation on a topological space X. We refer to the topology on $X/\sim$ from Lemma 1.65 (1) as the *quotient topology* and we refer to $X/\sim$ as a *quotient space* of X.

The following lemma is also so basic that later on we will no longer cite it explicitly.

[28] In other words, $U \subset X/\sim$ is open if and only if $p^{-1}(U) \subset X$ is open.

Lemma 1.66. *Let $\sim$ be an equivalence relation on a topological space X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map*

$$\begin{array}{rcl} g\colon X/\sim & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X/\sim \arrow[d, "g"] \\ & Y. \end{tikzcd}$$

Furthermore, if f is continuous, then g is also continuous.

Definition. Let $f\colon X \to Y$ be a map as in Lemma 1.66.

(1) We refer to the map $g\colon X/\sim \to Y$ as the *induced map*.
(2) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.
(3) Sometimes we also say that the map $f\colon X \to Y$ *descends* to a map $X/\sim \to Y$.

Proof $(*)$. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. By definition there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \ := \ f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property.

Now suppose that f is continuous. We need to show that g is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology we need to show that that $p^{-1}(g^{-1}(U))$ is open in X. We observe that

$$p^{-1}(g^{-1}(U)) \ = \ (g \circ p)^{-1}(U) \ = \ f^{-1}(U)$$

is open, since f is continuous. □

The following remark can be viewed as a sibling of Remark 1.54.

Remark 1.67. *We consider maps in and out of a quotient space $X/\sim$:*

(1) *In practice it is usually easy to show that a given map $X/\sim \to Y$ out of the quotient space is continuous. More precisely, in practice it is often fairly clear that the map $X \to X/\sim \to Y$ is continuous, but then it follows from Lemma 1.66 that the original map $X/\sim \to Y$ is continuous.*
(2) *There is usually no easy way to show that a map $Y \to X/\sim$ to a quotient space is continuous.*

Notation. Let X be a topological space and let $A \subset X$ be a subset. For $P, Q \in X$ we define

$$P \sim Q \quad :\Longleftrightarrow \quad P = Q \text{ or } P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim$ and we always equip X/A with the quotient topology.

The following lemma is elementary but useful. On numerous occasions it will allow us to slightly shift our point of view.

Lemma 1.68. $(*)$ *Let X be a topological space and let $A \subset X$ be a subset. Furthermore let Y be another topological space and let $y_0 \in Y$. The map*

$$\Phi\colon \left\{ \begin{array}{c} \textit{set of continuous maps } f\colon X \to Y \\ \textit{such that } f(A) = \{y_0\} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{set of continuous maps } X/A \to Y \\ \textit{such that } f([A]) = y_0 \end{array} \right\}$$
$$(f\colon X \to Y) \mapsto (\overline{f}\colon X/A \to Y)$$

is well-defined and it is a bijection.

Proof $(*)$. It follows immediately from Lemma 1.66 that given a continuous map $f\colon X \to Y$ with $f(A) = \{y_0\}$) the corresponding map $\overline{f}\colon X/A \to Y$ is continuous. This shows that the map Φ is well-defined. It is clear that Φ is injective. Finally suppose we are given a map $g\colon X/A \to Y$ with $f([A]) = y_0$. We denote by $f\colon X \to X/A \to Y$ the map that is given by precomposing g with the obvious projection $X \to X/A$. It follows from Lemma 1.65 (3) that f is continuous. Clearly we have $\Phi(f) = g$. This shows that Φ is also a surjection. $\square$

Now we can finally give our first examples of interesting quotient spaces.

Examples.

(1) Let $a < b$ be two real numbers. It follows easily from Lemma 1.65 (4), Lemma 1.66 and Proposition 1.40 that the map

$$\begin{array}{rcl} \Omega\colon [a,b]/\{a,b\} & \to & S^1 \\ [t] & \mapsto & \exp\left(i \cdot \frac{2\pi(t-a)}{b-a}\right) \end{array}$$

is a homeomorphism.

(2) It follows easily from Lemma 1.65 (4), Lemma 1.66 and Proposition 1.40 that the map[29]

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r \cdot v] & \mapsto & \cos(\pi r) \cdot (1, 0, \ldots, 0) + \sin(\pi r) \cdot (0, v), \text{ with } r \in [0,1], v \in S^{n-1} \end{array}$$

is a homeomorphism that sends the point $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to $(-1, 0, \ldots, 0)$. Note that for $n = 1$ and $n = 2$ this homeomorphism is illustrated in Figure 36. Throughout this course we will use this homeomorphism, or slight variations thereof, to identify $\overline{B}^n/S^{n-1}$ with S^n.[30]

[29] Why is Ω continuous?

[30] This homeomorphism has the following nice property: The map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is smooth and, if we equip B^n and S^n with the usual orientations, the map is orientation-preserving. We hope that

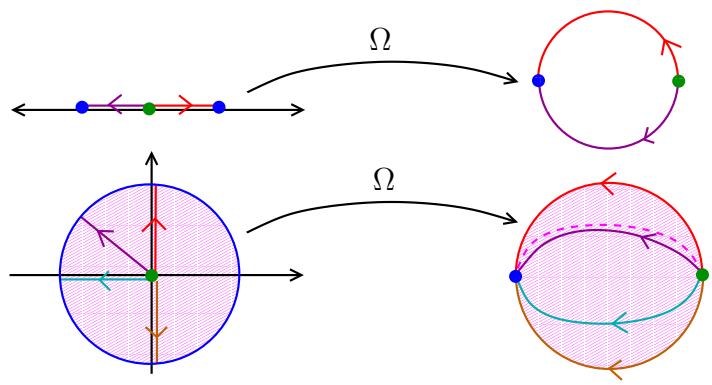

FIGURE 36.

(3) We consider the quotient of $X = \mathbb{R}$ by the compact interval $A = [-1, 1]$. This example has two interesting features:
   (a) The quotient $X/A = \mathbb{R}/[-1, 1]$ is actually homeomorphic to an old favorite, namely one can easily verify that the map

$$f\colon \mathbb{R} \to \mathbb{R}/[-1,1] \qquad\qquad g\colon \mathbb{R}/[-1,1] \to \mathbb{R}$$
$$x \mapsto \begin{cases} [x+1], \text{ if } x \geq 0 \\ [x-1], \text{ if } x < 0 \end{cases} \quad \text{and} \quad p \mapsto \begin{cases} x-1, \text{ if } p = [x] \text{ with } x \geq 1 \\ x+1, \text{ if } p = [x] \text{ with } x < -1 \end{cases}$$

   are continuous and inverses of one another.
   (b) Above we saw that projections $X \to X/\sim$ are basically by definition, continuous. One might think that projections should also be open. But in this example the projection $p\colon X = \mathbb{R} \to X/A = \mathbb{R}/[-1, 1]$ is *not* an open map, for example the image of the open set $(-1, 1) \subset \mathbb{R}$ is not open.

(4) We consider the quotient of $X = \mathbb{R}$ by the open interval $B = (-1, 1)$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/(-1, 1)$ the projection map. Even though this example looks very similar to the previous one, it shows very different behavior. For example the quotient $X/B = \mathbb{R}/(-1, 1)$ is not Hausdorff. More precisely consider $[-1] \in \mathbb{R}/(-1, 1)$. Any open neighborhood U of $[-1]$ in $\mathbb{R}/(-1, 1)$ has the property that $p^{-1}(U)$ is an open subset of $\mathbb{R}$ that contains -1. In particular $U = p(p^{-1}(U))$ contains the point $[0]$. Thus $[-1]$ and $[0]$ cannot be separated by open neighborhoods.

In the last example we saw that the quotient of a topological space that is Hausdorff is not necessarily again Hausdorff. It is pretty clear that if one puts enough conditions on the equivalence relation, then one can avoid such misdemeanor. This leads us to the following definition.

Definition. Let X be a set and let $\sim$ be an equivalence relation on X.

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the reader is not too fazed by the fact that we use some expressions that strictly speaking we will only introduce at a later stage.

(1) We refer to $\{(x,y) \in X \times X \mid x \sim y\}$ as the *graph of the equivalence relation.*
(2) Now suppose X is equipped with a topology. We say that the equivalence relation is *closed* if the graph is a closed subset of $X \times X$.

Now we can formulate the next lemma.

Lemma 1.69. $(*)$ *Let X be a topological space that is Hausdorff and compact. If $\sim$ is a closed relation, then $X/\sim$ is Hausdorff.*

Proof $(*)$. We denote by $p\colon X \to X/\sim$ the projection. We start out with the following claim.

Claim. Given any closed subset C of X the projection $p(C)$ is a closed subset of $X/\sim$.

Let C be a closed subset of X. We need to show that $p^{-1}(p(C))$ is closed in X. We denote by G the graph of the equivalence relation and we denote by $p_1, p_2\colon X \times X \to X$ the projection onto the first respectively the second factor. Then one easily sees that

$$p^{-1}(p(x)) \;=\; \{y \in X \mid \text{there exists an } x \in C \text{ with } y \sim x\} \;=\; p_2(p_1^{-1}(C) \cap G).$$

Note that $p_1^{-1}(C) = C \times X$ is a closed subset of $X \times X$.[31] Furthermore by our hypothesis G is a closed subset of $X \times X$. Thus we see that $p_1^{-1}(C) \cap G$ is closed. By Lemma 1.55 (2) and our hypothesis on X we know that $X \times X$ is compact. It follows from Lemma 1.18 (1) that $p_1^{-1}(C) \cap G$ is a compact subset of $X \times X$. But then it follows from Lemma 1.37 that $p_2(p_1^{-1}(C) \cap G)$ is a compact subset of X. But since X is Hausdorff we obtain that from Lemma 1.18 (2) that is also a closed subset of X. This concludes the proof of the claim.

Now let $[x], [y] \in X/\sim$ be two disjoint points in $X/\sim$. Since X is Hausdorff we know from Lemma 1.18 (2) that $\{x\}$ and $\{y\}$ are closed subsets of X. By the claim we see that $\{[x]\}$ and $\{[y]\}$ are closed subsets of $X/\sim$. Since the projection map $p\colon X \to X/\sim$ is continuous we see that $p^{-1}([x])$ and $p^{-1}([y])$ are two disjoint closed subsets of X. Since X is compact and Hausdorff we know that X is normal. Thus there exist open disjoint neighborhoods U and V of $p^{-1}([x])$ and $p^{-1}([y])$. We write $U' = X \setminus U$ and $V' = X \setminus V$. These are now closed subsets of X. By the claim their images are closed subsets of $X/\sim$. We now see that $(X/\sim) \setminus p(U')$ and $(X/\sim) \setminus p(V')$ are the desired open disjoint sets containing $[x]$ respectively $[y]$. $\square$

We conclude this section on quotient spaces with the following lemma.

Lemma 1.70. $(*)$ *Let X be a topological space and let $\sim$ be an equivalence relation on X. Furthermore let $\mathcal{B}$ be a basis of the topology of X. If the projection $X \to X/\sim$ is an open map, then*

$$p(\mathcal{B}) \;=\; \{p(B) \mid B \in \mathcal{B}\}$$

is a basis for the topology of $X/\sim$.

[31] This follows from the observation that $(X \times X) \setminus (C \times X) = (X \setminus C) \times X$ is the product of two open sets.

Proof $(*)$. First note that the sets in $p(\mathcal{B})$ are indeed open subsets of $X/\sim$ since we assume that the projection is open. We use the criterion provided by Lemma 1.28 (1) to show that $p(\mathcal{B})$ is a basis for the topology of $X/\sim$. Thus let $U \subset X/\sim$ be an open set and let $y \in U$. We need to show that there exists a $B \in \mathcal{B}$ with $y \in p(B) \subset U$.

We choose an $x \in X$ with $p(x) = y$. Since $\mathcal{B}$ is a basis for the topology of X and since $p^{-1}(U)$ is open in X there exists by Lemma 1.28 (1) a $B \in \mathcal{B}$ with $x \in B \subset p^{-1}(U)$. But then we also have that $y = p(x) \in p(B) \subset p(p^{-1}(U)) = U$. □

1.12. **Group actions on topological spaces.** We continue with the following definition that eventually will provide a rich source of examples of topological spaces.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An *action* of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (gh) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

(2) The action is called *free*, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(3) We say G acts *transitively*, if for any x and y in X there exists a $g \in G$ with $g \cdot x = y$.
(4) If X is a topological space, then we say that the action is *continuous*, if for every $g \in G$ the map

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$

is continuous.[32]

Remark. Sometimes what we call an "action" is referred to as a "left-action". A "right-action" is exactly the same as a left-action except that the second property gets replaced by $g \cdot (h \cdot x) = (hg) \cdot x$. Basically all the statements and definitions that we give for left-actions carry over, with minuscule modifications, to the setting of right-actions.

Examples.

(1) Given any group G the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & ghg^{-1} \end{array}$$

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

[32] If the action is continuous, then the map $x \mapsto g \cdot x$ is in fact a homeomorphism with inverse map given by $x \mapsto g^{-1} \cdot x$.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all bases of V. The map

$$\begin{array}{rcl} \mathrm{GL}(k,\mathbb{F}) \times V & \to & V \\ \big((a_{ij})_{i,j=1,\ldots,k}, \{v_1,\ldots,v_k\}\big) & \mapsto & \Big(\sum\limits_{i=1}^{k} a_{i1}v_i, \ldots, \sum\limits_{i=1}^{k} a_{ik}v_i\Big) \end{array}$$

is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free and continuous but it is not transitive.

We continue with the following definition.

Definition. Let X be a set and let G be a group. It is straightforward to verify that

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We write[33]

$$X/G \ := \ X/\sim .$$

If X is a topological space, then we view X/G as a topological space equipped with the quotient topology.

Lemma 1.71. *Let X be a topological space and let G be a group that acts on X.*

(1) *The projection map $X \to X/G$ is continuous.*
(2) *If G acts continuously on X, then the projection map is also open.*

Proof. The first statement follows immediately from Lemma 1.65 (3). We turn to the proof of the second statement. Thus let $U \subset X$ be an open subset. We need to show that $p(U)$ is open in X/G. Thus we need to show that $p^{-1}(p(U))$ is an open subset of X. It follows easily from the definitions that

$$p^{-1}(p(U)) \ = \ \bigcup_{g \in G} gU.$$

Since G acts continuously the map $x \mapsto gx$ is a homeomorphism. Thus we see that $p^{-1}(p(U))$ is the union of open sets, thus it is open itself. □

The following lemma discusses one example in greater detail.

Lemma 1.72. *The map*

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1,\ldots,t_n)] & \mapsto & \big(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\big) \end{array}$$

is a homeomorphism.

[33] It is arguably more logical to write $G\ X$ since we are dealing with a left-action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right-actions at all.

Convention. Often we use the homeomorphism of Lemma 1.72 to make the identification $\mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$.

Proof. We start out with the following observations:

(1) The map is continuous by Lemma 1.66 and it is easy to see that it is a bijection.
(2) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By Proposition 1.22 we know that $[0,1]^n$ is compact. It follows from Lemma 1.37 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.
(3) From Lemma 1.55 we obtain that $(S^1)^n$ is Hausdorff.

Now it follows easily from Proposition 1.40 that the given map is indeed a homeomorphism. □

We continue with more examples of group actions and the resulting quotient topological spaces.

Examples.

(1) Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$

defines an action that is free and continuous. We refer to the quotient space $S^n/\{\pm 1\}$ as the *n-dimensional real projective space* $\mathbb{R}\mathrm{P}^n$.[34] In Exercise 1.32 we will show that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1.

(2) Let $X = \mathbb{R}$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times \mathbb{R} & \to & \mathbb{R} \\ (\epsilon, x) & \mapsto & \epsilon \cdot x \end{array}$$

defines an action that is continuous. But this action is not free, since $(-1) \cdot 0 = 0$, but -1 is not the trivial element in the group $G = \{\pm 1\}$. In Exercise 1.33 we will show that $\mathbb{R}/\{\pm 1\}$ is homeomorphic to the half-open interval $[0, \infty)$.

We formulate the next example as a lemma.

Lemma 1.73. *Let $n \in \mathbb{N}$.*

(1) *The group $\mathrm{O}(n+1)$ acts continuously on S^n by the usual multiplication of a matrix and a vector $S^n \subset \mathbb{R}^{n+1}$.*
(2) *The action of $\mathrm{SO}(n+1)$ on S^n is free if and only if $n = 1$.*
(3) *The action of $\mathrm{O}(n+1)$ on S^n is transitive.*
(4) *If $n \geq 2$, then the action of $\mathrm{SO}(n)$ on S^n is also transitive.*
(5) *Given any $x, y \in S^n$ there exists a homeomorphism $\varphi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ with $\varphi(x) = y$.*

[34] For $n = 2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

Proof. The first two statements can be verified easily. The third and fourth statements fall in the realm of linear algebra. We leave it to the reader to undust their favorite linear algebra book and prove the statements. Finally the last statement follows from the third statement and the observation that multiplication by a matrix in $\mathrm{O}(n+1)$ defines a homeomorphism of $\overline{B}^{n+1}$. $\square$

We consider our next example in greater detail.

Examples.

(1) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. The map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$

defines an action that is free and continuous. This action is illustrated in Figure 37. We refer to the quotient $X/\mathbb{Z}$ as the *Möbius band*. The same argument as in the proof of Lemma 1.72 shows that the Möbius band is compact.

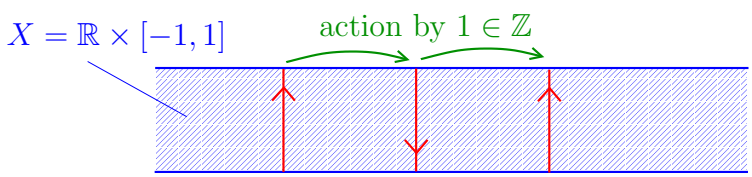

FIGURE 37.

The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. Given $\varphi, \psi \in \mathbb{R}$ and $r \in [-1,1]$ we set

$$A(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v(r,\psi) := \begin{pmatrix} 2 + r\sin\psi \\ 0 \\ r\cos\psi \end{pmatrix}.$$

We consider the map

$$\begin{array}{rcl} \Psi\colon (\mathbb{R} \times [-1,1])/\mathbb{Z} & \to & \mathbb{R}^3 \\ [(\varphi, r)] & \mapsto & A(2\pi \cdot \varphi) \cdot v\big(r, \pi\varphi\big). \end{array}$$

The image of Ψ is shown in Figure 38. The argument of Lemma 1.72 shows that Ψ is a homeomorphism onto its image. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subset of $\mathbb{R}^3$ that looks like what we had in mind.

(2) On occasion we will also be interested in the *open Möbius band* which is basically defined as above, except that we define it as the quotient of $\mathbb{R} \times (-1,1)$ instead of the quotient of $\mathbb{R} \times [-1,1]$.

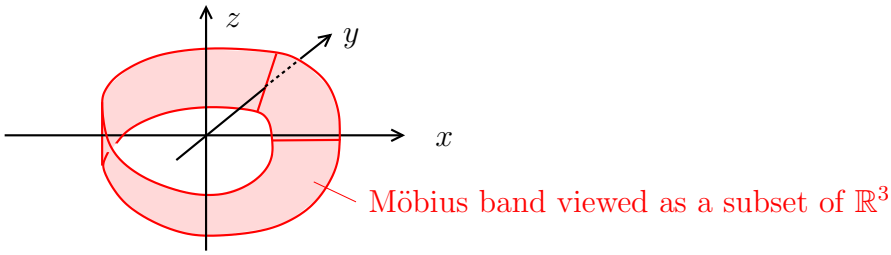

FIGURE 38.

If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise, see Exercise 1.34, to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

This example leads us to the following definition.

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts *properly* if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.

Examples.

(1) The action of any finite group is proper by definition.
(2) One can show easily that the action of $\mathbb{Z}$ on $\mathbb{R} \times [-1, 1]$ that we had considered on page 105 to define the Möbius band, is proper.
(3) It is straightforward to show that the action of $\mathrm{O}(n+1)$ on S^n from Lemma 1.73 is not proper.

We conclude this section with the following proposition.

Proposition 1.74. *Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.*

Unless one has an insatiable appetite for technical proofs it is best to move on to the next section.

The proof of Proposition 1.74 relies on the following two lemmas.

Lemma 1.75. $(*)$ *Let G be a group that acts continuously and properly on a topological space. If X is Hausdorff, then given any two points a and b in X there exist open neighborhoods A of a and B of b with the following property: for every $g \in G$ with $ga \neq b$ we also have $gA \cap B = \varnothing$.*

Proof of Lemma 1.75 $(*)$. Let G be a group that acts continuously and properly on a topological space X that is Hausdorff. Let a and b be two points in X. Since G acts properly on X there exist open neighborhoods U of a and V of b such that $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is a finite set. We denote the elements of this finite set by $g_1, \dots, g_r$. We set

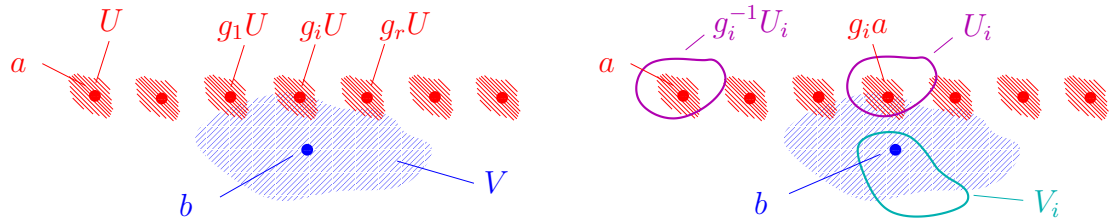

FIGURE 39. Illustration for the proof of Lemma 1.75 with $I = \{1, \dots, r\}$.

$$I \;:=\; \{i = 1, \dots, r \,|\, g_i a \neq b\}.$$

Since X is Hausdorff we can find for every $i \in I$ an open neighborhood U_i of $g_i a$ and an open neighborhood V_i of b such that $U_i \cap V_i = \varnothing$. We set

$$A \;:=\; U \cap \bigcap_{i \in I} g_i^{-1} U_i \quad \text{and} \quad B \;:=\; V \cap \bigcap_{i \in I} V_i.$$

The sets A and B are the intersection of finitely many open subsets,[35] thus they are open. In particular A is an open neighborhood of a and B is an open neighborhood of b.

Now one can easily verify that A and B have the desired properties. Indeed, let $g \in G$ with $ga \neq b$. If $g \notin \{g_1, \dots, g_r\}$, then $gU \cap V = \varnothing$, therefore $gA \cap B = \varnothing$. On the other hand, if $g \in \{g_1, \dots, g_r\}$, then there exists an $i \in I$ with $g = g_i$. It follows that $g_i(g_i^{-1}U_i) \cap V_i = U_i \cap V_i = \varnothing$, hence $g_i A \cap B = \varnothing$. □

Lemma 1.76. $(*)$ *Let G be a group that acts on a set X. We denote by $p\colon X \to X/G$ the projection. Let A and B be subsets of X. Then*

$$p(A) \cap p(B) \;=\; \varnothing \quad \Longleftrightarrow \quad \textit{for every } g \in G \textit{ we have } gA \cap B = \varnothing,$$

or equivalently,

$$p(A) \cap p(B) \;\neq\; \varnothing \quad \Longleftrightarrow \quad \textit{there exists a } g \in G \textit{ with } gA \cap B \neq \varnothing.$$

Proof of Lemma 1.76$(*)$. We prove the second statement. The first statement is easily seen to be equivalent to the second statement. We have

$$\begin{aligned} p(A) \cap p(B) \;\neq\; \varnothing \;&\Longleftrightarrow\; \text{there exist } a \in A \text{ and } b \in B \text{ with } p(a) = p(b) \in X/G \\ &\Longleftrightarrow\; \text{there exist } a \in A,\, b \in B \text{ and } g \in G \text{ with } ga = b \\ &\Longleftrightarrow\; \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing. \end{aligned}$$ □

Now we turn to the actual proof of Proposition 1.74.

Proof of Proposition 1.74 $(*)$. Let G be a group that acts on a topological space X. We assume that X is Hausdorff and that the action is proper. We want to show that the quotient space X/G is also Hausdorff. We denote by $p\colon X \to X/G$ the projection map.

Let x and y be two distinct points in X/G. We pick a and b in X with $p(a) = x$ and $p(b) = y$. Since $p(a) = x \neq y = p(b)$ we see that for every $g \in G$ we have $ga \neq b$. By

[35] Here we use that G acts continuously, since this implies that the sets $h^{-1}U_i$ are also open.

Lemma 1.75 there exist open neighborhoods A of a and B of b such that $gA \cap B = \varnothing$ for every $g \in G$. It follows from Lemma 1.76 that $p(A)$ and $p(B)$ are disjoint.

Since A and B are open it follows from Lemma 1.71 that $p(A)$ and $p(B)$ are open subsets of X/G, in particular they are open neighborhoods of $x = p(a)$ and $y = p(b)$. We have thus found the desired disjoint open neighborhoods of x and y in X/G. □

1.13. **Continuity of zeros** $(*)$**.** Group actions can also appear in unexpected situations. For example let us consider the following question.

Question 1.77. *Let $n \in \mathbb{N}$. Do the zeros of a polynomial of degree n vary continuously with the coefficients of the polynomial?*

The first question is, what does it mean that the "zeros of a polynomial" vary "continuously"? What are the topological spaces involved so that we can talk about continuity? The following proposition kills two birds at once: it makes the question precise and it gives a complete answer.

Proposition 1.78. *Let $n \in \mathbb{N}$. We consider the action by the permutation group S_n on $\mathbb{C}^n$ that is given by permuting the coefficients. The map*[36]

$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0,\dots,v_{n-1}) & \mapsto & \left[\begin{array}{c} \textit{set of zeros of the polynomial} \\ v_0 + v_1 z + \dots + v_{n-1}z^{n-1} + z^n, \\ \textit{counted with multiplicities} \end{array}\right] \end{array}$$

is continuous, in fact it is a homeomorphism.

Remark. For $n = 1, 2, 3, 4$ one can hope to prove Proposition 1.78 by the formulas for determining zeros of polynomials of degree n. But since no such formula exists for general polynomials of higher degree this approach is doomed for $n \geq 5$.

The proof of Proposition 1.78 makes use Rouché's Theorem from complex analysis.

Theorem 1.79. (Rouché's Theorem) *Let $U \subset \mathbb{C}$ be an open set, let $f, g\colon U \to \mathbb{C}$ be two holomorphic functions and let $z \in U$ and $r \in \mathbb{R}^+$ be such that $\overline{B}_r(z) = \{w \in \mathbb{C} \mid |w - z| \leq r\}$ is contained in U. If for all $w \in \partial\overline{B}_r(z)$ we have $|g(z)| < |f(z)|$, then f and $f + g$ have the same number of zeros (counted with multiplicities) within the open disk $B_r(z)$.*

Proof. This theorem is proved in every self-respecting book on complex analysis, see e.g. [Lan, Theorem VI.1.6]. □

Example. We consider $f(z) = z^5$ and the disk $\overline{B}_1(0)$. The polynomial $f(z)$ has one zero of multiplicity five in $\overline{B}_1(0)$. For any $a \in \mathbb{C}$ with $|a| < 1$ we have $|az| < |z^5|$ for all points $z \in \partial\overline{B}_1(0) = S^1$. Thus by Rouché's Theorem 1.79 we see that for such a the polynomial

[36]Slightly more rigorously the map f is defined as follows: given $(v_0,\dots,v_{n-1}) \in \mathbb{C}^n$ we define $f(v_0,\dots,v_{n-1}) := [(\alpha_1,\dots,\alpha_n)]$ where $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ are determined uniquely (up to reordering) by the condition that $v_0 + v_1 z + \dots + v_{n-1}z^{n-1} + z^n = (z - \alpha_1)\cdot\dots\cdot(z - \alpha_n) \in \mathbb{C}[z]$.

$z^5 + az$ also has five zeros (counted with multiplicities) in $\overline{B}_1(0)$. In fact for most values of a the polynomial will have five distinct zeros, each with multiplicity one.

We will also need the following definition:

Definition. Given $w = [(w_1, \dots, w_n)]$ and $z = [(z_1, \dots, z_n)]$ in $\mathbb{C}^n/S_n$ we define

$$d(w, z) \;=\; \min_{\sigma \in S_n} \max_{1 \le j \le n} \left| w_j - z_{\sigma(j)} \right|.$$

The following lemma summarizes two key statements regarding the above definition.

Lemma 1.80.

(1) *The map*

$$\begin{array}{rcl} d\colon \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n & \to & \mathbb{R}_{\ge 0} \\ (w, z) & \mapsto & d(w, z) \end{array}$$

is a metric on $\mathbb{C}^n/S_n$.

(2) *The topology defined by the metric* d *on* $\mathbb{C}^n/S_n$ *agrees with the quotient topology on* $\mathbb{C}^n/S_n$.

Proof. Both statements of the lemma are left as an exercise to the meticulous reader. □

Now we are in a position to prove Proposition 1.78.

Proof of Proposition 1.78. We consider the map[37]

$$\begin{array}{rcl} g\colon \mathbb{C}^n/S_n & \to & \mathbb{C}^n \\ [(\alpha_1, \dots, \alpha_n)] & \mapsto & \text{coefficients of the polynomial } (z - \alpha_1) \cdot \dots \cdot (z - \alpha_n). \end{array}$$

It is straightforward to check that g is continuous and that f and g are inverses of one another.

It remains to show that f is continuous. We introduce the following definitions: We denote by d the above metric on $\mathbb{C}^n/S_n$. It follows from Lemma 1.80 that we can view f as a map between the two metric spaces $(\mathbb{C}^n, \| - \|)$ and $(\mathbb{C}^n/S_n, d)$. Therefore it follows from Proposition 1.33 that it suffices to prove the following claim.

Claim.

$$\underset{v \in \mathbb{C}^n}{\forall} \; \underset{\epsilon > 0}{\forall} \; \underset{\delta > 0}{\exists} \; \underset{\substack{w \in \mathbb{C}^n \text{ with} \\ \|w\| < \delta}}{\forall} \; d(f(v), f(v + w)) < \epsilon.$$

Before we start with the proof of the claim we introduce the following definition: Given $v = (v_0, \dots, v_{n-1}) \in \mathbb{C}^n$ we denote by $p_v(z) := v_0 + v_1 z + \dots + v_{n-1} z^{n-1} + z^n$ the corresponding polynomial. Now we turn to the proof of the claim. Thus let $v = (v_0, \dots, v_{n-1}) \in \mathbb{C}^n$ and let $\epsilon > 0$. We denote by $\xi_1, \dots, \xi_k$ the *distinct* zeros of the corresponding polynomial $p_v(z)$. (Note that if $p_v(z)$ has zeros of higher multiplicity, then $k < n$.) After possibly

[37] Slightly more precisely the map g is defined as follows: given $[(\alpha_1, \dots, \alpha_n)] \in \mathbb{C}^n/S_n$ we have $f([(\alpha_1, \dots, \alpha_n)]) = (v_0, \dots, v_{n-1})$ where $v_0, \dots, v_{n-1} \in \mathbb{C}$ are uniquely determined by the condition that $v_0 + v_1 z + \dots + v_{n-1} z^{n-1} + z^n = (z - \alpha_1) \cdot \dots \cdot (z - \alpha_n) \in \mathbb{C}[z]$.

replacing ϵ by a smaller non-negative number we can assume, without loss of generality, that $\epsilon < \frac{1}{2} \cdot |\xi_i - \xi_j|$ for all $i \neq j$.

We denote by Γ the union of the circles of the radius ϵ around $\xi_1, \dots, \xi_k$. Furthermore we set

$$\eta \ := \ \inf_{z \in \Gamma} |p_v(z)| \ \in \ \mathbb{R}_{>0}.$$

It is elementary to show that there exists a $\delta > 0$ such that for all $w = (w_0, \dots, w_{n-1}) \in \mathbb{C}^n$ such that $\|w\| < \delta$ we have

$$|w_0 + w_1 + \cdots + w_{n-1} z^{n-1}| \ < \ \eta \text{ for all } z \in \Gamma.$$

We claim that this δ has the desired property. Thus let $w \in \mathbb{C}^n$ with $|w| < \delta$. It follows from the definition of δ and Rouché's Theorem 1.79 that the polynomials $p_v(z)$ and $p_{v+w}(z)$ have the same number of zeros (counted with multiplicities) in the disks $B_\epsilon(\xi_1), \dots, B_\epsilon(\xi_k)$. This means that the zeros of $p_v(z)$ and can be matched up with the zeros of $p_{v+w}(z)$ such that each zero is less then ϵ removed from its match. This statement implies immediately that $d(f(v), f(v+w)) < \epsilon$. $\square$

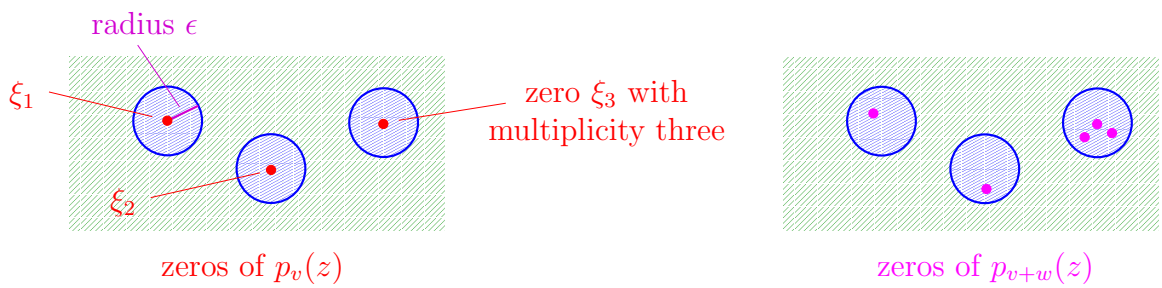

FIGURE 40. Illustration for the proof of Proposition 1.78.

Remark. With some effort one can also prove Proposition 1.78 without using Rouché's Theorem 1.79. We outline the key steps, using the notation that we introduced in the above proof of Proposition 1.78.

(1) By Lemma 1.39 it suffices to show that the map $g \colon \mathbb{C}^n/S_n \to \mathbb{C}^n$ is open.
(2) By Lemma 1.36 it suffices to show that for each $\epsilon > 0$ and each $z \in \mathbb{C}^n/S_n$ the image of the open ball $B_\epsilon(z) \subset (\mathbb{C}^n/S_n, d)$ under g is open.
(3) Thus let $\epsilon > 0$ and $z \in \mathbb{C}^n/S_n$.
(4) We pick $C \in \mathbb{R}_{\geq 0}$ such that $g(\overline{B}_\epsilon(z)) \subset B_C(0) \subset \mathbb{C}^n$.
(5) There exists a $D \in \mathbb{R}_{\geq 0}$ such that for any $v \in \mathbb{C}^n$ which satisfies $\|v\| \leq C$ we have $f(v) \in B_D(0) \subset \mathbb{C}^n/S_n$. (This statement requires some unraveling and a little bit of thought. It can be proved using Rouché's Theorem 1.79, but with a worse bound it can also be proved by hand.)
(6) By Proposition 1.40 we know that g restricted to $\overline{B}_D(0) \to g(\overline{B}_D(0))$ is a homeomorphism. Thus $g(B_\epsilon(z))$ is an open subset of $g(\overline{B}_D(0))$.

(7) By (5) and (6), and since f and g are inverses of one another, we know that $g(B_\epsilon(z))$ is also open in $B_D(0)$. But $B_D(0)$ is an open subset of $\mathbb{C}^n$, hence $g(B_\epsilon(z))$ is also open in $\mathbb{C}^n$.

1.14. **Projective spaces.** Let $n \in \mathbb{N}$. On page 104 we introduced the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$. In this section we will give an alternative definition and we will introduce the complex projective spaces. Even though these projective spaces appear initially rather arbitrary they will on several occasions play an essential role in these notes.

Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and let $G = \mathbb{R} \setminus \{0\}$, where we view G as a group via multiplication. The map

$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action that is free and continuous. It is straightforward to see that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n \;=\; S^n/\{\pm 1\} & \to & (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ [P] & \mapsto & [P] \end{array}$$

defines a homeomorphism.[38] Our new point of view on real projective spaces has the advantage that we can create a different class of examples of topological spaces by replacing $\mathbb{R}$ by $\mathbb{C}$. More precisely, this discussion leads us to the following definition:

Definition. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ and let $G = \mathbb{C} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P \end{array}$$

defines an action that is free and continuous. The quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ is called the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n$.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by $[x_0 : \ldots : x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.
(2) Given $m \leq n$ we will often view $\mathbb{R}\mathrm{P}^m$ as a subset of $\mathbb{R}\mathrm{P}^n$ via the map

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_m : 0 : \cdots : 0]. \end{array}$$

The same point of view applies to complex projective spaces.

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Lemma 1.81. *The real projective and the complex projective spaces are compact and Hausdorff.*

[38] For example an inverse is given by the map $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \to S^n/\{\pm 1\}$ that is defined by $x \mapsto [\frac{x}{\|x\|}]$.

Proof $(*)$. Let $n \in \mathbb{N}$. It follows from Lemma 1.37 and Proposition 1.74 that the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ is compact and Hausdorff.

For the complex projective space $\mathbb{C}\mathrm{P}^n$ we need to work just a little harder. In the remainder of this proof we take the view that

$$S^{2n+1} \;=\; \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \,|\, |z_1|^2 + \dots + |z_{n+1}|^2 = 1\}.$$

We consider the action

$$\begin{array}{rcl} S^1 \times S^{2n+1} & \to & S^{2n+1} \\ (w, (z_1, \dots, z_{n+1})) & \mapsto & (w \cdot z_1, \dots, w \cdot z_{n+1}). \end{array}$$

It is straightforward to verify that the maps

$$\begin{array}{l} f\colon S^{2n+1}/S^1 \to \mathbb{C}\mathrm{P}^n \\ [z_1, \dots, z_{n+1}] \mapsto [z_1 : \dots : z_{n+1}] \end{array} \quad \text{and} \quad \begin{array}{l} p\colon \mathbb{C}\mathrm{P}^n \to \mathbb{S}^{2n+1}/S^1 \\ [z_1 : \dots : z_{n+1}] \mapsto \left[\frac{1}{\sqrt{|z_1|^2 + \dots + |z_n|^2}}(z_1, \dots, z_{n+1})\right] \end{array}$$

are continuous and inverses of one another. Thus both maps are homeomorphisms and it suffices to show the statements for S^{2n+1}/S^1. Since S^{2n+1} is compact it follows again from Lemma 1.37 that S^{2n+1}/S^1 is compact.

It follows from Lemma 1.69 that S^{2n+1}/S^1 is Hausdorff once we have shown that the graph of the equivalence relation, which is given by

$$G \;=\; \{(v_1, v_2) \in S^{2n+1} \times S^{2n+1} \,|\, \text{there exists } w \in S^1 \text{ with } w \cdot v_1 = v_2\},$$

is a closed subset of $S^{2n+1} \times S^{2n+1}$. The graph G is the image of the map

$$\begin{array}{rcl} \Phi\colon S^{2n+1} \times S^1 & \to & S^{2n+1} \times S^{2n+1} \\ (v, w) & \mapsto & (v, w \cdot v). \end{array}$$

This map is continuous and the left-hand side is compact by Lemma 1.55 (2). Thus the image of Φ is compact by Lemma 1.37. By Lemma 1.55 (1) we know that $S^{2n+1} \times S^{2n+1}$ is Hausdorff. Thus we obtain from Lemma 1.18 that the image of Φ is a closed subset of $S^{2n+1} \times S^{2n+1}$. □

In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions.

Lemma 1.82.

(1) *Let $n \in \mathbb{N}$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial\overline{B}^n$, then the map*

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ [(x_1, \dots, x_n, x_{n+1})] & \mapsto & \begin{cases} [(x_1, \dots, x_n)], & \text{if } x_{n+1} \geq 0, \\ [-(x_1, \dots, x_n)], & \text{if } x_{n+1} < 0 \end{cases} \end{array}$$

is a homeomorphism.

(2) *The map*

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\cup\{\infty\} = \mathbb{R}^2\cup\{\infty\} \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0\cdot z_1^{-1}, & \textit{if } z_1 \neq 0, \\ \infty, & \textit{if } [z_0 : z_1] = [1:0] \end{cases} \end{array}$$

is a homeomorphism.

Proof. By Lemma 1.81 we know that each $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ is compact. Using Lemma 1.69 one can show that $\overline{B}^n/\sim$ is Hausdorff. Furthermore we know from Exercise 1.8 that $\mathbb{R}^2\cup\{\infty\}$ is Hausdorff. One can easily verify that both maps are continuous and bijections. Thus it follows from Proposition 1.40 that both maps are homeomorphisms. □

Convention.

(1) On many occasions we will use the homeomorphism from Lemma 1.82 to make the identification $\mathbb{R}\mathrm{P}^n = \overline{B}^n/\sim$.
(2) In Lemma 1.41 we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2\cup\{\infty\}$. Combining this homeomorphism with the homeomorphism from Lemma 1.82 we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we will often use to make the identification $S^2 = \mathbb{C}\mathrm{P}^1$.

We conclude this short section on projective spaces with two somewhat tangential remarks.

Remark.

(1) Here is a fun fact: if we consider $\mathbb{C}\mathrm{P}^2/\sim$, where $\sim$ denotes the equivalence relation defined by complex conjugation, i.e. where $[z : w] \sim [\overline{z} : \overline{w}]$, then the quotient space is homeomorphic (in fact diffeomorphic in a suitable sense) to S^4. This statement is known as the Arnold–Kuiper–Massey Theorem, see e.g. [**?**, Mas4, Sea].
(2) The definition of projective spaces makes sense for any field, even for fields of non-zero characteristic. For example, if $\mathbb{F}$ is a field of characteristic p we can still define the set $\mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1}\setminus\{0\})/(\mathbb{F}\setminus\{0\})$. A priori this is just a set, but together with the Zariski topology, see [Har, Lecture 2] or [SKKT, Chapter 1], it actually becomes an unusual, but interesting topological space.
For any field the map

$$\begin{array}{rcl} \mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1}\setminus\{0\})/(\mathbb{F}\setminus\{0\}) & \to & \{\text{all one-dimensional subspaces of } \mathbb{F}^{n+1}\} \\ [v] & \mapsto & \mathbb{F}\cdot v \end{array}$$

is a bijection. Thus we can view $\mathbb{F}\mathrm{P}^n$ as the set of all lines through the origin.

1.15. **Gluing two topological spaces.** In this section we consider a particularly important special case of a quotient topology which deserves its own section. Before we delve into topology we recall the following definition.

Definition. Suppose we are given a relation $\sim$ on a set X. We say $x, y \in X$ are *equivalent* if there exists a sequence $x = x_1, \ldots, x_k = y$ of elements in X such that for all $i = 1, \ldots, k-1$

the following holds: either $x_i \sim x_{i+1}$ or $x_{i+1} \sim x_i$. We write again $x \sim y$ if x and y are equivalent. It is straightforward to see that this is now indeed an equivalence relation. We say it is *generated* by the initial relation.

Example. The relations $x \sim (x+1)$ with $x \in \mathbb{R}$ on $\mathbb{R}$ generate the familiar equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

Definition. Let X and Y be topological spaces, let $A \subset X$ be a subset and let $\varphi\colon A \to Y$ be a continuous map.[39] (Note that φ does not need to be injective.) We write

$$X \cup_\varphi Y \ := \ X \cup_{\varphi\colon A \to Y} Y \ := \ (X \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation generated by $a \sim \varphi(a)$ for $a \in A$. We call $X \cup_\varphi Y$ the topological space obtained by *gluing X to Y via the map f*. The definition is illustrated in Figure 41.

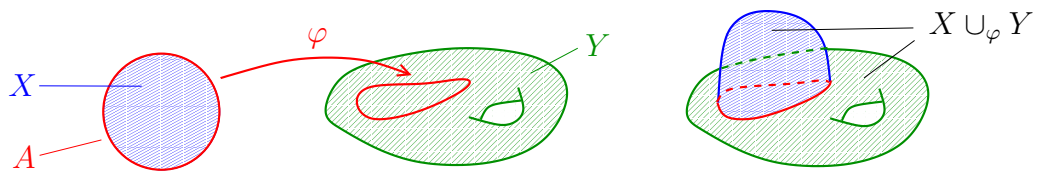

FIGURE 41.

Example. Let X be a topological space and let A be a subset. If Y is the topological space that consists of a single point, then there exists evidently precisely one map $f\colon A \to Y$ and it follows immediately from the definitions that $X \cup_\varphi Y = X/A$.

Lemma 1.83. $(*)$ *Let X and Y be two topological spaces, let $A \subset X$ be a subset and let $\varphi\colon A \to Y$ be a map. We denote by $i\colon X \to X \cup_\varphi Y$ and $j\colon Y \to X \cup_\varphi Y$ the obvious maps. The following statements hold:*

(1) *The maps $i\colon X \to X \cup_\varphi Y$ and $j\colon Y \to X \cup_\varphi Y$ are continuous.*
(2) *If A is a closed subset of X and if $\varphi(A)$ is a closed subset of Y, then $i(X)$ is a closed subset of $X \cup_\varphi Y$ and $j(Y)$ is a closed subset of $X \cup_\varphi Y$.*
(3) *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two maps to a topological space Z. If we have $g(\varphi(a)) = f(a)$ for all $a \in A$, then*

$$\begin{array}{rcl} h\colon X \cup_\varphi Y & \to & Z \\ p & \mapsto & \begin{cases} f(x) & \text{if } p = [x] \text{ for some } x \in X, \\ g(y) & \text{if } p = [y] \text{ for some } y \in Y \end{cases} \end{array}$$

is well-defined and it is the unique map $h\colon X \cup_\varphi Y \to Z$ with $h \circ i = f$ and $h \circ j = g$. Furthermore, if f and g are continuous, then h is also continuous.

[39] In principle the map φ need not be continuous, but in practice it always is, so for purely psychological reasons we put this condition in.

Proof $(*)$.

(1) This statement follows from Lemma 1.49 (1) and Lemma 1.65 (3).
(2) Suppose that A is a closed subset of X and that $\varphi(A)$ is a closed subset of Y. Let $p\colon X \sqcup Y \to X \cup_\varphi Y$ be the obvious projection map. We have $p^{-1}(i(X)) = X \sqcup \varphi(A)$. Thus it follows from Lemma 1.65 (2) that $i(A)$ is closed in $X \sqcup_\varphi A$. The proof that $j(Y)$ is closed in $X \sqcup_\varphi A$ is basically identical.
(3) The last statement is an immediate consequence of the definitions and Lemma 1.66.

□

The following lemma is quite subtle and gets used, often inadvertently, on many occasions.

Lemma 1.84. *Let X be a topological space and let A and B be two closed subsets of X with $X = A \cup B$. We denote by $\varphi\colon A \cap B \to A \cap B$ the identity map. The map*

$$\begin{array}{rcl} A \cup_{\varphi\colon A\cap B \to A\cap B} B & \to & X \\ [x] & \mapsto & x \end{array}$$

is a homeomorphism.

Remark.

(1) The statement of Lemma 1.84 also holds if A and B are both open subsets. But in all applications we will always deal with the case that A and B are both closed.
(2) In general the conclusion of the lemma does not hold without some assumptions on A and B. For example, consider $X = [-1, 1]$ together with the subsets $A = [-1, 0]$ and $B = (0, 1]$. In this case $A \cap B = \varnothing$, thus $A \cup_\varphi B$ is the disjoint union of A and B. It follows from Lemma 1.49 (5) and the Heine-Borel 1.21 Theorem that $A \sqcup B$ is not compact. On the other hand $X = [-1, 1]$ is compact by the Heine-Borel 1.21. Thus the two topological spaces $A \cup_\varphi B$ and X are not homeomorphic. We refer to Figure 42 for an illustration.

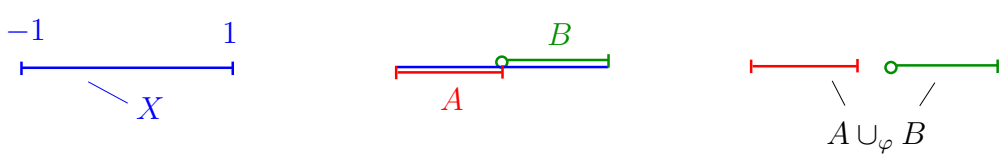

FIGURE 42.

Proof. It follows from Lemma 1.83 (3) that the given map $h\colon A \cup_{\varphi\colon A\cap B \to A\cap B} B \to X$ is continuous. Furthermore, basically by definition the map h is a bijection. By Lemma 1.39 it remains to show that the map h is closed. Thus let $W \subset A \sqcup_\varphi B$ be a closed subset. Note that $h(W) = (W \cap A) \cup (W \cap B)$. It follows from Lemma 1.65 (2), applied to the projection map

$$A \sqcup B \;\to\; A \cup_\varphi B = (A \sqcup B)/\sim,$$

that $A \cap W$ is closed in A and that $B \cap W$ is closed in B. Since A and B are by hypothesis closed in X we obtain from Lemma 1.35 (2) that $A \cap W$ and $B \cap W$ are closed in X. It follows that $h(W) = (W \cap A) \cup (W \cap B)$ is a closed subset of X. $\square$

We conclude this section with an amusing description of spheres. We introduce the following notation:

(1) We write $I = [-1, 1]$.
(2) Given $k \in \mathbb{N}_0$ we write[40]

$$\partial I^k \;=\; \{(x_1, \dots, x_k) \in I^k \mid \text{ at least one } x_i \text{ lies in } \{\pm 1\}\}.$$

(3) We recall that given $k \in \mathbb{N}_0$ there exists by Proposition 1.47 a canonical homeomorphism

$$\varphi_k \colon (\overline{B}^k, S^{k-1}) \;\xrightarrow{\cong}\; (I^k, \partial I^k)$$

with $\varphi_k(0) = 0$.

Now let $m, n \in \mathbb{N}_0$. An elementary set-theoretic argument shows that

$$((\partial I^m) \times I^n) \;\cup\; (I^m \times \partial I^n) \;=\; \partial I^{m+n}$$

and the two subsets to the left coincide in $(\partial I^m) \times (\partial I^n)$. (We refer to Figure 43 for an illustration.) It follows from Lemma 1.84 that the inclusions induce a homeomorphism

$$((\partial I^m) \times I^n) \cup_{\mathrm{id}_{\partial I^m \times \partial I^n}} (I^m \times \partial I^n) \;\xrightarrow{\cong}\; \partial I^{m+n}.$$

Together with the above homeomorphisms we obtain a canonical homeomorphism

$$(S^{m-1} \times \overline{B}^n) \cup_{\mathrm{id}_{S^{m-1} \times S^{n-1}}} (\overline{B}^m \times S^{n-1}) \;\xrightarrow{\cong}\; S^{m+n-1}.$$

For example, if we take $m = n = 2$, then we see that we can write S^3 as the union of two solid tori. We will discuss this particular case in greater detail later on.

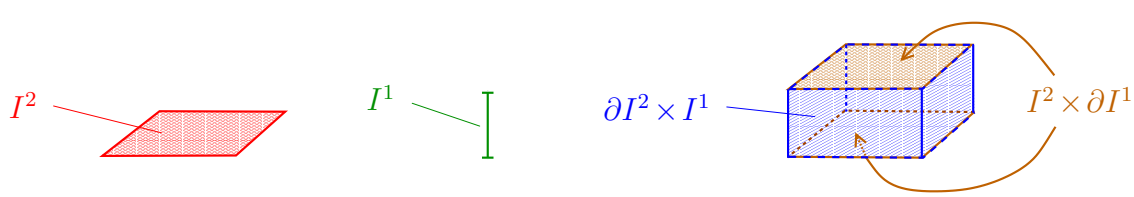

FIGURE 43.

[40]It is not difficult to see that ∂I^k is the boundary of I^k viewed as a subset of $\mathbb{R}^k$.

1.16. **Surfaces.** In this section we introduce some of my favorite topological spaces, namely surfaces.

(1) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the equivalence relation which is generated by

$$(x,0) \ \sim \ (x,1) \qquad \text{for all } x \in [0,1].$$

The quotient topological space $X/\sim$ is obtained from the square $X = [0,1] \times [0,1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. Put differently, "we glue the upper edge to the lower edge". Using Proposition 1.40 one can easily show that the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ (s,t) & \mapsto & \left(e^{2\pi i t}, s\right) \end{array}$$

is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the *cylinder* or the *annulus*. Sometimes we refer to $S^1 \times (0,1)$, which is homeomorphic to $(0,1) \times [0,1]/\sim$, as the *open cylinder* or *open annulus*.

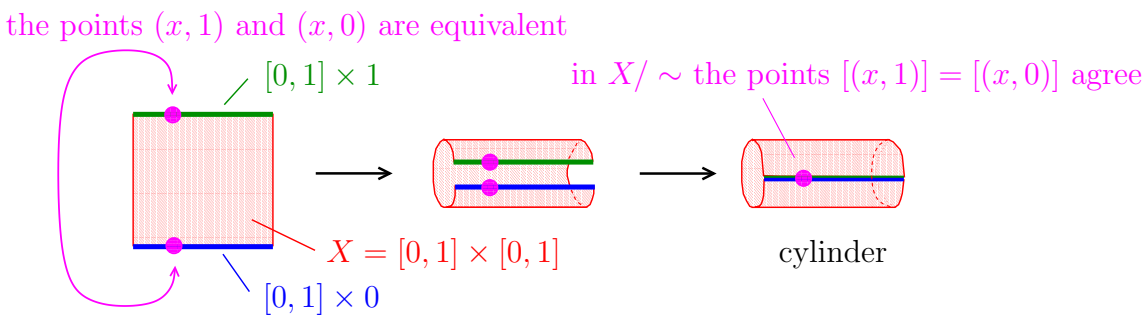

FIGURE 44. Construction of a cylinder by gluing two edges.

(2) Now we consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(0,y) \sim (1, 1-y) \qquad \text{for all } y \in [0,1].$$

The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist". We will now see that this topological space is homeomorphic to the Möbius band as defined on page 105. We consider the map

$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\mathbb{Z} \\ [(x,y)] & \mapsto & [(x, 2y-1)] \end{array}$$

The left-hand side is compact by Lemma 1.65 (4) and the right-hand side is Hausdorff by Proposition 1.74 and the discussion on page 106. The map is clearly continuous and a bijection. Thus it follows from the ever-popular Proposition 1.40 that the map is a homeomorphism. It is straightforward to see that this homeomorphism restricts

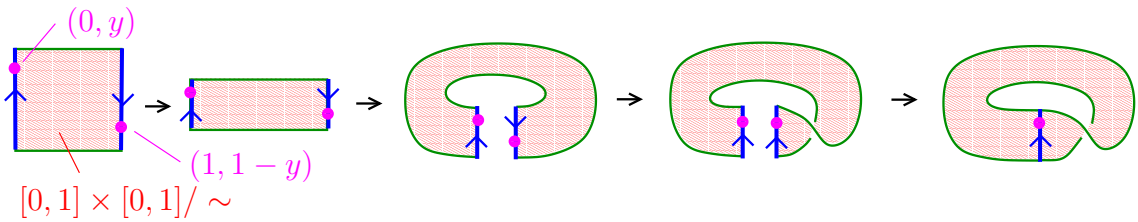

FIGURE 45. The Möbius band.

to a homeomorphism from $([0,1] \times (0,1))/(0,y) \sim (1, 1-y)$ to the open Möbius band $(\mathbb{R} \times (-1,1))/\sim$.

(3) Now we consider again $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, but this time with the equivalence relation which is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,y) \quad \text{for all } y \in [0,1].$$

Put differently, we obtain the quotient space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. Figure 46 illustrates that the quotient space $X/\sim$ is indeed a torus. As in the previous

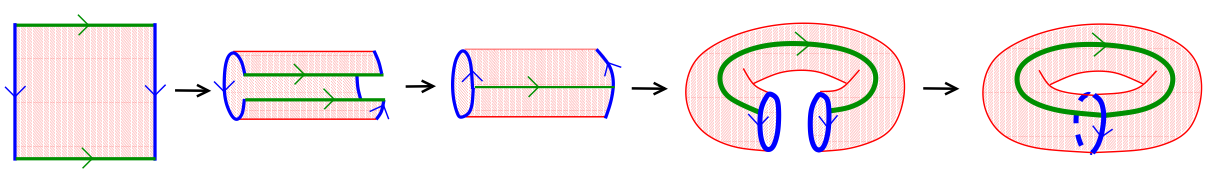

FIGURE 46. By gluing of the opposite edges of a square we obtain a torus.

example one can show that the inclusion $X \to \mathbb{R}^2$ descends to a homeomorphism $X/\sim \to \mathbb{R}^2/\mathbb{Z}^2$.

(4) Now we consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(x,0) \sim (x,1) \qquad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

Thus we obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. The resulting topological space is called the *Klein bottle*. In Figure 47 we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.

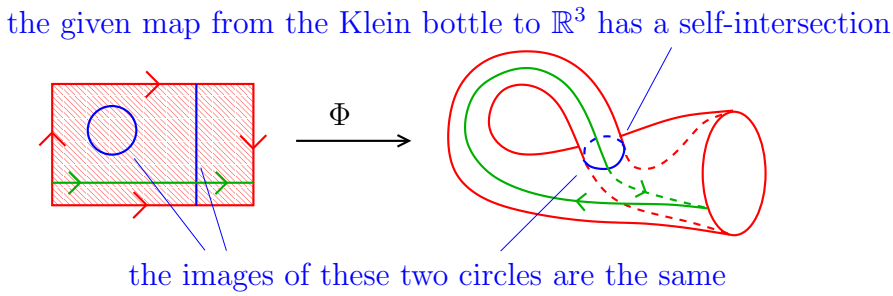

FIGURE 47. The definition of the Klein bottle.

(5) We denote by E_8 the regular octagon in $\mathbb{C}$ with the vertices $Q_k = e^{2\pi ik/16}$ where $k = 1, 3, \ldots, 15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$. Given two points $A, B \in \mathbb{R}^2 = \mathbb{C}$ we denote by $\overline{AB}$ the Euclidean segment from A to B. Furthermore, given $\varphi \in \mathbb{R}$ we denote by $r_\varphi \colon \mathbb{C} \to \mathbb{C}$ the reflection in the Euclidean line $\{te^{i\varphi} \,|\, t \in \mathbb{R}\}$. We denote by $\sim$ the equivalence relation on E_8 which is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0, 1, 4, 5$. These four relations are sketched in Figure 48. We refer to the topological space $E_8/\sim$ as the *surface of genus 2*. The equivalence relation is sketched in Figure 48. Furthermore, in Figure 49 we indicate how $E_8/\sim$ is related to the actual

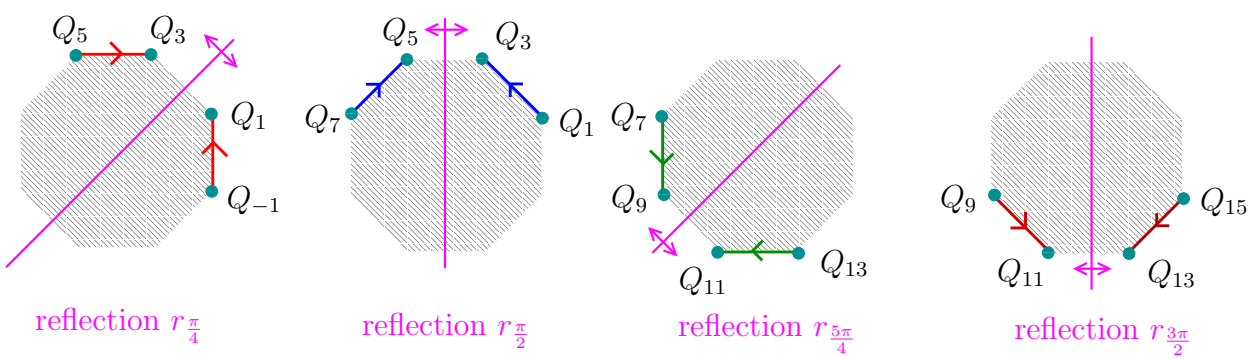

FIGURE 48.

"physical" surface of genus 2 in $\mathbb{R}^3$. The same way we can define the *surface of genus* g for any $g \geq 3$. More precisely, we take a regular $4g$-gon in $\mathbb{C}$ and for $j = 1, \ldots, g$ we identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. Here each identification is given by a reflection. For $g = 3$ this construction is sketched in Figure 50.

Finally we refer to the sphere S^2 as the *surface of genus 0* and we refer to the torus $S^1 \times S^1$ as the *surface of genus 1*.

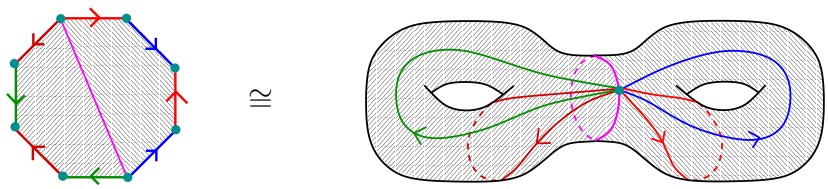

FIGURE 49.

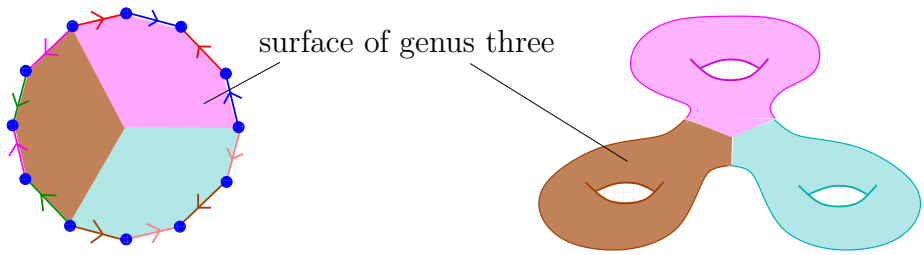

FIGURE 50. The surface of genus three.

We have now introduced many examples of topological spaces, some of them, for example the Möbius band, the torus and the surface of genus 2, can easily be pictured as subsets of "our universe", i.e. they can be viewed as subsets of $\mathbb{R}^3$. This raises the following question.

Question 1.85. *Can the real projective space* $\mathbb{R}\mathrm{P}^2$ *and the Klein bottle be viewed*[41] *as subsets of* $\mathbb{R}^3$*?*

1.17. **The two notions of connected topological spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable" topological spaces the two notions mercifully coincide.

Definition.

(1) We say that a topological space X is *path-connected* if given any two points x and y there exists a path from x to y, i.e. there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.
(2) We say that a topological space X is *connected* if the following holds[42]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open sets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

Examples.

(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.

[41]The word "view" is intentionally vague.

[42]Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.

(2) Next let $G \subset \mathbb{R}^2$ be the graph of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in Figure 51 we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[43] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x,y) \,|\, y < x^2\}$ and $V := \{(x,y) \,|\, y > x^2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round.

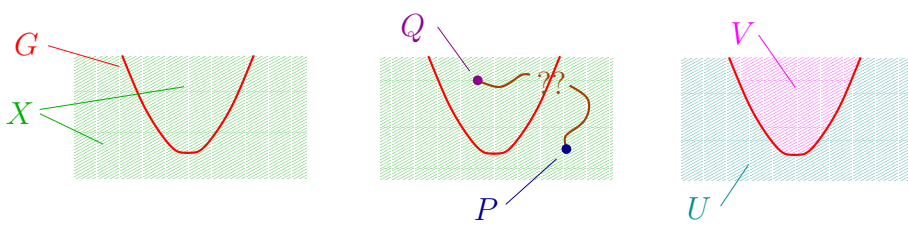

FIGURE 51.

It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0,1]$ is connected.

Proposition 1.86. *The interval* $[0,1]$ *is connected.*

Proof. Let U and V be two disjoint open subsets of $[0,1]$ with $U \cup V = [0,1]$. Without loss of generality we can assume that $0 \in U$. We want to show that $U = [0,1]$. We consider

$$A \;:=\; \big\{t \in [0,1] \,\big|\, [0,t] \subset U\big\}.^{44}$$

The set A contains 0, hence it is non-empty. Evidently the set is bounded thus we can consider $T := \sup A$. We need to show that $T = 1$ and that $T \in U$.

First let us suppose that $T \notin U$. This implies that $T \in V$. Since V is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V. In particular we have $(T-\epsilon, T+\epsilon) \cap A = \varnothing$ which implies that $\sup A \leq T - \epsilon$. Thus we have obtained a contradiction.

Now we know that $T \in U$. We still need to show that $T = 1$. Since U is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1] \subset U$. Thus $[0, T+\epsilon) \cap [0,1] \subset U$. In light of the definition of T that is only possible if $T = 1$. $\square$

[43]The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?

[44]Put differently,

$$A \;=\; \text{the set of all } t \in [0,1] \text{ such that the interval } [0,t] \text{ is contained in } U.$$

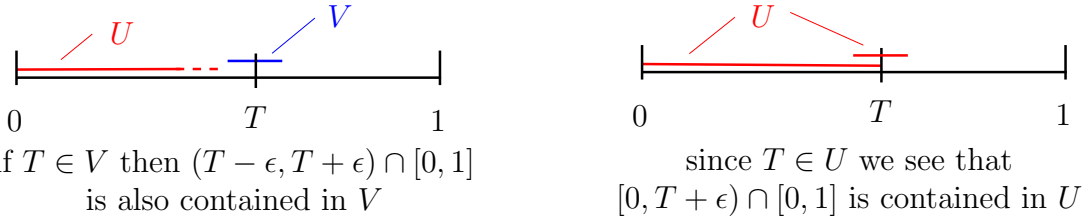

FIGURE 52. Illustration of the proof of Proposition 1.86.

Corollary 1.87. *Every path-connected topological space is also connected.*

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint non-empty open subsets. But that is a contradiction to Proposition 1.86. □

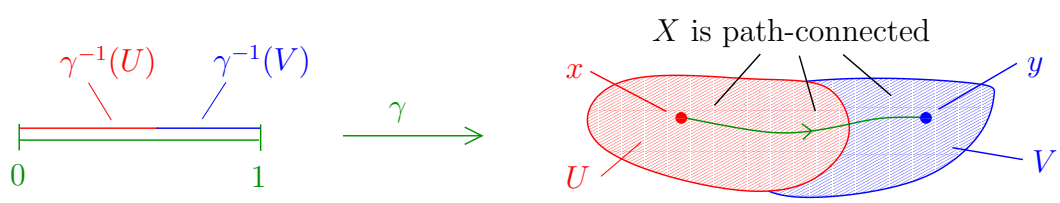

FIGURE 53. Illustration of the proof of Corollary 1.87.

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The following lemma comes in handy on several occasions.

Lemma 1.88. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If X is path-connected, then $f(X)$ is also path-connected.*
(2) *If X is connected, then $f(X)$ is also connected.*

Proof. We leave the elementary proof to the reader. □

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 1.89.

(1) *Let $n \geq 1$. Every open ball $B^n_r(y)$, every closed ball $\overline{B}^n_r(y)$ and all of $\mathbb{R}^n$ are path-connected and connected.*
(2) *Every sphere $S^n_r(y)$ with $n \geq 1$ is path-connected and connected.*

(3) *The annulus, the Möbius band, the torus and the surface of any genus are path-connected and connected.*

Proof. By Corollary 1.87 it suffices to show that all these topological spaces are path-connected. It is clear that convex subsets of $\mathbb{R}^n$ are path-connected. It follows from the example on page 78 that all balls (open or closed) are path-connected. The lemma now follows from the Lemma 1.88 (1) and the observation that all the remaining topological spaces are the homeomorphic image of convex subsets.[45] $\square$

For better or worse, the converse to Corollary 1.87 is unfortunately not always correct.

Example. Consider the topological space

$$X \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\pi]\big\} \;\subset\; \mathbb{R}^2.$$

The topological space X is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\pi]$ and an interval on the y-axis. We sketch the topological space X in Figure 54.[46] It is a very instructive exercise, see Exercise 1.42 to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected.

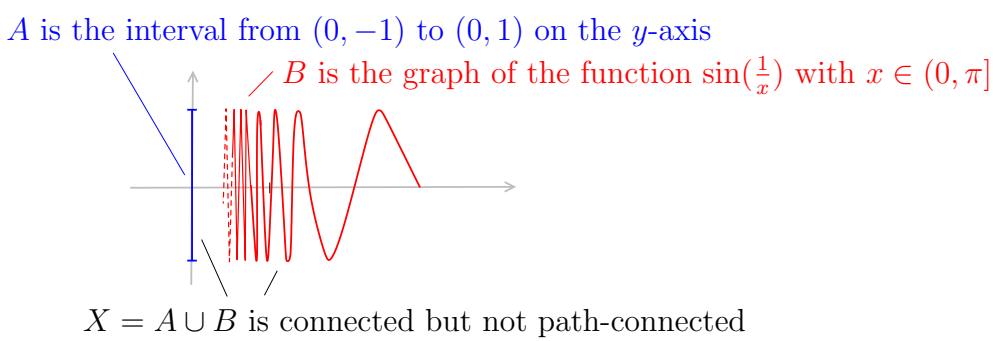


FIGURE 54.

Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree.

By a clever use of the notion of connectedness one can sometimes prove statements which at first glance have nothing to do with connectedness. For example we can now prove the following lemma.

[45] This is true by definition for all the stated topological spaces except for the spheres, but we saw on page 99 that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$.

[46] This topological space is sometimes referred to as the *topologist's sine curve.*

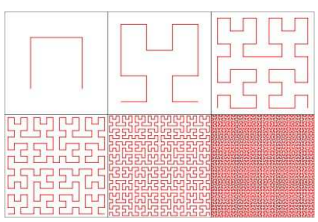
Lemma 1.90. *The topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.*

Remark. Later on we will study the question whether for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^2$. Let $P \in \mathbb{R}$ be a point. Then f restricts to a homeomorphism from $\mathbb{R} \setminus \{P\}$ to $\mathbb{R}^2 \setminus \{f(P)\}$. The topological space $\mathbb{R} \setminus \{P\} = (-\infty, P) \cup (P, \infty)$ is not path-connected[47], while one can easily verify that $\mathbb{R}^2 \setminus \{f(P)\}$ is path-connected. But as we remarked on page 72, homeomorphic spaces have the same topological properties. In particular two homeomorphic spaces are either both path-connected or they are both not path-connected. Thus we have obtained a contradiction. □

The proof of Lemma 1.90 also shows that the interval $[0, 1]$ is not homeomorphic to the square $[0, 1]^2$. One might think that this should surely be obvious since $[0, 1]^2$ is "clearly bigger" than $[0, 1]$. But the following proposition will disabuse the reader of this idea.

Proposition 1.91. *Let $n \in \mathbb{N}$. There exist maps[48] $f\colon [0, 1] \to [0, 1]^n$ that are surjective.*



first six stages in the construction
of the Hilbert space-filling curve

FIGURE 55. Illustration of Proposition 1.91.

Remark.

(1) A surjective map from $[0, 1]$ to $[0, 1]^n$, or from a one-dimensional manifold to some manifold of dimension ≥ 2, is often called a *space-filling curve*.
(2) It follows easily from Proposition 1.40 that for $n \geq 2$ an *injective* map $f\colon [0, 1] \to [0, 1]^n$ cannot be possibly be surjective. In other words, the maps promised by Proposition 1.91 are necessarily non-injective.

Proof.

(1) Many examples of such maps are given in [Sag], see e.g. [Sag, Theorem 2.1].
(2) This statement is an immediate consequence of [Wall, Lemma 4.11]. □

We continue with the following elementary but useful lemma.

Lemma 1.92. *Every map $f\colon X \to Y$ from a connected topological space X to a discrete topological space is constant.*

[47] Why is this topological space not path-connected? How can one show this?

[48] Just to make things clear, here we demand of course that f is continuous.

Remark. One could argue that Lemma 1.92 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of Lemma 1.92 is that any *non-constant* map $(a, b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic.

Proof. We need to show that for any $x \in X$ we have $f(x) = f(X)$. Since Y is a discrete topological space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. $\square$

In the remainder of this section we will discuss a few lemmas which will be useful for showing that certain topological spaces are (path-) connected. For example the next lemma gives a criterion for the union of (path-) connected subsets to be (path-) connected.

Lemma 1.93. $(*)$ *Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ is non-empty.*

(1) *If each B_i, $i \in I$ is path-connected, then $\bigcup_{i \in I} B_i$ is path-connected.*

(2) *If each B_i, $i \in I$ is connected, then $\bigcup_{i \in I} B_i$ is connected.*

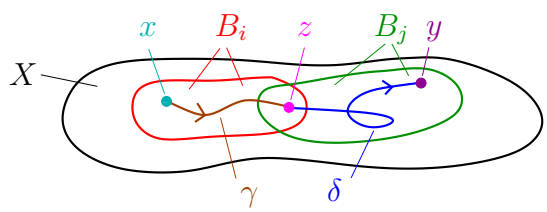


FIGURE 56. Illustration for the proof of Lemma 1.93 (1).

Proof $(*)$. Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ contains a point z. We write $Z := \bigcup_{i \in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma \colon [0, 1] \to B_i$ from x to z and we can pick a path $\delta \colon [0, 1] \to B_j$ from z to y. It is clear that

$$\begin{array}{rcl} [0,1] & \to & Z \\ t & \mapsto & \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \delta(2t-1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

is a path[49] from x to y.

(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U \;=\; \bigcup_{i\in I}(B_i \cap U) \;=\; \bigcup_{i\in I} B_i \;=\; Z. \qquad \square$$

We conclude this section with the following lemma.

Lemma 1.94. *Let X and Y be non-empty topological spaces. Then X and Y are (path)-connected if and only if $X \times Y$ is (path)-connected.*

Proof $(*)$. If $X \times Y$ is (path)-connected, then it follows from Lemma 1.88 applied to the projection maps $X \times Y \to X$ and $X \times Y \to Y$, that X and Y are (path)-connected.[50]

Now suppose that X and Y are path-connected. We need to show that $X \times Y$ is also path-connected. Thus let (x, y) and (x', y') be two points in $X \times Y$. Since X is path-connected we can find a path $p\colon [0,1] \to X$ from x to x' and similarly we can find a path $q\colon [0,1] \to Y$ from y to y'. Then

$$\begin{array}{rcl} [0,1] & \to & X \times Y \\ t & \mapsto & \begin{cases} (p(2t), y), & \text{if } t \in [0, \frac{1}{2}], \\ (x', q(2t-1)), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array}$$

is a path from (x, y) to (x', y').[51]

Finally suppose that X and Y are connected. We want to show that $X \times Y$ is also connected. We pick $a \in X$ and $b \in Y$. Let $x \in X$. Note that the subsets $\{x\} \times Y$ and $X \times \{b\}$ are connected by our hypothesis. Since these have the point (x, b) in common we deduce from Lemma 1.93 (2) that $(\{x\} \times Y) \cup (X \times \{b\})$ is also connected. Finally note that

$$X \times Y \;=\; \bigcup_{x\in X} (\{x\} \times Y) \cup (X \times \{b\}).$$

All the sets on the right are connected and they have the point (a, b) in common. Thus it follows again from Lemma 1.93 (2) that the union on the right-hand side is connected. We refer to Figure 57 for an illustration. $\square$

1.18. **The (path-) components of a topological space.**

Definition. Let x and y be two points in a topological space X. We say that the points x and y are *path-equivalent*, if there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

[49] It follows from Lemma 1.32 (2) that this map is indeed continuous.

[50] Note that the projection maps are surjective since we assume that X and Y are non-empty.

[51] It follows from Lemma 1.53 (3d) and Lemma 1.32 (2) that this map is indeed continuous.

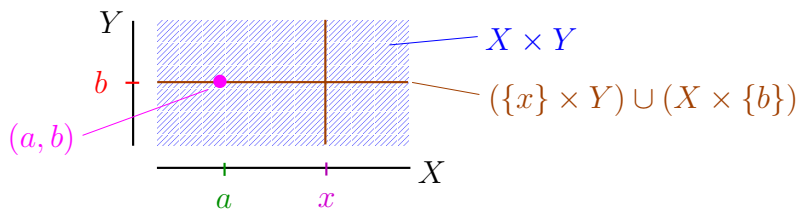


FIGURE 57. Illustration for the proof of Lemma 1.94.

Lemma 1.95. *For every topological space X the notion of path-equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive follows immediately from the argument in Lemma 1.93 (1). □

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the *path-components* of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[52] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.

Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ is in fact already a path-component of $\mathbb{Q}$.

Later on it will at times be useful to know the path-components of certain matrix groups. The relevant results are summarized in the following lemma.

Lemma 1.96.

(1) *The general linear group*

$$\mathrm{GL}(n,\mathbb{R}) \;=\; \{A \in M(n \times n, \mathbb{R}) \,|\, \det(A) \neq 0\} \;\subset\; M(n \times n, \mathbb{R}) \;=\; \mathbb{R}^{n^2}$$

has precisely two path-components, namely

$$\mathrm{GL}_+(n,\mathbb{R}) \;:=\; \{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(A) > 0\} \quad \textit{and} \quad \{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(A) < 0\}.$$

Put differently, two invertible matrices lie in the same path-component if and only if the signs of their determinants agree. The same statement also holds for $\mathrm{O}(n)$*, more precisely* $\mathrm{O}(n)$ *has precisely two path-components, namely*

$$\mathrm{SO}(n,\mathbb{R}) \;=\; \{A \in \mathrm{O}(n,\mathbb{R}) \,|\, \det(A) > 0\} \quad \textit{and} \quad \{A \in \mathrm{O}(n,\mathbb{R}) \,|\, \det(A) < 0\}.$$

(2) *The group* $\mathrm{SL}(n,\mathbb{R}) = \{A \in M(n \times n; \mathbb{R}) \,|\, \det(A) = 1\}$ *is path-connected.*

[52] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on why this is a perfectly sensible notation.

(3) *The unitary group* $\mathrm{U}(n)$ *and the special unitary group*

$$\mathrm{SU}(n) \;=\; \{A \in U(n) \mid \det(A) = 1\}$$

are both path-connected, thus they consist of a single path-component.

Proof. It is an instructive exercise in linear algebra to prove this lemma, see Exercise 1.45. Alternatively, the proof for both statements is given in [Bak, Chapter 9.2]. □

The following lemma follows immediately from the definitions and basic properties of equivalence relations.

Lemma 1.97. *Let X be a topological space. Then the following statements hold:*

(1) *as a set, X is the disjoint union of its path-components,*
(2) *each path-component of X is path-connected,*
(3) *each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.*

Our goal now is to define the "components" of a topological space that should play the same role as the path-components, i.e. we want to introduce the notion of "components" such that the statements of Lemma 1.97 hold, but with "path-connected" replaced by "connected" and "path-component" replaced by "component".

We start out with the following definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are *equivalent*, if there exists a connected subset of X that contains both x and y.

Lemma 1.98. *For every topological space X the above notion of equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except for transitivity. So suppose that x and y are equivalent and that y and z are equivalent. By definition there exists a connected subset A that contains x and y and there exists a connected subset B that contains y and z. By Lemma 1.93 (2) the union $A \cup B$ is connected and it thus follows that x and z are equivalent. □

Definition. Let X be a topological space. We call the equivalence classes of points on X the *components* of X.

Examples.

(1) The topological space illustrated in Figure 54 has two path-components but only a single component.
(2) It follows easily from the definitions that a topological space is totally disconnected if every component consists of a single point.

The following is now the promised analogue of Lemma 1.99.

Lemma 1.99. *Let X be a topological space. Then the following statements hold:*

(1) *as a set X is the disjoint union of its components,*
(2) *each component of X is connected,*
(3) *each component C of X is a maximal connected subset, i.e. if $C \subsetneq D$, then D is not connected.*

Remark. Let X be a topological space and let $\{X_i\}_{i\in I}$ be its set of components. Lemma 1.99 says that as a set we have $X = \bigcup_{i\in I} X_i$. But, if we equip the right-hand side with the disjoint union topology that we introduced on page 83, then this is in general *not* an equality (or homeomorphism) of topological spaces. For example, an easy counterexample is given by $X = \mathbb{Q}$.

Proof. Let X be a topological space. The first statement is clear. Now let C be a component of X. We pick $c \in C$. For each $d \in C$ there exists by definition a connected subset Y_d of X that contains c and d. We have

$$C = \bigcup_{c\in C}\{c\} \subset \bigcup_{c\in C} Y_c \underset{\substack{\uparrow \\ \text{by definition all points in } Y_d \text{ are equivalent to } c\text{, thus we have } Y_d \subset C}}{\subset} C \quad \text{which implies that} \quad C = \bigcup_{c\in C} Y_c.$$

It follows from Lemma 1.93 (2) that C is connected. Finally the last statement of the lemma follows immediately from the definitions. □

We wrap up the discussion of the components of a topological space with the following lemma.

Lemma 1.100. *Let X be a topological space.*
(1) *Every component of X is closed.*
(2) *If X has only finitely many components, then each component is also open.*

Example. We consider $X = \mathbb{Q}$. In this case every point is already a component. We see that every component is closed, but in this case no component is an open subset. Therefore we cannot drop the condition on the number of components in the second statement of the lemma.

Proof $(*)$. Let X be a topological space.

(1) Let C be a component of X. We need to show that $X \setminus C$ is open. By Lemma 1.8 it suffices to show that any $P \in X \setminus C$ admits a neighborhood that does not intersect C. Thus let $P \in X \setminus C$. By Lemma 1.99 we know that $C \cup \{P\}$ is not connected. Therefore there exists a decomposition $C \cup \{P\} = U \sqcup V$ with U and V open subsets such that both U and V are non-empty. Without loss of generality $C \cap U \neq \varnothing$. Since C is connected and since $C = (U \cap C) \sqcup (V \cap C)$ we see that $C = U \cap C$. Since V is also non-empty we see that $V = \{P\}$. By the definition of the subspace topology that means that there exists an open subset W of X such that $W \cap (C \cup \{P\}) = V = \{P\}$. But this shows that W is an open neighborhood of P that does not intersect C. We have thus shown that the complement of C is open. We refer to Figure 58 for an illustration of the above argument.

(2) Now suppose that X has only finitely many components. Let C be a component of X. We already know from (1) that C is closed. By definition the complement of C is the union of all other components. Since there are only finitely many and since they are closed by (1) it follows that the complement of C is closed, i.e. C is open. □

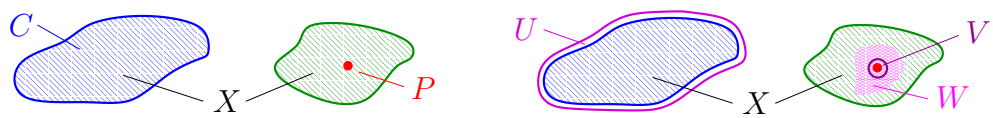


FIGURE 58. Illustration for the proof of Lemma 1.100 (1).

Remark. Let X be a topological space. We say that the points x and y are *quasi-equivalent*, if the following holds

$$\text{if } X = U \sqcup V \text{ is the disjoint union with } U, V \text{ open} \implies \text{either } x, y \in U \text{ or } x, y \in V.$$

It is straightforward to see that this indeed an equivalence relation on X. The equivalence classes of this equivalence relation are called *quasi-components*. It follows immediately from the definitions that components are contained in quasi-components. But the notions are different, for example it is shown in [SS, p. 137] that there exist topological spaces such that quasi-components are not connected. We will not make use of the concept of quasi-equivalence and quasi-components.

We conclude this section with the following lemma that can be viewed as a generalization of Lemma 1.88 and Lemmas 1.92.

Lemma 1.101. $(*)$ *Let X and Y be topological space. If X is (path-) connected and if Y consists of (path-) components $Y_i, i \in I$, then given any map $f\colon X \to Y$ there exists an $i \in I$ with $f(X) \subset Y_i$.*

Proof $(*)$. Again we leave the proof as a straightforward exercise to the reader. □

1.19. **Local properties.**

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say X is *locally P* if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.
(2) We say X is *o-locally P* if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

Remark.

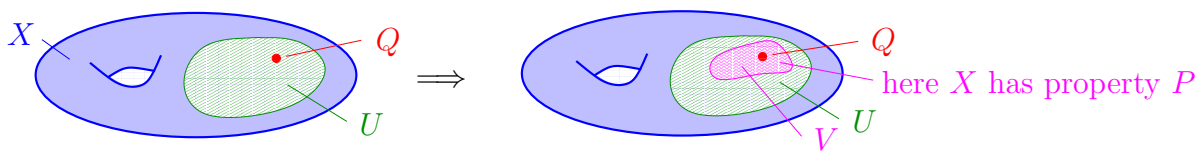


FIGURE 59.

(1) In the literature the term "locally P" is sometimes employed with subtly different definitions. The definition we use seems to be used in most books, see e.g. [Hat, Bre, Rot]. But the definition in [Mun3, p. 161] of "locally connected" is actually our definition of "o-locally connected".[53] So if one sees the notion "locally P" in the literature one needs to look up the definition to make sure what convention is used.
(2) Throughout the notes we are mostly interested in "o-local" properties. Since the term "locally P" has become fairly standard we are thus forced to carry along the prefix "o-" even though at times it makes for slightly awkward reading.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is o-locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$.The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.
(2) The space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is o-locally path-connected.[54] On the other hand $X = \mathbb{Q}$ is not even o-locally path-connected.
(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 53, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is o-locally Hausdorff.

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 1.102. *Let X be a topological space that is o-locally path-connected. Then*

$$X \text{ is connected} \iff X \text{ is path-connected.}$$

Proof. The "$\Leftarrow$"-direction is an immediate consequence of Corollary 1.87. In Exercise 1.46 we will show that if X is o-locally path-connected, then the converse also holds. □

In the following we discuss the notion of being locally compact.

Examples.

[53] It might not be apparent from [Mun3, p. 161] that the definition there of "locally connected" differs from ours, but the reason is that, as mentioned before on page 49, in [Mun3] a different notion of a neighborhood is being used.

[54] Why is that?

(1) Every open subset of $\mathbb{R}^n$ is locally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The closed ϵ-ball $\overline{B}^n_{\frac{\epsilon}{2}}(x)$ is a neighborhood[55] of x and it is compact by the Heine-Borel Theorem 1.21.
(2) Similar to (1) one can easily show that every interval is locally compact. In fact later on we will mostly need the fact that the interval $[0, 1]$ is locally compact.

In Exercise 1.48 we will see that there exists a topological space that is compact but not locally compact. The following lemma shows that this slightly unintuitive phenomenon cannot occur if we restrict ourselves to topological spaces that are Hausdorff.

Lemma 1.103. $(*)$ *Let X be a topological space. If X is compact and Hausdorff, then X is also locally compact.*

Proof. We will use Lemma 1.103 only once, in a rather inessential way. Therefore we leave the proof of this lemma as a not entirely trivial exercise to the reader. Alternatively we refer to [Wil, Theorem 18.2] for a proof. □

In the following definition we turn from properties of topological spaces to properties of maps.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

(1) We say that $f\colon X \to Y$ is *locally P* at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.
(2) We say that f is *locally P* if it is locally P at any $x \in X$.

Remark. Considering the language regarding properties of topological spaces one could argue that the above property of maps should be called "o-locally P". But since the term "o-locally" was born out of desperation we try to reduce its usage.

Examples.

(1) One easily verifies that the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a local homeomorphism at all points $x \neq 0$.
(2) The projection map $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a local homeomorphism. Indeed, let $Q \in \mathbb{R}$. We set $U = (Q - \frac{1}{2}, Q + \frac{1}{2})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore, by Lemma 1.71 (2) this is also open. It follows from Lemma 1.39 that $p\colon U \to V$ is a homeomorphism.

We will often use the statement of the following lemma without making reference to the lemma.

Lemma 1.104. *Let $f\colon X \to Y$ be a map between two topological spaces (for once we do not assume that f is continuous). If f is locally continuous, then f itself is continuous.*

[55] It is perhaps worth stressing that in the definition of a local property we need to find *neighborhoods* with a given property.

Proof. Let $f\colon X \to Y$ be a locally continuous map between two topological spaces. It follows from Lemma 1.30 (2) that given any $x \in X$ that there exists an open neighborhood U_x of x such that the restriction of f to U_x is continuous. It is an immediate consequence of Lemma 1.32 (1) that f itself is continuous. □

1.20. **Abstract graphs and spatial graphs.** We start out with the following rather abstract definition.

Definition.

(1) An *abstract graph* G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of* G and the elements of E are called the *edges of* G. Furthermore, given $e \in E$ the points in $\varphi(e)$ are called the *endpoints of* e.

(2) Let $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ be two abstract graphs. A *map* $G \to G'$ *of abstract graphs* consists of two maps $\Phi\colon V \to V'$ and $\Psi\colon E \to E'$ such that for any $e \in E$ we have $\varphi'(\Psi(e)) = \Phi(\varphi(e))$. If Φ and Ψ are bijections, then we refer to the map as an *isomorphism of abstract graphs.*

Examples.

(A) We take $V = \{A, B, C\}$, $E = \{e, f, g, h\}$ and φ to be the map that is defined by $\varphi(e) = \{A, B\}, \varphi(f) = \{B, C\}$, $\varphi(g) = \{B, C\}$ and $\varphi(h) = \{A\}$. Then (V, E, φ) is an abstract graph.

(B) Let $a, b \in \mathbb{Z}$. We denote by $G_{a,b}$ the abstract graph with vertex set $V = \mathbb{Z}$ and edge set $\{\{x, y\} \mid x, y \in \mathbb{Z} \text{ with } x = y \pm a \text{ or } x = y \pm b\}$. The map φ is given by sending each edge $e = \{x, y\}$ to $\{x, y\}$.

(C) We take V to be the set of all subway stations in München and we take E to be the set of all segments of the subway system. The map φ then assigns to each $e \in E$ the two stations that are connected by the segment e.

Definition. We say an abstract graph (V, E, φ) is *connected* if for any two vertices $v \neq w \in V$ there exist vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_0, \ldots, e_{n-1}$ with $\varphi(e_i) = \{v_i, v_{i+1}\}$ for $i = 0, \ldots, n-1$.

Examples. We want to determine whether or not the above examples (A), (B) and (C) of graphs are connected.

(A) This graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.

(B) Let $a, b \in \mathbb{Z}$. Basically by definition the graph $G_{a,b}$ is connected if and only if for any two integers $x, y \in \mathbb{Z}$ there exist $n, m \in \mathbb{Z}$ such that $x - y = ma + nb$. Elementary number theory says that this is the case if and only if a and b are coprime.

(C) This graph is connected since any two subway stations are connected by the subway system (one might have to change lines but that is of no concern to us).

So far this definition has little to do with topology. We will change this by introducing spatial graphs. Before we can do so we need the following definition.

Definition. An *arc* in $\mathbb{R}^n$ is a subset E of $\mathbb{R}^n$ for which there exists a map $\varphi\colon [0,1] \to \mathbb{R}^n$ with the following properties:

(1) the map φ is smooth, i.e. all derivatives are defined on the open interval $(0,1)$ and they extend to continuous maps on the closed interval $[0,1]$ that we also call derivatives,
(2) the first derivative $\varphi'(t)$ is non-zero for all $t \in [0,1]$,
(3) the restriction of φ to $(0,1)$ is injective,
(4) the set $\varphi((0,1))$ is disjoint from $\varphi(0)$ and $\varphi(1)$, and
(5) $\varphi((0,1)) = E$.

Given an arc E as above we refer to $\varphi(0)$ and $\varphi(1)$ as the *endpoints of* E.[56][57]

FIGURE 60.

Definition. A *spatial graph* G is a pair (V,E) with the following properties:

(1) V is a non-empty discrete subset of some $\mathbb{R}^n$.
(2) E is a subset of $\mathbb{R}^n$ with the following properties:
  (a) E is disjoint from V,
  (b) each component of E is an arc and the endpoints of each arc lie in V,
  (c) every bounded subset of $\mathbb{R}^n$ intersects only finitely many components of E.[58]

We refer to the points in V as the *vertices of* G and we refer to the components of E as the *edges of* G. Furthermore, given a spatial graph $G = (V,E)$ we write $|G| = V \cup E \subset \mathbb{R}^n$.

Convention. Sometimes, by a slight abuse of notation, we refer to the subset $|G|$ also a spatial graph. Furthermore, any topological space homeomorphic to such $|G|$ will also referred to as a spatial graph.

We sketch several examples of spatial graphs in Figure 61. The red dots correspond to the vertices and the blue segments correspond to the edges.[59]

[56]We leave it as an exercise to verify that the definition of "endpoints" is independent of the choice of φ. The key to proving this statement is that given E any two such maps φ and ψ differ by a reparametrization.

[57]Also note that according to our definition the endpoints of an arc do *not* lie on the arc.

[58]What is an example of (V,E) that satisfies all axioms except for 2 (c)?

[59]The starting point here are of course the pictures, in the sense that the definition is an attempt to turn the pictures of spatial graphs into a precise mathematical object.

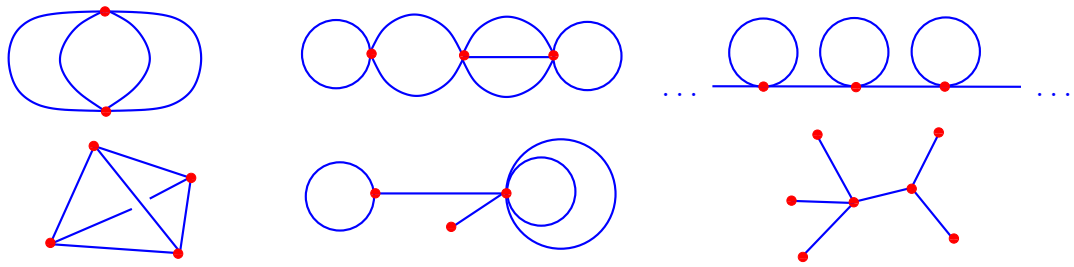


FIGURE 61.

Definition. Given a spatial graph (V, E) the *underlying abstract graph* is defined as the abstract graph (V', E', φ') given by setting

$$V := V', \quad E' := \text{the set of components of } E$$

and for $e \in E'$ we define $\varphi'(e)$ to be the set of endpoints of e. We say a spatial graph G is a *topological realization* of an abstract graph H, if the underlying abstract graph of G is isomorphic to H.

Examples. In Figure 62 we give two topological realizations of the graph from Example (A). In Figure 63 we show topological realizations of the graphs $G_{2,3}$ and $G_{0,2}$. Finally in Figure 64 we show a topological realization of the subway system of München.

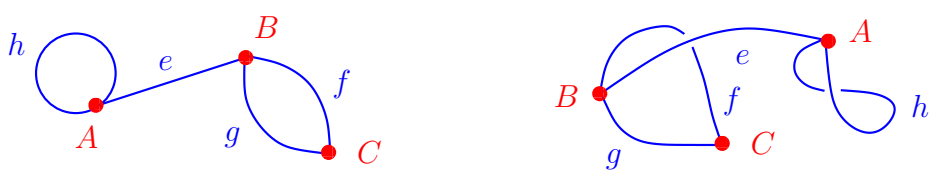
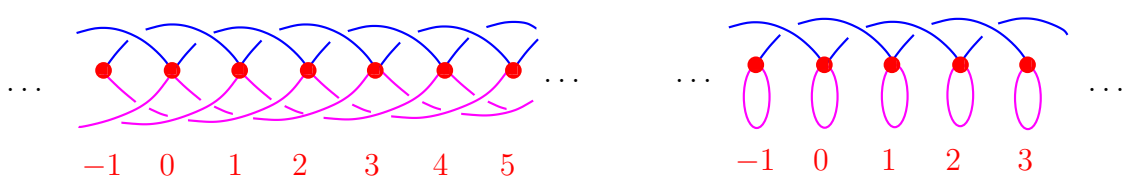


FIGURE 62.



topological realization of $G_{2,3}$ topological realization of $G_{0,2}$

FIGURE 63.

Remark. It is quite easy to show that every finite graph is the underlying abstract graph of a spatial graph in $\mathbb{R}^3$. We refer to [CELR, p. 2] for an elementary proof.

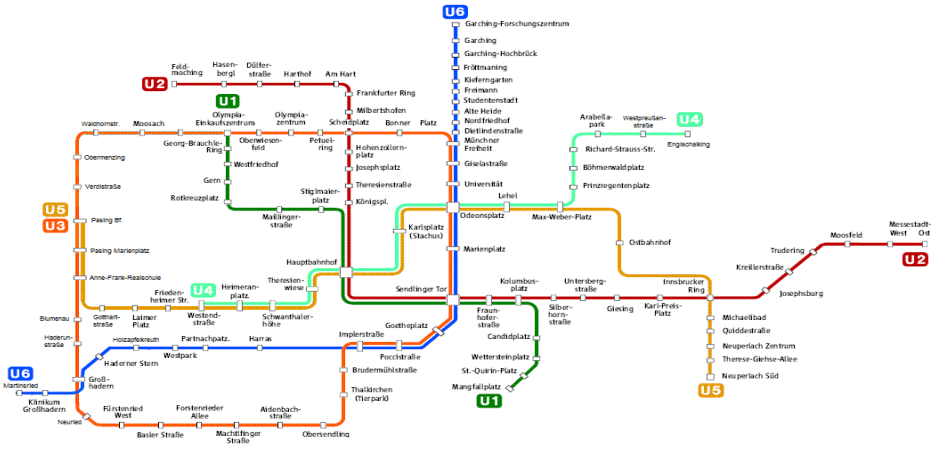


FIGURE 64. The subway system of München.

In the following we will often go back and forth between spatial graphs and abstract graphs without distinguishing these two different types of graphs in the notation. Also, very often, given a spatial graph $G = (V, E)$ we will not distinguish between G and the topological space $|G|$. Sometimes we will just talk about "graphs" and it will be clear from the context what type of graph we mean.

For the most part the two points of view are quite similar, for example given a spatial graph $G = G(V, E)$ the topological space $|G|$ is connected if and only if the underlying abstract graph is connected.[60]

Remark. The origins of graph theory go back to Euler and the problem on the seven bridges of Königsberg. The details of this story can be found on

`https://en.wikipedia.org/wiki/Seven_Bridges_of_Koenigsberg`

Euler's original article can be found here

`http://eulerarchive.maa.org//docs/originals/E053.pdf`

1.21. **Lebesgue's Lemma and uniformly continuous maps.** In this section we want to prove two statements that concern specifically maps whose domain is actually a metric space.

First we intend to formulate and prove Lebesgue's Lemma. Despite its humble name it does actually play an important role in topology. The formulation of Lebesgue's Lemma requires the following definition.

Definition. Let X be a metric space. The *diameter* of a non-empty subset A of X is defined as

$$\text{diameter}(A) \ := \ \sup\{d(a,b) \,|\, a, b \in A\} \ \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

[60]We will not prove this statement and we will therefore not make use of it.

FIGURE 65.

Example. One can easily show that

$$\text{diameter of the hyperrectangle } [a_1, b_1] \times \cdots \times [a_n, b_n] \;=\; \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

Now we can state Lebesgue's Lemma.

Lemma 1.105. (Lebesgue's Lemma) *Let K be a compact metric space and let $\{U_i\}_{i \in I}$ be an open covering of K. Then there exists a $\delta > 0$ such that for every subset A with* diameter$(A) < \delta$ *there exists an $i \in I$ with $A \subset U_i$.*

Remark. Any $\delta > 0$ that has the property stated in Lebesgue's Lemma is usually called a *Lebesgue number*.

Proof. Since K is compact we can cover K with finitely many of the U_i's. Put differently, without loss of generality we can assume that $I = \{1, \ldots, n\}$ is a finite set.

If there exists an $i \in \{1, \ldots, n\}$ with $K = U_i$, then any $\delta > 0$ has the desired property. Now suppose that this is not the case, i.e. we suppose that for every i we have $U_i \subsetneq K$.[61] We first prove the following claim.

Claim. The function

$$\begin{aligned} f\colon K &\to \mathbb{R} \\ x &\mapsto f(x) := \frac{1}{n} \sum_{i=1}^{n} d(x, K \setminus U_i) \end{aligned}$$

is continuous and positive.

It follows easily from the definitions and the triangle inequality that f is continuous. Now let $x \in K$. We want to show that $f(x) > 0$. Since $K = U_1 \cup \cdots \cup U_n$ there exists an i with $x \in U_i$. Since U_i is open there exists an $r > 0$ with $B_r(x) \subset U_i$. It follows that $d(x, K \setminus U_i) \geq r$, hence $f(x) \geq \frac{r}{n}$. This concludes the proof of the claim.

Since f is continuous and since K is compact it follows from Lemma 1.37 that the function f has a global minimum δ on K. It follows from the claim that this minimum δ is greater than 0. Now we want to show that this δ has the desired property. More precisely, we want to prove the following claim.

Claim. Let A be a subset of K with diameter$(A) < \delta$. Then there exists an $i \in \{1, \ldots, n\}$ with $A \subset U_i$.

[61] Where do we use in the subsequent argument that $U_i \subsetneq K$?

Let $x \in A$. We choose an $m \in \{1, \dots, n\}$ so that $d(x, K \setminus U_m)$ is maximal. We want to show that $A \subset U_m$. Since the diameter of A is less than δ we have $A \subset B_\delta(x)$. Thus it suffices to show that $B_\delta(x) \subset U_m$. Put differently, we need to show that $d(x, K \setminus U_m) \geq \delta$. Indeed we have

$$\begin{array}{ccccccc} d(x, K \setminus U_m) & \underset{\substack{\uparrow \\ \text{since } d(x,K\setminus U_m) \geq d(x,K\setminus U_i)}}{\geq} & \frac{1}{n}\sum_{i=1}^{n} d(x, K \setminus U_i) & = & f(x) & \underset{\substack{\uparrow \\ \text{by the choice of } \delta}}{\geq} & \delta. \end{array}$$

This concludes the proof of the claim. □

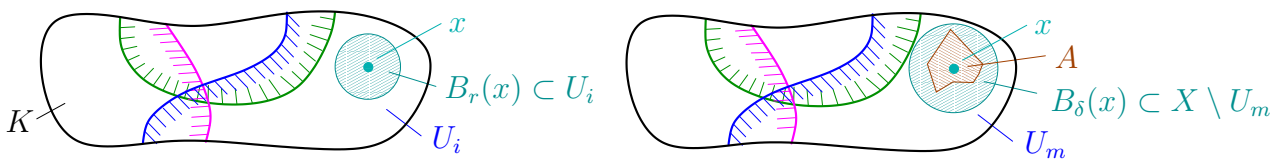


FIGURE 66. Illustration of the proof of Lebesgue's Lemma.

In many cases the following corollary is even more useful.

Corollary 1.106. *Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open covering of X. Then there exists an $N > 0$ such that for any $a_1, \dots, a_n \in \{0, \dots, N-1\}$ there exists an $i \in I$ such that*

$$f\big(\big[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_n}{N}, \tfrac{a_n+1}{N}\big]\big) \subset V_i.$$

More pictorially speaking the corollary says that if $f\colon [0,1]^n \to X$ is a map, and if we are given an open covering of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open sets covering X.

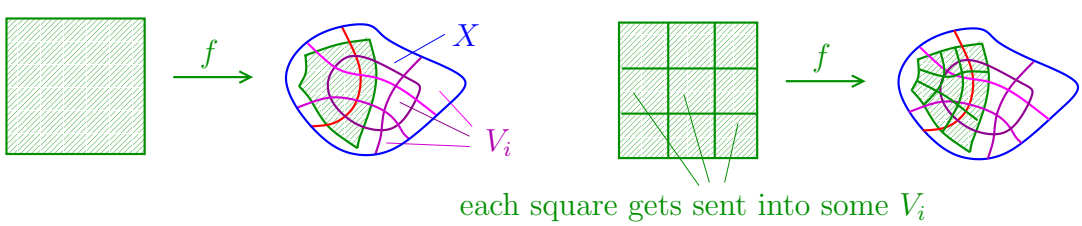


FIGURE 67. Illustration of Corollary 1.106

Proof. Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space and let $\{V_i\}_{i\in I}$ be an open covering of X. By Proposition 1.22 we know that $[0,1]^n$ is compact. Thus we can apply Lemma 1.105 to the open covering $U_i := f^{-1}(V_i)$, $i \in I$ of $[0,1]^n$ and we obtain a $\delta > 0$ such that for each subset A with diameter$(A) < \delta$ there exists an $i \in I$ with

$A \subset U_i = f^{-1}(V_i)$, which means that $f(A) \subset V_i$. It follows from the example on page 137 that the diameter of the cube of side length $\frac{1}{m}$ equals $\frac{1}{\sqrt{m}}$. Thus for sufficiently large N any cube of side length $\frac{1}{N}$ has diameter less than δ. $\square$

Definition. Let $f\colon X \to Y$ be map between two metric spaces X and Y. By Proposition 1.33 we know that

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x\in X}{\forall} \quad \underset{\epsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{y\in B_\delta(x)}{\forall} \quad d(f(x), f(y)) < \epsilon.$$

Now we define

$$f \text{ is } \textit{uniformly continuous} \quad :\Longleftrightarrow \quad \underset{\epsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{x\in X}{\forall} \quad \underset{y\in B_\delta(x)}{\forall} \quad d(f(x), f(y)) < \epsilon.$$

The following proposition is the second main result of this section.

Proposition 1.107. *Let $f\colon X \to Y$ be a continuous map between two metric spaces. If X is compact, then f is uniformly continuous.*

Proof $(*)$. Let $f\colon X \to Y$ be a continuous map between two metric spaces and let $\epsilon > 0$. The "$\Leftarrow$"-direction of Proposition 1.33 tells us in particular that given any $x \in X$ there exists a $\delta(x) > 0$ such that for all $y \in B_{\delta(x)}(x)$ we have $d(f(x), f(y)) < \frac{\epsilon}{2}$. Since the open balls $B_{\delta(x)/2}(x)$ form an open covering of X and since X is compact there exist $x_1, \dots, x_k$ such that $B_{\delta(x_1)/2}(x_1) \cup \dots \cup B_{\delta(x_1)/2}(x_1) = X$. We claim that

$$\delta \;:=\; \min\left\{\frac{\delta(x_1)}{2}, \dots, \frac{\delta(x_k)}{2}\right\}$$

has the desired properties. Thus let $x, y \in X$ with $d(x, y) < \delta$. There exists a $j \in \{1, \dots, k\}$ with $x \in B_{\delta(x_j)/2}(x_j)$. From the triangle inequality and from $\delta \le \frac{\delta(x_j)}{2}$ we obtain that $y \in B_{\delta(x_j)}(x_j)$. It follows that

$$d(f(x), f(y)) \;\underset{\substack{\uparrow \\ \text{triangle inequality}}}{\le}\; d(f(x_j), f(x)) + d(f(x_j), f(y)) \;\underset{\substack{\uparrow \\ \text{since } x \in B_{\delta(x_j)}(x_j) \text{ and } y \in B_{\delta(x_j)}(x_j)}}{<}\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

We have thus shown that δ does indeed have the desired properties. $\square$

1.22. **Topological groups.** In this section we consider the notion of a topological group which will crop up every now and then in these notes.

Definition. A *topological group* is a topological space X together with a group structure, such that the two maps[62]

$$\begin{array}{rcl} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous.

[62]Here we evidently equip $X \times X$ with the product topology.

The most basic example of a topological group is given by taking any group and equipping it with the discrete topology. As we will see, somewhat surprisingly even this banal example can turn out to be interesting.

The following lemma gives us much more interesting examples of topological groups.

Lemma 1.108. *For any $n \in \mathbb{N}$ the matrix groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are topological groups.*

Proof. As we pointed out on page 86, matrix multiplication is continuous. Furthermore, basic linear algebra, see e.g. [HJ, Chapter 0.8.2], says that taking the inverse is given by the map

$$\begin{array}{rcl} \mathrm{GL}(n,\mathbb{R}) & \to & \mathrm{GL}(n,\mathbb{R}) \\ A & \mapsto & \frac{1}{\det(A)} \cdot \underbrace{\begin{array}{c}\text{matrix whose } (i,j)\text{-entry is given by taking the determinant} \\ \text{of the matrix that is obtained from } A \text{ by removing} \\ \text{the } i\text{-th column and the } j\text{-th row of } A.\end{array}}_{=\text{called the adjugate of } A} \end{array}$$

But this map is clearly continuous. This shows that $\mathrm{GL}(n,\mathbb{R})$ is a topological group. Basically the same argument also shows that $\mathrm{GL}(n,\mathbb{C})$ is a topological group. □

We obtain more examples of topological groups by taking subgroups and products:

Lemma 1.109.

(1) *Let X be a topological group and let Y be a subgroup. If we equip Y with the subspace topology, then Y is also a topological group.*
(2) *Let $\{X_i\}_{i\in I}$ be a family of topological groups. If we equip $\prod_{i\in I} X_i$ with the product topology and the obvious group structure, then it is again a topological group.*

Proof.

(1) This statement follows immediately from Lemma 1.30 together with Lemma 1.53 (1b).
(2) This statement follows easily from Lemma 1.60 and Lemma 1.61 (2).

□

Example. It follows from Lemmas 1.108 and 1.109 (1) that $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

Lemma 1.110. *Let G be a topological group.*

(1) *Given any $g \in G$ the two maps*

$$\begin{array}{rcl} l_g\colon G & \to & G \\ h & \mapsto & g\cdot h \end{array} \quad \text{and} \quad \begin{array}{rcl} r_g\colon G & \to & G \\ h & \mapsto & h\cdot g \end{array}$$

are continuous.
(2) *If G is a topological group, then any two path-components of G are homeomorphic.*

Proof $(*)$.

(1) The map l_g is the composition of the two maps

$$\begin{array}{ccc} G & \to & G \times G \\ h & \mapsto & (g,h) \end{array} \quad \text{and} \quad \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & x \cdot y. \end{array}$$

The first map is continuous by Lemma 1.53 (3b) and the second map is continuous since G is a topological group. Thus l_g itself is continuous. The same argument shows that r_g is continuous.

(2) Let X, Y be two path-components of G. We pick $x \in X$ and $y \in Y$. We consider the map $\varphi\colon G \to G$ given by $g \mapsto yx^{-1}g$. By (1) we know that φ is continuous. Furthermore it follows from Lemma 1.101 that $\varphi(X) \subset Y$.

We also consider the map $\psi\colon G \to G$ given by $g \mapsto xy^{-1}g$. The same argument as above shows that ψ is continuous and that $\psi(Y) \subset X$. Clearly φ and ψ are inverses of one another. But this implies that φ and ψ are homeomorphisms between X and Y. $\square$

Exercises for Chapter 1.

Exercise 1.1. Show that for statements (1), (2) and (4) of Lemma 1.1 there are indeed examples such that the stated inclusion is indeed a strict inclusion.

Exercise 1.2. Let $f\colon X \to Y$ be a map between two sets. Let $A \subset X$ and let $B \subset Y$ be two subsets. Prove the following equivalence of statements:

$$f(A) \subset B \qquad \Longleftrightarrow \qquad A \subset f^{-1}(B).$$

Exercise 1.3.

(a) Prove Lemma 1.11.

(b) Let U be an open subset of $\mathbb{R}^2$. Is it true that the interior of $\overline{U}$ equals U?

Exercise 1.4. Let U be an open subset of $\mathbb{R}^n$. We denote by $C = \overline{U}$ the closure of U.

(a) Suppose that U is convex. Show that $U = \mathring{C}$.

(b) Given an example of an open subset U of $\mathbb{R}^n$ where $U \neq \mathring{C}$.

Exercise 1.5. Let $f\colon X \to Y$ be a map between topological spaces and let $B \subset Y$ be a subset.

(a) Show that $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$.

(b) Given an example that shows that in general we do not have an equality in (a).

(c) Show that $f(f^{-1}(\overline{B})) = \overline{B}$.

Exercise 1.6. Let $\{U_i\}_{i \in I}$ be a family of subsets. If I is finite, then we know by Lemma 1.12 that the following two equalities hold:

$$\text{(a)} \quad \text{interior of } \bigcap_{i \in I} U_i \;=\; \bigcap_{i \in I} \mathring{U}_i \qquad \text{and} \qquad \text{(b)} \quad \text{closure of } \bigcup_{i \in I} U_i \;=\; \bigcup_{i \in I} \overline{U_i}.$$

Do these equalities also hold if I is infinite?

Exercise 1.7. Let (X, d) be a compact metric space. A rather surprising theorem, see [Gros] or alternatively [Ris], says that there exists a unique $r \in \mathbb{R}_{\geq 0}$ such that for any finite sequence of points $x_1, \dots, x_n \in X$ there exists an $x' \in X$ with $\frac{1}{n}\sum_{i=1}^{n} d(x_i, x') = r$. This number is called the *rendezvous value of* (X, d).

(a) Determine the rendezvous value of the interval $X = [0, 1]$ equipped with the metric $d(x, y) = |x - y|$.
(b) Determine the rendezvous value of the circle $X = S^1$ equipped with the Euclidean metric of $\mathbb{R}^2$.

Exercise 1.8.

(a) Show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and compact.
(b) Show that "the line with two zeros" is neither Hausdorff nor compact.

Exercise 1.9.
Let $f\colon X \to Y$ be a map between topological spaces. Show that f is continuous if and only if for every $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.

Exercise 1.10. Let $\mathbb{R}^n \cup \{\infty\}$ be "$\mathbb{R}^n$ with a point at infinity". Let $v \in \mathbb{R}^n$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n \cup \{\infty\} & \to & \mathbb{R}^n \cup \{\infty\} \\ Q & \mapsto & \begin{cases} Q + v, & \text{if } Q \in \mathbb{R}^n, \\ \infty, & \text{if } Q = \infty \end{cases} \end{array}$$

is continuous.

Exercise 1.11. Let f be a complex polynomial of degree ≥ 1. Show that the map

$$\begin{array}{rcl} \Theta(f)\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ t & \mapsto & \begin{cases} f(t), & \text{if } t \in \mathbb{C}, \\ \infty, & \text{if } t = \infty \end{cases} \end{array}$$

is continuous.
Remark: By Lemma 1.41 we can make the identification $S^2 = \mathbb{C} \cup \{\infty\}$. Thus we see that complex polynomials of degree ≥ 1 give rise to interesting self-maps of S^2.
Remark: The analogous statement also holds for a polynomial f with real coefficients, it gives rise to a self-map $\Theta(f)$ of $\mathbb{R} \cup \{\infty\} = S^1$. We illustrate this situation in Figure 68.

Exercise 1.12. Let X be a topological space. We say a subset $U \subset X$ is *large* if U is open and dense. Show that the intersection of finitely many large subsets of X is again large.

Exercise 1.13. Let $f, g\colon X \to Y$ be two maps between two topological spaces. Let $A \subset X$ be a subset such that f and g agree on A. Show that if A is dense, if Y is Hausdorff and if f and g are continuous, then f and g agree on all of X.

Exercise 1.14.

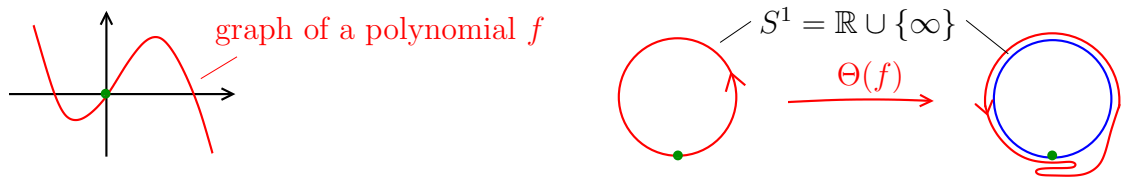


FIGURE 68. Illustration for Exercise 1.11.

(a) Give an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.
(b) Show that the half-open interval $[0,1)$ and the open interval $(0,1)$ are not homeomorphic.

Exercise 1.15. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Let $0 \leq a < b \leq 1$. Show that f can only go finitely many times from a to b. Show that it is not possible to find numbers $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < \dots$ in $[0,1]$ such that for each $i \in \mathbb{N}$ we have $f(x_i) = a$ and we have $f(y_i) = b$.

Exercise 1.16. Are the two spatial graphs G and H shown in Figure 69 homeomorphic?

FIGURE 69.

Exercise 1.17. Show that the topological space $[0,1] \cap \mathbb{Q}$ is not compact.

Exercise 1.18.

(a) Let A be a real $m \times n$-matrix. Show that there exists a $C \in \mathbb{R}$ such that for all $v \in \mathbb{R}^n$ we have $\|Av\| \leq C \cdot \|v\|$.
(b) Let $A \in \mathrm{GL}(n, \mathbb{R})$ be an invertible matrix. Show that there exists a $\lambda > 0$ such that $\|Av\| \geq \lambda \|v\|$ for all $v \in \mathbb{R}^n$.

Hint: In both cases use that S^{n-1} is compact.

Exercise 1.19. Let A be an invertible real $m \times m$-matrix. Show that the map

$$\begin{array}{rcl} \mathbb{R}^m \cup \{\infty\} & \to & \mathbb{R}^m \cup \{\infty\} \\ x & \mapsto & \begin{cases} A \cdot x, & \text{if } x \in \mathbb{R}^m, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is continuous.
Hint: use Exercise 1.18.

Exercise 1.20. Is every open map necessarily continuous? More precisely, if $f\colon X \to Y$ is a map between topological spaces with the property that for any open $U \subset X$ the image $f(U)$ is an open subset of Y, does it follow that f is continuous?

Exercise 1.21. Let $n \in \mathbb{N}$. We consider the stereographic projection

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \dots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1 \end{cases} \end{array}$$

that we introduced in Lemma 1.41. Show that Φ is continuous.

Exercise 1.22.

(a) Show that the topological space $[-1,0] \sqcup (0,1]$ is homeomorphic to the topological space $[-2,-1] \cup (0,1] \subset \mathbb{R}$.
(b) Show see that the topological space $[-1,0] \sqcup (0,1]$ is not homeomorphic to the topological space $[-1,1]$.

Exercise 1.23. Show that there are non-homeomorphic topological spaces A and B and a non-empty topological space X such that $A \times X$ is homeomorphic to $B \times X$.
Hint: you could take $A = \{0\}$ and $B = \{0,1\}$.

Exercise 1.24. Let X be a topological space. We consider the "diagonal"

$$\Delta \ := \ \{(x,x) \in X \times X \,|\, x \in X\}.$$

We equip $X \times X$ with the product topology and we equip Δ with the subspace topology. Show that the map $X \to \Delta$ given by $x \mapsto (x,x)$ is a homeomorphism.

Exercise 1.25. Let $\{X_i\}_{i \in I}$ be a family of topological spaces.

(a) Suppose that for each $i \in I$ we are given a closed subset $A_i \subset X_i$. If I is finite, then it follows from Lemma 1.51 that $\prod_{i \in I} A_i$ is a closed subset of $\prod_{i \in I} X_i$. Does this conclusion also hold if I is infinite?
(b) Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a subset $A_i \subset X_i$. Show that

$$\prod_{i \in I} \overline{A_i} \ = \ \overline{\prod_{i \in I} A_i} \ \subset \ \prod_{i \in I} X_i.$$

Exercise 1.26. Let (X,d) be a metric space. Show that the map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

is continuous.

Exercise 1.27. Let X be a metric space and let $B \subset X$ be a non-empty subset. Show that the map

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x,B) := \inf\{d(x,b) \,|\, b \in B\} \end{array}$$

is continuous.

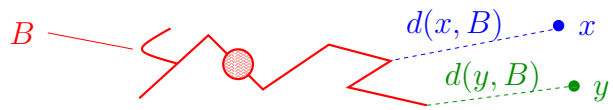


FIGURE 70. Illustration for Exercise 1.27.

Exercise 1.28. A topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. Now suppose that X is a metrizable topological space. Show that given any closed set $A \subset X$ there exists a map $f\colon X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = A$.
Hint: Use Exercise 1.27.

Exercise 1.29. Show that the map from $\prod_{i\in\mathbb{N}} \{0,2\}$ to the Cantor set, that we defined on page 95, is continuous.

Exercise 1.30. Let X be a topological space. Show that

$$X \text{ is Hausdorff} \quad \Longleftrightarrow \quad \text{the diagonal } \Delta = \{(x,x) \,|\, x \in X\} \text{ is a closed subset of } X \times X.$$

Exercise 1.31. Let $f, g\colon X \to Y$ be two maps between topological spaces. Suppose that Y is Hausdorff. Show that $\{x \in X \,|\, f(x) = g(x)\}$ is a closed subset of X.
Hint: Use Exercise 1.30.

Exercise 1.32.

(a) Show that the real projective space $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1.
(b) Do you think that the real projective space $\mathbb{R}\mathrm{P}^2 = (\mathbb{R}^3 \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is homeomorphic to S^2? As of now we lack the tools to answer this question. Nonetheless, what do you think the answer will be?

Exercise 1.33. Show that the topological space $\mathbb{R}/\{\pm 1\} = \mathbb{R}/x \sim -x$ is homeomorphic to the half-open interval $[0, \infty)$.

Exercise 1.34. Let the group $(\mathbb{Q}, +)$ act on the topological space $\mathbb{R}$ by addition. Show that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Exercise 1.35. Given $v, w \in \mathbb{R}^2 \setminus \{(0,0)\}$ we define

$$\sphericalangle(v,w) := \text{unique } \varphi \in (-\pi, \pi] \text{ such that } w = r \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \cdot w \text{ for some } r \in \mathbb{R}_{>0}.$$

(a) Show that the map

$$\begin{array}{rcl} \Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & (-\pi, \pi] \\ (v,w) & \mapsto & \sphericalangle(v,w) \end{array}$$

is not continuous.

(b) Show that the map

$$\begin{array}{rcl} \Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & \mathbb{R}/2\pi\mathbb{Z} \\ (v,w) & \mapsto & [\sphericalangle(v,w)] \end{array}$$

is continuous. (Here $\mathbb{R}/2\pi\mathbb{Z}$ denotes the quotient of the topological space $\mathbb{R}$ by the group $(2\pi\mathbb{Z},+)$.)

Exercise 1.36. We consider the following subsets of $[0,1]$:

$$A = \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}, \quad B = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \quad \text{and} \quad C = \{0\} \cup B.$$

(a) Show that $[0,1]/A$ is homeomorphic to a subset of $\mathbb{R}^2$. See e.g. Figure 71 for an idea.
(b) Is $[0,1]/B$ homeomorphic to a subset of $\mathbb{R}^2$?
(c) Is $[0,1]/C$ homeomorphic to a subset of $\mathbb{R}^2$?

FIGURE 71.

Exercise 1.37. Let X be topological space and let A and B be two subsets with $A \cup B = X$. We write $C := A \cap B$. We consider the disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ and we consider $Y = (A \sqcup B)/\sim$ where for each $c \in C$ we identify the point $(c,1) \in A \times \{1\}$ with the corresponding point $(c,2) \in B \times \{2\}$. We consider the map

$$p\colon Y = (A \sqcup B)/\sim \;\to\; X$$

that is given by

$$\begin{array}{rcll} [(a,1)] & \mapsto & a & \text{for } a \in A, \\ [(b,2)] & \mapsto & b & \text{for } b \in B. \end{array}$$

(a) Is the map p necessarily continuous?
(b) Is the map p necessarily open?
(c) Suppose that A and B are open subsets of X. Show that p is a homeomorphism.

Remark: The failure of p to be a homeomorphism can lead to nasty surprises.

Exercise 1.38. Let X be a topological space and let A and B be two disjoint subsets. We equip A with the subspace topology and we equip X/B with the quotient topology. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/B$ the projection.

(a) Show that the map $p \circ i\colon A \to X/B$ is continuous and injective.
(b) Show that the map $p \circ i\colon A \to X/B$ is not necessarily a homeomorphism onto its image.

(c) Suppose that there exists an open subset $U \subset X$ with $A \subset U$ and $U \cap B = \varnothing$. Show that the map $p \circ i \colon A \to X/B$ is a homeomorphism onto its image.

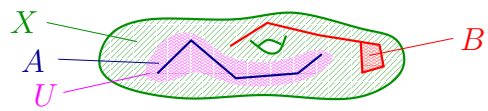


FIGURE 72. Illustration for Exercise 1.38.

Exercise 1.39. Similar to the definition of the Möbius band on page 105 we consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times \{-1, 1\}$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times \{-1,1\}) & \to & \mathbb{R} \times \{-1,1\} \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Show that $X/G = X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 1.40. We say a group G *acts discretely* on a topological space X, if it acts continuously and if for each $x \in X$ there exists an open neighborhood U so that

$$U \cap gU \;=\; \varnothing \text{ for all } g \neq e.$$

Can the group $\mathbb{Z}$ act discretely on S^2?

Exercise 1.41. Build a Möbius band out of a strip of paper and cut it along the central curve. What topological space do you obtain that way?

Exercise 1.42. Show that the topologist's sine curve, that we defined on page 123 is connected and show that it is *not* path-connected.

Exercise 1.43. A topological space X is called *arc-connected* if given any two points $P, Q \in X$ there exists a map $f \colon [0,1] \to X$ with $f(0) = P$, $f(1) = Q$ and which has the property that f is a homeomorphism onto its image.[63]

(a) Show that the line with two zeros is not arc-connected.
(b) Let $n \geq 2$ and let X be the "$\mathbb{R}^n$ with two origins", which is defined in the hopefully obvious way. Is X arc-connected?

Remark: A rather amazing (non-trivial!) theorem says that any topological space that is path-connected *and which is Hausdorff* is in fact arc-connected. We refer to [Wil, Corollary 31.6] and [Enge, p. 381].

Exercise 1.44.

[63]Sometimes, see e.g. [Wil, Enge], such topological spaces are also called *arcwise connected*. Rather confusingly some other books, e.g. [Bre], use the term *arcwise connected* for what we call path connected. The lesson is, that in any given book one always should look up the definitions.

(a) Let $n \in \mathbb{N}_{\geq 3}$. We consider

$$X_n \;:=\; \{r \cdot e^{2\pi i t k/n} \,|\, r \in [0,1) \text{ and } k \in \{0,\dots,n-1\}\}.$$

We write $P := 0$. Show that every self-homeomorphism $f\colon X_n \to X_n$ preserves P. We refer to Figure 73 for an illustration.

(b) Does the statement in (a) also hold for $n = 1$ and/or $n = 2$?

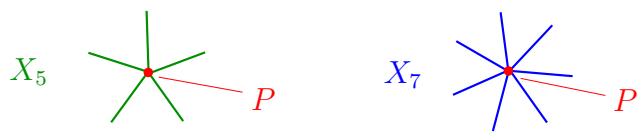


FIGURE 73.

Exercise 1.45. Let $n \in \mathbb{N}$. We consider the general linear group

$$\mathrm{GL}(n,\mathbb{R}) \;=\; \{A \in M(n \times n, \mathbb{R}) \,|\, \det(A) \neq 0\} \;\subset\; M(n \times n, \mathbb{R}) \;=\; \mathbb{R}^{n^2}.$$

Show that two matrices lie in the same path-component of $\mathrm{GL}(n,\mathbb{R})$ if and only if the signs of their determinants agree. (Put differently, the goal is to prove the first part of Lemma 1.96.)

Exercise 1.46. Let X be a topological space that is o-locally path-connected. Show that if X is connected, then it is also path-connected.

Exercise 1.47. Let X be a path-connected subset of $\mathbb{R}^2$. Is it also necessarily locally path-connected?

Exercise 1.48. We consider the set $X = (0,1)$ with the topology where the open sets are precisely the sets $\varnothing, X$ and the intervals $(0, 1 - \frac{1}{n})$. Show that X is compact and show that X is not locally compact.

Exercise 1.49. Given a topological graph $G = (V,E)$ we write $|G| = V \cup E$. Let $G = (V,E)$ and $G' = (V',E')$ be two topological graphs in $\mathbb{R}^3$. If $|G|$ is homeomorphic to $|G'|$, does it follow that the underlying abstract graphs are isomorphic?

2. General topology: some delicate bits (∗)

In this chapter we will introduce the compact-open topology and we will discuss the very delicate interplay between product and quotient topologies. In all likelihood the reader is better of with consulting the chapter only when the results of this chapter are actually used.

2.1. **The compact-open topology** (∗) **.** So far we had mostly studied topological spaces that arose more or less directly from geometry. In this section we will see that topological spaces also arise naturally from more abstract set ups.

First, recall that we saw on page 54 that a topology on a given set X makes it possible to talk of the convergence of a sequence in X. From real analysis we know that it is not only interesting to think about convergence of sequences of points but also to talk of convergence of functions. In fact, in real analysis one often encounters two different types of convergence that we now recall.

Definition. Let $U \subset \mathbb{R}^n$ be a subset, let $\{f_n \colon U \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions and let $f \colon U \to \mathbb{R}$ be a function. We define

$$(f_n)_{n\in\mathbb{N}} \textit{ converges pointwise to } f \quad :\Longleftrightarrow \quad \underset{x\in U}{\forall} \quad \underset{\epsilon>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n\geq N}{\forall} \quad |f_n(x) - f(x)| < \epsilon$$

and

$$(f_n)_{n\in\mathbb{N}} \textit{ converges uniformly to } f \quad :\Longleftrightarrow \quad \underset{\epsilon>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{x\in U}{\forall} \quad \underset{n\geq N}{\forall} \quad |f_n(x) - f(x)| < \epsilon.$$

Example. We consider the sequence of functions

$$\begin{aligned} f_n \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 0, & \text{if } t < 0 \text{ or } t > \frac{2}{n}, \\ n \cdot t, & \text{if } t \in [0, \frac{1}{n}], \\ 2 - n \cdot t, & \text{if } t \in (\frac{1}{n}, \frac{2}{n}]. \end{cases} \end{aligned}$$

We refer to Figure 74 for an illustration. One easily verifies that this sequence converges pointwise to the zero function but it does not converge uniformly to the zero function.

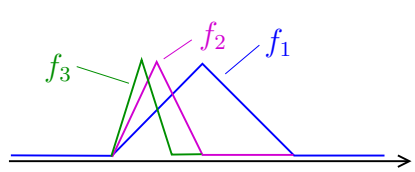


Figure 74.

At times one also wants to study convergence of sequences of continuous[64] maps between general topological spaces. This leads us to the following definition.

[64]Here and in the following we sometimes talk explicitly about "continuous" maps, even though it is usually understood in these notes that all maps are continuous.

Definition. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all continuous maps from Y to X.
(2) The *compact-open topology* on X^Y is the topology generated, in the sense of the definition on page 63, by all the sets of the form

$$M(K, U) \ := \ \{f \in X^Y \mid f(K) \subset U\}$$

where $K \subset Y$ is compact and where $U \subset X$ is open. This definition is illustrated in Figure 75.

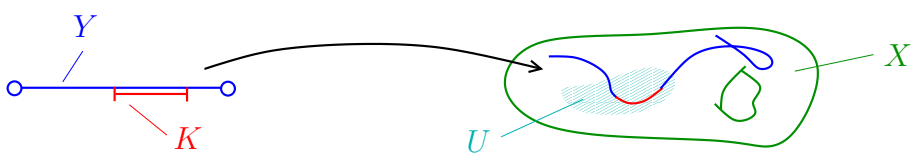


FIGURE 75.

Examples.

(1) Let $Y \subset \mathbb{R}$ be an interval. We want to consider the compact-open topology on $C(Y, \mathbb{R}) = \mathbb{R}^Y$. As usual we picture elements of $C(Y, \mathbb{R})$ by their graphs. In Figure 76 we show on the left an open set $M(K, U)$ as above and on the right we show the graph of some function $f \in C(Y, \mathbb{R})$ and an open neighborhood thereof.

Now let $f_n \in C(Y, \mathbb{R})$, $n \in \mathbb{N}$ be a sequence of functions and let $f \in C(Y, \mathbb{R})$. We leave it as an amusing, albeit somewhat lengthy, exercise to show that the following holds:

$$\begin{array}{c} (f_n)_{n\in\mathbb{N}} \text{ converges}^{65} \text{ to } f \text{ with respect} \\ \text{to the compact-open topology} \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \text{for each compact subset } K \subset Y \\ \text{the sequence } (f_n|_K)_{n\in\mathbb{N}} \\ \text{converges uniformly to } f|_K \end{array}$$

Alternatively a proof for this statement is given in [Wil, Theorem 43.7] or in [Jos, Theorem 15.1.2].

(2) We consider $Y = \{1, \dots, n\}$ which we equip with the discrete topology. In Exercise 2.4 we will see that for any topological space X we have a homeomorphism $X^Y \cong X^n$.

The final example deserves its own lemma.

Lemma 2.1. *Let $m, n \in \mathbb{N}$. We have the obvious inclusion*

$$\begin{array}{rcl} M(m \times n, \mathbb{R}) & \to & C(\mathbb{R}^n, \mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n} \\ A & \mapsto & \begin{pmatrix} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & Av \end{pmatrix}. \end{array}$$

[65] We refer to page 54 for the definition of convergence of a sequence in a topological space.

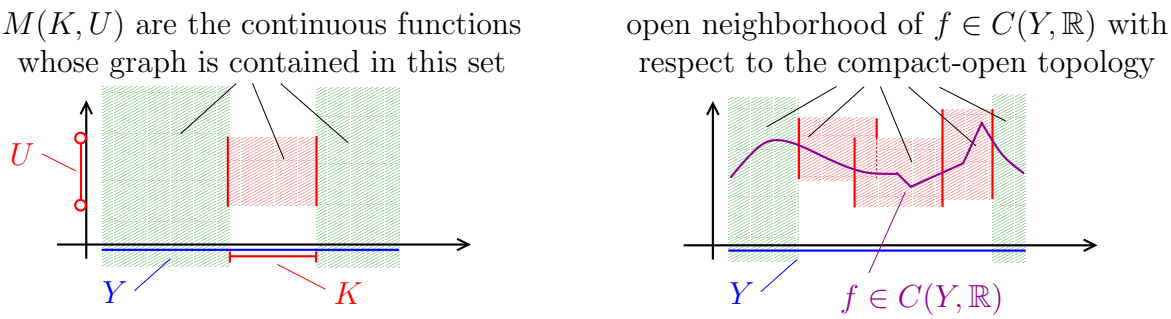


FIGURE 76.

The compact-open topology on $C(\mathbb{R}^n, \mathbb{R}^m)$ induces the usual topology on $M(n \times m, \mathbb{R})$ that we introduced on page 86.

Sketch of proof. This statement can be proved, with some effort, using Lemma 1.24. Alternatively the statement can also be deduced from [Hat, Proposition A.13] or [Mun3, Theorem 5.1]. □

To get some practice with the compact-open topology we will prove the following two lemmas in Exercises 2.1 and 2.2.

Lemma 2.2. *Let X and Y be topological spaces. If Y is Hausdorff, then X^Y is also Hausdorff.*

Lemma 2.3. *If X and Y are topological spaces, then the map*

$$\begin{array}{rcl} X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

is continuous.

We continue with a few technical statements on the compact-open topology. In most applications later on we will consider the case that $Y = [0,1]$. In particular the extra hypothesis in the subsequent propositions on the topological space Y will not be an issue in our future applications.

Proposition 2.4. *Let X and Y be topological spaces. If Y is locally compact, then the following two statements hold:*

(1) *The evaluation map*[66]

$$\begin{array}{rcl} e\colon X^Y \times Y & \to & X \\ ((f\colon Y \to X), y) & \mapsto & f(y) \end{array}$$

is continuous.

[66]Here we equip $X^Y \times Y$ with the product topology from page 84.

(2) *For any $y \in Y$ the evaluation map*

$$\begin{array}{rcl} e_y \colon X^Y & \to & X \\ (f \colon Y \to X) & \mapsto & f(y) \end{array}$$

is continuous.

Proof $(*)$.

(1) Let U be an open subset of X. We need to show that $e^{-1}(U)$ is open. By Lemma 1.8 it suffices to show that given $(f, y) \in e^{-1}(U)$ there exists a neighborhood W of (f, y) with $W \subset e^{-1}(U)$. Thus let $f \in X^Y$ and $y \in Y$ with $e(f, y) = f(y) \in U$. Since f is continuous and since Y is locally compact there exists a compact neighborhood K of y which is contained in the open neighborhood $f^{-1}(U)$ of y. In other words, we have found a compact neighborhood K of y with $f(K) \subset U$. Therefore we have $f \in M(K, U)$ and the map e sends $W := M(K, U) \times K$ into U. By the definition of the product topology the set $W = M(K, U) \times K$ is in fact a neighborhood of (f, y) in $X^Y \times Y$.

(2) This statement follows from (1) and Lemma 1.53 (3b). □

We continue with the following elementary but essential lemma.

Lemma 2.5. $(*)$ *Let X and Y be topological spaces.*

(1) *If $\varphi \colon X \to \widetilde{X}$ is a continuous map, then the induced map*

$$\begin{array}{rcl} \varphi^* \colon X^Y & \to & \widetilde{X}^Y \\ (f \colon Y \to X) & \mapsto & (\varphi \circ f \colon Y \to \widetilde{X}) \end{array}$$

is continuous.

(2) *If $\psi \colon Y \to \widetilde{Y}$ is a continuous map, then the induced map*

$$\begin{array}{rcl} \psi_* \colon X^{\widetilde{Y}} & \to & X^Y \\ (f \colon \widetilde{Y} \to X) & \mapsto & (f \circ \psi \colon Y \to X) \end{array}$$

is continuous.

Proof $(*)$.

(1) We check continuity using the continuity criterion of Proposition 1.34. Thus let $K \subset Y$ be a compact subset and let $U \subset \widetilde{X}$ be an open subset. We have

$$\begin{aligned} (\varphi^*)^{-1}(M(K,U)) &= \{f \in X^Y \mid \varphi^*(f) \in M(K,U)\} = \{f \colon Y \to X \mid \overbrace{(\varphi \circ f)(K)}^{=\varphi(f(K))} \subset U\} \\ &\underset{\substack{\uparrow \\ \text{by Exercise 1.2}}}{=} \{f \colon Y \to X \mid f(K) \subset \varphi^{-1}(U)\} = M(K, \varphi^{-1}(U)). \end{aligned}$$

Since φ is continuous we see that $\varphi^{-1}(U)$ is an open subset of X. Since the sets $M(K, U)$ form by definition a basis for the topology of $\widetilde{X}^Y$ we deduce from Proposition 1.34 that φ^* is indeed continuous.

(2) Let $K \subset Y$ be a compact subset and let $U \subset X$ be an open subset. We have

$$\begin{aligned}(\psi_*)^{-1}(M(K,U)) &= \{f \in X^{\widetilde{Y}} \mid \psi_*(f) \in M(K,U)\} \\ &= \{f\colon \widetilde{Y} \to X \mid \underbrace{(f\circ\psi)(K)}_{=f(\psi(K))} \subset U\} \;=\; M(\psi(K),U).\end{aligned}$$

Since ψ is continuous we obtain from Lemma 1.37 that $\psi(K)$ is a compact subset of $\widetilde{Y}$. Thus we obtain again from Proposition 1.34 that ψ_* is indeed continuous. □

On many occasions we will also need the following rather technical proposition.

Proposition 2.6. *Let Y be a topological space that is locally compact. Furthermore let X and T be two topological spaces. Let $H\colon Y \times T \to X$ be a map. Given $t \in T$ we denote by $H_t\colon Y \to X$ the map defined by $H_t(y) := H(y,t)$. Then*

$$H\colon Y\times T \to X \text{ is continuous} \iff \begin{array}{l}(1)\ \text{each } H_t\colon Y \to X \text{ is continuous, and}\\ (2)\ \text{the map}^{67}\ T \to X^Y \text{ given by } t \mapsto H_t \text{ is continuous.}\end{array}$$

Proof (∗). [68] First we prove the "⇐"-direction. We point out that $H\colon Y \times T \to X$ is the composition of the two maps

$$\begin{array}{rcl} Y\times T &\to& X^Y \times Y \\ (y,t) &\mapsto& (H_t, y)\end{array} \quad \text{and the evaluation map} \quad \begin{array}{rcl} X^Y \times Y &\to& X \\ (f,y) &\mapsto& f(y).\end{array}$$

The first map is continuous by (2) and Lemma 1.53. Furthermore, since Y is locally compact we obtain from Proposition 2.4 that the second map is continuous. Thus H itself is continuous.

Now we turn to the "⇒"-direction. Thus we suppose that $H\colon Y \times T \to X$ is continuous.

(1) Let $t \in T$. The map H_t is the composition of the inclusion $Y \to Y \times T$, $y \mapsto (y,t)$ with the map H. The first map is continuous by Lemma 1.53 (3b) and the second map is continuous by our hypothesis. Hence H_t is also continuous. In particular we now know that H_t defines an element in X^Y.

(2) We need to show that the map $T \to X^Y$ given by $t \mapsto H_t$ is continuous. By Proposition 1.34 and Lemma 1.8 it suffices to show that for any set of the form $M(K,U) \subset X^Y$ and any $t \in T$ with $H_t \in M(K,U)$ there exists a neighborhood W of $t \in T$ such that for all $t' \in W$ we have $H_{t'} \in M(K,U)$.

Thus let K be a compact subset of Y, let U be an open subset of X and let $t \in T$ with $H_t \in M(K,U)$. We start out with the following claim.

Claim. Given any $y \in K$ there exist open neighborhoods $V_y \subset Y$ of y and $W_y \subset T$ of t such that $H(V_y \times W_y) \subset U$.

[67] Note that by (1) we know that $H_t \in X^Y$.

[68] I strongly advise against reading the proof.

By the continuity of $H\colon Y \times T \to X$ we know that $H^{-1}(U)$ is an open subset of $Y \times K$. Note that $H(y,t) = H_t(y) \in U$ since $H_t \in M(K,U)$ and $y \in K$. Thus $H^{-1}(U)$ is an open neighborhood of $(y,t) \in Y \times K$. The existence of V_y and W_y is now a consequence of the definition of the product topology of $Y \times T$. This concludes the proof of the claim.

It follows from the compactness of K and Lemma 1.16 that there exist points $y_1, \dots, y_n \in K$ with $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. We set $W := W_{y_1} \cap \dots \cap W_{y_n} \subset T$. Note that this is an open neighborhood of t. Then

$$H(K \times W) \underset{\substack{\uparrow \\ \text{since } K \subset \bigcup_{i=1}^n V_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i} \times W\Big) \underset{\substack{\uparrow \\ \text{since } W \subset W_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i} \times W_{y_i}\Big) = \bigcup_{i=1}^n \underbrace{H(V_{y_i} \times W_{y_i})}_{\subset U} \subset U.$$

It follows from this observation and the definition of $H_{t'}$ and $M(K,U)$ that for any $t' \in W$ we have $H_{t'} \in M(K,U)$. □

On a later occasion we will also need the following equally technical proposition.

Proposition 2.7. *Let X, Y and Z be topological spaces. If Y and Z are both locally compact and Hausdorff, then the map*[69]

$$\begin{array}{rcl} X^{Y \times Z} & \to & (X^Y)^Z \\ (f\colon Y \times Z \to X) & \to & \left(\begin{array}{rcl} Z & \to & X^Y \\ z & \mapsto & (y \mapsto f(y,z)) \end{array} \right) \end{array}$$

is a homeomorphism.

Remark. In fact a slightly stronger version of Proposition 2.7 holds. By [Dug, Theorem 5.3] one only needs to assume that Y is locally compact, with no further restrictions on X, Y or Z. We will not need this stronger statement.

Once again no sane person will ever voluntarily read the proof. For those readers who are still around we first provide the proof of the following lemma.

Lemma 2.8. *Let X and Y be topological spaces. Furthermore let $\mathcal{V}$ be a subbasis for X. We consider*

$$\mathcal{C} = \big\{ M(K,V) \in \mathcal{P}\big(X^Y\big) \,\big|\, K \text{ a compact subset of } Y \text{ and } V \in \mathcal{V} \big\}.$$

If Y is Hausdorff, then $\mathcal{C}$ is subbasis[70] *for the topology of X^Y.*

Proof. Let K be a compact subset of Y, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y \to X$ is a map with $K \subset f^{-1}(U)$. By Lemma 1.28 (2) we need to show that there exist finitely many $C_1, \dots, C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^m C_i \subset M(K,U)$.

[69]Note that the map is well-defined, in the sense that the map $Z \to X^Y$ actually defines an element in $(X^Y)^Z$. More precisely, it follows from the "$\Rightarrow$"-direction of Proposition 2.6 and the fact that Y is locally compact that the map $Z \to X^Y$ given by $z \mapsto (y \mapsto f(y,z))$ actually lies in $(X^Y)^Z$.

[70]We refer to page 63 for the definition of a subbasis of a topological space.

Since $\mathcal{V}$ is a subbasis for the topology of X we can write $U = \bigcup_{i\in I} U_i$ where each U_i is of the form $U_i = \bigcap_{V\in P_i} V$ where P_i is a finite subset of $\mathcal{V}$. Since K is compact and since f is continuous it follows from Lemma 1.16 that there exists a finite subset $J \subset I$ such that $K \subset \bigcup_{j\in J} f^{-1}(U_j)$.

Since K is compact and Hausdorff we obtain from Lemma 1.42 that K is normal. It follows from Lemma 1.44, together with Lemma 1.18 (1) and the hypothesis that Y is Hausdorff, that there exist compact subsets $K_j \subset f^{-1}(U_j)$, $j \in J$, with $K \subset \bigcup_{j\in J} K_j$.

Thus we see that

$$\begin{array}{ccccccc} f & \in & \bigcap_{j\in J} M(K_j, U_j) & = & \bigcap_{j\in J} M\Big(K_j, \bigcap_{V\in P_j} V\Big) & \overset{\text{since } M(K,A\cap B) = M(K,A)\cap M(K,B)}{=} & \bigcap_{j\in J}\bigcap_{V\in P_j} \underbrace{M(K_j,V)}_{\in\mathcal{C}} \subset M(K,U). \\ & \uparrow & & \uparrow & & & \uparrow \\ & \text{since } K_j \subset f^{-1}(U_j) & & \text{definition of } P_j & & & \text{since } K \subset \bigcup_{j\in J} K_j \text{ and since each } V \text{ lies in } U \end{array}$$

The finitely many sets $\{M(K_j, V)\}_{j\in J, V\in P_j}$ in $\mathcal{C}$ thus do the trick. □

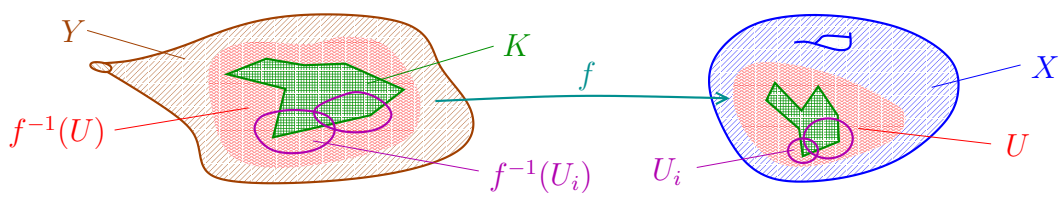


FIGURE 77. Illustration for the proof of Lemma 2.8.

Now we are finally ready to provide the proof of Proposition 2.7.

Proof of Proposition 2.7. Let X, Y and Z be topological spaces. We assume that Y and Z are both locally compact and Hausdorff. We need to show that the map

$$\begin{array}{rcl} \Phi\colon X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z) & \to & \left(\begin{array}{ccc} Z & \mapsto & X^Y \\ z & \mapsto & (y \mapsto f(y,z)) \end{array}\right) \end{array}$$

is a homeomorphism. We start out with the following claim.

Claim. The set

$$\mathcal{C} = \left\{ M(A\times B, U) \in \mathcal{P}\big(X^{Y\times Z}\big) \;\middle|\; \begin{array}{l} A \text{ a compact subset of } Y,\ B \text{ a compact subset of } Z \\ \text{and } U \text{ an open subset of } X \end{array} \right\}$$

is a subbasis for the topological space $X^{Y\times Z}$.

Let $K \subset Y \times Z$ be a compact subset, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y \times Z \to X$ is a map with $K \subset f^{-1}(U)$. By Lemma 1.28 (2) we need to show that there exist finitely many $C_1, \dots, C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^{m} C_i \subset M(K,U)$.

Let $k \in K \subset Y \times Z$. We write $k = (y,z)$. Since $f^{-1}(U)$ is an open subset of $Y \times Z$ it follows from the definition of the product topology on $Y \times Z$ that there exist open neighborhoods $V_k \subset Y$ of y and $W_k \subset Z$ of z such that $k = (y,z) \in V_k \times W_k \subset f^{-1}(U)$. Since Y and Z are locally compact we can find a compact neighborhood $A_k \subset Y$ of y and a compact neighborhood $B_k \subset Z$ of z such that $k \in A_k \times B_k \subset V_k \times W_k \subset f^{-1}(U)$. Since K is compact it follows from Lemma 1.16 that there exist finitely many points $k_1, \dots, k_m \in K$ such that $K \subset \bigcup_{i=1}^{m} A_{k_i} \times B_{k_i}$. We now see that

$$\begin{array}{ccccccc} & & & & \text{since } M(V \cup W, U) = M(V,U) \cap M(W,U) & & \\ & & & & \downarrow & & \\ f & \in & \bigcap_{i=1}^{m} \overbrace{M(A_{k_i} \times B_{k_I}, U)}^{\in \mathcal{C}} & = & M\Big(\bigcup_{i=1}^{m} A_{k_i} \times B_{k_i}, U\Big) & \subset & M(K,U). \\ & \uparrow & & & & \uparrow & \\ & \text{since } f(A_{k_i} \times B_{k_i}) \subset U & & & & \text{since } K \subset \bigcup_{i=1}^{m} A_{k_i} \times B_{k_i} & \end{array}$$

This concludes the proof of the claim.

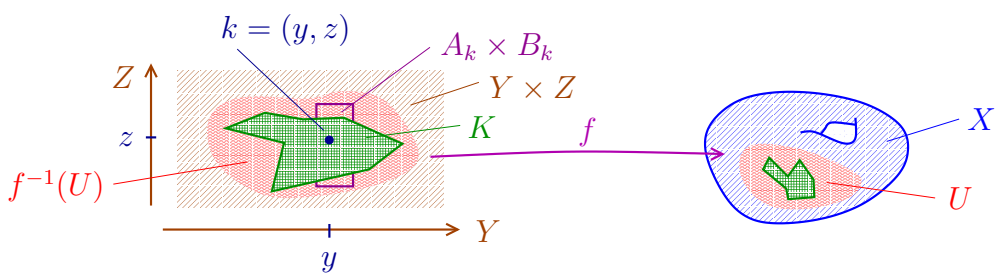


FIGURE 78. Illustration for the proof of Proposition 2.7.

The given map $\Phi\colon X^{Y \times Z} \to (X^Y)^Z$ is clearly a bijection. Furthermore under this bijection the sets $M(A \times B, U)$ are precisely the sets $M(B, M(A,U))$. By the claim the former sets form a subbasis for $X^{Y \times Z}$. Thus it suffices to show that the latter sets form a subbasis for $(X^Y)^Z$. But that is an immediate consequence of Lemma 2.8, applied twice, and our hypothesis that Y and Z are Hausdorff. $\square$

The following proposition is also somewhat technical.

Proposition 2.9. *Let X, Y and Z be topological spaces. If Y is locally compact, then the map*

$$\begin{array}{rcl} \Phi\colon X^Y \times Y^Z & \to & X^Z \\ (f,g) & \mapsto & f \circ g \end{array}$$

is continuous.

The proof of Proposition 2.9 rests on the following little lemma.

Lemma 2.10. $(*)$ *Let Y be a topological space that is locally compact. If K is a compact subset that is contained in an open subset U of Y, then there exists a compact subset L with $K \subset \mathring{L} \subset L \subset U$.*

Proof $(*)$. Let $x \in K$. Since Y is locally compact there exists a compact neighborhood $L_x \subset U$ of x. Since K is compact there exist $x_1, \ldots, x_m \in K$ with $K \subset \mathring{L}_{x_1} \cup \cdots \cup \mathring{L}_{x_m}$. We set $L := L_{x_1} \cup \cdots \cup L_{x_m}$. By Lemma 1.17 we know that L is compact. We have

$$K \;\subset\; \bigcup_{i=1}^{m} \mathring{L}_{x_i} \;\underset{\substack{\uparrow \\ \text{Lemma 1.12 (1)}}}{\subset}\; \text{interior of} \bigcup_{i=1}^{m} L_x \;=\; \mathring{L} \;\subset\; L \;\subset\; U.$$

□

Proof of Proposition 2.9 $(*)$. By Proposition 1.34 and Lemma 1.8 and the definition of the product topology on $X^Y \times Y^Z$ it suffices to prove the following claim.

Claim. Let K be a compact subset of Z and let V be an open subset of X. If $f \in X^Y$ and $g \in Y^Z$ satisfy $f \circ g \in M(K, V)$, then there exists an open subset $U \subset Y$ and a compact subset $L \subset Y$ with

$$\Phi\big(\underbrace{M(L,V)}_{\subset X^Y} \times \underbrace{M(K,U)}_{\subset Y^Z}\big) \;\subset\; M(K,V).$$

First note that $g(K) \subset Y$ is compact by Lemma 1.37 and that $f^{-1}(V) \subset Y$ is open since f is evidently continuous. Since Y is locally compact we obtain from Lemma 2.10 that there exists a compact subset L with $g(K) \subset \mathring{L} \subset L \subset f^{-1}(U)$. It is clear that $U = \mathring{L}$ and L have the desired properties. This concludes the proof of the claim. □

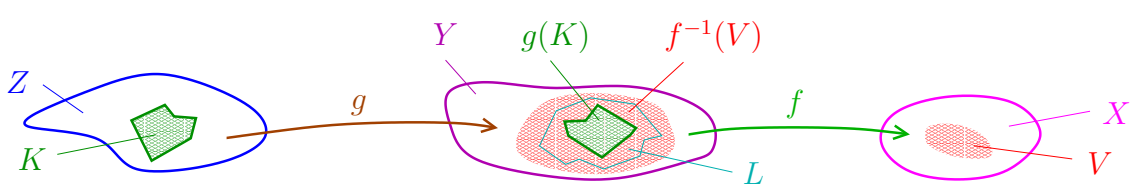


FIGURE 79. Illustration for the proof of Proposition 2.9.

The following corollary gives us a very long list of new examples of topological groups.

Corollary 2.11. *Let X be a topological space. If X is compact and Hausdorff, then the group*

$$\text{Homeo}(X) \;:=\; \textit{the set of all self-homeomorphisms of } X,$$

equipped with the subspace topology coming from X^X, is a topological group.

Remark. Let X be a topological space that is locally compact and Hausdorff. In light of Proposition 2.9 one might expect that in this case $\mathrm{Homeo}(X)$ forms a group. In [Dij] it is shown that this is in general *not* the case. On the other hand it is shown in [Are, Theorem 4] that if X is furthermore locally connected, then $\mathrm{Homeo}(X)$ is in fact a topological group.

Proof $(*)$. Let X be a topological space that is compact and Hausdorff. First note that by Lemma 1.103 we know that X is locally compact. Thus it follows from Proposition 2.9 that the multiplication map $\mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ is continuous. Therefore it suffices to show that the map

$$\begin{array}{rcl} \Phi\colon \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ f & \mapsto & f^{-1} \end{array}$$

is continuous. The proof of continuity rests on the following simple claim.

Claim. If $K \subset X$ and $U \subset X$ are subsets, then $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$.

Let $h \in \mathrm{Homeo}(X)$. We have

$$\begin{array}{rcl} h \in \Phi^{-1}(M(K,U)) & \overset{\text{since } \Phi \text{ is a bijection and } \Phi(h)=h^{-1}}{\Longleftrightarrow} & h^{-1} \in M(K,U) \quad \Longleftrightarrow \quad h^{-1}(K) \subset U \quad \overset{\text{since } h \text{ is a bijection}}{\Longleftrightarrow} \quad K \subset h(U) \\ & \Longleftrightarrow & \underbrace{X \setminus h(U)}_{=h(X\setminus U)} \subset X \setminus K \quad \Longleftrightarrow \quad h \in M(X \setminus U, X \setminus K). \end{array}$$

This concludes the proof of the claim.

By Proposition 1.34 it suffices to prove the following claim.

Claim. If $K \subset X$ is a compact subset and if $U \subset X$ is an open subset, then $\Phi^{-1}(M(K,U))$ is an open subset of X.

By the previous claim we know that $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$. Since U is open we know that $X \setminus U$ is closed. Since X is compact we obtain from Lemma 1.18 (1) that $X \setminus U$ is compact. Furthermore, since K is compact and since X is Hausdorff we obtain from Lemma 1.18 (2) that $X \setminus U$ is closed. Thus we see that $M(X \setminus U, X \setminus K)$ is an open subset of X^X. This concludes the proof of the claim. □

2.2. **Mixing product and quotient topologies** $(*)$**.** In this, rather technical section, we will see that mixing product and quotient topologies can be quite delicate. We start out with the following elementary lemma.

Lemma 2.12. *Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x,y) \sim_\times (x',y)$ whenever $x \sim x'$. The following map is a bijection and it is continuous:*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x,y)] & \mapsto & ([x],y). \end{array}$$

Proof $(*)$. It follows basically immediately from the definition of $\sim_\times$ that the map φ is a bijection. We will walk cautiously through the continuity argument.

We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ and $q\colon X \to X/\sim$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q \times \mathrm{id}_Y"] & \\ (X \times Y)/\sim_\times \arrow[rr, "\varphi"] & & (X/\sim) \times Y. \end{tikzcd}$$

Let $W \subset (X/\sim) \times Y$ be an open subset. We need to show that $\varphi^{-1}(W)$ is an open subset of $(X \times Y)/\sim_\times$. By definition of the quotient topology we need to show that $p^{-1}(\varphi^{-1}(W))$ is an open subset of $X \times Y$.

By Lemma 1.8 and the definition of the product topology it suffices to prove the following claim.

Claim. For each $(x, y) \in p^{-1}(\varphi^{-1}(W)) \subset X \times Y$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood V_y of $y \in Y$ such that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$.

Since $([x], y) \in W$ and since W is an open subset of $(X/\sim) \times Y$ we obtain, by definition of the product topology, an open neighborhood $\widetilde{U}_{[x]}$ of $[x] \in X/\sim$ and an neighborhood V_y of $y \in Y$ with $\widetilde{U}_{[x]} \times V_y \subset W$. We set $U_x := q^{-1}(\widetilde{U}_{[x]})$. By definition of the quotient topology this is an open subset of X. It remains to show that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$. In other words, we need to show that $\varphi(p(U_x \times V_y)) \subset W$. This inclusion follows from the following short argument:

$$\varphi(p(U_x \times V_y)) = (\varphi \circ p)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{by the above commutative triangle}}}{=} (q \times \mathrm{id}_Y)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{since } U_x = q^{-1}(\widetilde{U}_{[x]})}}{=} q(q^{-1}(\widetilde{U}_{[x]})) \times V_y = \widetilde{U}_{[x]} \times V_y \subset W.$$

This concludes the proof of the claim and thus also of the lemma. $\square$

One would expect that the map from Lemma 2.12 surely must be a homeomorphism. But perhaps rather shockingly, the next lemma shows that in general this is not the case:

Lemma 2.13. *There exist topological spaces X and Y and an equivalence relation $\sim$ on X with the following property: If we denote by $\sim_\times$ the equivalence relation on $X \times Y$ generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$, then the map*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and it is continuous, but it is not a homeomorphism.

Remark. We refer to [Fab, Theorem 1], [BrownR2, p. 111] and [Bou3, p. 128] for other examples which also show that mixing product and quotient topologies is in general a very dangerous idea.

Proof. We consider $X = \mathbb{R} \times \mathbb{N} = \{(x,n) \,|\, x \in \mathbb{R}, n \in \mathbb{N}\}$, equipped with the product topology. Let $\sim$ be the equivalence relation on X that is generated by $(0,m) \sim (0,n)$ for all $m,n \in \mathbb{N}$. We set $Y := \prod_{i\in\mathbb{N}} \mathbb{R}$, equipped with the product topology introduced on page 94. By Lemma 2.12 we know that the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x,y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and that it is continuous. It remains to show that φ is not a homeomorphism. We will do so by exhibiting an open subset V of $(X \times Y)/\sim_\times$ such that the image $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

Before we can proceed we need to introduce the following notation:

(1) We denote by $q\colon X \to X/\sim$ the obvious projection.
(2) We write $f = q \times \mathrm{id}_Y\colon X \times Y \to (X/\sim) \times Y$.
(3) Given $n \in \mathbb{N}$ we set

$$U_n \;:=\; \{((t,n),(y_1,y_2,y_3,\dots)) \in X \times Y \,|\, |t \cdot y_n| < 1\}.$$

and we set $U := \bigcup_{n\in\mathbb{N}} U_n$.

The key to proving the lemma is the following claim.

Claim. The set U has the following properties:

(1) U is an open subset of $X \times Y$,
(2) $f^{-1}(f(U)) = U$,
(3) $f(U)$ is not open in $(X/\sim) \times Y$.

We set out to prove that U does indeed have the three promised properties.

(1) We want to show that U is an open subset of $X \times Y$. It suffices to show that each U_n is open. First note that the map

$$\begin{array}{rcl} \gamma\colon X \times Y = \mathbb{R} \times \mathbb{N} \times \prod_{i\in\mathbb{N}} \mathbb{R} & \to & \mathbb{R} \\ (t,n,y_1,y_2,\dots) & \mapsto & t \cdot y_n \end{array}$$

is continuous by Lemma 1.60. Furthermore note that $\{n\}$ is an open subset of $\mathbb{N}$. We now see that

$$U_n \;=\; \gamma^{-1}((-1,1)) \cap \underbrace{\mathbb{R} \times \{n\} \times \prod_{i\in\mathbb{N}} \mathbb{R}}_{\text{open by definition of the product topology}}$$

is the intersection of two open sets, thus it is open itself.

(2) We need to show $f^{-1}(f(U)) = U$. In other words, we need to show that if v is an element of U with $f(v) \in f(U)$, then $v \in U$. But this statement follows from the following claim.

Claim. Let $v = ((s,m),(y_1,y_2,\dots))$ and $w = ((t,n),(z_1,z_2,\dots))$ be two elements in $X \times Y$. If $f(v) = f(w)$ and if $w \in U$, then we also have $v \in U$.

If $v = w$, then there is nothing to show. On the other hand, if $v \neq w$, then it follows from the definition of the map f and the equivalence relation $\sim$ that $s = t = 0$. But this implies that v and w lie in every U_n, in particular they both lie in U. This concludes the proof of the claim.

(3) We prove this statement by contradiction. Thus suppose that $f(U)$ is, contrary to our hopes, an open subset of $(X/\sim) \times Y$. Note that $((0,0),(0,0,\dots)) \in U$. This implies that $f((0,0),(0,0,\dots)) = ([(0,0)],(0,0,\dots))$ is contained in $f(U)$. By definition of the product topology of $(X/\sim) \times Y$ and of the product topology of $Y = \prod_{i\in\mathbb{N}} \mathbb{R}$ there exists an open subset $V \subset X/\sim$ and open subsets $W_i \subset \mathbb{R}$ with $([(0,0)],(0,0,\dots)) \in V \times \prod_{i\in\mathbb{N}} W_i \subset f(U)$ and such that all but finitely W_i are equal to $\mathbb{R}$. We pick an $N \in \mathbb{N}$ with $W_N = \mathbb{R}$. Since $[(0,0)] \in V$ we see that $q^{-1}(V)$ is an open subset of $X = \mathbb{R} \times \mathbb{N}$ that contains the subset $q^{-1}([(0,0)]) = \{0\} \times \mathbb{N}$. In particular $q^{-1}(V)$ contains some point (s,N) with $s \neq 0$. We pick a real number $y_N > \frac{1}{|s|}$ and we consider the point

$$P \;:=\; ((s,N),(0,\dots,0,\underset{\substack{\uparrow \\ \text{the } N\text{-th entry}}}{y_N},0,\dots)) \;\in X \times Y.$$

Note that

$$P \;\underset{\substack{\uparrow \\ \text{since } (s,N)\in q^{-1}(V) \\ \text{and since } W_N=\mathbb{R}}}{\in}\; q^{-1}(V) \times \prod_{i\in\mathbb{N}} W_i \;\underset{\substack{\uparrow \\ \text{since } f = q\times \mathrm{id}_Y}}{=}\; f^{-1}\Big(V \times \prod_{i\in\mathbb{N}} W_i\Big) \;\underset{\substack{\uparrow \\ \text{see above}}}{\subset}\; f^{-1}(f(U)) \;\underset{\substack{\uparrow \\ \text{by (2)}}}{=}\; U \;=\; \bigcup_{n\in\mathbb{N}} U_n.$$

By definition of the U_n's we see that the only possibility is that $P \in U_N$. But we have $|s \cdot y_N| > 1$, i.e. P cannot possibly lie in U_N. We have thus obtained a contradiction.

This concludes the proof of the claim.

It remains to prove the following claim.

Claim. We set $V := p(U)$.

(1) V is an open subset of $(X \times Y)/\sim_\times$,
(2) $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ the obvious projection map. We have the following commutative diagram

$$\begin{tikzcd} X \times Y \arrow[rr, "f=q\times \mathrm{id}_Y"] \arrow[dr, "p"'] & & (X/\sim) \times Y \\ & (X \times Y)/\sim_\times . \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Note that

$$\begin{array}{ccccccc} p^{-1}(V) & = & p^{-1}(p(U)) & = & f^{-1}(f(U)) & = & U & = & \text{an open subset of } X \times Y. \\ & & & \uparrow & & \uparrow & & \uparrow & \\ & & \text{since } p = \varphi^{-1} \circ f \text{ and since } \varphi \text{ is a bijection} & & & & \text{by the previous claim} & & \end{array}$$

By the definition of the quotient topology on $(X \times Y)/\sim_\times$ this means that V is indeed an open subset of $(X \times Y)/\sim_\times$. Furthermore we have $\varphi(V) = \varphi(p(U)) = f(U)$. But by the previous claim we know that $f(U)$ is not an open subset of $(X/\sim) \times Y$. This concludes the proof of the claim and thus of the lemma. □

The previous lemma leads us to the following cautionary remark:

Remark.

In general products and quotients of topological spaces do not mix well.

Definition. We say that a map $f\colon X \to Y$ between topological spaces is a *quotient map* if f is surjective and if a subset $U \subset Y$ is open in Y if and only if the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) We consider the map $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is shown in Figure 80. This map is continuous, it is *not* an open map, but the reader is invited to verify that the map is in fact a quotient map.

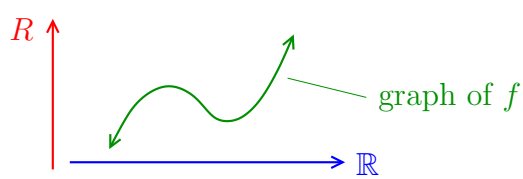


FIGURE 80.

(2) Let $X = \{0, 1\}$ be a set of two elements $0, 1$. Let $\mathcal{S}$ be the trivial topology and let $\mathcal{T}$ be the discrete topology. The identity map $(X, \mathcal{S}) \to (X, \mathcal{T})$ is a continuous bijection, but it is not a quotient map.

The following lemma summarizes a few elementary properties of quotient maps.

Lemma 2.14.

(1) *Let X be a topological space and let $\sim$ be an equivalence relation on X. The projection map $X \to X/\sim$ is a quotient map.*
(2) *If $f\colon X \to Y$ is a quotient map, then it is also continuous.*
(3) *If $f\colon X \to Y$ is a bijection between topological spaces and if it is a quotient map, then f is a homeomorphism.*

Proof.

(1) This statement follows immediately from the definition of the quotient topology on the quotient $X/\sim$.
(2) Let $f\colon X \to Y$ be a quotient map. If follows immediately from the "only if" bit of the definition of a quotient map that f is continuous.
(3) This statement follows immediately from (2) and Lemma 1.39. □

The following is the key technical theorem which in most reasonable situations will help us to navigate the choppy waters involving product and quotient maps.

Theorem 2.15. *Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space. If Y is locally compact (e.g if $Y = [0,1]$), then the map*

$$f \times \mathrm{id}_Y \colon X \times Y \;\to\; Z \times Y$$

is also a quotient map.

Remark. This theorem is sometimes called the "Whitehead Theorem" since it was first proved by John Whitehead [Whd8, Lemma 4] in 1948.

We will provide the proof of Theorem 2.15 in the next section. In the following we first state and prove some useful consequences of Theorem 2.15. In fairness to the reader one should perhaps say that it is arguably best to move on to the next chapter. It is only advisable to read the remainder of this section once one really faces acute technical problems involving the mix of product and quotient maps.

The following little lemma is the key to many applications of Theorem 2.15.

Lemma 2.16. *Suppose we are given a commutative diagram of maps between topological spaces:*

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "f"] & \\ Y \arrow[rr, "\varphi"'] & & Z. \end{tikzcd}$$

The following two statements hold:

(1) *If p and f are quotient maps, then φ is also a quotient map.*
(2) *If p is a quotient map and if f is continuous, then φ is continuous.*

Proof.

(1) Suppose that p and f are quotient maps. Let $V \subset Z$ be an open subset. We have the following implications:

$$\begin{array}{ccccccc} V \text{ is open} & \Leftrightarrow & f^{-1}(V) \text{ is open} & \Leftrightarrow & p^{-1}(\varphi^{-1}(V)) \text{ is open} & \Leftrightarrow & \varphi^{-1}(V) \text{ is open.} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \begin{array}{c}\text{since } f \text{ is a quotient}\\ \text{map}\end{array} & & \begin{array}{c}\text{since } f = \varphi \circ p \text{ we have}\\ f^{-1}(V) = p^{-1}(\varphi^{-1}(V))\end{array} & & \begin{array}{c}\text{since } p \text{ is a quotient map and}\\ \text{since } p(p^{-1}(\varphi^{-1}(V))) = \varphi^{-1}(V)\end{array} & \end{array}$$

Furthermore, it follows from the fact that the diagram commutes and the fact that f is surjective, that φ is surjective. We have thus verified that φ is a quotient map.

(2) Now we suppose that p is a quotient map and that f is continuous. We need to show that given an open subset $V \subset Z$ the preimage $\varphi^{-1}(V)$ is open. The argument is basically exactly the same as in (1), except that the first arrow is only a right arrow, since we only assume that f is continuous. $\square$

We have the following addendum to Lemma 2.12.

Lemma 2.17. *Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$.* If Y is locally compact*, then the following map is a homeomorphism:*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Proof $(*)$. As we pointed out in the proof of Lemma 2.12, it is basically clear that φ is a bijection. By Lemma 2.14 it remains to show that φ is a quotient map. We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ and $q\colon X \to X/\sim$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "f = q \times \mathrm{id}_Y"] & \\ (X \times Y)/\sim_\times \arrow[rr, "\varphi"] & & (X/\sim) \times Y. \end{tikzcd}$$

By Lemma 2.14 (1) we know that p and q are quotient maps. Since Y is locally compact we obtain from Theorem 2.15 that $q \times_Y$ is a quotient map. It follows from Lemma 2.16 (1) that φ is also a quotient map. $\square$

Later on we will use the following special case of Lemma 2.17.

Lemma 2.18. *Let X and Y be two topological spaces, let $A \subset X$ be a subset and let $f\colon A \to Y$ be a map. Using the notation from page 114, we see that the map*

$$\begin{array}{rcl} (X \times [0,1]) \cup_{f \times \mathrm{id}_{[0,1]}} (Y \times [0,1]) & \to & (X \cup_f Y) \times [0,1] \\ [(P, t)] & \mapsto & ([P], t) \end{array}$$

is a homeomorphism.

Proof. Once one unravels the definitions it is clear that this statement follows immediately from Lemma 2.17 together with the observation that $[0, 1]$ is locally compact. $\square$

We conclude this list of Theorem 2.15 with the following proposition that gives a fairly explicit illustration of how Theorem 2.15 can be used.

Proposition 2.19. $(*)$ *Let X be a topological space, let A be a subset of X and let Y be a topological space that is locally compact. Then the map*

$$\begin{array}{rcl} \varphi\colon (X/A) \times Y & \to & (X \times Y)/(A \times Y) \\ ([x], y) & \mapsto & [(x, y)] \end{array}$$

is a quotient map, in particular it is continuous.

Example. In Figure 81 we illustrate the map studied in Proposition 2.19 in the case that $X = \mathbb{R}$, $A = [a, b]$ and $Y = [0, 1]$.

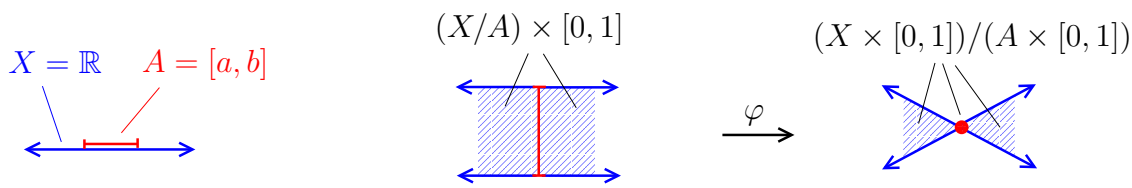


FIGURE 81.

Proof $(*)$. We denote by $p\colon X \times Y \to (X \times Y)/(A \times Y)$ and $q\colon X \to X/\sim$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "q \times \mathrm{id}_Y"'] \arrow[dr, "p"] & \\ (X/A) \times Y \arrow[rr, "\varphi"] & & (X \times Y)/(A \times Y). \end{tikzcd}$$

By Lemma 2.14 the maps p and q are quotient maps. Since Y is locally compact we obtain from Theorem 2.15 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 2.16 that φ is also a quotient map. By Lemma 2.14 (2) this implies that φ is indeed continuous. $\square$

2.3. **Proof of Theorem 2.15** $(*)$**.** Now that we procrastinated for as long as possible, we will finally run out of reasons not turn to the proof of Theorem 2.15. In fact we provide two proofs.

The first proof of Theorem 2.15 cleverly reduces the proof to Proposition 2.6. This approach pushes all the set-theoretic nastiness into the proof of Proposition 2.6. The second proof of Theorem 2.15 is a bare-hands argument which can be read independently of the previous section on the compact-open topology.

First proof of Theorem 2.15, using Proposition 2.6. We write $W = Z \times Y$ and we equip W with the topology that turns

$$\begin{array}{rcl} g\colon X \times Y & \to & W \\ (x, y) & \mapsto & (f(x), y) \end{array}$$

into a quotient map. In other words, as a set we have $W = Z \times Y$, but this time equipped with the topology that is defined by the condition that $U \subset W$ is open if and only if $g^{-1}(U)$ is open. Note that with this topology on W the map g is in particular continuous. Next we consider the following diagram

$$\begin{tikzcd} X \times Y \arrow[rr, "f \times \mathrm{id}_Y"] \arrow[dr, "g(x,y)=(f(x),y)"'] & & Z \times Y \arrow[dl, "h(z,y)=(z,y)"] \\ & W & \end{tikzcd}$$

where $h\colon Z \times Y \to W$ is the identity. Since g is by definition a quotient map it remains to show that h is a homeomorphism. Evidently h is a bijection. Furthermore note that every open set in $Z \times Y$ is open in W since $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that h is continuous.

We consider the maps

$$\begin{array}{rcl} \widehat{g}\colon X & \to & W^Y \\ x & \mapsto & \left(\begin{array}{rcl} Y & \to & W \\ y & \mapsto & g(x,y) \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widehat{h}\colon Z & \to & W^Y \\ z & \mapsto & \left(\begin{array}{rcl} Y & \to & W \\ y & \mapsto & h(z,y) \end{array}\right) \end{array}$$

These maps fit into the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "\widehat{g}"] & \\ Z \arrow[rr, "\widehat{h}"'] & & W^Y. \end{tikzcd}$$

We make the following observations:

(1) Since $g\colon X \times Y \to W$ is continuous and since Y is by hypothesis locally compact we obtain from Proposition 2.6 that $\widehat{g}\colon X \to W^Y$ is continuous.
(2) By our hypothesis the map f is a quotient map.
(3) It follows from (1) and (2), together with Lemma 2.16 (2), that the map $\widehat{h}\colon Z \to W^Y$ is continuous.
(4) It follows from Proposition 2.6 and (3) that h is indeed continuous. □

Our second proof of Theorem 2.15 is independent of earlier results. But it does mean that we will have to get our hands somewhat dirty.

Definition. Let $f\colon X \to Y$ be a map between two sets. We say $U \subset X$ is *saturated with respect to* f if $U = f^{-1}(f(U))$. If f is understood, then we just say U *is saturated*.

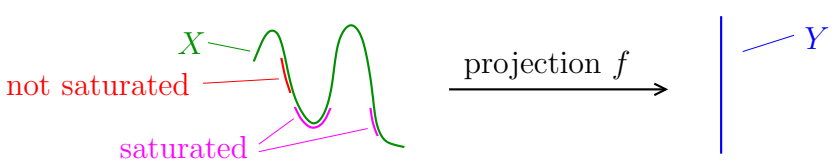


FIGURE 82.

Remark. Let $f\colon X \to Y$ be a map between two sets.

(1) Let $U \subset Y$ be a subset. By Lemma 1.1 (1) we know that $U \subset f^{-1}(f(U))$. Thus U is saturated if and only if $f^{-1}(f(U)) \subset U$.
(2) If f is an injection, then it follows from Lemma 1.1 (1) that every subset of X is saturated.

The following is, besides Lemma 1.1, the only purely set-theoretic lemma of these notes.

Lemma 2.20. *Let $f\colon X \to Z$ be a map between two sets and let Y be another set. Let $\widetilde{W}$ be a subset of $X \times Y$ that is saturated with respect to the map $f \times \mathrm{id}_Y\colon X \times Y \to Z \times Y$. Let $K \subset Y$. We consider the set*

$$U \;:=\; \{x' \in X \,|\, \{x'\} \times K \subset \widetilde{W}\}.$$

The set U is saturated with respect to f.

Proof. We write $h = f \times \mathrm{id}_Y$. By the above we need to show that $f^{-1}(f(U)) \subset U$. Thus let $x \in f^{-1}(f(U)) \subset X$. We have

$$h(\{x\}\times K) \underset{\substack{\uparrow \\ \text{definition of } h}}{=} (f\times \mathrm{id}_Y)(\{x\}\times K) = \{f(x)\}\times K \underset{\substack{\uparrow \\ \text{since } f(x)\in f(U) \text{ we know that} \\ f(x)=f(x') \text{ for some } x'\in U}}{=} \{f(x')\}\times K \underset{\substack{\uparrow \\ \text{definition of} \\ h = f\times \mathrm{id}_Y}}{=} h(\{x'\}\times K) \underset{\substack{\uparrow \\ \text{since } x'\in U}}{\subset} h(\widetilde{W}).$$

It follows that

$$\{x\}\times K \underset{\substack{\uparrow \\ \text{Lemma 1.1 (1)}}}{\subset} h^{-1}(h(\{x\}\times K) \underset{\substack{\uparrow \\ \text{by the above argument}}}{\subset} h^{-1}(h(\widetilde{W})) \underset{\substack{\uparrow \\ \text{since } \widetilde{W} \text{ is saturated}}}{=} \widetilde{W}.$$

We have thus shown that $x \in U$. □

We will also need the following lemma which is again in the realm of topological spaces.

Lemma 2.21. $(*)$ *Let X and Y be topological spaces, let $x \in X$ and let $K \subset Y$ be a compact subset. Let $\widetilde{W} \subset X \times Y$ be an open subset that contains $\{x\} \times K$. There exist open neighborhoods $U \subset X$ of x and $V \subset Y$ of K such that $\{x\} \times K \subset U \times V \subset \widetilde{W}$.*

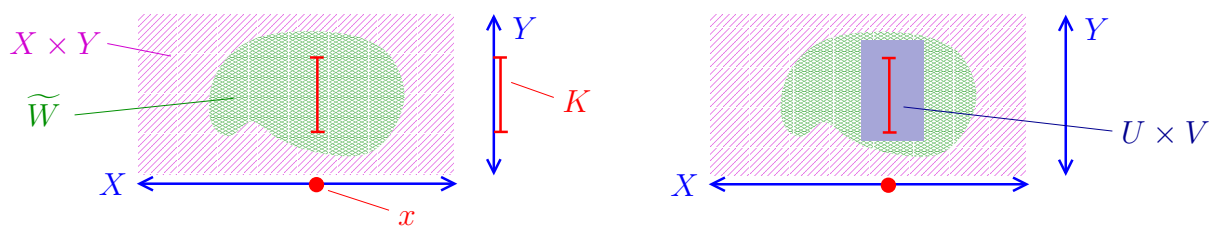


FIGURE 83.

Proof of Lemma 2.21 $(*)$. Since $\widetilde{W} \subset X \times Y$ is open and since $\{x\} \times K \subset \widetilde{W}$ we can find for any $k \in K$ open neighborhoods A_k of $x \in X$ and B_k of $k \in Y$ such that $A_k \times B_k \subset \widetilde{W}$. Since K is compact there exist $k_1, \dots, k_m \in K$ such that $K \subset V := B_{k_1} \cup \dots \cup B_{k_m}$. We define $U := U_{k_1} \cap \dots \cap U_{k_m}$. Since U is the intersection of finitely many open sets each of which contain x we see that U is an open neighborhood of x. Clearly we have the inclusions $\{x\} \times K \subset U \times V \subset \widetilde{W}$. □

For convenience we remind the reader of the following lemma.

Lemma 1.8. *Let X be a topological space and let $U \subset X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.*

Now we can provide our alternative proof of Theorem 2.15.

Second proof of Theorem 2.15. Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space that is locally compact. We need to show that the map

$$f \times \mathrm{id}_Y \colon X \times Y \;\to\; Z \times Y$$

is also a quotient map.

Since f is a quotient map we know by Lemma 2.14 (2) that f is continuous. Therefore it follows from Lemma 1.53 (3) that the map $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that if $W \subset Z \times Y$ is a subset such that the preimage $\widetilde{W} := (f \times \mathrm{id}_Y)^{-1}(W)$ is an open subset of $X \times Y$, then W itself is open.

We intend to use Lemma 1.8 to show that W is open. Thus let $(z, y) \in W$. We pick $x_0 \in f^{-1}(\{z\})$. We equip $\{x_0\} \times Y \subset X \times Y$ with the subspace topology. By Lemma 1.53 (3) the map

$$\begin{array}{rcl} i\colon Y & \to & \{x_0\} \times Y \\ y & \mapsto & (x_0, y) \end{array}$$

is a homeomorphism onto its image. In particular $Y_{x_0} := i^{-1}(\widetilde{W} \cap (\{x_0\} \times Y))$ is an open subset of Y. Note that $\{x_0\} \times Y_{x_0} \subset \widetilde{W}$. Since Y is locally compact we can find a compact neighborhood K_{x_0} of x_0 that is contained in Y_{x_0}. By definition of a neighborhood there exists an open subset Y'_{x_0} of Y with $x_0 \in Y'_{x_0} \subset K_{x_0}$. We define

$$U_{x_0} \;:=\; \{x \in X \mid \{x\} \times K_{x_0} \subset \widetilde{W}\}.$$

Claim. The set $f(U_{x_0})$ is an open subset of Z.

First we intend to use Lemma 1.8 to show that U_{x_0} is indeed an open subset of Y_{x_0}. Thus let $x \in U_{x_0}$. Since $\widetilde{W}$ is an open subset of $X \times Y$ and since K_{x_0} is compact it follows from Lemma 2.21 that there exist open subsets $A \subset X$ and $B \subset Y$ with $x \in A$ and $K_{x_0} \subset B$ such that $A \times B \subset \widetilde{W}$. Evidently this implies that $A \subset U_{x_0}$. Since $x \in A$ we have verified that the openness criterion of Lemma 1.8 is satisfied. We have thus shown that U_{x_0} is an open subset of Y_{x_0}.

Now we want to show that $V := f(U_{x_0})$ is an open subset of Z. By Lemma 2.20 we know that U_{x_0} is saturated with respect to f. In other words, we know that $U_{x_0} = f^{-1}(V)$. Now we use that by hypothesis $f\colon X \to Z$ is a quotient map. Since $f^{-1}(V) = U_{x_0}$ is open by (1) we obtain that V itself is open. But that is what we had wanted to show. This concludes the proof of the claim.

By definition of the product topology we obtain that $f(U_{x_0}) \times Y'_{x_0}$ is an open subset of $X \times Y$. Evidently we have $(x, y) \in f(U_{x_0}) \times Y'_{x_0}$. It remains to prove the following claim.

Claim. We have $f(U_{x_0}) \times Y'_{x_0} \subset W$.

The claim follows from the following equalities and inclusions:

$$f(U_{x_0}) \times Y'_{x_0} = (f \times \mathrm{id}_Y)(U_{x_0} \times Y'_{x_0}) \underset{\substack{\uparrow \\ \text{since } Y'_{x_0} \subset K_{x_0}}}{\subset} (f \times \mathrm{id}_Y)(U_{x_0} \times K_{x_0}) \underset{\substack{\uparrow \\ \text{definition of } U_{x_0}}}{\subset} (f \times \mathrm{id}_Y)(\widetilde{W}) \underset{\substack{\uparrow \\ \text{by Lemma 1.1 (2)} \\ \text{since by definition} \\ \widetilde{W} = (f \times \mathrm{id}_Y)^{-1}(W)}}{=} W.$$

This concludes the proof of the claim. □

Remark. A different proof of Theorem 2.15, based on Propositions 2.4 and 2.6, is given in [Hat, Lemma A.17].

Exercises for Chapter 2.

Exercise 2.1. Let X and Y be topological spaces. Suppose that Y is Hausdorff. Show that X^Y is also Hausdorff.

Exercise 2.2. Let X and Y be topological spaces. Show that the map

$$\begin{array}{rcl} X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

is continuous.

Exercise 2.3. Let $n \in \mathbb{N}$. We consider the following map

$$\begin{array}{rcl} \Theta\colon \mathrm{SO}(n) & \to & \mathrm{Homeo}(S^{n-1}) := \text{self-homeomorphisms of } S^{n-1} \\ A & \mapsto & (P \mapsto A \cdot P). \end{array}$$

(a) Show that the map Θ is continuous.
(b) Show that the map Θ is a homeomorphism onto its image.

Hint: Use Lemma 1.30 (4) and Lemma 2.1.

Exercise 2.4. Let $Y = \{1, \ldots, n\}$ be a finite discrete topological space with n elements and let X be a topological space. Show there exists a homeomorphism $X^Y \cong X^n$.

Exercise 2.5. Let X and Y be topological spaces and let $A \subset X$ be a subset. We denote by $p\colon X \to X/A$ the obvious projection map. Show that $A \subset X$ is compact and closed, then the map

$$p \times \mathrm{id}_Y\colon X \times Y \to (X/A) \times Y$$

is a quotient map, in particular it is continuous.
Remark: Despite appearances this exercise has little to do with Proposition 2.19. Instead one should try to give a bare-hands proof using Lemma 2.21.

3. Topological manifolds and manifolds

In this chapter we introduce topological manifolds and study some basic properties. We also introduce smooth structures on topological manifolds. A topological manifold equipped with a smooth structure will just be called a manifold. In the literature the term "manifold" can be used with quite different conventions and meanings. So we try to state clearly the various definitions used throughout these notes. At the end, just for fun, we give the classification of 1-dimensional (topological) manifolds.

3.1. **Topological manifolds: definition and examples.** In this section we give the definition of a topological manifold. In these notes the study of topological manifolds will be one of our main objectives.

Unfortunately we first have to get the following technical and slightly annoying definition out of the way.

Definition. A topological space is called *second-countable* if there exists a countable basis for the topology.

Of course not every topological space is second-countable. For example consider $X = \mathbb{R}$, but for once equipped with the discrete topology. Since $\mathbb{R}$ is uncountable one sees almost immediately that $X = \mathbb{R}$ is not second-countable. We will use the next lemma to see that most topological spaces that we are interested in are actually second-countable.

Lemma 3.1.

(1) *Every subset of some $\mathbb{R}^n$ is second-countable.*
(2) *If X is a topological space that is second-countable, then every subset is also second-countable.*
(3) *If $X_1, \dots, X_k$ are topological spaces that are second-countable, then $X_1 \times \cdots \times X_k$ is second-countable.*
(4) *Let X be a topological space. If there exists a countable family of open subsets $\{U_i\}_{i\in I}$ such that $\bigcup_{i\in I} U_i = X$ and if each U_i is second-countable, then X itself is second-countable.*
(5) *Let X be a topological space and let $\sim$ be an equivalence relation on X. If X is second-countable and if the projection $X \to X/\sim$ is an open map, then $X/\sim$ is also second-countable.*

Proof $(*)$.

(1) This follows from the discussion on page 92.
(2) This statement follows from Lemma 1.29.
(3) This statement follows from Lemma 1.59.
(4) For each U_i we pick a countable basis $\mathcal{B}_i$ of the topology of U_i. It is straightforward to verify that $\bigcup_{i\in I} \mathcal{B}_i$ is a countable basis for the topology of X.
(5) This statement follows from Lemma 1.70. □

Now we can move on to the definition of a topological manifold.

Definition. Let X be a topological space.

(1) An *n-dimensional chart for X at a point $x \in X$* is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $E_{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

   In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii). We refer to Figure 84 for an illustration.

(2) We say X is an *n-dimensional topological manifold*, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.

(3) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).[71] We denote by ∂X the set of all boundary points of X.

(4) We say a topological manifold X is *closed* if X is compact with $\partial X = \varnothing$.[72]

(5) An *n-dimensional atlas* for X is a family of n-dimensional charts $\{\Phi_i\colon U_i \to V_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = X$.

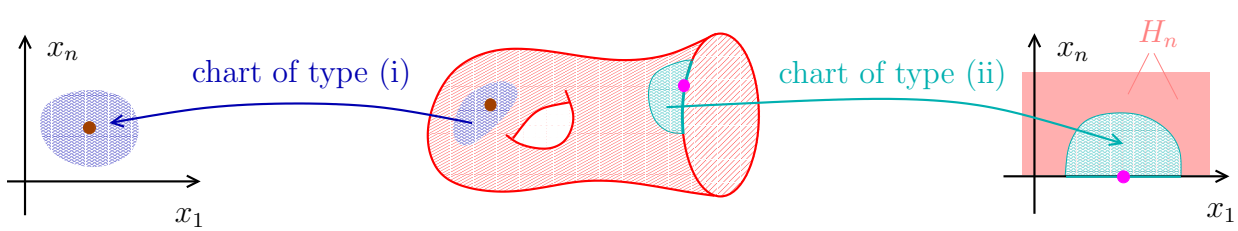


FIGURE 84.

In the following we will give several examples of topological manifolds. When it comes to verifying that a topological space is a topological manifold the main focus is usually on establishing that any point admits a chart. It is easy to forget that one also needs to verify that the topological space is second-countable and Hausdorff. In many cases this task is taken care of by the following lemma.

Lemma 3.2. *Let $n \in \mathbb{N}$. Every subset of $\mathbb{R}^n$ is second-countable and Hausdorff.*

Proof. Let A be a subset of $\mathbb{R}^n$. By Lemma 3.1 we know that A is second-countable. Furthermore, it follows from Proposition 1.13 and Lemma 1.14 that A is also Hausdorff. $\square$

[71] It might sound more natural to define a boundary point to be a point that admits a chart of type (ii). In fact we will see later that a point on a topological manifold admits *either* a chart of type (i) or a chart of type (ii). But the proof of this statement is not trivial and as of now we can not prove this statement.

[72] The latter condition just means that all points in X admit a chart of type (i).

For the record we also state the following lemma that is an immediate consequence of Lemma 3.1 (4).

Lemma 3.3. $(*)$ *If X is a topological space that admits a finite atlas, then X is second-countable.*

Now we turn to giving examples of topological manifolds.

Examples. Let $n \in \mathbb{N}$.

(1) We consider the n-dimensional sphere S^n. As usual we refer to $N := (0, \dots, 0, 1) \in S^n$ as the "North Pole" and similarly we refer to $S := (0, \dots, 0, -1) \in S^n$ as the "South Pole". It follows quite easily from slight modifications of the arguments in the proof of Lemma 1.41 that the stereographic projection $S^n \setminus \{N\} \to \mathbb{R}^n$ from the "North Pole" together with the stereographic projection $S^n \setminus \{S\} \to \mathbb{R}^n$ from the "South Pole" are two n-dimensional charts of type (i) that cover all of S^n. By Lemma 3.2 this shows that S^n is an n-dimensional topological manifold.

Note that both charts are of type (i), thus we see that the boundary of the topological manifold S^n is the empty set.[73] Finally note that S^n is compact, thus S^n is a closed n-dimensional topological manifold.

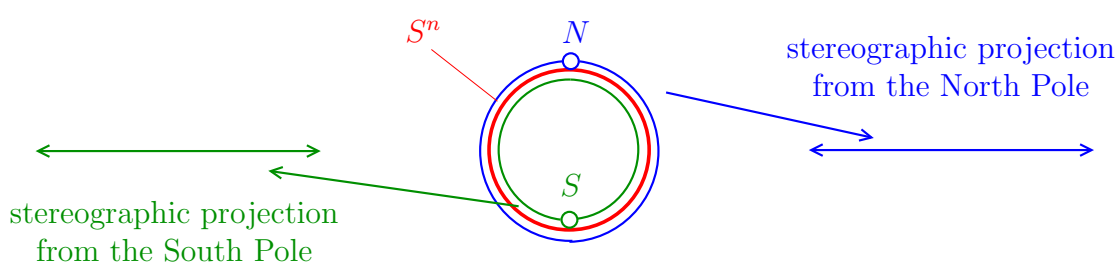


FIGURE 85.

(2) Every open subset U of $\mathbb{R}^n$ is a topological n-dimensional manifold. This follows again from Lemma 3.2 and the observation that the identity map is a chart of type (i) for all points. In particular we see that the boundary of U is empty.

(3) Let M be an open subset of the half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Again, using Lemma 3.2, it is straightforward to see that M is an n-dimensional topological manifold:

   (a) Given any point $X = (x_1, \dots, x_n) \in M$ with $x_n > 0$ the identity map on the open subset $U := \{(y_1, \dots, y_n) \in M \mid y_n > 0\}$ is a chart of type (i) around x.

[73] The vigilant reader will have noticed that we have committed a major sin: we introduced an object called "boundary" twice, with quite different meanings:

(a) On page 50 we introduced the boundary ∂A of a subset A of a topological space X.

(b) On page 171 we introduced the boundary ∂M of a topological manifold M.

Usually it is clear from the context what we mean when we write ∂.

(b) Given any point $(x_1, \ldots, x_n) \in M$ with $x_n = 0$ the identity map $M \to M$ is a chart of type (ii) around x.

In this example we run into a major, and rather painful, subtlety. Namely one would think that the boundary of M as a topological manifold should be precisely the points in M with $x_n = 0$. But at the moment we cannot draw this conclusion. After all, in principle, such points also might admit charts of type (i). Only after having introduced and studied homology groups we will see that no such charts of type (i) exist.

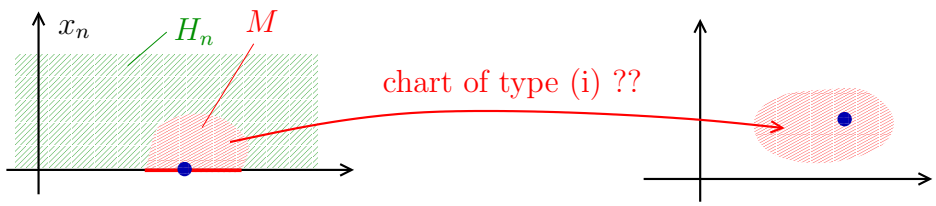


FIGURE 86.

(4) Next we want to show that the closed n-ball $\overline{B}^n$ is an n-dimensional topological manifold. We provide the following charts:

(a) We consider the chart of type (i) that is given by the identity on B^n.

(b) We set $U := \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n > 0\}$ and we consider the map

$$\begin{array}{rcl} U & \to & H_n \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_{n-1}, \sqrt{x_1^2 + \cdots + x_{n-1}^2} - x_n). \end{array}$$

This is a chart of type (ii) for all points in $\{(x_1, \ldots, x_n) \in S^{n-1} \,|\, x_n > 0\}$. We refer to Figure 87 for an illustration of this chart.

(c) We consider the obvious variations of the previous chart for $x_n < 0$ and for other coordinates.

These charts form an atlas for $\overline{B}^n$, together with Lemma 3.2 this shows that $\overline{B}^n$ is an n-dimensional topological manifold. We have found a chart of type (i) for each point in B^n and for the points on S^{n-1} we had found points of type (ii). Thus we have the strong suspicion that $\partial \overline{B}^n = S^{n-1}$ but as of right now we cannot confirm this.

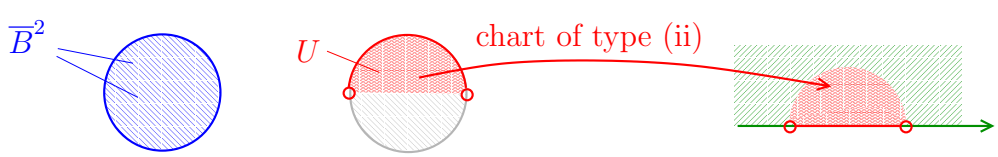


FIGURE 87.

(5) Given $a < b$ in $\mathbb{R}$ the interval $[a, b]$ is easily seen to be a 1-dimensional topological manifold. Every point in (a, b) admits a chart of type (i). Using Exercise 1.14 one can show fairly easily that the two points $a, b \in [a, b]$ do *not* admit charts of type (i). Thus the boundary of $[a, b]$ consists, as expected, of the two points a and b.

(6) In Exercise 3.1 we will verify that the line with two zeros admits a chart of type (i) for every point. It is also easy to see that the line with two zeros is second-countable. But since the line with two zeros is not Hausdorff we see that it is not a topological manifold.
(7) We consider the topological space X that is given by the set $X = \mathbb{R}$ but for once we equip $X = \mathbb{R}$ with the discrete topology. Every point of X admits a 0-dimensional chart and X is obviously Hausdorff. But X is not second-countable, thus X is not a topological manifold.
(8) We conclude this list of examples with the simplest of all examples: the empty set is a topological manifold of any dimension.

The following proposition gives a useful method for constructing more topological manifolds.

Proposition 3.4. *Let M be an m-dimensional topological manifold with possibly non-empty boundary and let N be an n-dimensional topological manifold with empty boundary. Then the following hold:*

(1) *The product $M \times N$ is an $(m+n)$-dimensional topological manifold.*
(2) *We have*[74] $\partial(M \times N) \subset \partial M \times N$.
(3) *If M and N are closed, then $M \times N$ is also closed.*

The above statements also hold with the roles of M and N reversed.

Examples.

(1) It follows from Proposition 3.4 and the above examples that for any $n \in \mathbb{N}$ the n-dimensional torus $(S^1)^n$ is a closed n-dimensional topological manifold.
(2) It follows from Proposition 3.4 and the above examples that the annulus $S^1 \times [0,1]$ is a 2-dimensional topological manifold and that $S^1 \times \overline{B}^2$ is a 3-dimensional topological manifold.

Proof $(*)$.

(1) First note that it follows from Lemma 1.55 (1) that $M \times N$ is Hausdorff and it follows from Lemma 3.1 that $M \times N$ is second-countable. One can now easily use products of charts to show that $M \times N$ is an $(m+n)$-dimensional topological manifold.
(2) The product of charts of type (i) is again a product of type (i). This immediately implies the second statement.
(3) This statement follows from (2) and Lemma 1.55 (2). □

We conclude this chapter with the following two lemmas.

Lemma 3.5. $(*)$ *Let M and N be two topological manifolds.*

(1) *If $f\colon M \to N$ is a local homeomorphism, then $f(\partial M) = (\partial N) \cap f(M)$.*

[74]Note that right now we can only prove the inclusion "$\subset$". Later on we will see that in fact we have an equality.

(2) *If $f\colon M \to N$ is in fact a homeomorphism, then f restricts to a homeomorphism $\partial M \to \partial N$.*

Proof $(*)$. The lemma follows from the elementary observation that the restriction of a chart for a point on a topological manifold has the same type as the original chart. We leave it to the reader to fill in the minuscule details. □

The following lemma is slightly more interesting than the previous lemma.

Lemma 3.6. $(*)$ *Let M be a topological manifold.*

(1) *Any union of components of ∂M is a closed subset of M. In particular ∂M itself is a closed subset of M.*
(2) *If M is compact, then ∂M is also compact.*

Proof $(*)$. Let M be a topological manifold.

(1) First note that, basically by definition, the set of points in M admitting a chart of type (i) is an open subset of M. It follows from the definition of ∂M that $M \setminus \partial M$ is an open subset. Thus the boundary ∂M is a closed subset of M. We denote by $\{Y_i\}_{i\in I}$ the components of ∂M. By Lemma 1.100 each Y_i is an open subset of ∂M. Thus for any $i \in I$ there exists, by definition of the subspace topology, an open set $Z_i \subset M$ with $Z_i \cap \partial M = Y_i$. For any $J \subset I$ we have
$$M \setminus \bigcup_{j\in J} Y_j \;=\; \underbrace{(M \setminus \partial M)}_{\text{open in } M} \;\cup\; \underbrace{\bigcup_{j\notin J} Z_j}_{\text{open in } M}.$$
We have thus shown that the union of components of ∂M corresponding to J is indeed a closed subset of M.
(2) Now suppose that M is compact. Thus ∂M is a closed subset of the compact topological space M. We obtain from Lemma 1.18 (1) that ∂M itself is compact. □

3.2. **Surfaces are topological manifolds.** The goal of this section is to prove the following proposition.

Proposition 3.7. *For any $g \in \mathbb{N}_0$ the surface of genus g is a closed 2-dimensional topological manifold.*

Proof $(*)$. We start out this proof with the discussion of the two cases $g = 0$ and $g = 1$ that we had basically dealt with already. First recall that on page 172 we saw that any S^n is a closed n-dimensional topological manifold. The surface of genus 0 is by definition the sphere S^2, thus we are done by the above. Furthermore, the surface of genus 1 is by definition the torus $S^1 \times S^1$. The desired statement thus follows from the above and Proposition 3.4.

It remains to deal with the case $g \geq 2$. To simplify the notation we prove the proposition for $g = 2$. The proof of the general case is totally analogous. We consider the octagon E_8 with the vertices $Q_k = e^{2\pi i k/16}$, $k = 1, 3, \ldots, 15$ and the equivalence relation $\sim$ that we

introduced on page 119. We illustrate the octagon together with the equivalence relation in Figure 88.

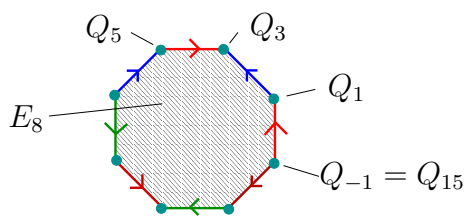


FIGURE 88.

The projection map $E_8 \to E_8/\sim$ is easily seen to be an open map. It follows from Lemma 3.1 (5) that $E_8/\sim$ is second-countable. We leave it as an exercise to verify that $E_8/\sim$ is Hausdorff. Note that E_8 is compact by the Heine-Borel Theorem 1.21. Thus we obtain from Lemma 1.37 that $E_8/\sim$ is also compact. To show that $E_8/\sim$ is a closed 2-dimensional topological manifold it remains to give an atlas for $E_8/\sim$ that consists only of charts of type (i).

First we consider the map

$$\begin{array}{rcl} \Phi\colon U := p\big(\mathring{E}_8\big) & \to & V = \mathring{E}_8 \ \subset \ \mathbb{R}^2 \\ p(X) & \mapsto & X. \end{array}$$

This is a chart of type (i) for each point in the interior of E_8.

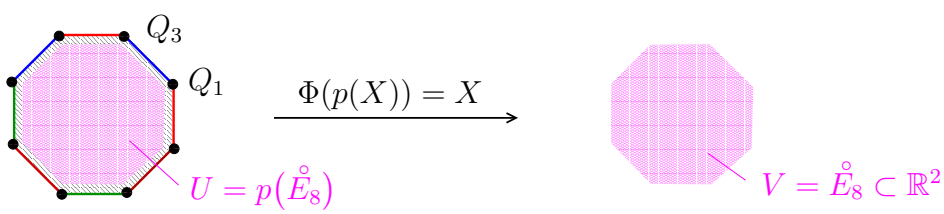


FIGURE 89.

Next we introduce charts that cover the interiors of the edges. First we consider the edge given by Q_{-1} and Q_1. We denote by U and U' the half-disks that are illustrated in Figure 90. (Note that it is understood that the line segment is included but that the circle is not included in U and U'.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line that goes through Q_{-1} and Q_1. It is straightforward to verify that the map

$$\begin{array}{rcl} \Psi_1\colon p(U \ \cup \ U') & \to & \text{the open disk } U \cup f(r_{\frac{\pi}{4}}(U)) \ \subset \ \mathbb{R}^2 \\ p(X) & \mapsto & \begin{cases} X, & \text{if } X \in U \\ f(r_{\frac{\pi}{4}}(X)), & \text{if } X \in U' \end{cases} \end{array}$$

is well-defined and that it is a homeomorphism. This is a chart of type (i) for all points in the interior of the edge from Q_{-1} to Q_1. Similarly we define charts Ψ_2, Ψ_3, Ψ_4 that take care of the other pairs of edges.

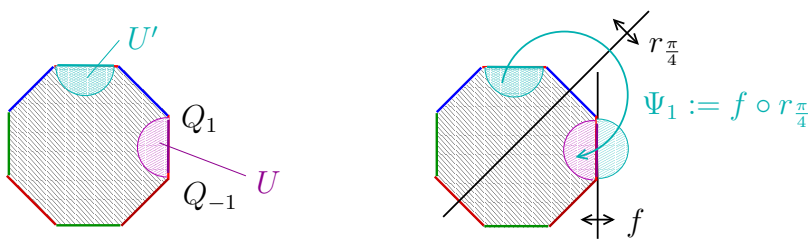


FIGURE 90.

It remains to find a chart of type (i) that takes care of the point P in $E_8/\sim$ that is represented by the vertices. (It is a fun exercise to verify that all vertices get identified.) We pick an $\eta > 0$ such that the η-disks centered at $Q_1, \dots, Q_{15}$ are disjoint. For $k = 1, 3, \dots, 15$ we set $U_k := B_\eta(Q_k) \cap E_8$. The sets $U_1, U_3, \dots, U_{15}$ are disjoint and one can easily verify that

$$p(U_1) \cup p(U_3) \cup \cdots \cup p(U_{15})$$

is an open neighborhood of $p(Q_1) = p(Q_3) = \cdots = p(Q_{15})$.

Before we write down the chart rigorously it might help to visualize the situation. To the left of Figure 91 we see the sets $U_1, \dots, U_{15}$. Next we arrange these "pieces of pie" as in the second second part of Figure 91. Thus we see the "pieces of pie" with an angle of $\frac{3\pi}{4}$ where we show the various identifications of the edges. The map from the second to the third part is given by compressing the angle of the "pieces of pie" to a third and by rotating the "pieces of pie". These eight "pieces of pie", each with an angle of $\frac{\pi}{4}$, can be combined to give a disk in $\mathbb{R}^2$.

Now we want to turn the above idea into formal mathematics. We consider the map

$$\begin{array}{rcl} \Omega\colon p(U_1) \cup \cdots \cup p(U_{15}) & \to & B^2_\eta(0) \\ p(\underbrace{Q_k + re^{i\alpha}}_{\in p(U_k)}) & \mapsto & re^{i\left(\frac{1}{3}(\alpha - \beta(k)) + \gamma(k)\right)}, \text{ for } k = 1, 3, \dots, 15, \end{array}$$

where $\beta(k)$ and $\gamma(k)$ are given by the following table:

k	1	3	5	7	9	11	13	15
$\beta(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$

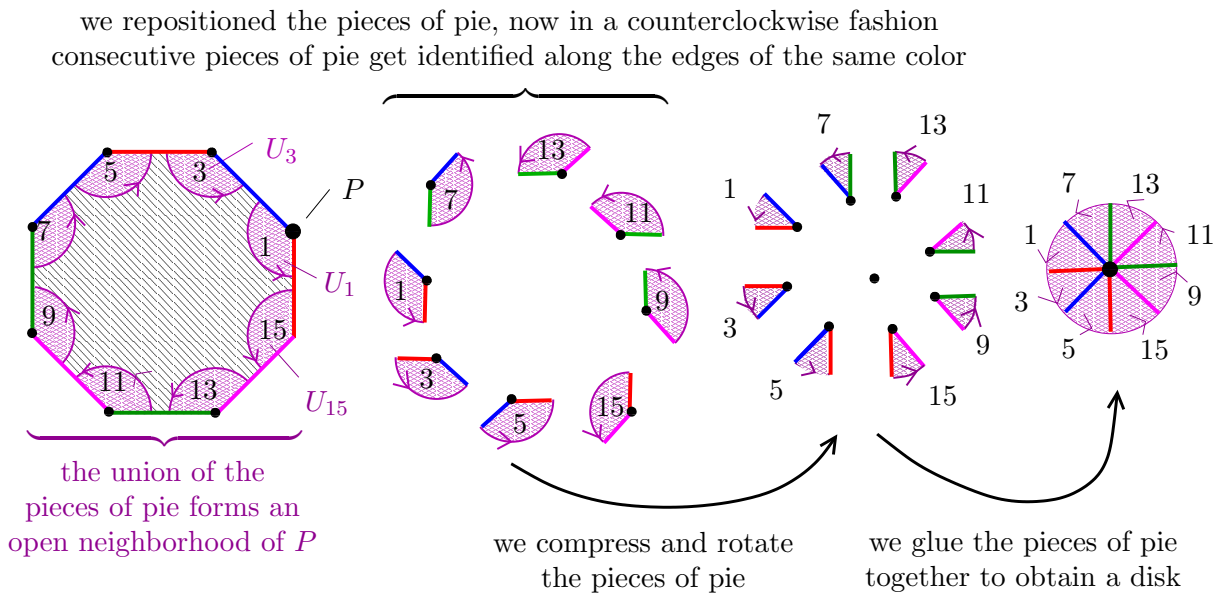


FIGURE 91.

It is fairly easily to verify that Ω is well-defined and that it is a homeomorphism. This is a chart of type (i) for P.

The eight charts $\Phi, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and Ω are now charts of type (i) that cover all of $E_8/\sim$. We have thus found the desired atlas. $\square$

3.3. **Properties of topological manifolds.** Topological manifolds are particularly important and also particularly nice topological spaces. For example the next lemma shows that when it comes to topological manifolds we do not have to worry about the two different definitions of connectedness.

Lemma 3.8.

(1) *Every topological manifold is o-locally path-connected.*
(2) *A topological manifold is connected if and only if it is path-connected.*

Throughout the lecture notes we will on many occasions make use of Lemma 3.8 (2) without referring to it explicitly.

Proof $(*)$. Let X be an n-dimensional topological manifold.

(1) Since X is o-locally homeomorphic to an open subset of $\mathbb{R}^n$ it follows easily from the fact, proved on page 131, that open subsets of $\mathbb{R}^n$ are o-locally path-connected, that X itself is o-locally path-connected. We leave it to the reader to fill in the straightforward details.
(2) The second statement is an immediate consequence of (1) and of Lemma 1.102. $\square$

On several occasions we will need the following, slightly technical lemma.

Lemma 3.9. *Let M be a topological manifold and let $\Phi\colon U \to V$ be a chart. Let $A \subset V$ be a subset that is compact. Then the corresponding subset $\Phi^{-1}(A)$ is a closed subset of M.*[75]

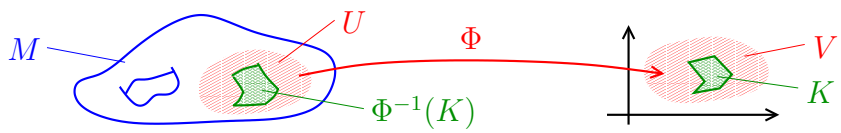


FIGURE 92. Illustration of Lemma 3.9.

Proof. Let M be a topological manifold and let $\Phi\colon U \to V$ be a chart. Let $\Psi = \Phi^{-1}\colon V \to U$ be the inverse map. Let $A \subset V$ be a subset that is compact. It follows from Lemma 1.37 that $\Psi(A)$ is a compact subset of M. But M is Hausdorff by definition of a topological manifold. Hence it follows from Lemma 1.18 (2) that $\Psi(A) = \Phi^{-1}(A)$ is closed in M. □

For completeness' sake we also mention the following much deeper theorem.

Theorem 3.10. *Every topological manifold X is* metrizable*, i.e. X admits a metric d such that the topology from the metric agrees with the given topology.*

Example. On S^1 we consider the metric that is defined by

$$d(z,w) \;=\; \min\{|s-t| \,|\, z = e^{si} \text{ and } w = e^{ti}\}.$$

Put differently, $d(z,w)$ is the length of the shortest arc from z to w. We leave it as an exercise to verify that this is a metric and that the topology defined by this metric agrees with the usual topology on S^1. Alternatively we could have taken the, much duller, Euclidean metric on $\mathbb{R}^2$, restricted to S^1.

Proof. We refer to [Mun2, Section 4.5] for a proof of the theorem. □

Next we specify what we mean by a submanifold.

Definition. Let M be an n-dimensional topological manifold.

(1) We say a subset $N \subset M$ is a *k-dimensional submanifold* if given any $P \in N$ one of the following holds:
   (α) there exists a chart $\Phi\colon U \to V$ of type (i) for M and P such that[76]
   $$\Phi(U \cap N) \;\subset\; \{(0,\ldots,0,x_1,\ldots,x_k) \,|\, x_i \in \mathbb{R}\},$$
   (β) or there exists a chart $\Phi\colon U \to V$ of type (ii) for M and P such that $\Phi(P)$ lies in E_{n-1} and
   $$\Phi(U \cap N) \;\subset\; \{(0,\ldots,0,x_1,\ldots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$

[75] It does not suffice that A is a closed subset of the topological space V. Indeed, $A = V$ is a closed subset of V, but $U = \Phi^{-1}(V)$ is in general not a closed subset of M.

[76] Why are the zeros to the left and not to the right?

(γ) or there exists a chart $\Phi\colon U \to V$ of type (i) for M and P such that $\Phi(P)$ lies in E_{n-1} and

$$\Phi(U \cap N) \ \subset \ \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$

We refer to such charts as *submanifold charts* of N.

(2) We say a submanifold N is *proper* if N is a closed subset of M and if for each $P \in N$ we can find submanifold charts of type (α) and (β).

We illustrate the three types of submanifold charts in Figure 93.

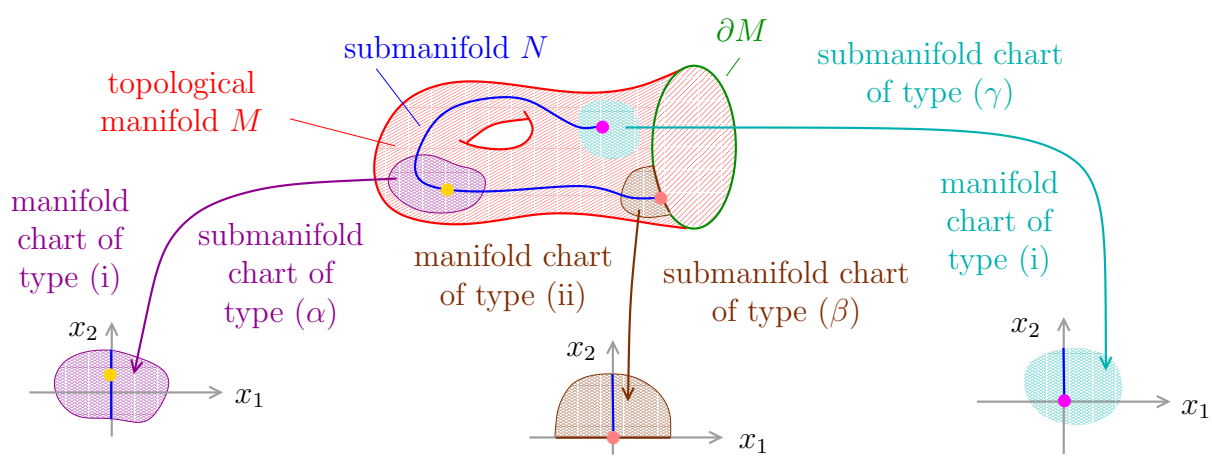


FIGURE 93.

Furthermore in Figure 94 we give typical examples of submanifolds, some of which are proper, some are not.

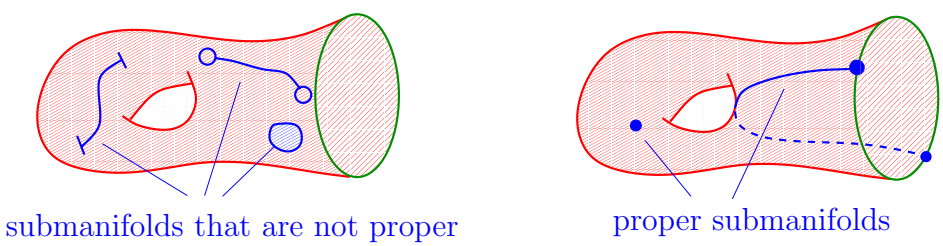


FIGURE 94.

Remark. The notion of a "submanifold" is used differently by different authors. Therefore, reading the literature it is a good idea to check what definition of a submanifold the authors use. In particular our definition of a "proper submanifold" is often used as the definition of a submanifold.

We conclude this section with the following lemma that is almost obvious.

Lemma 3.11. *Let M be a topological manifold. If $N \subset M$ is a k-dimensional submanifold, then it is also a k-dimensional topological manifold.*

Proof. It follows from Lemmas 1.14 and 3.1 that N is Hausdorff and second-countable. It is clear from the definitions that N admits a k-dimensional atlas. Thus N is indeed a k-dimensional topological manifold. □

3.4. **Manifolds.** Next we want to define smooth structures on topological manifolds. We will refer to a topological manifold that is equipped with a "smooth structure" as a manifold. One can easily fill many books with the theory of manifolds also. Since it is not our goal to write a book on manifolds we will only give a fairly quick introduction to manifolds. We refer to the many books on manifolds, e.g. [Le2, Jä3] for details.

Before we can define the notion of a smooth structure on a manifold we have to agree on what we mean by smooth maps on a subset of $\mathbb{R}^n$.

Definition.

(1) Let U be an open subset of $\mathbb{R}^n$. We say a map $f = (f_1, \dots, f_m) \colon U \to \mathbb{R}^m$ is *smooth*, if all partial derivatives, of arbitrarily large degree, of each coordinate function f_i exist.
(2) Let A be any subset of $\mathbb{R}^n$. We say a map $f \colon A \to \mathbb{R}^m$ is *smooth*, if given any $a \in A$ there exists an open neighborhood U (in $\mathbb{R}^n$) of a and a smooth map $\widetilde{f} \colon U \to \mathbb{R}^m$ which coincides with f on $A \cap U$.

Example.

(1) The function $f \colon [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is not smooth, since it cannot be extended to a smooth function on any open interval (a, b) that contains 0.
(2) One can easily show that any function $f \colon \mathbb{Z} \to \mathbb{R}$ is smooth.
(3) A careful reader might have noticed that we have now given two definitions for a map $[a, b] \to \mathbb{R}^n$ to be smooth, namely besides the definition we just gave, we had also given a definition on page 134. It turns out that by Borel's Theorem, see [Strg, Theorem 4.32], these two definitions are actually the same.
(4) In the following our main interest will lie in maps defined on open subsets of the half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Thus let U be an open subset of H_n and let $f \colon U \to \mathbb{R}^m$ be a function. By a more advanced version of Borel's Theorem, see [Hör, Theorem I.1.2.6], the map f is smooth in the above sense if and only if f is continuous, if $f|_{\mathring{H}_n \cap U}$ is smooth and if all the partial derivatives of $f|_{\mathring{H}_n \cap U}$ have continuous extensions to all of U. We will not make use of this fact.

Now is a good moment to remind ourselves of the following lemma that is proved in any self-respecting analysis course, we refer to [Kön, Chapter 9] for details.

Lemma 3.12. *If $a \leq b < c \leq d$ are four real numbers, then there exists a smooth function $f \colon [a, d] \to [0, 1]$ with $f|_{[a,b]} \equiv 0$ and $f|_{[c,d]} \equiv 1$ which is strictly monotonously increasing on $[b, c]$.*

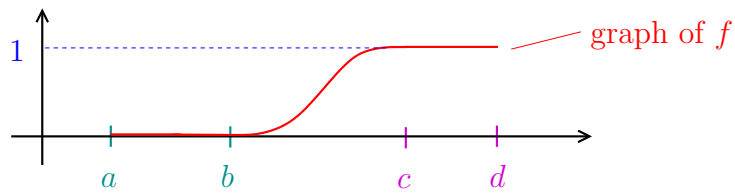


FIGURE 95.

Now we can continue with the definition of a smooth structure on a topological manifold.

Definition.

(1) An atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ for an n-dimensional topological manifold X is called *smooth* if for all $i, j \in I$ the *transition map*

$$\Phi_j \circ \Phi_i^{-1} \colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open set in } \mathbb{R}^n \text{ or } H_n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^n}$$

is smooth.

(2) An *n-dimensional manifold* is a pair $(X, \mathcal{A})$ consisting of an n-dimensional topological manifold X together with a smooth atlas $\mathcal{A}$ for X. (As with many other definitions we usually suppress the choice of a smooth atlas from the notation.)

(3) We define the *boundary of a manifold* $(X, \mathcal{A})$ as the set of all points that do not admit a chart of type (i) from the given atlas $\mathcal{A}$.[77]

Remark.

(1) In the literature "topological manifolds" are sometimes referred to as "manifolds", and what we now call "manifolds" are referred to as "smooth manifolds".

(2) Rather surprisingly in 1960 Michel Kervaire [Kerv2] showed that there exists a 10-dimensional topological manifold that does not admit a smooth atlas.

The following proposition gives us interesting examples of manifolds.

Proposition 3.13. *The following topological spaces admit the structure of a manifold:*

(1) *open subsets of* $\mathbb{R}^n$,
(2) *all closed balls* $\overline{B}^n$,
(3) *all spheres* $S^n \subset \mathbb{R}^{n+1}$,
(4) *the n-dimensional torus* $(S^1)^n$,
(5) *the surfaces of genus g as defined on page 119.*

More precisely, the atlases that we provided in Section 3.1 and in the proof of Proposition 3.7 are smooth.

[77] In fact later on we will see that the boundary of a manifold in the above sense agrees with the boundary of a topological manifold. But since so far we do not have the tools to prove this statement we prefer to use this definition.

Proof (∗). It is elementary to verify that the atlases for examples (1) to (4) that we provided in Section 3.1 are smooth. It remains to consider the surfaces of genus $g \geq 2$. The atlas that we provided in the proof of Proposition 3.7 looks more dubious. In fact all transition maps are compositions of (some of) the following types of maps:

(1) reflections in lines,
(2) translations,
(3) and for some $\eta > 0$ the map

$$\begin{array}{rcl} \{re^{i\alpha} \,|\, r \in (0,\eta), \alpha \in (0, \frac{\pi}{4})\} & \to & \{re^{i\alpha} \,|\, r \in (0,\eta), \alpha \in (0, \frac{3\pi}{4})\} \\ z = re^{i\varphi} & \mapsto & re^{3i\varphi} = \frac{1}{|z|^2} \cdot z^3, \end{array}$$

(4) the inverse of (3).

It is clear that the first three types of maps are smooth. It follows from Proposition 3.20 (2) below that the fourth map is also smooth. Thus the transition maps in the atlas are smooth. □

For completeness we state the following lemma which follows easily from the definitions.

Lemma 3.14. *Let M be a manifold and let X be an open subset of M. The restrictions of the charts of the smooth atlas of M to X define a smooth atlas for X.*

We will constantly use Lemma 3.14 to view open subsets of a manifold as manifolds in their own right.

We give a few more standard definitions regarding manifolds.

Definition.

(1) A map $f\colon M \to N$ between manifolds is called *smooth* if for any chart Φ of M and any chart Ψ of N, in the respective atlases, the concatenation $\Psi \circ f \circ \Phi^{-1}$ is smooth.[78]
(2) Given two manifolds M and N we denote by $C^\infty(M, N)$ the set of all smooth maps from M to N.
(3) We say a map $f\colon M \to N$ between manifolds is a *diffeomorphism* if f is smooth, if f is a bijection and if $f^{-1}\colon N \to M$ is also smooth. If such a diffeomorphism exists, then we say that M and N are *diffeomorphic*.
(4) We say that two smooth atlases for a given topological manifold are *equivalent*, if the identity map is a diffeomorphism between the two manifolds corresponding to the smooth atlases. A *smooth structure* on a topological manifold is a choice of an equivalence class. Usually we do not distinguish between a smooth atlas and the smooth structure it represents. In particular a manifold, which by definition is a topological manifold together with a smooth atlas, is the same as a topological manifold together with a smooth structure.

[78] If M is an m-dimensional manifold and if N is an n-dimensional manifold, then the map $\Psi \circ f \circ \Phi^{-1}$ is a map from an open subset of $\mathbb{R}^m$ or from an open subset of $H_m = \{(x_1, \ldots, x_m) \,|\, x_m \geq 0\}$ to $\mathbb{R}^n$, hence the notion of smooth makes sense for such a map.

(5) Let A be a subset of a manifold M (in most applications $M = \mathbb{R}^n$), let N be a manifold and let $f\colon A \to N$ be a map. We say f is *smooth* if given any $a \in A$ there exists an open neighborhood U in M of a and a smooth map $\widetilde{f}\colon U \to N$ which coincides with f on A.[79]

Remark.

(1) There exist manifolds that are homeomorphic but not diffeomorphic. In fact John Milnor [Mi1] showed in 1956 that there exist manifolds that are homeomorphic to S^7 but that are not diffeomorphic to S^7. In other words, Milnor showed that S^7 admits at least two different smooth structures.
(2) It follows from [KSi2, p. 251] together with [KeM, Theorem 1.2] that in dimension $n \neq 4$ every topological manifold admits at most a finite number of smooth structures. (We also refer to [AMB, Chapter 7] and especially to [Sco, p. 207-224] for more details.) Amazingly, this statement is wrong in dimension 4. More precisely there is a plethora of closed topological 4-dimensional manifolds that admit an infinite number of smooth structures, see e.g. [FS2] for details.

The following harmless lemma contains three basic statements about smooth maps between manifolds.

Lemma 3.15. *Let $f\colon M \to N$ be a map between two manifolds.*

(1) *If there exists an open covering $\{U_i\}_{i\in I}$ of M such that each restriction $f|_{U_i}\colon U_i \to N$ is smooth, then f itself is smooth.*
(2) *If $f\colon M \to N$ is smooth it is also continuous.*
(3) *If $f\colon M \to N$ is a bijection (e.g. if it is a homeomorphism) and if it is a local diffeomorphism, then it is in fact a diffeomorphism.*

Proof.

(1) This statement follows from the observation that smoothness is an open condition.
(2) This statement is a straightforward consequence of Lemma 1.32 and the well-known fact that given an open set $U \subset \mathbb{R}^n$, any smooth map $U \to \mathbb{R}^m$ is also continuous. We refer to [Le2, Proposition 2.4] for more details.
(3) Let $f\colon M \to N$ be a local diffeomorphism that is a bijection. We need to show that $f^{-1}\colon N \to M$ is also smooth. But that follows immediately from (1) and our hypothesis that f is a local diffeomorphism. □

We continue with the following definition.

Definition. Given a smooth atlas $\mathcal{A}$ for a topological manifold M we define the *maximal smooth atlas* of $(M, \mathcal{A})$ to be the set of all charts of type (i) and (ii) from page 171 that are smooth maps with respect to the given smooth structure.[80]

[79] This is of course just a generalization of the definition on page 181.

[80] Let $\Phi\colon U \to V$ be a chart for M. By the convention from Lemma 3.14 we view U as a manifold. Furthermore recall that we view any open subset of $\mathbb{R}^n$ and H_n as a manifold with the obvious smooth atlas given by the identity. Thus it makes sense to ask whether or not $\Phi\colon U \to V$ is smooth.

The verification of the following lemma is left to the reader.

Lemma 3.16. *Let $(M, \mathcal{A})$ be a manifold.*

(1) *The maximal atlas of $(M, \mathcal{A})$ is a smooth atlas for M that contains $\mathcal{A}$.*
(2) *The maximal atlas of $(M, \mathcal{A})$ equals the union of all smooth atlases of M that are equivalent to the atlas $\mathcal{A}$.*

In the previous section we introduced the notion of a submanifold of a topological manifold. We add the following convention.

Convention. If M is a manifold, then we demand that in the definition of a "submanifold" on page 179 the charts come from the maximal smooth atlas.

Remark. Sometimes the above convention can be ambiguous. For example, if we consider $M = \mathbb{R}^2$ it is not clear whether we view M as a topological manifold or as a manifold. If from the context it is not clear what we mean, then we distinguish in our language between a *topological submanifold* and a *smooth submanifold*. This distinction is illustrated in Figure 96.

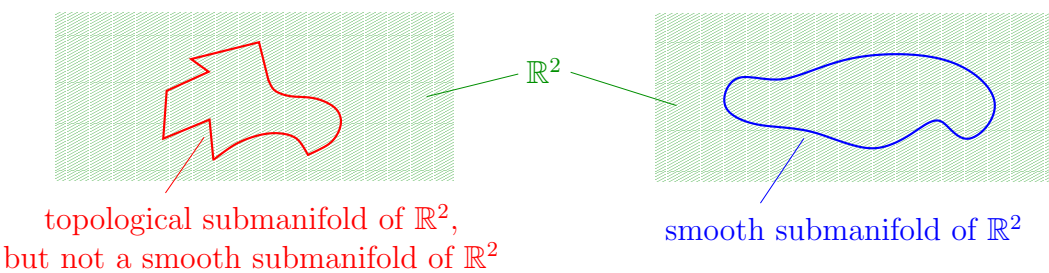


FIGURE 96.

Examples.

(1) We consider the topological manifold $M = \mathbb{R}^2$ with the smooth atlas $\mathcal{A}$ given by the identity $\Phi = \mathrm{id}_M$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. We consider the subset $N = \{(x, f(x) \,|\, x \in \mathbb{R}\}$. We cannot use the unique chart Φ in our atlas $\mathcal{A} = \{\Phi\}$ to show that N is indeed a submanifold of the manifold $(\mathbb{R}^2, \mathcal{A})$. But the chart $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi(x, y) = (x, y - f(x))$ lies in the maximal atlas of $(\mathbb{R}^2, \mathcal{A})$ and we can use it to verify that N is indeed a submanifold of the manifold $(\mathbb{R}^2, \mathcal{A})$.
(2) In Figure 97 we show $M = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ together with two subsets K and $N = \{(x, x^2) \,|\, x \geq 0\}$. In Exercise 3.3 we will see that both K and N are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a manifold.

Remark. As we remarked on page 180, there are various subtly different definitions of a "submanifold". This is even more pronounced when it comes to submanifolds of manifolds,

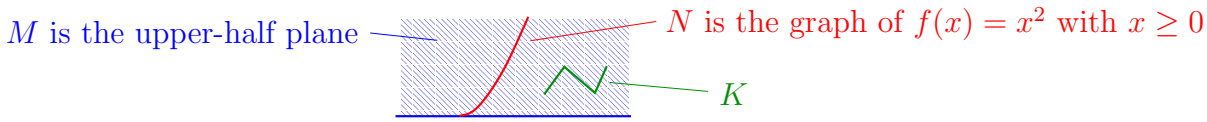


FIGURE 97.

especially if M and N are allowed to have boundary. There is no textbook that would consider K in Figure 97 as a submanifold of the manifold $M = \{(x, y) \,|\, y \geq 0\}$. As we remarked above, if we follow our convention or the convention of [Wall2, p. 31], then $N = \{(x, x^2) \,|\, x \geq 0\}$ is not a submanifold of M. On the other hand the definition used in [Le2, p. 98] or [Hirs, p. 30] views N as a submanifold of M. What we call a proper submanifold is called a neat submanifold in [Hirs, p. 30]. Sigh.

The following lemma follows almost immediately from Lemma 3.11 and the definitions. We leave it to the reader to fill in the details.

Lemma 3.17. *Let M be a manifold and let N be a submanifold of M. Then N admits a natural structure of a manifold which has the property that the inclusion map $N \to M$ is a smooth map.*

3.5. **Topological properties of manifolds.** The following proposition is usually proved in one one of the analysis courses. Since it is one of the reasons why dealing with manifolds is easier than dealing with topological manifolds we also give the proof.

Proposition 3.18.

(1) *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets.*
   (a) *If $f\colon U \to V$ is a diffeomorphism, then for any $P \in U$ the differential Df_P is an invertible matrix.*
   (b) *If U and V are diffeomorphic and non-empty, then $m = n$.*

(2) *Let M be a non-empty m-dimensional manifold and let N be an n-dimensional manifold. If M and N are diffeomorphic, then $m = n$.*

Example. Suppose U is an open subset, let's say connected, of $\mathbb{R}^n$ and that $f\colon U \to \mathbb{R}^n$ is a smooth map such that Df_P is invertible for every $P \in U$. One might be tempted to think that $f\colon U \to f(U)$ is already a diffeomorphism itself. But this is not necessarily the case, a counterexample is for example given by

$$\begin{array}{rcl} f\colon \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & z^2. \end{array}$$

One can easily verify that Df_z is invertible for every $z \in \mathbb{C} \setminus \{0\}$. But f is evidently not injective.

Proposition 3.18 thus says, in particular, that the dimension of a non-empty manifold is well-defined. Eventually, using homology groups, we will show that the analogous statements also holds for topological manifolds.

Proof.

(1) (a) Let $f\colon U \to V$ be a diffeomorphism between open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. We denote by $g\colon V \to U$ the inverse of f. Let $P \in U$. It follows from the chain rule that the matrices Df_P and $Dg_{f(P)}$ are inverses of one another, in particular Df_P is invertible.

(b) Since U is now assumed to be non-empty we can find a $P \in U$. By (1a) the $m \times n$-matrix Df_P is invertible, this means that the matrix is a square matrix, i.e. $m = n$.

(2) The second statement follows easily from the first statement by considering charts. □

The following theorem can be viewed as a partial inverse to Proposition 3.18 (1a).

Theorem 3.19. (Inverse Function Theorem) *Let $f\colon U \to V$ be a smooth map between two open subsets of $\mathbb{R}^n$ and let $P \in U$ be a point. If Df_P is invertible, then f is a local diffeomorphism at P, i.e. there exist open neighborhoods U' of P in U and V' of $f(P)$ in V such that $f|_{U'}\colon U' \to V'$ is a diffeomorphism.*

Proof. This theorem is proved in most courses on multivariable real analysis. A proof is for example given in [Le2, Corollary C.34]. □

The Inverse Function Theorem 3.19 is one of the main reasons why dealing with manifolds and smooth maps is often easier than dealing with topological manifolds and continuous maps. For example the proofs of the following two propositions rest heavily on the Inverse Function Theorem 3.19.

Proposition 3.20. (Smooth invariance of domain) *Let $U \subset \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential Df_P is invertible at every point P in the open set U. Then the following two statements hold:*

(1) *The map f is open, i.e. given any open subset $V \subset U$ the image $f(V)$ is an open subset of $\mathbb{R}^n$.*
(2) *If f is injective, then $f\colon U \to f(U)$ is a diffeomorphism.*

Proof. This proposition is a straightforward consequence of the Inverse Function Theorem 3.19. We leave it to the reader to fill in the few remaining details or alternatively we refer to [Le2, Corollary C.36] for a proof. □

We continue with the following, slightly subtle proposition.

Proposition 3.21. *Let X be an n-dimensional manifold, i.e. X is an n-dimensional topological manifold equipped with a smooth atlas. The following statements hold:*

(1) *Every point on X admits either a chart of type (i) from the given smooth atlas or it admits a chart of type (ii) from the given smooth atlas.*
(2) *The boundary $Y := \partial X$ has a canonical structure of an $(n-1)$-dimensional manifold with $\partial Y = \varnothing$. Furthermore the inclusion map $\partial X \to X$ is a smooth map.*

(3) (a) *Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.*
(b) *If X is compact, then ∂X is also compact.*

Here again we would expect that a similar statement should hold for topological manifolds. But since for topological manifolds we have fewer tools, e.g. right now we are missing an analogue of Proposition 3.20, we have to postpone a proof for topological manifolds to much later.

Proof.

(1) With a little bit of effort this statement can be deduced from Proposition 3.20. We refer to [Le2, Theorem 1.46] for the full details.
(2) The second statement is an easy consequence of the first statement.
(3) The proof of the third statement is identical to the proof of Lemma 3.6.[81] □

The following lemma is used frequently, often without noticing it.

Lemma 3.22. *Let M be an n-dimensional manifold and let X be an n-dimensional submanifold. If X is a closed subset of M (For example X could be a compact submanifold) with $X \cap \partial M = \varnothing$, then*

$$\textit{boundary of } X \textit{ as a subset of } M \;=\; \textit{boundary of } X \textit{ viewed as a manifold.}$$

In particular, the interior $\mathring{X}$ of X as a subset of M agrees with $X \setminus \partial X$ where ∂X denotes the boundary of X viewed as a manifold in its own right.

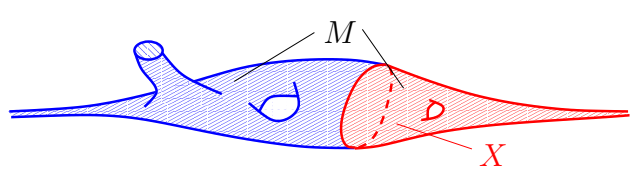


FIGURE 98. Illustration of Lemma 3.22.

[81]Note that it does not follow directly from Lemma 3.6 since our definition of the boundary of a manifold, given on page 182, is a priori different from the definition of the boundary of a topological manifold. Only later will we see that there is no difference.

Proof $(*)$. Let $P \in M$. The lemma follows from the following observations:

$$\begin{array}{ll} P \text{ lies in the boundary of } X & \overset{\text{since } X \text{ is closed}}{\overset{\downarrow}{\Longleftrightarrow}} \quad P \text{ lies on } X \text{ and every open neighborhood} \\ \text{as a subset of } M & \qquad\qquad \text{of } P \text{ contains a point in } M \setminus X \\ & \Longleftrightarrow \quad P \text{ admits a submanifold chart of type } (\gamma) \\ & \quad\uparrow \\ & \text{since } X \text{ is a submanifold of codimension zero} \\ & \Longleftrightarrow \quad P \text{ lies on the boundary of } X \text{ viewed as a manifold.} \\ & \quad\uparrow \\ & \text{Proposition 3.21 (1)} \end{array}$$

□

We continue with the following definition.

Definition. Let M be an n-dimensional manifold. We say M is *decomposed into submanifolds* $\{A_i\}_{i\in I}$ if the following conditions hold:

(1) each A_i is an n-dimensional submanifold of M,
(2) each A_i is a closed subset of M,
(3) for any $i, j \in I$ with $i \neq j$ the intersection $A_i \cap A_j$ is a union of boundary components of A_i and it is a union of boundary components of A_j,
(4) for every $i \in I$ we have

$$\text{the boundary of } A_i \text{ viewed as a manifold } = \bigcup_{j\neq i} A_i \cap A_j \ \cup \ A_i \cap \partial M.$$

This definition is illustrated in Figure 99.

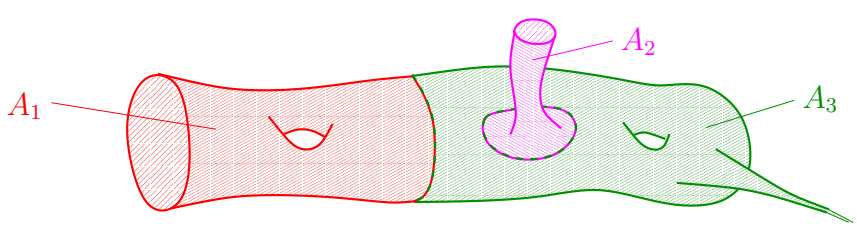


FIGURE 99.

The following lemma shows that the Klein bottle and the real projective plane admit some interesting decompositions into submanifolds.

Lemma 3.23.

(1) *We can decompose the Klein bottle into two submanifolds M and N where M and N are both diffeomorphic to the Möbius band.*
(2) *We can decompose the real projective plane into two submanifolds D and M where D is diffeomorphic to the disk $\overline{B}^2$ and M is diffeomorphic to the Möbius band.*

Proof.

(1) The proof of the first statement is sketched in Figure 100. In Figure 101 we attempt to sketch the equivalent statement that the Klein bottle is obtained from gluing two copies of the Möbius band along the common boundary.

(2) The proof of the second statement is sketched in Figure 102. Here, for $\mathbb{RP}^2$ we use the homeomorphism $\mathbb{RP}^2 \cong \overline{B}^2/\sim$ that we gave in Lemma 1.82. □

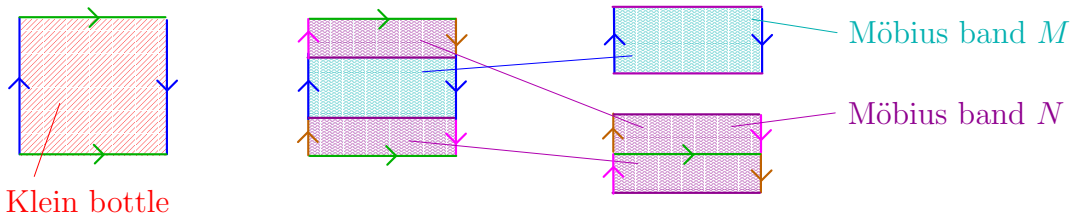


FIGURE 100.

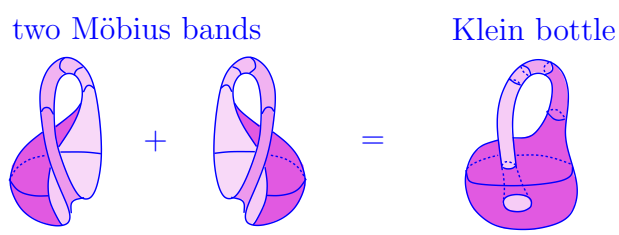


FIGURE 101.

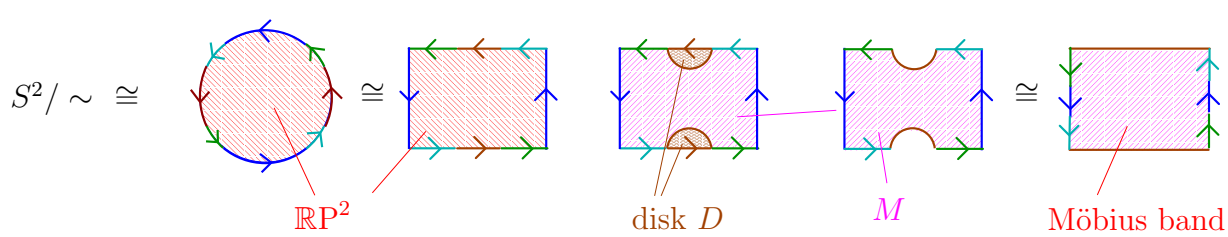


FIGURE 102.

The following proposition will be helpful on several occasions.

Proposition 3.24. *Let M be an n-dimensional manifold and let W be an n-dimensional submanifold of M with $W \cap \partial M = \varnothing$. If W is compact, then the manifold M is decomposed into the submanifolds M and $X := M \setminus \mathring{W}$. (Here the interior $\mathring{W}$ of W as a subset of M agrees with $W \setminus \partial W$ where ∂W denotes the boundary of W viewed as a manifold in its own right.)*

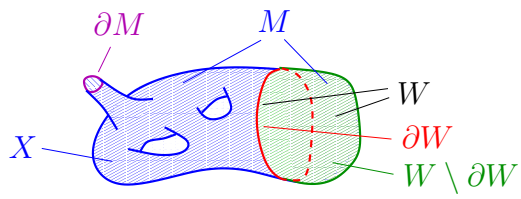


FIGURE 103. Illustration of Proposition 3.24.

Proof. First note that it is a consequence of Lemma 1.18 and the fact that manifolds are Hausdorff that W is a closed subset of M. The proposition is now a fairly immediate consequence of Lemma 3.22. We leave it to the reader to fill in the details. □

3.6. **(Topological) manifolds and group actions.** Now we return to group actions since these are one of the best ways to construct new topological spaces and (topological) manifolds.

Definition. Let M be a manifold with an action by a group G. We say that the action is *smooth* if for any $g \in G$ the map

$$\begin{array}{rcl} M & \to & M \\ x & \mapsto & g \cdot x \end{array}$$

is smooth.

Example. Basically all the actions on manifolds that we had written down so far are smooth. We will study some examples in greater detail after the next proposition.

The following proposition is the main result of this section.

Proposition 3.25. *If G is a group that acts freely, properly and continuously on an n-dimensional topological manifold M, then the following hold:*

(1) *The quotient space M/G is also an n-dimensional topological manifold and the projection map $p\colon M \to M/G$ is a local homeomorphism.*
(2) *If M is a manifold and if G acts smoothly, then there exists a unique smooth structure on M/G such that the projection map $p\colon M \to M/G$ is a local diffeomorphism.*

Furthermore, in both cases the following three statements hold:

(a) *The G-action restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.*
(b) *If M is compact, then M/G is compact.*
(c) *If M is closed, then M/G is closed.*

Examples.

(1) The action

$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$

from page 103 is free, proper and smooth. It follows from Proposition 3.25 that $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional manifold.

(2) Using the action of $G = \mathbb{Z}$ on $M = \mathbb{R} \times [-1, 1]$ from page 105 we obtain that the Möbius band $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is a 2-dimensional manifold. Furthermore, we have

$$\partial(\text{Möbius band}) \underset{\substack{\uparrow \\ \text{Proposition 3.25 (a)}}}{=} (\mathbb{R} \times \{-1, 1\})/\mathbb{Z} \underset{\substack{\uparrow \\ \text{see Exercise 1.39}}}{\cong} S^1.$$

(3) Finally, using the action from page 104 of $G = \{\pm 1\}$ on $M = S^n$ we obtain that the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ is a closed n-dimensional manifold. Now let $m \leq n$. On page 111 we introduced the convention that we view $\mathbb{R}\mathrm{P}^m$ as a subset of $\mathbb{R}\mathrm{P}^n$. We leave it as an exercise to verify that $\mathbb{R}\mathrm{P}^m$ is in fact a submanifold of $\mathbb{R}\mathrm{P}^n$.

The proof of Proposition 3.25 relies on the following two somewhat technical lemmas.

Lemma 3.26. $(*)$ *Let X be a topological space and let G be a group that acts freely, properly and continuously on X. If X is Hausdorff, then for every $x \in X$ there exists an open neighborhood U such that $gU \cap U = \varnothing$ for all $g \neq e$.*

Proof of Lemma 3.26 $(*)$. Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts freely, properly and continuously on X. Finally let $x \in X$. Since G acts freely we know that for every non-trivial $g \in G$ we have $gx \neq x$.

Since the action is proper and continuous we can apply Lemma 1.75 to $a = b = x$ and we obtain open neighborhoods A and B of x such that $gA \cap B = \varnothing$ for every non-trivial $g \in G$. Thus $U := A \cap B$ has the desired property □

Lemma 3.27. $(*)$ *Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts continuously on X. We denote by $p\colon X \to X/G$ the projection. If $U \subset X$ is an open subset such that the map $p\colon U \to X/G$ is injective, then the map $p\colon U \to p(U)$ is a homeomorphism.*

Proof of Lemma 3.27 $(*)$. By Lemma 1.71 the projection $p\colon X \to X/G$ is continuous. Thus the restriction of p to U is also continuous. By our hypothesis the map $p\colon U \to p(U)$ is injective and evidently it is surjective. Thus the map is a bijection. By Lemma 1.71 (2) the map is furthermore open. It follows from Lemma 1.39 that the map $f\colon U \to p(U)$ is in fact a homeomorphism. □

Now we are in a position to prove Proposition 3.25. For clarity we deal with topological manifolds and manifolds separately.

Proof of Proposition 3.25 for topological manifolds. Let M be an n-dimensional topological manifold and let G be a group that acts on M. We suppose that the action is continuous, free and proper. We want to show that M/G is an n-dimensional topological manifold. Thus we need to prove the following claim.

Claim.

(1) M/G is second-countable,

(2) M/G is Hausdorff, and
(3) every point $y \in M/G$ admits an n-dimensional chart.

The first statement of the claim follows immediately from Lemma 1.71 and 3.1 (5). The second statement follows from Proposition 1.74. Thus it remains to deal with the third statement. As usual we denote by $p\colon M \to M/G$ the projection. Let $y \in M/G$. We pick an $x \in M$ with $p(x) = y$. We make the following two observations:

(a) Since M is an n-dimensional topological manifold there exists an n-dimensional chart $\Phi\colon V \to W$ with $x \in V$.
(b) By Lemma 3.26 there exists an open neighborhood U of x such that $gU \cap U \neq \varnothing$ for every $g \neq e$. Note that this implies that the restriction of $p\colon M \to M/G$ to U is injective.

After possibly replacing U and V by $U \cap V$ we can assume that $U = V$. Note that by Lemma 1.71 (2) the projection $p(U)$ is an open neighborhood of x. Now we consider the maps

$$p(U) \xrightarrow{p^{-1}} U \xrightarrow{\Phi} W.$$

The map Φ is by definition a homeomorphism. Furthermore it follows from Lemma 3.27 that the map $p\colon U \to p(U) \subset M/G$ is a homeomorphism as well. Thus the above map $\Phi \circ p^{-1}\colon p(U) \to W$ is a homeomorphism, in particular it is an n-dimensional chart for M/G that contains y. This completes the proof of the claim and thus completes the proof that M/G is an n-dimensional topological manifold.

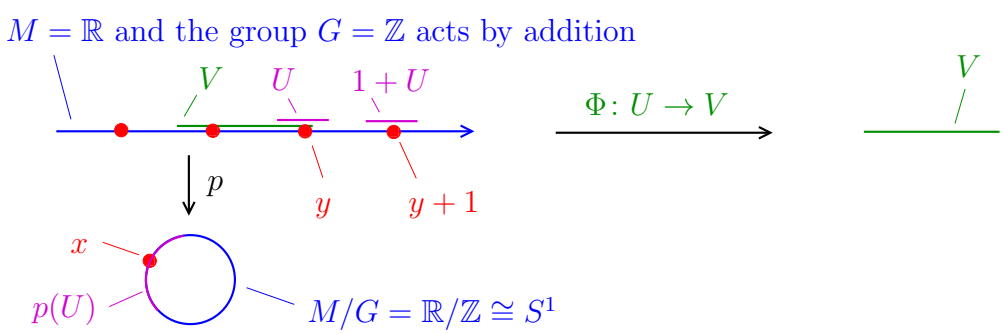


FIGURE 104. Illustration for the proof of Proposition 3.25.

It remains to prove the following claim.

Claim.

(1) The projection $p\colon M \to M/G$ is a local homeomorphism.
(2) The G-action on M restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(3) If M is compact, then M/G is compact.
(4) If M is closed, then M/G is closed.

We turn to the proofs of these three statements.

(1) Implicitly we had already proved the first statement, indeed, given $x \in M$ we run through the above argument and we see, with the same notation, that $p(U)$ is an open subset of M/G and that $p\colon U \to p(U)$ is a homeomorphism.
(2) It follows from Lemma 3.5 (1) that the G-action on M restricts to an action on ∂M. Furthermore it follows easily from Lemma 3.5 (2) and the above statement (1) that $\partial(M/G) = (\partial M)/G$. We leave it to the reader to fill in the details.
(3) The third statement is of course just an immediate consequence of Lemma 1.37.
(4) This statement is an immediate consequence of statements (2) and (3).

This completes the proof of the second claim. □

Proof of Proposition 3.25 for manifolds $(*)$. Let $(M, \mathcal{A})$ be an n-dimensional manifold. Let G be a group that acts freely, properly and smoothly on M. We need to show that M/G admits a smooth atlas such that the projection map $p\colon M \to M/G$ is a local diffeomorphism.

(1) We run through the proof of Proposition 3.25 (1), but this time only using charts for M that lie in $\mathcal{A}$, and we obtain an atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ for M/G. It is straightforward to see that any transition map $\Phi_j \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap U_j) \to \Phi_j(U_i \cap U_j)$ can be written as the composition of (some) the following maps:
   (a) charts[82] and their inverses,
   (b) restrictions to open subsets,
   (c) the restriction of the action of $g \in G$ on M to some open subset.
   All these maps are smooth maps between manifolds.[83] Thus the transition maps are smooth.
(2) In the above proof of Proposition 3.25 (1) we already saw that the projection $p\colon M \to M/G$ is a local homeomorphism. It is straightforward to verify that, with our choice of a smooth atlas for M/G, the projection is in fact a local diffeomorphism.

The proof of the remaining statements (a), (b) and (c) is basically identical to the above proof for topological manifolds.

Finally, the statement that the smooth structure on M/G is uniquely determined by the smooth structure on M and the property that the projection $M \to M/G$ is a local diffeomorphism is a consequence of Lemma 3.15 (1) and the fact that the action is proper. We leave it to the reader to fill in the details. □

We conclude this section with the following amusing lemma. It gives an example of a diffeomorphism that will pop up on several occasions.

[82]Recall that we view open subsets of $\mathbb{R}^n$ as manifolds. With this convention charts are in fact diffeomorphisms between manifolds.

[83]Here the manifolds are open subsets of $\mathbb{R}^n$ or of M.

Lemma 3.28. *Let $n \in \mathbb{N}$. If $A \in \mathrm{GL}(n, \mathbb{Z})$ is a matrix, then the map*

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a diffeomorphism.

Proof. We will provide the proof in Exercise 3.6. □

Example. We consider the map $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$. The corresponding self-diffeomorphism $f(A)$ of $\mathbb{R}^2/\mathbb{Z}^2 = [0,1] \times [0,1]/\sim$ is illustrated in Figure 105.

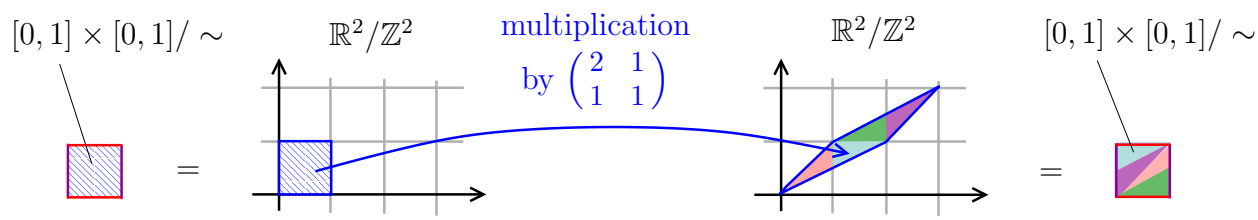


FIGURE 105.

3.7. **Tangent spaces of manifolds.** In this and the following section we introduce tangent spaces and orientations on manifolds. A more detailed account is given in most textbooks on manifolds. The following discussion is based on [Le2, Chapter 15], [Jä3, Chapter 4] and [Frie2].

Definition. Let M be a manifold and let $P \in M$.[84]

(1) A *derivation at P* is a linear map $v\colon C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M, \mathbb{R})$ we have

$$v(fg) \;=\; f(P) \cdot v(g) + v(f) \cdot g(P).$$

(2) The set of all derivations at P is called the *tangent space $T_P M$ at P*.

This definition of a tangent space might sound rather odd, but the following proposition should help in getting some intuition for the definition.

Proposition 3.29. *Let M be an open subset of $\mathbb{R}^n$ or of H_n. For any $P \in M$ the map*

$$\begin{array}{rcl} \mathbb{R}^n & \to & T_P M \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(M, \mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \frac{d}{dt}\big|_{t=0} f(P + tv)) \end{array} \right) \end{array}$$

is an isomorphism of vector spaces.

[84] We should stress that we allow P to be in the boundary of M.

Proof. The proof of this proposition is an amusing exercise in real analysis. A detailed proof for the case that M is an open subset of $\mathbb{R}^n$ is given in [Le2, Proposition 3.2]. The case that M is an open subset of H_n follows fairly easily from the definition of a smooth map on H_n, see page 181. □

Convention. Given an open subset M of $\mathbb{R}^n$ or of H_n and given $P \in M$ we will use the isomorphism $\mathbb{R}^n \xrightarrow{\cong} T_P M$ from Proposition 3.29 to make the identification $T_P M = \mathbb{R}^n$.

The following proposition summarizes some basic properties of the tangent space.

Proposition 3.30.

(1) *If $f\colon M \to N$ is a smooth map, then for any $P \in M$ the map*

$$\begin{array}{rcl} Df_P = f_*\colon T_P M & \to & T_{f(P)}N \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(N,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & v(g\circ f) \end{array} \right) \end{array}$$

is linear. We refer to Df_P as the induced map.

(2) *The induced maps have the following two properties:*[85]
   (a) *If* $\mathrm{id}\colon M \to M$ *is the identity map, then for any $P \in M$ the induced map $D\,\mathrm{id}_P\colon T_P M \to T_P M$ is also the identity.*
   (b) *For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between manifolds and any $P \in L$ we have*

$$D(g\circ f)_P \;=\; Dg_{f(P)} \circ Df_P\colon T_P L \;\to\; T_{(g(f(P))}N.$$

(3) *If $f\colon M \to N$ is a local diffeomorphism, then for any $P \in M$ the induced map $Df_P\colon T_P M \to T_{f(P)}N$ is an isomorphism.*

(4) *If M is an n-dimensional manifold, then for any $P \in M$ the tangent space is an n-dimensional subvector space of the vector space* $\mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R})$.

Sketch of proof. The first three statements follow fairly easily from the definitions. Using charts one can reduce the proof of Statement (4) to the case that M is an open subset of $\mathbb{R}^n$ or H_n. But we had dealt with this case in Proposition 3.29. We refer to [Le2, Proposition 3.10] and [Le2, Proposition 3.6] for more details. □

There are several alternative interpretations of the tangent space of a manifold. In the following we will discuss a commonly used definition for submanifolds of $\mathbb{R}^k$.

Definition.

(1) Let M be an n-dimensional submanifold of $\mathbb{R}^k$. Given any $P \in M \setminus \partial M$ we define the *visual tangent space*

$$V_P M \;:=\; \text{all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon (-\epsilon,\epsilon) \to M \text{ with } \gamma(0) = P.$$

This definition is illustrated in Figure 106.

[85] The reader who is already familiar with categories will notice that this statement just says that the tangent space at a point is a functor from the category of pointed manifolds to the category of real vector spaces.

(2) Let $f\colon M \to N$ be a smooth map between submanifolds of $\mathbb{R}^k$ respectively $\mathbb{R}^l$. Given any $P \in M \setminus \partial M$ with $f(P) \notin \partial N$ we define[86]

$$\begin{array}{rcl} Df_P = f_*\colon V_PM & \to & V_{f(P)}N \\ \gamma'(0) & \to & (f \circ \gamma)'(0) \end{array}$$

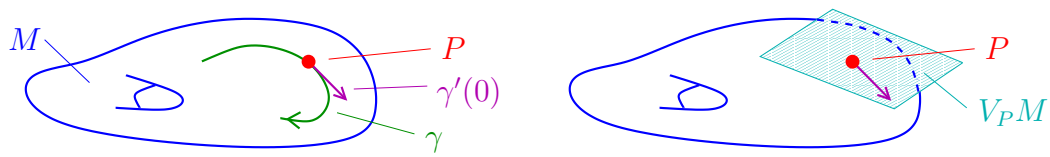


FIGURE 106.

Example.

(1) If M is an open subset of $\mathbb{R}^n$, then for any $P \in M$ we more or less evidently have $V_PM = \mathbb{R}^n$. Furthermore, if $f\colon M \to N$ is a smooth map between two open subsets of $\mathbb{R}^n$, then $Df_P\colon V_PM = \mathbb{R}^n \to V_PN = \mathbb{R}^n$ is given precisely by multiplication by the differential $Df_P \in M(n \times n, \mathbb{R})$.
(2) For $M = S^n \subset \mathbb{R}^{n+1}$ one easily shows that for any $P \in S^n$ we have the equality $V_PM = \{v \in \mathbb{R}^{n+1} \mid \langle v, P\rangle = 0\}$.

We formulate the next example as a lemma, so that later on it is easy to refer to the statement.

Lemma 3.31. $(*)$ *If $W \subset \mathbb{R}^n$ is a k-dimensional vector space, then W is a k-dimensional submanifold of $\mathbb{R}^n$ and for any $P \in W$ we have $V_PW = W$.*

Proof $(*)$. It follows from elementary linear algebra that W is a k-dimensional submanifold of $\mathbb{R}^n$. We leave it as an elementary exercise to the reader to verify that for any $P \in W$ we have $V_PW = W$. $\square$

The next proposition elucidates in particular the relationship between the tangent space T_PM and the visual tangent space V_PM of a submanifold M of $\mathbb{R}^k$.

Proposition 3.32.

(1) *The obvious analogues of the statements of Proposition 3.30 hold for the visual tangent spaces.*
(2) *Let M be an n-dimensional submanifold of $\mathbb{R}^k$ and let $P \in M \setminus \partial M$. The map*

$$\begin{array}{rcl} \Phi_M\colon V_PM & \to & T_PM \\ \gamma'(0) & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \frac{d}{dt}\big|_{t=0} f(\gamma(t)) \end{array} \right) \end{array}$$

[86]Using charts and the chain rule one can show fairly easily that this map is well-defined.

is an isomorphism of vector spaces. Furthermore, if $f\colon M \to N$ is a smooth map between submanifolds of $\mathbb{R}^k$ respectively $\mathbb{R}^l$ such that $f(P) \notin \partial N$ the following diagram commutes

$$\begin{array}{ccc} V_P M & \xrightarrow{\Phi_M} & T_P M \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ V_{f(P)} N & \xrightarrow{\Phi_N} & T_{f(P)} N. \end{array}$$

Proof.

(1) The proof that the obvious analogues of the statements for Proposition 3.30 hold for the visual tangent spaces is fairly elementary and is left to the reader.
(2) First note that it follows almost immediately from the definitions that the given diagram does indeed commute. It remains to show that the horizontal maps are isomorphisms. Using charts it suffices to show this for open subsets of $\mathbb{R}^n$. But in this special case that is an immediate consequence of Proposition 3.29. □

Convention. Given a submanifold M of some $\mathbb{R}^k$ and $P \in M \setminus \partial M$ we will use the isomorphism from Proposition 3.32 to make the identification $T_P M = V_P M$.

Remark.

(1) With a little bit of effort one could also define $V_P M$ for points $P \in \partial M$ and one could then extend Proposition 3.32 to this more general situation.
(2) The above reinterpretation of $T_P M$ can be generalized to any manifold. More precisely, if M is a manifold and $P \in M \setminus \partial M$ one can interpret $T_P M$ as equivalence classes of smooth paths, i.e. equivalence classes of smooth maps $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = P$. We refer to [Le2, p. 72] or [Jä3, Chapter 3.3] for details. The interpretation (or very often the definition) of the tangent space in terms of equivalence classes of smooth paths is very intuitive, but it has the disadvantage that it gets awkward to write down the definition of the tangent space at a point on the boundary.

The following theorem that gives a convenient criterion for a map between manifolds to be a (local) diffeomorphism.

Theorem 3.33. (Inverse Mapping Theorem) $(*)$ *Let $f\colon M \to N$ be a smooth map between two manifold.*

(1) *Let $P \in M \setminus \partial M$. If $f(P) \notin \partial N$ and if Df_P is invertible, then f is a local diffeomorphism at P, i.e. there exist open neighborhoods U of P and V of $f(P)$ such that the restriction of f to $f\colon U \to V$ is a diffeomorphism.*
(2) *If Df_P is invertible for every $P \in M$ and if f is a bijection, then f is a diffeomorphism.*

Remark. We consider the inclusion $i\colon H = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\} \to \mathbb{R}^2$ and the point $P = (0, 0) \in \partial H$. We see that the differential Di_P is an isomorphism, but the inclusion is

not a local diffeomorphism at P. This shows that in general we cannot drop the condition that $P \notin \partial M$.[87]

Proof.

(1) The first statement can be reduced fairly easily to the Inverse Function Theorem 3.19 that deals with maps between open subsets of some $\mathbb{R}^n$. We refer to [Le2, Theorem 4.5] for more details.
(2) The second statement follows immediately from (1) and Lemma 3.15 (3). □

3.8. **Orientations of manifolds.** In this section we introduce and study orientations of manifolds. First we need to give the following purely algebraic definition.

Definition. Let V be a finite-dimensional non-trivial real vector space. We write $k = \dim(V)$.

(1) We say that two bases[88] of V are *equivalent* if the determinant of the base change matrix is positive.
(2) An equivalence class of bases is called an *orientation* for V. An *oriented vector space* is a vector space together with an orientation.
(3) Let $\mathcal{O}$ be an orientation for V. We say $\{v_1, \dots, v_k\}$ is a *positive basis*, if $\{v_1, \dots, v_k\}$ lies in $\mathcal{O}$, otherwise we say that it is a negative basis.

Remark.

(1) It is straightforward to see that any finite-dimensional non-trivial real vector space admits precisely two orientations. Given an orientation $\mathcal{O}$ for some vector space V we denote by $-\mathcal{O}$ the other orientation.
(2) We always view $\mathbb{R}^n$ as equipped with the orientation such that the canonical basis $\{e_1, \dots, e_n\}$ is a positive basis.

Definition. Let V and W be oriented vector spaces. We say an isomorphism $\Phi\colon V \to W$ is *orientation-preserving* if the image of a positive basis of V is a positive basis of W. Otherwise we say that Φ is *orientation-reversing*.

The following lemma gets often used subconsciously:

Lemma 3.34. *Let $V_0, \dots, V_k$ be oriented vector spaces and let $f_i\colon V_i \to V_{i+1}$, $i = 0, \dots, k-1$ be isomorphisms. Then*

$$\begin{array}{c}\textit{the map } f_{k-1} \circ \cdots \circ f_0\colon V_0 \to V_k \textit{ is} \\ \textit{orientation-preserving}\end{array} \iff \begin{array}{c}\textit{the number of } f_i \textit{ which are} \\ \textit{orientation-reversing is even.}\end{array}$$

Proof. The lemma is an elementary exercise in linear algebra that is left to the reader for his or her enjoyment. □

Definition. Let M be a k-dimensional manifold with $k \geq 1$.

[87] Do we really need that $f(P) \notin \partial N$?

[88] Here and throughout the notes a basis of a finite-dimensional vector space is always understood to be an ordered basis.

(1) An *orientation for M* is a map which assigns to every tangent space T_PM, $P \in M$, an orientation such that the following condition is satisfied: [89]
   (∗) For every chart $\Phi\colon U \to V$ in the given atlas of the manifold M and any component U' of U the map

$$\begin{array}{ccccccc} \Phi_*\colon T_PM & \xleftarrow{\cong} & T_PU & \xrightarrow{\Phi_*} & T_{\Phi(P)}V & = & \mathbb{R}^k \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{isomorphisms by Proposition 3.30 (2)} & & & \text{convention on page 196} & & \end{array}$$

   is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.
(2) A manifold together with an orientation is called an *oriented manifold*.
(3) A manifold that admits an orientation is called *orientable*, otherwise we say that the manifold is *non-orientable*.

Convention.

(1) Given a picture of a 1-dimensional submanifold of $\mathbb{R}^2$ or $\mathbb{R}^3$ we indicate an orientation by drawing a positive vector (in this case a non-zero vector is already a basis for the tangent space).
(2) Given a picture of a 2-dimensional submanifold of $\mathbb{R}^3$ we often adopt one of the usual two conventions for indicating an orientation:
   (a) we draw a little circle with an arrow, this indicates that a positive basis is given by v, w where w is obtained by rotating v into the indicated direction by an angle in $(0, \pi)$,
   (b) alternatively we draw a normal vector n, in this case a basis v, w of the tangent space is positive if n, v, w is a positive basis for $\mathbb{R}^3$.

Examples.

(1) Let U be an open subset of $\mathbb{R}^n$. We equip U with the orientation where a positive basis for any tangent space $T_PU = \mathbb{R}^n$ is given by the standard basis for $\mathbb{R}^n$. Since an atlas is given by the identity map we see that this defines an orientation on U.
(2) In the following we will always view S^n as an oriented manifold with the convention that a basis $v_1, \dots, v_n \in T_PS^n = V_PS^n \subset \mathbb{R}^{n+1}$ is positive if $\det(P, v_1, \dots, v_n) > 0$.[90] For S^1 we obtain the "counterclockwise orientation". We sketch the orientations for S^1 and S^2 in Figure 107.

The next definition is basically self-explanatory.

Definition. Let $f\colon M \to N$ be a local diffeomorphism between two oriented manifolds. We say the map f is *orientation-preserving* if for any $P \in M$ the induced isomorphism $f_*\colon T_PM \to T_{f(P)}N$ is orientation-preserving.

[89] One can show fairly easily that it suffices to show that this condition holds for some family of charts that cover M. We do not need to verify it for all charts in the given atlas.

[90] How would you formulate this orientation convention purely in terms of derivations?

FIGURE 107.

The following lemma gives another approach to defining an orientation on a given manifold.

Lemma 3.35. $(*)$ *Let $(M, \mathcal{A})$ be a manifold. Suppose that all transition maps between charts in $\mathcal{A}$ are orientation-preserving. Given $P \in M$ we define*

$$\textit{orientation of } T_PM := (D\Phi_P)^{-1}(\textit{orientation of } T_{\Phi(P)}V = \mathbb{R}^n)$$

where $\Phi\colon U \to V$ is a chart around P from the given atlas. The following statements hold:

(1) *The definition of the orientation of T_PM does not depend on the choice of the chart.*
(2) *The orientations of the tangent space T_PM define an orientation for M.*

Sketch of proof $(*)$. The first statement follows easily from the hypothesis that the transition maps are orientation-preserving. The second statement is a consequence of the observation in Footnote 89. □

Example. We consider the surface Σ of genus 2. In Proposition 3.7 we gave an explicit atlas for Σ. In Proposition 3.13 we saw that the transition maps are smooth. We leave it to the reader to verify that all the transition maps are in fact orientation-preserving. Thus it follows from Lemma 3.35 that Σ is orientable. Basically the same argument shows that all surfaces of genus $g \geq 3$ are also orientable.

Lemma 3.36. *Let $f\colon X \to Y$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Given $P \in X$ we have*[91]

$$f_* = Df_P\colon T_PX \to T_{f(P)}Y \textit{ is orientation-preserving} \iff \det(Df_P) > 0.$$

Proof. By Proposition 3.32 we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n = V_PX \arrow[r, "\Phi_X"] \arrow[d, "f_*=Df_P"'] & T_PX \arrow[d, "f_*=Df_P"] \\
\mathbb{R}^n = V_{f(P)}Y \arrow[r, "\Phi_Y"] & T_{f(P)}Y.
\end{tikzcd}$$

[91]Unfortunately the symbol Df_P has several, subtly different meanings. To the left it stands for the induced map on tangent spaces. To the right it stands for the $n \times n$-matrix given by the partial derivatives of f.

where the horizontal maps are isomorphisms and where the horizontal maps are orientation-preserving by definition of the orientations of the oriented manifolds U and V. Thus it follows from Lemma 3.34 that the left-hand vertical map is orientation-preserving if and only if the right-hand vertical map is orientation-preserving. But by definition the left-hand vertical map is orientation-preserving if and only if $\det(Df_P) > 0$. □

The following lemma summarizes a few elementary properties of orientations.

Lemma 3.37. *Let M be an n-dimensional manifold*

(1) *If two orientations on M agree at a point and if M is path-connected, then the two orientations agree everywhere.*
(2) *Let $f\colon M \to N$ be a diffeomorphism. If M is path-connected, then for any $P \in M$ we have*

$$f \text{ is orientation-preserving} \iff Df_P\colon T_PM \to T_{f(P)}N \text{ is orientation-preserving.}$$

(3) *Suppose M is equipped with an orientation $\mathcal{O}$. Let $f\colon M \to M$ be a self-diffeomorphism. Then*

$$\begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } \mathcal{O} \end{array} \iff \begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } -\mathcal{O}. \end{array}$$

(4) *Let N be an open subset of M. Then N itself admits the structure of an n-dimensional manifold such that the inclusion map is a local diffeomorphism and N admits a unique orientation such that the inclusion map $N \to M$ is orientation-preserving.*
(5) *If $M \setminus \partial M$ admits an orientation, then M admits a unique orientation such that the inclusion $M \setminus \partial M \to M$ is orientation-preserving.*

Proof $(*)$.

(1) Suppose we are given two orientations on M. We denote by U the set of all points where the orientations agree and we denote by V the set of all points where the orientations disagree. It follows easily from the definitions that U and V are open. If the orientations agree at a point we see that $U \neq \varnothing$. It follows from Lemma 3.8 (2) that $U = M$.
(2) The proof of this statement is very similar to the proof of the first statement. We leave it to the reader to fill in the details.
(3) This statement follows immediately from the definitions and Lemma 3.34.
(4) This statement follows easily from Proposition 3.30.
(5) Using charts it suffices to prove this statement for manifolds that are open subsets of H_n, but in this case the statement is clear. □

The first part of the following proposition can be viewed as a refinement of Proposition 3.25.

Proposition 3.38. *Let G be a group that acts freely, properly and smoothly on an n-dimensional manifold M.*

(1) *If M is oriented and if for each $g \in G$ the map*

$$\begin{array}{rcl} M & \to & M \\ P & \mapsto & g \cdot P \end{array}$$

is orientation-preserving, then M/G admits an orientation such that the projection map $p\colon M \to M/G$ is orientation-preserving.

(2) *Conversely, if M/G admits an orientation, then there exists a unique orientation on M such that the projection map $p\colon M \to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.*

Remark. In Exercise 3.9 we will prove a partial converse to Proposition 3.9 (2).

Proof. We leave the fairly straightforward proof of the proposition as an exercise to the reader. □

Examples.

(1) Note that the differential of any translation map on $\mathbb{R}^n$ is the identity. Thus it follows from Lemma 3.37 (2) that the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition is orientation-preserving. It follows from Proposition 3.38 that $\mathbb{R}^n/\mathbb{Z}^n$ is orientable.

(2) In Exercise 3.4 we will see that the action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd. It follows from Proposition 3.38 that at least for n odd the manifold $\mathbb{R}\mathrm{P}^n = S^n/\pm 1$ is orientable.

Next we intend to give examples of manifolds that are non-orientable.

Lemma 3.39. *The following manifolds are non-orientable:*

(1) *the Möbius band,*
(2) *the Klein bottle, and*
(3) *the projective plane $\mathbb{R}\mathrm{P}^2$.*

Proof. Let $X = \mathbb{R} \times (-1, 1)$ and $G = \mathbb{Z}$. Very similar to the definition on page 105 we consider the action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times (-1,1)) & \to & \mathbb{R} \times (-1,1) \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

This action is again free, proper and smooth. As on page 105 we refer to the quotient as the open Möbius band.

Claim. The open Möbius band is non-orientable.

Suppose that the open Möbius band $X/\mathbb{Z}$ admits an orientation. It follows from Proposition 3.38 that $X = \mathbb{R} \times (-1, 1)$ admits an orientation such that the $\mathbb{Z}$-action is orientation-preserving. It follows from Lemma 3.37 (2) and (3) that this $\mathbb{Z}$-action preserves the

usual orientation on $X = \mathbb{R} \times (-1,1)$. Given $n \in \mathbb{Z}$ we consider the diffeomorphism of $X = \mathbb{R} \times (-1,1)$ that is given by the action by n, i.e. we consider the diffeomorphism

$$\begin{array}{rcl} \Phi_n \colon \mathbb{R} \times (-1,1) & \to & \mathbb{R} \times (-1,1) \\ (x,y) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Given any $(x,y) \in X$ the differential of Φ_n is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$. It follows from Lemma 3.36 that for n odd the diffeomorphism Φ_n is not orientation-preserving. We have thus obtained a contradiction. This concludes the proof of the claim.

We turn to the actual proof that the three given manifolds are non-orientable. By Lemma 3.37 (4) it suffices to prove the following claim.

Claim. The open Möbius band is diffeomorphic to open subsets of the Möbius band, of the Klein bottle and of the projective plane $\mathbb{R}\mathrm{P}^2$.

It is clear that the open Möbius band is an open subset of the Möbius band. The statements regarding the Klein bottle and the projective plane $\mathbb{R}\mathrm{P}^2$ follow immediately from Lemma 3.23. □

Remark. For most people starting out with mathematics the definition of a submanifold M of some $\mathbb{R}^n$ and the definition of a tangent space to M as a subvector space of $\mathbb{R}^n$ are quite natural and reasonable. But initially it is perhaps not clear why one should introduce "abstract" manifolds. Furthermore the definition on page 195 of the tangent space T_PM to such a manifold traditionally leads to a lot of head scratching.

The above proof of Lemma 3.39 is a good example why it is useful to introduce these abstract notions. In principle it is possible to show that Möbius band is non-orientable purely by working in the explicit 3-dimensional setting, see e.g. [Frie2, Lemma 9.16]. But this proof is rather messy and not very illuminating. In contrast, the fact that we can work with abstract manifolds, in this case with $(\mathbb{R} \times [-1,1])/\mathbb{Z}$, and their tangents spaces liberates us from the constraints of the 3-dimensional setting and we obtain a much clearer and much more elegant proof that the Möbius band is non-orientable.

We had just seen that odd-dimensional projective spaces are orientable and that the projective plane $\mathbb{R}\mathrm{P}^2$ is non-orientable. This raises the following question.

Question 3.40. *Is $\mathbb{R}\mathrm{P}^{2n}$ orientable for $n \geq 2$?*

In the next lemma we will see that the boundary of an oriented manifold inherits an orientation.

Lemma 3.41. *Let $n \geq 2$ and let M be an oriented n-dimensional manifold. Let $P \in \partial M$. We denote by $w \in T_PM$ a tangent vector which points into the "exterior" of M.*[92] *We say that a basis $v_1, \dots, v_{n-1}$ for $T_P(\partial M)$ is a positive basis of $T_P(\partial M)$ if $w, v_1, \dots, v_{n-1}$ is a positive basis of T_PM. The following two statements hold:*

[92] The notion of a "normal vector which points into the interior" can be made precise using charts. We leave it as an exercise to fill in the details.

(1) *The definition of a positive basis for P does not depend on the choice of w.*
(2) *These orientations of the tangent spaces of ∂M define an orientation on ∂M.*

Proof. We leave the fairly elementary verification of the lemma to the reader. Alternatively full details are given in [Le2, Proposition 15.24]. □

Convention. Let $n \geq 2$ and let M be an oriented n-dimensional manifold. Unless we say something else we always equip ∂M with the orientation given by Lemma 3.8. We refer to this orientation of ∂M as the *boundary orientation*.

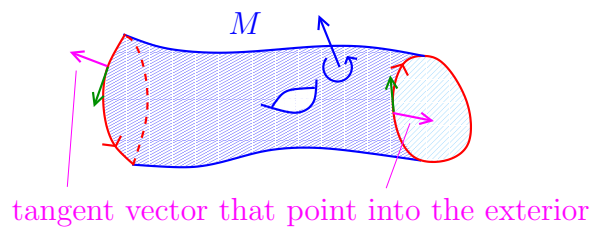


FIGURE 108.

Examples.

(1) If we equip the manifold $\overline{B}^{n+1}$ with the usual orientation coming from $\mathbb{R}^{n+1}$, then the orientation of $\partial\overline{B}^{n+1} = S^n$ agrees with the above orientation of S^n.
(2) Given an oriented manifold N we denote by $-N$ the manifold equipped with the opposite orientation. It follows easily from the definitions that $\partial(-N) = -\partial N$.

We conclude this section with the following proposition.

Proposition 3.42. $(*)$ *Let M be a manifold with possibly non-empty boundary and let N be a manifold with empty boundary. Given $Q \in N$ and $P \in M$ we consider the inclusion maps*

$$\begin{array}{rcl} i_Q\colon M & \to & M \times N \\ x & \mapsto & (x, Q) \end{array} \quad \text{and} \quad \begin{array}{rcl} j_P\colon N & \to & M \times N \\ y & \mapsto & (P, y). \end{array}$$

The following statements hold:

(1) *The product $M \times N$ admits a unique manifold structure such that products of smooth charts for M and N lie in the smooth atlas for $M \times N$.*
(2) *With the manifold structure as in* (1) *the following statements hold:*
   (a) *The projections from $M \times N$ onto M and onto N are smooth.*
   (b) *All the inclusion maps i_P and j_Q are smooth.*
(3) *If M and N are oriented, then we obtain an orientation on $M \times N$ by defining*

$$\textit{positive basis of } T_{(P,Q)}(M \times N) := \begin{array}{l} (i_Q)_*(\textit{positive basis of } T_P M), \textit{ followed by} \\ \qquad (j_P)_*(\textit{positive basis of } T_Q N). \end{array}$$

We refer to this orientation as the product orientation on $M \times N$.

(4) *We have*[93] $\partial(M \times N) = (\partial M) \times N$.
(5) *If M and N are oriented and if the dimension of M is at least two, then the equality $\partial(M \times N) = (\partial M) \times N$ from* (4) *is in fact an equality of oriented manifolds.*

All of the above statements, except for (5)*, hold also with the roles of M and N reversed.*[94]

Proof $(*)$.

(1),(2) By Proposition 3.4 we already know that $M \times N$ is a topological manifold. Furthermore it is very easy to see that the products of charts from the smooth atlases for M and for N define a smooth atlas for $M \times N$ which has the properties specified in (2).
(3) The elementary proof of this statement is left to the reader.
(4) First we want to argue that that we have the equality $\partial(M \times N) = \partial M \times N$. In Proposition 3.4 we could only show the inclusion "$\subset$". The situation is much better in the setting of manifolds since we have Proposition 3.21 (1) at our disposal. Using this proposition one can show[95] fairly easily that $\partial(M \times N) = M \times \partial N$. We also refer to [Le2, Proposition 1.45 and 2.12] for more details.
(5) It is elementary (albeit potentially slightly confusing) to deduce this statement regarding the orientations from the definitions. □

3.9. **Intersections of submanifolds**$(*)$**.** In this section we want to study the following question: given a manifold M and two submanifolds X and Y, is the intersection $X \cap Y$ again a submanifold of M? It is not difficult to show that in general the answer is negative. For example consider $M = \mathbb{R}^2$ with the two submanifolds X and Y that are shown in Figure 109. The intersection $X \cap Y$ has three components, one of which is a submanifold of dimension one, and the other two components consist of a single point. We see that $X \cap Y$ is not a manifold (after all, what would its dimension be?), in particular it is not a submanifold of M.

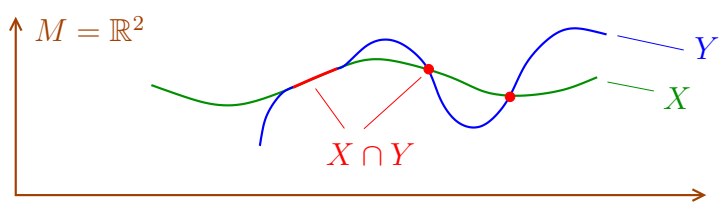


FIGURE 109.

[93]Note that here, when we are dealing with manifolds, we can determine the boundary. This is in contrast to the slightly unfortunate situation in Proposition 3.4.

[94] In Exercise 3.12 we will determine the relationship between the orientations of $\partial(M \times N)$ and $M \times \partial N$ if N is allowed to have non-empty boundary.

[95]Here we use our super subtle definition of the boundary of a manifold, see page 182.

If one spends some time pondering the question why in the previous example the intersection $X \cap Y$ is not a submanifolds, then one is inexorably led towards the following definition.

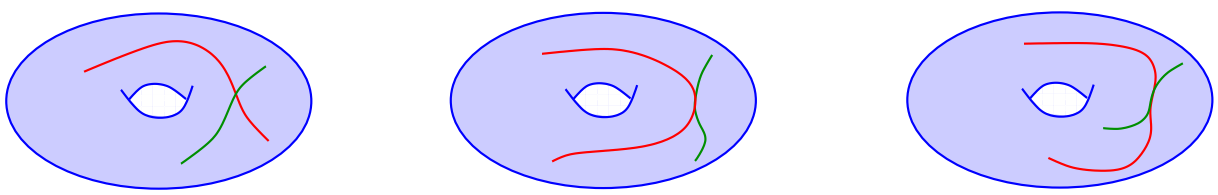
Definition. Let M be an m-dimensional manifold.[96] Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say P is a *transverse intersection point* if $P \notin \partial M$ and if X and Y *intersect transversally in* P, i.e. if the equality[97]

$$T_P X + T_P Y \;=\; T_P M$$

holds.

(2) We say that X and Y *intersect transversally* if each intersection point is a transverse intersection point.[98]



intersection point is transverse | intersection point is not transverse | intersection point is not transverse

FIGURE 110.

The following lemma is basically an amusing real analysis exercise.

Lemma 3.43. *Let M be an m-dimensional manifold, let $X \subset M$ a submanifold of codimension k and let Y be a submanifold of M of codimension l.*

(1) *If $P \in M$ is a transverse intersection point of X and Y, then there exists a chart $\Phi\colon U \to V$ around P with*

$$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k} \times \{0\}) \quad \textit{and} \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l}).$$

(2) *If the submanifolds X and Y intersect transversally, then $X \cap Y$ is a submanifold of codimension $k + l$.*

Proof.

(1) The first statement is proved in [Kos, Theorem IV.1.6] or implicitly also in [Le2, Theorem 6.30 (b)].

[96] Note that in the following we work with manifolds, we will need the smooth atlas in the definition of a transverse intersection point.

[97] Since X is a submanifold of M we can view the tangent space $T_P X$ as a subspace of $T_P M$.

[98] Note that if X and Y intersect transversally, then $X \cap Y \cap \partial M = \varnothing$.

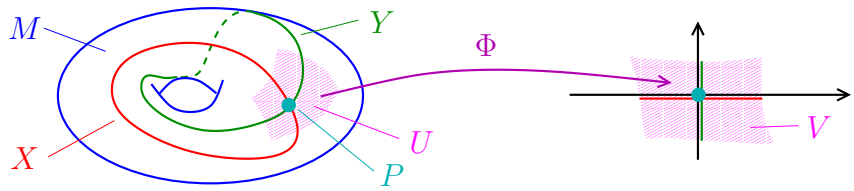


FIGURE 111. Illustration of Lemma 3.43.

(2) The second statement is an immediate consequence of the first statement. □

Example. In Figure 112 we see two infinite cylinders X and Y in the manifold $\mathbb{R}^3$. The cylinders are submanifolds of codimension one. They intersect transversally and the intersection consists of two circles, i.e. $X \cap Y$ is a one-dimensional submanifold of $\mathbb{R}^3$. Put differently, $X \cap Y$ is a submanifold of codimension two.

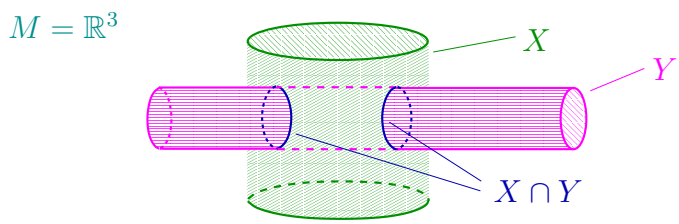


FIGURE 112.

3.10. **The Regular Value Theorem I.** In general it can be quite painful to show directly that a given object is a manifold. For example, we will shortly see that the set of orthogonal matrices $\mathrm{O}(n)$ form a manifold. But only the most enterprising reader will feel like writing down an explicit atlas.

In this section we will state the Regular Value Theorem which is a convenient way for showing that certain objects are indeed manifolds. Not surprisingly the formulation of the Regular Value Theorem is preceded by the definition of a regular value. This notion might be familiar to the reader from an earlier course on real analysis.

Definition. Let M and N be manifolds (possibly with boundary). Let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M \backslash \partial M$. We say P is a *regular point* if the differential $Df_P\colon T_PM \to T_{f(P)}N$ is an epimorphism. Otherwise we say that P is a *critical point*.
(2) Let $P \in \partial M$. We say P is a *regular point* if the differential $Df_P\colon T_P(\partial M) \to T_{f(P)}N$ is an epimorphism.[99] Otherwise we say that P is a *critical point*.

[99]Note that the restriction of a map on a manifold to its boundary is also a smooth map.

(3) We say $Q \in N$ is a *regular value* if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a *critical value*.

The following elementary observation will be used several times without explicitly mentioning it.

Observation. Let $f\colon M \to N$ be a smooth map between manifolds.

(1) If $Q \in N$ is a point with $f^{-1}(Q) = \varnothing$, then Q is a regular value.
(2) If $\dim(M) < \dim(N)$, then the converse to (1) holds, thus a point $P \in N$ is a regular value if and only if $f^{-1}(P) = \varnothing$.

The following theorem is often a convenient way for showing that a given subset of a manifold, e.g. a given subset of some $\mathbb{R}^n$, is in fact a manifold.

Theorem 3.44. (Regular Value Theorem) *Let M be an m-dimensional manifold, let N be an n-dimensional manifold without boundary, let $f\colon M \to N$ be a smooth map and let $s \in N$ be a regular value of f. Then the following statements hold:*

(1) *The preimage $X := f^{-1}(s)$ is a proper $(m-n)$-dimensional submanifold of M.*
(2) *If M is orientable, then X is also orientable.*
(3) *The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.*

Examples.

(1) In Figure 113 we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x,y,z) = z$. This function has precisely four critical points which are indicated in Figure 113 on the left. On the right we show a regular value s and its preimage and we show a critical value t and its preimage $f^{-1}(t)$ which in this case is *not* a submanifold.

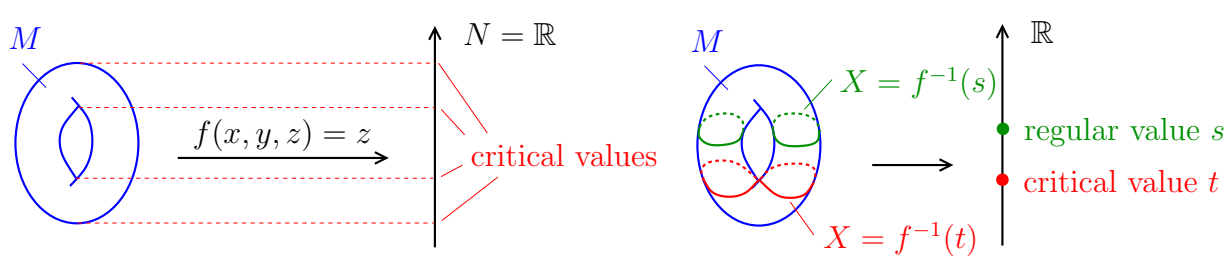


FIGURE 113. Illustration of the Regular Value Theorem 3.44.

(2) Similarly to (1) we again consider a manifold $M \subset \mathbb{R}^3$ together with the "height function". This time M has boundary and $f^{-1}(s)$ is a proper submanifold of M with non-empty boundary.

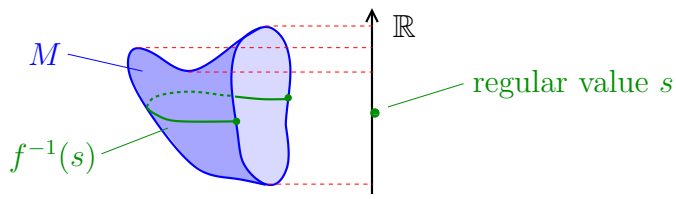


FIGURE 114. Illustration of the Regular Value Theorem 3.44.

(3) Let $g \in \mathbb{N}_0$. According to [Hirs, p. 28] there exists a polynomial function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that 0 is a regular value and such that $f^{-1}(0)$ is a surface of genus g. For example, if we take

$$f(x,y,z) \;=\; (4x^2(1-x^2)-y^2)^2 + z^2 - \tfrac{1}{4},$$

then [Hirs, p. 28] states that $f^{-1}(0)$ is a surface of genus 2.

Sketch of proof. The first statement is proved in basically every book on manifolds, see e.g. [Mi2, p. 11] or [Le2, Corollary 5.14]. The second statement is unfortunately not written down explicitly in the literature, but it can be proved without too much effort by modifying the proof of (1). The third statement follows from the observation that $f^{-1}(s)$ is a closed subset of the compact topological space M, hence it itself is compact. □

The following lemma can be viewed as a particularly interesting application of the Regular Value Theorem 3.44.

Lemma 3.45. *Let $n \in \mathbb{N}$. The set*

$$\mathrm{O}(n) \;=\; \{A \in M(n\times n,\mathbb{R}) \,|\, A^TA = \mathrm{id}\}$$

of orthogonal matrices (and thus also its component $\mathrm{SO}(n)$) is a $\frac{1}{2}n(n-1)$-dimensional submanifold of $M(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. Furthermore the set of unitary matrices

$$\mathrm{U}(n) \;=\; \{A \in M(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id}\}$$

is an n^2-dimensional submanifold of $M(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ and the set of special unitary matrices

$$\mathrm{SU}(n) \;=\; \{A \in M(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id} \;\; \textit{and} \;\; \det(A) = 1\}$$

is an (n^2-1)-dimensional submanifold of $M(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = (\mathbb{R}^2)^{n^2} = \mathbb{R}^{2n^2}$. All of the above examples, viewed as manifolds in their own right, are closed.

Proof. We start with a preamble: Throughout the proof we use Proposition 3.32 to identify the abstract tangent spaces T_PM with the visual tangent spaces V_PM.
Now we proceed to the actual proof of the lemma. First we prove the desired statements for $\mathrm{O}(n)$.

Claim.

(1) The set
$$\mathrm{Sym}(n,\mathbb{R}) \;=\; \{B \in M(n\times n,\mathbb{R}) \,|\, B^T = B\} \;=\; \{(b_{ij}) \in M(n\times n,\mathbb{R}) \,|\, b_{ij} = b_{ji}\}$$
is a submanifold of $M(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n+1)$.

(2) The map
$$\begin{array}{rcl} f\colon M(n\times n,\mathbb{R}) & \to & \mathrm{Sym}(n,\mathbb{R}) \\ A & \mapsto & A^TA \end{array}$$
is smooth and id is a regular value.

The first statement follows immediately from Lemma 3.31. We turn to the proof of the second statement. It should be clear that f is smooth and that f does indeed take values in $\mathrm{Sym}(n,\mathbb{R})$. So it remains to show that id is a regular value.

In the following we again use Lemma 3.31 which implies that the tangent spaces of the submanifolds $\mathrm{Sym}(n,\mathbb{R})$ and $M(n\times n,\mathbb{R})$ are just given by these subvector spaces. So let $A \in M(n\times n,\mathbb{R})$ with $A^TA = \mathrm{id}$ and let $C \in T_{\mathrm{id}}\,\mathrm{Sym}(n,\mathbb{R}) = \mathrm{Sym}(n,\mathbb{R})$. We need to show that there exists a B in $T_AM(n\times n,\mathbb{R}) = M(n\times n,\mathbb{R})$ with $Df_AB = C$.

We start out with the following little calculation: Given $B \in M(n,\times n,\mathbb{R})$ we have
$$\begin{aligned} \tfrac{d}{dt}\big|_{t=0} f(A+tB) &= \tfrac{d}{dt}\big|_{t=0}(A^T + tB^T)\cdot(A+tB) \\ &= \tfrac{d}{dt}\big|_{t=0}\big(A^TA + tA^TB + tB^TA + t^2B^TB\big) \;=\; A^TB + B^TA. \end{aligned}$$
Using that $A^T\cdot A = \mathrm{id}$ and $C = C^T$ it is now easy to see that $B := \frac{1}{2}A\cdot C$ satisfies $Df_AB = C$. This concludes the proof of the claim.

It follows from the claim, together with the Regular Value Theorem 3.44, that $\mathrm{O}(n)$ is a proper submanifold of $M(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. Since $\mathbb{R}^{n^2}$ has no boundary and since $\mathrm{O}(n)$ is a proper submanifold we see that the boundary of $\mathrm{O}(n)$ is also empty. Finally note that we already saw on page 86 that $\mathrm{O}(n)$ is compact. These last two observations imply that $\mathrm{O}(n)$ is in fact a closed manifold.

The proof for $\mathrm{U}(n)$ is almost verbatim the same as the proof for $\mathrm{O}(n)$. The proof for $\mathrm{SU}(n)$ requires just a little bit of extra thought. We will deal with it in Exercise 3.15. □

For later usage we introduce the notion of a submersion:

Definition. A map $f\colon M \to N$ between manifolds is called a *submersion* if f is smooth and if given any $P \in M$ the map $Df_P\colon T_PM \to T_{f(P)}N$ is an epimorphism.

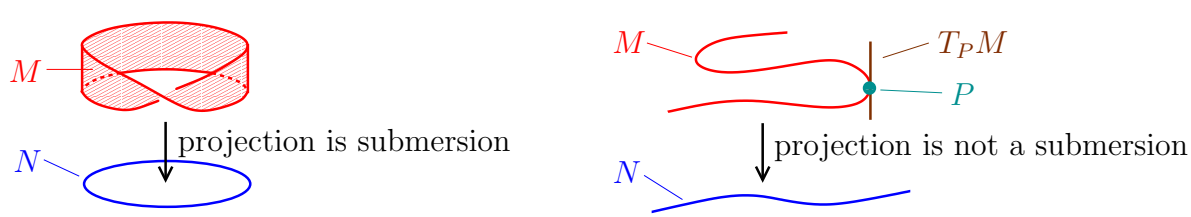


FIGURE 115.

Any projection of the form $f\colon M \times N \to N$ is evidently a submersion. The following theorem says that if C is a convex subset of $\mathbb{R}^n$, then given any compact manifold M any submersion $f\colon M \to N$ is basically precisely of that form.

Theorem 3.46. (Submersion Theorem) *Let M be a compact m-dimensional manifold, let C be an n-dimensional submanifold of $\mathbb{R}^n$ and let $f\colon M \to C$ be a surjective map. If C is a convex subset of $\mathbb{R}^n$ and if f is a submersion, then there exists a closed manifold Y and a diffeomorphism $\varphi\colon M \to C \times Y$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & C \times Y \arrow[dl, "(c,y)\mapsto c"] \\ & C. & \end{tikzcd}$$

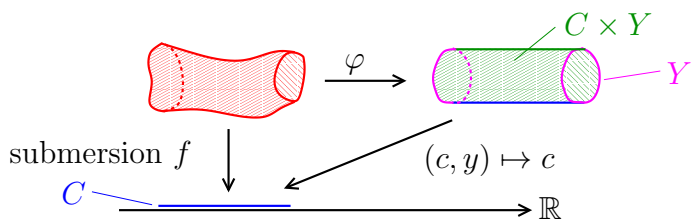


FIGURE 116. Illustration of the Submersion Theorem 3.46.

Proof. This theorem follows from the proof of [Dun, Theorem 8.5.10] or alternatively the proof of [Ebe, Theorem 4.1]. □

3.11. **Sets of measure zero and Sard's Theorem.** One can ask, whether a smooth map between manifolds always admits regular values. In this section we will formulate Sard's Theorem which gives an affirmative answer to this question.

To formulate the theorem we need the notion of a "subset of measure zero" that we will now introduce. The precise definition is not all that important, what matters in practice are the properties stated in the subsequent proposition.

Definition.

(1) A *cuboid* is any subset of $\mathbb{R}^m$ of the form $[c_1, d_1] \times \cdots \times [c_m, d_m]$. We define the volume of such a cuboid as
$$\mathrm{Vol}([c_1, d_1] \times \cdots \times [c_m, d_m]) \;=\; \prod_{i=1}^{m} (d_i - c_i).$$
(2) Let $X \subset \mathbb{R}^m$ be a subset. We say that X *has measure zero* if given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i\in I}$ of cuboids such that $X \subset \bigcup_{i\in I} A_i$ and such that $\sum_{i\in I} \mathrm{vol}(A_i) < \epsilon$. This definition is illustrated in Figure 117.
(3) Let M be an m-dimensional manifold.

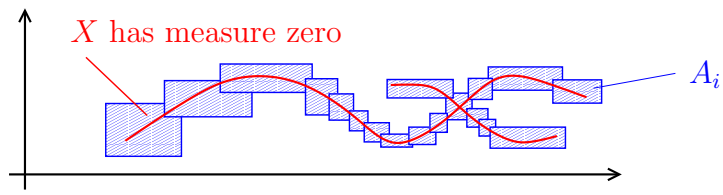


FIGURE 117.

(a) We say that a subset $X \subset M$ is a *subset of measure zero* if for any chart $\Phi\colon U \to V$ from the smooth atlas the set $\Phi(X \cap U)$ is a subset of measure zero of $\mathbb{R}^m$.
(b) We say that a subset $X \subset M$ is a *subset of full measure* if $M \setminus X$ is a subset of measure zero.

The following proposition summarizes key properties of subsets of measure zero.

Proposition 3.47. *Let M be an m-dimensional manifold.*

(1) (a) *The union of countably many subsets of measure zero is again a subset of measure zero.*
(b) *The intersection of countably many subsets of full measure is again a subset of full measure.*
(2) (a) *Given any subset X of measure zero of M the complement $M \setminus X$ is dense in M.*
(b) *Every subset of M of full measure is dense in M.*
(3) *Let X be a subset of M. If there exist charts $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ from the smooth atlas such that $\bigcup_{i\in I} U_i = M$ and such that each $\Phi_i(U_i \cap X)$ is a subset of measure zero of $\mathbb{R}^m$, then X is a subset of measure zero.*
(4) *If $f\colon M \to N$ is a smooth map between manifolds and if $\dim(M) < \dim(N)$, then $f(M)$ is a subset of measure zero in N. In other words, $N \setminus f(M)$ is a subset of full measure.*
(5) (a) *Let $f\colon M \to N$ be a smooth map between manifolds of the same dimension. If X is a subset of measure zero in M, then $f(X)$ has measure zero in N.*
(b) *If $f\colon M \to N$ is a diffeomorphism, then $X \subset M$ has full measure if and only if $f(X)$ does.*

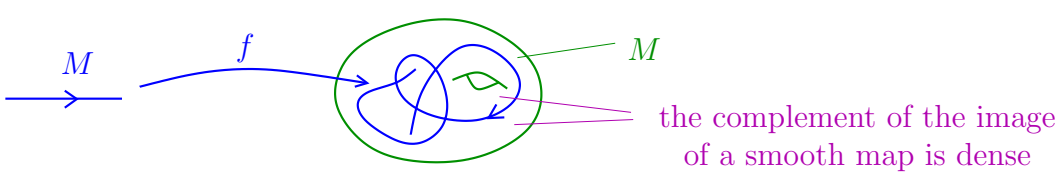


FIGURE 118. Illustration of Proposition 3.47 (2) and (4).

Sketch of proof.

(1) Let $\{X_i\}_{i\in I}$ be a countable family of subsets of measure zero of $\mathbb{R}^n$. Using the fact that $\sum_{i\in\mathbb{N}} \frac{1}{2^i} = 1$ it is straightforward to show that $\bigcup_{i\in I} X_i$ is also a subset of measure zero. Statement (1a) is an easy consequence of this observation. Statement (1b) is just a convenient reformulation of Statement (1a).

(2) Statement (2a) follows easily from the observation that a subset of $\mathbb{R}^m$ that contains an open ball $B^m_r(z)$ of radius $r > 0$ does not have measure zero. We refer to [Le2, Proposition 6.8] for more details. Once again Statement (2b) is just a reformulation of Statement (2a).

(3) This statement is proved in [Le2, Lemma 6.6].

(4) We write $m = \dim(M)$ and we write $n = \dim(N)$. We pick a countable family $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ of charts for M from the maximal smooth atlas for M such that $M = \bigcup_{i\in I} U_i$.[100] It follows easily from [Le2, Proposition 6.5] and the hypothesis that $m < n$ that every $f(\Phi_i(U_i))$ is a subset of measure zero of N. It follows from (1) that $f(M)$ is subset of measure zero of N.

(5) Statement (5a) is [Le2, Theorem 6.9]. Statement (5b) follows easily from Statement (5a). □

Now we can formulate Sard's Theorem which shows that a smooth map between manifolds always admits a plethora of regular values. The theorem is named after Arthur Sard [Sar] who first proved a similar statement in 1942.

Theorem 3.48. (Sard's Theorem) *If $f \colon M \to N$ is a smooth map between manifolds (possibly with boundary)*[101]*, then the set of critical values of f is of measure zero in N.*

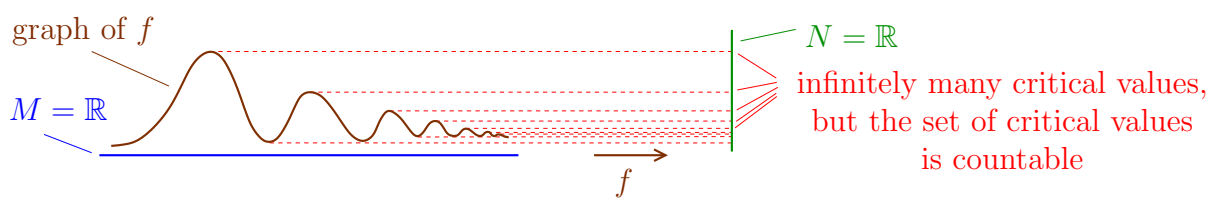


FIGURE 119. Illustration of Sard's Theorem 3.48.

Remark.

(1) As we pointed out on page 209, if $f \colon M \to N$ is a smooth map between manifolds with $\dim(M) < \dim(N)$, then a point $P \in N$ is a regular value if and only if $f^{-1}(P) = \varnothing$. It follows from this observation that Sard's Theorem 3.48 together with Proposition 3.47 (2) implies Proposition 3.47 (4).

[100] In Exercise 3.7 (2) we will show that such a countable atlas exists.

[101] We stress that M and N are allowed to have boundary, since (a) we need these cases, and (b) since many proofs in the literature assume that M and N do not have boundary.

(2) Sard's Theorem 3.48 can also be used to give one more proof of the fundamental theorem of algebra, we refer to [Bre, Corollary II.6.4] or [Mi2, p. 8] for details.

Proof. The theorem is a straightforward consequence of [Le2, Theorem 6.10]. □

3.12. **The classification of 1-dimensional (topological) manifolds** $(*)$**.** One of the goals of topology is to completely classify suitable classes of topological spaces. For example one could try to classify all connected topological manifolds of a given dimension. The following proposition gives a complete classification in the 1-dimensional case.

Proposition 3.49. *Let M be a non-empty connected 1-dimensional topological manifold. Then the homeomorphism type of M is given by the following table*[102]

	without boundary	*with boundary*
compact	*the circle* S^1	*the closed interval* $[0,1]$
non-compact	*the open interval* $(0,1)$	*the half-open interval* $[0,1)$.

Note that the adjectives "Hausdorff" and "second-countable" in the definition of a topological manifold are both necessary:

(1) The line with two zeros satisfies all axioms of a manifold except for the Hausdorff axiom.
(2) If one drops the condition that a topological manifold has to be second-countable, then there is one more example of a connected 1-dimensional topological manifold, namely the "long line", see [Gal2], [Kra, p. 66] and

`http://en.wikipedia.org/wiki/Long_line_(topology)`

Proposition 3.49 is proved in [Gal2]. In the following we only deal with the case that M is closed. Put differently, we prove the following proposition.

Proposition 3.50. *Let M be a non-empty compact connected closed 1-dimensional topological manifold. Then M is homeomorphic to S^1.*

In the proof of Proposition 3.50 we will need the following lemma.

Lemma 3.51. *Let M be a 1-dimensional topological manifold and let $\Phi\colon U \to (-2,2)$ be a chart. Let $[a,b] \subset (-2,-2)$ be a compact interval. Then $\Phi^{-1}([a,b])$ is a closed subset of M.*

Proof. Since Φ is a homeomorphism the map $\Phi^{-1}\colon (-2,2) \to U$ is also continuous. Since $[a,b]$ is compact it follows from Lemma 1.37 that $\Phi^{-1}([a,b])$ is also compact. But by Lemma 1.18 (2) any compact subspace of a Hausdorff space is closed. But M is a topological manifold, in particular M is Hausdorff. Therefore $\Phi^{-1}([a,b])$ is a closed subset of M. □

Definition. Let M be a 1-dimensional topological manifold.

[102] By Exercise 1.14 we know that the half-open interval $[0,1)$ is not homeomorphic to the open interval $(0,1)$.

(1) A *c-parametrization* for M is a homeomorphism $\Psi\colon [-1,1] \to C$ from the compact interval $[-1,1]$ to a closed subset C of M. We say the c-parametrization is *open* if $\Psi((-1,1))$ is an open subset of M.
(2) We say that two c-parametrizations Ψ and Ψ' are *almost disjoint* if
$$\Psi((-1,1)) \cap \Psi'((-1,1)) \ = \varnothing.$$
(3) A *c-atlas* is a family of c-parametrizations $\{\Psi_i\colon [-1,1] \to C_i\}_{i\in I}$ such that the union of the C_i's equals M. We call the c-atlas open if all c-parametrizations in the c-atlas are open. We say the c-atlas *almost disjoint* if any two different c-parametrizations in the c-atlas are almost-disjoint.

Using Lemma 3.51 one can prove the following lemma that is illustrated in Figure 120. We leave the proof as a voluntary exercise.

Lemma 3.52. *Let M be a 1-dimensional topological manifold and let $\Psi_1\colon [-1,1] \to C_1$ and $\Psi_2\colon [-1,1] \to C_2$ be two open c-parametrizations such that $C_1 \cap C_2$ is non-empty but such that C_2 is not contained in C_1. Then*
$$[-1,1] \setminus \Psi_1^{-1}\big(\Psi_2((-1,1)\cap C_1)\big) \ = \ [a,b]$$
for some $-1 \le a < b \le 1$.

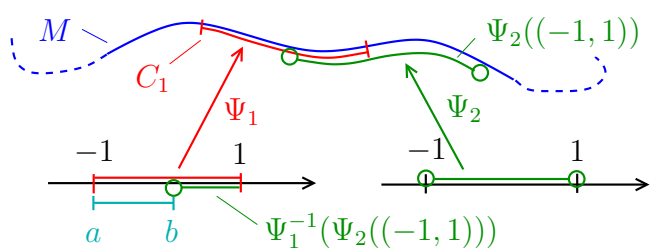


FIGURE 120. Illustration of Lemma 3.52.

Now we can provide the proof of Proposition 3.50.

Proof of Proposition 3.50. Let M be a compact connected closed 1-dimensional topological manifold. We start out with the following claim.

Claim. There exists a finite open almost-disjoint c-atlas $\Psi_i\colon [-1,1] \to C_i$ with $i = 1,\dots,k$.

Since M is assumed to be closed there exists for each $P \in M$ a chart $\Phi_P\colon U_P \to (-2,2)$ of type (i) such that $\Phi_P(P) = 0$. The open sets $\Phi_P^{-1}((-1,1))$ are an open covering of M.[103] Since M is compact there exist $P_1,\dots,P_k$ such that the open sets $\Phi_{P_i}^{-1}((-1,1))$, $i = 1,\dots,k$ are an open covering of M. It follows that the union of the sets $C_i := \Phi_{P_i}^{-1}([-1,1])$,

[103]Here we use implicitly the following fact: if U is open in M, and if V is open in U, then V is also open in M. Why is that?

$i = 1, \dots, k$ is also all of M. It follows from Lemma 3.51 that each C_i is a closed subset of M.

For $i = 1, \dots, k$ we now consider the homeomorphism Ψ_i that is given by

$$\Psi_i := \big(\Phi_{P_i}|_{C_i}\big)^{-1} \colon [-1,1] \ \xrightarrow{\cong}\ C_i.$$

By the above discussion these are open c-parametrizations. After possibly removing some c-parametrizations we can assume that no C_i is contained in any other C_j.

Now we make the following modifications to our c-atlas. If for some $i < j$ we have $\Psi_i((-1,1)) \cap \Psi_j((-1,1)) \neq \varnothing$, then it follows from Lemma 3.52 that

$$[-1,1] \setminus \Psi_i^{-1}\big(\Psi_j((-1,1)) \cap C_i\big) \ = \ [a,b]$$

for some $-1 \leq a < b \leq 1$. Now we replace Ψ_i by the map

$$[-1,1] \ \xrightarrow{\text{linear homeomorphism}}\ [a,b] \ \xrightarrow{\Psi_i}\ M.$$

Repeating this process we obtain finitely many open c-parametrizations that are furthermore almost-disjoint. These c-parametrizations form the desired c-atlas. This concludes the proof of the claim.

We say that a c-atlas is *simple* if given any $P \in M$ there exist at most two c-parametrizations that contain P.

Claim. Any almost-disjoint open c-atlas is simple.

This follows from the fact that any point P on a 1-dimensional topological manifold has the property, that given any open neighborhood U there exists an open neighborhood V of x, contained in U, such that $V \setminus \{P\}$ contains precisely two components. If a point $P \in M$ is contained in k c-parametrizations of an open almost-disjoint c-atlas, then one can easily show that P admits an open neighborhood U such that for any open neighborhood V of P contained in U the set $V \setminus \{P\}$ has at least k components. This concludes the proof of the claim.

Now let $\Psi_i \colon [-1,1] \to C_i$ and $\Psi_j \colon [-1,1] \to C_j$ be two c-parametrizations of our c-atlas such that $C_i \cap C_j$ consists of a single point. Since the c-parametrizations are almost disjoint we have $C_i \cap C_j \subset \Psi_i(\{\pm 1\})$ and $C_i \cap C_j \subset \Psi_j(\{\pm 1\})$. After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ and/or $\Psi_j(t)$ by $\Psi_j(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$. We consider the map

$$\begin{array}{rcl} \Psi \colon [-1,1] & \to & C_1 \cup C_2 \\ x & \mapsto & \begin{cases} \Psi_i(2x+1), & \text{if } x \in [-1,0], \\ \Psi_j(2x-1), & \text{if } x \in [0,1]. \end{cases} \end{array}$$

It is straightforward to verify that Ψ is a bijection and continuous. It follows from Proposition 1.40 that Ψ is in fact a homeomorphism.[104]

[104]Here we use that M is Hausdorff, hence the subset $C_i \cup C_j$ is also Hausdorff.

By iteratively applying the procedure from the previous claim to our c-atlas we obtain a finite almost-disjoint simple c-atlas[105] $\Psi_i\colon C_i \to [-1,1]$, $i = 1,\dots,k$ such that for any $i \neq j$ either $C_i \cap C_j$ is empty or it consists of two points.

Now pick a c-parametrization $\Psi_i\colon [-1,1] \to C_i$ from the c-atlas. Since M is closed and since $[-1,1]$ is not closed we see that Ψ_i is not a homeomorphism,[106] in particular $C_i \neq M$. Since M is connected[107] there exists another c-parametrization $\Psi_j\colon [-1,1] \to C_j$ such that $C_i \cap C_j \neq \varnothing$. By our choice of c-atlas the set $C_i \cap C_j$ consists of two points.

After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$ and $\Psi_i(-1) = \Psi_j(1)$. We consider the map

$$\begin{array}{rcl} \Psi\colon [-1,1]/-1\sim 1 & \to & C_i \cup C_j \\ x & \mapsto & \begin{cases} \Psi_i(2x+1), & \text{if } x \in [-1,0], \\ \Psi_j(2x-1), & \text{if } x \in [0,1]. \end{cases} \end{array}$$

It is straightforward to verify that ψ is a bijection[108] and continuous.[109] It follows from Proposition 1.40 that Ψ is in fact a homeomorphism.

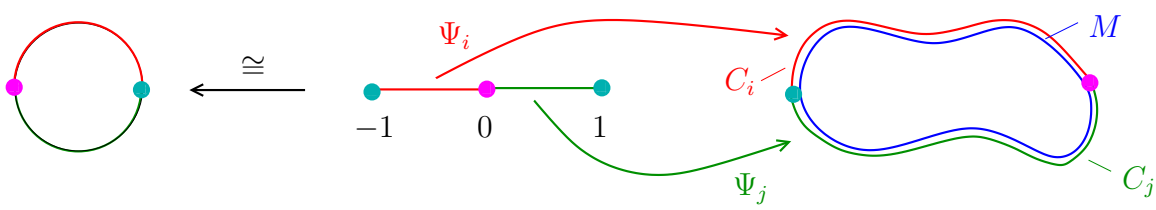


FIGURE 121.

Thus we have found a homeomorphism from $[-1,1]/-1\sim 1 = S^1$ to $C_i \cup C_j$. It remains to prove the following claim.

Claim. We have $C_i \cup C_j = M$.

The c-atlas we work with is simple and almost-disjoint. Since $\Psi_i(\pm 1) = \Psi_j(\mp 1)$ it follows that C_i and C_j did not intersect any other C_k. This shows that $C_i \cup C_j$ and $\bigcup_{k\neq i,j} C_k$ are disjoint. Since both are closed and since M is connected and since $C_i \cup C_j$ is non-empty it follows that $M = C_i \cup C_j$. □

If we consider manifolds instead of topological manifolds and diffeomorphisms instead of homeomorphisms we get the same answer. More precisely the following proposition holds.

[105]Note that we no longer claim that the resulting c-atlas is open.

[106]Careful reading shows that here we make use of the fact that a half-open interval is not homeomorphic to an open interval.

[107]Why does this follow from connectedness?

[108]Here we use that we identify -1 and 1 on the left-hand side.

[109]It is continuous at $[-1] = [1]$ since $\Psi_i(-1) = \Psi_j(1)$.

Proposition 3.53. *Let M be a non-empty connected 1-dimensional manifold. Then the diffeomorphism type of M is given by the following table*

	without boundary	*with boundary*
compact	*the circle* S^1	*the closed interval* $[0,1]$
non-compact	*the open interval* $(0,1)$	*the half-open interval* $[0,1)$.

If M has no boundary, then this proposition is proved in the appendix of [Mi2]. The case of manifolds with boundary can be reduced to the case of manifolds without boundary by first considering the interior of the manifold.

Exercises for Chapter 3.

Exercise 3.1. Show that the line with two zeros admits a chart of type (i) for every point.

Exercise 3.2. Does there exist a compact topological space that satisfies all axioms of a closed topological 1-dimensional manifold except for the Hausdorff axiom?

Exercise 3.3. In Figure 97 we show $M = \{(x,y) \,|\, y \geq 0\}$ together with two subsets K and $N = \{(x,x^2) \,|\, x \geq 0\}$. Show that both K and N are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a manifold.
To make the exercise more explicit you could take K to be the union of the line segments connecting the points $(1,1)$ and $(2,2)$ and connecting the points $(2,2)$ with $(3,1)$.

Exercise 3.4. Show that the action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd.
Hint: Use Proposition 3.32 (2) and Lemma 3.37 (2).

Exercise 3.5.

(a) Let $p\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from Lemma 1.41. It restricts to a diffeomorphism $\varphi\colon S^n \setminus \{(0,\dots,0,1)\} \to \mathbb{R}^n$. We equip both sides with the usual orientation. Show that φ is orientation-preserving if and only if n is odd.

(b) We consider the projection

$$\begin{array}{rcl} \psi\colon S^n_{>0} = \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} > 0\} & \to & B^n \\ (x_1,\dots,x_{n+1}) & \to & (x_1,\dots,x_n). \end{array}$$

We equip both sides with the usual orientation. Show that ψ is orientation-preserving if and only if n is even.

Exercise 3.6. Provide a proof for Lemma 3.28.

Exercise 3.7.

(a) Let M be a topological manifold. Show that M admits a countable atlas.
Hint: Use that topological manifolds are second countable and use Lemma 1.26.

(b) Let M be a manifold. Show that M admits a countable atlas such that each chart lies in the maximal smooth atlas of M.

Exercise 3.8. Let M be a compact topological manifold and let $A \subset M$ be a non-empty closed subset with $A \neq M$. Show that $M \setminus A$ is not compact.
Hint: Use Theorem 3.10 and Exercise 1.28.

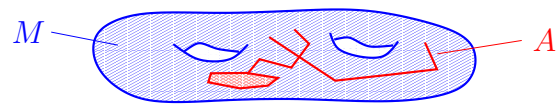


FIGURE 122. Illustration for Exercise 3.8.

Exercise 3.9. Let G be a group that acts freely, properly and smoothly on an n-dimensional oriented manifold M. Suppose that M/G is orientable.

(a) Suppose that M is path-connected. Show that every $g \in G$ acts on M in an orientation-preserving way.
Hint: Use Proposition 3.38 (2).
(b) Show that the conclusion of (a) does not hold if M is not path-connected.

Remark: This exercise can be viewed as a converse to Proposition 3.38 (1).

Exercise 3.10. Let M be a manifold. Is the boundary ∂M a submanifold of M?

Exercise 3.11. Let $M := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \leq 0\}$, $N := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}$ and $H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 = 0\}$. Let $X \subset M$ be a k-dimensional topological submanifold with $\partial X = X \cap H$ and let $Y \subset N$ be a k-dimensional topological submanifold with $\partial Y = Y \cap H$. We suppose that $X \cap H = Y \cap H$. Show that $X \cup Y$ is a k-dimensional topological submanifold of $\mathbb{R}^n$.

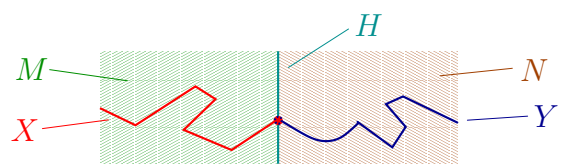


FIGURE 123. Illustration for Exercise 3.11.

Exercise 3.12. Let M be an m-dimensional manifold with possibly non-empty boundary and let N be an n-dimensional manifold with empty boundary. We equip the product manifolds $M \times N$ and $N \times M$ with the product orientations as defined in Proposition 3.4.

(a) Show that the diffeomorphism

$$\begin{array}{rcl} M \times N & \to & N \times M \\ (x, y) & \mapsto & (y, x) \end{array}$$

is orientation-preserving if and only if $m \cdot n$ is even.

(b) We equip ∂M and $N \times M$ with the boundary orientation and we equip $N \times \partial M$ with the product orientation. Show that we have the following equality of *oriented* manifolds:

$$\partial(N \times M) \;=\; \begin{cases} N \times \partial M, & \text{if } n \text{ is even,} \\ -(N \times \partial M), & \text{if } n \text{ is odd.} \end{cases}$$

Exercise 3.13. Let $f\colon M \to N$ be a smooth map between two manifolds.

(a) Show that the set of regular points of f is an open subset of M.

(b) Let $v \in N$ be a regular value such that $f^{-1}(v)$ is finite. We assume that M is compact. Show that there exists an open neighborhood U of v such that each point in U is a regular value.

(c) Show that in general the set of regular values of f is not an open subset of N.

Exercise 3.14. Let $n \in \mathbb{N}$.

(a) Show that

$$\mathrm{GL}(n, \mathbb{R}) \;=\; \{A \in M(n \times n, \mathbb{R}) \mid \det(A) \neq 0\}$$

is a manifold of dimension n^2.

(b) Show that

$$\mathrm{SL}(n, \mathbb{R}) \;=\; \{A \in M(n \times n, \mathbb{R}) \mid \det(A) = 1\}$$

is a manifold of dimension $n^2 - 1$.

Exercise 3.15. Show that

$$\mathrm{SU}(n, \mathbb{R}) \;=\; \left\{A \in M(n \times n, \mathbb{C}) \,\middle|\, \overline{A}^T A = \mathrm{id} \text{ and } \det(A) = 1\right\}$$

is a manifold of dimension $n^2 - 1$.

4. How can we show that two topological spaces are not homeomorphic?

In the previous two sections we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

But these criteria are of no use when it comes to studying more subtle examples. For example they do not suffice to address the following questions:

(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \in \mathbb{Z}_{\geq 0}$ we denote by U the complement of k points in $\mathbb{R}^2$. If $k \neq l$, is it possible that U_k and U_l are homeomorphic?
(3) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?

In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?

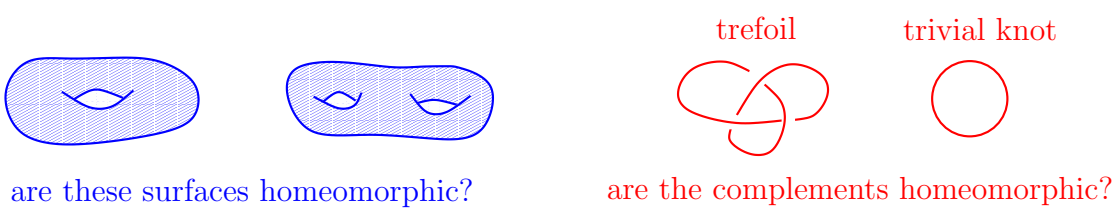


Figure 124.

The reader might have encountered some ideas for tackling these questions in one or both of the following topics:

(a) Differential forms on manifolds, in particular de Rham cohomology, as developed say in [BT, Le2].
(b) Complex analysis, see e.g. [Lan, Jä2].

For the reader who has seen differential forms and de Rham cohomology we recall that each manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $H^k_{dR}(M)$. It

follows more or less from the definition that diffeomorphic manifolds have isomorphic de Rham cohomology groups. This approach has several disadvantages:

(1) the de Rham cohomology groups are only defined for manifolds,
(2) the de Rham cohomology groups can only be used to show that two manifolds are not *diffeomorphic*, a priori they cannot be used to show that two manifolds are not *homeomorphic*,
(3) at least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given manifold.

Later on we will introduce "singular cohomology" which will take care of all of these three issues.

Next we give some definitions that are familiar to the reader who has taken a course on complex analysis.

(1) Given a subset U of $\mathbb{C}$, we defined a *path* in U to be a map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a *loop*.
(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called *path-homotopic in* U, if there exists a map
$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$
with the following properties
   (a) for every $t \in [a,b]$ we have
$$\Gamma(t,0) \;=\; \gamma_0(t) \text{ and } \Gamma(t,1) \;=\; \gamma_1(t),$$
   (b) for every $s \in [0,1]$ we have
$$\Gamma(a,s) \;=\; P \quad \text{ and } \Gamma(b,s) \;=\; Q.$$
   Put differently, a path-homotopy between two paths consists of a "continuous" family of paths $\{\Gamma(-,s)\}_{s\in[0,1]}$ from P to Q which interpolates between the paths γ_0 and γ_1. In Figure 125 we illustrate a path-homotopy between two paths.
(3) A loop $\gamma\colon [a,b] \to U$ is called *null-homotopic* if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.
(4) We say U is *simply connected* if U is non-empty and if each loop is null-homotopic.

Example. Every loop in $\mathbb{R}^2 = \mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with starting and endpoint P. Then
$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s) + P\cdot s \end{array}$$
is a path-homotopy between the loop γ and the constant path at P. The fact that every loop in $\mathbb{C}$ is null-homotopic is illustrated in Figure 126.

Now we recall the following definition from complex analysis.

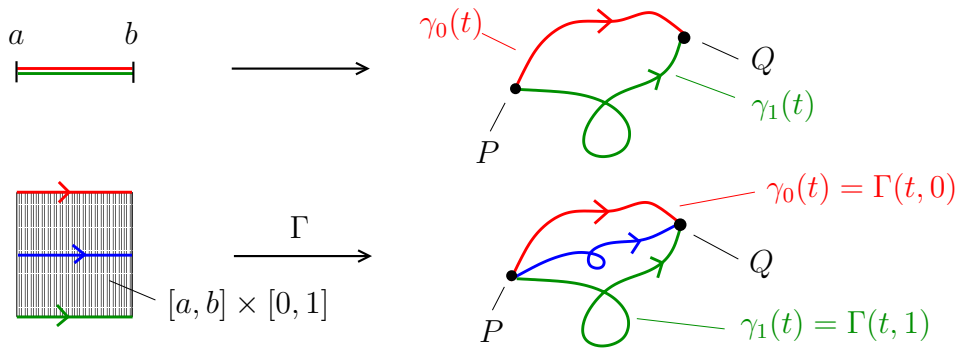


FIGURE 125. Schematic image of a path-homotopy Γ between two paths γ_0 and γ_1 from P to Q.

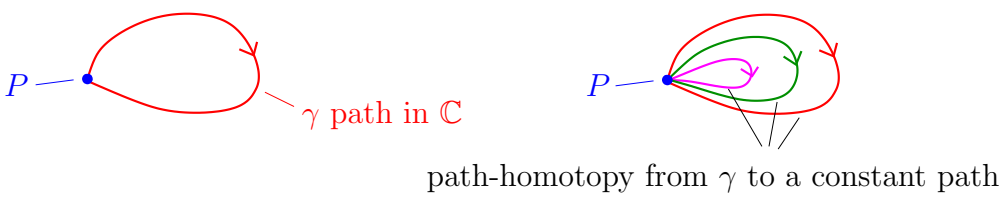


FIGURE 126.

Definition. Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma\colon [a,b] \to U$ be a smooth path. is defined as

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} f(\gamma(t))\cdot\gamma'(t)\,dt \;\in \mathbb{C}.$$

Examples.

(1) Let $U = \mathbb{C}\setminus\{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the loop given by $\gamma(t) = e^{2\pi i t}$, i.e. γ is a loop that "goes once around the origin". Then

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \tfrac{1}{e^{2\pi i t}}\cdot e^{2\pi i t}\cdot 2\pi i\,dt \;=\; 2\pi i.$$

(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

The following proposition is one of the main results in complex analysis, see e.g. [Lan, Theorem III.5.1].

Proposition 4.1. *Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then*

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

The following corollary is a consequence of Proposition 4.1 and of the above calculations.

Corollary 4.2. *The loop $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ given by $\gamma(t) = e^{2\pi i t}$ is* not *path-homotopic to a constant path, i.e. γ is not null-homotopic.*

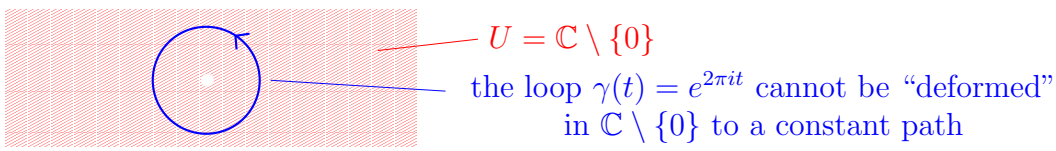


FIGURE 127.

Remark. In the following Chapter 4.2 we will use Corollary 4.2 so that we have at least one non-trivial topological statement in our toolbox. Somewhat later, after somewhat lengthy preparations, we will proof a general theorem, which in particular will provide us with a new proof of Corollary 4.2.

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. From Corollary 4.2 it now follows immediately that $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic.

This second approach to showing that two topological spaces are different also has several disadvantages over the first approach:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$,
(2) this approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definitions make sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

In particular, when it comes to distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$, then naively we would say that V has "more" loops that are not null-homotopic than U. In the subsequent chapter we will in particular make precise what "more" means.

5. The fundamental group

5.1. **Path-homotopy classes of paths.** In this section we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces.

Definition. Let X be a topological space.

(1) A *path* in X is a[110] map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the *starting point of* γ and we call $\gamma(b)$ the *endpoint of* γ. Often we say that γ is a *path from* $\gamma(a)$ *to* $\gamma(b)$.
(2) A *loop* is a path for which the starting point and the endpoint coincide.

Convention. In the following, if we do not specify the domain of a path or a loop, then it is understood to be $[0,1]$.

Definition. Given a path $\gamma\colon [0,1] \to X$ we refer to the path $\overline{\gamma}\colon [0,1] \to X$ that is given by $\overline{\gamma}(t) = \gamma(1-t)$ as the *inverse path of* γ.

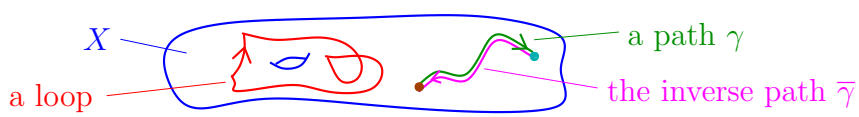


Figure 128.

Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A *path-homotopy between the paths* γ_0 *and* γ_1 is a map
$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$
with the following properties
   (a) for every $t \in [a,b]$ we have
$$\Gamma(t,0) \;=\; \gamma_0(t) \text{ and } \Gamma(t,1) \;=\; \gamma_1(t),$$
   (b) for every $s \in [0,1]$ we have
$$\Gamma(a,s) \;=\; P \quad \text{ and } \Gamma(b,s) \;=\; Q.$$
(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are *path-homotopic*, and we write $\gamma_0 \simeq \gamma_1$.
(3) We say a loop $\gamma\colon [a,b] \to X$ is *null-homotopic*, if it is path-homotopic to the constant path δ given by $\delta(t) = \delta(a)$ for all $t \in [a,b]$.

Examples.

[110] Recall that all maps are understood to be continuous, unless we say something else.

(A) Let $\gamma_0, \gamma_1 \colon [a,b] \to \mathbb{R}^n$ be two paths in $\mathbb{R}^n$ with same starting point and the same endpoint. Then

$$\begin{array}{rcl} \Gamma \colon [a,b] \times [0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma_0(t) \cdot (1-s) + \gamma_1(t) \cdot s \end{array}$$

is a path-homotopy between γ_0 and γ_1. We had already illustrated this fact in Figure 126.

(B) Let $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$. As we had just seen in Corollary 4.2, the path

$$\begin{array}{rcl} \gamma \colon [0,1] & \to & X \\ t & \mapsto & e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t)) \end{array}$$

is not null-homotopic in X.

(C) Let $Y = S^1 \times (0,2)$ and consider the loop

$$\begin{array}{rcl} \widetilde{\gamma} \colon [0,1] & \to & Y = S^1 \times (0,2) \\ t & \mapsto & (e^{2\pi i t}, 1). \end{array}$$

This loop is not null-homotopic. This can be seen as follows: we consider the map

$$\begin{array}{rcl} \Phi \colon Y = S^1 \times (0,2) & \to & X = \mathbb{C} \setminus \{0\} \\ (z,t) & \mapsto & z \cdot t. \end{array}$$

If $\widetilde{\gamma}$ was null-homotopic in Y, then there would exist a path-homotopy

$$\widetilde{F} \colon [0,1] \times [0,1] \to Y$$

from $\widetilde{\gamma}$ to the constant path $\widetilde{\delta}(t) := (1,1)$, $t \in [0,1]$. But then

$$F := \Phi \circ \widetilde{F} \colon [0,1] \times [0,1] \to X$$

would be a path-homotopy in $X = \mathbb{C} \setminus \{0\}$ from the path $\gamma := \Phi \circ \widetilde{\gamma}$ to the constant path $\delta := \Phi \circ \widetilde{\delta}$. But $\gamma = \Phi \circ \widetilde{\gamma}$ is precisely the path γ from example (B) which we showed not to be null-homotopic in $X = \mathbb{C} \setminus \{0\}$.

(D) We consider the two loops in Figure 129. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[111]

(E) We consider the real projective space $\mathbb{R}\mathrm{P}^2$. Recall that in Lemma 1.82 we gave an identification $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$ where $\sim$ is the equivalence relation on $\overline{B}^2$ generated by $P \sim -P$ for $P \in S^1 = \partial \overline{B}^2$. Then

$$\begin{array}{rcl} \gamma \colon [0,1] & \to & \overline{B}^2/\sim \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t))] \end{array}$$

is a loop in $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$, see Figure 130.[112] But is it null-homotopic?

[111] For example the given loop in $\mathbb{C} \setminus \{0,1\}$ is actually null-homotopic in $\mathbb{C} \setminus \{0\}$ and it is null-homotopic in $\mathbb{C} \setminus \{1\}$. In particular our trick of using Corollary 4.2 does not work in this case.

[112] Why is it a loop?

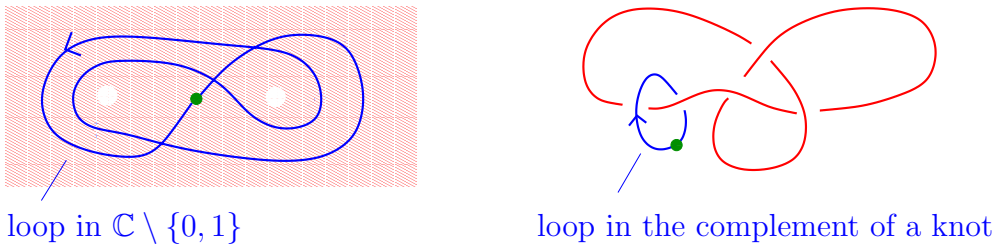


FIGURE 129.

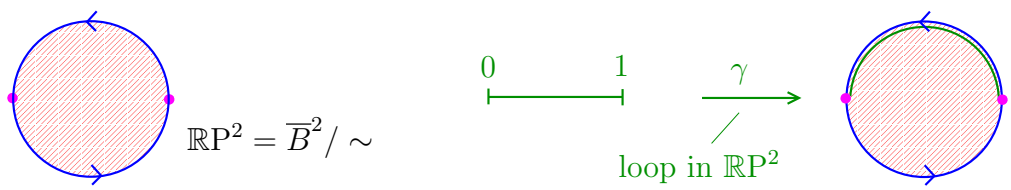


FIGURE 130.

(F) On page 66 we considered the pseudocircle, namely the set $X = \{A, B, C, D\}$ with a slightly unusual topology. Furthermore we gave an example of a continuous non-constant loop $f\colon S^1 \to X$. The question arises, is this loop null-homotopic?

Remark. Let X be a topological space and let $x_0 \in X$. It follows from Lemma 1.68 that the maps

$$\begin{array}{rcl} \{\text{loops } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = \gamma(1) = x_0)\} & \Leftrightarrow & \{\text{maps } f\colon S^1 \to X \text{ with } f(1) = x_0\} \\ (\gamma\colon [0,1] \to X) & \mapsto & (z = e^{2\pi i t} \mapsto \gamma(t)) \\ (t \mapsto f(e^{2\pi i t})) & \leftarrow\!\shortmid & (f\colon S^1 \to X) \end{array}$$

are evidently bijections. Thus we can think of loops in X where the starting and endpoint is x_0 as maps from S^1 to X that send 1 to x_0. We will use this bijection to go back and forth between these two points of view.

The following lemma gives a useful criterion for a loop to be null-homotopic.

Lemma 5.1. *Let X be a topological space and let $x_0 \in X$. Let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We denote by*

$$\begin{array}{rcl} \varphi\colon S^1 & \to & X \\ z = e^{2\pi i t} & \mapsto & \gamma(t) \end{array}$$

the corresponding map from S^1 to X. Then γ is null-homotopic if and only if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$.

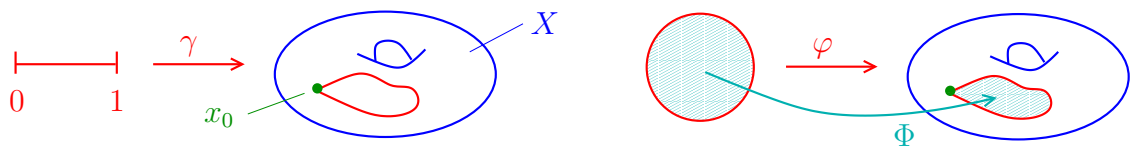


FIGURE 131. Illustration of Lemma 5.1.

Proof $(*)$. We start out with a few preparations. We write $A := ([0,1]\times\{0\})\cup(\{0,1\}\times[0,1])$ and we consider the map

$$\begin{array}{rcl} g\colon [0,1]\times[0,1] & \to & \overline{B}^2 \\ (t,s) & \mapsto & se^{2\pi it}+1-s. \end{array}$$

Note that this map has the property that all points in A get sent to $1 \in S^1$. Using this

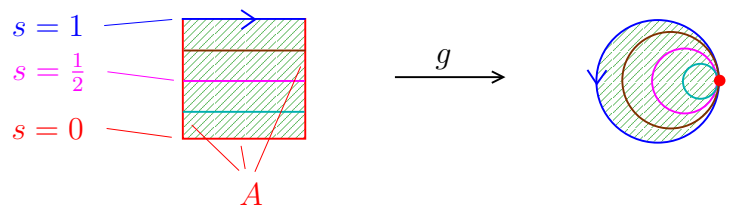


FIGURE 132.

observation, Lemma 1.66 and Proposition 1.40 we see that the map

$$\begin{array}{rcl} h\colon ([0,1]\times[0,1])/A & \to & \overline{B}^2 \\ [P] & \mapsto & g(P) \end{array}$$

is a homeomorphism.

Now we turn to the actual proof of the lemma. Thus let X be a topological space, let $x_0 \in X$ and let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0)=\gamma(1)=x_0$. We define $\varphi\colon S^1 \to X$ via $\varphi(e^{2\pi it}) = \gamma(t)$.

First we suppose that γ is null-homotopic. This means that there exists a path-homotopy $F\colon [0,1]\times[0,1] \to X$ such that $F(t,0)=x_0$ and $F(t,1)=\gamma(t)$ for all $t\in[0,1]$ and such that $F(0,s)=F(1,s)=x_0$ for all $s\in[0,1]$. In other words, F is constant on A. We consider the map[113]

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 \xleftarrow[\cong]{h} ([0,1]\times[0,1])/A & \to & X \\ [P] & \mapsto & F(P). \end{array}$$

One can easily verify that we have $\Phi|_{S^1} = \varphi$.

[113]Note that we implicitly use Lemma 1.66 to argue that the map $([0,1]\times[0,1])/A \to X$ is well-defined and continuous.

Conversely, if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$, then one can easily verify that $\Phi \circ g\colon [0,1] \times [0,1] \to X$ is a path-homotopy from the loop γ to the constant loop c_{x_0}.[114] □

Before we continue with the examples we need to gather a few more elementary facts about path-homotopies of paths.

Proposition 5.2. *Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.*

The proof of Proposition 5.2 makes use of the following lemma which we will use, implicitly or explicitly, surprisingly often.

Lemma 5.3. $(*)$ *Let X be a topological space and let $a < b < c$ be real numbers. Let $f\colon X \times [a,c] \to Y$ be a map to some topological space Y. If the restrictions of f to $X \times [a,b]$ and to $X \times [b,c]$ are continuous, then f itself is continuous.*

Proof of Lemma 5.3. This statement is an immediate consequence of Lemma 1.32 (2) and the observation, see Lemma 1.51, that $X \times [a,b]$ and $X \times [b,c]$ are in fact closed subsets of $X \times [a,c]$. □

Proof of Proposition 5.2. In order to simplify the notation we only work with paths defined on the interval $[0,1]$. We have to show that "path-homotopy" satisfies the three properties of an equivalence relation. Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths from x to y.

(1) The "constant path-homotopy" given by $\Gamma(t,s) := \gamma(t)$ defines a path-homotopy between γ and γ.

(2) Let Γ be a path-homotopy from α to β. Then the map Γ' defined by
$$\Gamma'(t,s) \ := \ \Gamma(t, 1-s)$$
is a path-homotopy from β to α.

(3) Let Γ be a path-homotopy from α to β and let Δ be a path-homotopy from β to γ. We consider the map
$$\begin{array}{rcl} [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \Gamma(t, 2s), & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \Delta(t, 2s-1), & \text{if } s \in \left(\frac{1}{2}, 1\right] \end{cases} \end{array}$$
It follows from Lemma 5.3 that this map is continuous. It is now clear that this map is a path-homotopy from α to γ. This part of the proof is illustrated in Figure 133. □

Definition. Given a path $f\colon [0,1] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the *path-homotopy class of f*.

[114] Why did we not just define g to be the map that sends the horizontal lines to concentric circles where each of them is centered around the origin?

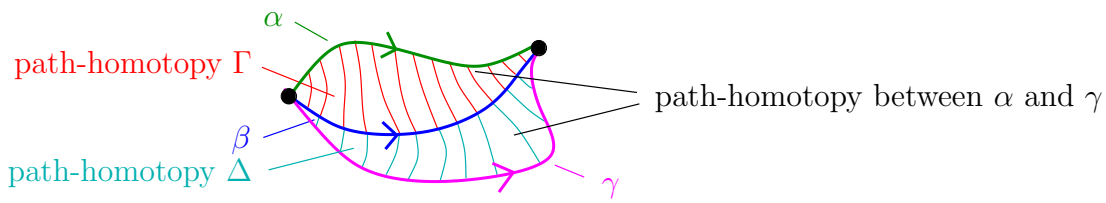


FIGURE 133.

Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map[115] with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi\colon [a,b] \to X$ as a *reparametrization of* γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

Lemma 5.4. *Let $\gamma\colon [a,b] \to X$ be a path in a topological space. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.*

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map with $\varphi(a) = a$, $\varphi(b) = b$. Then a path-homotopy between γ and $\gamma \circ \varphi$ is given by[116]

$$\begin{array}{rcl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$

□

It is pictorially clear that we can concatenate two paths if the endpoint of the first path agrees with the starting point of the second path. We use this observation to define the product of two paths.

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the *product of* α *and* β as the path $\alpha * \beta$ which is given by[117]

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

The product of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.

The proof of the following lemma is elementary and is left as an exercise. The statement of the lemma is illustrated in Figure 135.

[115] Note that φ is not required to be a homeomorphism.

[116] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?

[117] It follows from Lemma 1.32 (2) and our hypothesis $\alpha(b) = \beta(c)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.

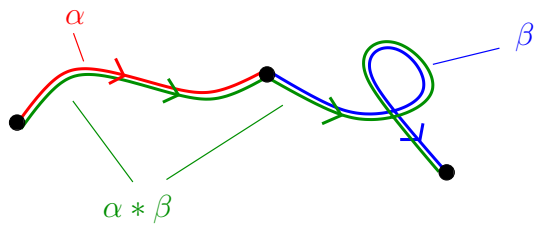
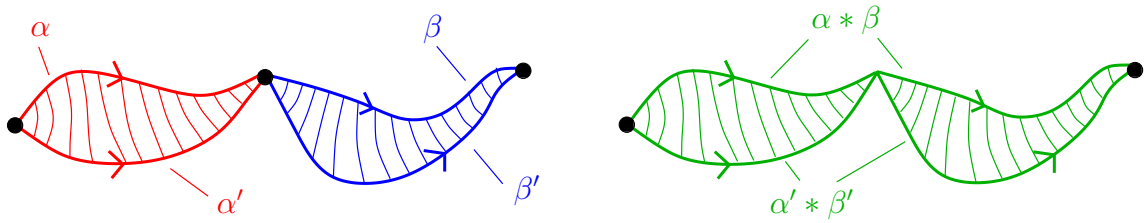


FIGURE 134. Definition of the product of two paths.

Lemma 5.5. *Let $\alpha, \alpha' \colon [a,b] \to X$ and $\beta, \beta' \colon [c,d] \to X$ be two pairs of paths in a topological space X such that $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then*

$$\alpha \simeq \alpha' \text{ and } \beta \simeq \beta' \quad \Longrightarrow \quad \alpha * \beta \simeq \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.



α is path-homotopic to α'
and β is path-homotopic to β'

then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$

FIGURE 135. Illustration of Lemma 5.5.

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the[118] *product of the equivalence classes* $[\alpha]$ *and* $[\beta]$ as

$$[\alpha] \cdot [\beta] \;=\; [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 5.6. *Let $\alpha, \beta, \gamma \colon [0,1] \to X$ be three paths in a topological space X. Then the following hold:*

(1) *If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then*[119]

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \;=\; ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

[118]According to Lemma 5.5 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

[119]Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

(2) *For $x \in X$ we denote by e_x the constant path that is given by $e_x(t) := x$, $t \in [0,1]$. Then the following equality holds*

$$[e_{\alpha(0)}] \cdot [\alpha] \;=\; [\alpha] \;=\; [\alpha] \cdot [e_{\alpha(1)}].$$

(3) *We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold*

$$[\alpha] \cdot [\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \textit{and} \quad [\overline{\alpha}] \cdot [\alpha] \;=\; [e_{\alpha(1)}].$$

Proof $(*)$. Let $\alpha, \beta, \gamma \colon [0,1] \to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We need to show that

$$\alpha * (\beta * \gamma) \;\simeq\; (\alpha * \beta) * \gamma.$$

We consider the map

$$\begin{array}{rcl} \Phi \colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(3t), & \text{if } t \in [0, \frac{1}{3}], \\ \beta(3t-1), & \text{if } t \in (\frac{1}{3}, \frac{2}{3}], \\ \gamma(3t-2), & \text{if } t \in (\frac{2}{3}, 1]. \end{cases} \end{array}$$

It follows from our hypothesis and Lemma 1.32 (2) that this map is indeed continuous. We consider the maps

$$\begin{array}{rcl} p \colon [0,1] & \to & [0,1] \\ t & \mapsto & \begin{cases} \frac{2}{3}t, & \text{if } t \in [0, \frac{1}{2}], \\ \frac{4}{3}t - \frac{1}{3}, & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} q \colon [0,1] & \to & [0,1] \\ t & \mapsto & \begin{cases} \frac{4}{3}t, & \text{if } t \in [0, \frac{1}{2}], \\ \frac{2}{3}t + \frac{1}{3}, & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array}$$

Then

$$\begin{array}{ccccccc} \alpha * (\beta * \gamma) & = & \Phi \circ p & \simeq & \Phi & \simeq & \Phi \circ q & = & (\alpha * \beta) * \gamma \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{by definition} & & \text{by Lemma 5.4} & & & & \text{by definition} & \end{array}$$

(2) We consider the map

$$\begin{array}{rcl} q \colon [0,1] & \to & [0,1] \\ t & \mapsto & \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}], \\ 2t - 1, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases} \end{array}$$

Then

$$\begin{array}{ccccc} e_{\alpha(0)} * \alpha & = & \alpha \circ q & \simeq & \alpha. \\ & \uparrow & & \uparrow & \\ & \text{by definition} & & \text{by Lemma 5.4} & \end{array}$$

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{array}{rcl} F \colon [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \alpha(2t \cdot s), & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \alpha(2s - 2t \cdot s), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

It follows from Lemma 5.3 that this map is indeed continuous. For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \overline{\alpha}$. For any $s \in [0, 1]$ we furthermore have $F(0, s) = F(1, s) = P$. Thus F is a path-homotopy from the constant path to $\alpha * \overline{\alpha}$ which implies that $[e_{\alpha(0)}] = [\alpha * \overline{\alpha}] = [\alpha] \cdot [\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha} * \alpha] = [e_{\alpha(1)}]$. □

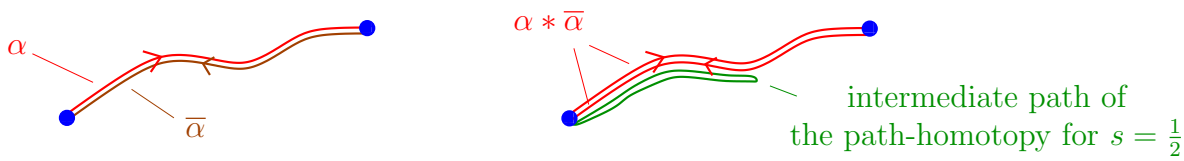


FIGURE 136. Illustration of the proof of Proposition 5.6 (3).

5.2. **The fundamental group of a pointed topological space.** In Proposition 5.6 we saw that the product of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes of paths do not form a group, since the product of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops.

Definition. Let X be a topological space and let $x_0 \in X$ be a point. A *loop in* (X, x_0) is a path $f\colon [0, 1] \to X$[120] with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f.

Proposition 5.7. *Let X be a topological space and $x_0 \in X$. The set*

$$\pi_1(X, x_0) \ := \ \{\textit{path-homotopy classes of loops in } (X, x_0)\}$$

together with the product map $[\alpha] \cdot [\beta] := [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

Proof. It is clear that the product of any two loops in (X, x_0) is defined. According to Lemma 5.5 the product of paths descends to a well-defined map

$$\begin{array}{rcl} \pi_1(X, x_0) \times \pi_1(X, x_0) & \to & \pi_1(X, x_0) \\ ([\alpha], [\beta]) & \mapsto & [\alpha] \cdot [\beta] := [\alpha * \beta]. \end{array}$$

According to Proposition 5.6 (1) this product map satisfies the associativity axiom. The trivial element is according to Proposition 5.6 (2) given by the path-homotopy class of the constant loop $[e_{x_0}]$. Furthermore according to Proposition 5.6 (3) the inverse of the path-homotopy class of a loop $\gamma\colon [0, 1] \to X$ in x_0 is given by the path-homotopy class of the loop defined by $\overline{\gamma}(t) := \gamma(1 - t)$, $t \in [0, 1]$. □

Definition. We call $\pi_1(X, x_0)$ the *fundamental group of X with respect to the base point x_0*.

[120]Note that part of the definition of "loop in (X, x_0)" is that the domain is the interval $[0, 1]$.

One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier sections.

Examples.

(1) We consider $X = \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be an arbitrary point. As we pointed out on page 227, all loops in (X, x_0) are null-homotopic. More precisely, if $\gamma\colon [0,1] \to \mathbb{R}^n$ is a loop in $(\mathbb{R}^n, x_0)$, then

$$\begin{array}{rcl} F\colon [0,1] \times [0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t) \cdot (1-s) + x_0 \cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X, x_0) = 0$.[121]

(2) We say that a subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists an $x \in X$ such that for all $y \in X$ the segment $\{xt + y(1-t) \,|\, t \in [0,1]\}$ lies in X. For example open and closed balls and open and closed rectangles are star-shaped. Furthermore all convex subsets are star-shaped. Precisely the same proof as in (1) shows that the $\pi_1(X, x) = 0$.

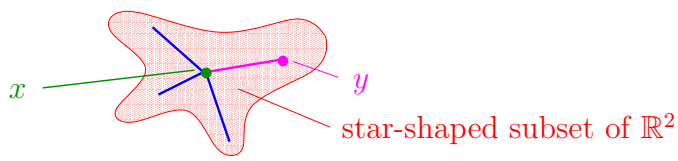


FIGURE 137.

Next we state the following theorem from complex analysis.

Theorem 5.8. *Let X be a proper subset*[122] *of $\mathbb{C}$, let $x_0 \in X$ be a point and let $w \in \mathbb{C} \setminus X$ be a point in the complement of X. The map*

$$\begin{array}{rcl} \pi_1(X, x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{z-w}\, dz \end{array}$$

is a well-defined homomorphism.

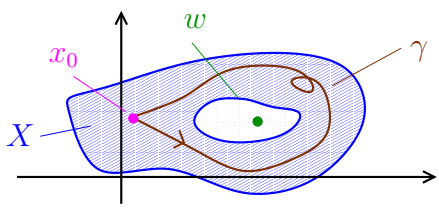
Remark. As in the case of Corollary 4.2, we do not really need Theorem 5.8 and its subsequent Corollary 5.9. If one has seen complex analysis, or if one is willing to take Theorem 5.8 on face value, it allows us to show that some fundamental groups are non-trivial, before we have developed the full machinery for calculating fundamental groups. But eventually we

[121] If $\pi_1(X, x_0)$ is the trivial group, then we usually write $\pi_1(X, x_0) = 0$. This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X, x_0)$, and it would therefore make more sense to write $\pi_1(X, x_0) = 1$ if the group $\pi_1(X, x_0)$ is trivial.

[122] We say A is a proper subset of B if $A \subset B$ and if $A \neq B$.

will supplant Corollary 5.9 by a much stronger statement proved using purely topological methods.

Proof. A vigilant reader might first have noticed that the statement of Theorem 5.8 contains a slight cheat. We use the path integral over a continuous path γ, whereas so far on page 224 we gave only the definition for the integral over a smooth path. The more general definition of a path integral is given in [Lan, Chapter III.4]. The fact that the given integral takes values in $\mathbb{Z}$ is proved in [Lan, Lemma IV.4.1]. Furthermore the fact that this integral defines a well-defined map on the fundamental group is a consequence of [Lan, Theorem III.5.1]. Finally it follows almost immediately from the definitions that the given map is a homomorphism. □



the map

$$\begin{array}{rcl} \pi_1(X,x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi i}\int\limits_\gamma \frac{1}{z-w}\,dz \end{array}$$

is a homomorphism

FIGURE 138.

The previous example together with Theorem 5.8 give us the following neat corollary.

Corollary 5.9. *There exists an epimorphism $\pi_1(S^1,1)\to\mathbb{Z}$.*

Proof. The corollary follows immediately from Theorem 5.8 and the calculation on page 224. □

This raises the following question.

Question 5.10. *Is the above epimorphism $\pi_1(S^1,1)\to\mathbb{Z}$ in fact an isomorphism?*

The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 5.11. *Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon[0,1]\to X$ be a path from x_0 to x_1. Then the following two statements hold:*

(1) *The map*

$$\begin{array}{rcl} p_*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}] \end{array}$$

is well-defined and it is a group isomorphism.

(2) *If $x_0=x_1$, then the map p_* is just conjugation by $[p]\in pi_1(X,x_0)$, i.e. for any $g\in\pi_1(X,x_0)=\pi_1(X,x_1)$ we have*

$$p_*(g) \;=\; [p]\cdot g\cdot[p]^{-1} \;\in\; \pi_1(X,x_0).$$

(3) *If $f\colon X \to Y$ is a map to another topological space, then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "p_*"] \arrow[d, "f_*"] & \pi_1(X,x_1) \arrow[d, "f_*"] \\ \pi_1(Y,f(x_0)) \arrow[r, "(f\circ p)_*"] & \pi_1(Y,f(x_1)). \end{tikzcd}$$

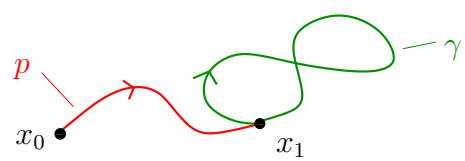


FIGURE 139. Schematic sketch of Proposition 5.11 (1).

Proof $(*)$. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1] \to X$ from x_0 to x_1.

(1) Let γ be a loop in (X,x_1). Then $p*\gamma*\overline{p}$ is evidently a loop in (X,x_0). From Lemma 5.5 and Proposition 5.6 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}] \end{array}$$

is well-defined. Furthermore for two loops γ,δ in (X,x_1) we have

$$\begin{array}{rclcl} \Phi([\gamma])*\Phi([\delta]) & = & [p*\gamma*\overline{p}]*[p*\delta*\overline{p}] & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & = & [p*\gamma*e_{x_1}*\delta*\overline{p}] & = & [p*(\gamma*\delta)*\overline{p}] \;=\; \Phi([\gamma*\delta]). \\ & \uparrow & & \uparrow & \\ & \text{Proposition 5.6 (3) and Lemma 5.5} & & \text{Proposition 5.6 (2) and Lemma 5.5} & \end{array}$$

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by[123]

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [\delta] & \mapsto & [\overline{p}*\delta*p]. \end{array}$$

(2),(3) These two statements follow basically immediately from the definitions. □

We obtain the following corollary:

Corollary 5.12. *Let X be a path-connected topological space and let $x_0,x_1 \in X$. Then the groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic.*

[123]Why is $\Phi\circ\Psi$ the identity on $\pi_1(X,x_0)$ and why is $\Psi\circ\Phi$ the identity on $\pi_1(X,x_1)$?

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x \in X$ such that $\pi_1(X, x) = 0$. It now follows from Corollary 5.12 that $\pi_1(X, y) = 0$ for any point $y \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial.

If X is a path-connected topological space, then Corollary 5.12 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. Sometimes we suppress the base point from the notation and we denote by $\pi_1(X)$ the isomorphism type of the fundamental group.

Definition. Let X be a non-empty path-connected topological space. If $\pi_1(X) = 0$[124], then we say that X is *simply connected.*[125]

The following lemma will be proved in Exercise 5.2.

Lemma 5.13. *Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths from P to Q are path-homotopic.*

We already saw on page 235 that $\mathbb{R}^n$ and more generally, all star-shaped subsets of $\mathbb{R}^n$ are simply connected. The following statement is much more interesting:

Proposition 5.14. *For any $n \geq 2$ the sphere S^n is simply connected.*

In the proof of Proposition 5.14 we will need the following lemma.

Lemma 5.15. *For any $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.*

Proof. In Lemma 1.41 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We set $Q := f^{-1}(\infty)$. Thus f restricts to homeomorphism $f\colon S^n \setminus \{Q\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists a homeomorphism[126] $g\colon S^n \to S^n$ with $g(P) = Q$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. But $\mathbb{R}^n$ is simply connected, hence $S^n \setminus \{P\}$ is also simply connected. □

Now we can turn to the proof of Proposition 5.14.

Proof of Proposition 5.14.

> Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By Lemma 5.15 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.
> This sounds convincing, except for one crucial gap: we did not justify why there

[124]Since X is a path-connected topological space it is irrelevant which base point we consider.

[125]It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as on page 223.

[126]The existence of the homeomorphism follows from Proposition 10.23. But one can also easily give an elementary proof of that fact. More precisely, using basic facts from linear algebra one can show that there exists an orthogonal matrix A with $A \cdot P = Q$. Then the map $g\colon S^n \to S^n$ that is given by multiplication by A has the desired property.

exists a point P that does not lie in the image of γ. In fact, as we saw in Proposition 1.91, such a point does not need to exist since there are surjective maps from an interval to any S^n.

We write

$$N \;:=\; (0,\dots,0,1) \quad \text{“North Pole”} \quad \text{and} \quad U \;:=\; S^n \setminus \{N\},$$

and also

$$S \;:=\; (0,\dots,0,-1) \quad \text{“South Pole”} \quad \text{and} \quad V \;:=\; S^n \setminus \{S\}.$$

We fix a base point x_0 on S^n that is neither N nor S. By Lemma 5.15 it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in $U = S^n \setminus \{N\}$.

So let γ be a loop in (S^n, x_0).

Claim. There exists a subdivision

$$0 \,=\, t_0 \,<\, t_1 \,<\, \dots \,<\, t_{k-1} \,<\, t_k \,=\, 1,$$

such that the following hold:

(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \neq N$ and $\gamma(t_i) \neq S$.

By Corollary 1.106 applied to $\gamma\colon [0,1] \to S^n$ and $S^n = U \cup V$ there exists a k such that for any $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) = N$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ can be contained in $U = S^n \setminus \{N\}$. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. Iterating this process we can assume that for each every i we have $\gamma(t_i) \neq N$. Similarly we can arrange for every i we have $\gamma(t_i) \neq S$. This concludes the proof of the claim.

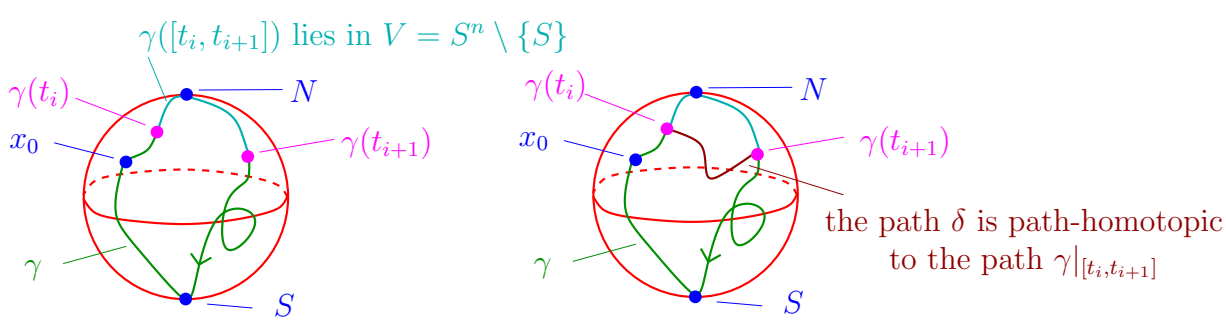


FIGURE 140. Illustration of Proposition 5.14.

As mentioned before, by Lemma 5.15 it now suffices to prove the following claim.

Claim. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path that does not hit N.

We pick a subdivision as in the previous claim. Suppose there exists an $i \in \{0, \dots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ hits N. By the choice of our subdivision we know that $\gamma([t_i, t_{i+1}])$ lies entirely in $V = S^n \setminus \{S\}$.[127] Since $S^n \setminus \{N, S\}$ is path-connected and since $\gamma(t_i) \neq N, S$, $\gamma(t_{i+1}) \neq N, S$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V = S^n \setminus \{N, S\}$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i, t_{i+1}]}$ and δ both lie in the simply connected set $V = S^n \setminus \{S\}$ they are path-homotopic by Lemma 5.13. In the loop γ we now replace $\gamma([t_i, t_{i+1}])$ by δ. We obtain a new loop that is path-homotopic to the original loop and that does not hit N on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that does not hit N. This concludes the proof of the claim.[128] $\square$

Exercises for Chapter 5.

Exercise 5.1. Consider the torus $T = S^1 \times S^1$ and consider the loop

$$\begin{array}{rcl} [0,1]/0 \sim 1 & \to & T = S^1 \times S^1 \\ t & \mapsto & (e^{2\pi i t}, 1). \end{array}$$

Is the loop null-homotopic? If yes, provide a path-homotopy to a constant path. If not, prove that the path is not null-homotopic.

Exercise 5.2. Prove Lemma 5.13.

Exercise 5.3.

(1) We consider the topological space

$$X \ := \ \big\{(0,y) \,\big|\, y \in [-1,1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x})) \,\big|\, x \in (0,\pi]\big\} \ \subset \ \mathbb{R}^2.$$

We let $x_0 = (0,0)$. Is $\pi_1(X, x_0)$ trivial or non-trivial?

(2) Let $f\colon X \to Y$ be a surjective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ necessarily an epimorphism?

(3) Let $f\colon X \to Y$ be an injective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ necessarily a monomorphism?

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[127] We know it lies in U or V, but since it hits N it cannot lie in U.

[128] Where did we use in the proof that $n \geq 2$?

6. Categories, functors and natural transformations

In this section we introduce the notion of a category,of a functor and of a natural transformation that many readers will already have encountered in different courses. These notions might initially appear to rather abstract, but they play an essential role in algebraic topology and they are regularly used in all branches of pure mathematics.

6.1. **Definition and examples of categories.**

Definition. A *category* $\mathcal{C}$ consists of the following data:

(1) A class[129] $\mathrm{Ob}(\mathcal{C})$ of mathematical objects which are called the *objects of the category*,
(2) for each pair (X, Y) of objects there exists a set[130] of *morphisms* $\mathrm{Mor}_{\mathcal{C}}(X, Y)$,
(3) for any three objects X, Y and Z there exists a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

that satisfies the following axioms:

(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W, X), g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W, Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ with the property that

$$\begin{array}{rcl} \mathrm{id}_X \circ f & = & f \quad \text{for all } f \in \mathrm{Mor}_{\mathcal{C}}(Z,X), \quad \text{and} \\ f \circ \mathrm{id}_X & = & f \quad \text{for all } f \in \mathrm{Mor}_{\mathcal{C}}(X,Y). \end{array}$$

If the category is clear from the context, then we just write $\mathrm{Mor}(X, Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X, Y)$.

Example.

(a) We call the category $\mathcal{C}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & := & \text{all sets}, \\ \mathrm{Mor}(X,Y) & := & \text{all maps from } X \text{ to } Y, \end{array}$$

together with the usual composition of maps the *category of all sets*.

[129] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to [Ci, FPr] for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.

[130] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.

(b) We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{all sets}, \\ \mathrm{Mor}(X,Y) &:= \begin{cases} \{\mathrm{id}_X\}, & \text{if } X = Y \\ \varnothing, & \text{if } X \neq Y. \end{cases} \end{aligned}$$

The composition map only needs to be defined if $X = Y = Z$[131], and in this case we define $\mathrm{id}_X \circ \mathrm{id}_X := \mathrm{id}_X$. One can easily convince oneself that this defines a category.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{all } \mathbb{K}\text{-vector spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the *category of* $\mathbb{K}$*-vector spaces*.

(d) We refer to the category $\mathcal{G}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{G}) &:= \text{all groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the *category of groups*.

(d) We refer to the category $\mathcal{A}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{A}) &:= \text{all abelian groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the *category of abelian groups*.

(f) We refer to the category $\mathcal{T}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{T}) &:= \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathcal{C}(X,Y) := \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the *category of topological spaces*. A *pointed topological space* is a pair (X, x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{P}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{P}) &:= \text{all pointed topological spaces}, \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the *category of pointed topological spaces*.

(g) We refer to the category $\mathcal{TG}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{TG}) &:= \text{all topological groups}, \\ \mathrm{Mor}(X,Y) &:= \text{all homomorphisms from } X \text{ to } Y \text{ that are continuous}, \end{aligned}$$

with the usual composition of maps as the *category of topological groups*.

[131] If $X \neq Y$ then $\mathrm{Mor}(X,Y) = \varnothing$, but then $\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z)$ is also the empty set for any Z.

(h) We refer to the category $\mathcal{M}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{M}) &:= \text{all manifolds,} \\ \mathrm{Mor}(X,Y) &:= \mathcal{C}(X,Y) := \text{all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the *category of manifolds*. We define the category of *pointed manifolds* the same way as we defined the category of *pointed topological spaces*.

(i) A *pair of topological spaces* is a pair (X, A), where X is a topological space and A is a subset of X. We refer to the category $\mathcal{Q}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Q}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the *category of pairs of topological spaces*. Sometimes it is helpful to exclude the case that $A = \varnothing$. We refer to a pair (X, A) of topological spaces with $A \neq \varnothing$ as a *non-trivial* pair of topological spaces. The same was a s above we obtain the *category $\mathcal{Q}'$ of non-trivial pairs of topological spaces*

There exist also more "exotic" examples of categories.

(j) Let G be an arbitrary group. We consider

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \{\text{a set with a unique element } *\}, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := gf \end{aligned}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be easily deduced from the group axioms of G.

The name "morphisms" suggests that morphisms are necessarily maps. But as we will later see, this is not necessarily always the case.

Convention. If the category we are working with at a given moment is understood, then we just write map instead of morphisms. For example we might write "let $f\colon X \to Y$ be a map", if X and Y are groups, then it is understood that the map is in fact a homomorphism, whereas if X and Y are topological spaces, then it is understood that f is continuous. This convention thus generalizes the convention that we introduced on page 75.

6.2. **Functors.**

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$F\colon \mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X),F(Y)),$$

such that following axioms are satisfied

(F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$,
(F2) for $\phi \in \mathrm{Mor}(X,Y)$ and $\psi \in \mathrm{Mor}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) \;=\; F(\psi) \circ F(\phi) \;\in \mathrm{Mor}(F(X), F(Z)).$$

Convention. For a covariant functor F and $\phi \in \mathrm{Mor}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The second axiom then becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Now we give some examples of functors.

Examples.

(1) Let $\mathcal{V}$ be the category of real vector spaces and let W be a real vector space. Then the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}) & \to & \mathrm{Ob}(\mathcal{V}) \\ V & \mapsto & \mathrm{Hom}_{\mathbb{R}}(W,V), \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(U,V) & \mapsto & \mathrm{Mor}(F(U),F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}(W,U) & \to & \mathrm{Hom}(W,V) \\ f & \mapsto & \phi \circ f \end{pmatrix} \end{array}$$

defines a covariant functor.

(2) Let $\mathcal{P}$ be the category of pointed manifolds and let $\mathcal{V}$ be the category of real vector spaces. By Proposition 3.30 (2) we know that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{P}) & \to & \mathrm{Ob}(\mathcal{V}) \\ (M,P) & \mapsto & T_P M, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}((M,P),(N,Q)) & \mapsto & \mathrm{Mor}(T_P M, T_Q N) \\ f & \mapsto & f_* = Df_P \end{array}$$

is a covariant functor from the category of pointed manifolds to the category of real vector spaces.

(3) Let $\mathcal{X}$ be the category of topological spaces and let $X \in \mathrm{Ob}(\mathcal{X})$. It follows easily from Lemma 1.53 (3c) that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{X}) & \to & \mathrm{Ob}(\mathcal{X}) \\ Y & \mapsto & X \times Y, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(Y,Z) & \mapsto & \mathrm{Mor}(X \times Y, X \times Z) \\ f & \mapsto & \mathrm{id}_X \times f \end{array}$$

is a covariant functor from the category of topological spaces to itself.

(4) Let $\mathcal{X}$ be the category of topological spaces and let $\mathcal{C}$ be the category of all sets. It follows easily from Lemma 1.101 that the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{X}) & \to & \operatorname{Ob}(\mathcal{C}) \\ X & \mapsto & \pi_0(X), \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}(Y,Z) & \mapsto & \operatorname{Mor}(X\times Y, X\times Z) \\ C & \mapsto & f(C) \end{array}$$

is a covariant functor from the category of topological spaces to the category of sets.

We also have the following definition which looks almost the same as the previous definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *contravariant functor* $F\colon \mathcal{C}\to\mathcal{D}$ consists of a map

$$F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$\operatorname{Mor}(X,Y) \to \operatorname{Mor}(F(Y),F(X))$$

such that following axioms are satisfied:

(F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \operatorname{Ob}(\mathcal{C})$,
(F2) for any $\phi \in \operatorname{Mor}(X,Y)$ and $\psi \in \operatorname{Mor}(Y,Z)$ we have

$$F(\psi\circ\phi) \;=\; F(\phi)\circ F(\psi) \;\in \operatorname{Mor}(F(Z),F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of maps" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom are swapped.

Convention. For a contravariant functor F and $\phi \in \operatorname{Mor}(X,Y)$ we often write ϕ^* instead of $F(\phi)$. The second axiom then becomes $(\psi\circ\phi)^* = \phi^*\circ\psi^*$.

Examples.

(1) Let $\mathcal{V}$ be the category of real vector spaces. Then the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{V}) & \to & \operatorname{Ob}(\mathcal{V}) \\ V & \mapsto & V^* := \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}(U,V) & \mapsto & \operatorname{Mor}(V^*,U^*) \\ (\phi\colon U\to V) & \mapsto & \left(\begin{array}{rcl} \operatorname{Hom}(V,\mathbb{R}) & \to & \operatorname{Hom}(U,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{array}\right) \end{array}$$

is a contravariant functor.

(2) Let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{V}$ be the category of real vector spaces. Then the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{T}) & \to & \operatorname{Ob}(\mathcal{V}) \\ X & \mapsto & C(X,\mathbb{R}) := \text{set of continuous functions } X \to \mathbb{R}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}(X,Y) & \mapsto & \operatorname{Mor}(C(Y,\mathbb{R}), C(X,\mathbb{R})) \\ (\phi\colon X \to Y) & \mapsto & \left(\begin{array}{rcl} C(Y,\mathbb{R}) & \to & C(X,\mathbb{R}) \\ f & \mapsto & f \circ \phi \end{array} \right) \end{array}$$

is a contravariant functor.

(3) Let $\mathcal{M}$ be the category of manifolds, let $\mathcal{V}$ be the category of real vector spaces and let $k \in \mathbb{Z}_{\geq 0}$. In [BT, Chapter 2] or [Le2, Corollary 17.3] it is shown that the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{M}) & \to & \operatorname{Ob}(\mathcal{V}) \\ M & \mapsto & H^k_{\mathrm{dR}}(M) := k\text{-th de Rham cohomology group of } M, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}(M,N) & \mapsto & \operatorname{Mor}(H^k_{\mathrm{dR}}(N), H^k_{\mathrm{dR}}(M)) \\ (f\colon M \to N) & \mapsto & \left(\begin{array}{rcl} f^*\colon H^k_{\mathrm{dR}}(N) & \to & H^k_{\mathrm{dR}}(M) \\ \omega & \mapsto & f^*\omega \end{array} \right) \end{array}$$

is a contravariant functor.

We will not see any other examples of contravariant functors in the near future. We will come back to them when we study cohomology groups.

6.3. **The fundamental group as functor.** Now, after recalling the definition of categories and functors, we turn back to the study of fundamental groups.

Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It is straightforward to see that

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class.[132] Furthermore it follows easily from the definitions[133] that f_* is a group homomorphism. We call f_* the *induced map*. It also follows easily from the definitions that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X,x_0), \\ (g \circ f)_* & = & g_* \circ f_*, & \text{for all maps } f\colon (X,x_0) \to (Y,y_0) \text{ and } g\colon (Y,y_0) \to (Z,z_0). \end{array}$$

[132]Indeed, suppose that γ and δ are two loops in (X,x_0) that are path-homotopic. Let F be a path-homotopy between γ and δ. Then $f \circ F$ is a path-homotopy between $f \circ \gamma$ and $f \circ \delta$. We had used almost the same argument on page 227.

[133]If γ and δ are loops in x_0, then it follows immediately from the definitions that $(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta)$.

Summarizing we just convinced ourselves of the validity of the following proposition.

Proposition 6.1. *Let $\mathcal{P}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. Then the map*

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{P}) & \to & \mathrm{Ob}(\mathcal{G}) \\ (X, x_0) & \mapsto & \pi_1(X, x_0) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}((X, x_0), (Y, y_0)) & \to & \mathrm{Mor}(\pi_1(X, x_0), \pi_1(Y, y_0)) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X, x_0) & \to & \pi_1(Y, y_0) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array} \right) \end{array}$$

is a covariant functor.

Next we will see that the functoriality of the fundamental groups can be helpful for studying certain questions.

Definition. Let X be a topological space and let $A \subset X$ be a subspace. We say A is a *retract of* X if there exists a retraction $r\colon X \to A$, i.e. a map[134] with $r(a) = a$ for all $a \in A$.

Examples.

(1) Let X be a topological space. Then any point $x_0 \in X$ is a retract of X, indeed, the map $r(x) = x_0$ for $x \in X$ is a retraction.
(2) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z, w) = (z, 1)$.
(3) The circle S^1 is a retract of $\overline{B}^2 \setminus \{(0, 0)\}$, indeed, the map

$$\begin{array}{rcl} \overline{B}^2 \setminus \{(0,0)\} & \to & S^1 \\ z & \mapsto & \frac{z}{|z|} \end{array}$$

is a retraction. (Here we consider $\overline{B}^2$ as subset of $\mathbb{C}$.)
(4) The set $S^0 = \{-1, 1\}$ is not a retract of $\overline{B}^1 = [-1, 1]$. Indeed, since $[-1, 1]$ is connected and since S^0 is discrete it follows from Lemma 1.92 that any map from $[-1, 1]$ to S^0 has to be constant. Thus there does not exist a map $f\colon [-1, 1] \to S^0$ with $f(-1) = -1$ and $f(1) = 1$.

Now we can push the last example up by one dimension.

Lemma 6.2. *The circle S^1 is not a retract of $\overline{B}^2$.*

[134] As always, a map is understood to be continuous. If we did not demand continuity, then any non-empty subset of X would be a retract of X.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of maps between topological spaces

$$\begin{tikzcd} & \overline{B}^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "r\circ i = \mathrm{id}_{S^1}"'] & & S^1 \end{tikzcd} \quad \text{which induces} \quad \begin{tikzcd} & \pi_1(\overline{B}^2, 1) \arrow[dr, "r_*"] & \\ \pi_1(S^1, 1) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_* = (\mathrm{id}_{S^1})_*"'] & & \pi_1(S^1, 1). \end{tikzcd}$$

It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 238 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in Corollary 5.9 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group.[135] Thus we have obtained a contradiction. □

The last example and the last lemma now raise the following question.

Question 6.3. *Let $n \geq 2$. Is S^n a retract of $\overline{B}^{n+1}$?*

At the moment we cannot answer this question. We will first need to develop new invariants before we can return to this question. We will not be able to answer this question for many months.

Almost exactly the same argument as in the proof of Lemma 6.2 gives a proof of the following lemma which we will prove in Exercise 6.2.

Lemma 6.4. *Let X be a topological space and let $A \subset X$ be a retract of X. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.*

1. *There exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.*
2. *The inclusion induced map $\pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.*

Example. We had just seen on page 247 that the circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$. Furthermore we saw in Corollary 5.9 that $\pi_1(S^1, 1)$ is not the trivial group. It follows from Lemma 6.4 that the fundamental group of the torus is also non-trivial. In Proposition 5.14 we saw that $\pi_1(S^2) = 0$. Thus we see that the torus $S^1 \times S^1$ and the sphere S^2 are not homeomorphic.

Lemma 6.5. $(*)$ *Let E_8 be the regular octagon from the definition of the surface of genus 2, see page 119 for details, and let $\Sigma = E_8/\sim$ be the surface of genus 2. Let P and Q be two adjacent vertices of E_8. Then the following hold:*

[135] Why not?

(1) *The map*

$$\begin{array}{rcl} \gamma\colon S^1 & \to & E_8/\sim \\ e^{2\pi i t} & \mapsto & [t\cdot P + (1-t)\cdot Q] \end{array}$$

is a homeomorphism onto its image.

(2) *There exists a map* $\Phi\colon E_8/\sim \to S^1\times S^1$ *such that the following diagram commutes*

$$\begin{tikzcd} & S^1 \arrow[dl, "\gamma"'] \arrow[dr, "z\mapsto(z,1)"] & \\ E_8/\sim \arrow[rr, "\Phi"] & & S^1\times S^1. \end{tikzcd}$$

(3) *The subset* $\gamma(S^1)$ *is a retract of* $\Sigma = E_8/\sim$.

(4) *There exists an epimorphism* $\pi_1(\Sigma, P)\to\mathbb{Z}$ *that sends* $[\gamma]$ *onto* 1.[136]

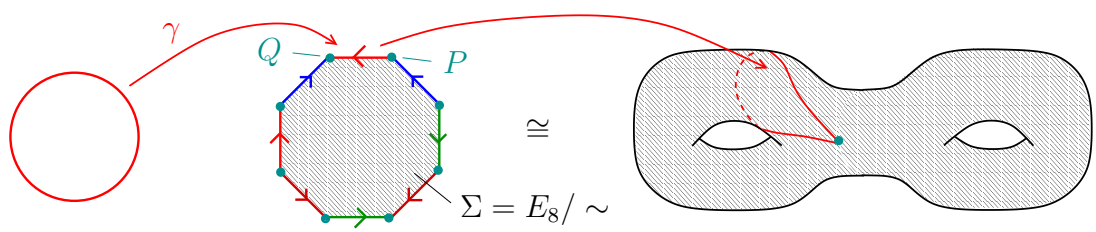


FIGURE 141. Illustration of Lemma 6.5.

Remark. The lemma shows in particular that the four loops that we drew in Figure 49 on the surface of genus 2 represent non-trivial elements in the fundamental group. Almost the same way one can show that in the surface of genus g, $g\geq 2$, defined as a quotient of a regular $4g$-gon, each edge gives rise to a non-trivial element of the fundamental group.

Proof $(*)$.

(1) It is clear that γ is continuous and injective. By Proposition 3.7 we know that $\Sigma = E_8/\sim$ is Hausdorff. It follows from Proposition 1.40 that γ is a homeomorphism onto its image.

(2) The map Φ is sketched in Figure 142 on the top. It is given by sending all points in the lower pentagon to a single point, such a map can be extended continuously to the upper pentagon in such a way that the remaining four sides of the upper

[136]Recall that by the bijection on page 228 we view maps $f\colon S^1\to\Sigma$ with $f(1)=P$ as elements in $\pi_1(\Sigma,P)$.

pentagon get sent homeomorphically to the four sides of the square.[137] [138] This map Φ has the desired property.

(3) The retraction is now given by the concatenation of the maps

$$\Sigma \xrightarrow{\Phi} S^1 \times S^1 \xrightarrow{(z,w)\mapsto z} S^1 \xrightarrow{\gamma} \Sigma.$$

(4) This statement is a consequence of (3), Lemma 6.4 and Corollary 5.9. □

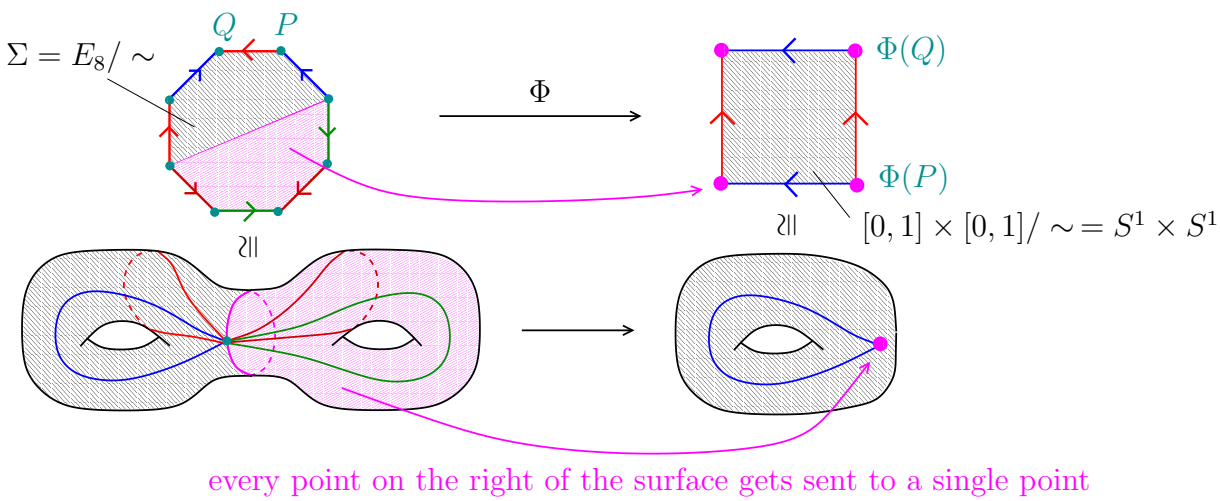


FIGURE 142.

This lemma concludes our discussion of some of the basic properties of fundamental groups. So far we could either show that the fundamental group of a given topological space is trivial, or we could show it is non-trivial by invoking Corollary 5.9. In the next section we will start with a much more systematic discussion of fundamental groups which will allow us to determine the fundamental groups of many topological spaces.

6.4. **Natural transformations.** In this last section we introduce the notion of a natural transformation between two functors. This notion will initially not play a major role, but once we introduce homology groups this notion will become an increasingly important to

[137] It is clearly slightly painful to write down this map explicitly. It is easier to write down a similar map, which maps a square X onto a triangle Y in such a way that one side of the square gets sent to a vertex of the triangle and such that the remaining three sides of the square get sent to the three sides of the triangle. For example the map

$$\begin{array}{rcl} X := [0,1]\times[0,1] & \to & Y := \{(x,y)\in\mathbb{R}^2 \,|\, x\in[0,1], y\in[0,x]\} \\ (x,y) & \mapsto & (x,xy) \end{array}$$

is of that form.

[138] In the lower part of Figure 142 we illustrate the map for the surfaces viewed as subsets of $\mathbb{R}^3$. There the map Φ is given by crushing the right-hand half of the surface of genus 2 to a single point.

state results precisely. As of right now we do not have enough functors at our hand to perhaps truly appreciate the value of the notion of a natural transformation.

After this long preamble, let us give the definition of natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors. A *natural transformation between the functors F and G* assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F_*(f)"] \arrow[d, "\Phi_X"'] & F(Y) \arrow[d, "\Phi_Y"] \\ G(X) \arrow[r, "G_*(f)"] & G(Y). \end{tikzcd}$$

If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X\colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a *natural isomorphism.* Given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are *naturally* isomorphic.[139] The same way we can of course also define the notion of a natural transformation between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:

$$\begin{tikzcd} & F(X) \arrow[rr, "F_*(f)"] \arrow[dd, "\Phi_X"] & & F(Y) \arrow[dd, "\Phi_Y"] \\ X \arrow[ur, squiggly] \arrow[rr, "f"] \arrow[dr, squiggly] & & Y \arrow[ur, squiggly] \arrow[dr, squiggly] & \\ & G(X) \arrow[rr, "G_*(f)"] & & G(Y). \end{tikzcd}$$

Here the wiggly arrows are not maps but they are just meant to remind us which objects a functor associates to the objects X and Y.

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Examples.

(1) Let $\mathcal{R}$ the category of commutative rings and let $\mathcal{G}$ be the category of groups. Let $n \in \mathbb{N}$. We consider the functors

$$\begin{array}{rcl} F\colon \mathcal{R} & \to & \mathcal{G} \\ R & \mapsto & \mathrm{GL}(n, R) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{R} & \to & \mathcal{G} \\ R & \mapsto & R^* := \text{units in } R. \end{array}$$

[139] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical isomorphism" is not a well-defined mathematical concept.

For each commutative ring R we consider the group homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathrm{GL}(n,R) & \to & R^* \\ A & \mapsto & \det(A). \end{array}$$

We claim that these group homomorphisms define a natural transformation from F to G. Thus let $f\colon R \to S$ be a homomorphism between commutative rings and let $A \in F(R)$, i.e. $A = (a_{ij})$ is a matrix in $\mathrm{GL}(n,R)$. We have

$$\begin{aligned} \Phi_A(F_*(f)(A)) &= \det((f(a_{ij}))_{i,j=1,\dots,n}) \overset{\text{Leibniz formula}}{\overset{\downarrow}{=}} \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\cdot\prod_{i=1}^n f(a_{i,\sigma(i)}) \\ &\underset{\substack{\uparrow \\ f \text{ is a ring homomorphism}}}{=} f\Big(\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\cdot\prod_{i=1}^n a_{i,\sigma(i)}\Big) \underset{\substack{\uparrow \\ \text{Leibniz formula}}}{=} f(\det(A)) = G_*(f)(\Phi_R(A)). \end{aligned}$$

But that means, by definition, that the Φ_R define a natural transformation from F to G.

(2) Let $\mathcal{V}$ be the category of real vector spaces and consider the functor F given by[140]

$$\text{real vector space } V \mapsto (V^*)^*$$

and

$$(f\colon V \to W) \mapsto ((f^*)^*\colon (V^*)^* \to (W^*)^*).$$

For each real vector V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{R} \\ (\varphi\colon V \to \mathbb{R}) & \mapsto & \varphi(v) \end{array}\right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor to the functor F.[141]

(3) On page 196, given a submanifold M of some $\mathbb{R}^n$ and a point $P \in M$, we introduced the visual tangent space V_PM which is a subvector space V_PM of $\mathbb{R}^n$. Both the usual tangent space and the visual tangent space define a functor from the category of pointed submanifolds of some $\mathbb{R}^n$ to the category of real vector spaces. Proposition 3.32 (2) says that there exists a natural isomorphism between these two functors.

The above examples are perhaps not particularly impressive. But as we will see, the notion of a natural transformation will eventually become one of the key notions in this course.

[140]As usual, given a real vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ the dual space. Therefore $(V^*)^*$ is the dual space of the dual space of V. The functor given by $V \mapsto V^*$ is contravariant, but the "double functor" $V \mapsto (V^*)^*$ is again covariant.

[141]Is this in fact a natural isomorphism, i.e. is V always isomorphic to $(V^*)^*$?

Exercises for Chapter 6.

Exercise 6.1. Show that $\pi_1(S^1 \times S^1)$ admits an epimorphism onto $\mathbb{Z}^2$.

Exercise 6.2. Prove Lemma 6.4.

Exercise 6.3. Let X be a topological space and let A be a retract.

(a) Suppose that X is Hausdorff. Show that A is a closed subset of X.
(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that X is Hausdorff.

Exercise 6.4. Let $\varphi\colon X \to Y$ be a map between topological spaces. Show that if $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $\varphi_*\colon \pi_1(X, x_1) \to \pi_1(Y, \varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).

7. Fundamental groups and coverings

We have now introduced the fundamental group of a (pointed) topological space and we have proved several basic facts. But the only proof that fundamental groups are non-trivial relied on a fact from complex analysis. In particular, so far we have not yet managed to compute a single fundamental group that is not the trivial group. For example, we know that $\pi_1(S^1)$ is non-trivial, but we do not know what group it is.

In this chapter we will compute many non-trivial fundamental groups using the theory of covering spaces. Before we delve into the theory of covering spaces we insert a short section on the cardinality of sets.

7.1. **The cardinality of sets** $(*)$**.** In this short section we introduce the notion of the cardinality of a set. This notion plays a role in all branches of mathematics.

Definition.

(1) We say that two given sets S and T are *equinumerous* if there exists a bijection $\varphi\colon S \to T$.[142]
(2) Given a set S we define the *cardinality* $\#S$ *of* S as the equinumerosity equivalence class of S in the class of all sets. Put differently we have
$$\#S \ := \ \text{class of all sets } T \text{ that are equinumerous with } S.$$
(3) If S is a finite set, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ which satisfies $\#S = \#\{1, \dots, n\}$. If S infinite, then sometimes we write $\#S = \infty$.
(4) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or $\#S = \#\mathbb{N}$, then we say that S is *countable*, otherwise we say that S is *uncountable*.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by
$$\begin{array}{rcl} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}_0, \\ x-1, & \text{if } x \in \mathbb{N}. \end{cases} \end{array}$$

Remark.

(1) Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T.
(2) In almost any proper course on real analysis it is shown that $\mathbb{Q}$ is countable and that $\mathbb{R}$ is not countable. Thus we $\#\mathbb{Q} \neq \#\mathbb{R}$. This example also shows that the above notation $\#S = \infty$ for an infinite set S is rather dangerous, since we have $\#\mathbb{Q} = \infty$ and $\#\mathbb{R} = \infty$ even though $\#\mathbb{Q} \neq \#\mathbb{R}$.
(3) If S is countable, then any subset T is also countable.[143]

[142] It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.

[143] After excluding the case that S is finite and the case that T is finite one sees that one has to prove the following: given any infinite subset T of $\mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.

Definition. Let S and T be two sets.

(1) We write $\#S \leq \#T$ if there exists a monomorphism $\varphi\colon S \to T$.
(2) We write $S < T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Examples.

(1) We have $\#\mathbb{Q} \leq \#\mathbb{R}$ but we had just pointed out that $\#\mathbb{Q} \neq \#\mathbb{R}$ since $\#\mathbb{Q} = \#\mathbb{N}$ but $\#\mathbb{R} \neq \#\mathbb{N}$. Therefore $\#\mathbb{Q} < \#\mathbb{R}$.
(2) It follows from the previous remark that a set S is countable if and only if $S \leq \#\mathbb{N}$.
(3) Given a set X we denote by $P(X)$ its power set, i.e. the set of all subsets of X. For every X we have $\#X < \#P(X)$.[144] Indeed, the map

$$\begin{array}{ccc} X & \to & P(X) \\ x & \mapsto & \{x\} \end{array}$$

is an injection, hence $\#X \leq \#P(X)$. Now let $f\colon X \to P(X)$ be a map. We have to show that it cannot be surjective. Indeed, the set

$$Y \ := \ \{x \in X \mid x \notin f(x)\}$$

is not in the range of f: if there did exist a $z \in X$ with $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, which is a contradiction.

Theorem 7.1. (Bernstein-Schröder)[145] *Let S and T be two sets. Then*

$$\#S \leq \#T \text{ and } \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

Example. Let U be an open non-empty subset of $\mathbb{R}$.[146] Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective. Furthermore, since U is open there exists an open interval (a, b) that is a subset of U. But any open interval is homeomorphic to $\mathbb{R}$, so there exists an injective map $\mathbb{R} \xrightarrow{\cong} (a, b) \subset U$. It follows from the Bernstein-Schröder-Theorem that there exists a bijection from U to $\mathbb{R}$.

Sketch of proof $(*)$. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[147] We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $g \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of f there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

[144] This statement is known as Cantor's Theorem.

[145] The theorem was initially stated by Cantor without a proof and it was first proved by Felix Bernstein (1878-1956) and Ernst Schröder (1841-1902) in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder theorem.

[146] Every open subset of $\mathbb{R}$ is the disjoint union of open intervals (a_i, b_i), $i \in I$. The number of intervals can be finite, but it can also be countably infinite and in fact it can also be uncountably infinite. For example the complement of the Cantor set consists of uncountably many open intervals.

[147] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

Since $A \sqcup B$ is the disjoint union of all equivalence classes it suffices to prove the following claim.[148]

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

So let I be an equivalence class and let $a \in A \cap I$.[149] The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[150]

$$\begin{array}{lcccccccccl}
 & & & A & B & A & B & A & B & & \\
(1) & a' & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots & \text{with } a' \in A \setminus g(B) \\
(2) & b' & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots & \text{with } b' \in B \setminus f(A) \\
(3) & \dots & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots &
\end{array}$$

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$. This concludes the proof of the claim. $\square$

Remark. The *continuum hypothesis*, first formulated by Cantor in 1878, states that there exists no set S with $\#\mathbb{Q} < \#S < \#\mathbb{R}$. Any attempt at proving this statement relies on first of all making precise what "sets" are supposed to be or what axioms they are supposed to satisfy. As wikipedia writes "the answer to this problem is independent of ZFC set theory (that is, Zermelo-Fraenkel set theory with the axiom of choice included), so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent." This independence was proved in 1963 by Paul Cohen for which he obtained a Fields medal in 1966. More useful information can as always be found on wikipedia:

`https://en.wikipedia.org/wiki/Continuum_hypothesis`

7.2. **Covering spaces.** Now we return to the study of topological spaces.

Definition. Let $p\colon X \to B$ be a map between topological spaces.

(1) We say an open subset $U \subset B$ is *uniformly covered*, if $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property, that the restriction of p to each subset V_i is a homeomorphism.

[148]Indeed, recall that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B = \bigsqcup_{j \in J} (B \cap I_j).$$

Therefore it suffices to give a bijection $A \cap I_j \to B \cap I_j$ for every $j \in J$.

[149]There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[150]Note that we do not claim that the elements in the given sequences are distinct, for example $f\colon A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

(2) We say the map $p\colon X \to B$ is a *covering*, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.

Remark. If U is a uniformly covered subset of B, then clearly any open subset of U is also uniformly covered. For example it follows that if $p\colon X \to B$ is a covering and if B is a manifold, then any point $b \in B$ admits also a connected uniformly covered neighborhood.[151]

Examples.

(a) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & e^{i\varphi} \end{array}$$

is a covering. Indeed, let $P = e^{i\alpha}$ be a point in S^1. We pick the open neighborhood

$$U \;:=\; \left\{ e^{i\varphi} \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right) \right\}.$$

Then

$$p^{-1}(U) \;:=\; \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism. This example is illustrated in Figure 143.

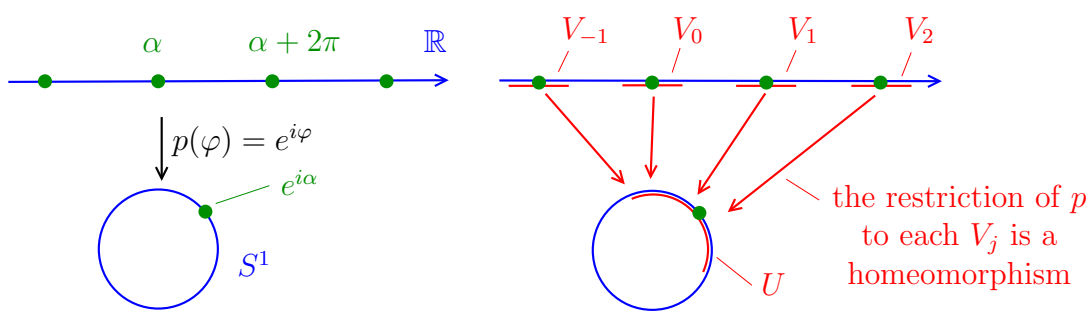


FIGURE 143.

(b) We consider the "infinite helix"

$$H \;:=\; \left\{ (re^{it}, t) \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R} \right\}$$

together with the "annulus"

$$A \;:=\; \left\{ re^{it} \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R} \right\}.$$

The map

$$\begin{array}{rcl} p\colon H & \to & A \\ (re^{it}, t) & \mapsto & re^{it} \end{array}$$

is a covering. This statement is illustrated in Figure 144. The proof of the statement is similar to the proof in the previous example.

[151] Why does this follow for manifolds? Why does this not hold for all topological spaces B?

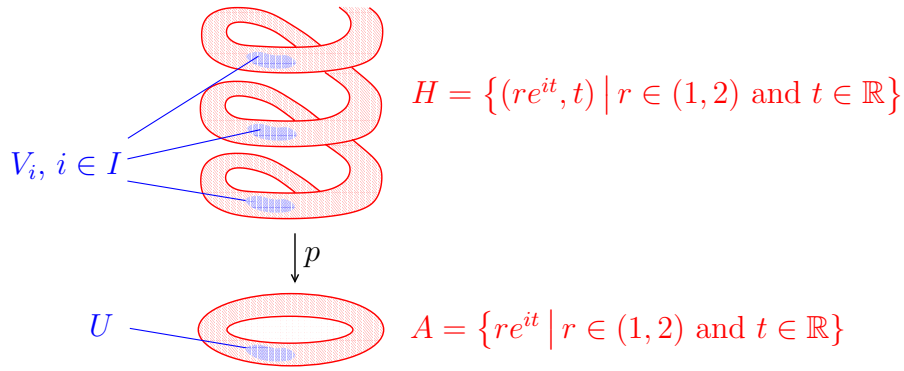


FIGURE 144.

(c) The map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x] = x + \mathbb{Z}^n \end{array}$$

is a covering. Indeed, let $[x] = [(x_1, \dots, x_n)]$ be a point in $\mathbb{R}^n/\mathbb{Z}^n$. We consider

$$W \ := \ \left\{(y_1, \dots, y_n) \,\middle|\, |y_i - x_i| < \tfrac{1}{2}\right\} \ \subset \ \mathbb{R}^n$$

and we put $U := p(W)$. Then U is an open[152] neighborhood of $[x]$. Furthermore

$$p^{-1}(U) \ = \ \bigcup_{i \in \mathbb{Z}^n} (W + i),$$

and for each i the restriction of p to $W + i \subset \mathbb{R}^n$ is a homeomorphism.[153]

(d) For every $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n = S^n/x \sim -x \\ x & \mapsto & [x] = \{x, -x\} \end{array}$$

is a covering. Indeed, for $[x] \in \mathbb{R}\mathrm{P}^n$ we set

$$W \ = \ \{y \in S^n \mid \underbrace{x \cdot y}_{\text{scalar product}} > 0\}.$$

Then $p(W)$ is an open neighborhood of $[x]$ and it is uniformly covered by the two open sets W and $-W$.

(e) The projection map

$$\begin{array}{rcl} p\colon S^1 & \to & [-1, 1] \\ (x, y) & \mapsto & x \end{array}$$

[152]The set U is open since the projection map $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is open by Lemma 1.71 (2).

[153]It is straightforward to verify that the map $p\colon V_i := W + i \to U$ is a bijection. It follows from Lemma 1.71 (1) that the projection map p is continuous and as we had just mentioned in the previous footnote, the projection map is also open. It follows that the map $p\colon V_i = W + i \to U$ is a homeomorphism.

is not a covering. Indeed, the point $b = 1 \in [-1, 1]$ has no open neighborhood that is uniformly covered. This example is illustrated in Figure 145.

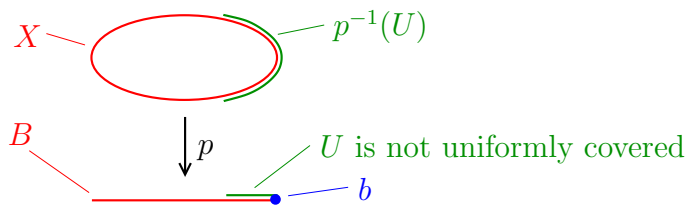


FIGURE 145.

Now we can formulate the following lemma that will be proved in Exercise 7.1.

Lemma 7.2. *Let $p\colon X \to B$ be a covering of topological spaces. If B is a path-connected topological space, then for any two points P and Q in B we have*

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Example. For a general map $p\colon X \to B$ between topological spaces the number of preimages of a point depends of course on the point chosen in B. For instance in the above example (e) we have $\#p^{-1}(0) = 2$ but $\#p^{-1}(1) = 1$.

Definition. Let $p\colon X \to B$ be a covering of a path-connected topological space.

(1) We define the *degree* $[X : B]$ *of* p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.[154]
(2) If the degree is a finite number n, then we say that p is an *n-fold covering* of B. Otherwise we say that p is an *infinite covering*. If the cardinality is countable, then sometimes we call it a *countable covering*.

Examples.

(1) We had just seen that the map $p\colon S^n \to S^n/\sim$ is a 2-fold covering of the real projective space $S^n/\sim = \mathbb{R}\mathrm{P}^n$.
(2) Let $n \in \mathbb{N}$. As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Then the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is a covering. Indeed, let $z = e^{2\pi i\varphi}$ be a point in S^1. Then
$$U \;:=\; \big\{e^{2\pi i\psi} \,\big|\, \psi \in \big(\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4}\big)\big\}$$
has the desired property. Indeed, we have
$$p^{-1}(U) \;=\; \bigsqcup_{j=1}^{n} \underbrace{\big\{e^{2\pi i\psi} \,\big|\, \psi \in \big(\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n}\big)\big\}}_{=:V_j},$$
and the restriction of p to each set V_j is easily seen to be a homeomorphism. This shows that p is a covering and it is clear that p is an n-fold covering.

[154] It follows from Lemma 7.2 that this definition does not depend on the choice of b.

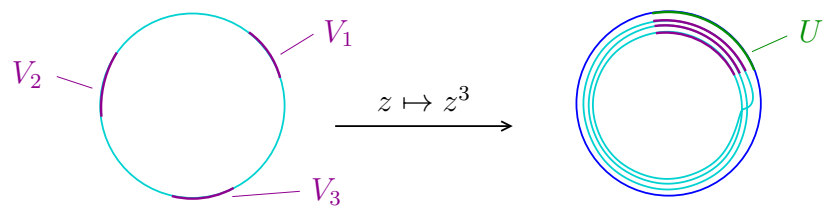


FIGURE 146.

(3) Let $n \in \mathbb{N}$. Basically the same argument as before shows that the map $p\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ given by $p(z) = z^n$ is an n-fold covering.
(4) We consider the map

$$\begin{array}{rcl} p\colon \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \to & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band. This covering is illustrated in Figure 147.

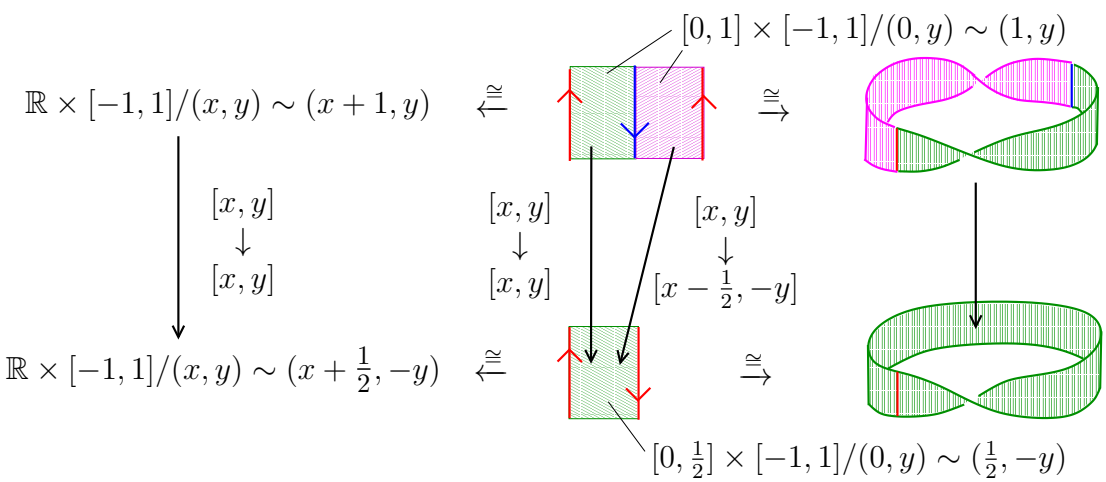


FIGURE 147.

(5) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1\sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a degree 2 covering map from the torus to the Klein bottle.

(6) In Figure 148 we sketch three coverings of spatial graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[155]

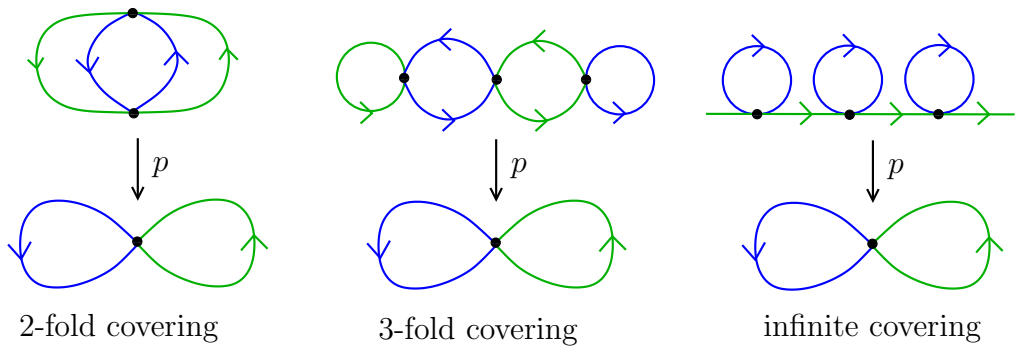


FIGURE 148.

Question 7.3. *We had just seen that there exists a covering map $S^2 \to \mathbb{R}\mathrm{P}^2$ of degree two and that there exists a covering map from the torus to the Klein bottle of degree two. This raises the question: between which surfaces do there exist covering maps? More precisely we can ask the following:*

(1) *Does there exist a covering map from the torus $S^1 \times S^1$ to the sphere S^2?*
(2) *Does there exist a covering map from the sphere S^2 to the torus $S^1 \times S^1$?*

The following lemma summarizes some basic facts about coverings.

Lemma 7.4. *Let $p\colon X \to B$ be a covering of topological spaces. Then the following hold:*

(1) *The map $p\colon X \to B$ is open.*[156]
(2) *For any $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.*
(3) *If B is Hausdorff, then X is also Hausdorff.*
(4) *If p is a finite covering and if B is compact, then X is also compact.*
(5) *If p is an infinite covering, then X is non-compact.*

Proof $(*)$. Let $p\colon X \to B$ be a covering of topological spaces.

(1) Let U be an open subset of X. We need to show that $p(U)$ is open. It suffices to show that for each $P \in p(U)$ there exists an open neighborhood V such that $P \in V \subset U$. Since p is a covering there exists an open neighborhood W of P such that $p^{-1}(W)$ is the union of disjoint open subsets $\{W_i\}_{i\in I}$ with the property that the restriction of p to each subset W_i is a homeomorphism.

Since $P \in p(U)$ there exists a $Q \in U$ with $p(Q) = U$. There exists an $i \in I$ with $Q \in W_i$. Since W_i is open and since U is open the set $U \cap W_i$ is also open in X. Since

[155]It is a good exercise to convince oneself, that the indicated maps are in fact coverings.
[156]Recall that this means that the image of each open set in X is open in B.

$p\colon W_i \to W$ is a homeomorphism and since $U \cap W_i$ is open in W_i the set $p(U \cap W_i)$ is also open in $p(W_i) = W$. But since W is open the set $V := p(U \cap W_i)$ is also an open subset of B. It is the desired open neighborhood of P with $P \in V \subset U$.

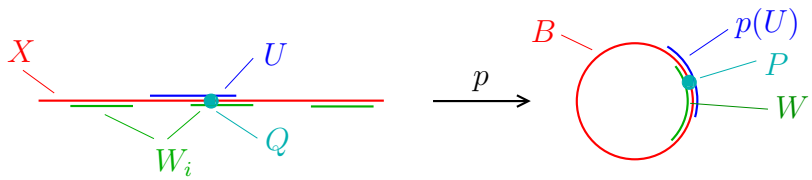


FIGURE 149. Illustration for the proof of Lemma 7.4 (1).

(2) Let $b \in B$. Since $p\colon X \to B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. For each $i \in I$ there exists a unique point $x_i \in V_i$ with $p(x_i) = b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i, $i \in I$. Since each V_i is open in X it follows from the definition of the subspace topology that each $x_i \in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X.

(3) This statement is relatively easy to prove and it in Exercise 7.4.

(4) Let $p\colon X \to B$ be an n-fold covering of a compact space B. Let $\{U_i\}_{i\in I}$ be an open covering of X. Let $b \in B$. We pick an open connected neighborhood M of b that is uniformly covered. For each $x \in p^{-1}(b)$ we denote by N_x the component of $p^{-1}(M)$ that contains x. For each point in $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We write

$$W_b \ := \bigcap_{x\in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open covering $\{U_i\}_{i\in I}$.

Since M is uniformly covered the map $p\colon N_x \to M$ is a homeomorphism for any $x \in p^{-1}(y)$. We denote by $p_x\colon N_x \to M$ this homeomorphism. We write $F := p^{-1}(b)$. We then have

$$p^{-1}(W_b) \ = \bigcup_{y\in F} p_y^{-1}\Big(\bigcap_{x\in F} p_x(U_{i_x} \cap N_x)\Big) \ = \bigcup_{y\in F} \overbrace{\bigcap_{x\in F} \underbrace{p_y^{-1}(p_x(U_{i_x}))}_{=U_{iy} \text{ if } y=x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{=N_y}}^{\subset U_{i_y}\cap N_y} \subset \bigcup_{y\in F} U_{i_y}.$$

This concludes the proof of the claim.

By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by

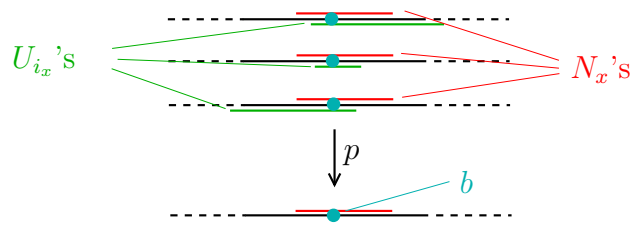


FIGURE 150. Illustration for the proof of Lemma 7.4 (4).

finitely many of these open sets, i.e. there exist $b_1, \ldots, b_k \in B$ with $B = W_{b_1} \cup \cdots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \ldots, p^{-1}(W_{b_k})$ cover X. By the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open covering $\{U_i\}_{i \in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i \in I}$.

(5) Let $p\colon X \to B$ be an infinite covering. If B is non-compact, then it follows from Lemma 1.37 that X is also non-compact. So now suppose that B is compact. By definition we can cover B by open sets that are uniformly covered. Since B is compact we can cover B by finitely many open sets $U_1, \ldots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \cdots \cup U_{k-1}$.[157] Now let P be a point in $U_k \setminus (U_1 \cup \cdots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that the restriction of p to each V_i is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open covering

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i \in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open sets V_i, $i \in I$ from the open covering. Since I is infinite it follows that X does not satisfy the definition of a compact set, i.e. X is not compact. □

We conclude this section with the following lemma that we will prove in Exercise 7.3.

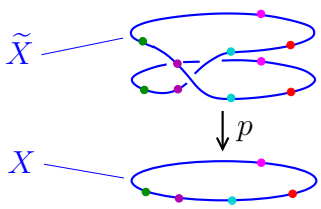
Lemma 7.5. *Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define*

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \textit{the unique other point in } p^{-1}(p(Q)). \end{array}$$

[157] Indeed, if U_k was contained in $U_1 \cup \cdots \cup U_{k-1}$, then we could remove U_k from the collection of open sets and we continue with the collection $U_1, \ldots, U_{k-1}$.

(*The definition of f is illustrated in Figure 151*). *This map is continuous and it has the property that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$



$f\colon \widetilde{X} \to \widetilde{X}$ swaps the points of the same color

FIGURE 151. Illustration of Lemma 7.5.

7.3. **Group actions and covering spaces.** We recall two definitions about actions from pages 102 and 106 and we introduce one new definition.

Definition. Let G be a group which acts continuously on a topological space X.

(1) We say G acts *freely* if $g \cdot x = x$ for $g \in G$ and $x \in X$ implies that $g = e$.
(2) We say G acts *properly* if for any two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.
(3) We say G *acts discretely*, if for each $x \in X$ there exists an open neighborhood U so that
$$U \cap gU \;=\; \varnothing \text{ for all } g \neq e.$$

The following lemma summarizes the relationship between these different definitions.

Lemma 7.6. *Let G be a group which acts continuously on a topological space X.*

(1) *If G is finite, then the action is proper.*
(2) *If G acts discretely, then it also acts freely.*
(3) *If X is a Hausdorff space and if the action by G is free and proper, then the action is also discrete.*

Proof. The first two statements follow immediately from the definitions. The third statement is the content of Lemma 3.26. □

Example.

(A) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the open neighborhood
$$U \;=\; \big\{y = (y_1, \dots, y_n) \in \mathbb{R}^n \,\big|\, \max\{|x_1 - y_1|, \dots, |x_n - y_n|\} < \tfrac{1}{2}\big\}$$
has the desired property.

(B) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. The action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n,(-1)^n y) \end{array}$$

is discrete. Indeed, for any $(x,y) \in X$ the open neighborhood

$$U \;=\; \left\{(a,b) \,\middle|\, |a-x| < \tfrac{1}{2} \text{ and } b \in [-1,1]\right\}$$

has the desired property.

(C) Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$

defines an action that is discrete. Indeed, for any $P \in S^n$ the open hemisphere

$$U \;=\; \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$$

has the desired property.[158]

(D) Every discrete action is free, but as we will see in this example, the converse does not hold. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and $G = \mathbb{R} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action. It is straightforward to see that the action is free. But the action is not discrete. Indeed, it is easy to see that for each open non-empty set U there exist uncountably many real numbers $r \neq 1$ with $U \cap rU \neq \varnothing$.

(E) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means[159]

$$G \;=\; \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

The group G acts by definition on $\mathbb{R}^2$. In Exercise 7.6 we will see that this action of G on $\mathbb{R}^2$ is discrete.

[158] Alternatively one can argue, that the action is obviously free and S^n is Hausdorff, hence the action is discrete by Lemma 7.6 (1) and (3).

[159] For example the map $B \circ A \circ A \circ B^{-1} \circ A$ lies in G.

We consider again the square $X = [0,1] \times [0,1]$ with the equivalence relation that is generated by $(0,y) \sim (1, 1-y)$ and $(x,0) \sim (x,1)$. It is straightforward to see that the map

$$\begin{array}{rcl} X/\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$

is a homeomorphism. Put differently, $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle.

(F) Let

$$X \;=\; S^3 \;=\; \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$$

and let $G = \mathbb{Z}_p$ for some $p \in \mathbb{N}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Then the map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1,z_2)) & \mapsto & (e^{2\pi i k/p} z_1, e^{2\pi i k q/p} z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.[160] We denote by $L(p,q) = S^3/\mathbb{Z}_p$ the quotient space. It follows from Propositions 3.25 and that $L(p,q)$ is again a closed 3-dimensional manifold. Furthermore it follows from Proposition 3.38 that $L(p,q)$ is orientable, in fact it has an orientation such that the projection map $p\colon S^3 \to L(p,q)$ is orientation-preserving. We will always view $L(p,q)$ as a manifold equipped with this orientation. We refer to the manifolds $L(p,q)$ as *lens spaces*.[161] We consider two special cases:

(a) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1,q)$ is therefore just the original sphere S^3.

(b) For $p = 2$ we have q is odd, so for $k = 0$ we have $e^{2\pi i k/2} = e^{2\pi i k q/2} = 1$ and for $k = 1$ we have $e^{2\pi i k/2} = e^{2\pi i k q/2} = -1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2 = \{\pm 1\}$ that we had used in the definition of the real projective space $\mathbb{R}\mathrm{P}^n$. This shows that $L(2,q) = \mathbb{R}\mathrm{P}^n$.

We discuss the last type of examples in slightly more detail. For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p,q) = 1$ we just introduced a new topological space, namely the lens space $L(p,q)$. It is natural to ask, for which parameters (p,q) are these topological spaces homeomorphic?

Lemma 7.7. *Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,*[162] *then $L(p,q)$ and $L(p,r)$ are diffeomorphic.*[163]

[160] It is straightforward to verify that the map does indeed define an action. It is clear that the action is smooth and a direct calculation shows that the action is orientation-preserving. One can also easily verify that the action is free. It follows from Lemma 7.6 (1) and (3) that the action is discrete.

[161] The name "lens space" comes from an alternative description of these manifolds.

[162] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s \bmod p$. We write $r^{-1} := s$.

[163] In particular any lens space is of the form $L(p,q)$ with $q \in \{0, \dots, p-1\}$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. We define a map

$$L(p,q) \quad \to \quad L(p,r)$$

as follows:

$$\begin{array}{lrcl} \text{Case 1: } q \equiv r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,z_2)] \\ \text{Case 2: } q \equiv -r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,\overline{z_2})] \\ \text{Case 3: } q \equiv r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,z_1)] \\ \text{Case 4: } q \equiv -r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,\overline{z_1})] \end{array}$$

In all four cases it is straightforward to verify that the given map is well-defined and that it is in fact a diffeomorphism. □

In the proof of Lemma 7.7 it was quite straightforward to write down the homeomorphism. This raises the question whether there are homeomorphisms between more lens spaces that are harder to find, or whether all other pairs of lens spaces are non-homeomorphic. More precisely, we have the following question.

Question 7.8. *Let $p, \widetilde{p} \in \mathbb{N}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(\widetilde{p},\widetilde{q}) = 1$. If $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, does it follow that $p = \widetilde{p}$ and $q \equiv \pm\widetilde{q}^{\pm 1} \bmod p$?*

This question will keep us busy for a very long time.

We had just seen on page 265 that the group $\{\pm 1\}$ acts freely and continuously on any sphere. We had also seen that any finite cyclic group $\mathbb{Z}_p$ acts freely and continuously on the 3-sphere. This raises the question, whether such an action by $\mathbb{Z}_p$ also exists on the 2-sphere. More generally we can ask the following question.

Question 7.9. *Given $n \in \mathbb{N}$, which finite cyclic groups can act freely and continuously on the n-dimensional sphere S^n?*[164]

We will come back to this question once we have introduced homology groups.

The following proposition connects the notion of discrete actions to the theory of covering spaces.

Proposition 7.10. *Let X be a topological space together with a discrete and continuous action by a group G. Then the canonical projection $p\colon X \to X/G$ is a covering.*

Example. This proposition, combined with examples (A) and (C), gives in particular new proofs that the maps $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ and $S^n \to S^n/\sim = \mathbb{R}\mathrm{P}^n$ are covering maps.

Proof. Let G be a group which acts discretely and continuously on a topological space X. We want to show that the projection map $p\colon X \to X/G$ is a covering. Let $[x] \in X/G$. Since the group acts discretely there exists an open neighborhood V of $x \in X$ such that

$$V \cap gV \;=\; \varnothing \text{ for all } g \neq e.$$

[164] Any cyclic group $\mathbb{Z}_k$, $k \neq 0$ admits a non-trivial action on S^n where $j \in \mathbb{Z}_k$ acts by rotation around the x-axis by the angle $2\pi\frac{j}{k}$. But this action is not free, since the points $(-1,0,\dots,0)$ and $(0,\dots,0,1)$ are fixed.

We put $U := p(V)$. It follows from Lemma 1.71 that the set U is an open neighborhood of $[x] = p(x)$. The preimage $p^{-1}(U)$ is the disjoint union of the open subsets[165] $gV, g \in G$. The restriction of the map p to any gV is a continuous, bijective[166] map $gV \to U$ and according to Lemma 1.71 this map is furthermore open. It follows that the restriction of p to any gV is in fact a homeomorphism. Thus we see that U is uniformly covered. Since this holds for any point in X/G we have shown that the projection map $p\colon X \to X/G$ is a covering. $\square$

The following proposition is a slight generalization of the previous proposition.

Proposition 7.11. *Let X be a topological space together with a discrete and continuous action by a group G. Furthermore let H be a subgroup of G. Then the canonical projection*

$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering.

For H the trivial group we recover the statement of Proposition 7.10. In fact the proof of Proposition 7.11 is almost identical to the proof of Proposition 7.10. We leave the details to the reader.

Examples.

(1) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & \big(x + \frac{1}{2}n, (-1)^n y\big). \end{array}$$

Let $H = 2\mathbb{Z}$. According to Proposition 7.11 the projection map $p\colon X/H \to X/G$ is a covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc} (\mathbb{R} \times [-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R} \times [-1,1])/\mathbb{Z} \\ \downarrow = & & \downarrow = \\ (\mathbb{R} \times [-1,1])/(x,y) \sim (x + \frac{1}{2}2k, (-1)^{2k}y) & \longrightarrow & (\mathbb{R} \times [-1,1])/(x,y) \sim (x + \frac{1}{2}k, (-1)^k y) \\ \downarrow = & & \downarrow = \\ \underbrace{(\mathbb{R} \times [-1,1])/(x,y) \sim (x+k, y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R} \times [-1,1])/(x,y) \sim (x + \frac{1}{2}k, (-1)^k y)}_{=\text{Möbius band}}. \end{array}$$

Each point on X/G has precisely two preimages, so the covering is in fact a 2-fold covering. Thus we recover the result from page 260 that the annulus is a 2-fold covering of the Möbius band.

[165] Why is $p^{-1}(U) = \bigcup_{g \in G} gV$ and why are the sets gV disjoint?

[166] Why is the map bijective?

(2) We consider again the group G of self-homeomorphisms of $\mathbb{R}^2$ that is generated by

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We let H be the subgroup of G generated by A^2 and B. Note that A^2 is the map given by

$$\begin{array}{rcl} A^2\colon \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \mapsto \underbrace{((x+1)+1, (1-(1-y)).}_{=(x+2,y)} \end{array}$$

Note that A^2 and B are two homeomorphisms that commute, so any element in H is in fact of the form

$$\begin{array}{rcl} A^{2k} \circ B^l\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+2k, y+l) \end{array}$$

for some $k, l \in \mathbb{Z}$. By Proposition 7.11 the projection map $p\colon \mathbb{R}^2/H \to \mathbb{R}^2/G$ is a covering map. We already know that $\mathbb{R}^2/G$ is the Klein bottle. It follows immediately from the above description of H, $\mathbb{R}^2/H = \mathbb{R}^2/(2\mathbb{Z}\oplus\mathbb{Z})$, so $\mathbb{R}^2/H$ is just the 2-dimensional torus. As on page 260 we see that we can view the 2-dimensional torus as a 2-fold covering of the Klein bottle.

7.4. **Lifting of paths.**

Definition. Let $p\colon X \to B$ be a map between topological spaces and let $f\colon [a,b] \to B$ be a path. Let $x \in X$ be a point with $p(x) = f(a)$. A path $g\colon [a,b] \to X$ is called a *lift of f to the starting point x*, if $g(a) = x$ and if $p \circ g = f$, i.e. if the following diagram of maps commutes:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ {[a,b]} \arrow[ur, "g"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Examples.

(1) We consider the map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & e^{it} \end{array}$$

and the path

$$\begin{array}{rcl} f\colon [0,b] & \to & S^1 \\ t & \mapsto & e^{2it} \end{array}$$

with starting point $f(0) = 1$. Let $x = 4\pi$. Then $p(x) = p(4\pi) = 1$ and

$$\begin{array}{rcl} g\colon [0,b] & \to & \mathbb{R} \\ t & \mapsto & 4\pi + 2t \end{array}$$

is a lift of f to the starting point 4π. Note that if $b = \pi$, then f is a loop in $(S^1, 1)$, but the lift is not a loop in $(\mathbb{R}, 4\pi)$. Put differently, the lift of a loop is not necessarily again a loop. This example is illustrated in Figure 152.

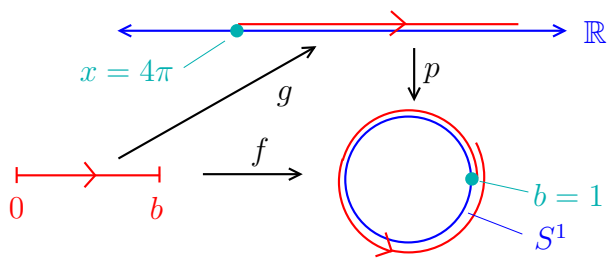


FIGURE 152.

(2) In Figure 153 we illustrate the lifting of a path in the covering $p\colon H \to A$ of an infinite helix over an annulus that we had considered on page 257.

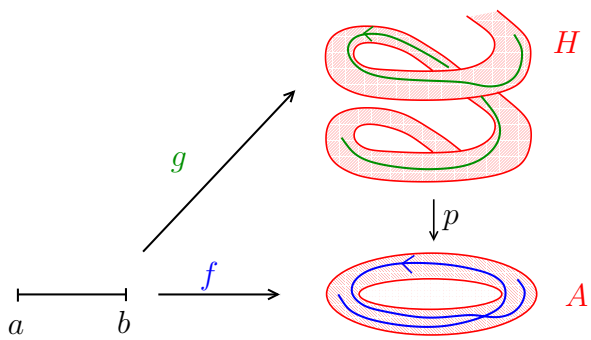


FIGURE 153.

Proposition 7.12. *Let $p\colon X \to B$ be a covering and let $f\colon [0,1] \to B$ be a path. Then given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $g\colon [0,1] \to X$ of f to the starting point x.*

Proof. Let $p\colon X \to B$ be a covering, let $f\colon [0,1] \to B$ be a path and let x be a point in X with $p(x) = f(0)$. We first show the existence of a lift and then we prove the uniqueness of the lift.

(a) We consider

$$S \;:=\; \big\{ s \in [0,1] \,\big|\, \text{there exists a lift of } f|_{[0,s]} \text{ to the starting point } x \big\}.$$

Note that S is non-empty since 0 has the desired property. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. It suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S. We set $b := f(t)$. Since p is a covering there

exists an open neighborhood U of $b \in B$, which is uniformly covered. As a reminder, this means that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$.

By the definition of t there exists an $s \in (t-\epsilon, t] \cap S$. We denote by g the lift of $f|_{[0,s]}$ the starting point x. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i \colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

We consider the map

$$\begin{array}{rcl} h\colon [0, t+\epsilon) \cap [0,1] & \to & X \\ z & \mapsto & \begin{cases} g(z), & \text{if } z \in [0,s], \\ p_i^{-1}(f(z)), & \text{if } z \in (s, t+\epsilon) \cap [0,1]. \end{cases} \end{array}$$

Since $g(s) = p_i^{-1}(f(s))$ we see that this map is indeed continuous. It is clear that this defines a lift of f to the starting point x.

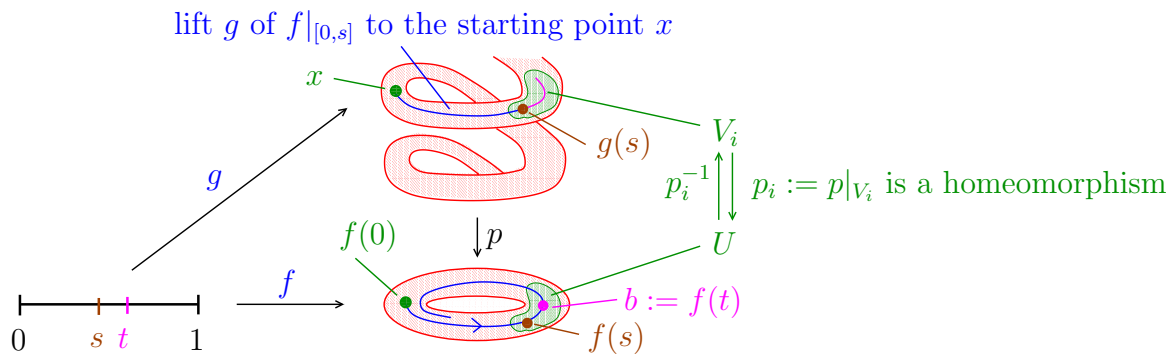


FIGURE 154.

(b) Now we show the uniqueness of the lift. The proof of that statement is similar to the proof of the existence of a lift. So suppose that $g, h\colon [0,1] \to X$ are two lifts of f to the starting point x. We want to show that $g = h$. We consider

$$S \;:=\; \{s \in [0,1] \,|\, g(t) = h(t) \text{ for all } t \in [0,s]\}\,.$$

Note that $0 \in S$, i.e. S is non-empty. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. Again it suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S.

We put $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$ which is uniformly covered. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$. We pick an $s \in (t-\epsilon, t] \cap [0,1]$. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i \colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

Since $g(s) = h(s)$ it follows that $h(s) \in V_i$. Then it follows that $g(z) \in V_i$ and $h(z) \in V_i$ for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. It follows from the definitions that

$$g(z) \;=\; p_i^{-1}(f(z)) \quad \text{and} \quad h(z) \;=\; p_i^{-1}(f(z))$$

for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. In particular $[t, t+\epsilon) \cap [0,1]$ lies in S. □

7.5. **Lifting of path-homotopies.** In the last chapter we saw that given a covering map $p\colon X \to B$ any path $f\colon [0,1] \to B$ can be lifted to a path $g\colon [0,1] \to X$. In this chapter we want to study the question whether the lifts of path-homotopic paths are still path-homotopic.

Before we start this discussion we first generalize the notion of a lift.

Definition. Let $p\colon X \to B$ be a covering and let $f\colon Y \to B$ be a map between topological spaces. A *lift of f to X* is a map[167] $\tilde{f}\colon Y \to X$ such that $p \circ \tilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Proposition 7.13. *Let $p\colon X \to B$ be a covering, let B be a topological space, let Y be a topological manifold*[168] *and let $f\colon Y \times [0,1] \to B$ be a map. Furthermore let $\tilde{f}\colon Y \times 0 \to X$ be a lift of $f|_{Y\times 0}$. Let*

$$\tilde{f}\colon Y \times [0,1] \;\to\; X$$

be the map that is uniquely determined[169] *by the property that for each $y \in Y$ the path $t \mapsto \tilde{f}(y,t)$ is the lift of the path $t \mapsto f(y,t)$ to the starting point $\tilde{f}(y,0)$. Then the map $\tilde{f}\colon Y \times [0,1] \to X$ is continuous.*

The statement of Proposition 7.13 is sketched in Figure 155. One way of formulating Proposition 7.13 is that a "continuous" family of paths in B lifts again to a "continuous" family of paths in X.

Proof $(*)$. Let $p\colon X \to B$ be a covering, let $f\colon Y \times [0,1] \to B$ be a continuous map and let $\tilde{f}\colon Y \times 0 \to X$ be a continuous lift of $f|_{Y\times 0}$. (Here and throughout this proof we do not suppose that all maps are continuous.) We begin with the following definitions:

(1) A *good set* $U \times I$ consists of a connected subset $U \subset Y$ and an open interval[170] $I \subset [0,1]$ such that $f(U \times I)$ lies in a uniformly covered subset of B.

[167] Once again, all maps are understood to be continuous.

[168] Later on we will apply the proposition to the case that $Y = [0,1]$ and that $Y = \overline{B}^n$ for some $n \in \mathbb{N}_0$.

[169] The existence and the uniqueness of this map is an immediate consequence of Proposition 7.12.

[170] We say $A \subset [0,1]$ is an open interval, if A is an interval and if A is open with respect to the subspace topology of $[0,1]$. Put differently, $A = (a,b)$, or $A = [0,a)$ or $A = (b,1]$ or $A = [0,1]$ for $0 \le a \le b \le 1$.

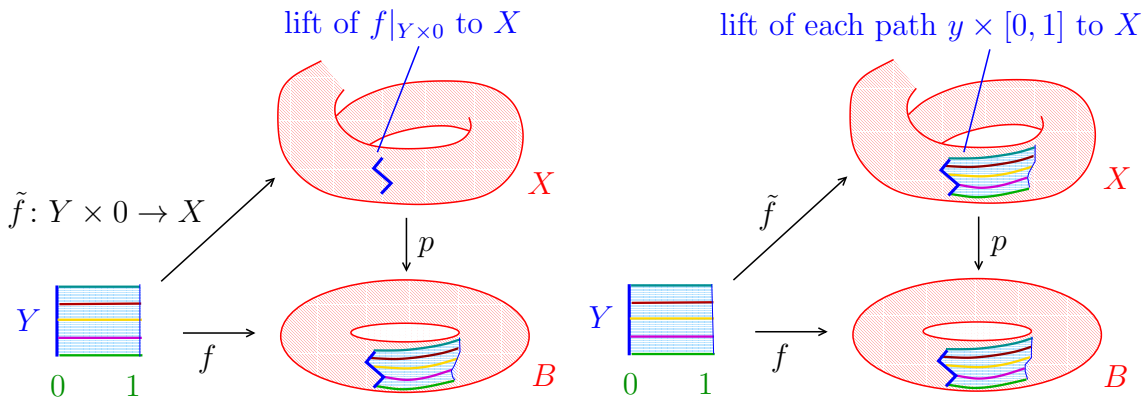


FIGURE 155. Schematic image for the lift of f.

(2) We say a good set $U\times I$ is *very good*, if there exists a $t\in I$ such that the restriction of $\tilde{f}$ to $U\times\{t\}$ is continuous.
(3) Let $y\in Y$ and $t\in[0,1]$. A (*very*) *good neighborhood of* (y,t) is a (very) good set $U\times I$ with $y\in U$ and $t\in I$.

It follows easily[171] from the continuity of f, the hypothesis that Y is a topological manifold and the fact that $p\colon X\to B$ is a covering that each point (y,t) admits a good neighborhood.[172]

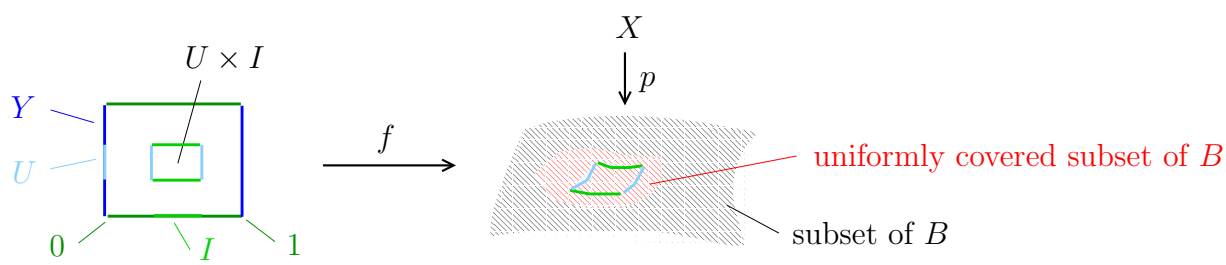


FIGURE 156. Schematic image of a good neighborhood.

Claim. Let $U\subset Y$ open and let $I\subset[0,1]$ be an open interval such that $U\times I$ is very good. Then $\tilde{f}$ is continuous on $U\times I$.

[171]Why does this follow?

[172]Here we use the fact, established in Corollary 1.87 and Lemma 3.8, that the topological manifold Y is o-locally connected.

Let $U \subset Y$ be an open connected subset and let $I \subset [0,1]$ be an interval such that $U \times I$ is very good. This means in particular that there exists a $t \in I$ such that

$$\begin{array}{rcl} U & \to & X \\ y & \mapsto & \tilde{f}(y,t) \end{array}$$

is continuous.

According to our hypothesis there exists a uniformly covered subset $W \subset B$ such that $f(U \times I) \subset W$. Since $U \times I$ is connected we can without loss of generality suppose that W is connected. Since W is uniformly covered the preimage $p^{-1}(W)$ is the union of disjoint open subsets $\{V_j\}_{j \in J}$, with the property, that the restriction of p to each subset V_j is a homeomorphism.

Now we want to show that there exists a $j \in J$ with $\tilde{f}(U \times I) \subset V_j$. We choose a $y \in U$. Then there exists a unique $j \in J$ such that $\tilde{f}(y,t) \in V_j$. Since $\tilde{f}$ is continuous $U \times \{t\}$ and since U is connected it follows from Lemma 1.101[173] that $\tilde{f}(U \times \{t\}) \subset V_j$. Now let $z \in U$. We just showed that $\tilde{f}(z,t) \in V_j$. By the construction of $\tilde{f}$ the map $\tilde{f}$ is continuous on $\{z\} \times I$. Since I is connected it follows again from Lemma 1.101 that $\tilde{f}(\{z\} \times I) \subset V_j$. We have thus shown that $\widetilde{f}(U \times I) \subset V_i$. This part of the argument is sketched in Figure 157.

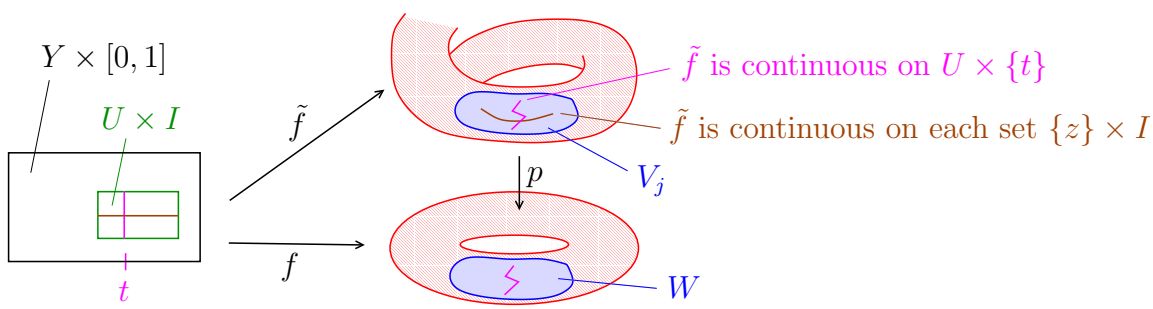


FIGURE 157.

We denote by $p_j \colon V_j \to W$ the homeomorphism which is given by the restriction of p to V_j. Since $\tilde{f}(U \times I) \subset V_j$ it follows that on $U \times I$ the map $\tilde{f}$ is given by $\tilde{f} = p_j^{-1} \circ f$. In particular the map $\tilde{f}$ is a composition of continuous maps, so it is continuous on $U \times I$. Thus we have proved the claim.

Claim. Let $y \in Y$. We consider

$$T := \{t \in [0,1] \,|\, \text{there exists a very good neighborhood of } (y,t)\}.$$

We claim that $T = [0,1]$.

Thus let $y \in Y$. Since $[0,1]$ is connected it suffices to show the following three statements:

[173] We apply Lemma 1.101 to the continuous map $U \times \{t\} \to p^{-1}(W)$ from the connected set U to the set $p^{-1}(W)$ with components V_j, $j \in J$.

(1) $T \neq \varnothing$,
(2) T is open, and
(3) T is closed.

Now we will prove these three statements:

(1) As we pointed out above, every point in $Y \times I$, in particular the point $(y, 0)$, admits a good neighborhood. By our hypothesis $\tilde{f}$ is continuous on $Y \times \{0\}$. It follows that the good neighborhood is in fact very good.
(2) It is easy to see that T is open. Indeed, suppose that we are given a very good neighborhood $U \times I$ of (y, t). Then $U \times I$ is also a very good neighborhood for any $(y, s) \in U \times I$. Therefore the open neighborhood I of t also lies in T.
(3) It remains to show that T is closed. Put differently, we need to show that T equals its closure $\overline{T}$. Thus let $t \in \overline{T}$. As we already pointed out, there exists a good neighborhood $U \times I$ of (y, t). Since $t \in \overline{T}$ there exists an $s \in T \cap I$. Since $s \in T$ it follows from the above claim that there exists a very good open neighborhood $V \times J$ of (y, s). By the previous claim the restriction of $\tilde{f}$ to $V \times J$ is continuous, in particular it is continuous on $(U \cap V) \times \{s\}$. We denote by W the component of $U \cap V$ that contains y. Then $W \times I$ is a very good neighborhood of (y, t). This step is illustrated in Figure 158.

This concludes the proof of the claim.

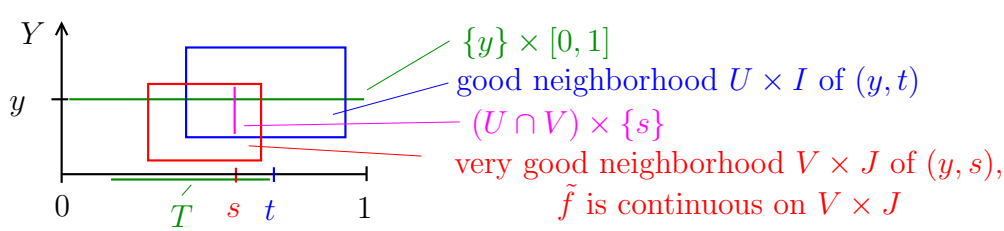


FIGURE 158.

We have just shown that given any $y \in Y$ and any $t \in [0,1]$ there exists a very good neighborhood $U_{y,t}$ of (y,t). By the first claim the map $\tilde{f}$ is continuous on each $U_{y,t}$. But the open subsets $U_{y,t}$ cover all of $Y \times [0,1]$. It follows from Lemma 1.104 that $\tilde{f}$ is continuous on $Y \times [0,1]$. $\square$

Corollary 7.14. *Let $p\colon X \to B$ be a covering and let $f, g\colon [0,1] \to B$ be two paths. Let $\tilde{f}, \tilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\tilde{f}$ and $\tilde{g}$ agree and the paths $\tilde{f}$ and $\tilde{g}$ are path-homotopic.*

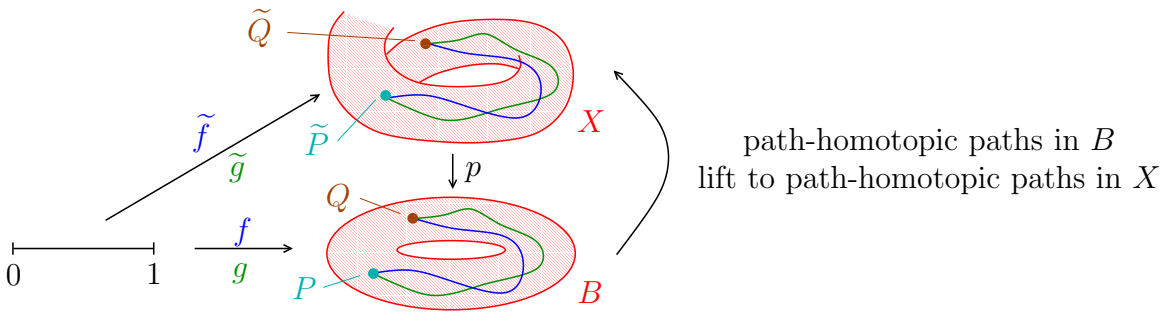


FIGURE 159. Illustration of Corollary 7.14.

Proof $(*)$. We write $P := f(0) = g(0)$ and $Q := f(1) = g(1)$. We denote by $\tilde{P} := \tilde{f}(0) = \tilde{g}(0)$ the common starting point of the lifts and we write $\tilde{Q} := \tilde{f}(1)$. Let

$$\begin{array}{rcl} H\colon [0,1]\times[0,1] & \to & B \\ (t,s) & \mapsto & H(t,s) \end{array}$$

be a path-homotopy between the paths f and g. As a reminder, H is a map such that

$$H(t,0) \;=\; f(t) \quad \text{and} \quad H(t,1) \;=\; g(t) \quad \text{for all } t\in[0,1],$$

and such that

$$H(0,s) \;=\; P \quad \text{and} \quad H(1,s) \;=\; Q \quad \text{for all } s\in[0,1].$$

On $\{0\}\times[0,1]$ we consider the lift $\widetilde{H}$ of H which is given by $\widetilde{H}(0,s) = \widetilde{P}$. Furthermore we denote by $\widetilde{H}\colon [0,1]\times[0,1]\to X$ the map which is defined as follows: for each $s\in[0,1]$ the path $t\mapsto \widetilde{H}(t,s)$ is the unique lift of the path $t\mapsto H(t,s)$ to the starting point $\tilde{P}$ lift. According to Proposition 7.13 the map $\widetilde{H}$ is in fact continuous. By definition of $\widetilde{H}$ we have

$$\widetilde{H}(t,0) \;=\; \tilde{f}(t) \quad \text{and} \quad \widetilde{H}(t,1) \;=\; \tilde{g}(t) \text{ for all } t\in[0,1],$$

For each $s\in[0,1]$ we have $p(\widetilde{H}(1,s)) = Q$. Since $\widetilde{H}$ is continuous we obtain a map

$$\begin{array}{rcl} [0,1] & \to & p^{-1}(Q) \;\subset\; X \\ s & \mapsto & \widetilde{H}(1,s). \end{array}$$

Since $[0,1]$ is connected it follows from Lemma 1.92 and Lemma 7.4 that $\widetilde{H}(1,s) = \tilde{Q}$ for all $s\in[0,1]$. In particular we obtain that

$$\tilde{f}(1) \;=\; \widetilde{H}(1,0) \;=\; \tilde{Q} \;=\; \widetilde{H}(1,1) \;=\; \tilde{g}(1).$$

Thus it follows that $\widetilde{H}$ is indeed a path-homotopy between the paths $\tilde{f}$ and $\tilde{g}$. □

Definition. Let $p\colon X\to B$ be a map between topological spaces and let $x_0\in X$ and $b_0\in B$. We say a map $p\colon X\to B$ is *a covering of pointed topological spaces*, if $p\colon X\to B$ is a covering and if $p(x_0) = b_0$.

Corollary 7.15. *For any covering $p\colon (X,x_0) \to (B,b_0)$ of pointed topological spaces the induced map $p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)$ is a monomorphism.*

Proof. We have[174]

$$\begin{aligned} [f] \in \ker(\pi_1(X,x_0) \to \pi_1(B,b_0)) \quad &\Longrightarrow \quad [p\circ f] = e \in \pi_1(B,b_0) \quad \Longrightarrow \quad p\circ f \simeq e_{b_0} \\ &\underset{\uparrow}{\Longrightarrow} \quad f \simeq e_{x_0} \qquad\qquad\qquad \Longrightarrow \quad [f] = e \in \pi_1(X,x_0). \end{aligned}$$

by Corollary 7.14, since f is the lift of $p\circ f$ and e_{x_0} is the lift of e_{b_0} to the starting point x_0 □

Example. In Question 7.3 (1) we had asked whether there exists a covering map from the torus $S^1 \times S^1$ to the sphere S^2. By Proposition 5.14 we know that $\pi_1(S^2) = 0$ whereas we saw on page 248 that $\pi_1(S^1 \times S^1)$ is non-trivial. Using Corollary 7.15 we can now answer Question 7.3 (1) in the negative: there is no covering map from $S^1 \times S^1$ to S^2.

We conclude the section with the following lemma.

Lemma 7.16. *Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces.*

(1) *Let f be a loop in (B,b_0). We denote by $\tilde{f}$ the lift of f to the starting point x_0. Then the following holds:*

$$\tilde{f} \text{ is a loop in } (X,x_0) \quad \Longleftrightarrow \quad [f] \text{ lies in } p_*(\pi_1(X,x_0)) \subset \pi_1(B,b_0).$$

(2) *Now we suppose that X is simply connected. Let f and g loops in (B,b_0) and let $\tilde{f}$ and $\tilde{g}$ be the lifts of f and g to the starting point x_0. Then the following holds:*

$$f \text{ and } g \text{ are path-homotopic loops} \quad \Longleftrightarrow \quad \text{the endpoints of } \widetilde{f} \text{ and } \widetilde{g} \text{ agree.}$$

(3) *If X is path-connected, then $[X : B] = [\pi_1(B,b_0) : p_*(\pi_1(X,x_0))]$.*

Remark. In Exercise 7.7 we use Lemma 7.16 (3) to answer Question 7.3 (2) in the negative.

Proof (∗).

(1) We prove the "⇒"-direction. So let f be a loop in (B,b_0). We denote by $\tilde{f}$ the lift of f to the starting point x_0. If $\tilde{f}$ is a loop in (X,x_0), then it defines an element in $\pi_1(X,x_0)$ and we have $[f] = [p\circ\tilde{f}] = p_*([\tilde{f}]) \in p_*(\pi_1(X,x_0))$. The proof of the "⇐"-direction is Exercise 7.7.[175]
(2) The "⇒"-direction is an immediate consequence of Corollary 7.14. The proof of the "⇐"-direction is done once again in Exercise 7.7. For that direction we need that X is simply connected.
(3) This statement is proved in Exercise 7.7. □

[174] We use the notation from page 233: for $y \in Y$ we denote by e_y the constant path $e_y(t) := x$, $t \in [0,1]$.

[175] Note that $[f]$ lies in $p_*(\pi_1(X,x_0)) \subset \pi_1(B,b_0)$ means that there exists a loop g in (X,x_0) such that $p\circ g$ is *path-homotopic* to f. A priori it does not mean that there exists a loop g with $p\circ g = f$.

7.6. **Group actions and fundamental groups.** The following proposition finally allows us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous sections.

Theorem 7.17. *Let X be a simply connected topological space. Let G be a group which acts continuously and discretely on X. We choose an $x \in X$. We denote by $p\colon X \to X/G$ the canonical projection map. Then the map*

$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x]) \\ g & \mapsto & [p \circ (\textit{path in } X \textit{ from } x \textit{ to } g \cdot x)] \end{array}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

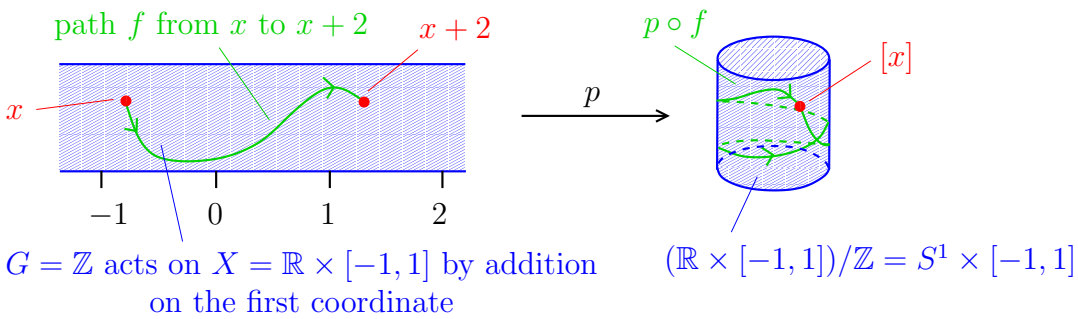


FIGURE 160. Illustration of Theorem 7.17.

Before we provide the proof of Theorem 7.17, let us impatiently use it to finally confirm our suspicion that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proposition 7.18.

(1) *The maps*

$$\Phi\colon \mathbb{Z} \to \pi_1(\mathbb{R}/\mathbb{Z}, 0), \quad n \mapsto \underbrace{\left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ t & \mapsto & p(nt) = [nt] \end{array}\right]}_{\substack{\text{the map } t \mapsto nt \text{ is a} \\ \text{path from } 0 \text{ to } 0+n}} \quad \textit{and} \quad \Psi\colon \mathbb{Z} \to \pi_1(S^1, 1), \quad n \mapsto \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & e^{2\pi i n t} \end{array}\right]$$

are isomorphism of groups.

(2) *Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. The induced map on $\pi_1(S^1, 1)$ is given by multiplication by n.*

(3) *The map*

$$\begin{array}{rcl} \Theta\colon \pi_1(S^1, 1) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz \end{array}$$

is an isomorphism.

Proof.

(1) We consider the action of $G = \mathbb{Z}$ on the simply connected topological space $X = \mathbb{R}$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the canonical projection map. It follows immediately from Theorem 7.17 that the given map $\Phi\colon \mathbb{Z} \to \pi_1(\mathbb{R}/\mathbb{Z}, 0)$ is an isomorphism. We denote by $\phi\colon \mathbb{R}/\mathbb{Z} \to S^1$ the homeomorphism given by $\phi([t]) = e^{2\pi i t}$. If we compose the above isomorphism with the isomorphism $\phi_*\colon \pi_1(\mathbb{R}/\mathbb{Z}, 0) \to \pi_1(S^1, 1)$, then we see that the second map $\Psi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ is also an isomorphism.

(2) This statement follows easily from the definitions. To practice the definitions we will prove this statement in Exercise 7.10.

(3) It is straightforward to see that $\Theta \circ \Psi\colon \mathbb{Z} \to \pi_1(S^1, 1) \to \mathbb{Z}$ is the identity. By (1) the first map Ψ is an isomorphism. Thus $\Theta = \Phi^{-1}$ is also an isomorphism. □

We will provide many more examples for Theorem 7.17 immediately after the proof of Theorem 7.17.

Proof of Theorem 7.17 (∗). [176] Let X be a simply connected topological space. Let G be a group which acts continuously and discretely on X. We choose an $x \in X$. We denote by $p\colon X \to X/G$ the canonical projection map. We consider the map

$$\begin{array}{rcl} \Phi\colon G & \to & \pi_1(X/G, [x]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } gx)]. \end{array}$$

We need to prove the following four statements:

(1) Φ is well-defined,
(2) Φ is a group homomorphism,
(3) Φ is surjective,
(4) Φ is injective.

Now we provide the proofs for these four claims.

(1) We first show that the map Φ is well-defined. Let $g \in G$. Since X is in particular path-connected there exists a path from x to gx. Now let f be an arbitrary path in X from x to gx. From $p(x) = [x] = [gx] = p(gx)$ it follows that $p \circ f$ is a loop in $(X/G, [x])$. In particular $p \circ f$ does indeed define an element in $\pi_1(X/G, [x])$.

Now we need to show that this element does not depend on the choice of f. So let f' be another path in X from x to gx. Since X is simply connected it follows from Lemma 5.13 that the paths f and f' are path-homotopic in X, i.e. there exists a path-homotopy $H\colon [0,1] \times [0,1] \to X$ from f to f'. But then $p \circ H\colon [0,1] \times [0,1] \to X/G$ is a path-homotopy from the loop $p \circ f$ to the loop $p \circ f'$. This shows that the map Φ is well-defined.

(2) Now we show that Φ is a group homomorphism. Let $g, h \in G$ and pick be a path k from x to gx and pick a path l from x to hx. Then gl[177] is a path from gx to ghx

[176] It is perhaps helpful to read the proof with the example of the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ in mind.

[177] Here gl denotes the path $[0,1] \to X$ that is given by $t \mapsto g \cdot l(t)$.

and $k * gl$ is a path from x to ghx. It follows that

$$\begin{aligned}\Phi(gh) \underset{\substack{\uparrow \\ \text{since } k*gl \text{ is a path from } x \text{ to } ghx}}{=} [p \circ (k * gl)] &= [(p \circ k) * (p \circ gl)] \overset{\substack{\text{since } p(gx) = p(x) \in X/G \text{ for all } x \in X \\ \downarrow}}{=} [(p \circ k) * (p \circ l)] \\ &\underset{\substack{\uparrow \\ \text{definition of } \cdot \text{ in } \pi_1(X/G,[x])}}{=} [p \circ k] \cdot [p \circ l] = \Phi(g) \cdot \Phi(h).\end{aligned}$$

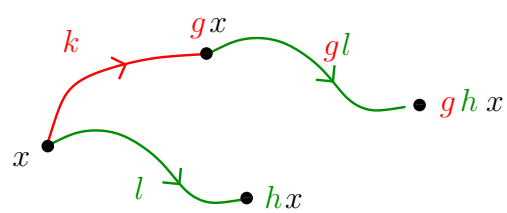


FIGURE 161. Schematic image for the proof that Φ is a group homomorphism.

Before we continue we recall that according to Proposition 7.10 the projection map $p\colon X \to X/G$ is a covering.

(3) Now we prove that Φ is surjective. We choose an arbitrary element $c \in \pi_1(X/G,[x])$ and represent it by a loop f in $(X/G,[x])$. According to Proposition 7.12 there exists a lift $\tilde{f}\colon [0,1] \to X$ of the path f to the starting point x. By definition we have $p \circ \tilde{f} = f$. In particular we obtain for the endpoint of $\tilde{f}$ that $p(\tilde{f}(1)) = [x]$, therefore it follows that $\tilde{f}(1) = gx$ for some $g \in G$. Thus we have proved that

$$\Phi(g) = \big[p \circ (\text{path from } x \text{ to } gx)\big] = [p \circ \tilde{f}] = [f] = c.$$

(4) We conclude the proof of the proposition by showing that Φ is injective. Let $g \in G$. Then the following holds:

$$\begin{aligned}\Phi(g) = e &\implies [p \circ (\text{path } f \text{ from } x \text{ to } gx)] = e \in \pi_1(X/G,[x]) \\ &\implies p \circ f \simeq e_{[x]} \text{ in } X/G \\ &\underset{\substack{\uparrow \\ \text{by Corollary 7.14, since } f \text{ is a lift of } p\circ f \text{ and } e_x \\ \text{is a lift of } e_{[x]} \text{ to the starting point } x}}{\implies} f(1) = e_x(1) \implies gx = x \underset{\substack{\uparrow \\ \text{since the action is discrete,} \\ \text{hence free}}}{\implies} g = e.\end{aligned}$$

This shows that Φ is indeed injective.

We have thus shown that Φ is a well-defined isomorphism. □

Now we can determine the fundamental groups of many of the topological spaces that we have encountered so far.

Corollary 7.19. *We have the following isomorphisms of fundamental groups*

$$
\begin{array}{lrcll}
\text{(A)} & \pi_1(\textit{n-dimensional torus}) & = & \pi_1(\mathbb{R}^n/\mathbb{Z}^n) & \cong \mathbb{Z}^n \\
 & \textit{in particular} & & & \\
 & \pi_1(S^1) & = & \pi_1(\mathbb{R}/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(B)} & \textit{for any } n \neq 0,1 & & & \\
 & \pi_1(\textit{real projective space } \mathbb{R}\mathrm{P}^n) & = & \pi_1(S^n/\{\pm 1\}) & \cong \mathbb{Z}_2, \\
\text{(C)} & \pi_1(\textit{Möbius band }) & = & \pi_1((\mathbb{R}\times[-1,1])/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(D)} & \pi_1(\textit{lens space } L(p,q)) & = & \pi_1(S^3/\mathbb{Z}_p) & \cong \mathbb{Z}_p.
\end{array}
$$

Proof $(*)$. By Proposition 5.14 we know that for any $n \geq 2$ the sphere S^n is simply connected. Furthermore we know by the discussion on page 235 that $\mathbb{R}^n$, $n \geq 1$, and that the strip $\mathbb{R}\times[-1,1]$ are simply connected. We saw on page 264 that all the actions are discrete. The actions are of course also continuous. The corollary is now an immediate consequence of Theorem 7.17. □

Thus we have now finally determined the fundamental groups of many spaces. In Proposition 5.14 we saw that $\pi_1(S^2) = 0$ whereas we have now seen that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. The fundamental groups are not isomorphic, hence the sphere S^2 and the torus $S^1 \times S^1$ are not homeomorphic.

We also obtained a partial answer to Question 7.8: if two lens spaces $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, then $p = \widetilde{p}$. But as of right now we cannot say anything about the parameters q and $\widetilde{q}$.

Examples. Theorem 7.17 not only gives us the fact that $\pi_1(X/G)$ is isomorphic to the group G, but it also gives us an explicit isomorphism. Now we will study this isomorphism in two examples to explicitly determine non-trivial elements in the fundamental group of the torus and the real projective space.

(1) As a reminder, the torus is defined as[178]

$$T \;=\; \mathbb{R}^2/\mathbb{Z}^2 \;=\; [0,1]\times[0,1]/(x,0)\sim(x,1) \text{ and } (0,y)\sim(1,y).$$

[178]Here we are slightly generous in our usage of the equality sign. More precisely, strictly speaking $\mathbb{R}^2/\mathbb{Z}^2$ is not the same as $[0,1]\times[0,1]/\sim$. But there exists a canonical homeomorphism between these two spaces that we use to identify these spaces.

The explicit isomorphism of Theorem 7.17

$$\Phi\colon \mathbb{Z}^2 \quad \to \quad \pi_1(T,(0,0))$$

assumes in particular the following values

$$\Phi((1,0)) \quad = \quad x \quad := \quad \underbrace{\left[\begin{array}{rcl} [0,1] & \to & [0,1]\times[0,1]/\sim \\ t & \mapsto & [(t,0)] \end{array}\right]}_{t\mapsto(t,0) \text{ is a path from } (0,0) \text{ to } (1,0)}$$

and

$$\Phi((0,1)) \quad = \quad y \quad := \quad \underbrace{\left[\begin{array}{rcl} [0,1] & \to & [0,1]\times[0,1]/\sim \\ t & \mapsto & [(0,t)] \end{array}\right]}_{t\mapsto(0,t) \text{ is a path from } (0,0) \text{ to } (0,1)}.$$

In the description of the torus as a subset of $\mathbb{R}^3$, see Figure 162, these two loops correspond to the "longitude" and the "meridian".

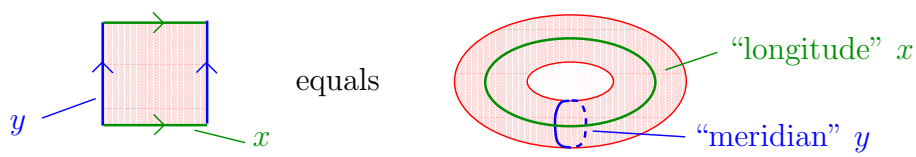


FIGURE 162.

(2) Now we consider the real projective space $\mathbb{RP}^2 = S^2/\{\pm 1\} = S^2/x \sim -x$ and as usual we denote by $p\colon S^2 \to \mathbb{RP}^2$ the projection. We put $Q := (1,0,0)$. We consider the following path in S^2:

$$\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(\pi t), \sin(\pi t), 0). \end{array}$$

This is a path from Q to $-Q = (-1)\cdot Q$. Since -1 is the one non-trivial element of the group $G = \{\pm 1\}$ it follows from the explicit isomorphism of Theorem 7.17 that the equivalence class of $p \circ f$ is precisely the one non-trivial element of the fundamental group $\pi_1(\mathbb{RP}^2, p(Q)) \cong \{\pm 1\}$.[179] Since $\pi_1(\mathbb{RP}^2, p(Q))$ is isomorphic to $\mathbb{Z}_2$ we see that $p \circ f$ represents an element of order 2, i.e. $(p\circ f) * (p \circ f)$ represents the trivial element in the fundamental group. Put differently, $(p\circ f) * (p\circ f)$ is null-homotopic. This one can see quite explicitly. More precisely, we have

$$\begin{aligned} [p\circ f]\cdot[p\circ f] \quad &= \quad [(p\circ f)*(p\circ f)] \quad = \quad [(p\circ(-f))*(p\circ f)] \quad = \quad [p\circ(f*(-f))] \\ &= \quad \left[p \circ \underbrace{\left(\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(2\pi t), \sin(2\pi t), 0) \end{array}\right)}_{\text{loop } g \text{ in } S^2}\right]. \end{aligned}$$

[179] In particular we have now shown that the loop in $\mathbb{RP}^2$ given on page 227 is not null-homotopic.

The loop g in (S^2, Q) is clearly null-homotopic.[180] But then the loop $p \circ g$ is also null-homotopic. This example is illustrated in Figure 163.

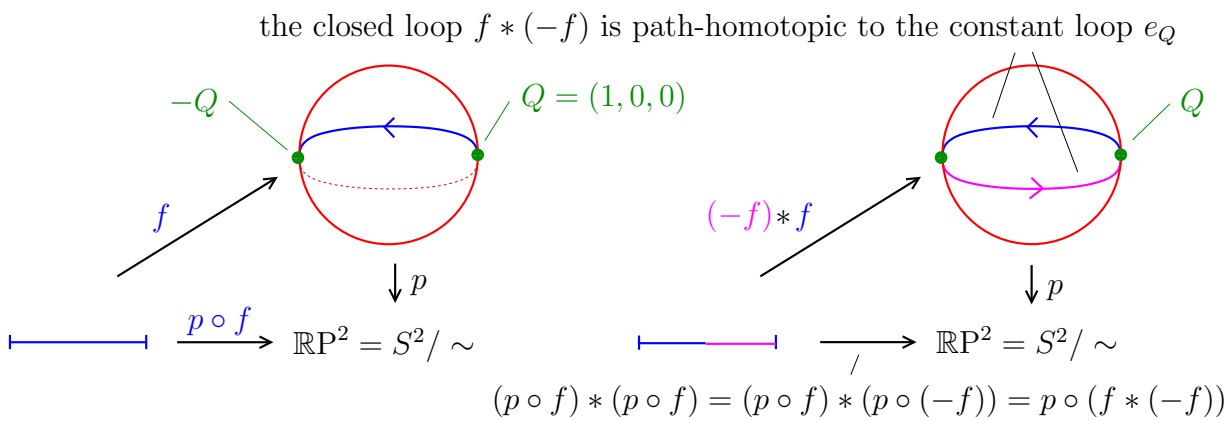


FIGURE 163.

Example. We consider the Klein bottle. Recall that on page 265 we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses. In Exercise 7.6 we will see G acts discretely on $\mathbb{R}^2$ and we saw on page 265 that $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle. It follows from Theorem 7.17 that

$$\pi_1(\text{Klein bottle}) \;\cong\; G = \text{ group that is generated by the homeomorphisms } A \text{ and } B.$$

The maps A and B do not commute. Indeed, the map $A \circ B\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(x,y) \;\mapsto\; (x+1, (1-(y+1))) \;=\; (x+1, -y)$$

whereas the map $B \circ A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(x,y) \;\mapsto\; (x+1, (1-y)+1) \;=\; (x+1, 2-y).$$

Thus we see that the fundamental group of the Klein bottle is a non-abelian group.

In Figure 164 we show the two elements b and c of the fundamental group of the Klein bottle $K = [0,1] \times [0,1]/\sim$ which correspond to B and $C = B^{-1} \circ A$ under the isomorphism of Theorem 7.17 and the homeomorphism from page 265. Since the two homeomorphisms $B(x,y) = (x, y+1)$ and $C(x,y) = (x+1, -y)$ do not commute, the corresponding elements $[b]$ and $[c]$ of $\pi_1(K)$ do not commute either.

[180] More precisely, it is straightforward to write down an explicit path-homotopy from g to the constant path e_Q. Alternatively one can also deduce that g is null-homotopic from our calculation $\pi_1(S^2, Q) = 0$.

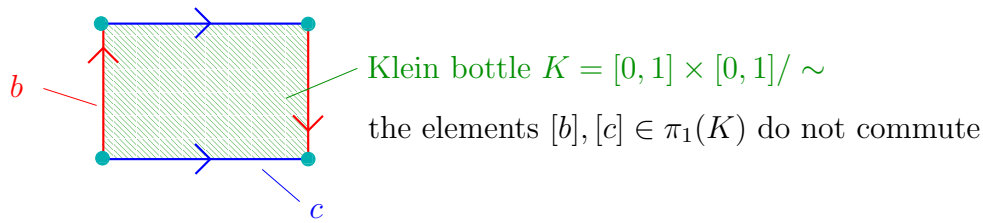


FIGURE 164.

The previous example showed that fundamental groups can be non-abelian groups. In fact, as we will see later, "most" fundamental groups are non-abelian. This will pose several major problems. For example, what is a good way to describe a non-abelian group? How can we decide whether or not two non-abelian groups are isomorphic?

We conclude this section with a more unusual topological space. Namely we remark in Exercise 7.9 we will use Theorem 7.17 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 66, is isomorphic to $\mathbb{Z}$. In particular we show that the loop given on page 66 is not null-homotopic. This answers the question we had posed on page 228.

7.7. **The fundamental group of the product of two topological spaces.** The next proposition says that the fundamental groups of the product $A \times B$ of two topological spaces A and B is the direct product of the fundamental groups of A and B. Before we state the proposition explicitly we recall the definition of the direct product of two groups.

Definition. Let G and H two groups. We can form the cartesian product

$$G \times H \ := \ \{(g,h) \,|\, g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1, h_1) \cdot (g_2, h_2) \ := \ (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H*.

We start out with the following lemma, which strictly speaking we will not need, but which is worth knowing.

Lemma 7.20. $(*)$ *Let $\varphi_1 \colon G_1 \to \pi$ and $\varphi_2 \colon G_2 \to \pi$ be two group homomorphisms. If for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute, i.e. if we have $\varphi_1(g_1) \cdot \varphi_2(g_2) = \varphi_2(g_2) \cdot \varphi_1(g_1)$, then the map*

$$\begin{array}{rcl} \varphi_1 \times \varphi_2 \colon G_1 \times G_2 & \to & \pi \\ (g_1, g_2) & \mapsto & \varphi_1(g_1) \cdot \varphi_2(g_2) \end{array}$$

is a homomorphism.[181]

Proof (∗). We write $\Phi = \varphi_1 \times \varphi_2$. Let $g_1, g_1' \in G_1$ and let $g_2, g_2' \in G_2$. We have

$$\begin{array}{rcl} \Phi(g_1,g_2)\cdot\Phi(g_1',g_2') & = & \varphi_1(g_1)\cdot\varphi_2(g_2)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2') \\ & = & \varphi_1(g_1)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2)\cdot\varphi_2(g_2') \\ & \uparrow & \\ & \text{by our hypothesis} & \end{array} \begin{array}{rcl} & & \\ = & \varphi_1(g_1g_1')\cdot\varphi_2(g_2g_2') \\ = & \Phi(g_1g_1',g_2g_2'). \end{array}$$

□

Now we can formulate the promised proposition.

Proposition 7.21. *Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the inclusion maps*

$$\begin{array}{rcl} i\colon A & \to & A\times B \\ a & \mapsto & (a,b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon B & \to & A\times B \\ b & \mapsto & (a_0,b) \end{array}$$

and we consider the projection maps

$$\begin{array}{rcl} p\colon A\times B & \to & A \\ (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A\times B & \to & B \\ (a,b) & \mapsto & b. \end{array}$$

Then the maps

$$\begin{array}{rcl} \Phi\colon \pi_1(A,a_0)\times\pi_1(B,b_0) & \to & \pi_1(A\times B,(a_0,b_0)) \\ (x,y) & \to & i_*(x)\cdot j_*(y) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \pi_1(A\times B,(a_0,b_0)) & \to & \pi_1(A,a_0)\times\pi_1(B,b_0) \\ z & \to & (p_*(z),q_*(z)) \end{array}$$

are both homomorphisms[182] *and they are inverses of one another, in particular both are isomorphisms.*

Example. It follows from Proposition 7.21 and the above calculations of fundamental groups that

$$\pi_1(S^1\times L(p,q)) \;\cong\; \pi_1(S^1)\times\pi_1(L(p,q)) \;\cong\; \mathbb{Z}\times\mathbb{Z}_p.$$

Of course we can also apply Proposition 7.21 several times and we see once again that

$$\pi_1(\underbrace{S^1\times\cdots\times S^1}_{n\text{ times}}) \;\cong\; \underbrace{\pi_1(S^1)\times\cdots\times\pi_1(S^1)}_{n\text{ times}} \;\cong\; \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{n\text{ times}} \;\cong\; \mathbb{Z}^n.$$

Finally it follows from Propositions 5.14 and 7.21 that for any $k,l \geq 2$ we have

$$\pi_1(S^k\times S^l) \;\cong\; \pi_1(S^k)\times\pi_1(S^l) \;\cong\; 0\times 0 \;=\; 0.$$

In particular we see that the fundamental group cannot distinguish the products of spheres of dimension ≥ 2. For example at the moment we cannot show that the two 6-dimensional manifolds $S^3\times S^3$ and $S^2\times S^2\times S^2$ are not homeomorphic.

[181] In fact the converse also holds, if $\varphi_1\times\varphi_2$ is a homomorphism, then the for every $g_1\in G_1$ and $g_2\in G_2$ the images $\varphi_1(g_1)\in\pi$ and $\varphi_2(g_2)\in\pi$ commute. Why is that?

[182] Note that a priori it is not clear that Φ is a homomorphism since it is not clear whether the hypothesis of Lemma 7.20 is satisfied.

In the proof of Proposition 7.21 we will need the following purely group-theoretic lemma.

Proof of Proposition 7.21 $(*)$. It is clear that $\Psi\colon \pi_1(A\times B,(a_0,b_0)) \to \pi_1(A,a_0)\times\pi_1(B,b_0)$ is a homomorphism. Thus it suffices to show that Φ and Ψ are inverses of one another. Once we have shown that, it follows for free that Φ is also a homomorphism.

To show that Φ and Ψ are inverses of one another we introduce another map, namely we consider[183]

$$\begin{array}{rcl} \Theta\colon \pi_1(A,a_0)\times\pi_1(B,b_0) & \to & \pi_1(A\times B,(a_0,b_0)) \\ ([f\colon (S^1,1)\to(A,a_0)],[g\colon (S^1,1)\to(B,b_0)]) & \to & \left[\begin{array}{rcl} (S^1,1) & \to & (A\times B,(a_0,b_0)) \\ z & \mapsto & (f(z),g(z)) \end{array}\right]. \end{array}$$

It is almost a tautology that Θ and Ψ are inverses of one another. Thus it remains to prove the following claim.

Claim. The two maps $\Phi,\Theta\colon \pi_1(A,a_0)\times\pi_1(B,b_0)\to\pi_1(A\times B,(a_0,b_0))$ agree.

We use the convention from page 228 to change our view of fundamental groups. Thus let $f\colon[0,1]\to A$ be a loop in (A,a_0) and $g\colon[0,1]\to B$ be a loop in (B,b_0). We need to show that

$$i_*([f])\cdot j_*([g]) \;=\; [(t\mapsto(f(t),g(t))] \;\in\; \pi_1(A\times B,(a_0,b_0).$$

In other words, we need to show that the two loops

$$\begin{array}{rcl} [0,1] & \to & A\times B \\ t & \mapsto & \begin{cases} (f(2t),b_0), & \text{if } t\in[0,\frac12] \\ (a_0,g(2t-1)), & \text{if } t\in[\frac12,1] \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} [0,1] & \to & A\times B \\ t & \mapsto & (f(t),g(t)) \end{array}$$

are path-homotopic. In fact an explicit path-homotopy is given by the map[184]

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & A\times B \\ (t,s) & \mapsto & \begin{cases} \big(f\big(\frac{2t}{1+s}\big),g(st)\big), & \text{if } t\in[0,\frac12] \\ \big(f(st),g\big(\frac{2t-1+s}{1+s}\big)\big), & \text{if } t\in[\frac12,1]. \end{cases} \end{array}$$

□

Summarizing we have now finally managed to calculate many non-trivial fundamental groups. Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) graphs,
(2) knot complements,
(3) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C}\setminus\{0,1\}$,
(4) surfaces of genus ≥ 2.

In the following chapters we will introduce more methods for determining the fundamental group of a given topological space. But first we give in the next section two applications of our results.

[183] One can easily verify that Θ is well-defined.

[184] Note that this is indeed a path-homotopy of loops.

7.8. **Applications: the Fundamental Theorem of Algebra and the Borsuk-Ulam Theorem.** As a first application of our methods we give a proof of the Fundamental Theorem of Algebra. Somewhat surprisingly, contrary to the standard proof [Lan, Corollary III.7.6], this proof does not make any use of complex analysis.

Theorem 7.22. (Fundamental Theorem of Algebra) *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

Proof. Let $q(z)$ be a polynomial with complex coefficients with $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$. We consider the path

$$\begin{array}{rcl} f_r^{q(z)} \colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(re^{2\pi i s})/q(r)}{\left|q(re^{2\pi i s})/q(r)\right|} \end{array}$$

Note that $f_r^{q(z)}(s=0) = f_r^{q(z)}(s=1) = 1$, i.e. $f_r^{q(z)}$ is a loop in $(S^1, 1)$. Now let

$$p(z) \quad = \quad a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 1$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. For $t \in [0,1]$ we also consider the polynomial

$$p_t(z) \quad := \quad a_n z^n + t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n z^n$. Since $a_n \neq 0$ there exists an $s > 0$ such that[185]

$$|a_n z^n| \ > \ |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

for any z with $|z| = s$. This inequality implies that for any $t \in [0,1]$ and any z with $|z| = s$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = s$.

We obtain the following sequence of path-homotopic loops:

$$\text{constant loop at } 1 \ \overset{\substack{\text{follows from the definition} \\ \downarrow}}{=} \ f_0^{p(z)} \ \underset{\substack{\uparrow \\ \text{the maps } f_r^{p(z)} \text{ with } r \in [0,s] \text{ are defined,} \\ \text{since the polynomials } p(z) \text{ has no zeros} \\ \text{and they form a path-homotopy from } f_0^{p(z)} \text{ to } f_s^{p(z)}}}{\simeq} \ f_s^{p(z)} \ = \ f_s^{p_1(z)} \ \underset{\substack{\uparrow \\ \text{the maps } f_s^{p_t(z)} \text{ with } t \in [0,1] \text{ are defined,} \\ \text{since the polynomials } p_t(z) \text{ have no zeros on } |z| = s, \\ \text{and they form a path-homotopy from } f_r^{p_1(z)} \text{ to } f_r^{p_0(z)}}}{\simeq} \ f_s^{p_0(z)} \ \overset{\substack{\text{follows from } p_0(z) = a_n z^n \\ \downarrow}}{=} \ \text{the loop} \ \begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & e^{2\pi i n t}. \end{array}$$

By Proposition 7.18 we had know that the loop on the right defines a non-trivial element in the fundamental group $\pi_1(S^1, 1)$, but we had just shown that it is path-homotopic to the constant path. Thus we have obtained a contradiction. $\square$

The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984).[186] It was first proved in 1932.

[185] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $s \in \mathbb{R}$ such that $s \cdot |a_n| > |a_{n-1}| + \cdots + |a_0|$ will work.

[186] Ulam became much more famous through his work on the hydrogen bomb.

Theorem 7.23. (Borsuk-Ulam Theorem) *For every map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.*

Example.

(1) It follows from the Borsuk-Ulam Theorem that there exist two antipodal points on earth with the same temperature and the same barometric pressure.
(2) The Borsuk-Ulam Theorem implies in particular that there is no injective map from $S^2 \to \mathbb{R}^2$.
(3) The statement of the Borsuk-Ulam Theorem also makes sense for 2 replaced by any $n \in \mathbb{N}$. For $n = 1$ one can prove the statement using elementary methods.[187] We will address the problem for $n \geq 3$ once we have introduced homology groups.

Proof. We prove the theorem by contradiction. So suppose there exists a map $f\colon S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all $x \in S^2$. This allows to define the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^1 \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

Note that $g(-x) = -g(x)$ for all $x \in S^2$. Now let η be the loop that is defined by

$$\begin{array}{rcl} \eta\colon [0,1] & \to & S^2 \\ s & \mapsto & \eta(s) = (\cos(2\pi s), \sin(2\pi s), 0) \end{array}$$

and let $h = g \circ \eta\colon [0,1] \to S^1$ be the composition of η with g. We pick a $c \in \mathbb{R}$ such that $h(0) = e^{2\pi i c}$. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$, the usual covering map. We note that for $z, w \in \mathbb{R}$ we have

$$p(z) = p(w) \iff z - w \in \mathbb{Z} \quad \text{and} \quad p(z) = -p(w) \iff z - w \in \tfrac{1}{2} + \mathbb{Z}.$$

By Proposition 7.12 we can lift the path $h\colon [0,1] \to S^1$ to a path $\widetilde{h}\colon [0,1] \to \mathbb{R}$ with $\widetilde{h}(0) = c$. We refer to Figure 165 for an illustration of all the definitions.

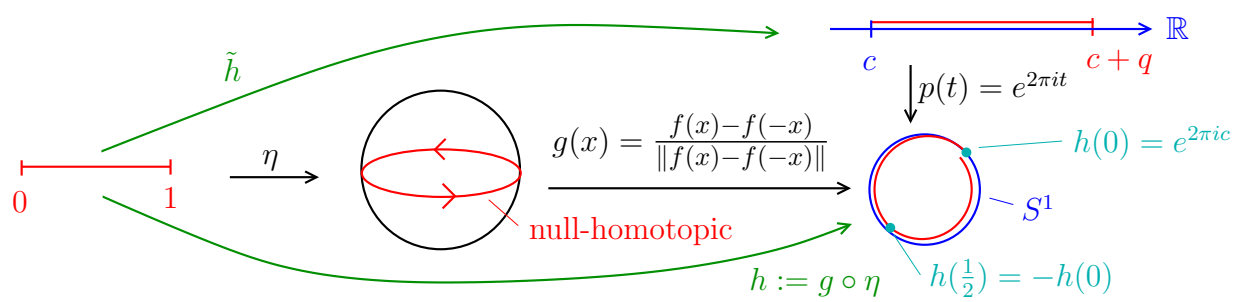


FIGURE 165. Illustration of the proof of the Borsuk-Ulam Theorem 7.23

[187] How would you prove the statement?

Given any $s \in [0, \frac{1}{2}]$ we have

$$p\big(\widetilde{h}\big(s+\tfrac{1}{2}\big)\big) \;=\; h\big(s+\tfrac{1}{2}\big) \;=\; g\big(\eta\big(s+\tfrac{1}{2}\big)\big) \underset{\substack{\uparrow \\ \text{by the definition of } \eta}}{=} g(-\eta(s)) \underset{\substack{\uparrow \\ \text{since } g(-x)=-g(x) \text{ for all } x\in S^2}}{=} -g(\eta(s)) \;=\; -h(s) \;=\; -p\big(\widetilde{h}(s)\big).$$

By the above discussion of p this implies that

$$\widetilde{h}\big(s+\tfrac{1}{2}\big) - \widetilde{h}(s) \;=:\; \frac{q(s)}{2}$$

with $q(s)$ an odd integer. The map $q\colon [0, \frac{1}{2}] \to \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from Lemma 1.92 that it is constant. We denote this constant value by q. Since q is odd it is in particular non-zero.

It follows from the above that

$$\widetilde{h}(1) \;=\; \widetilde{h}\big(\tfrac{1}{2}\big) + \tfrac{q}{2} \;=\; \widetilde{h}(0) + \tfrac{q}{2} + \tfrac{q}{2} \;=\; \widetilde{h}(0) + q.$$

Since q is non-zero we obtain from Lemma 7.16 that $h = g \circ \eta$ is not null-homotopic. But h was the composition $h = g \circ \eta\colon [0,1] \to S^2 \to S^1$. By Proposition 5.14 the loop η is null-homotopic in S^2 which implies by Proposition 6.1 that $h = g \circ \eta$ is null-homotopic in S^1. Thus we have obtained a contradiction. □

Exercises for Chapter 7.

Exercise 7.1. Prove Lemma 7.2.

Exercise 7.2.

(a) Let $p\colon X \to B$ be a map between topological spaces that is two-to-one, i.e. such that for any $b \in B$ the preimage $p^{-1}(b)$ consists of precisely two points. Is p necessarily a covering map?

(b) Is the map

$$\begin{array}{rcl} \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

a covering map? Hint: $z = x + iy$ with $x, y \in \mathbb{R}$.

(c) Let $p\colon X \to B$ be a map between topological spaces that is a local homeomorphism and that is surjective. Is p necessarily a covering map? Give a short justification for your answer.

(d) Let $p\colon X \to B$ be a surjective map between topological spaces. Let $f\colon [0,1] \to B$ be a map. Does there always exist a lift from f to X, i.e. does there exist a map $g\colon [0,1] \to X$ such that $p \circ g = f$? Give a short justification for your answer.

Exercise 7.3. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

Show that this map is continuous.

Exercise 7.4. Prove Lemma 7.4 (3).

Exercise 7.5. Consider the 5-fold covering map

$$\begin{array}{rcl} p\colon X = S^1 & \to & B = S^1 \\ z & \mapsto & z^5. \end{array}$$

Let $f\colon [0,1] \to B = S^1$ be the loop given by $t \mapsto e^{2\pi i k t}$. By Proposition 6.11 this loop lifts to a path $g\colon [0,1] \to X = S^1$. For which k is the lift g again a loop?

Exercise 7.6. We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means

$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

(a) For $n \in \mathbb{Z}$, what is $A^{-n}BA^n$? (Perhaps start out with $n = 1$.)
(b) We write $C = A^2$. We denote by H the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by C and B. This means

$$H = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } C, B, C^{-1} \text{ and } B^{-1}. \end{array}$$

Does A lie in H?
Hint: Consider which self-diffeomorphisms of $\mathbb{R}^2$ are orientation-preserving.
(c) Show that any element X of G lies in H or it is of the form $G = AY$ with $Y \in H$.
(d) Show that the group G acts discretely on $\mathbb{R}^2$.

Exercise 7.7.

(a) Prove the "$\Leftarrow$"-direction of Lemma 7.16 (1).
(b) Prove the "$\Leftarrow$"-direction of Lemma 7.16 (2).
(c) Prove Lemma 7.16 (3).
(d) Answer Question 7.3 (2) in the negative.

Exercise 7.8.

(a) Does there exist a covering map $p\colon K \to T$ from the Klein bottle to the 2-dimensional torus?
(b) Let X be a topological space and let $b \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1, 1) \to (X, x_0)$ with $b \in f_*(\pi_1(S^1, 1))$?
(c) Let X be a topological space and let $b, c \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1 \times S^1, (1,1)) \to (X, x_0)$ with $b, c \in f_*(\pi_1(S^1 \times S^1, (1,1)))$?

Exercise 7.9.

(a) Use Theorem 7.17 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 66, is isomorphic to $\mathbb{Z}$.
(b) Let $f\colon S^1 \to X$ be the map that we introduced on page 66. Show that the induced map $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, A)$ is an isomorphism.

Exercise 7.10. Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. Show that the induced map on $\pi_1(S^1, 1)$ is given by multiplication by n.

Exercise 7.11. Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the maps

$$\begin{array}{rcl} f\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon B & \to & A \times B \\ b & \mapsto & (a_0, b) \end{array}$$

and the map

$$\begin{array}{rcl} \Phi\colon \pi_1(A, a_0) \times \pi_1(B, b_0) & \to & \pi_1(A \times B, (a_0, b_0)) \\ (x, y) & \to & f_*(x) \cdot g_*(y) \end{array}$$

(a) Show that Φ is a monomorphism, i.e. that it is injective.
Hint: Consider the maps

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a, b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a, b) & \mapsto & b. \end{array}$$

(b) Show that Φ is an epimorphism, i.e. that it is surjective.
Hint: Let $h(t) = (\alpha(t), \beta(t))$ be a loop in $(A \times B, (a_0, b_0))$. Consider the map

$$\begin{array}{rcl} \Psi\colon [0,1] \times [0,1] & \to & A \times B \\ (s, t) & \mapsto & (\alpha(s), \beta(t)). \end{array}$$

Hint: You might want to use ideas similar to the proof that the map Φ is a homomorphism.

Exercise 7.12. Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. In Lemma 3.28 we saw that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a homeomorphism. We identify $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0)$ with $\mathbb{Z}^n$ as in Theorem 7.17. Show that the following diagram

$$\begin{tikzcd} \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) \arrow[r, "f(A)_*"] \arrow[d, "="] & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) \arrow[d, "="] \\ \mathbb{Z}^n \arrow[r, "v \mapsto Av"] & \mathbb{Z}^n \end{tikzcd}$$

commutes.

8. Covering spaces and manifolds

8.1. **Coverings of manifolds.** In this section we show that covering spaces of (topological) manifolds are themselves (topological) manifolds.

Proposition 8.1. *Let $p\colon \widetilde{X} \to X$ be a countable covering*[188] *of a connected topological space X.*

(1) *If X is an n-dimensional topological manifold, then $\widetilde{X}$ is also an n-dimensional topological manifold. Furthermore, $\partial\widetilde{X} = p^{-1}(\partial X)$.*
(2) (a) *If X admits a smooth atlas, then $\widetilde{X}$ also admits a smooth atlas such that p is a local diffeomorphism,*[189] *in particular such that p is a smooth map.*
  (b) *If X is equipped with a smooth structure*[190] *then $\widetilde{X}$ admits a* unique *smooth structure such that $p\colon \widetilde{X} \to X$ is a local diffeomorphism.*
(3) *If X is an oriented manifold, then $\widetilde{X}$ also admits an orientation such that $p\colon \widetilde{X} \to X$ is orientation-preserving.*

Convention. If X is a manifold, i.e. if X is a topological manifold equipped with a smooth structure, and if we have constructed a finite covering $p\colon \widetilde{M} \to M$, then we always equip $\widetilde{M}$ with the unique smooth structure given by Proposition 8.1 (2b).

In the proof of Proposition 8.1 we will need the following technical lemma.

Lemma 8.2. *Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. If X is second-countable, then $\widetilde{X}$ is also second-countable.*

Proof. Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. Recall that we say $U \subset X$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{\widetilde{U}_i\}_{i\in I}$ with the property that the restriction of p to each subset $\widetilde{U}_i$ is a homeomorphism. If U is uniformly covered, then every open subset of U is also uniformly covered.

Let $\mathcal{B}$ be a countable basis for the topology of X. We denote by $\mathcal{U} = \{U_j\}_{j\in J}$ the family of all open subsets of X that are uniformly covered. Since p is a covering we have $\bigcup_{j\in J} U_j = X$. It follows from Lemma 1.26 that

$$\mathcal{B}' := \{B \in \mathcal{B} \mid \text{ there exists a } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

Now let $B \in \mathcal{B}'$. Since open subsets of uniformly covered subsets are uniformly covered we know that $p^{-1}(B)$ is the union of disjoint open sets B_i, $i \in I_B$ such that the restriction

[188] Recall that a covering $p\colon \widetilde{X} \to X$ of a connected topological space is called countable if the index, i.e. the cardinality of the preimage of a point in X, is countable.

[189] Recall that a map $f\colon M \to N$ between two manifolds is called a *local diffeomorphism* if given any $P \in M$ there exists an open neighborhood U of P and open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a diffeomorphism.

[190] We refer to page 183 for the definition of a smooth structure on a manifold.

of p to each B_i is a homeomorphism. Since p is a countable covering the index set I_B is countable.

Claim.

$$\widetilde{\mathcal{B}} := \{B_i \mid B \in \mathcal{B}' \text{ and } i \in I_B\}$$

is a basis for the topology of $\widetilde{X}$.

It is straightforward to see that $\widetilde{\mathcal{B}}$ has the basis property. Every set in $\widetilde{\mathcal{B}}$ is of course open in $\widetilde{X}$. By Lemma 1.25 it suffices to show that given any open set W of $\widetilde{X}$ and any point $x \in W$ there exists a set B_i in $\widetilde{\mathcal{B}}$ with $x \in B \subset W$. So let $x \in W$. Since $\mathcal{B}'$ is a basis for the topology of X there exists a $B \in \mathcal{B}'$ with $p(x) \in B \subset p(W)$. But then there exists an $i \in I_B$ with $x \in B_i \subset W$. This completes the proof of the claim.

Since $\mathcal{B}'$ is countable and since each I_B is countable we see that $\widetilde{\mathcal{B}}$ consists of countably many open sets. This shows that $\widetilde{X}$ is second-countable. □

Now we can provide the proof of Proposition 8.1.

Proof of Proposition 8.1. Let $p\colon \widetilde{X} \to X$ be a countable covering of a connected n-dimensional topological manifold X.

(1) First we note that it follows from the hypothesis that X is a topological manifold that X is Hausdorff. It follows from Lemma 7.4 (3) that $\widetilde{X}$ is also Hausdorff. Similarly it follows from our hypothesis that X is a topological manifold and that p is a countable covering together with Lemma 8.2 that $\widetilde{X}$ is second-countable.

Since $p\colon \widetilde{X} \to X$ is a covering there exists a family of open subsets $\{U_i\}_{i\in I}$ of X such that each U_i is uniformly covered. Furthermore, since X is an n-dimensional manifold there exists an atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. After possibly replacing each V_j by the family $\{V_j \cap U_i\}_{i\in I}$ we can without loss of generality assume that each V_j is uniformly covered. Thus for each j the preimage $p^{-1}(V_j)$ is the union of disjoint open subsets $\{\widetilde{V}_{ji}\}_{i\in I_j}$ with the property, that the restriction of p to each subset $\widetilde{V}_{ji}$ is a homeomorphism. It is now clear that $\{\Phi_j \circ p\colon \widetilde{V}_{ji} \to W_j \mid j \in J \text{ and } i \in I_j\}$ is an atlas for $\widetilde{X}$.

Finally we intend to determine the boundary of $\widetilde{X}$. Let $y \in \widetilde{X}$. Using the fact that $p\colon \widetilde{X} \to X$ is a local homeomorphism one can easily show that the point $y \in \widetilde{X}$ does not admit a chart of type (i) in the sense of page 171 if and only if $p(y) \in X$ does not admit such a chart. This implies that $\partial\widetilde{X} = p^{-1}(\partial X)$.

(2) (a) The transition maps of the atlas from (1) are precisely given by restrictions to open subsets of the transition maps of the atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. Thus if we start out with a smooth atlas for X, then the atlas we just constructed for $\widetilde{X}$ is also smooth.

We also know that a covering map is a local homeomorphism. It follows immediately from the definitions that with this smooth atlas for $\widetilde{X}$ these homeomorphisms are in fact diffeomorphisms.

(b) This statement follows easily from (a), together with Lemma 3.15. We leave it to the reader to fill in the details.

(3) The last statement follows easily from the fact that p is a local diffeomorphism. $\square$

8.2. **The orientation covering of a non-orientable manifold.** We saw that many of our examples of non-orientable manifolds admit a 2-fold covering such that the covering space is an orientable manifold.

(1) For example the projection map $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\}$ shows that the real projective plane is covered by the sphere.

(2) On page 260 we had considered the map

$$\begin{array}{rcl} p\colon \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+\frac{1}{2},1-y)}^{\text{Möbius band}} \\ [(x,y)] & \to & [(x,y)] \end{array}$$

and we remarked this is a 2-fold covering. Thus the non-orientable Möbius band admits a 2-fold covering such that the covering space is orientable.

(3) Almost the same map as in (2) also shows that the torus is a 2-fold covering of the Klein bottle.

Now we will see that any non-orientable manifold admits a unique 2-fold covering such that the covering space is orientable.

Proposition 8.3. *Let M be a connected non-orientable manifold. The following two statements hold:*

(1) *There exists a connected 2-fold covering $p\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable.*

(2) *If $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ are two connected 2-fold coverings such that $\widetilde{M}$ and $\widehat{M}$ are orientable, then there exists a diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \widehat{M} \arrow[dl, "q"] \\ & M. & \end{tikzcd}$$

Definition. Let M be a connected non-orientable manifold. We refer to the covering $p\colon \widetilde{M} \to M$ from Proposition 8.3 (1) as the *orientation covering of M*. By Proposition 8.3 (2) this covering is unique in an appropriate sense, which justifies speaking of *the* orientation covering.

Proof. We start out with a general construction. Let W be an n-dimensional manifold. We consider the set

$$\widetilde{W} \;=\; \{(Q,O) \mid Q\in W \text{ and } O \text{ is an orientation of } T_QW\}$$

together with the map

$$\begin{array}{rcl} p\colon \widetilde{W} & \to & W \\ (Q,O) & \mapsto & Q. \end{array}$$

Now we have to put a topology on $\widetilde{W}$. We use the following notation:

(1) We denote by $O_{\mathbb{R}^n}$ the standard orientation of $\mathbb{R}^n$.
(2) Given an orientation O for an n-dimensional vector space we denote by $-O$ the unique other orientation.
(3) Given an isomorphism $\varphi\colon V \to W$ between vector spaces and given an orientation O on W we denote by φ^*O the corresponding orientation on V.

Given a chart $\Phi\colon U \to V$ of W and $\epsilon \in \{-,+\}$ we define[191]

$$B_\epsilon(\Phi) \ := \ \{(Q, \epsilon \cdot \Phi^* O_{\mathbb{R}^n}) \mid Q \in U\} \ \subset \ \widetilde{W}.$$

It is straightforward to show that the collection of these sets $B_\pm(\Phi)$ satisfies the basis properties (B1) and (B2) from page 60.[192] Now we endow $\widetilde{W}$ with the topology generated by the sets $B_\pm(\Phi)$. We leave it as an exercise to show that $p\colon \widetilde{W} \to W$ is continuous. Next we show that p is a 2-fold covering. So let $Q \in W$. We pick a connected chart[193] $\Phi\colon U \to V$ around Q. Then $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. It follows from Lemma 3.37 (1) applied to the connected manifold U that $B_-(\Phi)$ and $B_+(\Phi)$ are disjoint. Furthermore it is straightforward to see that the two projections maps $p\colon B_\pm(\Phi) \to U$ are homeomorphisms. This shows that U is uniformly covered by two sets, i.e. p is a 2-fold covering.

In the proof of Proposition 8.1 we saw that a smooth atlas for $\widetilde{W}$ is given by all maps $B_\pm(\Phi) \xrightarrow{p} U \xrightarrow{\Phi} V$ for all connected charts $\Phi\colon U \to V$ of W. In particular $\widetilde{W}$ is again an n-dimensional manifold and with this atlas the projection map $p\colon \widetilde{W} \to W$ is a local diffeomorphism.

Now we show that $\widetilde{W}$ is orientable. So let $(Q,O) \in \widetilde{W}$. The map $p\colon \widetilde{W} \to W$ is a local diffeomorphism. Now we equip $T_{(Q,O)}\widetilde{W}$ with the orientation p^*O.[194] By Footnote 89 it suffices to consider the continuity of the orientation with respect to the above atlas. So let $\Phi\colon U \to V$ be a connected chart of W. Let $\epsilon \in \{-,+\}$ and denote by $\Phi_\epsilon\colon B_\pm(\Phi) \to V$ the corresponding chart of $\widetilde{W}$. But then by construction we know that Φ_ϵ is orientation-preserving for all points if $\epsilon = +$ and it is orientation-reversing for all points if $\epsilon = -$.

We conclude this discussion of $\widetilde{W}$ with the following claim.

Claim. Suppose W is connected. Then $\widetilde{W}$ is connected if and only if W is non-orientable.

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[191] Strictly speaking we should write $(d_P\Phi)^*O_{\mathbb{R}^n}$ instead of just $\Phi^*O_{\mathbb{R}^n}$.

[192] OK, so why does it satisfy (B2)?

[193] Here we say a chart $\Phi\colon U \to V$ is connected if U is connected.

[194] Put differently, using p we can identify $T_{(Q,O)}\widetilde{W}$ with T_QW and we equip it with the orientation at hand, namely O.

If W is orientable, then it is straightforward to see that $\widetilde{W}$ consists of two disjoint components each of which is diffeomorphic to W. Now suppose that W is non-orientable. We need to show that $\widetilde{W}_{,}$ is connected. If it was not connected, then it would consist of at least two components $\{\widetilde{W}_{,i}\}_{i\in I}$. Since $W_{,}$ is connected the restriction of each $p\colon \widetilde{W}_{,} \to W_{,}$ to $p_i\colon \widetilde{W}_{,i} \to W_{,}$ would also be a covering. Clearly $2 = [\widetilde{W}_{,} : W_{,}] = \sum_{i\in I}[\widetilde{W}_{,i} : W_{,}]$. It follows that $\#I = 2$ and that each $p_i\colon \widetilde{W}_{,i} \to W_{,}$ is a covering of degree one, i.e. each p_i is a homeomorphism. Since p_i is a local diffeomorphism it follows that $p_i\colon \widetilde{W}_{,i} \to W_{,}$ is in fact a diffeomorphism. But that is not possible since $W_{,}$ was assumed to be non-orientable whereas $\widetilde{W}_{,i}$ is orientable. This concludes the proof of the claim.

Finally we turn to the actual proof of the proposition. Thus let M be a connected non-orientable manifold.

(1) It follows from the above discussion and the above claim that $\widetilde{M}$ is a oriented connected manifold and that $p_M\colon \widetilde{M} \to M$ is a 2-fold covering.
(2) Suppose that $q\colon N \to M$ is also a connected 2-fold covering such that N is orientable. It follows easily from the construction of $\widetilde{M}$ and $\widetilde{N}$ and the fact that q is a local diffeomorphism that there exists a 2-fold covering $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ which makes the following diagram commute

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\widetilde{q}"] \arrow[d, "p_N"'] & \widetilde{M} \arrow[d, "p_M"] \\ N \arrow[r, "q"] & M. \end{tikzcd}$$

From the above claim and the fact that N is oriented connected we obtain that $\widetilde{N}$ consists of two components $\widetilde{N}_1$ and $\widetilde{N}_2$. The restriction of p_N to $\widetilde{N}_1$ and the restriction of $\widetilde{q}$ to $\widetilde{N}_1$ are 1-fold coverings, hence diffeomorphisms. It is clear that $\widetilde{q} \circ (p_N|_{\widetilde{N}_1})^{-1}\colon N \to \widetilde{M}$ has the desired properties. □

Corollary 8.4.

(1) *If M is a connected non-orientable manifold, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.*
(2) *Any simply connected manifold is orientable.*

Proof. The second statement of the corollary is an immediate consequence of the first statement. Thus it suffices to prove the first statement.

Let M be a connected non-orientable manifold. By Proposition 8.3 there exists a connected 2-fold covering $p\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. It follows from Lemma 7.16 that $p_*(\pi_1(\widetilde{M}))$ is an index 2 subgroup of $\pi_1(M)$. In Exercise 8.1 we will show that every index 2 subgroup of a group π is normal. In particular $p_*(\pi_1(\widetilde{M}))$ is a normal index 2

subgroup of $\pi_1(M)$. Thus we obtain an epimorphism $\pi_1(M) \to \Gamma := \pi_1(M)/p_*(\pi_1(\widetilde{M}))$. The group Γ has two elements, but any group with two elements is isomorphic to $\mathbb{Z}_2$. Thus we found the desired epimorphism $\pi_1(M) \to \mathbb{Z}_2$. $\square$

Remark. A similar proof of Proposition 8.3 (1) with slightly more detail is also given in [Le4, Chapter 8.7].

We conclude this section with a discussion of orientation-preserving and reversing loops.

Definition. Let M be a connected non-orientable manifold, let $x_0 \in M$ be a point and let $\gamma\colon [0,1] \to M$ be a loop in x_0. We denote by $p\colon \widetilde{M} \to M$ the orientation covering provided by Proposition 8.3. We pick $\widetilde{x}_0 \in p^{-1}(x_0)$ and we denote by $\widetilde{g}\colon [0,1] \to \widetilde{M}$ the unique lift of γ, provided by Proposition 7.12, to the starting point x_0. We say γ is *orientation-preserving* if $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)$, otherwise we say that γ is *orientation-reversing*.[195]

Example. Let M be the Möbius band and let $\widetilde{M}$ be the annulus. As on page 260 we consider the 2-fold covering $p\colon \widetilde{M} \to M$ which is just an incarnation of the orientation covering. We pick a point in M and we denote by γ a loop that goes "once around M" The lift of γ is not a loop, hence γ is orientation-reversing. We refer to Figure 166 for an illustration.

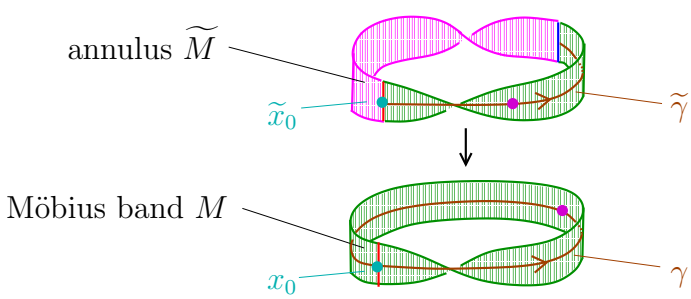


FIGURE 166.

We conclude this chapter with the following lemma which gives an alternative proof to Corollary 8.4 (1).

Lemma 8.5. *Let M be a connected non-orientable manifold and let $x_0 \in M$ be a point. The map*

$$\begin{array}{rcl} \Phi\colon \pi_1(M, x_0) & \to & \{\pm 1\} \\ [\gamma\colon [0,1] \to M] & \mapsto & \begin{cases} 1, & \text{if } \gamma \text{ is orientation-preserving,} \\ -1, & \text{if } \gamma \text{ is orientation-reversing} \end{cases} \end{array}$$

is well-defined and it is an epimorphism.

Proof. We denote by $p\colon \widetilde{M} \to M$ the orientation covering. The following three easy observations imply the lemma.

[195] It follows easily from Lemma 7.5 that this definition does not depend on the choice of $\widetilde{x}_0$.

(1) It follows immediately from Corollary 7.14 that Φ is well-defined.
(2) Using Lemma 7.5 one easily verifies that Φ is a homomorphism.
(3) We denote by $\widetilde{x}_0'$ the other point in $p^{-1}(x_0)$. By Proposition 8.3 there exists a path $\delta\colon [0,1] \to \widetilde{M}$ from $\widetilde{x}_0$ to $\widetilde{x}_0'$. It follows basically immediately from the definitions that $\Phi([p \circ \delta]) = -1$. This shows that Φ is an epimorphism. $\square$

Exercises for Chapter 8.

Exercise 8.1. Let π be a group. Show that every index 2 subgroup of π is normal.

9. Homotopy equivalent topological spaces

In this chapter we will see that "homotopy equivalent" topological spaces have isomorphic fundamental groups. In some instances this fact will allow us to reduce the calculation of fundamental groups to some of our earlier calculations.

The definition of being "homotopy equivalent" is slightly subtle and rests on the notion of two maps being homotopic.

9.1. **Homotopic maps.** In its most basic form a homotopy is a continuous deformation of maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation.

Definition. Let X and Y be two topological spaces and let $a < b$ be two real numbers. Given a map $F\colon X \times [a,b] \to Y$[196] and given $t \in [a,b]$ we sometimes denote by F_t the map[197]

$$\begin{array}{rcl} F_t\colon X & \to & Y \\ x & \mapsto & F(x,t). \end{array}$$

Now we move on to the actual definition we are interested in.

Definition. Let X and Y be two topological spaces.

(1) A *homotopy from X to Y* is a map $F\colon X \times [0,1] \to Y$.
(2) Let $f_0, f_1\colon X \to Y$ be two maps. A *homotopy between the maps f_0 and f_1* is a homotopy $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are *homotopic* and sometimes we write $f_0 \simeq f_1$.

Examples.

(1) Let $X \subset \mathbb{R}^n$ be a star-shaped subset. Recall that this means that there exists an $x_0 \in X$ such that for all $x \in X$ the segment $\{x_0 \cdot (1-t) + x \cdot t \,|\, t \in [0,1]\}$ lies in X. For example all balls and also all of $\mathbb{R}^n$ are star-shaped. We consider the maps

$$\begin{array}{rcl} f_0\colon X & \to & X \\ x & \mapsto & x_0 \end{array} \quad \text{and} \quad \begin{array}{rcl} f_1 = \mathrm{id}\colon X & \to & X \\ x & \mapsto & x. \end{array}$$

Put differently, f_0 is the constant map which sends every point to the point x_0 and f_1 is the identity. Then

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \underbrace{x_0 \cdot (1-t) + x \cdot t}_{\in X \text{ since } X \text{ is star-shaped}} \end{array}$$

is a homotopy between f_0 and f_1.

[196] Recall that all maps between topological spaces are understood to be continuous.
[197] Note that the map F_t is indeed continuous.

(2) In Figure 167 we show four maps α, β, γ and δ from the circle S^1 to the annulus $X = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic, whereas the maps γ and δ look as if they are not homotopic.

the maps α and β are homotopic the maps γ and δ appear not to be homotopic

FIGURE 167.

Remark. Let $\gamma_0, \gamma_1 \colon [a,b] \to Y$ be two maps from an interval to a topological space Y. We just said that the two maps are homotopic if there exists a map $\Gamma \colon [a,b] \times [0,1] \to Y$ such that

(i) $$\Gamma(t,0) = \gamma_0(t) \quad \text{and} \quad \Gamma(t,1) = \gamma_1(t) \quad \text{for all } t \in [a,b].$$

We can also view these two maps as paths. If they have the same same starting point $\gamma_0(a) = \gamma_1(a) =: P$ and the same endpoint $\gamma_0(b) = \gamma_1(b) =: Q$, then we said on page 223 that the paths are path-homotopic if there exists a map $\Gamma \colon [a,b] \times [0,1] \to Y$ that satisfies (i) and that satisfies also

(ii) $$\Gamma(a,s) = P \quad \text{and} \quad \Gamma(b,s) = Q \quad \text{for all } s \in [0,1].$$

In the literature sometimes "path-homotopic" is also called "homotopic". To distinguish the two notions of homotopies what we call "homotopic" is sometimes called "freely homotopic". Clearly two path-homotopic paths are also homotopic. But in general the converse does not hold. For example, consider the two paths

$$\begin{array}{rcl} \alpha \colon [0,1] & \to & Y = \mathbb{C} \setminus \{0\} \\ x & \mapsto & e^{2\pi i x} \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta \colon [0,1] & \to & Y = \mathbb{C} \setminus \{0\} \\ x & \mapsto & 1. \end{array}$$

Here α goes counterclockwise once around the origin and β is the constant path. We saw in Corollary 4.2 that α and β are not path-homotopic. On the other hand they are homotopic. Indeed a homotopy is given by

$$\begin{array}{rcl} F \colon [0,1] \times [0,1] & \to & Y = \mathbb{C} \setminus \{0\} \\ (x,t) & \mapsto & e^{2\pi i x(1-t)}. \end{array}$$

This free is illustrated in Figure 168.

The difference between a path-homotopy and a homotopy of paths is that in the former context we want to "keep the endpoints fixed". Sometimes when we study other maps between topological spaces we also might want to keep a suitable subset fixed under a homotopy. This leads us to the following definition.

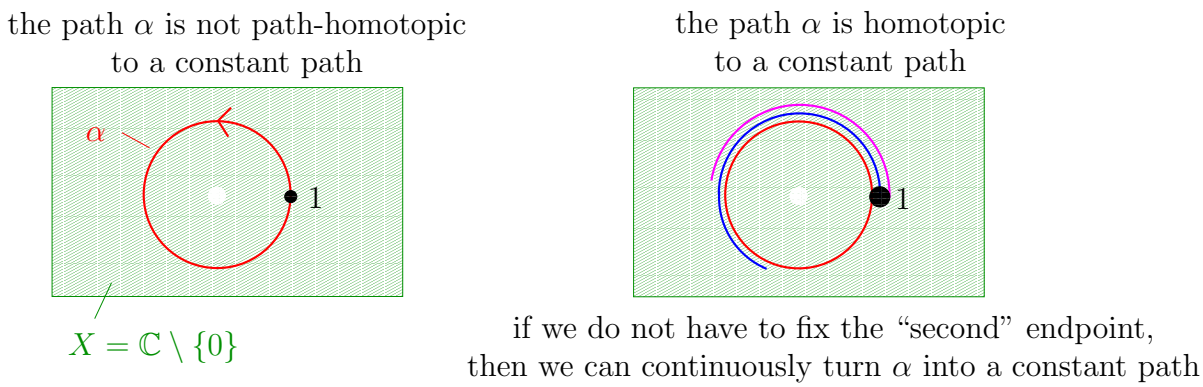


FIGURE 168.

Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.

(1) We say a map $F\colon X \times [0,1] \to Y$ is a *homotopy rel* A if for all $x \in A$ and $s, t \in [0,1]$ we have $F(x,s) = F(x,t)$.
(2) We say that two maps $f, g\colon X \to Y$ with $f|_A = g|_A$ are *homotopic rel* A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

Examples.

(1) By definition two paths $\alpha, \beta\colon [0,1] \to X$ in a topological space X are path-homotopic if and only if the two maps are homotopic rel $\{0,1\}$.
(2) We consider the two maps $f, g\colon \overline{B}^2 \to S^2$ that are shown in Figure 169. These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately right now we lack the tools for showing the latter statement.

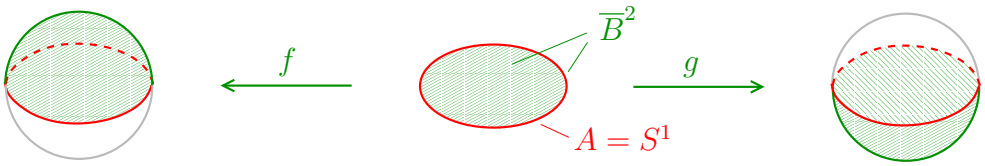


FIGURE 169.

We continue with the next variation on the notion of homotopy.

Definition. Let X and Y be two topological spaces.

(1) An *isotopy from* X *to* Y is a homotopy $F\colon X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t\colon X \to Y$ is a homeomorphism onto its image.

(2) Let $f_0, f_1 \colon X \to Y$ be two maps that are homeomorphisms onto their images. We say f_0 and f_1 are *isotopic* if there exists an isotopy $F \colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$.

Examples.

(1) In Figure 170 we show two maps $f_0, f_1 \colon [0,1] \to \mathbb{R}^2$ that are homeomorphisms onto their image. We also indicate two homotopies F and G between f_0 and f_1. The map F is an isotopy, the map G is not.

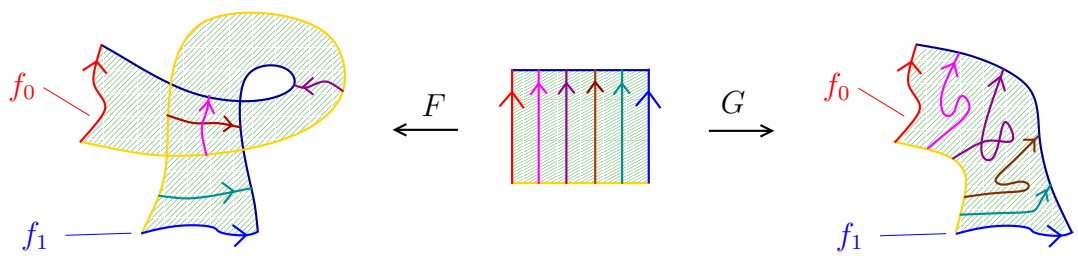


FIGURE 170.

(2) Let $n \in \mathbb{N}$. Let $f \colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism. If $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, then f is isotopic to $\mathrm{id}_{\overline{B}^n}$. Indeed, in Exercise 9.1 we will see that the map

$$\begin{array}{rcl} H \colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \le \|x\| < t, \\ x, & \text{otherwise} \end{cases} \end{array}$$

is continuous and that each H_t is a homeomorphism. It is clear that $H_0 = \mathrm{id}$ and $H_1 = f$. Thus we see that H is the desired isotopy. The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as *Alexander's trick*.[198] The definition of the homotopy H is illustrated in Figure 171.

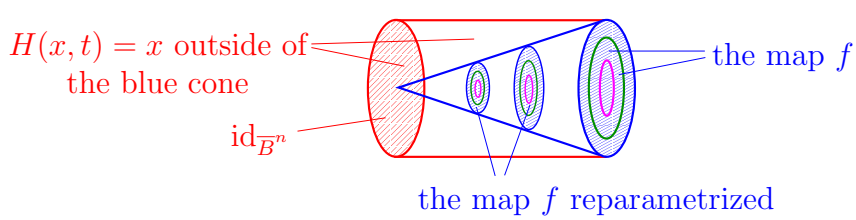


FIGURE 171.

Now we can formulate the following lemma that is a refined version of Proposition 1.47 (2).

[198]The existence of such a homotopy was first observed by a James Alexander in 1923, see [Al].

Lemma 9.1. $(*)$ *Let A be a bounded closed convex subset of $\mathbb{R}^n$ with non-empty interior. Let Q be a point in the interior of A. There exists a canonical isotopy $F\colon A \times [0,1] \to \mathbb{R}^n$ with the following three properties:*

(1) *we have $F_0 = \mathrm{id}_A$,*
(2) *the map F_1 is a homeomorphism from A to $\overline{B}^n$ and this map restricts to a homeomorphism $\partial A \to S^{n-1}$,*
(3) *for any $t \in [0,1]$ we have $F(Q,t) = (1-t) \cdot Q$.*[199]

Proof $(*)$. First consider the case that Q is the origin. In the following we use the notation that we introduced in the proof of Proposition 1.48 (2). Similar to the discussion on page 81 we consider the map

$$\begin{array}{rcl} F\colon A \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x \cdot \frac{1}{(1-t)+t\cdot\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map has all the desired properties.

We leave it as an exercise to modify the above proof to deal with the case that Q is not the origin. □

Sometimes it is useful to combine two homotopies or isotopies. This leads us to the following definition.

Lemma 9.2. $(*)$ *Let X and Y be topological spaces and let $A \subset X$ be a (possibly empty) subset of X. Let $G, H\colon X \times [0,1] \to Y$ be two homotopies rel A. If $G_1 = H_0$, then the map*

$$\begin{array}{rcl} G * H\colon X \times [0,1] & \to & Y \\ (P,t) & \mapsto & \begin{cases} G(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(P,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1. Furthermore, if G and H are isotopies, then $G * H$ is also an isotopy. We refer to $G * H$ as the combination of G and H.

Proof $(*)$. We only need to verify that $G*H$ is indeed continuous, but that is a consequence of Lemma 5.3 together with our hypothesis that $G_1 = H_0$. □

In Proposition 5.2 we showed that being path-homotopic is an equivalence relation on the set of paths between two points. The first part of the following lemma is the obvious generalization to "homotopy of maps".

Lemma 9.3.

(1) *Let X be and Y topological spaces and let A be a (possibly empty) subset of X. Then "homotopy rel A of maps" forms an equivalence relation on the set of maps from X to Y.*

[199] In particular, if Q is already the origin, then F is a homotopy rel the origin.

(2) *Let $f, g\colon X \to Y$, let $\alpha\colon W \to X$ and $\beta\colon Y \to Z$ be maps between topological spaces. The following conclusion holds:*

$$f \simeq g \quad \Longrightarrow \quad f \circ \alpha \sim g \circ \alpha \quad \text{and} \quad \beta \circ f \sim \beta \circ g.$$

Proof.

(1) Not surprisingly the proof of Lemma 9.3 is basically identical to the earlier proof of Proposition 5.2. The only change is that we need to use Lemma 9.2 to show that the combination of two homotopies is indeed a homotopy, i.e. continuous.

(2) This statement is basically obvious, we just need to precompose and postcompose a homotopy H between f and g by α respectively β. □

Definition. Given two topological spaces X and Y and a map $f\colon X \to Y$ we denote by $[f]$ the equivalence class of f. Furthermore we denote by $[X, Y]$ the set of homotopy equivalence classes of maps from X to Y.

Example. We write $I = [0, 1]$. A slight generalization of the argument on page 299 shows that for any path-connected topological space X the set $[I, X]$ consists of a single element, i.e. all maps $I = [0, 1] \to X$ are freely homotopic. In Exercise 9.7 we generalize this statement to show that if A is a convex subset of $\mathbb{R}^n$ and if X is a path-connected topological space, then any two maps $f, g\colon A \to X$ are homotopic.

We continue with a rather classical type of an example.

Definition. Let $n \in \mathbb{N}$ and let $V \subset \mathbb{R}^n$ be a hyperplane, i.e. V is a subvector space of codimension one. We pick a vector v of unit length orthogonal to V. We refer to the map

$$\begin{array}{rcl} \rho_V\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ w & \mapsto & w - 2 \cdot \langle v, w \rangle \cdot v \end{array}$$

as the *reflection in the hyperplane* V. We refer to Figure 172 for an illustration.

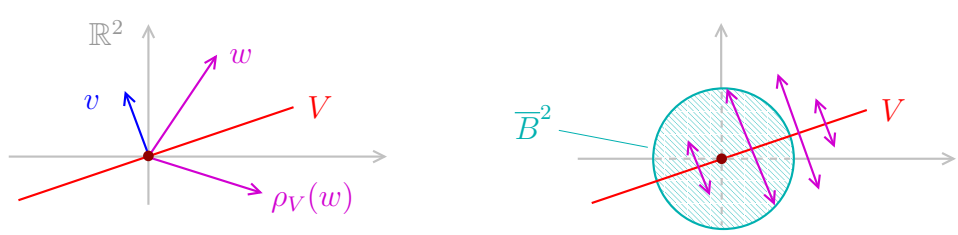


FIGURE 172.

Lemma 9.4. *Let $n \in \mathbb{N}$.*

(1) *Any reflection in a hyperplane of $\mathbb{R}^n$ has the following properties:*
   (a) *it is a diffeomorphism,*
   (b) *it is an isometry (i.e. it preserves the Euclidean distance) and preserves the origin,*

(c) *it restricts to diffeomorphisms of* $\overline{B}^n$ *and* S^{n-1}.

(2) *Any two reflections in hyperplanes are related by an isotopy* $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ *such that each* F_t *is also a reflection. In particular the isotopies are isotopies rel the origin.*

Proof. Let $n \in \mathbb{N}$.

(1) Let ρ be a reflection in a hyperplane of $\mathbb{R}^n$. Clearly ρ is smooth and it satisfies $\rho \circ \rho = \mathrm{id}_{\mathbb{R}^n}$. This implies immediately that ρ is a diffeomorphism. A short amusing calculation shows that ρ preserves the scalar product, hence length. Since $\rho(0) = 0$ we see that ρ restricts to diffeomorphisms of $\overline{B}^n$ and S^{n-1}.

(2) Suppose we are given two hyperplanes V and W of $\mathbb{R}^n$. We pick an orthogonal matrix $A \in \mathrm{SO}(n)$ with $A \cdot V = W$.[200] By Lemma 1.96 (2) there exists a path $\gamma\colon [0,1] \to \mathrm{O}(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. One can now easily verify that the map

$$\begin{array}{rcl} \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (P,t) & \mapsto & \rho_{\gamma(t)\cdot V}(P) \end{array}$$

is the desired isotopy. □

We conclude this section with the following lemma that we will prove in Exercise 9.8.

Lemma 9.5. *Let* X, Y *and* Z *be topological spaces. Let* $f, f'\colon X \to Y$ *and* $g, g'\colon Y \to Z$ *be maps.*

(1) *If* f *and* f' *are homotopic and if* g *and* g' *are homotopic, then the map* $g \circ f$ *is homotopic to* $g' \circ f'$.

(2) *If there exists a subset* A *of* X *such that* f *and* f' *are homotopic rel* A *and if there exists a subset* B *of* Y *such that* $f(A) = f'(A) \subset B$ *and such that* g *and* g' *are homotopic rel* B*, then* $g \circ f$ *is homotopic to* $g' \circ f'$ *rel* A.

Now we define the *homotopy category* $\mathcal{H}$ *of topological spaces* by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{H}) & := & \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition[201]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

Note that in this category the morphisms are not maps but they are equivalence classes of maps.

[200] Why does such a special orthogonal matrix A exist?

[201] The fact that this map is well-defined is a consequence of Lemma 9.5.

9.2. **Closed oriented curves and fundamental groups.** (∗) The following definition gets used frequently throughout the notes.

Definition. Let M be a manifold. A *closed* (*oriented*) *curve in* M is a closed (oriented) connected 1-dimensional submanifold either of ∂M or of M.

Convention. Often we just write "curve" instead of "closed curve".

Remark.

(1) Note that the boundary ∂M a manifold M is *not* a submanifold of M. In particular, if M is a 2-dimensional manifold and C is a closed boundary component, then C is not a submanifold of M, but it is a closed curve in M in the above sense.
(2) On occasions, when we deal with explicit examples, we will use gentle generalizations of the above notion of a closed oriented curve, which in practice should not cause any confusion.

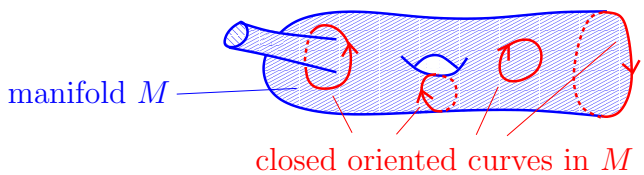


FIGURE 173.

Closed (oriented) curves in a given manifold M have the nice feature that they are easy to draw. But strictly speaking they do not represent elements in $\pi_1(M)$ since elements in $\pi_1(M)$ are represented by *maps* $S^1 \to M$ and not just the images of such maps. Thus we need to clarify the connection between curves and homotopy classes of maps.

The key to this issue is the following lemma, which is also of independent interest.

Lemma 9.6.

(1) *Let* $f\colon [0,1] \to [0,1]$ *be a self-homeomorphism.*[202]
   (a) *If* f *is monotonously increasing, then* f *is isotopic to the map* $f(x) = x$.
   (b) *If* f *is monotonously decreasing, then* f *is isotopic to the map* $f(x) = -x$.
(2) *Every self-homeomorphism of* S^1 *is isotopic to the map* $f(z) = z$ *or to the map* $g(z) = \overline{z}$.

Proof. We will provide the proof in Exercise 9.3. □

Corollary 9.7. *Let* M *be a manifold and let* $C \subset M$ *be a closed oriented curve in* M. *Furthermore let* $x_0 \in C$. *For any two orientation-preserving diffeomorphisms* $\alpha, \beta\colon S^1 \to C$ *with* $\alpha(1) = \beta(1) = x_0$ *we have* $[\alpha] = [\beta] \in \pi_1(M, x_0)$.

[202]It is an elementary real analysis exercise to show that every self-homeomorphism $f\colon [0,1] \to [0,1]$ is either monotonously increasing or monotonously decreasing.

Proof. It is fairly straightforward to prove the corollary using Lemmas 9.6 (1) and 5.4. We leave it as an exercise to the reader to fill in the details. □

Convention. Let M be a manifold and let $C \subset M$ be a closed oriented curve in M. Throughout these notes we use that Corollary 9.7 allows us to view C unambiguously as an element in $\pi_1(M, x_0)$ for any base point $x_0 \in C$.

9.3. **Homotopy equivalent topological spaces.** The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

Definition. Let X be and Y topological spaces.

(1) A map $f\colon X \to Y$ is a *homotopy equivalence between X and Y*, if f admits a *homotopy inverse*, i.e. a map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are *homotopy equivalent*.
(3) We say X is *contractible*, if X is homotopy equivalent to a topological space which consists of precisely one point.

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. This is indeed the case.

Lemma 9.8.

(1) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are homotopy equivalences, then $g \circ f\colon X \to Z$ is also a homotopy equivalence.*
(2) *The relation "homotopy equivalence" is an equivalence relation.*[203]
(3) *Let $f, g\colon X \to Y$ be two maps. If f is a homotopy equivalence and if g is homotopic to g, then g is also a homotopy equivalence.*

Proof $(*)$.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces that admit homotopy inverses $f'\colon Y \to X$ and $g'\colon Z \to Y$. We have
$$(f' \circ g') \circ (g \circ f) \;=\; f' \circ (g' \circ g) \circ f \;\underset{\substack{\uparrow \\ \text{by Lemma 9.3 (2) since } g' \circ g \simeq \mathrm{id}_Y}}{\simeq}\; f' \circ \mathrm{id}_Y \circ f \;=\; f' \circ f \;\simeq\; \mathrm{id}_X\,.$$
(2) It is clear that the only property of an equivalence relation that requires any thought is transitivity. But this follows from (1).
(3) We leave the elementary verification of the last statement to the reader. □

[203]The perspicacious reader might notice that here things get logically a little dicey. More precisely, on page 96 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [FPr, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret the second statement of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

Now we will give a long list of examples of homotopy equivalent spaces.

Examples.

(1) Let X be a star-shaped subset of $\mathbb{R}^n$ and let $x_0 \in X$ be a point such that for any $x \in X$ the segment from x_0 to x lies in X. We consider the maps

$$\begin{array}{rcl} f\colon X & \to & \{x_0\} \\ x & \mapsto & x_0 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \{x_0\} & \to & X \\ x_0 & \mapsto & x_0. \end{array}$$

Then $f \circ g = \mathrm{id}_{\{x_0\}}$ and we had just seen on page 299 that $g \circ f \simeq \mathrm{id}_X$. Thus we showed that any star-shaped subspace of $\mathbb{R}^n$, e.g. any convex subset of $\mathbb{R}^n$ or $\mathbb{R}^n$ itself, is homotopy equivalent to a point, i.e. it is contractible.

(2) It follows from the previous example together with Lemma 9.8 that for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent.

(3) We consider the Möbius band

$$X = [0,1] \times [-1,1] \ / \ (0,y) \sim (1,-y)$$

and the circle

$$Y = [0,1] \ / \ 0 \sim 1.$$

We consider the maps

$$\begin{array}{rcl} f\colon X & \to & Y \\ [(x,y)] & \mapsto & [x] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon Y & \to & X \\ [x] & \mapsto & [(x,0)]. \end{array}$$

Then $f \circ g = \mathrm{id}_Y$ and furthermore we have $g \circ f \simeq \mathrm{id}_X$ as can be seen using the following homotopy

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ ([(x,y)],t) & \mapsto & [(x, y \cdot (1-t))]. \end{array}$$

The Möbius band is thus homotopy equivalent to the circle.

We leave the proof of the following lemma as an edifying exercise to the reader, see Exercise 9.11.

Lemma 9.9. *Let $f\colon X \to Y$ be a map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $\pi_0(X) \to \pi_0(Y)$. In particular, X is path-connected if and only if Y is path-connected.*

Definition. Let X be a topological space and let A be a subset.

(1) A *deformation retraction* from X to A is a map $F\colon X \times [0,1] \to X$ with the following properties:
   (a) for every $x \in X$ we have $F(x,0) = x$ and $F(x,1) \in A$,
   (b) for every $a \in A$ we have $F(a,t) = a$ for all $t \in [0,1]$.
   Put differently, a deformation retraction is a homotopy $F\colon X \times [0,1] \to X$ rel A such that $F_0 = \mathrm{id}$ and $F_1(X) \subset A$.[204]

[204]Note that $F_1\colon X \to A$ is retraction from X to A.

(2) We say A is a deformation retract of X if there exists a deformation retraction from X to A.[205]
(3) We say a retraction $r\colon X \to A$ is *realized by a deformation retraction* if there exists a deformation retraction F from X to A with $F_1 = r$.

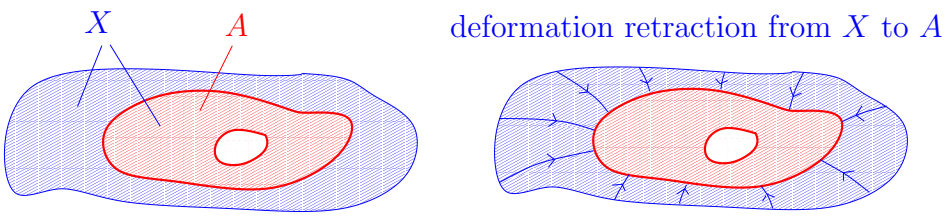


FIGURE 174. Schematic picture of a deformation retraction.

Examples.
(1) For any topological space X and any interval $I \subset \mathbb{R}$ containing 0 the subspace $X \times \{0\}$ is a deformation retract of $X \times I$. In fact the deformation retraction is given by
$$\begin{array}{rcl} (X \times I) \times [0,1] & \to & X \times I \\ ((x,s),t) & \mapsto & (x, s \cdot (1-t)). \end{array}$$
(2) In Exercise 9.12 we will see that for any $n \in \mathbb{N}$ the sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
(3) It follows easily from the previous examples that the central circle $[0,1] \times \{0\}$ of the Möbius band is a deformation retract of the Möbius band $[0,1] \times [-1,1]/\sim$.

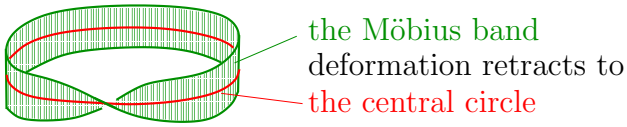


FIGURE 175.

In Exercise 9.13 we will prove the following elementary lemma.

[205]Unfortunately in the literature the term "deformation retract" is used differently by different authors. Our notion of a "deformation retract" agrees with the definition in [Hat, p. 2] and [Mun, p. 208]. What we call a "deformation retract" is called a "strong deformation retract" in [Bre, p. 45] and [Rot, p. 209]. In these books this notion is used to distinguish it from the notation of a "weak deformation retract", which means that there exists a map $F\colon X \times [0,1] \to X$ such that
(a) for every $x \in X$ we have $F(x,0) = x$ and $F(x,1) \in A$,
(b) for every $a \in A$ we have $F(a,1) = a$.
As always it is fun to find examples of pairs (X, A) of topological spaces such that A is a weak deformation retract but not a strong deformation retract of X.

Lemma 9.10. *Let A be a subset of a topological space X which admits a deformation retraction F from X to A. Then the inclusion map*

$$\begin{array}{ccc} A & \to & X \\ a & \mapsto & a \end{array} \quad \text{and the map} \quad \begin{array}{ccc} X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are both homotopy equivalences. In particular the deformation retract A is homotopy equivalent to the total space X.

The lemma often gives a convenient way for showing that two spaces are homotopy equivalent. For example the lemma, together with the previous example implies that the sphere S^{n-1} and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent.

The notions of a topological space being contractible and admitting a deformation retraction to a point sound quite similar. The following lemma gives the precise relationship between these two notions.

Lemma 9.11.

(1) *If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.*
(2) *If a topological space X is contractible, then given any point $x_0 \in X$ there exists a homotopy $H\colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_1(x) = x_0$ for all $x_0 \in X$.*
(3) *The "infinite zigzag comb" shown in Figure 176 is contractible, but it does not admit a deformation retraction to a point.*

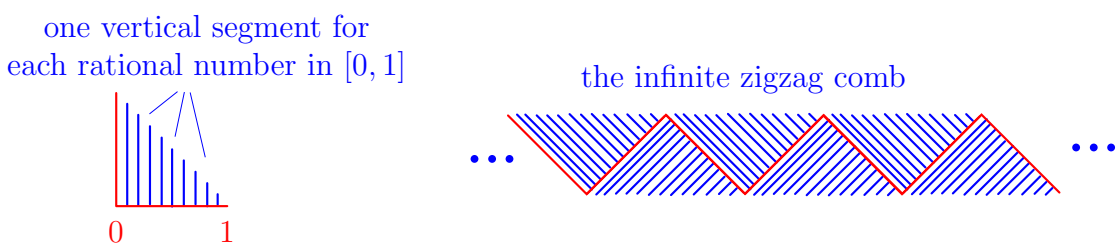


FIGURE 176.

Proof.

(1) This statement follows immediately from Lemma 9.10.
(2) Suppose X is a topological space that is contractible. This implies that there exists map $g\colon \{*\} \to X$ and a homotopy $G\colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and such that $G_1(x) = g(*)$ for all $x \in X$. We set $x_1 = g(*)$. Note that we know from Lemma 9.9 that X is path-connected. Thus there exists a path $\gamma\colon [0,1] \to X$ from

x_1 to x_0. The desired homotopy is now given by the map[206]

$$\begin{array}{rcl} H\colon X\times[0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G(x,2\cdot t), & \text{if } t\in[0,\frac{1}{2}], \\ \gamma(2\cdot t-1), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$

(3) This statement will be proved in Exercise 9.6. □

9.4. **The fundamental groups of homotopy equivalent topological spaces.** Now we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will often greatly simplify the calculation of fundamental groups.

Proposition 9.12.

(1) *Let $f\colon X\to Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0\in X$ be a point. We put $y_0:=f(x_0)$. Then the induced map*

$$f_*\colon \pi_1(X,x_0) \;\to\; \pi_1(Y,y_0)$$

is an isomorphism.

(2) *If X is contractible topological space, then for any $x_0\in X$ we have $\pi_1(X,x_0)=0$.*

Example. It follows from Lemma 9.10, Proposition 9.12 (1), Corollary 7.19, Proposition 5.14 and the previous example that for any $n\geq 2$ and any $Q\in\mathbb{R}^n$ we have

$$\pi_1(\mathbb{R}^n\setminus\{Q\}) \;\cong\; \pi_1(\mathbb{R}^n\setminus\{0\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=2, \\ 0, & \text{if } n\geq 3. \end{cases}$$

Proof (∗). The second statement of the proposition is just a frequently used special case of the first statement. Thus let us prove the first statement. Let $f\colon X\to Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0\in X$ be a point. We put $y_0=f(x_0)$. We pick a homotopy inverse $g\colon Y\to X$ of f. We put $x_1=g(y_0)$. According to our hypothesis we have $\mathrm{id}_X\simeq g\circ f$, i.e. there exists a homotopy H between id_X and $g\circ f$.[207]

We denote by

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & X \\ t & \mapsto & H(x_0,t) \end{array}$$

the path with which the homotopy H connects the points

$$H(x_0,0) \;=\; (\mathrm{id}_X)(x_0) \;=\; x_0 \quad\text{and}\quad H(x_0,1) \;=\; (g\circ f)(x_0) \;=\; x_1.$$

[206]This map is continuous by Lemma 1.32 (2).

[207]Can we already conclude from this fact that $g\circ f$ induces the identity map on fundamental groups? Unfortunately not, for example $g\circ f$ does not even induce a map on $\pi_1(X,x_0)$, since $g\circ f$ sends the base point x_0 to the point x_1, which in general is different from x_0.

Claim. The maps

$$\begin{array}{rcl} (g\circ f)_*\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [s] & \mapsto & [(g\circ f)\circ s] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [s] & \mapsto & [\overline{\alpha}*s*\alpha] \end{array}$$

are identical.

Let $s\colon [0,1]\to X$ be a loop in (X,x_0). We want to show that the loops $(g\circ f)\circ s$ and $\overline{\alpha}*s*\alpha$ are path-homotopic. We consider the map[208]

$$\begin{array}{rcl} G\colon [0,1]\times[0,1] & \to & X \\ (r,t) & \mapsto & \begin{cases} \overline{\alpha}(4r), & \text{if } r\in\left[0,\frac{1}{4}t\right], \\ H\left(s\left(\dfrac{r-\frac{1}{4}t}{1-\frac{1}{2}t}\right),1-t\right), & \text{if } r\in(\frac{1}{4}t,1-\frac{1}{4}t), \\ \alpha(4r-3), & \text{if } r\in\left[1-\frac{1}{4}t,1\right]. \end{cases} \end{array}$$

Using Lemma 5.3 it is straightforward to verify that G is indeed a continuous map with

$$G(0,t) \;=\; G(1,t) \;=\; x_1 \quad \text{for all } t\in[0,1]$$

and which satisfies

$$G(r,0) \;=\; ((g\circ f)\circ s)(r) \quad \text{and} \quad G(r,1) \;=\; (\overline{\alpha}*s*\alpha)(r) \quad \text{for all } r\in[0,1].$$

This shows that G is in fact a path-homotopy between the loops $(g\circ f)\circ s$ and $\overline{\alpha}*s*\alpha$[209]

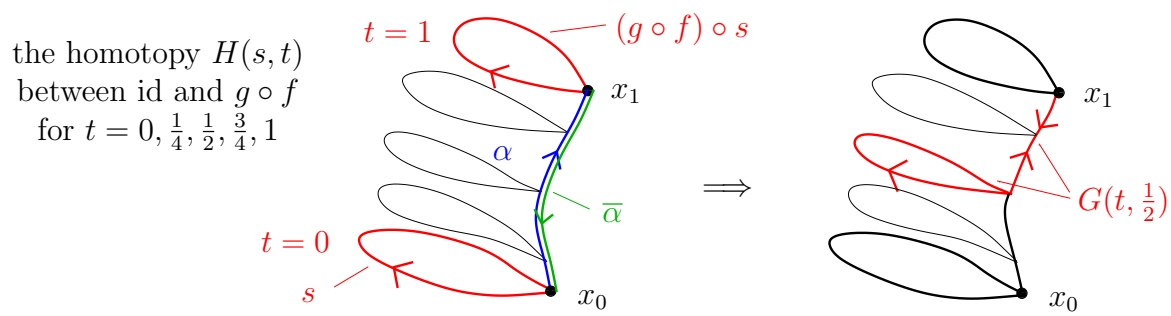


FIGURE 177. Schematic image of H and G.

in (X,x_1). Thus we have proved the claim.

[208]The map G is defined as follows: for $t\in[0,1]$ the loop $r\mapsto G(r,t)$ is given by traveling along $\overline{\alpha}$ from x_1 to $\overline{\alpha(t)}=\alpha(1-t)$, then one follows the loop given by $H(s(-),t)$, and then one returns along α back to x_1.

[209]To be precise, G is a homotopy between $(g\circ f)\circ s$ and a suitable parametrization of $\overline{\alpha}*s*\alpha$.

We consider the following diagram:

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "{[s]\mapsto[\overline{\alpha}*s*\alpha]}"] \arrow[d, "f_*"] & \pi_1(X,x_1) \arrow[d, "f_*"] \\
\pi_1(Y,y_0) \arrow[r, "{[t]\mapsto[(f\circ\overline{\alpha})*t*(f\circ\alpha)]}"'] \arrow[ur, "g_*"] & \pi_1(Y,y_1).
\end{tikzcd}$$

We make the following observations:

(1) By the claim and Proposition 5.11 the map $g_* \circ f_* \colon \pi_1(X,x_0) \to \pi_1(X,x_1)$ is an isomorphism.
(2) This implies that the map $f_* \colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is a monomorphism.
(3) If we replace the roles of f and g, then the argument of the claim and Proposition 5.11 shows that the map $f_* \circ g_* \colon \pi_1(Y,y_0) \to \pi_1(Y,y_1)$ is also an isomorphism.
(4) This implies that the map $f_* \colon \pi_1(X,x_1) \to \pi_1(Y,y_1)$ is an epimorphism.
(5) We consider the diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "{[s]\mapsto[\overline{\alpha}*s*\alpha]}", "\cong"'] \arrow[d, hook, "f_*"] & \pi_1(X,x_1) \arrow[d, two heads, "f_*"] \\
\pi_1(Y,y_0) \arrow[r, "\cong", "{[t]\mapsto[(f\circ\overline{\alpha})*t*(f\circ\alpha)]}"'] & \pi_1(Y,y_1).
\end{tikzcd}$$

The horizontal maps are isomorphisms by Proposition 5.11. It follows immediately from the definitions of the maps that the diagram commutes.
(6) Since the right vertical map is an epimorphism and since the horizontal maps are isomorphisms we now see that the left vertical map is also an epimorphism,
(7) The combination of (2) and (6) shows that the left vertical map is an isomorphism.

□

In Lemma 1.90 we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 9.13. *The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.*

Unfortunately we are still not able to settle the question whether for $k \neq l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic. Before we can settle this question we will have to develop a completely new set of tools.

Proof. Let us suppose that there exists a homeomorphism $f \colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Then f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\begin{array}{ccccccc}
\mathbb{Z} & \cong & \pi_1(\mathbb{R}^2 \setminus \{P\}) & \cong & \pi_1(\mathbb{R}^3 \setminus \{Q\}) & = & 0. \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{page 311} & & \text{since } f \text{ is a homeomorphism} & & \text{page 311} &
\end{array}$$

Thus we obtained a contradiction. □

9.5. **Homotopies on quotients and on gluings** (∗)**.** In this section we collect two slightly subtle lemmas which show that, under the right circumstances, homotopies descend to quotient spaces and that given a "gluing $X \cup_f Y$" we can also glue homotopies. This section is much less pleasant to read than the previous section and it is arguably best if the reader only takes a peak once the results are actually used.

Before we state the two technical lemmas, we point out that the proof of the lemmas eventually boils down to the rather subtle and delicate Theorem 2.15. One might view both lemmas "obvious", but they are not.

After this preamble, let us move to the first lemma which shows that homotopies descend to quotients.

Lemma 9.14. *Let X be a topological space. Furthermore let $A \subset B$ be two subsets of X. If $F\colon B \times [0,1] \to X$ is a homotopy with $F(A \times [0,1]) \subset A$, then the induced map*[210]

$$\begin{array}{rcl} \varphi\colon (B/A) \times [0,1] & \to & X/A \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Proof. We denote by $p\colon B \to B/A$ and $q\colon X \to X/A$ the obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & B \times [0,1] \arrow[dl, "p\times \mathrm{id}_{[0,1]}"'] \arrow[dr, "q\circ F"] & \\ (B/A) \times [0,1] \arrow[rr, "\varphi", "{([x],t)\mapsto[F(x,t)]}"'] & & X/A. \end{tikzcd}$$

By Lemma 2.14 (1) the map p is a quotient map. Since $[0,1]$ is locally compact we obtain from Theorem 2.15 that $p \times \mathrm{id}_{[0,1]}$ is also a quotient map. Since $q \circ F$ is continuous we obtain from Lemma 2.16 (1) that φ is continuous. □

It is perhaps easier to appreciate the following useful corollary.

Corollary 9.15. *Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A. In particular X/A is contractible.*

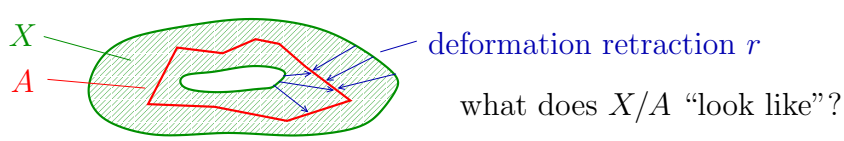


FIGURE 178. Illustration for Corollary 9.15.

[210]Since F is a homotopy rel A we see that the map is well-defined.

Proof. Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. Since r is deformation retraction we obtain from Lemma 9.14 that the map

$$\begin{array}{rcl} (X/A) \times [0,1] & \to & X/A \\ ([x],t) & \mapsto & [r(x,t)]. \end{array}$$

is well-defined and continuous. Now it is clear that this map is a deformation retraction from X/A to A/A. □

The second lemma shows that we can "glue homotopies".

Lemma 9.16. *Let X and Y be two topological spaces, let $A \subset X$ be a subset and let $f\colon A \to Y$ be a map. Finally let Z be another topological space. Suppose that we are given two homotopies $G\colon X \times [0,1] \to Z$ and $H\colon Y \times [0,1] \to Z$ such that for any $a \in A$ and any $t \in [0,1]$ we have $G(a,t) = H(f(a),t)$. Then the map*[211]

$$\begin{array}{rcl} F\colon (X \cup_f Y) \times [0,1] & \to & Z \\ ([P],t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in X, \\ H(P,t), & \text{if } P \in Y \end{cases} \end{array}$$

is well-defined and it is continuous.

Proof. One can easily verify that the given map F is well-defined. Recall that by Lemma 2.18 we know that the map

$$\begin{array}{rcl} \Psi\colon (X \times [0,1]) \cup_{f \times \mathrm{id}_{[0,1]}} (Y \times [0,1]) & \to & (X \cup_f Y) \times [0,1] \\ [(P,t)] & \mapsto & ([P],t) \end{array}$$

is a homeomorphism. Thus it suffices to show that the map

$$\begin{array}{rcl} F \circ \Psi\colon (X \times [0,1]) \cup_{f \times \mathrm{id}_{[0,1]}} (Y \times [0,1]) & \to & Z \\ [(P,t)] & \mapsto & \begin{cases} G(P,t), & \text{if } (P,t) \in X \times [0,1], \\ H(P,t), & \text{if } (P,t) \in Y \times [0,1] \end{cases} \end{array}$$

is continuous. But this is an immediate consequence of the hypothesis on G and H and Lemma 1.83 □

9.6. **The wedge of topological spaces.**

Definition. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and suppose that for each $i \in I$ we are given a point $x_i \in X_i$.

(1) We refer to

$$\bigvee_{i \in I} X_i \ := \ \Big(\bigsqcup_{i \in I} X_i\Big)/\sim \quad \text{where } x_i \sim x_j \text{ for all } i,j \in I$$

as the *wedge of the X_i*. We refer to the point given by the identifications as the *wedge point*. If we do not say anything else it will be denoted by $*$.

[211] We refer to page 114 for the definition of $X \cup_f Y$.

(2) Let Y be a topological space and suppose that we are given a family of maps $\{f_i\colon X_i \to Z\}_{i\in I}$ such for every $i, j \in I$ we have $f_i(x_i) = f_j(x_j)$. We define the map

$$\begin{array}{rcl} \bigvee_{i\in I} f_i\colon \bigvee_{i\in I} X_i & \to & Z \\ [y] & \mapsto & f_i([y]) \text{ if } y \in X_i. \end{array}$$

It follows from Lemma 1.66 that this map is indeed continuous.

(3) If $i = \{i_1, \dots, i_k\}$ is a finite family, then often we just write $X_{i_1} \vee \cdots \vee X_{i_k}$ and similarly we just write $f_{i_1} \vee \cdots \vee f_{i_k}$.

Remark. The notation in the definition of the wedge of two topological spaces does not mention that the definition depends on the choice of the points $x_i \in X_i$. If we pick disconnected topological spaces X_i, then it is easy to see in examples that in general the homeomorphism type of the wedge $\bigvee_{i\in I} X_i$ depends on the choice of components from which we take the x_i.

But fortunately in many situations the homeomorphism type of the wedge does not depend on the choice of the x_i. For example, if the X_i are connected manifolds, then it follows from Proposition 10.23 that the homeomorphism type of the wedge does not depend on the choices of the x_i, as long as we pick these two points away from the boundary.

Examples.

(1) The wedge $A \vee B$ of two topological spaces A and B should be viewed as the result of gluing A and B along the points a and b. The wedge of two circles is shown in Figure 179.

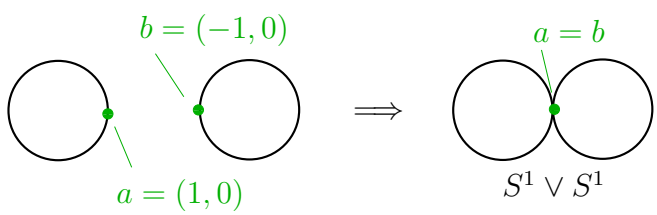


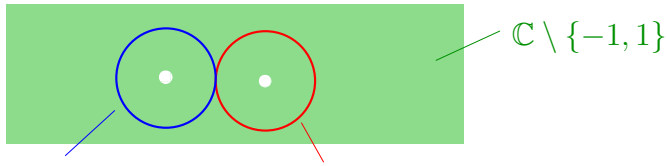
FIGURE 179. The wedge of two circles.

(2) If G is a connected spatial graph with one vertex and n edges, then it is straightforward to see that G is homeomorphic to the wedge of n circles.

(3) In Exercise 9.14 we will see that the wedge of two circles is homotopy equivalent to $\mathbb{C} \setminus \{\pm 1\}$. More precisely, we will show that

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.



$$\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\} \;=\; S^1 \vee S^1$$

FIGURE 180.

Definition. Let $\{X_j\}_{j \in J}$ be a family of topological spaces and suppose that for each $j \in J$ we are given a point $x_j \in X_j$. Given $k \in I$ we refer to the maps

$$\begin{aligned} i_k \colon X_k &\to \bigvee_{j \in J} X_j \\ x &\mapsto [x] \end{aligned} \qquad \text{and} \qquad \begin{aligned} p_k \colon \bigvee_{j \in J} X_j &\to X_k \\ [x] &\mapsto \begin{cases} x, & \text{if } x \in X_k, \\ x_k, & \text{otherwise} \end{cases} \end{aligned}$$

as the *natural inclusion* respectively the *natural projection.*

Lemma 9.17. $(*)$ *Let $\{X_j\}_{j \in J}$ be a family of topological spaces and suppose that for each $j \in J$ we are given a point $x_j \in X_j$. Let $k \in K$.*

(1) *The natural inclusion i_k is a homeomorphism onto its image.*
(2) *A map $f \colon \bigvee_{j \in J} X_j \to Y$ to a topological space is continuous if and only if for each $k \in J$ the map $f \circ i_k \colon X_k \to Y$ is continuous.*
(3) *The natural projection p_k is continuous.*

Remark. In most cases Lemma 9.17 allows us to safely view each X_k as a subset of the wedge $\bigvee_{j \in J} X_j$.

Proof $(*)$.

(2) This statement follows easily from Lemma 1.49 and Lemma 1.66.
(3) This is an immediate consequence of (2).
(1) It follows easily from Lemma 1.49 and Lemma 1.65 (3) that the natural inclusion i_k is continuous. It is clear that the natural inclusion i_k is an injection. Evidently we have $p_k \circ i_k = \mathrm{id}_{X_k}$. Since the natural projection p_k is continuous we see that the natural inclusion i_k is indeed a homeomorphism onto its image. $\square$

We conclude this section with two technical lemmas which can safely be skipped at a first reading.

Lemma 9.18. $(*)$ *Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces and let (Z, z_0) be a pointed topological space. Let $\{f_i \colon (X_i, x_0) \to (Z, z_0)\}_{i \in I}$ and $\{g_i \colon (X_i, x_0) \to (Z, z_0)\}_{i \in I}$ be two families of maps. The following statement holds:*

for every $i \in I$ there exists a homotopy rel x_i from f_i to g_i $\iff$ *the maps $\bigvee_{i \in I} f_i$ and $\bigvee_{i \in I} g_i$ from $\bigvee_{i \in I} X_i$ to Z are homotopic rel the wedge point.*

Proof. The "$\Leftarrow$"-direction of the lemma follows immediately from Lemma 9.17 (1). If the index set I has two elements, then the "$\Rightarrow$"-direction lemma is a straightforward consequence of Lemma 9.16 which eventually reduces the problem to Lemma 2.17. For a general family the "$\Rightarrow$"-direction lemma can also be reduced easily to Lemma 2.17.

It is worth pointing out that the proof of the "$\Rightarrow$"-direction is not obvious. As we just pointed out, it all boils down to Lemma 2.17. This lemma in turn relies on the delicate, frequently overlooked, underappreciated Theorem 2.15. □

The following lemma gives in particular a nice little application of Lemma 9.18.

Lemma 9.19. $(*)$ *Let (A, a_0) and $(B, b_0$ be two pointed topological spaces. We form the wedge $A \vee B$ using the points a_0 and b_0. If $\{a_0\}$ is a deformation retract of A, then $B = \{a_0\} \vee B$ is a deformation retract of $A \vee B$.*

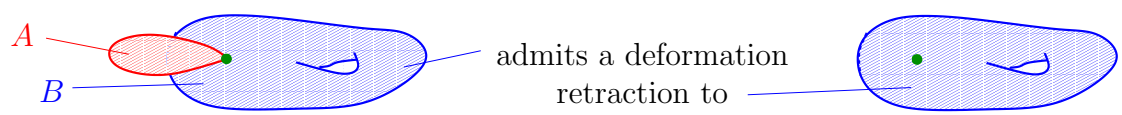


FIGURE 181. Illustration of Lemma 9.19.

Proof $(*)$. Let $F\colon A \times [0,1] \to A$ be a deformation retraction from A to $\{a_0\}$. Let G be the trivial homotopy on B. It follows again from Lemma 9.16 that we can combine the two homotopies to get the desired deformation retraction. □

9.7. **Spatial graphs.** In this section we have another look at graphs.

Definition. Let $G = (V, E)$ be a finite graph, i.e. a graph such that V and E are finite.[212]

(1) We refer to
$$\chi(G) \;=\; \#V \;-\; \#E \;=\; \text{number of vertices} \;-\; \text{number of edges}$$
as the *Euler characteristic of* G.

(2) A *tree* is a finite connected graph with Euler characteristic 1.

(3) Given a vertex v and an edge e we write
$$f(v,e) \;:=\; \begin{cases} 0, & \text{if no endpoint of } e \text{ equals } v, \\ 1, & \text{if } e \text{ has two endpoints, one of which equals } v, \\ 2, & \text{if } e \text{ has one endpoint and this one endpoint equals } v. \end{cases}$$

(4) The *valence of a vertex* v is defined as
$$\text{valence}(v) \;:=\; \sum_{\text{edge } e} f(v,e).$$

In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".

[212]Note that the subsequent definition makes sense for abstract graphs and for spatial graphs.

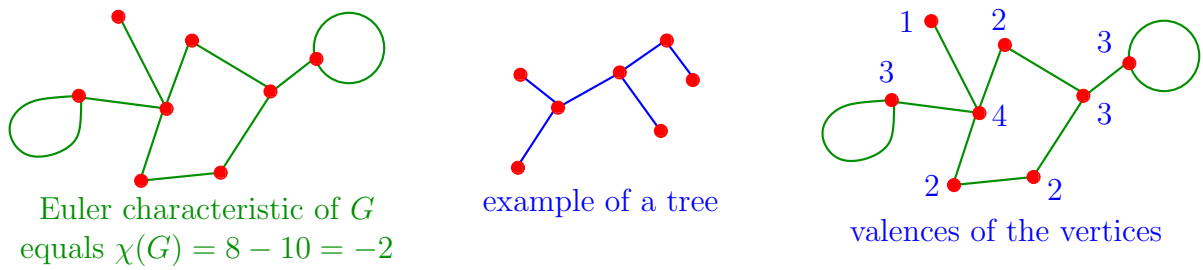


FIGURE 182.

Proposition 9.20. *Let T be a spatial graph that is a tree. Then the following hold:*

(1) *If T has at least one edge, then it admits a vertex of valence one.*
(2) *The tree T is contractible.*
(3) *The fundamental group $\pi_1(T)$ is trivial.*

Proof $(*)$. Let T be a tree. Then the following hold:

(1) We suppose that T has at least one edge. Note that a vertex with valence 0 is an isolated point. Since T has at least one edge and since it is by definition connected, we see that the tree T does not have a vertex of valence 0. Furthermore we have

$$\sum_{\text{vertex } v} \text{valence}(v) = \sum_{\text{vertex } v} \sum_{\text{edge } e} f(v,e) \underset{\substack{\uparrow \\ \text{each edge contributes 2 to the sum}}}{=} 2 \cdot \#\text{ edges} \underset{\substack{\uparrow \\ \text{since } \chi(T)=1}}{=} 2 \cdot \#\text{vertices} - 2.$$

This equality can only hold if there is at least one vertex with valence less than 2. By the above the valence has to be at least one, so there exists a vertex of valence one.

(2) We prove that any tree is contractible by induction on the number of edges. If a tree has zero edges, then it consists of precisely one point, in particular it is even homeomorphic to a point. Now suppose that we know that all trees with n edges are contractible. Let $T = (V, E)$ be a tree with $n+1$ edges. By (1) it admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We write $T' = (V \setminus \{v\}, E \setminus e)$. We reduced the number of vertices and of edges by one, so the Euler characteristic is unchanged, i.e. $\chi(T') = \chi(T) = 1$, so T' is again a tree.

We denote by u the other endpoint of e. We consider the maps

$$\begin{aligned} f\colon T &\to T' \\ x &\mapsto \begin{cases} u, & \text{if } x \in \overline{e}, \\ x, & \text{otherwise.} \end{cases} \end{aligned} \qquad \text{and} \qquad \begin{aligned} g\colon T' &\to T \\ x &\mapsto x. \end{aligned}$$

We claim that g is a homotopy inverse to f. Evidently $f \circ g = \mathrm{id}_{T'}$. We need to show that $g \circ f \simeq \mathrm{id}_T$. We pick a homeomorphism $\varphi\colon [0,1] \to \overline{e}$ with $\varphi(0) = u$ and

$\varphi(1) = v$. Then

$$\begin{array}{rcl} H\colon T \times [0,1] & \to & T \\ (x,t) & \mapsto & \begin{cases} \varphi(\varphi^{-1}(x)\cdot t), & \text{if } x \in \overline{e}, \\ x, & \text{otherwise.} \end{cases} \end{array}$$

is a homotopy from $g \circ f$ to id_T.

We have thus shown that T is homotopy equivalent to T'. By our induction hypothesis T' is homotopy equivalent to a point. So it follows from Lemma 9.8 that T itself is also homotopy equivalent to a point, i.e. T is contractible.

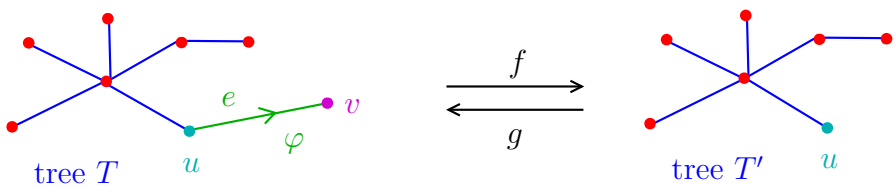


FIGURE 183. Illustration for the proof that every tree is contractible.

(3) It follows from (2) and Proposition 9.12 (2) that the fundamental group of T is trivial.

□

Now we provide a few more definitions on graphs.

Definition. Let $G = (V, E)$ be a graph.

(1) A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and such that E' is a union of components of E.
(2) We say an edge e is *adjacent to a subgraph* G' if e has two endpoints where one of the endpoints lies on G' and the other does not.
(3) A *spanning tree* for G is a subgraph that is a tree and which does not admit an adjacent edge.

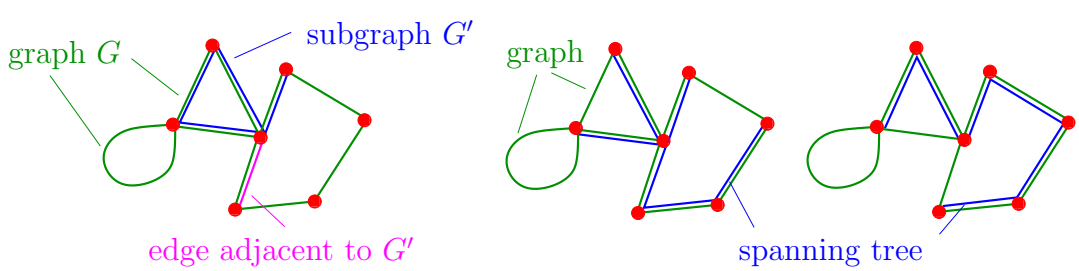


FIGURE 184.

Remark.

(1) If an edge e is adjacent to a subgraph G', then it is not part of the subgraph. Indeed, by definition of a subgraph, for any edge that is part of the subgraph also its endpoints have to lie on the subgraph.
(2) The example of Figure 184 shows that a graph usually has many different spanning trees.
(3) A spanning tree is often also called a *spanning tree*.

We conclude this discussion on graphs with the following proposition.

Proposition 9.21. *Let $G = (V, E)$ be a finite connected spatial graph.*

(1) *The graph G admits a spanning tree.*
(2) *Every spanning tree of G contains all vertices of G.*
(3) *The graph G is homotopy equivalent to the wedge of $1 - \chi(G)$ circles.*
(4) *Let T be a spanning tree and let $e_1, \dots, e_k$ be the edges that do not lie in T. The quotient map $G \to G/T$ is a homotopy equivalence and G/T is the wedge of the k circles given by identifying the endpoints of $e_1, \dots, e_k$ to a single point.*
(5) *Two finite connected spatial graphs with the same Euler characteristic are homotopy equivalent.*

We will study later the question, whether the converse to statement (4) holds: if two finite connected graphs are homotopy equivalent, does it follow that the Euler characteristics are the same?

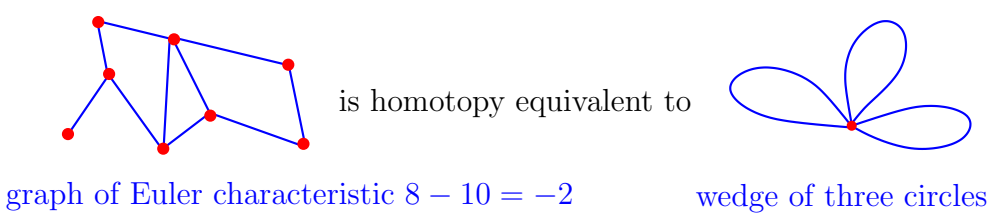


FIGURE 185. Illustration of Proposition 9.21 (3).

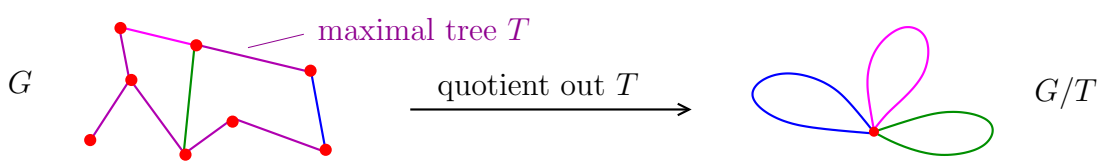


FIGURE 186. Illustration of Proposition 9.21 (4).

We only provide a sketch of the proof.

Sketch of the proof $(*)$. Let $G = (V, E)$ be a finite connected spatial graph.

(1) We pick a vertex v. We define $T_0 = \{v\}$ and view it as a subtree of G. Now we define a sequence of trees $T_0, \dots, T_k$ by iteratively attaching an adjacent edge to the previous tree. This process stops after finitely many steps since G is finite. We always add an edge and a vertex, thus we see that $\chi(T_k) = \cdots = \chi(T_0) = 1$, i.e. T_k is a tree. By definition it is in fact a spanning tree. This argument is illustrated in Figure 187 on the left.

(2) Let T be a subgraph of G that is a tree. Suppose that there exists a vertex w that does not lie in T. We will show that T is not maximal. Let v be a vertex in T. Since G is connected there exists a sequence of edges $e_0, \dots, e_n$ with the following properties:
   (a) v is a vertex of e_0,
   (b) for each $i \in \{0, \dots, n-1\}$ a vertex of e_i is also a vertex of e_{i+1}, and
   (c) w is a vertex of e_n.

   Then there exists an $i \in \{0, \dots, n-1\}$ such that a vertex of e_i lies in T and the other one does not. But then e_i is an edge adjacent to T. This shows that T was not maximal. This argument is illustrated in Figure 187 on the right.

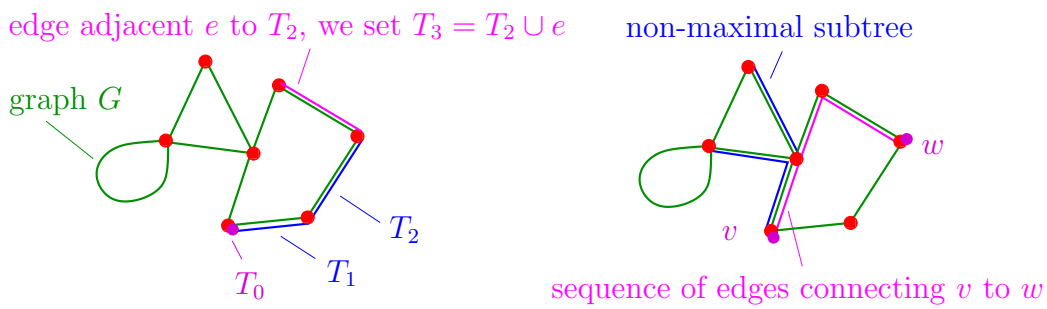


FIGURE 187. Illustration for the proofs of the first two statements of Proposition 9.21.

(3) Let G be a finite spatial graph. We write $n = 1 - \chi(G)$. We prove the statement by induction on the number of edges in a spanning tree. If a spanning tree has zero edges, then it consists of a single vertex. By (2) this means that the original graph has only one vertex and it has n edges. It follows from the discussion on page 316 that the graph is in fact homeomorphic to the wedge of n circles.

So suppose that we know the desired statement whenever the spanning tree has k edges. Now suppose that G has a spanning tree T with $k+1$ edges.

By Proposition 9.20 the tree $T = (W, F)$ has a vertex v of valence one. We denote by e the one edge in T such that v is an endpoint of e. We denote by u the other endpoint of v. By the definition of an edge there exists a homeomorphism $\varphi\colon [0,1] \to e$ such that $\varphi(0) = u$ and $\varphi(1) = v$. We denote by $f_1, \dots, f_k$ the edges in $G \setminus T$ that have v as an endpoint.

We first consider the case that for $j = 1, \dots, k$ the edge f_j has another endpoint w_j. We pick homeomorphisms $\psi_j\colon [0,1] \to f_j$, $j = 1, \dots, n$ such that $\psi_j(0) = v$ and $\psi_j(1) = w_j$. Now we denote by G' a graph that has the vertices $V \setminus \{v\}$, the edges

$E \setminus \{e, f_1, \dots, f_k\}$ and which has new edges f'_j, $j = 1, \dots, k$ with endpoints u and w_j.[213] We pick homeomorphisms $\psi'_j \colon [0,1] \to f'_j$, $j = 1, \dots, n$ such that $\psi'_j(0) = u$ and $\psi'_j(1) = w_j$. We consider the maps[214]

$$g \colon G \to G' \qquad x \mapsto \begin{cases} \psi'_j(\psi_j^{-1}(x)), & \text{if } x \in f_j, \\ u, & \text{if } x \in e, \\ x, & \text{otherwise.} \end{cases} \quad \text{and} \quad g' \colon G' \to G \qquad x \mapsto \begin{cases} \varphi(2t), & \text{if } x = \psi'_j(t), t \in [0, \frac{1}{2}], \\ \psi_j(2t-1), & \text{if } x = \psi'_j(t), t \in [\frac{1}{2}, 1], \\ x, & \text{otherwise.} \end{cases}$$

We leave it as an exercise to show that $g \circ g' \colon G' \to G'$ and $g' \circ g \colon G \to G$ are homotopic to the identity map.[215] This shows that G is homotopy equivalent to G' for which $T' = (W \setminus \{v\}, F \setminus e)$ is a spanning tree. Since T' has one edge less than T it follows from our induction hypothesis that G' is homotopy equivalent to the wedge of $1 - \chi(G') = 1 - \chi(G) = n$ circles. It follows from Lemma 9.8 that G itself is homotopy equivalent to the wedge of n circles.

The case that there are edges in $G \setminus E$ for which v is the only endpoint is dealt with in a very similar way. We leave this case as an exercise to the reader. A related statement will be proved in Exercise 9.17.

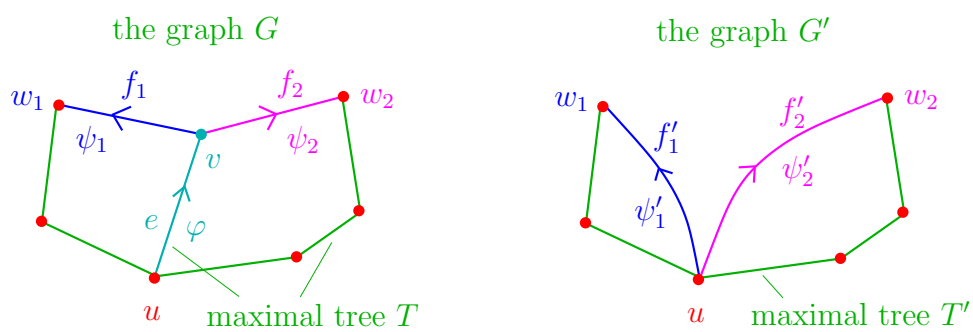


FIGURE 188. Illustration for the proof of the third statement of Proposition 9.21.

(4) This statement was proved implicitly in (3). We leave it to the reader to fill in the details.
(5) The last statement is an immediate consequence of (3) and Lemma 9.8. □

9.8. **Homotopy classes of loops** (∗)**.** In the last section of this chapter we want to study the question, when are two maps $S^1 \to X$ to a topological space homotopic? In fact the following proposition gives in some sense a complete answer to the above question.

[213] At that point one has to argue why such a graph G' exists. One way of seeing this is to take the graph $G \subset \mathbb{R}^n$ and view it as a graph in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. In $\mathbb{R}^{n+1}$ we can connect the points u and w_j by disjoint edges, without intersecting the previous graph.

[214] Using Lemma 1.32 (2') one can easily verify that these maps are in fact continuous.

[215] Note that if $n = 0$, then this is precisely the construction from the proof of Proposition 9.20.

Proposition 9.22. *Let X be a topological space and let $f_0, f_1 \colon S^1 \to X$ be two maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.*

(1) *The following statement holds:*[216]

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0).\end{array}$$

(2) *If $x_0 = x_1$, then the following statement holds:*

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff [f_0] \text{ and } [f_1] \text{ are conjugate in } \pi_1(X, x_0).$$

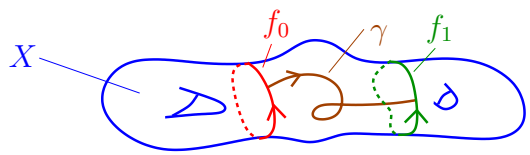


FIGURE 189. Illustration of Proposition 9.22.

Proof. Note that the second statement is an immediate consequence of the first statement and the observation that if $x_0 = x_1$ any path from x_0 to $x_1 = x_0$ is in fact a loop and thus defines an element in $\pi_1(X, x_0)$. Therefore it suffices to prove the first statement.

Now let X be a topological space. We take the view that maps $S^1 \to X$ are the same as loops, i.e. maps $[0,1] \to X$ such that 0 and 1 get sent to the same point. Thus let $f_0, f_1 \colon [0,1] \to X$ be two loops. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.

"$\Rightarrow$" Suppose there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ with $F(0,t) = F(1,t)$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = f_1$ and. We denote by $\gamma \colon [0,1] \to X$ the path that is given by $\gamma(t) = F(0,t)$ for $t \in [0,1]$. We consider the surjective map $\Phi \colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in Figure 190. The map $\Phi \circ F$ is easily seen to be a homotopy between the loops f_0 and $\gamma * f_1 * \overline{\gamma}$. But that is exactly what we needed to show.

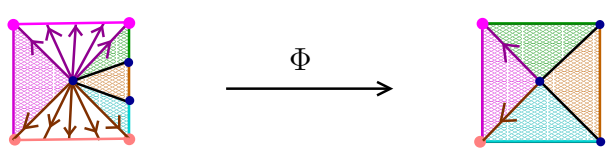


FIGURE 190. First illustration for the proof of Proposition 9.22.

[216]Sometimes two maps $S^1 \to X$ that are homotopic are called *freely homotopic* to stress the fact that no base points are fixed.

"$\Leftarrow$" Now suppose that there exists a path γ from x_0 to x_1 such that $[\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0)$. This means that there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ with $F(0,t) = F(1,t) = x_0$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = \gamma * f_1 * \overline{\gamma}$. We consider the homeomorphism $\Psi\colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in Figure 191. The map $\Psi \circ F$ is easily seen to be a homotopy between $c_{x_0} * f_0 * c_{x_0}$ and f_1, viewed as maps $S^1 \to X$. But the map $c_{x_0} * f_0 * c_{x_0}$ is easily seen to be homotopic to the map f_0. Since being homotopic is transitive we have indeed shown that f_0 and f_1 are homotopic. $\square$

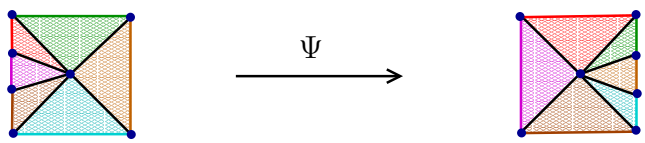


FIGURE 191. Second illustration for the proof of Proposition 9.22.

Exercises for Chapter 9.

Exercise 9.1. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism with $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. We consider the map

$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \leq \|x\| < t, \\ x, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that H is continuous.
(b) Show that for each $t \in [0,1]$ the map $H_t\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism.

Exercise 9.2. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism. Show that there exists a homeomorphism $F\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ with $F|_{S^n} = f$.
Remark: This statement is also sometimes referred to as Alexander's trick.

Exercise 9.3.

(a) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.
  (i) Show that f is monotonously increasing or monotonously decreasing.
  (ii) Suppose f is monotonously increasing. Show that f is isotopic to the map $f(x) = x$.
  (iii) Suppose f is monotonously decreasing. Show that f is isotopic to the map $f(x) = -x$.
(b) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.
  Hint: Let $f\colon S^1 \to S^1$ be a self-homeomorphism. First consider the case that $f(1) = 1$.

Exercise 9.4. Let h be a complex polynomial of degree ≥ 1. In Exercise 1.11 we showed that the map

$$\begin{array}{rcl} \Theta(h)\colon \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} \\ z & \mapsto & \begin{cases} h(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

is continuous. Now let f and g be two polynomials with complex coefficients of degree ≥ 1. Suppose that $\deg(f)=\deg(g)$. Show that the maps $\Theta(f)$ and $\Theta(g)$ are homotopic.
Remark: It is quite easy to write down a candidate for a homotopy. But why is the map you wrote down actually continuous?

Exercise 9.5. Using Lemma 1.41 we can make the identification $S^1=\mathbb{R}\cup\{\infty\}$. Now let f be a polynomial with real coefficients of degree $n\geq 1$. In Exercise 1.11 we showed that the map

$$\begin{array}{rcl} F\colon S^1=\mathbb{R}\cup\{\infty\} & \to & \mathbb{R}\cup\{\infty\}=S^1 \\ t & \mapsto & \begin{cases} f(t), & \text{if } t\in\mathbb{R}, \\ \infty, & \text{if } t=\infty \end{cases} \end{array}$$

is continuous. (See Figure 68 for an illustration.) Show that F is homotopic to one of the following three types of self-maps of S^1:

$$(1)\qquad a(z)\;=\;z \qquad\qquad (2)\qquad b(z)\;=\;\overline{z} \qquad\qquad (3)\qquad c(z)\;=\;1.$$

Exercise 9.6.

(a) We consider the topological space

$$X \;=\; \{(x,y)\,|\,x\in[0,1]\cap\mathbb{Q}, y\in[0,1-x]\}\cup([0,1]\times\{0\})$$

that is shown in Figure 192 to the left.
  (i) Show that given any point P in $[0,1]\times\{0\}$ there exists a deformation retraction from X to P.
  (ii) Show that for any point Q in X that does not lie on $[0,1]\times\{0\}$ there is no deformation retraction from X to Q.

(b) We consider the infinite zigzag comb Z that is shown in Figure 192 to the right.
  (i) Show that Z is contractible.
  (ii) Show that Z does not admit a deformation retraction to any point in Z.

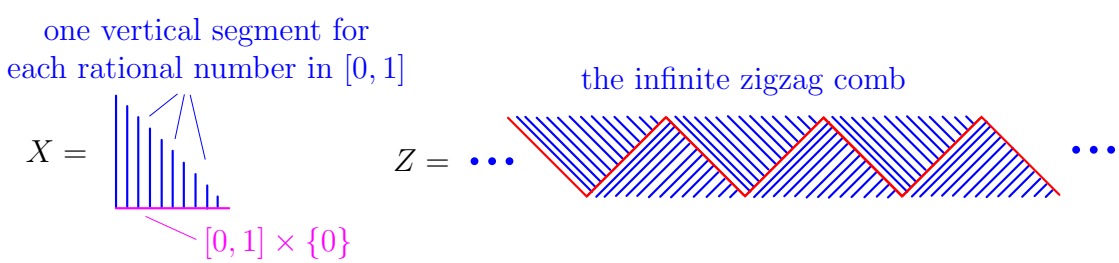


FIGURE 192.

Exercise 9.7. Let A be a convex subset of $\mathbb{R}^n$ and let X be a path-connected topological space. Show that any two maps $f, g\colon A \to X$ are homotopic.

Exercise 9.8. Prove Lemma 9.5.

Exercise 9.9. Show that the topological spaces

$$X = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\} \quad \text{and} \quad Y = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{-1\}$$

are not homotopy equivalent.

FIGURE 193. Illustration of Exercise 9.9.

Exercise 9.10. Let $0 < r < s$ be two real numbers. Show that the obvious inclusion map $B^n_r(0) \setminus \{0\} \to B^n_s(0) \setminus \{0\}$ is a homotopy equivalence,

Exercise 9.11. Prove Lemma 9.9.

Exercise 9.12. Show that for any $n \in \mathbb{N}$ the sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Exercise 9.13. Prove Lemma 9.10.

Exercise 9.14. Show that

$$S^1 \vee S^1 = \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.

Exercise 9.15. Let $X := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$. We denote by $\| - \|$ the usual Euclidean norm on X. We denote by $P = (0, 0)$ the center of X. The *Paris metric* on X is given by

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (A, B) & \mapsto & \|A - P\| + \|B - P\|. \end{array}$$

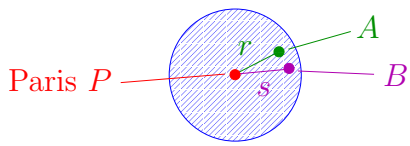
This metric takes its name from the fact that all (OK, many) train lines in France go towards Paris. So if you want to go from A to B you invariably (OK, often) have to go through Paris.

(a) Show that the map

$$\begin{array}{rcl} \varphi\colon \bigvee_{z \in S^1} [0,1]_z & \to & X \\ [r] & \mapsto & r \cdot z \quad \text{if } r \in [0,1]_z. \end{array}$$

is not homeomorphism.

(b) Does there exist a homeomorphism between $\bigvee_{z\in S^1}[0,1]_z$ and X?



in the Paris metric the distance between P and Q equals $r+s$

FIGURE 194. Illustration of Exercise 9.15.

Exercise 9.16. Let (X,x_0) and (Y,y_0) be two pointed topological spaces. We use the base points to define the wedge $X\vee Y$.

(a) Show that the map

$$\begin{array}{rcl} f\colon X\vee Y & \to & X\times Y \\ [P] & \mapsto & \begin{cases} (P,y_0), & \text{if } P\in X, \\ (x_0,P), & \text{if } P\in Y \end{cases} \end{array}$$

is a homeomorphism onto its image.
Remark: We do not assume that X and Y have any nice properties.

(b) Using (a) we can view $X\vee Y$ as a subspace of $X\times Y$. We refer to the quotient

$$X\wedge Y \;:=\; (X\times Y)/(X\vee Y)$$

as the *smash product of* (X,x_0) *and* (Y,y_0). Show that for any $m,n\in\mathbb{N}_0$ the smash product $S^m\wedge S^n$ is homeomorphic to S^{m+n}.

We refer to Figure 195 for an illustration.

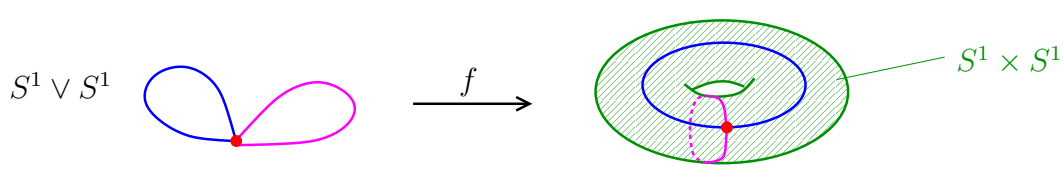


FIGURE 195. Illustration for Exercise 9.16.

Exercise 9.17. Let

$$X \;:=\; \underbrace{\{z\in\mathbb{C}\,|\,|z-2|=1\}}_{=:A} \;\cup\; [-1,1] \;\cup\; \underbrace{\{z\in\mathbb{C}\,|\,|z+2|=1\}}_{=:B}$$

and

$$Y \;:=\; \underbrace{\{z\in\mathbb{C}\,|\,|z-1|=1\}}_{=:C} \;\cup\; \underbrace{\{z\in\mathbb{C}\,|\,|z+1|=1\}}_{=:D}.$$

Show that X and Y are homotopy equivalent.

FIGURE 196. Illustration for Exercise 9.17.

Exercise 9.18. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$ with the property that $\{x_i\}$ is a closed subset of X_i. We use these points to form the wedge $\bigvee_{i\in I} X_i$. Let $K \subset \bigvee_{i\in I} X_i$ be a compact subset. Show that there exists a finite subset $J \subset I$ such that $K \subset \bigvee_{i\in J} X_i$.

Exercise 9.19. We consider the subset X of $\mathbb{R}^2$ that is shown in Figure 197. It is the union of infinitely many circles $A_i \subset \mathbb{R}^2$, $i \in \mathbb{N}$ that meet in the origin. Furthermore let $Y = \bigvee_{i\in\mathbb{N}} S^1$. There exists a fairly obvious bijection $f\colon X \to Y$ which sends each A_i to the i-th factor of Y.

(a) Is the map f continuous?
(b) Is the map f^{-1} continuous?

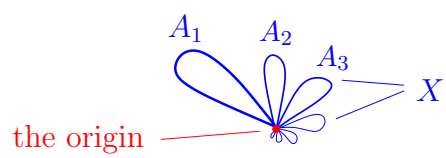


FIGURE 197. Illustration of Exercise 9.19.

Exercise 9.20. Let $f, g\colon (X, x_0) \to (Y, y_0)$ be two maps between pointed topological spaces.

(a) Show that if f and g are homotopic rel $\{x_0\}$, then $f_* = g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
(b) Does the conclusion of (a) hold if we only demand that f and g are homotopic, but we no longer demand that they are homotopic rel the base point?

Exercise 9.21. Let $f\colon X \to Y$ be a map between two topological spaces. Let $x_0 \in X$ be a base point. We write $y_0 = f(x_0)$ and we denote by $i\colon f(X) \to Y$ the inclusion map. Suppose that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Does it follow that $i_*\colon \pi_1(f(X), y_0) \to \pi_1(Y, y_0)$ is also an isomorphism?

10. Differential topology

Differential topology is the study of manifolds and their submanifolds viewed as topological spaces. In this section we will first give several basic definitions in differential topology. Afterwards we will state, without proof but with precise references, some technical results which later on will make our life much easier.

10.1. **Immersions and embeddings.** We start out with some definitions that will crop up frequently throughout these lecture notes.

Definition. Let N be a compact topological manifold and let M be any topological manifold. A map $\varphi\colon N \to M$ is called *proper* if $\varphi^{-1}(\partial M) = \partial N$.

We illustrate the definition of a proper map in Figure 198.

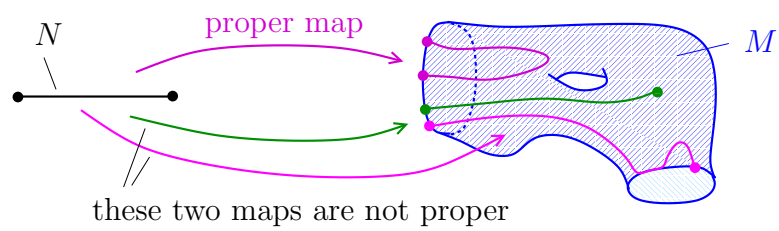


Figure 198.

The following definition is related in spirit to the notion of a submersion that we had introduced on page 211.

Definition. A map $\varphi\colon N \to M$ between two manifolds is called an *immersion* if the following conditions are satisfied:

(1) φ is smooth,
(2) for each $P \in N$ the induced map $\varphi_*\colon T_P N \to T_{\varphi(P)} M$ is a monomorphism,
(3) for each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$,
(4) for each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\varphi_*(T_P N)$ is *not* contained in $T_{\varphi(P)}(\partial M)$.[217]

An immersion that also satisfies the following extra condition

(5) $\varphi\colon N \to \varphi(N)$ is a homeomorphism,[218][219]

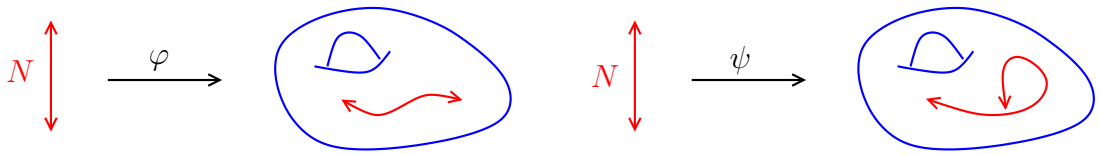
is called an *embedding*.

The following proposition relates the notion of embeddings to the notion of submanifolds.

[217]Another way of expressing this condition is that the map $T_P N \to T_{\varphi(P)} M / T_{\varphi(P)} \partial M$ is an epimorphism.

[218]Here $\varphi(N) \subset M$ is endowed with the subspace topology.

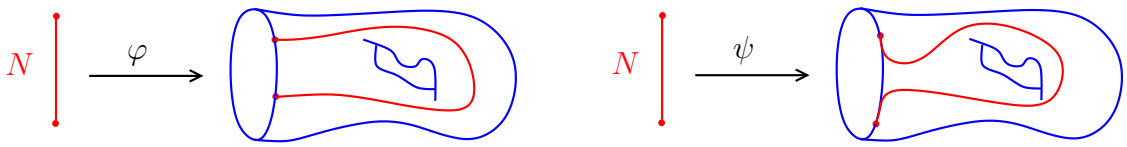
[219]If N is compact, then it follows from Proposition 1.40 that it suffices to verify that φ is injective and continuous.



φ is an embedding

ψ is not an embedding since $\varphi\colon N \to \varphi(N)$ is not a homeomorphism

FIGURE 199.



φ is an embedding

ψ is not an embedding since condition (3) is not satisfied

FIGURE 200.

Proposition 10.1. *Let N be an n-dimensional manifold and let M be a manifold.*

(1) *Let $\varphi\colon N \to M$ be an embedding, then $\varphi(N)$ is an n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.*
(2) *Suppose N is compact. If $\varphi\colon N \to M$ is a proper embedding, then $\varphi(N)$ is a proper n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.*

Proof. For (2) we need to make the following observation: If $\varphi\colon N \to M$ is a map and N is compact, then it follows from Lemma 1.37 and Lemma 1.18 that $\varphi(N)$ is a closed subset of M.

For the actual meat of the proof we have to refer to the literature. For manifolds without boundary the proposition is proved in [Le2, Proposition 5.2] and [BG, Corollary 2.6.11]. The proof for manifolds with boundary is quite similar. We leave it to the reader to fill in the details. Note that [Le2, Proposition 5.2] actually deals with manifolds with boundary. But as we pointed out on page 186, the definition of a submanifold with boundary in [Le2] differs from our definition and also the notion of an embedding differs from our definition. □

Example. It is straightforward to verify that the map $\Psi\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^3$ given in Lemma 1.58 restricts to an embedding $\Psi\colon S^1 \times S^1 \to \mathbb{R}^3$ and that it also restricts to an embedding $\Psi\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$. By Proposition 10.1 this allows us to view $S^1 \times S^1$ and $\overline{B}^2 \times S^1$ as submanifolds of $\mathbb{R}^3$.

In the following proposition we consider immersions and embeddings between manifolds of the same dimension.

Proposition 10.2. *Let M and N be manifolds of the same dimension and furthermore let $\varphi\colon N \to M \setminus \partial M$ be a map.*

(1) *If $\varphi\colon N \to M \setminus \partial M$ is an immersion, then $\varphi(N \setminus \partial N)$ is an open subset of M.*
(2) *If $\varphi\colon N \to M \setminus \partial M$ is an embedding and if N is compact, then M is decomposed into the submanifolds $\varphi(N)$ and $X := M \setminus \varphi(N \setminus \partial N)$ and the boundary of $M \setminus \varphi(N \setminus \partial N)$ as a manifold in its own right is given by $\partial M \cup \varphi(\partial N)$. (We refer to Figure 201 for an illustration.)*

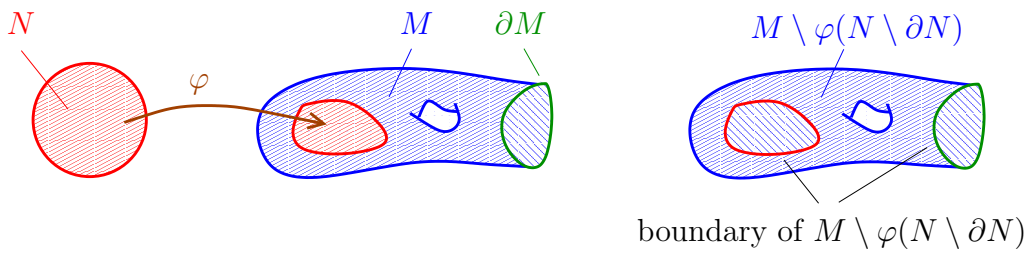


FIGURE 201. Illustration of Proposition 10.2 (2).

Sketch of proof.

(1) Let $\varphi\colon N \to M \setminus \partial M$ be an immersion. It is not difficult to deduce from Proposition 3.20 that $\varphi(N \setminus \partial N)$ is an open subset of M. We leave it to the reader to fill in the details.
(2) This statement follows immediately from Propositions 10.1 and 3.24. □

We conclude this discussion of immersions and embeddings with the following instructive example which is dressed up as a lemma.

Lemma 10.3. *Given $\lambda \in \mathbb{R}$ we consider the following map*

$$\begin{array}{rcl} \varphi_\lambda\colon \mathbb{R} & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ t & \mapsto & [(t, \lambda \cdot t)]. \end{array}$$

The following statements hold:

(1) *The map φ_λ is an immersion if and only if $\lambda \neq 0$.*
(2) *The map φ_λ is injective if and only if λ is irrational.*
(3) *If λ is irrational, then $\varphi_\lambda\colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is* not *a homeomorphism onto its image, i.e. φ_λ is not an embedding.*
(4) *Suppose λ is rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. Then the map*

$$\begin{array}{rcl} \varphi_\lambda\colon S^1 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ e^{2\pi i t} & \mapsto & [(qt, \lambda \cdot qt)] \end{array}$$

is an embedding.

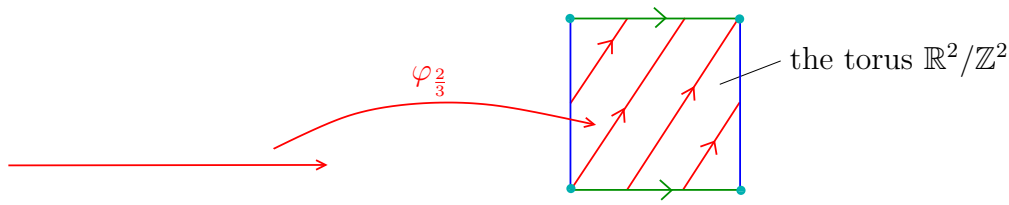


FIGURE 202.

Proof. Statements (1), (2) and (4) are amusing exercises. Statement (3) requires the simple number theoretic fact that given any $\alpha \in \mathbb{R}$ and any $N \in \mathbb{N}$ there exist $n, m \in \mathbb{Z}$ with $1 \leq n \leq N$ and with $|n\alpha - m| < \frac{1}{N}$.[220] We leave the details to the reader. □

10.2. **Smooth homotopies.** Now suppose we are given two smooth maps $f_0, f_1 \colon M \to N$ between manifolds. On page 301 we defined a homotopy between f_0 and f_1 to be a map $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$. As we will see shortly, on many occasions this definition is not very useful. It is much more reasonable to consider "smooth" homotopies, namely we want F itself to be a smooth map.

But this leads to a slightly annoying question: What does it mean for $F \colon M \times [0,1] \to N$ to be smooth? If M is closed, then by Proposition 3.42 (1) we can view $M \times [0,1]$ as a manifold and we know what it means for a map $F \colon M \times [0,1] \to N$ to be smooth. But what should we do if M has non-empty boundary?

This leads us to the following definition.

Definition. Let M and N be manifolds and let $a < b$ be two real numbers. We say a map $\varphi \colon M \times [a,b] \to N$ is *smooth* if the following conditions are satisfied:

(1) The map $(M \setminus \partial M) \times [a,b] \to N$ is smooth.
(2) Given any $x \in \partial M$ there exists an open neighborhood U of x in M, an $\epsilon \in (0, b-a)$ and a smooth map $\psi \colon U \times (a-\epsilon, a+\epsilon) \to N$ that agrees with φ on $U \times [a, a+\epsilon)$.
(3) Given any $x \in \partial M$ there exists an open neighborhood U of x in M, an $\epsilon \in (0, b-a)$ and a smooth map $\psi \colon U \times (b-\epsilon, b+\epsilon) \to N$ that agrees with φ on $U \times (b-\epsilon, b]$.

Note that in all three cases it makes sense to talk of a map being smooth since by Proposition 3.42 (4) we know that $(M \setminus \partial M) \times [a,b]$ is a manifold with boundary $(M \setminus \partial M) \times \{a,b\}$, that $U \times (a-\epsilon, a+\epsilon)$ is a manifold with boundary $\partial U \times (a-\epsilon, a+\epsilon)$ and finally $U \times (b-\epsilon, b+\epsilon)$ is a manifold with boundary $\partial U \times (b-\epsilon, b+\epsilon)$.

Remark.

(1) If M is a manifold without boundary, then the cases (2) and (3) do not occur and the above definition of a smooth map $\varphi \colon M \times [a,b] \to N$ is precisely the one we had always used.

[220] This statement is known as Dirichlet's Approximation Theorem, but it is actually elementary to prove, see e.g. [Le2, Lemma 4.21].

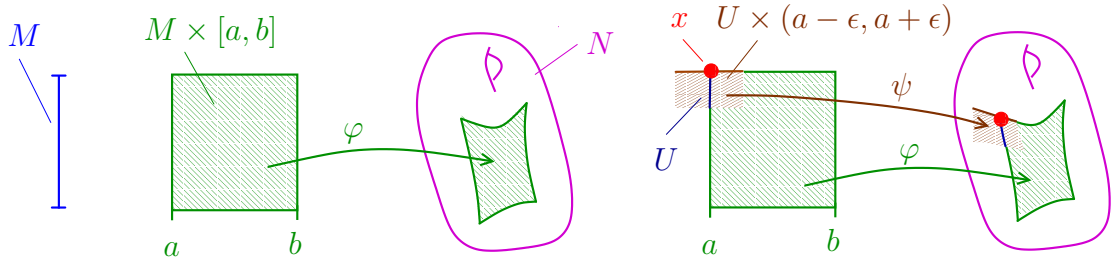


FIGURE 203.

(2) Let M be a manifold. We could view $M \times [0,1]$ as a manifold with "corners", see e.g. [Wall2, p. 30] or [Le2, Chapter 16] for a definition. The definition of a smooth map on a manifold with corners is very similar to the definition for manifolds with boundary, which in turn relies on the definition of a smooth map on an arbitrary subset of $\mathbb{R}^n$ that we gave on page 181. It follows easily from the definitions that these two definitions of a smooth map on $M \times [0,1]$ agree.

Now we can define the notion of a smooth homotopy.

Definition. Let M and N be manifolds.

(1) A *smooth homotopy* is a smooth map $F\colon M \times [0,1] \to N$.
(2) We say that two smooth maps $f_0, f_1\colon M \to N$ are *smoothly homotopic* if there exists a smooth homotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.

Example. Let M and N be manifold and let $F\colon M \times [0,1] \to N$ be a homotopy. If there exists an $\epsilon > 0$ and a smooth map $M \times (-\epsilon, 1+\epsilon) \to N$ that restricts to F on $M \times [0,1]$, then F is by definition smooth. For example, it follows easily from this observation that the map

$$\begin{array}{rcl} S^1 \times [0,1] & \to & \mathbb{R}^2 \\ (z,t) & \mapsto & z \cdot (1-t) \end{array}$$

is a smooth homotopy from the inclusion map $S^1 \to \mathbb{R}^2$ to a constant map.

We continue with our next definition.

Definition. Let M and N be manifolds.

(1) A (*proper*) *smooth isotopy of M in N* is a smooth[221] map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that for each $t \in [0,1]$ the map $F_t\colon M \to N$ is a (proper) embedding.

[221] We would like to stress that here we want a smooth map, not just a continuous map. Also note that we defined on page 333 what it means for the map $F\colon M \times [0,1] \to N$ to be smooth.

(2) We say that two (proper) embeddings $f_0, f_1 \colon M \to N$ are (*properly*) *smoothly isotopic* if there exists a (proper) smooth isotopy $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.

If $M = N$ and if each F_t is a diffeomorphism, then we refer to a smooth isotopy as a *diffeotopy* and we say *diffeotopic* instead of *smoothly isotopic*.

The following lemma gives in particular an example of a diffeotopy.

Lemma 10.4. *Let $n \in \mathbb{N}$.*

(1) *If $\gamma \colon [0,1] \to \mathrm{O}(n)$ is a smooth path, then the map*

$$\begin{array}{rcl} F \colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$

is a diffeotopy rel the origin.

(2) *Let $A \in \mathrm{SO}(n)$. If $n \geq 2$, then there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.*

The same statement holds with $\overline{B}^n$ replaced by $\mathbb{R}^n$ and S^{n-1}.

Proof.

(1) The first statement is basically obvious.

(2) Suppose that $n \geq 2$ and let $A \in \mathrm{SO}(n)$. By Lemma 1.96 the manifold $\mathrm{SO}(n)$ is path-connected. Therefore, by Exercise 10.11 there exists a smooth path in $\mathrm{SO}(n)$ from id to A. The statement now follows from (1). □

Remark. The term "diffeotopic" is (alas!) used differently by different authors. Our definition agrees with the definition given in most textbooks, see e.g. [Wall2, p. 49] and [BJ, Definition 9.3]. But it differs from the definition provided in [Hirs, Chapter 8], there the map on $M \times [0,1]$ only needs to be continuous, but each F_t needs to be a diffeomorphism. At least a priori this is a weaker condition.

We leave the proof of the following lemma as a voluntary exercise to the eager reader.

Lemma 10.5. *Let M be an oriented connected manifolds and let $F \colon M \times [0,1] \to N$ be a diffeotopy. Then either all maps $F_t \colon M \to M$ are orientation-preserving or all F_t are orientation-reversing.*

For later purposes we state the following lemma.

Lemma 10.6. $(*)$ *Let N be a manifold and let M be another manifold and let $f, g, h \colon M \to N$ be proper embeddings. If f is smoothly isotopic to g and g is smoothly isotopic to h, then f is also smoothly isotopic to h. The same statement also holds for "diffeotopic" instead of "smoothly isotopic".*

Proof $(*)$. Let $\Phi \colon M \times [0,1] \to N$ be a smooth isotopy from f to g and let $\Psi \colon M \times [0,1] \to N$ be a smooth isotopy from g to h. The first idea is of course to "stack" the two smooth

isotopies together, i.e. to consider the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, 2\cdot t), & \text{if } t \in [0, \frac{1}{2}], \\ \Psi(P, 2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

By Lemma 5.3 we know that this map is continuous. The problem with this approach is that this map is not necessarily smooth at $M \times \{\frac{1}{2}\}$. Thus we have to modify the above approach slightly. It follows from Lemma 3.12 that there exist a smooth function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$ such that $f(t) = 1$ for all $t \in [\frac{1}{2}, 1]$ and a smooth function $g\colon [0,1] \to [0,1]$ with $g(1) = 1$ such that $g(t) = 0$ for all $t \in [0, \frac{1}{2}]$. It is now straightforward to verify that the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, f(2\cdot t)), & \text{if } t \in [0, \frac{1}{2}], \\ \Psi(P, g(2\cdot t - 1)), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

is a smooth isotopy from f to h. The proof for "diffeotopic" instead of "smoothly isotopic" is verbatim the same. □

The following lemma gives an interesting example of a smooth deformation retraction.

Lemma 10.7. $(*)$ *Let $n \in \mathbb{N}_0$ and let $k \in \{1, \dots, n\}$. We write*

$$\begin{array}{rcl} \mathrm{GL}(n,k) & = & \textit{set of } k\textit{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) & = & \textit{set of } k\textit{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{array}$$

(Note that for $k = n$ we have $\mathrm{GL}(n,k) = \mathrm{GL}(n)$ and $\mathrm{O}(n,k) = \mathrm{O}(n)$.) We denote by $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ the map given by the Gram-Schmidt orthonormalization process.[222] *The following statements hold:*

(1) *The above map $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is realized by a smooth deformation retraction*[223] *$R\colon \mathrm{GL}(n,k) \times [0,1] \to \mathrm{O}(n,k)$ which has the following extra property:*
   $(*)$ *For any k-dimensional subvector space V of $\mathbb{R}^n$ and any $t \in [0,1]$ we have*

$$R(\textit{basis of } V, t) \subset \textit{basis of } V.$$

   In particular r is smooth and it is a retraction.

(2) *The inclusion $\mathrm{O}(n,k) \to \mathrm{GL}(n,k)$ is a homotopy equivalence.*

The analogous statement hold if we replace the pair $\mathrm{O}(n,k)$ and $\mathrm{GL}(n,k)$ by one of the following four pairs:

(a) $\mathrm{SO}(n)$ *and* $\mathrm{GL}_+(n,\mathbb{R})$,
(b) $\mathrm{U}(n)$ *and* $\mathrm{GL}(n,\mathbb{C})$,
(c) $\mathrm{SO}(n)$ *and* $\mathrm{SL}(n,\mathbb{R})$.
(d) $\mathrm{SU}(n)$ *and* $\mathrm{SGL}(n,\mathbb{C})$.

[222] This map is defined in any course on linear algebra, see e.g. [Cu, Theorem 15.9].

[223] In Exercise 10.6 we will see that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$. Therefore the word "smooth" makes sense in the most naive sense.

Sketch of proof Proof $(*)$.

(1) One easily verifies that each of the $2k$ steps of the Gram-Schmidt process is realized by a smooth deformation retraction that satisfies $(*)$. The proof of Lemma 10.6 shows that we can combine these smooth deformation retractions to obtain the desired smooth deformation retraction that satisfies $(*)$.
(2) This statement follows immediately from (1) and Lemma 9.10.

We consider the four extra pairs :

(a) The given smooth deformation retraction restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$.
(b) The proof in this case is basically identical to the proof concerning the pair $\mathrm{O}(n)$ and $\mathrm{GL}(n,\mathbb{R})$.
(c) The given smooth deformation retraction does *not* restrict to a deformation retraction from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$. The problem is that in the Gram-Schmidt process we have to successively normalize the column vectors to length one. This process does not preserve the determinant. The solution is that when we normalize the i-th column vector by a factor r, then we compensate by simultaneously rescaling the $(i+1)$-st column vector by r^{-1}. This issue does not arise when we have arrived at the last column vector, since starting out with a matrix of determinant one there is no need to rescale the last column vector.
(d) The proof in this case is basically identical to the proof of (c). □

For general culture we cite the following difficult theorem due to Jean Cerf [Cer2]. We could not find results in the literature that deal with the lower dimensional cases.

Theorem 10.8. $(*)$ *Let $n \geq 6$. Every orientation-preserving diffeomorphism of $\overline{B}^n$ is diffeotopic to the identity.*

We conclude this dense section with a short discussion of submanifolds.

Definition. Let M_0, M_1 be two (proper) submanifolds of a manifold N.

(1) A (*proper*) *smooth isotopy* is a (proper) smooth isotopy $F\colon M_0 \times [0,1] \to N$ with $F_0 = \mathrm{id}_{M_0}$ and with $F_1(M_0) = M_1$.
(2) We say that the two submanifolds are (*properly*) *smoothly isotopic* if there exists a (proper) smooth isotopy between M_0 and M_1.

If M_0 and M_1 are oriented manifolds, then we demand that the map $F_1\colon M_0 \to M_1$ is orientation-preserving.

Example. In Figure 204 we consider some oriented 1-dimensional submanifolds of a surface Σ. In some cases it is clear that submanifolds are smoothly isotopic, in other cases this looks dubious.

10.3. **The Collar Neighborhood Theorem.** After the preparations from the last two sections we now have the language to state many useful results from differential topology.

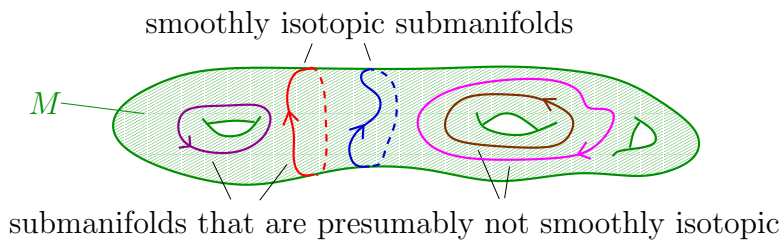


FIGURE 204.

Definition. Let M be a manifold. A *collar neighborhood* of ∂M is defined as the image of an embedding $f\colon \partial M \times [0,1] \to M$ that is the identity map on $\partial M \times \{0\} = \partial M$.

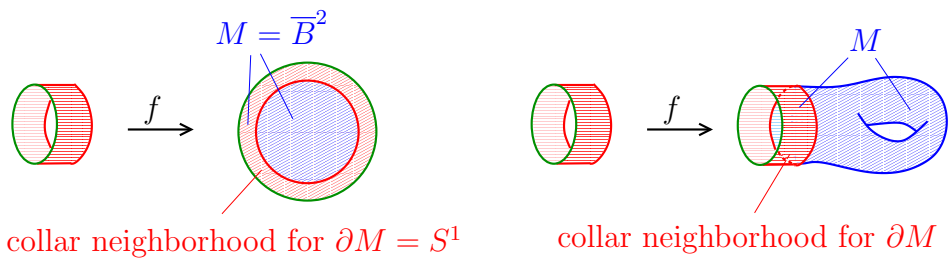


FIGURE 205.

Convention. We continue with the notation from the definition. By abuse of notation we usually denote the image of f by $\partial M \times [0,1]$ as well. In particular we make the identification $\partial M \times \{0\} = \partial M$. Sometimes we refer to $\partial M \times [0,1)$ as an *open collar neighborhood* of ∂M.

The following lemma justifies the names "collar neighborhood" and "open collar neighborhood".

Lemma 10.9. *Let M be a manifold and let $\partial M \times [0,1]$ be a collar neighborhood of ∂M. For any $t \in (0,1]$ the set $\partial M \times [0,t)$ is an open subset of M, in particular it is in fact a neighborhood of ∂M in the sense of the definition on page 48.*

Proof. The proof of this lemma is quite similar to the proof of Proposition 10.2, i.e. the proof is a reasonably straightforward consequence of Proposition 3.20. Once again we leave it to the reader to fill in the details. □

Now we can state the Collar Neighborhood Theorem.

Theorem 10.10. (Collar Neighborhood Theorem) *Every manifold admits a collar neighborhood.*

Proof. This statement is proved in basically any book on differential topology, see [Wall2, Theorem 1.5.5] or alternatively [Le2, Theorem 9.25], [BJ, Theorem 13.6], [Muk, Theorem 7.2.13]. □

The following proposition says, that under a fairly mild hypothesis, collar neighborhoods are essentially unique.

Proposition 10.11. *Let M be a manifold and let $f_0, f_1 \colon \partial M \times [0,1] \to M$ be two embeddings that are the identity map on $\partial M \times \{0\} = \partial M$. If ∂M is compact*[224]*, then there exists a diffeotopy*

$$H \colon M \times [0,1] \;\to\; M$$

rel ∂M[225] *from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1 \colon M \to M$ with $\Phi \circ f_0 = f_1$. In particular Φ restricts to a diffeomorphism*

$$\Phi \colon f_0(\partial M \times [0,1])) \;\to\; f_1(\partial M \times [0,1]).$$

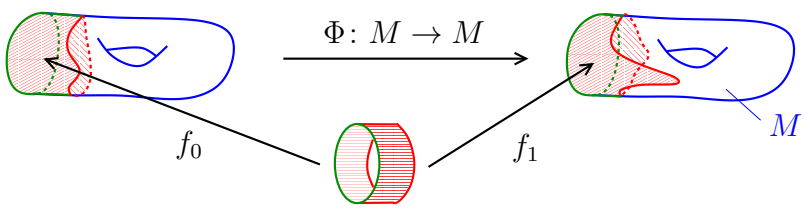


FIGURE 206.

Proof. The proposition is proved in [Wall2, Proposition 2.5.7]. A very similar statement is also proved in [BJ, Theorem 13.7]. □

We continue with the following proposition.

Proposition 10.12. *Let M be an n-dimensional manifold (we do not assume that M is connected). Let A and B be two distinct boundary components of M. Furthermore let $f \colon A \to B$ be a diffeomorphism.*[226] *Then the following statements hold:*

(1) *The topological space*

$$M/a \sim f(a) \qquad \text{for } a \in A$$

admits the structure of an n-dimensional manifold such that $M \setminus (A \cup B)$ is a submanifold. If A and B are compact, then the diffeomorphism type of this manifold is canonical.[227]

[224]It is not so clear to me whether one really needs that the boundary of M is compact, but both references prefer to assume that the boundary is compact. Fortunately for our purposes it is irrelevant whether a more general version of Proposition 10.11 holds.

[225]Recall that this means that $H(P,t) = P$ for all $P \in \partial M$ and $t \in [0,1]$.

[226]Recall that by Proposition 3.21 the boundary components A and B are canonically $(n-1)$-dimensional manifolds.

[227]The slightly annoying restriction to compact A and B is due to the fact that we need to apply Proposition 10.11. It might well be that the statement holds without any restrictions on A and B. Also note that in practice one deals almost always with the case of compact boundary components.

(2) *If M consists of two components X and Y and if $A \subset X$ and $B \subset Y$, then the images of X and Y in $M/a \sim f(a)$ are codimension zero submanifolds.*
(3) *If M is compact, then $M/a \sim f(a)$ is also compact.*
(4) *The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.*
(5) *If M is oriented, if A and B are equipped with the boundary orientations and if f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.*

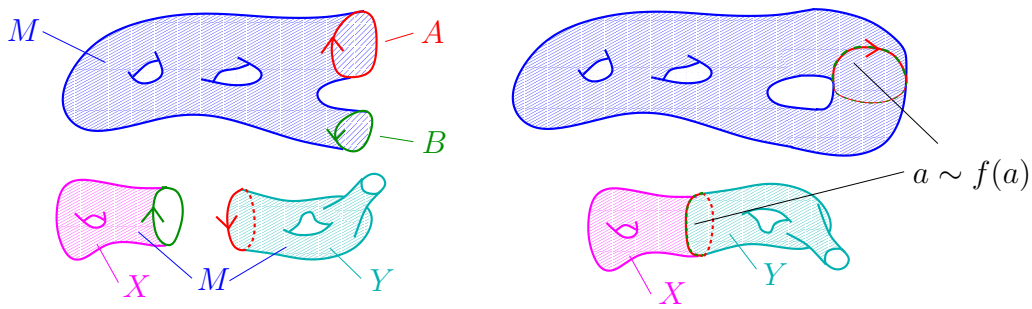


FIGURE 207. Illustration of Proposition 10.12.

Sketch of proof (∗). We first prove statement (1) of the proposition. In the following we write $W := M/a \sim f(a)$ and $M' := M \setminus (A \cup B)$. We denote by $p\colon M \to W$ the projection map. We make the following two observations:

(a) It follows from Proposition 3.21 (3a) that M' is an open subset of M and of W.
(b) It follows from the Collar Neighborhood Theorem 10.10 that there exists a collar neighborhood $\partial M \times [0,1]$ of ∂M.

First we intend to provide a smooth atlas for W. We do so as follows:

(α) We restrict all the charts in the given atlas of M to the open subset M'.
(β) We equip A with the smooth $(n-1)$-dimensional atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ that is provided by Proposition 3.21. Given $i \in I$ we define

$$\begin{aligned} p(U_i \times [0,1)) \cup p(f(U_i) \times [0,1)) &\xrightarrow{\Psi_i} V_i \times (-1,1) \\ x &\mapsto \begin{cases} (\Phi_i(y), t), & \text{if } x = p(y,t) \text{ with } y \in U_i \text{ and } t \in [0,1), \\ (\Phi_i(f^{-1}(y)), -t), & \text{if } x = p(y,t) \text{ with } y \in f(U_i) \text{ and } t \in [0,1). \end{cases} \end{aligned}$$

Using (a) and (b) one easily verifies that the charts from (α) and (β) define a smooth n-dimensional atlas for W. (We refer to Figure 208 for an illustration.)

Now assume that A and B are compact. Using Proposition 10.11 one can show, with some more effort, that the smooth structure that we had just defined on W does not depend

on the choice of the collar neighborhood. Put differently, the smooth structure on W is canonical in the sense that it does not depend on any choices.

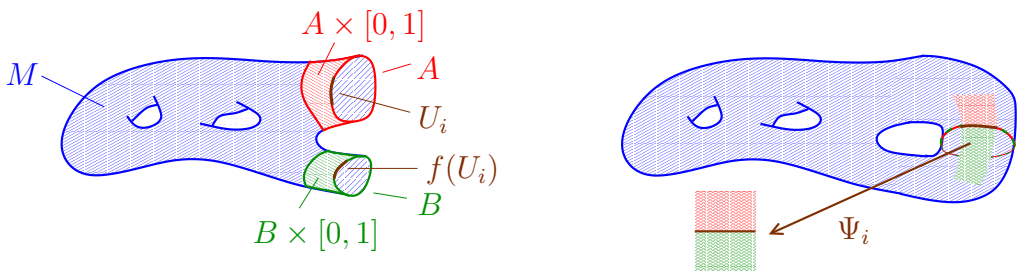


FIGURE 208.

As usual we still have to deal with the slightly annoying task of showing that W is Hausdorff and second-countable:

(i) We leave it as an exercise to show, using (a) and (b) above, that W is Hausdorff.

(ii) Let $\mathcal{B} = \{B_i\}_{i\in I}$ be a countable basis for the topology of M. Once again we leave it as an exercise to verify, using (a) and (b) above, that

$$\big\{B_i \cap M'\big\}_{i\in I} \;\cup\; \big\{p\big(((B_i \cap A) \cup (f(B_i \cap A))) \times [0, \tfrac{1}{2})\big)\big\}_{i\in I}$$

is a basis for the topology of W. Since this is a countable family of sets we see that W is second-countable.

This concludes the proof of statement (1) of the proposition.

We turn to the proofs of the remaining statements:

(2) This statement follows almost immediately from inspecting the atlas we had just constructed. We leave the details to the reader.

(3) This statement follows immediately from Lemma 1.65 (4).

(4) The atlas that we gave above for $W = M/a \sim f(a)$ has a chart of type (i) for every point in $M \setminus (\partial M \setminus (A \cup B))$ and a chart of type (ii) for every point in $\partial M \setminus (A \cup B)$. Therefore it follows from Proposition 3.21 (1) that the boundary of $W = M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.

(5) The final statement can be proved once again using the Collar Neighborhood Theorem 10.10. We leave the verification of this statement as a slightly annoying exercise to the dedicated reader. □

For many purposes and constructions it does not make a difference if we modify a map by a diffeotopy. For example we have the following lemma which often gets used implicitly.

Lemma 10.13. *Let M be an n-dimensional manifold (we do not assume that M is connected) with compact boundary. Let A and B be two distinct compact boundary components of M. Furthermore let f_0 and f_1 be two diffeomorphisms from A to B. If f_0 and f_1 are diffeotopic, then the results of gluing via f_0 respectively via f_1 are diffeomorphic, i.e. we have*

a diffeomorphism

$$M/x \sim f_0(x) \;\cong\; M/x \sim f_1(x).$$

Furthermore, given a neighborhood of A and given a neighborhood B one can find a diffeomorphism that is the identity outside of the chosen neighborhoods.

Sketch of proof. Since f_0 and f_1 are diffeotopic and since they are diffeomorphisms there exists a diffeotopy $F\colon A \times [0,1] \to A$ from $f_1 \circ f_0^{-1}$ to id_A. With the same approach as in the proof of Lemma 10.6 we can arrange that $F_t = \mathrm{id}_A$ for $t \in [\frac{1}{2}, 1]$.

By the Collar Neighborhood Theorem 10.10 we can pick a collar neighborhood $\partial M \times [0,1]$. (Since A is compact we can, after possibly rescaling, arrange that $A \times [0,1]$ is contained in a given neighborhood of A.) One can now easily verify that

$$\begin{array}{rcl} M/x \sim f_0(x) & \to & M/x \sim f_1(x) \\ y & \mapsto & \begin{cases} y, & \text{if } y \in M \setminus (A \times [0,1]), \\ F(a,t) & \text{if } y = (a,t) \in A \times [0,1] \end{cases} \end{array}$$

is a diffeomorphism. □

10.4. **Gluing submanifolds** $(*)$**.** In the previous section we introduced the notion of a collar neighborhood of the boundary of a given manifold M. In many situations one is also given a proper submanifold N and one would like the collar neighborhood and the submanifold to live in harmony. In this section we will deal with this situation and show that harmony can indeed be achieved. The results in this section are even more technical than in the previous section. Thus it is best to skip this section at first reading and only to look at it when we actually use it later on.

In Proposition 10.12 we saw that if we are given a manifold M (which in many examples will be disconnected) and a diffeomorphism $f\colon \partial M \to \partial M$, then any choice of a collar neighborhood of ∂M lets us view $M/x \sim f(x)$ again as a manifold. Now suppose that we are given a proper submanifold N of M with $f(\partial N) = \partial N$. We would like to view $N/x \sim f(x)$ as a submanifold of $M/x \sim f(x)$. But it is easy to find examples where this is not the case, since in all likelihood $N/x \sim f(x)$ will not be smooth at the gluing points. We refer to Figure 209 for an illustration where we consider $M = ((-\infty, 0] \times \mathbb{R}) \sqcup ([1, \infty) \times \mathbb{R})$ with the obvious gluing and the obvious collar neighborhoods.

If one thinks about this issue for a few minutes then one is almost inexorably lead to the following definition.

Definition. Let M be a manifold with a collar neighborhood $\partial M \times [0,1]$, let N be a proper submanifold of M and let $\epsilon \in [0,1]$. We say that N *is a product with respect to* $\partial M \times [0, \epsilon]$ if the map $\partial N \times [0,\epsilon] \to M \times [0,\epsilon]$ defines a diffeomorphism $\partial N \times [0,\epsilon] \to (M \times [0,\epsilon]) \cap N$.

Example. We consider $M = \mathbb{R} \times [0, \infty)$ with the obvious collar neighborhood $\mathbb{R} \times [0,1]$. Furthermore we consider the proper submanifold N that is illustrated in Figure 210. Note that N is a product with respect to $\mathbb{R} \times [0, \frac{1}{2}]$ but that it is not a product with respect to $\mathbb{R} \times [0,1]$.

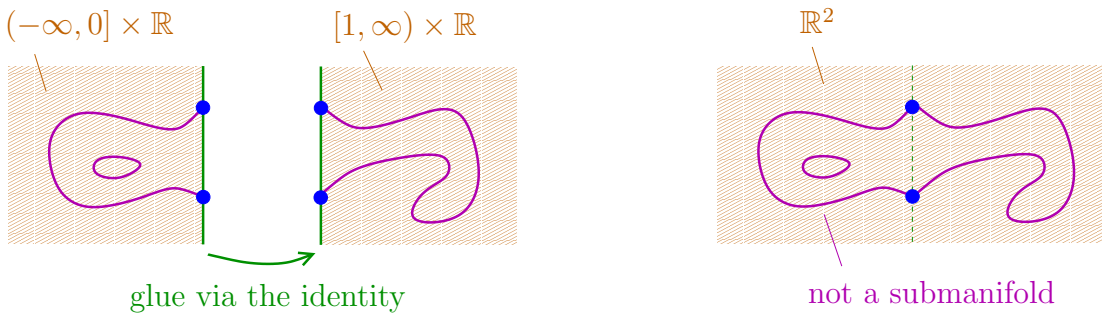


FIGURE 209.

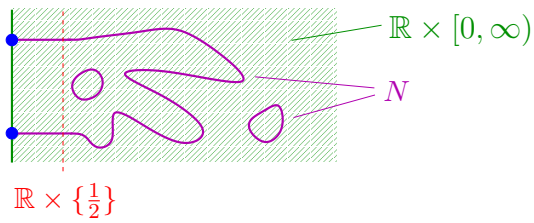


FIGURE 210.

The following proposition gives an approach to circumventing this issue. For better readability it is formatted as if it was statement (6) of Proposition 10.12.

Proposition 10.14. *Let M be an n-dimensional manifold (we do not assume that M is connected). Let A and B be two distinct boundary components of M. Furthermore let $f\colon A \to B$ be a diffeomorphism.*

(6) *Let N be a proper submanifold of M such that f restricts to a diffeomorphism $A \cap N \to B \cap N$. If there exists a collar neighborhood of M such that N is a product with respect to $\partial N \times [0, \epsilon]$ for some $\epsilon > 0$ and if we use this collar neighborhood to define the manifold structure on $M/a \sim f(a)$, then $N/a \sim f(a)$ is a submanifold of $M/a \sim f(a)$.*

Proof. The proof follows easily from the definition of the smooth structure on $M/a \sim f(a)$ as provided in the proof of Proposition 10.12 (1). We will not bore the reader with unnecessary details. □

The question arises whether we can always find such a nice collar neighborhood. For the following theorem, which is a refined version of the above Collar Neighborhood Theorem 10.15, gives an affirmative answer.

Theorem 10.15. (Collar Neighborhood Theorem) *Let M be a manifold. If N is a proper submanifold of M, then M admits a collar neighborhood $\partial M \times [0, 1]$ such that N is a product with respect to $\partial M \times [0, 1]$.*

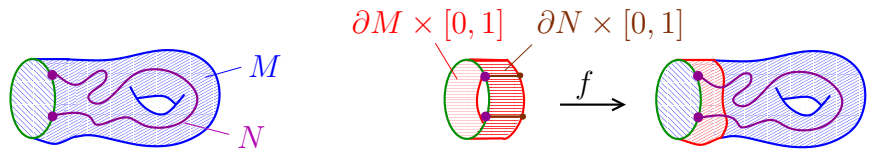


FIGURE 211. Illustration of the Collar Neighborhood Theorem 10.15.

Proof. This statement is precisely [Wall2, Proposition 1.5.6 (ii)]. □

At times one is given a collar neighborhood for the manifold M that one is quite happy with. The following proposition says that we can modify the given submanifold to still obtain the desired statement.

Proposition 10.16. $(*)$ *Let M be a manifold, let N be a compact proper submanifold of M and let $\partial M \times [0,1]$ be a collar neighborhood for M. There exists a diffeotopy F of M rel ∂M and an $\epsilon > 0$ such that $F_0 = \mathrm{id}$ and such that $F_1(N)$ is a product with respect to $\partial M \times [0,\epsilon]$.*

Sketch of proof $(*)$. If ∂M is compact, then we can take $\epsilon = 1$ and the statement follows immediately from the above Collar Neighborhood Theorem 10.15 and the uniqueness statement given by Proposition 10.11.

It ∂M is non-compact (and later on we will be interested in such cases), then we can not use the above "of the shelf argument" and we need to work a little harder. Note that it follows from the hypothesis that N is a proper submanifold that the map

$$p\colon N \cap (\partial M \times [0,1]) \xrightarrow{\text{inclusion}} \partial M \times [0,1] \xrightarrow{(x,t)\mapsto t} [0,1]$$

is a submersion in a neighborhood of ∂N. Since N is compact we see that there exists an $\eta > 0$ such that p is in fact a submersion on $p^{-1}([0,\eta])$. It follows from the Submersion Theorem 3.46 there exists an embedding $\varphi\colon \partial N \times [0,\eta] \to \partial M \times [0,\eta]$ with the following properties:

(1) $\varphi(\partial N \times [0,\eta]) = V \cap (\partial M \times [0,\eta])$,
(2) for each $t \in [0,\eta]$ we have $\varphi(\partial N \times \{t\}) \subset \partial M \times \{t\}$.

Similar to Lemma 3.12 there exists a smooth function $f\colon [0,\eta] \to [0,\eta]$ such that $f(t) = 0$ for $t \in [0, \frac{\eta}{4}]$ and such that $f(t) = t$ for $t \in [\frac{3\eta}{4}, \eta]$. One can now easily verify that

$$N' \;:=\; \{(\varphi(P, f(t)), t) \,|\, P \in \partial N \text{ and } t \in [0,\eta]\} \cup (N \setminus (N \cap (\partial M \times [0,\eta))))$$

is a submanifold that is product with respect to $\partial M \times [0, \frac{\eta}{4}]$. We refer to Figure 212 for an illustration of the argument.

It is not difficult to write down an explicit proper isotopy $E\colon N \times [0,1] \to M$ rel ∂N with $E_0 = \mathrm{id}$ and $E_1(N) = N'$. The existence of the desired isotopy F of M is now a consequence of the Isotopy Extension Theorem 10.21 (2) below. □

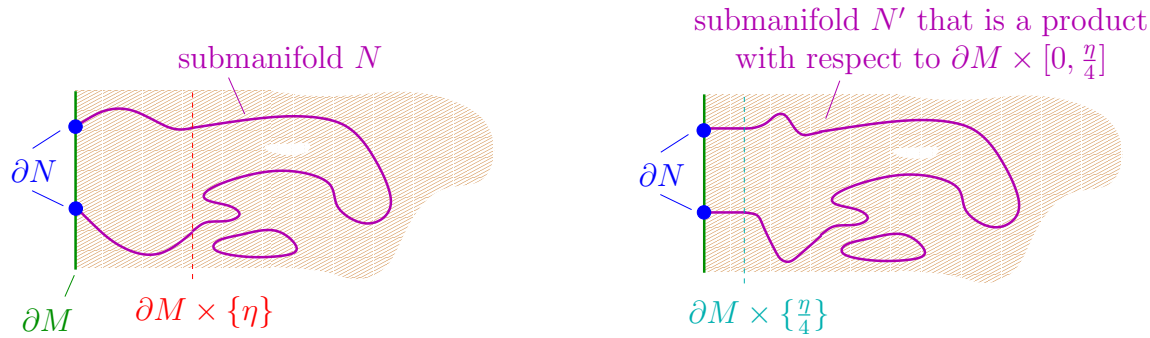


FIGURE 212.

The above results now allow us to prove the following proposition which will be used on several occasions.

Proposition 10.17. *Let M and N be two n-dimensional manifolds with boundary components $A \subset M$ and $B \subset N$ and let $f\colon A \to B$ be a diffeomorphism. Suppose we have chosen collar neighborhoods for ∂M and ∂N so that $M \cup_f N$ comes with a manifold structure. Furthermore let $X \subset M$ and $Y \subset N$ be two proper m-dimensional submanifolds such that $f(X \cap A) = Y \cap B$. Then there exists a proper submanifold $Z \subset M \cup_f N$ with the following three properties:*

(1) *$Z \cap A = X \cap A$ (and thus also $Z \cap B = Y \cap B$),*
(2) *outside of any previously chosen neighborhood of $A = B \subset M \cup_f N$ we have the equality $Z = X \cup_f Y$, and*
(3) *the submanifold Z is homeomorphic to $X \cup_f Y$.*

We refer to Figure 213 for an illustration.

Proof. The proposition can be deduced quite easily from Propositions 10.14 and 10.16. We leave it to the reader to fill in the details. □

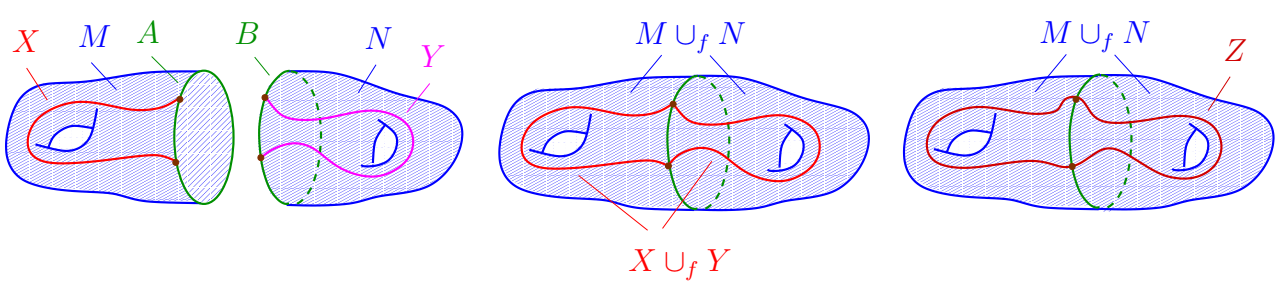


FIGURE 213. Illustration of Proposition 10.17.

10.5. **The push-in of a submanifold** $(*)$**.** In this short section we introduce the following quite natural definition.

Definition. Let W be a manifold with compact boundary and let X be a compact submanifold of ∂W. By the Collar Neighborhood Theorem 10.10 we can pick a collar neighborhood $\mu\colon \partial W \times [0,1] \to W$ and we can pick a collar neighborhood $\nu\colon \partial X \times [0,1] \to X$. Furthermore by Lemma 3.12 we can pick a smooth monotonously increasing function $\varphi\colon [0,1] \to [0,1]$ with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\varphi(t) = 1$ for $t \in [\frac{1}{2}, 1]$. We consider the map

$$\begin{array}{rcl} f\colon X & \to & W \\ P & \mapsto & \begin{cases} \mu(P, \varphi(t)), & \text{if } P = \nu(Q,t) \text{ for some } Q \in \partial X \text{ and some } t \in [0,1], \\ \mu(P,1), & \text{otherwise.} \end{cases} \end{array}$$

We call $f(X)$ the *push-in of* X.[228] We refer to Figure 214 for an illustration.

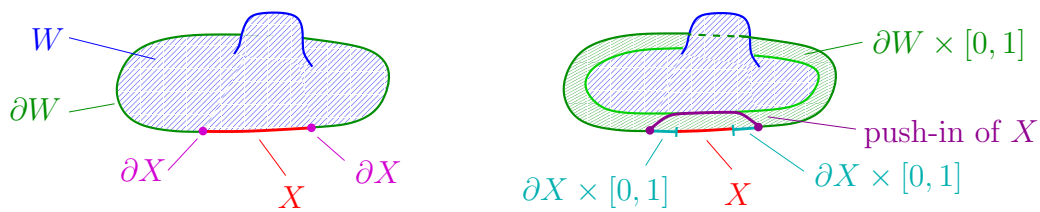


FIGURE 214.

The following lemma summarizes the key properties of push-ins.

Lemma 10.18. *Let W be a manifold with compact boundary and let X be a compact submanifold of ∂W. The push-in Y of X is a compact proper submanifold of W whose boundary equals $\partial Y = \partial X$.*

Proof. We continue with the notation from the definition of the push-in. First we claim that f is smooth. This can be seen as follows: The map f is evidently smooth on $\partial X \times [0,1)$ and $X \setminus (\partial X \times [0, \frac{3}{4}))$. Since these sets are open and since they cover all of X we see that f is indeed smooth.

Now it is straightforward to see that f is a proper embedding. It follows from Proposition 10.1 (2) that $f(X)$ is a proper submanifold. It is now straightforward to verify that $Y := f(X)$ has the desired properties. □

10.6. **The Tubular Neighborhood Theorem I.** The following definition resembles the definition of the collar of the boundary of a manifold.

Definition. Let M be an m-dimensional manifold and let N be a closed[229] n-dimensional submanifold.

[228] It follows from Proposition 10.11 that the push-in of X is unique in a suitable sense.

[229] Here by "closed" we mean that N is closed as a manifold, i.e. N is compact and $\partial N = \varnothing$.

(1) A *trivial tubular neighborhood for* N is an embedding $F\colon \overline{B}^{m-n} \times N \to M$ such that for any $P \in N$ we have $F(0, P) = P$.
(2) If M and N are oriented, then we say that a trivial tubular neighborhood for N is *orientation-preserving* if the embedding $F\colon \overline{B}^{m-n} \times N \to M$ is orientation-preserving where we equip $\overline{B}^{m-n} \times N$ with the product orientation introduced on page 205.

Example. We consider $M = S^1 \times S^1$ together with the submanifold $N = S^1 \times \{1\}$. The map

$$\begin{array}{ccccc} \overbrace{[-1,1]}^{=\overline{B}^1} & \times & \overbrace{S^1}^{=N} & \to & S^1 \times S^1 \\ & & (t, z) & \mapsto & (e^{it}, z) \end{array}$$

is a trivial tubular neighborhood for N, we refer to Figure 215 for an illustration.

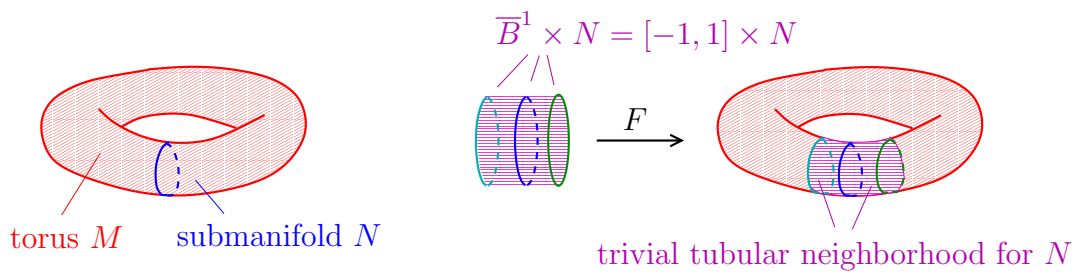


FIGURE 215.

The following Tubular Neighborhood Theorem guarantees the existence of (orientation-preserving) trivial tubular neighborhoods in some situations.

Theorem 10.19. (Tubular Neighborhood Theorem) *Let M be an m-dimensional manifold and let N be a closed n-dimensional submanifold. Suppose we are in one of the following four situations:*

(1) *N is zero-dimensional,*
(2) *N is one-dimensional and M is orientable,*
(3) *N is of codimension one and M and N are both orientable,*
(4) *we have $M = \mathbb{R}^m$ or $M = S^m$ and N is a closed orientable submanifold of codimension two,*

then N admits a trivial tubular neighborhood $F\colon \overline{B}^{m-n} \times N \to M$. Furthermore, if M and N are oriented, then we can find an orientation-preserving trivial tubular neighborhood $F\colon \overline{B}^{m-n} \times N \to M$.[230]

[230] If N is 0-dimensional, then $\overline{B}^{m-n} \times N$ is just a finite union of copies of $\overline{B}^m$ and we give it the obvious orientation.

Example. In Figure 216 we illustrate the first three of the above cases. Note that in the third example we consider the case that M is a 3-dimensional ball and N is a 2-dimensional disk. In particular in this case N actually has a boundary. This artistic freedom is due to the fact that it is rather challenging to draw a 3-dimensional manifold together with a tubular neighborhood of a closed 2-dimensional submanifold.

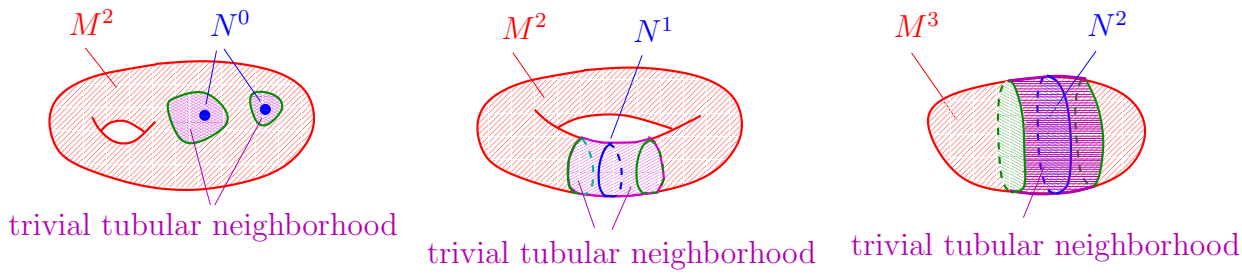


FIGURE 216.

Sketch of proof. In the following we provide a sketch of the argument if $M = \mathbb{R}^n$ and if N is a closed k-dimensional submanifold of $\mathbb{R}^n$. In the remainder of this discussion we use Proposition 3.32 to think of the tangent spaces of any submanifold of some $\mathbb{R}^m$ as a subvector space of $\mathbb{R}^m$. Given $P \in N$ we write

$$(T_P N)^\perp \;=\; \{v \in \mathbb{R}^n \,|\, \langle v, w\rangle = 0 \text{ for all } w \in T_P N\}$$

and we write $VN := \{(v, P) \,|\, P \in N \text{ and } v \in (T_P N)^\perp\}$. Without too much effort one can equip VN with an "obvious" structure of an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. We consider the map

$$\begin{array}{rcl} f\colon VN & \to & \mathbb{R}^n \\ (v, P) & \mapsto & P + v. \end{array}$$

One can easily verify that this map is smooth and that for each $P \in N$ the differential of f at $(0, P)$ is an isomorphism. It follows from the Inverse Mapping Theorem 3.33 that for each $P \in N$ there exists an open neighborhood U_P of P and an $\epsilon_P > 0$ such that f restricted to $\{(Q, w) \,|\, Q \in U_P, \text{ and } w \in (T_Q N)^\perp \text{ with } \|w\| < \epsilon_P\}$ is an embedding. But since N is compact this implies that there exists an $\epsilon > 0$ such that the restriction of f to

$$W \;:=\; \{(P, w) \,|\, P \in N, \text{ and } w \in (T_P N)^\perp \text{ with } \|w\| \le \epsilon\}$$

is an embedding. We refer to Figure 217 for an illustration.

We claim that W is our trivial tubular neighborhood. To show this we need to use our hypothesis on the (co-) dimensions and orientability of M and N. In fact it follows from [Hau, Proposition 9.2.3] and [MiS, Axiom 3] that in our situations there exists a diffeomorphism

$$\mathbb{R}^{n-k} \times N \;\to\; VN$$

such that for each P the map restricts to an isomorphism $\mathbb{R}^{n-k} \times \{P\} \to (T_P N)^\perp \times \{P\}$ of vector spaces. Using this diffeomorphism one can now fairly easily show that W is diffeomorphic to $\overline{B}^{n-k} \times N$.

The proof of the general case of the tubular neighborhood theorem follows along the same lines. But in general one needs to equip M with a "Riemannian structure" and instead of adding normal vectors (which no longer makes any sense) one needs to study the "exponential map" defined by geodesics. This approach constructs the analogue of W as above, see [Wall2, Theorem 2.3.3] for details. (See also [Kos, Section III.2], [Bre, Theorem II.11.14] and [Le2, Theorem 6.24] for related and similar results.) Once again appealing to [Hau, Proposition 9.2.3] and [MiS, Axiom 3] (for the last case we also need [Er, p. 100] and [Mi12, p. 44], see also [Bre, Corollary VII.14.6]) one sees that W is diffeomorphic to $\overline{B}^{n-k} \times N$.[231] □

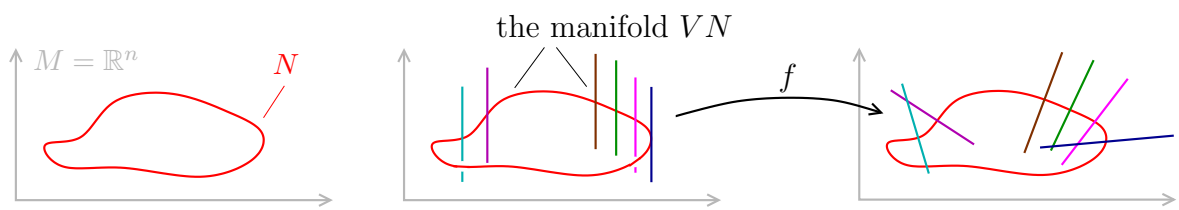


FIGURE 217. Illustration for the proof of the Tubular Neighborhood Theorem 10.19.

Remark. As many readers might already know, there exists a generalization of the Tubular Neighborhood Theorem 10.19 that deals with any compact proper submanifold N. The precise definition of a tubular neighborhood is in general somewhat lengthy and technical (and frankly often done imprecisely in the literature). Thus we postpone the formulation to a later point when we really need the more general case.

The following proposition tells us to what degree trivial tubular neighborhoods are unique.

Proposition 10.20. (Uniqueness of trivial tubular neighborhoods) *Let M be an m-dimensional manifold and let N be a closed n-dimensional submanifold. Furthermore let $F, G\colon \overline{B}^{m-n} \times N \to M$ be two trivial tubular neighborhoods for N. Then there exists a diffeotopy*

$$H\colon M \times [0,1] \;\to\; M$$

rel N[232] *from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1$ such that the following two statements hold:*

[231] The alert reader will correctly suspect that right now we are swiping lots of technical stuff under the rug.

[232] Recall that this means that $H(P,t) = P$ for all $P \in N$ and $t \in [0,1]$.

(1) *The diffeomorphism* $\Phi = H_1$ *restricts to diffeomorphisms*

$$\begin{aligned} \Phi\colon F(\overline{B}^{m-n} \times N) &\to G(\overline{B}^{m-n} \times N) \\ \Phi\colon M \setminus F(B^{m-n} \times N) &\to M \setminus G(B^{m-n} \times N). \end{aligned}$$

(2) *For all* $P \in N$ *the diffeomorphism* $\Phi = H_1$ *restricts to a boundary preserving diffeomorphism*

$$\Phi\colon F(\overline{B}^{m-n} \times \{P\}) \to G(\overline{B}^{m-n} \times \{P\})$$

and the self-diffeomorphism $G^{-1} \circ \Phi \circ F$ *of* $\overline{B}^{m-n} \times \{P\} = \overline{B}^{m-n}$ *is given by multiplication by a matrix* $A \in \mathrm{O}(m-n)$.

If M *and* N *are oriented and if* F *and* G *are orientation-preserving, then all the above diffeomorphisms are orientation-preserving, in particular the matrix* A *mentioned in* (2) *lies in* $\mathrm{SO}(m-n)$.

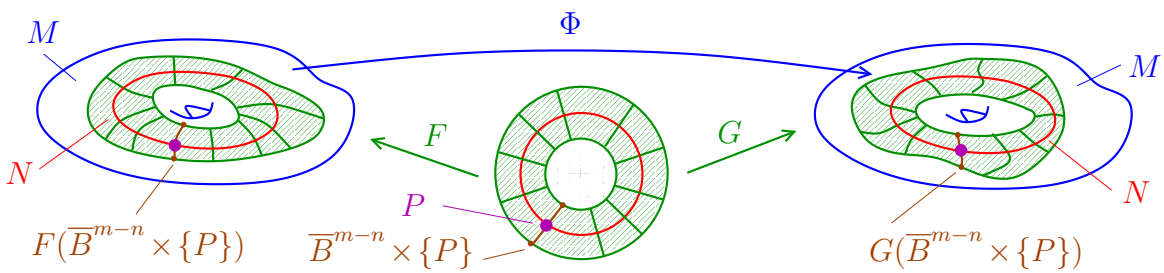


FIGURE 218. Illustration of Proposition 10.20.

Proof. A proof for the proposition is given in [Wall2, Theorem 2.5.5 and Theorem 2.5.8]. We refer to [BJ, Theorem 12.13], [Kos, Corollary III.3.2] and [Lan3, Theorem IV.6.2] for alternative approaches, which, with a little bit of mental gymnastics, can also be used to prove the proposition.

The last statement that, under the given hypotheses, the diffeomorphisms are orientation-preserving are a consequence of Lemma 10.5 and the fact that id_M is orientation-preserving. □

Convention. In the following, by a slight abuse of language, we often refer to $F(\overline{B}^{m-n} \times N)$ as a *trivial tubular neighborhood of* N. It follows immediately from Proposition 10.2 that a trivial tubular neighborhood of N is in fact a neighborhood of N in the sense of the definition on page 48.

Before we can state the next theorem we need to introduce one more innocuous definition.

Definition. Let M be a manifold. We say that a diffeotopy $F\colon M \times [0,1] \to M$ has *compact support* if there exists a compact subset $K \subset M$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Examples.

(1) Evidently every diffeotopy of a compact manifold has compact support.
(2) The diffeotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ given by $F(x,t) = x + t$ does *not* have compact support.
(3) In Figure 219 we sketch a non-trivial diffeotopy of $\mathbb{R}$ with compact support.

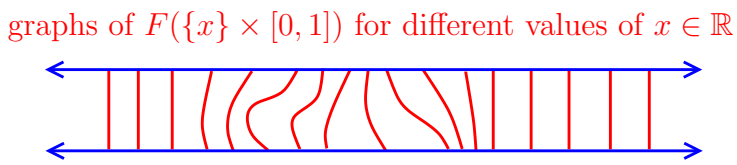


FIGURE 219.

The following theorem is one of the key technical tools for dealing with manifolds.

Theorem 10.21. (Isotopy Extension Theorem) *Let M be a manifold and let N be a compact manifold.*

(1) *Let $f\colon N \to M \backslash \partial M$ be an embedding. If $F\colon N \times [0,1] \to M \backslash \partial M$ is a smooth isotopy from f to some embedding f', then there exists a diffeotopy $G\colon M \times [0,1] \to M$ rel some neighborhood of ∂M with $G_0 = \mathrm{id}$ such that*
$$G(f(x),t) \;=\; F(x,t) \quad \text{for all } x \in N \text{ and } t \in [0,1].$$
(2) $(*)$ *Let $f\colon N \to M$ be a proper embedding. If $F\colon N \times [0,1] \to M$ is a proper smooth isotopy from f to some proper embedding f', then there exists a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ such that*
$$G(f(x),t) \;=\; F(x,t) \quad \text{for all } x \in N \text{ and } t \in [0,1].$$
Furthermore, if F is an isotopy rel ∂N, then we can choose G to be a diffeotopy rel ∂M.

In both cases we can furthermore arrange that the diffeotopy G has compact support.

Examples.

(1) Let M be a manifold. The Isotopy Extension Theorem 10.21 often gets applied as follows: N is a submanifold in $M \setminus \partial M$ and we start out with the inclusion map $i\colon N \to M$. The Isotopy Extension Theorem 10.21 thus loosely says, that if we deform N in M, then we can do so by "dragging M" along.
(2) In fact often we apply (1) to the special case that $N = \{P\}$ consists of a single point $P \in M \setminus \partial M$. In this case a smooth isotopy of P is just a smooth path $F\colon [0,1] \to M \setminus \partial M$ with $F(0) = P$. The Isotopy Extension Theorem 10.21 says that there exists a diffeotopy $G\colon M \times [0,1] \to M$ rel a neighborhood of ∂M from the identity to a diffeomorphism G_1 with $G(P,t) = F(t)$ for all $t \in [0,1]$. We refer to Figure 220 for an illustration.

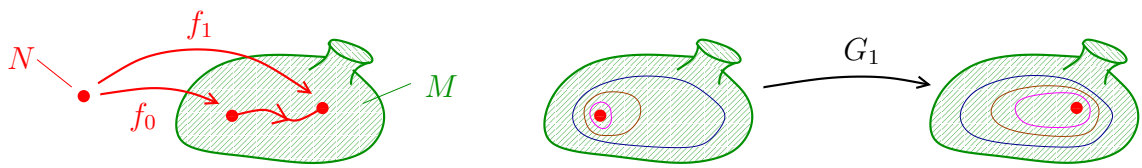


FIGURE 220. Illustration of the Isotopy Extension Theorem 10.21.

Proof of the Isotopy Extension Theorem 10.21. The theorem follows almost immediately from the statements and the proofs of Theorem 2.4.2 and Theorem 2.4.6 of [Wall2]. Note that the results of [Wall2] do not explicitly say that we can assume that $G_0 = \mathrm{id}$, but a quick look at the proof of [Wall2, Theorem 2.4.2] shows that this is the case. Similarly in (2) the fact that in the given situation we can choose G to be a diffeotopy rel ∂M is not mentioned in [Wall2], but it does follow from the proof. We also refer to [Kos, Theorem II.5.2], [Hirs2, Theorem 8.1.3] and [Pal3, Chapter 6] for related results. □

The following corollary to the Isotopy Extension Theorem 10.21 is quite useful.

Corollary 10.22. *Let M be a manifold without boundary and let N be a compact manifold. Let $f\colon N \to M$ and $g\colon N \to M$ be two embeddings. If there exists a smooth isotopy $F\colon N \times [0,1] \to M$ from f to g, then there exists a diffeomorphism*[233] *from $M \setminus f(N)$ to $M \setminus g(N)$.*

Proof. By the Isotopy Extension Theorem 10.21 there exists a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ such that $G_1(f(N)) = g(N)$. But this means that $G_1\colon M \to M$ is a diffeomorphism with $G_1(f(N)) = g(N)$. It follows that G_1 restricts to a diffeomorphism $M \setminus f(N) \to M \setminus g(N)$. □

Example. The statement of Corollary 10.22 and the statement of the Isotopy Extension Theorem 10.21 are quite subtle. In Figure 221 we illustrate two embeddings $f, g\colon S^1 \to \mathbb{R}^3$ and we illustrate a map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$, $F_1 = g$ and with the property that every single $F_t\colon S^1 \to \mathbb{R}^3$ is an embedding. But as should be pretty clear (and we will give a rigorous proof later on) the complements $\mathbb{R}^3 \setminus f(S^1)$ and $\mathbb{R}^3 \setminus g(S^1)$ are *not* diffeomorphic. This shows that in the formulations of Corollary 10.22 and the Isotopy Extension Theorem 10.21 it is essential that the map F is smooth on all of $N \times [0,1]$.

We also have the following proposition which is at times quite useful.

Proposition 10.23. *Let M be a connected n-dimensional manifold.*

(1) *Given any two points P and Q in $M \setminus \partial M$ there exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P) = Q$.*

[233] Since N is compact we know by Lemmas 1.37 and 1.18 that $M \setminus f(N)$ and $M \setminus g(N)$ are open subsets of M, in particular they are manifolds in their own right.

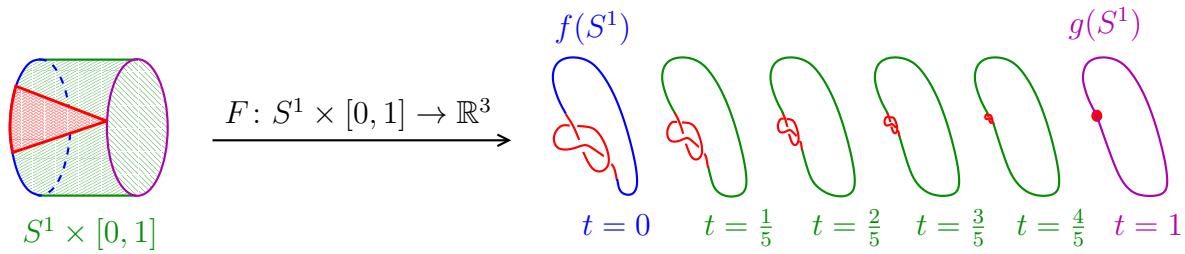


FIGURE 221.

(2) *If $n \geq 2$, then given distinct points $P_1, \ldots, P_m$ and distinct points $Q_1, \ldots, Q_m$ in $M \setminus \partial M$ there exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P_i) = Q_i$ for $i = 1, \ldots, m$.*

In both cases we can furthermore arrange that the diffeotopy has compact support. (Note that if M is oriented, then it follows from Lemma 10.5 that in both cases Φ is in fact orientation-preserving.)

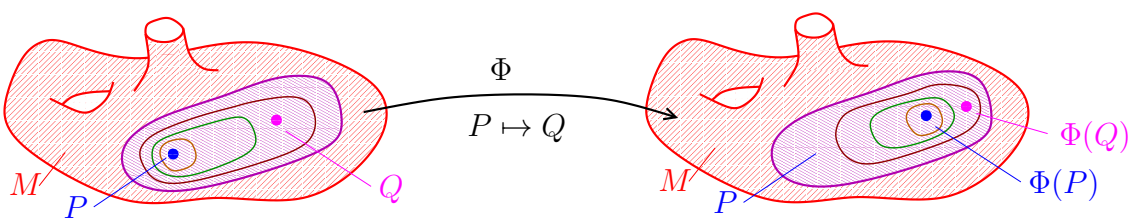


FIGURE 222. Illustration of Proposition 10.23.

In the proof of Proposition 10.23 we will make use of the following lemma which will be useful in its own right.

Lemma 10.24. *Let M be a manifold.*

(1) *Any path $\gamma\colon [0,1] \to M \setminus \partial M$ is path-homotopic in $M \setminus \partial M$ to a smooth path. (Here recall "path-homotopic" means that we keep the endpoints fixed).*
(2) *If M is connected, then any two points P, Q in $M \setminus \partial M$ can be connected by a smooth path in $M \setminus \partial M$.*

Proof of Lemma 10.24.

(1) The first statement follows from Exercise 10.11 applied to $W := M \setminus \partial M$.
(2) Now suppose that M is connected. From Lemma 3.8 and Exercise 10.5 we obtain that $M \setminus \partial M$ is path-connected. Thus the second statement follows from the first statement. □

Now we can move on to the actual proof of Proposition 10.23.

Proof of Proposition 10.23. Let M be a connected n-dimensional manifold.

(1) Let P, Q be two points in $M \setminus \partial M$. By Lemma 10.24 there exists a smooth path $F\colon [0,1] \to M \setminus \partial M$ from P to Q. The desired statement follows now from the Isotopy Extension Theorem 10.21, using the above example.

(2) $(*)$We prove this statement by induction on m. We dealt with the case $m = 1$ in (1). Now suppose that we know the statement for some $m \in \mathbb{N}_{\geq 2}$. Thus let $P_1, \dots, P_{m+1}$ and $Q_1, \dots, Q_{m+1}$ be two sets of distinct points in $M \setminus \partial M$. By induction we can without loss of generality already assume that $P_i = Q_i$ for $i = 1, \dots, m$.[234] Now pick disjoint closed n-balls around $P_1, \dots, P_m$ that are disjoint from $P := P_{m+1}$ and $Q := Q_{m+1}$. We denote by W the manifold obtained from M by removing the corresponding open balls.[235] Since $n \geq 2$ we obtain from Exercise 10.7 (2) that W is still connected. Thus we can apply (1) to P and Q in W to obtain a diffeotopy of W with all the desired properties. Since the diffeotopy is the identity near the boundary of W we can extend it via the identity to a diffeotopy of all of M. □

On one occasion later on we will need the following proposition.

Proposition 10.25. $(*)$ *Let M be a connected* non-orientable *n-dimensional manifold, let $P_1, \dots, P_m$ be points in $M \setminus \partial M$ and let $I \subset \{1, \dots, m\}$ be a subset. There exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi\colon M \to M$ with the following properties:*

(1) *for all $i \in \{1, \dots, m\}$ we have $\Phi(P_i) = P_i$,*
(2) *the diffeotopy fixes all P_i with $i \notin I$,*
(3) *the map $D\Phi_{P_i}\colon T_{P_i}M \to T_{P_i}M$ is orientation-preserving if and only if $i \in I$.*

Furthermore can assume that the diffeotopy has compact support.

Proof. We leave it to the reader to use the paths from Lemma 8.5 to modify the proof of Proposition 10.23 to obtain a proof of the proposition. □

We conclude with the following theorem which follows easily from [Wall2, Proposition 2.4.7].

Theorem 10.26. (Extension Theorem)$(*)$ *Let $f\colon N \to M$ be an embedding of a manifold with boundary into a manifold M such that $f(N) \subset M \setminus \partial M$. Then there exists an embedding*[236]

$$f\colon \big(N \sqcup (\partial N \times [0,1])\big)/_{\partial N = \partial N \times 0} \;\to\; M$$

which restricts to the original f on N and such that the image lies in $M \setminus \partial M$.

[234]The super vigilant reader will have noticed that implicitly we are using Lemma 10.6.

[235]Here we implicitly use Lemma 3.22.

[236]Here the topological space $\big(N \sqcup (\partial N \times [0,1])\big)/_{\partial N = \partial N \times 0}$ is understood to be the topological space $(N \sqcup (\partial N \times [0,1]))/\sim$ where $P \in \partial N$ is equivalent to $P \times \{0\}$ on $\partial N \times [0,1]$.

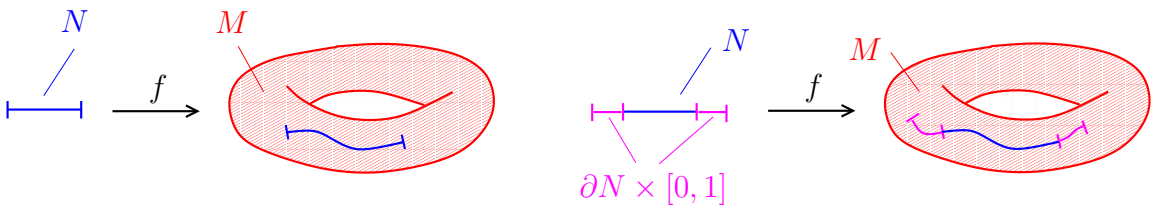


FIGURE 223. Illustration of the Extension Theorem.

10.7. **The connected sum operation.** In this section we introduce the connected sum of two manifolds. As we will see this construction combines features from the last two sections.

Definition. Let $n \in \mathbb{N}$. Let M and M' be two connected non-empty n-dimensional manifolds.

(1) Suppose that at least one of M or M' is non-orientable. We pick two embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ and we define the *connected sum of M and M'* as

$$M\#M' \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big) \,/\, \varphi(P) = \varphi'(P) \text{ for all } P \in S^{n-1}.$$

(2) If both M and M' are orientable, then we pick orientations for M and M'. Let $\varphi\colon \overline{B}^n \to M\setminus\partial M$ and $\varphi'\colon \overline{B}^n \to M'\setminus\partial M'$ be embeddings such that φ is orientation-preserving and such that φ' is orientation-reversing. As before we define the *connected sum of the oriented manifolds M and M'* as

$$M\#M' \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big) \,/\, \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

For completeness we define the connected sum of an empty manifold with another connected manifold M to be M, i.e. $\varnothing\#M := M$.[237]

Example.

(1) In Figure 224 we sketch that the connected sum of two tori is diffeomorphic to the surface of genus two.[238]

(2) The connected sum of an n-dimensional manifold M with S^n is diffeomorphic to M. Indeed, this can be seen as follows: in the construction of the connected sum $M\#S^n$ we consider the ball $S^n_{\geq 0}$ in S^n. Since $S^n \setminus S^n_{>0} = S^n_{\leq 0}$ is again a closed ball it follows easily that $M\#S^n$ is diffeomorphic to M.

[237]The reader will surely benignly overlook the fact that in some of the proofs down the line we do not deal with this special separately.

[238]The picture should be pretty convincing, but it would be fairly painful to try to prove the statement rigorously using the official definition of the surface of genus 2 as the quotient of an octagon. This is a typical example where one is quite content to work with a convincing picture instead of providing a rigorous, but much less instructive argument.

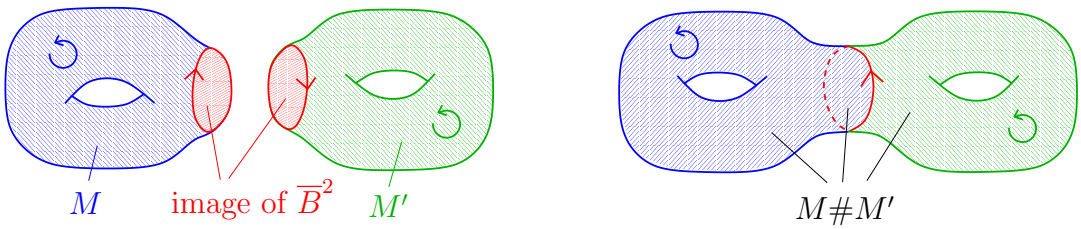


FIGURE 224.

Proposition 10.27. *Let $n \in \mathbb{N}$. Let M and M' be two connected n-dimensional manifolds. (If both M and M' are orientable, then we demand that M and M' are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be embeddings. If M and M' are orientable, then we demand that φ is orientation-preserving and that φ' is orientation-reversing. The following three statements hold:*

1. *The resulting connected sum $M\#M'$ is canonically an n-dimensional manifold.*
2. *The diffeomorphism type of $M\#M'$ does not depend on the choice of φ and φ'.*
3. *If M and M' are oriented, then the manifold $M\#M'$ admits a unique orientation that coincides with the orientations of $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$.*

In the proof of Proposition 10.27 we will need the following theorem which was originally proved by Richard Palais [Pal] in 1960.

Theorem 10.28. *Let M be an n-dimensional manifold. (If M is orientable, then we pick an orientation for M.) Let $\varphi_1, \dots, \varphi_m\colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_m\colon \overline{B}^n \to M \setminus \partial M$ be two sets of m embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $m = 1$, then there exists a diffeotopy*

$$F\colon M \times [0,1] \;\to\; M$$

rel some neighborhood of ∂M from the identity $F_0 = \mathrm{id}_M$ to a diffeomorphism $F_1\colon M \to M$ such that for each $i \in \{1, \dots, m\}$ the following diagram commutes:

$$\begin{tikzcd} & \overline{B}^n \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_i"] & \\ M \arrow[rr, "F_1", "\cong"'] & & M. \end{tikzcd}$$

Furthermore we can assume that F has compact support. (Note that if M is orientable, then the diffeomorphism F_1 is in fact orientation-preserving by Lemma 10.5).

Proof of Theorem 10.28. Before we start with the actual proof we want to point out that on several occasions we implicitly use Lemma 10.5 which allows us to "compose" smooth isotopies and diffeotopies. We also introduce the notation that given $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

For $i = 1, \dots, m$ we write $P_i := \varphi_i(0)$ and $Q_i := \psi_i(0)$. If $n \geq 2$ or if $m = 1$, then we can apply Proposition 10.23 so that we can arrange that $P_i = Q_i$ for $i = 1, \dots, m$.

For a short period we consider the cases that M is orientable and non-orientable separately.

(1) Suppose that M is orientable. Recall that in this case we assume that M is in fact oriented. Furthermore we assume that all the φ_i, ψ_i are either orientation-preserving or orientation-reversing. In particular for each $i \in \{1, \dots, m\}$ the map $D(\psi_i^{-1} \circ \varphi_i) \colon T_0\overline{B}^n \to T_0\overline{B}^n$ is orientation-preserving.

(2) Suppose that M is non-orientable. We denote by $I \subset \{1, \dots, m\}$ the set of i's for which $D(\psi_i^{-1} \circ \varphi_i) \colon T_0\overline{B}^n \to T_0\overline{B}^n$ is orientation-reversing. We apply Proposition 10.25 to I. Using the resulting diffeotopy we can henceforth assume that each $D(\psi_i^{-1} \circ \varphi_i) \colon T_0\overline{B}^n \to T_0\overline{B}^n$ is in fact orientation-preserving.

We continue with the common discussion.

We observe that $\varphi_1, \dots, \varphi_m$ and $\psi_1, \dots, \psi_m$ are trivial tubular neighborhoods for the closed 0-dimensional submanifold $N = \{P_1, \dots, P_m\}$. Thus it follows from Proposition 10.20 that there exist matrices $A_1, \dots, A_m \in \mathrm{O}(n)$ and a smooth isotopy rel $N = \{P_1, \dots, P_m\}$ from $\varphi_1 \cup \dots \cup \varphi_m$ to $\psi_1 \circ \rho(A_1) \sqcup \dots \sqcup \psi_m \circ \rho(A_m)$. Since each $D(\psi_i^{-1} \circ \varphi_i) \colon T_0\overline{B}^n \to T_0\overline{B}^n$ is orientation-preserving we see that each A_i lies in fact in $\mathrm{SO}(n)$.

Therefore it follows from Lemma 10.4 that there exists a smooth isotopy from the map $\psi_1 \circ \rho(A_1) \sqcup \dots \sqcup \psi_m \circ \rho(A_m)$ to the map $\psi_1 \sqcup \dots \sqcup \psi_m$. Thus we have found a smooth isotopy from $\varphi_1 \sqcup \dots \sqcup \varphi_m$ to $\psi_1 \sqcup \dots \sqcup \psi_m$. The existence of the desired diffeotopy follows from the Isotopy Extension Theorem 10.21. □

Now we can provide the proof of Proposition 10.27.

Proof of Proposition 10.27.

(1) We write $W := M \setminus \varphi(B^n)$ and $W' := M \setminus \varphi'(B^n)$. By Proposition 10.2 (2) these are both n-dimensional manifolds with boundary components $\varphi(S^{n-1})$ and $\varphi'(S^{n-1})$. Note that by definition we have

$$M \# M' \;=\; (W \sqcup W')/a \sim \varphi'(\varphi^{-1}(a)) \quad \text{where } a \in \varphi(S^{n-1}).$$

It follows from Proposition 10.12 that $M \# M'$ admits a canonical structure of an n-dimensional manifold.

(3) Now we suppose that M and M' are oriented. Recall that in this case we suppose that the map $\varphi \colon \overline{B}^n \to M \setminus \partial M$ is orientation-preserving and furthermore that the map $\varphi' \colon \overline{B}^n \to M' \setminus \partial M'$ is orientation-reversing. We continue with the notation from (1). It follows from Lemma 3.37 that W and W' inherit an orientation from M and M'. The gluing map $\varphi' \circ \varphi^{-1} \colon \varphi(S^{n-1}) \to \varphi'(S^{n-1})$ is orientation-reversing. Thus the desired statement follows from Proposition 10.12.

(2) We need to show that the definition of $M \# M'$ is independent of the choices of the embeddings φ and φ'. Given embeddings $\varphi \colon \overline{B}^n \to M \setminus \partial M$ and $\varphi' \colon \overline{B}^n \to M' \setminus \partial M'$

we now introduce the following cumbersome notation:

$$(M\#M')(\varphi,\varphi') \;:=\; \big(M\setminus\varphi\big(B^n\big)\big)\sqcup\big(M'\setminus\varphi'\big(B^n\big)\big)\,/\,\varphi(P)\sim\varphi'(P) \text{ for all } P\in S^{n-1}.$$

Below we show independence of φ and φ'.

(a) First we consider the case that both M and M' are orientable. Therefore let $\varphi,\psi\colon \overline{B}^n\to M\setminus\partial M$ be two orientation-preserving embeddings and furthermore let $\varphi',\psi'\colon \overline{B}^n\to M'\setminus\partial M'$ be two orientation-reversing embeddings. By Theorem 10.28 there exist in particular diffeomorphisms $F\colon M\to M$ with $F\circ\varphi=\psi$ and $F'\colon M'\to M'$ with $F'\circ\varphi'=\psi'$. One can now easily verify that the map

$$\begin{array}{rcl}(M\#M')(\varphi,\varphi') & \to & (M\#M')(\psi,\psi')\\ x & \mapsto & \begin{cases} F(x) & \text{if } x\in M\setminus\varphi(B^n),\\ F'(x) & \text{if } x\in M'\setminus\varphi'(B^n)\end{cases}\end{array}$$

is a diffeomorphism.

(b) Next we consider the case that M and M' are both non-orientable. Then basically the same argument as in (a) applies. The only difference is that we now apply Theorem 10.28 to the case of non-orientable manifolds, so we no longer have to make any assumptions on whether or not φ,ψ and φ',ψ' are orientation-preserving.

(c) Now consider the case that M is orientable and that M' is non-orientable. Let $\varphi,\psi\colon \overline{B}^n\to M\setminus\partial M$ be two embeddings and let $\varphi',\psi'\colon \overline{B}^n\to M'\setminus\partial M'$ be two embeddings. If φ and ψ are both orientation-preserving or if both are orientation-reversing, then the same argument as in (a) and (b) applies. Now suppose that this is not the case. We denote by $\tau\colon \overline{B}^n\to\overline{B}^n$ any orientation-reversing diffeomorphism (e.g. τ could be the reflection in one of the coordinate axes.) Then as before we see that $(M\#M')(\varphi,\varphi')$ is diffeomorphic to $(M\#M')(\psi\circ\tau,\psi'\circ\tau)$. But $(M\#M')(\psi\circ\tau,\psi'\circ\tau)$ is, basically by definition, the same as $(M\#M')(\psi,\psi')$.

(d) The case that M is non-orientable and that M' is orientable is of course dealt with the same way as in (d). □

Remark. The fact that the diffeomorphism type of the connected sum is well-defined is rather subtle

(1) If M and M' are two *orientable* manifolds, then the diffeomorphism type of the connected sum is defined only if both manifolds are in fact *oriented*, i.e. they are equipped with an orientation. In general the definition does depend on the choice of orientation. More precisely, there exist oriented manifolds M and M' such that $M\#M'$ is not diffeomorphic (in fact not even homeomorphic) to $M\#(-M')$, where $-M'$ denotes as usual the manifold M' with the opposite orientation. We will see such examples later once we have introduced the cup product on cohomology groups.

(2) Often the connected sum operation gets abbreviated as follows: *Given two manifolds we remove an open ball from each manifold and then glue M and M' together,*

along the boundary spheres that we had just created, in such a way that the orientations of the boundaries are opposite. This description is rather dangerous since the diffeomorphism type of the connected sum is only well-defined if we glue the boundary components via the diffeomorphism given by the embeddings of the closed balls.

The name "connected sum" suggests that the connected sum $M\#M'$ is connected. From the pictures this sounds reasonable, but funnily enough, it is not necessarily correct: One can easily see that the connected sum $\mathbb{R}\#\mathbb{R}$ of $\mathbb{R}$ with itself is the disjoint union of two copies of $\mathbb{R}$, thus disconnected! Fortunately, for all intents and purposes that is the only counterexample. More precisely, we have the following lemma.

Lemma 10.29. *If $n \geq 2$, then the connected sum of two connected n-dimensional manifolds is connected.*

Proof $(*)$. The key to proving the lemma is the following claim.

Claim. Let M be an n-dimensional manifold and let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be an embedding. If M is connected and if $n \geq 2$, then $M \setminus \varphi(B^n)$ is also connected.

We leave the proof of this claim to the reader, see Exercise 10.7 (b). The reader will notice that it takes an effort to provide a completely rigorous proof.

Once we have proved this claim, the lemma is an immediate consequence of the claim, Lemma 1.93 and the fact that for $n \geq 2$ the sphere S^{n-1} is connected. □

We conclude this section with the following lemma.

Lemma 10.30. *Let $n \geq 1$.*

(1) *Given two connected non-empty n-dimensional manifolds X and Y the connected sums $X\#Y$ and $Y\#X$ are diffeomorphic.*
(2) *Given three connected non-empty n-dimensional manifolds X, Y and Z the connected sums $(X\#Y)\#Z$ and $X\#(Y\#Z)$ are diffeomorphic.*

Remark. Let $n \geq 1$. It follows from Lemma 10.30 that given connected n-dimensional manifolds $M_1, \ldots, M_k$ we can unambiguously define the diffeomorphism type of the connected sum $M_1\#\ldots\#M_k$ without having to worry in what order we perform the connected sum of two manifolds.[239]

Proof. The first statement is just an immediate consequence of the definitions, whereas the second statement is a fairly straightforward consequence of Theorem 10.28. We leave it to the reader to fill in the details. □

[239]This argument is familiar from algebra: if a multiplication is associative, then we can unambiguously define the product of finitely many elements using any choice of parenthesis. This sounds reasonable and it is correct, but if one thinks about it, one realizes that this requires a proof, which is for example provided in [DF, p. 19].

10.8. **Knots and their complements.** Before we introduce knots we recall several different ways how we can describe the 3-sphere S^3. More precisely, by definition and Lemma 1.41 we have

$$S^3 = \big\{(z,w) \in \mathbb{C}^2 \big| |w|^2 + |z|^2 = 1\big\} = \big\{(w,x,y,z) \in \mathbb{R}^4 \big| w^2 + x^2 + y^2 + z^2 = 1\big\} \underset{\substack{\uparrow \\ \text{via the stereographic projection from Lemma 1.41}}}{\cong} \mathbb{R}^3 \cup \{\infty\}.$$

We will go back and forth between these models without mentioning these maps. In particular we will always view $\mathbb{R}^3$ as a subset of $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Definition. A *knot* is a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1. We say that the knot is *oriented* if it is oriented as a 1-dimensional manifold.

Remark.

(1) It follows from Proposition 10.1 that a subset of S^3 is a knot if and only if it is the image of an embedding $S^1 \to S^3$.[240]

(2) The definition of a knot is supposed to model the "physical objects" that we have in mind and that are sketched in Figure 225. It is therefore perhaps at first not clear why we consider knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. The reason is that topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$.

In Figure 225 we give three examples of knots. More precisely, we show 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} = S^3$.[241]

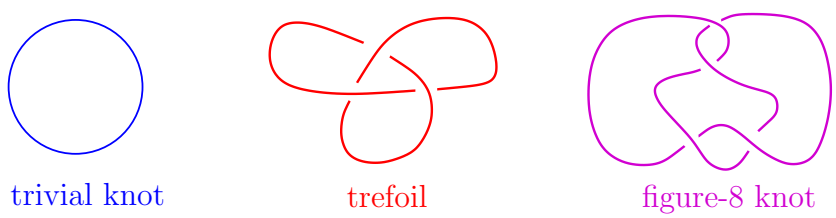


FIGURE 225. The trivial knot, the trefoil knot and the figure-8 knot.

[240]To be more precise, the "if"-direction is a consequence of Proposition 10.1. On the other hand, if $K \subset S^3$ is a knot, then by definition there exists a diffeomorphism $\varphi\colon S^1 \to K \subset S^3$ and it follows easily from the definitions that the map φ viewed as a map from $S^1 \to S^3$ is an embedding whose image is precisely K.

[241]In principle it is possible to give a precise description of these three knots. For example the trivial knot is defined as $\{(x,y,0) \,|\, x^2+y^2=1\} \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Later on we will give a precise definition of the trefoil. It is also clear that one can give a precise description of the figure-8 knot, but this description would be painful to write down and it would not add to our understanding. We therefore stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise description in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description.

Now we want to say that two knots are "the same" if one can be "deformed" into the other. Fortunately we already have the language to make this picture precise: The notion of a smooth isotopy between submanifolds that we introduced on page 337 makes this notion precise.

For example the three knots shown in Figure 226 are smoothly isotopic. Usually we do not distinguish two knots if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial knotted*.

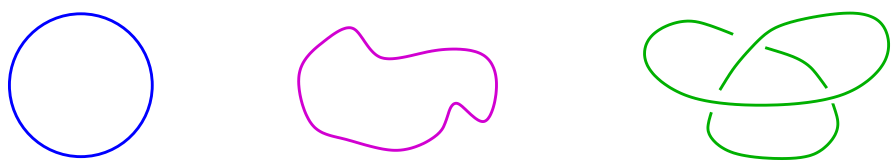


FIGURE 226. The three knots are smoothly isotopic.

Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is smoothly isotopic to the trivial knot. This arouses the suspicion, that the trefoil is in fact not smoothly isotopic to the trivial knot. This raises the following question.

Question 10.31. *How can we show that the trefoil is not smoothly isotopic to the trivial knot?*

More generally we have the following question.

Question 10.32. *Given a knot K*[242]*, how can we determine whether or not K is smoothly isotopic to the trivial knot?*

The following proposition allows us to turn these two questions into the question, whether two topological spaces are homeomorphic.

Proposition 10.33. *Let K and J be two knots. If they are smoothly isotopic, then the following two statements hold:*

(1) *There exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(K) = J$. If K and J are oriented, then Φ restricts to an orientation-preserving diffeomorphism $K \to J$.*
(2) *The knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic, in particular homeomorphic.*

Remark. In fact the converse to Proposition 10.33 (2) also holds. If K and J are two knots such that there exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are smoothly isotopic. One might think that

[242]The phrase "given a knot K" is consciously slightly vague.

this should be quite straightforward to show, but it turns out to be surprisingly difficult. A proof was given only in 1989 by Cameron Gordon and John Luecke [GL]. The proof builds on the work of William Thurston [Thu] for which he got the fields medal in 1982.

Proof. Let

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

be a smooth isotopy from a knot K to a knot J. By the Isotopy Extension Theorem 10.21 we can extend the smooth isotopy F to a diffeotopy of S^3. This means in particular that there exists a diffeotopy

$$\begin{array}{rcl} G\colon S^3 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & G(z,t) \end{array}$$

from the identity to a diffeomorphism $\Phi = G_1\colon S^3 \to S^3$ with $\Phi|_K = F_1$, in particular with $\Phi(K) = J$. By Lemma 10.5 the diffeomorphism Φ is orientation-preserving. The map Φ restricts to an orientation-preserving diffeomorphism $\Phi\colon S^3 \setminus K \to S^3 \setminus J$. □

We conclude this discussion of knots with a definition.

Definition. Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem 10.19 there exists an orientation-preserving embedding

$$F\colon \overline{B}^2 \times K \;\to\; S^3$$

such that $F(0,P) = P$ for all $P \in K$. For any $P \in K$ we refer to the oriented submanifold[243] $F(S^1 \times \{P\})$ as a *meridian* of K.

It follows easily from Proposition 10.20 that any two meridians of a given knot K are smoothly isotopic in the knot complement $S^3 \setminus K$. More casually speaking, a meridian is a closed curve which circles once around a knot where the orientation is given by the right-hand-rule, i.e. if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian. This rule is illustrated in Figure 228.

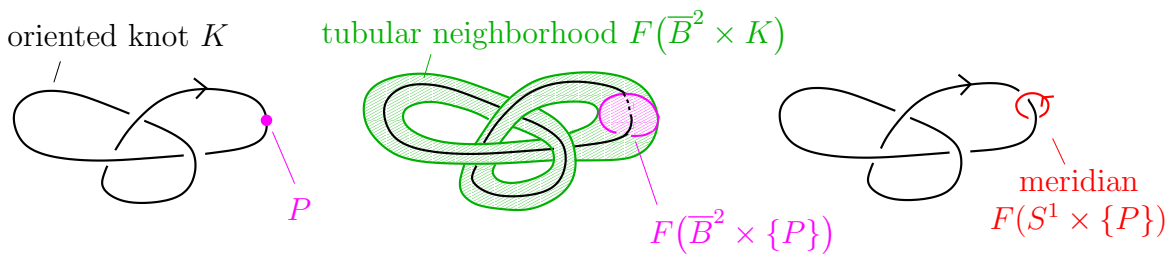


FIGURE 227.

[243]It follows easily from Proposition 10.1 that $F(S^1 \times \{P\})$ is indeed a submanifold of $S^3 \setminus K$.

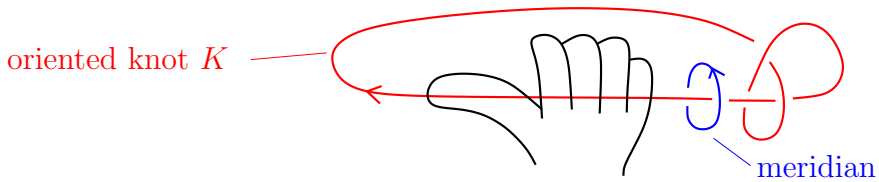


FIGURE 228.

Exercises for Chapter 10.

Exercise 10.1. Let M and N be manifolds and let $f\colon M \to N$ be an immersion. We denote by $\Delta = \{(x,x) \in M \times M \mid x \in M\}$ the "diagonal" of $M \times M$.

(a) Show that there exists an open neighborhood U of the diagonal Δ such that for any $(x,y) \in U \setminus \Delta$ we have $f(x) \neq f(y)$.
(b) We suppose that M is compact. Show that f admits only finitely many double points, i.e. that there exist only finitely many distinct pairs $(x_1,y_1),\dots,(x_n,y_n)$ in $M \times M \setminus \Delta$ such that $f(x_i) = f(y_i)$ for $i = 1,\dots,n$.

Exercise 10.2. Let $f\colon M \to N$ be an immersion between two manifolds without boundary. Show that given any $P \in M$ there exists an open neighborhood U of P such that the restriction $f|_U\colon U \to M$ is an embedding.
Hint: Use the Inverse Mapping Theorem 3.33.

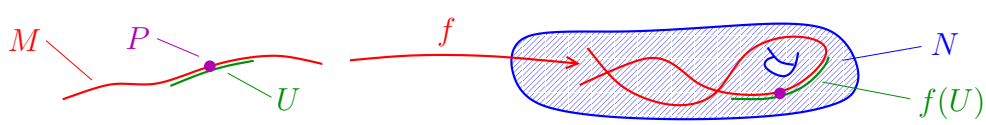


FIGURE 229. Illustration of Exercise 10.2.

Exercise 10.3. Let M and N be two manifolds with empty boundary. We suppose that $\dim(M) < \dim(N)$. Furthermore let W be a submanifold of N without boundary. Suppose that for any $P \in M$ with $f(P) \in W$ we have the equality $Df_P(T_PM) + T_{f(P)}W = T_{f(P)}N$. Show that $W \setminus f(M)$ has full measure in W.
Hint: Use Exercise 10.2, Lemma 3.43 (2) and Proposition 3.47.

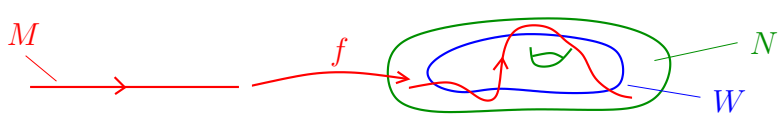


FIGURE 230.

Exercise 10.4.

(a) Show that the two oriented 1-dimensional submanifolds A and B of $S^1 \times S^1$ that are shown in Figure 231 on the left are not smoothly isotopic.
(b) We consider $\mathbb{R}^2$ with the two oriented 1-dimensional submanifolds X and Y that are shown in Figure 231 on the right. Are these X and Y smoothly isotopic?

Remark: Beware of base points.

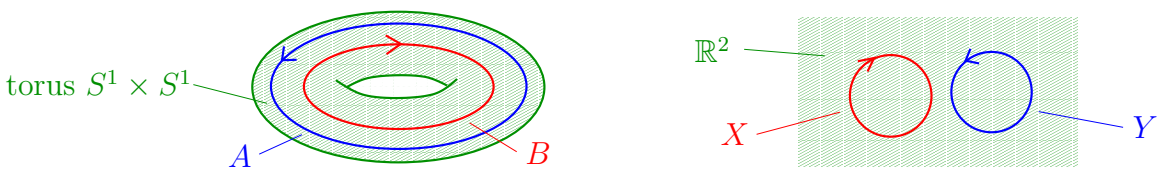


FIGURE 231.

Exercise 10.5. Let M be a manifold. Show that M is path-connected if and only if $M \setminus \partial M$ is path-connected.
Hint: use the Collar Neighborhood Theorem 10.10.

Exercise 10.6. Let $n \in \mathbb{N}_0$ and let $k \in \{1, \dots, n\}$. We write

$$\mathrm{GL}(n,k) \;=\; \underbrace{\text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n}_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{nk}}.$$

Show that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$.

Exercise 10.7. Let M be a connected n-dimensional manifold.

(a) Let $P \in M$. Show that if $n \geq 2$, then $M \setminus \{P\}$ is also connected.
(b) Let $\varphi \colon \overline{B}^n \to M \setminus \partial M$ be an embedding. Show that if $n \geq 2$, then $M \setminus \varphi(B^n)$ is also connected.

Exercise 10.8. Does the statement of the Tubular Neighborhood Theorem 10.19 (2) also hold if the manifold M is non-orientable? More precisely, let M be an non-orientable m-dimensional manifold and let N be a 1-dimensional submanifold. Does there exist an embedding

$$F \colon \overline{B}^{m-1} \times N \;\to\; M$$

such that $F(0,P) = P$ for all $P \in N$?

Exercise 10.9. Let $K = \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be the trivial knot in $\mathbb{R}^3 \cup \{\infty\} = S^3$. Let $X = S^3 \setminus K$ and let $Y = \mathbb{R}^3 \setminus K$. One of the two topological spaces X or Y is homotopy equivalent to S^1. Which one?

Exercise 10.10. We consider the disk $\overline{B}^2$ with two distinct points P and Q. By Proposition 10.23 there exists an orientation-preserving diffeomorphism f of $\overline{B}^2$ that swaps P and Q and that is the identity on S^1. Try to visualize such a map. (See also Figure 232 for a mental aid.)

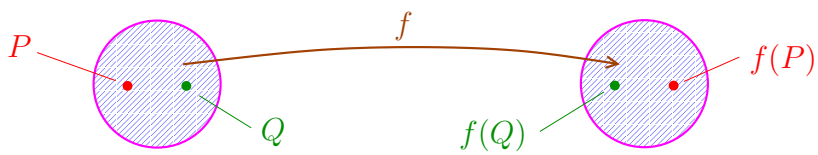


FIGURE 232.

Exercise 10.11. Let M be a manifold without boundary. We say that a path $\gamma\colon [a,b] \to M$ is *piecewise smooth* if there exists a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0, \ldots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth.

(a) Show that any path $\alpha\colon [0,1] \to M$ is path-homotopic to a piecewise smooth path β.

(b) Show that any piecewise smooth path $\beta\colon [0,1] \to M$ is path-homotopic to a smooth path $\gamma\colon [0,1] \to M$.
Hint: Use Lemmas 3.12 and 5.4

Exercise 10.12. Show that there exists a smooth deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SL}(n,\mathbb{R})$.

11. Basics of group theory

Before we continue with developing the theory of fundamental groups of topological spaces we want to introduce several basic constructions in group theory and we want state several facts which we will use throughout the remainder of this course.

11.1. **Free abelian groups and finitely generated abelian groups.** The following definition will stay with us throughout these notes.

Definition.

(1) Let S be a non-empty set. We write

$$\mathbb{Z}^S := \text{ all maps from } S \text{ to } \mathbb{Z}.$$

Furthermore, for S the empty set we write $\mathbb{Z}^S := \{0\}$. We consider $\mathbb{Z}^S$ with the obvious group structure, i.e. the group structure that is given by

$$\begin{array}{rcl} \mathbb{Z}^S \times \mathbb{Z}^S & \to & \mathbb{Z}^S \\ (f,g) & \mapsto & \left(\begin{array}{rcl} S & \to & \mathbb{Z} \\ s & \mapsto & f(s)+g(s) \end{array} \right). \end{array}$$

(2) Given any set S, we are mostly interested in the subgroup

$$\mathbb{Z}^{(S)} := \text{ all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$

We refer to $\mathbb{Z}^{(S)}$ as the *free abelian group generated by* S or as the *free abelian group on the generating set* S.

(3) We refer to the cardinality of S as the *rank of* $\mathbb{Z}^{(S)}$.

When working with free abelian groups it is usually helpful to use the following notation.

Notation. Let S be a set.

(1) Given $s \in S$ we denote by s also the map

$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

This map lies of course in $\mathbb{Z}^{(S)}$. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

(2) Given $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 s_1 + \dots + n_k s_k$ of $\mathbb{Z}^{(S)}$. We refer to $n_1 s_1 + \dots + n_k s_k$ as a *formal linear combination* of elements in S. Every element of $\mathbb{Z}^{(S)}$ can be written as such a formal linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can write[244]

$$f = \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

[244] It is straightforward to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

The following lemma says that free abelian group generated by a set S has a "universal property". The lemma also says that possessing such a universal property classifies free abelian groups up to isomorphism.

Lemma 11.1.

(1) *Let S be a set and let $\phi\colon S \to G$ be a map to an abelian group G. Then there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ that makes the following diagram commute*

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\phi"'] & \mathbb{Z}^{(S)} \arrow[d, "\psi"] \\ & G. \end{tikzcd}$$

(2) *Conversely let π be a group that admits a subset S with the following universal property: given any map $\phi\colon S \to G$ to an abelian group G there exists a unique homomorphism $\psi\colon \pi \to G$ that makes the following diagram commute*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\phi"'] & \pi \arrow[d, "\psi"] \\ & G. \end{tikzcd}$$

Then π is isomorphic to $\mathbb{Z}^{(S)}$.

Definition.

(1) We say that a group π is a *free abelian group on a generating set* S, if π has the universal property with respect to S. In other words, by Lemma 11.1, G needs to be isomorphic to $\mathbb{Z}^{(S)}$.

(2) We say that a group π is *free abelian* if it is a free abelian group on some generating set S.

Proof.

(1) We consider the map[245]

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ f = \sum_{i=1}^{n} n_i s_i & \mapsto & \psi(f) := \sum_{i=1}^{n} n_i \cdot \phi(s_i). \end{array}$$

It is straightforward to verify that ψ has the desired property and that it is the unique such homomorphism.

(2) Let π be a group that has the universal property with respect to S. We need to show that π is isomorphic to $\mathbb{Z}^{(S)}$. It is more elegant to show that it is isomorphic to any other group π' that has the universal property with respect to S.

This argument is similar to many other "universal property" proofs in algebra. We consider the inclusion maps $\varphi\colon S \to \pi$ and $\varphi'\colon S \to \pi'$. By the universal

[245]Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.

property of φ and φ' there exist homomorphisms $\psi\colon \pi \to \pi'$ and $\psi'\colon \pi' \to \pi$ that make the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \pi \arrow[d, "\psi"] \\ & \pi' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[dr, "\varphi"'] & \pi' \arrow[d, "\psi'"] \\ & \pi. \end{tikzcd}$$

In particular we get the commutative diagrams

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi'"] \arrow[ddr, "\varphi"'] & \pi \arrow[d, "\psi"] \\ & \pi' \arrow[d, "\psi'"] \\ & \pi \end{tikzcd} \quad \text{which simplifies to} \quad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \pi \arrow[d, "\psi'\circ\psi"] \\ & \pi \end{tikzcd} \quad \text{and of course we also have the commutative diagram} \quad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \pi \arrow[d, "\mathrm{id}_\pi"] \\ & \pi. \end{tikzcd}$$

It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\psi' \circ \psi = \mathrm{id}_\pi$. The same way one shows that $\psi \circ \psi' = \mathrm{id}_{\pi'}$. In particular ψ and ψ' are isomorphisms. □

Lemma 11.2.

(1) *Any subgroup G of a free abelian group F is again a free abelian group.*
(2) *If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is a monomorphism, then*[246] $\#S \le \#T$.
(3) *If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is an epimorphism, then $\#S \ge \#T$.*

Proof. The first two statements are proved in [Rot, Theorem 9.3]. Now let $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ be an epimorphism. For each $t \in T$ there exists therefore some $\phi(t) \in S$ with $\varphi(\phi(t)) = t$. By Lemma 11.1 there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(T)} \to \mathbb{Z}^{(S)}$ with $\psi(t) = \phi(t)$ for all $t \in T$. Thus we have $\varphi(\psi(t)) = \varphi(\phi(t)) = t$ for all t, so $\varphi \circ \psi = \mathrm{id}$, in particular ψ is a monomorphism. It follows from (2) that $\#T \le \#S$. □

Definition.

(1) If F is a free abelian group, then we define the *rank of F* as the cardinality of any set S with $F \cong \mathbb{Z}^{(S)}$.[247]
(2) Given an abelian group G we say that $S \subset G$ is a *generating set* if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an epimorphism.[248]

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[246] Here we denote by $\#S$ and $\#T$ the cardinalities of the sets S and T, as defined on page 254.

[247] It follows from Lemma 11.2 and the Bernstein-Schröder Theorem 7.1 that the rank of a free abelian group is well-defined.

[248] Here we mean by "canonical map $\mathbb{Z}^{(S)} \to G$" the unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ from Lemma 11.1 that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, hook, "\phi"'] & \mathbb{Z}^{(S)} \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

(3) Given an abelian group G we say that $S \subset G$ is a *basis* if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an isomorphism.
(4) We say that an abelian group G is *finitely generated* if it admits a finite generating set.

Example. Let S be a finite set with m elements $s_1, \ldots, s_m$. Let $\phi\colon S \to \mathbb{Z}^m$ be the map given by $\phi(s_i) = e_i$. Then the resulting homomorphism $\Psi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^m$ is an isomorphism. Put differently, $\mathbb{Z}^m$ is a free abelian group of rank m.

We leave the proof of the following useful characterizations of generating sets as an exercise to the reader.

Lemma 11.3. *Let G be an abelian group and let $S \subset G$ be a subset. Then the following two statements are equivalent:*
(1) *S is a generating set.*
(2) *Any subgroup H of S that contains S is already equal to G.*

We use this opportunity to recall the classification of finitely generated abelian groups which had been proved in the algebra course. A proof is for example also provided in [Hun, Chapter II.2].

Theorem 11.4. (Classification of finitely generated abelian groups) *Let A be a finitely generated abelian group. Then there exist non-zero natural numbers $a_1, \ldots, a_k$ with $a_i | a_{i+1}$ for $i = 1, \ldots, k-1$ and an $r \in \mathbb{N}_0$ such that*

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Furthermore the numbers $a_1, \ldots, a_k$ and r are uniquely determined by A.

Definition. Given a finitely generated abelian group A we denote the direct sum of the cyclic groups on the right-hand side of Theorem 11.4 as the *canonical form of A*.

Example. The Chinese Remainder Theorem says that for any coprime m and n we have an isomorphism

$$\begin{array}{rcl} \mathbb{Z}_{mn} & \xrightarrow{\cong} & \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_m \oplus \mathbb{Z}_n \\ k + mn\mathbb{Z} & \mapsto & (k + m\mathbb{Z}, k + n\mathbb{Z}). \end{array}$$

For example consider $A = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. It is not in the form of Theorem 11.4, but we have the isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$, which is of the desired form. Thus $\mathbb{Z}_{15}$ is the canonical form of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$.

We will not provide a proof for Theorem 11.4, but we will sketch a proof for the following lemma. The third statement of the lemma is a somewhat weaker version of Theorem 11.4.

Lemma 11.5.

where $S \to G$ is the inclusion map.

(1) *Let V be an $m \times n$-matrix over $\mathbb{Z}$. For any two matrices $A \in \mathrm{GL}(m,\mathbb{Z})$ and $B \in \mathrm{GL}(n,\mathbb{Z})$ we have an isomorphism*

$$\mathbb{Z}^m/AVB\mathbb{Z}^n \;\cong\; \mathbb{Z}^m/V\mathbb{Z}^n.$$

(2) *Let V be an $m \times n$-matrix over $\mathbb{Z}$. Then there exist two matrices $A \in \mathrm{GL}(m,\mathbb{Z})$ and $B \in \mathrm{GL}(n,\mathbb{Z})$*[249] *such that*[250]

$$A \cdot V \cdot B \;=\; \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \;\in\; M(m \times n, \mathbb{Z})$$

for some $p_1, \dots, p_t \in \mathbb{N}$.

(3) *Let A be a finitely generated abelian group. Then there exists an $r \geq 0$ and $p_1, \dots, p_t \in \mathbb{N}$ such that*

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}_{p_i}.$$

As mentioned before, we will only provide sketch for the proof of the lemma.

Sketch of proof.

(1) The proof of the first statement is elementary. It is left as an exercise to the reader.

(2) First note that if an $m \times n$-matrix P is obtained from an $m \times n$-matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m \times m$-matrix of the form[251]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$-matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $m \times m$-matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

[249] The matrices A and B are invertible over $\mathbb{Z}$, i.e. their determinants are ± 1.

[250] Here the last row and column of zeros can be arbitrarily large.

[251] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

where the extra 1 sits in the (k,l)-entry. Both these $m \times m$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let V be an $m \times n$-matrix over $\mathbb{Z}$. If V is the zero matrix, then there is nothing to show. So suppose that V is not the zero matrix. We perform the following steps:

(a) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.

(b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.

(c) The same way as in (2) we can arrange that that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.

Now we have to distinguish two cases:

(a) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.

(b) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$V = \begin{pmatrix} v_{11} & 0 \\ 0 & V' \end{pmatrix}$$

for some $(m-1)\times(n-1)$-matrix V' and we perform the same operations on the smaller matrix V'.

(3) Let A be a finitely generated abelian group. By the definition of a finitely generated abelian group there exists an epimorphism $\psi\colon \mathbb{Z}^m \to A$. Since $\mathbb{Z}$ is a Noetherian ring we know that $\ker(\psi)$ is again a finitely generated free abelian group of rank $n \le m$. In particular there exists an isomorphism $v\colon \mathbb{Z}^n \to \ker(\psi)$. We denote by V the $m\times n$-matrix that represents the resulting map $\mathbb{Z}^n \to \ker(\psi) \subset \mathbb{Z}^m$ with respect to the standard bases. Then $A \cong \mathbb{Z}^m/V\mathbb{Z}^n$. We pick matrices A and B as in (2). From (1) it follows that $\mathbb{Z}^m/V\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m/AVB\mathbb{Z}^n$. But the latter is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_t} \oplus \mathbb{Z}^{m-n}$. □

Definition. Let A be an abelian group.

(1) If A is finitely generated, then we refer to the r in the statement of Theorem 11.4 as the *rank* of the finitely generated abelian group A. We denote it by $\mathrm{rank}(A)$.[252]

[252] If A is a finitely generated free abelian group, then this definition of rank agrees with the definition given for free abelian groups on page 366.

(2) We refer to

$$\operatorname{tor}(A) \;=\; \{a \in A \mid \text{ there exists an } n \in \mathbb{N} \text{ with } na = 0\}$$

as the *torsion subgroup of* A.[253] If $\operatorname{tor}(A)$ is the trivial subgroup, then we say that A is *torsion-free*.

In the following three lemmas we collect several basic facts about finitely generated abelian groups. Most of the statements follow more or less directly from Theorem 11.4. We will not prove the lemmas and we leave it as an exercise to deduce the statements from Theorem 11.4 and the algebra course you took.

Lemma 11.6. *Let*

$$0 \;\to\; A \;\to\; B \;\to\; C \;\to\; 0$$

be a short exact sequence of abelian groups[254] *such that at least two of these groups are finitely generated. Then the following holds:*

(1) *all three groups are finitely generated abelian groups,*
(2) $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$,
(3) *if A and C are torsion-free, then B is also torsion-free.*

Lemma 11.7. *Let $A \in M(n \times n, \mathbb{Z})$ be a matrix. Then*

$$\mathbb{Z}/A\mathbb{Z} \text{ is finite} \quad\Longleftrightarrow\quad \det(A) \neq 0.$$

Furthermore, if $\det(A) \neq 0$, *then*

$$0 \;\to\; \mathbb{Z}^n \;\xrightarrow{A\cdot}\; \mathbb{Z}^n \;\to\; \mathbb{Z}^n/A\mathbb{Z}^n \;\to\; 0$$

is a short exact sequence and $\mathbb{Z}^n/A\mathbb{Z}^n$ is a finite group with $|\det(A)|$ elements.

We also have the following lemma.

Lemma 11.8.

(1) *If A, B and C are finitely generated abelian groups, then*

$$A \oplus C \cong B \oplus C \quad\Longrightarrow\quad A \cong B.$$

(2) *If A is a subgroup of a finitely generated abelian group B, then* $\operatorname{tor}(A)$ *is a subgroup of* $\operatorname{tor}(B)$ *and* $\operatorname{rank}(A) \leq \operatorname{rank}(B)$.
(3) *If A is a subgroup of $\mathbb{Z}^r$, then A is a free abelian group with rank less or equal than r.*[255]
(4) *If A is a subgroup of $\mathbb{Z}^r$ with* $\operatorname{rank}(A) < r$, *then A is a proper subgroup, i.e. $A \neq \mathbb{Z}^r$.*
(5) *Let $\varphi\colon A \to B$ be a homomorphism between two free abelian groups of the same finite rank. If φ is an epimorphism, then φ is already an isomorphism.*

[253] Why is it a subgroup?

[254] Recall that this means that the map $A \to B$ is injective, that $B \to C$ is surjective and that the image of $A \to B$ agrees with the kernel of $B \to C$.

[255] In fact this statement is already contained in Lemma 11.2.

(6) *If A is a finitely generated abelian group and $\varphi\colon A \to B$ is an epimorphism, then B is also finitely generated and* $\operatorname{rank}(A) \geq \operatorname{rank}(B)$.

Abelian groups that are not finitely generated are not classified and they tend to be significantly harder. For example the following lemma shows that not every torsion-free abelian group is a free abelian group.

Lemma 11.9. *The abelian group $(\mathbb{Q}, +)$ is torsion-free but not free abelian.*

Proof. Clearly $(\mathbb{Q}, +)$ is torsion-free. If it was free abelian, then there would exist an epimorphism to $\mathbb{Z}$. In Exercise 11.1 to show that such an epimorphism cannot exist. □

In Exercise 11.2 (1) we will see that the conclusion of Lemma 11.8 (1) does not hold for general abelian groups. But the following statement actually does hold.

Proposition 11.10. *Let A and B be any two abelian groups. If C is a finitely generated abelian group, then*

$$A \oplus C \cong B \oplus C \quad \Longrightarrow \quad A \cong B.$$

Proof. We will consider a special case in Exercise 11.2 (b). The general case is proved in [Cohn, Walka]. □

11.2. **The free product of groups.** We first recall the definition of the direct product of two groups G and H that we already gave on page 284. The direct product of two groups G and H is defined as

$$G \times H \ := \ \{(g, h) \mid g \in G \text{ and } h \in H\}$$

together with the group structure that is given by

$$(g_1, h_1) \cdot (g_2, h_2) \ := \ (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H*. Via the monomorphisms

$$\begin{array}{ccc} G & \to & G \times H \\ g & \mapsto & (g, e) \end{array} \quad \text{and} \quad \begin{array}{ccc} H & \to & G \times H \\ h & \mapsto & (e, h) \end{array}$$

we can view G and H as subgroups of $G \times H$. Note that these subgroups commute, i.e. for every $g \in G$ and $h \in H$ we have the equality $(g, e)(e, h) = (g, h) = (e, h)(g, e)$.

Now we will introduce the "free product" $G * H$ of two groups G and H. The definition is perhaps initially harder than the definition of the "direct product" $G \times H$. But for many, perhaps even most applications, the free product is more useful.

So let G be and H be two groups. We consider the set $G * H$ of all finite sequences $(x_1, \ldots, x_m)$ such that the following conditions are satisfied:

(1) each x_i lies in one of the groups G or H,
(2) no x_j is the neutral element of G or of H,
(3) any two consecutive x_j's lie in two different groups.

Here we also allow the "empty sequence" ().

Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $G * H$. We define the product of these two elements as follows:

(1) If x_m and y_1 lie in different groups, then we define

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \ := \ (x_1, \dots, x_m, y_1, \dots, y_n).$$

(2) If x_m and y_1 lie in the same group, then there exists a j such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$, and such that $x_{m-j} \neq y_{j+1}^{-1}$. We define[256]

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \ := \ (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$

Another way of stating the product formula is as follows: given two sequences $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we stack them together $(x_1, \dots, x_m, y_1, \dots, y_n)$ and then we delete any occurrence of a subsequence of the form a, a^{-1} for $a \in G$ or $a \in H$ and if a subsequence is of the form a, b with $a, b \in G$ or $a, b \in H$, then we replace it by ab.

We henceforth refer to $G * H$ together with this product structure as the *free product* of G and H. We have the following important lemma.

Lemma 11.11. *The free product of two groups G and H is again a group. The neutral element of $G * H$ is hereby given by the empty sequence* () *and the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by*

$$(x_1, \dots, x_m)^{-1} \ = \ (x_m^{-1}, \dots, x_1^{-1}) \ \in \ G * H.$$

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by $(x_m^{-1}, \dots, x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W = G * H$ and we denote by $(P(W), \circ)$ the group of all permutations of W, i.e. $P(W)$ is the set of all bijections from W to W and the group structure is given by composition of maps. For $g \in G$ we consider the map[257]

$$\begin{aligned} \lambda_g \colon W \ &\to \ W \\ (x_1, \dots, x_m) \ &\mapsto \ \begin{cases} (g, x_1, x_2, \dots, x_m), & \text{if } x_1 \in H, \\ (gx_1, x_2, \dots, x_m), & \text{if } x_1 \in G \text{ and } g \neq x_1^{-1}, \\ (x_2, \dots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{aligned}$$

It is now straightforward to see that for any $g, g' \in G$ and any $(x_1, \dots, x_m) \in W$ we have $g(g' \cdot (x_1, \dots, x_m)) = (gg') \cdot (x_1, \dots, x_m)$. This means that $\lambda_g \circ \lambda_{g'} = \lambda_{gg'}$. It is now

[256]Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

[257]It is straightforward to see that the sequences on the right do indeed lie in $G * H$.

straightforward to see that the map

$$\begin{array}{rcl} \lambda\colon G & \to & P(W) \\ g & \mapsto & \lambda_g \end{array}$$

is a group homomorphism. Similarly we define $\lambda\colon H \to P(W)$. Now we consider the map

$$\begin{array}{rcl} \lambda\colon W & \to & P(W) \\ (x_1,\dots,x_n) & \to & \lambda_{x_1}\circ\cdots\circ\lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1,\dots,x_n)$ applied to the empty word $()$ gives the word $(x_1,\dots,x_n)$. Hence $\lambda(x_1,\dots,x_n)$ is not the identity map from $W \to W$, i.e. $\lambda(x_1,\dots,x_n)$ is non-trivial in $P(W)$.

Furthermore the map $\lambda\colon W \to P(W)$ satisfies $\lambda(x\cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda\colon (W,\ \cdot\ ) \to (P(W),\circ)$ that preserves the product structure. But since $(P(W),\circ)$ satisfies the associativity law we now see that $(W,\ \cdot\ )$ also satisfies the associativity law. $\square$

The following lemma summarizes two main properties of the free product of two groups.

Lemma 11.12. *Let G and H groups. Then the following holds:*

(1) *The maps*

$$\begin{array}{rcl} i\colon G & \to & G*H \\ g & \mapsto & \begin{cases} (g), & \text{if } g\neq e, \\ (), & \text{if } g=e \end{cases} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon H & \to & G*H \\ h & \mapsto & \begin{cases} (h), & \text{if } h\neq e, \\ (), & \text{if } h=e \end{cases} \end{array}$$

are monomorphisms.

(2) *For any two group homomorphisms $\alpha\colon G\to A$ and $\beta\colon H\to A$ there exists a unique group homomorphism $\gamma\colon G*H\to A$ such that the following diagram commutes*

$$\begin{tikzcd} G*H & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & A. \end{tikzcd}$$

Example. Let G and H be two groups. We denote by $\alpha\colon G\to G\times H$ and $\beta\colon H\to G\times H$ the obvious inclusion maps. It follows from Lemma 11.12 there exists a unique homomorphism $\gamma\colon G*H\to G\times H$ which makes the following diagram commute:

$$\begin{tikzcd} G*H \arrow[dr, dashed, "\gamma"] & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & G\times H. \end{tikzcd}$$

Remark. In the literature the direct product of two groups often gets introduced in a slightly different way, namely by a "universal property". More precisely, let G be and H be two groups. The *direct product of G and H* is then defined as a group K together with two

homomorphisms $i\colon G \to K$ and $j\colon G \to K$ which satisfy the following universal property: for every two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon H \to A$ there exists a *unique* group homomorphism $\gamma\colon K \to A$ such that the following diagram commutes

$$\begin{tikzcd} K \arrow[dr, dashed, "\gamma"] & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & A. \end{tikzcd}$$

Lemma 11.12 can now be reinterpreted as saying that the direct product of two groups always exists. An argument that is very similar to the one we gave on page 368 shows that the direct product is unique in the following sense: If $(K, i\colon G \to K, j\colon H \to K)$ and $(K', i'\colon G \to K', j'\colon H \to K')$ both satisfy the above universal property, then there exists a unique isomorphism $\Phi\colon K \to K'$ such that $i' = \Phi \circ i$ and $j' = \Phi \circ j$. [258]

Proof. It is clear that the maps i and j are injective, furthermore it follows immediately from the definition of the group structure on $G * H$ that the maps are group homomorphisms.

Now let $\alpha\colon G \to A$ and $\beta\colon H \to A$ be two group homomorphisms. For an element $(g_1, h_1, \dots, g_k, h_k) \in G * H$ with $g_i \in G, h_i \in H$, $i = 1, \dots, n$ we define

$$\gamma(g_1, h_1, \dots, g_k, h_k) \ := \ \alpha(g_1) \cdot \beta(h_1) \cdot \dots \cdot \alpha(g_k) \cdot \beta(h_k).$$

The same way we define γ for all other elements in $G * H$. Now one can show easily that γ is a group homomorphism.

It remains to show the uniqueness of γ. Any element in $G * H$ is a product of elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. It follows that any homomorphism $\delta\colon G * H \to A$ is already uniquely determined by the values of δ on all elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. Put differently, there is at most one homomorphism $\delta\colon G * H \to A$ with $\delta(g) = \alpha(i(g))$ for all $g \in G$ and $\delta(h) = \beta(j(h))$ for all $h \in H$. $\square$

Remark.

(1) We will use the monomorphisms from Lemma 11.12 to view G and H as subgroups of $G * H$.
(2) If there is no danger of confusion, then we often shorten $(x_1, \dots, x_m) \in G * H$ to the notation $x_1 \dots x_m$. This notation is coherent with the convention (1). Note that if $g \in G$ and $h \in H$ are non-trivial elements, then the sequences (g, h) and (h, g) both lie in $G * H$ and they are different. With the convention that we just introduced that means that $g \cdot h \neq h \cdot g$. Put differently, a non-trivial element of G never commutes with a non-trivial element of H.
(3) The construction of the free product of two groups generalizes in an obvious way to the free group of finitely many, in fact even to infinitely many groups.

[258] Why does $K = G \times H$ with the obvious maps $i\colon G \to K$ and $j\colon H \to K$ not satisfy this universal property? Or does it?

Let t be a symbol. We define

$$\langle t\rangle \;:=\; \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$

The set $\langle t\rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group is called *the infinite cyclic group generated by* t. The map $\mathbb{Z} \to \langle t\rangle$, $n \mapsto t^n$ is evidently an isomorphism. Now we come to one of the main definitions in group theory.

Definition. For a set S we refer to

$$\langle S\rangle \;:=\; \text{free product of the groups } \langle s\rangle,\ s \in S$$

as the *free group on the* (*generating*) *set* S. Via the map $s \mapsto (s)$ we can and will view S as a subset of $\langle S\rangle$.

Example. If $S = \{t_1, \dots, t_k\}$, then we obtain the group

$$\langle t_1, \dots, t_k\rangle \;:=\; \langle t_1\rangle * \cdots * \langle t_k\rangle$$

which we also refer to as the *free group on the generators* $t_1, \dots, t_k$. Each element of the free group $F(t_1, \dots, t_k)$ is thus of the form

$$t_{r_1}^{s_1}\, t_{r_2}^{s_2} \,\dots\, t_{r_l}^{s_l}$$

with $r_i \neq r_{i+1}$ for $i = 1, \dots, l-1$ and $s_i \neq 0$ for $i = 1, \dots, l$.

Convention. Sometimes we denote the free group on k generators just by F_k.

Example. We consider the free group $\langle x, y\rangle$ on generators x and y. Elements in this group are of the form $xy^3x^{-2}y$ or $y^{-2}xy$. The multiplication is the "obvious one", for example

$$(xy^3x^{-1}y^2) \cdot (y^{-2}x^3yxy^{-1}) \;=\; xy^3x^2yxy^{-1}.$$

The group is not abelian since $xy \neq yx$.[259]

The following innocent looking lemma summarizes the key property of free groups.

Lemma 11.13. *Let S be a set, let G be an arbitrary group and let $g\colon S \to G$ be a map. Then there exists a unique homomorphism $\varphi\colon \langle S\rangle \to G$ with $\varphi(s) = g(s)$ for every $s \in S$.*

Proof. For $s \in S$ the map

$$\begin{array}{ccc} \langle s\rangle & \to & G \\ s^n & \mapsto & g(s)^n \end{array}$$

is evidently the unique homomorphism $\varphi_s\colon \langle s\rangle \to G$ with $\varphi_s(s) = g(s)$. It now follows from the obvious generalization of Lemma 11.12 to the case of the free product of arbitrarily many groups that there exists a unique homomorphism $\varphi\colon \langle S\rangle \to G$ with $\varphi(s) = \varphi_s(s) = g(s)$ for $s \in S$. $\square$

[259] Why is $xy \neq yx$?

Remark. Sometimes in the literature free groups are defined via a universal property. More precisely, let S be a set. A *free group on* S is sometimes defined as a group K together with a map $f\colon S \to K$ which satisfies the following universal property: for any map $g\colon S \to G$ from the set S to a group G[260] there exists a unique group homomorphism $\varphi\colon K \to G$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & K \arrow[d, "\varphi"] \\ & G. \end{tikzcd}$$

In this language our above Lemma 11.13 says that the obvious map $S \to \langle S \rangle$ is a free group on S in the above sense. As before, a slight variation of the argument on page 368 shows that the free group on S is unique in an appropriate sense.

Definition.

(1) A group G is called *free*, if it is isomorphic to the free group $\langle S \rangle$ on some set S.
(2) Given a group G we say that $S \subset G$ is a *generating set* if the canonical homomorphism $\langle S \rangle \to G$ is an epimorphism.[261] Sometimes we also say that the elements in S are *generators of* G.
(3) We say that a group G is *finitely generated* if it admits a finite generating set.

Example.

(1) As on page 265 we consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and we denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that

$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

We saw on page 265 that $\mathbb{R}^2/G$ equals the Klein bottle K and using Theorem 7.17 we saw on page 283 that $\pi_1(\text{Klein bottle}) \cong G$. The group G is finitely generated in the above sense. More precisely, it follows immediately from the definition of G that $S = \{A, B\} \subset G$ is a generating set for G.

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[260] Note that in contrast to the Lemma 11.1 group G is now not necessarily abelian.

[261] Here we mean by "canonical map $\langle S \rangle \to G$" the unique homomorphism $\psi\colon \langle S \rangle \to G$ from Lemma 11.13 that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, hook, "\phi"'] & \langle S \rangle \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

where $S \to G$ is the inclusion map.

(2) For any set S there exists by Lemma 11.13 a unique homomorphism $\langle S\rangle \to \mathbb{Z}^{(S)}$ that sends $s \in S$ to s viewed as an element in $\mathbb{Z}^{(S)}$. Since the set $S \subset \mathbb{Z}^{(S)}$ generates $\mathbb{Z}^{(S)}$ we see that this homomorphism is an epimorphism. It follows that any finitely generated abelian group in the sense of page 369 is also a finitely generated group in the above sense.

The subgroup of a free abelian group is again a free abelian group. Does the same hold for free groups? We record this as a question for future reference.

Question 11.14. *Is every subgroup of a free group again a free group?*

The next lemma says in particular that free groups on two sets are isomorphic if and only if the cardinalities of the sets are the same.

Lemma 11.15. *Let S and T be two sets.*

(1) *If there exists an epimorphism $\langle S\rangle \to \langle T\rangle$, then $\#S \geq \#T$.*
(2) *If $\langle S\rangle$ is isomorphic to $\langle T\rangle$, then $\#S = \#T$.*
(3) *If given $n \in \mathbb{N}$ we denote by F_n the free group on n generators, then for $k \neq l$ the groups F_k and F_l are not isomorphic.*

Proof.

(1) We first recall that we had just seen in the previous example that for any set U there exists a unique epimorphism $\psi_U\colon \langle U\rangle \to \mathbb{Z}^{(U)}$ that sends $u \in U$ to u viewed as an element in $\mathbb{Z}^{(U)}$.

Now let S and T be two sets and let $\varphi\colon \langle S\rangle \to \langle T\rangle$ be an epimorphism. By Lemma 11.1 there exists a unique homomorphism $\widetilde{\varphi}\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ that sends $s \in S$ to $\psi_T(\varphi(s)) \in \mathbb{Z}^{(T)}$. We obtain the following diagram of homomorphisms

$$\begin{tikzcd} \langle S\rangle \arrow[r, two heads, "\varphi"] \arrow[d, two heads, "\psi_S"'] & \langle T\rangle \arrow[d, two heads, "\psi_T"] \\ \mathbb{Z}^{(S)} \arrow[r, "\widetilde{\varphi}"] & \mathbb{Z}^{(T)}. \end{tikzcd}$$

It follows immediately from the definition of $\tilde{\varphi}$ that this diagram commutes.[262] Since the top horizontal and since the vertical maps are epimorphisms we see that the bottom horizontal map is also an epimorphism. It follows from Lemma 11.2 that $\#S \geq \#T$.

(2) Let $f\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism. We apply (1) to f and to f^{-1} and we obtain that $\#S \geq \#T$ and $\#T \geq \#S$. It follows from the Bernstein-Schröder Theorem 7.1 that $\#S = \#T$.

(3) This statement is an immediate consequence of (2).[263] □

[262] Here we use that a homomorphism $\gamma\colon \langle S\rangle \to G$ is uniquely determined by the values of γ on S. Put differently, if $\alpha\colon \langle S\rangle \to G$ and $\beta\colon \langle S\rangle \to G$ are two homomorphisms with $\alpha(s) = \beta(s)$ for all $s \in S$, then it follows immediately from the uniqueness statement of Lemma 11.13 that $\alpha = \beta$.

[263] Note that if we just want to prove statement (3), then in the proof of (2) we could replace the slightly esoteric Lemma 11.2 by the more conventional Lemma 11.6 that was proved in the Algebra course.

11.3. **An alternative definition of the free product of groups.** For some purposes it is sometimes useful to work with a different model for the free product of two groups.

Let A and B be two groups. We set[264]

$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1,\dots,g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}$$

and we denote by $\sim$ the equivalence relation on $S(A,B)$ that is generated by the relations

$$(g_1,\dots,g_l,e,g_{l+2},\dots,g_k) \;\sim\; (g_1,\dots,g_l,g_{l+2},\dots,g_k)$$

and by the relations

$$(g_1,\dots,g_l,g_{l+1},\dots,g_k) \;\sim\; (g_1,\dots,g_lg_{l+1},\dots,g_k)$$

if both g_l and g_{l+1} lie in A or both lie in B. It is now straightforward to verify that the map

$$\begin{array}{rcl} S(A,B)/\sim \;\times\; S(A,B)/\sim & \to & S(A,B)/\sim \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)] \end{array}$$

is well-defined and that it defines a group structure on $S(A,B)/\sim$.[265]

Proposition 11.16. *Let A and B be two groups. Then the map*[266]

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A*B \\ [(g_1,\dots,g_k)] & \mapsto & g_1\cdot\dots\cdot g_k \end{array}$$

is well-defined and it is an isomorphism. The inverse map is given by

$$\begin{array}{rcl} \Psi\colon A*B & \to & S(A,B)/\sim \\ (g_1,\dots,g_k) & \mapsto & [(g_1,\dots,g_k)]. \end{array}$$

In the proof of Proposition 11.16 we will need the following little lemma.

Lemma 11.17. *Let $\Phi\colon G\to H$ and $\Psi\colon H\to G$ be two group homomorphisms such that $\Phi\circ\Psi=\mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.*

The lemma can be summarized as follows, if we are given a commutative diagram of the form

$$\begin{tikzcd} & G \arrow[dr, "\Phi"] & \\ H \arrow[ur, two heads, "\Psi"] \arrow[rr, "\mathrm{id}"'] & & H, \end{tikzcd}$$

then Φ is an isomorphism.[267]

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[264]This definition is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

[265]The neutral element is again represented by the empty sequence $(\,)$ and the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. This time associativity is not a problem. Why not?

[266]Recall that we use the canonical monomorphisms from Lemma 11.12 to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(g_1,\dots,g_k)]\mapsto(g_1)\cdot\dots\cdot(g_k)$.

[267]Is Ψ also necessarily an isomorphism?

Proof. It follows from $\Phi \circ \Psi = \mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi \circ \Psi = \mathrm{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. □

Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A * B \\ [(g_1,\dots,g_k)] & \mapsto & g_1 \cdot \dots \cdot g_k \end{array}$$

is well-defined and that it is a homomorphism. This homomorphism is surjective since $A * B$ is a subset of $S(A,B)$. Next we consider the map[268]

$$\begin{array}{rcl} \Psi\colon A * B & \to & S(A,B)/\sim \\ (g_1,\dots,g_k) & \mapsto & [(g_1,\dots,g_k)]. \end{array}$$

It is straightforward to see that Ψ is a homomorphism. It is clear that $\Phi \circ \Psi = \mathrm{id}_{A*B}$. It is not so obvious that $\Psi \circ \Phi = \mathrm{id}_{S(A,B)/\sim}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 11.17, to show that Ψ is surjective. This in turn follows from the following claim.

Claim. Any element in $S(A,B)$ is equivalent to an element in $A * B$.

We prove the claim by induction on the length of a sequence in $S(A,B)$. Clearly the claim holds for all sequences of length 0. Suppose we already know the claim for all sequences of length $\leq k-1$. Let $(g_1,\dots,g_k)$ be a sequence of length k. If $(g_1,\dots,g_k) \in A * B$, then we are done. Now suppose that this is not the case. Then one of the following occurs:

(1) either there exists an $i \in \{1,\dots,k\}$ with $g_i = e$, or
(2) there exists an $i \in \{1,\dots,k-1\}$ with $g_i, g_{i+1} \in A$, or
(3) there exists an $i \in \{1,\dots,k-1\}$ with $g_i, g_{i+1} \in B$.

In the first case the sequence is equivalent to $(g_1,\dots,g_{i-1},g_{i+1},\dots,g_k)$. In the second and third case the sequence is equivalent to $(g_1,\dots,g_{i-1}\cdot g_{i+1},\dots,g_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. This concludes the proof of the claim. □

Let A and B be two groups. In the following we will use the isomorphism of Proposition 11.16 to identify the group $S(A,B)/\sim$ with the free product $A * B$. Both groups have their advantages:

(1) In the group $S(A,B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A * B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a,b) and (b,a) of $A * B$ are by definition different. But from the definition of $S(A,B)/\sim$ it is not immediately clear that $[(a,b)]$ and $[(b,a)]$ are indeed different.

[268]Put differently, $A * B \to S(A,B)/\sim$ is the homomorphism that is given by the universal property of $A * B$.

11.4. **The Grushko-Neumann Theorem.** We start out with the following natural definition.

Definition. Let G be a finitely generated group. We denote by $d(G) \in \mathbb{N}_0$ the minimal number of elements in a generating set for G.

Remark. It is an unfortunate fact of life that in the literature the number $d(G)$ is usually referred to as the rank of the group G. But for abelian groups the invariants $\operatorname{rank}(G)$ and $d(G)$ can differ quite dramatically. For example for $G = (\mathbb{Z}_2)^n$ we have $\operatorname{rank}(G) = 0$ but $d(G) = n$.

The following theorem was proved by Igor Grushko [Gru] in 1940 and independently by Bernhard Neumann [Neum] in 1940.

Theorem 11.18. (Grushko-Neumann Theorem) *Given any two finitely generated groups A and B we have*

$$d(A * B) \;=\; d(A) + d(B).$$

Proof. The proof is evidently given in the above two references, but it can also be found in any self-respecting book on combinatorial group theory, see e.g. [LS, Corollay IV.1.9] and [MKS, p. 192]. □

We already saw that some groups can be written as a direct product of two groups in different ways. For example it follows from the Chinese Remainder Theorem that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{35}$ and $\mathbb{Z}_5 \times \mathbb{Z}_{21}$ are isomorphic. The following theorem says that the situation is much more rigid for free products. Before we can state the theorem we have to introduce the following definition.

Definition. A group G is *indecomposable* if it is not isomorphic to the free product of two non-trivial groups.

The following lemma will be proved in Exercise 11.4.

Lemma 11.19. *Abelian groups and finite groups are indecomposable.*

Now we can state the Grushko[269] Decomposition Theorem.

Theorem 11.20. *Let G be a finitely generated group.*

(1) *The group G is isomorphic to the free product $G = A_1 * \cdots * A_k$ of non-trivial indecomposable groups.*[270]
(2) *The isomorphism types of $A_1, \dots, A_k$ are unique up to permutation. More precisely, if $A_1, \dots, A_k$ and $B_1, \dots, B_l$ are indecomposable finitely generated groups such that*

$$A_1 * \cdots * A_k \;\cong\; B_1 * \cdots * B_l,$$

then $k = l$ and furthermore there exists a permutation $\sigma \in S_k$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, \dots, k$.

[269] Igor Grushko was a Russian mathematician. The theorem was proved in 1940.

[270] If G is trivial then $k = 0$ and the free product of zero groups is by convention the trivial group.

Proof.

(1) This statement follows easily from induction on $d(G)$ and the Grushko-Neumann Theorem 11.18. Indeed, if $d(G) = 0$ then there is nothing to prove. Next suppose that the statement is known for all groups G with $d(G) < k$. Let G be a group with $d(G) = k$. If G is indecomposable, we are done. If G is not indecomposable, then we can write G as the free product of two non-trivial groups A and B. By the Grushko-Neumann Theorem 11.18 we have $d(G) = d(A) + d(B)$. Since A and B are non-trivial we have $d(A) \geq 1$ and $d(B) \geq 1$. Thus we have $d(A) < k$ and $d(B) < k$ and by induction we can find decompositions for A and B. These two decompositions put together give the desired decomposition for $G = A * B$.

(2) The uniqueness statement follows from a refinement of Grushko's proof due to John Stallings [Sta2, p. 168f]. See also [Nel, Theorem 4.1] for a more direct reference. □

Example. It follows from Lemma 11.19 and from the Grushko Decomposition Theorem 11.20 that the free product $\mathbb{Z}_3 * \mathbb{Z}_{35}$ is not isomorphic to the free product $\mathbb{Z}_5 * \mathbb{Z}_{21}$.

Exercises for Chapter 11.

Exercise 11.1. Show that $\mathbb{Q}$ does not admit an epimorphism onto $\mathbb{Z}$.

Exercise 11.2.

(a) Show that there exists an abelian group G with $G \oplus G \cong G$.

(b) Let A be a finitely generated abelian group. Show that if B is an abelian group with $B \oplus A \cong A$, then $B \cong 0$.
Hint: Arguably it is easiest to start out with the special case that $A = \mathbb{Z}$.

Exercise 11.3. Let M be a connected manifold, let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be points on M. Does there exist an action by a free group F on M such that for any $i \in \{1, \ldots, n\}$ there exists a $g \in F$ with $g \cdot P_i = Q_i$?

Exercise 11.4. Show that abelian groups and finite groups are indecomposable.

Exercise 11.5. Let $p, q, r \in \mathbb{N}_{\geq 2}$ be coprime. By the Chinese Remainder Theorem we know that $\mathbb{Z}_{pq} \times \mathbb{Z}_r \cong \mathbb{Z}_p \times \mathbb{Z}_{qr}$. Show, without making use of the Grushko Decomposition Theorem, that $\mathbb{Z}_{pq} * \mathbb{Z}_r$ is not isomorphic to $\mathbb{Z}_p * \mathbb{Z}_{qr}$.
Perhaps it is notationally easier to write $A = \{e, a, \ldots, a^{pq-1}\}$, $B = \{e, b, \ldots, b^{r-1}\}$, $C = \{e, c, \ldots, c^{p-1}\}$ and $D = \{e, d, \ldots, d^{qr-1}\}$, where A, B, C and D have the obvious multiplicative group structure. Then the problem is to show that $A * B$ is not isomorphic to $C * D$.

Exercise 11.6. Let S and T be sets. If there exists a monomorphism $\varphi\colon \langle S \rangle \to \langle T \rangle$, does it follow that $\#S \leq \#T$?

Exercise 11.7. A group G is called *locally free* if every finitely generated subgroup is free. Is every locally free group also a free group?

12. The basic Seifert-van Kampen Theorem

We are still lacking the computation of fundamental groups of even fairly simple topological spaces. Now we will take a new approach to calculating fundamental groups. Namely in the following two chapters we will formulate and prove results which allow us to calculate the fundamental group of a given topological space from the fundamental groups of suitable subspaces.

12.1. **The formulation of the basic Seifert–van Kampen Theorem.** The idea of the Seifert-van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X.

Let X be a topological space and let $U, V \subset X$ be two subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d, "i"] \\ V \arrow[r, "j"'] & X = U \cup V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d, "i_*"] \\ \pi_1(V, x_0) \arrow[r, "j_*"'] & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{tikzcd}$$

Our goal is to determine the fundamental group of the total space $X = U \cup V$ in terms of the fundamental groups of U, V, $U \cap V$ and the induced maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

We start out with the following lemma:

Lemma 12.1. *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is path-connected, then the homomorphism*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \ \to \ \pi_1(X, x_0),$$

which is induced by the homomorphisms $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and by Lemma 11.12 (2), is surjective.

Example.

(1) We consider the sphere

$$X \ = \ S^2 \ = \ \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > -\tfrac{1}{2}\}, \\ V &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z < \tfrac{1}{2}\}. \end{aligned}$$

We pick the base point $x_0 = (1, 0, 0) \subset U \cap V$. Here the intersection $U \cap V$ is path-connected, therefore we can apply Lemma 12.1. It says that we have an epimorphism

$$\pi_1(U, x_0) * \pi_1(V, x_0) \ \to \ \pi_1(S^2, x_0).$$

The subspaces U and V are homeomorphic to an open disk,[271] thus the fundamental groups are trivial. It follows that the trivial group[272] surjects onto $\pi_1(S^2, x_0)$, thus $\pi_1(S^2, x_0)$ itself is trivial. Precisely the same argument shows that $\pi_1(S^n) = 0$ for $n \geq 3$.

We had already obtained this result in Proposition 5.14. In fact, as we will see shortly, the proof of Lemma 12.1 has clear parallels to the proof of Proposition 5.14.

(2) We consider the sphere

$$X \;=\; S^1 \;=\; \{e^{it} \mid t \in \mathbb{R}\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &:= \{e^{it} \mid t \in (-\tfrac{\pi}{4}, \tfrac{5\pi}{4})\} \\ V &:= \{e^{it} \mid t \in (\tfrac{3\pi}{4}, \tfrac{9\pi}{4})\}. \end{aligned}$$

We pick the base point $x_0 = 1$. The subsets U and V are both homeomorphic to an open interval, hence $\pi_1(U, x_0) = \pi_1(V, x_0) = 0$, but of course we know from Corollary 7.19 that $\pi_1(X, x_0) \cong \mathbb{Z}$. Thus we see that $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is not an epimorphism. This is not a contradiction to Lemma 12.1, since in this example the intersection $U \cap V$ is not path-connected.

Now we turn to the proof of Lemma 12.1.

Proof of Lemma 12.1. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. It suffices to prove the following claim.[273]

Claim. Every loop $s\colon [0,1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the product of finitely many loops in (U, x_0) or in (V, x_0).

So let $s\colon [0,1] \to X = U \cup V$ be a loop in x_0. Since U and V are open we can use Corollary 1.106 to find a subdivision

$$0 \;=\; t_0 \;<\; t_1 \;<\; t_2 \;<\; \ldots \;<\; t_k \;<\; t_{k+1} \;=\; 1$$

of the interval $[0,1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V. The same argument as in the proof of Proposition 5.14 shows that we can assume that each $s(t_i)$ lies in $U \cap V$.[274] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected we can find for each $i \in \{1, \ldots, k\}$ a path p_i in $U \cap V$ which connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$.

[271] A homeomorphism is for example given by stereographic projection.

[272] Here we use the obvious fact that the free product of two times the trivial group is again the trivial group.

[273] Why does it suffice to prove this claim?

[274] Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1}, t_i])$ and $s([t_i, t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1}, t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision.

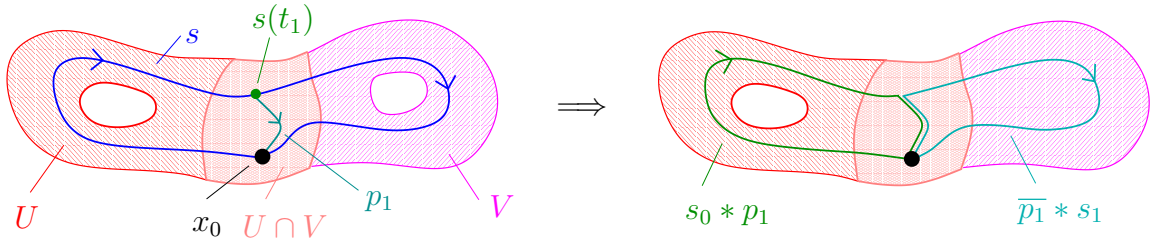

FIGURE 233. Schematic image for the proof of Lemma 12.1.

Now we have the following path-homotopies:

$$\begin{array}{ccccl}
 & & \text{by Lemma 5.4, since } s \text{ is a reparametrization of } s_0 * \cdots * s_k & & \text{since each } p_i * \overline{p_i} \text{ is null-homotopic} \\
 & & \downarrow & & \downarrow \\
s & \simeq & s_0 * \cdots * s_k & \simeq & s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k \\
 & & & \simeq & (\underbrace{s_0 * p_1}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_1} * s_1 * p_2}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_2} * s_2 * p_3}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).
\end{array}$$

Thus we have shown that s is path-homotopic to a product of loops in (U, x_0) and (V, x_0).[275] $\square$

Theorem 12.2. (Seifert-van Kampen)[276] *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is simply connected, then the map*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and Lemma 11.12 (2), is an isomorphism.

Remark. One might wonder whether the hypothesis that $U \cap V$ is simply connected is really necessary. But this is indeed the case. For example we could take $X = S^1 \times [-1, 1]$ with $U = S^1 \times [-1, \frac{1}{2})$ and $V = S^1 \times (-\frac{1}{2}, 1]$. In this case we have $\pi_1(X) \cong \mathbb{Z}$ which is not isomorphic to the non-abelian group $\pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$.

We postpone the proof of the Seifert-van Kampen Theorem 12.2 to the next section and we first discuss several examples and applications of the theorem.

[275]The proof of Lemma 12.1 is arguably easier to follow than the proof of Proposition 5.14. The reason is that in the special case Proposition 5.14 we introduced more notation, e.g. North Pole and South Pole, than was perhaps strictly necessary. All we really needed was that we could cover S^2 by two open simply connected subsets U and V such that $U \cap V$ is path-connected. The precise choice of U and V is irrelevant.

[276]The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be two points. Recall that on page 315 we defined the wedge $A \vee B$ of A and B as

$$A \vee B \ := \ A \sqcup B \ / \ a = b.$$

We illustrate this definition in Figure 234. In the following we will often identify A and B with their respective images in $A \vee B$. Another way of thinking about the wedge of A and B is to say that the wedge $X = A \vee B$ can be decomposed into two subsets A and B such that $A \cap B$ is a point. Since $A \cap B$ is a point it is in particular simply connected. Thus it sounds like one should be able to apply the Seifert-van Kampen Theorem 12.2. This is almost correct, except that in the Seifert-van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. We therefore need to introduce a somewhat technical definition.

FIGURE 234. The wedge of a circle with a torus.

Definition. We say a point x in a topological space X is *good*, if the following two conditions are satisfied:

(1) $\{x\}$ is a closed subset of X and
(2) there exists an open neighborhood U of x such that x is a deformation retract of U.

Remark.

(1) As we pointed out on page 55, if X is Hausdorff, then any finite subset, in particular any subset consisting of a single point, is closed. Thus the first condition in the definition of a good point is satisfied "for free".
(2) What we call a "good point" is sometimes called a "nondegenerate point", see e.g. [Le3, p. 212]. On the other hand, the term "nondegenerate point" is defined in a subtly different way, see e.g. [Sp, p. 380] or [Bre, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [Hat, p. 114].

Examples.

(1) Let X be an n-dimensional topological manifold (e.g. a sphere). We claim that every point is good. Since X is by definition Hausdorff we only need to show that Condition (2) is always satisfied. Thus let $x \in X$. Using a chart around x one easily sees that x admits an open neighborhood that is homeomorphic to the open ball B^n or to the "open half-ball" $\{(x_1, \ldots, x_n) \in B^n \,|\, x_n \geq 0\}$. In both cases x is evidently a deformation retract of that open neighborhood.

(2) We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, "since any open neighborhood of 0 has non-trivial fundamental group". In Exercise 12.1 we will provide a proper proof that $x = 0$ is not a good point of X.

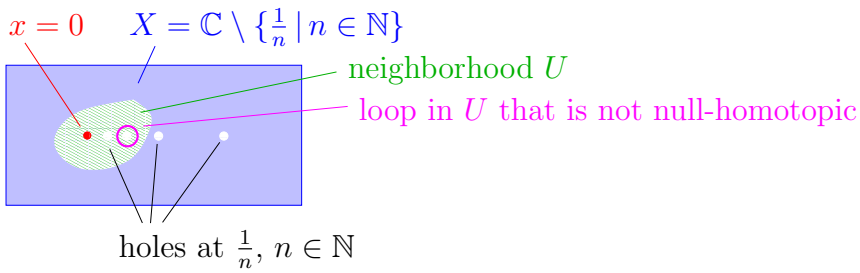

FIGURE 235. Example of a point that is not good.

Proposition 12.3. *Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. Then the inclusion maps induce an isomorphism*[277]

$$\pi_1(A, a) * \pi_1(B, b) \xrightarrow{\cong} \pi_1(A \vee B, a = b).$$

Remark. In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of Proposition 12.3.

Proof. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. By Lemma 9.17 we can view A and B as subsets of $X = A \vee B$ in the obvious way.

We pick an open neighborhood C in A of a that deformation retracts to a. Furthermore we pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$, each viewed as subsets of $X = A \vee B$.

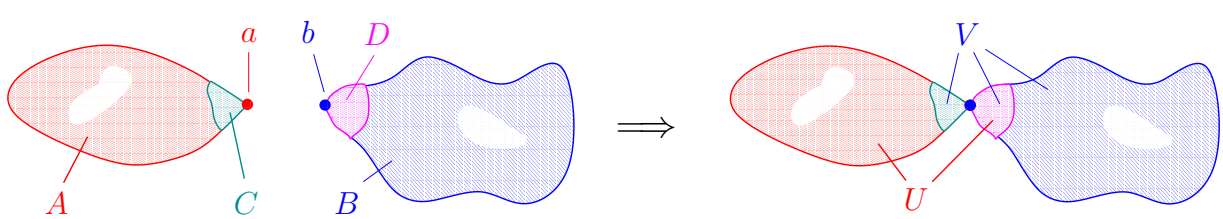

FIGURE 236. Schematic image for the proof of Proposition 12.3.

[277]More precisely, the two natural inclusions $A \to A \vee B$ and $B \to A \vee B$ induce homomorphisms $\pi_1(A, a) \to \pi_1(A \vee B, a = b)$ and $\pi_1(B, b) \to \pi_1(A \vee B, a = b)$. Combined with Lemma 11.12 (2) we obtain the homomorphism $\pi_1(A, a) * \pi_1(B, b) \to \pi_1(A \vee B, a = b)$.

Now we consider the following diagram

$$\begin{array}{ccc} \pi_1(A,a) * \pi_1(B,b) & \longrightarrow & \pi_1(X,x_0) \\ \downarrow & & \downarrow = \\ \pi_1(U,x_0) * \pi_1(V,x_0) & \longrightarrow & \pi_1(X,x_0). \end{array}$$

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) It follows from the definition of the quotient topology on $A \vee B = (A \sqcup B)/\sim$ that U and V are open subsets of X.
(3) It follows from Lemma 9.19 that A is a deformation retract of $U = A \vee C$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee D$.
(5) Furthermore, very similar to the argument in (3), we see that $A \cap B$ is a deformation retract of $U \cap V = C \vee D$. It follows from Lemma 9.10 and Proposition 9.12 (1) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert-van Kampen Theorem 12.2 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.
(7) It follows from (3), (4) and Lemma 9.10 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (6) together with Proposition 9.12 (1) that the two inclusion induced maps $\pi_1(A,a) \to \pi_1(U,a)$ and $\pi_1(B,b) \to \pi_1(V,b)$ are isomorphisms, which implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We deduce that the top horizontal map is, as desired, also an isomorphism. □

Examples.

(1) We consider again the wedge $S^1 \vee S^1$ with $a = (1,0)$ and $b = (-1,0)$. These points are simple.[278] By Proposition 7.18 we know that $\pi_1(S^1, a) = \langle x \rangle$[279] and

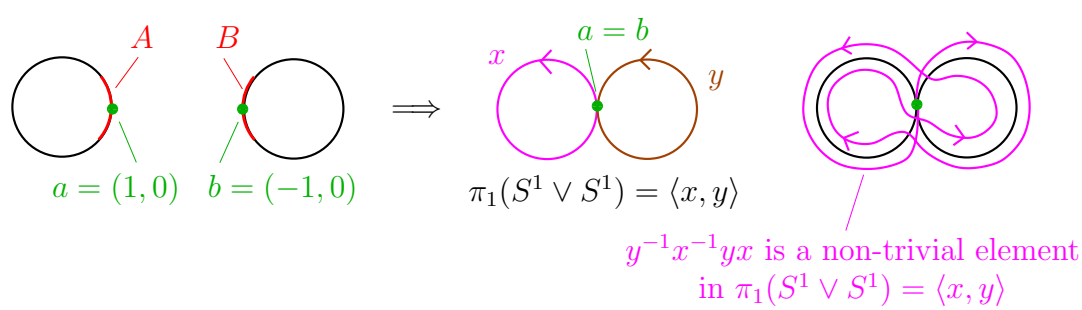

FIGURE 237. The wedge of two circles.

[278]This follows either from the general statement that any point on a topological manifold is simple, or more simply, by taking A and B to be small "intervals" around a and b.

$\pi_1(S^1, b) = \langle y \rangle$ where x and y are represented by loops that go "once around the circle", as illustrated in Figure 237. Proposition 12.3 says that the inclusion maps induce an isomorphism

$$\langle x, y \rangle \;=\; \langle x \rangle * \langle y \rangle \;=\; \pi_1(S^1, a) * \pi_1(S^1, b) \;\xrightarrow{\cong}\; \pi_1(S^1 \vee S^1, a = b).$$

(2) We consider the loop γ from page 227 in $\mathbb{C}\setminus\{\pm 1\}$ that is also illustrated in Figure 238 on the left. We had already pointed out on page 316 that there exists a deformation retraction f from $\mathbb{C} \setminus \{\pm 1\}$ to the wedge of two circles that is given by

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \mid |z+1| = 1\} \cup \{z \in \mathbb{C} \mid |z-1| = 1\}}_{\text{two circles that meet at the origin}}.$$

We then have $f_*([\gamma]) = [f \circ \gamma] = \langle y^{-1}x^{-1}yx \rangle \in \langle x, y \rangle = \pi_1(S^1 \vee S^1)$, which is non-trivial. Thus we see that $f \circ \gamma$ is not null-homotopic in $S^1 \vee S^1$ which implies that γ is not null-homotopic in $\mathbb{C} \setminus \{\pm 1\}$.[280]

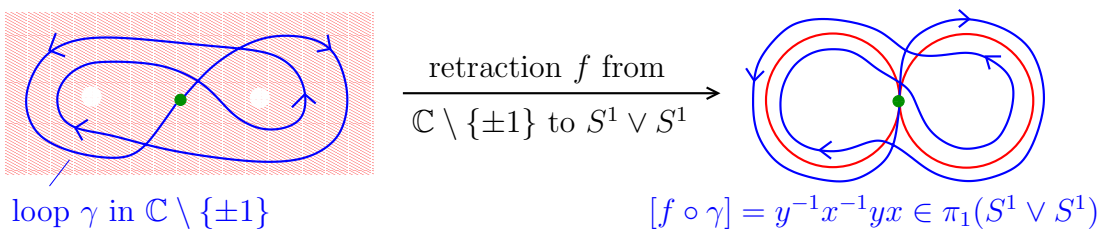

FIGURE 238.

Before we continue we condense the above example into a lemma.

Lemma 12.4. *Let $k \in \mathbb{N}_0$. The map*

$$\langle x_1, \dots, x_k \rangle \;\to\; \pi_1\Big(\bigvee_{i=1}^{k} S^1, *\Big)$$

that is given by

$$x_i \;\mapsto\; \begin{bmatrix} [0,1] & \to & S^1_i \\ t & \mapsto & e^{2\pi i t} \end{bmatrix}$$

is an isomorphism.

Convention. We refer to the images of $x_1, \dots, x_k$ under the isomorphism of Lemma 12.4 as the *standard generating set* of $\pi_1\Big(\bigvee_{i=1}^{k} S^1, *\Big)$.

[279]Recall that $\langle x \rangle$ denotes the free group generated by x, i.e. $\langle x \rangle = \{\dots, x^{-1}, e = x^0, x, x^2, \dots\}$.

[280]We could also have applied Seifert-van Kampen Theorem 12.2 to the covering $\mathbb{C} \setminus \{\pm 1\} = U \cup V$ where $U = \{z \in \mathbb{C} \setminus \{-1\} \mid \mathrm{Re}(z) < \frac{1}{2}\}$ and $V = \{z \in \mathbb{C} \setminus \{1\} \mid \mathrm{Re}(z) > -\frac{1}{2}\}$. Both sets U and V are open and they are homotopy equivalent to a circle.

Proof. Similar to the above example the lemma follows from applying Proposition 12.3 altogether $k-1$ times. □

Lemma 12.5.

(1) *Let G be a finite connected spatial graph with v vertices and e edges. Then*

$$\pi_1(G) \;\cong\; \textit{free group on } 1-\chi(G) = 1+e-v \textit{ generators.}$$

(2) *Two finite connected graphs are homotopy equivalent if and only if they have the same Euler characteristic.*

Proof.

(1) Let G be a finite connected spatial graph with v vertices and e edges. In Proposition 9.21 (3) we showed that the graph G is homotopy equivalent to the wedge of $1-\chi(G) = 1+e-v$ circles. Thus it follows from Propositions 9.12 (1) and 12.3 that $\pi_1(G)$ is isomorphic to the free group on $1-\chi(G) = 1+e-v$ generators.

(2) Let G and G' be two finite connected graphs. If $\chi(G) = \chi(G')$, then we already saw in Proposition 9.21 (5) that G and G' are homotopy equivalent. On the other hand, if $\chi(G) \neq \chi(G')$, then it follows from (1) together with Lemma 11.15 that G and G' are not homotopy equivalent. □

As a last example in this section we would like to determine the fundamental group of

$$X \;=\; \big(\mathbb{R} \sqcup (\mathbb{Z}\times S^2)\big) \,/\, n \sim (n,(0,0,-1)).$$

The space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. The space X is illustrated in Figure 239. Given $k \in \mathbb{N}$ we denote by X_k the

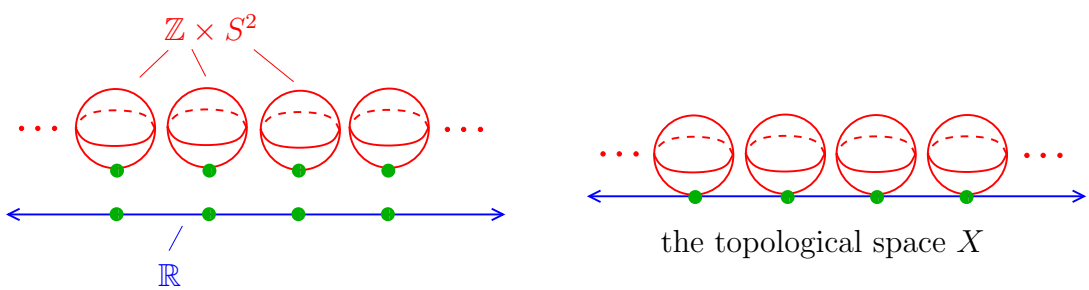

FIGURE 239.

topological space that is obtained by X by restricting oneself to the interval $(-k-\frac{1}{2}, k+\frac{1}{2})$, i.e.

$$X_k \;=\; \Big(\big(-k-\tfrac{1}{2}, k+\tfrac{1}{2}\big) \sqcup (\{-k,\dots,k-1,k\}\times S^2)\Big) \,/\, n \sim (n,(0,0,-1)).$$

Note that X_k can be viewed as the wedge of an open interval with $2k+1$ spheres. It follows immediately from Proposition 5.14, Proposition 12.3 and a straightforward induction argument that $\pi_1(X_k) = 0$ for all k. Furthermore one can easily convince oneself that the sets X_k are open. It now follows from the following lemma that $\pi_1(X)$ is also trivial.

Lemma 12.6. *Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:*

(1) *each X_k is open,*
(2) *each X_k is simply connected,*
(3) *the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,*
(4) *we have $\bigcup_{k\in\mathbb{N}} X_k = X$.*

Then $\pi_1(X, x_0) = 0$.

The proof of the lemma, which will make essential use of Lemma 1.38, is the content of Exercise 12.2.

12.2. **Proof of the Seifert-van Kampen Theorem 12.2.** After this long list of applications we finally provide the proof of the Seifert-van Kampen Theorem 12.2.

Proof. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$ such that $U \cap V$ is simply connected. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U, x_0)$ and $B := \pi_1(V, x_0)$. We need to show that the map

$$\Phi\colon A * B \;=\; \pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by the inclusions $U \to X$ and $V \to X$ is an isomorphism. Taking into account Lemma 12.1 it suffices to show that the map is injective. To prove injectivity it is enough to show that there exists a map $\Psi\colon \pi_1(X, x_0) \to A * B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*B}$. We will prove the existence of such a map Ψ in three steps:

(1) we first construct a map $\overline{\Psi}\colon \{\text{loops in } (X, x_0)\} \to A * B$,
(2) then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X, x_0) & \to & A * B \\ [g] & \mapsto & \overline{\Psi}(g) \end{array}$$

(3) and finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*B}$.

For the remainder of the proof we will identify $A * B$ with $S(A, B)/\sim$ using the isomorphism given by Proposition 11.16.

We start out with step (1). Let $g\colon [0, 1] \to X$ be a loop in (X, x_0). We apply Corollary 1.106 to the map $g\colon [0, 1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \dots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say a subinterval of $[0, 1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say a interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$. This definition is illustrated in Figure 240.

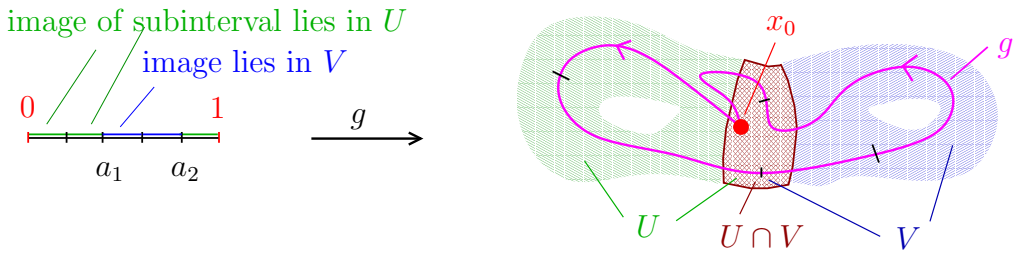

FIGURE 240.

(c) Given a coloring α we denote by $0 < a_1 < \cdots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. This definition is also illustrated in Figure 240. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.

(d) A *basing* for a coloring α is a collection of paths $p_1, \ldots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[281]

Now given a coloring α and a basing $p_1, \ldots, p_{k-1}$ we define[282] [283]

$$\overline{\Psi}(g) \;:=\; \Big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in } (U,x_0) \text{ or}\\ \text{loop in } (V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in } (U,x_0) \text{ or}\\ \text{loop in } (V,x_0)}}\big], \ldots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in } (U,x_0) \text{ or}\\ \text{loop in } (V,x_0)}}\big]\Big] \;\in\; S(A,B)/\sim = A * B.$$

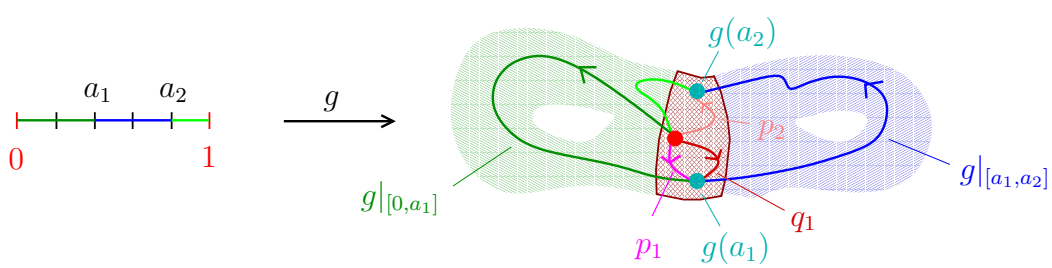

FIGURE 241.

Our next goal is to prove the following claim.

Claim. The definition of $\overline{\Psi}(g)$ does not depend on any of the choices made.

We need to prove the following three statements:

[281] Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

[282] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U, x_0) or (V, x_0), so each path represents an element in $\pi_1(U, x_0)$ or $\pi_1(V, x_0)$. Thus we obtain a sequence of elements in $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. The equivalence class of this sequence defines an element in $S(A, B)/\sim = A * B$.

[283] This definition is almost the same as in the proof of Lemma 12.1.

(a) given N and a coloring the definition does not depend on the choice of the basings,
(b) given N the definition does not depend on the choice of the coloring,
(c) the definition does not depend on the choice of N.

We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. First we pick a different path q_1 from x_0 to $g(a_1)$ instead of the original path p_1. (See Figure 241 for an illustration.) In this case we have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;= \\
&\qquad = \big[[g|_{[0,a_1]} * \overline{p}_1],\, [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\, [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\qquad \underset{\uparrow}{=} \big[[g|_{[0,a_1]} * \overline{p}_1 * q_1 * \overline{q}_1],\, [q_1 * \overline{q}_1 * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\, [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\qquad \text{since } q_1 * \overline{q}_1 \text{ is null-homotopic} \\
&\qquad = \big[[g|_{[0,a_1]} * \overline{q}_1],\, [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\, [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\qquad \uparrow \\
&\qquad \overline{p}_1 * q_1 \text{ and } \overline{q}_1 * p_1 \text{ are null-homotopic since } U \cap V \text{ is simply connected}^{284} \\
&\qquad = \text{ definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Precisely the same argument shows that the choices of the other paths p_i are also irrelevant. This concludes the proof of (a).

Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single i where the values for $\alpha(i)$ and $\beta(i)$ differ.

We consider the case, sketched in Figure 242, with $\alpha(i) =$ blue and $\beta(i) =$ green such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[285]

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition for α. We pick a basing $p_1, \dots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m + \frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually

[284]This is the only time that in the proof we use the hypothesis that $U \cap V$ is simply connected. In fact if one studies the argument carefully one sees that we only used that the maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ are trivial.

[285]In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it as an exercise to the reader to perform that argument.

in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \;= \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \dots\big] \\
&\quad \underset{\uparrow}{=} \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \dots\big] \\
&\quad \text{since } g_{[a_{m-1},a_m]} \simeq g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} \text{ and } g_{[a_m,a_m+\frac{1}{N}]} \simeq q * g_{[a_m+\frac{1}{N},a_{m+1}]} \\
&\quad = \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}.
\end{aligned}$$

(We refer to Figure 242 for an illustration of the proof.) This concludes the proof of (b).

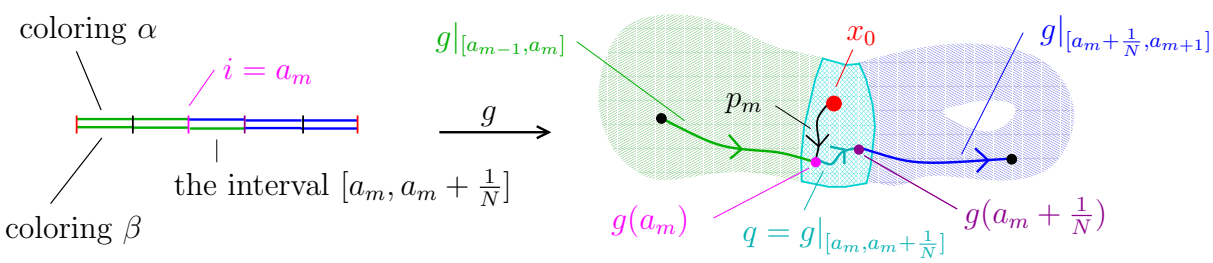

FIGURE 242.

It remains to prove (c). First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A * B$. Now if had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A * B$. This concludes the proof of (c). Summarizing, we are now done with the proof of the claim.

We turn to step (2) of the above program. We need to prove the following claim.

Claim. If $g, g' \colon [0,1] \to X$are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{aligned} H \colon [0,1] \times [0,1] &\to X \\ (t,s) &\mapsto H^s(t). \end{aligned}$$

from the loop g to the loop g'. We apply Corollary 1.106 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for each choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V. Given $j \in \{0, \frac{1}{N}, \dots, 1\}$ we now define

$$\begin{aligned} g(j) \colon [0,1] &\to X \\ t &\mapsto H^j(t) \end{aligned}$$

Note that each $g(j)$ is a loop in x_0 and that $g(0) = g$ and $g(1) = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(g(j)) = \overline{\Psi}(g(j + \frac{1}{N}))$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V. We pick a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, \frac{j+1}{N}]$ can be colored. Note that α also defines a coloring for the loops $g(j)$ and $g(j + \frac{1}{N})$.

We define $0 = a_0 < a_1 < \dots < \dots a_k = 1$ as above for the coloring α. Furthermore we pick a basing p_i, $i = 1, \dots, k-1$ for the loop $g(j)$. For $i = 0, \dots, k$ we write $q_i = H^{[j, j+\frac{1}{N}]}|_{a_i}$. Put differently, q_j connects a point on the loop $g(j)$ to the corresponding point on the loop $g(j + \frac{1}{N})$. Now we have

$$\begin{aligned}
\overline{\Psi}(g(j)) &= \overline{\Psi}(g(j)) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\
&= \big[\cdots * [p_m * \underbrace{H^j|_{[a_m, a_{m+1}]}}_{=g(j)|_{[a_m, a_{m+1}]}} * \overline{p}_{m+1}] * \dots\big] \\
&= \big[\cdots * [p_m * q_m * \overline{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} * \overline{q}_{m+1} * \overline{p}_{m+1}] * \dots\big] \\
&\uparrow \\
&\text{since each } q_m * \overline{q}_m \text{ is null-homotopic} \\
&= \big[\cdots * [p_m * q_m * \underbrace{H^{j+\frac{1}{N}}|_{[a_m, a_{m+1}]}}_{=g(j+\frac{1}{N})|_{[a_m, a_{m+1}]}} * \overline{q}_{m+1} * \overline{p}_{m+1}] * \dots\big] \\
&\uparrow \\
&\text{since } H \text{ restricts to path-homotopies from } \overline{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} \\
&\quad \text{to } H^{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \dots, k-1 \\
&= \overline{\Psi}(g(j + \tfrac{1}{N})) \text{ defined using } \alpha \text{ and the basing } p_1 * q_1, \dots, p_{k-1} * q_{k-1}.
\end{aligned}$$

This last part of the proof is illustrated in Figure 243. This concludes the proof of the claim and thus of part (2) of our three step program.

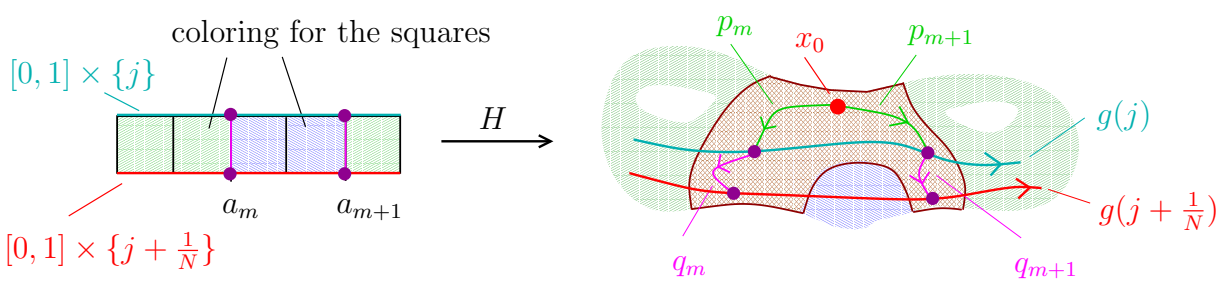

FIGURE 243.

Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & A*B \\ [g] & \mapsto & \big[\,\overline{\Psi}(g)\big]. \end{array}$$

It remains to prove the following claim.

Claim. We have $\Psi\circ\Phi = \mathrm{id}_{A*B}$.

So let $[y_1]\cdot\dots\cdot[y_k]\in A*B$ where $y_1,\dots,y_k$ are loops in (U,x_0) and (V,x_0). Then

$$\begin{array}{rcl} (\Psi\circ\Phi)([y_1]\cdot\dots\cdot[y_k]) & \overset{\text{definition of } \Phi\colon A*B=\pi_1(U,x_0)*\pi_1(V,x_0)\to\pi_1(X,x_0)}{\overset{\downarrow}{=}} & \Psi([y_1*\dots*y_k]) \;=\; \big[\,\overline{\Psi}(y_1*\dots*y_k)\big] \;\underset{\substack{\text{by definition of } \overline{\Psi}\text{, where we use } N=k \text{ and where} \\ \text{we can use the constant paths as basings since} \\ y_1,\dots,y_k \text{ are already loops in } (U,x_0) \text{ and } (V,x_0)}}{\underset{\uparrow}{=}}\; [y_1*\dots*y_k] \\ & = & [y_1]\cdot\dots\cdot[y_k]. \end{array}$$

This concludes the proof of the claim and thus also the proof of the Seifert-van Kampen Theorem 12.2 □

12.3. **More examples: surfaces and the connected sum of manifolds.** Given $g\in\mathbb{N}_0$ we denote by Σ_g the surface of genus g. In this section we first want to prove the following proposition.

Proposition 12.7. *For $g\neq h$ the surface Σ_g of genus g is not homeomorphic to the surface Σ_h of genus h.*

As on several earlier occasions we want to prove this result by considering what happens if we remove a point from the topological spaces.

Lemma 12.8. *Let $g\in\mathbb{N}_0$. We denote by Σ_g the surface of genus g.*

(1) *For any point P the complement $\Sigma_g\setminus\{P\}$ is homotopy equivalent to the wedge of $2g$ circles. In particular*

$$\pi_1(\Sigma_g\setminus\{P\}) \;\cong\; \langle x_1,y_1,\dots,x_g,y_g\rangle.$$

(2) *Let $f\colon\overline{B}^2\to\Sigma_g$ be an embedding. The complement $\Sigma_g\setminus f(B^2)$ is homotopy equivalent to the wedge of $2g$ circles. Furthermore, if $Q\in f(S^1)$ is a point, then there exists an isomorphism*

$$\pi_1(\Sigma_g\setminus f(B^2),Q) \;\overset{\cong}{\to}\; \langle x_1,y_1,\dots,x_g,y_g\rangle$$

with

$$[f\colon S^1\to\Sigma_g\setminus f(B^2)] \;\mapsto\; [x_1,y_1]\cdot\dots\cdot[x_g,y_g]$$

where $[x_i,y_i]:=x_iy_ix_i^{-1}y_i^{-1}$ is the commutator of x_i and y_i. In particular the map $f\colon S^1\to\Sigma_g\setminus f(B^2)$ represents a non-trivial element in $\pi_1(\Sigma_g\setminus f(B^2),Q)$.

FIGURE 244. Illustration of Lemma 12.8.

Proof of Lemma 12.8. We prove the lemma for the surface Σ of genus 2. The proof for $g \geq 3$ is almost identical. So let $\Sigma = E_8/\sim$ as sketched in Figure 245.

(1) We denote by O the origin in $E_8 \subset \mathbb{C}$. We first show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles.

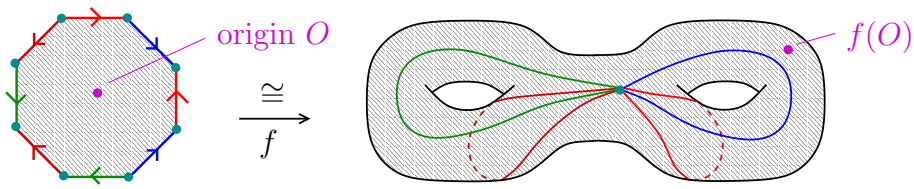

FIGURE 245.

Claim. There exists a deformation retraction from $\Sigma \setminus \{O\} = (E_8 \setminus \{O\})/\sim$ to $\partial E_8/\sim$.

First we consider the map[286][287]

$$\begin{array}{rcl} r\colon E_8 \setminus \{O\} & \to & \partial E_8 \\ re^{i\varphi} & \mapsto & \text{unique point } se^{i\varphi} \text{ on } \partial E_8. \end{array}$$

This map "projects points radially outward towards the boundary". The deformation retraction is now given by "pushing outward radially", i.e. it is given by the map

$$\begin{array}{rcl} ((E_8 \setminus \{O\})/\sim) \times [0,1] & \mapsto & (E_8 \setminus \{O\})/\sim \\ ([P], t) & \mapsto & [P \cdot (1-t) + r(P) \cdot t]. \end{array}$$

This concludes the proof of the claim.

Claim. The space $\partial E_8/\sim$ is homeomorphic to the wedge of 4 circles.

A homeomorphism is given by sending the point on $\partial E_8/\sim$ that corresponds to the vertices to the common point of the four circles, and by sending the edges that get

[286]Note that this map cannot be extended continuously to O.

[287]Here we use polar coordinates, i.e. r, s are positive real numbers.

identified to the same circle. We refer to Figure 246 for an illustration. We leave it as an exercise to write down the proof in detail.

The two claims put together show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles. We now obtain that

$$\pi_1(\Sigma_2 \setminus \{P\}) \underset{\substack{\uparrow \\ \text{Proposition 9.12 (1)}}}{\cong} \pi_1(\text{wedge of 4 circles}) \underset{\substack{\uparrow \\ \text{by Lemma 12.4}}}{\cong} \langle x_1, y_1, x_2, y_2 \rangle.$$

Now let P be some other point on Σ. By Proposition 10.23 there exists a homeomorphism $\Phi\colon \Sigma \to \Sigma$ with $\Phi(P) = O$.[288] This shows that $\Sigma \setminus \{P\}$ is homeomorphic to $\Sigma \setminus \{O\}$. We can thus appeal to the above calculation to get the desired result.

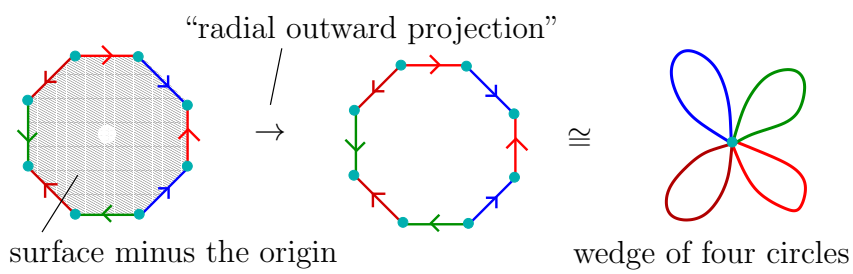

FIGURE 246.

(2) The proof of this statement is very similar to the proof of the first statement. First remove an open disk centered at O and we perform almost the same calculation as above. One easily sees that the boundary curve corresponds to the product of commutators. It follows from Theorem 10.28 that the same calculation also works for any other disk. We leave the details to the reader. □

Now we can turn to the proof of Proposition 12.7.

Proof of Proposition 12.7. Let S and T be two surfaces of genus g and h. Suppose that there exists a homeomorphism $f\colon S \to T$. Pick a point $P \in S$. We write $Q = f(P)$. The homeomorphism f restricts to a homeomorphism $f\colon S \setminus \{P\} \to T \setminus \{Q\}$. It now follows that

$$\langle x_1, \ldots, x_{2g} \rangle \underset{\substack{\uparrow \\ \text{Lemma 12.8 (1)}}}{\cong} \pi_1(S \setminus \{P\}) \xrightarrow[f_*]{\cong} \pi_1(T \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{Lemma 12.8 (1)}}}{\cong} \langle y_1, \ldots, y_{2h} \rangle.$$

We obtain from Lemma 11.15 that $g = h$. □

For many applications the following theorem, which is a variation on the Seifert-van Kampen Theorem 12.9, is very useful.

[288] Note that here we use Proposition 3.13 which says that Σ is a manifold with empty boundary.

Theorem 12.9. (Seifert-van Kampen Theorem for manifolds) *Let M be an m-dimensional manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:*

1. *$M = R \cup S$,*
2. *$R \cap S$ is a component of ∂R and it is a component of ∂S,*
3. *$R \cap S$ is non-empty and simply connected,*
4. *R and S are closed subsets of M. (Note that by Lemma 1.18 (2) this condition is satisfied if R and S are compact.)*

Then for any base point $x_0 \in R \cap S$ the inclusion induced map

$$\pi_1(R, x_0) * \pi_1(S, x_0) \ \to \ \pi_1(M, x_0)$$

is an isomorphism.

Example. Let $n \geq 3$ and let $S^n_{\geq 0} := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ be the upper hemisphere and define similarly the lower hemisphere $S^n_{\leq 0}$. Then $S^n_{\geq 0}, S^n_{\leq 0}$ are submanifolds of S^n that satisfy the hypothesis of Theorem 12.9. We saw on page 73 that both hemispheres are homeomorphic to closed balls. In particular they are compact and thus closed subsets of S^n. Therefore we obtain once again that $\pi_1(S^n) \cong \pi_1(S^n_{\geq 0}) * \pi_1(S^n_{\leq 0}) = \{e\}$.

We will of course prove Theorem 12.9 by reducing it to Theorem 12.2. As we will see, the proof is very similar to the proof of Proposition 12.3 where we determined the fundamental group of a wedge of two spaces.

The proof of Theorem 12.9 relies on the following lemma which we will use on several occasions.

Lemma 12.10. $(*)$ *Let X be a topological space and let A and B be two subsets with $X = A \cup B$. If $A \cap B$ is a deformation retract of B and if A and B are closed subsets of X, then A is a deformation retract of X.*

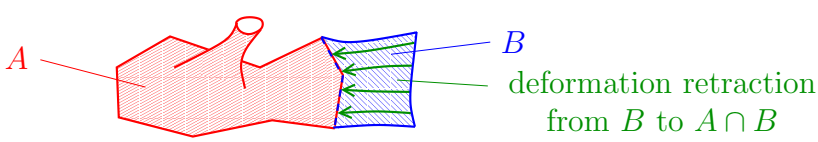

FIGURE 247. Illustration of Lemma 12.10.

Proof $(*)$. Lemma 12.10 can also be viewed as a consequence of Lemma 1.84 and the slightly scary Lemma 9.16. Since our situation is fairly straightforward we also give a direct, fairly elementary proof.

We pick a deformation retraction $F\colon B \times [0,1] \to B$ from B to $A \cap B$. We consider the map

$$\begin{array}{rcl} G\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in A, \\ F(x,t), & \text{if } x \in B \setminus (A \cap B). \end{cases} \end{array}$$

We claim that this map is the desired deformation retraction from X to A. Evidently all we need to do is to convince ourselves that G is continuous. It is clear that the restriction of G to $A \times [0,1]$ is continuous. Furthermore, since F is a deformation to $A \cap B$ we see that the restriction of G to $B \times [0,1]$ are continuous. By our hypothesis A and B are closed subsets of X. It follows from Lemma 1.51 that $A \times [0,1]$ and $B \times [0,1]$ are closed subsets of $X \times [0,1]$. Thus we conclude from Lemma 1.32 (2) that G is indeed continuous. $\square$

Remark. On two occasions we are going to struggle a lit bit to verify the hypothesis of Lemma 12.10 that A and B need to be closed. Thus it is good to remind ourselves of the fact that hypothesis are usually added not to vex the reader but since they are necessary. For example, let us consider the situation shown in Figure 248. Here $A \cap B$ is a deformation retract of B, but it should be pretty clear that A is not a deformation retract of $A \cup B = S^1$.

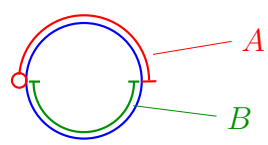

FIGURE 248.

Proof of Theorem 12.9 $(*)$. Let M be an m-dimensional manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:

(1) $M = R \cup S$,
(2) $R \cap S$ is a union of components of ∂R and it is a union of components of ∂S,
(3) $R \cap S$ is simply connected,
(4) R and S are closed subsets of M.

We pick a base point[289] $x_0 \in R \cap S$. Furthermore we pick maps $f\colon \partial R \times [0,1] \to R$ and $g\colon \partial S \times [0,1] \to S$ that are provided by the Collar Neighborhood Theorem 10.10. It follows from (2) that we can set

$$U \;:=\; R \cup \underbrace{g((R \cap S) \times [0,1))}_{\subset S \times [0,1)} \quad \text{and} \quad V \;:=\; S \cup \underbrace{f((R \cap S) \times [0,1))}_{\subset R \times [0,1)}.$$

We consider the following diagram

$$\begin{array}{ccc} \pi_1(R, x_0) * \pi_1(S, x_0) & \longrightarrow & \pi_1(M, x_0) \\ \downarrow & & \downarrow{=} \\ \pi_1(U, x_0) * \pi_1(V, x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of Proposition 12.3.

(1) Since the vertical maps are induced by inclusions we see that the diagram commutes.

[289]Recall that being simply connected entails being non-empty, hence such a point exists.

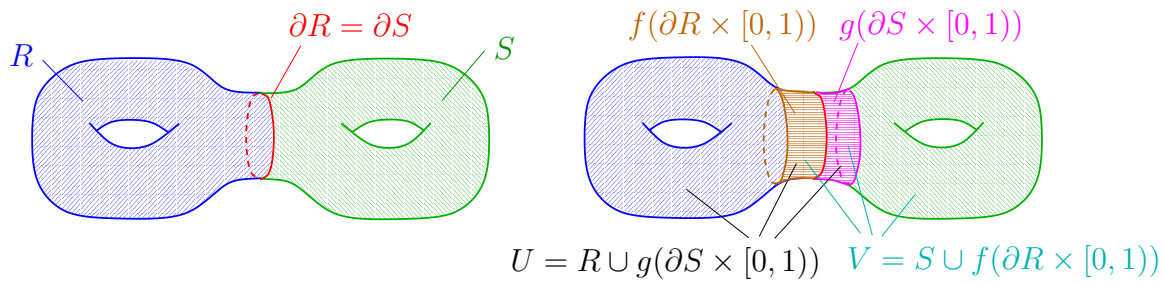

FIGURE 249. Illustration of the proof of Theorem 12.9.

(2) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (4) closed subsets of M we obtain from Lemma 1.7 (2a) it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R. Furthermore, $U \cap S = g((R \cap S) \times [0,1])$ is an open subset of S by (2) and Lemma 10.9.
(3) We want to use Lemma 12.10 to show that R is a deformation retract of U. We need to verify that the hypotheses of Lemma 12.10 are satisfied, i.e. we need to show that R and $g((R \cap S) \times [0,1))$ are closed subsets of U. By our hypothesis R and S are a closed subsets of M. It follows from Lemma 1.6 (2) that $R = R \cap U$ and $g((R \cap S) \times [0,1]) = S \cap U$ are closed subsets of U.
(4) The obvious analogues of (2) and (3) hold for V and S.
(5) We obtain from (2), (3), (4) and Proposition 9.12 that the vertical map on the left is an isomorphism.
(6) Similar to (3) one can show that $R \cap S$ is a deformation retract of $U \cap V$. It follows from (3), Lemma 9.10 and Proposition 9.12 (1) that $U \cap V$ is simply connected.
(7) It follows from the Seifert-van Kampen Theorem 12.2 that the bottom horizontal map of the above commutative diagram is an isomorphism.
(8) Since the vertical maps are isomorphisms we see that the top horizontal map is also an isomorphism. □

We conclude this section by considering the connected sum of two manifolds of dimension ≥ 3. More precisely, we have the following proposition:

Proposition 12.11. *Let $n \geq 3$ and let M and M' be two oriented connected non-empty n-dimensional manifolds. Then*[290]

$$\pi_1(M \# M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Examples.

[290]Note that by Lemma 10.29 we know that $M \# M'$ is also connected, thus we do not need to worry about base points.

(1) Let p, q be two coprime integers and let r, s be another two coprime integers. We can then consider the lens spaces $L(p,q)$ and $L(r,s)$. It follows from Proposition 12.11 and Corollary 7.19 that
$$\pi_1(L(p,q)\#L(r,s)) \;\cong\; \mathbb{Z}_p * \mathbb{Z}_r.$$
It follows for example from this calculation of fundamental groups, from Lemma 11.19 and from the Grushko Decomposition Theorem 11.20 that the two topological spaces $L(3,1)\#L(35,1)$ and $L(5,1)\#L(21,1)$ are not homeomorphic.

(2) Let $n \geq 3$ and let M and N be two oriented connected n-dimensional manifolds. We denote by $-N$ the same manifold as N but with the opposite orientation. It follows from Proposition 12.11 that $\pi_1(M\#N) \cong \pi_1(M\#(-N))$.

Proof of Proposition 12.11. Let M and M' be two oriented connected non-empty n-dimensional manifolds and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two embeddings where φ is orientation-preserving and φ' is orientation-reversing. Recall that the connected sum of M and M' is defined as
$$M\#M' \;:=\; \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big) \,/\, \varphi(P) = \varphi'(P) \text{ for } P \in S^{n-1}.$$
We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.

Claim. The inclusion induced maps $\pi_1(X,x_0) \to \pi_1(M,x_0)$ and $\pi_1(X',x_0) \to \pi_1(M',x_0)$ are isomorphisms.

We show that the inclusion induced map $\pi_1(X,x_0) \to \pi_1(M,x_0)$ is an isomorphism. We write $Y = \varphi(\overline{B}^n)$. It follows from Proposition 10.1 that X and Y are n-dimensional submanifolds of M. Note that $X \cup Y = M$ and that $X \cap Y = \partial Y \cong S^{n-1}$. Since $n \geq 3$ we obtain from Proposition 5.14 that $X \cap Y$ is simply connected. We obtain that
$$\pi_1(X,x_0) \underset{\substack{\uparrow \\ \text{since } \pi_1(\overline{B}^n,x_0)=\{e\}}}{\xrightarrow{\cong}} \pi_1(X,x_0) * \pi_1\big(\overline{B}^n,x_0\big) \underset{\substack{\uparrow \\ \text{Theorem 12.9}}}{\xrightarrow{\cong}} \pi_1(M).$$
The same way we can show that the inclusion induced map $\pi_1(X',x_0) \to \pi_1(M',x_0)$ is an isomorphism. This concludes the proof the claim.

Now we see that[291]
$$\pi_1(M\#M,x_0) \underset{\substack{\uparrow \\ \text{by Theorem 12.9, since} \\ X\cap X' \cong S^{n-1} \text{ is simply connected}}}{\xleftarrow{\cong}} \pi_1(X,x_0) * \pi_1(X',x_0) \underset{\substack{\uparrow \\ \text{above claim}}}{\xrightarrow{\cong}} \pi_1(M,x_0) * \pi_1(M',x_0).$$
□

Remark. We cannot apply the Seifert-van Kampen Theorem 12.2 to the connected sum of two surfaces, since in this case the fundamental group of the overlap $R \cap R'$ in the proof of Proposition 12.11 is not simply connected. In particular we cannot use this approach

[291] Here the directions of the arrow indicate in which direction the obvious inclusion induced maps point.

to determine the fundamental group of the surface of genus 2, which by Figure 224 is the connected sum of two tori. Our next goal is to formulate and prove a generalization of the Seifert-van Kampen Theorem 12.2 which will allow us to determine the fundamental group of the surface of genus g for $g \geq 2$.

Exercises for Chapter 12.

Exercise 12.1. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. Show that the point $x = 0$ is not good, i.e. show that there does not exist an open neighborhood U of $x = 0$ such that $x = 0$ is a deformation retract of U.

Exercise 12.2.

(a) Prove Lemma 12.6.
Hint: use Lemma 1.38.
(b) Does the conclusion of Lemma 12.6 also hold if the sets X_k are not open?

Exercise 12.3. Let M be a *topological* manifold of dimension $n \geq 3$. Let P be a point and let $x_0 \in M \setminus \{P\}$ be a base point. Show that the inclusion induced map

$$\pi_1(M \setminus \{P\}, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

Exercise 12.4.

(a) How can one hang a painting with two nails provided, such that if one removes either of the two nails, the painting falls down?
(b) What is a solution to (a) if we use three nails instead of two nails?

Hint: What does this have to do with free groups?

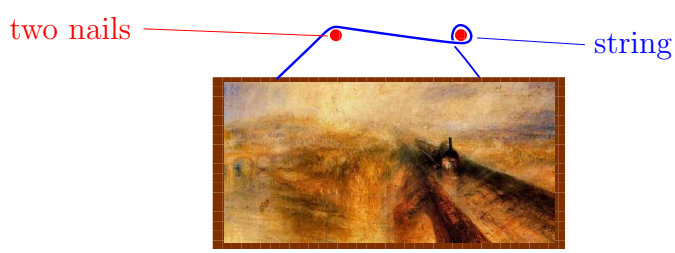

FIGURE 250. Illustration of Exercise 12.4.

Exercise 12.5. Show that condition (4) in the statement of the Seifert-van Kampen Theorem 12.9 for manifolds cannot be dropped. More precisely, give an example of an m-dimensional manifold and m-dimensional submanifolds R and S such that conditions (1), (2) and (3) are satisfied, but such that $\pi_1(R, x_0) * \pi_1(S, x_0)$ is not isomorphic to $\pi_1(M, x_0)$. Hint: The dimension m does not have to be taken particularly large.

Exercise 12.6.

(a) Let $n \geq 3$ and furthermore let $K \subset \mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $x_0 \in \mathbb{R}^n \setminus K$. Show that the inclusion induced map $\pi_1(\mathbb{R}^n \setminus K, x_0) \to \pi_1(S^n \setminus K, x_0)$ is an isomorphism.

(b) Does the conclusion of (a) also hold for $n = 2$?

(c) Does the conclusion of (c) also hold for non-compact subsets of $\mathbb{R}^n$?

Exercise 12.7. We consider the two topological spaces

$$X = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, 0) \,|\, n \in \mathbb{N}\} \text{ and } \quad \text{ and } Y = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, y) \,|\, n \in \mathbb{N}, y \in [-1, 1]\}.$$

We write $x_0 = (0, 0)$. We refer to Figure 251 for an illustration.

(a) Show that $\pi_1(X, x_0)$ is uncountable.
Hint: For $n \in \mathbb{N}$ we write $Z_n := \mathbb{R}^2 \setminus \{\frac{1}{n}, 0\}$ and we denote by $f_n \colon X \to Z_n$ the inclusion map. Use the induced maps $f_{n*} \colon \pi_1(X, x_0) \to \pi_1(Z_n, x_0) \cong \mathbb{Z}$ to find a homomorphism from $\pi_1(X, x_0)$ to a group such that the image is uncountable.

(b) Show that $\pi_1(Y, x_0)$ is countable.
Hint: Use Exercise 1.15 to find a monomorphism from $\pi_1(Y, x_0)$ into a countable group.

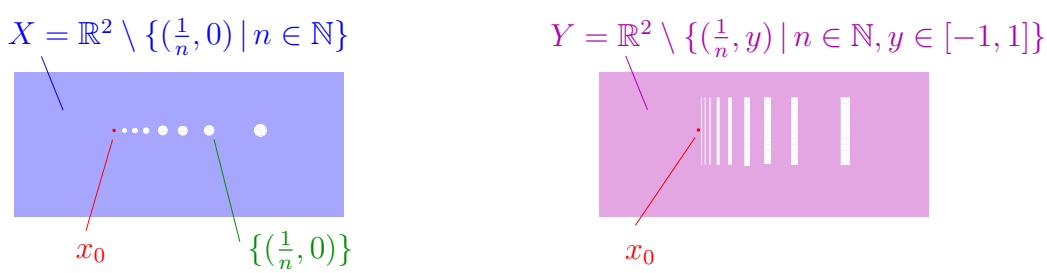

FIGURE 251.

13. Presentations of groups and amalgamated products

We have now made some progress, for example we have shown that surfaces of different genus are not homeomorphic. But this proof is perhaps slightly unsatisfactory in so far as we used the trick of removing a point, instead of dealing directly with the fundamental groups of surfaces.

In the next section we will cover the generalized Seifert-van Kampen Theorem. It will allow us to compute many more fundamental groups. In particular we will determine the fundamental groups of surfaces and knot complements.

Before we can state the generalized Seifert-van Kampen Theorem we need to introduce more concepts from group theory. We will do so in this chapter.

13.1. **Basic definitions in group theory.** Let G be a group. Recall that a subgroup H is called *normal* if for any $g \in G$ we have $g^{-1}Hg = H$. If $H \subset G$ is normal, then we often write $H \lhd G$. If H_i, $i \in I$ is a family of normal subgroups of G, then it is straightforward to see that the intersection $\bigcap_{i\in I} H_i$ is again a normal subgroup of G.

Let H be a subgroup of G. The subgroup H defines an equivalence relation on G by defining $g \sim \widetilde{g}$ if there exists an $h \in H$ with $gh = \widetilde{g}$. Put differently, $g \sim \widetilde{g}$ if $\widetilde{g}^{-1}g \in H$. The equivalence classes are precisely the sets of the form $gH = \{gh \mid h \in H\}$ for some $g \in G$. We denote the set of equivalence classes by G/H.

Remark. Let H be a normal subgroup of G.

(1) For any $h \in H$ and $a, b \in G$ we have

$$ahbH \;=\; ab \cdot \underbrace{b^{-1}hb}_{\text{in } H \text{ since } H \text{ is normal}} H \;=\; abH.$$

(2) If $aH = bH$, then for any $c, d \in G$ we have

$$cadH \;=\; cb\underbrace{(b^{-1}a)}_{\in H}dH \;\underset{\substack{\uparrow \\ \text{above remark (1)}}}{=}\; cbdH.$$

Now we will see that the group structure on G descends to a group structure on G/H. More precisely, given aH and bH we set

$$aH \cdot bH \;:=\; abH.$$

It follows immediately from the above remark (1) that this definition does not depend on the representatives a and b. It is now straightforward to see that this defines indeed a group structure on G/H. The following lemma plays the same role as Lemma 1.66 does for topological spaces.

Lemma 13.1. *Let G be a group and let H be a normal subgroup. Let $\varphi\colon G \to \Gamma$ be a group homomorphism with the property that $H \subset \ker(\varphi)$. Then there exists a unique*

homomorphism $\psi\colon G/H \to \Gamma$ *such that the following diagram commutes*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/H \arrow[d, "\psi"] \\ & \Gamma. \end{tikzcd}$$

Proof. For $gH \in G/H$ we put $\psi(gH) := \varphi(g)$. Since $H \subset \ker(\varphi)$ this definition is independent of the choice of the representatives g.[292] It is straightforward to verify that ψ is a homomorphism and that it is uniquely determined by the desired property. □

13.2. **Presentations of groups.**

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \textit{ generated by } A \;:=\; \langle A \rangle \;:=\; \bigcap_{\substack{A \subset H \subset G \\ H \text{ subgroup}}} H,$$

i.e. $\langle A \rangle$ is the intersection of all subgroups H of G which contain A. This is again a subgroup of G, namely the smallest[293]subgroup of G which contains A. Similarly we define

$$\text{subgroup of } G \textit{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{A \subset H \lhd G} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all *normal* subgroups H of G which contain A. This is again a normal subgroup of G, namely the smallest normal subgroup of G which contains A.

The following lemma follows immediately from the definitions and from the fact that in an abelian group every subgroup is normal.

Lemma 13.2. *Let G be a group and let $A \subset G$ be a subset.*

(1) *If A is a subgroup, then $\langle A \rangle = A$.*
(2) *If A is a normal subgroup, then $\langle\langle A \rangle\rangle = A$.*
(3) *If G is abelian, then $\langle\langle A \rangle\rangle = \langle A \rangle$.*
(4) *For any $B \subset G$ with $A \subset G \subset \langle\langle A \rangle\rangle$ we have $\langle\langle B \rangle\rangle = \langle\langle A \rangle\rangle$.*
(5) *Suppose that for each $a \in A$ we are given $g_a \in A$. Then $\langle\langle \{g_a a g_a^{-1}\}_{a \in A} \rangle\rangle = \langle\langle A \rangle\rangle$.*

The following lemma gives a more explicit description of $\langle A \rangle$ and $\langle\langle A \rangle\rangle$.

Lemma 13.3. *Let G be a group and let $A \subset G$ be a subset.*

(1) *Then $\langle A \rangle$ equals the set of all elements of G which can be written as products of elements in A and their inverses, i.e.*

$$\langle A \rangle \;=\; \big\{ a_1^{\epsilon_1} \cdot \dots \cdot a_k^{\epsilon_k} \mid a_1, \dots, a_k \in A \textit{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \big\}.$$

[292]Indeed, if $gH = \tilde{g}H$, then $g = \tilde{g}h$ for some $h \in H$, therefore $\varphi(g) = \varphi(\tilde{g}h) = \varphi(\tilde{g}) \cdot \varphi(h) = \varphi(\tilde{g})$.

[293]Here "smallest" means that if B is another subgroup that contains A, then $A \subset B$. This statement is obvious, since B appears in the definition of $\langle A \rangle$ as one of the H's.

(2) *The subset* $\langle\langle A\rangle\rangle$ *equals the set of all elements of* G *which can be written as products of conjugates of elements in* A *and their inverses, i.e.*

$$\langle\langle A\rangle\rangle = \big\{g_1 a_1^{\epsilon_1} g_1^{-1} \dots g_k a_k^{\epsilon_k} g_k^{-1} \mid a_1, \dots, a_k \in A, g_1 \dots, g_k \in G \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\}\big\}.$$

Example. On page 265 we introduced a subgroup of the group H of all self-homeomorphisms of $\mathbb{R}^2$ that was generated by two given self-homeomorphisms A and B. The lemma shows that the definition of "being generated" by a subset used on page 265 agrees with the above more formal definition.

Proof. We only prove the second statement. The proof of the first statement is very similar. So let G be a group and let $A \subset G$ be a subset. The subgroup $\langle\langle A\rangle\rangle$ contains all elements of A and their inverses. The normal subgroup $\langle\langle A\rangle\rangle$ then contains also all elements of the form $ga^{\pm 1}g^{-1}, g \in G, a \in A$. Since $\langle\langle A\rangle\rangle$ is a subgroup it is also closed under multiplication, i.e. it contains products of such elements. It follows that $\langle\langle A\rangle\rangle$ contains the right-hand side of (2). Conversely it is straightforward to see that the right-hand side is indeed a normal[294] subgroup of G that contains A, hence it is contained in the left-hand side of (2). □

Lemma 13.4. *Let* G *be a group and let* $A \subset G$ *be a subset. Let* $\alpha\colon G \to \Gamma$ *be a group homomorphism such that* $\alpha(a) = e \in \Gamma$ *for all* $a \in A$. *Then there exists a unique group homomorphism* $\beta\colon G/\langle\langle A\rangle\rangle \to \Gamma$ *such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\alpha"'] & G/\langle\langle A\rangle\rangle \arrow[d, "\beta"] \\ & \Gamma. \end{tikzcd}$$

Proof. Let G be a group and let $A \subset G$ be a subset. Furthermore let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup it follows from the definition of $\langle\langle A\rangle\rangle$ that $\langle\langle A\rangle\rangle \subset \ker(\alpha)$. The proposition now follows immediately from Lemma 13.1. □

The following definition will subsequently play an important role:

Definition. Let be X be a set and let R be a subset of the free group $\langle X\rangle$ generated by the set X. We define

$$\langle X \mid R\rangle \;:=\; \langle X\rangle \,/\, \langle\langle R\rangle\rangle.$$

We refer to the elements of X as *generators of* $\langle X \mid R\rangle$ and we refer to the elements of R as the *relators of* $\langle X \mid R\rangle$.

Examples.

(1) For any $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k\rangle \;=\; \langle g\rangle/\langle\langle g^k\rangle\rangle \;\underset{\substack{\uparrow \\ \text{since } \langle g\rangle \text{ is abelian it follows from Lemma 13.2 that } \langle\langle g^k\rangle\rangle = \langle g^k\rangle}}{=}\; \langle g\rangle/\langle g^k\rangle \;\cong \mathbb{Z}_k.$$

[294] Why is the right-hand side normal?

(2) The group $\langle x_1, \dots, x_k \mid \rangle$ which has no relators is just the free group on the generators $x_1, \dots, x_k$.

(3) We consider the group

$$\pi \;=\; \langle x, y \mid [x, y] \rangle$$

where $[x, y] := xyx^{-1}y^{-1}$ is the commutator of x and y. What can we say about this group? Modulo $\langle\langle [x, y] \rangle\rangle$ we have[295]

$$\underset{\substack{\uparrow \\ \text{Remark (1) on page 406}}}{yx \;=\; xyx^{-1}y^{-1} \cdot yx \;=\; xy} \qquad \text{and} \qquad \underset{\substack{\uparrow \\ \text{calculation on the left and Remark (2) on page 406}}}{xy^{-1} \;=\; y^{-1} \cdot yx \cdot y^{-1} \;=\; y^{-1} \cdot xy \cdot y^{-1} \;=\; y^{-1}x.}$$

Similarly one can show that the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$ hold in $\pi = \langle x, y \mid [x, y] \rangle$. This shows that in π the elements $x^{\pm 1}$ and $y^{\pm 1}$ commute. This means that in π an arbitrary product of powers of x and y can be reordered to equal a product of the $x^m y^n$. For example in π we have the equality

$$x^3y^2x^{-1}y \;= x^3y \cdot yx^{-1} \cdot y \;\underset{\substack{\uparrow \\ \text{since } yx^{-1} = x^{-1}y}}{=}\; x^3y \cdot x^{-1}y \cdot y = x^3 \cdot yx^{-1} \cdot y^2 \;\underset{\substack{\uparrow \\ \text{since } yx^{-1} = x^{-1}y}}{=}\; x^3 \cdot x^{-1}y \cdot y^2 = x^2y^3.$$

By Lemma 11.13 there exists a unique homomorphism $\alpha\colon \langle x, y \rangle \to \mathbb{Z}^2$ which satisfies $\alpha(x) = (1, 0)$ and $\alpha(y) = (0, 1)$. Note that

$$\alpha([x, y]) \;=\; \alpha(xyx^{-1}y^{-1}) \;=\; \alpha(x) + \alpha(y) - \alpha(x) - \alpha(y) \;=\; 0.$$

Thus by Lemma 13.4 there exists a unique group homomorphism

$$\beta\colon \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \;\to\; \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ and $\beta(y) = (0, 1)$. This homomorphism is evidently an epimorphism. It is also a monomorphism, indeed:

(a) we had just seen in the previous example that any element in π is equivalent to an element of the form $x^m y^m$,

(b) furthermore $\beta(x^m y^n) = (m, n)$ is zero if and only if $x^m y^n = x^0 y^0$ is the trivial element in π.

Frequently we will write relators as equations, for example we often write

$$\langle x, y \mid xyx^{-1}y^{-1} \rangle \;=\; \langle x, y \mid xyx^{-1}y^{-1} = e \rangle \;=\; \langle x, y \mid xy = yx \rangle.$$

Because of this notation one often refers to a relator also as a relation.

(4) The same way as in the above example one can show that there exists an isomorphism

$$\beta\colon \langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } i, j \in \{1, \dots, m\} \rangle \;\xrightarrow{\cong}\; \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

[295] Here we follow the standard convention that in the notation we do not distinguish between an element in $\langle X \rangle$ and an element in $\langle X \mid R \rangle$, the same way as usually we do not distinguish in the notation between an element in $\mathbb{Z}$ and an element in $\mathbb{Z}_5$.

(5) We consider the following three self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{A}\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, -t). \end{array}$$

We already know the first two maps from page 265. We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. We are interested in the group G since $\mathbb{R}^2/G$ is the Klein bottle K and by Theorem 7.17 we have $\pi_1(K) \cong G$.

Note that $\widetilde{A} = B^{-1}A$. Let $X = \{x, y\}$ be the set that consists of two elements. By Lemma 11.13 there exists a unique homomorphism $\alpha\colon \langle x, y\rangle \to G$ with $\alpha(x) = \widetilde{A}$ and $\alpha(y) = B$. Now let R be the subset of $\langle X\rangle = \langle x, y\rangle$ that consists of the single element $yxyx^{-1}$. We want to consider $\alpha(yxyx^{-1}) \in G$, which is by definition a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. For any point $(s,t) \in \mathbb{R}^2$ this map is given by

$$\begin{aligned} \alpha(yxyx^{-1})(s,t) \;&=\; B\widetilde{A}B\widetilde{A}^{-1}(s,t) \;=\; B\widetilde{A}B(s-1,-t) \\ &=\; B\widetilde{A}(s-1,-t+1) \;=\; B(s,t-1) = (s,t). \end{aligned}$$

This shows that $\alpha(yxyx^{-1})$ is the identity map of $\mathbb{R}^2$, i.e. it is the trivial element in G. It follows from Lemma 13.4 that α descends to a homomorphism

$$\beta\colon \langle x, y\rangle / \langle\langle yxyx^{-1}\rangle\rangle = \langle x, y \mid yxyx^{-1}\rangle \;\to\; G.$$

This homomorphism is an epimorphism, since the image of β is the group generated by $\widetilde{A} = B^{-1}A$ and B, but this group is the same as the group generated by A and B, i.e. the image of β is G. Is the homomorphism β also a monomorphism? We will come back to this question later on.

We continue with the following definitions.

Definition.

(1) A *presentation* is a set X together with a subset $R \subset \langle X\rangle$. We call the elements of X the *generators* and we call the elements of R the *relators*[296] of the presentation.
(2) Let π be a group. A *presentation for* π is a presentation (X, R) together with an isomorphism
$$\pi \;\xrightarrow{\cong}\; \langle X \mid R\rangle.$$
Very often one just says $\langle X \mid R\rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.
(3) We say that a group π is *finitely generated*, if π admits a presentation with finitely many generators.[297]
(4) We say a group π is *finitely presented*, if π admits a presentation with finitely many generators and finitely many relators.

Now we give some examples of finitely generated and finitely presented groups.

[296]The relators are often also called relations.

[297]This definition of a finitely generated group is equivalent to the one we gave on page 378. Why is that?

Examples.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π. By Lemma 11.13 there exists a unique homomorphism $\alpha\colon \langle X\rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then

$$\langle X \mid R\rangle \;=\; \langle X\rangle/\langle\langle R\rangle\rangle \;=\; \langle X\rangle/\langle\langle \ker(\alpha)\rangle\rangle \;\underset{\substack{\uparrow \\ \text{by Lemma 13.2 since } R=\ker(\alpha) \text{ is a normal subgroup}}}{=}\; \langle X\rangle/\ker(\alpha) \;\xrightarrow[\alpha]{\cong}\; G.$$

(2) We had just seen on page 409 that

$$\mathbb{Z}^n \;\cong\; \langle x_1,\dots,x_n \mid [x_i,x_j] = e \text{ with } i,j \in \{1,\dots,n\}\rangle$$

which shows that $\mathbb{Z}^n$ is a finitely presented group.

(3) In fact every finitely generated abelian group is also finitely presented. Indeed, let A be a finitely generated abelian group. Then

$$A \;\underset{\substack{\uparrow \\ \text{Theorem 11.4} \\ \text{for some } a_i \in \mathbb{N}_0}}{\cong}\; \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i} \;\underset{\substack{\uparrow \\ \text{as in (2)}}}{=}\; \langle x_1,\dots,x_n \mid [x_i,x_j] = e \text{ for } i,j = 1,\dots,n, x_i^{a_i} = e \text{ for } i = 1,\dots,n\rangle.$$

(4) There exist groups which are finitely generated but not finitely presented. In fact there are countably many isomorphism types of finitely presented groups,[298] whereas there are uncountably many isomorphism types of finitely generated groups.[299] A proof for both statements is given in Chapter 14 of [Rob].

(5) We saw on page 378 that the fundamental group of the Klein bottle is finitely generated. But is it also finitely presented?

The following lemma shows that every finitely presented group admits infinitely many finite presentations.

Lemma 13.5. *Let $X = \{x_1,\dots,x_k\}$ be a finite set and furthermore let $r = \{r_1,\dots,r_l\}$ be a finite subset of $\langle X\rangle = \langle x_1,\dots,x_k\rangle$. Then for any word s in $r_1,\dots,r_l$, i.e. any element s of $\langle r_1,\dots,r_l\rangle$, we have the equality*[300]

$$\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \;=\; \langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle \tag{1}$$

and for any symbol x and any two words s,t in $x_1,\dots,x_k$ we have an isomorphism

$$\begin{array}{rcl} \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle & \xrightarrow{\cong} & \langle x_1,\dots,x_k,x \mid r_1,\dots,r_l,s\cdot x\cdot t\rangle. \\ x_i & \mapsto & x_i \end{array} \tag{2}$$

[298] That statement is pretty obvious.

[299] This statement is much less obvious. The idea is that in a free group $\langle x_1,\dots,x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1,r_2,\dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

[300] Note that these two presentations are different, since $\{r_1,\dots,r_l\}$ and $\{r_1,\dots,r_l,s\}$ are two different subsets of $\langle x_1,\dots,x_k\rangle$.

Proof.

(1) It is straightforward to see that $\langle\langle r_1, \dots, r_l \rangle\rangle$ and $\langle\langle r_1, \dots, r_l, s \rangle$ are precisely the same subsets of $\langle\langle x_1, \dots, x_k \rangle\rangle$. This shows that the two groups in (1) are identical.

(2) The fact that the map in (2) induces an isomorphism will be proved in Exercise 13.1. □

The above two methods, and their inverses, of obtaining new presentations out of a given one are called *Tietze transformations.*[301] Any two finite presentations for some given group are in fact related by a finite sequence of these two Tietze transformations. This fact is for example proved in Proposition II.2.2 of [LS].

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition.

Definition. Let $\{x_1, \dots, x_k\}$ be a set and let $r \in \langle X \rangle$. Furthermore let G be a group and let $g_1, \dots, g_k$. We say that $g_1, \dots, g_k$ *satisfy the relation* r if the homomorphism[302] $\langle x_1, \dots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \dots, k$ sends r to the trivial element.

Examples.

(1) We consider $\{x_1, x_2\}$ and the word $r = x_1 x_2 x_1^{-1} x_2^{-1}$. If a group is abelian, then any two elements g_1, g_2 satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.

(2) Let G be a finite group with d elements. We consider the set $\{x\}$. It follows from Lagrange's Theorem, see e.g. [KuS, p. 8], that every element in G satisfies the relation x^d.

Lemma 13.6. *Let π be a group with a finite presentation $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$. The map*

$$\begin{array}{rcl} \mathrm{Hom}(\pi, G) & \to & \{\textit{all } (g_1, \dots, g_k) \in G^k \textit{ that satisfy } r_1, \dots, r_l\} \\ f & \mapsto & (f(x_1), \dots, f(x_k)) \end{array}$$

is a bijection.

Remark. Note that if G is a finite group, then the right-hand side of Lemma 13.6 is a finite set. In practice it is usually straightforward, albeit tedious, to determine the right-hand side.

Proof. This lemma follows immediately from Lemma 13.4 and Lemma 11.13. □

The following lemma often works well in practice to show that two presentations correspond to isomorphic groups.

Lemma 13.7. *Let $X = \{x_1, \dots, x_k\}$ be a finite set and furthermore let $r = \{r_1, \dots, r_l\}$ be a finite subset of $\langle X \rangle = \langle x_1, \dots, x_k \rangle$. Suppose that we can write each $r_i(x_1, \dots, x_k)$ as a word $s_i(x_1, \dots, x_{k-1}, x_k \cdot w)$ where w is a word in $x_1, \dots, x_{k-1}$, then the homomorphism*

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; \langle x_1, \dots, x_{k-1}, t \,|\, s_1, \dots, s_l \rangle$$

[301] Heinrich Tietze (1880-1964) was an Austrian mathematician.

[302] The fact that there exists a unique such homomorphism is a consequence of Lemma 11.13.

that is induced by $x_i \mapsto x_i$, $i = 1, \dots, k-1$ and $x_k \mapsto tw^{-1}$ is an isomorphism.

Definition. We refer to the isomorphism of Lemma 13.7 as the *isomorphism given by the substitution $t = x_k \cdot w$.*

Example. A picky reader might decide that the formulation of Lemma 13.7 is not completely precise. But the following example should make the meaning clear:

$$\langle a, b \,|\, a^{-1}ba^{-1}bab^{-1} \rangle \;=\; \langle a, b \,|\, \underbrace{a^{-1}b \cdot a^{-1}b \cdot a \cdot (a^{-1}b)^{-1} \cdot a^{-1}}_{=a^{-1}ba^{-1}bab^{-1}} \rangle \;\underset{\substack{\uparrow \\ \text{substitution } c = a^{-1}b}}{=}\; \langle a, c \,|\, c \cdot c \cdot a \cdot c^{-1} \cdot a^{-1} \rangle.$$

Proof. It follows from Lemma 13.6 that there exists in fact a unique homomorphism

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; \langle x_1, \dots, x_{k-1}, t \,|\, s_1, \dots, s_l \rangle$$

that has the property that $x_i \mapsto x_i$, $i = 1, \dots, k-1$ and that $x_k \mapsto tw^{-1}$. Furthermore, we have a map from the right to left by the reverse substitution $t \mapsto x_k \cdot w$. It is straightforward to verify that these homomorphisms are inverses of one another, hence both are isomorphisms. □

We conclude this section with the following lemma.

Lemma 13.8. $(*)$ *Let H be a finitely presented group and let $\varphi\colon H \to G$ be a group homomorphism. If there exists a group homomorphism $\rho\colon G \to H$ such that $\rho \circ \varphi = \mathrm{id}_H$, then H is also finitely presented.*

Proof $(*)$. Let $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation for the group G. We write $F = \langle g_1, \dots, g_k \rangle$ and we denote by $p\colon F \to G$ the obvious epimorphism. For each $i \in \{1, \dots, k\}$ we pick $w_i \in F$ with $p(w_i) = (\varphi \circ \rho \circ p)(g_i) \in G$. It suffices to prove the following claim.

Claim. A finite presentation for H is given by $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1}w_1, \dots, g_k^{-1}w_k \rangle$.

Let L be the group that is defined by the finite presentation of the claim. We denote by $u\colon G \to L$ and $q\colon F \to L$ the obvious projections. Note that for each $j \in \{1, \dots, l\}$ we have $\rho(p(r_i)) = \rho(e) = e$ and for each $i \in \{1, \dots, k\}$ we have

$$\rho(p(g_i^{-1}w_i)) \;=\; \rho(p(g_i^{-1})) \cdot \rho(p(w_i)) \;\underset{\substack{\uparrow \\ \text{since } p(w_i) = (\varphi \circ \rho \circ p)(g_i) \text{ and since } \rho \circ \varphi = \mathrm{id}_H}}{=}\; \rho(p(g_i^{-1})) \cdot \rho(p(g_i)) \;=\; e.$$

It follows from Lemma 13.4 that the homomorphism $p \circ \varphi\colon F \to H$ factors through a homomorphism $v\colon L \to H$.

We want to show that $u \circ \varphi$ is an isomorphism. First note that

$$v \circ (u \circ \varphi) \;=\; (v \circ u) \circ \varphi \;\underset{\substack{\uparrow \\ \text{since } \rho = v \circ u}}{=}\; \rho \circ \varphi \;=\; \mathrm{id}_H\,.$$

By Lemma 11.17 it remains to show that $u \circ \varphi$ is an isomorphism. Note that for $i = 1, \dots, k$ we have

$$(u \circ \varphi)(\underbrace{(\rho \circ p)(w_i)}_{\in H}) \;\underset{\substack{\uparrow \\ \text{since } \varphi \circ \rho = \mathrm{id}_H}}{=}\; u(p(w_i)) \;=\; q(w_i) \;\underset{\substack{\uparrow \\ \text{since } q(w_i) = g_i \in L}}{=}\; g_i \;\in\; L.$$

Since $g_1, \dots, g_k$ is a generating set for L we see that $u \circ \varphi$ is indeed an epimorphism.

For convenience we summarize all objects in the following diagram

$$\begin{tikzcd}
H \arrow[r, "\varphi"] & G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle \arrow[l, bend right, "\rho"'] \arrow[d, "u"] & F = \langle g_1, \dots, g_k \rangle \arrow[l, "p"'] \arrow[dl, "q"] \\
& L := \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1} w_1, \dots, g_k^{-1} w_k \rangle. \arrow[ul, bend left, "v"] &
\end{tikzcd}$$

□

13.3. **The abelianization of a group.** The following definition should already be familiar from the algebra course.

Definition. Let π be a group. We refer to

$$[\pi, \pi] \;:=\; \langle\!\langle \{ \underbrace{[x, y]}_{= xyx^{-1}y^{-1}} \mid x, y \in \pi \} \rangle\!\rangle$$

as the *commutator subgroup* of π.

We start out with the following elementary lemma.

Lemma 13.9. *Let π be a group. Then the following hold:*

(1) *The group $\pi/[\pi, \pi]$ is abelian.*
(2) *We have*

$$[\pi, \pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \dots, x_n, y_n \in \pi \right\}.$$

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $\pi = \langle a, b, c, d \rangle$ is the free group on four generators a, b, c, d, then we have $[a, b] \cdot [c, d] \in [\pi, \pi]$, but it is not a commutator.[303]

Proof. Let π be a group.

(1) For any two elements $x, y \in \pi$ we have

$$xy\,[\pi, \pi] \;=\; yx \cdot x^{-1}y^{-1}xy\,[\pi, \pi] \;=\; yx \cdot \underbrace{[x^{-1}, y^{-1}]}_{\in [\pi, \pi]}\,[\pi, \pi] \;=\; yx\,[\pi, \pi].$$

This shows that $\pi/[\pi, \pi]$ is indeed abelian.

[303] Why not?

(2) Note that the inverse of a commutator is again a commutator, i.e. given $x, y \in \pi$ we have

$$[x,y]^{-1} \;=\; (xyx^{-1}y^{-1})^{-1} \;=\; yxy^{-1}x^{-1} \;=\; [y,x].$$

Furthermore a conjugate of a commutator is again a commutator. Indeed, given $x, y, g \in \pi$ we have

$$\begin{aligned} g[x,y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]. \end{aligned}$$

It follows from this discussion and from Lemma 13.3 that each element of the commutator subgroup can be written as a product of commutators, i.e.

$$[\pi,\pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in \pi \right\}.$$

□

Definition. Let π be a group. We call

$$\pi_{\mathrm{ab}} \;:=\; \pi/[\pi,\pi]$$

the *abelianization* of π.[304]

The following proposition summarizes some of the key properties of the abelianization of groups.

Proposition 13.10.

(1) *If π is abelian, then $\pi_{\mathrm{ab}} = \pi$.*

(2) *Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:*

$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, "\alpha"'] & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[d, "\beta"] \\ & H. \end{tikzcd}$$

In particular for any group π and any abelian group G the map

$$\begin{array}{ccc} \mathrm{Hom}(\pi_{\mathrm{ab}}, G) & \to & \mathrm{Hom}(\pi, G) \\ (\alpha\colon \pi_{\mathrm{ab}} \to G) & \mapsto & \big(\pi \to \pi_{\mathrm{ab}} \xrightarrow{\alpha} G\big) \end{array}$$

is a bijection.

(3) *Let G and H be two groups. There exists a unique isomorphism*

$$(G * H)_{\mathrm{ab}} \;\xrightarrow{\cong}\; G_{\mathrm{ab}} \times H_{\mathrm{ab}}$$

[304]The abelianization of a group π is sometimes also written as $H_1(\pi)$. The reason for that somewhat strange looking notation will become clear later on.

which makes the following diagram commute:

$$\begin{tikzcd} G * H \arrow[r, two heads] \arrow[d, two heads] & G \times H \arrow[d, two heads] \\ (G * H)_{\mathrm{ab}} \arrow[r, "\cong"] & G_{\mathrm{ab}} \times H_{\mathrm{ab}}, \end{tikzcd}$$

where the other three maps are the obvious projection maps.

(4) *Let $F = \langle x_1, \dots, x_m \rangle$ be the free group on m generators $x_1, \dots, x_m$. Then the map*[305]

$$\begin{array}{rcl} \Phi\colon F_{\mathrm{ab}} = F/[F,F] & \to & \mathbb{Z}^m \\ [x_i] & \mapsto & e_i := \textit{the } i\textit{-th standard basis vector} \end{array}$$

is an isomorphism.

(5) *Let $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a finitely presented group. We denote by*

$$\Phi\colon \langle x_1, \dots, x_m \rangle \to \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism

$$\Psi\colon \pi/[\pi,\pi] \xrightarrow{\cong} \mathbb{Z}^m/\langle \Phi(r_1), \dots, \Phi(r_n) \rangle$$

such that the following diagram commutes

$$\begin{tikzcd} \langle x_1, \dots, x_m \rangle \arrow[rr, two heads, "x_i \mapsto e_i", "\Phi"'] \arrow[d, two heads] & & \mathbb{Z}^m \arrow[dd, two heads] \\ \pi = \langle x_1, \dots, x_m \rangle / \langle\langle r_1, \dots, r_n \rangle\rangle \arrow[d, two heads] & & \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[rr, "\Psi", "\cong"'] & & \mathbb{Z}^m/\langle \Phi(r_1), \dots, \Phi(r_n) \rangle. \end{tikzcd}$$

(6) *We denote by $\mathcal{G}$ the category of groups and we denote by $\mathcal{A}$ the category of abelian groups. Then the maps*

$$G \mapsto G_{\mathrm{ab}} = G/[G,G]$$

and

$$(\varphi\colon G \to H) \mapsto \left(\begin{array}{rcl} \varphi_*\colon G_{\mathrm{ab}} = G/[G,G] & \to & H_{\mathrm{ab}} = H/[H,H] \\ [g] & \mapsto & [\varphi(g)] \end{array} \right)$$

define a covariant functor from $\mathcal{G}$ to $\mathcal{A}$.

(7) *Let $\mathcal{G}$ be the category of groups. Given a group G we denote by*

$$\begin{array}{rcl} \Phi_G\colon G & \to & G_{\mathrm{ab}} = G/[G,G] \\ g & \mapsto & [g] \end{array}$$

[305]By Lemma 11.13 there exists a unique homomorphism $F \to \mathbb{Z}^m$ with $x_i \mapsto e_i$. By (1) this map descends to a homomorphism $F/[F,F] \to \mathbb{Z}^m$. The way we write down the map is a little bit sloppy, since we only specify the map on the generators of $F/[F,F]$.

the projection map. These maps Φ_G define a natural transformation from the identity functor $\mathcal{G} \to \mathcal{G}$ to the abelianization functor $\mathcal{G} \to \mathcal{G}$ that we had introduced in (6).[306]

(8) *Isomorphic groups have isomorphic abelianizations.*

(9) *If $\varphi\colon A \to B$ is an epimorphism, then the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism.*

Examples.

(1) By Proposition 13.10 (1) and (3) we have

$$(\mathbb{Z}_3 * \mathbb{Z}_{35})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_3 \times \mathbb{Z}_{35} \quad \text{and} \quad (\mathbb{Z}_5 * \mathbb{Z}_{21})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_5 \times \mathbb{Z}_{21}.$$

It follows from the Chinese Remainder Theorem that the abelianizations are isomorphic. On the other hand on page 383 we had already observed that the Grushko Decomposition Theorem 11.20 implies that the groups $\mathbb{Z}_3 * \mathbb{Z}_{35}$ and $\mathbb{Z}_5 * \mathbb{Z}_{21}$ are not isomorphic. Thus we have seen that non-isomorphic groups can have isomorphic abelianizations.

(2) We give an explicit example for statement (5). Let

$$\pi \;=\; \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle.$$

Let $\Phi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1, 0)$ and $\Phi(y) = (0, 1)$. Then the abelianization of π is given by[307]

$$\pi_{\mathrm{ab}} \;=\; \mathbb{Z}^2/\langle \underbrace{\Phi(x^3y^{-2})}_{=\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}}\rangle \;=\; \mathbb{Z}^2/\begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix}\mathbb{Z}^2 \;\cong\; \mathbb{Z}_8.$$

(3) Let S be a set. It is straightforward to show that the abelianization of the free group $\langle S\rangle$ generated by S is isomorphic to the free abelian group $\mathbb{Z}^{(S)}$ generated by S.

(4) Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In the algebra course it was shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself.[308] The commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group.

Now we provide the proof of Proposition 13.10.

Proof of Proposition 13.10.

[306] Super-picky readers might argue that in (6) we introduced a functor from $\mathcal{G}$ to $\mathcal{A}$, but of course we can also view it as a functor from $\mathcal{G}$ to $\mathcal{G}$.

[307] Why is the group isomorphic to $\mathbb{Z}_8$?

[308] A proof of this statement is also given in any self-respecting algebra book, see e.g. [Rob, p. 69].

(1) The first statement is obvious.[309]
(2) Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then $[\pi,\pi]$ lies in the kernel of α. The desired statement now follows immediately from Lemma 13.1.
(3) The maps

$$\begin{array}{rcl} \Phi\colon (G*H)_{\mathrm{ab}} & \to & G_{\mathrm{ab}} \times H_{\mathrm{ab}} \\ [(x_1,\dots,x_k)] & \mapsto & \left(\left[\prod_{x_i\in G} x_i\right], \left[\prod_{x_i\in H} x_i\right]\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon G_{\mathrm{ab}} \times H_{\mathrm{ab}} & \to & (G*H)_{\mathrm{ab}} \\ ([g],[h]) & \mapsto & [g\cdot h] \end{array}$$

are easily seen to be well-defined, to be homomorphisms and to be inverses of one another.
(4) This statement follows immediately from the statements (1) and (3) together with the observation that $\langle x_1,\dots,x_m\rangle = \langle x_1\rangle * \cdots * \langle x_m\rangle$, that $\langle x_i\rangle_{\mathrm{ab}} \cong \mathbb{Z}$ and that $\mathbb{Z}^m = \mathbb{Z}\times\cdots\times\mathbb{Z}$.
(5) It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \ \to\ \mathbb{Z}^m/\langle \Phi(r_1),\dots,\Phi(r_n)\rangle$$

that makes the diagram of statement (5) commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1,\dots,x_m\rangle$ and we write $G = \langle y_1,\dots,y_n\rangle$.We consider the following diagram

$$\begin{tikzcd} G_{\mathrm{ab}} = G/[G,G] \arrow[r, "{[y_i]\mapsto e_i}", "\cong"'] \arrow[d, "{[y_i]\mapsto[r_i]}"'] & \mathbb{Z}^n \arrow[d, "{e_i\mapsto\Phi(r_i)}"] \\ F_{\mathrm{ab}} = F/[F,F] \arrow[r, "{\Phi([x_i])=e_i}", "\cong"'] \arrow[d, two heads] & \mathbb{Z}^m \arrow[d, two heads] \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[r, "\Psi"] & \dfrac{\mathbb{Z}^m}{\langle \Phi(r_1),\dots,\Phi(r_n)\rangle}. \end{tikzcd}$$

Note that the diagram commutes. By (4) the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism.[310] We leave the details to the reader.

[309] Why?

[310] More precisely, we use the following fact: suppose we are given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "\cong"] \arrow[d, two heads] & B' \arrow[d, two heads] \\ C \arrow[r] & C', \end{tikzcd}$$

(6) This statement follows easily from the definitions and the observation that the image of a commutator under a group homomorphism is once again a commutator.
(7) This statement is basically obvious once one has internalized the definition of a natural transformation.
(8) This statement is an immediate consequence of (6).
(9) We will prove this statement in Exercise 13.3. □

13.4. **The amalgamated product of groups.** In the formulation of the generalized version of the Seifert-van Kampen theorem we will need to replace the free product of two groups by the "amalgamated product" of two groups.

In Section 11.2 we had first defined the free product via an explicit construction. We had then explained that one can also characterize the free product via a universal property. In this section we will proceed the other way round, we will first state the desired universal property and then we will explicitly construct a group which satisfies the universal property.

Proposition 13.11. *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. Then there exists a triple*

$$(A *_G B, \varphi\colon A \to A *_G B, \psi\colon B \to A *_G B)$$

where $A *_G B$ is a group and φ and ψ are homomorphisms with the following properties:

(a) *the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & A *_G B, \end{tikzcd}$$

(b) *if $(H', \varphi'\colon A \to H', \psi'\colon B \to H')$ is another triple which satisfies property* (a)*, then there exists a unique homomorphism $\Theta\colon A *_G B \to H'$ such that $\varphi' = \Theta \circ \varphi$ and $\psi' = \Theta \circ \psi$, i.e. such that the following diagram commutes*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\ B \arrow[r, "\psi"'] \arrow[drr, bend right, "\psi'"'] & A *_G B \arrow[dr, dashed, "\Theta"] & \\ & & H'. \end{tikzcd}$$

Proof. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. As usual we view A and B as subgroups of $A * B$. We put

$$A *_G B \;:=\; A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase". This statement can also be viewed as a special case of the "five lemma".

and we define $\varphi\colon A \to A *_G B$ as the composition of the obvious homomorphisms

$$A \;\hookrightarrow\; A * B \;\twoheadrightarrow\; A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle,$$

and precisely the same way we define the homomorphism $\psi\colon B \to A *_G B$. It follows from the construction of $A *_G B$ that $\varphi(\alpha(g)) = \psi(\beta(g))$ for all $g \in G$. Therefore the triple $(A *_G B, \varphi\colon A \to A *_G B, \psi\colon B \to A *_G B)$ satisfies property (a).

Now let $(H', \varphi'\colon A \to H', \psi'\colon B \to H')$ be another triple which satisfies property (a). We consider the following diagram

$$\begin{tikzcd}
G \arrow[rr, "\alpha"] \arrow[d, "\beta"'] & & A \arrow[dl, hook] \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\
& A * B \arrow[r, two heads] \arrow[dr, hook] & A *_G B = \dfrac{A * B}{\langle\langle\alpha(g)\beta(g)^{-1}\rangle\rangle} \arrow[dr, dashed, red] & \\
B \arrow[ur, hook] \arrow[rr, "\psi"'] \arrow[rrr, bend right, "\psi'"'] & & & H'
\end{tikzcd}$$

where $\Phi\colon A * B \to H'$ is the unique homomorphism determined by ψ' and φ'. Since $\varphi' \circ \alpha = \psi' \circ \beta$ it follows that for any $g \in G$ we have

$$\Phi(\alpha(g)\cdot\beta(g)^{-1}) \;=\; \Phi(\alpha(g))\cdot\Phi(\beta(g))^{-1} \;=\; \varphi'(\alpha(g))\cdot\psi'(\beta(g))^{-1} \;=\; e.$$

Put differently, all elements of the form $\alpha(g)\beta(g)^{-1}$ in $A * B$ lie in the kernel of Φ. By Lemma 13.4 there exists a unique homomorphism $\Theta\colon A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle \to H'$ which makes the following diagram commute

$$\begin{tikzcd}
A * B \arrow[r] \arrow[dr, "\Phi"'] & A *_G B = A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle \arrow[d, "\Theta"] \\
& H'.
\end{tikzcd}$$

This proves the existence of Θ. The uniqueness follows from the observation that another choice of Θ' would give another choice for Φ, but this homomorphism is unique. $\square$

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle$$

and we refer to $A *_G B$ as the *amalgamated product of A and B with amalgam G.*[311]

[311]The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

Remark. A variation of the argument on page 368 shows that if

$$(H, \varphi\colon A \to H, \psi\colon B \to H)$$

is a triple that has both the properties (a) and (b) of Proposition 13.11, then there exists a unique isomorphism $\Theta\colon A *_G B \xrightarrow{\cong} H$ such that $\varphi(a) = \Theta(a)$ for all $a \in A$ and $\psi(b) = \Theta(b)$ for all $b \in B$. Often we will use this uniquely defined isomorphism to identify H with the amalgamated product $A *_G B$.

Example. Let $G = \mathbb{Z}_2, A = \mathbb{Z}_4$ and $B = \mathbb{Z}_6$ and consider the homomorphisms

$$\begin{array}{rcl} \alpha\colon G & \to & A \\ [k] & \mapsto & [2k] \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon G & \to & B \\ [k] & \mapsto & [3k]. \end{array}$$

Furthermore we consider the homomorphisms[312]

$$\begin{array}{rcl} \varphi\colon A = \mathbb{Z}_4 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^k \end{array} \quad \text{and} \quad \begin{array}{rcl} \varphi\colon B = \mathbb{Z}_6 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^k. \end{array}$$

It is straightforward to verify that[313] $\varphi \circ \alpha = \psi \circ \beta$, therefore there exists by Proposition 13.11 a homomorphism

$$A *_G B \;=\; \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \;\to\; \mathrm{SL}(2,\mathbb{Z}).$$

More surprisingly, this homomorphism is in fact an isomorphism. We refer to [Rosb, p. 218] and [Ser, p. 35] for a proof.

The following lemma summarizes two important special cases of the amalgamated product of groups.

Lemma 13.12.

(1) *For two groups A and B and $G = \{e\}$ the trivial group, the corresponding amalgamated product is the usual free product, i.e.*

$$A *_{\{e\}} B \;= A * B.$$

(2) *If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then*

$$A *_G \{e\} \;=\; A/\langle\langle \alpha(G) \rangle\rangle.$$

(3) *If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to B$ is an isomorphism, then the identity on A and $\beta^{-1}\colon B \to G$ induce an isomorphism*

$$A *_G B \;\xrightarrow{\cong}\; A.$$

[312] Are these really homomorphisms?

[313] Indeed, the one non-trivial element of $G = \mathbb{Z}_2$ gets sent under both maps to $-\mathrm{id}$.

Proof. The first two statements follow immediately from the definition

$$A *_G B \ := \ A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle.$$

For the third statement it is arguably best to work with the definition of the amalgamated product via the universal property. With this approach one quickly obtains that the given map is an isomorphism. □

The following lemma gives us a convenient description of the amalgamated product of two finitely presented groups.

Lemma 13.13. *Let*

$$A \ = \ \langle x_1, \dots, x_k \mid r_1, \dots, r_p \rangle$$

and

$$B \ = \ \langle y_1, \dots, y_l \mid s_1, \dots, s_q \rangle$$

be two finitely presented groups. Furthermore let $\alpha\colon G \to A$ *and* $\beta\colon G \to B$ *be two group homomorphisms and let* $g_1, \dots, g_m$ *be generators of* G*. Then*

$$A *_G B \ = \ \langle x_1, \dots, x_k, y_1, \dots, y_l \mid r_1, \dots, r_p, s_1, \dots, s_q \text{ and } \alpha(g_i) = \beta(g_i), i = 1, \dots, m \rangle.$$

Examples.

(1) Let $A_1 = \langle x_1, y_1 \rangle$ and $A_2 = \langle x_2, y_2 \rangle$ be two free groups and for $i = 1, 2$ let $\varphi_i\colon \langle t \rangle \to A_i$ be the homomorphism that is given by $\varphi_i(t^n) := [x_i, y_i]^n$. Then we obtain from Lemma 13.13 that

$$A_1 *_{\langle t \rangle} A_2 \ = \ \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle.$$

(2) Let $A = \langle a \rangle$ and let $B = \langle b \rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y] \rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then

$$A *_G B \ \underset{\substack{\uparrow \\ \text{Lemma 13.13}}}{=} \ \langle a, b \mid \alpha(x) = \beta(x), \alpha(y) = \beta(y) \rangle \ = \ \langle a, b \mid a = e, e = b \rangle \ \underset{\substack{\uparrow \\ \text{Tietze transformation (2), see page 412}}}{=} \ \langle | \rangle \ = \ \{e\}.$$

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial.

Proof of Lemma 13.13. We need to show that

$$\underbrace{\frac{A*B}{\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle}}_{\text{definition of } A*_G B} = \underbrace{\frac{A*B}{\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle}}_{\text{right-hand side of the proposition}}.$$

Put differently, we need to prove the following claim.

Claim. In the free product $A * B$ we have

$$\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle \ = \ \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle.$$

It is clear that the right-hand side is contained in the left-hand side. To show the reverse inclusion we need to show that for each $g \in G$ we have $\alpha(g)\beta(g)^{-1} \in \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$. We start out with two observations:

(1) We first consider the case $g = g_i^{-1}$ for some i. Then
$$\alpha(g)\beta(g)^{-1} \;=\; \alpha(g_i^{-1})\beta(g_i^{-1})^{-1} \;=\; \alpha(g_i)^{-1}\big(\alpha(g_i)\beta(g_i^{-1})\big)^{-1}\alpha(g_i),$$
thus $\alpha(g)\beta(g)^{-1}$ is a conjugate of the inverse of $\alpha(g_i)\beta(g_i^{-1})$, in particular it lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$.

(2) Note that if a, b and c are elements in a group π such that ac and b lie in a normal subgroup Γ, then $abc = ac \cdot c^{-1}bc$ also lies in Γ.

Now we turn to the proof of the claim. We define the *length* of $g \in G$ as
$$\ell(g) \;=\; \min\{n \mid \text{ we can write } g = g_{i_1}^{\epsilon_1} \cdot \dots \cdot g_{i_n}^{\epsilon_n} \text{ for } \epsilon_j \in \{-1,1\} \text{ and } i_j \in \{1,\dots,m\}\}.$$
Since $g_1, \dots, g_m$ are generators of G any $g \in G$ has a length.

We prove the claim by induction on the length of $g \in G$. For $\ell(g) = 0$ there is nothing to prove. Now suppose we have shown the claim for all elements of length $n-1$. Now let $g \in G$ be an element of length n. Thus we can write $g = g_{i_1}^{\epsilon_1} \cdot \dots \cdot g_{i_n}^{\epsilon_n}$ for $\epsilon_1, \dots, \epsilon_n \in \{-1,1\}$ and $i_1, \dots, i_n \in \{1,\dots,m\}$. We then have
$$\begin{aligned}\alpha(g)\beta(g)^{-1} \;=\; \alpha(g)\beta(g^{-1}) \;&=\; \alpha\big(g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n} \cdots g_{i_1}^{-\epsilon_1}\big) \\ &=\; \underbrace{\alpha\big(g_{i_1}^{\epsilon_1}\big) \cdots \alpha\big(g_{i_{n-1}}^{\epsilon_{n-1}}\big)}_{=:a} \cdot \underbrace{\alpha\big(g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n}\big)}_{=:b} \cdot \underbrace{\beta\big(g_{i_{n-1}}^{-\epsilon_{n-1}}\big) \cdots \beta\big(g_{i_1}^{-\epsilon_1}\big)}_{=:c}.\end{aligned}$$
Note that b lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by (1) and ac lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by induction. Thus $abc = \alpha(g)\beta(g)^{-1}$ lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by (2). □

In the previous example we saw that the amalgamated product of two non-trivial groups can be trivial. We conclude this introduction to amalgamated products with the following proposition that is proved in [Ser, p. 3] or alternatively [vdW]. It gives a criterion for the amalgamated product to be non-trivial.

Proposition 13.14. *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. If α and β are monomorphisms, then the canonical homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also monomorphisms.*

Exercises for Chapter 13.

Exercise 13.1.

(a) Show that
$$\langle x_1, \dots, x_k \mid x_k \rangle \;\cong\; \langle x_1, \dots, x_{k-1} \rangle.$$

(b) Show that in the group $\pi = \langle x, y \mid xyx^{-1}y^{-1} \rangle$ we have $yx^{-1} = x^{-1}y$.

Exercise 13.2.

(a) Let $k \geq 5$. Determine the abelianization of the permutation group S_k.
(b) Let G be an abelian group, let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be epimorphisms. Show that $A *_G B$ is also abelian.

Exercise 13.3. Given a group G we denote by G_{ab} its abelianization.

(a) Let $\varphi\colon A \to B$ be an epimorphism. Show that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism.
(b) Let $\varphi\colon A \to B$ be a monomorphism. Is the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ also a monomorphism?

Exercise 13.4. Let $\varphi\colon G \to A$ and $\psi\colon G \to B$ group monomorphisms which we use to define the amalgamated product $A *_G B$. Suppose that the map $A \to A *_G B$ is an isomorphism. Show that ψ is an isomorphism.

14. The general Seifert-van Kampen Theorem

14.1. **The formulation of the general Seifert-van Kampen Theorem.** With the group theoretic preparations from the previous chapter we can now formulate and prove the general Seifert-van Kampen Theorem.

Theorem 14.1. (Seifert–van Kampen) *Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two open subsets U and V such that $U \cap V$ is non-empty and path-connected. Let $x_0 \in U \cap V$. Then there exists an isomorphism*

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(V, x_0) \arrow[r] \arrow[drr, bend right] & \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \arrow[dr, "\Phi", "\cong"'] & \\
& & \pi_1(X, x_0).
\end{tikzcd}$$

Here all other maps are the obvious inclusion induced homomorphisms.

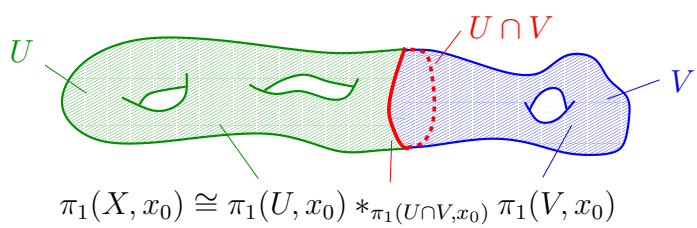

Figure 252. Illustration of Theorem 14.1.

The proof of the general Seifert-van Kampen Theorem is a modification of the proof of the more "basic" Seifert-van Kampen Theorem 12.2 where we had assumed that $U \cap V$ is in fact simply connected. We only sketch the proof of the general Seifert-van Kampen Theorem 14.1. The full details are for example given in Section 1.2 of [Hat].

Sketch of the proof. The various inclusions of topological spaces give rise to the following commutative diagram:

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U, x_0) \arrow[ddr] & \\
\pi_1(V, x_0) \arrow[drr] & & \\
& & \pi_1(X, x_0).
\end{tikzcd}$$

By Proposition 13.11 there exists a unique homomorphism

$$\Phi\colon \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \;\to\; \pi_1(X,x_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U\cap V,x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U,x_0) \arrow[d, "k"] \arrow[ddr, bend left] & \\
\pi_1(V,x_0) \arrow[r, "l"] \arrow[drr, bend right] & \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[dr, "\Phi"] & \\
& & \pi_1(X,x_0).
\end{tikzcd}$$

It remains to show that Φ is surjective and that Φ is injective.

We first show surjectivity. We consider the following commutative diagram[314] of homomorphisms

$$\begin{tikzcd}
& \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[dr, "\Phi"] & \\
\pi_1(U,x_0) * \pi_1(V,x_0) \arrow[ur] \arrow[rr] & & \pi_1(X,x_0).
\end{tikzcd}$$

By Lemma 12.1 the horizontal map is surjective. It follows that the diagonal map Φ on the right is also surjective.

Now we turn to the proof of the injectivity of Φ. We write $A = \pi_1(U,x_0), B = \pi_1(V,x_0)$ and $G = \pi_1(U\cap V,x_0)$. As on page 380 we consider the set

$$S(A,B) \;:=\; \{\text{all finite sequences } (x_1,\dots,x_k) \text{ with each } x_i \text{ in } A \text{ or } B\}.$$

Furthermore we denote by $\sim_G$ the equivalence relation on $S(A,B)$ generated by the relation

$$(x_1,\dots,x_l,e,x_{l+2},\dots,x_k) \;\sim\; (x_1,\dots,x_l,x_{l+2},\dots,x_k)$$

where e is trivial in A or B, by the relation

$$(x_1,\dots,x_l,x_{l+1},\dots,x_k) \;\sim\; (x_1,\dots,x_lx_{l+1},\dots,x_k)$$

if both x_l and x_{l+1} lie in A or both lie in B, and also by the relation

$$(x_1,\dots,x_l,i(g),x_{l+2},\dots,x_k) \;\sim\; (x_1,\dots,x_l,j(g),x_{l+2},\dots,x_k)$$

for $g\in G$. Similar to Proposition 11.16 one can now show that the map

$$\begin{array}{rcl} S(A,B)/\sim_G & \to & A *_G B \\ [(x_1,\dots,x_k)] & \mapsto & x_1\cdot\dots\cdot x_k \end{array}$$

is a well-defined bijection. We will use this isomorphism to identify $S(A,B)/\sim_G$ with $A *_G B$.

The proof of the present theorem is now almost identical to the proof of Theorem 12.2, one just needs to replace the discussion involving $S(A,B)/\sim$ by the corresponding objects in $S(A,B)/\sim_G$. More precisely, in Footnote 284 we pointed out where in the proof of

[314] What are the maps, and why does the diagram commute?

Theorem 12.2 we used the hypothesis that $U \cap V$ is simply connected. At that point in the proof of Theorem 12.2 we need to replace $A * B = S(A,B)/\sim$ with $A *_G B = S(A,B)/\sim_G$. Full details for the proof can be found for example in Section 1.2 of [Hat]. $\square$

The same way that we deduced Theorem 12.9 from Theorem 12.2 we can now obtain the following theorem from Theorem 14.1.

Theorem 14.2. (Seifert-van Kampen for manifolds) *Let M be an m-dimensional manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:*

(1) *$M = R \cup S$,*
(2) *$R \cap S$ is a component of ∂R and it is a component of ∂S,*
(3) *R and S are closed subsets of M.*

We pick a base point $x_0 \in R \cap S$. Then the inclusion induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \ \to \ \pi_1(M, x_0)$$

is an isomorphism.

Example. On page 400 we saw that the earlier Seifert-van Kampen Theorem 12.9 can be applied to show that $\pi_1(S^n) = 0$ for $n \geq 3$. If we replace Theorem 12.9 by the more general Theorem 14.2, then we can also determine the fundamental group of $\pi_1(S^2)$ using the "Seifert-van Kampen method". More precisely, we denote by $S^2_{\geq 0} := \{(x,y,z) \in S^2 \mid z \geq 0\}$ the upper hemisphere and similarly we define the lower hemisphere $S^2_{\leq 0}$. Then $S^2_{\geq 0}, S^2_{\leq 0}$ are submanifolds of S^2 that satisfy the hypothesis of Theorem 14.2. By the discussion on page 73 both hemispheres are homeomorphic to $\overline{B}^2$, and we obtain once again that $\pi_1(S^2) \cong \pi_1(S^2_{\geq 0}) *_{\mathbb{Z}} \pi_1(S^2_{\leq 0}) = \{e\} *_{\mathbb{Z}} \{e\} = \{e\}$.

14.2. **The fundamental groups of surfaces.** Now we want to use the Seifert-van Kampen Theorem 14.1 to determine the fundamental groups of surfaces of genus $g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.[315]

(1) We first consider the torus $T = [0,1] \times [0,1]/\sim$. In the following we will use the following notation, see also Figure 253 for an illustration:
   (a) We denote by Q the point on T corresponding to the four vertices of the square.
   (b) We write $T = U \cup V$ where
   $$U \ = \ T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2}, \tfrac{1}{2})} \subset T \quad \text{ and } V \ = \ B_{\frac{1}{4}}(\tfrac{1}{2}, \tfrac{1}{2}) \subset T,$$
   i.e. U is the complement of a closed disk and V is an open disk and $U \cap V$ is an open annulus.
   (c) We let $P = (r,r) \in U \cap V$ and we denote by p the direct path from Q to P.

[315]The following examples can also be handled using Theorem 14.2 instead of Theorem 14.1. The advantage of using Theorem 14.1 at this point is that in the illustrations one can draw $U \cap V$ better if U and V are open subsets.

(d) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus" and that represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.[316]

(e) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.

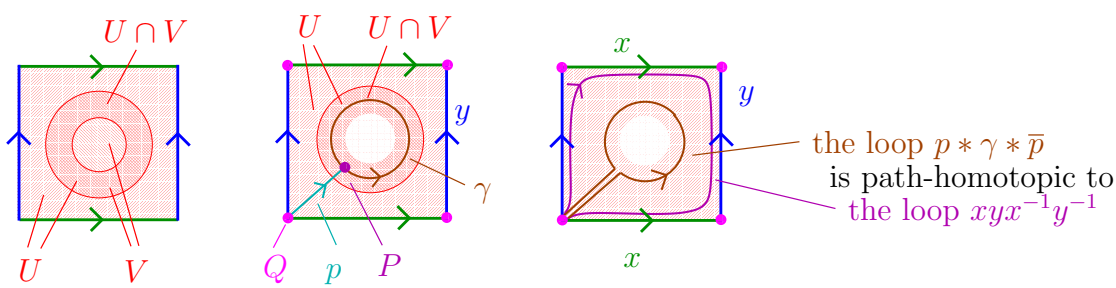

FIGURE 253.

Now we have

$$\pi_1(T,P) \underset{\substack{\uparrow \\ \text{Seifert-van Kampen Theorem 14.1}}}{\cong} \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) \underset{\substack{\uparrow \\ V \text{ is a disk, hence } \pi_1(V,P)=\{e\}}}{=} \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\}$$

$$\underset{\substack{\uparrow \\ \text{by Lemma 13.12 we have } A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle}}{=} \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle \underset{\substack{\uparrow \\ \text{since } \pi_1(U\cap V,P)=\langle[\gamma]\rangle}}{=} \pi_1(U,P)/\langle\langle[\gamma]\rangle\rangle$$

$$\underset{\substack{\uparrow \\ \text{by Proposition 5.11 the path } p \text{ gives} \\ \text{rise to an isomorphism } p_*}}{\overset{\cong}{\longrightarrow}} \pi_1(U,Q)/\langle\langle[p*\gamma*\overline{p}]\rangle\rangle \underset{\substack{\uparrow \\ \text{since } p*\gamma*\overline{p} \text{ is path-homotopic to } xyx^{-1}y^{-1}}}{=} \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle = (*).$$

As in the proof of Lemma 12.8 we can show that $W := \partial([0,1]\times[0,1])/\sim$ is a deformation retract of U and that W is homeomorphic to the wedge of two circles. By Lemma 12.4 we can therefore make the identification $\pi_1(W,P) = \langle x, y\rangle$, see also Figure 254. Now we continue with the above calculation of $\pi_1(T,P)$. We have

$$(*) = \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \underset{\substack{\uparrow \\ \text{by Lemma 9.10 and Proposition 9.12 (1)}}}{\overset{\cong}{\longrightarrow}} \pi_1(W,P)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle$$

$$\underset{\substack{\uparrow \\ \text{since } W \text{ is the wedge of the circles } x \text{ and } y}}{=} \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle = \langle x, y \mid [x,y]\rangle \underset{\substack{\uparrow \\ \text{see page 409}}}{\cong} \mathbb{Z}^2.$$

Thus we have reproved that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

[316] Since $U \cap V$ is an open annulus its fundamental group is isomorphic to $\mathbb{Z}$.

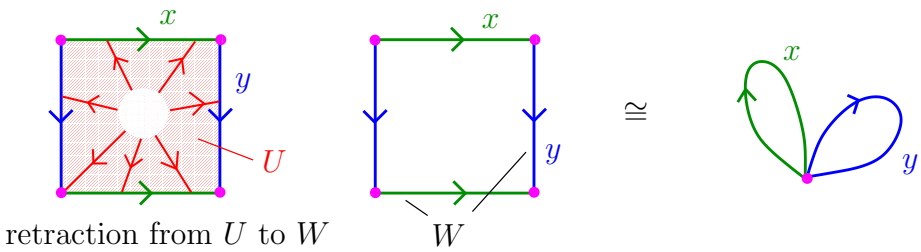

FIGURE 254.

(2) We turn to the Klein bottle K. We perform almost the same argument as for the torus, we just replace the situation illustrated in Figure 254 by the situation illustrated in Figure 255. The same argument as above shows that[317]

$$\pi_1(K) \;\cong\; \langle x, y \mid yxyx^{-1}\rangle.$$

Using this presentation and using Proposition 13.10 (5) we obtain in particular that the abelianization of $\pi_1(K)$ is given by

$$\pi_1(K)_{\mathrm{ab}} \;\cong\; \langle x, y \mid yxyx^{-1}\rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2/\begin{pmatrix}0\\2\end{pmatrix}\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}_2.$$

On page 283 we already gave a very different calculation of $\pi_1(K)$. More precisely, we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1) \end{array}$$

and their inverses and we showed that $\pi_1(K) \cong G$. On page 410 we already showed that there exists a homomorphism $\Phi\colon \langle x, y \mid yxyx^{-1}\rangle \to G$ with $\Phi(x) = B^{-1}A$ and $\Phi(y) = B$. This homomorphism is in fact an isomorphism.[318]

[317] In particular this shows that $\pi_1(K)$ is finitely presented. This answers the question we had raised on page 411.

[318] This can be seen as follows. Denote by H the group of all self-homeomorphisms of $\mathbb{R}^2$. We consider the diagram

$$\begin{tikzcd} \langle x, y \mid yxyx^{-1}\rangle \arrow[r, "\cong"] \arrow[rr, bend right, "\Phi", "x\mapsto B^{-1}A, y\mapsto B"'] & \pi_1(K) & \text{subgroup of } H \text{ generated by } A, B. \arrow[l, "\cong"'] \end{tikzcd}$$

Here the isomorphisms to $\pi_1(K)$ are the ones given by the two calculations of the fundamental group. If one goes carefully through the definitions of the two horizontal maps, then one can see that the diagram commutes. Since two of the three maps are isomorphisms and since the diagram commutes it follows that Φ is also an isomorphism.

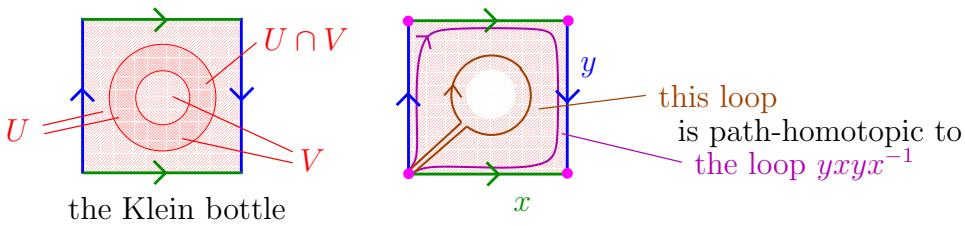

FIGURE 255.

(3) Next we consider again the real projective space $\mathbb{R}\mathrm{P}^2$. Here we use the description of $\mathbb{R}\mathrm{P}^2$ from Lemma 1.82 as

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}\mathrm{P}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}\mathrm{P}^2$ and we denote by α the loop

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & \mathbb{R}\mathrm{P}^2 \\ t & \mapsto & [e^{i\pi t}]. \end{array}$$

By a slight abuse of notation we denote by α also the corresponding element in $\pi_1(\mathbb{R}\mathrm{P}^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{R}\mathrm{P}^2$. Evidently α is a generator of $\pi_1(W, Q)$. Then, with similar arguments as in the torus case, we see that

$$\begin{array}{ccccccccc} \pi_1(\mathbb{R}\mathrm{P}^2) & \cong & \pi_1(U,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle & \cong & \pi_1(W,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle & \cong & \langle\alpha\rangle/\langle\alpha^2\rangle & \cong & \mathbb{Z}_2. \\ & \uparrow & & \uparrow & & \uparrow & & & \\ & \text{Seifert-van Kampen Theorem 14.1} & & \text{since } W \text{ is a deformation} & & \text{since } \pi_1(W,Q) = \langle\alpha\rangle & & & \\ & \text{and Lemma 13.12 (2)} & & \text{retract of } U & & \text{and } [p*\gamma*\overline{p}] = \alpha^2 & & & \end{array}$$

In particular we obtain the same result, as we should, as in Corollary 7.19.

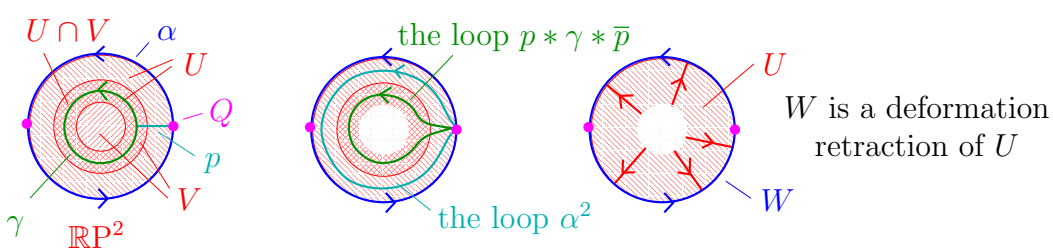

FIGURE 256.

(4) Now that we have gained some confidence that the approach of determining fundamental groups using Seifert-van Kampen Theorem 14.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

So now we consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. In the following we use basically the same notation as in the torus case or alternatively we refer to Figures 257 and 258 for the notation. We write $W = \partial E_8/\sim$. By the proof of Proposition 12.7 we know that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. See also Figure 258 for an illustration. Basically

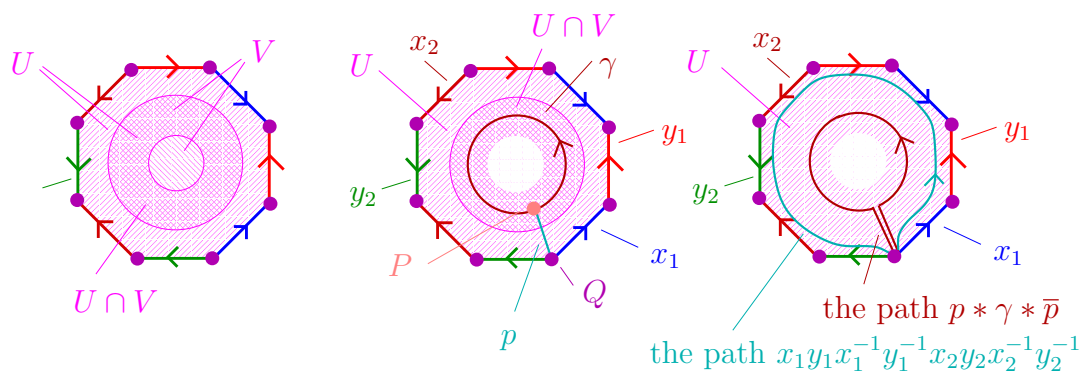

FIGURE 257.

exactly the same argument as in the torus case shows that

$$\begin{aligned}
\pi_1(E_8/\sim, P) &\cong \pi_1(U,Q)/\langle\langle \overbrace{\underbrace{x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}}_{=[x_1,y_1][x_2,y_2]}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\
&\cong \pi_1(W,Q)/\langle\langle [x_1,y_1][x_2,y_2] \rangle\rangle \\
&\cong \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1,y_1][x_2,y_2] \rangle\rangle \;=\; \langle x_1, y_1, x_2, y_2 \mid [x_1,y_1][x_2,y_2] \rangle.
\end{aligned}$$

Thus we have finally succeeded in calculating the fundamental group of the surface of genus 2.

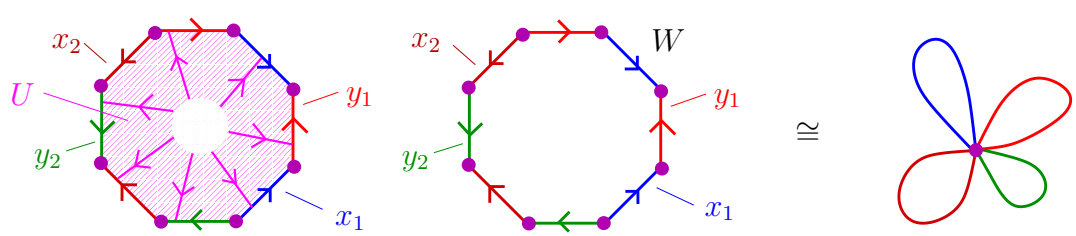

FIGURE 258.

The following gives in particular a new proof for Proposition 12.7.

Proposition 14.3.

(1) *For any* $g \in \mathbb{N}_0$ *we have*

$$\pi_1(\textit{surface of genus } g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$$

and we have

$$\pi_1(\textit{surface of genus } g)_{\mathrm{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) *For* $g \neq h$ *the surfaces of genus* g *and* h *are not homeomorphic.*

Proof. Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from Proposition 5.14 and for $g = 1$ we obtain the presentation from the discussion on page 428. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition.

The stated abelianization of the fundamental group is an immediate consequence of Proposition 13.10 (5).

Finally for $g \neq h$ we obtain from the above calculation and from the classification of finitely generated abelian groups that we gave in Theorem 11.4 that the abelianizations of the fundamental groups of the surfaces of genus g and h are not isomorphic. Hence the surfaces are not homeomorphic. □

Remark.

(1) In Figure 259 we show loops that represent the generators x_1, y_1, x_2, y_2 of the fundamental group for the surface of genus 2.

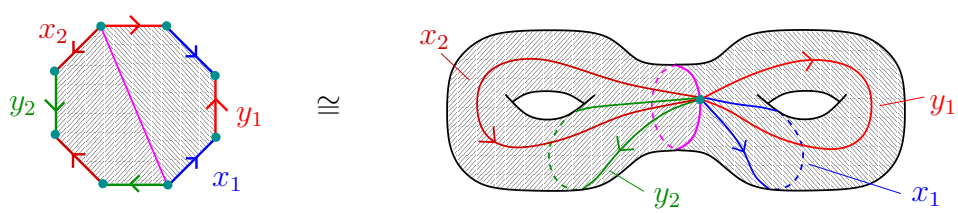

FIGURE 259.

(2) As explained in Figure 224 we can view the surface Σ of genus 2 as the connected sum of two copies of the torus T. We can also use this point of view to determine the fundamental group of the surface Σ of genus 2. More precisely, we have $\Sigma = F_1 \cup F_1$ with $F_1 = T \setminus B^2$ and $F_2 = T \setminus B^2$ where $F_1 \cap F_2 = \partial F_1 = \partial F_2 \cong S^1$ and we obtain that [319]

$$\begin{array}{lcl} & \text{Seifert-van Kampen Theorem 14.2} & \\ & \downarrow & \\ \pi_1(\Sigma) & \cong & \pi_1(F_1) *_{\pi_1(F_1 \cap F_2)} \pi_1(F_2) \\ & \cong & \langle x_1, y_1 \rangle *_{\langle t \rangle} \langle x_2, y_2 \rangle \qquad \cong \; \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle. \\ & \uparrow & \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ & \text{by Lemma 12.8 (2) where} & \qquad\qquad \text{see page 422} \\ & t \mapsto [x_1, y_1] \text{ and } t \mapsto [x_2, y_2] & \end{array}$$

[319] Why is this group isomorphic to the above group $\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle$?

This approach to calculating the fundamental group of the surface of genus 2, or any higher genus, actually comes with a bonus: From Lemma 12.8 (2) we know that the curve $F_1 \cap F_2$ represents non-trivial elements in $\pi_1(F_1)$ and $\pi_1(F_2)$. But from the above description of $\pi_1(\Sigma)$ it follows from Proposition 13.14 that the curve also represents a non-trivial element in $\pi_1(\Sigma)$. An analogous statement holds for similar curves on surfaces of higher genus. We refer to Figure 260 for an illustration.

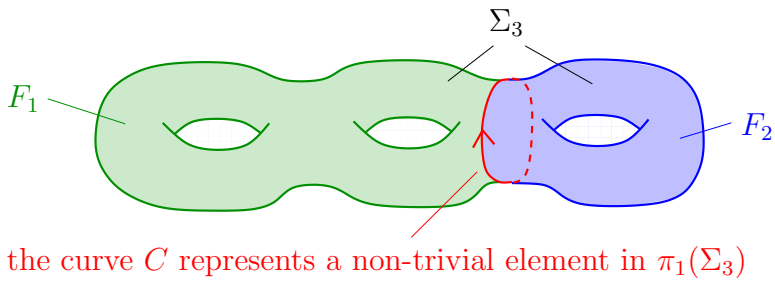

FIGURE 260.

Let $g \geq 2$ and let π be the fundamental group of the surface of genus g. We have just shown that $\pi_{\mathrm{ab}} \cong \mathbb{Z}^{2g}$. This raises the question, whether perhaps π is isomorphic to $\mathbb{Z}^{2g}$ or to the free group F_{2g} on $2g$ generators. The proof of the following lemma is Exercise 14.1.

Lemma 14.4. *The group $\pi = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$ is non-abelian.*

The lemma says in particular that π is not isomorphic to $\mathbb{Z}^{2g}$. It is much harder to determine whether or not π is a free group. We record this as a question for later.

Question 14.5. *Let π be the fundamental group of the surface of genus g. Is π isomorphic to the free group F_{2g} on $2g$ generators?*

14.3. **Non-orientable surfaces.** We recall the definition of the connected sum of two manifolds in the case that one of the two manifolds is $\mathbb{R}\mathrm{P}^2$. In the following we always use Lemma 1.82 which gives us the identification

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z$$

and we view $\frac{1}{2}\overline{B}^2 = \{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$ as a subset of $\mathbb{R}\mathrm{P}^2$. Now let F be a connected 2-dimensional manifold. We pick an embedding $\varphi\colon \overline{B}^2 \to F$. According to the definition on page 355 the connected sum of F with $\mathbb{R}\mathrm{P}^2$ is defined as

$$F\#\mathbb{R}\mathrm{P}^2 \;:=\; \big(F \setminus \varphi(B^2)\big) \sqcup \big(\mathbb{R}\mathrm{P}^2 \setminus \tfrac{1}{2}B^2\big) \,/\, \big\{\varphi(z) = \tfrac{1}{2}z \,\big|\, z \in S^1\big\}.$$

The following proposition summarizes some key aspects of the connected sum in this particular example.

Proposition 14.6. *Let F be a connected 2-dimensional manifold.*

(1) *The diffeomorphism type of* $F\#\mathbb{RP}^2$ *is well-defined.*
(2) *The connected sum* $F\#\mathbb{RP}^2$ *is a connected, non-orientable 2-dimensional manifold.*

Proof. By Proposition 10.27 we only have to show that $F\#\mathbb{RP}^2$ is non-orientable. The connected sum $F\#\mathbb{RP}^2$ contains $\mathbb{RP}^2 \setminus \frac{1}{2}B^2$, i.e. it contains the real projective plane minus an open disk. We already saw in Figure 102 that $\mathbb{RP}^2$ minus an open disk is a Möbius band. For convenience we recall the argument in Figure 261. Thus we see that $F\#\mathbb{RP}^2$ contains a Möbius band, i.e. it contains a non-orientable submanifold of codimension 0. Therefore $F\#\mathbb{RP}^2$ cannot be orientable. □

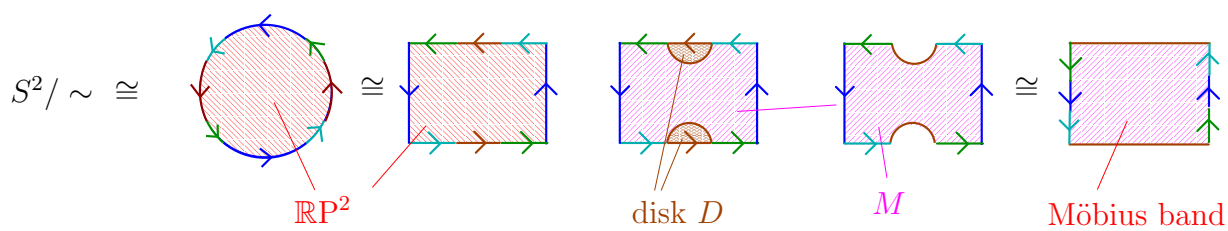

FIGURE 261.

Now we obtain many new examples of closed non-orientable 2-dimensional manifolds beyond the Klein bottle. For example the connected sum of the surface of genus g with $\mathbb{RP}^2$ or the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2$ are closed non-orientable 2-dimensional manifolds.

Example. By Proposition 14.6 the connected sum $F = \Sigma_g\#\mathbb{RP}^2$ of the surface of genus g with the real projective plane is non-orientable. So according to Proposition 8.3 it admits a 2-fold covering $\widetilde{F} \to F$ that is orientable. This raises the following question: what is the genus of $\widetilde{F}$? We will return to this question at a later stage.

We want to do one more calculation of a fundamental group, namely we want to determine the fundamental group of $\mathbb{RP}^2\#\mathbb{RP}^2$. We record the result in the following lemma.

Lemma 14.7. *We have*

$$\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \ \cong\ \langle x, y \mid x^2 = y^2\rangle.$$

Proof. In order to distinguish the two copies of $\mathbb{RP}^2$ we write $P = \mathbb{RP}^2$ and $Q = \mathbb{RP}^2$. The proof is of course a variation of the proof of Proposition 12.11. We have

$$\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \ = \pi_1(P\#Q) \ \underset{\substack{\uparrow \\ \text{Seifert-van Kampen Theorem 14.2}}}{\cong}\ \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1)} \pi_1\big(Q \setminus \tfrac{1}{2}B^2\big).$$

We pick a generator g for $\pi_1(\frac{1}{2}S^1)$. In the calculation of the fundamental group of the real projective plane on page 430 we saw that $\pi_1(P \setminus \frac{1}{2}B^2) \cong \langle x\rangle$ and that the inclusion induced

map $\pi_1(\frac{1}{2}S^1) \to \pi_1(P \setminus \frac{1}{2}B^2)$ is given by $g \mapsto x^2$. The same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) = \langle y \rangle$. Summarizing we see that

$$\begin{aligned}
\pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) &\cong \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1)} \pi_1\big(Q \setminus \tfrac{1}{2}B^2\big) \\
&\cong \text{amalgamated product of } \begin{pmatrix} \langle g \rangle & \to & \langle x \rangle \\ g & \mapsto & x^2 \end{pmatrix} \text{ and } \begin{matrix} \langle g \rangle & \to & \langle y \rangle \\ g & \mapsto & y^2 \end{matrix} \Big) \\
&\underset{\substack{\uparrow \\ \text{Lemma 13.13}}}{\cong} \langle x, y \mid x^2 = y^2 \rangle \;=\; \langle x, y \mid x^2 y^{-2} \rangle.
\end{aligned}$$

□

Now the question is whether $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is a "new" 2-dimensional manifold or whether we have already seen it before. A first step towards understanding the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) = \langle x, y \mid x^2 = y^2 \rangle$ is to determine its abelianization. Using Proposition 13.10 (5) we see that

$$\pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2)_{\mathrm{ab}} \;\cong\; \langle x, y \mid x^2 y^{-2} \rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2 / \begin{pmatrix} 2 \\ -2 \end{pmatrix} \mathbb{Z} \;\cong\; \mathbb{Z} \oplus \mathbb{Z}_2.$$

Interestingly we obtain the same abelianization as we got on page 429 for the abelianization of the fundamental group of the Klein bottle. This raises the question whether $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle K.

At first glance it is not clear what the answer should be. We had just seen that

$$\pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x, y \mid x^2 = y^2 \rangle$$

whereas we saw on page 429 that

$$\pi_1(K) \;\cong\; \langle x, y \mid yxyx^{-1} \rangle.$$

So at the very least there is no obvious isomorphism between these two groups. We therefore decide to have a closer look at the two topological spaces involved.

(1) We had just seen in Figure 261 that removing an open disk from $\mathbb{R}\mathrm{P}^2$ we obtain the Möbius band. This shows that $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is given by gluing two Möbius bands along their common boundary.
(2) In Figure 262 we see a closed curve in the Klein bottle that decomposes the Klein bottle into two Möbius bands.

Put differently, we see that both $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ and the Klein bottle are the result of gluing two Möbius bands along their boundary. It is now straightforward to see that the Klein bottle is homeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$.[320]

Given $k \in \mathbb{N}$ we now denote by $k \cdot \mathbb{R}\mathrm{P}^2$ the connected sum of k copies of the real projective plane.[321] By Proposition 14.6 the topological space $k \cdot \mathbb{R}\mathrm{P}^2$ is a non-orientable 2-dimensional manifold.

We conclude this section with the following lemma.

[320]Now we know that the groups $\pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2) \cong \langle x, y \mid x^2 = y^2 \rangle$ and $\pi_1(K) \cong \langle x, y \mid yxyx^{-1} \rangle$ are isomorphic. But what is an explicit isomorphism between these two groups?

[321]To be more precise, we define $0 \cdot \mathbb{R}\mathrm{P}^2$" $= S^2$ and iteratively $k \cdot \mathbb{R}\mathrm{P}^2 := (k-1) \cdot \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ for $k \in \mathbb{N}$.

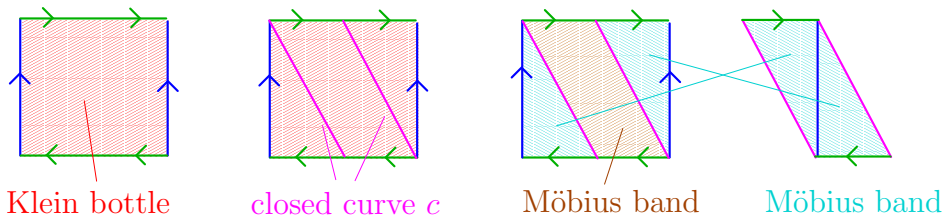

FIGURE 262.

Lemma 14.8. *Let $k \in \mathbb{N}$. Then*

$$\pi_1(k \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \dots, x_k \mid x_1^2 \cdot \dots \cdot x_k^2 \rangle \quad \textit{and} \quad \pi_1(k \cdot \mathbb{R}\mathrm{P}^2)_{\mathrm{ab}} \;\cong\; \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2.$$

Proof. A straightforward induction argument and a generalization of the argument of Lemma 14.7 gives the desired presentation for $\pi_1(k \cdot \mathbb{R}\mathrm{P}^2)$.[322] The abelianization can be determined easily from the presentation using Proposition 13.10 (5). We leave the details to the reader. □

14.4. **The classification of closed 2-dimensional (topological) manifolds.** The following proposition now gives the complete classification of connected closed topological 2-dimensional manifolds up to homeomorphism.

Theorem 14.9. *Every non-empty closed connected 2-dimensional topological manifold is homeomorphic to either*

(1) *a surface of genus g for some unique $g \in \mathbb{N}_0$ or*
(2) *to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$.*

Proof. By Radó's Theorem [Rad] every closed connected 2-dimensional topological manifold can be "triangulated". A proof that every non-empty closed connected "triangulated" 2-dimensional topological manifold is homeomorphic to a surface of genus g or to $k \cdot \mathbb{R}\mathrm{P}^2$ for some k is for example given in [ST, ST2] or alternatively in [FW, Gl] or [Mas2, Theorem 5.1] or in [GX, Theorem 6.2].

It remains to prove the uniqueness statement. This means that we have to show that the topological spaces listed in (1) and (2) are pairwise non-homeomorphic. It follows from Proposition 14.3 and Lemma 14.8 together with the classification of finitely generated groups, see Theorem 11.4, that the abelianizations of the fundamental groups of the topological manifolds listed in (1) and (2) are pairwise non-isomorphic. But then it follows from Proposition 13.10 (8) that also the fundamental groups are pairwise non-isomorphic, which in turn implies that the given topological manifolds are pairwise not homeomorphic. □

The following theorem gives the classification of closed connected 2-dimensional manifolds up to diffeomorphism.

Theorem 14.10. *Let Σ be a non-empty closed connected 2-dimensional manifold.*

[322] Why is for $k = 2$ this group isomorphic to the group given in Lemma 14.7?

(1) *If Σ is orientable, then it is diffeomorphic to a surface of genus g for some unique $g \in \mathbb{N}_0$. Furthermore, if Σ is oriented, then we can chose the diffeomorphism to be orientation-preserving.*
(2) *If Σ is non-orientable, then it is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$.*

Proof. The fact that any non-empty closed connected 2-dimensional manifold is diffeomorphic to one of the given manifolds is [Hirs, Theorems 9.3.5 and 9.3.10]. The uniqueness statement for g and k is an immediate consequence of the uniqueness statement of Theorem 14.9. □

Examples.

(1) It is perhaps tempting do dismiss Theorem 14.10 (1) as "obvious". After all, what other closed connected 2-dimensional manifold could there possibly be? It is thus instructive to consider Figure 263. It shows a hexagon and a decagon where *opposite sides* are identified. The same proof that showed that the $4g$-gon, with appropriate sides identified, is a closed orientable 2-dimensional manifold also shows that the topological spaces shown in Figure 263 are closed orientable 2-dimensional manifolds. To which of the standard surfaces are they diffeomorphic to? We will come back to this question at a later stage.

hexagon with opposite sides identified dodecagon with opposite sides identified

FIGURE 263.

(2) Let $g \in \mathbb{N}$ and denote by Σ_g the surface of genus g. The connected sum $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ contains a Möbius band as a submanifold of codimension 0,[323] hence $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is non-orientable. It follows from Theorem 14.10 that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$. In Exercise 14.2 we will determine k as a function of g.

14.5. **The classification of 2-dimensional (topological) manifolds with boundary.** Now we turn to the classification of compact 2-dimensional (topological) manifolds with boundary. We first prove the following proposition.

[323]This follows from the observation that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ contains $\mathbb{R}\mathrm{P}^2$ minus a disk, which by Figure 102 is a Möbius band.

Proposition 14.11. *If F is a 2-dimensional topological manifold, then the corresponding boundary $Y := \partial F$ is a 1-dimensional topological manifold which has no boundary, i.e. such that $\partial Y = \varnothing$.*

Remark. It might come as a surprise that we do not yet know the statement of the proposition. In fact in Proposition 3.21 we saw that the boundary of any n-dimensional manifold is an $(n-1)$-dimensional manifold. But we did not show the corresponding statement for topological manifolds. In fact at the moment we can prove the statement only for topological manifolds of dimension *two*. We will deal with the case of higher-dimensional topological manifolds after we will have introduced and studied homology groups.

The key ingredient in the proof of Proposition 14.11 is the following lemma.

Lemma 14.12. *Let $H = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the closed upper half-space in $\mathbb{R}^2$. If V is an open subset of H with $V \cap \partial H \neq \varnothing$, then V is not homeomorphic to an open subset of $\mathbb{R}^2$.*

For example it follows from the lemma that the half-open square $(0,1) \times [0,1)$ is not homeomorphic to any open subset of $\mathbb{R}^2$.

Proof. Let V be an open subset of the closed upper half-space H which contains a point $Q \in \partial H = \{(x,0) \mid x \in \mathbb{R}\}$. Furthermore let U be an open subset of $\mathbb{R}^2$. Suppose there exists a homeomorphism $\Phi\colon U \to V$. We write $\Psi := \Phi^{-1}$ and we write $P = \Psi(Q)$.

Since U is open in $\mathbb{R}^2$ we can pick a $\sigma > 0$ with $B_\sigma(P) \subset U$. Since V is open in H and since Ψ is continuous there exists an $\epsilon > 0$ with $B_\epsilon(Q) \cap H \subset V$ and with $\Psi(B_\epsilon(Q)) \subset B_\sigma(P)$. Furthermore, by the continuity of Φ there exists an $\eta > 0$ with $\Phi(B_\eta(P)) \subset B_\epsilon(Q))$. We refer to Figure 264 for an illustration. We consider the following commutative diagram of maps

$$\begin{tikzcd}
 & (B_\epsilon(Q) \setminus \{Q\}) \cap H \arrow[dr, "\Psi = \Phi^{-1}"] & \\
B_\eta(P) \setminus \{P\} \arrow[ur, "\Phi"] \arrow[rr, "i"'] & & B_\sigma(P) \setminus \{P\},
\end{tikzcd}$$

where $i\colon B_\eta(P) \setminus \{P\} \to B_\sigma(P) \setminus \{P\}$ denotes the inclusion map. Now we pick a base point $x \in B_\eta(P) \setminus \{P\}$. By the functoriality of fundamental groups, the above commutative diagram of maps gives rise to the following commutative diagram

$$\begin{tikzcd}
 & \pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) \arrow[dr, "\Psi_* = \Phi_*^{-1}"] & \\
\pi_1(B_\eta(P) \setminus \{P\}, x) \arrow[ur, "\Phi_*"] \arrow[rr, "i_*"'] & & \pi_1(B_\sigma(P) \setminus \{P\}, x).
\end{tikzcd}$$

We make the following observations:

(1) the topological spaces on the left and on the right are homotopy equivalent to S^1, in particular the fundamental groups are non-trivial,

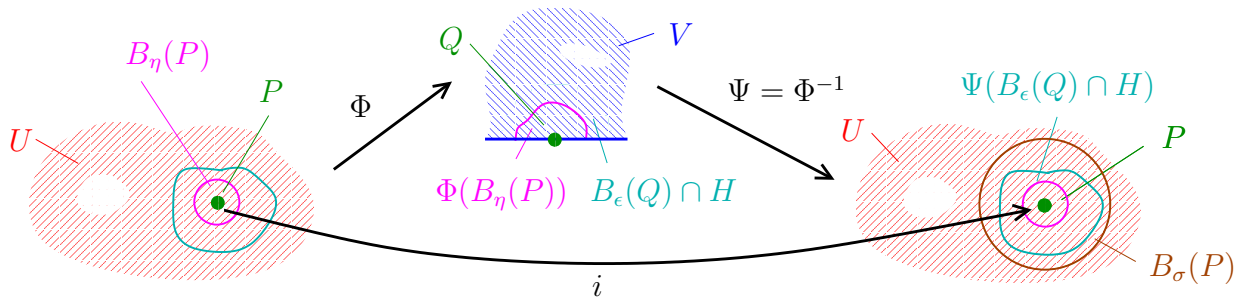

FIGURE 264.

(2) by Exercise 9.10 the inclusion map $B_\eta(P) \setminus \{P\} \to B_\sigma(P) \setminus \{P\}$ is a homotopy equivalence, therefore it follows from Proposition 9.12 (1) that the inclusion induced map $i_*\colon \pi_1(B_\eta(P) \setminus \{P\}, x) \to \pi_1(B_\sigma(P) \setminus \{P\}, x)$ is an isomorphism,
(3) the topological space $(B_\epsilon(Q) \setminus \{Q\}) \cap H$ is star-shaped[324], which implies by the discussion on page 238 that $\pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) = 0$.

Summarizing we just showed that an isomorphism of non-trivial groups factors through the trivial group. This is not possible and hence we have obtained the desired contradiction. □

Now we can turn to the proof of Proposition 14.11.

Proof of Proposition 14.11. Let F be a 2-dimensional topological manifold with $\partial F \neq \varnothing$. We first recall some definitions.

(a) A 2-*dimensional chart for* F *at a point* $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^2$ or
  (ii) V is an open subset of the half-space $H := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and $\Phi(x)$ lies on $\partial H = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.
  In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).
(b) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.

We start our actual proof with the following claim.

Claim. Let x be a point on F. Then x admits either a chart of type (i) or it admits a chart of type (ii).

If a point did admit a chart $\Phi\colon U \to V$ of type (i) and a chart $\Psi\colon X \to Y$ of type (ii), then $\Phi \circ \Psi^{-1}\colon \Psi(U \cap X) \to \Phi(U \cap X)$ would give rise to a homeomorphism from an open set of H containing a point on $\partial H = \{(x, 0) \mid x \in \mathbb{R}\}$ to an open set in $\mathbb{R}^2$. But by Lemma 14.12 such a homeomorphism cannot exist. This concludes the proof of the claim.

[324]Why is that?

The following claim recalls the statement that we actually want to prove.

Claim. The boundary ∂F is a 1-dimensional topological manifold which has no boundary, i.e. such that $\partial Y = \varnothing$.

Let $P \in F$ be a point on the boundary. Let $\Phi\colon U \to V$ be a chart of type (ii) for P. It follows from the claim and the definitions that $\partial F \cap U = \Phi^{-1}(\partial H \cap V)$. Thus we see that Φ restricts to a chart $\Phi\colon \partial F \cap U \to \partial H \cap V$ of type (i), here we view $\partial H \cap V$ as an open subset of $\partial H = \mathbb{R}$. This shows that ∂F admits a 1-dimensional atlas where all charts are charts of type (i). Since F is second-countable and Hausdorff we obtain from Lemma 3.1 (2) and Lemma 1.14 that ∂F is also second-countable and Hausdorff. This completes the proof that ∂F is indeed a 1-dimensional topological manifold with no boundary points. □

After this rather technical discussion we go back to more hands-on topology.

Definition. Given a non-empty closed connected 2-dimensional manifold Σ and $n \in \mathbb{N}_0$ we define Σ *minus n open disks* as the result of removing the interiors of n disjoint closed disks.[325]

The following lemma captures some of the main features of the above definition.

Lemma 14.13. *Let Σ be a non-empty closed connected 2-dimensional manifold Σ and let $n \in \mathbb{N}_0$.*

1. *The topological space Σ minus n open disks is a compact connected submanifold of Σ with n boundary components.*
2. *The diffeomorphism type of Σ minus n open disks does not depend on the choice of the disks.*
3. *We denote the boundary components of Σ minus open disks by $C_1, \dots, C_n$. Given any permutation $\sigma \in S_n$ there exists a diffeomorphism f of Σ minus n open disks with $f(C_i) = C_j$, $i = 1, \dots, n$. Furthermore, if Σ is oriented, then f can be chosen to be orientation-preserving.*

Sketch of proof.

1. This statement follows from Proposition 10.2 (2).
2. This statement follows easily from Theorem 10.28.
3. Conveniently enough this statement also follows almost immediately from Theorem 10.28. □

This leads us to the following notation:

Notation. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Sometimes we refer to g as the genus of $\Sigma_{g,n}$. Furthermore for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$R_{k,n} \;:=\; \text{the manifold } k \cdot \mathbb{R}\mathrm{P}^2 \text{ minus } n \text{ open disks.}$$

[325] Why did we not make the definition as "the result of removing n disjoint open disks"? Wouldn't that be much easier?

Examples.

(1) The surface $\Sigma_{0,2}$ is the result of removing two open disjoint disks from the 2-sphere. The resulting topological space is homeomorphic to a cylinder. We refer to Figure 265 for an illustration of this statement.

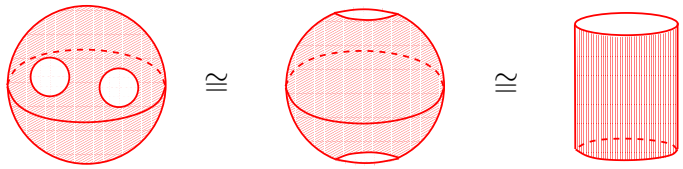

FIGURE 265.

(2) We saw in Figure 261 that the result of removing an open disk from $\mathbb{R}\mathrm{P}^2$ is the Möbius band, i.e. $R_{1,1}$ is the Möbius band.

Theorem 14.14. *Every non-empty compact connected 2-dimensional topological manifold is homeomorphic to either*

(1) *the surface $\Sigma_{g,n}$ for unique $g, n \in \mathbb{N}_0$ or*
(2) *to the surface $R_{k,n}$ for unique $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Sketch of the proof. Let F be a non-empty compact connected 2-dimensional topological manifold. By Lemma 3.6 and Proposition 14.11 the boundary ∂F is a compact 1-dimensional topological manifold with no boundary. Since ∂F is compact it has finitely many components that we denote by $C_1, \dots, C_n$. (Note that we allow that case that $n = 0$.) Each component C_i is compact and has no boundary. Thus it follows from Propositions 3.49 that for each $i = 1, \dots, n$ there exists a homeomorphism $f_i \colon S^1 \to C_i$.

We denote by $\widehat{F}$ the result of gluing a disk to each boundary component. More precisely, we consider

$$\widehat{F} \ := \ \Big(F \sqcup \bigsqcup_{i=1}^{n} \overline{B}^2 \times \{i\}\Big) \,/\, \sim \ \text{ where } f_i(P) \sim (P, i) \text{ for } P \in S^1 \text{ and } i \in \{1, \dots, n\}.$$

The resulting topological space $\widehat{F}$ is a non-empty closed connected, 2-dimensional topological manifold[326]. Furthermore the homeomorphism type of $\widehat{F}$ does not depend on the choices of the homeomorphisms $f_1, \dots, f_n$.[327]

Now we apply Theorem 14.9. There are two possible conclusions:

[326]This statement requires more justification, we skip the details, hence we refer to it only as a "sketch of proof".

[327]This statement also needs a justification. This is the second reason why it is called "sketch of a proof".

(1) Either there exists exists a homeomorphism $\Phi\colon \Sigma_g \to \widehat{F}$ from the surface of genus g to $\widehat{F}$ for a unique $g \in \mathbb{N}_0$. Then Φ restricts to a homeomorphism

$$\Phi\colon \Sigma_{g,n} \;=\; \Sigma_g \setminus \bigcup_{i=1}^{n} \Phi^{-1}\big(B^2 \times \{i\}\big) \;\to\; F.$$

(2) Or there exists a homeomorphism $\Phi\colon k \cdot \mathbb{RP}^2 \to \widehat{F}$ for a unique $k \in \mathbb{N}$. Then Φ restricts to a homeomorphism

$$\Phi\colon R_{k,n} \;=\; k \cdot \mathbb{RP}^2 \setminus \bigcup_{i=1}^{n} \Phi^{-1}\big(B^2 \times \{i\}\big) \;\to\; F.$$

In both cases n is uniquely determined as the number of boundary components of F. Thus we have shown that F is homeomorphic to either $\Sigma_{g,n}$ or $R_{k,n}$ for some $g \in \mathbb{N}_0$ or $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ which are furthermore unique. □

The following theorem gives us the analogous classification statement for manifolds with boundary up to *diffeomorphism*.

Theorem 14.15. *Let Σ be a non-empty compact connected 2-dimensional manifold.*

(1) *If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some unique $g, n \in \mathbb{N}_0$. If Σ is oriented, then the diffeomorphism can be chosen to be orientation-preserving.*
(2) *If Σ is non-orientable, then Σ is diffeomorphic to the surface $R_{k,n}$ for some unique $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Proof. The theorem is proved in a very similar fashion as Theorem 14.14. One needs to replace Proposition 3.49 by Proposition 3.53. Furthermore we have to use Theorem 14.10 instead of Theorem 14.9. Full details are given in [Hirs, Theorem 9.3.7]. □

The following corollary is an immediate, arguably rather striking, consequence of Theorem 14.15 and Lemma 14.13.

Corollary 14.16. *Let Σ be a compact connected 2-dimensional manifold. We denote the boundary components of Σ by $C_1, \dots, C_n$. Given any permutation $\sigma \in S_n$ there exists a diffeomorphism f of Σ minus n open disks with $f(C_i) = C_j$, $i = 1, \dots, n$. Furthermore, if Σ is oriented, then we can choose f to be orientation-preserving.*

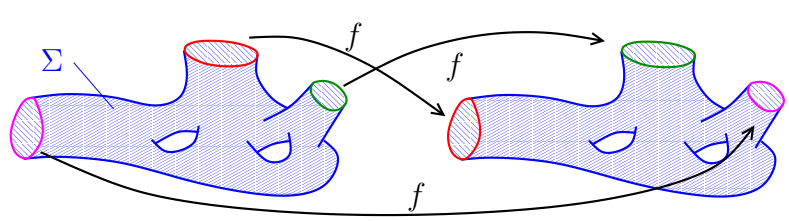

FIGURE 266. Illustration of Corollary 14.16.

It is natural to ask whether we can also classify non-compact 2-dimensional manifolds. For example a first guess would be that perhaps all non-compact orientable 2-dimensional manifolds without boundary are "the surface of infinite genus". But as we see in Figure 267 there exist several 2-dimensional manifolds for which the name "infinite genus" seems appropriate. In Exercise 14.3 we will show that the "surfaces of infinite genus" X and Y shown in Figure 267 not homeomorphic. This already shows that the classification of non-compact 2-dimensional manifolds can get a little tricky. In fact the classification for orientable 2-dimensional manifolds without boundary was first given by Ian Richards [Ric] in 1963, building on earlier work of Béla Kerékjártó [Kere] from 1923. In [PM] this result was extended to the case of 2-dimensional manifolds with non-empty boundary. As a side

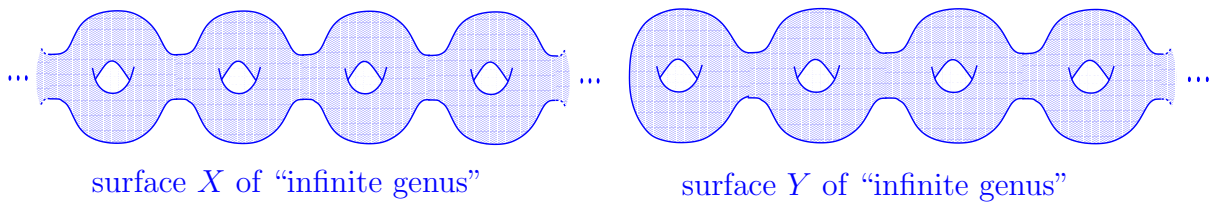

FIGURE 267.

we point out that the surface Y of infinite genus shown in Figure 267 is sometimes called the Loch Ness monster. The reason is shown in Figure 268: the visible part of an infinite Loch Ness monster, together with the surface of the infinite Loch Ness form a 2-dimensional manifold that is diffeomorphic to Y. More information on the mathematical Loch Ness monster and an explicit rigorous construction are given in [ARM].

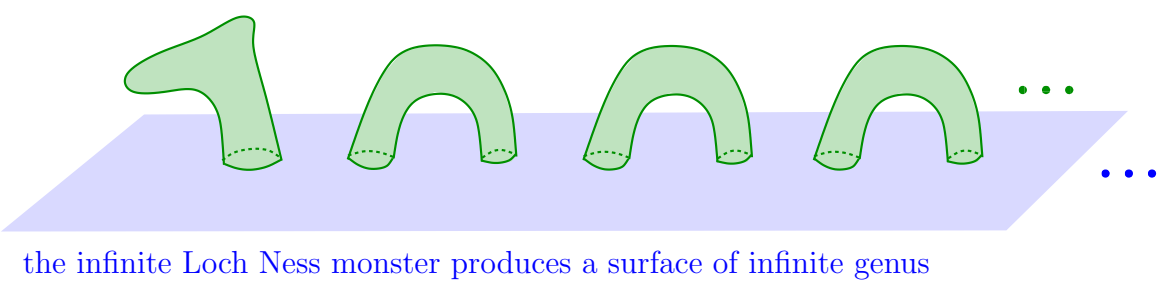

FIGURE 268.

We conclude this chapter with a short discussion of the properties of fundamental groups of 2-dimensional manifolds. We recall that the fundamental group of the real projective space $\mathbb{R}\mathrm{P}^2$ is isomorphic to $\mathbb{Z}_2$, therefore it is torsion and in particular it is not torsion-free. It is less clear whether or not the fundamental groups of the other 2-dimensional manifolds are torsion-free. More precisely, we have the following question.

Question 14.17.

(1) *Let $g \in \mathbb{N}$. Is the fundamental group*

$$\pi_1(\textit{surface of genus } g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$$

torsion-free?

(2) *Let $k \geq 2$. Is the fundamental group*

$$\pi_1(k \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \dots, x_k \mid x_1^2 \cdot \dots \cdot x_k^2 \rangle$$

torsion-free?

We will come back to this question at a much later stage.

14.6. **Retractions onto boundary components of 2-dimensional manifolds.** Let X be a topological space and let $A \subset X$ be a subset. Recall that we say that A is a *retract of* X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$. We also recall the statement of Lemma 6.2.

Lemma 6.2. *The circle S^1 is not a retract of $\overline{B}^2$.*

We quickly recall the proof of Lemma 6.2.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$, i.e. a map with $r \circ i = \mathrm{id}_{S^1}$. We obtain the commutative diagram

$$\begin{tikzcd} & \overline{B}^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "r\circ i = \mathrm{id}_{S^1}"'] & & S^1 \end{tikzcd}$$

which induces the commutative diagram

$$\begin{tikzcd} & \underbrace{\pi_1(\overline{B}^2, 1)}_{=0} \arrow[dr, "r_*"] & \\ \underbrace{\pi_1(S^1, 1)}_{\cong\mathbb{Z}} \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_* = (\mathrm{id}_{S^1})_*"'] & & \underbrace{\pi_1(S^1, 1)}_{\cong\mathbb{Z}}. \end{tikzcd}$$

But a commutative diagram as given on the right-hand side cannot exist since an isomorphism of a non-trivial group cannot factor through the trivial group. □

Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we define, as on page 440

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

For example $\Sigma_{0,1}$ is the result of removing an open disk from S^2, i.e. $\Sigma_{0,1}$ is a disk. In light of Lemma 6.2 the following question arises:

Question 14.18. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. Let C be a boundary component of $\Sigma_{g,n}$. Is C a retract of $\Sigma_{g,n}$?*

Lemma 6.2 says that the answer to Question 14.18 is no if $g = 0$ and $n = 1$. The proof of Lemma 6.2 suggests that if we want to answer Question 14.18, then given a boundary component C of $\Sigma_{g,n}$ we should try to understand the inclusion induced map $\pi_1(C) \to \pi_1(\Sigma_{g,n})$.

Lemma 14.19. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \ldots, C_n$. Then the following statements hold:*

(1) *Let $* \in \Sigma_{g,n} \setminus \partial\Sigma_{g,n}$ be a base point. We pick an orientation for $\Sigma_{g,n}$. There exists a presentation*

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \cdot z_1 \cdot \cdots \cdot z_n \rangle$$

with the following property: for any $i \in \{1, \ldots, n\}$ the element z_i is represented by a curve of the form $\alpha * \gamma * \overline{\alpha}$ where α is a path from $*$ to a point on C_i and γ is a path that goes once around C_i in an orientation-preserving way.[328] *We refer to Figure 269 for an illustration.*

(2) *If $g \geq 1$, then for any $i \in \{1, \ldots, n\}$ the inclusion induced map $\pi_1(C_i) \to \pi_1(\Sigma_{g,n})$ is a monomorphism.*

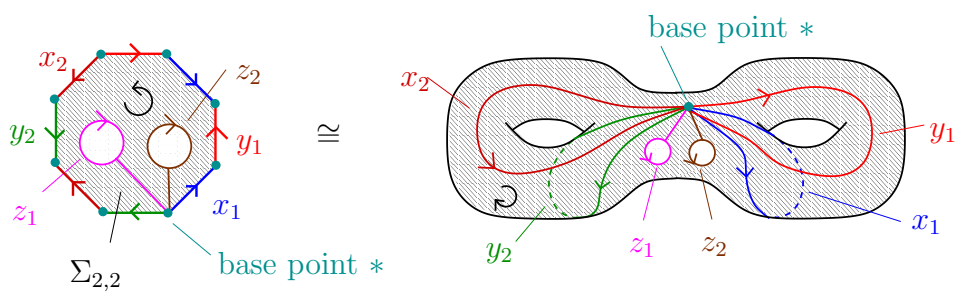

FIGURE 269. Illustration of the curves in Lemma 14.19.

Sketch of the proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

(1) First note that by Proposition 10.23 we can without loss of generality assume that the base point corresponds to any of the vertices of the $4g$-gon as shown in Figure 270. The following argument is very similar to the argument on the pages 428 and 431 and it relies heavily on the notation from Figure 270. We have

$$\begin{aligned}
\pi_1(\Sigma_{g,n}) &\overset{\text{Seifert-van Kampen Theorem 14.1}}{=} \pi_1(U) *_{\pi_1(U \cap V)} \underbrace{\pi_1(V)}_{=\{e\}} \overset{\text{Lemma 13.12}}{\cong} \pi_1(U) / \langle\langle \pi_1(U \cap V) \rangle\rangle \\
&\underset{\substack{\text{followings from the deformation retraction shown in Figure 270} \\ \text{together with Lemma 9.10 and Proposition 9.12 (1)}}}{=} \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \rangle / \langle\langle [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \cdot z_1 \cdot \cdots \cdot z_n \rangle\rangle.
\end{aligned}$$

[328] Since we picked an orientation for $\Sigma_{g,n}$ each boundary component also inherits an orientation.

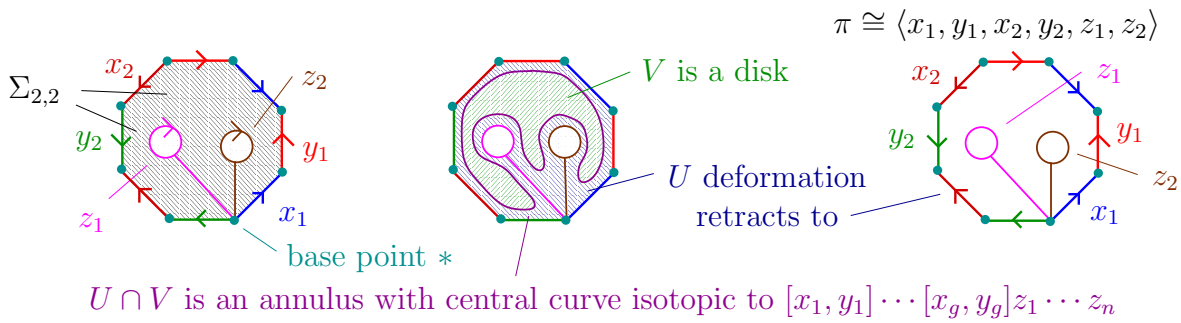

FIGURE 270. Illustration for the proof of Lemma 14.19.

(2) Now suppose that $g \geq 1$. Let $i \in \{1, \dots, n\}$. The image of $\pi_1(C_i)$ in $\pi := \pi_1(\Sigma_{g,n})$ is the subgroup generated by z_i. It suffices to show that z_i is an element of infinite order in π. It follows immediately from Lemmas 11.13 and 13.4 that there exists a homomorphism

$$\langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdots [x_g, y_g] \cdot z_1 \cdots z_n \rangle \to \langle x_1, y_1, \dots, x_g, y_g \rangle$$

with

$$\begin{array}{rcll} x_j & \mapsto & x_j & \text{for } j = 1, \dots, g \\ y_j & \mapsto & y_j & \text{for } j = 1, \dots, g \end{array} \quad \text{and} \quad \begin{array}{rcll} z_i & \mapsto & ([x_1, y_1] \cdots [x_g, y_g])^{-1} & \\ z_j & \mapsto & e & \text{for } j \neq i. \end{array}$$

The image of z_i has infinite order in the free group $\langle x_1, y_1, \dots, x_g, y_g \rangle$,[329] hence z_i is already an element of infinite order in π. □

We continue with the notation from Lemma 14.19. We consider the case $n \geq 1$, i.e. we consider the case that we have non-empty boundary. In this case it follows from Lemma 13.5 (2) that for any $i \in \{1, \dots, n\}$ the obvious map

$$\langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, \widehat{z_i}, \dots, z_n \rangle \to \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \dots [x_g, y_g] \cdot z_1 \dots z_n \rangle$$

is an isomorphism. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators. The following lemma gives an alternative, more topological approach to seeing that $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

Lemma 14.20. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. Given any $i \in \{1, \dots, n\}$ there exists a homotopy equivalence*[330]

$$\Sigma_{g,n} \to C_1 \vee \dots \vee \widehat{C_i} \vee \dots \vee C_n \vee \underbrace{S^1 \vee \dots \vee S^1}_{2g \text{ copies}}$$

which is the identity on $C_1, \dots, \widehat{C_i}, \dots, C_n$. In particular, by Proposition 9.12 (1) and the discussion on page 390 we see that $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

[329] Why does it have infinite order?

[330] Here $\widehat{C_i}$ means that we leave out C_i.

Sketch of proof. In Figure 271 we show that, starting from any boundary component of $\Sigma_{g,n}$, we can push "inward" to obtain a deformation retraction to a subset that is a spatial graph G with a unique spanning tree. By Proposition 9.21 (4), is homotopy equivalent to a wedge of $2g+n-1$ circles where the circles correspond to $x_1, y_1, \ldots, x_g, y_g$ and the remaining boundary components.[331][332] □

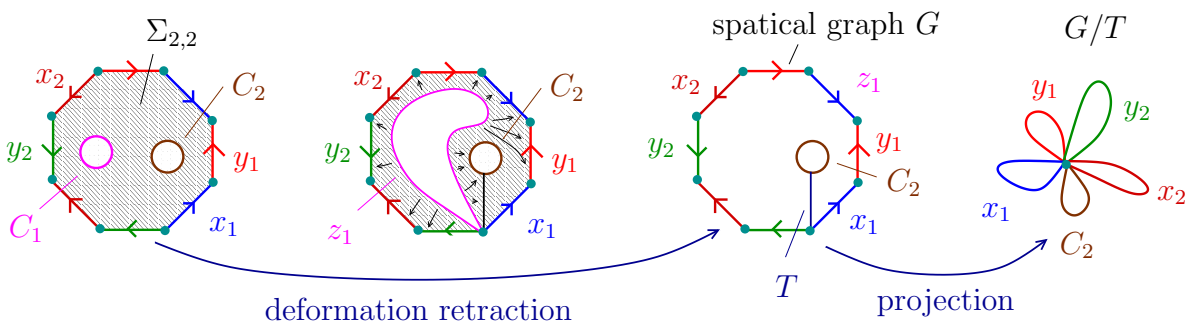

FIGURE 271.

The following lemma gives an answer to Question 14.18.

Lemma 14.21. *Let $g \in \mathbb{N}_0$.*

(1) *The unique boundary component of $\Sigma_{g,1}$ is not a retract of $\Sigma_{g,1}$.*
(2) *For $k \geq 2$ every boundary component of $\Sigma_{g,k}$ is a retract of $\Sigma_{g,k}$.*

Proof. Let $g \in \mathbb{N}_0$.

(1) The case $g = 0$ was dealt with in Lemma 6.2. Now we assume that $g \geq 1$. We denote by C the unique boundary component of $\Sigma := \Sigma_{g,1}$. We had just seen in Lemma 14.19 that $\pi_1(C) \to \pi_1(\Sigma)$ is a monomorphism if $g \geq 1$. So we cannot apply the same argument as in Lemma 6.2 where we studied the induced map on the level of fundamental groups. The idea now is to study instead the induced map on the *abelianizations* of the fundamental groups.

We denote by $i\colon C \to \Sigma$ the inclusion map and we suppose that there exists a retraction $r\colon \Sigma \to C$, i.e. a map with $r \circ i = \mathrm{id}_C$. We write $\pi_1(C) = \langle t \rangle$ and we use the isomorphism from Lemma 14.19 (1) to make the identification

$$\pi_1(\Sigma) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \cdot z \rangle$$

[331] We leave it to others with more time and energy to write down a rigorous argument.
[332] This argument is evidently closely related to the proof of Lemma 12.8.

where z corresponds to t under the inclusion $C = \partial\Sigma \to \Sigma$. We obtain the commutative diagram

$$\begin{tikzcd} & \Sigma \arrow[dr, "r"] & \\ C \arrow[ur, "i"] \arrow[rr, "r\circ i = \mathrm{id}_C"'] & & C \end{tikzcd}$$

which induces by Proposition 13.10 (6) the commutative diagram

$$\begin{tikzcd} & \pi_1(\Sigma)_{\mathrm{ab}} \arrow[dr, "r_*"] & \\ \underbrace{\pi_1(C)_{\mathrm{ab}}}_{=\langle t\rangle} \arrow[ur, "i_*", "t\mapsto z"'] \arrow[rr, "(r\circ i)_* = \mathrm{id}"'] & & \underbrace{\pi_1(C)_{\mathrm{ab}}}_{=\langle t\rangle}. \end{tikzcd}$$

In $\pi_1(\Sigma)$ we have

$$z \;=\; ([x_1, y_1] \cdot \dots \cdot [x_g, y_g])^{-1} \;\in\; \pi_1(\Sigma).$$

This shows that z lies in the commutator subgroup of $\pi_1(\Sigma)$, i.e. z represents the trivial element in $\pi_1(\Sigma)_{\mathrm{ab}}$. Thus we have shown that the left diagonal map is the zero map. But the bottom horizontal map is an isomorphism of a non-trivial group and the diagram commutes. We have thus obtained a contradiction.

(2) This statement follows immediately from Lemma 14.20 since we can project the wedge of circles onto any circle. Alternatively, a retraction from $\Sigma_{2,n}$ to a boundary component is sketched in Figure 272. The general case is treated the same way.[333] □

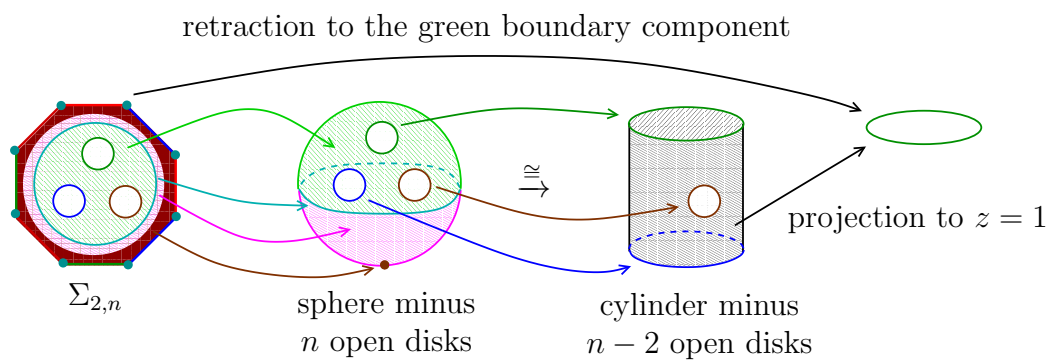

FIGURE 272.

Exercises for Chapter 14.

Exercise 14.1. Provide a proof for Lemma 14.4. Put differently, let $g \in \mathbb{N}_{\geq 2}$. Show that the group

$$\pi \;:=\; \pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \,|\, [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$$

is not abelian.
Hint: You could try to show that the group admits an epimorphism onto a non-abelian group.

[333] The proof is somewhat similar to the proof of Lemma 6.5.

Exercise 14.2. Let $g \in \mathbb{N}$. By Theorem 14.10 we know that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$. Determine k as a function of g.

Exercise 14.3. Prove that the two surfaces shown in Figure 267 are not homeomorphic.
Remark: Evidently a solution to a problem can only be as rigorous as its formulation. In this case one could perhaps consider the complements of suitable compact subsets.

Exercise 14.4.

(a) Let M be a oriented connected m-dimensional manifold and let N be a 1-dimensional submanifold *that is diffeomorphic to* S^1. Let $x_0 \in M \setminus N$. Suppose that $m \geq 4$. Show that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is an isomorphism.
Hint: Use the Tubular Neighborhood Theorem 10.19.

(b) Show that the conclusion (a) does not hold for $m = 3$, i.e. give an example of a connected 3-dimensional manifold M and a connected 1-dimensional submanifold N such that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is not an isomorphism.
Remark: the manifold M in the example does not have to be closed.

Exercise 14.5. Let F be the sphere minus k open disks. (Alternatively, let F be the disk $\overline{B}^2$ minus $k-1$ open disks.) We denote by $C_1, \ldots, C_k$ the boundary components of F. We pick a base point $x_0 \in F$. Furthermore for each $i \in \{1, \ldots, k\}$ we pick a path γ_i from a point on C_i to x_0. Finally let $\Gamma \subsetneq \pi_1(F, x_0)$ be a proper normal subgroup. Show that there exists an $i \in \{1, \ldots, k\}$ such that C_i, viewed as an element in $\pi_1(F, x_0)$ via the path γ_i, does not lie in Γ.
Hint: You might want to start out with your preferred base point x_0 and your preferred paths $\gamma_1, \ldots, \gamma_k$, e.g. as shown in Figure 273.

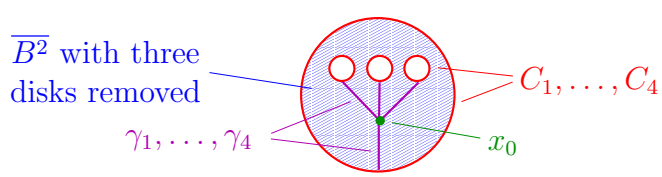

FIGURE 273.

15. Cones, suspensions, cylinders

In this section we first introduce two systematic ways to construct new topological spaces out of a given topological space and we present three systematic ways to construct new topological spaces out of a map $f\colon X \to Y$ between given topological spaces. Initially it might not be entirely apparent why we should be interested, but all five examples will play an important role at various points. The main excuse for presenting these constructions at this very moment is that they allow us to play a little bit with our newly acquired abilities for computing fundamental groups.

15.1. **The cone and suspension of a topological space.**

In this section we define the cone and suspension of a topological space. The cone of a topological space is a particularly easy construction.

Definition. Given a topological space X the *cone on* X is defined to be the topological space

$$\mathrm{Cone}(X) := (X\times[0,1])/(X\times\{0\}) = (X\times[0,1])/\sim \text{ where } (x,0)\sim(x',0) \text{ for all } x,x'\in X.$$

We refer to the point $X\times\{0\}$ as the *tip of the cone* $\mathrm{Cone}(X)$ or alternatively as the *cone point*.

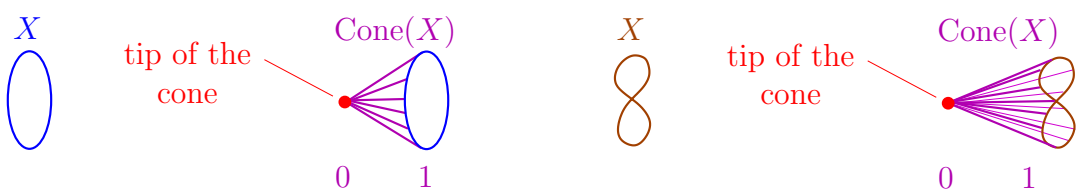

Figure 274.

Examples.

(1) Let $n\in\mathbb{N}_0$. It follows from Lemma 1.66 that the map

$$\begin{array}{rcl}\mathrm{Cone}(S^n) & \to & \overline{B}^{n+1}\\ [(v,t)] & \mapsto & v\cdot t\end{array}$$

is continuous. One can easily verify that the map is a bijection. It follows immediately from Lemmas 1.37 and 1.55 that $\mathrm{Cone}(S^n)$ is compact. Since $\overline{B}^{n+1}$ is Hausdorff we obtain from Proposition 1.40 that the given map is a homeomorphism.

(2) Let X be a subset of $\mathbb{R}^n$. We define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) := \text{ union of all line segments from } \underbrace{(x,0)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,1)}_{\in\mathbb{R}^{n+1}} \subset \mathbb{R}^{n+1}.$$

(a) In Exercise 15.1 we will see that if X is a compact subset of $\mathbb{R}^n$, then the map

$$\begin{array}{rcl} f_X\colon \operatorname{Cone}(X) & \to & \widetilde{\operatorname{Cone}}(X) \\ [(x,t)] & \mapsto & t\cdot(x,0)+(1-t)\cdot(0,1) \end{array}$$

is a homeomorphism.

(b) After looking at Figure 274 and considering Examples (1) and (2a) one might walk away with the feeling that one has a pretty good understanding of the cone construction. But this optimism is potentially ill-founded. For example in Exercise 15.1 we see will see that for $X=(0,1)\subset\mathbb{R}$ the above map $f_X\colon \operatorname{Cone}(X)\to\widetilde{\operatorname{Cone}}(X)$ from (2a) is *not* a homeomorphism.

This definition does not seem to be particularly interesting, but we have one lemma whose proof is perhaps slightly more subtle than one might initially think.

Lemma 15.1. *Let X be a topological space. The tip of the cone* $\operatorname{Cone}(X)$ *is a deformation retract of* $\operatorname{Cone}(X)$.

Proof. We consider the "obvious" map

$$\begin{array}{rcl} \varphi\colon \overbrace{(X\times[0,1])/(X\times\{0\})\times[0,1]}^{=\operatorname{Cone}(X)\times[0,1]} & \to & \overbrace{(X\times[0,1])/(X\times\{0\})}^{=\operatorname{Cone}(X)} \\ ([(x,s)],t) & \mapsto & [(x,s\cdot(1-t))] \end{array}$$

One might think that this map "obviously" does the trick. The wary reader will not have failed to notice that here we are mixing the product and the quotient topologies, which, as we have learned the painful way in Section 2.2, can be quite dangerous and can lead to non-continuous maps.

So let us show that the map φ is indeed continuous.[334] Since there is an abundance of intervals we now write $S=[0,1]$ and $T=[0,1]$. We consider the following two maps

$$\begin{array}{rcl} H\colon (X\times S)\times T & \to & X\times S \\ ((x,s),t) & \mapsto & (x,s\cdot(1-t)) \end{array} \quad\text{and}\quad \begin{array}{rcl} p\colon X\times S & \to & \operatorname{Cone}(X) \\ (x,s) & \mapsto & [(x,s)]. \end{array}$$

Note that H and p are continuous.[335] We have the following commutative diagram

$$\begin{tikzcd} & (X\times S)\times T \arrow[dl, "p\times\mathrm{id}_T"'] \arrow[dr, "p\circ H"] & \\ \operatorname{Cone}(X)\times T \arrow[rr, "\varphi"] & & \operatorname{Cone}(X). \end{tikzcd}$$

By Lemma 2.14 the map p is a quotient map. Since T is locally compact we obtain from Theorem 2.15 that $p\times\mathrm{id}_T$ is a quotient map. By the discussion above we know that $p\circ H$ is continuous. Thus it follows from Lemma 2.16 (2) that φ is continuous. $\square$

[334]The fact that one actually needs to prove that the map φ is continuous seems to get swept under the carpet in every single book on (algebraic) topology that I have consulted.

[335]The map p is continuous by definition of the topology on $\operatorname{Cone}(X)$. But what is a super-formal argument for showing that H is continuous?

We move on to the next definition.

Definition. Let X be a topological space. We define the *suspension of* X to be the topological space

$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.

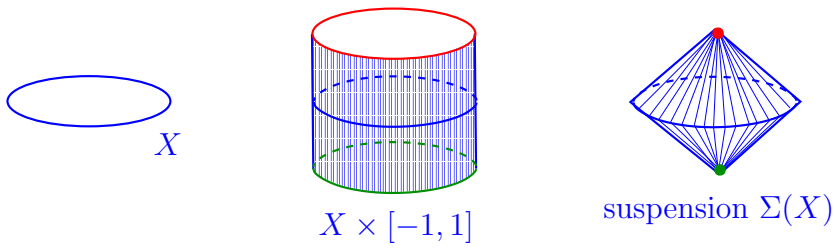

FIGURE 275.

Before we give an example we state a basic lemma regarding suspensions.

Lemma 15.2. *Let X be a topological space.*

(1) *If X is compact, then* $\mathrm{Cone}(X)$ *and the suspension* $\Sigma(X)$ *are also compact.*
(2) *If X is Hausdorff, then* $\mathrm{Cone}(X)$ *and the suspension* $\Sigma(X)$ *are also Hausdorff.*
(3) $(*)$ *The injections*

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are homeomorphisms onto their images. We will use these maps to view X as a subset of $\mathrm{Cone}(X)$ *and to view* $\mathrm{Cone}(X)$ *as a subset of* $\Sigma(X)$*. In particular we can and will view X as a subset of* $\Sigma(X)$*.*

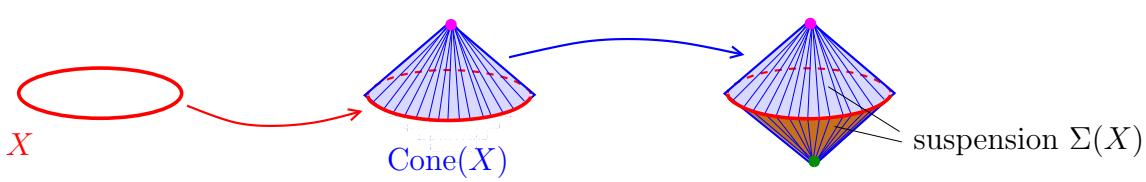

FIGURE 276. Illustration of Lemma 15.2 (3).

Proof.

(1) This statement follows from Lemmas 1.37 and 1.55.
(2) In Exercise 15.2 we sill show that $\Sigma(X)$ is Hausdorff, if X is Hausdorff. The proof for $\mathrm{Cone}(X)$ is basically identical.

(3) We will prove this statement in Exercise 15.3. □

Example. It is pretty clear from Figure 275 that the suspension of S^n is homeomorphic to S^{n+1}. In fact throughout the course we will use the following homeomorphism[336] to identify $\Sigma(S^n)$ with S^{n+1}:

$$\begin{array}{rcl} \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P\cos\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}}\big). \end{array}$$

Note that this homeomorphism sends the "North Pole" of $\Sigma(S^n)$ to the "North Pole" of S^{n+1}.

For future reference we record the following lemma.

Lemma 15.3. *If X is a path-connected topological space, then its suspension $\Sigma(X)$ is simply-connected.*

Proof $(*)$. Let $p\colon X \times [-1,1] \to \Sigma(X)$ be the obvious projection map. We consider the subsets $U = p(X \times [-1, \frac{1}{2})$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subset \Sigma(X)$. We calculate that

$$\pi_1(\Sigma(X),x) \underset{\substack{\uparrow \\ \text{Seifert–van Kampen} \\ \text{Theorem 14.1}}}{\cong} \pi_1(U,x) *_{\pi_1(U\cap V,x)} \pi_1(V,x) \underset{\substack{\uparrow \\ \text{as in Lemma 15.1 we see that } U \text{ and } V \text{ are contractible} \\ \text{hence their fundamental groups are trivial by Proposition 9.12 (2)}}}{=} \begin{array}{c}\text{amalgamated product} \\ \text{of trivial groups}\end{array} = \text{trivial group.}$$

We have thus shown that $\Sigma(X)$ is indeed simply connected. □

We conclude this discussion of suspensions with the fairly obvious notion of the suspension of a map.

Definition. Given a map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the *suspension of f*.

We conclude this section with the following lemma.

Lemma 15.4. $(*)$

(1) *If $f\colon X \to Y$ is a map between topological spaces, then $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also continuous.*
(2) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces, then the maps $\Sigma(g \circ f)$ and $\Sigma(g) \circ \Sigma(f)$ from $\Sigma(X)$ to $\Sigma(Z)$ agree.*

[336] Using Proposition 1.40 and Lemma 15.2 (1) one easily verifies that this map is indeed a homeomorphism.

(3) *The maps*

$$\begin{array}{rcl} X & \mapsto & \Sigma(X) \\ (f\colon X \to Y) & \mapsto & (\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \end{array}$$

define a covariant functor from the category of topological spaces to the category of topological spaces. We refer to this functor as the suspension functor*.*

(4) *Let $f, g\colon X \to Y$ be two maps between topological spaces. If f and g are homotopic rel a (possibly empty) subset A, then $\Sigma(f)$ and $\Sigma(g)$ are homotopic rel $\Sigma(A)$.*

(5) *If $f\colon X \to Y$ is a homotopy equivalence of topological spaces, then the suspension $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also a homotopy equivalence.*

Proof $(*)$.

(1) We consider the following commutative diagram

$$\begin{tikzcd} X \times [-1,1] \arrow[r, "f\times \mathrm{id}_{[-1,1]}"] \arrow[d] \arrow[dr, dashed] & Y \times [-1,1] \arrow[d] \\ \Sigma(X) \arrow[r, "\Sigma(f)"'] & \Sigma(Y). \end{tikzcd}$$

By Lemma 1.53 (3c) we know that the top horizontal map is continuous. By Lemma 1.65 (3) the right vertical map is continuous. It follows that the diagonal map is continuous. Finally we obtain from Lemma 1.66 that the bottom horizontal map is continuous.

(2) This statement follows immediately from (1) and (2) and the definitions.

(3) Clearly it suffices to prove the following claim.

Claim. If $F\colon X \times [0,1] \to Y$ is a map, then the map

$$\begin{array}{rcl} G\colon \overbrace{((X \times [-1,1])/\sim)}^{=\Sigma(X)} \times [0,1] & \to & \overbrace{(Y \times [-1,1])/\sim}^{=\Sigma(Y)} \\ ([(x,t)], s) & \mapsto & [(F(x,s),t)] \end{array}$$

is continuous.

Verifying this claim is more subtle than one might have thought.[337] We denote by $\sim_\times$ the equivalence relation on $(X \times [-1,1]) \times [0,1]$ that is generated by $((x,t),s) \sim_\times$

[337] At the very least it is more subtle than the author of these modest notes had been thinking for 20 years.

$((x',t'),s)$ whenever $(x,t) \sim (x',t')$. We consider the following diagram:

$$\begin{tikzcd}
((X\times[-1,1])/\sim)\times[0,1] \arrow[ddrr, "G"'] & & ((X\times[-1,1])\times[0,1])/\sim_\times \arrow[ll, "{([(x,t)],s)\leftarrow[((x,t),s)]}"'] \arrow[d, "{[(x,t,s)]\mapsto(F(x,s),t)}"] \\
& & Y\times[-1,1] \arrow[d, "{(y,t)\mapsto[(y,t)]}"] \\
& & (Y\times[-1,1])/\sim .
\end{tikzcd}$$

The top map is a homeomorphism by Lemma 2.17 and the fact that $[0,1]$ is locally compact. The top vertical map is continuous by Lemma 1.66 and the bottom vertical map is continuous by Lemma 1.65 (3). It is clear that the diagram commutes. It now follows that G is continuous. This concludes the proof of the claim.

(4) The last statement is an immediate consequence of the proof of (3). □

15.2. **Mapping cylinder, mapping cone and mapping torus.** In this section we associate to a map $f\colon X \to Y$ between two topological manifold three different topological spaces. The main interest in these topological spaces is that these topological spaces encode some of the information contained in the actual map. On numerous occasions this will allow us to translate questions about maps between topological spaces into questions about topological spaces.

We start out with the definition of a mapping cylinder.

Definition. Let $f\colon X \to Y$ be map between topological spaces. We define the corresponding *mapping cylinder* to be the topological space

$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon X\to Y) := (X\times[0,1]\sqcup Y)/\sim \quad \text{where } (x,1)\sim f(x) \text{ for all } x\in X.$$

We illustrate the definition in Figure 277.

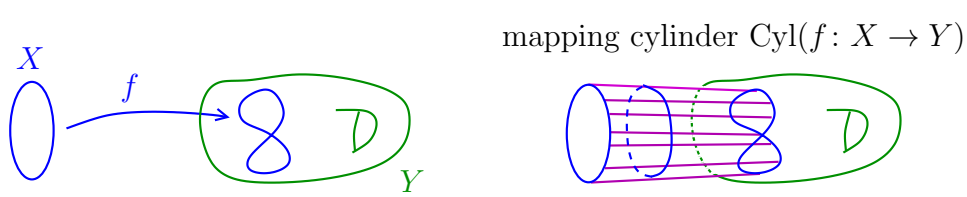

FIGURE 277.

Example. If X is a topological space and if $f\colon X \to \{y\}$ is the constant map to the topological space consisting of a single element y, then the mapping cylinder $\mathrm{Cyl}(f)$ is, basically by definition, homeomorphic to $\mathrm{Cone}(X)$.

Definition. Let $f\colon X \to Y$ be map between topological spaces. We refer to the maps

$$\begin{array}{rcl} X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

as the *natural inclusions*. Sometimes we use these inclusions to identify X and Y with the corresponding subsets of the mapping cylinder.

The following lemma says that the natural inclusions are homeomorphisms onto their images. This means that we can legitimately identify X and Y with their images in the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$. On many occasions we will implicitly use the lemma without actually referring to it.

Lemma 15.5. $(*)$ *If $f\colon X \to Y$ is a map between topological spaces, then the natural inclusions $X \to \mathrm{Cyl}(f\colon X \to Y)$ and $Y \to \mathrm{Cyl}(f\colon X \to Y)$ are homeomorphisms onto their images, in particular they are continuous.*

Proof. We will provide the proof in Exercise 15.4. □

We continue with a (non-) example. We will not really make use of this lemma, but Lemma 15.6 should warn the reader that the topology on a mapping cylinder is not necessarily what one might think it is.

Lemma 15.6. $(*)$ *Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map and we consider the map*

$$\begin{array}{rcl} \Theta\colon\ \mathrm{Cyl}(i) = ((A \times [0,1]) \sqcup X)/(a,1) \sim i(a) & \to & \overbrace{(A \times [0,1]) \cup (X \times \{0\})}^{\substack{\text{equipped with the subspace topology} \\ \text{coming from the product } X \times [0,1]}} \\ [P] & \mapsto & \begin{cases} P, & \text{if } P \in A \times [0,1], \\ (P,1), & \text{if } P \in X. \end{cases} \end{array}$$

The following two statements hold:

(1) *The map Θ is continuous and it is a bijection.*
(2) *If A is a closed subset of X, then Θ is a homeomorphism.*
(3) *There exist situations, e.g. $X = [0,\infty)$ and $A = [0,1)$, in which Θ is* not *a homeomorphism.*

Proof.

(1) It follows easily from Lemma 1.66 that the given map Θ is continuous. Furthermore it is basically clear that Θ is a bijection.
(2) Suppose that A is a closed subset of X. Note that $A \times \{1\}$ is a closed subset of $A \times [0,1]$. It follows basically from Lemma 1.84 that Θ is a homeomorphism.
(3) We consider the case $X = [0,\infty)$ and $A = [0,1)$. We denote by

$$p\colon (A \times [0,1)) \sqcup X \ \to\ \mathrm{Cyl}(i) = ((A \times [0,1)) \sqcup X)/\sim$$

the projection map. We consider the open set $U := \{(x,t) \in [0,1) \times [0,1] \,|\, x > t\}$ of $A \times [0,1)$. Furthermore we consider the open subset $V := [0,2)$ of $X = \mathbb{R}$. Note

that $p^{-1}(p(U \sqcup V)) = U \sqcup V$. Thus it follows immediately from the definition of the quotient topology on $\mathrm{Cyl}(i(= ((A \times [0,1)) \sqcup X)/\sim$ that $p(U \sqcup V)$ is an open subset of $\mathrm{Cyl}(i)$. On the other hand we have

$$\Theta(p(U \sqcup V)) \;=\; U \cup \{(x,1) \,|\, x \in V\},$$

but considering the point $(1,1)$ one can easily verify that this is *not* an open subset of $(A \times [0,1]) \cup (X \times \{0\})$. □

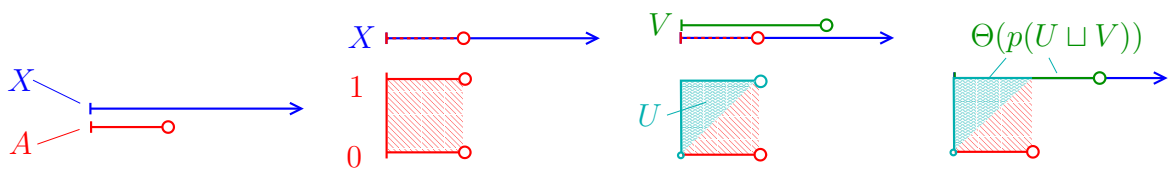

FIGURE 278. Illustration of the proof of Lemma 15.6 (3).

The following lemma summarizes some basic properties of the mapping cylinder.

Lemma 15.7. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *The subspace Y is a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$, in fact an explicit deformation retraction is given by the following map*

$$\begin{aligned} H\colon \mathrm{Cyl}(f\colon X \to Y) \times [0,1] \;&\to\; \mathrm{Cyl}(f\colon X \to Y) \\ (P,t) \;&\mapsto\; \begin{cases} P, & \text{if } P \in Y, \\ [(Q, s\cdot(1-t)+t)], & \text{if } P = [(Q,s)] \text{ where,} \\ & \quad Q \in X \text{ and } s \in [0,1]. \end{cases} \end{aligned}$$

(2) *We denote by $r = H_1$ the retraction $\mathrm{Cyl}(f\colon X \to Y) \to Y$ given by (1). If we denote by $i\colon X \to \mathrm{Cyl}(f)$ and $j\colon Y \to \mathrm{Cyl}(f)$ the natural inclusions, then the following statements hold:*
   (a) $r \circ j = f\colon X \to Y$,
   (b) *the map $j\colon Y \to \mathrm{Cyl}(f\colon X \to Y)$ is a homotopy equivalence,*
   (c) *the map $i\colon X \to \mathrm{Cyl}(f\colon X \to Y)$ is a homotopy equivalence if and only if the map $f\colon X \to Y$ is a homotopy equivalence.*
   (d) *Let $y_0 \in Y$ be a point. The inclusion induced map $\pi_1(Y, y_0) \to \pi_1(\mathrm{Cyl}(f), y_0)$ is an isomorphism.*

Remark. On page 455 we saw that the mapping cylinder of a constant map $f\colon X \to \{y\}$ is homeomorphic to the cone on X. Thus we see that Lemma 15.7 (1) can be viewed as a generalization of Lemma 15.1.

Proof.

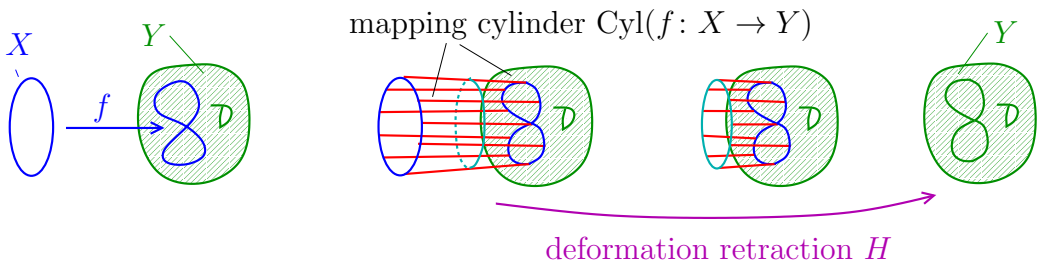

FIGURE 279. Illustration of Lemma 15.7 (1).

(1) We need to show that the map H is continuous. First note that the two maps

$$\begin{array}{rcl} (X\times[0,1])\times[0,1] & \to & \mathrm{Cyl}(f) \\ (Q,t) & \mapsto & [(Q, s\cdot(1-t)+t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} Y\times[0,1] & \to & \mathrm{Cyl}(f) \\ (P,t) & \mapsto & P \end{array}$$

are continuous[338]. But this implies, by the somewhat delicate Lemma 9.16, that

$$H\colon ((X\times[0,1])\cup_{(x,1)\mapsto f(x)} Y)\times[0,1] \ \to\ \mathrm{Cyl}(f)$$

itself is continuous. Once continuity is out of the way it is basically clear that H is a deformation retraction. Therefore it follows from Lemma 9.10 that the inclusion $Y\to\mathrm{Cyl}(f\colon X\to Y)$ is a homotopy equivalence.

(2) It is clear that $r\circ j = f$. It follows from Lemma 9.10 that j and r are homotopy equivalences. Therefore (c) is a consequence of Lemma 9.8 (1). Finally (d) follows immediately from (2b) together with Proposition 9.12 (1). □

In the above discussion we associated to a map $f\colon X\to Y$ between topological spaces a new topological space, namely the mapping cylinder $\mathrm{Cyl}(f\colon X\to Y)$. But we see in Lemma 15.7 (1) that the mapping cylinder, as a topological space, is a little, it is basically the same as Y. Therefore we move on to the definition of the mapping cone which is a much more interesting construction.

Definition. Given a map $f\colon A\to X$ between topological spaces the corresponding *mapping cone* is defined as

$$\mathrm{Cone}(f\colon A\to X) \ :=\ (\mathrm{Cone}(A)\sqcup X)/\sim \ \text{ where } [(a,1)]\sim f(a) \text{ for all } a\in A.$$

We refer to the point $[A\times\{0\}]$ as the *cone point* of the mapping cone. We illustrate the definition in Figure 280.

Remark. Let $f\colon A\to X$ be a map between topological spaces. It follows immediately from the definitions that

$$\mathrm{Cone}(f\colon A\to X) \ =\ \mathrm{Cyl}(f\colon A\to X)/(A\times\{0\}).$$

The following lemma summarizes some properties of mapping cones.

[338]For once this is basically obvious

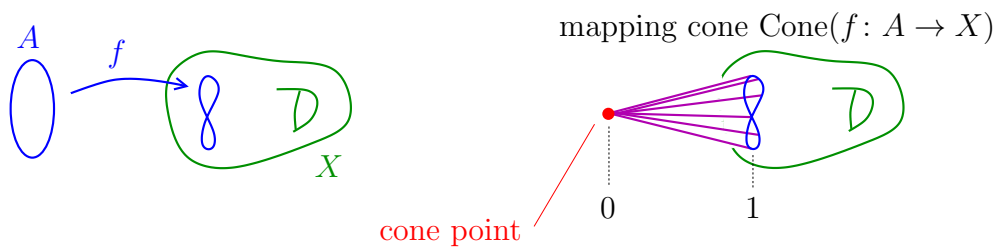

FIGURE 280.

Lemma 15.8. (∗) *Let $f\colon A \to X$ be a map between topological spaces.*

(1) *The natural inclusion*

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(f\colon A \to X) \\ x & \mapsto & [x] \end{array}$$

is a homeomorphism onto its image. Often we will use this natural inclusion to identify X with its image in $\mathrm{Cone}(f\colon A \to X)$.

(2) *The map*

$$\begin{array}{rcl} \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ [(a,t)] & \mapsto & [(a, 1-2t)] \end{array}$$

is a natural homeomorphism.

(3) *The cone point defines a closed subset of* $\mathrm{Cone}(f\colon A \to X)$.

(4) *The subset X is a deformation retract of the complement of the cone point.*

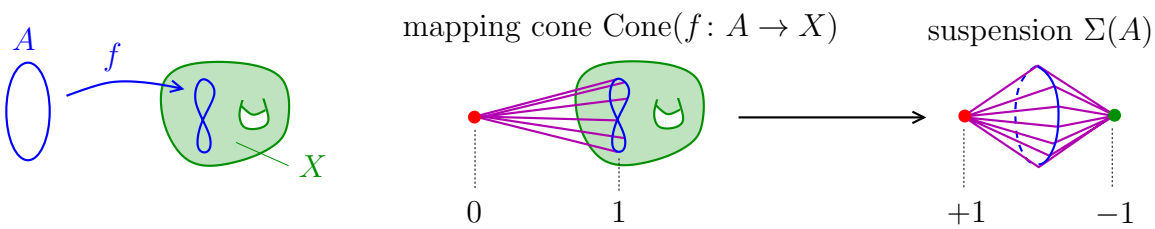

FIGURE 281. Illustration of Lemma 15.8 (2).

Sketch of proof.

(1) This statement can be proved easily by hand or it can be deduced from Lemma 15.5 together with Exercise 1.38 (3).

(2) It is clear that the given map is a bijection. One can easily verify, without any extra wizardry, that the given map is indeed a homeomorphism.

(3) This statement follows quite easily from the definition of the topologies involved.

(4) The proof of this statement is basically identical to the proof of Lemma 15.7 (1). □

So what have we done so far? We have associated to a map $f\colon X \to Y$ between topological spaces the topological space $\mathrm{Cone}(f\colon X \to Y)$. This construction will allow us to turn questions about maps between topological spaces into questions about topological spaces. As always in algebraic topology, not only do we want a map

$$\text{set of maps between topological spaces} \;\to\; \text{set of topological spaces},$$

but we really want a functor. It is clear what the category should be on the right. It is a little less clear what the category should be on the left. Here it is.

Definition. We refer to the category $\mathcal{M}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{M}) &:= \text{all maps between topological spaces,} \\ \mathrm{Mor}(f\colon X \to Y, \widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) &:= \begin{array}{c} \text{all pairs of maps} \\ (\varphi\colon X \to \widetilde{X}, \psi\colon Y \to \widetilde{Y}) \\ \text{such that the diagram} \end{array} \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array} \text{ commutes} \end{aligned}$$

with the (hopefully) obvious composition of morphisms as the *category of maps between topological spaces.*

The following lemma says that we can view "taking the mapping cone" and "taking the mapping cylinder" as covariant functors.

Lemma 15.9. $(*)$

(1) *The maps*

$$(f\colon X \to Y) \mapsto \mathrm{Cyl}(f\colon X \to Y)$$

and

$$(\varphi\colon X \to \widetilde{X}, \psi\colon Y \to \widetilde{Y}) \mapsto \left(\begin{array}{l} \mathrm{Cyl}(f\colon X \to Y) \to \mathrm{Cyl}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) \\ \qquad\qquad [P] \mapsto \begin{cases} [(\varphi(x), t)], & \text{if } P = (x,t) \text{ with,} \\ & \quad x \in X,\ t \in [0,1] \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array} \right).$$

define a covariant functor from the category $\mathcal{M}$ of maps between topological spaces to the category $\mathcal{T}$ of topological spaces.

(2) *If, in the setting of* (1)*, the maps φ and ψ are homeomorphisms onto their images, then the corresponding map* $\mathrm{Cyl}(f\colon X \to Y) \to \mathrm{Cyl}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y})$ *is also a homeomorphism onto its image.*

(3) *The analogues of* (1) *and* (2)*, with obvious modifications, hold if we replace mapping cylinders by mapping cones.*

(4) *Given a morphism*[339]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array}$$

the map from (3) *for mapping cones fits into the following commutative diagram:*

$$\begin{array}{ccccc} Y & \hookrightarrow & \mathrm{Cone}(f\colon X \to Y) & \longrightarrow & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \hookrightarrow & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \longrightarrow & \Sigma(\widetilde{X}) \end{array}$$

where the horizontal maps to the left are the inclusions maps introduced in Lemma 15.8 (1) and where the horizontal maps are given by the projection and the homeomorphism from Lemma 15.8 (2).

Proof (∗). One needs to convince oneself that the map between the mapping cylinders and the mapping cones is in fact continuous. For the mapping cylinders that is an immediate consequence of Lemma 1.83 (3). For the mapping cones first note that the same argument as in the proof of Lemma 15.4 (1) shows that the map

$$\begin{array}{rcl} \mathrm{Cone}(X) & \to & \mathrm{Cone}(\widetilde{X}) \\ [(x,t)] & \mapsto & [(\varphi(x),t)] \end{array}$$

is continuous. The continuity of the map between the mapping cones now follows again from Lemma 1.83 (3).

The second statement will be proved in Exercise 15.6. Finally statement (4) of the lemma follows basically from the definitions. □

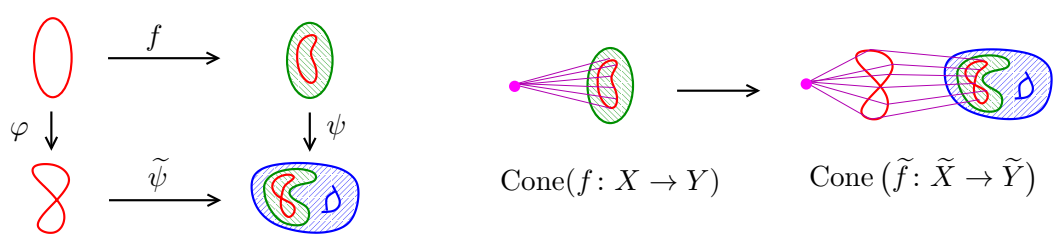

FIGURE 282. Illustration of Lemma 15.9.

The following lemma shows in particular that homotopic maps give rise to homotopy equivalent mapping cones. In particular at times this lemma allows us to translate the question whether two maps are homotopic to the question whether two topological spaces are homotopy equivalent.

[339]In fancy speak this commutative diagram says that the inclusion maps into the mapping cone and the projection from the mapping cone to the suspension are natural transformations between suitable functors.

Proposition 15.10. *Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. If f_0, f_1 are homotopic, then there exists a homotopy equivalence* $\mathrm{Cone}(f_0) \to \mathrm{Cone}(f_1)$ *rel the union of X and the cone point.*

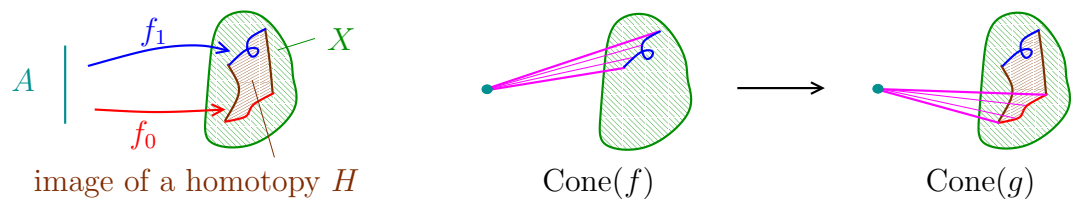

FIGURE 283. Illustration of Proposition 15.10.

It is not that difficult to write down explicit maps between the cones and to write down explicit homotopies. But we prefer to go take a more conceptual approach. Namely, instead of proving Proposition 15.10 directly we will prove the following more technical lemma.

Lemma 15.11. $(*)$ *Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. Let $j \in \{0,1\}$. We denote by $i_j \colon A \to A \times \{j\}$ the obvious inclusion map.*

(1) *The inclusion map i_j induces by Lemma 15.9(3) a map* $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$. *This map is a homeomorphism onto its image.*
(2) *If we use the map from* (1) *to view* $\mathrm{Cone}(f_j)$ *as a subset of* $\mathrm{Cone}(H)$, *then it is a deformation retract of* $\mathrm{Cone}(H)$.

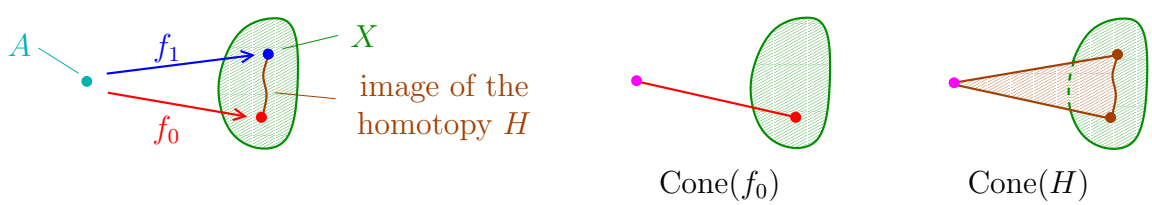

FIGURE 284. Illustration of Lemma 15.11.

Proof of Proposition 15.10 assuming Lemma 15.11 $(*)$. Let H be a homotopy between f_0 and f_1. By Lemma 15.11 we can view $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ as deformation retracts of $\mathrm{Cone}(H)$. It follows from Lemma 9.10 and Lemma 9.8 that $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent. A quick inspection of the maps involved shows that the two cones are homotopy equivalent rel the union of X and the cone point. $\square$

Proof of Lemma 15.11 (∗). Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. For notational convenience we only deal with the case $j = 0$. We denote by $i \colon A \to A \times \{0\}$ the obvious inclusion map. We get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_0} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle =} \\ A \times [0,1] & \xrightarrow{H} & X. \end{array}$$

Thus we get from Lemma 15.9 (3) a map $\mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$. It follows almost immediately from Lemma 15.9 (3) that this map is a homeomorphism onto its image.

Since this proof suffers from an abundance of intervals playing different roles we rescale two of the intervals:

(1) We use the interval $[0,1]$ for the homotopy H.
(2) We use the interval $[0,2]$ for the definition of the mapping cone.
(3) We use the interval $[0,3]$ for the definition of the desired deformation retraction.

With this convention out of the way we pick a deformation retraction

$$\varphi \colon ([0,1] \times [0,2]) \times [0,3] \ \to \ [0,1] \times [0,2]$$

rel $([0,2] \times \{0\}) \cup (\{2\} \times [0,1])$ such that $\varphi(r,s,t) = (r, \frac{1}{3}st)$ whenever $r \in [0, \frac{1}{2}]$. It is elementary to see that such a deformation retraction exists. We refer to Figure 285 for an illustration.

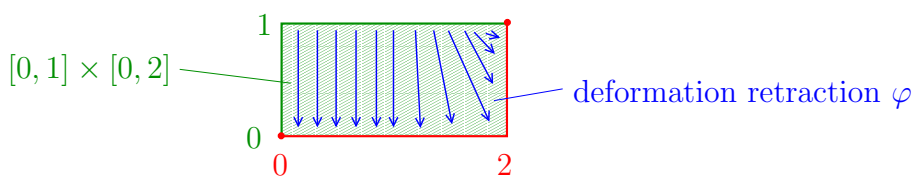

FIGURE 285.

Now we consider the map

$$\begin{array}{rcl} G \colon (\overbrace{((A \times [0,1] \times [0,2]) \sqcup X)/\sim}^{=\mathrm{Cone}(H\colon A\times[0,1]\to X)}) \times [0,3] & \to & \mathrm{Cone}(H) \\ ([P], z) & \mapsto & \begin{cases} [(a, r(x,y,z))], & \text{if } P = (a,x,y), \\ P, & \text{if } P \in X. \end{cases} \end{array}$$

Using Lemma 1.65 (3) and Lemma 9.16 one can easily show that this map G is continuous. It is straightforward to verify that this map has the desired properties. □

For completeness sake we also state the following corollary.

Corollary 15.12. $(*)$ *Let $f\colon A \to X$ be a map between topological space and let $g\colon X \to Y$ be a homotopy equivalence. The map*

$$\begin{array}{rcl} \mathrm{Cyl}(f\colon A \to X) & \to & \mathrm{Cyl}(g \circ f\colon A \to Y) \\ [P] & \mapsto & \begin{cases} [(a,t)], & \text{if } P = (a,t) \in A \times [0,1], \\ [g(P)], & \text{if } P \in X \end{cases} \end{array}$$

is a homotopy equivalence rel $A \times \{0\}$. The same way one sees that the mapping cones of f and g are homotopy equivalent.

Example. Let $f\colon S^1 \to \mathbb{R}^2$ be the inclusion and let $g\colon \mathbb{R}^2 \to \{*\}$ be the retraction to the origin. In Figure 286 we show the corresponding mapping cones, which we now know are homotopy equivalent by Corollary 15.12.

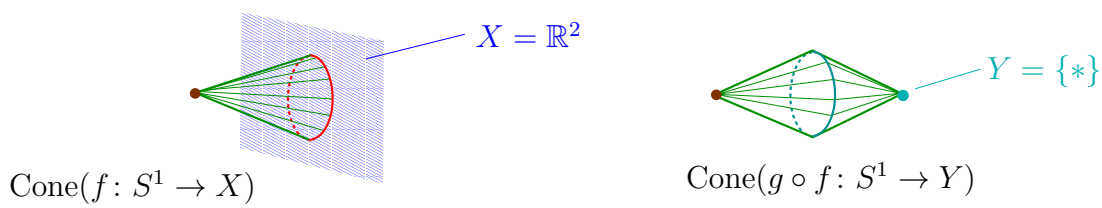

FIGURE 286.

Proof. The corollary is a straightforward consequence of Proposition 15.10. We leave it to the reader to fill in the details. □

The following lemma calculates the fundamental groups of mapping cones.

Lemma 15.13. $(*)$ *Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. If A is path-connected, then the inclusion induces an isomorphism*

$$\pi_1(X, x_0)/f_*(\pi_1(A, a_0)) \;\to\; \pi_1(\mathrm{Cone}(f\colon A \to X), [x_0]).$$

Example. Let $n \in \mathbb{N}$ and let $A = X = S^1$. We consider the map $f\colon A = S^1 \to X = S^1$ that is given by $f(z) = z^n$. It follows from Lemma 15.13 and Proposition 7.18, perhaps counter-intuitively, that $\pi_1(\mathrm{Cone}(f\colon A \to X)) \cong \mathbb{Z}_n$. We refer to Figure 287 for an attempt at an illustration of this example.

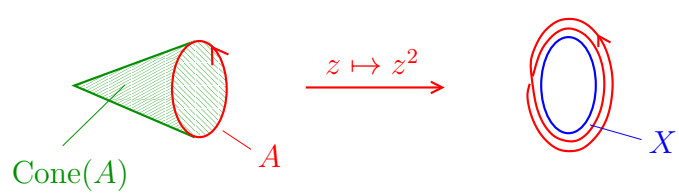

FIGURE 287.

Proof. We will provide the proof in Exercise 15.7. □

We conclude our discussion of mapping cones with the following proposition.

Proposition 15.14. (∗) *Let X be a topological space and let A be a subset of X. We denote by $i\colon A \to X$ the inclusion map. We use Lemma 15.8 (1) to view X as a subset of* $\mathrm{Cone}(i\colon A \to X)$.

(1) *If A is contractible, then given any $a_0 \in A$ there exists a deformation retraction G from* $\mathrm{Cone}(i\colon A \to X)$ *to X such that $G_1(\text{cone point } [A \times \{0\}]) = a_0$.*
(2) *If $a_0 \in A$ is a deformation retract of A, then there exists a deformation retraction G from* $\mathrm{Cone}(i\colon A \to X)$ *to X such that for any $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$.*

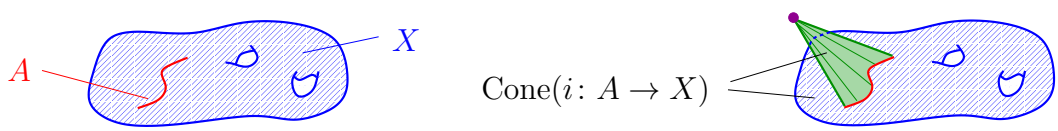

FIGURE 288. Illustration of Proposition 15.14.

Proof. We first prove Statement (1). Thus let $a_0 \in A$. Since A is contractible it follows from Lemma 9.11 (2) that there exists a map $r\colon A \times [0,1] \to A$ such that $r_0 = \mathrm{id}$ and such that $r_1(a) = a_0$ for all $a \in A$. We pick continuous functions $x, y\colon [0,1] \times [0,1] \to [0,1]$ with the following properties:

(a) For any $t \in [0,1]$ we have $x(0,t) = 1$ or $y(0,t) = 0$.
(b) We have $x(s,0) = 0$ and $y(s,0) = s$ for all $s \in [0,1]$.
(c) We have $y(s,1)$ for all $s \in [0,1]$.
(d) We have $x(1,t) = 0$ and $y(1,t) = 1$ for all $t \in [0,1]$.

We sketch graphs of such functions x and y in Figure 289. Now we consider the map

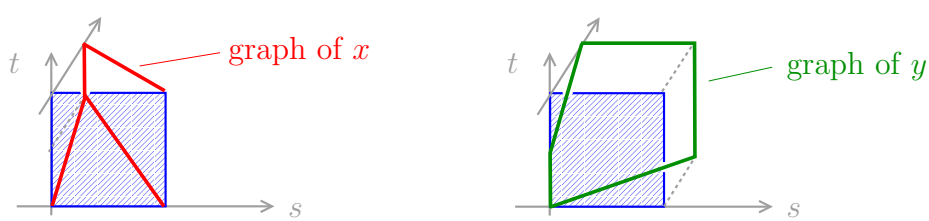

FIGURE 289.

$$\begin{array}{rcl} F\colon \overbrace{((A \times [0,1])/(A \times \{0\}))}^{=\mathrm{Cone}(A)} \times [0,1] & \to & \overbrace{(A \times [0,1])/(A \times \{0\})}^{=\mathrm{Cone}(A)} \\ ([(a,s)],t) & \mapsto & [(r(a,x(s,t)), y(s,t))]. \end{array}$$

We make the following observations:

(0) Since x and y are continuous and since $[0,1]$ is locally compact we obtain from Lemmas 1.66 and Lemma 2.17 that F is continuous.
(1) It follows from (a) that F is actually well-defined.
(2) It follows from (b) that $F_0 = \mathrm{id}$.
(3) It follows from (c) that F_1 takes values in $A \times \{1\}$.
(4) It follows from (d) that F is a homotopy rel $A \times \{1\}$.

In summary this shows that F is a deformation retraction from $\mathrm{Cone}(A)$ to $A \times \{1\}$. Now we consider the map

$$\begin{array}{rcl} G\colon \overbrace{(\mathrm{Cone}(A) \cup_{(a,1)=a} X)}^{=\mathrm{Cone}(i\colon A\to X)} \times [0,1] & \to & \mathrm{Cone}(i\colon A \to X) \\ ([P],t) & \mapsto & \begin{cases} [F(P,t)] & \text{if } P \in \mathrm{Cone}(A), \\ [P], & \text{if } P \in X. \end{cases} \end{array}$$

It follows from Lemma 9.16 that this map is continuous. Clearly G is a deformation retract from $\mathrm{Cone}(i\colon A \to X)$ to X. By construction it sends the cone point to a_0.

It remains to prove Statement (2). If we start out with a deformation retraction r form A to a_0, then the above homotopy F has the desired property. □

15.3. **The mapping cone of a suspension** $(*)$**.** In this section we show that under favorable circumstances the operation of taking a mapping cone (or mapping cylinder) commutes with the operation of taking the suspension of a map. This technical proposition will play a useful role later on. The proposition is also a good way to test whether we really understand the various topologies involved in the definition of mapping cones, mapping cylinders and suspensions.

Proposition 15.15. *If $f\colon X \to Y$ is a map between topological spaces, then the map*[340]

$$\begin{array}{rcl} \Phi\colon \Sigma(\mathrm{Cone}(f\colon X \to Y)) & \to & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \\ P & \mapsto & \begin{cases} [([(x,t)],s)], & \textit{if } P = [([(x,s)],t)] \textit{ with } x \in X,\ t \in [-1,1] \\ & \qquad \textit{and with } s \in [0,1], \\ [([y],t)], & \textit{if } P = [([y],t)] \textit{ with } y \in Y \textit{ and } t \in [-1,1]. \end{cases} \end{array}$$

is a homeomorphism.

In the proof of Proposition 15.15 we will need the following lemma which is quite similar to Proposition 2.19.

Lemma 15.16. *Let X be a topological space. If Y is a locally compact topological space (e.g. $Y = [0,1]$), then the map*

$$\begin{array}{rcl} \varphi\colon \Sigma(X) \times Y & \to & \Sigma(X \times Y) \\ ([(x,t)],y) & \mapsto & [((x,y),t)] \end{array}$$

is continuous.

[340] It takes a major effort to disentangle the notation. We write $[\,-\,]$ to indicate the equivalence class corresponding to a suspension and we write $[\,-\,]$ to indicate the equivalence class corresponding to the equivalence relation used to define mapping cones.

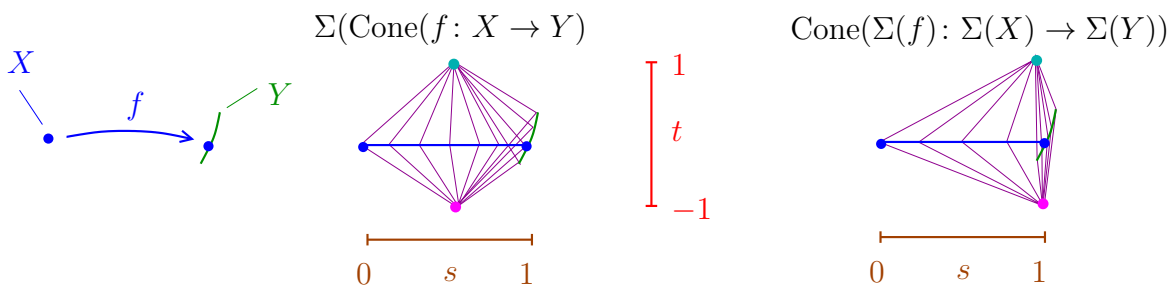

FIGURE 290. Illustration of Proposition 15.15.

In Figure 291 we illustrate Lemma 15.16. This cute picture is not meant to hide the grim reality that for arbitrary topological spaces X it actually requires a non-trivial input to prove Lemma 15.16.

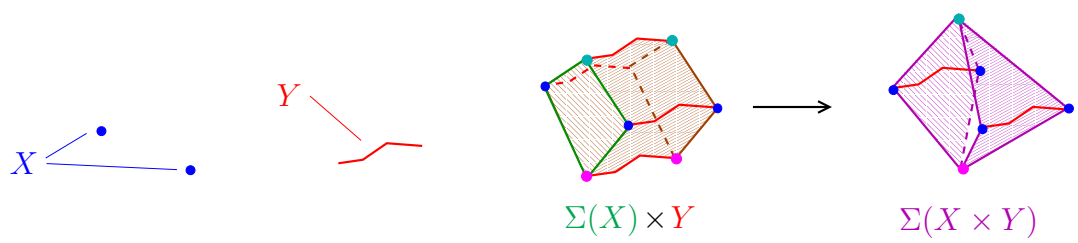

FIGURE 291. Illustration of Lemma 15.16.

Proof. We denote by $p\colon (X \times Y) \times [-1,1] \to \Sigma(X \times Y)$ and $q\colon X \times [-1,1] \to \Sigma(X)$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd}
(X \times [-1,1]) \times Y \arrow[r, "(x,t,y)\mapsto(x,y,t)", "\cong"'] \arrow[d, "q\times \mathrm{id}_Y"'] & (X \times Y) \times [-1,1] \arrow[d, "p"] \\
\Sigma(X) \times Y \arrow[r, "\varphi"] & \Sigma(X \times Y).
\end{tikzcd}$$

By Lemma 1.53 the top horizontal map is a homeomorphism. By Lemma 2.14 the maps p and q are quotient maps. Since Y is locally compact we obtain from Theorem 2.15 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 2.16 that φ is also a quotient map. By Lemma 2.14 (2) this implies that φ is indeed continuous. $\square$

Proof of Proposition 15.15. Let $f\colon X \to Y$ be a map between topological spaces. One can easily verify that the given map Φ is a bijection. It remains to show that Φ and Φ^{-1} are continuous. We will do so in the following two claims. The domain and the target of Φ are both a scary mix of product and quotient topologies. Thus we need to tread super carefully.

Claim. The map Φ is continuous.

We introduce the following notation:

(a) We consider the map $\varphi\colon X \times \{1\} \to Y$ that is given by $\varphi(x,1) = f(x)$.
(b) We denote by $p\colon \operatorname{Cone}(f\colon X \to Y) \times [-1,1] \to \Sigma(\operatorname{Cone}(f\colon X \to Y))$ the projection map.

We consider the following diagram:

$$\begin{tikzcd}
(X \times [0,1]) \times [-1,1] \arrow[d, "i"] & Y \times [-1,1] \arrow[d, "j"] & \\
((X \times [0,1]) \times [-1,1]) \cup_{\varphi \times \mathrm{id}_{[-1,1]}} \arrow[d, "{[(P,t)] \mapsto ([P],t)}"] & (Y \times [-1,1]) \arrow[ddr, bend left, "=:\Psi"] & \\
((X \times [0,1]) \cup_{\varphi} Y) \times [-1,1] \arrow[d, "="] & & \\
\operatorname{Cone}(f\colon X \to Y) \times [-1,1] \arrow[d, "q", "{(P,t) \mapsto [(P,t)]}"'] & & \\
\Sigma(\operatorname{Cone}(f\colon X \to Y)) \arrow[rr, "\Phi"] & & \operatorname{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)).
\end{tikzcd}$$

We make the following observations:

(1) By Lemma 2.18 we know that the vertical map between the second and the third row is a homeomorphism.
(2) It follows from (1) together with Lemma 1.66, applied to the projection map q, that Φ is continuous if Ψ is continuous.
(3) It follows from Lemma 1.83 (3) that Ψ is continuous if $\Psi \circ i$ and $\Psi \circ j$ are continuous.
(4) We have the following commutative diagram:

$$\begin{tikzcd}
Y \times [-1,1] \arrow[rr, "\Psi \circ i"] \arrow[dr, "{(P,t) \mapsto [(P,t)]}"'] & & \operatorname{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)). \\
& \Sigma(Y) \arrow[ur, "{Q \mapsto [Q]}"'] &
\end{tikzcd}$$

The diagonal maps are continuous by Lemma 1.65 (3) and Lemma 1.83 (3). Thus we see that $\Psi \circ i$ is continuous. The same argument shows that $\Psi \circ j$ is continuous.

The combination of (2), (3) and (4) shows that Φ is indeed continuous. This concludes the proof of the claim.

Claim. The map

$$\Phi^{-1}\colon \overbrace{\mathrm{Cone}(\Sigma(f)\colon \Sigma(X)\to\Sigma(Y))}^{=((\Sigma(X)\times[0,1])\sqcup\Sigma(Y))/\sim} \to \Sigma(\mathrm{Cone}(f\colon X\to Y))$$
$$P \mapsto \begin{cases} [([(x,s)],t)], & \text{if } P=[([(x,t)],s)] \text{ with } s\in[0,1] \text{ and} \\ & \text{with } [(x,t)]\in\Sigma(X), \text{ i.e. } x\in X,\ t\in[-1,1], \\ [([y],t)], & \text{if } P=[([y],t)] \text{ with } t\in Y,\ t\in[-1,1]. \end{cases}$$

is continuous.

By Lemma 1.83 it suffices to show that the maps

$$\begin{array}{rcl} \Sigma(X)\times[0,1] & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y)) \\ ([(x,t)],s) & \mapsto & [([(x,s)],t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Sigma(Y) & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y)) \\ [(y,t)] & \mapsto & [([y],t)] \end{array}$$

are continuous. The second map is continuous by Lemma 15.4 (1). Furthermore it follows from the combination of Lemma 15.16 with Lemma 15.4 (1) that the first map is continuous. This concludes the proof of the claim. □

15.4. **The mapping torus.** In this last section we discuss the mapping torus. This construction gives rise to many interesting topological spaces. We will also see that determining the fundamental group of a mapping torus is somewhat challenging, but not impossible.

Definition. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism.[341] We refer to

$$M(X,f) \;:=\; (X\times[0,1])\,/\,(x,0)\sim(f(x),1)$$

as the *mapping torus of* (X,f). We refer to the map

$$\begin{array}{rcl} M(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & e^{2\pi i t} \end{array}$$

as *the canonical projection onto* S^1. Sometimes X is called the *fiber* of the mapping torus and f is called the *monodromy* of the mapping torus.

The following lemma takes care of our two favorite properties of topological spaces, namely compactness and being Hausdorff.

Lemma 15.17. *Let X be a topological space and let $f\colon X\to X$ be a homeomorphism.*

(1) *If X is Hausdorff, then the mapping torus $M(X,f)$ is Hausdorff.*
(2) *If X is compact, then the mapping torus $M(X,f)$ is compact.*

Proof.

(1) This statement will be proved in Exercise 15.8.
(2) This statement follows immediately from Lemma 1.55 (2) and Lemma 1.37. □

[341] In principle one could also work with maps $f\colon X\to X$ that are not homeomorphisms. But for simplicity we only discuss the case that f is in fact a homeomorphism.

Next we will see that several well-established topological spaces are actually mapping tori.

Examples.

(1) We consider the topological space $[0,1]$ and let $f\colon [0,1] \to [0,1]$ be the self-homeomorphism of $[0,1]$ that is given by $f(w) = 1-w$. It follows immediately from the definitions that the corresponding mapping torus $M([0,1], f)$ is precisely the Möbius band as defined on page 117. We refer to Figure 292 for an illustration.

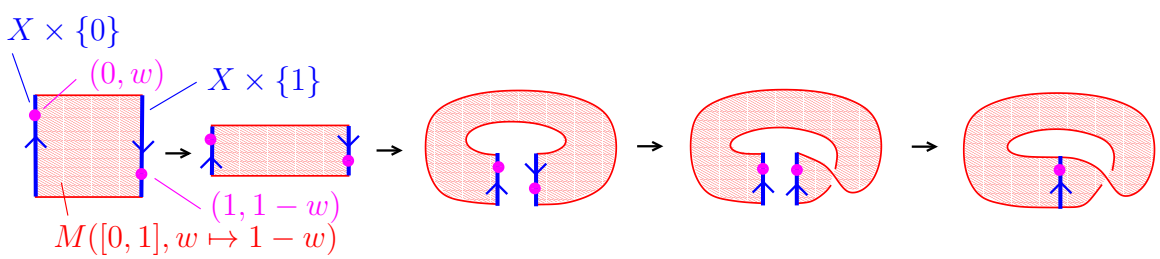

FIGURE 292. The Möbius band as a mapping torus.

(2) Let $X = S^1$ and let $f\colon X \to X$ be the homeomorphism that is given by $f(z) = \overline{z}$. It follows easily from Proposition 1.40 and Lemma 15.17 (1) that the map

$$\begin{array}{rcl} \overbrace{([0,1]\times[0,1])\,/\,(x,0)\sim(x,1),(0,y)\sim(1,1-y)}^{\text{Klein bottle}} & \to & M(S^1,f) \\ [(x,y)] & \mapsto & [(e^{2\pi i y},x)] \end{array}$$

is a homeomorphism from the Klein bottle to the mapping torus $M(S^1, f)$.[342] This homeomorphism is also illustrated in Figure 293.

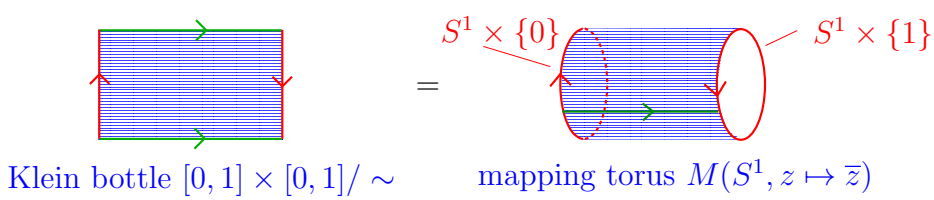

FIGURE 293.

We slow down a little to consider the case $\varphi = \mathrm{id}$ in more detail. On page 117 we already saw that $M(S^1, \mathrm{id})$ is homeomorphic to the torus $S^1 \times S^1$. Thus the following lemma does not come as a surprise.

[342] What does the mapping torus $M(S^1, z \mapsto -z)$ look like?

Lemma 15.18. *Given any topological space X the map*

$$\begin{array}{rcl} \Phi\colon M(X,\mathrm{id}) & \to & X\times S^1 \\ [(x,t)] & \mapsto & (x,e^{2\pi it}) \end{array}$$

is a homeomorphism.

Remark. This lemma is often viewed as "obvious". But even though it is indeed pretty clear that Φ is a bijection and continuous, it is much less clear why the map is actually a homeomorphism since this is another instance where we mix product and quotient topologies. Fortunately in our present situation it is not difficult to prove "by hand" that the map is a homeomorphism.

Proof $(*)$. It is easy to verify that Φ is a bijection. Using Remark 1.67 (1) one can easily show that Φ is continuous. It still remains to show that Φ is actually a homeomorphism.

Let us first consider the special, and perhaps most common case, that X is compact and Hausdorff. By Lemma 15.17 we know that $M(X,\mathrm{id})$ is compact. Furthermore it follows from Lemma 1.55 (1) that $X\times S^1$ is Hausdorff. Thus we obtain from Proposition 1.40 that Φ is a homeomorphism.

Now let us consider the general case which requires more effort. Those readers who have solved Exercise 2.5 (using that $\{0,1\}$ is a compact and closed subset of $[0,1]$) can apply Lemma 2.16 to deduce that Φ^{-1} is continuous. On the other hand the readers who have not solved Exercise 2.5 need to stick around a little longer.

By Lemma 1.39 it remains to show that Φ is an open map. So let U be an open subset of $M(X,\mathrm{id}) = (X\times[0,1])/\sim$. Let $P := [(x,t)]\in U$ be a point. We can assume that $t\in[0,1)$. We need to show that $Q := \Phi([(x,t)])$ admits an open neighborhood Y that is contained in $V := \Phi(U)$. In the following we denote by $p\colon X\times[0,1]\to M(X,\mathrm{id})$ the projection map.

If $t\in(0,1)$, then by the definition of the topology on $M(X,\mathrm{id})$ there exists an open subset $W\subset X$ and an open interval $(a,b)\subset(0,1)$ with $P\in W\times(a,b)\subset U$. We set $Y := \Phi(W\times(a,b)) = W\times\{e^{2\pi it}\,|\,t\in(a,b)\}$. Note that $p^{-1}(Y)=Y$ is an open subset of $X\times[0,1]$. Thus Y has the desired property

Now suppose that $t=0$. In $X\times[0,1]$ we pick an open neighborhood $W_1\times[0,\epsilon_1)$ of the point $(x,0)$ that is contained in $p^{-1}(U)$. Similarly we pick an open neighborhood $W_2\times(1-\epsilon_2,1]$ of the point $(x,1)$ that is also contained in $p^{-1}(U)$. It is now straightforward to see that $Y := (W_1\cap W_2)\times\{e^{2\pi is}\,|\,s\in(-\epsilon_2,\epsilon_1)\}$ has the desired property. $\square$

Now we continue the discussion of mapping tori with a few more examples.

Examples.

(1) In [Rol, Chapter 10.I.10] and [BZH, Section 5.C] it is shown that the complement of the trefoil knot and the complement of the figure-8 knot are homeomorphic to a mapping torus, where the fiber is given by removing a point from the torus.
(2) If one wants to construct new interesting topological spaces out of a given topological space we need to find "non-trivial" self-homeomorphisms. For example let Σ be an

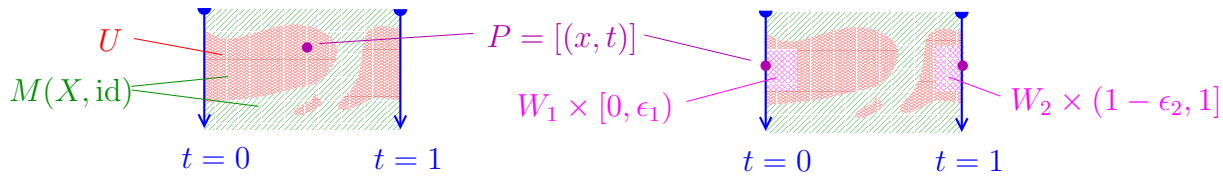

FIGURE 294.

oriented surface and let $C \subset \Sigma$ be a closed oriented curve on Σ, i.e. C is an oriented submanifold diffeomorphic to S^1. By the Tubular Neighborhood Theorem 10.19 there exists an orientation-preserving embedding $\Phi\colon [-1,1] \times S^1 \to \Sigma$ such that $\Phi\colon \{0\} \times S^1 \to C$ is an orientation-preserving diffeomorphism. We refer to the homeomorphism[343]

$$\begin{array}{ccc} \Sigma & \to & \Sigma \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \Phi([-1,1] \times S^1), \\ \Phi\big(t, ze^{2\pi \frac{t+1}{2} i}\big) & \text{if } P = \Phi(t,z) \text{ for some } z \in S^1, t \in [-1,1] \end{cases} \end{array}$$

as the *Dehn*[344] *twist along the curve* C. The definition of a Dehn twist is illustrated in Figure 295.

At times it is convenient to replace the function $\frac{t+1}{2}$ in the exponent by a smooth transition function as in Lemma 3.12, this gives rise to a diffeomorphism (why?) that we refer to as *smooth Dehn twist*.

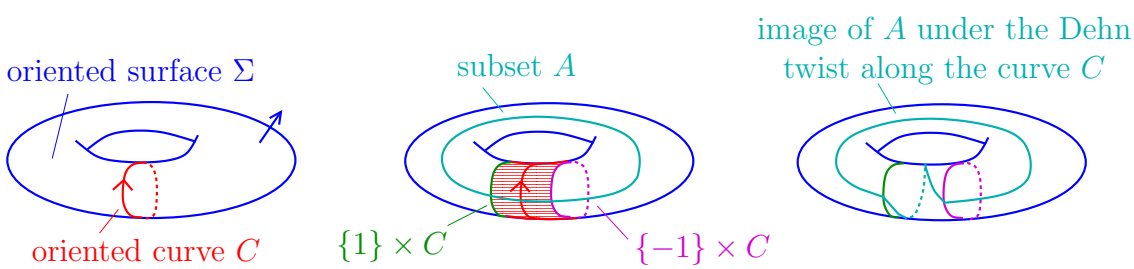

FIGURE 295.

For completeness we also state the following lemma.

Lemma 15.19. *Let X be an n-dimensional manifold and let $f\colon X \to X$ be a diffeomorphism. Then the following statements hold:*

[343]Using Lemma 1.32 (2') and Proposition 10.2 it is straightforward to see that the map is indeed continuous. Clearly the map is also a bijection. What is the inverse map?

[344]Max Dehn (1878-1952) was a German–Jewish mathematician. In 1939 he fled to Norway. After the German occupation of Norway he fled to America via Siberia and Japan. He died in 1952 in Black Mountain, North Carolina.

(1) *the mapping torus $M(X,f)$ is an $(n+1)$-dimensional manifold such that the obvious inclusion map $X \times (0,1) \to M(X,f)$ is a smooth map,*
(2) *the boundary $\partial M(X,f)$ is given by the mapping torus $M(\partial X, f|_{\partial X})$, in particular, if X is closed, then $M(X,f)$ is also closed,*
(3) *if X is orientable and if f is orientation-preserving, then $M(X,f)$ is also an orientable manifold.*

Proof $(*)$. First consider the case that X has no boundary. By Proposition 3.42 (4) we know that $X \times [0,1]$ is an $(n+1)$-dimensional manifold with boundary given by $M \times \{0,1\}$. Note that we obtain the mapping torus $M(X,f)$ from the manifold $M \times [0,1]$ by identifying the boundary component $M \times \{0\}$ with $M \times \{1\}$. Thus the statements are an immediate consequence of Proposition 10.12.

We will not make use of Lemma 15.19 when X has non-empty boundary. Thus we leave it to the curious reader to modify the above argument to the general case. □

The lemma says in particular that self-diffeomorphisms of surfaces give rise to 3-dimensional manifolds. This construction is an important source of examples of 3-dimensional manifolds.

Definition. Let X and Y be topological spaces. Two homeomorphisms $f, g\colon X \to Y$ are called *isotopic*, if there exists an *isotopy* from f to g, i.e. a map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) We have $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$,
(2) for each $t \in [0,1]$ the map $X \to Y$ given by $x \mapsto H(x,t)$ is a homeomorphism.

Examples.

(1) We consider $X = [0,2]$ and the two self-homeomorphisms given by $f(x) = x$ and by $g(x) = \frac{1}{2} \cdot x^2$. Then[345]

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & f(x) \cdot (1-t) + g(x) \cdot t \end{array}$$

is an isotopy from f to g.
(2) Lickorish [Lic] showed that any self-homeomorphism of a compact orientable 2-dimensional manifold is isotopic to the concatenation of finitely many Dehn twists.

The following lemma will be proved in Exercise 15.9.

Lemma 15.20. *Let X be a topological space. If $f, g\colon X \to X$ are two homeomorphisms that are isotopic, then the mapping tori $M(X,f)$ and $M(X,g)$ are homeomorphic.*

[345] Let $t \in [0,1]$. Why is the map $[0,2] \to [0,2]$ given by $x \mapsto H(x,t)$ a homeomorphism?

Example. We consider $X = S^1$ and the self-homeomorphism that is given by $g(z) = -z$. The map $H(z,t) = e^{\pi i t} z$ defines an isotopy from $f = \mathrm{id}$ to g. It follows from Lemma 15.20 that the mapping torus $M(S^1, g(z) = -z)$ is homeomorphic to the torus $M(S^1, \mathrm{id}) = S^1 \times S^1$.

Now we can determine which surfaces are mapping tori.

Proposition 15.21. *The annulus, the Möbius band, the torus and the Klein bottle are, up to homeomorphism, the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus $M(X, f)$ of a connected 1-dimensional topological manifold X.*

Proof. By the examples (1), (2) on page 470 and by Lemma 15.18 we know that there exist the following types of homeomorphisms:

$$\begin{array}{rcl} \text{annulus } S^1 \times [0,1] & \cong & M([0,1], \mathrm{id}) \\ \text{Möbius band} & \cong & M([0,1], x \mapsto 1-x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{torus } S^1 \times S^1 & \cong & M(S^1, \mathrm{id}) \\ \text{Klein bottle} & \cong & M(S^1, z \mapsto \overline{z}). \end{array}$$

In particular all these four manifolds are homeomorphic to mapping tori. It remains to show that these are the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus.

It follows from Proposition 3.49 that any compact connected topological 1-dimensional manifold is homeomorphic to $[0,1]$ or to S^1. We know from Lemma 9.6 that any self-homeomorphism is isotopic to one of the self-homeomorphisms that we had written down above. But then it follows from Lemma 15.20 that any mapping torus is homeomorphic to one from the above list. □

For proving statements about mapping tori the following lemma gives a useful alternative point of view. We leave the elementary proof as an exercise to the reader.

Lemma 15.22. *Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Then the following hold:*

(1) *The map*[346]

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x,t)) & \mapsto & (f^n(x), t+n) \end{array}$$

defines a continuous, proper and discrete action[347] *of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.*

(2) *The map*

$$\begin{array}{rcl} M(X,f) = X \times [0,1])/\sim & \to & (X \times \mathbb{R})/\mathbb{Z} \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism.

---

[346]Here for $n \geq 1$ we write $f^n(x) = (\overbrace{f \circ \cdots \circ f}^{n\text{-times}})(x)$, for $n = 0$ we write $f^0(x) = x$ and for $n \leq -1$ we write $f^n(x) = (\underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{|n|\text{-times}})(x)$.

[347]Why is it an action?

In the following, given a topological space X and a homeomorphism $f\colon X \to X$ we denote the corresponding quotient space by $(X \times \mathbb{R})/\mathbb{Z}_f$ and we will identify the mapping torus $M(X, f)$ with $(X \times \mathbb{R})/\mathbb{Z}_f$ using the homeomorphism of Lemma 15.22.

Remark. It follows from Lemma 15.22 together with Proposition 3.25 that if X is a topological n-dimensional manifold and f is a homeomorphism, then the corresponding mapping torus $M(X, f) = (X \times \mathbb{R})/\mathbb{Z}_f$ is a topological $(n+1)$-dimensional manifold. Furthermore, if X is in fact an n-dimensional manifold and $f\colon X \to X$ is a diffeomorphism, then almost the same argument shows that the mapping torus $M(X, f) = (X \times \mathbb{R})/\mathbb{Z}_f$ is an $(n+1)$-dimensional manifold.

Lemma 15.23. *Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Then the map*

$$\begin{array}{rcl} X \times \mathbb{R} & \to & (X \times \mathbb{R})/\mathbb{Z}_f = M(X, f) \\ (x, t) & \mapsto & [(x, t)] \end{array}$$

is a covering of infinite degree. Furthermore for $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} M(X, f^k) = (X \times \mathbb{R})/\mathbb{Z}_{f^k} & \to & (X \times \mathbb{R})/\mathbb{Z}_f = M(X, f) \\ [(x, t)] & \mapsto & [(x, k \cdot t)] \end{array}$$

is a k-fold covering.

Proof. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. The first statement is an immediate consequence of Lemma 15.22 and Proposition 7.11.

Now let $k \in \mathbb{N}$. We consider the following commutative diagram of maps:

$$\begin{tikzcd} (X \times \mathbb{R})/\mathbb{Z}_{f^k} \arrow[rr, "{[(x,t)]\mapsto[(x,k\cdot t)]}"] \arrow[dr, "{[(x,t)]\mapsto[(x,k\cdot t)]}"'] & & (X \times \mathbb{R})/\mathbb{Z}_f. \\ & (X \times \mathbb{R})/(k \cdot \mathbb{Z})_f \arrow[ur, "{[(x,t)]\mapsto[(x,t)]}"'] & \end{tikzcd}$$

The left diagonal map is easily seen to be well-defined and to be a homeomorphism. By Proposition 7.11 the right diagonal map is a k-fold covering. It follows that the top horizontal map is also a k-fold covering. □

Example.

(1) We apply Lemma 15.23 to $X = [0, 1]$ and the homeomorphism that is given by $f(x) := 1 - x$. Note that $f^2(x) = f(f(x)) = f(1-x) = 1-(1-x) = x$ for all x, i.e. $f^2 = \mathrm{id}$. Thus we recover the statement from page 260 that the annulus $M([0, 1], \mathrm{id})$ is a 2-fold covering of the Möbius band $M([0, 1], f(x) = 1 - x)$.

(2) We apply Lemma 15.23 to $X = S^1$ and the self-homeomorphism of $X = S^1$ that is given by $f(z) := \overline{z}$. As in the previous example we see that $f^2 = \mathrm{id}$. Thus we get, after the discussion on page 269, a new proof that the torus $M(S^1, \mathrm{id})$ is a 2-fold covering of the Klein bottle $M(S^1, f(z) = \overline{z})$.

Our next goal is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$.[348] We define the *semidirect product of N and $\mathbb{Z}_k$ with respect to φ* as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$,[349] but where the group multiplication[350] is given by

$$(h, m) \cdot (h', m') \;:=\; (h \cdot \varphi^m(h'), m + m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_\varphi \mathbb{Z}_k$.

Examples.

(1) Let N be a group and $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.
(2) Let $G = \mathbb{Z}_2$ and let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the *$n$-th dihedral group D_n*. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation group S_3 via the isomorphism [351]

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h, m) & \mapsto & (1\ 2\ 3)^h \cdot (1\ 2)^m. \end{array}$$

Remark. In Exercise 15.10 we will see that $N = \{(h, 0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \lhd N \rtimes \mathbb{Z}_k$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

We have the following elementary lemma.

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[348] Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[349] In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot k$.

[350] It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}$ with the given multiplication is indeed a group.

[351] First note that a straightforward calculation shows that $(1\,2)^{-1} \cdot (1\,2\,3) \cdot (1\,2) = (1\,3\,2) = (1\,2\,3)^{-1}$. This implies that also for any $m \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\,2)^{-m} \cdot (1\,2\,3)^s \cdot (1\,2)^m = (1\,2\,3)^{s(-1)^m}$. Now for any $(h, m), (h', m') \in \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ we have

$$\begin{aligned} \Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2\,3)^{h'} \cdot (1\,2)^{m'} = (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2)^{-m} \cdot (1\,2\,3)^{(-1)^m h'} \cdot (1\,2)^m \cdot (1\,2)^{m'} \\ &= (1\,2\,3)^{h+(-1)^m h'} \cdot (1\,2)^{m+m'} = \Psi(h + (-1)^m h', m + m') = \Psi((h,m) \cdot (h',m')). \end{aligned}$$

We have thus shown that Ψ is a homomorphism.

Lemma 15.24. *Let G be a group and let $\gamma\colon G \to \mathbb{Z}$ be an epimorphism. We pick an element $t \in G$ with $\gamma(t) = 1$ and we write $N := \ker(\gamma)$. We denote by φ the isomorphism*

$$\begin{array}{rcl} \varphi\colon N & \to & N \\ h & \mapsto & tht^{-1}. \end{array}$$

Then the map[352]

$$\begin{array}{rcl} \Psi\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(gt^{-\gamma(g)}, \gamma(g)\big) \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Ψ is injective and bijective. It remains to show that Ψ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned} \Psi(g \cdot h) \;=\; (ght^{-\gamma(gh)}, \gamma(gh)) \;&=\; (gt^{-\gamma(g)} \cdot \underbrace{t^{\gamma(g)} \cdot ht^{-\gamma(h)} \cdot t^{-\gamma(g)}}_{=\varphi^{\gamma(g)}(ht^{-\gamma(h)})}, \gamma(g) + \gamma(h)) \\ &=\; (gt^{-\gamma(g)}, \gamma(g)) \cdot (ht^{-\gamma(h)}, \gamma(h)) \;=\; \Psi(g) \cdot \Psi(h). \end{aligned}$$

□

The following proposition shows that the fundamental group of a mapping torus $M(X, f)$ is isomorphic to semidirect product of $\pi_1(X)$ with $\mathbb{Z}$.

Proposition 15.25. *Let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. Then there exists an isomorphism $\psi\colon \pi_1(M(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ such that the following diagram commutes*

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d] & \pi_1(M(X,f)) \arrow[r] \arrow[d, "\cong"', "\psi"] & \pi_1(S^1) \arrow[d, "\cong"] \\ \pi_1(X) \arrow[r] & \pi_1(X) \rtimes \mathbb{Z} \arrow[r] & \mathbb{Z}. \end{tikzcd}$$

Here the top-left horizontal map is induced by the inclusion $X = X \times \{0\} \to M(X, f)$ and the top-right horizontal map is induced by the canonical projection map $M(X, f) \to S^1$. The bottom-left horizontal map is given by the natural inclusion into the first term of the semidirect product and the bottom-right horizontal map is given by the projection onto the second term of the semidirect product.[353]

In the proof of Proposition 15.25 we will need the following lemma which is a straightforward generalization of Theorem 7.17 and the proof thereof. We leave the details of the proof to the reader.

Lemma 15.26. *Let X be a path-connected*[354] *topological space. Let G be a group which acts continuously and discretely on X. We choose an $x \in X$. We denote by $p\colon X \to X/G$*

[352]Note that $\gamma(gt^{-\gamma(g)}) = \gamma(g) + \gamma(t)(-\gamma(g)) = 0$, i.e. $gt^{-\gamma(g)}$ does indeed lie in $N = \ker(\gamma)$.

[353]Note that both maps on the bottom are group homomorphisms.

[354]In Theorem 7.17 we had assumed that X is simply connected.

the corresponding covering map.[355] *Then the map*

$$\begin{array}{rcl} \pi_1(X/G,[x]) & \to & G \\ & & \textit{unique } g \in G \textit{ such that for the unique lift}^{356} \\ [\gamma\colon [0,1] \to X/G] & \mapsto & \widetilde{\gamma}\colon [0,1] \to X \textit{ of the path } \gamma \textit{ with } \widetilde{\gamma}(0) = x \\ & & \textit{we have } \widetilde{\gamma}(1) = g \cdot x. \end{array}$$

is an epimorphism with kernel isomorphic to $\pi_1(X,x)$.

Now we can give the proof of Proposition 15.25.

Proof of Proposition 15.25. We have

$$\begin{array}{ccccccc} \pi_1(M(X,f)) & = & \pi_1((X\times\mathbb{R})/\mathbb{Z}_f) & \cong & \pi_1(X\times\mathbb{R})\rtimes\mathbb{Z} & \cong & \pi_1(X)\rtimes\mathbb{Z}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 15.22} & & \text{Lemmas 15.26 and 15.24} & & \text{Proposition 9.12} & \end{array}$$

Thus we have obtained an isomorphism $\psi\colon \pi_1(M(X,f)) \xrightarrow{\cong} \pi_1(X)\rtimes\mathbb{Z}$. We leave the verification that the given diagram commutes to the reader. □

Remark. A careful reading of the proof of the proposition shows that we can identify the automorphism in the semidirect product. More precisely, let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. Let x be a base point and let γ be path from $f(x)$ to x. We denote by φ the automorphism of $\pi_1(X,x)$ that is given by

$$\begin{tikzcd} \pi_1(X,x) \arrow[r, "f_*"] \arrow[rr, bend right, "=:\varphi"'] & \pi_1(X,f(x)) \arrow[r, "\cong", "\text{Prop. 5.11}"'] & \pi_1(X,x). \end{tikzcd}$$

Then $\pi_1(M(X,f)) \cong \pi_1(X,x)\rtimes_\varphi \mathbb{Z}$.

We conclude this section with a discussion of several examples.

Examples.

(1) Let $K = M(S^1, f(z) := \overline{z})$ be the Klein bottle. Then the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the previous remark[357] that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z}\rtimes_\varphi\mathbb{Z}.$$

(2) Let $A \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix. In Exercise 7.12 we showed that the map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v] \mapsto [Av]$ is a homeomorphism. Furthermore we know that, under the usual identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) = \mathbb{Z}^n$ as in Theorem 7.17, the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) = \mathbb{Z}^n$ is given by multiplication by A. It follows from the above remark

[355] Recall that by Proposition 7.10 the map $p\colon X \to X/G$ is a covering.

[356] Recall that by Proposition 7.12 there exists a unique lift $\widetilde{\gamma}$ of γ to the starting point $\widetilde{\gamma}(0) = x$.

[357] On page 429 we saw that $\pi_1(K) \cong \langle x,y \,|\, yxyx^{-1}\rangle$. What is an explicit isomorphism between the groups $\langle x,y \,|\, yxyx^{-1}\rangle$ and $\mathbb{Z}\rtimes_\varphi\mathbb{Z}$?

that $\pi_1(M(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Note that this implies that $\pi_1(M(\mathbb{R}^n/\mathbb{Z}^n, f(A)))$ is solvable.[358]

(3) Let X be a connected topological space. We calculate that

$$\pi_1(X \times S^1) \underset{\substack{\uparrow \\ \text{Lemma 15.18}}}{=} \pi_1(M(X, \mathrm{id}_X)) \underset{\substack{\uparrow \\ \text{by the above remark}}}{\cong} \pi_1(X) \rtimes_{\mathrm{id}} \mathbb{Z} = \pi_1(X) \times \mathbb{Z} = \pi_1(X) \times \pi_1(S^1).$$

So in this special case we obtain the same result as in Proposition 7.21.

Exercises for Chapter 15.

Exercise 15.1. Given a subset X of $\mathbb{R}^n$ we define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) := \text{union of all line segments from } \underbrace{(x,1)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,0)}_{\in\mathbb{R}^{n+1}} \subset \mathbb{R}^{n+1}$$

and we consider the map

$$\begin{aligned} f_X\colon \mathrm{Cone}(X) &\to \widetilde{\mathrm{Cone}}(X) \\ [(x,t)] &\mapsto t\cdot(x,1) + (1-t)\cdot(0,1). \end{aligned}$$

(a) Let X be a compact subset of $\mathbb{R}^n$. Show that the map f_X is a homeomorphism.

(b) Let $X = (0,1) \subset \mathbb{R}$. Show that the map f_X is not a homeomorphism.
Remark: It is actually true that f_X is continuous and a bijection. To show that f_X is not a homeomorphism we need to show that there exists an open subset U of $\mathrm{Cone}(X)$ such that the image under f_X is not open. You could try your hand at

$$U := \mathrm{Cone}(X) \setminus \{[(\tfrac{1}{n}, \tfrac{1}{n})] \mid n \in \mathbb{N}_{\geq 2}\}.$$

We refer to Figure 296 for an illustration.

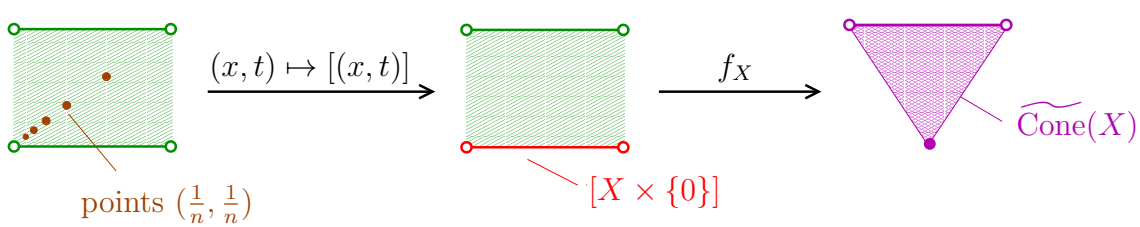


FIGURE 296.

Exercise 15.2. Let X be a topological space that is Hausdorff. Show that the suspension $\Sigma(X)$ is also Hausdorff.

[358] Recall that a group G is solvable if there exist subgroups $G_0 = \{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ such that for $i = 0, \ldots, k-1$ the group G_i is normal in G_{i+1} and such that G_{i+1}/G_i is abelian. In our case the filtration is given by $\{0\} \subset \mathbb{Z}^n \subset \mathbb{Z}^n \rtimes_A \mathbb{Z}$.

Exercise 15.3. Let X be a topological space. Show that the injections

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are homeomorphisms onto their images.

Exercise 15.4. Let $f\colon X \to Y$ be a map between topological spaces. Show that the natural inclusions

$$\begin{array}{rcl} X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & (x,0) \end{array} \quad \text{and} \quad \begin{array}{rcl} Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

are homeomorphisms onto their images.

Exercise 15.5. Let X be a topological space and let A be a subset of X.

(a) We denote by $i\colon A \to X$ the inclusion. Show that the map

$$\begin{array}{rcl} A \times [0,1] & \to & \mathrm{Cyl}(i) = ((A \times [0,1]) \sqcup X)/\sim \\ (a,t) & \mapsto & [(a,t)] \end{array}$$

is a homeomorphism onto its image.

(b) Let $f\colon A \to X$ be a homeomorphism onto its image. Is the map $A \times [0,1] \to \mathrm{Cyl}(f)$ necessarily a homeomorphism onto its image?

Exercise 15.6. Give a proof of Lemma 15.9 (3).

Exercise 15.7. Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. We suppose that A is path-connected. Show that the natural inclusion map $X \to \mathrm{Cone}(f\colon A \to X))$ induces an isomorphism

$$\pi_1(X,x_0)/f_*(\pi_1(A,a_0)) \;\to\; \pi_1(\mathrm{Cone}(f\colon A \to X)).$$

Exercise 15.8. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We assume that X is Hausdorff. Show that the mapping torus $M(X,f)$ is also Hausdorff.

Exercise 15.9. Prove Lemma 15.20. More precisely, let X be a topological space. Show that if $f,g\colon X \to X$ are two homeomorphisms that are isotopic, then the mapping tori $M(X,f)$ and $M(X,g)$ are homeomorphic.

Exercise 15.10. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$. We define the *semidirect product of N and $\mathbb{Z}_k$ with respect to φ* as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$, and where the multiplication is given by

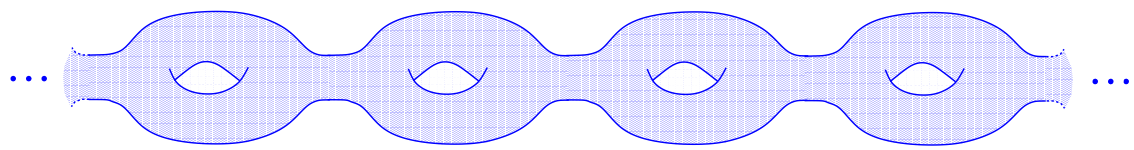
$$(h,m)\cdot(h',m') \;:=\; (h\cdot\varphi^m(h'), m+m').$$

(a) Show that $N \rtimes_\varphi \mathbb{Z}_k$ is indeed a group.
(b) Show that $N = \{(h,0) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, h \in N\}$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$.
(c) Is $\mathbb{Z}_k = \{(0,n) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, n \in \mathbb{Z}_k\} \subset N \rtimes_\varphi \mathbb{Z}_k$ necessarily a normal subgroup?

Exercise 15.11. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Show that the mapping torus $M(X, f\circ f)$ is a 2-fold covering of the mapping torus $M(X,f)$.

16. Limits

The goal of this section is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in Figure 297.[359] One idea might be to view the surface of "infinite genus" as the limit of a sequence given by compact surfaces. But then we need to make precise what we mean by a "limit of topological spaces" and "limit of the corresponding fundamental groups". We will do so in the following sections.



surface of "infinite genus"

Figure 297.

16.1. **Preordered and directed sets.**

Definition. Let X be a set and let $\leq$ be a relation on X.

(1) The relation is called *reflexive* if for any $x \in X$ we have $x \leq x$.
(2) The relation is called *transitive* if for any $x, y, z \in X$ we have
$$x \leq y \text{ and } y \leq z \quad \Longrightarrow \quad x \leq z.$$
(3) A reflexive and transitive relation is called a *preorder*.
(4) A set together with a preorder is called a *preordered set*.
(5) A *directed set* is a preordered set such that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(2) Given any set X the relation "$\leq$" where $x \leq y$ only if $x = y$ is a preorder, which we call the *trivial preorder* on X.
(3) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subseteq$", defines a preorder on $\mathcal{P}(M)$. Thus $(\mathcal{P}(M), \subseteq)$ is a preordered set, in fact it is straightforward to see that it is even a directed set.
(4) The set $X = \{0, x, y\}$ with the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$ is a preordered set. We refer to $(\{0, x, y\}, \leq)$ as the *push-pull set*. Note that this is not a directed set, since there is no $z \in X$ with $x \leq z$ and $y \leq z$.

[359] Actually we also need to make precise what we mean by the surface of "infinite genus".

(5) We consider the set $\mathbb{N}$. The divisibility relation $k|l$ for $k, l \in \mathbb{N}$ is a preorder on $\mathbb{N}$ and with this preorder $\mathbb{N}$ is in fact a directed set.[360]

16.2. **The direct limit of a direct system.**

Definition. Let $(I, \leq)$ be a preordered set[361] and let $\mathcal{C}$ be a category. A *direct system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij}\colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ik} & = & f_{jk} \circ f_{ij} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Examples.

(A) We consider the directed set $I = (\mathbb{N}, \leq)$ and the category of groups. For $i \in \mathbb{N}$ we consider the group $\mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ij}\colon \mathbb{Z}^i & \to & \mathbb{Z}^j \\ (x_1, \dots, x_i) & \mapsto & (x_1, \dots, x_i, 0, \dots, 0) \end{array}$$

the inclusion map. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ij}\}_{i\leq j})$ forms clearly a direct system.

(B) Again we consider $I = (\mathbb{N}, \leq)$ and the category of groups. Given $i \in \mathbb{N}$ we define

$$F_i \ := \ \langle x_1, \dots, x_i \rangle \ = \ \text{the free group on } i \text{ generators}$$

and given $i \leq j$ we set

$$\begin{array}{rcl} f_{ij}\colon F_i & \to & F_j \\ x_k & \mapsto & x_k, \ \text{for } k = 1, \dots, i. \end{array}$$

This is clearly a direct system of groups.

(C) Given $r \in \mathbb{N} \cup \{\infty\}$ we set

$$X_r \ := \ \left(0, r + \tfrac{1}{2}\right) \ \cup \ \bigcup_{j=1}^{r} \underbrace{\left\{ z \in \mathbb{C} \,\middle|\, \left|z - (j + \tfrac{1}{4}i)\right| = \tfrac{1}{4} \right\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}i \in \mathbb{C}} \ \subset \ \mathbb{C}.$$

For example, for $r \in \mathbb{N}$ the set X_r is the subset of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We refer to Figure 298 for an illustration.

Now we consider the directed set $(\mathbb{N}, \leq)$ and for $r \leq s$ we denote by $f_{rs}\colon X_r \to X_s$ the obvious inclusion map. This is clearly a direct system of topological spaces.

(D) We can generalize example (C) as follows. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. Given $i \leq j$ we denote by $\iota_{ij}\colon X_i \to X_j$ the inclusion map. Then $(\{X_i\}_{i\in I}, \{\iota_{ij}\}_{i\leq j})$ is a preordered system in the category of topological spaces.

[360] Indeed, given $k, l \in \mathbb{N}$ we have $k|kl$ and $l|kl$.

[361] In the literature one often demands that I is a directed set, see e.g. [Hat, p. 243], but this is not necessary.

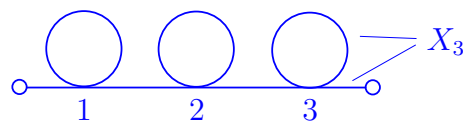


FIGURE 298.

(E) Let $(\{0, x, y\}, \leq)$ be the push-pull set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over the preordered set $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y, the identity morphisms $\mathrm{id}_{A_0}, \mathrm{id}_{A_x}, \mathrm{id}_{A_y}$ and morphisms $f_x \colon A_0 \to A_x$ and $f_y \colon A_0 \to A_y$. We refer to such a system as a *pushout system* and often we arrange the morphisms in a diagram as follows

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \\ A_y. & \end{tikzcd}$$

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ in the category $\mathcal{C}$. A *direct limit*[362] of the direct system is an object Y in $\mathcal{C}$ together with morphisms $g_i \colon X_i \to Y, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ij} = g_i \colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd} X_j \arrow[dr, "g_j"] & \\ & Y. \\ X_i \arrow[uu, "f_{ij}"] \arrow[ur, "g_i"'] & \end{tikzcd}$$

(2) If we are given another object Y' and morphisms $g'_i \colon X_i \to Y', i \in I$ that satisfy (1), then there exists a unique morphism $F \colon Y \to Y'$ such that for all $i \leq j$ the following diagram commutes[363]

$$\begin{tikzcd} X_j \arrow[dr, "g_j"] \arrow[drr, bend left, "g'_j"] & & \\ & Y \arrow[r, dashed, "F", "\exists!"'] & Y'. \\ X_i \arrow[uu, "f_{ij}"] \arrow[ur, "g_i"'] \arrow[urr, bend right, "g'_i"'] & & \end{tikzcd}$$

[362]Somewhat confusingly a "direct limit" is often also called a "projective limit" or "colimit of a direct system".

[363]The notation $\exists!$ means that there exists a uniquely defined morphism that makes the diagram commute.

The "usual argument" for objects that satisfy a universal property[364] shows, that if the direct limit exists, then it is unique in an appropriate sense. We denote the direct limit by $\varinjlim X_i$ or sometimes by $\varinjlim_I X_i$.[365]

Example. Now we discuss the direct limits of the direct systems we had discussed above.

(A) We consider the above direct system in the category of groups that is given by the free abelian groups $\mathbb{Z}^i$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $\mathbb{Z}^i \to \mathbb{Z}^j$ for $i \leq j$. We claim that[366]

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $\mathbb{Z}^j \to \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious inclusion maps. Indeed, given a group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that satisfy condition (1) we define

$$\begin{array}{rcl} F \colon \varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} & \to & Y' \\ (x_1, \dots, x_j, 0, \dots) & \mapsto & g'_j(x_1, \dots, x_j). \end{array}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_j.[367] It is clear that the diagram (2) commutes. Furthermore, since $(x_1, \dots, x_j, 0, \dots) = g_j(x_1, \dots, x_j)$ it is clear that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have now shown that the direct limit is the free abelian group on an infinite countable generating set.

(B) We consider the above direct system in the category of groups that is given by the free groups $F_i := \langle x_1, \dots, x_i \rangle$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $F_i \to F_j$ for $i \leq j$. Similar to the proof of (A) one can show that

$$\varinjlim F_i \;=\; \langle x_1, x_2, x_3, \dots \rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \varinjlim F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious maps. Put differently, the direct limit is the free group on an infinite countable generating set.

(C) We continue with the direct system $(X_r, r \in \mathbb{N})$ in the category of topological spaces from above. Perhaps not surprisingly the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r \;=\; X_\infty \;=\; \big(0, \infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{ z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}i)\big| = \tfrac{1}{4} \big\} \;\subset\; \mathbb{C},$$

where the maps $X_s \to \varinjlim X_r = X_\infty$, $s \in \mathbb{N}$ are the obvious inclusion maps. We leave the verification of this statement as an exercise.

[364] See for instance the discussion on page 368 for an example of such an argument.

[365] Here we ignore the maps f_{ij} and g_i in the notation. Usually these maps are clear from the context.

[366] Why is the direct limit not given by $\mathbb{Z}^{\mathbb{N}}$, i.e. the group of all infinite sequences?

[367] Here we did not demand that $x_j \neq 0$.

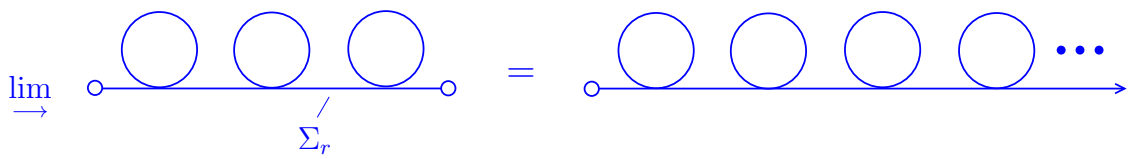


FIGURE 299.

(D) Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. In the category of topological spaces we have

$$\varinjlim X_i \;=\; X \;:=\; \bigcup_{i\in I} X_i,$$

where the topology on $X = \bigcup_{i\in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore the maps $g_j \colon X_j \to X = \bigcup_{i\in I} X_i, j \in \mathbb{N}$ are the obvious inclusion maps.[368] The verification of this statement is Exercise 16.1.

Definition. If the direct limit of a pushout system

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \\ A_y & \end{tikzcd}$$

exists, then we refer to it as the *pushout* of the *pushout system*. The defining property of the pushout Z can be summarized in the following commutative diagram[369]

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \arrow[d, "g_x"] \arrow[ddr, bend left, "g'_x"] & \\ A_y \arrow[r, "g_y"] \arrow[drr, bend right, "g'_y"'] & \text{pushout } Z \arrow[dr, dashed, "\exists!"] & \\ & & Z'. \end{tikzcd}$$

Examples.

(1) We first consider the pushout in the category of groups. By Proposition 13.11 the pushout of a pushout system in the category of groups

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \\ B & \end{tikzcd} \qquad \text{is given by} \qquad \begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & A *_G B = \dfrac{A * B}{\langle\langle \{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G} \rangle\rangle}. \end{tikzcd}$$

[368]Why are the maps g_j continuous, i.e. why are they morphisms in the category of topological spaces.

[369]Here we suppress the morphism $g_0 \colon A_0 \to Z$ from the diagram since it is given by $g_x \circ f_x = g_y \circ f_y$.

(2) Now we consider the pushout in the category of topological spaces. In Exercise 16.3 we will see that the pushout of the pushout system of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z := (Y \sqcup Z)/f(x) \sim g(x). \end{tikzcd}$$

Here we equip $Y \sqcup_X Z$ with the quotient topology induced from the obvious topology on $Y \sqcup Z$.

(3) The direct limit of a direct system does not always exist. For example, denote by $\mathcal{C}$ the category with two objects, namely the trivial group and the group $\mathbb{Z}_2$, and where the morphisms are the usual group homomorphisms. Then the pushout system

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \end{tikzcd} \quad \begin{array}{c} \text{does not have a direct limit} \\ \text{since there is no commutative} \\ \text{diagram of the form} \end{array} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \\ \mathbb{Z}_2 \arrow[r, "\alpha"] & C \end{tikzcd}$$

with $C = 0$ or $C = \mathbb{Z}_2$ such that in the following two diagrams there exists the desired diagonal maps from C to $Y' = \mathbb{Z}_2$:[370]

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \arrow[ddr, bend left, "x \mapsto x"] & \\ \mathbb{Z}_2 \arrow[r, "\alpha"] \arrow[drr, bend right, "x \mapsto 0"'] & C \arrow[dr] & \\ & & Y' = \mathbb{Z}_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \arrow[ddr, bend left, "x \mapsto 0"] & \\ \mathbb{Z}_2 \arrow[r, "\alpha"] \arrow[drr, bend right, "x \mapsto x"'] & C \arrow[dr] & \\ & & Y' = \mathbb{Z}_2. \end{tikzcd}$$

Proposition 16.1. *The direct limit of any direct system exists in the following categories:*

(0) *the category of sets,*
(1) *the category of topological spaces,*
(2) *the category of groups,*
(3) *the category of abelian groups,*
(4) *the category of rings,*[371]
(5) *the category of R-modules, where R is a commutative ring.*

Sketch of proof. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be one of the six given categories. Furthermore let $(\{X_i\}_{i \in I}, \{f_{ij} \colon X_i \to X_j\}_{i \leq j})$ be a direct system in $\mathcal{C}$.

(0),(1) If we work in the category of sets, then we define the direct limit as the set

$$\varinjlim X_i \ := \ \Big(\bigsqcup_{i \in I} X_i \Big) / \sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i.$$

[370] Why is there no such diagram with $C = 0$ or $C = \mathbb{Z}_2$ and homomorphisms α and β?

[371] Throughout the lecture notes a *ring* is understood to be associative and it is understood to have a multiplicatively neutral element, unless we say explicitly otherwise. We do not demand that a ring is commutative.

If we work in the category of topological spaces, then $\bigsqcup_{i\in I} X_i$ is equipped with the topology where a subset U is open if and only if $U \cap X_i$ is open for all i.[372]

(2) If we work in the category of groups, then we cannot use the definition of (0) and (1), since the union of groups is not a group. Therefore we define the direct limit via the "smallest group that contains all X_i", i.e. we define it as the group

$$\varinjlim X_i \;:=\; \Big(\mathop{\ast}_{i\in I} X_i\Big) / \langle\!\langle \{ f_{ij}(x)\cdot x^{-1} \mid i \le j \text{ and } x \in X_i \} \rangle\!\rangle.$$

(3),(4),(5) If we work in the category of abelian groups, or in the category of rings or in the category of R-modules, then we define[373]

$$\varinjlim X_i \;:=\; \Big(\bigoplus_{i\in I} X_i\Big) / \{ f_{ij}(x) - x \mid \; i \le j \text{ and } x \in X_i \}.$$

Furthermore in all six cases we denote by $g_j \colon X_j \to \varinjlim X_i$, $j \in I$ the obvious maps. It is now straightforward to verify that in all five categories we provided a direct limit. We leave the details to the reader. □

Remark.

(1) Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij} \colon X_i \to X_j\}_{i\le j})$ in any of the four algebraic categories mentioned in the previous proposition. Now we assume that $(I, \le)$ is not only a preordered set, but that it is in fact a directed set. Then, as for direct limits of sets and topological spaces, we can also define

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big) / \sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \le j,\, x \in X_i$$

and we denote by $g_j \colon X_j \to \varinjlim X_i$, $j \in I$, the obvious maps. If we work in the category of groups, then $\varinjlim X_i$ is naturally a group as follows: if we are given $a \in X_i$ and $b \in X_j$, by definition of a directed set there exists $k \in I$ with $i \le k$ and $j \le k$, and we define

$$[a]\cdot[b] \;:=\; \big[\underbrace{f_{ik}(a)}_{\in X_k} \cdot \underbrace{f_{jk}(b)}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed a group structure on $\varinjlim X_i$. Similarly we deal with the case that we work in the other three algebraic categories.

[372]Note that for the above example of the pushout we do *not* get the same topological space (why not?) but we get a topological space that is homeomorphic (why?) to the above pushout. Since we only care about the direct limit up to homeomorphism this makes no difference.

[373]Here we really mod out by the subgroup generated by the differences $f_{ij}(x) - x$.

(2) An important special case is given by $I = \mathbb{N}_0$ and where the maps $f_{ij}\colon X_i \to X_j$ are inclusion maps. In this case one can in fact take

$$\varinjlim X_i \;:=\; \bigcup_{i\in\mathbb{N}_0} X_i.$$

In the above algebraic categories the algebraic structure are defined as in the previous remark. In the topological category the topology on $X := \bigcup_{i\in\mathbb{N}_0} X_i$ is given, as in the proof of Proposition 16.1, by the condition that $U \subset X$ is open if and only if each $U \cap X_i$ is an open subset of X_i.

We conclude this section with the following lemma that gives a particularly simple but commonly used calculation of a direct limit.

Lemma 16.2. *Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Then $\varinjlim X_n$ exists and it is naturally isomorphic to X_N, more precisely, a direct limit is given by $\varinjlim X_n := X_N$ and the maps*

$$\begin{array}{rcll} f_{N-1}\circ\circ f_{n+1}\circ f_n\colon X_n & \to & X_N & \text{if } n \leq N, \\ (f_n\circ\cdots\circ f_{N+1}\circ f_N)^{-1}\colon X_n & \to & X_n & \text{if } n > N \end{array}$$

define a direct limit.

Proof. We will provide the proof in Exercise 16.2. □

16.3. **Fundamental groups and direct limits.** In this section we want to address the following question:

Question 16.3. *Let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system of topological spaces. Now we can form the direct limit $\varinjlim X_i$ of the topological spaces. The maps f_{ij} also induce homomorphisms on fundamental groups. The question is, when does "taking limits commute with taking fundamental groups", i.e. under what circumstances do we have*[374]

$$\pi_1\big(\varinjlim X_i\big) \;=\; \varinjlim \pi_1(X_i)\ ?$$

Example. We have already seen two instances where the equality of the question holds.

(1) Suppose X is a topological space that is the union of two open subsets U and V such that U, V and $U \cap V$ are path-connected. We fix a base point in $U \cap V$. Then

$$\pi_1\left(\varinjlim \begin{array}{c} U\cap V \to U \\ \downarrow \\ V \end{array}\right) \underset{\substack{\uparrow \\ \text{see page 486}}}{=} \pi_1(X) \underset{\substack{\uparrow \\ \text{Seifert-van Kampen} \\ \text{Theorem 14.1}}}{=} \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \underset{\substack{\uparrow \\ \text{see page 485}}}{=} \varinjlim \begin{array}{c} \pi_1(U\cap V) \to \pi_1(U) \\ \downarrow \\ \pi_1(V) \end{array}$$

[374]As it is this question is not well-phrased, since fundamental groups depend on base points and we ignored them in the above discussion.

(2) Let X be a path-connected topological space and let X_k, $k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:
   (a) each X_k is open,
   (b) each X_k is simply connected,
   (c) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
   (d) we have $\bigcup_{k\in\mathbb{N}} X_k = X$.

   Then for any $w \in X_1$ we have[375]

$$\pi_1\big(\varinjlim X_i, w\big) \underset{\substack{\uparrow \\ \text{see page 488}}}{=} \pi_1\Big(\bigcup_{i\in\mathbb{N}} X_i, x_0\Big) \underset{\substack{\uparrow \\ \text{Lemma 12.6}}}{=} \{e\} = \varinjlim \{e\} = \varinjlim \pi_1(X_i, w).$$

The following proposition generalizes the second example.

Proposition 16.4. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that the following holds:*

(1) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$,*

and such that one of the following two conditions holds:

(2) *every compact subset of $\varinjlim X_i$ is already contained in one of the X_i, or*

(3) *each X_i is open in X_{i+1}.*

Then for any $w \in X_1$ we have

$$\pi_1\big(\varinjlim X_i, w\big) = \varinjlim \pi_1(X_i, w).$$

Remark. We will see that the proof of Proposition 16.4 is a generalization of the ideas behind the proof of Lemma 12.6.

The following lemma shows that the hypothesis (3) in Proposition 16.4 implies in fact hypothesis (2).

Lemma 16.5. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that each X_i is open in X_{i+1}. Then every compact subset of $\varinjlim X_i$ is already contained in one of the X_i.*

Proof of Lemma 16.5. Let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that each X_i is open in X_{i+1}.

Claim. For every $i \in \mathbb{N}$ the subset X_i is open in X.

By the definition of the topology on X we have to show that for each $j \in \mathbb{N}$ the intersection $X_i \cap X_j$ is open in X_j. Since the sequence is nested it follows that for $j \leq i$ we have $X_i \cap X_j = X_j$, in particular $X_i \cap X_j$ is open in X_j. Now we need to show that X_i is open

[375]Recall that by the discussion on page 482 the topological spaces $\{X_i\}_{i\in\mathbb{N}}$ together with the inclusion maps form a direct system in the category of topological spaces. The direct limit is given by the union of all X_i's, where the union is given a suitable topology. Furthermore note that the inclusion induced maps $\pi_1(X_i, w) \to \pi_1(X_j, w)$ turn the groups $\{\pi_1(X_i, w)\}_{i\in\mathbb{N}}$ into a direct system in the category of groups.

in any X_{i+k} with $k \in \mathbb{N}_0$. We do so by induction on k. For $k = 0$ the statement is clear. Suppose we already know that X_i is open in X_{i+k}. Then X_i is an open subset in X_{i+k} and X_{i+k} is in turn an open subset of X_{i+k+1} by our hypothesis. It follows that X_i is open in X_{i+k+1}.[376] This concludes the proof of the claim.

Now let $K \subset \varinjlim X_i$ be a compact subset. From the claim we know that each X_i is open in $X = \varinjlim X_i$ we obtain immediately from Lemma 1.38 and the hypothesis that the X_i's are nested that there exists a $k \in \mathbb{N}$ such that $K \subset X_i$. $\square$

Proof of Proposition 16.4. By Lemma 16.5 we only have to prove the proposition under the hypothesis (2). Thus let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that every compact subset of $\varinjlim X_i$ is already contained in one of the X_i. Let $w \in X_1$. We write $X = \bigcup_{i \in I} X_i = \varinjlim X_i$.

Given $k \leq l$ we denote by φ_{kl} the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X_l, w)$. We denote by $\Phi \colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)$ the unique homomorphism that makes the following diagram commute for any $k \leq l$:

$$\begin{tikzcd}
\pi_1(X_l, w) \arrow[dr, "\psi_l"] \arrow[drr, bend left] & & \\
& \varinjlim \pi_1(X_i, w) \arrow[r, dashed, "\Phi", "\exists!"'] & \pi_1(X, w). \\
\pi_1(X_k, w) \arrow[uu, "\varphi_{kl}"] \arrow[ur, "\psi_k"] \arrow[urr, bend right] & &
\end{tikzcd}$$

We need to show that Φ is an isomorphism. First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$. We represent g by a loop $\gamma \colon [0,1] \to X$. By Lemma 1.37 and our hypothesis (2) there exists a $k \in \mathbb{N}$ such that the image of γ lies in X_k. We consider the following commutative diagram

$$\begin{tikzcd}
& \varinjlim \pi_1(X_i, w) \arrow[r, "\Phi"] & \pi_1(X, w). \\
\pi_1(X_k, w) \arrow[ur] \arrow[urr, bend right, "{[\gamma] \mapsto g}"'] & &
\end{tikzcd}$$

By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.

Now we show that the map Φ is injective. So let $g \in \varinjlim \pi_1(X_i, w)$. Since $(\mathbb{N}, \leq)$ is a directed set we can use the description of the direct limit from page 487, i.e. we can write

$$\varinjlim \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i \in \mathbb{N}} \pi_1(X_i, w) \Big) / \sim \quad \text{where } g \sim \varphi_{kl}(g) \text{ for all } k \leq l,\ g \in \pi_1(X_k, w).$$

[376] Here we use the following general fact: if U is an open subset in V and V is an open subset of W, then U is an open subset of W. Why is that true?

In particular we see that there exists a k such that g lies in the image of the map $\psi_k\colon \pi_1(X_k,w) \to \varinjlim \pi_1(X_k,w)$. We pick a loop $\gamma\colon [0,1] \to X_k$ such that $\psi_k([\gamma]) = g$. Now suppose that $\Phi(g)$ is trivial. By definition this means that there exists a path-homotopy $H\colon [0,1]\times[0,1] \to X$ from the path γ to the constant path. As above we can apply Lemma 1.37 and our hypothesis (2) to deduce that there exists an $l \geq k$ such that the image of H lies in X_l. This implies that $\varphi_{kl}([\gamma])$ is trivial in $\pi_1(X_l,w)$. We obtain the following diagram

$$\begin{tikzcd}
\pi_1(X_l,w) \arrow[dr] & & \\
& \varinjlim \pi_1(X_i,w) \arrow[r, "g\mapsto e"] & \pi_1(X,w). \\
\pi_1(X_k,w) \arrow[uu, "{[\gamma]\mapsto e}"] \arrow[ur, "{[\gamma]\mapsto g}"'] & &
\end{tikzcd}$$

It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. $\square$

Examples.

(1) As on page 482 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We then have [377]

$$\pi_1\big(X_\infty,w\big) \underset{\substack{\uparrow\\ \text{see page 484}}}{=} \pi_1\big(\varinjlim X_r,w\big) \underset{\substack{\uparrow\\ \text{Proposition 16.4}}}{=} \varinjlim \pi_1\big(X_r,w\big) \underset{\substack{\uparrow\\ \text{Lemma 12.5}}}{=} \varinjlim \langle x_1,\dots,x_r\rangle \underset{\substack{\uparrow\\ \text{see page 484}}}{=} \langle x_1,x_2,\dots\rangle.$$

Furthermore the proof of Lemma 12.5 and the above argument show that the generators $x_1,x_2,\dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n\in\mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".

(2) Now we consider

$$\widetilde{X} \;:=\; \mathbb{R} \;\cup\; \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z\in\mathbb{C}\,\big|\,\big|z-(j+\tfrac{1}{4}i)\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}i\,\in\,\mathbb{C}} \;\subset\; \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real axis with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X})$ is the free group on infinitely many generators x_n, $n\in\mathbb{Z}$. In Figure 300 we illustrate a covering map $p\colon \widetilde{X} \to X = S^1\vee S^1$.

[377] The third equality follows from Lemma 12.5, to be more precise, it follows from the fact that for any $r\leq s$ the inclusion map $\iota_{rs}\colon X_r \to X_s$ and the isomorphisms of Lemma 12.5 give rise to a commutative diagram

$$\begin{tikzcd}
\pi_1(X_r,w) \arrow[r, "\cong"] \arrow[d, "(\iota_{rs})_*"'] & \langle x_1,\dots,x_r\rangle \arrow[d, "x_i\mapsto x_i"] \\
\pi_1(X_s,w) \arrow[r, "\cong"] & \langle x_1,\dots,x_s\rangle.
\end{tikzcd}$$

The fact that this diagram commutes follows easily from the proof of Lemma 12.5. We leave the details to the reader.

By Corollary 7.15 the map $\pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely generators and by the discussion on page 390 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{rcl} \pi_1(\widetilde{X}) = \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots \rangle & \to & \langle s, t \rangle = \pi_1(X) \\ x_n & \mapsto & s^n t s^{-n} \end{array}$$

is a monomorphism. In particular we see that the free group $\langle s, t \rangle$ contains a subgroup that is "much larger" than the original free group itself.

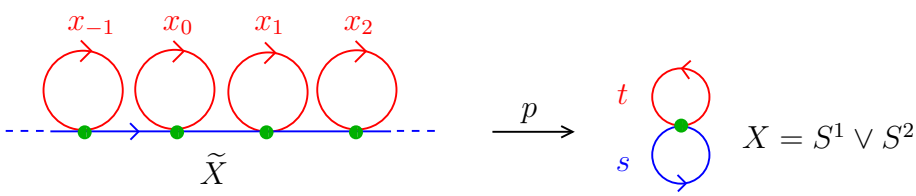


FIGURE 300.

(3) An argument similar to (1) shows that $\pi_1(\mathbb{C} \setminus \mathbb{Z})$ is a free group generated by $x_n, n \in \mathbb{Z}$ where x_n is a loop which goes "once around the n-th hole". We leave the details as an exercise to the reader. We refer to Figure 301 for an illustration.

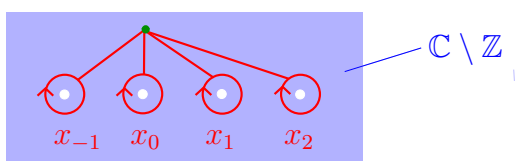


FIGURE 301.

Finally we return to the initial goal of this section, namely to determine the fundamental group of the surface of "infinite genus" that we illustrate in Figure 302. Before we can calculate its fundamental group we have to give a precise description of the surface of infinite genus. We write

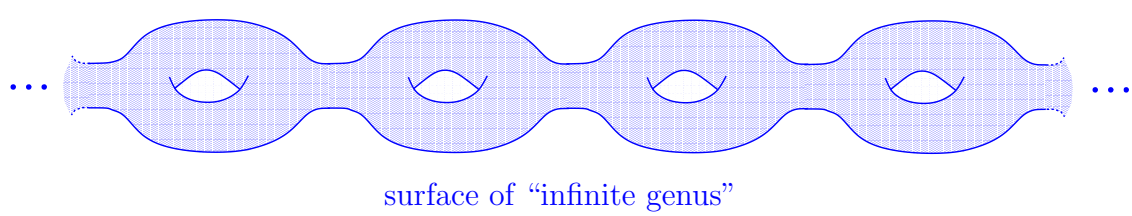


FIGURE 302.

$$F \;:=\; \text{the torus minus two open disks.}$$

We define $X_0 = S^1 \times [-1,1]$ and for $i \geq 0$ we define iteratively

$$X_{i+1} \;:=\; X_i \text{ with a copy of } F \text{ attached to each of the two boundary components of } X_i.$$

Note that each X_i is a surface of genus $2i$ with two open disks removed. We define

$$\text{surface of infinite genus} \;:=\; \varinjlim X_i \;=\; \bigcup_{i\in\mathbb{N}_0} X_i.$$

It should be clear that this definition gives a rigorous description of the surface that we had sketched in Figure 302.

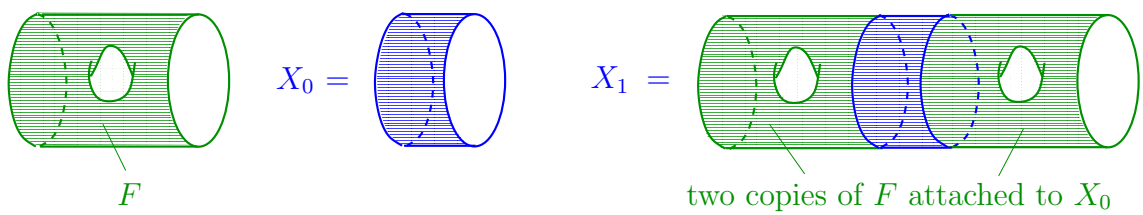


FIGURE 303.

Lemma 16.6. *The fundamental group of the surface of infinite genus is the free group on infinitely many generators.*

Remark. More generally, the fundamental group of any non-compact 2-dimensional manifold is a free group, this is for example proved in [Sti, p. 142].

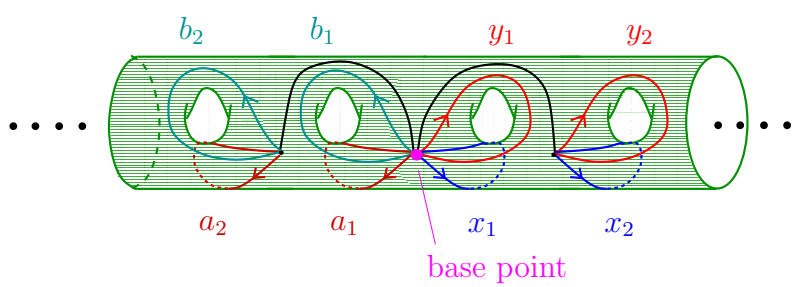


FIGURE 304. A set of generators for the fundamental group of a surface of infinite genus.

Proof. By Lemma 14.19 we have[378]

$$\pi_1(F) \;=\; \langle a,b,c,c' \mid c = [a,b]\cdot c'\rangle \;=\; \langle x,y,z,z' \mid z = [x,y]\cdot z'\rangle$$

[378]This is indeed a consequence of Lemma 14.19, the only change is that we reversed the orientation of one of the boundary curves.

where the generators z, z' and c, c' correspond to the two boundary components. We refer to Figure 305 for an illustration. An induction argument, using the Seifert-van Kampen

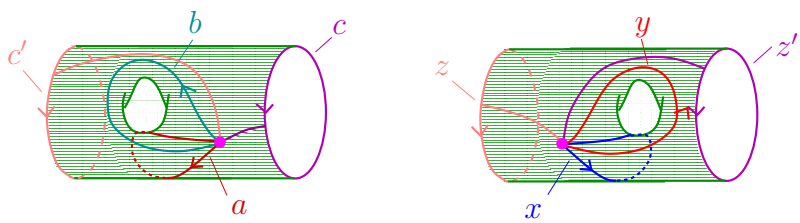


FIGURE 305.

Theorem 14.2 for manifolds together with Lemma 13.13 and the above presentations of $\pi_1(F)$ shows that for any $l \geq k$ we have a commutative diagram

$$\begin{array}{ccc}
\pi_1(X_k) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, ..., x_k, y_k \\ a_1, b_1, ..., a_k, b_k \end{matrix} z_k, c_k \;\middle|\; \prod_{j=1}^{k} [a_j, b_j] c_k = \prod_{j=1}^{k} [x_j, y_j] z_k \right\rangle \\
\Big\downarrow & & \Big\downarrow {\scriptstyle c_k \mapsto \prod_{j=k+1}^{l} [a_j, b_j] c_l \;\; z_k \mapsto \prod_{j=k+1}^{l} [x_j, y_j] z_l} \\
\pi_1(X_l) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, ..., x_l, y_l \\ a_1, b_1, ..., a_l, b_l \end{matrix} z_l, c_l \;\middle|\; \prod_{j=1}^{l} [a_j, b_j] c_l = \prod_{j=1}^{l} [x_j, y_j] z_l \right\rangle
\end{array}$$

where the left vertical map is induced by the inclusion $X_k \to X_l$. We leave the details to the energetic reader. For any $l \geq k$ we also have the following commutative diagram of

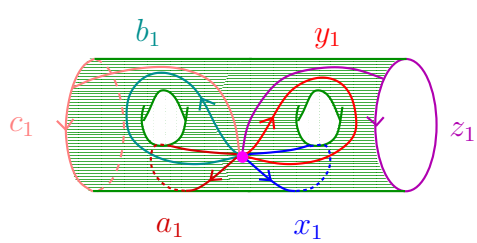


FIGURE 306.

homomorphisms:

$$\begin{tikzcd}
\left\langle \begin{matrix} x_1, y_1, ..., x_k, y_k \\ a_1, b_1, ..., a_k, b_k \end{matrix} p \right\rangle \arrow[r, "\cong", "p \mapsto \prod_{j=1}^{k} [a_j, b_j] c_k"'] \arrow[d, hook] & \left\langle \begin{matrix} x_1, y_1, ..., x_k, y_k \\ a_1, b_1, ..., a_k, b_k \end{matrix} z_k, c_k \;\middle|\; \prod_{j=1}^{k} [a_j, b_j] c_k = \prod_{j=1}^{k} [x_j, y_j] z_k \right\rangle \arrow[d, "c_k \mapsto \prod_{j=k+1}^{l} [a_j, b_j] c_l \quad z_k \mapsto \prod_{j=k+1}^{l} [x_j, y_j] z_l"] \\
\left\langle \begin{matrix} x_1, y_1, ..., x_l, y_l \\ a_1, b_1, ..., a_l, b_l \end{matrix} p \right\rangle \arrow[r, "\cong", "p \mapsto \prod_{j=1}^{l} [a_j, b_j] c_l"'] & \left\langle \begin{matrix} x_1, y_1, ..., x_l, y_l \\ a_1, b_1, ..., a_l, b_l \end{matrix} z_l, c_l \;\middle|\; \prod_{j=1}^{l} [a_j, b_j] c_l = \prod_{j=1}^{l} [x_j, y_j] z_l \right\rangle
\end{tikzcd}$$

The horizontal homomorphisms are in fact isomorphisms, this is for example a consequence of the Tietze transformations from page 412.[379]

We denote by Σ_∞ the surface of infinite genus. Furthermore given any $i \in \mathbb{N}_0$ we write $U_i = X_i \setminus \partial X_i$. Note that each U_i is in fact an open subset of Σ_∞. We have

$$\pi_1(\Sigma_\infty) \underset{\substack{\uparrow \\ \text{since } \Sigma_\infty = \bigcup_{i \in \mathbb{N}_0} U_i}}{=} \pi_1\big(\varinjlim U_i\big) \overset{\substack{\text{Proposition 16.4} \\ \downarrow}}{=} \varinjlim \pi_1(U_i) \underset{\substack{\uparrow \\ \text{since } \pi_1(U_i) \xrightarrow{\cong} \pi_1(X_i)}}{=} \varinjlim \pi_1(X_i) \overset{\substack{\text{by the above two} \\ \text{commutative diagrams} \\ \downarrow}}{=} \varinjlim \left\langle \begin{matrix} x_1, y_1, ..., x_i, y_i \\ a_1, b_1, ..., a_i, b_i \end{matrix} p \right\rangle \underset{\substack{\uparrow \\ \text{see example on} \\ \text{page 484}}}{=} \left\langle \begin{matrix} x_1, y_1, ... \\ a_1, b_1, ... \end{matrix} p \right\rangle.$$

This concludes the proof of the lemma. □

16.4. **The inverse limit of an inverse system.** For completeness' sake we also introduce the inverse limit of an inverse system, even though it will not play a role in the intermediate future.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. An *inverse system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[380] such that the following two conditions are satisfied:

(1)	f_{ii}	$=$	id_{X_i}	for all $i \in I$,
(2)	f_{ki}	$=$	$f_{ji} \circ f_{kj}$	for all $i, j, k \in I$ with $i \leq j \leq k$.

Examples.

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[379]Indeed, we write $X = \{x_1, y_1, \ldots, x_m, y_m, a_1, b_1, \ldots, a_m, b_m\}$, $u = \prod_{j=1}^{m} [a_j, b_j]$, $v = \prod_{j=1}^{m} [x_j, y_j]$, $c = c_m$ and $z = z_m$, then

$$\langle X, z, c \mid uc = vz \rangle = \langle X, z, c \mid z = v^{-1}uc \rangle \xleftarrow{\cong} \langle X, c \rangle \xrightarrow{\cong} \langle X, c, p \mid p = uc \rangle = \langle X, c, p \mid X, c = u^{-1}p \rangle \xleftarrow{\cong} \langle X, p \rangle$$

where all three isomorphisms are the isomorphisms of the Tietze transformation (2). It is straightforward to see that the concatenation equals for $m = k, l$ precisely the horizontal maps in the diagram.

[380]So here for $i \leq j$ we have a morphism $X_j \to X_i$ instead of $X_i \to X_j$ in a direct system.

(A) For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}^j & \to & \mathbb{Z}^i \\ (x_1, \dots, x_j) & \mapsto & (x_1, \dots, x_i) \end{array}$$

the projection onto the first i coordinates. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of groups.

(B) Let p be a prime. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}_{p^i}$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}_{p^j} & \to & \mathbb{Z}_{p^i} \\ n + p^j\mathbb{Z} & \mapsto & n + p^i\mathbb{Z} \end{array}$$

the obvious projection map. Then $(\{\mathbb{Z}_{p^i}\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of rings.

(C) We let $I = \mathbb{N}$, but this time we use the preorder given by divisibility, i.e. we write $k \leq l$ if and only if $k|l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}_k$ and for $k|l$ we denote by

$$\begin{array}{rcl} f_{lk}\colon \mathbb{Z}_l & \to & \mathbb{Z}_k \\ n + l\mathbb{Z} & \mapsto & n + k\mathbb{Z} \end{array}$$

the obvious projection map. Then $(\{\mathbb{Z}_k\}_{k\in\mathbb{N}}, \{f_{lk}\}_{k|l})$ forms an inverse system in the category of rings.

The following definition is the "mirror" image of the definition of the direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Furthermore suppose we are given an inverse system $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ in the category $\mathcal{C}$. An *inverse limit*[381] of the inverse system is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon Y \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $f_{ji} \circ g_j = g_i\colon Y \to X_i$, i.e. for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd} & X_j \arrow[d, "f_{ji}"] \\ Y \arrow[ur, "g_j"] \arrow[r, "g_i"'] & X_i. \end{tikzcd}$$

(2) If we are given another object Y' and morphisms $g_i'\colon Y' \to X_i$, $i \in I$, that satisfy (1), then there exists a unique morphism $F\colon Y' \to Y$ such that for all $i \leq j$ the

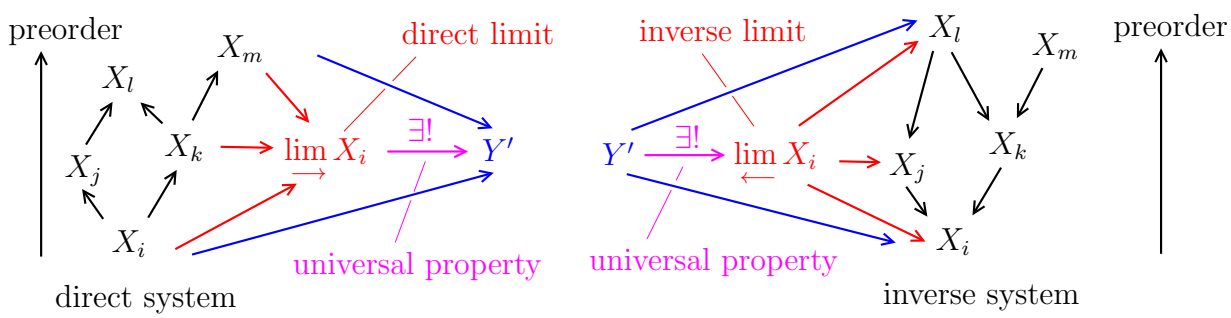
[381]Somewhat confusingly an "inverse limit" is often also called an "inductive limit" or "limit of an inverse system".

following diagram commutes

$$\begin{tikzcd}
 & & X_j \arrow[dd, "f_{ji}"] \\
Y' \arrow[urr, bend left, "g'_j"] \arrow[r, dashed, "F", "\exists!"'] \arrow[drr, bend right, "g'_i"'] & Y \arrow[ur, "g_j"] \arrow[dr, "g_i"] & \\
 & & X_i.
\end{tikzcd}$$

The "usual argument" for objects that satisfy a universal property shows, that if the inverse limit exists, then it is unique in an appropriate sense. We denote the inverse limit by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

In the following diagram we compare schematically the definition of the direct limit of a direct system to the inverse limit of an inverse system:[382]



Examples. In order to get an idea of how to construct inverse limits we first consider the special case that the preorder on I is trivial, in the sense that we have $i \leq j$ if and only if $i = j$.

(D) Let I be a set with the trivial preorder. Suppose that for any $i \in I$ we are given a set X_i. Together with the identity maps of the X_i this defines an inverse system over I

[382]The difference between a direct limit and an inverse limit is not the direction of the arrows between the X_i. In fact given a preorder "$\leq$" one could define a preorder "$\widetilde{\leq}$" via $a \widetilde{\leq} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the limit, or whether the limit maps to the X_i.

in the category of sets. We claim that the inverse limit is given by[383]

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

together with the maps g_i that are given by the projection onto the i-component. Now we verify this statement. Since we only have $i \le j$ if $i = j$ there is nothing to verify in part (1) of the definition of the inverse limit. Now suppose we are given a set Y' and maps $g_i' \colon Y' \to X_i$. Then we define

$$\begin{array}{rcl} F\colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \prod_{i\in I} X_i \\ i & \mapsto & g_i'(y) \end{array}\right). \end{array}$$

It is straightforward to verify that F is uniquely determined and that it has the desired property. We have thus determined the inverse limit $\varprojlim X_i$.

(E) Let I be a set with the trivial preorder and suppose that for any $i \in I$ we are given a topological space X_i. Together with the identity maps of the X_i this defines an inverse system over I in the category of topological spaces. As in (D) we define

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

and each map g_i is again given by the projection onto the i-component.

To conclude that we have found an inverse limit in the category of topological spaces we still have to equip $\prod_{i\in I} X_i$ with a topology which has the following two properties:

(i) the maps $g_i \colon \prod_{i\in I} X_i \to X_i$ are continuous,

(ii) given a topological space Y' and continuous maps $g_i' \colon Y' \to X_i$ the map F, defined as in (D), is continuous.[384]

We set

$$\mathcal{B} \;:=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}.$$

[383]Recall that the product $\prod_{i\in I} X_i$ is defined as the set of all maps $f \colon I \to \bigsqcup_{i\in I} X_i$ with the property that $f(i) \in X_i$ for all $i \in I$. Such a map f is often written as $(f(i))_{i\in I}$ and often we refer to $f(i)$ as the *i-component* of f. Furthermore we refer to the map

$$\begin{array}{rcl} \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

as the *projection onto the i-component*. Note that if the X_i's are groups, rings or modules, then the product inherits this algebraic structure by component-wise multiplication and addition.

[384]On the product $\prod_{i\in I} X_i$ we could also consider the topology $\mathcal{T}$ that is generated by all sets of the form $\prod_{i\in I} U_i$ with U_i open in X_i. This might be a more "natural" topology on $\prod_{i\in I} X_i$. But with this topology $\mathcal{T}$ the map F does not necessarily have to be continuous. The problem is that $\mathcal{T}$ has "too many" open sets.

It is straightforward to verify that $\mathcal{B}$ has the basis property that we introduced on page 60. We equip $\prod_{i\in I} X_i$ with the topology generated by this basis and we refer to this topology as the *product topology.*[385] Now we need to verify that we found the correct topology on the direct limit.

(i) It follows immediately from the definitions that the maps g_i are continuous with respect to the product topology.

(ii) Suppose we are given Y and g_i' as above. We denote by F the map defined as above. We claim that F is continuous. By Proposition 1.34 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I}(g_i')^{-1}(U_i) = \bigcap_{U_i\neq X_i}\underbrace{(g_i')^{-1}(U_i)}_{\text{open in } Y'} = \begin{array}{c}\text{intersection of finitely}\\ \text{many open subsets of } Y'.\end{array}$$

This concludes the proof that F is continuous.

Now we can prove fairly easily that inverse limits exist in many categories that are of interest to us.

Proposition 16.7. *The inverse limit of any inverse system exists in the following categories:*

(0) *the category of sets,*
(1) *the category of topological spaces,*
(2) *the category of groups,*
(3) *the category of topological groups,*
(4) *the category of abelian groups,*
(5) *the category of rings,*
(6) *the category of R-modules, where R is a commutative ring.*

Furthermore, the direct limits of (0),...,(6) *are "coherent". For example if $\{X_i\}_{i\in I}$ is a direct system of topological groups, then the direct limit is the direct system of the corresponding topological spaces together together with an appropriate group structure.*

We will see that the proof of Proposition 16.7 is a relatively straightforward generalization of the examples (D) and (E) above.

Proof. Let $(I,\leq)$ be a preordered set and let $\mathcal{C}$ be one of the seven given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system in the category $\mathcal{C}$. We define

$$\varprojlim X_i \;:=\; \big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\big|\, f_{ji}(x_j) = x_i \text{ for all } i\leq j\big\}.$$

Furthermore we denote by $g_i\colon \varprojlim X_i \to X_i$ the obvious projection maps. It follows immediately from the definitions that for any $i\leq j$ we have $f_{ji}\circ g_j = g_i$. Furthermore, given an

[385]If the index set I is finite, i.e. if we consider the product of finitely many topological spaces, then this definition of the product topology agrees with the definition given on page 84.

object Y' and morphisms $g_i'\colon Y' \to X_i$, $i \in I$ which satisfy the condition $f_{ji} \circ g_j' = g_i'$ for all $i \in I$ we define

$$\begin{array}{rcl} F\colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \prod\limits_{i\in I} X_i \\ i & \mapsto & g_i'(y) \end{array} \right). \end{array}$$

It is straightforward to verify that $F(Y')$ lies in $\varprojlim X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.[386]

For the remaining six categories we will use the same definition of the direct limit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

(1) Suppose that we work in the category of topological spaces. We equip $\prod\limits_{i\in I} X_i$ with the product topology that we defined in Example (E) above. We equip $\varprojlim X_i$ with the subspace topology. The same argument as in Example (E) shows that the maps g_i and F are continuous.

(2) In the category of groups we view $\prod\limits_{i\in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\varprojlim X_i$ is a subgroup of $\prod\limits_{i\in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category of groups.

(3) We equip the set $\varprojlim X_i$ with the topological structure and the group structure as in the above statements (1) and (2). It follows from Lemma 1.109 that this turns $\varprojlim X_i$ into a topological group, which turns out to be the inverse limit of the inverse system of topological groups.

(4),(5),(6) These categories are dealt with the same way as (2).

For all seven categories we have now provided an inverse limit. Since we always equipped the same set with different structures we see that the direct limits are coherent. □

Remark. We consider the important special case that the preordered set $(I, \leq) = (\mathbb{N}, \leq)$. Furthermore let

$$(\{X_n\}_{n\in\mathbb{N}}, \{f_{nm}\colon X_n \to X_m\}_{m\leq n})$$

[386]Now we will see that we can use the "same" inverse limit for all the categories, this is very different from the construction of the direct limit for the same categories. There, in the proof of Proposition 16.1, we had to use several different construction.

be an inverse system over $(\mathbb{N}, \leq)$ in any of the above seven categories. Given $n \in \mathbb{N}_{\geq 2}$ we write $g_n := f_{n,n-1} \colon X_n \to X_{n-1}$. We have

$$\begin{aligned} \varprojlim X_i &= \big\{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } f_{nm}(x_n) = x_m \text{ for all } m \leq n\big\} \\ &= \big\{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } g_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\big\} \\ &\uparrow \end{aligned}$$

follows from $f_{nm} = f_{n,n-1} \circ \cdots \circ f_{m+1,m} = g_n \circ \cdots \circ g_{m+1}$

$$= \text{all infinite sequences of the form } \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ x_3 & \in & X_3 \\ \downarrow & & \downarrow g_3 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_2 \\ x_1 & \in & X_1 \end{array}$$

This discussion shows that we can view of elements in the inverse limit as "upward going sequences".

Now we return to our initial examples of inverse systems and their inverse limits.

Examples.

(A) It follows quite easily from the previous remark that

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}^n = \{(x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{Z}\} \cong \mathbb{Z}^{\mathbb{N}}.$$

Note that this is an uncountable group. Recall that on page 484 we had determined that

$$\varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i \text{ are non-zero}\}$$

which is a countable group.[387] Thus we see that in this particular case the inverse limit is "much larger" than the direct limit.[388]

(B) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}_{p^i}$ of the rings $\mathbb{Z}_{p^i} = \mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p-adic numbers*.[389] The p-adic numbers play a very prominent role in many branches of pure mathematics.

[387] Why is $\mathbb{Z}^{(\mathbb{N})}$ countable?

[388] Note though that we are comparing limits of different system, first of a direct system and then of an inverse system.

[389] In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p-adic numbers by $\mathbb{Z}_p$. If the expression "$\mathbb{Z}_p$" appears in a paper or a book it is a good idea to check the convention used by the author.

(C) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$, is called the *ring of the profinite integers*. Using the Chinese Remainder Theorem one can show that

$$\widehat{\mathbb{Z}} \cong \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

Now we continue with three examples that we had not considered yet.

(D) We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{aligned} f_{ji} \colon X_j = \mathbb{N} &\to X_i = \mathbb{N} \\ n &\mapsto n + (j - i). \end{aligned}$$

Then $(\{X_i = \mathbb{N}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of sets. In Exercise 16.6 we will see that in this case the inverse limit is the empty set.

(E) Let X be a topological space. We denote by K the set of all compact subsets of X. We consider K with the preorder given by inclusion, i.e. we have $A \leq B$ if $A \subseteq B$. Given $A \in K$ we define $X_A := X \setminus A$. Furthermore given $A \leq B$ we denote by[390] $f_{AB} \colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It is straightforward to verify that the sets $\{\pi_0(X_A)\}_{A \in K}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\operatorname{ends}(X) := \varprojlim \pi_0(X_A)$ are called the *ends of* X. In Exercise 16.8 we will determine the number of ends of $X = \mathbb{R}^n$.

(F) Given a topological space Y we denote by $C(Y, \mathbb{R})$ the abelian group of all continuous functions on Y. Note that for each subset $W \subset Y$ we have a homomorphism $C(Y, \mathbb{R}) \to C(W, \mathbb{R})$ given by $f \mapsto f|_W$, i.e. given by restricting a function on Y to a function on W.

Let X be a topological space and let $\{U_i\}_{i \in \mathbb{N}}$ be sequence of open subsets such for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$ and such that $\bigcup_{i \in \mathbb{N}} U_i = X$. Then the groups $\{C(U_i, \mathbb{R})\}_{i \in \mathbb{N}}$ together with the restriction maps form an inverse system. We leave it as an exercise to verify that the restriction maps from X to the U_i gives rise to an isomorphism

$$C(X, \mathbb{R}) \to \varprojlim C(U_i, \mathbb{R}).$$

Definition. Let $(\{0, x, y\}, \leq)$ be the push-pull set as defined on page 481, i.e. the preordered set with the non-trivial relations $0 \leq x$ and $0 \leq y$, and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y, the identity morphisms $\mathrm{id}_{A_0}, \mathrm{id}_{A_x}, \mathrm{id}_{A_y}$ and morphisms $f_x \colon A_x \to A_0$ and $f_y \colon A_y \to A_0$. We refer to such a system as a *pullback system* and often we arrange the morphisms in a diagram as follows

[390] Recall that on page 127, given a topological space Y we defined $\pi_0(Y)$ as the set of path-components of Y. A map $f \colon Y \to Z$ between two topological spaces induces a map $f_* \colon \pi_0(Y) \to \pi_0(Z)$ in an obvious way.

$$\begin{tikzcd} & A_y \arrow[d, "f_y"] \\ A_x \arrow[r, "f_x"'] & A_0. \end{tikzcd}$$

If the inverse limit of such a pullback system exists, then we refer to it as the *pullback* of the *pullback system*. The defining property of the pullback Z can be summarized in the following diagram

$$\begin{tikzcd} Z' \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g'_y"] \arrow[ddr, bend right, "g'_x"'] & & \\ & \text{pullback } Z \arrow[r, "g_x"'] \arrow[d, "g_y"] & A_y \arrow[d, "f_y"] \\ & A_x \arrow[r, "f_x"'] & A_0. \end{tikzcd}$$

The following lemma deals with the pullback in the category of topological spaces.

Lemma 16.8. *Let $p\colon X \to B$ be a map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a map.*

(1) *In the category of topological spaces the pullback of the pullback system*

$$\begin{tikzcd} & X \arrow[d, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd} \quad \textit{is given by} \quad \begin{tikzcd} \{(c,x) \in C \times X \mid f(c) = p(x)\} \arrow[r, "(c,x)\mapsto x"] \arrow[d, "(c,x)\mapsto c"'] & X \arrow[d, "p"] \\ C \arrow[r, "f"'] & B. \end{tikzcd}$$

Often the pullback is denoted by f^*X.

(2) *If $p\colon X \to B$ is a covering, then the pullback $f^*X \to C$ is also a covering of the same degree.*

Proof.

(1) This statement follows quite easily from the definitions or alternatively from the proof of Proposition 16.7.

(2) This statement will be proved in Exercise 16.9. □

We conclude this section on inverse limits with the following lemma.

Lemma 16.9. *Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces. If all the X_i's are compact and Hausdorff, then the inverse limit $\varprojlim X_i$ is also compact.*[391]

[391]Note that this is very different from the situation of the direct limit. The direct limit of topological spaces is not necessarily compact, for example the direct limit of the direct system $[0, n]$, $n \in \mathbb{N}$ is given by the non-compact space $\bigcup_{n\in\mathbb{N}} [0, n] = [0, \infty)$.

Example. Let p be a prime. The abelian group $\widehat{\mathbb{Z}}_p$ is the inverse limit of finite, hence compact groups. It follows from Proposition 16.10 that $\widehat{\mathbb{Z}}_p$ is also a compact topological group. On the other hand $\widehat{\mathbb{Z}}_p$ is infinite,[392] which shows that the topology on $\widehat{\mathbb{Z}}_p$ is *not* the discrete topology.

Proof. Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of compact topological spaces that are Hausdorff. Given $i \in I$ we denote by $p_i\colon \prod_{j\in J} X_j \to X_i$ the projection map and we denote by $D_i = \{(x,x) \mid x \in X_i\} \subset X_i \times X_i$ the diagonal. By Exercise 1.30 the hypothesis that X_i is Hausdorff implies that the diagonal D_i is a closed subset of $X_i \times X_i$. By the proof of Proposition 16.7 we have

$$\begin{aligned}\varprojlim X_i &:= \big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\big|\, f_{lk}(x_l) = x_k \text{ for all } k \leq l\big\}\\ &= \bigcap_{k\leq l}\big\{x \in \prod_{i\in I} X_i \,\big|\, \underbrace{((f_{lk}\circ p_l)\times p_k)}_{\substack{(f_{lk}\circ p_l)\times p_k \text{ is a map}\\ \prod_{i\in I} X_i \to X_k \times X_k}}(x) \in D_k\big\} \;=\; \bigcap_{k\leq l} \underbrace{((f_{lk}\circ p_l)\times p_k)^{-1}(D_k)}_{\text{closed since } D_k \text{ closed}}.\end{aligned}$$

Thus we see that $\varprojlim X_i$ is the intersection of closed subsets of $\prod_{i\in I} X_i$, hence $\varprojlim X_i$ itself is a closed subset. But $\prod_{i\in I} X_i$ is compact by Tychonoff's Theorem 1.63, hence the closed subset $\varprojlim X_i$ is also compact. □

16.5. **The profinite completion of a group** $(*)$**.** Note that we can view any group as a topological group by equipping it with the discrete topology. With this topology a group is compact if and only if it is finite. Furthermore any homomorphism of groups equipped with the discrete topology is also a morphism in the category of topological groups.

Proposition 16.10. *Let $(I, \leq)$ be a preordered set and let $(\{G_i\}_{i\in I}, \{f_{ji}\colon G_j \to G_i\}_{i\leq j})$ be an inverse system of finite groups. Then the inverse limit $\varprojlim G_i$ exists in the category of topological groups and it is a compact topological group.*

Proof. It follows from Proposition 16.7 that the inverse limit $\varprojlim G_i$ exists in the category of topological groups. Since all the G_i's are finite, in particular compact, it follows from Lemma 16.9 that the inverse limit $\varprojlim G_i$ is also compact. □

Now let π be a group. We consider the set of all finite index normal subgroups $\{\pi_i\}_{i\in I}$. We equip the index set I with the preorder defined by the convention that $i \leq j$ if and only if $\pi_j \subset \pi_i$.[393] Given $i \leq j$ we denote by $\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i$ the obvious projection map. Then $(\{\pi/\pi_i\}_{i\in I}, \{\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i\}_{i\leq j})$ forms an inverse system. We refer to

$$\widehat{\pi} := \varprojlim \pi/\pi_i$$

[392] In fact $\widehat{\mathbb{Z}}_p$ is uncountable (why?).

[393] Note that we wrote $\pi_j \subset \pi_i$ instead of $\pi_i \subset \pi_j$.

as the *profinite completion of* π. By Proposition 16.10 the profinite completion $\widehat{\pi}$ is a compact topological group. It follows immediately from the definition of an inverse limit that there exists a unique homomorphism $\pi \to \widehat{\pi}$ such that for any i the following diagram commutes:

$$\begin{tikzcd} \pi \arrow[r] \arrow[drr, bend right] & \widehat{\pi} \arrow[dr] & \\ & & \pi/\pi_i. \end{tikzcd}$$

Examples.

(1) The profinite completion $\widehat{\mathbb{Z}}$ of the group $\mathbb{Z}$ equals precisely the profinite integers that we had encountered on page 502.
(2) The profinite completion of a finite group G is the group itself, i.e. $\widehat{G} = G$.[394]
(3) Let $G = \mathbb{Q}$. This group has no proper subgroups of finite index.[395] It follows immediately from the definitions that $\widehat{\mathbb{Q}} = \{0\}$.

Now let $\varphi\colon A \to B$ be a group homomorphism. We denote by $\{B_i\}_{i\in I}$ the finite index normal subgroups of B. For $B_j \subset B_i$ we denote again by $p_{ji}\colon B/B_j \to B/B_i$ the projection map. For every $i \in I$ the preimage $\varphi^{-1}(B_i)$ is a finite index normal subgroup of A.[396] Now we define $\psi_i\colon \widehat{A} \to A/\varphi^{-1}(B_i) \xrightarrow{\varphi} B/B_i$. Note that ψ_i is continuous since the first map is continuous by definition and the second map is a map between two topological spaces that are equipped with the discrete topology. It follows immediately from the definitions that for $B_j \subset B_i$ we have $p_{ji} \circ \psi_j = \psi_i$. It now follows from the definition of the inverse limit in the category of topological groups that there exists a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$ such that for any i the following diagram commutes

$$\begin{tikzcd} \widehat{A} \arrow[r, "\widehat{\varphi}"] \arrow[drr, bend right, "\psi_i"'] & \widehat{B} \arrow[dr] & \\ & & B/B_i. \end{tikzcd}$$

It follows easily from the definitions and Proposition 16.10 that the maps

$$A \mapsto \widehat{A}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(A, B) & \to & \mathrm{Mor}(\widehat{A}, \widehat{B}) \\ (\varphi\colon A \to B) & \mapsto & (\widehat{\varphi}\colon \widehat{A} \to \widehat{B}) \end{array}$$

define a functor from the category of groups to the category of compact topological groups.

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[394] Why is that?
[395] Why not?
[396] Why is it a finite index subgroup?

Exercises for Chapter 16.

Exercise 16.1. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. We define

$$\varinjlim X_i \;=\; X \;:=\; \bigcup_{i\in I} X_i,$$

where the topology on $X = \bigcup_{i\in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore we denote by $g_j \colon X_j \to X = \bigcup_{i\in I} X_i, j \in \mathbb{N}$ the obvious inclusion maps. Show that $\varinjlim X_i$ together with the maps g_j is the direct limit of the direct system in the category of topological spaces.

Exercise 16.2.

(a) For $i \in \mathbb{N}$ we define $F_i = \langle x_i, x_{i+1}\rangle$. For $i \in \mathbb{N}$ we denote by $g_i \colon F_i \to F_{i+1}$ the homomorphism that is given by $g_i(x_i) = e, g_i(x_{i+1}) = x_{i+1}$. Furthermore for $i < j$ we define $f_i \colon F_i \to F_j$ to be the homomorphism $g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i$. What is the direct limit $\varinjlim F_i$ in the category of groups?

(b) Given $k \in \mathbb{N}$ we define $X_k = \mathbb{N}$ to be the topological space where the open sets are $\{1, \dots, k\}, \{k+1\}, \{k+2\}, \dots$. For $k \leq l$ we denote by $f_{kl} \colon X_k \to X_l$ the identity map. (Note that with the choice of our topologies on X_k and X_l this map is in fact continuous). What is the direct limit $\varinjlim X_k$ in the category of topological spaces?

Remark: as a set the direct limit is again $\mathbb{N}$, but what is the topology?

Exercise 16.3. Show that the pushout of the pushout system of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "g"'] & & Y \arrow[d] \\ Z \arrow[rr] & & Y \sqcup_X Z := (Y \sqcup Z)/f(x) \sim g(x). \end{tikzcd}$$

Here we equip $Y \sqcup_X Z$ with the quotient topology induced from the obvious topology on $Y \sqcup Z$.

Exercise 16.4. Suppose we are given two pushouts

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, dashed] \\ Z \arrow[r, dashed] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, dashed] \\ \widetilde{Z} \arrow[r, dashed] & \widetilde{P}. \end{tikzcd}$$

Furthermore suppose we are given a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dd] \arrow[dr] & & Y \arrow[dr] \arrow[dd, dashed] & \\ & \widetilde{X} \arrow[rr] \arrow[dd, dashed] & & \widetilde{Y} \arrow[dd, dashed] \\ Z \arrow[rr, dashed] \arrow[dr] & & P \arrow[dr, dashed, "\exists!"] & \\ & \widetilde{X} \arrow[rr, dashed] & & \widetilde{P}. \end{tikzcd}$$

An example for such a situation is sketched in Figure 307

(a) Show that there exists a unique map $P \to \widetilde{P}$ which makes the cube commute.

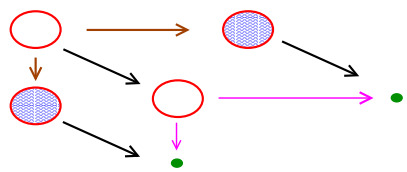


FIGURE 307.

(b) Suppose that the maps $X \to \widetilde{X}$, $Y \to \widetilde{Y}$ and $Z \to \widetilde{Z}$ are homotopy equivalences. Does it follows that the map $P \to \widetilde{P}$ is a homotopy equivalence?

Exercise 16.5. Proof Lemma 16.2. It is best to prove it using the definition of direct limits instead of using any of the above explicit descriptions of direct limits.

Exercise 16.6. We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{array}{rcl} f_{ji} \colon X_j = \mathbb{N} & \to & X_i = \mathbb{N} \\ n & \mapsto & n + (j - i). \end{array}$$

Then $(\{X_i = \mathbb{N}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of sets. Show that the inverse limit is the empty set.

Exercise 16.7. Let

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \\ C & & \end{array}$$

be a pushout system in the category of groups, R-modules or topological spaces such that $f \colon A \to B$ is an isomorphism (respectively homeomorphism). We consider the direct limit, i.e. the pushout

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\ C & \xrightarrow{s} & X. \end{array}$$

Show that $r \colon B \to X$ is also an isomorphism (respectively homeomorphism).

Exercise 16.8. Let X be a topological space. We denote by K the set of all compact subsets of X. We consider K with the preorder given by inclusion, i.e. we have $A \leq B$ if $A \subseteq B$. Given $A \in K$ we define[397] $X_A := \pi_0(X \setminus A)$. Furthermore given $A \leq B$ we denote by $f_{AB} \colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It follows immediately from the discussion on page 245 that the sets $\{\pi_0(X_A)\}_{A \in K}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets.

[397]Recall that by the definition on page 127, given a topological space Y we had denoted $\pi_0(Y)$ by the set of path-components of X.

The elements of $\varprojlim \pi_0(X_A)$ are called the *ends of* X.
Let $n \in \mathbb{N}$. Determine the number of ends of $X = \mathbb{R}^n$.

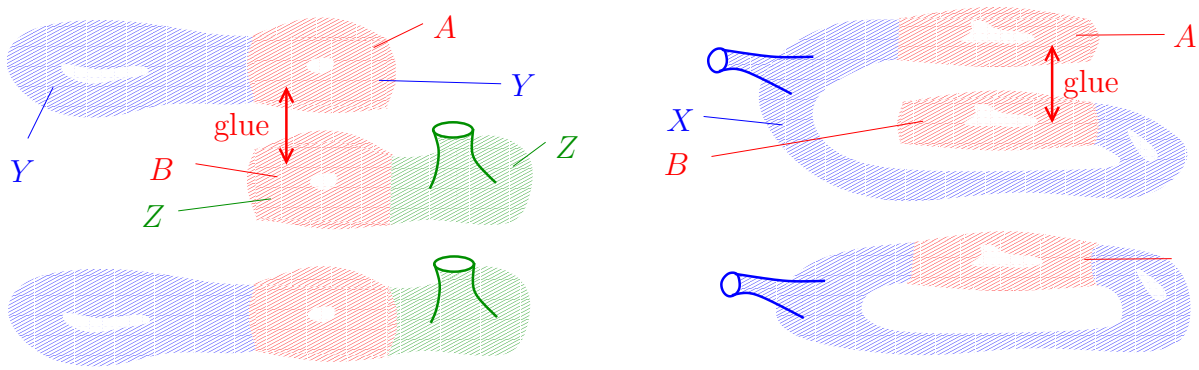
Exercise 16.9. Prove Lemma 16.8 (2), i.e. show that the pullback $f^*X \to C$ of a covering $p\colon X \to B$ under a map $f\colon C \to B$ is again a covering of the same degree.

17. The HNN-Seifert-van Kampen Theorem

One of our main goals at the moment is to calculate fundamental groups of interesting topological spaces. As we have discovered, one of the most powerful tools is the Seifert-van Kampen Theorem 14.1.

In light of Exercise 1.37 (c) we can say, modulo technical details, that the Seifert-van Kampen Theorem 14.1 allows us to calculate the fundamental group of a topological space $X = Y \cup_f Z$ that is obtained from gluing two path-connected topological spaces Y and Z along path-connected open subsets $A \subset X$ and $B \subset Y$. This situation is illustrated in Figure 308 to the left.

Sometimes one is given the situation that is shown in Figure 308 to the right. Namely we are given a path-connected topological space X and we glue two disjoint path-connected open subsets A and B together. In this chapter we will prove the HNN-Seifert-van Kampen Theorem which will allow us to calculate the fundamental group in the latter situation.



$\pi_1(Y \cup_f Z) = \pi_1(Y) *_{\pi_1(A)} \pi_1(Z)$ $\qquad$ $\pi_1 =???$

Figure 308.

We start out with the following notation.

Notation. Let X be a topological space, let A and B be two subsets of X and let $f\colon A \to B$ be a homeomorphism. We write

$$X(f) \;:=\; X(f\colon A \to B) \;:=\; X/\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A.$$

Examples.

(1) Let X be the torus with two open disks removed. We denote by A and B the two boundary components and we endow them with the orientation coming from a fixed orientation of the torus. Let $f\colon A \to B$ be an orientation-reversing diffeomorphism. Then, with some effort, one sees that $X(f\colon A \to B)$ is diffeomorphic[398] to the surface of genus 2. This statement is illustrated in Figure 309.[399]

[398] Note that by Lemma 10.13 we know that $X(f\colon A \to B)$ is indeed a manifold.

[399] What topological space do we obtain if f is chosen to be orientation-preserving?

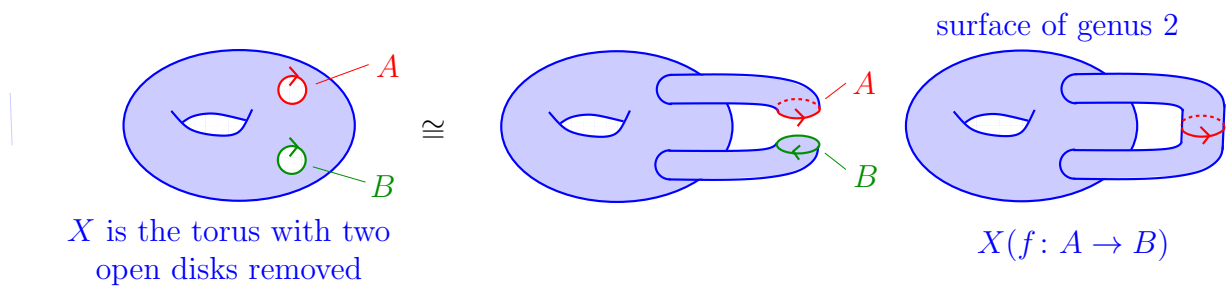


FIGURE 309.

(2) Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider $X = Y \times [0,1]$ and by a slight abuse of notation we denote by $f\colon Y \times \{0\} \to Y \times \{1\}$ also the map defined by $f(y,0) = (f(y),1)$. It follows immediately from the definitions that
$$X(f) \;=\; (Y \times [0,1])(f) \;=\; M(X,f),$$
where $M(X,f)$ denotes the mapping torus that we introduced on page 469.

Our goal now is to determine $\pi_1(X(f))$ in terms of the fundamental groups of X, A and the induced map $f_*\colon \pi_1(A) \to \pi_1(B)$. Perhaps not surprisingly, we again need a new construction from group theory.

Definition. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to
$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g\in\Gamma} \rangle\rangle$$
as the *HNN-extension of* (π, α, β).[400]

Remark.

(1) If $\pi = \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ is a finitely presented group and if $g_1, \dots, g_m$ is a generating set for Γ, then it follows easily from the definitions that the HNN-extension of (π, α, β) is given by
$$\langle x_1, \dots, x_k, t \mid r_1, \dots, r_l, \alpha(g_i) = t\beta(g_i)t^{-1} \text{ for } i = 1, \dots, m \rangle.$$

(2) If $\alpha, \beta\colon \Gamma \to \pi$ are monomorphisms, then it follows from [Ser, p. 45] that the obvious map $\pi \to \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$ is also a monomorphism.

Examples.

(1) Let $\pi = \Gamma$, let $\alpha = \mathrm{id}$ and let $\beta\colon \pi \to \pi$ be an isomorphism. Then it is straightforward to see that the map
$$\begin{array}{rcl} \Phi\colon \pi \rtimes_\beta \mathbb{Z} & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ (g,n) & \mapsto & g \cdot t^{-n} \end{array}$$

[400]The concept of such a group was introduced by Graham Higman, Bernhard Neumann and Hannah Neumann [HNN] in 1950, whence the name.

is an isomorphism.[401] Put differently, we have now shown that a semidirect product is a special case of an HNN-extension.

(2) Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$.

(3) Let π and Γ be two groups and let $\alpha, \beta \colon \Gamma \to \pi$ be two homomorphisms. Then there exists an epimorphism

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \ \to \ \mathbb{Z}$$

that is uniquely determined by $g \mapsto 0$ if $g \in \pi$ and $t^n \mapsto n$.

The following proposition, which is proved in [Ser, p. 45], is an analogue of Proposition 13.14.

Proposition 17.1. $(*)$ *Let π and Γ be two groups and let $\alpha, \beta \colon \Gamma \to \pi$ be two homomorphisms. If α and β are monomorphisms, then the obvious inclusion map*

$$\begin{array}{rcl} \pi & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \ := \ (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g \in \Gamma} \rangle\rangle \\ h & \mapsto & h \end{array}$$

is also a monomorphism.

We now return to topological spaces. The following theorem can be considered a sibling of the Seifert-van Kampen Theorem 14.1.

Theorem 17.2. (HNN-Seifert-van Kampen Theorem) *Let X be a path-connected topological space, let A and B be two open path-connected subsets of X and let $f \colon A \to B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. We write $\Gamma = \pi_1(A, x)$. Furthermore we denote by*

$$\alpha \colon \Gamma = \pi_1(A, x) \ \to \ \pi_1(X, x)$$

the inclusion induced map and we denote by

$$\beta \colon \Gamma = \pi_1(A, x) \ \xrightarrow{f_*} \ \pi_1(B, f(x)) \ \to \ \pi_1(X, f(x)) \ \xrightarrow{\gamma_*} \ \pi_1(X, x)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from Proposition 5.11, using the path γ. Then there exists an isomorphism

$$\Phi \colon \pi_1(X(f), x) \ \xrightarrow{\cong} \ \langle \pi_1(X, x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle.$$

which has the following two properties:

---

[401] As a reality check, here is the calculation that the given map is indeed a homomorphism: let (g, m) and (h, n) be two elements in $\pi \rtimes_\beta \mathbb{Z}$, then

$$\begin{array}{rcl} \Phi(g,m)\Phi(h,n) \ = \ gt^{-m} \cdot ht^{-n} \ \underset{\substack{\uparrow \\ \text{since } h = \alpha(h) = t\beta(h)t^{-1} = t \cdot t\beta(\beta(h))t^{-1} \cdot t^{-1} = \dots}}{=} \ gt^{-m} \cdot t^m\beta^m(h)t^{-m} \cdot t^{-n} \ = \ g\beta^m(h)t^{-m-n} & = & \Phi(g\beta^m(h), m+n) \\ & = & \Phi((g,m)(h,n)). \end{array}$$

This shows that Φ is indeed a homomorphism.

(1) *The diagram*

$$\begin{tikzcd} & \pi_1(X,x) \arrow[dl] \arrow[dr] & \\ \pi_1(X(f),x) \arrow[rr, "\Phi"] & & \langle \pi_1(X,x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \end{tikzcd}$$

with the obvious diagonal maps commutes.

(2) *The closed loop in $X(f)$ defined by γ corresponds under the isomorphism Φ to t.*

Example. As on page 509 we consider the surface X that is given by removing two open disks from a torus. We refer to the two boundary components as A and B. We equip A and B with the orientations given in Figure 310. We pick a diffeomorphism $f\colon A \to B$ that preserves the orientation. Then $X(f)$ is the surface of genus two.

We pick a base point $x \in A$. We pick a loop a in (A,x) we pick loops c,d in (X,x), a path γ in X from x to $f(x)$ and a loop b in $(B,f(x))$ as indicated indicated in Figure 310. Finally we write $\tilde{b} = \gamma * b * \overline{\gamma}$. By Lemma 14.19 and as in the proof of Lemma 16.6 we have an isomorphism

$$\pi_1(X) \;=\; \langle c,d,a,\tilde{b} \mid a = [c,d]\cdot\tilde{b}\rangle.$$

We write $\Gamma = \pi_1(A,x)$. Note that $\Gamma = \langle a\rangle$ and $\alpha(a) = a$ and $\beta(a) = \tilde{b}$. It follows from Theorem 17.2 and the remark on page 510 that

$$\begin{array}{ccccc} \pi_1(X(f),x) & \underset{\substack{\uparrow \\ \text{Theorem 17.2}}}{\cong} & \langle \pi_1(X), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle & \underset{\substack{\uparrow \\ \text{remark on page 510}}}{=} & \langle c,d,a,\tilde{b},t \mid a = [c,d]\cdot\tilde{b}, a = t\tilde{b}t^{-1}\rangle \\ & \underset{\substack{\uparrow \\ \text{Tietze transformations}}}{=} & \langle c,d,\tilde{b},t \mid [c,d]\cdot\tilde{b} = t\tilde{b}t^{-1}\rangle & = & \langle c,d,\tilde{b},t \mid \underbrace{[c,d]\cdot\tilde{b}t\tilde{b}^{-1}t^{-1}}_{=[c,d][\tilde{b},t]}\rangle \end{array}$$

Thus we have now obtained the same presentation that we had already obtained in Proposition 14.3.

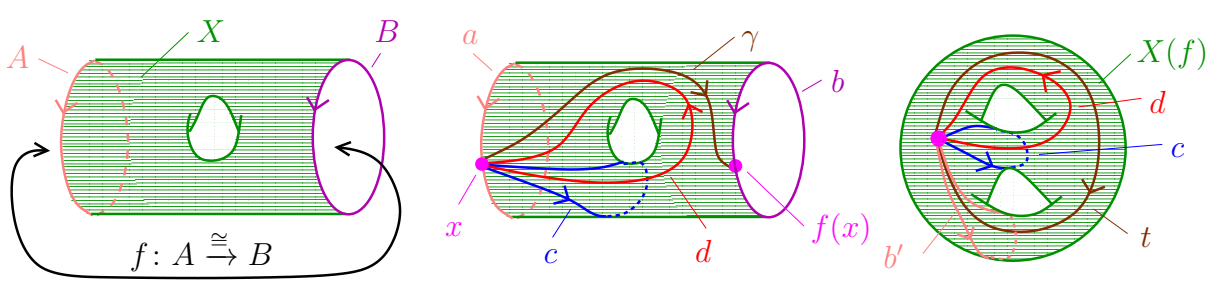


FIGURE 310.

Sketch of the proof of Theorem 17.2(∗). [402] Let X be a path-connected topological space, let A and B be two open path-connected subsets of X and let $f\colon A \to B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. For $i \in \mathbb{Z}$ we now define $X_i := X \times \{i\}$, $A_i := A \times \{i\}$, $B_i := B \times \{i\}$ and $x_i := \{x\} \times \{i\}$. Furthermore we define

$$\widetilde{X} \;:=\; \Big(\bigcup_{i\in\mathbb{Z}} X_i\Big) \,/\, (a,i) \sim (f(a), i-1) \qquad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

There is an obvious action of $\mathbb{Z}$ on $\widetilde{X}$ given by $n \cdot [(x,i)] := [(x, i+n)]$ for $(x,i) \in X_i$. It is straightforward to see that this action is discrete and continuous. It follows easily from the definitions that the map

$$\begin{array}{rcl} \Phi\colon X(f) & \to & \widetilde{X}/\mathbb{Z} \\ [x] & \mapsto & [(x,0)] \end{array}$$

is a homeomorphism. We refer to Figure 311 for an illustration.

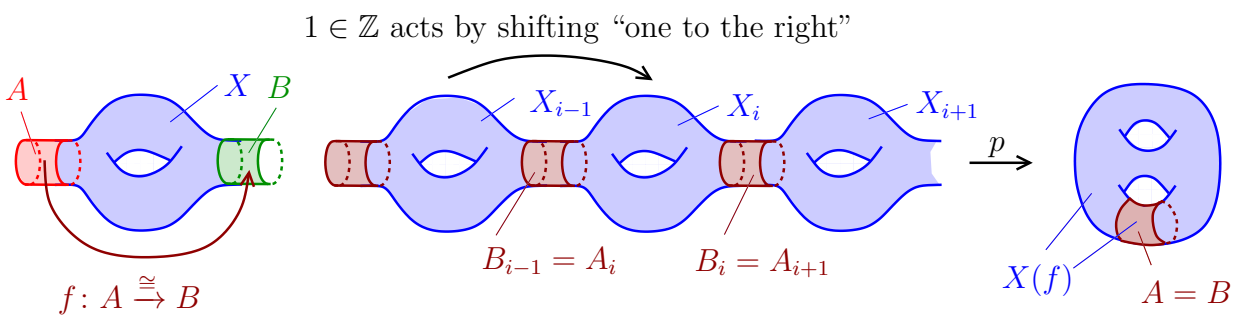


FIGURE 311.

Given $i \in \mathbb{Z}$ we write $\pi_i := \pi_1(X_i, x_i)$ and $\Gamma_i = \pi_1(A_i, x_i)$.

Claim. There exists an isomorphism[403]

$$\pi_1(\widetilde{X}) \;\cong\; \ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots$$

Given any $n \in \mathbb{N}$ we write

$$\widetilde{X}_n \;:=\; \Big(\bigcup_{i=-n}^{n} X_i\Big) \,/\, (a,i) \sim (f(a), i-1) \qquad \text{for } a \in A, i \in \{-n+1, \ldots, n\}.$$

It follows from the Seifert-van Kampen Theorem 14.1 applied iteratively $2n$ times that

$$\pi_1\big(\widetilde{X}_n\big) \;\cong\; \pi_{-n} \underset{\Gamma_{-n+1}}{*} \pi_{-n+1} \ldots \pi_{n-1} \underset{\Gamma_n}{*} \pi_n.$$

[402] As we saw on pages 510 and 511, we can view Theorem 17.2 as a generalization of Proposition 15.25. In fact the subsequent proof is partly modelled on the proof of Proposition 15.25.

[403] This claim is somewhat imprecise, we do not explain what exactly we mean by "…" and we are not careful with the base points. This is one reason why this argument is referred to as a "sketch" of a proof.

Now we see that

$$\pi_1\big(\widetilde{X}\big) \;=\; \pi_1\big(\lim_{\longrightarrow}\widetilde{X}_n\big) \;\underset{\substack{\uparrow\\ \text{Proposition 16.4}}}{=}\; \lim_{\longrightarrow}\pi_1\big(\widetilde{X}_n\big) \;\underset{\substack{\uparrow\\ \text{above calculation}}}{=}\; \dots\pi_{-2}\underset{\Gamma_{-1}}{*}\pi_{-1}\underset{\Gamma_0}{*}\pi_0\underset{\Gamma_1}{*}\pi_1\underset{\Gamma_2}{*}\pi_2\dots$$

This concludes the proof of the claim.

Now we denote by φ the automorphism of the infinite amalgamated product of the claim that "shifts every term by one to the right". Then we obtain an isomorphism[404]

$$\pi_1(X) \;\overset{\substack{\text{above homeomorphism}\\ \downarrow}}{\cong}\; \pi_1\big(\widetilde{X}/\mathbb{Z}\big) \;\overset{\substack{\text{Lemma 15.24 and the above claim}\\ \downarrow}}{\cong}\; \big(\dots\pi_{-1}\underset{\Gamma_0}{*}\pi_0\underset{\Gamma_1}{*}\pi_1\dots\big)\rtimes_\varphi\mathbb{Z} \;\overset{\cong}{\to}\; \langle\pi_1(X,x),t\mid\alpha(\Gamma)=t\beta(\Gamma)t^{-1}\rangle$$
$$(g_i,n) \;\mapsto\; gt^{-i}\cdot t^n,\ \text{where } g_i\in\pi_i$$

□

Remark. One of the key inputs in the proof of Theorem 17.2 was the Seifert-van Kampen Theorem 14.1. If we use instead the Seifert-van Kampen Theorem for manifolds, see Theorem 14.2, then we obtain a "manifold version" of Theorem 17.2. We leave it as an exercise to formulate that version and to make the necessary changes to the proof.

In many situations the following corollary to the HNN-Seifert-van Kampen Theorem 17.2 is quite useful.

Corollary 17.3. *Let X be a topological space which can be written as a union $X = Y\cup Z$ such that the following conditions are satisfied:*

(1) *Y and Z are open subsets,*
(2) *Y is path-connected,*
(3) *Z is simply connected,*
(4) *$Y\cap Z$ consists of two simply-connected path components A and B, each of which is open in X.*

Let y be a point in Y and let γ be the concatenation of a path in Y from y to a point $a\in A$, a path in Z from a to a point $b\in B$ and a path in Y from b to y. Then the map

$$\pi_1(Y,y)*\langle t\rangle \;\to\; \pi_1(X,y)$$

that is given by the inclusion induced map $\pi_1(Y,y)\to\pi_1(X,y)$ and given by $t\mapsto[\gamma]$ is an isomorphism.

Example. We consider the topological space X that is obtained from "gluing an interval"" to the torus. Using the notation from Figure 313 one can show quite easily, using Corollary 17.3 that $\pi_1(X,x_0)\cong\langle x,y,t\,|\,[x,y]\rangle\cong\mathbb{Z}^2*\langle t\rangle$. We will work out the details of this example in Exercise 17.1.

[404]Here we also skip many details.

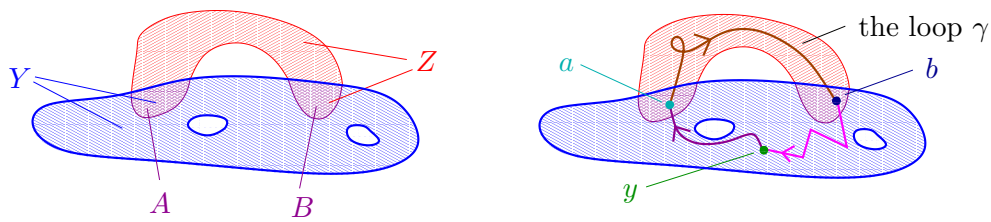


FIGURE 312.

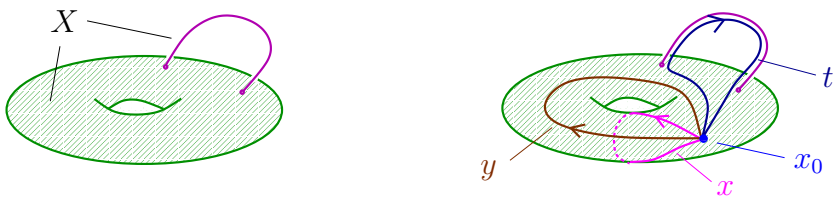


FIGURE 313.

Proof. Let Z' be a disjoint copy of Z (e.g. we could set $Z' = Z \times \{0\}$). We consider $X' = (Y \sqcup Z')/a \sim a'$ where we identify each point $a \in A$ with the corresponding point in Z'. It follows from our hypothesis that A, Y, Z are open in X that Y and Z' are open subsets of X'. Thus we can apply Seifert-van Kampen Theorem 12.2. Since $Z = Z'$ and A are simply connected we obtain that the inclusion induced map $\pi_1(Y, y) \to \pi_1(X', y)$ is an isomorphism.

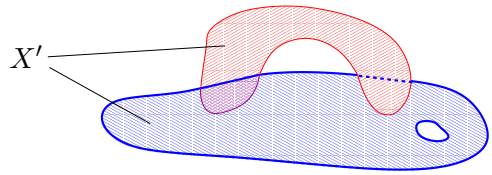


FIGURE 314. Illustration for the proof of Corollary 17.3.

We denote by B' the copy of B in Z' and thus also the copy in X'. Thus we see that we see that B and its doppelgänger B' are contained in X'. If $f\colon B' \to B$ is the obvious homeomorphism, then the desired statement follows immediately from our hypothesis that $B = B'$ is simply connected and the HNN-Seifert-van Kampen Theorem 17.2. $\square$

We conclude this chapter with the following lemma.

Lemma 17.4. $(*)$ *Let $n \in \mathbb{N}$ and let M be the 2-sphere with n open disks removed. We enumerate the boundary components by $C_1, \ldots, C_n$. For $i = 1, \ldots, n$ we pick a path $\beta_i\colon [0,1] \to C_i$ that goes one around C_i. We pick a base point $* \in M \setminus \partial M$. We pick paths $\alpha_1, \ldots, \alpha_n\colon [0,1] \to M$ with the following properties:*

(1) *each* α_i *is an embedding,*
(2) *each* α_i *is a path from* $*$ *to the point* $\beta_i(0)$*,*
(3) *for each* i *we have* $\alpha_i([0,1)) \subset M \setminus \partial M$*,*
(4) *the images of the paths only intersect at* $*$*.*

For any $j \in \{1, \dots, n\}$ *the homomorphism*

$$\begin{array}{rcl} \langle y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n \rangle & \to & \pi_1(M, *) \\ y_i & \mapsto & [\alpha_i * \beta_i * \overline{\alpha_i}] \end{array}$$

is an isomorphism.

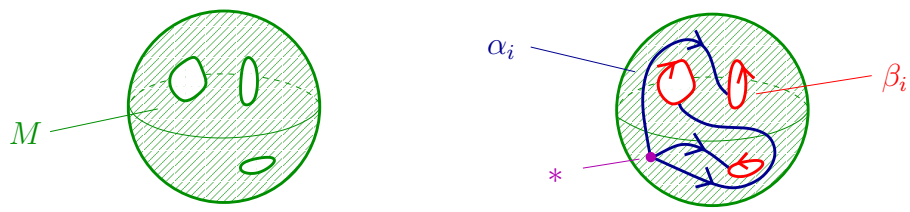


FIGURE 315. Illustration of Lemma 17.4.

Remark.

(1) This proposition should ring a bell, it sounds quite like Lemma 14.19. The difference is that in Lemma 14.19 we only said that certain paths β_i such that the given map is an isomorphism. The above lemma gives a more precise statement what type of paths work.
(2) One might think that perhaps one can use any old paths α_i to obtain a basis for the free group $\pi_1(M, *)$. But as we will see in Exercise 17.3, this is not the case.

We conclude this chapter with a sketch of a proof of Lemma 17.4. Perhaps the most friendly words that can be uttered regarding the proof are that the proof shows that it can be quite hard to write down rigorous arguments once one leaves the friendly word of submanifolds and instead tries to deal with more general subsets.

Sketch of a proof. Without loss of generality we can assume that $j = n$. We consider $N := M \setminus \alpha_n([0,1])$. This is an open subset of the manifold M, thus N itself is a manifold. Furthermore by Exercise 64.2 we know that the inclusion $M \setminus \alpha_n([0,1]) \to M$ is a homotopy equivalence.[405] For $i = 1, \dots, n-1$ we write $X_i := \alpha_i((0,1]) \subset N$. It follows from (1) and

[405]The alert reader will not have failed to notice that we are referring to an exercise that is dealt with later in the notes. Fortunately we do not need to be afraid of circular logic. The exercise builds on an extension of the Tubular Neighborhood Theorem 10.19 to submanifolds with boundary. This formulation, see Theorem 64.7, is just a little awkward and was thus postponed to later on. But if one uses a naive extension of the Tubular Neighborhood Theorem 10.19 to the case of submanifolds with boundary, then one can easily prove Exercise 64.2 with our present knowledge.

Proposition 10.1 (1) that $X = X_1 \cup \cdots \cup X_{n-1}$ is a submanifold of N. By the Tubular Neighborhood Theorem 10.19 we can pick a tubular neighborhood $[-1,1] \times X$. We write $W := N \setminus (-1,1) \times X$, we pick a base point $\star \in W$ and we pick path p in M from $*$ to $\star$.

Using Corollary 17.3 one can prove[406] the following:

(1) there exists loops $\eta_1, \ldots, \eta_{n-1}$ in $(N, \star)$ such that the obvious map

$$\pi_1(W, \star) * \langle [\eta_1], \ldots, [\eta_{n-1}] \rangle \ \to \ \pi_1(N, \star)$$

is an isomorphism,

(2) for each $i \in \{1, \ldots, n-1\}$ the loop $p * \eta_i * \overline{p}$ is path-homotopic to $\alpha_i * \beta_i * \overline{\alpha_i}$.

Putting everything together we see that we have an isomorphism

$$\pi_1(W) * \langle [\alpha_1 * \beta_1 * \overline{\alpha_1}], \ldots, [\alpha_{n-1} * \beta_{n-1} * \overline{\alpha_{n-1}}] \rangle \ \xrightarrow{\cong} \ \pi_1(M).$$

By Lemma 14.19 we know that the right-hand side is a free group of rank $n-1$. It follows from the Grushko-Neumann Theorem 11.18 that $\pi_1(W)$ is trivial. We have thus obtained the desired isomorphism.[407] □

Exercises for Chapter 17.

Exercise 17.1. Give full details for the calculation of the fundamental group of the topological space shown in Figure 313.

Exercise 17.2. Let X be a path-connected topological space. Furthermore let $f \colon X \to X$ be a homeomorphism. In Proposition 15.25 we saw that there exists an isomorphism $\psi \colon \pi_1(M(X,f)) \cong \pi_1(X) \rtimes \mathbb{Z}$. Give an alternative proof of this statement using the HNN-Seifert-van Kampen Theorem 17.2.

Exercise 17.3. We consider the permutation group S_4 where given $\alpha, \beta \in S_4$ we write $\alpha \cdot \beta = \alpha \circ \beta$. We consider the cycles[408]

$$\sigma = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \ \mu = (1\,2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \ \text{ and } \nu = (1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

(a) Show that σ and μ form a generating set for S_4.

---

[406] Here is the point where the demanding reader might feel that not enough details are provided.

[407] The esthetically-inclined reader might object to the use of the Grushko-Neumann Theorem 11.18. But any other approach also opens up a can of worms. For example one might be tempted to show that W is contractible.

[408] Recall that for pairwise disjoint $a_1, \ldots, a_k \in \{1, \ldots, n\}$ we denote by $(a_1\ a_2\ \ldots\ a_k) \in S_n$ the permutation that is given by

$$\begin{array}{rcl} \{1,\ldots,n\} & \to & \{1,\ldots,n\} \\ m & \mapsto & \begin{cases} a_{i+1}, & \text{if } m = a_i \text{ for some } i \in \{1,\ldots,k-1\}, \\ a_1 & \text{if } m = a_k, \\ m, & \text{otherwise.} \end{cases} \end{array}$$

Put differently, the numbers $(a_1, \ldots, a_k)$ get cyclically permuted and all other numbers stay the same.

(b) Show that σ and ν do not form a generating set for S_4.
(c) Show that $(\mu \cdot \sigma) \cdot \mu \cdot (\mu \cdot \sigma)^{-1} = \nu$.
(d) Show that the endomorphism $\langle x, y\rangle \to \langle x, y\rangle$ that is given by $x \mapsto (yx)^{-1}x(yx)$ and $y \mapsto y$ is not an isomorphism.
(e) We consider M which is defined as the disk $\overline{B}^2$ minus two open disks[409] We consider the base point $*$ and the curves β_1 and β_2 as shown in Figure 316. Show that for $i = 1, 2$ there exist paths α_i from $*$ to the starting point of β_i such that $[\alpha_1 * \beta_1 * \overline{\alpha_1}]$ and $[\alpha_2 * \beta_2 * \overline{\alpha_2}]$ do *not* form a generating set for $\pi_1(M, *)$.

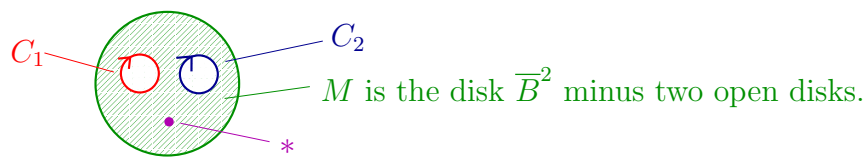


FIGURE 316.

[409]It follows easily from stereographic projection that M is diffeomorphic to S^2 minus three open disks.

18. An excursion into knot theory

In this chapter we will study the fundamental groups of the complements of knots. In particular we will use these calculations to show that the trivial knot, the trefoil and the figure-8 knot are pairwise different, i.e. pairwise they are not smoothly isotopic.

18.1. **One more avatar of the 3-sphere.** We first recall that we have different descriptions of the 3-dimensional sphere S^3, i.e. we have

$$S^3 = \big\{(w,x,y,z)\,\big|\,w^2+x^2+y^2+z^2=1\big\} = \big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1\big\} \underset{\substack{\uparrow \\ \text{via the stereographic projection as defined in Lemma 1.41} \\ \text{note that in Exercise 3.5 we saw that this map} \\ \text{is (lucky us!) orientation-preserving}}}{=} \mathbb{R}^3\cup\{\infty\}.$$

Furthermore, on page 116 we had given a canonical homeomorphism

$$(\overline{B}^2\times S^1)\sqcup_{S^1\times S^1}(S^1\times\overline{B}^2)\ \xrightarrow{\cong}\ S^3.$$

Even though the homeomorphism is canonical, it is not particularly explicit. Furthermore it has the disadvantage that, if one looks up the definition, one sees that the restriction to the two solid tori on the left is not smooth. The following lemma rectifies these two problems.

Lemma 18.1. *We consider the following two maps*[410]

$$\begin{aligned}\Phi\colon \overline{B}^2\times S^1 &\to A := \big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1 \text{ and } |z|^2\geq\tfrac12\big\}\\ (a,b) &\mapsto \left(\frac{b}{\sqrt{|a|^2+|b|^2}},\frac{a}{\sqrt{|a|^2+|b|^2}}\right)\end{aligned}$$

and

$$\begin{aligned}\Psi\colon S^1\times\overline{B}^2 &\to B := \big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1 \text{ and } |w|^2\geq\tfrac12\big\}\\ (a,b) &\mapsto \left(\frac{b}{\sqrt{|a|^2+|b|^2}},\frac{a}{\sqrt{|a|^2+|b|^2}}\right).\end{aligned}$$

These maps have the following properties:

(1) *The maps $\Phi\colon\overline{B}^2\times S^1\to A$ and $\Phi\colon S^1\times\overline{B}^2\to B$ are orientation-preserving diffeomorphisms.*[411]

(2) *The map*

$$\begin{aligned}\Theta\colon (\overline{B}^2\times S^1)\sqcup_{S^1\times S^1}(S^1\times\overline{B}^2) &\to S^3\\ [P] &\mapsto \begin{cases}\Phi(P), & \text{if } P\in\overline{B}^2\times S^1,\\ \Psi(P), & \text{if } P\in S^1\times\overline{B}^2\end{cases}\end{aligned}$$

is well-defined and it is a homeomorphism.

[410]The flipping of the order of the coordinates is not a typo but it is in fact done on purpose.

[411]Note that A and B are submanifolds of codimension zero of the oriented manifold S^3, thus they inherit an orientation from S^3.

(3) *The submanifolds A and B of S^3 have the following properties:*
   (a) *A and B are diffeomorphic to the solid torus $S^1 \times \overline{B}^2$,*
   (b) *$S^3 = A \cup B$,*
   (c) *$A \cap B = \partial A = \partial B$ is a torus, i.e. diffeomorphic to $S^1 \times S^1$.*

Convention. We use the homeomorphism Θ from Lemma 18.1 (2) to add the topological space $(\overline{B}^2 \times S^1) \sqcup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$ to our list of avatars of S^3.

Remark. We continue with the notation from Lemma 18.1 but we now view A and B as subsets of $S^3 = \mathbb{R}^3 \cup \{\infty\}$. In this setting we have

$$\text{“central curve of } A\text{”} \;=\; \Phi(\{0\} \times S^1) \;=\; S^1 \times \{0\} \;=\; \text{the trivial knot}$$

and we have

$$\text{“central curve of } B\text{”} \;=\; \Psi(S^1 \times \{0\}) \;=\; \text{the } z\text{-axis} \cup \{\infty\}.$$

We visualize the solid torus A and the central curve of the solid torus B in Figure 317. The solid torus B is more difficult to illustrate. It consists of a closed disk attached to each point on the central curve. For example the closed disk attached to the origin is just the “obvious” closed disk in the xy-plane that touches the torus $\Phi(S^1 \times S^1)$.[412]

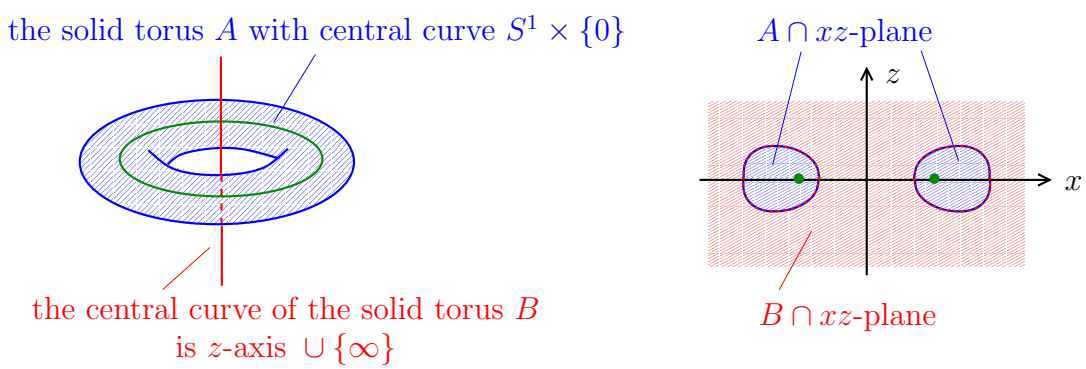


FIGURE 317.

Sketch of proof $(*)$.

(1) First we show that the map $\Phi\colon \overline{B}^2 \times S^1 \to A$ is an orientation-preserving diffeomorphism. We leave it to the reader to verify that the image of Φ does indeed lie in A. Next we consider the map

$$\begin{aligned} \widetilde{\Phi}\colon A = \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \tfrac{1}{2}\right\} &\to \overline{B}^2 \times S^1 \\ (z,w) &\mapsto \left(\tfrac{w}{|z|}, \tfrac{z}{|z|}\right). \end{aligned}$$

One can easily verify that $\widetilde{\Phi}$ and Φ are inverses of one another and that Φ and $\widetilde{\Phi}$ are smooth. This observation implies that $\Phi\colon \overline{B}^2 \times S^1 \to A$ is a diffeomorphism. It

[412] What is the disk attached to the point ∞?

remains to show that Φ is orientation-preserving. By Lemma 3.37 it suffices to show that at the point $(0,1) \in \overline{B}^2 \times S^1$ the differential $D\Phi_{(0,1)}$ is orientation-preserving. This can be done by hand easily. Almost the same argument shows that the map $\Psi\colon S^1 \times \overline{B}^2 \to B$ is an orientation-preserving diffeomorphism.

(2) It is clear that Φ and Ψ agree on $S^1 \times S^1$. Therefore it follows from Lemma 1.66 that the given map $\Theta\colon (\overline{B}^2 \times S^1) \sqcup_{S^1 \times S^1} (S^1 \times \overline{B}^2) \to S^3$ is well-defined and continuous. Using (1) one easily verifies that the map is a bijection. It follows from Proposition 1.40 that the map Θ is a homeomorphism.

(3) The last statement is an immediate consequence of (1) and (2). □

Lemma 1.58 and Lemma 18.1 give us two embeddings of the 2-dimensional torus $S^1 \times S^1$ into $\mathbb{R}^3$. The following lemma says that these two embeddings are essentially the same.

Lemma 18.2. $(*)$ *We denote by $\Omega\colon S^3 \setminus \{(0,1)\} \to \mathbb{R}^3$ the stereographic projection as defined in Lemma 1.41. We consider the two embeddings*[413]

$$\begin{array}{rcl} \overline{B}^2 \times S^1 & \xrightarrow{\Theta} & \mathbb{R}^3 \\ ((x,y), e^{i\varphi}) & \mapsto & \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{x}{2} \\ 0 \\ -\frac{y}{2} \end{pmatrix} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \overline{B}^2 \times S^1 & \xrightarrow{\Phi} & S^3 \setminus \{(0,1)\} \xrightarrow{\Omega} \mathbb{R}^3 \\ (z,w) & \mapsto & \left(\frac{w}{\sqrt{|z|^2+|w|^2}}, \frac{z}{\sqrt{|z|^2+|w|^2}} \right) \end{array}$$

provided by Lemma 1.58 and provided by Lemma 18.1. There exists a smooth diffeotopy G of S^3 rel $\{(0,0)\} \times S^1$ with $G_0 = \mathrm{id}$ and $G_1 \circ \Theta = \Omega \circ \Phi$.

Convention. In all the future examples we will not distinguish between the two embeddings of the solid torus $\overline{B}^2 \times S^1 \to \mathbb{R}^3$ that are given in Lemma 18.2.

Sketch of proof $(*)$. By the Isotopy Extension Theorem 10.21 it suffices to show that there exists a smooth isotopy rel $\{(0,0)\} \times S^1$ from Θ to $\Omega \circ \Phi$.

We write $H = \{(x,0,z) \in \mathbb{R}^3 \,|\, x > 0\}$. Let us first consider the restrictions of Θ and $\Omega \circ \Phi$ to the disk $\overline{B}^2 \times \{1\}$. In both cases the image lies in H. These maps are illustrated in Figure 318. We leave it to the dear reader to show that there exists a smooth isotopy $F\colon \overline{B}^2 \times \{1\} \times [0,1] \to H$ rel $\{(0,0,1)\}$ from Θ to $\Omega \circ \Phi$.

The desired isotopy is given by "rotating F" around the z-axis. Slightly more precisely it is given by the map

$$\begin{array}{rcl} \overline{B}^2 \times S^1 \times [0,1] & \to & \mathbb{R}^3 \\ ((x,y), e^{i\varphi}, t) & \mapsto & \text{rotation of } F((x,y),t) \in \mathbb{R}^3 \text{ by } \varphi \text{ around the } z\text{-axis.} \end{array}$$ □

18.2. **The fundamental group of the complement of the trefoil.** Now we want to finally turn to the study of knots. In particular we would like to study the knots that are illustrated in Figure 319. In Question 10.31 we had raised the question, how can one show that the trefoil is not trivial?

[413] As a reality check note that it follows from Lemma 1.41 that Θ and $\Omega \circ \Phi$ agree on $\{(0,0)\} \times S^1$.

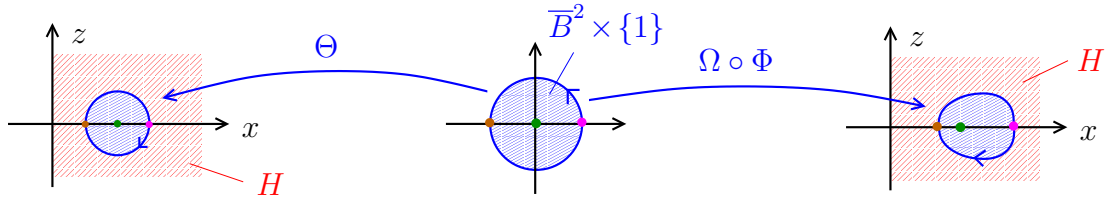


FIGURE 318. Illustration for the proof of Lemma 18.2.

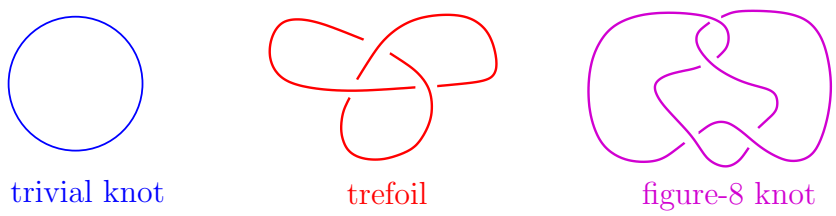


FIGURE 319. The trivial knot, the trefoil knot and the figure-8 knot.

Now we have some tools to attack this question. We saw in Proposition 10.33 that if K and J are smoothly isotopic knots, then the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic. In particular if $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ are not isomorphic, then K and J are not smoothly isotopic. Thus we arrive at the following variation of Question 10.31.

Question 18.3. *Can the fundamental group be used to distinguish the trefoil from the trivial knot?*

Before we continue we want to give a precise description of the trivial knot and the trefoil know. We define the *trivial knot* as the knot[414]

$$U := \{(e^{it}, 0) \mid t \in \mathbb{R}\} \subset \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1\right\}$$

or equivalently

$$U := \{(\cos t, \sin t, 0) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3 \cup \{\infty\}.$$

Now we want to give a rigorous definition of the trefoil knot. We consider the curve $C = \{(2t, 3t) \mid t \in \mathbb{R}\}$ on the torus $T = [0, 2\pi] \times [0, 2\pi]/\sim$. We define the trefoil K as the image of C under the map

$$\begin{aligned} \Phi\colon T = [0, 2\pi] \times [0, 2\pi]/\sim &\to S^3 \\ (s, t) &\mapsto \left(\tfrac{1}{\sqrt{2}} e^{is}, \tfrac{1}{\sqrt{2}} e^{it}\right). \end{aligned}$$

This definition of the trefoil is illustrated in Figure 320.

Now we turn to the calculation of the fundamental groups of the complements of the trivial knot and the trefoil knot.

[414]The name U comes from the fact that the trivial knot is often also called the *unknot*.

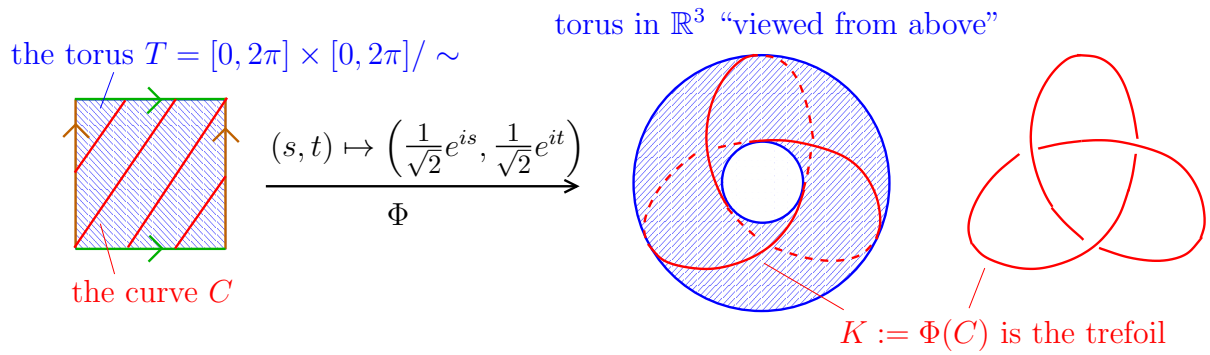


FIGURE 320.

Proposition 18.4. *If U is the trivial knot, then $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$, in particular $\pi_1(S^3 \setminus U) \cong \mathbb{Z}$.*

Proof. Note that

$$\begin{aligned} S^3 \setminus U &= \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \setminus \{(z,0) \mid |z| = 1\} \\ &= \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \text{ and } |w| > 0\}. \end{aligned}$$

Now basically the same argument as in the proof of Lemma 18.1 shows that the map

$$\begin{aligned} \Phi\colon S^1 \times \mathbb{C} &\to S^3 \setminus U = \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 > 0\right\} \\ (a,b) &\mapsto \left(\frac{b}{\sqrt{|a|^2+|b|^2}}, \frac{a}{\sqrt{|a|^2+|b|^2}}\right) \end{aligned}$$

is a diffeomorphism. Since $S^1 \times \mathbb{C}$ is homotopy equivalent to S^1 we obtain from the above and from Proposition 9.12 (1) that $\pi_1(S^3 \setminus U) = \pi_1(S^1 \times \mathbb{C}) = \pi_1(S^1) \cong \mathbb{Z}$. □

Remark. Let U be the trivial knot. A careful reading of the proof of Proposition 18.4 shows that a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian that we defined on page 362. We refer to Figure 321 for an illustration of the trivial knot and a meridian.

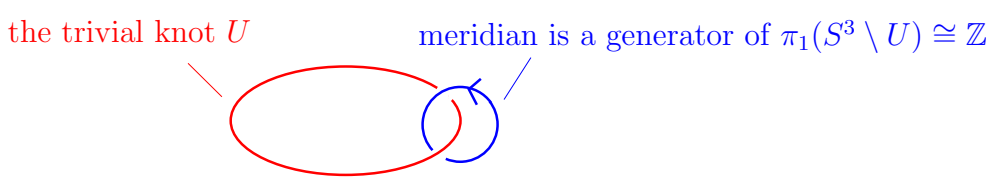


FIGURE 321.

Proposition 18.5. *If K is the trefoil, then*

$$\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle.$$

Proof. By Lemma 18.1 there exists a diffeomorphism

$$\Phi\colon S^3 \xrightarrow{\cong} \underbrace{\left(S^1 \times \overline{B}^2\right)}_{=:A} \cup_{S^1\times S^1 = S^1\times S^1} \underbrace{\left(\overline{B}^2 \times S^1\right)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $T := S^1 \times S^1$. We identify the trefoil with its image under Φ, i.e. we have

$$K \;=\; \{(e^{2si}, e^{3si}) \mid s \in \mathbb{R}\} \;\subset\; T.$$

We write $A_K = A \backslash K, B_K = B \backslash K$ and $T_K = T \backslash K$. Note that A_K and B_K are submanifolds of $X_K := S^3 \setminus K$ and with[415] $A_K \cap B_K = \partial A_k = \partial B_K = T_K$ and with $A_K \cup B_K = X_K$. Note that $x := S^1 \times \{0\}$ is a deformation retract of A_K and that $y := \{0\} \times S^1$ is a deformation retract of B_K. Finally note that T_K retracts onto a parallel copy of K. More precisely, for $\epsilon > 0$ sufficiently small, the subset T_K retracts onto the loop

$$z \;=\; \{(e^{2si+\epsilon}, e^{3si}) \mid s \in \mathbb{R}\} \;\subset\; T_K.$$

By an abuse of notation we denote by x, y and z also the loops corresponding to the circles x, y and z with the obvious orientation. Then the inclusion induced maps

$$\begin{array}{ccccc} \pi_1(A_K) & \leftarrow & \pi_1(T_K) & \rightarrow & \pi_1(B_K) \\ i_*(g) & \leftarrow\!\shortmid & g & \mapsto & j_*(g) \end{array} \quad \text{become} \quad \begin{array}{ccccc} \langle x\rangle & \leftarrow & \langle z\rangle & \rightarrow & \langle y\rangle \\ x^2 & \leftarrow\!\shortmid & z & \mapsto & y^3. \end{array}$$

Thus we see that

$$\begin{array}{ccccccc} \pi_1(X_K) & = & \pi_1(A_K) *_{\pi_1(T_K)} \pi_1(B_K) & \cong & \langle x\rangle *_{x^2=y^3} \langle y\rangle & = & \langle x, y \mid x^2 = y^3\rangle. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Seifert-van Kampen Theorem 14.2} & & \text{above discussion} & & \text{Lemma 13.13} & \end{array}$$

□

Now the question is whether or not $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$. As usual we first consider the abelianization of $\pi_1(S^3 \setminus K)$, but it is straightforward to see, using Proposition 13.10 that the abelianization is isomorphic to $\mathbb{Z}$.[416][417] Despite this slight setback the following lemma does fortunately show that the group $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is not isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$.

Lemma 18.6.

(1) *The group $\langle x, y \mid x^2 = y^3\rangle$ admits an epimorphism onto the permutation group S_3, in particular the group $\langle x, y \mid x^2 = y^3\rangle$ is not isomorphic to $\mathbb{Z}$.*
(2) *The trefoil is not smoothly equivalent to the trivial knot.*

Proof.

[415] Here we mean by ∂A_K and ∂B_K the boundary of A_K and B_K viewed as submanifolds of S^3.

[416] In fact that is not a coincidence, using homology groups we will soon see that for any knot J the abelianization of $\pi_1(S^3 \setminus J)$ is isomorphic to $\mathbb{Z}$.

[417] If one goes very carefully through the definitions and calculations then one can see that a meridian once again becomes a generator of the abelianization.

(1) It follows from Lemma 13.4 and a straightforward calculation that there exists a unique homomorphism

$$\Phi\colon \langle x, y \mid x^2 = y^3\rangle \quad \to \quad S_3 = \text{permutation group on three elements}$$

with

$$\Phi(x) \quad = \quad \sigma \ := \ (1\,2) \ = \ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$\Phi(y) \quad = \quad \tau \ := \ (1\,2\,3) \ = \ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3.[418] On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto an non-abelian group[419].

(2) It follows from (1) and above discussion that the trefoil is non-trivial. □

18.3. **Knot diagrams.** In the previous section we gave a completely rigorous calculation of the isomorphism type of the fundamental group of the complement of a trefoil and we used this to show that the trefoil is not smoothly isotopic to the trivial knot. In the next section we will give a practical algorithm for determining the isomorphism type of the fundamental group of the complement of a given knot. The key to formulating the algorithm is the notion of a knot diagram that we will study in this section.

Before we start with knot theory we introduce one bit of notation.

Notation. Given a smooth map $f\colon S^1 \to \mathbb{R}^n$ and $z = e^{is} \in S^1$ we write

$$f'(z) \ := \ \text{derivative at the point } s \text{ of the function } \mathbb{R} \to \mathbb{R}^n \text{ given by } t \mapsto f(e^{it}).$$

Definition. A map $\gamma\colon S^1 \to \mathbb{R}^2$ is called *diagrammatic* if the following conditions are satisfied:

(a) the map is an immersion,
(b) if $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent,
(c) given any $P \in \gamma(S^1)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Furthermore any $z \in S^1$ for which there exists a $w \neq z \in S^1$ with $\gamma(z) = \gamma(w)$ is called a *double point of* γ. The definitions are illustrated in Figure 322.

The following lemma will be proved in Exercise 18.2.

Lemma 18.7. *Every diagrammatic map has only finitely many double points.*

Definition.

[418] In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3.

[419] Why not?

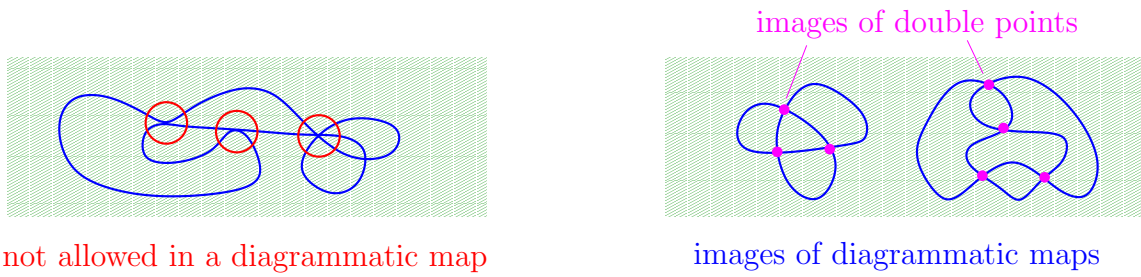


FIGURE 322.

(1) A *knot diagram* is a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$ together with a map
$$c\colon \{\text{double points of } \gamma\} \ \to\ \{\pm 1\}$$
which has the property that for $z \neq w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.
(2) Let (γ, c) be a knot diagram. Let z be a double point. If $c(z) = +1$, then we say that the double point is an *overcrossing*, otherwise we call it an *undercrossing*.

The next lemma shows that knot diagrams give rise to knots.

Lemma 18.8. *Let $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ be a knot diagram. We can pick disjoint compact segments $I_1, \dots, I_k$ of S^1 and a smooth function $\eta\colon S^1 \to [-1, 0]$ with the following properties:*
(a) *Each segment contains a unique undercrossing and each undercrossing is contained in some segment.*
(b) *No segment contains an overcrossing.*
(c) *The value of η outside the segments is equal to 0.*
(d) *On each segment the function has a unique local minimum, namely at the corresponding undercrossing where the value of η is equal to -1.*

The following two statements hold:
(1) *The image of S^1 under the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$ is a knot.*
(2) *Any two choices of segments and η as above give rise to smoothly isotopic knots.*

If we equip S^1 with the standard orientation, then in the above statements we can also replace "knot" by "oriented knot".

Lemma 18.8 shows that a knot diagram gives rise to an essentially unique (oriented) knot. We refer to this knot as the *knot associated to the knot diagram*. At times we will rather blur the difference between a knot diagram and the associated knot.

Example. In Figure 323 we show to the left a knot diagram. To the right we show the associated knot. It is pretty clear that the resulting knot is smoothly isotopic to the trefoil.

Proof. It follows quite easily from Lemma 3.12 that such segments $I_1, \dots, I_k$ and that such a function η exist. Given such a function it is clear that the map $z \mapsto (\gamma(z), \eta(z))$ is injective. Since γ is an immersion it follows that the map $z \mapsto (\gamma(z), \eta(z))$ is in fact an

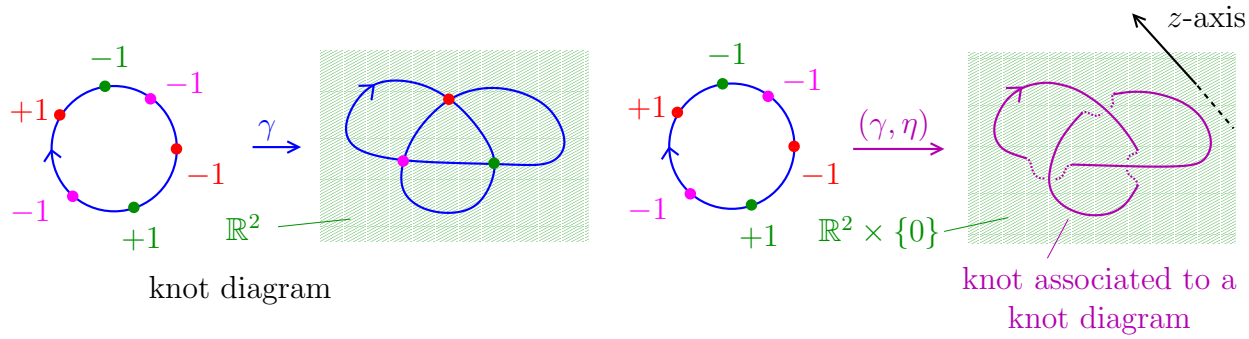


FIGURE 323.

embedding. Thus it follows from Proposition 10.1 (1) that the image is a one-dimensional submanifold of $\mathbb{R}^3 \cup \{\infty\} = S^3$ that is diffeomorphic to S^1. In other words, the image is a knot.

Finally suppose we are given functions η and η' with the properties stated as above. We consider the map

$$\begin{array}{rcl} H\colon S^1 \times [0,1] & \to & \mathbb{R}^3 \cup \{\infty\} = S^3 \\ (z,t) & \mapsto & (\gamma(z), \eta(z) \cdot (1-t) + \eta'(z) \cdot t). \end{array}$$

This map is smooth. By construction we know that for any two distinct points $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta'(z) < \eta'(w)$. Using this observation one can easily verify that each H_t is an embedding. In other words, the map H_t is a smooth isotopy between the two (oriented) knots corresponding to η and η'. $\square$

In practice, i.e. in future examples, we do what every other sane topologist does, namely we draw a suggestive picture and we interpret it as a knot diagram and a corresponding knot in the obvious way. For example in Figure 324 we show the first knots in the standard table of knots. We refer to the standard textbooks on knot theory [Rol, BZH] for a continuation of the table. But it should already be pretty clear from this table that knots form a straightforward way to produce an almost endless list of interesting topological problems.

In Lemma 18.8 we saw that any knot diagram gives rise to a knot. The following proposition gives us the converse:

Proposition 18.9. *Given any knot $K \subset S^3$ there exists a knot diagram such that K is smoothly isotopic to the knot associated to the knot diagram.*

The proof of Proposition 18.9 mostly rests on the next lemma. The formulation of this lemma requires the following notation.

Notation. Given $v \in S^2$ we denote by

$$\begin{array}{rcl} \pi_v\colon \mathbb{R}^3 & \to & v^\perp := \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\} \\ w & \mapsto & w - \langle v, w\rangle v \end{array}$$

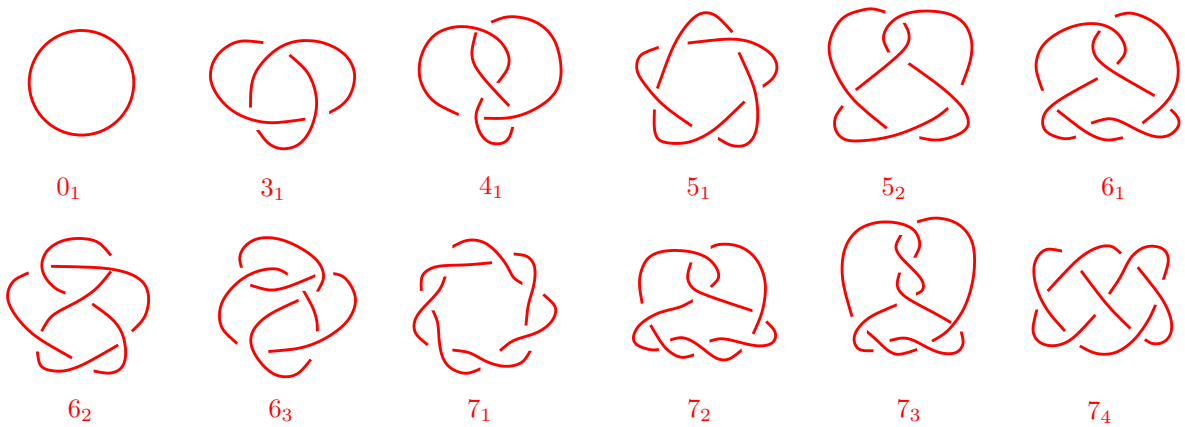


FIGURE 324.

the projection onto the orthogonal complement of v.

Now we can formulate the lemma.

Lemma 18.10. *Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f\colon S^1 \to K$. We consider the following three properties of a vector $v \in S^2$:*

(a) *the map $\pi_v \circ f\colon S^1 \to v^\perp = \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\}$ is an immersion,*
(b) *whenever $z \neq w \in S^1$ satisfy $(\pi_v \circ f)(z) = (\pi_v \circ f)(w)$, then $(\pi_v \circ f)'(z)$ and $(\pi_v \circ f)'(w)$ are linearly independent,*
(c) *for every $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of either one or two points.*

The set

$$\{v \in S^2 \mid v \text{ satisfies (a), (b) and (c)}\}$$

has full measure in S^2.[420]

Remark. In Proposition 3.47 (2) we showed that a subset of full measure is dense. The chronically bored reader can try to show that the set $\{v \in S^2 \mid v$ satisfies (a), (b) and (c)$\}$ considered in Lemma 18.10 is open.

Proof. Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f\colon S^1 \to K$. We say that a subset of a manifold W is *large* if it is open and if it has full measure in W. It follows from Proposition 3.47 (5b) that the intersection of two large subsets is again large. Furthermore, it follows from Proposition 3.47 (1) that any self-diffeomorphism of S^2 sends large sets to large sets.

It is convenient to consider an extra property of vectors in S^2 which is somewhat weaker than the above property (c):

[420]We refer to page 213 for the definition of a subset of full measure.

(c′) given any $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of finitely many points.

We write

$$V(a) \ := \ \{v \in S^2 \mid \ v \text{ satisfies (a)}\}.$$

Similarly we define $V(b), V(c), V(a,b), V(a,b,c')$ and so on. Note that with this language we need to show that $V(a,b,c)$ has full measure.

Let $v \in S^2$. We start out with three elementary observations regarding the projection maps π_v.

(1) Given $u \in \mathbb{R}^3 \setminus \{0\}$ we have

$$\pi_v(u) = 0 \quad \Longleftrightarrow \quad u \in \mathbb{R} \cdot v \quad \Longleftrightarrow \quad \tfrac{u}{\|u\|} \in \{v, -v\}.$$

(2) The projection map π_v is linear, hence for any $P \in \mathbb{R}^3$ we have $(D\pi_v)_P = \pi_v$.

(3) It follows from (2) and the chain rule that given any $z \in S^1$ we have the equality $(\pi_v \circ f)'(z) = \pi_v(f'(z))$.

Next we consider the three maps[421] [422]

$$\begin{aligned} \varphi\colon S^1 &\to S^2 \\ z &\mapsto \tfrac{f'(z)}{\|f'(z)\|} \end{aligned} \qquad \begin{aligned} \psi\colon \overbrace{\{(z,w) \in S^1 \times S^1 \mid z \neq w\}}^{=:M} &\to S^2 \\ (z,w) &\mapsto \tfrac{f(z)-f(w)}{\|f(z)-f(w)\|} \end{aligned} \quad \text{and} \quad \begin{aligned} \rho\colon S^2 &\to S^2 \\ v &\mapsto -v. \end{aligned}$$

We make the following observations:

(4) If X is a subset of full measure (respectively large subset) of S^2, then it follows from the above discussion that $X \cap \rho(X)$ is also a subset of full measure (respectively large).

(5) Since f is smooth we see that the map φ is also smooth.

(6) Given $v \in S^2$ it follows easily from (1) and (3) that $\pi_v \circ f$ is an immersion if and only if neither v nor $-v$ lies in $\varphi(S^1)$. In other words, we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus \varphi(S^1))$.

(7) Note that $M = \{(z,w) \in S^1 \times S^1 \mid z \neq w\}$ is an open subset of the manifold $S^1 \times S^1$, thus it is a manifold in an obvious way. With this manifold structure the map $\psi\colon M \to S^2$ is easily seen to be smooth.

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[421] Since f is an immersion we see that the map φ is well-defined, i.e. we do not divide by zero. Similarly, since f is in particular injective we see that the map ψ is well-defined, i.e. once again we do not divide by zero.

[422] The astute reader will notice that throughout the argument it might be more reasonable to work with maps φ and ψ that take values in $\mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ instead of taking values in S^2. We stick with the maps to S^2 since S^2 has the advantage that its tangent spaces can be described easily as subvector spaces of $\mathbb{R}^3$, which makes it easy to write down differentials. Also, the approach of working with maps to $\mathbb{R}\mathrm{P}^2$ instead also creates an extra layer of notation which is as annoying as our notation which requires the extra map ρ.

(8) For each $(x, y) \in S^1$ we make the identification[423] $T_{(x,y)}S^1 = \mathbb{R}$ via the basis vector $(-y, x)$. Given $(z, w) \in M$ we use the above identification to make the identification $T_{(z,w)}M = T_{(z,w)}(S^1 \times S^1) = \mathbb{R}^2$. Furthermore given $v \in S^2$ we already saw on page 197 that we have the equality $T_vS^2 = v^\perp = \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\} \subset \mathbb{R}^3$.

(9) Using the identifications from (8) an elementary calculation[424] shows that for a point $(z, w) \in M$ we can write the differential $D\psi_{(z,w)}$, viewed as a homomorphism $T_{(z,w)}M = \mathbb{R}^2 \to T_{\psi(z,w)}S^2 \subset \mathbb{R}^3$, in the following way as a (3×2)–matrix:

$$D\psi_{(z,w)} \;=\; \tfrac{1}{\|f(z)-f(w)\|} \cdot \underbrace{\big(\pi_{\psi(z,w)}(f'(z)) \quad -\pi_{\psi(z,w)}(f'(w))\big)}_{(3\times 2)\text{-matrix}}.$$

(10) It follows from (9) that $(z, w) \in M$ is a regular point of ψ if and only if the vectors $\pi_{\psi(z,w)}(f'(z))$ and $\pi_{\psi(z,w)}(f'(w))$ are linearly independent.

(11) Let $v \in S^2$. It follows immediately from (1) and the linearity of π_v that we have $\pi_v(f(z)) = \pi_v(f(w))$ if and only if $\psi(z, w) = v$ or $\psi(z, w) = -v$.

(12) We denote by $\sigma\colon S^1 \times S^1$ the diffeomorphism given by $(z, w) \mapsto (w, z)$. It follows immediately from the definitions that $\psi \circ \sigma = \rho \circ \psi$.

(13) It follows easily from (3), (10), (11) and (12) that

$$V(b) \;=\; \{\text{regular values of } \psi\}.$$

After these initial remarks we turn to our first claim.

Claim.

(α) The sets $V(a)$ and $V(a, b)$ are large.

(β) We have $V(a) = V(a, c')$ and $V(a, b) = V(a, b, c')$.

We prove the two statements of the claim in several steps.

(i) Since S^1 is compact we obtain from Lemma 1.37 and Lemma 1.18 that $S^2 \setminus \varphi(S^1)$ is an open subset of S^2. Furthermore, since φ is smooth and since $\dim(S^1) < \dim(S^2)$ we obtain from Proposition 3.47 (5) that $S^2 \setminus \varphi(S^1)$ is a subset of full measure. In summary, we have shown that $S^2 \setminus \varphi(S^1)$ is large. It follows from (5) and (6) that $V(a) = (S^2 \setminus (\varphi(S^1)) \cap \rho(S^2 \setminus (\varphi(S^1)))$ is large.

(ii) It follows from Sard's Theorem 3.48 that the set of regular values of ψ is a subset of full measure. It follows from (i) and (13), together with Proposition 3.47 (1) that $V(a, b) = V(a) \cap V(b)$ has full measure.

(iii) Let $v \in V(a)$. It follows from (11), the fact that S^1 is compact and Exercise 10.1 that $\psi^{-1}(v)$ is finite. In particular we see that $V(a, c') = V(a)$ and thus also $V(a, b, c') = V(a, b)$.

[423] As on many other occasions we use Proposition 3.32 to view the tangent space of a submanifold of some $\mathbb{R}^n$ as a subvector space of $\mathbb{R}^n$.

[424] This calculation can be performed easily using the chain rule and by writing ψ as the composition of the map $(z, w) \mapsto f(z) - f(w)$ followed by the map $P \mapsto \frac{P}{\|P\|}$.

(iv) It remains to show that $V(a,b)$ is open. Let $v \in V(a,b) = V(a) \cap V(b)$. From (13) and (iii), together with Exercise 3.13 (2), we obtain an open neighborhood U of $v \in S^2$ that is still contained in $V(b)$. Since $V(a)$ is open we see that $U \cap V(a)$ is an open neighborhood of $v \in S^2$ that is still contained in $V(a,b)$. It follows from Lemma 1.8 that $V(a,b)$ is indeed open.

This concludes the proof of the claim.

Before we continue it is perhaps helpful to summarize what we have shown so far: In the claim we have seen that $V(a,b,c') = V(a,b)$ is large, in particular that it has full measure. But our actual goal is to show that $V(a,b,c)$ has full measure.

To do so we consider the map

$$\begin{array}{rcl} \Xi\colon \overbrace{\{(x,y,z) \in S^1 \times S^1 \times S^1 \,|\, x \neq y \text{ and } x \neq z\}}^{=:N} & \to & S^2 \times S^2 \\ (x,y,z) & \mapsto & (\psi(x,y), \psi(x,z)). \end{array}$$

We make the following simple observations:

(14) We view N as a manifold in an obvious way. Evidently the map Ξ is smooth.

(15) It follows easily from (1) that we have $v \in V(c)$ if and only if none of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$.

(16) If one of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$, then so does (v,v). This can be seen as follows: If $\Xi(x,y,z) \in \{(\pm v, \pm v)\}$, then $f(x), f(y), f(z)$ lie on the line $\mathbb{R} \cdot v$. After possibly permuting x, y, z we can assume that $f(x) = f(y) + r \cdot v$ and that $f(x) = f(z) + s \cdot v$ with $r, s > 0$. But with this permutation we have $\Xi(x,y,z) = (v,v)$.

Claim. We consider the "partial diagonal"

$$\Delta \;:=\; \{(v,v) \in S^2 \times S^2 \,|\, v \in V(a,b)\}.$$

The complement of $\Xi(N) \cap \Delta$ has full measure in Δ.

In the previous claim we saw that $V(a,b)$ is an open subset of S^2. Using this fact it is straightforward to see that Δ is a 2-dimensional submanifold of $S^2 \times S^2$. Thus, by Exercise 10.3 (b) it suffices to prove that if $(x,y,z) \in N$ satisfies $\Xi(x,y,z) = (v,v)$ for some $v \in V(a,b)$, then Ξ intersects Δ transversally, i.e. we have the equality

$$(D\Xi_{(x,y,z)})(T_{(x,y,z)}N) + T_{(v,v)}\Delta \;=\; T_{(v,v)}(S^2 \times S^2) \;\subset\; \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6.$$

We write $p = \pi_v(f'(x))$, $q = \pi_v(f'(y))$ and $r = \pi_v(f'(z))$ and we write $\mu = \frac{1}{\|f(x)-f(y)\|}$ and $\nu = \frac{1}{\|f(x)-f(z)\|}$. By (10) and (13), and by the fact that $v \in V(b)$, we know that p, q and r are pairwise linearly independent. In particular q and r form a basis for T_vS^2. Note that this implies that $\begin{pmatrix} q \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ r \end{pmatrix}$ form a basis for $T_{(v,v)}(S^2 \times S^2)$. Thus, using (9) we see that

$$(D\Xi_{(x,y,z)})(T_{(x,y,z)}N) + T_{(v,v)}\Delta \;=\; \text{span of the columns of } \begin{pmatrix} \mu \cdot p & -\mu \cdot q & 0 & q & r \\ \nu \cdot p & 0 & -\nu \cdot r & q & r \end{pmatrix}.$$

Using that p, q and r are pairwise linearly independent it is not hard to see that the five columns of the matrix on the right span a 4-dimensional subspace of $\mathbb{R}^6$. Since this subspace is contained in the 4-dimensional subspace $T_{(v,v)}(S^2 \times S^2)$ we see that it equals $T_{(v,v)}(S^2 \times S^2)$. This concludes the proof of the claim.

We consider the "diagonal map" $d\colon V(a,b) \to \Delta \subset S^2 \times S^2$ that is given by $d(v) = (v,v)$. One can easily show that $d\colon V(a,b) \to \Delta$ is a diffeomorphism. It follows from (16) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$. By the claim $\Delta \setminus \Xi(N)$ is a subset of full measure of Δ. Since d is a diffeomorphism we obtain from Proposition 3.47 (5b) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$ has full measure in $V(a,b)$. We saw above that $V(a,b)$ has full measure in S^2. It follows easily from Proposition 3.47 (5a) that $\Delta \setminus \Xi(N)$ has full measure in S^2. □

Now we can provide the proof of Proposition 18.9.

Proof of Proposition 18.9. Let $K \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a knot. It follows from Lemma 1.73 (4) and Lemma 10.4 (2) that, possibly after applying a smooth isotopy, we can assume that $K \subset \mathbb{R}^3$.

We pick $v \in S^2$ as in Lemma 18.10. It follows from Lemma 1.73 (4) and Lemma 10.4 (2) that, possibly after applying a smooth isotopy, we can assume that $v = e_3$. We define $\gamma := \pi_{e_3} \circ f\colon S^1 \to \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$. It follows immediately from the fact that $v = e_3$ has the three properties (a), (b) and (c) stated in Lemma 18.10 that γ is diagrammatic. Next we consider the function

$$\begin{array}{rcl} \epsilon\colon S^1 & \to & \mathbb{R} \\ w & \mapsto & z\text{-coordinate of } f(w). \end{array}$$

Furthermore, given $z_1 \neq z_2 \in S^1$ with $\gamma(z_1) = \gamma(z_2)$ we define

$$c(z_i) \ := \ \begin{cases} +1, & \text{if } \epsilon(z_i) > \epsilon(z_{3-i}), \\ -1, & \text{if } \epsilon(z_i) < \epsilon(z_{3-i}). \end{cases}$$

It is clear that (γ, c) is a knot diagram. We pick a function $\eta\colon S^1 \to [-1,0]$ as in Lemma 18.8. We consider the map

$$\begin{array}{rcl} H\colon S^1 \times [0,1] & \to & \mathbb{R}^3 \\ (z,t) & \mapsto & (\gamma(z), (1-t)\cdot\epsilon(t) + t\cdot\eta(t)). \end{array}$$

Clearly this map is smooth. Given any $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta(z) < \eta(w)$. From this observation we deduce that each H_t is actually an embedding. Thus we see that H is a smooth isotopy from K to the knot associated to the knot diagram (γ, η). □

We have now seen that every knot, up to smooth isotopy, arises from a knot diagram. The question arises, when do two knot diagrams give rise to smoothly isotopic knots. The following theorem gives a complete answer. We formulate the theorem in a slightly informal way since we will not make use of it.

Theorem 18.11. *Two knot diagrams give rise to smoothly isotopic knots if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and Reidemeister moves. The three Reidemeister moves are illustrated in Figure 325.*

Proof. The theorem, not surprisingly, goes back to work of Kurt Reidemeister [Rei2, Chapter 3] in 1932. The formulation and proof in [Rei2] is in terms of "polygonal knots". The statement of the theorem is well known in our setting, where knots are defined as submanifolds of S^3 and it appears in any textbook on knot theory. A satisfactory proof can be found in [OSS, Theorem B.1.1]. □

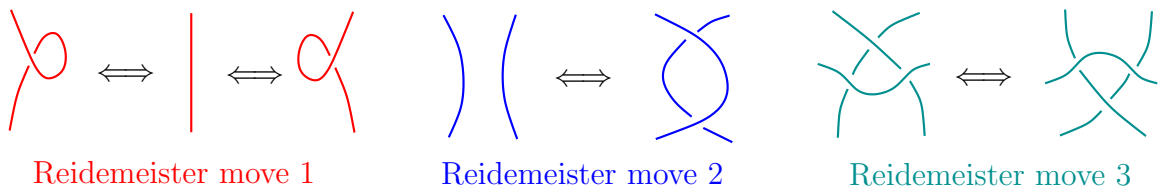


FIGURE 325.

Remark. Throughout these lecture notes we will see on several occasions that the tools of algebraic topology, e.g. fundamental groups, linking pairings and Reidemeister torsion can be used fruitfully to study knots. Proposition 18.9 and Theorem 18.11 open up an alternative route to studying knots. More precisely, these two results show that there exists a bijection

$$\begin{array}{c}\text{knot diagrams up to smooth isotopy}\\ \text{and Reidemeister moves}\end{array} \xrightarrow{\cong} \text{knots up to smooth isotopy}$$

This bijection lies at the heart of the definition of several knot invariants, like the Jones polynomial [Jon] that was introduced in 1984 and the HOMFLY-PT polynomial [FYHLMO, PT] that was discovered in 1985. These polynomial invariants are quite different from any of the invariants that are obtained through algebraic-topological methods. We refer to [Lic3, Chapters 3 and 15] and [BZH, Chapter 17] for more information on these invariants.

18.4. **The Wirtinger presentation.** In the previous section we saw that we can associate a knot diagram to any given knot. In this section we will see that we give an explicit algorithm that, given a knot diagram, produces a presentation for the fundamental group of the complement $S^3 \setminus K$ of a knot K.

In this section we will take several liberties and we will not attempt to give full details in the proofs. The issue is that any attempt to make the proofs completely rigorous renders them totally unreadable.

Before we can state the promised algorithm we need to introduce a few more definitions.

Definition. Let K be an oriented knot that is associated to a knot diagram $(f\colon S^1 \to \mathbb{R}^2, c)$.

(1) We refer to the images of the double points as the *crossings of* K.

(2) Let $x \in f(S^1)$ be a crossing. Let $z, w \in S^1$ with $f(z) = f(w) = x$ and such that z is an undercrossing and w is an overcrossing. We say x is a *positive crossing* if the ordered basis $\{f'(w), f'(z)\}$ is a positive basis for $\mathbb{R}^2$. Otherwise we say that x is a *negative crossing*.

(3) We refer to the components of $K \cap (\mathbb{R}^2 \times \{0\})$ as the *strands of* K.

These definitions are illustrated in Figures 326 and 327.

Remark. In a picture of a diagram the strands are understood to correspond precisely to the segments drawn.

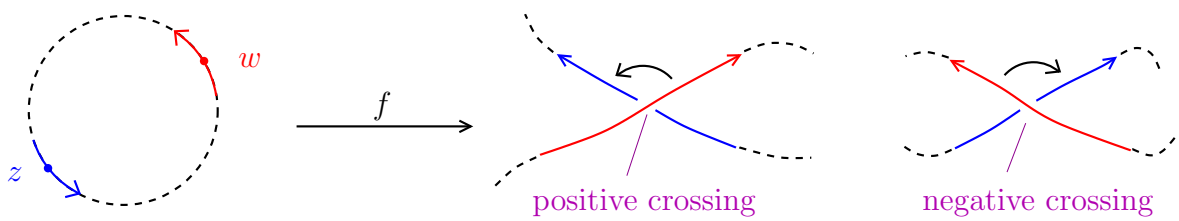


FIGURE 326.

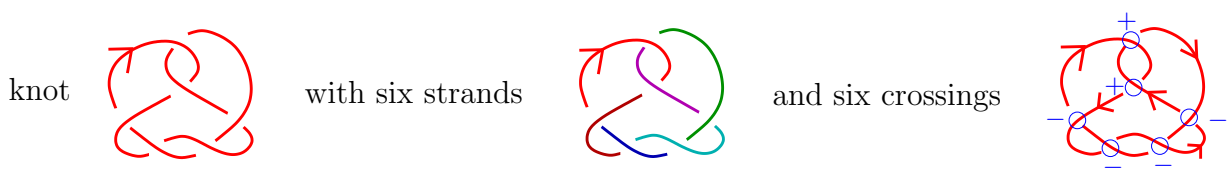


FIGURE 327.

The following lemma will be proved in Exercise 18.5.

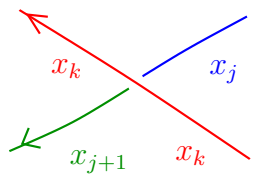
Lemma 18.12. *If a knot diagram has at least one crossing, then the number of crossings equals the number of strands.*

The following proposition shows how to determine the fundamental group of a knot that is associated to a knot diagram. We will sacrifice some rigor in the formulation in an attempt to make it more readable. We leave it to the reader to make the proposition totally rigorous.

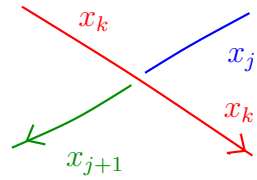
Proposition 18.13. *Let K be a knot that is associated to a given knot diagram that has at least one crossing. We enumerate the strands cyclically by $x_1, \dots, x_n$ where we make the identifications $x_0 = x_n$ and $x_{n+1} = x_1$. For the crossing between the strands x_j and x_{j+1} we define a relation r_j as shown in Figure 328. Then we have an isomorphism*

$$\pi_1(S^3 \setminus K) \;\cong\; \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle.$$

Furthermore we can drop any one of the relations and we still obtain an isomorphism.



the positive crossing gives rise to the relation $r_j = x_k \cdot x_{j+1} \cdot x_k^{-1} \cdot x_j^{-1}$



the negative crossing gives rise to the relation $r_j = x_k^{-1} \cdot x_{j+1} \cdot x_k \cdot x_j^{-1}$

FIGURE 328.

Remark. Given a knot K, any presentation with n generators and $n-1$ relators as obtained from Proposition 18.13 is called a *Wirtinger presentation of* $\pi_1(S^3 \setminus K)$.

In the following we first explain why Proposition 18.13 is plausible. Afterwards we will provide the technical proof of Proposition 18.13.

"Plausibility argument for Proposition 18.13". Suppose we are given a knot diagram that has n crossings. We denote by K the corresponding knot. We work with the base point $* = (0,0,2) \in \mathbb{R}^3$ which lies "above" the knot K. We enumerate the strands cyclically. For the j-th strand we denote by x_j the oriented triangle that starts at $*$ and "circles once around the j-th strand" according to the "right-hand rule". We illustrate the definition in Figure 329.

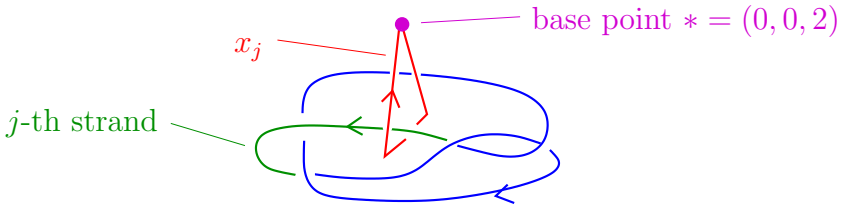


FIGURE 329.

Next we consider the situation at a positive crossing. By "sliding" various x_j along the strands and by some further path-homotopies we can assume that we are in the situation in Figure 330 on the left. If one follows the loop $x_k * x_{j+1} * \overline{x_k} * \overline{x_j}$ on its journey one realizes that on three occasions one goes back and forth to the base point. Thus the loop is actually path-homotopic to the loop shown in Figure 330 on the right. But that loop is clearly null-homotopic. Almost the same logic applies to negative crossings.

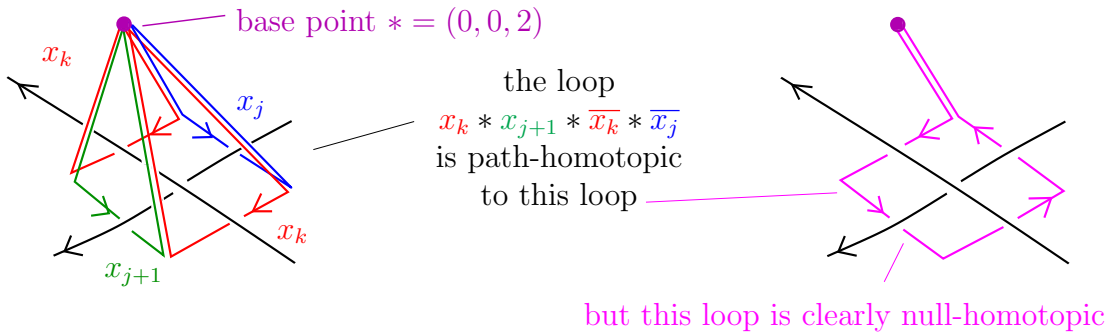


FIGURE 330.

This discussion, together with Lemmas 11.13 and 13.4, shows that there exists a unique group homomorphism

$$\langle x_1, \dots, x_n \,|\, \text{relations corresponding to crossings}\rangle \;\to\; \pi_1(S^3 \setminus K, *)$$

with

$$x_i \;\mapsto\; [x_i].$$

It remains to show that this group homomorphism is in fact an isomorphism. We will do so next in the proper proof of Proposition 18.13. □

Sketch of a proof for Proposition 18.13. Suppose once again that we are given a knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ that has $n \geq 1$ crossings. We denote by $P_1, \dots, P_n$ the double points of γ. We pick a smooth function $\eta\colon S^1 \to [-1, 0]$ as in the statement of Lemma 18.8. We denote by K the image of the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$. It follows from property (d) of Lemma 18.8 that

$$K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) \;=\; K \cap (\mathbb{R}^2 \times \{-1\}) \;=\; \{(P_1, -1), \dots, (P_n, -1)\}.$$

In contrast to the above plausibility argument we now actually work with the base point $\infty \in S^3 = \mathbb{R}^3 \cup \{\infty\}$. We set

$$A \;:=\; (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}) \setminus K,$$

and

$$B \;:=\; (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}) \setminus K.$$

We start out with the following claim.

Claim 1. The inclusion maps induce an isomorphism

$$\pi_1(A, \infty) *_{\pi_1(A \cap B, \infty)} \pi_1(B, \infty) \;\cong\; \pi_1(S^3 \setminus K).$$

Evidently the statement of the claim smells like it should be a consequence of the Seifert–van Kampen 14.1. The problem is that A and B are not open subsets of $S^3 \setminus K$. We rectify

this problem by replacing A and B by suitable open subsets. We define

$$U := A \cup (\mathbb{R}^2 \setminus \{P_1, \dots, P_n\}) \times (-2, -1)$$

and

$$\begin{aligned} V := B &\cup (\mathbb{R}^2 \setminus \gamma(S^1)) \times [-1, 0) \\ &\cup \{(z,t) \mid \text{ there exists } w \in S^1 \text{ with } z = \gamma(w) \text{ and } t \in [-1, \eta(z))\}. \end{aligned}$$

One can easily verify that U and V are open subsets of $S^3 \setminus K$. Furthermore the "obvious retractions" of the blue sets to the corresponding subsets of $\mathbb{R}^2 \times \{-1\}$ show that A is a deformation retract of U, that B is a deformation retract of V and that $A \cap B$ is a deformation retract of $U \cap V$. Now we see that the inclusion maps induce isomorphisms

$$\begin{array}{ccccc} \pi_1(S^3 \setminus K) & \overset{\cong}{\leftarrow} & \pi_1(U, \infty) *_{\pi_1(U \cap V, \infty)} \pi_1(V, \infty) & \overset{\cong}{\rightarrow} & \pi_1(A, \infty) *_{\pi_1(A \cap B, \infty)} \pi_1(B, \infty). \\ & \uparrow & & \uparrow & \\ & \text{by the Seifert–van Kampen Theorem 14.1} & & \begin{array}{c}\text{by the above, Lemma 9.10} \\ \text{and Proposition 9.12 (1)}\end{array} & \end{array}$$

This concludes the proof of the claim.

Our next goal is to understand the groups $\pi_1(A, \infty)$, $\pi_1(B, \infty)$ and $\pi_1(A \cap B, \infty)$. We handle these three groups consecutively. To better understand $\pi_1(A, \infty)$ we introduce the following objects.

(1) As in the plausibility argument we set $* = (0, 0, 2)$. We fix once and for all a path connecting $*$ to ∞.
(2) We write $D := \{(\gamma(z), w) \mid z \in S^1 \text{ and } w \in [-1, \eta(z))\}$.
(3) We cyclically enumerate the strands of the knot.
(4) For $j \in \{1, \dots, n\}$ we denote by D_j the unique component of D that contains the j-th strand in its closure.
(5) For $j = 1, \dots, n$ we denote by x_j a loop in $*$ that is the boundary of a triangle that intersects D_j precisely once in a positive[425] direction. We use the path from (1) to view each x_j as a loop in ∞.

Some of the above objects are illustrated on the left of Figure 331.

Claim 2. The obvious map $\langle x_1, \dots, x_n \rangle \to \pi_1(A, \infty)$ is an isomorphism.

Note that each D_j is contractible and hence simply connected. Furthermore one can easily show that $A \setminus D$ is contractible. We can now use Corollary 17.3 iteratively n times to prove the claim. We leave it to the assiduous reader to fill in the details.[426] This concludes the proof of the claim.

Fortunately the next group $\pi_1(B, \infty)$ on our list is easy to handle.

Claim 3. The group $\pi_1(B, \infty)$ is trivial.

[425]Here by "positive direction" we mean that the orientation vectors of the triangle followed by the orientation vector of the knot should form a positive basis for $\mathbb{R}^3$.

[426]The details involve replacing the set D_j by a suitable open subset to be able to apply Corollary 17.3.

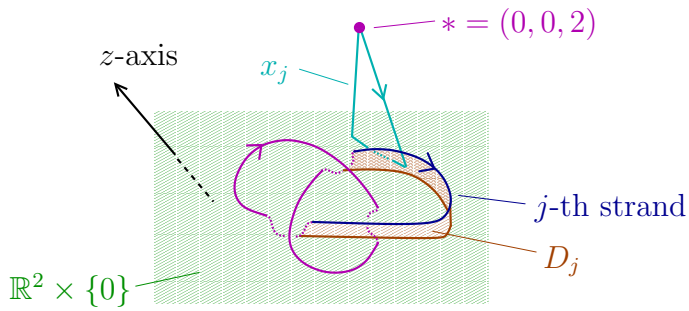


FIGURE 331.

Using the stereographic projection from Lemma 1.41 and using the map defined on page 73 it is not hard to see that there exists a homeomorphism

$$f\colon (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\} \;\to\; \overline{B}^3$$

which sends the points in $\mathbb{R}^2 \times \{-1\}$ to S^2. Since $K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) = \{(P_1,-1),\dots,(P_n,-1)\}$ we see that B is homeomorphic to $\overline{B}^3$ with n points on its boundary removed. But this does not change the fact that the origin is a deformation retract. We have thus shown that $f(B)$, and hence also B, is contractible. But this implies by Proposition 9.12 (2) that $\pi_1(B,\infty)$ is indeed trivial. This concludes the proof of the claim.

Finally we turn to the group $\pi_1(A \cap B, \infty)$. We pick orientation-preserving embeddings $\varphi_i\colon \overline{B}^2 \to \mathbb{R}^2 \times \{-1\}$, $i = 1,\dots,n$ with disjoint images and such that for each i we have $\varphi_i(0) = (P_i,-1)$. It follows almost immediately[427] from Lemma 17.4 that we have an equality

$$\langle z_1,\dots,z_n \mid z_1 \cdot \dots \cdot z_n \rangle \;=\; \pi_1(A \cap B, \infty)$$

where for any $i \in \{1,\dots,n\}$ the element z_i is represented by a curve of the form $\alpha_i * \gamma_i * \overline{\alpha}_i$ where α_i is a suitable path from ∞ to a point on $\varphi_i(S^1)$ and γ_i is a path that goes once around $\varphi_i(S^1)$ in an orientation-preserving way.

Now we can finally prove the desired statement. Indeed, we have

$$\begin{array}{ccccc}
\pi_1(S^3 \setminus K) & \cong & \pi_1(A,\infty) *_{\pi_1(A\cap B,\infty)} \pi_1(B,\infty) & \cong & \pi_1(A,\infty)/\langle\langle \pi_1(A \cap B,\infty)\rangle\rangle \\
 & \uparrow & & \uparrow & \\
 & \text{by Claim 1} & & \text{by Claim 3 and Lemma 13.12 (2)} & \\
 & \cong & \langle x_1,\dots,x_n\rangle/\langle\langle z_1,\dots,z_n\rangle\rangle & \cong & \langle x_1,\dots,x_n\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle. \\
 & \uparrow & & \uparrow & \\
 & \begin{array}{c}\text{by Claim 2, Lemma 13.2 (4) and the} \\ \text{above fact that } \{z_1,\dots,z_n\} \text{ is a} \\ \text{generating set for } \pi_1(A\cap B,\infty)\end{array} & & \begin{array}{c}\text{the “plausibility argument shows that each} \\ z_i \text{ is conjugate to } r_i\text{, thus it follows from} \\ \text{Lemma 13.2 (5) that } \langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle r_1,\dots,r_n\rangle\rangle\end{array} &
\end{array}$$

[427]The slightly cautious “almost” refers to the fact that we also make use of the fairly obvious fact that $((\mathbb{R}^2 \times \{-1\}) \cup \{\infty\}) \setminus \bigcup_{i=1}^{n} \varphi_i(B^2)$ is a deformation retract of $A \cap B = ((\mathbb{R}^2 \times \{-1\}) \cup \{\infty\}) \setminus \{P_1,\dots,P_n\}$.

It remains to show that we can drop any one of the relations. Thus let $j \in \{1, \dots, n\}$. It follows easily from $z_1 \cdot \dots \cdot z_n = e$ that $z_j \in \langle z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n \rangle$. But by Lemma 13.2 (4) this implies that $\langle\langle z_1, \dots, z_n \rangle\rangle = \langle\langle z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n \rangle\rangle$. Therefore we see that in the above equality we can drop the relation r_j. $\square$

As mentioned above, we continue with two examples, namely the trefoil and the figure-8 knot. First we consider the knot diagram for the trefoil K that is shown in Figure 332 to the left. We number the strands as shown. To the right we show how the three crossings give rise to three relations. As pointed out in Proposition 18.13, we can drop any one of the relations.

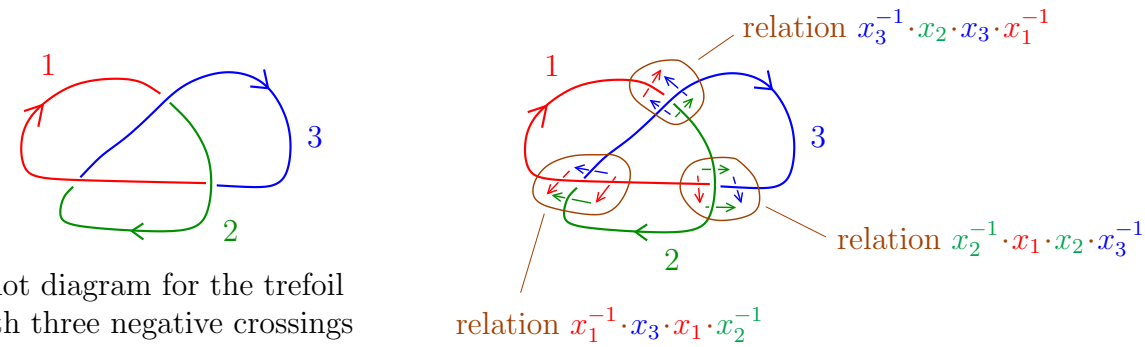


FIGURE 332.

This discussion shows that we have proved the following lemma.

Lemma 18.14. *Let K be the trefoil. There exists an isomorphism*

$$\pi_1(S^3 \setminus K) \;\cong\; \langle x_1, x_2, x_3 \,|\, x_3^{-1} x_2 x_3 x_1^{-1}, x_2^{-1} x_1 x_2 x_3^{-1} \rangle.$$

Remark. Let K be the trefoil. In Proposition 18.5 we had also shown that there exists an isomorphism $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$. In particular we have now shown that the two groups $\langle x_1, x_2, x_3 \,|\, x_3^{-1} x_2 x_3 x_1^{-1}, x_2^{-1} x_1 x_2 x_3^{-1} \rangle$ and $\langle x, y \mid x^2 = y^3 \rangle$ are isomorphism. In Exercise 18.6 we will give a purely algebraic proof of this statement.

The following corollary to Proposition 18.13 gives us the disappointing news that our old trick of using abelianizations of fundamental groups cannot be used to distinguish knots.

Corollary 18.15. *Given any knot K the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.*

Proof. We will prove the corollary in Exercise 18.3. Later on, once we have introduced homology groups we will give a different, arguably more conceptual proof of the lemma. $\square$

After this sobering corollary we move on to the next example.

Lemma 18.16. *Let J be the figure-8 knot. There exists an isomorphism*

$$\pi_1(S^3 \setminus J) \;\cong\; \langle x_1, x_2, x_3, x_4 \,|\, x_3 x_2 x_3^{-1} x_1^{-1}, x_4^{-1} x_3 x_4 x_2^{-1}, x_1 x_4 x_1^{-1} x_3^{-1} \rangle.$$

Proof. We leave it to the reader to use the knot diagram shown in Figure 333 to prove the lemma. □

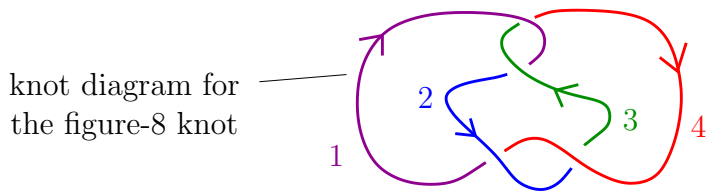


FIGURE 333.

On its own Lemma 18.16 is not particularly enlightening. It is initially not clear whether this group is isomorphic to $\mathbb{Z}$ or to the fundamental group of the complement of the trefoil. By Corollary 18.15 we know that there is no hope that the abelianization will tell us something interesting.

So we need a different approach. Recall that in Lemma 18.6 we used the (non-) existence of an epimorphism onto the symmetric group S_3 to distinguish the trefoil from the trivial knot. The idea now is to repeat this trick with a different finite group. Recall that on page 476 we introduced the dihedral groups D_n which have $2n$ elements. It is encouraging that $D_3 \cong S_3$ thus we try our luck with D_5.[428]

Using Lemma 13.6 and enough time on our hands we obtain the following table:

	trivial knot	trefoil	figure-8 knot
epimorphism onto S_3	✗	✓	✗
epimorphism onto the dihedral group D_5	✗	✗	✓

This table shows that we have proved the following corollary.

Corollary 18.17. *The trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.*

This approach of trying to distinguish knots by the (non-) existence of epimorphisms onto finite groups is rather ad hoc, but not without its merits. For example the following proposition is proved in [BF, BR].

Proposition 18.18. *Let K be either the trivial knot, the trefoil or the figure-8 knot. Furthermore let J be another knot. If for all finite groups G the group $\pi_1(S^3 \setminus K)$ admits an epimorphism onto G if and only if $\pi_1(S^3 \setminus J)$ does, then J is smoothly isotopic to K or to the mirror image of K, i.e. the reflection of K in the xy-plane.*

We conclude this section with the following question which is still open.

[428]The reader might ask, why we do not try our luck with D_4. But for reasons we cannot get into right now, given a knot K and an even number $2n$ there cannot be an epimorphism $\pi_1(S^3 \setminus K) \to D_{2n}$.

Question 18.19. *Let K and J be two knots in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. Suppose that the finite groups onto which $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ admit epimorphisms are the same. Does this imply that K and J are smoothly isotopic, up to possibly taking a reflection in the xy-plane?*

18.5. **The Reidemeister-Schreier process.** Even though Corollary 18.15 was quite disappointing, using a clever jiu-jitsu move we will see in this section that we can actually turn Corollary 18.15 to our advantage. The key to doing so is contained in the following definition and the subsequent lemma.

Definition. Let π be a group and let $n \in \mathbb{N}$. We say that a subgroup $\Gamma \subset \pi$ is *cocyclic of order n* if Γ is a normal subgroup such that π/Γ is isomorphic to $\mathbb{Z}_n$.

Lemma 18.20. *If π is a group whose abelianization is isomorphic to $\mathbb{Z}$, then given any $n \in \mathbb{N}$ there exists a unique subgroup that is cocyclic of order n. It is given by the kernel of any epimorphism $\pi \to \mathbb{Z}_n$.*

Proof $(*)$. Let π be a group whose abelianization is isomorphic to $\mathbb{Z}$ and let $n \in \mathbb{N}$. We denote by $\psi\colon \pi \to \pi_{\mathrm{ab}}$ the obvious projection and we pick an isomorphism $\varphi\colon \pi_{\mathrm{ab}} \xrightarrow{\cong} \mathbb{Z}$. It is clear that $(\varphi \circ \psi)^{-1}(n\mathbb{Z})$ is a subgroup of π that is cocyclic of order n. On the other hand, suppose that Γ is a subgroup of π that is cocyclic of order n. We pick an isomorphism $\theta\colon \pi/\Gamma \cong \mathbb{Z}_n$. It follows from Proposition 13.10 that there exists a unique epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi \arrow[r, "\psi", "\cong"'] \arrow[d, two heads, "g \mapsto g\Gamma"'] & \pi_{\mathrm{ab}} \arrow[r, "\varphi", "\cong"'] & \mathbb{Z} \arrow[d, two heads, "\alpha"] \\
\pi/\Gamma \arrow[rr, "\theta", "\cong"'] & & \mathbb{Z}_n.
\end{tikzcd}$$

It follows immediately that $\Gamma = (\varphi \circ \psi)^{-1}(n\mathbb{Z})$. $\square$

Corollary 18.15 and Lemma 18.20 allow us to make the following definition.

Definition. Given a knot K and $n \in \mathbb{N}$ we denote by $\pi(K, n)$ the unique subgroup of $\pi_1(S^3 \setminus K)$ that is cocyclic of order n.

Example. If K is the trivial knot, then we saw in Proposition 18.4 that we can make an identification $\pi_1(S^3 \setminus K) = \mathbb{Z}$. It follows easily that for any $n \in \mathbb{N}$ we have $\pi(K, n) = n \cdot \mathbb{Z}$, in particular $\pi(K, n)$ is isomorphic to $\mathbb{Z}$.

We have now a new tool for trying to distinguish knots:

Lemma 18.21. *Let K and J be two knots. If K and J are smoothly isotopic, then given any $n \in \mathbb{N}$ the groups $\pi(K, n)$ and $\pi(J, n)$ are isomorphic. In particular the abelianizations of these groups are isomorphic.*

The question arises, given a knot K and $n \in \mathbb{N}$, how can we determine $\pi(K,n)$ from a given presentation for $\pi_1(S^3 \setminus K)$? Fortunately the following proposition gives us an explicit algorithm. To facilitate the uptake we formulate the following proposition in a slightly informal way.

Proposition 18.22. *Let $\pi = \langle x_1, \dots, x_m, t \,|\, r_1, \dots, r_l \rangle$ be a group and let $\phi\colon \pi \to \mathbb{Z}_n$ be an epimorphism such that $\phi(x_1) = \dots = \phi(x_m) = 0$ and $\phi(t) = 1$. Then a presentation for the subgroup $\ker\{\phi\colon \pi \to \mathbb{Z}_n\}$ is given by*

$$\langle y_{1,0}, \dots, y_{1,n-1}, \dots, y_{m,0}, \dots, y_{m,n-1}, u \,|\, s_1^0, \dots, s_1^{n-1}, \dots, s_l^0, \dots, s_l^{n-1} \rangle$$

where each s_i^j is obtained as follows:

(1) *For $i = 1, \dots, m$ and $j = 0, \dots, n-1$ we write $t^j r_i t^{-j}$ as an expression in*

$$x_1, tx_1t^{-1}, \dots, t^{n-1}x_1t^{-n+1}, \dots, x_m, tx_mt^{-1}, \dots, t^{n-1}x_mt^{-n+1} \text{ and } t^n.$$

(2) *Then we replace each term $t^j x_i t^{-j}$ by $y_{i,j}$ and we replace t^n by u. We obtain a word in the new generators and denote it by s_i^j.*

The inclusion map $\ker\{\phi\colon \pi \to \mathbb{Z}_n\} \to \pi$ is then given by sending each $y_{i,j}$ to $t^j x_i t^{-j}$ and by sending u to t^n.

Proof. As we had mentioned above, the proposition follows fairly easily from the more general Reidemeister-Schreier process that is explained in [LS, Chapter II.4], [MKS, Chapter 2.3] or [Bog, Chapter 2.9]. □

Instead of studying the proof of Proposition 18.22 our time is better spent trying to understand two explicit examples.

Example. Let $K \subset S^3$ be the trefoil. We write $\pi = \pi_1(S^3 \setminus K)$. In Proposition 18.5 we showed that we can make the identification $\pi = \langle x, t \mid \underbrace{x^3 \cdot t^{-2}}_{=r} \rangle$. By Lemma 13.6 there exists a unique epimorphism $\phi\colon \pi \to \mathbb{Z}_2$ with $\phi(x) = 0$ and $\phi(t) = 1$. Evidently $\pi(K,2) = \ker\{\phi\colon \pi \to \mathbb{Z}_2\}$. Now we will use Proposition 18.22 to find a presentation $\langle y_0, y_1, u \,|\, s_0, s_1 \rangle$ for $\ker\{\phi\colon \pi \to \mathbb{Z}_2\}$. We write

$$t^0 r t^{-0} = \underbrace{x^3}_{=y_0^3} \cdot \underbrace{t^{-2}}_{=u^{-1}} \text{ and more interestingly } t^1 r t^{-1} = tx^3t^{-3} = (txt^{-1})^3 \cdot t^{-2} = (\underbrace{txt^{-1}}_{y_1^3})^3 \cdot \underbrace{t^{-2}}_{=u^{-1}}.$$

Thus we see that

$$\pi(K,2) = \ker\{\phi\colon \pi \to \mathbb{Z}_2\} \underset{\substack{\uparrow \\ \text{Proposition 18.22}}}{=} \langle y_0, y_1, u \,|\, y_0^3 u^{-1}, y_1^3 u^{-1} \rangle \underset{\substack{\uparrow \\ \text{by the Tietze transformations of Lemma 13.5}}}{=} \langle y_0, y_1 \,|\, y_0^3 = y_1^3 \rangle.$$

Using Proposition 13.10 we can now easily compute that the abelianization of the above group $\pi(K,2) = \ker\{\phi\colon \pi \to \mathbb{Z}_2\}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$.

This calculation, together with Lemma 18.21 and the above discussion of the trivial knot, gives a new proof of Lemma 18.6.

The previous example was fairly straightforward since we had started out with a presentation where one generator got sent to one, and the other ones got sent to zero. Unfortunately this is not the case for Wirtinger presentations, so we will need to work somewhat harder. In the following example we consider the figure-8 knot K and we carry out the first steps needed to calculate the isomorphism type of $\pi(K,2)$.

Example. Let J be the figure-8 knot. In Lemma 18.16 we saw that we can make the identification

$$\pi_1(S^3 \setminus J) \;=\; \langle y_1, y_2, y_3, y_4 \,|\, y_1^{-1}y_4y_1y_3^{-1}, y_3^{-1}y_2y_3y_1^{-1}, y_2^{-1}y_1y_2y_4^{-1}\rangle.$$

We denote by $\phi\colon \pi_1(S^3 \setminus J) \to \mathbb{Z}_2$ the epimorphism that is given by sending each y_i to 1. We set $t := y_1$. The goal is to replace the generators y_1, y_2, y_3 by generators x_1, x_2, x_3 which satisfy $\phi(x_i) = 0$. In fact we have

$$\begin{aligned}
\pi_1(S^3 \setminus J) &= \langle t, y_2, y_3, y_4 \,|\, t^{-1}y_4ty_3^{-1}, y_3^{-1}y_2y_3t^{-1}, y_2^{-1}ty_2y_4^{-1}\rangle \\
&= \langle t, y_2, y_3, y_4 \,|\, \underbrace{t^{-1}y_4t^{-1}tty_3^{-1}}_{=t^{-1}y_4ty_3^{-1}}, \underbrace{t^{-1}ty_3^{-1}y_2t^{-1}ty_3t^{-1}}_{=y_3^{-1}y_2y_3t^{-1}}, \underbrace{t^{-1}ty_2^{-1}ty_2t^{-1}ty_4^{-1}}_{=y_2^{-1}ty_2y_4^{-1}}\rangle \\
&= \langle t, x_2, x_3, x_4 \,|\, t^{-1}x_4tx_3^{-1}, t^{-1}x_3^{-1}x_2tx_3, t^{-1}x_2^{-1}tx_2x_4^{-1}\rangle. \\
&\uparrow \\
&\text{substitution } x_i = y_it^{-1},\ x_i^{-1} = ty_i^{-1}, \text{ see Lemma 13.7}
\end{aligned}$$

At this point we have a presentation to which we can apply Proposition 18.22. A heroic calculation shows that $\pi(J,2)_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}_5$. Thus we obtain a new proof that the figure-8 knot and the trefoil are not smoothly isotopic.

The above procedure for distinguishing knots works fairly well in practice. For example, with enough time at one's hand, or alternatively a convenient computer implementation, one can use the above methods to show that all the knots shown in Figure 324 are in fact pairwise different.

Nonetheless, the above approach is not overly satisfying. It would be much better to have invariants of knots that are much quicker to calculate. We will come back to this challenge at a much later point.

The above discussion, in particular Proposition 18.22 raises the following purely algebraic question.

Question 18.23. *Let π be a group with a given finite presentation and let $\phi\colon \pi \to G$ be an epimorphism onto a finite group.*

(1) *Is the kernel* $\ker\{\phi\colon \pi \to G\}$ *a finitely presented group?*
(2) *If the answer to* (1) *is yes, how can we determine a finite presentation for the kernel in terms of the given data?*

It turns out that the answer to the first question is yes.

Proposition 18.24. *If π is a finitely presented group and $\phi\colon \pi \to G$ is an epimorphism onto a finite group, then* $\ker\{\phi\colon \pi \to G\}$ *is also finitely presented.*

The proof of the proposition is given in most books on combinatorial group theory, see e.g. [LS, Chapter II.4], [MKS, Chapter 2.3] or [Bog, Chapter 2.9]. In fact the proof is algorithmic, in particular it also gives an affirmative answer to the second question. The algorithm is called the *Reidemeister-Schreier process*. The general Reidemeister-Schreier process is somewhat messy to formulate. Thus we refer to the above literature for details.

Remark. In the proof of Lemma 18.6 we showed that the group $\langle x, t \,|\, x^3 \cdot t^{-2} \rangle$ admits an epimorphism α onto the permutation group S_3. In [Bog, Example 2.9.3] the general Reidemeister-Schreier process is used to determine a presentation for the kernel of $\alpha \colon \langle x, t \,|\, x^3 \cdot t^{-2} \rangle \to S_3$.

Exercises for Chapter 18.

Exercise 18.1. We consider the two knot diagrams shown in Figure 334. Use Reidemeister moves to show that the associated knots are smoothly isotopic.

FIGURE 334.

Exercise 18.2. Show that every diagrammatic map has only finitely many double points.

Exercise 18.3. Let K be a knot. Use Proposition 18.13 to show that the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Exercise 18.4. We defined the sign of a crossing of a diagram of an oriented knot. How does the sign change if we reverse the orientation of the knot?

Exercise 18.5. Show that if a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

Exercise 18.6. Give a purely algebraic proof that the two groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\langle a, b, c \,|\, c^{-1}bca^{-1}, b^{-1}abc^{-1} \rangle$ are isomorphic.
Remark: You could use the Tietze transformations introduced in Lemma 13.5 and you could use Lemma 13.7.

Exercise 18.7. Let $K \subset S^3$ be the trefoil. We consider the "cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subset\; \overline{B}^4.$$

(a) Show that C is homeomorphic to the disk $\overline{B}^2$.
(b) Suppose that F is a proper 2-dimensional topological submanifold of $\overline{B}^4$. Show that given any point $P \in F \cap B^4$ there exists a closed neighborhood N of $P \in B^4$ and a homeomorphism $\Phi \colon N \to \overline{B}^4$ such that $\Phi(N \cap F)$ is the trivial knot in S^3.

(c) Show that C is not a *topological submanifold* of $\overline{B}^4$.
Hint: Evidently the idea is to apply (b) to the point $P = 0 \in C$. The difficulty is that N does not have to be a standard closed ball of the form $\overline{B}^4_r(0)$. But what is true is that given any such N there exist $r, s \in (0,1)$ with $\overline{B}^4_r(0) \subset N \subset \overline{B}^4_s(0)$. How can this information be put to good use?

FIGURE 335. Illustration for Exercise 18.7.

Exercise 18.8. Let K be the trefoil. Compute the abelianization of the group $\pi(K, 3)$.

19. Decision problems

In the last chapters we had developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. graphs, surfaces, knot complements and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces. Furthermore we will see later on that the fundamental group of every compact manifold admits a *finite* presentation.

The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 19.1.

(1) *Does there exist an algorithm that can decide whether or not a given finite presentation* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *represents the trivial group?*
(2) *Does there exist an algorithm that can decide whether or not two given finite presentations* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *and* $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ *represent isomorphic groups?*[429]
(3) *Does there exist an algorithm that can determine, given*
   (a) *a finite presentation* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *and*
   (b) *a word* w *in* $g_1, \dots, g_k$*, i.e. an element* $w \in \langle g_1, \dots, g_k \rangle$*,*
   whether or not w *represents the trivial element in the group* π*?*[430]

Remark. The above question is formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [CZ, Chapter 7].

On page 412 we introduced the Tietze transformations on presentations. We pointed out that two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that we do not know how many Tietze transformations are required to turn a presentation of a trivial group into a trivial presentation. In particular at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:[431]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,

[429] The question whether such an algorithm exists is often called the *isomorphism problem.*
[430] The question whether such an algorithm exists is often called the *word problem.*
[431] For elements g, h of $\langle a, b, c \rangle$ we write $g \sim h$ if they represent the same element in π.

(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement.

Now suppose we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. We can modify the presentation using Tietze transformations. If at some point we end up with the trivial presentation we know that the given presentation corresponds to the trivial group. But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation. Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a non-trivial group, actually verifies that the group is non-trivial?

Unfortunately the answer is quite sobering.

Theorem 19.2. *The answer to all three questions of Question 19.1 is no.*

Remark. Here "no" means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm.

Proof.
(1) Adyan [Ady] and Rabin [Rab] showed in 1955 that the answer to the first question is no.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.
(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either. □

Remark. The statement of Theorem 19.2 that the answer to all three questions of Question 19.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example

Mikhail Gromov [Grom][432] introduced the notion of a "word hyperbolic group" for which the answer to all three questions turns out to be yes. Examples of word hyperbolic groups are given by

(1) finite groups,
(2) free groups, fundamental groups of surfaces of genus ≥ 2,
(3) fundamental groups of hyperbolic manifolds of any dimension,
(4) fundamental groups of "Riemannian manifolds with negative sectional curvature".

In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [Ol] for more details.

The topic of word hyperbolic groups is a very important topic of mathematics in its own right. See e.g. [Lö] for an introduction to word hyperbolic groups.

Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example, given $n \in \mathbb{N}$ we would like to classify closed, orientable connected n-dimensional manifolds up to diffeomorphism. For $n = 1$ we did so in Proposition 3.53 and for $n = 2$ we saw the classification in Theorem 14.9. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

The situation is very different in higher dimensions. First of all we have the following proposition, which is actually fairly easy to prove, see e.g. [CZ, Theorem 5.1.1] for details.[433]

Proposition 19.3. *Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional manifold with fundamental group π.*

Markov [Mar] used Proposition 19.3 and Theorem 19.2 to prove the following theorem.[434]

Theorem 19.4. *Let $n \geq 4$. Then there is no algorithm that can decide whether or not two closed orientable n-manifolds are homeomorphic.*

This leaves us with the case of 3-dimensional manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 19.5. *There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.*

In this case the algorithm is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [Thu] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 19.5. We refer to the survey paper [AFW, Theorem 4.27] for details and precise references.

[432]Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

[433]We will provide a proof in "Introduction to Knot Theory II".

[434]Is Theorem 19.4 an immediate consequence of Proposition 19.3 and Theorem 19.2?

20. The universal covering of reasonable topological spaces

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of covering maps. One key goal of this section is to address the following question.

Question 20.1. *Does every path-connected topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected?*

We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? Or does it exist for the complement $S^3 \setminus K$ of a knot $K \subset S^3$? How about the wedge $S^1 \vee S^1$ of two circles?

20.1. **Local properties of topological spaces.** Before we can address Question 20.1 we need to recall the definition of a local property of topological spaces that we already gave in Section 1.19.

Definition. Let P be a property of topological spaces. We say a topological space X is *o-locally P* if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P.

Examples.

(1) Every simply connected topological space is by definition path-connected, thus every o-locally simply connected topological space is also o-locally path-connected.
(2) On page 131 we saw that every open subset of $\mathbb{R}^n$ is o-locally simply connected.
(3) According to Lemma 3.8 every topological manifold is o-locally path-connected. In fact almost the same proof shows that every topological manifold is o-locally simply connected.
(4) It is straightforward to see that each spatial graph is o-locally simply connected. Similarly it is easy to verify that the wedge of two manifolds is o-locally simply connected.
(5) We consider
$$X \;:=\; \mathbb{R}^2 \;\setminus\; \bigcup_{n=1}^{\infty} \tfrac{1}{n} \times [-1,1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not o-locally path-connected. Indeed, we consider the point $(0,0) \in X$ and the neighborhood $U = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U. We refer to Figure 336 for an illustration.
(6) We consider
$$X \;:=\; \mathbb{R}^2 \;\setminus\; \bigcup_{n=1}^{\infty} \big(\tfrac{1}{n}, 0\big),$$
with the subspace topology coming from $\mathbb{R}^2$. Then X is path-connected, o-locally path-connected but it is not o-locally simply connected. Indeed the origin admits

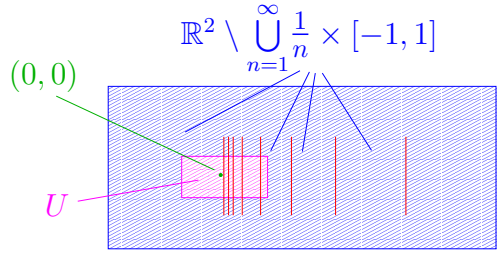


FIGURE 336.

no neighborhood of the desired form, since any neighborhood of the origin contains "infinitely many holes". We made this statement precise in Exercise 12.1.

20.2. **Lifting maps to coverings.** Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z,z_0) \to (B,b_0)$ be a map. Recall that on page 272 we said that f *lifts*, if there exists a *lift* $\widetilde{f}\colon (Z,z_0) \to (X,x_0)$, i.e. a map $\widetilde{f}\colon Z \to X$ with $\widetilde{f}(z_0) = x_0$ and which makes the following diagram commute

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B,b_0). \end{tikzcd}$$

For example, if $f\colon ([0,1],0) \to (B,b_0)$ is a map from the interval $[0,1]$ to B, i.e. if f is a path with starting point b_0, then we saw in Proposition 7.12 that there exists a unique lift of f to a map $\widetilde{f}\colon ([0,1],0) \to (X,x_0)$.

Example. Consider the following maps:

$$\begin{tikzcd} & (S^1,1) \arrow[d, "p"', "z\mapsto z^3"] \\ (S^1,1) \arrow[ur, dashed, "???\ \text{lift}\ \widetilde{f}\ ???"] \arrow[r, "z\mapsto z^2", "f"'] & (S^1,1). \end{tikzcd}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\widetilde{f}$?

To answer the question we need to study lifts of maps a little more carefully. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and let $f\colon (Z,z_0) \to (B,b_0)$ be a map. Suppose that there exists a lift $\widetilde{f}\colon (Z,z_0) \to (X,x_0)$. We then obtain the commutative diagram

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B,b_0) \end{tikzcd}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p \circ \widetilde{f}\big)_* = p_* \circ \widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{tikzcd} & \pi_1(X,x_0) \arrow[d, "p_*"] \\ \pi_1(Z,z_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(B,b_0). \end{tikzcd}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}\left(\left(p \circ \widetilde{f}\right)_*\right) \;=\; \operatorname{im}\left(p_* \circ \widetilde{f}_*\right) \;\subset\; \operatorname{im}(p_*).$$

So summarizing, if there exists a lift of f, then

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

Example. In the above example we have $\pi_1(S^1, 1) = \mathbb{Z}$ and $\operatorname{im}(f_*) = 2\mathbb{Z}$ whereas we have $\operatorname{im}(p_*) = 3\mathbb{Z}$. So we see that f does not lift.

The following proposition says, that the above conclusion has a converse, for "reasonably" topological spaces. More precisely, we have the following proposition which, under a modest technical hypothesis, gives a complete criterion for when a map lifts.

Proposition 20.2. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a map.*

(1) *If Z is o-locally path-connected, then we have the following equivalence of statements:*

$$\begin{tikzcd} & (X, x_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[r, "f"'] \arrow[ur, dashed, "\text{there exists a lift } \widetilde{f}"] & (B, b_0) \end{tikzcd} \qquad \Longleftrightarrow \qquad \operatorname{im}(f_*) \subset \operatorname{im}(p_*).$$

(2) *There exist at most one such lift, in other words, any two such lifts agree.*[435]

Example. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in Proposition 20.2 is automatically satisfied if Z is simply connected.
(2) Now let $f\colon (S^n, s_0) \to (B, b_0)$ be a map. The topological manifold S^n is path-connected and o-locally path-connected. By Proposition 5.14 the sphere S^n is simply connected if $n \geq 2$. Thus it follows from (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n, s_0) \to (X, x_0)$ of the map $f\colon (S^n, s_0) \to (B, b_0)$.

In the proof of Proposition 20.2 we will need the following lemma.

Lemma 20.3. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0. Then*

$$\begin{array}{c}\textit{endpoint of the lift of } \gamma * \delta \\ \textit{to the starting point } x_0\end{array} \quad = \quad \begin{array}{c}\textit{endpoint of the lift of } \delta \\ \textit{to the starting point } \widetilde{\gamma}(1).\end{array}$$

[435]Note that for (2) we do not need the annoying hypothesis that Z is o-locally path-connected.

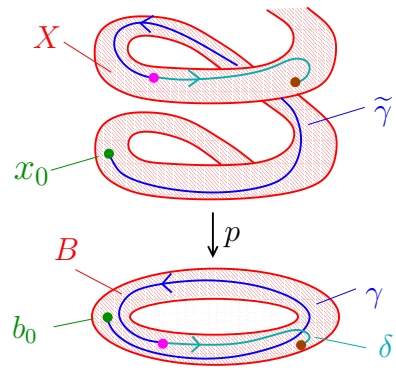


FIGURE 337. Illustration of Lemma 20.3

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0 and we denote by $\widetilde{\delta}$ the lift of δ to the starting point $\widetilde{\gamma}(1)$. Finally we denote by $\widetilde{\gamma * \delta}$ the lift of $\gamma * \delta$ to the starting point x_0. For $t \in [0,1]$ we write $\beta(t) = (\gamma * \delta)(\frac{1}{2}(t+1))$ and $\widetilde{\beta}(t) = (\widetilde{\gamma * \delta})(\frac{1}{2}(t+1))$. Note that by definition we have $\beta(t) = \delta(t)$ for $t \in [0,1]$ and that $\widetilde{\beta}$ and $\widetilde{\delta}$ are both lifts of $\beta = \delta$ to the starting point $\beta(0) = \gamma(1) = \delta(0)$. Thus it follows that

$$\text{endpoint of } \widetilde{\gamma * \delta} \;=\; (\widetilde{\gamma * \delta})(1) \;=\; \widetilde{\beta}(1) \;\underset{\substack{\uparrow \\ \text{by Proposition 7.12}}}{=}\; \widetilde{\delta}(1) \;=\; \text{endpoint of } \widetilde{\delta}.$$

□

Now we can give the proof of Proposition 20.2. To make the proof slightly more readable we prove the two statements (1) and (2) separately.

Proof of Proposition 20.2 (1). Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and o-locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a map.

By the discussion on page 551 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to Proposition 7.12 we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0,1] \to X$ with starting point x_0. We define

$$\widetilde{f}(z) \;:=\; \widetilde{(f \circ \alpha)}(1).$$

This definition is illustrated in Figure 338.

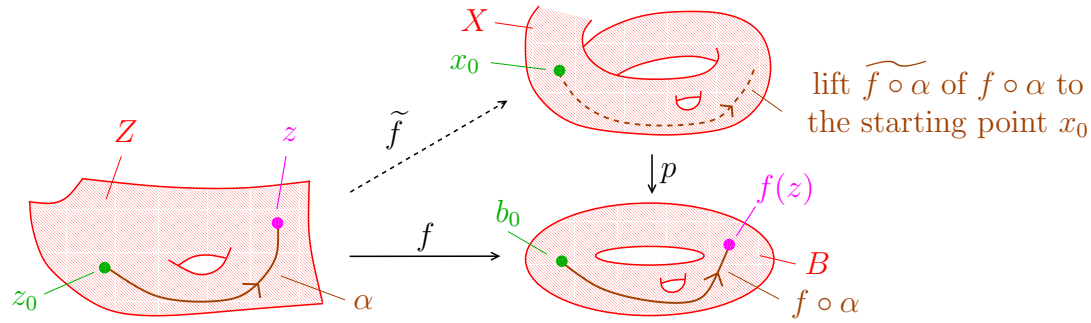


FIGURE 338.

We have to verify the following statements:

(1) the map $\widetilde{f}\colon Z \to X$ is well-defined,
(2) the map $\widetilde{f}$ is continuous,
(3) the map $\widetilde{f}$ is a lift.
(4) and the map $\widetilde{f}$ is the unique lift.

We will do so in the following:

(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$ and $\beta(1) = z$. (This situation is illustrated in Figure 339.) It follows that

$$\begin{aligned}
\widetilde{(f \circ \alpha)}(t=1) &= (\text{lift of } f \circ \alpha \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&\uparrow \\
&\text{by Corollary 7.14 since } f \circ \alpha \text{ and } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ are path-homotopic} \\
&= (\text{lift of } (f \circ (\underbrace{\alpha * \overline{\beta}}_{=:\gamma,\ \text{a loop in } (Z, z_0)})) * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } f \circ \beta \text{ to the starting point } (\widetilde{f \circ \gamma})(1))(t=1) \\
&\uparrow \\
&\text{Lemma 20.3, where } \widetilde{f \circ \gamma} \text{ denotes the lift of } f \circ \gamma \text{ to the starting point } x_0 \\
&= (\text{lift of } f \circ \beta \text{ to starting point } x_0)(t=1) \;=\; \widetilde{(f \circ \beta)}(t=1). \\
&\uparrow \\
&\text{by our hypothesis we have } [f \circ \gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subset p_*(\pi_1(X, x_0)), \\
&\text{hence } \widetilde{f \circ \gamma} \text{ is a loop in } x_0 \text{ by Lemma 7.16}^{436}
\end{aligned}$$

Thus we have shown that the map $\widetilde{f}$ is well-defined.

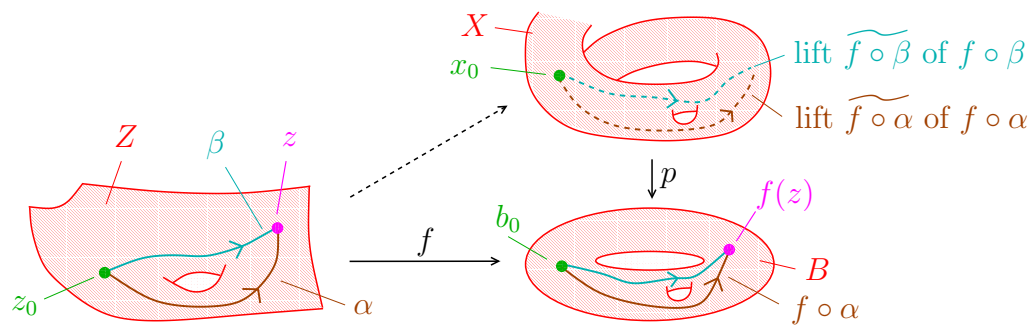


FIGURE 339.

(2) Now we want to show that $\widetilde{f}$ is continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is a homeomorphism onto its image. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is o-locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following claim.

Claim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q \circ f$.

So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

$$\begin{aligned}
\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha * \delta) = (f\circ\alpha) * (f\circ\delta) \text{ to the starting point } x_0 \\
&= \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z) \\
&\uparrow \\
&\text{by Lemma 20.3 and the definition of } \widetilde{f}(z) \text{ as the endpoint of } \widetilde{f\circ\alpha} \\
&= \text{endpoint of } q\circ(f\circ\delta) \;=\; (q\circ f)(\delta(1)) \;=\; (q\circ f)(w). \\
&\uparrow \\
&\text{since } q = p^{-1} \text{ we know that } q\circ(f\circ\delta) \text{ is the lift of } f\circ\delta \text{ to the starting point } f(z)
\end{aligned}$$

(3) We need to show that $\widetilde{f}$ is a lift, i.e. we need to show that $p \circ \widetilde{f} = f$. So let $z \in Z$. Then

$$p\big(\widetilde{f}(z)\big) \;\underset{\uparrow}{=}\; p\big(\big(\widetilde{f\circ\alpha}\big)(1)\big) \;=\; (f\circ\alpha)(1) \;=\; f(\alpha(1)) \;=\; z.$$

pick path α from z_0 to z and denote by $\widetilde{f\circ\alpha}$ the lift of $f\circ\alpha$

[436]Lemma 7.16 (1) says the following: if δ is a loop in (B, b_0) and if $\widetilde{\delta}$ denotes the lift of δ to the starting point x_0, then the following holds:

$$\widetilde{\delta} \text{ is a loop in } (X, x_0) \quad\Longleftrightarrow\quad [\delta] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

We apply the lemma to $\delta = f\circ\gamma$.

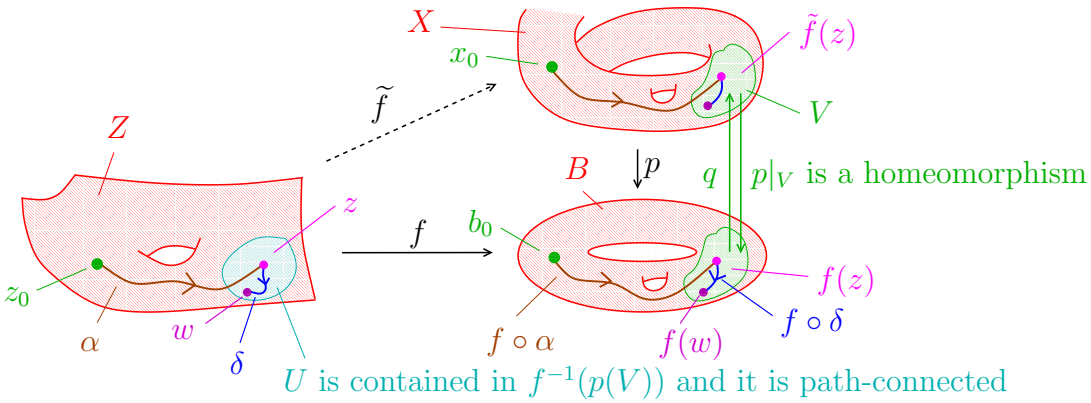


FIGURE 340.

□

Proof of Proposition 20.2 (2). Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space. Finally suppose we are given a map $f\colon (Z, z_0) \to (B, b_0)$.

We will now see, using the uniqueness of the lift of paths established in Proposition 7.12, that any two lifts $\widetilde{f}, \widetilde{f'}\colon (Z, z_0) \to (X, x_0)$ agree. Thus let $z \in Z$. Since Z is path-connected there exists a path $\alpha\colon [0,1] \to Z$ from z_0 to z. Then $f \circ \alpha\colon [0,1] \to B$ is a path and $\widetilde{f} \circ \alpha$ and $\widetilde{f'} \circ \alpha$ are both lifts of $f \circ \alpha$ to the starting point x_0. By Proposition 7.12 we have $\widetilde{f} \circ \alpha = \widetilde{f'} \circ \alpha$. In particular the endpoints agree. But this implies the desired statement that $\widetilde{f}(z) = (\widetilde{f} \circ \alpha)(1) = (\widetilde{f'} \circ \alpha)(1) = \widetilde{f'}(z)$. □

In Exercise 20.2 we will prove the following corollary.

Corollary 20.4. *Let $\pi\colon X \to B$ be a covering of topological spaces. Then any open simply connected and o-locally path-connected subset $U \subset B$ is uniformly covered.*

Proposition 20.2 can also be useful in dealing with situations where it is initially not apparent why covering spaces should be useful. For example we can now prove the following proposition.

Proposition 20.5. *Let $f\colon S^1 \to S^1$ be a diffeomorphism. If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.*[437]

Proof. As usual we identify S^1 with $\mathbb{R}/\mathbb{Z}$. Thus let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a diffeomorphism. First we consider the case that f is orientation-preserving. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

[437]Note that in Lemma 9.6 we proved an analogous statement for self-homeomorphisms of S^1. The case of self-homeomorphisms is technically easier since it is often easier to write down a continuous map than to write down a smooth map.

the usual covering map. By Proposition 20.2 there exists a map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "f"] & \mathbb{R}/\mathbb{Z}. \end{tikzcd}$$

We make the following observations regarding $\widetilde{f}$:

(1) Since p is a local diffeomorphism and since f is a smooth map we see that $\widetilde{f}$ is also smooth.
(2) Since f is orientation-preserving and since p is a local diffeomorphism we see that $\widetilde{f}$ is also orientation-preserving. This implies[438] that $\widetilde{f}$ is a strictly monotonously increasing function.
(3) Since f is a diffeomorphism and since the above diagram commutes we see that for any $x \in \mathbb{R}$ the restriction of $\widetilde{f}$ to the interval $[x, x+1)$ defines a bijection with $[f(x), f(x+1))$.
(4) For any $x \in \mathbb{R}$ we have by the commutativity of the diagram that $f(x+1) - f(x) \in \mathbb{Z}$.
(5) It follows easily from (2), (3) and (4) that for any $x \in \mathbb{R}$ we have $f(x+1) = f(x)+1$.

Now we consider the map

$$\begin{aligned} \widetilde{F}\colon \mathbb{R} \times [0,1] &\to \mathbb{R} \\ (x,t) &\mapsto \widetilde{f}(x) \cdot (1-t) + x \cdot t. \end{aligned}$$

This map is evidently smooth. Since $x \mapsto x$ is a strictly monotonously increasing diffeomorphism of $\mathbb{R}$ one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (1) to (5). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{aligned} F\colon (\mathbb{R}/\mathbb{Z}) \times [0,1] &\to \mathbb{R}/\mathbb{Z} \\ (x,t) &\mapsto \widetilde{f}(x) \cdot (1-t) + x \cdot t \end{aligned}$$

where each $F_t\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since p is a local diffeomorphism we obtain from Lemma 3.15 (3) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that f is orientation-reversing. In this case $r \circ g$ is orientation-preserving. We apply the above result to $r \circ g$ and we easily obtain the desired diffeotopy from f to r. □

The following corollary is an immediate consequence of Proposition 20.5.

Corollary 20.6. *If C and D are two oriented manifolds that are diffeomorphic to S^1, then any two orientation-preserving diffeomorphisms $C \to D$ are diffeotopic.*

[438] Why?

We conclude this section with the following corollary.

Corollary 20.7. *Let M be a 2-dimensional manifold (possibly disconnected) and let A and B be two distinct boundary components of M. Suppose A and B are equipped with orientations. Given any two orientation-preserving diffeomorphisms $f, g\colon A \to B$ the resulting manifolds*

$$M/f(a) \sim a \quad \text{and} \quad M/g(a) \sim \quad (\text{in both cases with } a \in A)$$

are diffeomorphic.

Loosely speaking, the corollary says that if we glue two oriented boundary components of a surface together in an orientation-preserving way, then the diffeomorphism type of the resulting manifold does not depend on the specific choice of the orientation-preserving diffeomorphism.

Proof. It follows from Proposition 3.21 and Proposition 3.53 that A and B are both diffeomorphic to S^1. The corollary is now an immediate consequence of Lemma 10.13 and Corollary 20.6. □

20.3. **Existence of covering spaces.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In Corollary 7.15 we saw that

$$p_*\colon \pi_1(X, x_0) \ \to \ \pi_1(B, b_0)$$

is an injective map. Thus we can view $\pi_1(X, x_0)$ as a subgroup of $\pi_1(B, b_0)$.

Generalizing Question 20.1 one can now ask, whether to each subgroup Γ of $\pi_1(B, b_0)$ there exists a corresponding covering with $p_*(\pi_1(X, x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is o-locally simply connected.

In the following the main property we will be interested in is being o-locally simply connected. To save us working with this tongue-twister we introduce the following definition.

Definition. A topological space that is o-locally simply connected, is called *reasonable.*[439]

The following proposition gives in particular an affirmative answer to Question 20.1 under rather modest hypotheses.

Proposition 20.8. *Let Y be a path-connected reasonable topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.*

(1) *There exists a path-connected covering*[440] *$p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.*
(2) *For the covering p from (1) we have*

$$[X : Y] \ = [\pi_1(Y, y_0) : \Gamma].$$

[439]The name is chosen since mos topological spaces that are "reasonable" in a colloquial sense will also be reasonable in the technical sense.

[440]Here we say that a covering $p\colon X \to Y$ is path-connected if X is path-connected.

Example.

(1) We consider $(Y, y_0) = (S^1, 1)$. We pick an identification[441] $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1, 1)$. Then the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1, 1)) = n\mathbb{Z}$. Another example is given by the covering

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & e^{2\pi i t}. \end{array}$$

Here we also have $\operatorname{im}(q_*) = n\mathbb{Z}$.

(2) Let Z be a simply connected topological space and let $z_0 \in Z$. Furthermore let π be a group that acts on Z continuously and discretely. We consider $Y = Z/\pi$, we denote by $q\colon Z \to Y = Z/\pi$ the projection map and we write $y_0 = q(z_0)$. Using the isomorphism of Theorem 7.17 we can identify π with $\pi_1(Y, y_0)$.

Now let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. Then the group Γ also acts on Z and we can consider the projection map

$$p\colon X := Z/\Gamma \ \to \ Y := Z/\pi$$

We denote by x_0 the image of z_0 under the projection map $Z \to X = Z/\Gamma$. One can show easily that $p\colon X = Z/\Gamma \to Y = Z/\pi$ is a covering map with $p_*(\pi_1(X, x_0)) = \Gamma$. We summarize the situation in the following diagram

$$\begin{tikzcd} Z \arrow[dr] \arrow[dd, "q"'] & & \\ & X = Z/\Gamma \arrow[dl, "p"] & \\ Y = Z/\pi & & \end{tikzcd} \qquad \begin{tikzcd} \pi_1(Z, z_0) = \{e\} \arrow[d] \\ \pi_1(X, x_0) = \Gamma \arrow[d, hook, "p_*"'] \\ \pi_1(Y, y_0) = \pi. \end{tikzcd}$$

Proof of Proposition 20.8. Let Y be a path-connected and reasonable[442] topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. We only need to show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of Lemma 7.16 (3).

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second, that we have such an X. Pick x_0 with $p(x_0) = y_0$. Then, somewhat similar to the statement of Theorem 7.17, we

[441] This means that we pick an isomorphism $\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

[442] Several times we will make use of the fact that a reasonable topological is in particular o-locally path-connected.

have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of Theorem 7.17 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y, y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

We first consider[443]

$$W := \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in W$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on W. The definitions of W and $\sim$ are illustrated in Figure 341. We write $X := W/\sim$ and $x_0 := [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

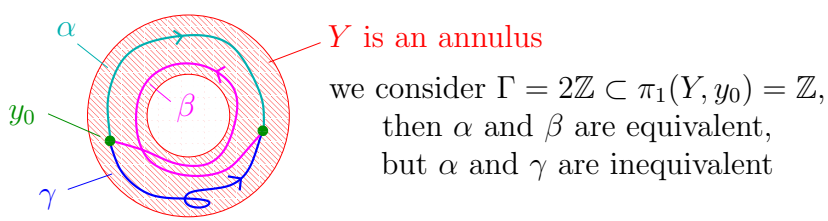


FIGURE 341.

Now we consider the map

$$\begin{array}{rcl} p\colon X = W/\sim & \to & Y \\ [v] & \mapsto & v(1). \end{array}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v.

What remains to be done is to verify that our construction gives the desired object. We still need to do the following steps:

(1) we have to put a suitable topology on X,
(2) we have to show that with the topology from (1) the above map $p\colon X \to Y$ is continuous and open,
(3) we need to prove that the map $p\colon X \to Y$ is a covering,

[443] At a first reading of the proof of the proposition it is perhaps best to just think of the case that $Y = S^1 \times [-1, 1]$ and that $\Gamma \subset \pi_1(S^1)$ is the trivial group.

(4) we have to show that $X = W/\sim$ with the topology from (1) is path-connected,
(5) finally we need to show that we have the desired equality $p_*(\pi_1(X, x_0)) = \Gamma$.

Now we perform these steps.

(1) Let $x = [f] \in X = W/\sim$ and let V be an open path-connected neighborhood of $p(x) = f(1) \in Y$. We consider

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = W/\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X. We have to verify that these sets satisfy the "basis property" that we introduced on page 60.

(B1) For each $x \in X$ there exists an open path-connected neighborhood V.[444] Clearly we have $x \in U(x, V)$. This shows that $\mathcal{B}$ satisfies (B1).

(B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Since Y is in particular o-locally path-connected there exists an open path-connected neighborhood V of $f(1)$ with $V \subset A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$. We refer to Figure 342 for an illustration.

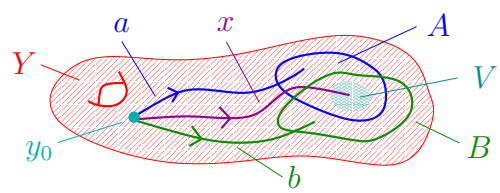


FIGURE 342.

We henceforth consider X as topological space with the topology generated by $\mathcal{B}$.[445]

(2) Now we want to show that the map $p\colon X = W/\sim \to Y$ is continuous and open. We start with the following observation: for every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x, V)) = V$.[446]

Now we turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x, B) \subset p^{-1}(A)$. Since Y is o-locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x, B)) = B \subset A$, i.e. $U(x, B) \subset p^{-1}(A)$.

Finally we show that p is also open. By Lemma 1.36 it suffices to verify that images of the basis of the topology of X are open. But this statement we had just verified in the above observation.

[444] Here we use that Y is reasonable, in particular it is o-locally path-connected.

[445] If we equip $W \subset Y^{[0,1]}$ with the compact-open topology from page 150, then one can see fairly easily that the above topology on X is just the resulting quotient topology. We could have immediately equipped X with this topology, but we will need later on that the sets $U(x, V)$ form a basis for this topology.

[446] By definition we have $p(U(x, V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x, V))$.

(3) Now we want to prove that $p\colon X = W/\sim \to Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is reasonable, i.e. since it is o-locally simply connected, there exists an open simply connected neighborhood V of y. We claim that V already has the desired property.

Claim.

$$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y}\in p^{-1}(y)} U(\widetilde{y}, V).$$

It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] \;=\; [g * h * \overline{h}] \;\in\; U(\underbrace{[g * h]}_{\in p^{-1}(y)}, V).$$

We refer to Figure 343 for an illustration.

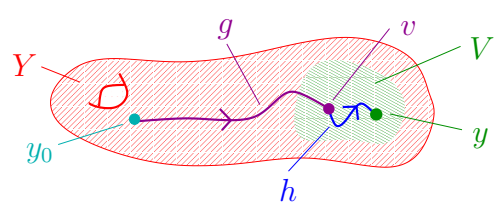


FIGURE 343.

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'], V) \cap U([v''], V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\substack{\uparrow \\ u''*\overline{u}' \text{ is null-homotopic since } V \text{ is simply connected}}}{=}\; [v''].$$

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. We refer to Figure 344 for an illustration. This concludes the proof of the claim.

Since the sets $U(\widetilde{y}, V)$ are open and disjoint it now suffices to prove the following claim.

Claim. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection map $p\colon X = W/\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

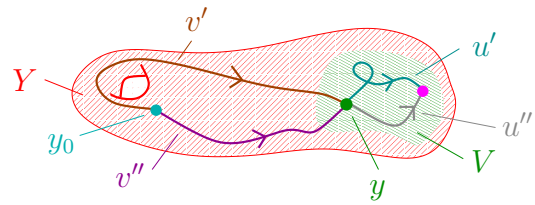


FIGURE 344.

So let $\widetilde{y} \in p^{-1}(y)$. As we had just pointed out in (2), the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that V is simply connected it also follows immediately that p is injective. Furthermore we had just verified in (2) that p is continuous and open. Therefore $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

(4) Now we want to show that $X = W/\sim$ is path-connected. We start out with the following claim.

Claim. Let $\alpha\colon [0,1] \to Y$ be a path. Then

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = W/\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

is a continuous map.

By Proposition 1.34 it suffices to show that for each $U(x, V)$ in the basis $\mathcal{B}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x, V))$ is open in $[0,1]$. So suppose we are given $U(x, V)$. We denote by $\{I_j\}_{j \in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0,1]$. Note that for each $j \in J$ we either have[447] $\widetilde{\alpha}(I_j) \subset U(x, V)$ or $\widetilde{\alpha}(I_j) \cap U(x, V) = \varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x, V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j) \subset U(x, V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x, V))$ is a union of open subsets of $[0,1]$, hence the preimage itself is open. This concludes the proof of the claim.

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x. So let $x = [\alpha] \in X$. Then such a path is given by $\widetilde{\alpha}\colon [0,1] \to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

(5) It remains to show that $p_*(\pi_1(X, x_0)) = \Gamma$. Let $g \in \pi_1(Y, y_0)$. We choose a representatives $\alpha\colon [0,1] \to Y$ of g. We consider the following path[448] in $X = W/\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = W/\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

[447] This might require a few minutes of thought.

[448] Note that this map is continuous by the claim in (4).

Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\substack{\uparrow \\ \text{Lemma 7.16 (1)}}}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = W/\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\substack{\uparrow \\ \text{definition of } \sim \text{ on } W}}{\Leftrightarrow} \alpha \in \Gamma. \qquad \square$$

Now the question arises, to what degree is the covering corresponding to the subgroup Γ unique. To answer that question we need an appropriate notion of two coverings being equivalent.

Definition. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say p and q are *equivalent*, if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & e^{it} \end{array}$$

are equivalent. In fact the desired map $\Phi\colon S^1 \to \mathbb{R}/n\mathbb{Z}$ is given by $e^{2\pi i t} \mapsto [nt]$.

Now we can formulate the following proposition:

Proposition 20.9. *Let B be a topological space, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B,b_0)$ be a subgroup.*

(1) *Let $p\colon (X,x_0) \to (B,b_0)$ be a path-connected and o-locally path-connected covering such that $p_*(\pi_1(X,x_0)) = \Gamma$. This covering has the following universal property: for any other path-connected covering $q\colon (Y,y_0) \to (B,b_0)$ with $\Gamma \subset q_*(\pi_1(Y,y_0))$ there exists a unique covering $r\colon (X,x_0) \to (Y,y_0)$ such that the following diagram commutes:*

$$\begin{tikzcd} (X,x_0) \arrow[d, "p"'] \arrow[dr, dashed, "\exists!\, r"] & \\ (B,b_0). & (Y,y_0) \arrow[l, "q"] \end{tikzcd}$$

(2) *If B is path-connected reasonable, then there exists, up to equivalence, a unique path-connected covering $p\colon (X,x_0) \to (B,b_0)$ of pointed topological spaces such that $p_*(\pi_1(X,x_0)) = \Gamma$.*

Proof. Let B be a topological space and let $b_0 \in B$.

(1) Let $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ be two path-connected coverings such that $p_*(\pi_1(X,x_0)) \subset q_*(\pi_1(Y,y_0))$. We assume furthermore that X is o-locally path-connected. It follows from Proposition 20.2 that there exists a unique map $r\colon (X,x_0) \to (Y,y_0)$ which makes the following diagram commute

$$\begin{tikzcd} & (Y,y_0) \arrow[d, "q"] \\ (X,x_0) \arrow[ur, "r"] \arrow[r, "p"'] & (B,b_0). \end{tikzcd}$$

It remains to show that r is a covering. This will be done in Exercise 20.3.

(2) The existence of such a covering is just the statement of Proposition 20.8. The uniqueness statement is an immediate consequence of (1)[449] and the usual proof of uniqueness for an object satisfying a universal property. We leave the details to the reader. □

Definition. Let B be a topological space, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B,b_0)$ be a subgroup. We refer to any path-connected covering $p\colon (X,x_0) \to (B,b_0)$ of pointed topological spaces with $p_*(\pi_1(X,x_0)) = \Gamma$ as *the covering of B corresponding to* Γ. Put differently, in our language we usually suppress the fact that the covering provided by Proposition 20.9 (2) is well-defined only up to equivalence.

Proposition 20.9 can be viewed as a "machine" that constructs covering spaces of a given topological space X. In the next section we discuss the case that X is a surface in somewhat greater detail to get a better feeling for what is happening.

20.4. **Covering spaces of surfaces.** Suppose we start out with a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, then by Proposition 20.8 we obtain a corresponding finite covering $p\colon \widetilde{F} \to F$. By Proposition 8.1 and Lemma 7.4 (4) the topological space $\widetilde{F}$ is again a closed orientable connected 2-dimensional manifold. From Theorem 14.10 we know that $\widetilde{F}$ is again a surface of some genus h. How can we determine h from g and Γ?

For future reference we record the question.

Question 20.10. *Given a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, what is the genus of the corresponding finite cover?*

It will take a while till we can give a satisfactory answer. In fact we will obtain a complete answer only once we have introduced the Euler characteristic.

But we want to discuss one example in detail. We consider the surface F of genus two that we view as the connected sum of two tori $S = S^1 \times S^1$ and $T = S^1 \times S^1$. We denote by u, v the standard generators for $\pi_1(S)$ given by the two copies of S^1 in each torus. Similarly we denote by x, y the standard generators for $\pi_1(T)$. We pick an embedded closed disk

[449] If one does the argument carefully, then one sees, that here we use the following fact: if B is reasonable, then also every covering space of B is reasonable, in particular every covering space of B is o-locally path-connected.

in S and an embedded closed disk in T which both lie in the complement of the standard generators. We denote the corresponding open disks by D and E and we write $S^* = S \setminus D$ and $T^* \setminus E$. We build the connected sum $S\#T$ by gluing S^* to T^* along the boundary.[450] Furthermore, by a slight abuse of notation, we denote by u, v and x, y also the generators of $\pi_1(S^*) = \langle u, v\rangle$ and $\pi_1(T^*) = \langle x, y\rangle$.

By the discussion on page 432 the abelianization of $\pi_1(S\#T) = \pi_1(S^* \cup_{\partial S^* = \partial T^*} T^*)$ is the free abelian group on u, v, x, y. Now we consider the epimorphism

$$\varphi\colon \pi_1(S\#T) \;\to\; \pi_1(S\#T)/[\pi_1(S\#T), \pi_1(S\#T)] \;\to\; \mathbb{Z}_3$$

given by $u \mapsto 1, v \mapsto 0, x \mapsto 0$ and $y \mapsto 0$. Our goal in the following discussion is to visualize the covering $p\colon \widetilde{F} \to F = S\#T$ corresponding to $\ker(\varphi)$. Since φ is an epimorphism onto a group of order 3 we see that $\ker(\varphi)$ is a subgroup of index 3. By Lemma 7.16 this implies that p is a 3-fold covering.

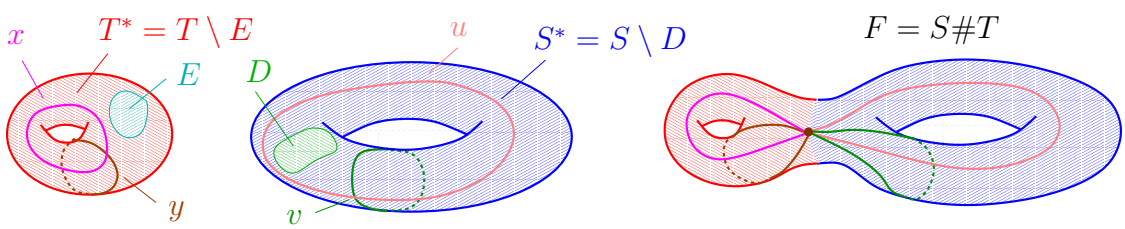


FIGURE 345.

We first consider the torus $S = S^1 \times S^1$ and furthermore we consider the epimorphism $\psi\colon \pi_1(S) = \langle u, v \mid [u, v]\rangle \to \mathbb{Z}_3$ given by $\psi(u) = 1$ and $\psi(v) = 0$. The covering corresponding to $\ker(\psi) \subset \pi_1(S)$ is equivalent to the covering[451]

$$\begin{array}{rcl} \widetilde{S} = S^1 \times S^1 & \to & S = S^1 \times S^1 \\ (a, b) & \mapsto & (a^3, b). \end{array}$$

Now we consider the epimorphism $\psi^*\colon \pi_1(S^*) \to \mathbb{Z}_3$ given by $\psi^*(u) = 1$ and $\psi^*(v) = 0$. By the argument of Footnote 451 the covering corresponding to $\ker(\psi^*)$ is then given by $p\colon \widetilde{S}^* := \widetilde{S} \setminus p^{-1}(D) \to S^* = S \setminus D$. Recall that we perform the connected sum operation $S\#T$ by gluing $T^* := T \setminus E$ to $S^* = S \setminus D$ to obtain $F = S\#T$. The same way we can

[450]It follows from Corollary 20.7 that we can be a bit casual in our language, i.e. we can talk about "gluing surfaces" without specifying explicit diffeomorphisms of boundary components, as long as we take into account orientations.

[451]Indeed, since $\widetilde{S} \to S$ is a connected 3-fold covering we know from Lemma 7.16 that $p_*(\pi_1(\widetilde{S}))$ is a subgroup of $\pi_1(S)$ of index three. Furthermore it is contained in the subgroup $\ker\left(\pi_1(S) \xrightarrow{\psi} \mathbb{Z}_3\right)$ which also has index three, thus $p_*(\pi_1(\widetilde{S})) = \ker(\psi\colon \pi_1(S) \to \mathbb{Z}_3)$. But by Proposition 20.9 all path-connected coverings that correspond to the same subgroup are equivalent.

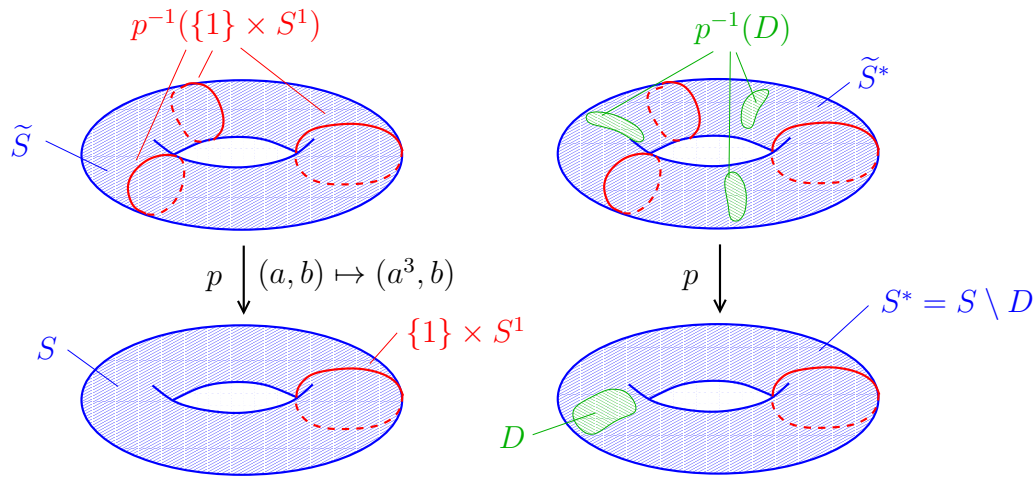


FIGURE 346.

also glue three copies of T^*, i.e. $T^* \times \{1,2,3\}$, to the three boundary components of $\widetilde{S}^*$ to obtain a closed manifold $\widetilde{F}$. It follows from Lemma 1.32 (2') that the map

$$\begin{array}{rcl} q\colon \widetilde{F} & \to & F \\ P & \mapsto & \begin{cases} p(P), & \text{if } P \in \widetilde{S}^*, \\ Q, & \text{if } P = Q \times \{n\} \text{ with } Q \in T^* \text{ and } n \in \{1,2,3\} \end{cases} \end{array}$$

is continuous. Similar to Footnote 451 one can now see that $q\colon \widetilde{F} \to F$ is indeed the covering of F corresponding to $\ker(\varphi)$. This construction is illustrated in Figure 347 which also shows that $\widetilde{F}$ is a surface of genus 4.

20.5. **The universal covering of a reasonable topological space.** In Proposition 20.9 we saw that subgroups of fundamental groups give rise to covering spaces. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to Proposition 20.9.

Corollary 20.11. *Let X be a path-connected and reasonable topological space and let $x_0 \in X$. Then the following holds:*

(1) *There exists, up to equivalence, a unique path-connected covering*

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

such that $\widetilde{X}$ is simply connected.

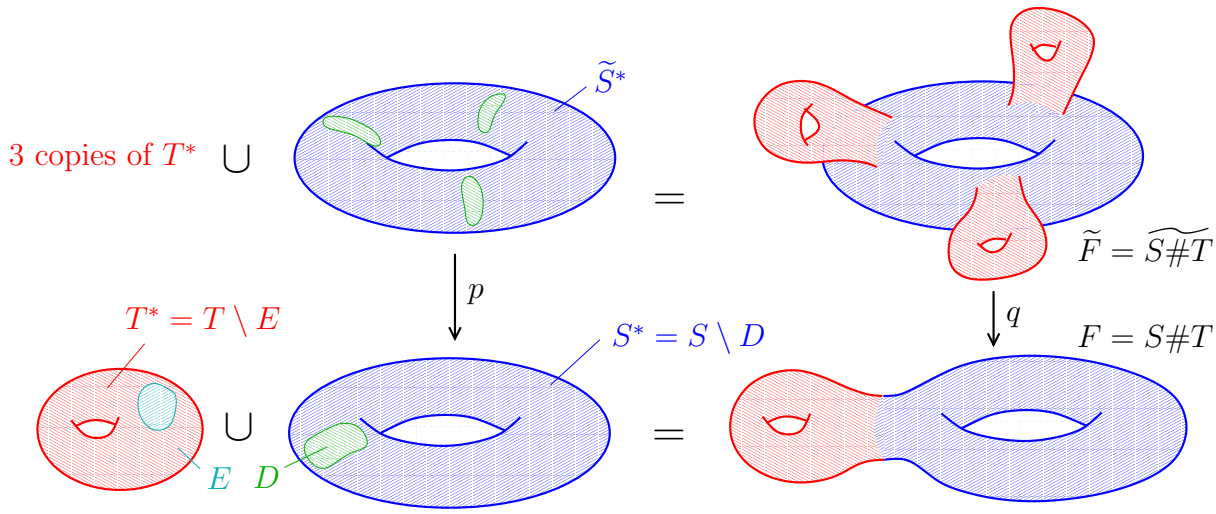


FIGURE 347.

(2) *Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. Then there exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:*

$$\begin{tikzcd} (\widetilde{X}, \widetilde{x}_0) \arrow[dr, "r"] \arrow[dd, "p"'] & \\ & (Y, y_0) \arrow[dl, "q"] \\ (X, x_0). & \end{tikzcd}$$

Proof. Let X be a path-connected and reasonable topological space and let $x_0 \in X$.

(1) We apply Proposition 20.9 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. But then it follows from Corollary 7.15 that $\widetilde{X}$ is simply connected.

(2) The second statement is also an immediate consequence of Proposition 20.9. $\square$

Definition. Let X be a path-connected and reasonable topological space. We refer to the covering $p\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is simply connected as the *universal covering* of X. As in the previous definition we suppress the fact that the universal covering is well-defined only up to equivalence.[452]

Remark. Recall that most "reasonable" topological spaces are reasonable in the technical sense. For example we saw on page 549 that every topological manifold is reasonable. Thus, given a path-connected topological manifold we can talk of is universal covering.

[452]In the literature the universal covering of a topological space is usually indicated by the same symbol but decorated with a tilde, e.g. the universal covering of X is usually denoted by $\widetilde{X}$.

Examples.

(1) The map
$$\begin{array}{rcl} p\colon \mathbb{R} & \to & \mathbb{R}/\mathbb{Z} = S^1 \\ t & \mapsto & e^{2\pi i t} \end{array}$$
is the universal covering of S^1.

(2) More generally, the map
$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1, \dots, t_n) & \mapsto & \left(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}\right) \end{array}$$
is the universal covering of the n-dimensional torus $(S^1)^n$.

(3) The projection map $p\colon S^3 \to S^3/\mathbb{Z}_p = L(p,q)$ is the universal covering of the lens space $L(p,q)$.

Remark. The universal covering is often also just called the *universal cover*. Also very often one suppresses the actual map p from the notation. For example, often one says that "$\mathbb{R}^n$ is the universal covering of $(S^1)^n$". The name "universal cover(ing)" stems from the fact, proved in Corollary 20.11, that the universal cover(ing) of a topological space X "covers" every other covering of X.

We have now shown that for every reasonable topological space X there exists a universal covering $p\colon \widetilde{X} \to X$. So far we have given an abstract construction, but if possible we would like to visualize it. We have just given an explicit description of the universal covering of the n-dimensional torus $(S^1)^n$ and the lens spaces. Now we will describe the universal covers of $X = S^1 \vee S^2$ and $X = S^1 \vee S^1$.

First we consider $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0,0,-1) \in S^2$. As on page 391 we consider the topological space
$$\widetilde{X} \;=\; \left(\mathbb{R} \cup (\mathbb{Z} \times S^2)\right) / \, n \sim (n, (0,0,-1)), \quad n \in \mathbb{Z}.$$
It is straightforward to see that the map
$$\begin{array}{rl} p\colon \left(\mathbb{R} \cup (\mathbb{Z} \times S^2)\right) / \, n \sim (n,(0,0,-1)) & \to S^1 \vee S^2 \\ P & \mapsto \begin{cases} e^{i(2\pi P + \frac{\pi}{2})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2 \end{cases} \end{array}$$
is well-defined and in fact a covering. We saw on page 391 that $\widetilde{X}$ is simply connected. Thus $p\colon \widetilde{X} \to X$ is the universal covering of X.

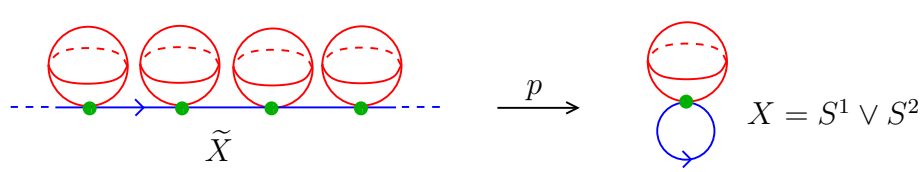


FIGURE 348.

Now we turn to $X = S^1 \vee S^1$. In the following we give a semi-rigorous description of the universal covering of $S^1 \vee S^1$ as a subset of $\mathbb{R}^2$. Iteratively we build a sequence of topological spaces Y_k, $k \in \mathbb{N}_0$ as follows:

(1) The space Y_0 is the cross centered at the origin where each segment has length 1.
(2) Given Y_k we attach to each endpoint of Y_k three segments of length $\frac{1}{4^k}$[453] such that each endpoint of Y_k now has a neighborhood that is a cross. We denote the resulting space by Y_{k+1}.

Finally we put $Y = \bigcup_{k \in \mathbb{N}_0} Y_k$. This construction is illustrated in Figure 349. We summarize

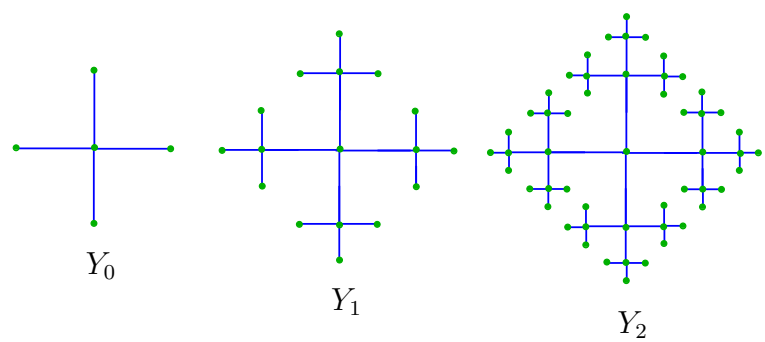


FIGURE 349.

some of the key properties of the construction in the following claim.

Claim.

(1) Each Y_k is a tree.[454]
(2) Y is simply connected.

Sketch of proof. First note that $\chi(Y_0) = 5 - 4 = 1$. For $k \geq 1$ we obtain Y_k from Y_{k-1} by adding three vertices and three edges, hence $\chi(Y_k) = \chi(Y_{k-1}) = \cdots = \chi(Y_0) = 1$. This shows that each Y_k is a tree. It follows from Proposition 9.20 that each Y_k is simply connected.

Given $k \in \mathbb{N}$ we denote by U_k the result of removing the 4^{k+1} endpoints of Y_k. Note that each U_k is open in Y_k, hence each U_k is open in Y. Furthermore each U_k deformation retracts to Y_{k-1}. Hence it follows from the fact that each Y_k is simply connected that also each U_k is simply connected.

It follows as in Lemma 12.6 that $Y = \bigcup_{k \in \mathbb{N}} Y_k = \bigcup_{k \in \mathbb{N}} U_k$ itself is simply connected. $\square$

Now we construct a covering $p\colon Y \to S^1 \vee S^1$. To distinguish the two copies of S^1 we name them A and B, which are wedged together along $a \in A$ and $b \in B$. We give both

[453] In Figure 349 and 350 the lengths of the segments is $\frac{1}{2^k}$ since these are easier to draw.

[454] Recall that a tree is a graph with finitely many vertices V and edges E, such that the Euler characteristic $\chi(G) = \#V - \#E = 1$.

A and B the usual counterclockwise orientation. Furthermore we orient each edge of the infinite graph Y as indicated in Figure 350. We consider the map $p\colon Y \to A \vee B$ that is

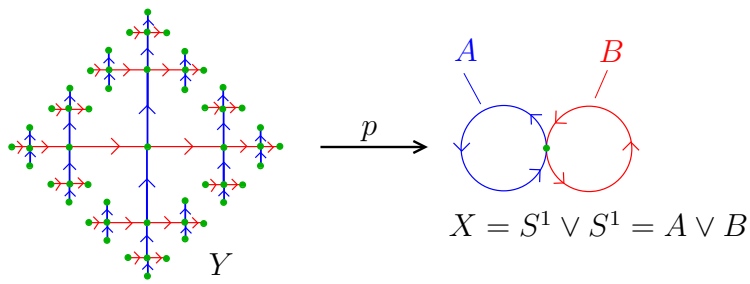


FIGURE 350.

given as follows:

(1) each vertex of the infinite graph Y gets sent to the point $a = b \in A \vee B$,
(2) each horizontal edge of Y gets sent in an orientation-preserving way to $A \setminus \{a\}$,
(3) each vertical edge of Y gets sent in an orientation-preserving way to $B \setminus \{b\}$.

This map is illustrated in Figure 350. It is relatively straightforward to convince oneself that $p\colon Y \to X$ is in fact a covering.[455] Since Y is simply connected, this is the universal covering of $X = A \vee B = S^1 \vee S^1$.

We have now determined explicitly the universal covers of several topological spaces. The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Therefore it is natural to ask the following question.

Question 20.12. *What does the universal covering of the surface of genus $g \geq 2$ look like?*

We conclude this section with a more general statement. Recall that in Theorem 7.17 we saw that if X is a simply connected topological space and if G is a group that acts continuously and discretely on X, then the fundamental group of the quotient space X/G is isomorphic to G. Now we will show that, under modest extra hypotheses, the converse holds. More precisely, we have the following proposition.

Proposition 20.13. $(*)$*Let Y be a path-connected, reasonable topological space and let $y_0 \in Y$. We write $\pi = \pi_1(Y, y_0)$. Then there exists a simply connected topological space X, namely the universal covering of Y, and a continuous and discrete action of π on X such that Y is homeomorphic to X/π.*

[455] We have to check that each point in $X = A \vee B$ admits an open neighborhood that is uniformly covered. It is clear that such a point exists for all points that are not the wedge point. For the wedge point itself the desired open neighborhood is given by small open intervals on A and B. Then each component of the preimage consists of a little cross with center P, with one horizontal edge going into P, one horizontal edge going out of P, one vertical edge going into P and one vertical edge going out of P.

Proof $(*)$. Let Y be a path-connected, reasonable topological space and let $y_0 \in Y$ be a base point. In Proposition 20.8 we already showed that Y admits a simply connected covering. We recall the explicit construction given in the proof of Proposition 20.8. Thus we consider

$$W \;:=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in W$ we write

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] = e \in \pi_1(Y, y_0)$$

and we put $X = W/\sim$. As on page 560, given $x = [f] \in X = W/\sim$ and an open path-connected neighborhood V of $f(1)$ we define

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \;\subset\; Y = W/\sim .$$

Furthermore, as in the proof of Proposition 20.8 we endow X with the topology generated by these sets $U(x, V)$.

Claim. The map

$$\begin{array}{rcl} \pi_1(Y, y_0) \times X & \to & X \\ ([g], [u]) & \mapsto & [g * u] \end{array}$$

is a well-defined action.

We first show that the map is independent of the choice of u. So suppose that $u, v \in W$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition in the definition of $\sim$. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{=e \in \pi_1(Y, y_0)} * [g]^{-1} \;=\; e \;\in \pi_1(Y, y_0).$$

It is straightforward to see that the map does not depend on the choice of representative g. Finally we want to show that it is indeed an action. So let $[g], [h] \in \pi_1(Y, y_0)$. Then

$$[g] \cdot ([h] \cdot [u]) \;=\; [g] \cdot [h * u] \;=\; [g * (h * u)] \;=\; [(g * h) * u] \;=\; [g * h] \cdot [u] \;=\; ([g] \cdot [h]) \cdot [u].$$

This concludes the proof of the claim.

Now we need to show the following:

(1) the group $\pi_1(Y, y_0)$ acts continuously and discretely on $X = W/\sim$,
(2) there exists a homeomorphism $f\colon X/\pi_1(Y, y_0) \to Y$.

We do this in the following two steps:

(1) It follows immediately from the definition of the topology on $X = W/\sim$ and from Proposition 1.34 that the action is continuous.

Now we need to show that the action is discrete. So let $x = [w] \in W/\sim$. Since W is reasonable we can pick a simply connected neighborhood V around the endpoint P of w. It suffices to show that for $[g] \in \pi_1(Y, y_0)$ we have

$$[g] \cdot U(x, V) \cap U(x, V) \neq \varnothing \quad \Longrightarrow \quad [g] = e \in \pi_1(Y, y_0).$$

So suppose that $[g] \cdot U(x,V) \cap U(x,V) \neq \varnothing$. This means that there exist paths a and b in U, starting from P such that $g * w * a$ and $w * b$ have the same endpoint and they are path-homotopic. But then

$$[g] \;=\; [g] \cdot \underbrace{[w] \cdot \overbrace{[a * \bar{b}]}^{=e\in\pi_1(Y,y_0)=\{e\}} * [w]^{-1}}_{=e\in\pi_1(Y,y_0)} \;=\; [(g * w * a) * \overline{(w * b)}] \;=\; e \in \pi_1(Y, y_0).$$

We refer to Figure 351 for an illustration.

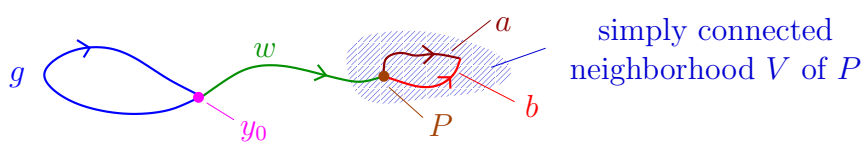


FIGURE 351.

(2) We consider the map

$$\begin{aligned} f\colon X/\pi_1(Y,y_0) &\to Y \\ [w] &\mapsto w(1). \end{aligned}$$

It is straightforward to see that this map is a bijection. It follows immediately from Lemma 1.66 that the map is continuous and it follows from Lemma 1.36 that the map is open. Thus it is a homeomorphism. □

20.6. **Preimages of subsets under coverings** $(*)$**.** Let (B, b_0) be a pointed topological space that is path-connected reasonable. In Proposition 20.8 (2) we saw that subgroups of $\pi_1(B, b_0)$ give rise to coverings of B. Arguably the most common source of subgroups is given by considering kernels of group homomorphisms $\pi_1(B, b_0) \to G$.

Example. Let B be the 2-dimensional sphere S^2 minus three open disks. We pick a base point b_0 for B and we pick loops u, v, w as shown in Figure 352. From Lemma 14.19 we know that $\pi_1(B, b_0) = \langle u, v, w \,|\, uvw \rangle$. By Lemma 13.4 there exists a unique homomorphism $\varphi\colon \pi_1(B, b_0) \to \mathbb{Z}_6$ with $\varphi(u) = 2$, $\varphi(v) = 3$ and $\varphi(w) = -5$. Let $p\colon X \to B$ be the covering corresponding to $\ker(\varphi)$. It is pretty clear, and we will see it in the subsequent Proposition8.1, that X is again a compact orientable 2-dimensional manifold with boundary given by $\partial X = p^{-1}(\partial B)$. Thus we know from Theorem 14.14 that there exist unique $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that X is homeomorphic to $\Sigma_{g,n}$, i.e. such that X is homeomorphic to the surface of genus g minus n open disks.

Therefore the following question arises:

Question 20.14. *In the previous example, what are g and n?*

The final proposition of this chapter will allow us to answer one half of Question 20.14, namely we can determine n. We will determine g at a later stage.

FIGURE 352.

Proposition 20.15. *Let (B, b_0) be a pointed topological space that is path-connected reasonable. Furthermore let $\varphi\colon \pi_1(B, b_0) \to G$ be a group epimorphism and let $p\colon (X, x_0) \to (B, b_0)$ be the covering with $p_*(\pi_1(X, x_0)) = \ker(\varphi)$. Finally let A be a path-connected subset of B and let $a_0 \in A$. We denote by $i\colon A \to B$ the inclusion map and we pick a path γ from b_0 to a_0. We consider the homomorphism*

$$\psi := \varphi \circ \gamma_* \circ i_*\colon \pi_1(A, a_0) \xrightarrow{i_*} \pi_1(B, a_0) \xrightarrow{\gamma_*} \pi_1(B, b_0) \xrightarrow{\varphi} G.$$

With these preparations the following two statements hold:

(1) *Each path-component of $p^{-1}(A)$ is the covering of A corresponding to the subgroup $\ker\{\psi\colon \pi_1(A, a_0) \to G\}$.*
(2) *If G is a finite group, then*

$$b_0(p^{-1}(A)) \;=\; [G : \mathrm{im}\{\psi\colon \pi_1(A, a_0) \to G\}].$$

Example. As promised we can now determine the number n of boundary components of the covering X in the previous example. More precisely, denote by U, V, W the three boundary components of B. We make the following three observations:

(1) The image of $\pi_1(U)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(u) = 2$, thus the subgroup has index two, which by Proposition 20.15 implies that $p^{-1}(U)$ consists of two components.
(2) The image of $\pi_1(U)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(v) = 3$, thus $p^{-1}(V)$ consists of three components.
(3) The image of $\pi_1(U)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(v) = -5$. Since -5 is coprime to 6 we see that this subgroup is all of $\mathbb{Z}_6$, thus $p^{-1}(W)$ consists of precisely one component.

Altogether we see that X has $2 + 3 + 1 = 6$ boundary components.

Remark. The reader who is in need of a brain teaser might want to come up with an appropriate generalization of Proposition 20.15 to the case where we consider coverings of B coming from a non-normal subgroup.

Proof. To simplify the notation a little bit we assume that $a_0 = b_0$, in particular we ignore the role of γ. Throughout the proof we refer to Figure 353 for an illustration.

(1) Let W be a path-component of $p^{-1}(A)$. We pick a point $w_0 \in W$ with $p(w_0) = a_0$. We start out with the following slightly subtle claim.

Claim.
$$p_*(\pi_1(X, w_0)) \;=\; \ker\{\varphi\colon \pi_1(B, a_0) \to G\}.$$

We pick a path γ in X from x_0 to w_0. Since $p \circ \gamma$ is a loop in a_0 we can write $h := [p \circ \gamma] \in \pi_1(B, a_0)$. We consider the following diagram

$$\begin{tikzcd}
\pi_1(X, w_0) \arrow[r, "\gamma_*", "\cong"'] \arrow[d, "p_*"] & \pi_1(X, x_0) \arrow[d, "p_*"] \\
\pi_1(B, a_0) \arrow[r, "g \mapsto hgh^{-1}", "\cong"'] \arrow[d, "\varphi"] & \pi_1(B, a_0) \arrow[d, "\varphi"] \\
G \arrow[r, "g \mapsto \varphi(h) g \varphi(h)^{-1}", "\cong"'] & G.
\end{tikzcd}$$

The top square commutes by Proposition 5.11 (2) and (3). Evidently the bottom square also commutes. By definition of (X, x_0) we have
$$p_*(\pi_1(X, x_0)) \;=\; \ker\{\varphi\colon \pi_1(B, a_0) \to G\}.$$
The claim now follows from the fact that the diagram commutes and that the horizontal maps are isomorphisms.

In the following we denote by $j\colon W \to X$ the inclusion map. We consider the following diagram

$$\begin{tikzcd}
\pi_1(W, w_0) \arrow[r, "j_*"] \arrow[d, "p_*"] & \pi_1(X, w_0) \arrow[d, "p_*"] \\
\pi_1(A, a_0) \arrow[r, "i_*"] \arrow[dr, "\psi"'] & \pi_1(B, a_0) \arrow[d, "\varphi"] \\
& G.
\end{tikzcd}$$

Recall that we need to show the equality $p_*(\pi_1(W, w_0)) = \ker(\psi)$. We show the two inclusions separately.

"$\subset$" This inclusion follows immediately from the definitions, from the commutativity of the above diagram and from the claim.

"$\supset$" Thus let $g \in \ker(\psi)$. We represent it by a loop γ in (A, a_0). By the above claim and Lemma 7.16 the loop $i \circ \gamma$ in (B, a_0) lifts to a loop $\widetilde{\gamma}$ in (X, w_0). By definition the image of $\widetilde{\gamma}$ lies in $p^{-1}(A)$ and it contains w_0. In particular the image of $\widetilde{\gamma}$ lies in the path-component of $p^{-1}(A)$ that contains w_0, in other words, $\widetilde{\gamma}$ is actually a loop in (W, w_0). Since $g = p_*([\widetilde{\gamma}])$ we obtain that $g \in p_*(\pi_1(W, w_0))$.

(2) We write $n = |G|$ and $k = |\operatorname{im}\{\psi\colon \pi_1(A, a_0) \to G\}|$. It follows from Lemma 7.16 and (1) that each path-component of $p^{-1}(A)$ is a k-fold covering of A. By Lemma 7.16 the map $p\colon X \to B$ is an n-fold covering. Evidently $p\colon p^{-1}(A) \to A$ is also an

n-fold covering. It follows almost immediately that $p^{-1}(A)$ has necessarily n/k path-components. But $n/k = |G|/|\operatorname{im}(\psi)|$ is by definition the degree of $\operatorname{im}(\psi)$ in G. $\square$

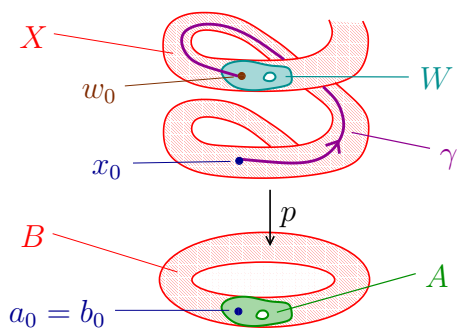


FIGURE 353. Illustration for the proof of Proposition 20.15.

Exercises for Chapter 20.

Exercise 20.1. Show that there is no algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ represents a group that is isomorphic to $\mathbb{Z}$.

Exercise 20.2. Prove Corollary 20.4.

Exercise 20.3. Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings of topological spaces such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. It follows from Proposition 20.2 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ which makes the following diagram commute

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "q"] \\ (X, x_0) \arrow[ur, "r"] \arrow[r, "p"'] & (B, b_0). \end{tikzcd}$$

Show that r is a covering.

Exercise 20.4. We consider Figure 354.

(a) Let X be the wedge of two circles and let $p\colon Y \to X$ be the universal covering that is illustrated in the figure. Denote the usual generators of $\pi_1(X)$ by a and b. By an abuse of notation we denote by a and b also representing loops. What does the lift of the loop $a * b * \overline{a} * \overline{b}$ to Y to the starting point y_0 look like?

(b) We continue with the notation from (1). Now consider the loop

$$\gamma := a * a * b * a * a * \overline{b} * \overline{a} * b * b * \overline{a} * \overline{a} * \overline{b} * \overline{a} * \overline{b}.$$

We denote by $\widetilde{\gamma}$ the lift of γ to the starting point y_0. Is y_0 the endpoint of $\widetilde{\gamma}$?

(c) Does there exist a homeomorphism $f\colon Y \to Y$ with $f(y_0) = z_0$?

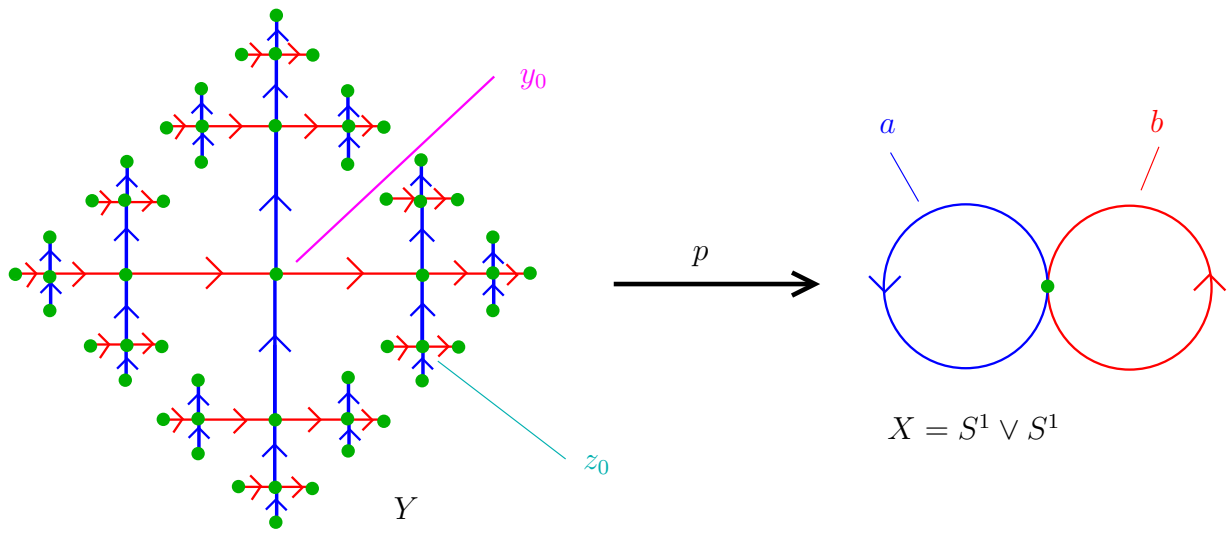


FIGURE 354.

Exercise 20.5.

(a) Let M be the Möbius band. We pick an identification $\pi_1(M) = \mathbb{Z}$. Given $k \in \mathbb{N}$ we denote by Γ_k the kernel of the epimorphism $\pi_1(M) = \mathbb{Z} \to \mathbb{Z}_k$. Let $M_k \to M$ be the covering corresponding to Γ_k. What well-known topological space is M_k? You do not have to justify your answer.

(b) Draw the universal covering of $\mathbb{R}\mathrm{P}^2 \vee S^1$ as a subset of $\mathbb{R}^3$.

(c) Let $X = A \vee B$ be the wedge of two circles. Let $\Gamma \subset \pi_1(X) = \langle a, b\rangle$ be the commutator subgroup. What does the covering $p\colon \widetilde{X} \to X$ corresponding to Γ look like? You can draw it as a subset of $\mathbb{R}^2$.
Hint: You have seen $\widetilde{X}$ many times in your life.

Exercise 20.6. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We denote by $M := M(X, f) := (X \times [0,1])/(x,0) \sim (f(x),1)$ the corresponding mapping torus (see page 469.) Let $p\colon \widetilde{M} \to M$ be a finite covering. Show that $\widetilde{M}$ is homeomorphic to a mapping torus.

Exercise 20.7. We say two manifolds M and N are *commensurable* if M and N admit finite covers $\widetilde{M}$ and $\widetilde{N}$ that are diffeomorphic.

(a) Show that the torus and the surface of genus two are not commensurable.

(b) Let $g, h \in \mathbb{N}_{n\geq 2}$. Let M be the surface of genus g and let N be the surface of genus h. Show that M and N are commensurable.

(c) Determine which compact connected 2-dimensional manifolds are commensurable.

Exercise 20.8. Let $f\colon S^1 \to S^1$ be a homeomorphism with $f(1) = 1$. Let $p\colon \mathbb{R} \to S^1$ be the covering that is given by $t \mapsto e^{it}$.

(a) Show that there exists a unique map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "f"] & S^1. \end{tikzcd}$$

(b) Show that the map $\widetilde{f}$ from (a) is a homeomorphism.

21. Complex manifolds

Eventually our goal will be to give an explicit description of the universal covering of the surfaces of genus $g \geq 2$. This will require a surprisingly long journey through complex manifolds, Riemannian manifolds and hyperbolic geometry. In this chapter we start out with a very short introduction to complex manifolds.

First we recall the definition of a holomorphic function.

Definition. Let $U \subset \mathbb{C}$ be an open subset. A function $f\colon U \to \mathbb{C}$ is *holomorphic* if for any $z_0 \in U$ the limit

$$\frac{d}{dz} f(z_0) \;:=\; f'(z_0) \;:=\; \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\in \mathbb{C}$$

exists.

Example. In any textbook on complex analysis it is shown that polynomials, the exponential function, and more generally functions defined by power series are holomorphic. Moreover products, fractions, sums and compositions of holomorphic functions are again holomorphic.

The definition of a complex function being holomorphic is almost verbatim the same as the definition of a real function being smooth. But rather surprisingly being holomorphic is significantly more restrictive, as is shown in the following two standard results in complex analysis, see e.g. [Jä2] or [Lan].

Theorem 21.1. (Liouville's Theorem) *Every bounded holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is constant.*

Theorem 21.2. (Maximum Principle) *Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.*

The following proposition also belongs to the usual repertoire of a course on complex analysis, see e.g. [Lan, Chapter IX.1].

Proposition 21.3. (Schwarz Reflection Principle) *Let U be an open subset of the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ and let $f\colon U \to \mathbb{C}$ be a continuous function with the following properties:*

1. *f is holomorphic on $\mathring{U} = \{z \in U \mid \operatorname{Im}(z) > 0\}$,*
2. *f only assumes real values on $U \cap \mathbb{R}$.*

We set $U' := \{\overline{z} \mid z \in U\}$, i.e. U' is the reflection of U in the x-axis. Then the function

$$\begin{aligned} \widetilde{f}\colon U \cup U' &\to \mathbb{C} \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\overline{z})}, & \text{if } z \in U', \end{cases} \end{aligned}$$

is holomorphic.

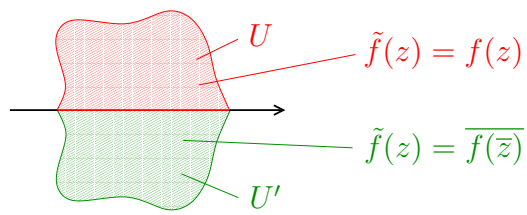


FIGURE 355. Illustration of the Schwarz reflection principle.

Given a topological manifold we now introduce the notion of a holomorphic atlas and a complex manifold. The definition is almost the same as for a smooth atlas and a manifold, except that we have to replace "smooth" by "holomorphic".

Definition.

(1) Let $U \subset \mathbb{C}^m$ be open. We say that a map $f = (f_1, \dots, f_n) \colon U \to \mathbb{C}^n$ is *holomorphic* if for any $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and any $(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m) \in \mathbb{C}^{m-1}$ the map

$$\begin{array}{rcl} \overbrace{\{z \in \mathbb{C} \mid (w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m) \in U\}}^{\text{open subset of } \mathbb{C}} & \to & \mathbb{C} \\ z & \mapsto & f_j(w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m) \end{array}$$

is holomorphic

(2) Let M be a $2n$-dimensional topological manifold M without boundary. An atlas $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ for M is called *holomorphic* if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1} \colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n} = \mathbb{C}^n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^{2n} = \mathbb{C}^n}$$

is holomorphic.

(3) An *n-dimensional complex manifold* is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M without boundary, together with a holomorphic atlas for M.

(4) We define the *maximal holomorphic atlas* of a complex manifold the same way as we defined the maximal smooth atlas of a manifold, see page 184.

Remark. Since holomorphic maps are in particular smooth we see that every n-dimensional complex manifold is in particular a $2n$-dimensional manifold.

Examples.

(1) Let U be an open subset of $\mathbb{C}^n$. The identity map is an atlas for U and it is evidently a holomorphic atlas. Thus U is an n-dimensional complex manifold.

(2) We consider the action

$$\begin{array}{rcl} (\mathbb{Z} \oplus \mathbb{Z}i) \times \mathbb{C} & \to & \mathbb{C} \\ ((a + bi), z) & \mapsto & z + a + bi. \end{array}$$

The same argument as in the proof of Proposition 3.25 shows that $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is a 1-dimensional complex manifold. The topological space $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is evidently diffeomorphic to the torus $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Thus we can view the torus as a complex manifold.

The following proposition shows in particular that the Klein bottle and the real projective plane $\mathbb{R}\mathrm{P}^2$ are *not* complex manifolds.

Proposition 21.4. *An n-dimensional complex manifold, viewed as a $2n$-dimensional real manifold, comes with a canonical orientation. More precisely, we can equip any complex manifold with an orientation which has the following two properties:*

(1) *any open subset of $\mathbb{C}^n = \mathbb{R}^{2n}$ inherits the usual orientation,*
(2) *the orientations are preserved under local biholomorphisms.*

Before we can prove Proposition 21.4 we need the following definition and the subsequent lemma.

Definition. Let V be an n-dimensional complex vector space. We say that a set of vectors $\{v_1, \dots, v_n\}$ in V is *totally real* if the vectors $v_1, iv_1, \dots, v_n, iv_n$ form a basis of V, viewed as a $2n$-dimensional real vector space.

Lemma 21.5. *Let V be an n-dimensional complex vector space.*

(1) *V admits a set of totally real vectors.*
(2) *If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are two sets of totally real vectors for a complex vector space V, then the bases $\{v_1, iv_1, \dots, v_n, iv_n\}$ and $\{w_1, iw_1, \dots, w_n, iw_n\}$ are equivalent in the sense of page 199.*

Given a finite dimensional complex vector space we refer to the orientation introduced in Lemma 21.5 as the *canonical* orientation.

Proof. The first statement follows from the observation that V is isomorphic, as a complex vector space, to $\mathbb{C}^n$, and for $\mathbb{C}^n$ a totally real basis is given by vectors $e_i = (0, \dots, 1, \dots, 0)$, $i = 1, \dots, n$.

We turn to the proof of the second statement. First we consider the case $n = 1$. In this case we are given two non-zero vectors v and w. Then $v = (a + ib)w$ for some $a + bi \neq 0$. The base change matrix from the real basis $\{v, iv\}$ to the real basis $\{w, iw\}$ is given by the real 2×2-matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Evidently the determinant of this matrix is positive. This concludes the proof of the claim for $n = 1$. The general statement is basically just an exercise in linear algebra. This

exercise though turns out to be slightly harder than one might expect.[456] A proof is for example given in [CCL, Theorem 7.2.1]. □

Now we sketch a proof of Proposition 21.4.

Sketch of a proof of Proposition 21.4. Let M be an n-dimensional complex manifold. It is relatively straightforward to see that each tangent space T_PM is canonically an n-dimensional complex vector space. Now we equip it with the above canonical orientation. These orientations of the tangent spaces define a canonical orientation on the $2n$-dimensional real manifold M. One can easily verify that the orientations have the desired properties (1) and (2). □

The next lemma gives us an example of a complex manifold that will play a major role later on.

Lemma 21.6. *Let $n \in \mathbb{N}$. On page 111 we introduced the complex projective space*

$$\mathbb{C}\mathrm{P}^n \;=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

For $i = 0, \dots, n$ we consider the map

$$\begin{array}{rcl} \Phi_i \colon \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\} & \to & \mathbb{C}^n \\ [z_0 : \cdots : z_n] & \mapsto & \big(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\big). \end{array}$$

These maps form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$ that turn the complex projective space $\mathbb{C}\mathrm{P}^n$ into an n-dimensional compact complex manifold, in particular we can view $\mathbb{C}\mathrm{P}^n$ as a closed $2n$-dimensional oriented real manifold.

Proof. In Lemma 1.81 we saw that each $\mathbb{C}\mathrm{P}^n$ is compact and Hausdorff. As we will show below, $\mathbb{C}\mathrm{P}^n$ admits a finite atlas, thus it follows from Lemma 3.3 that $\mathbb{C}\mathrm{P}^n$ is second-countable. For the remaining statements it is easy to overlook that first one needs to prove the following claim.

Claim. If $i \in \{0, \dots, n\}$, then

$$V_i \;:=\; \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, z_i \neq 0\}$$

is an open subset of $\mathbb{C}\mathrm{P}^n$.

The claim is equivalent to showing that the complement $\mathbb{C}\mathrm{P}^n \setminus V_i$ is a closed subset of $\mathbb{C}\mathrm{P}^n$. We note that

$$\mathbb{C}\mathrm{P}^n \setminus V_i \;=\; \text{the image of the map } \begin{array}{rcl} f \colon \mathbb{C}\mathrm{P}^{n-1} & \to & \mathbb{C}\mathrm{P}^n \\ [w_0 : \cdots : w_{n-1}] & \mapsto & [w_0 : \cdots : w_{i-1} : 0 : w_i : \cdots : w_{n-1}]. \end{array}$$

[456] Basically the problem is as follows: Let A and B be two real $n \times n$-matrices such that $\det(A + iB) \in \mathbb{C}$ is non-zero. Why does it follow that the determinant of the real matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is positive?

The map f is easily seen to be continuous. Since $\mathbb{C}\mathrm{P}^{n-1}$ is compact we obtain from Lemma 1.37 that $f(\mathbb{C}\mathrm{P}^{n-1})$ is compact. But since $\mathbb{C}\mathrm{P}^n$ is Hausdorff we obtain from Lemma 1.18 (2) that $f(\mathbb{C}\mathrm{P}^{n-1}) = \mathbb{C}\mathrm{P}^n \setminus V_i$ is in fact a closed subset of $\mathbb{C}\mathrm{P}^n$. This concludes the proof of the claim.

Using this claim it is now straightforward to verify that the given maps $\Phi_0, \dots, \Phi_n$ form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$. This shows that $\mathbb{C}\mathrm{P}^n$ is a complex n-dimensional manifold. Note that the charts are of type (i). This shows that $\mathbb{C}\mathrm{P}^n$ has no boundary points, in particular since it is compact, it is closed Finally the last statement is an immediate consequence of Proposition 21.4. □

Remark. The notion of a submanifold also extends in the obvious way to the setting of complex manifolds. Let $m \leq n$. On page 111 we introduced the convention that we view $\mathbb{C}\mathrm{P}^m$ as a subset of $\mathbb{C}\mathrm{P}^n$. Using the above atlas one can easily that $\mathbb{C}\mathrm{P}^m$ is in fact a complex submanifold of $\mathbb{C}\mathrm{P}^n$.

As we had already pointed out on page 113, the one-dimensional projective space $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2. This shows that S^2 can be viewed as a 1-dimensional complex manifold. We had just seen that the 2-sphere and the torus are complex manifolds and we saw that non-orientable manifolds cannot be complex manifolds. The following question now arises:

Question 21.7. *Let $g \geq 2$. Does the surface of genus g have the structure of a complex manifold?*

We will discuss this question later on.

Remark. It is often very hard to determine whether or not a given orientable even-dimensional manifold can also be a complex manifold. For example, Armand Borel and Jean-Pierre Serre [BS][457] showed in 1953 that for $n \neq 2, 6$ the n-dimensional sphere S^n cannot be a complex manifold. On the other hand it is still an open question whether S^6 is a complex manifold.

Definition. We say that a map $f\colon M \to N$ between two complex manifolds is *holomorphic* if f is continuous and if for every chart $\Phi\colon U \to V$ in the holomorphic atlas of M and every chart $\Psi\colon W \to Y$ in the holomorphic atlas of N the map

$$\Phi(f^{-1}(W) \cap U) \xrightarrow{\Psi \circ f \circ \Phi^{-1}} Y$$

is holomorphic.

The following proposition will be proved in Exercise 21.1.

Proposition 21.8. *Let M be a compact 1-dimensional complex manifold. Then any holomorphic function $f\colon M \to \mathbb{C}$ is constant.*

[457] Armand Borel (1923–2003) was a Swiss mathematician, Jean-Pierre Serre (*1926) is a French mathematician who was awarded a Fields medal in 1954.

Remark. If $M = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$, then we can also give another, arguably easier proof. More precisely, let $f\colon \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i) \to \mathbb{C}$ be a holomorphic function. Since $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is compact the function f is bounded. We denote by $p\colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ the projection map. Then the composition $f \circ p\colon \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function, thus constant by the Liouville Theorem 21.1. Since p is surjective it follows that f is also constant.

In the next definition we generalize the notion of (local) diffeomorphism in the obvious way to the complex setting.

Definition.

(1) We say that a map $f\colon M \to N$ between two complex manifolds is a *biholomorphism* if f is a bijection and if both f and f^{-1} are holomorphic.
(2) We say that a map $f\colon M \to N$ between two complex manifolds is a *local biholomorphism* if given any $P \in M$ there exists an open neighborhood U of P and an open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a biholomorphism.

Examples.

(1) In some cases one can explicitly write down a biholomorphism. For example, for the upper half-plane $\mathbb{H} = \{x + iy \mid y > 0\}$ and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ it is straightforward to verify that the maps
$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-i}{z+i} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{i+iw}{1-w} \end{array}$$
are biholomorphisms that are inverse to one-another.
(2) The proof of Proposition 3.25 shows that $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ has the structure of a complex 1-dimensional manifold in such a way that the projection map $\mathbb{C} \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is a local biholomorphism.
(3) Let $a, b \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$. We write $T(a, b) = \mathbb{C}/(\mathbb{Z}a + \mathbb{Z}b)$. The same argument as in (2) shows that this is a 1-dimensional complex manifold such that the projection map $p\colon \mathbb{C} \to T(a, b)$ is a local biholomorphism. Now suppose that $v, w \in \mathbb{C}$ are also two complex numbers that are linearly independent over $\mathbb{R}$. We can form the torus $T(v, w) = C/(\mathbb{Z}v + \mathbb{Z}w)$. The tori $T(a, b)$ and $T(v, w)$ are diffeomorphic. But the argument of [Bal, p. 15] shows that the 1-dimensional complex manifolds $T(a, b)$ and $T(v, w)$ are biholomorphic only if $ab^{-1} = vw^{-1} \in \mathbb{C}$ or if $ab^{-1} = wv^{-1} \in \mathbb{C}$.

Lemma 21.9.

(1) *Let $p\colon \widetilde{X} \to X$ be a countable covering of a complex manifold X. Then $\widetilde{X}$ is also a complex manifold in such a way that p is a local biholomorphism.*
(2) *Let $p\colon \widetilde{X} \to X$ be a covering of complex manifolds such that p is a local biholomorphism. If Z is a complex manifold and if $f\colon Z \to X$ is a holomorphic map, then any lift $\widetilde{f}\colon Z \to \widetilde{X}$ is also holomorphic.*

Proof. The first statement is proved almost the same way as Proposition 8.1 (2). The second statement is an immediate consequence of the fact that p is a local biholomorphism and that being holomorphic is a local condition. We leave the details to the reader. $\square$

In general it is hard to write down an explicit biholomorphism between two complex manifolds. One of the main sources of biholomorphisms is the Riemann Mapping Theorem.

Theorem 21.10. (Riemann Mapping Theorem) *Let $U \subset \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$. If $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

This theorem is proved in [Jä2] or alternatively in [Lan, Chapter X].

In many applications it is also useful to know that the biholomorphism from the Riemann Mapping Theorem extends to the boundary of U and $\mathbb{D}$. To state the corresponding theorem we need one more definition.

Definition.

(1) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we equip with the topology that we defined on page 53.
(2) Given $U \subset \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of U in $\overline{\mathbb{C}}$.
(3) A *Jordan curve* is the image of an injective map $S^1 \to \overline{\mathbb{C}}$.

The following theorem was proved in 1913 by Constantin Carathéodory.[458] A proof is for example provided in [GM, Chapter I.3].

Theorem 21.11. (Carathéodory Theorem) *Let $U \subsetneq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$ and let $\phi\colon U \to \mathbb{D}$ be a biholomorphism. If $\partial\overline{U}$ is a Jordan curve, then ϕ extends to a homeomorphism $\overline{U} \to \mathbb{C}$.*

Example. We consider the open subset U of $\mathbb{C}$ that is shown in Figure 356. It is a simply connected open proper subset of $\mathbb{C}$ and the boundary is clearly a Jordan curve. Thus it follows from the Riemann Mapping Theorem and from Carathéodory's Theorem that there exists a homeomorphism $\phi\colon \overline{U} \to \overline{\mathbb{D}}$ which restricts to a biholomorphism $\phi\colon U \to \mathbb{D}$.

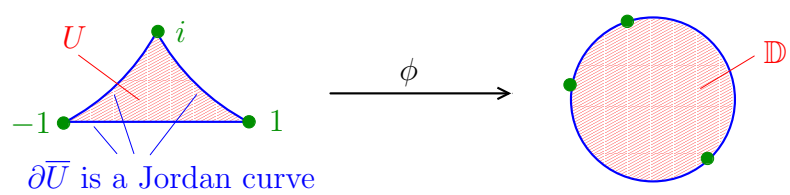


FIGURE 356.

[458]Constantin Carathéodory (1873-1950) was a Greek mathematician who was in particular a professor at the LMU in Munich.

Exercise for Chapter 21.

Exercise 21.1. Let M be a compact connected 1-dimensional complex manifold. Show that any holomorphic function $f\colon M \to \mathbb{C}$ is constant.
Hint: Use the Maximum Principle, i.e. use Theorem 21.2.

22. Hyperbolic geometry

22.1. **Hyperbolic space.** We start out with the following definition.

Definition. Let V be a real vector space. A *symmetric form on* V is a map[459]

$$g\colon V \times V \to \mathbb{R}$$

that is bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in \mathbb{R}$ we have

$$\begin{aligned} g(\lambda u + \mu v, w) &= \lambda g(u, w) + \mu g(v, w) \\ g(u, \lambda v + \mu w) &= \lambda g(u, v) + \mu g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

We say g is *positive-definite* if for all $v \in V \setminus \{0\}$ we have

$$g(v, v) > 0.$$

Now we can recall that following key definition from the theory of manifolds, see e.g. [Le2, Chapter 17].

Definition. A *Riemannian structure* on a manifold M is a smooth[460] map g which assigns to each point P a positive-definite symmetric form g_P on the tangent space $T_P M$.

In this chapter and the following chapter we are mostly interested in Riemannian structures on surfaces, and more specifically, in hyperbolic structures. Now we recall the definition of hyperbolic space.

Definition.

(1) The *hyperbolic half-plane* is the manifold

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

together with the Riemannian structure that assigns to the tangent space $T_z\mathbb{H} = \mathbb{R}^2$ at a point $z = x + iy \in \mathbb{H}$ the positive-definite symmetric form

$$\begin{aligned} g\colon \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (v, w) &\mapsto \frac{1}{y^2}\langle v, w\rangle \end{aligned}$$

where $\langle v, w\rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

(2) A *Möbius transformation* of $\mathbb{H}$ is a map of the form[461]

$$\begin{aligned} \mathbb{H} &\to \mathbb{H} \\ z &\mapsto \frac{az + b}{cz + d} \end{aligned}$$

[459] Sometimes such a structure is called a *bilinear symmetric form*, but we abbreviate this to *symmetric form*.

[460] The notion of "smooth" in this context is made precise in [Le2, Chapter 17]. Once one has digested the definition of a "smooth map" it is straightforward to see that all examples below satisfy this condition.

[461] An elementary calculation shows that the image of a Möbius transformation $\mathbb{H} \to \mathbb{C}$ does indeed lie again in $\mathbb{H}$.

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

Example. In Figure 357 we see a decomposition of $\mathbb{H}$ into angels and devils. With respect to the hyperbolic metric all the angels have the same size and all the devils have the same size, even though they differ in the usual Euclidean sense.

FIGURE 357.

Examples. We give three special types of Möbius transformations of $\mathbb{H}$:

(1) given $r > 0$ the scalar multiplication $z \mapsto rz = \frac{\sqrt{r}\cdot z+0}{0\cdot z+\frac{1}{\sqrt{r}}}$ is a Möbius transformation,
(2) given $d \in \mathbb{R}$ the horizontal translation $z \mapsto z + d = \frac{1\cdot z+d}{0\cdot z+1}$ is a Möbiustransformation,
(3) the map $z \mapsto -\frac{1}{z} = \frac{0\cdot z+1}{(-1)\cdot z+0}$ is a Möbius transformation.

The third Möbius transformation is the composition of taking the inverse $z \mapsto \frac{1}{z}$ and the reflection $z \mapsto -z$ in the origin. We illustrate this map in Figure 358. For simplicity we refer to the map $z \mapsto -\frac{1}{z}$ as an inversion.

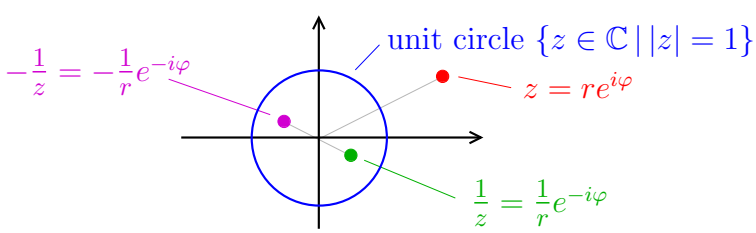


FIGURE 358.

Proposition 22.1. *Every Möbius transformation of $\mathbb{H}$ is the composition of scalar multiplications, horizontal translations and inversions.*

Proof. Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and denote by

$$\Phi(z) \;=\; \frac{az+b}{cz+d}$$

the corresponding Möbius transformation. If $c = 0$, then

$$\Phi(z) \;=\; \frac{a}{d}z + \frac{b}{d}$$

is the composition of the scalar multiplication[462] $z \mapsto \frac{a}{d}z$ with the horizontal translation $z \mapsto \frac{b}{d} + z$. If $c \neq 0$, then

$$\Phi(z) \;=\; \frac{az+b}{cz+d} \;=\; \frac{acz+bc}{c^2z+dc} \;\underset{\substack{\uparrow \\ \text{since } ad-bc=1}}{=}\; \frac{acz+ad-1}{c^2z+cd} \;=\; \frac{-1+a(cz+d)}{c^2z+cd} \;=\; \frac{-1}{c^2z+dc} + \frac{a}{c}.$$

Thus we see that Φ is the composition of the following maps

$$z \mapsto c^2z, \quad z \mapsto z + dc, \quad z \mapsto \frac{-1}{z} \quad \text{and } z \mapsto z + \frac{a}{c}.$$

□

Definition.

(1) The *hyperbolic disk*[463]is the manifold

$$\mathbb{D} \;=\; \{z \in \mathbb{C} \mid |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $T_z\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$\begin{array}{rcl} g_z\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v, w) & \mapsto & \frac{2}{(1-|z|^2)^2}\langle v, w\rangle \end{array}$$

where, as above, $\langle v, w\rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

(2) A *Möbius transformation* of $\mathbb{D}$ is a map of the form[464]

$$\begin{array}{rcl} \mathbb{D} & \to & \mathbb{D} \\ z & \mapsto & e^{i\theta} \cdot \frac{z-a}{1-\overline{a}z} \end{array}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

Examples.

(1) In Figure 359 we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense.

---

[462]Note that $c = 0$ implies that $1 = ad - bc = ad$, in particular $ad > 0$, hence $\frac{a}{d} > 0$.

[463]The hyperbolic disk is often also called the *Poincaré model for hyperbolic space*.

[464]An elementary calculation shows that the image of a Möbius transformation $\mathbb{D} \to \mathbb{C}$ does indeed lie again in $\mathbb{D}$.

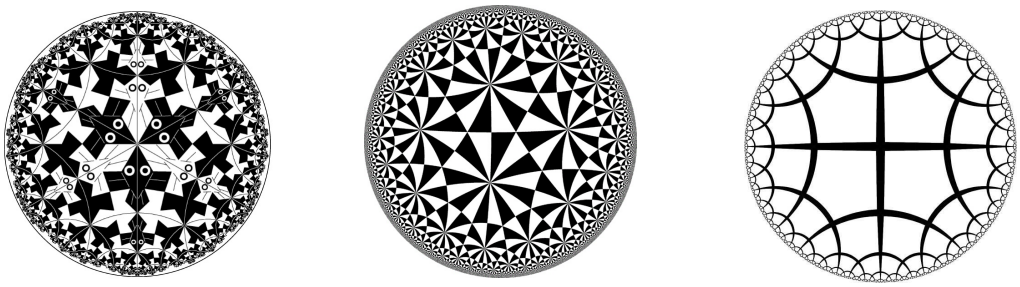


FIGURE 359.

(2) If we set $a = 0$ in the definition of a Möbius transformation of $\mathbb{D}$, then we obtain precisely the rotation by the angle θ. Put differently, the Möbius transformations preserving 0 are precisely given by rotations around the origin.

Before we can state the next proposition we give a few more definitions regarding maps between (Riemannian) manifolds.

Definition.

(1) Given a smooth map $f\colon M \to N$ between manifolds and given $P \in M$ we denote by $df_P = f_*\colon T_PM \to T_{f(P)}N$ the differential, i.e. the induced homomorphism from the tangent space T_PM to the tangent space $T_{f(P)}N$.
(2) Let $f\colon M \to N$ be a local diffeomorphism. Given a Riemannian structure h on N we denote by f^*h the Riemannian structure on M that at each point $P \in M$ is given by $(f^*h)(v, w) = h(f_*v, f_*w)$.[465]
(3) We say that a map $f\colon (M, g) \to (N, h)$ between Riemannian manifolds is a (*local*) *isometry* if f is a (local) diffeomorphism and if $g = f^*h$.
(4) We say that Riemannian manifolds (M, g) and (N, h) are *isometric* if there exists an isometry from (M, g) to (N, h).

The following proposition will allow us to go back and forth between the hyperbolic half-plane and the hyperbolic disk.

Proposition 22.2. *The maps*

$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-i}{z+i} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{i+iw}{1-w} \end{array}$$

have the following properties:

(1) *Φ and Ψ are inverses to one another, in particular they are biholomorphisms,*
(2) *they extend to homeomorphisms between $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$*[466] *and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$,*

[465] It follows from [Le2, Proposition 13.9] that f^*h is indeed a Riemannian structure on M. Note that in the proof, to show that f^*h is indeed non-degenerate, one needs that f is a local diffeomorphism.

[466] Here $\overline{\mathbb{H}}$ has the same topology as defined on page 53.

(3) *they are orientation-preserving,*[467]
(4) *they are isometries,*
(5) *they give a bijection between the Möbius transformations of* $\mathbb{D}$ *and* $\mathbb{H}$.[468]

Proof. All five statements follow from a straightforward calculation. The statements can also be deduced, more or less directly, from any book that deals with Möbius transformations, see e.g. [Schwe, Chapter 6] and [And, Chapter 2]. □

Remark. Proposition 22.2 says that $\mathbb{D}$ and $\mathbb{H}$ are two equivalent ways to describe hyperbolic space. In fact there are several other ways to describe hyperbolic space, for example we could also use the Beltrami-Klein model

`https://en.wikipedia.org/wiki/Beltrami-Klein_model`

or the hyperboloid model

`https://en.wikipedia.org/wiki/Hyperboloid_model`.

Some of the connections between these models are explained in the following video of Henry Segerman and Saul Schleimer:

`https://www.youtube.com/watch?v=eGEQ_UuQtYs`.

The following proposition summarizes some of the key properties of Möbius transformations.

Proposition 22.3. [469]

(1) *Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.*
(2) *Möbius transformations are biholomorphisms, they are isometries and they are orientation-preserving.*
(3) *Möbius transformations act transitively, i.e. given any two points* P *and* Q *there exists a Möbius transformation which sends* P *to* Q.
(4) *Given* $P, Q \in \mathbb{D}$ *and non-zero vectors* $v \in T_P\mathbb{D}$ *and* $w \in T_Q\mathbb{D}$ *with* $\|v\|_g = \|w\|_g$ *there exists a Möbius transformation* ϕ *with* $\phi(P) = Q$ *and* $d\phi_P(v) = w$.
(5) *Let* ϕ *and* ψ *be two Möbius transformations. Suppose there exists a* $P \in \mathbb{D}$ *such that* $\phi(P) = \psi(P)$ *and such that* $d\phi_P = d\psi_P$. *Then* $\phi = \psi$.

Proof. By Proposition 22.2 it suffices to prove the statements for $\mathbb{D}$ or for $\mathbb{H}$, whichever is more convenient.

(1),(2) The first two statements follow from a straightforward calculation, see e.g. [Schwe, p. 43].

[467]The hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disk $\mathbb{D}$ are both subsets of $\mathbb{C} = \mathbb{R}^2$ and inherit from $\mathbb{C} = \mathbb{R}^2$ the canonical orientation.

[468]More precisely, if α is a Möbius transformation of $\mathbb{H}$, then $\Phi \circ \alpha \circ \Psi$ is a Möbius transformation of $\mathbb{D}$ and conversely, if α is a Möbius transformation of $\mathbb{D}$, then $\Psi \circ \alpha \circ \Phi$ is a Möbius transformation of $\mathbb{H}$.

[469]The statement of the proposition applies to Möbius transformations of $\mathbb{D}$ and to Möbius transformations of $\mathbb{H}$.

(3) We prove this statement for $\mathbb{D}$. By (1) it suffices to show that given any $P \in \mathbb{D}$ there exists a Möbius transformation ψ with $\psi(P) = 0$. But such a Möbius transformation is given by
$$\psi(z) \;=\; \tfrac{z-P}{1-\overline{P}z}.$$
(4) Using (3) we see that we only have to deal with the case that $P = Q = 0$. But then the desired Möbius transformation is given by a rotation.
(5) By (3) there exists a Möbius transformation θ such that $\theta(0) = P$ and there exists a Möbius transformation ω such that $\omega(\phi(P)) = \omega(\psi(P)) = 0$. We consider the Möbius transformations $\phi' = \omega \circ \phi \circ \theta$ and $\psi' = \omega \circ \psi \circ \theta$. It follows from our hypothesis that both ϕ' and ψ' fix 0. As we pointed out above, the only Möbius transformations of $\mathbb{D}$ that fix 0 are rotations.

By our hypothesis we have $d\phi'_0 = d\psi'_0$, which means that the rotations ϕ' and ψ' agree. But this implies that ϕ and ψ also agree. □

Note that Möbius transformations extend in an obvious way uniquely to homeomorphisms of $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. We will denote the extension of a Möbius transformation with the same symbol.

Lemma 22.4. *Let $\{P, Q, R\}$ and let $\{S, T, U\}$ be two sets of three distinct points on the boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Then there exists a Möbius transformation ψ of $\mathbb{H}$ with $\psi(P) = S$ and $\psi(\{Q, R\}) = \{T, U\}$.*

Proof. Let P, Q, R be three distinct points on $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. It suffices to show that there exists a Möbius transformation ψ with $\psi(P) = \infty$ and with $\psi(\{Q, R\}) = \{0, 1\}$. We proceed as follows:

(1) We first claim that there exists a Möbius transformation α with $\alpha(P) = \infty$. Indeed, if $P = \infty$ then we just take $\alpha = \mathrm{id}$. Otherwise we can combine a horizontal translation with the inversion to obtain a Möbius transformation α with $\alpha(P) = \infty$. Given such α we write $Q' = \alpha(Q)$ and $R' = \alpha(R)$. Note that the points Q', R' lie on $\mathbb{R} = \overline{\mathbb{H}} \setminus \{\infty\}$. Without loss of generality we can assume that $Q' < R'$.
(2) Next we pick a horizontal translation such that $\beta(Q') = 0$. Note that we still have $(\beta \circ \alpha)(P) = \beta(\infty) = \infty$. Also note that it follows from $Q' < R'$ that $\beta(R') > 0$.
(3) Finally we pick a scalar multiplication γ such that $(\gamma \circ \beta)(R') = 1$. Note that γ fixes 0 and ∞. Thus we see that $\psi = \gamma \circ \beta \circ \alpha$ has the desired property. □

Definition. A *hyperbolic line* is a subset of $\mathbb{D}$ of one of the following two types:

(1) it is the intersection of $\mathbb{D}$ with a Euclidean line through the origin,
(2) it is the intersection of $\mathbb{D}$ with a Euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the Euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the *endpoints* of the hyperbolic line. We refer to Figure 360 for an illustration.

Lemma 22.5.

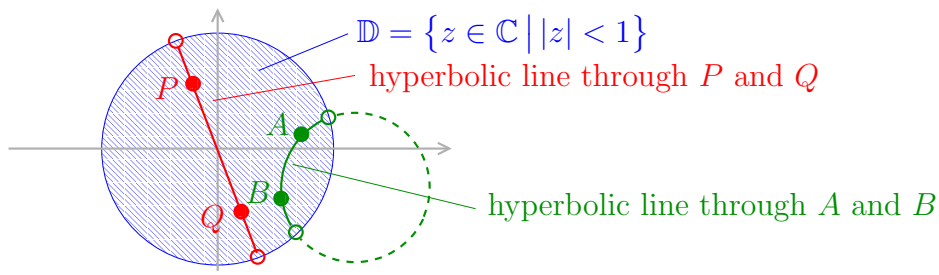


FIGURE 360.

(1) *For any two distinct points P and Q in $\mathbb{D}$ there exists a unique hyperbolic line that contains P and Q.*
(2) *For any two distinct points P and Q in $\partial\overline{\mathbb{D}}$ there exists a unique hyperbolic line with P and Q as endpoints.*
(3) *Let P and Q be two distinct points in $\partial\overline{\mathbb{D}}$. Every Möbius transformation ϕ sends the hyperbolic line with endpoints P and Q to the hyperbolic line with endpoints $\phi(P)$ and $\phi(Q)$.*
(4) *Möbius transformations act transitively on hyperbolic lines, i.e. given any two hyperbolic lines g and h there exists a Möbius transformation ϕ with $\phi(g) = h$.*

Proof. In principle one can prove the lemma directly in $\mathbb{D}$. But it is easier to prove it using $\mathbb{H}$. In $\mathbb{H}$ the hyperbolic lines are defined to be vertical lines $\{x + iy \mid y > 0\}$ and Euclidean half-circles such that the origin lies on $\mathbb{R}$. It is straightforward to see that the diffeomorphisms of Proposition 22.2 define a bijection between the hyperbolic lines in $\mathbb{D}$ and in $\mathbb{H}$.

The corresponding statements for hyperbolic lines in $\mathbb{H}$ are straightforward to prove. They can also be obtained from any textbook on hyperbolic geometry, see e.g. [And] and [Bon, Chapter 2]. □

22.2. **Angles in Riemannian manifolds.** We continue our quick introduction to Riemannian manifolds and hyperbolic space with a short discussion of angles.

Definition.

(1) Let g be a positive-definite symmetric form on a real vector space V. Given $v, w \in V \setminus \{0\}$ we define[470]

$$\text{angle between } v \text{ and } w \ := \ \text{unique } \alpha \in [0, \pi] \text{ with } g(v, w) = \cos(\alpha) \cdot \|v\| \cdot \|w\|.$$

(2) Given a Riemannian manifold (M, g) and non-zero vectors $v, w \in T_P M$ we use the form g_P to define the angle between v and w.

Examples.

[470]We have $|g(v, w)| \leq \big|\|v\| \dot{\|} w\|\big|$ by the Cauchy-Schwarz inequality.

(1) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}_{>0}$ be a smooth function. For any $P \in U$ we can define a positive-definite symmetric form g_P on $T_PU = \mathbb{R}^n$ via
$$g_P(v,w) \;=\; f(P)\cdot\langle v,w\rangle,$$
where $\langle\,,\,\rangle$ denotes the usual scalar product. A straightforward calculation shows that in this Riemannian structure the angle at any point is the usual Euclidean angle in $\mathbb{R}^n$. This argument implies in particular to the Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$, in both cases the hyperbolic angle, defined by the hyperbolic Riemannian structure, is the same as the Euclidean angle.

(2) A hyperbolic triangle is defined to be a compact subset of $\mathbb{H}$ whose boundary is given by segments of three hyperbolic lines. One of the many elegant theorems in hyperbolic geometry, see [And, Theorem 5.16], says that given such a hyperbolic triangle its angle sum equals π minus its "area". Even without the definition of "area" it is pretty clear that this result implies that the angle sum of a hyperbolic triangle is always less than π. This statement is illustrated in Figure 361.

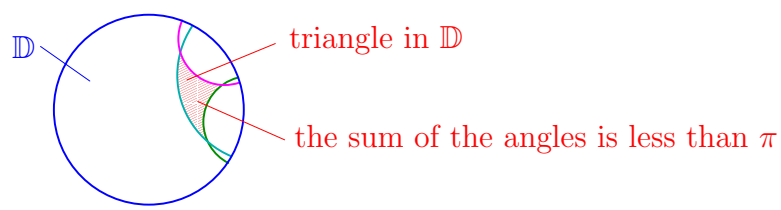


FIGURE 361.

The following lemma follows easily from the definitions and from Proposition 22.3 (2).

Lemma 22.6. *Local isometries, in particular Möbius transformations, preserve angles.*

Definition.

(1) Let $P \in \mathbb{D}$ and let g be a hyperbolic line through P. Let Q be one of the two endpoints of g. We refer to the set of points between P and Q as a *ray emanating from P*.

(2) Let $P \in \mathbb{D}$ and let r, s be two rays emanating from P. We refer to the angle between the tangent vectors of r and s at P as the *angle between the rays r and s*.

Examples.

(1) Given $\varphi \in [0, 2\pi)$ the set $S_\varphi := \{re^{i\varphi} \mid r \in [0,1)\}$ is a ray emanating from 0.

(2) It follows from the above discussion of angles in $\mathbb{D}$, that given $\varphi, \theta \in [0, 2\pi)$ the angle between the two rays S_φ and S_θ in $\mathbb{D}$ is the usual Euclidean angle.

Lemma 22.7. *Let r, s be two rays emanating from $P \in \mathbb{D}$ and let r', s' be two rays emanating from $P' \in \mathbb{D}$. Then the following holds:*
$$\begin{array}{c}\textit{there is a Möbius transformation } \phi \\ \textit{with } \phi(r\cup s) = r'\cup s'\end{array} \quad\Longleftrightarrow\quad \textit{angle between } r,s \;=\; \textit{angle between } r',s'.$$

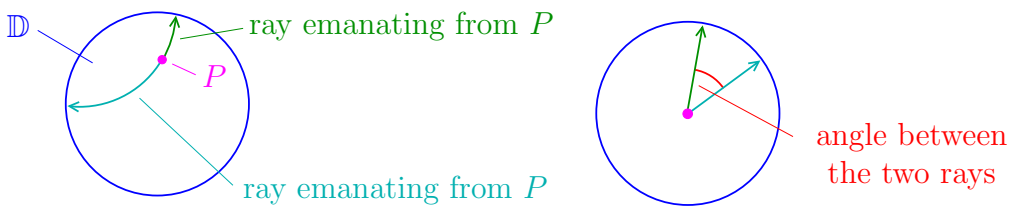


FIGURE 362.

Furthermore, if such a Möbius transformation exists, then it is unique.

Proof. By Lemma 22.6 Möbius transformations preserve angles and by Proposition 22.3 Möbius transformations act transitively on $\mathbb{D}$. Therefore we can without loss of generality assume that $P = P' = 0$. The lemma follows from the observation, made on page 589, that Möbius transformations preserving 0 are precisely given by rotations around the origin. $\square$

22.3. **The distance metric of a Riemannian manifold.**

Definition. Let (M, g) be a connected Riemannian manifold.

(1) Given $v \in T_P M$ we write $\|v\|_g = \sqrt{g_P(v, v)}$. If g is understood from the context, then we just write $\|v\|$.
(2) Let $\gamma\colon [a, b] \to M$ be a piecewise smooth path. By definition we can pick a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0, \dots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth. We define the *length of* γ as[471]
$$\ell_{(M,g)}(\gamma) \ := \ \sum_{i=0}^{m-1} \int_{t=s_i}^{t=s_{i+1}} \|\gamma'(t)\|_g \, dt.$$
If (M, g) is understood from the context, then we write $\ell(\gamma)$ instead of $\ell_{(M,g)}(\gamma)$.
(3) Let P and Q be two points on M.
$$d_{(M,g)}(P, Q) \ := \ \inf\{\ell(\gamma) \mid \underbrace{\gamma \text{ is a smooth path in } M \text{ from } P \text{ to } Q}_{\text{exists by Lemma 10.24}}\}.$$
If (M, g) is understood from the context, then we just write $d(P, Q)$.

We start our discussion of the distance function d on Riemannian manifolds with the following somewhat technical proposition.

Proposition 22.8. *Let (M, g) be a connected Riemannian manifold. Then the following hold:*

(1) *the pair $(M, d_{(M,g)})$ is a metric space,*
(2) *the topology defined by the metric $d_{(M,g)}$ agrees with the topology of M,*

[471]Recall that given a smooth path $\delta\colon [a, b] \to M$ and given any $t \in [a, b]$ the path defines, basically by definition of the tangent space $T_{\delta(t)}M$, an element in $T_{\delta(t)}M$. We denote this vector by $\delta'(t)$.

(3) *given any $P \in M$ and given any open neighborhood U of P there exists an $r > 0$ such that*

$$B_r^{(M,g)}(P) \;:=\; \{Q \in M \mid d_{(M,g)}(P,Q) < r\} \;\subset\; U.$$

Proof. Let (M,g) be a connected n-dimensional Riemannian manifold. We first show that $(M, d_{(M,g)})$ is a metric space.

It is clear that $d = d_{(M,g)}$ is symmetric. It follows easily from the definitions and the fact that the concatenation of two piecewise smooth paths is again piecewise smooth that d satisfies the triangle inequality.

Now let $P \neq Q$ be two distinct points in M. We need to show that $d(P,Q) > 0$. Since M is Hausdorff and since it is o-locally diffeomorphic to open subsets of $\mathbb{R}^n$ we can find a chart $\Phi\colon U \to B_{2r}(0)$ for P with $\Phi(P) = 0$ and such that $Q \notin U$. We write $\Psi = \Phi^{-1}$. Before we continue we introduce the following definitions:

(1) We write $U_r = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| = r\})$ and given a subset $I \subset [0, 2r)$ we write $U_I = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| \in I\})$.

(2) For a positive-definite bilinear form h on $\mathbb{R}^n$ we set

$$\|h\| \;:=\; \min\Big\{\sqrt{h(v,v)}\;\Big|\; v \in S^{n-1}\Big\}.$$

Note that h is positive-definite, that $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that S^{n-1} is compact. It follows from Lemma 1.37 that the minimum exists and that $\|h\| > 0$. It is straightforward to see that for any $v \in \mathbb{R}^n$ we have

$$h(v,v) \;\geq\; \|h\| \cdot \|v\|_{\text{eucl}}.$$

(3) We set

$$C \;:=\; \min\Big\{\|(\Psi^* g)_P\|\;\Big|\; P \in \overline{B_r(0)}\Big\}.$$

Since $P \mapsto \|(\Psi^* g)_P\|$ is continuous and since $\overline{B_r(0)}$ is compact it follows from Lemma 1.37 that $C > 0$.

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. There exists a $t \in [0,1]$ with $\gamma(t) \in U_r$.

Suppose there is no $t \in (0,1)$ with $\gamma(t) \in U_r$. We consider the sets $A = U_{[0,r)}$ and $B = M \setminus U_{[0,r]}$. The set A is clearly open. Furthermore it follows from Lemma 3.9 that $U_{[0,r]}$ is closed, hence B is open. Since $\gamma([0,1]) \cap U_r = \varnothing$ we have $[0,1] = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$. But both sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open by the continuity of γ and both are non-empty since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. But this contradicts the fact, established with some effort in Proposition 1.86, that $[0,1]$ is connected. This concludes the proof of the claim.

Now it suffices to prove the following claim.

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. Then $\ell_g(\gamma) \geq C \cdot r$.

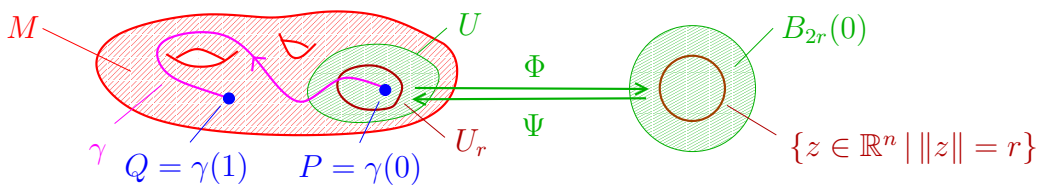


FIGURE 363. Illustration of the proof of Proposition 22.8

Let $T := \inf\{t \in (0,1) \mid \gamma(t) \in U_r\}$. By the above claim this definition makes sense since we take the infimum of a non-empty set. By Lemma 3.9 the set U_r is a closed subset of M. Since γ is continuous we also have $\gamma(T) \in U_r$. Now we have

$$\begin{aligned}
\ell_g(\gamma) \;\geq\; \ell_g(\gamma|_{[0,T]}) \;&=\; \ell_{\Psi^* g}(\Phi \circ \gamma|_{[0,T]}) &&=\; \int_{t=0}^{t=T} \|(\Phi\circ\gamma)'(t)\|_{\Psi^* g}\, dt \\
&\underset{\substack{\uparrow \\ \text{for any } v \in T_P\overline{B_r(0)} = \mathbb{R}^n \text{ we have} \\ \|v\|_{(\Psi^* g)_P} \;\geq\; \|(\Psi^* g)_P\| \cdot \|v\|_{\text{eucl}}}}{\geq} \int_{t=0}^{t=T} C \cdot \|(\Phi\circ\gamma)'(t)\|_{\text{eucl}}\, dt &&\underset{\substack{\uparrow \\ \text{[Rudi, Theorem 6.13]}}}{\geq}\; C \cdot \Big\| \int_{t=0}^{t=T} (\Phi\circ\gamma)'(t)\, dt \Big\|_{\text{eucl}} \\
&=\; C \cdot \|\underbrace{(\Phi\circ\gamma)(T)}_{\|-\|_{\text{eucl}}=r} - \underbrace{(\Phi\circ\gamma(0))}_{=0}\|_{\text{eucl}} \;=\; C \cdot r.
\end{aligned}$$

This concludes the proof of the claim and therefore the proof of statement (1) of the proposition.

Next we turn to the proof of statement (2). This means that we need to show that the topology defined by d is the same as the original topology of M. The proof is again somewhat technical and uses ideas similar to the proof of (1). We refer to [Le1, Lemma 6.2] for the proof.

Finally statement (3) is just an immediate consequence of the definitions and of statement (2). □

Lemma 22.9. *Let (M,g) and (N,h) be two connected Riemannian manifolds. Furthermore let $f\colon (M,g) \to (N,h)$ be a local isometry. Then the following hold:*

(1) *For any piecewise smooth path $\gamma\colon [a,b] \to M$ we have*
$$\ell_{(N,h)}(f\circ\gamma) \;=\; \ell_{(M,g)}(\gamma).$$

(2) *For any $P, Q \in M$ we have*
$$d_{(N,h)}(f(P), f(Q)) \;\leq\; d_{(M,g)}(P,Q).$$

Proof. Let $f\colon (M,g) \to (N,h)$ be a local isometry. We start out with the following claim.

Claim.

(1) For a smooth path $\gamma\colon [c,d] \to M$ and $t \in (c,d)$ we have $\|(f\circ\gamma)'(t)\|_{(N,h)} = \|\gamma'(t)\|_{(M,g)}$.

(2) For any piecewise smooth path $\gamma\colon [a,b] \to M$ we have $\ell_{(N,h)}(f \circ \gamma) = \ell_{(M,g)}(\gamma)$.

Now we provide the proof of the two statements.

(1) The first statement is an immediate consequence of the definitions and the hypothesis that f is a local isometry.
(2) By the additivity of the lengths of piecewise smooth paths it suffices to consider the case that $\gamma\colon [a,b] \to M$ is a smooth path. But in that case the equality follows immediately from the definitions and from (1).

This concludes the proof of the claim.

Now let P and Q be two points on M. For any piecewise smooth path p in M from P to Q we have

$$\begin{array}{ccccc} d_{(N,h)}(f(P), f(Q)) & \underset{\substack{\uparrow \\ \text{by definition of } d_{(N,h)}}}{\leq} & \ell_{(N,h)}(f \circ p) & \underset{\substack{\uparrow \\ \text{by the claim}}}{=} & \ell_{(M,g)}(p). \end{array}$$

It follows that the left-hand side is also less or equal than the infimum over all paths on the right-hand side. Thus $d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P, Q)$. □

22.4. **The hyperbolic distance function.** We start out with the following definition.

Definition. Let (M, g) be a Riemannian manifold. A path $\gamma\colon [0,a] \to M$ is called a *geodesic* if the following conditions are satisfied:

(1) γ is smooth,
(2) $\|\gamma'(t)\|_g = 1$ for every $t \in [0,a]$ and
(3) $\ell(\gamma) = d_{(M,g)}(\gamma(0), \gamma(a))$.

The following proposition summarizes the key facts about the distance metric on $\mathbb{D}$.

Proposition 22.10.

(1) *Let $r \in (0,1)$ and $\varphi \in \mathbb{R}$. In the Riemannian manifold $\mathbb{D}$ the path*

$$\begin{array}{rcl} \gamma\colon [0,r] & \to & \mathbb{D} \\ t & \mapsto & te^{i\varphi} \end{array}$$

is, up to reparametrization, the unique shortest path from 0 to $re^{i\varphi}$.

(2) *We have*

$$d_{(\mathbb{D},g)}\big(0, re^{i\varphi}\big) \;=\; \ell(\gamma) \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-|\gamma(t)|^2} \|\gamma'(t)\|_{\text{eucl}}\, dt \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-t^2}\, dt.$$

(3) *Given $\varphi \in \mathbb{R}$ we have*

$$\lim_{r \to 1} d_{(\mathbb{D},g)}\big(0, re^{i\varphi}\big) \;=\; \infty.$$

(4) *Given any P and Q in $\mathbb{D}$ there is a unique geodesic from P to Q.*

Proof. The first statement can be deduced easily from [Bon, Lemma 2.4 and Proposition 2.22]. The second statement follows immediately from the first statement, and the third statement is a straightforward consequence of the second statement. For the fourth statement, note that it suffices to prove it for $P = 0$. In that case it is a straightforward consequence of (1). $\square$

Lemma 22.11. *Let (P, Q) and (S, T) be two pairs of distinct points on $\mathbb{D}$ such that $d(P, Q) = d(S, T)$. Then there exists a unique Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.*

Example. Let $r \in (0, 1)$, $\varphi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$ and $\gamma \in (0, \pi)$. We consider the pairs of points $P = re^{i\varphi}, Q = re^{i(\varphi+\gamma)}$ and $S = re^{i\psi}, T = re^{i(\psi-\gamma)}$. We then have

$$d(S, T) \;=\; d(T, S) \;=\; d(re^{i(\psi-\gamma)}, re^{i\psi}) \;\underset{\substack{\uparrow \\ \text{since rotation by } \varphi+\gamma-\psi \text{ is an isometry}}}{=}\; d(re^{i\varphi}, re^{i(\varphi+\gamma)}).$$

By Lemma 22.11 there exists a Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$. This Möbius transformation is illustrated in Figure 364. Recall that from Proposition 22.3 and Lemma 22.5 we know that Möbius transformations preserve hyperbolic lines and that they are orientation-preserving. In particular ϕ sends the hyperbolic line through P and Q to the hyperbolic line through S and T. Since ϕ is orientation-preserving it "swaps" the two sides of the hyperbolic line. We refer to Figure 364 for an illustration.

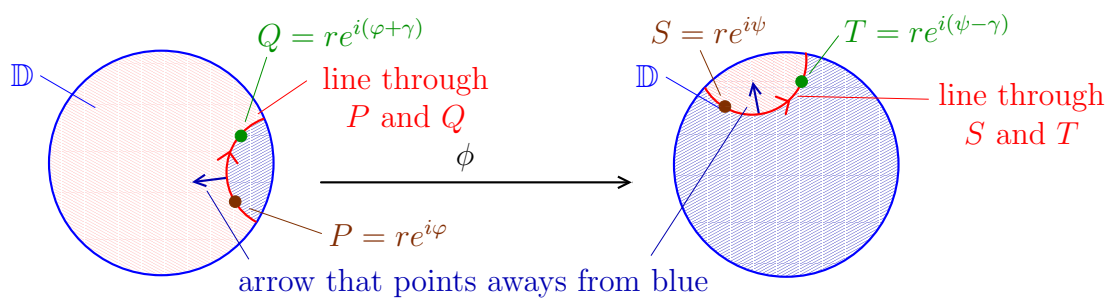


FIGURE 364.

Proof of Lemma 22.11. It follows from Proposition 22.3 (3) that, without loss of generality, we can assume that $P = 0$ and $S = 0$. We write $Q = re^{i\varphi}$ and $T = se^{i\psi}$. It is a direct consequence of Proposition 22.10 (1) and our hypothesis that

$$d(0, re^{i\varphi}) \;=\; d(P, Q) \;=\; d(S, T) \;=\; d(0, se^{i\psi})$$

that we have $r = s$. But this means that we can apply the rotation by the angle $\varphi - \psi$, which is a Möbius transformation, to turn (P, Q) into (S, T). This proves the existence of ϕ. The uniqueness follows from the observation, that we had made already on page 589, that the only Möbius transformations that fix 0 are precisely rotations. $\square$

Lemma 22.12. *Let $P \in \mathbb{D}$ and $r > 0$. Let $S, T \in B_r(P)$. Then the image of the unique geodesic in $\mathbb{D}$ from S to T lies in $B_r(P)$.*

Proof. As usual, this time by Propositions 22.3, 22.5 and 22.10, it suffices to prove the lemma for $P = 0$. In that case it follows from an elementary argument. We leave the details of the proof as an exercise. We also refer to Figure 365 for an illustration. □

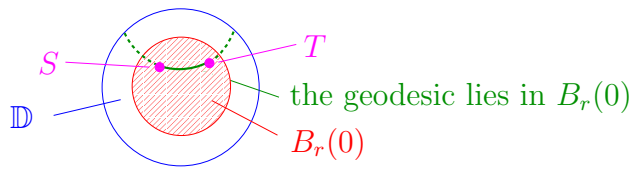


FIGURE 365.

22.5. **Complete metric spaces.** We conclude this chapter with a short discussion of complete metric spaces.

Definition.

(1) We say that a metric space (X, d) is *complete* if every Cauchy sequence in (X, d) converges.
(2) We say that a connected Riemannian manifold (M, g) is *complete* if the corresponding metric space $(M, d_{(M,g)})$ is complete.

Examples.

(1) The metric space given by $\mathbb{R}^n$ and the Euclidean metric is complete.
(2) The metric space given by the open ball B^n and the Euclidean metric is not complete. Indeed the sequence $a_n = 1 - \frac{1}{n}$ is a Cauchy sequence, but it does not converge in the open ball B^n. Similarly we see that for example the metric space $\mathbb{C} \setminus \{0, 1\}$ with the usual Euclidean metric is not complete.

The following proposition gives a useful criterion for a metric space to be complete.

Proposition 22.13. *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). We want to show that the sequence converges, i.e. we want to show that

$$\underset{x\in X}{\exists} \ \underset{\epsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{n\geq N}{\forall} \ d(a_n, x) < \epsilon.$$

Let us assume that no such x exists. Thus we negate the above statement and we obtain that

$$\underset{x\in X}{\forall} \ \underset{\epsilon>0}{\exists} \ \underset{N\in\mathbb{N}}{\forall} \ \underset{n\geq N}{\exists} \ d(a_n, x) \geq \epsilon.$$

Given $x \in X$ we pick $\epsilon_x > 0$ as above and we write

$$U_x \ := \ B_{\epsilon_x}(x) \quad \text{and} \quad V_x \ := \ B_{\epsilon_x/2}(x).$$

Note each U_x has the property that it does not contain all but finitely many of the a_n.

The family of open sets $\{V_x\}_{x\in X}$ covers X. Since X is compact there exist $x_1,\dots,x_k$ such that $X = V_{x_1}\cup\dots\cup V_{x_k}$. Now let $\epsilon := \frac{1}{2}\min\{\epsilon_{x_1},\dots,\epsilon_{x_k}\}$. For this $\epsilon>0$ we pick $N\in\mathbb{N}$ as in the definition of a Cauchy sequence. This means in particular that for all $m\geq N$ we have $d(a_N,a_m)<\epsilon$.

Since $X = V_{x_1}\cup\dots\cup V_{x_k}$ there exists an $i\in\{1,\dots,k\}$ with $a_N\in V_{x_i}$. It suffices to prove the following claim to obtain the desired contradiction.

Claim. For any $m\geq N$ we have $a_m\in U_{x_i}$.

Let $m\geq N$. We see that

$$\begin{array}{ccccccccc} d(x_i,a_m) & \underset{\uparrow}{\leq} & d(x_i,a_N)+d(a_N,a_m) & \underset{\uparrow}{<} & \frac{1}{2}\epsilon_{x_i}+\epsilon & \underset{\uparrow}{\leq} & \frac{1}{2}\epsilon_{x_i}+\frac{1}{2}\epsilon_{x_i} & = & \epsilon_{x_i}. \\ & \text{triangle inequality} & & \text{since } a_N\in V_{x_i} \text{ and } d(a_N,a_m)<\epsilon & & \text{since } \epsilon\leq\frac{1}{2}\epsilon_{x_i} & & & \end{array}$$

This implies that $a_m\in B_{\epsilon_{x_i}}(x_i) = U_{x_i}$. This concludes the proof of the claim. □

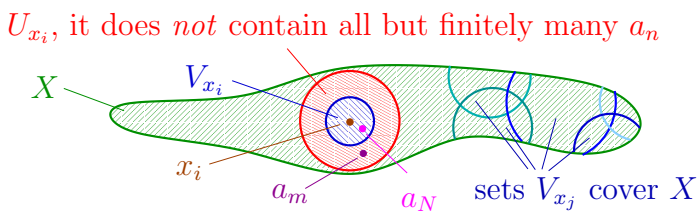


FIGURE 366. Illustration of the proof of Proposition 22.13.

The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are of course not compact, but nonetheless, as we will now see, we can use Proposition 22.13 to show that $\mathbb{D}$ and $\mathbb{H}$ are complete.

Lemma 22.14. *The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are complete.*

Proof. By Proposition 22.2 the two Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are isometric. Therefore it suffices to show that $\mathbb{D}$ is complete. We denote by $d = d_{(\mathbb{D},g)}$ the metric on $\mathbb{D}$ corresponding to the Riemannian structure. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. An elementary argument using the triangle inequality and the definition of a Cauchy sequence shows that there exists a $C\in\mathbb{R}$ such that $d(0,a_n)\leq C$ for all $n\in\mathbb{N}$.

It follows from Proposition 22.10 that there exists an $r\in(0,1)$ such that all a_n are contained in the closed Euclidean ball $\overline{B}_r^{eucl}(0)$. By the Heine-Borel Theorem 1.21 this ball is a compact subset of $\mathbb{R}^2$, viewed with the usual topology. It follows from Proposition 22.8 that $\overline{B}_r^{eucl}(0)$ is also compact in the topology defined by d. Now we see that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $(\overline{B}_r^{eucl}(0),d)$. It follows from Proposition 22.13 that the sequence converges. □

Exercises for Chapter 22.

Exercise 22.1. Does the manifold $B^2 \setminus \{0\}$ admit a complete Riemannian metric?

23. The universal covering of surfaces

Throughout this chapter we mean by a *surface* a connected 2-dimensional manifold. For example the surface of genus g, $g \geq 0$, but also $\mathbb{C}$, $\mathbb{C} \setminus \{0,1\}$ and the result of removing finitely many points from Σ_g are surfaces.

Definition. Let M be a surface. We say that a Riemannian structure g on M is *hyperbolic* if the following two conditions are satisfied:

(1) (M,g) is o-locally isometric to $\mathbb{D}$, i.e. if given any $P \in M$ there exists an isometry $\Phi\colon U \to V$ from an open neighborhood U of P to an open subset V of $\mathbb{D}$,
(2) (M,g) is complete.

If g is hyperbolic, then we refer to (M,g) as a hyperbolic surface.

Remark. By Proposition 22.13 the condition (2) of the definition is automatically satisfied if M is a closed surface.

In this chapter we will pursue the following three goals:

(1) we want to show that for any $g \geq 2$ the surface of genus g is a complex manifold,
(2) we intend to prove that for any $g \geq 2$ the surface of genus g is hyperbolic,
(3) given $g \geq 2$ we want to determine the universal covering of the surface of genus g.

23.1. **Hyperbolic surfaces.** In this section we will show that the surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic. In the subsequent subsection we will then consider more general types of surfaces.

Definition. Let M be a 2-dimensional topological manifold without boundary. A *Möbius structure* for M is a family of homeomorphisms $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ from open subsets of M to open subsets of $\mathbb{D}$ such that $\bigcup_{i\in I} U_i = M$ and such that for any $i,j \in I$ the transition map

$$\underbrace{\Phi_i(U_i \cap U_j)}_{\subset \mathbb{D}} \xrightarrow{\left(\Phi_i|_{U_i\cap U_j}\right)^{-1}} U_i \cap U_j \xrightarrow{\Phi_j|_{U_i\cap U_j}} \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a manifold together with a Möbius structure as a *Möbius manifold*.

Example.

(1) We will always view $\mathbb{D}$ as equipped with the Möbius structure that is given by the chart $\mathrm{id}\colon \mathbb{D} \to \mathbb{D}$.
(2) Any open subset of a Möbius manifold is also a Möbius manifold.

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a manifold is a complex manifold and we can use them to show that a manifold has a hyperbolic structure.

Lemma 23.1. *Let M be a surface and let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M. Then the following hold:*

(1) *The charts form a holomorphic atlas for M, in particular M is a complex 1-dimensional manifold.*
(2) (a) *The manifold M admits a unique Riemannian structure g such that all the charts in the atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ are isometries.*
(b) *If the Riemannian structure from* (a) *is complete, then (M, g) is hyperbolic.*

In the following we will always view a Möbius manifold as a complex 1-dimensional manifold and a Riemannian manifold with the structures coming from Lemma 23.1.

Proof. Let M be a surface and let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M.

(1) The first statement is an immediate consequence of the definitions and of the fact, proved in Proposition 22.3 (2), that Möbius transformations are biholomorphisms.
(2) (a) We denote by g the Riemannian structure on $\mathbb{D}$. We define a Riemannian structure on M as follows. Let $P \in M$. We pick a chart $\Phi_i\colon U_i \to V_i$ from our Möbius structure and we define $h_P := \Phi_i^*(g_{\Phi_i(P)})$. Now suppose that we had picked a different chart Φ_j from the Möbius structure. The transition map from Φ_i to Φ_j is by definition a Möbius transformation, which is an isometry by Proposition 22.3 (2). It follows easily that Φ_j gives rise to the same definition of h_P. It is now straightforward to verify that this defines a Riemannian structure on M and that it has the desired properties.
(b) This statement is an immediate consequence of (a) and of the definition of a hyperbolic surface. □

The following proposition gives in particular an affirmative answer to Question 21.7.

Proposition 23.2. *Let $g \geq 2$. Then the following hold:*
(1) *The surface of genus g admits a Möbius structure.*
(2) *The surface of genus g is a complex 1-dimensional manifold.*
(3) *The surface of genus g is hyperbolic.*

Proof. We prove the proposition for the case that $g = 2$. The general case is proved by almost the same argument.

We first show that the surface of genus 2 admits a Möbius structure. By [Bon, Lemma 5.3] (see also [Frie2, Lemma 21.5]) there exists an $r > 0$ such that the interior angle at any vertex of the hyperbolic octagon H_8 in $\mathbb{D}$, defined by $Q_k = re^{2\pi ik/16}$, $k = 1, 3, \ldots, 15$, equals $\frac{\pi}{4}$.[472] This hyperbolic octagon is illustrated in Figure 367.

Let $k \in \{0, 1, 4, 5\}$. By Lemma 22.11 and the example on page 598 there exists a unique Möbius transformation Φ_k with $\Phi_k(Q_{2k-1}) = Q_{2k+5}$ and $\Phi_k(Q_{2k+1}) = Q_{2k+3}$. The Möbius transformations Φ_0 and Φ_1 are illustrated in Figure 368.

[472]Here is a sketch for the proof: For $r \to 0$ the interior angle converges to the interior angle of a Euclidean regular octagon, namely $\frac{3\pi}{4}$, whereas for $r \to \infty$ the interior angle converges to 0. The interior angle changes continuously with r (that is sort of clear, except that it is somewhat painful to provide a rigorous proof), so by the Intermediate Value Theorem there exists an r such that the interior angle equals $\frac{\pi}{4}$.

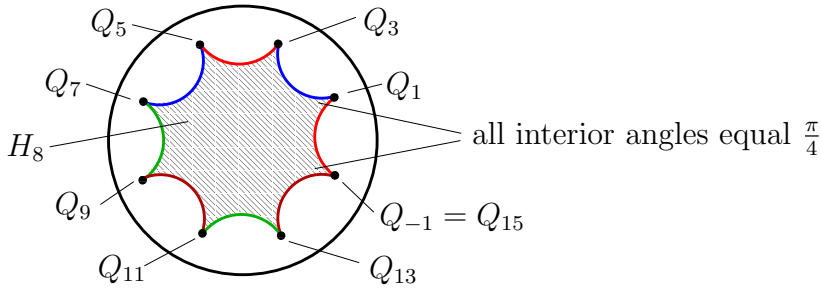


FIGURE 367.

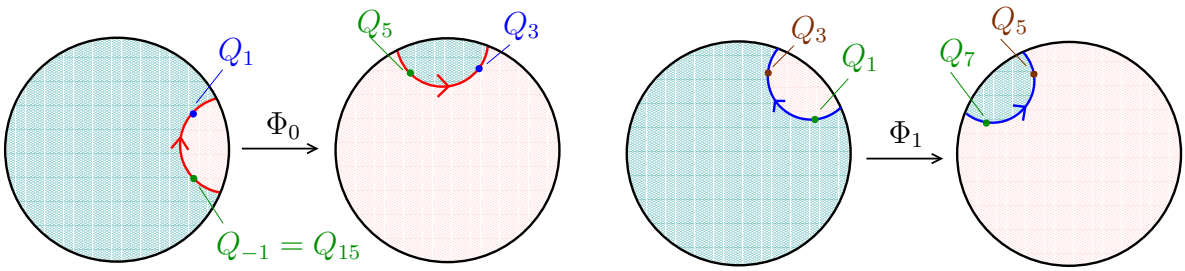


FIGURE 368.

Given $k \in \{0, 1, 4, 5\}$ we declare any point P on the edge connecting Q_{2k-1} and Q_{2k+1} to be equivalent to $\Phi_k(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences. It is straightforward to see that $H_8/\sim$ is homeomorphic to the surface of genus 2 that we defined on page 119 as $E_8/\sim$. In the following we denote by $p\colon H_8 \to H_8/\sim$ the obvious projection map.

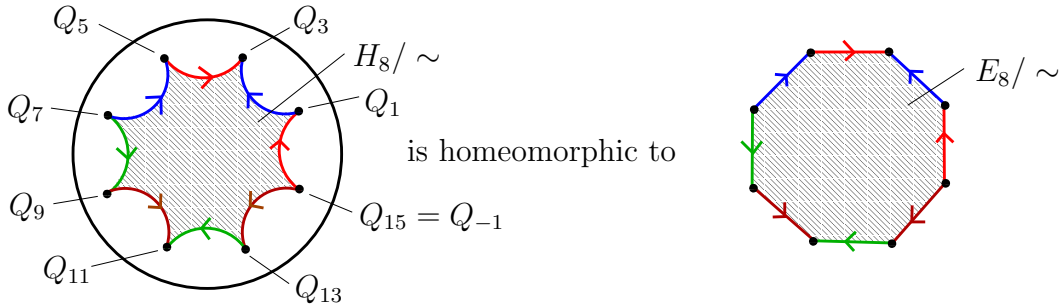


FIGURE 369.

Now we will give an explicit Möbius structure for $H_8/\sim$. More precisely, given any $P \in H_8/\sim$ we will give a chart ψ_P such that the transitions maps are given by Möbius transformations. So let $P \in H_8/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_8$, then the map

$$\begin{array}{rcl} \psi_P \colon p(\mathring{H}_8) & \to & \mathring{H}_8 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies in the interior of one of the eight edges of ∂H_8. In the following we deal with the case that z lies on the edge from Q_{-1} to Q_1. We pick an $r > 0$ such that $B_{2r}(z)$ does not contain any of the vertices $Q_1, Q_3, \ldots, Q_{15}$ and does not hit any of the other edges. Then we consider the map

$$\begin{array}{rcl} \psi_P \colon p(B_r(z) \cap H_8) \cup p(\Phi_0(B_r(z)) \cap H_8) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_8 \\ \Phi_0^{-1}(w), & \text{if } w \in \Phi_0(B_r(z)) \cap H_8. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$. We refer to Figure 370 for an illustration of the definition of the map ψ_P.

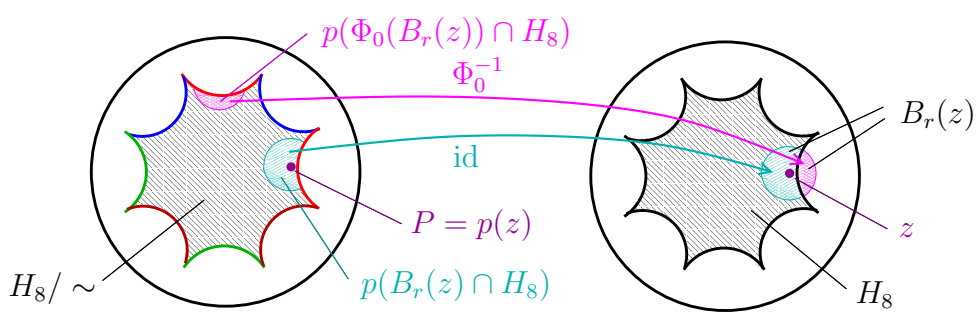


FIGURE 370.

(3) Finally suppose that $P = p(Q_1) = \cdots = p(Q_{15})$. We choose $r \in (0, \frac{1}{2}d(Q_{-1}, Q_1))$. Given $k \in \{1, 3, \ldots, 15\}$ we define $\gamma(k)$ via the following table:

k	1	3	5	7	9	11	13	15
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$.

Recall that we had chosen the octagon H_8 such that the angle at any vertex equals $\frac{\pi}{4}$. For $k = 1, 3, \ldots, 15$ we can therefore apply Lemma 22.7 to find a unique Möbius transformation θ_k with the following two properties:

(a) $\theta_k(Q_k) = 0$,

(b) θ_k applied to the two rays emanating from Q_k equals the two rays $\{se^{i\gamma(k)} \mid s \geq 0\}$ and $\{se^{i(\gamma(k)+\frac{\pi}{4})} \mid s \geq 0\}$ emanating from 0.

We consider the map

$$\begin{array}{rcl} \psi_P \colon p(B_r(Q_1) \cap H_8) \cup \cdots \cup p(B_r(Q_{15}) \cap H_8) & \to & B_r(0) \\ p(z) & \mapsto & \theta_k(z), \quad \text{if } z \in B_r(Q_k) \cap H_8. \end{array}$$

We leave it as an exercise to verify that ψ_P is well-defined and that it is indeed a homeomorphism. This map is a chart around $P = p(z)$. We refer to Figure 371 for an illustration of the definition of the map ψ_P.

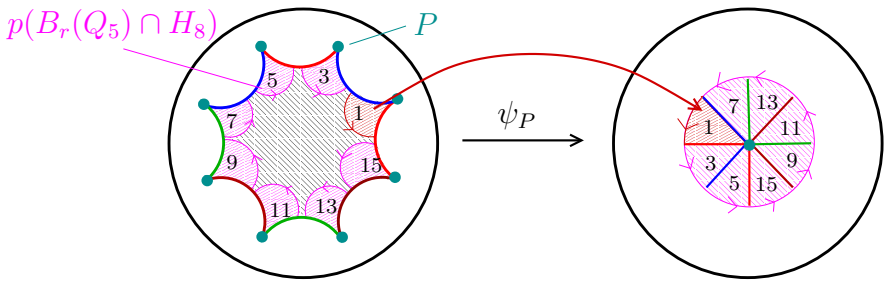


FIGURE 371.

It is straightforward to verify that all the transition maps between the above maps are given by Möbius transformations. We have thus shown that the maps $\{\psi_P\}_{P \in H_8}$ form a Möbius structure for $H_8/\sim$. This concludes the proof of (1).

Now we turn to the proof of (2). The existence of a Möbius structure together with Lemma 23.1 (2) shows immediately that the surface of genus 2 is a complex 1-dimensional manifold.

Finally we provide the proof of (3). In light of (1) and Lemma 23.1 (2) it suffices to show that the Riemannian manifold $(H_8/\sim, g)$ is complete. By Proposition 22.8 the topology on M defined by the metric $d_{(M,g)}$ agrees with the given topology of M. Since M is compact it now follows from Proposition 22.13 that (M, g) is complete. □

23.2. **More hyperbolic structures on the surfaces of genus $g \geq 2$ $(*)$.** Now we want to study the question, how many hyperbolic metrics does a surface of genus g admit. To make this question more precise, we introduce the following definition.

Definition. Let M be a surface of genus $g \geq 2$ and let h, h' be two hyperbolic metrics on M. We say h and h' are *equivalent* if there exists an isometry from (M, h) to (M, h').

This raises the question, whether a surface of genus $g \geq 2$ admits non-equivalent hyperbolic metrics. We return to the construction of a hyperbolic metric in Proposition 23.2. In our construction we started out with a regular $4g$-gon with interior angle $\frac{2\pi}{4g}$ and we identified pairs of sides. Two facts were crucial to obtain a hyperbolic metric:

(1) the sides were identified via an isometry,
(2) the sum of the interior angles equals 2π.

Or put differently, any choice of a $4g$-gon with side lengths $k_1, \ldots, k_{4g}$ and interior angles $\alpha_1, \ldots, \alpha_{4g}$ that satisfy the equations

$$k_1 = k_3,\ k_2 = k_4,\ \ldots,\ k_{4g-3} = k_{4g-1},\ k_{4g-2} = k_{4g} \quad \text{and} \quad \alpha_1 + \cdots + \alpha_{4g} = 2\pi$$

gives rise to a hyperbolic metric on a topological space that is homeomorphic to the surface of genus g. It is of course not clear whether different parameters give rise to non-equivalent hyperbolic structures. One way to distinguish them, in principle at least, is to look at the shortest length of a closed curve that is not null-homotopic. We will not elaborate on this procedure.

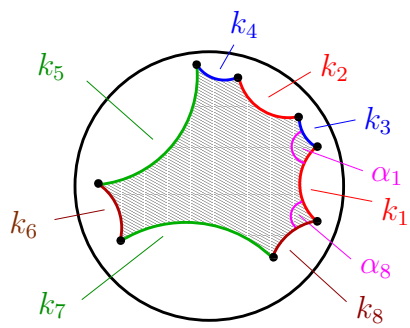


FIGURE 372.

How many such $4g$-gons are there? On the one hand we have parameters $k_1, k_3, \ldots, k_{4g-1}$ and $\alpha_1, \ldots, \alpha_{4g}$. On the other hand we have the condition that $\alpha_1 + \cdots + \alpha_{4g} = 2\pi$. Furthermore we want a $4g$-gon, so the last angle and the length of the last edge are determined by the others. Thus we obtain two other conditions. Summarizing we have $2g + 4g = 6g$ parameters that need to satisfy 3 conditions. Of course it is not clear which of these Riemannian structures are isometric. This naive discussion suggests that the space of hyperbolic metrics is $(6g-3)$-dimensional. Surprisingly this is not too far off the correct answer, in fact in [BP, Chapter B.4] the following proposition is proved.

Proposition 23.3. *The space of hyperbolic metrics on a surface of genus $g \geq 2$ can be viewed as a manifold of dimension $6g - 6$.*

We will not make use of this proposition. The space of hyperbolic metric is called the *Teichmüller space*, more information can be found at

`https://en.wikipedia.org/wiki/Teichmueller_space`.

The theory is named after Oswald Teichmüller (1913-1943) who was a brilliant mathematician and an ardent Nazi, see

`https://en.wikipedia.org/wiki/Oswald_Teichmueller`

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Teichmuller.html`

and

`https://de.wikipedia.org/wiki/Oswald_Teichmueller`.

23.3. **More examples of hyperbolic surfaces.** In this section we will generalize the discussion from the previous section to surfaces that are non-compact.

In the following let $Q_1 = 1, Q_2 = i, Q_3 = -1$ and $Q_0 = Q_4 = -i$. We denote by H_4 the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints Q_k, Q_{k+1} where $k = 0, 1, 2, 3$. We refer to Figure 374 for an illustration.

We consider the Möbius transformations[473]

$$\Phi_1(z) := (-i) \cdot \frac{z - \frac{-1+i}{2}}{1 - \frac{-1-i}{2} z} \quad \text{and} \quad \Phi_2(z) := (-i) \cdot \frac{z + \frac{-1-i}{2}}{1 + \frac{-1+i}{2} z}.$$

We record a few key properties of Φ_1 and Φ_2 in a lemma.

Lemma 23.4.

(1) *We have $\Phi_1(Q_2) = Q_4$ and $\Phi_1(Q_3) = Q_3$ and we have $\Phi_2(Q_2) = Q_4$ and $\Phi_1(Q_1) = Q_1$.*
(2) *The restriction of Φ_1 to the hyperbolic line with endpoints Q_3 and Q_2 is given by reflection in the x-axis.*[474]
(3) *The restriction of Φ_2 to the hyperbolic line with endpoints Q_1 and Q_2 is given by reflection in the x-axis.*

Proof. The first statement follows immediately from plugging in the points. A straightforward, albeit slightly painful calculation shows that for any $\alpha \in \mathbb{R}$ we have

$$\Phi_1\big(-1 + i + e^{-i\alpha}\big) = -1 - i + e^{i\alpha}.$$

This proves the second statement. The third statement is proved by a similarly tedious calculation. □

Remark. We consider the Möbius transformation Φ_1 in more detail. It sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. In particular it sends the two halves determined by the hyperbolic lines to one-another. Which half gets sent to which half is determined by the fact that Φ_1 is orientation-preserving. We refer to Figure 373 for an illustration of the Möbius transformations Φ_1 and Φ_2.

Given $k \in \{1, 2\}$ we declare any point P on the line with the endpoints Q_k and Q_{k+1} to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

Proposition 23.5.

(1) *The topological space $H_4/\sim$ admits a Möbius structure. In particular it is a 1-dimensional complex manifold.*

[473]Recall that a Möbius transformation of $\mathbb{D}$ is a map of the form $\psi(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

[474]The Möbius transformation Φ_1 sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. There are many such Möbius transformations, but the Möbius transformation we picked has the extra property, that on the hyperbolic line it is the "obvious map".

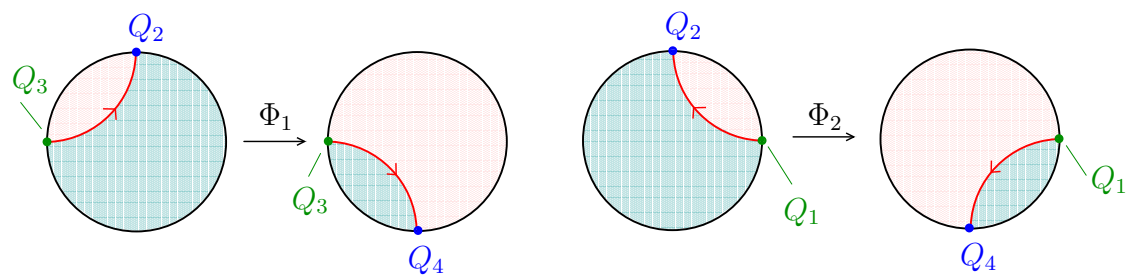


FIGURE 373.

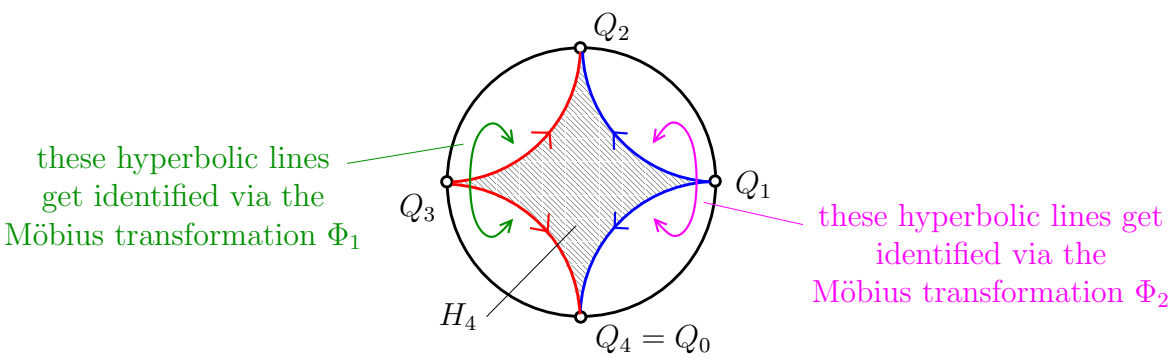


FIGURE 374.

(2) *The Riemannian structure on $H_4/\sim$ coming from* (1) *and Lemma 23.1 is complete.*
(3) *The topological space $H_4/\sim$ admits a hyperbolic metric.*

Proof. We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. The following argument is almost identical to the proof of Proposition 23.2 (1).

We denote by $p\colon H_4 \to H_4/\sim$ the projection map. In the following let $P \in H_4/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_4\big) & \to & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies on ∂H_4. In the following we deal with the case that z lies on the edge from Q_1 to Q_2. All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B_{2r}(z)$ intersects no other component of ∂H_4. We consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_4) \cup p(\Phi_2(B_r(z)) \cap H_4) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_1^{-1}(B_r(z)) \cap H_4. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps ψ_P form a Möbius structure for $H_4/\sim$. As in the proof of Proposition 23.2 it follows that $H_4/\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of Lemma 23.1. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 23.1 is complete. The subsequent argument is a variation of the proof of Proposition 22.13.

We denote by d the metric corresponding to the Riemannian structure on $H_4/\sim$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_4/\sim$ with respect to d. As in the proof of Proposition 22.13 we see that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

Claim. There exists an $r \in [0,1)$ such that for any $z \in H_4$ with $d(0, p(z)) \leq C$ we have $|z| \leq r$.

Given $s \in [0,1)$ we define

$$g(s) \;:=\; \int\limits_{t=0}^{t=s} \frac{\sqrt{2}}{1-t^2}\, dt.$$

Clearly we have $\lim\limits_{s\to 1} g(s) = \infty$. In particular there exists an $r \in [0,1)$ such that $g(s) > C+1$ for all $s \geq \frac{1}{2}r$.

We claim that r has the desired property. So let $z = x + iy \in H_4$ with $|z| > r$. Since H_4 is a subset of the square spanned by ± 1 and $\pm i$ we see that we have $|z| \leq |x| + |y|$. Put differently, we have $|x| \geq \frac{1}{2}r$ or $|y| \geq \frac{1}{2}r$. Without loss of generality we can assume that $|x| \geq \frac{1}{2}r$.

We need to show that $d(0, p(z)) \geq C + 1$. It suffices to show that for any piecewise smooth path $\gamma\colon [0,a] \to H_4/\sim$ with $\gamma(0) = 0$ and $\gamma(a) = p(z)$ we have $\ell(\gamma) \geq C + 1$.

We denote by

$$\begin{array}{rcl} q\colon H_4/\sim & \to & (-1,1) \\ [x+iy] & \mapsto & x \end{array}$$

the projection onto the real part. It follows from Lemma 23.4 that this map is indeed well-defined. A straightforward calculation shows that for all $t \in [0,a]$ we have

$$\big\|\gamma'(t)\big\|_{H_4/\sim} \;\geq\; \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}.$$

It follows that

$$\ell(\gamma) \;=\; \int\limits_0^a \big\|\gamma'(t)\big\|_{H_4/\sim}\, dt \;\underset{\substack{\uparrow \\ \text{above inequality}}}{\geq}\; \int\limits_0^a \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}\, dt \;\underset{\substack{\uparrow \\ \text{definition of } d_{(\mathbb{D},g)}}}{\geq}\; d_{(\mathbb{D},g)}(0, \overbrace{q(\gamma(a))}^{=x}) \;\underset{\substack{\uparrow \\ \text{Proposition 22.10}}}{=}\; g(|x|) \;\overset{\substack{\text{since } |x| \geq \frac{1}{2}r \\ \downarrow}}{\geq}\; C+1.$$

This concludes the proof of the claim.

By the above claim the sequence $\{a_n\}_{n\in\mathbb{N}}$ lies in the compact set $p(\{z \in H_4 \mid |z| \leq r\})$. But by Proposition 22.13 the metric space $p(\{z \in H_4 \mid |z| \leq r\})$ is complete, so the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges in $p(\{z \in H_4 \mid |z| \leq r\})$, in particular it converges in $p(H_4) = H_4/\sim$. Thus we have proved that $H_4/\sim$ is complete. This concludes the proof of (2).

Finally statement (3) is an immediate consequence of the definition of a hyperbolic metric and of statement (2). □

We had just seen in Proposition 23.5 that $H_4/\sim$ admits a Möbius structure, in particular it is a 1-dimensional complex manifold. It is relatively straightforward to show that $H_4/\sim$ is homeomorphic to the sphere minus three points, or equivalently, to $\mathbb{C}$ minus two points. Now we show the much stronger statement that $H_4/\sim$ is actually biholomorphic to $\mathbb{C} \setminus \{0, 1\}$.

Proposition 23.6. *There exists a biholomorphism* $H_4/\sim \to \mathbb{C} \setminus \{0, 1\}$.

Remark. We equip $H_4/\sim$ with the Riemannian metric g coming from the Möbius structure and we equip $\mathbb{C} \setminus \{0, 1\}$ with the usual Euclidean Riemannian metric that we denote by h.

By Proposition 23.5 the metric space $(H_4/\sim, g)$ is complete whereas $(\mathbb{C} \setminus \{0, 1\}, h)$ is evidently not complete. Thus we see that the biholomorphism $H_4/\sim \to \mathbb{C} \setminus \{0, 1\}$ from Proposition 23.6 is *not* an isometry.

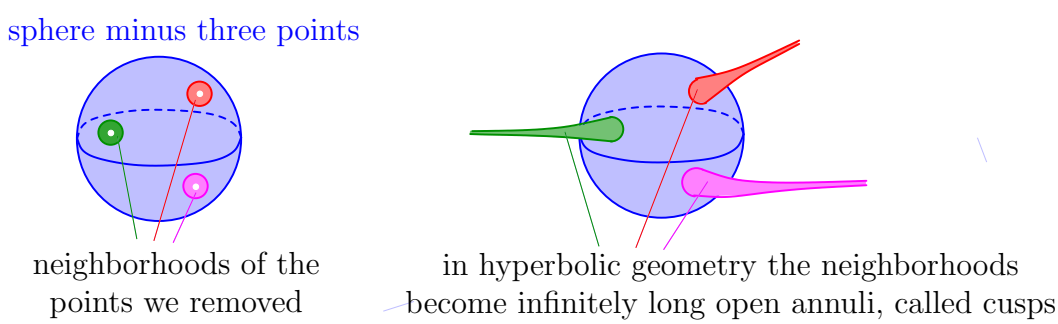


FIGURE 375.

Proof of Proposition 23.6. We denote by Δ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and i. (We refer to Figure 376 for an illustration of Δ.) It follows from applying the Riemann Mapping Theorem 21.10 together with Carathéodory's Theorem 21.11 that there exists a biholomorphism $\Psi\colon \Delta \to \mathbb{D}$ that extends to a homeomorphism $\Psi\colon \overline{\Delta} \to \overline{\mathbb{D}}$. We combine Ψ with the biholomorphism from $\mathbb{D}$ to $\mathbb{H}$ coming from Proposition 22.2. This way we obtain a biholomorphism $\Phi\colon \Delta \to \mathbb{H}$ that extends to a homeomorphism $\Phi\colon \overline{\Delta} \to \overline{\mathbb{H}}$.

Using Lemma 22.4 we can, if necessary, compose Φ with a Möbius transformation of $\mathbb{H}$, to arrange that $\Phi(i) = \infty$ and that $\Phi(\{-1, 1\}) = \{0, 1\}$.

Note that Φ restricts to a homeomorphism $\Phi\colon \partial\overline{\Delta} \to \partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. By the above we have $\Phi(\partial\overline{\Delta}) = \mathbb{R} \setminus \{0,1\}$. Now we consider the map

$$\begin{array}{rcl} \Psi\colon H_4 & \to & \mathbb{C} \setminus \{0,1\} \\ z & \mapsto & \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\overline{z})}, & \text{if } \overline{z} \in \overline{\Delta}, \end{cases} \end{array}$$

Note that by the above we have $f((-1,1)) = (0,1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 21.3, that the restriction of Ψ to the interior of H_4 is holomorphic. It follows from Lemma 23.4 that if P and Q are two equivalent points on ∂H_4, then $\Psi(P) = \Psi(Q)$. Therefore the map Ψ factors through a map $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$. Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step as an exercise to the reader. □

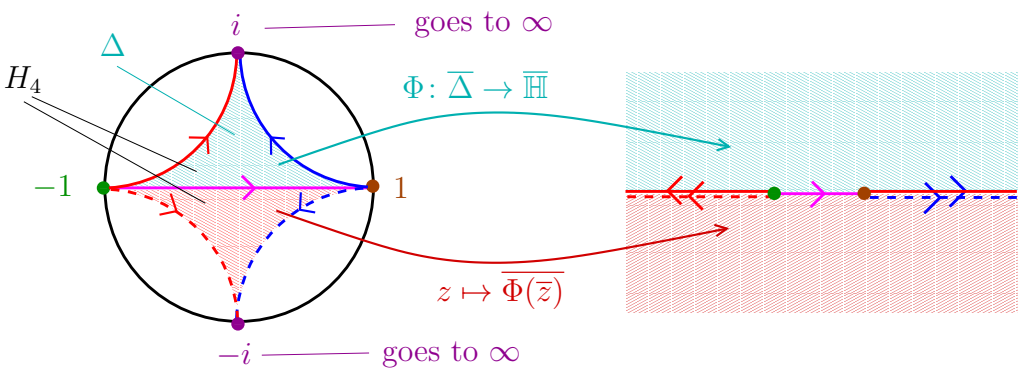


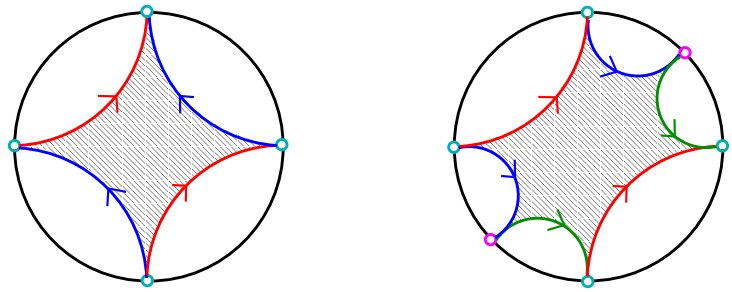
FIGURE 376. Illustration of the proof of Proposition 23.6

Definition. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we now denote by $F_{g,n}$ the surface that is obtained from the surface of genus g by removing n points. We refer to $F_{g,n}$ as the *n-punctured surface of genus g*.

For example the three-punctured sphere $F_{0,3}$ is the sphere with three points removed. As we pointed out above, $F_{0,3}$ is diffeomorphic to $\mathbb{C} \setminus \{0,1\}$. In particular, as we have just seen, the three-punctured sphere admits a hyperbolic structure. The following theorem says that "most" surfaces admit a hyperbolic structure.

Theorem 23.7. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g+n-2 > 0$, then $F_{g,n}$ admits a hyperbolic structure.*

Sketch of a proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g+n-2 > 0$ then variations on the argument of Propositions 23.2 and 23.5 show that $F_{g,n}$ admits a hyperbolic metric. In Figure 377 we sketch the constructions needed to show that the once-punctured and the twice-punctured torus have a Möbius structure. The proof that the corresponding Riemannian manifolds are complete is similar to the proof of Proposition 23.5. □



once punctured torus twice punctured torus

FIGURE 377.

23.4. **The universal covering of surfaces.**

Definition. Let $f\colon M \to N$ be a map between two Möbius manifolds.

(1) We say that f is a *Möbius map* if for any chart $\Phi\colon U \to V$ in the Möbius structure of M and any chart $\Psi\colon X \to Y$ in the Möbius structure of N the restriction of the map

$$\underbrace{\Phi(U \cap f^{-1}(X))}_{\subset \mathbb{D}} \xrightarrow{\Psi \circ f \circ \Phi^{-1}} \underbrace{\Psi(X \cap f(U))}_{\subset \mathbb{D}}$$

to any component of $\Phi(U \cap f^{-1}(X))$ is given by a Möbius transformation.

(2) We say that f is a *Möbius isomorphism*, if f is a *Möbius map*, if f is a bijection, and if f^{-1} is also a *Möbius map*.

The following lemma is an immediate consequence of the definitions and the fact, proved in Proposition 22.3, that Möbius transformations are biholomorphisms and isometries.

Lemma 23.8. *Let M and N be two Möbius manifolds and let $f\colon M \to N$ be a Möbius map. Then the following two statements hold:*

(1) *the map f is a local biholomorphism,*
(2) *the map f is a local isometry.*

The following theorem is the main result of this chapter.

Theorem 23.9. (Hadamard's Theorem) *Let $g \geq 2$, and let $M = H_{4g}/\sim$ be the surface of genus g with the Möbius structure constructed in Proposition 23.2. Then there exists a covering map $p\colon \mathbb{D} \to M$ with the following properties:*

(1) *p is a Möbius map,*
(2) *p is a local isometry,*
(3) *p is a local biholomorphism.*

The hyperbolic disk $\mathbb{D}$ is of course simply connected. The theorem thus says that for any $g \geq 2$ the hyperbolic disk $\mathbb{D}$ is the universal covering of the surface of genus g. Thus we have now answered Question 20.12.

The proof of Theorem 23.9 will require the following two sections.

23.5. **Proof of Theorem 23.9 I.** In this section we will lay the technical foundations for the proof of Theorem 23.9.

In the following lemma we first summarize a few properties of Möbius maps.

Lemma 23.10.

(1) *If $U \subset \mathbb{D}$ is a connected subset and if $f\colon U \to \mathbb{D}$ is a Möbius map, then f is given by a Möbius transformation.*
(2) *Let $f\colon M \to N$ be a map between two Möbius manifolds. If f is o-locally a Möbius map, then f is a Möbius map.*
(3) *The composition of two Möbius maps is again a Möbius map.*
(4) *A bijective Möbius map is a Möbius isomorphism.*

Proof.

(1) Let $U \subset \mathbb{D}$ be a connected subset and let $f\colon U \to \mathbb{D}$ be a Möbius map. It follows from Proposition 22.3 (4) that given $P \in U$ there exists a unique Möbius transformation ϕ_P such that $f = \phi_P$ in an open neighborhood of P. We consider the map
$$\begin{array}{rcl} \phi\colon U & \to & \text{set of all Möbius transformations} \\ P & \mapsto & \phi_P. \end{array}$$
We equip the right-hand side of the map ϕ with the discrete topology. It follows from the definitions that this map is continuous. But then it follows from Lemma 1.92 that this map is constant. But that is exactly what we had tried to show.
(2) This statement follows immediately from (1) and the definitions.
(3) This statement is an immediate consequence of definitions and the fact, proved in Proposition 22.3, that the compositions of two Möbius transformations is again a Möbius transformation.
(4) Let $f\colon M \to N$ be a bijective Möbius map. In Proposition 22.3 we saw that the inverse of a Möbius transformation is again a Möbius transformation. It follows that f^{-1} is o-locally a Möbius map. But then it follows from (2), that f^{-1} is in fact a Möbius map. □

Lemma 23.11. *Let U be an open connected subset of $\mathbb{D}$ and let M be a Möbius manifold. Furthermore let $\Phi, \Psi\colon U \to M$ be two Möbius maps.*[475] *Suppose one of the following two conditions is satisfied:*

(1) *There exists an open non-empty subset of U on which Φ and Ψ agree, or*
(2) *there exists a point $Q \in U$ with $\Phi(Q) = \Psi(Q)$ and a non-zero vector $v \in T_Q\mathbb{D}$ such that $d\Phi_Q(v) = d\Psi_Q(v)$,*

then the Möbius maps Φ and Ψ agree on all of U.

[475]Here we view $U \subset \mathbb{D}$ as equipped with the obvious Möbius structure.

Proof. Let U be an open connected subset of $\mathbb{D}$, let M be a Möbius manifold and let $\Phi, \Psi \colon U \to M$ be two Möbius maps. We consider one more statement:

(3) There exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $d\Phi_Q = d\Psi_Q$.

Clearly we have (1) $\Rightarrow$ (3). We also have (2) $\Rightarrow$ (3). Indeed, the differentials $d\Phi_Q$ and $d\Psi_Q$ are homomorphisms between two 1-dimensional complex vector spaces. So if they are agree on one non-zero vector, then they agree on all of the vector space $T_Q\mathbb{D}$.

So it suffices to show that if (3) is satisfied, then the maps Φ and Ψ agree on all of U. So suppose that there exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $d\Phi_Q = d\Psi_Q$. We consider

$$Y \ := \ \{P \in U \mid \Phi(P) = \Psi(P) \text{ and } d\Phi_P = d\Psi_P\}.$$

By our hypothesis we know that Y contains Q, in particular Y is non-empty. Since Φ and Ψ are smooth it follows that Y is closed. It remains to show that Y is open.

So let $P \in Y$. We pick a map $\Theta \colon W \to \mathbb{D}$ from the Möbius structure of M such that W is an open neighborhood of $\Phi(P) = \Psi(P)$. We denote by V the component of $\Phi^{-1}(W) \cap \Psi^{-1}(W)$ that contains P. Since Φ and Ψ are Möbius maps it follows from Lemma 23.10 (3) that the maps $\Theta \circ \Phi \colon V \to \mathbb{D}$ and $\Theta \circ \Psi \colon V \to \mathbb{D}$ are also Möbius maps. Since V is connected it follows from Lemma 23.10 (1) that both maps are given by Möbius transformations. These two Möbius transformations agree at P and they have the same differential at P. It follows from Proposition 22.3 (5) that Φ and Ψ agree on an open neighborhood of P. This shows that Y is also open. $\square$

Definition. Let M be a Möbius manifold and let $Q \in M$ and $r > 0$. We say $B_r^M(Q)$ is *small* if there exists a Möbius isomorphism $\Omega \colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ with $\Omega(0) = Q$.

Proposition 23.12. *Let M be a Möbius manifold and let $Q \in M$. Then there exists an $r > 0$ such that $B_r^M(Q)$ is small.*

In the proof of Proposition 23.12 we will make use of the following lemma.

Lemma 23.13. *Let (X, g) and (Y, h) be Riemannian manifolds. Let $Q \in X$ and $r > 0$. Let $f \colon B_{3r}^X(Q) \to (Y, h)$ be a local isometry and let $S, T \in B_r^X(Q)$. Then we have*

$$d_Y(f(S), f(T)) \ \leq \ d_X(S, T).$$

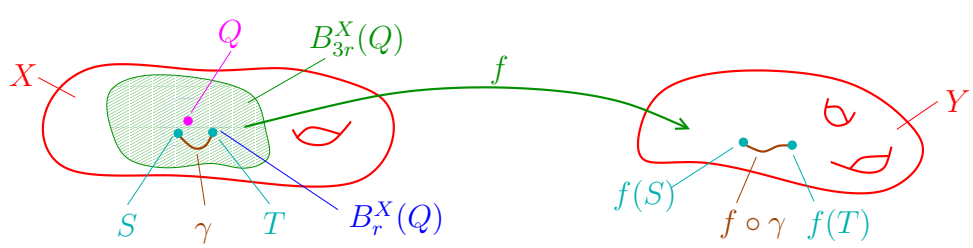


FIGURE 378. Illustration for Lemma 23.13.

Proof. Let $S, T \in B_r^X(Q)$. Furthermore let $\epsilon > 0$. By the triangle inequality we have $d_X(S,T) < 2r$. In particular there exists a piecewise smooth path γ in X from S to T such that $\ell_X(\gamma) < 2r$ and with $\ell_X(\gamma) < d_X(S,T) + \epsilon$. Given any t we have

$$d_X(Q, \gamma(t)) \;\le\; d_X(Q,S) + d_X(S, \gamma(t)) \;<\; r + \ell_X(\gamma) \;<\; 3r.$$

Thus we have shown that the image of γ lies in $B_{3r}^X(Q)$. In particular the image of γ lies in the domain of f. Thus we see that

$$d_Y(f(S), f(T)) \;\underset{\substack{\uparrow \\ \text{since } f\circ\gamma \text{ is a path from} \\ f(S) \text{ to } f(T)}}{\le}\; \ell_Y(f\circ\gamma) \;\underset{\substack{\uparrow \\ \text{by Lemma 22.9, since} \\ f \text{ is a local isometry}}}{=}\; \ell_X(\gamma) \;<\; d_X(S,T) + \epsilon.$$

Since this inequality holds for all $\epsilon > 0$ we see that $d_Y(f(S), f(T)) \le d_X(S,T)$. □

Now we can provide the proof of Proposition 23.12.

Proof of Proposition 23.12. Let M be a Möbius manifold and let $Q \in M$. We pick a map $\Phi\colon U \to V$ from the Möbius structure of M such that U is an open neighborhood of Q. By Proposition 22.3 (3), after possibly composing Φ with a Möbius transformation, we can furthermore assume that $\Phi(Q) = 0$. We write $\Omega := \Phi^{-1}$. By Proposition 22.8 (3) and by the openness of V there exists an $r > 0$ such that $B_{3r}^M(Q) \subset U$ and such that $B_{3r}^{\mathbb{D}}(0) \subset V$.[476]

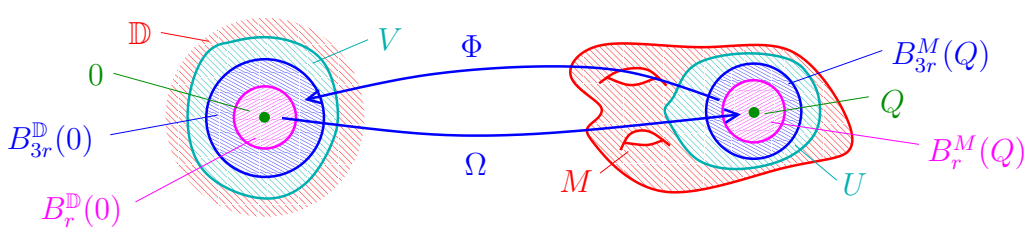


FIGURE 379. Illustration for the proof of Proposition 23.12.

It follows immediately from Lemmas 23.8 and 23.13 and from our choice of r that the following holds:

(1) For any $S, T \in B_r^M(Q)$ we have $d_{\mathbb{D}}(\Phi(S), \Phi(T)) \le d_M(S,T)$.
(2) For any $S, T \in B_r^{\mathbb{D}}(Q)$ we have $d_M(\Omega(S), \Omega(T)) \le d_{\mathbb{D}}(S,T)$.

It follows immediately from these two inequalities, from $\Phi(Q) = 0$ and from the fact that $\Phi = \Omega^{-1}$ that Ω restricts to an isometry $\Omega\colon (B_r^{\mathbb{D}}(0), d_{\mathbb{D}}) \to (B_r^M(Q), d_M)$ of metric spaces which is furthermore a bijection. Furthermore, since Φ is a Möbius isomorphism it follows that the inverse map $\Omega = \Phi^{-1}\colon (B_r^{\mathbb{D}}(0), d_{\mathbb{D}}) \to (B_r^M(Q), d_M)$ is also a Möbius isomorphism. □

[476]Note the discrete factor 3 in the radius of the ball.

Lemma 23.14. *Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$. Then the following hold:*

1. *For any $X \in \Phi^{-1}(Q)$ the map Φ restricts to a Möbius isomorphism $B_r^{\mathbb{D}}(X) \to B_r^M(Q)$.*
2. *Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$. Then there exists an $X \in \Phi^{-1}(Q)$ with $Y \in B_s^{\mathbb{D}}(X)$.*

Proof. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. We pick a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ as in the definition of "small". Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$.

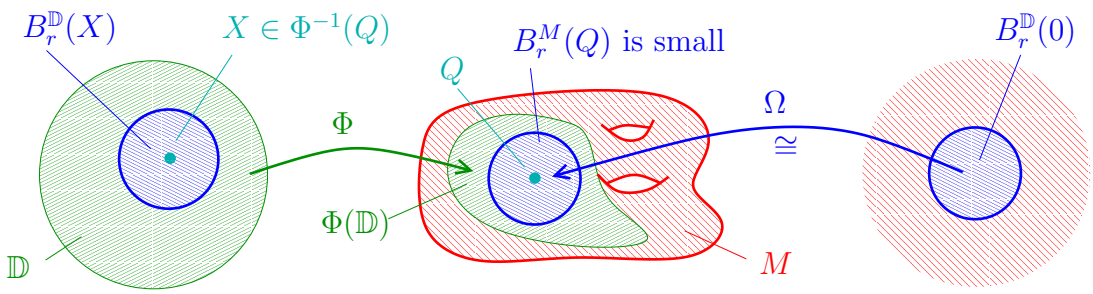


FIGURE 380. Illustration for the proof of Lemma 23.14 (1).

(1) Let $X \in \Phi^{-1}(Q)$. It follows immediately from the fact that Möbius maps are local isometries and from Lemma 23.13 that $\Phi(B_r^{\mathbb{D}}(X)) \subset B_r^M(Q)$. Therefore we can consider the Möbius map $\Omega^{-1} \circ \Phi\colon B_r^{\mathbb{D}}(X) \to B_r^{\mathbb{D}}(0)$. By Lemma 23.10 this Möbius map is the restriction of a Möbius transformation. In particular it is a Möbius isomorphism. But since Ω is also a Möbius isomorphism it follows that $\Phi\colon B_r^{\mathbb{D}}(X) \to B_r^M(Q)$ is also a Möbius isomorphism.

(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$, i.e. such that $d_M(Q, \Phi(Y)) < s$. By the triangle inequality we have
$$B_s^M(\Phi(Y)) \;\subset\; B_{2s}^M(Q) \;\subset\; B_r^M(Q).$$
The same argument as in (1) shows that the map Φ restricts to a Möbius isomorphism $B_s^{\mathbb{D}}(Y) \to B_s^M(\Phi(Y))$. We have $d_M(Q, \Phi(Y)) < s$, i.e. we have $Q \in B_s^M(\Phi(Y))$. Hence there exists an $X \in B_s^{\mathbb{D}}(Y)$ with $\Phi(X) = Q$. We refer to Figure 381 for an illustration. □

23.6. **Proof of Theorem 23.9 II.** In this section we will now finally provide the proof of Theorem 23.9. We will only cover the case $g = 2$, the proof of the general case is almost identical.

Throughout this section let $M = H_8/\sim$ be the surface of genus 2 with the Möbius structure constructed in Proposition 23.2. In the following we will show that there exists

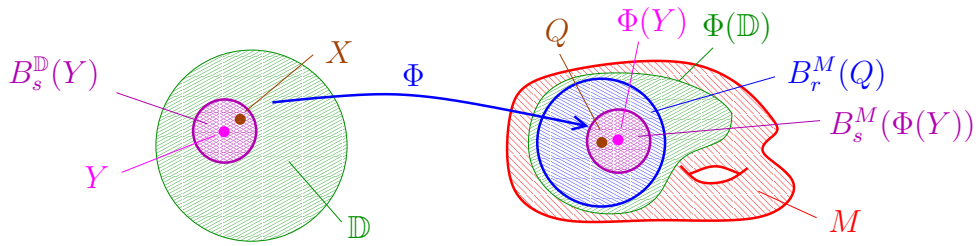


FIGURE 381. Illustration for the proof of Lemma 23.14 (2).

a Möbius map $p\colon \mathbb{D} \to M$ that is in fact a covering map. It then follows from Lemma 23.8 that p is also a local isometry and a local biholomorphism.

For the remainder of the proof we adopt the following notation and conventions.

(1) We view $\mathring{H}_8$ as a subset of $\mathbb{D}$ and also of $H_8/\sim$.
(2) Given $Z \in \mathbb{D}$ we denote by

$$\begin{array}{rcl} \gamma_Z\colon [0,1] & \to & \mathbb{D} \\ t & \mapsto & t \cdot Z \end{array}$$

the radial path from the origin $0 \in \mathbb{D}$ to Z.
(3) We say that $U \subset \mathbb{D}$ is *convex* if it is convex in the usual Euclidean sense, i.e. if given any $P, Q \in U$ the points $t \cdot P + (1-t) \cdot Q$, $t \in [0,1]$ also lie in U. On several occasions throughout the proof we will make use of the following elementary observations:
   (a) convex sets are path-connected,
   (b) the intersection of two convex sets is also convex,
   in particular, combining (a) and (b) we get
   (c) the intersection of two convex sets is path-connected.

Our first goal is to construct a suitable map $p\colon \mathbb{D} \to M = H_8/\sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say $t \in [0,1]$ is *good* if there exists a convex open neighborhood U of $\gamma([0,t])$ and a map $\Psi\colon U \to M$ which has the following properties:

(i) Ψ is a Möbius map,
(ii) it restricts to the identity on $\mathring{H}_8 \cap U \subset M$.

We start out with the following lemma.

Lemma 23.15. *The point $t = 1$ is good.*

Proof. We set

$$T := \{t \in [0,1] \mid t \text{ is good}\}.$$

We start out with the following observations:

(1) We have that $0 \in T$ since the identity on the open neighborhood $\mathring{H}_8$ of 0 has the required properties.[477]

[477]Here we use that for each point $Z \in \mathring{H}_8$ the identity map $\mathrm{id}\colon \mathring{H}_8 \to \mathring{H}_8$ is part of the Möbius structure of M.

(2) Being good is clearly an open condition, hence $T \subset [0,1]$ is open.

We want to show that $T = [0,1]$. By the above, and since $[0,1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since T is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of good numbers $t_n \in [0,t)$ that converges to s. Since each t_n is good we can pick for each n a convex open neighborhood U_n of $\gamma([0,t_n])$ and a map $\Psi_n \colon U_n \to M$ that has the desired properties (i) and (ii).

Let $m, n \in \mathbb{N}$. The Möbius maps Ψ_m and Ψ_n agree on the non-empty open connected subset $\overset{\circ}{H}_8 \cap U_n \cap U_m$. It follows from Lemma 23.11 that the maps Ψ_m and Ψ_n agree on $U_n \cap U_m$.

By the continuity of γ the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ converges to $\gamma(t)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. It follows from Lemma 22.9 that $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d_M.[478] It follows from Proposition 23.2 that the metric space $(M = H_8/\sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ converges to a point $S \in M$.

It follows from Proposition 23.12 that there exists an $r > 0$ and a Möbius isomorphism $\Omega \colon B_r^{\mathbb{D}}(0) \to B_r^M(S)$ such that $\Omega(0) = S$. We pick an n such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$. We write $t = t_n, \Psi = \Psi_n, U = U_n$ and we write $X = \gamma(t)$. Let $v \in T_X\mathbb{D}$ be a non-zero vector. We let $w := d(\Omega^{-1} \circ \Psi)_X(v)$. The Möbius maps Ω and Ψ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from Proposition 22.3 (4) that there exists a Möbius transformation Θ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.

The situation is illustrated in Figure 382. The Möbius maps $\Omega \circ \Theta \colon B_r^{\mathbb{D}}(X) \to M$ and $\Psi \colon U \to M$ agree at $X = \gamma(t)$ and we have $d(\Omega \circ \Theta)_X(v) = d\Psi_X(v)$. It follows from Lemma 23.11 that $\Omega \circ \Theta$ and Ψ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

Now we consider the map

$$\begin{array}{rcl} \Phi \colon U \cup B_r^{\mathbb{D}}(\gamma(t)) & \to & M \\ Q & \mapsto & \left\{ \begin{array}{rl} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{array} \right. \end{array}$$

By the above discussion this map is well-defined and it is o-locally a Möbius map. It follows from Lemma 23.10 (3) that this map is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$ and the fact that γ is a geodesic in $\mathbb{D}$ that $\gamma([t,s]) \subset B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0,t]) \cup \gamma([t,s]) = \gamma([0,s])$. It is now straightforward

[478] Indeed, suppose we have $\epsilon > 0$. Since $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an N such that $d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon$ for all $m, n \geq N$. Now let $m, n \geq N$. Without loss of generality we have $m > n$. Then

$$\begin{array}{ccccc} d_M(\Psi_m(\gamma(t_m)), \Psi_n(\gamma(t_n))) & \underset{\uparrow}{=} & d_M(\Psi_m(\gamma(t_m)), \Psi_m(\gamma(t_n))) & \underset{\uparrow}{\leq} & d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) \ < \ \epsilon. \\ & \text{since } \Psi_m \text{ and } \Psi_n \text{ agree on } \gamma|_{[0,t_n]} & & \text{Lemma 22.9} & \end{array}$$

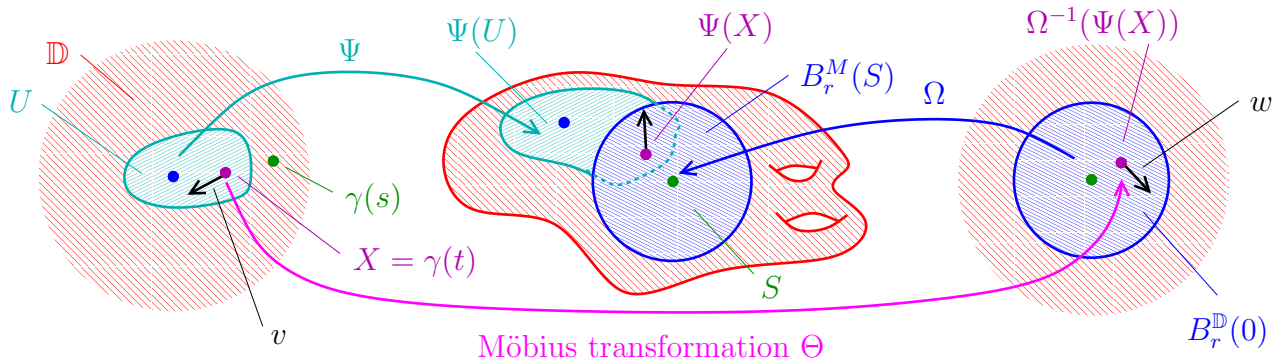


FIGURE 382.

to see that we can find a *convex* open subset W of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0,s])$. The Möbius map $\Phi\colon W \to M$ now certifies that s is also good. $\square$

Thus we have now shown that there exists a convex open neighborhood U of $\gamma([0,1])$ and a local isometry $\Psi\colon U \to H_8/\sim$ which agrees with the identity map on $U \cap \mathring{H}_8$. We define $p(Z) := \Psi(Z)$. By Lemma 23.11 this definition does not depend on the choice of U and Ψ.

Lemma 23.16. *The map $p\colon \mathbb{D} \to M$ is a Möbius map.*

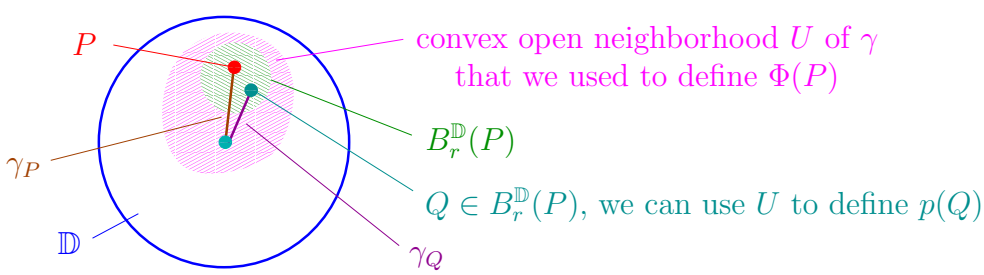


FIGURE 383. Illustration of the proof of Lemma 23.16.

Proof. By Lemma 23.10 (2) it suffices to show that p is o-locally a Möbius map. This means that it suffices to show that every point $P \in \mathbb{D}$ admits an open neighborhood W such that the restriction of p to W is a Möbius map. Thus let $P \in \mathbb{D}$. By the above claim there exists a convex open neighborhood U of $\gamma_P([0,1])$ and a Möbius map $\Psi\colon U \to H_8/\sim$ which restricts to the identity on $U \cap \mathring{H}_8$. Since U is open we can find an $r > 0$ such that for all $Q \in B_r^{\mathbb{D}}(P)$ the image of the radial path γ_Q from 0 to Q also lies in U. (We refer to Figure 383 for an illustration.) This implies that we can use the Möbius map $\Psi\colon U \to M$ to define p for any $Q \in B_r^{\mathbb{D}}(P)$. Put differently, $p = \Psi$ on $B_r^{\mathbb{D}}(P)$. We have thus shown that the restriction of p to the open neighborhood $W = B_r^{\mathbb{D}}(P)$ of P is a Möbius map. $\square$

The following lemma now concludes the proof of Theorem 23.9

Lemma 23.17. *The map $p\colon \mathbb{D} \to M = H_8/\sim$ is a covering map.*

Proof. First we show that the map $p\colon \mathbb{D} \to M = H_8/\sim$ is surjective. Recall that by construction p is the identity on $\mathring{H}_8$. Since p is continuous[479] it follows that the restriction of p to $H_8 \to M = H_8/\sim$ is surjective.[480]

It remains to show that every $Q \in M$ admits a uniformly covered neighborhood. Let $Q \in M$. By Proposition 23.12 there exists an $r > 0$ such that $B^M_{2r}(Q)$ is small.[481] We claim that $B^M_r(Q)$ is uniformly covered. More precisely, we make the following claims:

(1) $p^{-1}(B^M_r(Q)) = \bigcup_{X \in p^{-1}(Q)} B^{\mathbb{D}}_r(X)$,
(2) for $X \neq X' \in p^{-1}(Q)$ we have $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') = \varnothing$,
(3) given any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a homeomorphism.

We apply Lemma 23.14 to the map $p\colon \mathbb{D} \to M$. It gives us the following three facts:

(a) If $Y \in \mathbb{D}$ satisfies $p(Y) \in B^M_r(Q)$, then there exists an $X \in p^{-1}(Q)$ with $Y \in B^{\mathbb{D}}_r(X)$.
(b) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a Möbius isomorphism, in particular it is a homeomorphism.
(c) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_{2r}(X)$ is injective.

Now the desired properties (1) and (3) follow immediately from (a) and (b). It remains to prove (2). Suppose that we have $X \neq X' \in p^{-1}(Q)$ with $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') \neq \varnothing$. We need to show $X = X'$. We pick $Y \in B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X')$. The triangle inequality implies that $X' \in B^{\mathbb{D}}_{2r}(X)$. But p restricted to $B^{\mathbb{D}}_{2r}(X)$ is injective by (c). Hence $X = X'$. □

23.7. **Picard's Theorem.** We also have the following variation of Theorem 23.9.

Proposition 23.18. *Let $H_4/\sim$ be the three-punctured sphere with the Möbius structure constructed in Proposition 23.5. Then there exists a covering map $p\colon \mathbb{D} \to H_4/\sim$ with the following properties:*

(1) *p is a Möbius map,*
(2) *p is a local isometry,*
(3) *p is a local biholomorphism.*

Proof. The proof of Proposition 23.18 is almost the same as the proof of Theorem 23.9. In the proof of Theorem 23.9 we used that $H_8/\sim$ is complete. In the present case it follows from Proposition 23.5 (3) that $H_4/\sim$ is complete. The construction of the Möbius map $\Phi\colon \mathbb{D} \to H_4/\sim$ is verbatim the same as the construction of the Möbius map $\Phi\colon \mathbb{D} \to H_8/\sim$.

[479]The map p is a Möbius map, in particular it is a local biholomorphism which implies that it is a local homeomorphism. Thus it follows from Lemma 1.104 that p itself is continuous.

[480]This is the only part of the proof where we actually use that we work with $H_8/\sim$, the remainder of the proof works for any manifold M with a Möbius structure such that the corresponding Riemannian manifold is complete.

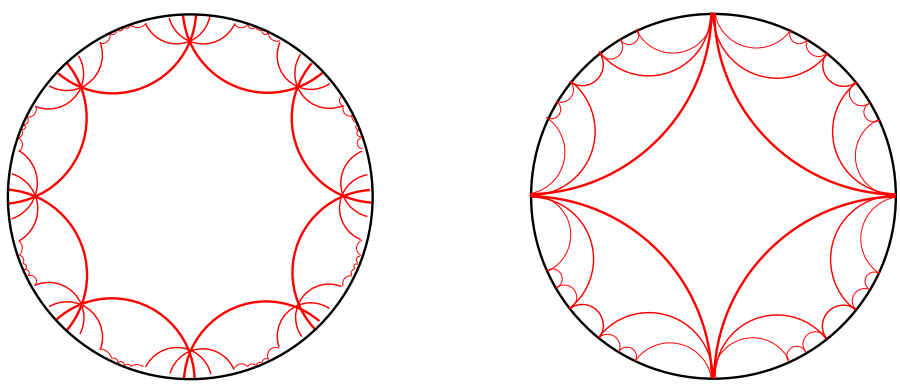
[481]Note the factor 2.

The same argument as before shows that $\Phi\colon \mathbb{D} \to H_4/\sim$ is surjective and that it is a covering map. $\square$

Remark. We consider again the covering $p\colon \mathbb{D} \to H_8/\sim$ that we had just constructed. It follows from Corollary 20.4 that the interior of H_8 is uniformly covered. In particular we see that

$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_8)}_{\text{“1-dimensional”}} = \bigsqcup \text{ copies of } \mathring{H}_8.$$

Put differently, up to the “one-dimensional” subset $p^{-1}(\partial H_8)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in Figure 384 on the left. The same argument also applies to the covering $\mathbb{D} \to H_4/\sim$ that we also had just constructed. The corresponding tessellation of $\mathbb{D}$ is shown in Figure 384 on the right.



preimages of $\mathring{H}_8$ under the universal cover $p\colon \mathbb{D} \to H_8/\sim$

preimages of $\mathring{H}_4$ under the universal cover $p\colon \mathbb{D} \to H_4/\sim$

FIGURE 384.

At the end of this chapter we return to complex analysis. There are different types of holomorphic functions defined on $\mathbb{C}$.

(1) There are the (rather dull) constant functions.
(2) There are holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ that are surjective. For example it is an easy consequence of the fundamental theorem of algebra that all maps defined by non-constant polynomials are surjective.
(3) There are also non-constant maps that are not surjective, for example the exponential function $z \mapsto \exp(z)$ is never zero[482], in fact its image is $\mathbb{C} \setminus \{0\}$.

On the other hand it is fairly easy prove, using Liouville's Theorem 21.1, that the image of a non-constant holomorphic function on $\mathbb{C}$ is dense.

[482]Why not?

Now we can prove Picard's Theorem[483] which is a significantly stronger statement than the above statement that the image is dense.

Theorem 23.19. (Picard's Theorem) *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f.*

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers a, b that do not lie in the image of f. We need to show that f is constant. We consider the biholomorphism

$$\begin{array}{rcl} \alpha\colon \mathbb{C} \setminus \{a,b\} & \to & \mathbb{C} \setminus \{0,1\} \\ z & \mapsto & \dfrac{z-a}{b-a}. \end{array}$$

Furthermore we denote by $\beta\colon \mathbb{C} \setminus \{0,1\} \to H_4/\sim$ the biholomorphism from Proposition 23.6. Now we consider the following diagram of maps

$$\begin{tikzcd} & & & \mathbb{D} \arrow[d, "\Phi"] \\ \mathbb{C} \arrow[r, "f"] & \mathbb{C} \setminus \{a,b\} \arrow[r, "\alpha", "\cong"'] & \mathbb{C} \setminus \{0,1\} \arrow[r, "\beta", "\cong"'] & H_4/\sim \end{tikzcd}$$

where $\Phi\colon \mathbb{D} \to H_4/\sim$ is the covering map from Proposition 23.18. Recall that Φ is a Möbius map which implies by Lemma 23.8 that Φ is a local biholomorphism.

Since $\mathbb{C}$ is simply connected we can appeal to Proposition 20.2 to obtain a lift of the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to H_4/\sim$ to the universal covering. More precisely, there exists a map $\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ such that the following diagram commutes

$$\begin{tikzcd} & & & \mathbb{D} \arrow[d, "\Phi"] \\ \mathbb{C} \arrow[r, "f"'] \arrow[urrr, "\widetilde{\beta\circ\alpha\circ f}"] & \mathbb{C} \setminus \{a,b\} \arrow[r, "\alpha", "\cong"'] & \mathbb{C} \setminus \{0,1\} \arrow[r, "\beta", "\cong"'] & H_4/\sim . \end{tikzcd}$$

As we pointed out in Lemma 21.9, it follows from the fact that $\beta \circ \alpha \circ f$ is holomorphic and the fact that Φ is a local biholomorphism, that the map $\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's Theorem 21.1 that $\widetilde{\beta \circ \alpha \circ f}$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since α and β are biholomorphisms this implies that f itself is already constant. $\square$

Remark. In the literature Theorem 23.19 is sometimes referred to as Picard's Little Theorem, to distinguish it from Picard's Great Theorem which says that if z_0 is an essential singularity of a holomorphic function, then in any neighborhood of z_0 the function f assumes all, but possibly one, value *infinitely* many times. We refer to [Con, Section 12] for details.

[483]Emile Picard (1856-1941) was a French mathematician. Many readers will have encountered him already in real analysis, as one of the two namesakes of the Picard-Lindelöf Theorem.

24. The deck transformation group (∗)

Definition. Let $p\colon X \to B$ be a covering.

(1) A *deck transformation* is a homeomorphism $d\colon X \to X$ such that $p(d(x)) = p(x)$ for all x, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "d", "\cong"'] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & B. & \end{tikzcd}$$

The group of deck transformations[484] is called the *deck transformation group* of the covering $p\colon X \to B$ and we denote it by $D(p\colon X \to B)$.

(2) We say that the covering is *regular* if for any $b \in B$ the deck transformation group D acts transitively on $p^{-1}(b)$, i.e. if given any two points x, x' with $p(x) = p(x') = b$ there exists a deck transformation d with $d(x) = x'$. Otherwise we say that the covering is *irregular*.

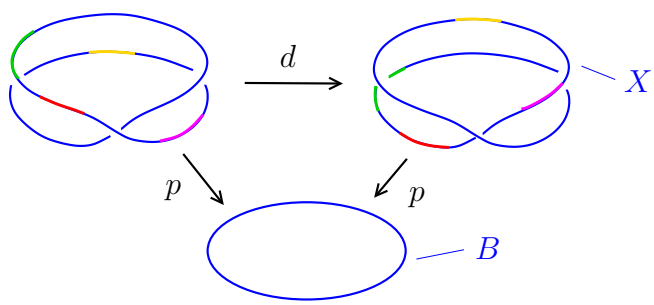


Figure 385. Schematic image of a deck transformation.

Example.

(1) We consider the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^2. \end{array}$$

Then the map $d\colon S^1 \to S^1$ given by $d(z) := -z$ is a deck transformation. It is now clear that the covering is regular.

(2) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

In Lemma 7.5 we saw that f is a deck transformation. It follows immediately that any 2-fold covering is regular.

[484] Why do deck transformations form a group?

(3) As an important special case of (2) consider a non-orientable path-connected manifold M. As in the proof of Proposition 8.3 we consider the 2-fold covering

$$\begin{array}{rcl} \widetilde{M} := \{(Q,O) \mid Q \in M \text{ and } O \text{ an orientation of } T_QM\} & \to & M \\ (Q,O) & \mapsto & Q. \end{array}$$

It is easy to show that the map $d\colon \widetilde{M} \to \widetilde{M}$ given by $d((Q,O)) := (Q,-O)$ is a deck transformation of p.

(4) We consider again the first two coverings of graphs from Figure 148 which are also shown again in Figure 386. The covering maps are the ones suggested by the colors and the directions. We first make the following observations. A deck transformation d does the following:
   (a) it sends a vertex again to a vertex,
   (b) it sends a blue edge to a blue edge, preserving the orientation,
   (c) it sends a red edge to a red edge, preserving the orientation,
   (d) if e is an edge with endpoints P and Q, then $d(e)$ is an edge with endpoints $d(P)$ and $d(Q)$, in particular if e is a loop, then $d(e)$ is also a loop.

   Now we consider the two examples.
   (a) The example on the left admits one non-trivial deck transformation, namely the map that is given by "rotation by π" around the center of the graph. The covering is now easily seen to be regular.
   (b) The loop a_3 in the graph on the right is the only red loop, so any deck transformation has to send it to itself. It follows that there can be no deck transformation that sends the vertex to the right to any of the other vertices.[485] Thus the covering on the right is irregular.

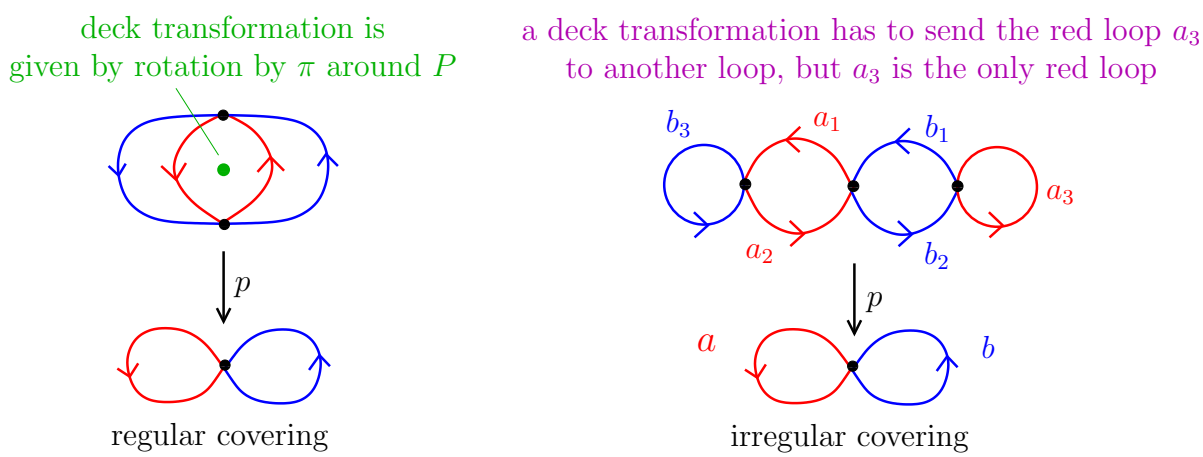


FIGURE 386.

[485] Does this covering admit any deck transformation?

(5) Finally we consider the 4-fold coverings illustrated in Figure 387. They are both regular coverings. For the graph on the left the deck transformation group is isomorphic to $\mathbb{Z}_4$, a generator is given by "rotation by $\frac{\pi}{2}$". The deck transformation group on the right is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, here one $\mathbb{Z}_2$ summand is given by "rotation by π" and the other one corresponds to "swapping the inner and the outer circle".

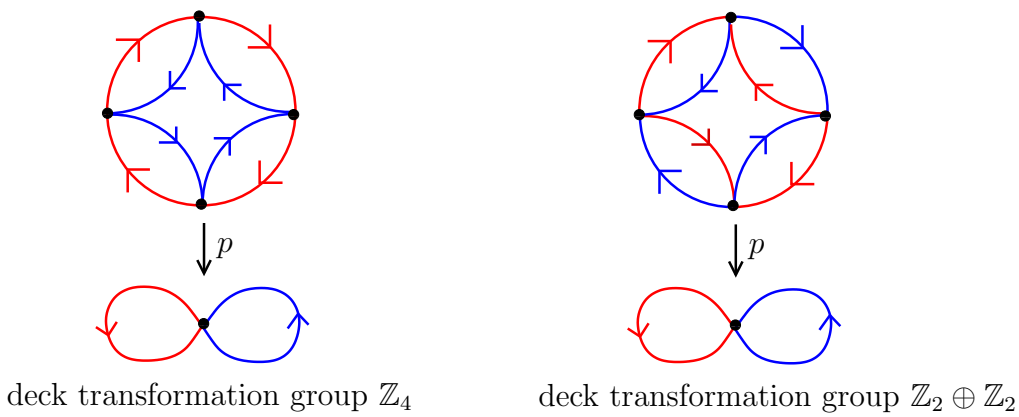


FIGURE 387.

Lemma 24.1. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces and let d_1 and d_2 be two deck transformations of p. Then the following holds*

$$d_1 \;=\; d_2 \quad \Longleftrightarrow \quad d_1(x_0) \;=\; d_2(x_0).$$

Proof. Evidently we only need to show the "$\Leftarrow$"-direction. Thus let d be a deck transformation of p. One way of stating that d is a deck transformation is to state that the following diagram commutes:

$$\begin{tikzcd} & (X, d(x_0)) \arrow[d, "p"] \\ (X, x_0) \arrow[ur, "d"] \arrow[r, "p"'] & (B, b_0). \end{tikzcd}$$

Put differently, the map d can be viewed as a lift of the map $(X, x_0) \to (B, b_0)$ to the covering $(X, d(x_0)) \to (B, b_0)$. By the uniqueness of lifts[486], that we showed in Proposition 20.2 (2), we now see that two deck transformations d_1 and d_2 of p with $d_1(x_0) = d_2(x_0)$ agree. □

Example. We let $B = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$, i.e. B is the wedge of two circles and we consider

$$X \;=\; (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/(e^{2\pi i n/5} \sim (1, n)) \text{ for } n \in \mathbb{Z}_5,$$

[486]Here we use that X is path-connected.

i.e. X is a circle with five circles attached at the points $e^{2\pi ik/5}$, $k = 0, \dots, 4$. Then the map

$$p\colon (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/\sim \to\ G = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1,a) \sim (1,b)$$

that is given by $p(z) = (z^5, a)$ for $z \in S^1$ and $p((z,n)) = (z,b)$ for $(z,n) \in S^1 \times \mathbb{Z}_5$, is a covering map of degree five. This covering illustrated in Figure 388.

We consider the map

$$\begin{array}{rcl} \mathbb{Z}_5 & \to & D(p\colon X \to B) \\ m & \mapsto & \left(\begin{array}{rcll} X & \to & X & \\ z & \mapsto & e^{2\pi im/5}z, & \text{for } z \in S^1, \\ (z,n) & \mapsto & (z, m+n) & \text{for } (z,n) \in S^1 \times \mathbb{Z}_5 \end{array}\right) \end{array}$$

In Figure 388 this corresponds to "rotation of X by the angle $\frac{2\pi m}{5}$". It is straightforward to show that this map is well-defined, i.e. the maps on the right are deck transformations, and using Lemma 24.1 one can easily see that this map is an isomorphism. Also one can now convince oneself easily that the covering $p\colon X \to B$ is regular.

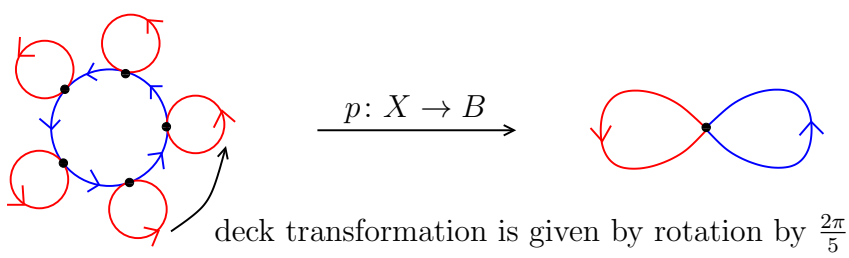


FIGURE 388.

Lemma 24.2. *Let X be a path-connected topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p\colon X \to X/G$. Then $p\colon X \to X/G$ is regular and the map*

$$\begin{array}{rcl} \Phi\colon G & \mapsto & D(p\colon X \to X/G) \\ g & \mapsto & \begin{pmatrix} X & \mapsto & X \\ x & \mapsto & gx \end{pmatrix} \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to see that the maps $x \mapsto gx$ are indeed deck transformations. It follows almost immediately that the covering $p\colon X \to X/G$ is regular. Also it follows easily from the axioms of a group action that Φ is a homomorphism of groups. Furthermore it is clear that Φ is a monomorphism.

It remains to show that Φ is an epimorphism. So let $d \in D(p\colon X \to X/G)$ be a deck transformation. We pick $x \in X$. Since $p(d(x)) = p(x)$ we have $d(x) = gx$ for some $g \in G$. Since X is path-connected it follows from Lemma 24.1 that $d(y) = gy$ for all $y \in X$. This shows that Φ is an epimorphism. $\square$

Lemma 24.3. *Any 2-fold covering of path-connected topological spaces is regular and the deck transformation group contains precisely two elements.*

Proof. Let $p\colon X \to B$ be a 2-fold covering of path-connected topological spaces. By Lemma 7.5 we know that

$$\begin{array}{rcl} d\colon X & \to & X \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)) \end{array}$$

is a deck transformation. Evidently it is regular. It follows easily from Lemma 24.1 and our hypothesis that X, B are path-connected that any deck transformation of p is either the identity or it equals the above deck transformation d. □

We recall the following important, albeit somewhat technical, definition.

Definition. Let G be a group and let $H \subset G$ be a subgroup. The *normalizer* $N(H)$ of the subgroup $H \subset G$ is defined as

$$N(H) \ := \ \{g \in G \mid g^{-1}Hg = H\}.$$

The following lemma summarizes a few elementary properties of the normalizer of a subgroup.

Lemma 24.4. *Let G be a group and let $H \subset G$ be a subgroup.*

(1) *The subgroup H of G is normal if and only if $N(H) = G$.*
(2) *The normalizer $N(H)$ of $H \subset G$ is a subgroup of G.*
(3) *The normalizer $N(H)$ of $H \subset G$ is the largest subgroup of G which contains H as a normal subgroup, in the sense that if A is a subgroup of G which contains H as a normal subgroup, then $A \subset N(H)$.*

Proof. Both statements are easily verified. Alternatively see [Isa, p. 26] for the elementary proof. □

We return to topology and we consider one more example in detail.

Example. Let Y be a path-connected and reasonable topological space and let $y_0 \in Y$. Furthermore let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. By Proposition 20.8 there exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. We recall the construction of X. We first considered

$$W \ := \ \{\text{all paths in } Y \text{ with starting point } y_0\}$$

and for $u, v \in W$ we wrote

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

We defined $X = W/\sim$ and the projection map $p\colon W/\sim \to Y$ that is given by $[w] \mapsto w(1)$. We then showed that, with an appropriate topology on $W/\sim$, the map $p\colon W/\sim \to Y$ is a path-connected covering with $p_*(\pi_1(W/\sim, x_0)) = \Gamma$.

As in the proof of Proposition 20.13 we now would like to construct deck transformations by precomposing elements in W by a loop in y_0. More precisely, let g be a loop in (Y, y_0). We consider the map

$$\begin{array}{rcl} W & \to & W \\ w & \mapsto & g * w \end{array}$$

We need to check that the map respects our equivalence relation $\sim$ on W. So suppose that $u, v \in W$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{\in \Gamma} * [g]^{-1}.$$

In general it is not the case that the right-hand side lies again in Γ. But it is true if $[g]$ lies in the normalizer of Γ. As in the proof of Proposition 20.13 it is now straightforward to see that the map

$$\begin{array}{rcl} \Psi\colon N(\Gamma)/\Gamma & \to & D(p\colon X \to Y) \\ [g] & \mapsto & \begin{pmatrix} W/\sim & \to & W/\sim \\ [u] & \mapsto & [g * u] \end{pmatrix} \end{array}$$

is a well-defined homomorphism. We will see in Proposition 24.5 that this in fact an isomorphism of groups.

Now we want to consider the general case of a covering. So let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$.[487] Now the question arises whether there is a connection between the groups π, Γ and $D(p\colon X \to B)$.

Similar to the proof of Theorem 7.17 we can consider the following map[488]

$$\begin{array}{rcl} \Phi\colon D(p\colon X \to B) & \mapsto & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

It is straightforward to see that this map is well-defined, i.e. it does not depend on the choice of the path.

The question now arises, whether this map Φ is a group isomorphism. But in general this is not the case, after all, if Γ is not normal in π, then π/Γ is not even a group. But we have the following proposition:

Proposition 24.5. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. We suppose that X and B are path-connected and o-locally path-connected. We write*

[487] Recall that by Corollary 7.15 the induced map $p_*\colon \Gamma \to \pi$ is injective, thus $\Gamma \cong \pi_1(X, x_0)$.

[488] We denote hereby with $[p \circ f]$ not only the equivalence class of the loop $p \circ f$ in π, but also its equivalence class π/Γ, since the notation $[p \circ f]\Gamma$ is too cumbersome.

$\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$. The above map

$$\begin{array}{rcl} \Phi\colon D(p\colon X \to B) & \mapsto & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\textit{path in } X \textit{ from } x_0 \textit{ to } d(x_0))\big]. \end{array}$$

restricts to an isomorphism of groups

$$\Phi\colon D(p\colon X \to B) \xrightarrow{\cong} N(\Gamma)/\Gamma.$$

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D := D(p\colon X \to B)$.

(A) First we have to show that $\Phi(D)$ lies indeed in $N(\Gamma)/\Gamma$. Thus let $d \in D$. We need to show that $\Phi(d)$ lies in $N(\Gamma)/\Gamma$. Thus let f be a loop in (X, x_0) and and let g be a path in X from x_0 to $d(x_0)$. We have to verify that $[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] \in \Gamma$. Indeed the following holds

$$[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] \;=\; \big[\overline{p \circ f} * p \circ g * p \circ f\big] \;=\; \big[p \circ \big(\underbrace{\overline{f} * g * f}_{\text{loop in } (X, x_0)}\big)\big] \;\in\; p_*(\pi_1(X, x_0)) = \Gamma.$$

(B) Now we have to show that Φ is a group homomorphism. So let d and e be two deck transformations. We pick a path g from x_0 to $d(x_0)$ and we pick a path h from x_0 to $e(x_0)$. Then

$$\Phi(d) \cdot \Phi(e) \;\underset{\substack{\uparrow \\ \text{definition of } \Phi(e) \text{ and } \Phi(d)}}{=}\; [p \circ g] \cdot [p \circ h] \;\underset{\substack{\uparrow \\ \text{since } p \circ d = p}}{=}\; [p \circ g] \cdot [p \circ (d \circ h)] \;\overset{\substack{\text{endpoint of } g \text{ equals starting point of } d \circ h \\ \downarrow}}{=}\; [g * (d \circ h)] \;\underset{\substack{\uparrow \\ \text{since } g * (d \circ h) \text{ is a path} \\ \text{from } x_0 \text{ to } d(e(x_0)), \text{ see Figure 389}}}{=}\; \Phi(e \circ d).$$

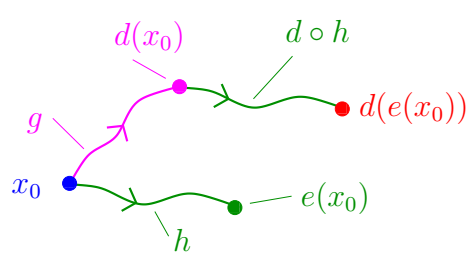


FIGURE 389.

(C) Now we will construct a map $\Psi\colon N(\Gamma)/\Gamma \to D$ in the inverse direction. Thus let $[f] \in N(\Gamma) \subset \pi = \pi_1(B, b_0)$ where f is a loop in (B, b_0). Let x_1 be the endpoint of the lift $\widetilde{f}$ of f to the starting point x_0.[489] We have $p(x_1) = p\big(\widetilde{f}(1)\big) = f(1) = x_0$. Thus we

[489] According to Corollary 7.14 the definition of x_1 does not depend on the choice of the representative of $[f] \in N(\Gamma)$.

obtain the following diagram of maps of pointed topological spaces:

$$\begin{array}{ccc} & & (X, x_1) \\ & & \big\downarrow p \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

Then

$$\begin{array}{rcl} p_*(\pi_1(X, x_1)) & = & \left\{ \left[p \circ \left(\overline{\widetilde{f}} * g * \widetilde{f} \right) \right] \,\middle|\, [g] \in \pi_1(X, x_0) \right\} \\ & \uparrow & \\ \text{by Proposition 5.11 since } \widetilde{f} \text{ is a path from } x_0 \text{ to } x_1 & & \end{array} \begin{array}{rl} = & [f]\, p_*(\pi_1(X, x_0))\, [f]^{-1} \\ = & p_*(\pi_1(X, x_0)). \\ \uparrow & \\ \text{since } [f] \in N(\Gamma) & \end{array}$$

It follows from Proposition 20.2 that there exists a lift $\widetilde{p}\colon (X, x_0) \to (X, x_1)$ of p, i.e. there exists a map $\widetilde{p}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\widetilde{p}}{\nearrow} & (X, x_1) \\ & & \big\downarrow p \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

The map is $\widetilde{p}$ is now by definition a deck transformation.

We have thus constructed a map $N(\Gamma) \to D$. We still need to show that this map descends to a map $N(\Gamma)/\Gamma \to D$. So let $[f] \in N(\Gamma)$ and let $[g] \in \Gamma$. We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Similarly we define $\widetilde{g * f}$. By the above we only need to show that the endpoints of $\widetilde{f}$ and $\widetilde{g * f}$ agree. In fact we have

$$\begin{array}{rcl} \widetilde{g * f}(1) & = & \text{endpoint of the lift of } f \\ & & \text{to the starting point } \widetilde{g}(1) \\ & \uparrow & \\ & \text{Lemma 20.3} & \end{array} \begin{array}{rcl} & = & \text{endpoint of the lift of } f \\ & & \text{to the starting point } x_0 \\ & \uparrow & \\ & \text{by Lemma 7.16 (2) (1) since } [g] \in p_*(\pi_1(X, x_0)) & \end{array} \begin{array}{rl} = & \widetilde{f}(1) \\ & \\ & \\ & \end{array}$$

(D) Now we want to show that Φ is a bijection. It suffices to show that $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$.

First let $d \in D$. Then

$$\begin{array}{rcl} \Psi(\Phi(d)) & = & \Psi(\text{loop } f \text{ in } (B, b_0) \text{ that lifts to a path from } x_0 \text{ to } d(x_0)) \\ & = & \text{deck transformation that sends } x_0 \text{ to } d(x_0) \;=\; d \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Lemma 24.1} \end{array}$$

This shows that $\Psi \circ \Phi$ is the identity.

Now let $g \in N(\Gamma)/\Gamma$. We pick a representative loop s in b_0 and we denote by $\widetilde{s}$ the lift of s to the starting point x_0. We denote by x_1 the endpoint of $\widetilde{s}$. Then

$$\Phi(\Psi(g)) \;=\; \big[p \circ \big(\text{path in } X \text{ from } x_0 \text{ to } \overbrace{\Psi(g)(x_0)}^{=x_1}\big)\big] \;\underset{\substack{\uparrow \\ \widetilde{s} \text{ is a path from } x_0 \text{ to } x_1}}{=}\; \big[p \circ \widetilde{s}\big] \;=\; [s] \;=\; g.$$

This shows that $\Phi \circ \Psi$ is the identity. □

In most applications we will use the following special case of Proposition 24.5.

Proposition 24.6. *Let X be a path-connected and reasonable topological space and let $x_0 \in X$. Let $\gamma\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a group G. We denote by $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ the covering corresponding to $\ker(\gamma)$ that is given by Proposition 20.8 and that is unique up to covering equivalence by Proposition 20.9. Then the map*

$$\begin{array}{rcl} \Phi\colon D(p\colon \widetilde{X} \to X) & \to & \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \xrightarrow{\gamma} G. \\ d & \mapsto & \big[p \circ (\textit{path from } x_0 \textit{ to } d(x_0))\big] \end{array}$$

is an isomorphism.

Proof. First note that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \ker(\gamma)$ is a normal subgroup of $\pi_1(X, x_0)$. Therefore the normalizer of $\Gamma := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ equals $\pi_1(X, x_0)$. It now follows immediately from Proposition 24.5 that the homomorphism on the left is an isomorphism. The homomorphism on the right is an isomorphism since $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \ker(\gamma)$. □

The following lemma now gives a complete characterization of regular coverings.

Corollary 24.7. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. Then the following statements are equivalent:*

(1) *The covering is regular.*
(2) *The deck transformation group $D(p\colon X \to B)$ acts transitively on $p^{-1}(b_0) \subset X$.*
(3) *The subgroup $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(B, b_0)$.*
(4) *There exists a group G which acts discretely on X and there exists a homeomorphism $f\colon X/G \to B$ such that the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X/G \arrow[rr, "f"'] & & B \end{tikzcd}$$

where $q\colon X \to X/G$ is the canonical projection map.

If p is a finite covering then the above statements are furthermore equivalent to

(5) $\#D(p\colon X \to B) \;=\; [X : B]$.

Remark. Let $p\colon (X,x_0) \to (B,b_0)$ be a 2-fold covering of path-connected pointed topological spaces. In Lemma 24.3 we saw that p is regular, which by Corollary 24.7 means that the index two subgroup $p_*(\pi_1(X,x_0))$ of $\pi_1(B,b_0)$ is normal. In fact we already showed in Exercise 8.1 that any index 2 subgroup Γ of a group π is normal.[490]

Proof. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B,b_0)$, $\Gamma := p_*(\pi_1(X,x_0))$ and $D = D(p\colon X \to B)$. The following four steps prove the equivalence of (1), (2) (3) and (4).

(1) $\Leftrightarrow$ (2) Recall that a covering is regular if for any $b \in B$ the group D acts transitively on $p^{-1}(b)$. Thus (1) $\Rightarrow$ (2) is trivial. Now suppose that d acts transitively on $p^{-1}(b_0)$. Let b be some other point in B. Since B is path-connected there exists a path s from b to b_0.

Now let x, x' be two points in $p^{-1}(b)$. We denote by $\widetilde{s}$ the lift of s to the starting point x and similarly we define $\widetilde{s}'$. We denote by x_0 and x_0' the endpoints of $\widetilde{s}$ and $\widetilde{s}'$. Since D acts transitively on $p^{-1}(b_0)$ there exists a deck transformation d with $d(x_0) = x_0'$. By the uniqueness of lifts to a starting point we then also have $d \circ \widetilde{s} = \widetilde{s}'$. But then we also have that d turns the starting point of $\widetilde{s}$ into the starting point of $\widetilde{s}'$, i.e. we have $d(x) = x'$.

(2) $\Leftrightarrow$ (3) We consider again the map

$$\begin{array}{rcl} \Phi\colon D(p\colon X \to B) & \mapsto & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

We first prove the following claim.

Claim. We have the following equivalence of statements:

$$D \text{ acts transitively on } p^{-1}(b_0) \subset X \quad \Leftrightarrow \quad \Phi\colon D \to \pi/\Gamma \text{ is surjective.}$$

First suppose that D acts transitively on $p^{-1}(b_0) \subset X$. Let $g \in \pi/\Gamma$ and pick a loop f that represents g. We denote by $\widetilde{f}$ the lift of f to the starting point x_0 and we denote by $\widetilde{x}_0$ the endpoint of $\widetilde{f}$. Since D acts transitively there exists a deck transformation d with $d(x_0) = \widetilde{x}_0$. Now

$$\Phi(d) = [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))] \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a path from } x_0 \text{ to } d(x_0)}}{=} [p \circ (d \circ \widetilde{f})] \underset{\substack{\uparrow \\ \text{since } p \circ d = p}}{=} [p \circ \widetilde{f}] = [f] = g.$$

Now suppose that $\Phi\colon D \to \pi/\Gamma$ is surjective. Let $\widetilde{x}_0 \in p^{-1}(b_0)$. We pick a path $\widetilde{f}$ from x_0 to $\widetilde{x}_0$ and we denote by f the loop $p \circ \widetilde{f}$. Since Φ is surjective there exists a deck transformation d with $\Phi(d) = [f]$. But then $d(x_0) = \widetilde{x}_0$. This concludes the proof of the claim.

[490] We already made use of this fact in the proof of Corollary 8.4.

Now we have the following equivalences

$$\begin{array}{rcl} D \text{ acts transitively on } p^{-1}(b_0) \subset X & \Longleftrightarrow & \Phi\colon D \to \pi/\Gamma \text{ is surjective} \\ & \underset{\substack{\uparrow \\ \text{Proposition 24.5}}}{\Longleftrightarrow} & \pi = N(\Gamma) \underset{\substack{\uparrow \\ \text{by the definitions}}}{\Longleftrightarrow} \Gamma \text{ is normal in } \pi. \end{array}$$

(1) ⇒ (4) We will show that the deck transformation group D has the desired properties.

We first show that the deck transformation group acts discretely on X. So let $x \in X$. We write $b = p(x)$. Since $p\colon X \to B$ is a covering there exists a connected open neighborhood U of b that is uniformly covered. This means that the restriction of p to each component of $p^{-1}(U)$ is a homeomorphism. Now let V be the component of $p^{-1}(U)$ that contains x. Then $gV \cap V = \varnothing$ for each non-trivial g. This shows that the action of D is discrete.

The map $p\colon X \to B$ has the property that $p(gx) = p(x)$ for all $g \in D$. Therefore the map p factors through a map $f\colon X/D \to B$. The map f is clearly surjective. Furthermore it is injective since we suppose that D acts transitively on each $p^{-1}(b)$. Finally the map f is not only continuous but, as follows easily from Lemma 7.4 (1), it is also open. Thus f is a homeomorphism.

(4) ⇒ (1) Now suppose that such a homeomorphism $f\colon X/G \to B$ exists. First note that for each $g \in G$ the map $x \mapsto gx$ not only defines a deck transformation of the covering $q\colon X \to X/G$ but also of the covering $p\colon X \to B$. Now let $b \in B$ and let $x, x' \in X$ with $p(x) = p(x')$. Since f is a homeomorphism we also have $q(x) = q(x')$. Hence there exists a $g \in G$ with $gx = x'$.

Now we suppose that p is a finite covering. We show the following equivalence:

(5) ⇔ (3) We have the following equivalences:[491]

$$\begin{array}{rcl} \#D = [X : B] & \underset{\substack{\text{Lemma 7.16 (3)} \\ \downarrow}}{\Longleftrightarrow} & \#D = [\pi : \Gamma] \underset{\substack{\text{Proposition 24.5} \\ \downarrow}}{\Longleftrightarrow} [N(\Gamma) : \Gamma] = [\pi : \Gamma] \\ & \Longleftrightarrow & N(\Gamma) = \pi \quad \Longleftrightarrow \Gamma \subset \pi \text{ is a normal subgroup.} \end{array}$$

□

We conclude this chapter with the following proposition which often, rather conveniently, allows us to replace an irregular covering by a regular covering.

Proposition 24.8. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces that are path-connected reasonable. If p is a finite-degree covering, then there exists a covering $q\colon (Y, y_0) \to (X, x_0)$ such that $p \circ q\colon (Y, y_0) \to (B, b_0)$ is a finite-degree* regular *covering.*

[491] Where do we actually use in the argument that the covering is finite?

For convenience this statement is summarized in the following diagram:

$$\begin{tikzcd}
(Y, y_0) \arrow[d, "\text{(2) there exists a covering } q"'] \arrow[dd, bend left, "\text{(3) such that } p \circ q \text{ is a finite-degree regular covering}"] \\
(X, x_0) \arrow[d, "\text{(1) given a finite-degree covering } p"'] \\
(B, b_0).
\end{tikzcd}$$

The proof of Proposition 24.8 rests on the following group-theoretic concept.

Definition. Let G be a group and let H be a subgroup. We refer to

$$H^c \ := \ \bigcap_{g \in G} gHg^{-1}$$

as the *normal core of* G.

The following lemma summarizes a few properties of the normal core.

Lemma 24.9. *Let G be a group and let H be a subgroup.*

(1) *The normal core of H is a normal subgroup of G that is contained in H.*
(2) *If X is a subgroup of G that is contained in H and that is normal in G, then $X \subset H^c$.*
(3) *If H is of finite-index in G, then H^c is also of finite-index in G.*

Proof of Lemma 24.9.

(1) The proof of the first statement is a routine exercise.
(2) Let X be a subgroup of G that is contained in H and that is normal in G. Then

$$\begin{array}{ccccccc} X & = & \bigcap_{g \in G} gXg^{-1} & \subset & \bigcap_{g \in G} gHg^{-1} & = & H^c. \\ & & \uparrow & & \uparrow & & \\ & & \text{since } X \text{ normal in } G & & \text{since } X \subset H & & \end{array}$$

(3) The proof of the last statement is arguably more interesting. Thus let H be a finite-index subgroup of G. We consider the set G/H of right coset of G with respect to H. We denote by $\mathrm{Perm}(G/H)$ the group of permutations of G/H, i.e. it is the group of all bijections of the set G/H. Note that G/H and thus also $\mathrm{Perm}(G/H)$ are finite. We consider the map

$$\begin{array}{rcl} \varphi \colon G & \mapsto & \mathrm{Perm}(G/H) \\ g & \mapsto & \left(\begin{array}{ccc} G/H & \to & G/H \\ xH & \mapsto & gxH \end{array} \right). \end{array}$$

This map is easily seen to be a homomorphism. Since $\mathrm{Perm}(G/H)$ is finite we see that $\ker(\varphi)$ is a finite-index normal subgroup of G. We claim that $\ker(\varphi) \subset H$. Thus let $g \in \ker(\varphi)$. This implies in particular that $\varphi(g)(H) = H$, but that means that $gH = H$, i.e. that $g \in H$. We have thus shown that indeed $\ker(\varphi) \subset H$. From (2)

and the above observation that $\ker(\varphi)$ is normal in G we obtain that $\ker(\varphi) \subset H^c$. But since $\ker(\varphi)$ has finite-index in G we see that H^c also has finite-index in G. $\square$

Now we can provide the proof of Proposition 24.8.

Proof of Proposition 24.8. Let $p\colon (X, x_0) \to (B, b_0)$ be a finite-degree covering of pointed topological spaces that are path-connected and reasonable. In the following we write $G := \pi_1(X, x_0)$ and $H := p_*(\pi_1(B, b_0))$. By Lemma 7.16 we know that H is a finite-index subgroup of G. It follows from Lemma 24.9 that the normal core H^c of H in G is a finite-index normal subgroup of G that is contained in H. Since B is reasonable we obtain from Proposition 20.8 a finite-degree path-connected covering $r\colon (Y, y_0) \to (B, b_0)$ of pointed topological spaces such that $r_*(\pi_1(Y, y_0)) = H^c$.

The existence of the desired covering map $q\colon (Y, y_0) \to (X, x_0)$ with $r = p \circ q$ is an immediate consequence of Proposition 20.9 and the fact that H^c is contained in H. $\square$

25. Related constructions in algebraic geometry and Galois theory (∗)

In this short chapter we indicate some connections between the topics that we have covered so far to algebraic geometry and Galois theory. Since the author of these notes does not know much about either field the following discussion should be taken with a grain of salt. In particular for an expert the subsequent discussion will certainly look too simplistic.

25.1. **The fundamental group of an algebraic variety** (∗).

Definition. Let $\mathbb{K}$ be a field.[492]

(1) Given a set of multivariable polynomials S, i.e. given a subset $S \subset \mathbb{K}[t_1, \dots, t_n]$ we denote by

$$V(S) \;=\; \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}$$

the *vanishing set of* S, i.e. the set of all points in $\mathbb{K}^n$ where all the polynomials in S, viewed as maps $\mathbb{K}^n \to \mathbb{K}$, vanish.

(2) We say $U \subset \mathbb{K}^n$ is *Zariski open*,[493] if there exists a subset S of $\mathbb{K}[t_1, \dots, t_n]$ such that $U = \mathbb{K}^n \setminus V(S)$.

Remark. Let S be a subset of $\mathbb{K}[t_1, \dots, t_n]$. Since the ring $\mathbb{K}[t_1, \dots, t_n]$ is Noetherian there exists a finite set $\{f_1, \dots, f_k\} \subset \mathbb{K}[t_1, \dots, t_n]$ such that $V(S) = V(f_1, \dots, f_k)$.

Lemma 25.1. *Let $\mathbb{K}$ be a field. The Zariski open subsets form a topology on $\mathbb{K}^n$.*

Remark.

(1) The topology defined by the Zariski open subsets is of course called the *Zariski topology on* $\mathbb{K}^n$.

(2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every Zariski open subset is also open with respect to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n = \mathbb{R}^{2n}$. But for $n \geq 1$ the converse does not hold, in fact "most" subsets that are open with respect to the usual topology are not open with respect to the Zariski topology.

Proof. Let $\mathbb{K}$ be a field.

(1) The vanishing set of the empty set, i.e. $V(\varnothing)$ is all of $\mathbb{K}^n$, hence the empty set $\varnothing = \mathbb{K}^n \setminus V(\varnothing)$ is open.

(2) The vanishing set of the constant polynomial $f = 1$ is the empty set, therefore $\mathbb{K}^n = \mathbb{K}^n \setminus V(1)$ is open.

(3) We need to show that the intersection of finitely many Zariski open sets is again Zariski open. By induction it suffices to consider the case of two Zariski open sets.

[492] For example we could take $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ but $\mathbb{K}$ could also be a field of non-zero characteristic, e.g. for a prime p we could take $\mathbb{K} = \mathbb{F}_p$ or $\mathbb{K}$ could be the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

[493] Oscar Zariski (1899-1986) was a Russian-American mathematician working in algebraic geometry.

So suppose we are given Zariski open sets $X = \mathbb{K}^n \setminus V(S)$ and $Y = \mathbb{K}^n \setminus V(T)$. Then

$$\mathbb{K}^n \setminus (X \cap Y) \;=\; (\mathbb{K}^n \setminus X) \cup (\mathbb{K}^n \setminus Y) \;=\; V(S) \cup V(T) \;=\; V\big(\{s \cdot t\}_{s \in S, t \in T}\big).$$

(4) We need to show that the union of Zariski open sets is again Zariski open. So suppose we are given a family of Zariski open sets $X_i = \mathbb{K}^n \setminus V(S_i)$, $i \in I$. Then

$$\mathbb{K}^n \setminus \bigcup_{i \in I} X_i \;=\; \bigcap_{i \in I} (\mathbb{K}^n \setminus X_i) \;=\; \bigcap_{i \in I} V(S_i) \;=\; V\Big(\bigcup_{i \in I} S_i\Big).$$

Thus we have verified that the Zariski open sets have all the properties of a topology. □

Definition. Let $\mathbb{K}$ be a field.

(1) A *variety over* $\mathbb{K}$ is defined as the vanishing set of a set of multivariable polynomials.
(2) The *Zariski topology on a variety* $V \subset \mathbb{K}^n$ is defined as the subspace topology on V induced from the Zariski topology on $\mathbb{K}^n$.

Example. Let $\mathbb{K}$ be a field and let and $f(s,t) = s^2 + t^2 - 1$. We define

$$S^1_{\mathbb{K}} \;:=\; V(f) \;=\; \{(x,y) \in \mathbb{K}^2 \mid f(x,y) = x^2 + y^2 - 1 = 0\} \;=\; \{(x,y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}.$$

For $\mathbb{K} = \mathbb{R}$ we obtain the usual circle S^1.

Any variety V is again a topological space, so in principle one could study the usual fundamental group. But in many cases, e.g. if the characteristic of $\mathbb{K}$ is non-zero, one will see that all continuous maps $S^1 \to V$ are constant, hence the usual fundamental group is trivial.

The correct analogue of a finite covering $p\colon X \to B$ between topological spaces for varieties is a finite étale morphism. The precise definition of a finite étale morphism is irrelevant for our purpose.[494] A heuristic fact is that varieties tend to have many finite étale morphisms, but there is no well-behaved analogue of an infinite covering in the context of varieties. In particular usually there is no concept of a "universal" étale morphism. The question is now, whether one can recover the notion of a fundamental group from the knowledge of finite covers.

Given a variety V we can consider all finite regular étale morphisms $p_i\colon W_i \to V$, $i \in I$. The corresponding deck transformations groups $D(p_i\colon W_i \to V)$, $i \in I$ are finite groups that form an inverse system. Now we can define the étale fundamental group of V as

$$\pi_1^{\text{ét}}(V) \;:=\; \varprojlim_i D(p_i\colon W_i \to V).$$

By Proposition 16.10 the étale fundamental group $\pi_1^{\text{ét}}(V)$ is naturally a compact topological group. In algebraic geometry the étale fundamental group of a variety plays a role similar to the role of the fundamental group of a topological space in topology.

If V is a complex variety without singularities, then the étale fundamental group is isomorphic to the profinite completion[495] of the usual topological fundamental group $\pi_1(V)$,

[494] Furthermore it is unknown to the author of these lecture notes.

[495] Recall that on page 505 we introduced the profinite-completion $\widehat{\pi}$ of a group π.

i.e. we have

$$\pi_1^{\text{ét}}(V) \;\cong\; \widehat{\pi_1(V)}.$$

We refer to [Groth] for a proof of this statement.

More information on this topic can be found at

`https://en.wikipedia.org/wiki/etale_fundamental_group`

25.2. **Galois theory** $(*)$**.** The theory of coverings of topological spaces has some similarities to the Galois theory of field extensions.

Throughout this section let K be a field of characteristic zero[496] and let L/K be a field extension of K, i.e. L is a field which contains K as a subfield. The dimension of L as a K vector space is denoted by $[L : K]$. We consider the group

$$\mathrm{Gal}(L/K) = \text{ all field automorphisms of } L \text{ which are the identity on } K.$$

Put differently, a field automorphism $f\colon L \to L$ lies in $\mathrm{Gal}(L/K)$ if and only if the following diagram commutes:

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ L \arrow[rr, "f"] & & L. \end{tikzcd}$$

We consider two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt{2})\colon \mathbb{Q}] = 2$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ consists of the identity and the homomorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt[3]{2})\colon \mathbb{Q}] = 3$ and the group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ consists only of the identity.

Given a field K we denote by $\overline{K}$ the algebraic closure of K. A field extension L/K is called *Galois*, if for each $f \in \mathrm{Gal}(\overline{K}/K)$ we have $f(L) = L$.[497] In the usual course on Galois theory one shows that L/K is Galois if and only if each polynomial in $K[x]$ which admits a zero in L splits over L, i.e. all other zeros of f lies also in L. For a Galois extension L/K we refer to $\mathrm{Gal}(L/K)$ as the *Galois group* of the field extension.

We return to the above two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois field extension.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois field extension, since $\sqrt[3]{2}$ is a zero of x^3-2, but the complex zero $\sqrt[3]{2}e^{2\pi i/3}$ of f does not lie in $\mathbb{Q}(\sqrt[3]{2})$.[498]

If L/K is a finite extension, then the main theorem of Galois theory says that the following two statements are equivalent:

(1) The field extension L/K is Galois.

[496] This restriction to characteristic zero is not necessary, but it simplifies the discussion.

[497] Strictly speaking this is the definition of a normal extension, but since K is a field of characteristic zero, a normal extension is already Galois.

[498] We put $z = \sqrt[3]{2}$ and $w = e^{2\pi i/3}$. It is a good exercise to show that $\mathbb{Q}(z, w)/\mathbb{Q}$ is a normal field extension with $[\mathbb{Q}(z, w)\colon \mathbb{Q}] = 6$. The group $\mathrm{Gal}(\mathbb{Q}(z, w)/\mathbb{Q})$ is isomorphic to the permutation group S_3.

(2) $\#\operatorname{Gal}(L/K) = [L : K]$.

The statement is formally very similar to the statement of Corollary 24.7 which says that for some finite covering $p\colon X \to B$ of topological spaces the following statements are equivalent:

(1) The covering $p\colon X \to B$ is regular.
(2) $\#D(p\colon X \to B) = [X : B]$.

The analogy between Galois theory and covering theory goes considerably further. Many statements about the Galois group of a Galois extension have a corresponding statement for the deck transformation group of a normal covering. For example, given a finite extension L/K the (normal) subgroups of Galois group $\operatorname{Gal}(L/K)$ correspond to (normal) intermediate fields and (normal) subgroups of the deck transformation group correspond to (regular) intermediate coverings.

Now let L/K be a Galois extension of infinite degree. All finite intermediate Galois extensions of K, i.e. all finite Galois extensions F/K with $F \subset L$, form a preordered set via the inclusion. The corresponding Galois groups form an inverse set. Indeed, given two finite Galois extensions F/K and G/K with $F \subseteq G$ the restriction of an automorphism of G to an automorphism of F defines a group homomorphisms $\operatorname{Gal}(G/K) \to \operatorname{Gal}(F/K)$. Therefore we can form the inverse limit of these Galois groups. By [FJ, Section 1.3] there exists a canonical isomorphism of groups

$$\operatorname{Gal}(L/K) \;=\; \varprojlim_{\substack{F/K \text{ finite} \\ \text{Galois extension}}} \operatorname{Gal}(F/K).$$

On the right-hand side we consider the inverse limit of an inverse system of finite groups. It follows from Proposition 16.10 that the right-hand side, and thus also the Galois group $\operatorname{Gal}(L/K)$, is naturally a compact topological group. This topology on $\operatorname{Gal}(L/K)$ is called the *Krull topology*.

Example. Let p be a prime. We denote by $\mathbb{F}_p$ the finite field with p elements and we denote by $\overline{\mathbb{F}}_p$ its algebraic closure. It is shown in [FJ, Section 1.5] that there exists a canonical isomorphism

$$\operatorname{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right) \;\cong\; \widehat{\mathbb{Z}}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$, or equivalently the profinite integers that we introduced on page 502.

26. CW-complexes I: Definitions, basic properties and constructions

In this chapter we will introduce CW-complexes. Initially the definition of a CW-complex might look rather confusing and artificial. But in this and the subsequent chapter we see that the notion of a CW-complex strikes the right balance:

(1) CW-complexes are topological space that have many desirable properties,
(2) the notion is flexible enough so that most of the topological spaces that we are interested in are actually CW-complexes.

26.1. **A word on homotopies.** Before we introduce CW-complexes we have to spend a minute talking about the concept of homotopies. Recall that on page 299 we defined what it means for two maps $f, g\colon X \to Y$ between topological spaces to be homotopic. Before we do anything else we want to discuss two relative versions of "homotopy". The first version we already saw on page 301. For the reader's convenience we recall the definition.

Definition. Let $f, g\colon X \to Y$ be two maps between topological spaces and let $A \subset X$ be a subset such that $f|_A = g|_A$.

(1) A *homotopy between f and g rel A* is a homotopy $H\colon X \times [0,1] \to Y$ between f and g such that $H(a,t) = f(a) = g(a)$ for all $t \in [0,1]$ and all $a \in A$.
(2) We say *f and g are homotopic rel A*, if such a homotopy rel A exists.

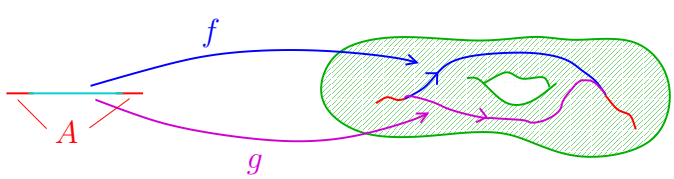


Figure 390. The maps f and g are homotopic, but they are not homotopic rel A.

We continue with another notion of a "relative homotopy".

Definition. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) A map $f\colon (X,A) \to (Y,B)$ is defined as a map[499] $f\colon X \to Y$ such that $f(A) \subset B$.

Now let $f, g\colon (X,A) \to (Y,B)$ be two maps.

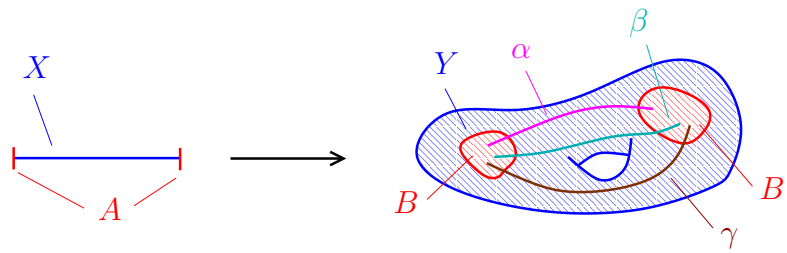
(2) A *homotopy between f and g* is a map[500]

$$H\colon X \times [0,1] \to Y,$$

such that the following two conditions are satisfied:
(a) $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$, and
(b) $H(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.

[499] As usual all maps between topological spaces are understood to be continuous.
[500] For $A = B = \varnothing$ we get the usual notion of a homotopy between maps $f, g\colon X \to Y$.

(3) If there exists a homotopy between f and g, then we say that f and g are *homotopic* and we write $f \simeq g$.



$\alpha, \beta\colon (X, A) \to (Y, B)$ are homotopic but $\alpha, \gamma\colon (X, A) \to (Y, B)$ are not homotopic

FIGURE 391.

For completeness we also mention the following definition which is the obvious relative analogue of the concepts that we introduced on page 307.

Definition. Let (X, A) and (Y, B) be two pairs of topological spaces.

(1) A map $f\colon (X, A) \to (Y, B)$ is a *homotopy equivalence between* (X, A) *and* (Y, B), if there exists a map $g\colon (Y, B) \to (X, A)$ such that the following hold:
   (a) $g \circ f$ and id_X are homotopic as maps of (X, A) to itself,
   (b) $f \circ g$ and id_Y are homotopic as maps of (Y, B) to itself.

(2) If there exists a homotopy equivalence between (X, A) and (Y, B), then we say that (X, A) and (Y, B) are *homotopy equivalent*.

Example. The two pairs of topological spaces (X, A) and (Y, B) shown in Figure 392 are homotopy equivalent.

FIGURE 392.

The above notions can be generalized even further. For example, one could define triples of topological spaces and there is an obvious definition of what it means for maps $f, g\colon (X, A, B) \to (Y, C, D)$ between triples of topological spaces to be homotopic. Similarly, given pairs of topological spaces (X, A) and (Y, B) and a subset Z of A there is an obvious definition for what it means for two maps $f, g\colon (X, A) \to (Y, B)$ to be homotopic rel Z. We will not spell out all the definitions. From the context it should always be clear what we mean.

26.2. **Definition of finite-dimensional CW-complexes and examples.** We start out with the definition of a finite-dimensional CW-complex and its topological realization.

Definition.

(a) A *CW-complex X of dimension* ≤ -1 is defined to be the empty set. Its *topological realization* $|X|$ is also defined to be the empty set.

(b) We define inductively higher dimensional CW-complexes. Let $n \in \mathbb{N}_0$. A *CW-complex X^n of dimension* $\leq n$ consists of the following:
   (a) a CW-complex X^{n-1} of dimension $\leq n-1$, with topological realization denoted by $|X^{n-1}|$,
   (b) together with a set of maps $\{\varphi_i \colon S^{n-1} \to |X^{n-1}|\}_{i \in I_n}$.

   The *topological realization* $|X^n|$ *of* X^n is defined as the topological space

$$|X^n| \ := \ \Big(|X^{n-1}| \ \sqcup \ \bigsqcup_{i \in I_n} \overline{B}^n\Big) / \sim$$

   where for each $i \in I_n$ we use the map φ_i to identify the points on the corresponding copy of $S^{n-1} = \partial \overline{B}^n$ with the corresponding points in $|X^{n-1}|$.

(c) A *CW-structure* for a topological space Y is a CW-complex X together with a homeomorphism $f \colon |X| \to Y$.

Remark.

(1) Note that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$. It follows from the above definition that a CW-complex of dimension ≤ 0 is exactly the same as set of points with the discrete topology.

(2) CW-complexes were introduced by John H. C. Whitehead[501]. The name comes from "C" for "closure-finite", and the "W" for "weak topology"[502]. Some people also say it goes back to the initials C. W. of Whitehead.

(3) In the notation we usually do not distinguish between the CW-complex and its topological realization, i.e. we usually denote the topological realization of a CW-complex X again by X.

Before we discuss a long list of examples of CW-complexes we want to introduce a few more definitions.

Definition. Let X be a CW-complex as above. We continue with the notation used in the previous definition

(1) The images of the $\overline{B}_i^k$, $i \in I_k$, in X are called the *k-dimensional cells*, or often short *k-cells*, of X.

(2) The images of the B_i^k, $i \in I_k$, in X are called the *open k-cells* of X.

(3) The maps $\varphi_i \colon S_i^{k-1} \to X$ are called the *attaching maps* of the k-cells and the maps $\overline{B}_i^k \to X$ are referred to as the *characteristic maps* of the k-cells.

[501] John Henry Constantine Whitehead (1904-1960) was a British mathematician.

[502] We will explain later on what "closure-finite" and "weak topology" mean.

(4) The union of the cells of dimension $\leq k$ is called the *k-skeleton* of X and we denote it by X^k.
(5) If X is non-empty, then we define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ as the maximal dimension of a cell.

Example. We consider the topological space which is sketched in Figure 393, namely

$$Y \;:=\; [-1,1] \times [-1,1] \cup \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + (y-1)^2 = 1 \text{ and } y \geq 1\}.$$

The topological space X thus consists of a square together together with a semicircle.

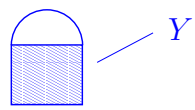


FIGURE 393.

Our goal is to find a CW-structure for X. First we consider the 0-dimensional CW-complex which consists of the 4 vertices of the square. Next we add a 1-cell. To do so we consider the map φ which sends $\partial\overline{B}^1 = \{\pm 1\}$ to the two points $\pm 1 \times -1$. The topological space $X^0 \cup_\varphi \overline{B}^1$ consists of the four points $\{\pm 1, \pm 1\}$ together with an edge which connects the lower two points. This topological space is sketched in the center of Figure 394. The

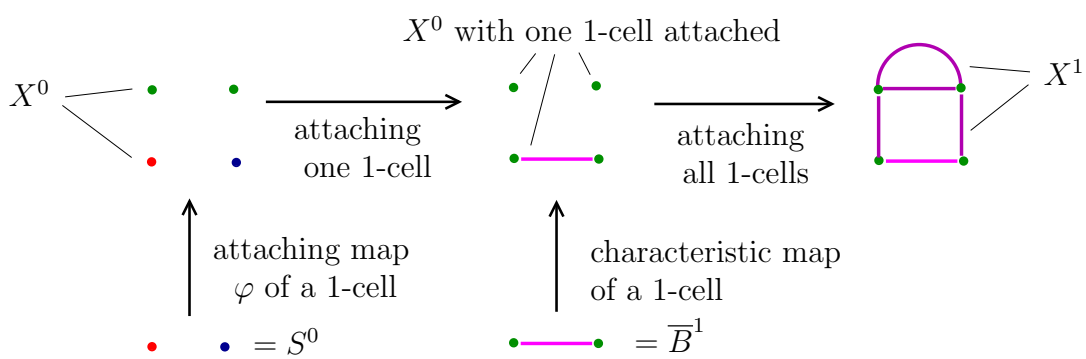


FIGURE 394. The CW-complex X^1.

same way we attach four more 1-cells and we obtain the 1-dimensional CW-complex which is sketched on the right of Figure 394. Finally we attach a 2-cell to X^1 with the attaching map shown in Figure 395 and we obtain a CW-complex X^2 whose topological realization is homeomorphic to Y.

Now we will see that most of the topological spaces that we have encountered so far can be described as CW-complexes.

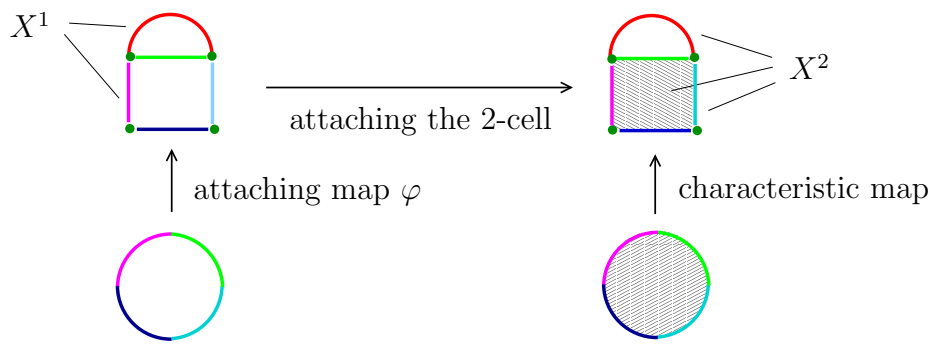


FIGURE 395. Attaching the 2-cell to the 1-skeleton X^1.

(1) Any spatial graph, as defined on page 134, is a one-dimensional CW-complex where the vertices correspond precisely to the 0-cells and the edges correspond precisely to the open 1-cells. In the following we refer to every one-dimensional CW-complex also as a topological graph. Here we refer to the 0-cells as the vertices and to the open 1-cells as the edges.[503]

(2) As a special case of (1) we see that S^1 can be viewed as a CW-complex with one 0-cell and one 1-cell. Furthermore we can view the wedge $S^1 \vee S^1$ as a CW-complex with one 0-cell and two 1-cells. Finally $\mathbb{R}$ can be viewed as a CW-complex where the 0-cells are given by the integers $\mathbb{Z}$ and the 1-cells are given by the intervals $[k, k+1]$ with $k \in \mathbb{Z}$. We refer to Figure 396 for an illustration.

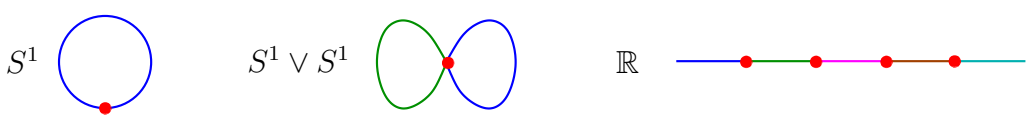


FIGURE 396.

(3) Now we consider the torus $T = [0,1] \times [0,1]/\sim$. We denote by $p\colon [0,1] \times [0,1] \to T$ the obvious projection map. We can view T as a CW-complex with one 0-cell corresponding to the point $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to $p(0 \times [0,1]) = p(1 \times [0,1])$ and to $p([0,1] \times 0) = p([0,1] \times 1)$ and one 2-cell corresponding to $p([0,1] \times [0,1])$. Note that since we have only one 0-cell there is a unique choice of attaching map for the 1-cells. Finally the attaching map for the 2-cell is illustrated

[503]In fact, if X is a CW-complex with finitely many 0-cells and 1-cells, then it is relatively straightforward to show that there exists a graph G in the sense of page 134 such that G is homeomorphic to X.

in Figure 397.[504] In Figure 397 we sketch the CW-structure of the torus and also the corresponding CW-structure of the torus viewed as a subset of $\mathbb{R}^3$. We refer to it as the *standard CW-structure* of the torus.

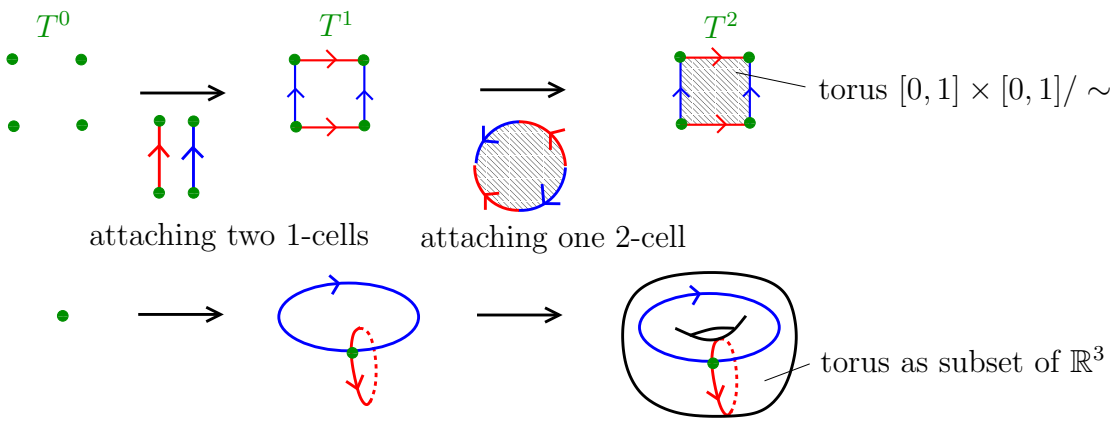


FIGURE 397. The torus as CW-complex.

(4) Next we consider the Klein bottle $K = [0,1]\times[0,1]/\sim$. As usual we denote by $p\colon [0,1]\times[0,1] \to K$ the projection map. Similar to the torus we can view K as CW-complex with one 0-cell corresponding to $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to $p(0\times[0,1]) = p(1\times[0,1])$ and to $p([0,1]\times 0) = p([0,1]\times 1)$ and one 2-cell corresponding to $p([0,1]\times[0,1])$ with the attaching map sketched in Figure 398.[505]
The 1-skeleta of the torus and the Klein bottle are both wedges of two circles. Furthermore both CW-complexes have precisely one 2-cell. The difference between the torus and the Klein bottle is that the attaching maps for the 2-cells are different.
(5) In Figure 399 we show that we can endow the torus with different CW-structures.
(6) In Figure 400 we sketch how we can view the projective plane $\mathbb{R}\mathrm{P}^2$, the Möbius band and the surface of genus 2 as CW-complexes.[506]

[504]Strictly speaking the attaching map is given by

$$\begin{array}{ccl} S^1 & \mapsto & T^1 \\ e^{it} & \mapsto & \begin{cases} p(\frac{2}{\pi}t, 0), & \text{if } t\in[0,\frac{\pi}{2}] \\ p(1, \frac{2}{\pi}(t-\frac{\pi}{2})), & \text{if } t\in[\frac{\pi}{2},\pi] \\ p(1-\frac{2}{\pi}(t-\pi), 1), & \text{if } t\in[\pi,\frac{3\pi}{2}] \\ p(0, 1-\frac{2}{\pi}(t-\frac{3}{2}\pi)), & \text{if } t\in[\frac{3\pi}{2},2\pi]. \end{cases} \end{array}$$

But writing out the attaching map with explicit formulas is clearly not very instructive.

[505]Evidently, as in Footnote 504, one could spell out the attaching map with precise formulas, but there is no point in doing it.

[506]What does the 1-skeleton of the given CW-structure for the surface of genus two look like, if we view the surface Σ as a "physical object" in $\mathbb{R}^3$?

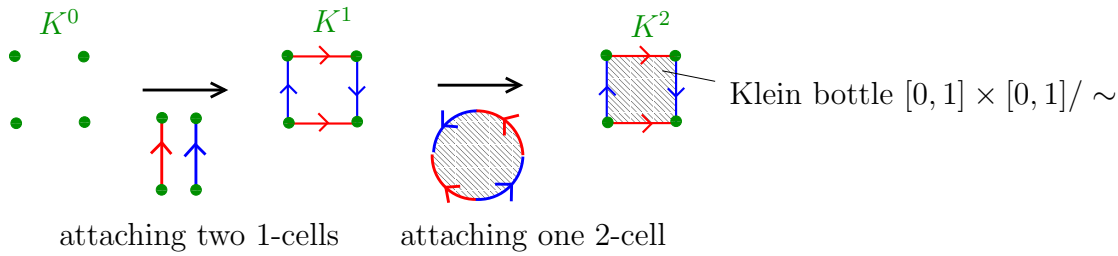


FIGURE 398. The Klein bottle as CW-complex.

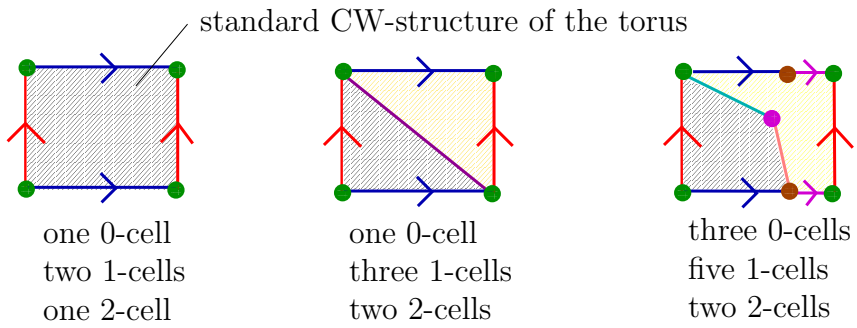


FIGURE 399. The torus with three different CW-structures.

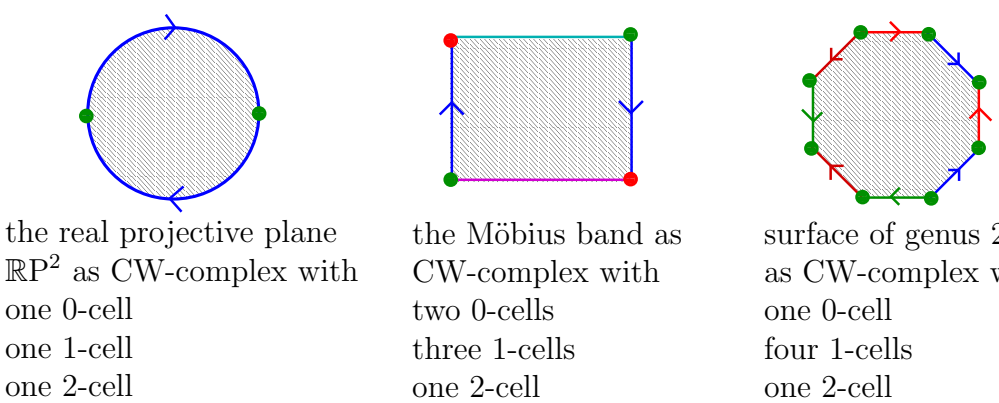


FIGURE 400.

(7) We conclude this discussion of surfaces with the 2-sphere. We can describe the 2-sphere as a CW-complex X as follows: We take a single 0-cell given by a point P and one 2-cell where the attaching map $\varphi\colon S^1 \to X^0 = \{P\}$ sends the whole boundary S^1 to

the point P. The topological realization of X is thus given by

$$X^0 \cup_\varphi \overline{B}^2 \;=\; \{P\} \cup_{P=\partial\overline{B}^2} \overline{B}^2 \;=\; \overline{B}^2/S^1 \;\underset{\substack{\uparrow \\ \text{see page 99}}}{\cong}\; S^2.$$

The example of the 2-sphere can be generalized to spheres of arbitrary dimensions: given any $n \geq 1$ we can view S^n as a CW-complex with one 0-cell and one n-cell.

We continue with the following basically self-explanatory definition.

Definition. A *subcomplex* of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X.

Example. In Figure 401 we consider the topological space X that is given by removing an open disk from a torus. We see that X has the structure of a CW-structure such that the boundary ∂X is a CW-subcomplex.

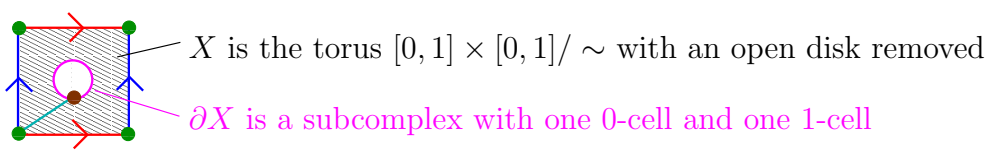


FIGURE 401.

The following lemma shows that some straightforward constructions provide examples of CW-complexes.

Lemma 26.1. *If A is a subcomplex of a CW-complex X, then the CW-structure on X restricts to a CW-structure on A.*

Proof. We set $|A^{-1}| = \varnothing$. For $n = 0, 1, 2, 3, \dots$ we iteratively consider the attaching maps $\{\varphi_i \colon S^{n-1} \to |A^{n-1}|\}_{i\in I}$ from the CW-structure of X whose image lies in $|A^{n-1}|$ and we set $|A^n| \;:=\; \Big(|A^{n-1}| \sqcup \bigsqcup_{i\in I} \overline{B}^n\Big)/\sim$. One can easily verify that this defines a CW-structure on A. $\square$

We continue the discussion of CW-complexes with two more explicit examples, namely the real projective spaces and the complex projective spaces which we had already defined on page 111. In the following let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$. We denote by $\sim$ the equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$ given by multiplication by an element in $\mathbb{A} \setminus \{0\}$ and we define

$$\mathbb{A}\mathrm{P}^n \;:=\; (\mathbb{A}^{n+1} \setminus \{0\})/\sim .$$

On page 192 and Lemma 21.6 we saw that $\mathbb{A}\mathrm{P}^n$ is a manifold. Given a point $(v_0, \dots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \dots : v_n]$ its equivalence class. For $k \leq n$ the map

$$\begin{array}{rcl} \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \dots : v_k] & \mapsto & [v_0 : \dots : v_k : 0 : \dots : 0] \end{array}$$

is easily seen to be an embedding and we use it to view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^n$.

We can give an explicit description of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as CW-complexes.

Lemma 26.2. *Let $n \in \mathbb{N}_0$.*

(1) *We can view $\mathbb{R}\mathrm{P}^n$ as a CW-complex with the following properties:*
   (a) *the CW-structure has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells,*
   (b) *for each $k \leq n$ the k-skeleton of $\mathbb{R}\mathrm{P}^n$ is given by $\mathbb{R}\mathrm{P}^k$.*

(2) *We can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex with the following properties:*
   (a) *the CW-structure has exactly one cell in the dimensions $0, 2, \ldots, 2n$ and no other cells,*
   (b) *for each $k \leq n$ the $2k$-skeleton of $\mathbb{C}\mathrm{P}^n$ is given by $\mathbb{C}\mathrm{P}^k$.*

Proof. We give the proof for $\mathbb{R}\mathrm{P}^n$. The proof for $\mathbb{C}\mathrm{P}^n$ is basically identical. We prove the lemma by induction on n. For $n = 0$ the topological space $\mathbb{R}\mathrm{P}^0$ consists of a single point which means that it is a CW-complex with a single 0-cell. Now suppose that we have equipped $\mathbb{R}\mathrm{P}^{n-1}$ with a CW-structure with the properties stated in (a) and (b). We view $\mathbb{R}\mathrm{P}^{n-1}$ as the subspace of $\mathbb{R}\mathrm{P}^n$ that is given by setting the last coordinate to zero. We consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n : 0]. \end{array}$$

It remains to show that there exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ that restricts to the identity on $\mathbb{R}\mathrm{P}^{n-1}$. We consider the map[507]

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ \underbrace{[(rx_1, \ldots, rx_n)]}_{\substack{\text{with } r \in [0,1] \text{ and} \\ (x_1, \ldots, x_n) \in S^{n-1}}} & \mapsto & [x_1 : \ldots : x_n : 1 - r]. \end{array}$$

It is clear that the restriction to $\mathbb{R}\mathrm{P}^{n-1}$ is the identity. Using Lemmas 1.81 and 1.66 and Proposition 1.40 one can easily verify that the map is a homeomorphism. □

We conclude this section with the following result which is proved in [Hirs, Section 6.4], see also [Mi4, Theorem 3.5] or [Ni, pages 21 and 41] or [Mun2, Theorem 10.6]. The second statement follows from the argument of the proof of [Mc, Theorem 5.1].

Theorem 26.3. *Every compact connected n-dimensional manifold M admits a finite CW-structure with the following properties:*

(1) *the maximal dimension of a cell is n,*
(2) *there is precisely one cell of dimension n,*
(3) *the boundary ∂M is a subcomplex.*

[507] Here we use that φ is a surjection, i.e. any point in $\mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n$ is represented by a point in $\overline{B}^n$.

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

Remark. The theorem is stated for manifolds and not for topological manifolds. The situation for topological manifolds is more complicated and we will deal with it later on in 45.1.

In the above discussion we have seen that arguably most of the topological spaces we are interested in can be viewed as CW-complexes. But note that not every topological space can be described as a CW-complex. For example consider the topological space X which consists of precisely two points P and Q and which is endowed with the trivial topology. (Recall that this means that the only open sets are the empty set and all of X.) Then we cannot view X as CW-complex since any CW-complex with finitely many points consists only of 0-cells together with the *discrete* topology. For the same reason $\{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not a CW-complex.

Remark. Let X be a topological space and let $f, g \colon S^{n-1} \to X$ be two maps. If these two maps are homotopic, then the topological spaces $X \cup_f \overline{B}^n$ and $X \cup_g \overline{B}^n$ are homotopy equivalent. We leave the proof of this statement as a slightly challenging exercise.

26.3. **Two topologies on $\mathbb{R}^\infty$.** In the next section we want to introduce infinite-dimensional CW-complexes. Before we do so we want to get more acquainted with $\mathbb{R}^\infty$ and topologies on the set $\mathbb{R}^\infty$. First of all we write

$$\mathbb{R}^\infty \;:=\; \mathbb{R}^{(\mathbb{N})} \;:=\; \{(x_1, x_2, \dots) \,|\, \text{only finitely many } x_i \text{ are non-zero}\}.$$

As in the finite-dimensional case we can equip $\mathbb{R}^\infty$ with the Euclidean metric[508]

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \;:=\; \sqrt{\textstyle\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

By Lemma 1.3 the metric d defines a topology on $\mathbb{R}^\infty$, more precisely

$$\mathcal{S} \;:=\; \{U \subset \mathbb{R}^\infty \,|\, \text{ for every } P \in U \text{ there exists an } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

is a topology on $\mathbb{R}^\infty$.

This topology might feel like it is the "right topology" on $\mathbb{R}^\infty$. But the next two examples might sow some doubts.

Examples.

(1) We claim that the function

$$\begin{aligned} \alpha \colon \mathbb{R}^\infty &\to \mathbb{R} \\ (x_1, x_2, \dots) &\mapsto \sum_{i=1}^{\infty} 2^i \cdot x_i \end{aligned}$$

[508]Note that the series on the right is in fact a finite sum, since all but finitely many terms are zero. It is again straightforward to verify that d is a metric.

is not continuous with respect to $\mathcal{S}$. To show this claim we consider the open subset $(-1,1) \subset \mathbb{R}$, then $\alpha^{-1}((-1,1))$ contains the origin 0. We claim that there is no $\epsilon > 0$ with $B_\epsilon(0) \subset \alpha^{-1}((-1,1))$. Indeed, suppose such $\epsilon > 0$ exists. We pick $i \in \mathbb{N}$ with $\frac{1}{2^i} < \epsilon$ and we consider the point

$$Q \;=\; (0,\dots,0,\tfrac{1}{2^i},0,\dots).$$

Then $Q \in B_\epsilon(0)$ but $\alpha(Q) = 1$, i.e. $Q \notin \alpha^{-1}((-1,1))$.

(2) Let B be the infinite diagonal matrix where the diagonal entries are given by 2^i for $i = 1,2,\dots$, i.e. let

$$B \;:=\; \begin{pmatrix} 2 & 0 & 0 & \dots \\ 0 & 4 & 0 & \dots \\ 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that the map

$$\begin{array}{rcl} \beta\colon \mathbb{R}^\infty & \to & \mathbb{R}^\infty \\ x & \mapsto & B\cdot x \end{array}$$

is not continuous with respect to $\mathcal{S}$. Indeed, an argument similar to (1) shows that the preimage of the open set $U = B_1(0)$ is not open.

The maps α and β feel quite natural, so we might want to look around for a different topology on $\mathbb{R}^\infty$ for which these two maps are in fact continuous. Of course that would be achieved by equipping $\mathbb{R}^\infty$ with the discrete topology, but clearly that's not what we want. Another reasonable condition on the desired topology is that for any $n \in \mathbb{N}_0$ the inclusion map

$$\begin{array}{rcl} \iota_n\colon \mathbb{R}^n & \hookrightarrow & \mathbb{R}^\infty \\ (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_n,0,\dots) \end{array}$$

should also be continuous. This immediately eliminates the discrete topology as a contender for the "right topology".

In fact this last condition leads us almost immediately to an alternative topology on $\mathbb{R}^\infty$. We saw in the proof of Proposition 16.1 that in the category of topological spaces we have

$$\varinjlim \mathbb{R}^n \;=\; \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty$ is equipped with the "weak topology", i.e. with the topology

$$\mathcal{T} \;:=\; \{U \subset \mathbb{R}^\infty \,|\, \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Put differently, a map $f\colon \mathbb{R}^\infty \to X$ to a topological space X is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}$ the map $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty \xrightarrow{f} X$ is continuous.

Examples.

(1) By definition of $\mathcal{T}$ the inclusion map $\mathbb{R}^n \to \mathbb{R}^\infty$ is continuous with respect to $\mathcal{T}$.

(2) The above map $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}$ is continuous with respect to the topology $\mathcal{T}$. Indeed, for any $n \in \mathbb{N}$ the restriction $\mathbb{R}^n \to \mathbb{R}^\infty \to \mathbb{R}$ is a linear map, hence continuous.
(3) In Exercise 26.2 we will see that $\mathcal{S} \subset \mathcal{T}$. This implies in particular that the identity map defines a continuous map $(\mathbb{R}^\infty, \mathcal{T}) \to (\mathbb{R}^\infty, \mathcal{S})$.
(4) Let a_i, b_i, $i \in \mathbb{N}$ be real numbers. The infinite hyperrectangle

$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \dots$$

is easily seen to be an open subset of $(\mathbb{R}^\infty, \mathcal{T})$.[509]

The following lemma shows in particular that the above map $\beta\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the topology $\mathcal{T}$.

Lemma 26.4. *Let $C = (c_{ij})_{i,j\in\mathbb{N}}$ be an infinite diagonal matrix. Then the map*[510]

$$\begin{array}{rcl} \gamma\colon \mathbb{R}^\infty & \to & \mathbb{R}^\infty \\ x & \mapsto & C \cdot x \end{array}$$

is continuous with respect to the topology $\mathcal{T}$.

Proof. Let $U \in \mathcal{T}$. We need to show that $\gamma^{-1}(U)$ lies in $\mathcal{T}$. So given any $n \in \mathbb{N}$ we need to show that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. So let $n \in \mathbb{N}$. Given $m \in \mathbb{N}$ we denote by $C_{m,n}$ the submatrix of C given by the top left $m \times n$-submatrix of C and we denote by $\gamma_{m,n}\colon \mathbb{R}^n \to \mathbb{R}^m$ the map given by $x \mapsto C_{m,n} \cdot x$. We have[511]

$$\begin{aligned} \gamma^{-1}(U) \cap \mathbb{R}^n &= \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in U\big\} &&= \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in \bigcup_{m\in\mathbb{N}} (U \cap \mathbb{R}^m)\big\} \\ &= \bigcup_{m\in\mathbb{N}} \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in U \cap \mathbb{R}^m\big\} &&= \bigcup_{m\in\mathbb{N}} \big\{x \in \mathbb{R}^n \,\big|\, C_{m,n} \cdot x \in U \cap \mathbb{R}^m\big\}. \\ &= \bigcup_{m\in\mathbb{N}} \underbrace{\gamma_{m,n}^{-1}(\overbrace{U \cap \mathbb{R}^m}^{\text{open in } \mathbb{R}^m})}_{\substack{\text{open in } \mathbb{R}^n \text{ since} \\ \gamma_{m,n} \text{ is continuous}}}. \end{aligned}$$

This shows that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is a union of open subsets of $\mathbb{R}^n$, hence itself is open. □

This discussion shows hopefully that the following convention is reasonable.

Convention. Unless we say something else we will view $\mathbb{R}^\infty$ as a topological space with the topology $\mathcal{T}$. Put differently, we think of $\mathbb{R}^\infty$ as $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$. Similarly we define $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ as topological spaces.

---

[509] This example is less obvious. The topology $\mathcal{T}$ has more open sets than $\mathcal{S}$ which made it easier in (2) to argue that the map α is continuous. But here in (3) it also makes it harder, since we have to check the continuity for more open sets on the right-hand side.

[510] Does this also work for any infinite-square matrix?

[511] For each $k \in \mathbb{N}$ we view $\mathbb{R}^k$ as a subset of $\mathbb{R}^\infty$ via the inclusion ι_k.

Remark. As a warm-up exercise the reader can verify that the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1+iy_1, x_2+iy_2, \dots) & \to & (x_1, y_1, x_2, y_2, \dots) \end{array}$$

is a homeomorphism. At times we will use this homeomorphism to make the identification $\mathbb{C}^\infty = \mathbb{R}^\infty$.

In the future, the more we work with $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$, the more we will see that the "direct limit topology" is in fact the "right" topology to work with.

To get acquainted with this new topological space we want to prove the following lemma.

Lemma 26.5. *Given any compact subset K of $\mathbb{R}^\infty$ there exists an $n \in \mathbb{N}$ with $K \subset \mathbb{R}^n$.*

Proof. So let $K \subset \mathbb{R}^\infty$ be a compact set. Suppose that there is no $n \in \mathbb{N}$ such that $K \subset \mathbb{R}^n$. Then there exists an infinite increasing sequence $n_1, n_2, n_3, \dots$ of natural numbers and points $x_1, x_2, x_3, \dots$ in K such that $x_i \in \mathbb{R}^{n_i} \setminus \mathbb{R}^{n_i-1}$. We denote by X the set whose elements are precisely the x_i.

Claim 1. The set X is a discrete subset of $\mathbb{R}^\infty$.

To simplify the notation we assume that $n_i = i$ for all $i \in \mathbb{N}$. For each i we denote by z_i the i-coefficient of x_i. Put differently, z_i is the highest non-zero coefficient of x_i. Furthermore for each $i \in \mathbb{N}$ we pick $\epsilon_i \in (0, \frac{\|z_i|}{2})$. Finally, given $x \in \mathbb{R}$ and $\epsilon > 0$ we write $I_\epsilon(x) = (x-\epsilon, x+\epsilon)$. We define

$$\begin{array}{rcccccccccc} U_1 & = & I_{\epsilon_1}(z_1) & \times & I_{\epsilon_2}(0) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(0) & \times & \dots \\ U_2 & = & \mathbb{R} & \times & I_{\epsilon_2}(z_2) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(z_4) & \times & \dots \\ U_3 & = & \mathbb{R} & \times & \mathbb{R} & \times & I_{\epsilon_3}(z_3) & \times & I_{\epsilon_4}(0) & \times & \dots \\ \vdots & & & & & & & & & & \end{array}$$

As we remarked on page 652, each U_i is an open subset of $\mathbb{R}^\infty$. Evidently U_i contains x_i and thus each U_i is an open neighborhood of x_i. Furthermore the sets U_i, $i \in \mathbb{N}$ are easily seen to be disjoint. But this shows, by definition of the subspace topology, that X is discrete. This concludes the proof of the claim.

Claim 2. The set X is a closed subset of K.

We need to show that $X \setminus K$ is open. By definition of the subspace topology it suffices to show that $\mathbb{R}^\infty \setminus X$ is open. In the following we continue with the notation from the previous claim. Thus let $y \in \mathbb{R}^\infty$. We pick $n \in \mathbb{N}$ such that $y \in \mathbb{R}^n$. In $\mathbb{R}^n$ we can and will pick an open neighborhood U of $(y_1, \dots, y_n)$ that does not contain $x_1, \dots, x_n \in \mathbb{R}^n$. One can now verify easily that

$$V := U \times I_{\epsilon_{n+1}}(0) \times I_{\epsilon_{n+2}}(0) \times \dots$$

is an open neighborhood of $y \in \mathbb{R}^\infty$ that does not intersect X. This concludes the proof of the claim.

It follows from Claim 1, Claim 2 and Lemma 1.19 (2) that X is indeed finite. But that is a contradiction to our assumption that there is no $n \in \mathbb{N}$ such that K is contained in $\mathbb{R}^n$. $\square$

26.4. **Infinite-dimensional CW-complexes.**

Definition.

(a) A *CW-complex* X is defined as a sequence

$$X^{-1} \;:=\; \varnothing \subset X^0 \subset X^1 \subset X^2 \subset \dots$$

of CW-complexes (in the sense of the definition of page 643) such that for each $m \leq n$ the CW-complex X^m is the m-skeleton of X^n. The cells of the X^n are called the cells of X.

(b) We define the *topological realization* $|X|$ of such a CW-complex X as[512]

$$|X| \;:=\; \varinjlim |X^n|.$$

In other words, a subset $U \subset X$ is open if and only if for every $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is an open subset of X^n. As before, in the notation we usually do not distinguish between the notation for a CW-complex and its topological realization.

(c) We define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty CW-complex X as the maximal dimension of a cell.[513]

(d) We say a CW-complex is *finite* if it has only finitely many cells and we say it is *countable* if it has only countably many cells.

(e) A *CW-structure* for a topological space Y is a CW-complex X together with a homeomorphism $f\colon X \to Y$.

Example. We consider the infinite sphere

$$S^\infty \;=\; \{x \in \mathbb{R}^\infty \,|\, d(x, \text{origin}) = 1\} \;=\; \Big\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\Big|\, \sum_{i=1}^\infty x_i^2 = 1\Big\}$$

which equip with the subspace topology coming from $\mathbb{R}^\infty$. It follows easily from the fact that $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ that we have the equality $S^\infty = \varinjlim S^n$. We consider iteratively the spheres S^n as CW-complexes, where S^0 consists of two 0-cells and where S^n is built starting from S^{n-1} by attaching two n-cells ("the upper hemisphere and the lower hemisphere") to S^{n-1}. We then obtain an ascending sequence

$$S^0 \;\subset\; S^1 \;\subset\; S^2 \;\subset\; \dots$$

[512]This topology on a union of topological spaces is called the *weak topology*. This explains the "W" of the name CW-complex.

[513]Here we define $\dim(X) := \infty$ if X admits cells of arbitrarily large dimensions.

of CW-complexes which satisfies precisely the required properties of a CW-complex as defined above. This shows that the sphere S^∞ is an infinite-dimensional CW-complex. We refer to Figure 402 for an illustration.[514]

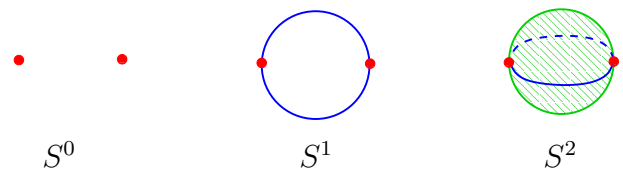


FIGURE 402.

FIGURE 403. An ascending sequence of CW-complexes $S^0 \subset S^1 \subset S^2 \dots$.

The following definition and notation is the obvious infinite-dimensional analogue of the finite-dimensional definitions introduced on page 111.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) We define the *infinite-dimensional projective space* $\mathbb{A}\mathrm{P}^\infty$ as the set
$$\mathbb{A}\mathrm{P}^\infty \;=\; (\mathbb{A}^\infty \setminus \{0\})/(\mathbb{A} \setminus \{0\})$$
together with the quotient topology.

(2) Given a point $(x_1, x_2, \dots) \in \mathbb{A}^\infty \setminus \{0\}$ we denote its equivalence class in $\mathbb{A}\mathrm{P}^\infty$ by $[x_1 : x_2 : \dots]$.

(3) Given $n \in \mathbb{N}$ we view $\mathbb{A}\mathrm{P}^n$ as a subset of $\mathbb{A}\mathrm{P}^\infty$ via the inclusion map
$$\begin{array}{ccc} \mathbb{A}\mathrm{P}^n & \to & \mathbb{A}\mathrm{P}^\infty \\ [x_0 : \dots : x_n] & \mapsto & [x_0 : \dots : x_n : 0 : \dots]. \end{array}$$

The following lemma shows in particular that we can view the real infinite projective space $\mathbb{R}\mathrm{P}^\infty$ as an infinite-dimensional CW-complex.

Lemma 26.6.

(1) *The inclusion maps $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ define a homeomorphism $\varinjlim \mathbb{R}\mathrm{P}^n \xrightarrow{\cong} \mathbb{R}\mathrm{P}^\infty$.*

(2) *The topological space $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure such that for every $n \in \mathbb{N}$ the n-skeleton is given by $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ where $\mathbb{R}\mathrm{P}^n$ is equipped with the CW-structure coming from Lemma 26.2.*

(3) *The infinite real projective space $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure which has precisely one n-cell for every $n \in \mathbb{N}_0$.*

(4) (a) *The action $\{\pm 1\} \times S^\infty \to S^\infty$ given by $(\epsilon, v) \mapsto \epsilon \cdot v$ is continuous and discrete.*
(b) *The obvious projection $S^\infty \to S^\infty/x \sim -x$ is a 2-fold covering.*

[514] If we equip S^∞ with the topology $\mathcal{S}$ coming from the Euclidean metric on $\mathbb{R}^\infty$, then this argument does not work and it is not clear (and presumably wrong), that $(S^\infty, \mathcal{S})$ is a CW-complex.

(c) *The map*

$$\begin{array}{rcl} S^\infty/x \sim -x & \to & \mathbb{R}\mathrm{P}^\infty \\ [(x_1, x_2, \dots)] & \mapsto & [x_1 : x_2 : \dots] \end{array}$$

is a homeomorphism.

Proof (∗).

(1) Given $n \in \mathbb{N}$ we denote by $p_n \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n$ the obvious projection map and we denote by $p \colon \mathbb{R}^\infty \setminus \{0\} \to \mathbb{R}\mathrm{P}^\infty$ the obvious projection map. Let $\mathcal{S}$ be the topology of $\mathbb{R}\mathrm{P}^\infty$. By definition $U \subset \mathbb{R}\mathrm{P}^\infty$ is open if and only if for every $n \in \mathbb{N}$ the set $p^{-1}(U) \cap (\mathbb{R}^{n+1} \setminus \{0\})$ is open. Next note that it follows from the discussion on page 488 that $\varinjlim \mathbb{R}\mathrm{P}^n = \mathbb{R}\mathrm{P}^\infty$ with the topology $\mathcal{T}$ that is defined by the condition that $U \subset \mathbb{R}\mathrm{P}^\infty$ is an open subset if and only if $U \cap \mathbb{R}\mathrm{P}^n$ is open for every $n \in \mathbb{N}$, i.e. U is open if and only if $p_n^{-1}(U \cap \mathbb{R}\mathrm{P}^n)$ is open for every $n \in \mathbb{N}$. Finally we make the observation that for every $U \in \mathbb{R}\mathrm{P}^\infty$ and every $n \in \mathbb{N}$ we have the equality

$$p_n^{-1}(U \cap \mathbb{R}\mathrm{P}^n) \;=\; p^{-1}(U) \cap (\mathbb{R}^{n+1} \setminus \{0\}).$$

It follows from this observation that $\mathcal{S} = \mathcal{T}$.

(2) This statement follows immediately from (1) and the definition of a CW-complex on page 654.

(3) This statement is an immediate consequence of (2) and the definition of the CW-structure on eahc $\mathbb{R}\mathrm{P}^n$.

(4) (a) The statement is not difficult to prove, but on needs to argue carefully with the given topology on S^∞. We will fill in the details in Exercise 26.3.
(b) This statement follows from (a) and Proposition 7.10.
(c) It is straightforward to verify that the given map is a bijection. We leave it as an exercise to the reader to verify that the map is continuous and that its inverse is continuous. □

The following lemma is the complex sibling of Lemma 26.6.

Lemma 26.7.

(1) *The inclusion maps* $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ *define a homeomorphism* $\varinjlim \mathbb{C}\mathrm{P}^n \xrightarrow{\cong} \mathbb{C}\mathrm{P}^\infty$.

(2) *The topological space* $\mathbb{C}\mathrm{P}^\infty$ *admits a CW-structure such that for every* $n \in \mathbb{N}$ *the* $2n$*-skeleton is given by* $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ *where* $\mathbb{C}\mathrm{P}^n$ *is equipped with the CW-structure coming from Lemma 26.2.*

(3) *The infinite complex projective space* $\mathbb{C}\mathrm{P}^\infty$ *admits a CW-structure which has precisely one* n*-cell for every even* $n \in \mathbb{N}_0$.

Proof. The proof of this lemma is basically verbatim the same as the proof of the corresponding parts of Lemma 26.6. □

The following lemma gives a convenient approach to showing that a map starting at a CW-complex is continuous.

Lemma 26.8. *Let X be a CW-complex. We denote by $\{\Phi_i \colon \overline{B}^{n_i} \to X\}_{i\in I}$ the characteristic maps of all the cells of X. The following two statements hold:*

(1) *A subset $U \subset X$ is open if and only if for each $i \in I$ the preimage of U under the characteristic map Φ_i is open, i.e. if each $\Phi_i^{-1}(U) \subset \overline{B}^{n_i}$ is an open subset of $\overline{B}^{n_i}$.*
(2) *A map $f \colon X \to Y$ to a topological space Y is continuous if and only if each map $f \circ \Phi_i \colon \overline{B}^{n_i} \to Y$ is continuous.*

Proof. Once one has unraveled the definition of the topology on a CW-complex one sees that the first statement of the lemma is obvious. The second statement is an immediate consequence of the first statement. □

For completeness we reiterate the definition of a subcomplex that we had initially given on page 648 for finite-dimensional CW-complex but which also makes sense in the more general setting of arbitrary CW-complexes.

Definition. A *subcomplex* of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X.

Lemma 26.9. *Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup_{i\in I} X_i = X$. The following three statements hold:*

(1) *A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.*
(2) *A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.*
(3) *A map $f \colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i} \colon X_i \to Y$ is continuous.*

Proof.

(1) This statement is an easy consequence of Lemma 26.8 (1).
(2) This statement follows from (1) and the elementary observation that for $A \subset X$ we have $(X \setminus A) \cap X_i = X_i \setminus (X_i \cap A)$.
(3) This statement is an easy consequence of Lemma 26.8 (2). □

The following lemma shows in particular that some straightforward constructions provide examples of subcomplexes.

Lemma 26.10. *Let X be a CW-complex.*

(1) *If A is a subcomplex of A, then the CW-structure on X restricts to a CW-structure on A.*
(2) *Every subcomplex of X is a closed subset of X.*
(3) *For any $n \in \mathbb{N}_0$ the n-skeleton X^n is a subcomplex of X.*
(4) *The union and also the intersection of arbitrarily many subcomplexes is again a subcomplex.*
(5) *Every path-component of a complex is again a subcomplex.*

Proof.

(1) The proof is basically identical to the proof of Lemma 26.1.
(2) This statement follows quite easily from the definition of the topology on A.
(3) This statement is clear.
(4) This statement is also clear.
(5) This statement is basically obvious since each closed ball is path-connected. □

Proof. The proof is basically identical to the proof of Lemma 26.10, we just need Lemma 26.9 (2) to take care of statement (2), i.e. of the statement that every subcomplex is a closed subset. □

26.5. **Properties of CW-complexes.** In the previous section we saw that, as promised, most examples of topological spaces that we are interested in are indeed CW-complexes. The following, somewhat technical, proposition shows that CW-complexes have, again as promised, many nice topological properties.

Proposition 26.11.

(1) *Every point on a CW-complex is closed.*
(2) *CW-complexes are Hausdorff.*
(3) *Every cell of a CW-complex X is compact and it is a closed subset of X.*
(4) *Let X be a CW-complex. Given any $x \in X$ and given any open neighborhood U of x there exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.*
(5) *Every point on a CW-complex is good in the sense of the definition on page 387.*
(6) *CW-complexes are are o-locally*[515] *contractible, o-locally path-connected and o-locally simply-connected.*[516]
(7) *A CW-complex is connected if and only if it is path-connected.*
(8) *There exists a map Φ which assigns to each subcomplex A a subset $\Phi(A) \subset X$ such that the following statements hold:*
   (a) *for each subcomplex A the set $\Phi(A)$ is an open neighborhood of A such that A is a deformation retract of $\Phi(A)$,*
   (b) *for any two subcomplexes A_1 and A_2 we have*
$$\Phi(A_1 \cap A_2) = \Phi(A_1) \cap \Phi(A_2) \quad \text{and} \quad \Phi(A_1 \cup A_2) = \Phi(A_1) \cup \Phi(A_2),$$
   (c) *for any two subcomplexes A and B with $A \subset B$ we have $\Phi(A) \subset \Phi(B)$.*

Remark.

(1) In a slightly simplified form Proposition 26.11 (8) says that we can assign to each subcomplex A in a "systematic way" an open neighborhood $\Phi(A)$ that deformation retracts to A.

[515] We refer to page 130 for the definition of "o-local".
[516] Recall that on page 557 we called a topological space *reasonable* if it is o-locally simply connected.

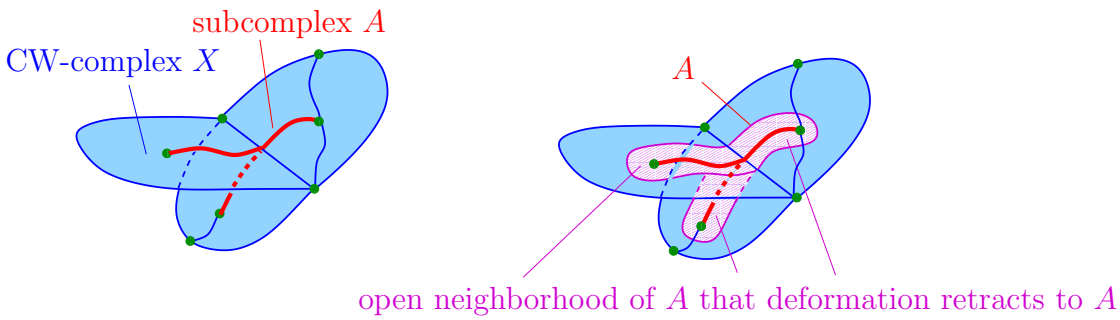


FIGURE 404. Illustration of Proposition 26.11 (8).

(2) In Proposition 26.11 (8) it is essential that A is a subcomplex. For example, if we consider $X = \mathbb{R}$ and $A = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, then A does not admit an open neighborhood U such that A is a deformation retract of U.[517]
(3) It follows from Proposition 26.11 (3) that every finite CW-complex is compact.
(4) Proposition 26.11 (6) sounds somewhat technical, but it ensures that later on we can apply Proposition 20.8 to obtain coverings of CW-complexes.
(5) Many other useful properties of CW-complexes are proved in [LW, Section II.3].

Proof $(*)$. [518][519] We start out with the following notation. Given a subset A of $\overline{B}^n$ and $\epsilon > 0$ we denote by

$$N_\epsilon(A) \;:=\; \big\{x \in \overline{B}^n \,\big|\, \text{ there exists an } y \in A \text{ with } \|x - y\| < \epsilon\big\}$$

the ϵ-neighborhood around A. Furthermore given two subsets X and Y of $\overline{B}^n$ we write

$$d(X, Y) \;:=\; \inf\{\|x - y\| \,|\, x \in X \text{ and } y \in Y\}.$$

Now let X be a CW-complex. We denote by Γ the set of cells of X and given $n \in \mathbb{N}_0$ we denote by Γ_n the set of n-cells. Given a cell γ we denote by Φ_γ the characteristic map.

Let $A \subset X$ be a subset and let ϵ be a function that assigns to each $\gamma \in \Gamma$ a positive number ϵ_γ. Given $n \in \mathbb{N}_0$ we define a subset $N^n_\epsilon(A)$ inductively as follows:

(1) For $n = 0$ we define $N^0_\epsilon(A) := A \cap X^0$.

[517] Why not?

[518] Proofs marked with $(*)$ were only sketched or not done at all in the lectures. The proofs marked with $(*)$ tend to be technical and not overly enlightening. It might make sense to browse through the proof to get an idea of what methods are used. Only if you have lots of time would I recommend to read the proofs in detail.

[519] Since I did not do the proofs in the lectures the probability that there are mistakes is also higher.

(2) Suppose that $N_\epsilon(A)^n$ has already been defined. We define

$$N_\epsilon^{n+1}(A) \;:=\; N_\epsilon^n(A) \cup \bigcup_{\gamma\in\Gamma_{n+1}} \Phi_\gamma\Big(N_{\epsilon_\gamma}(\overbrace{\Phi_\gamma^{-1}(A)}^{\subset \overline{B}^{n+1}}) \cup \underbrace{(1-\epsilon_\gamma, 1] \times \overbrace{\Phi_\gamma^{-1}(N_\epsilon^n(A))}^{\subset S^n}}_{\text{radial coordinates on } \overline{B}^{n+1}}\Big).$$

It is straightforward to verify that each $N_\epsilon^n(A)$ is an open neighborhood of $A \cap X^n$ in X^n. The definition of $N_\epsilon^{n+1}(A)$ is illustrated in Figure 405. Finally we set $N_\epsilon(A) := \bigcup_{n\in\mathbb{N}} N_\epsilon^n(A)$.

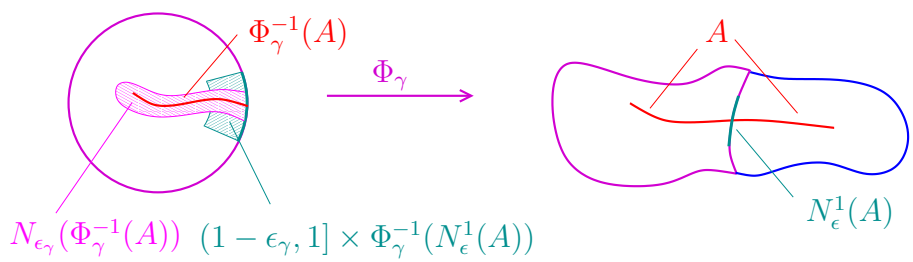


FIGURE 405.

It follows easily from the definition of the topology of a CW-complex that $N_\epsilon(A)$ is an open neighborhood of A in X. The definition of $N_\epsilon(A)$ is illustrated in Figure 406.

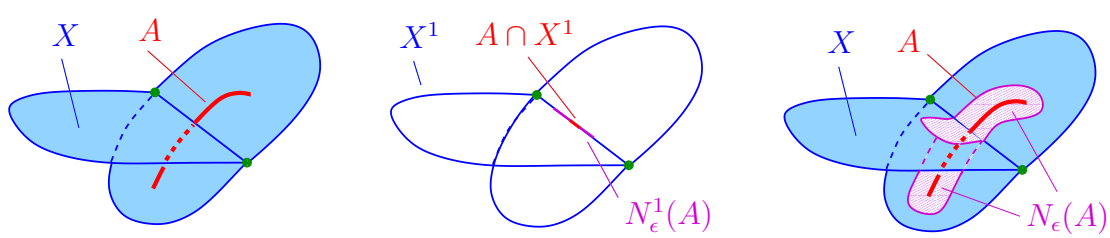


FIGURE 406.

Now we turn to the proofs of the seven statements:

(1) It follows easily from the definition of the topology on a CW-complex that every point is closed. We will fill in the details in Exercise 26.1.

(2) We will prove the following statement: given any two disjoint closed subsets A and B of X there exist open disjoint neighborhoods of A and B. By (1) this implies that X is Hausdorff.
So let A and B be two disjoint closed subsets of X. We will find ϵ as above such that $N_\epsilon(A)$ and $N_\epsilon(B)$ are disjoint. We will define the values of ϵ inductively on the dimension of the cells. For 0-cells we take $\epsilon = 1$. Now suppose that we have already found disjoint open sets $N_\epsilon^n(A)$ and $N_\epsilon^n(B)$. Let γ be an $(n+1)$-cell with characteristic map $\Phi_\gamma\colon \overline{B}^{n+1} \to X$.

Claim. We have

$$\begin{aligned} \mu_1 := d(\Phi_\gamma^{-1}(N_\epsilon^n(A)), \Phi_\gamma^{-1}(B)) &> 0, \\ \mu_2 := d(\Phi_\gamma^{-1}(N_\epsilon^n(B)), \Phi_\gamma^{-1}(A)) &> 0, \\ \mu_3 := d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) &> 0. \end{aligned}$$

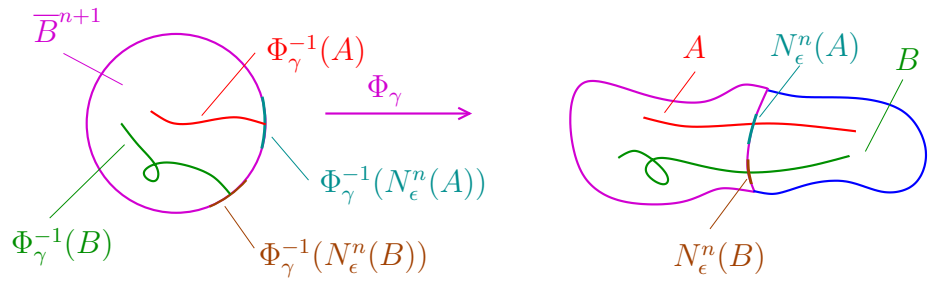


FIGURE 407.

We first consider μ_1. Suppose that $\mu_1 = 0$. If this was the case then there would exist a sequence of points in $\Phi_\gamma^{-1}(B)$ that converges to a point of $\Phi_\gamma^{-1}(B)$ in $\partial\overline{B}^{n+1}$ of distance zero from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$. But this is impossible since $\Phi_\gamma^{-1}(N_\epsilon^n(B))$ is a neighborhood of $\Phi_\gamma^{-1}(B)) \cap \partial\overline{B}^{n+1}$ in $\partial\overline{B}^{n+1}$ which is disjoint from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$. The second inequality $\mu_2 > 0$ is proved exactly the same way. Finally we have $\mu_3 > 0$ since $\Phi_\gamma^{-1}(A)$ and $\Phi_\gamma^{-1}(B)$ are disjoint closed, hence compact, subsets of $\overline{B}^{n+1}$.[520] This concludes the proof of the claim.

We set $\epsilon_\gamma := \frac{1}{2}\min\{\mu_1, \mu_2, \mu_3\}$. We do this for all $(n+1)$-cells γ. It then follows immediately from the definitions that $N_\epsilon^{n+1}(A)$ and $N_\epsilon^{n+1}(B)$ are disjoint.

(3) Every k-cell is the image of the compact set $\overline{B}^k$, hence it is compact by Lemma 1.37. It follows from Lemma 1.18 (2) and the fact, proved in (2), that X is Hausdorff that every k-cell is also a closed subset of X.

(4) Let $x \in X$ and let U be an open neighborhood of x. We leave it as an exercise to show that there exists an ϵ such that $N_\epsilon(x)$ is contained in U.

It remains to show that $N_\epsilon(x)$ admits a deformation retraction to $\{x\}$. We denote by m the unique natural number with $x \in X^m \setminus X^{m-1}$. It follows easily from the definitions that $N_\epsilon^m(x)$ is an open ball around x, in particular $N_\epsilon^m(x)$ deformation retracts onto $\{x\}$. Therefore it suffices to find a deformation retraction from $N_\epsilon(x)$ to $N_\epsilon^m(x)$.

First note that for $n > m$ we can construct a deformation retraction r_n from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$ by sliding outward along radial segments in the n-cells. We refer to Figure 408 for an illustration. A deformation retraction of $N_\epsilon(x)$ onto $N_\epsilon^m(x)$ is then obtained by performing the deformation retraction r_n from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$ during

[520] Here we use the following general fact: if K and L are two compact disjoints subsets of $\mathbb{R}^n$, then $d(K, L) > 0$. How would you prove this?

the "time-interval" $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ where the points of $N_\epsilon(x) \setminus N_\epsilon^{n-1}(x)$ stay fixed during that time interval.

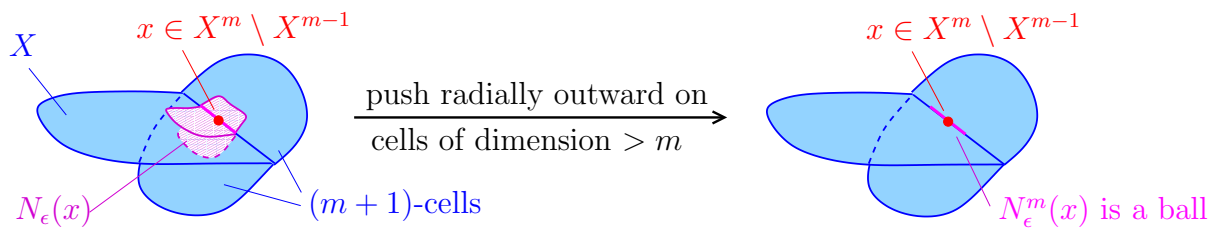


FIGURE 408.

(5) This statement follows immediately from (2) and (4).
(6) This statement follows immediately from (4) since every topological space that admits a deformation retraction to a point is contractible, and since every contractible topological space is path-connected and simply connected.
(7) By Lemma 1.102 a o-locally path-connected topological space is connected if and only if it is path-connected.
(8) We pick an $\epsilon \in (0,1)$. Given a subcomplex A of X we set $\Phi(A) := N_\epsilon(A)$. In each cell of $X \setminus A$ the set $N_\epsilon(A)$ is a product neighborhood of the boundary of this cell. So a deformation retraction of $N_\epsilon(A)$ onto A can be constructed as in (4). We leave the details to the reader. It is clear that Φ has all the other desired properties. □

Remark. Recall that on page 75 we introduced the definition that a topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by neighborhoods, i.e. if there exist disjoint neighborhoods U of A and V of B. In the proof of Proposition 26.11 (2) we actually showed that CW-complexes are normal.

One of the curses of the working topologist is the unfortunate fact that a bijection continuous map between topological spaces does not need to be a homeomorphism. The following lemma gives a nice criterion for a map between CW-complexes to be a homeomorphism.

Lemma 26.12. *Let $f\colon X \to Y$ be a map between CW-complexes. Suppose the following three conditions are satisfied:*

(1) *f is a bijection,*
(2) *f is continuous,*
(3) *the images of the cells of X are precisely the cells of Y.*

Then $f\colon X \to Y$ is a homeomorphism.

Proof (∗). We denote by $g = f^{-1}\colon Y \to X$ the inverse of f. It remains to show that g is continuous. Let $\{\Phi_i\colon \overline{B}^{m_i} \to X\}_{i\in I}$ be the characteristic maps of the CW-complex X and let $\{\Phi_j\colon \overline{B}^{n_j} \to Y\}_{j\in J}$ be the characteristic maps of the CW-complex Y. By Lemma 26.8 it suffices to show that for each $j \in J$ the map $g \circ \Psi_j\colon \overline{B}^{n_j} \to X$ is continuous. We consider the cell $\Psi_j(\overline{B}^{n_j})$. By hypothesis (3) there exists an $i \in I$ such that $f(\Phi_i(\overline{B}^{m_i})) = \Psi_j(\overline{B}^{n_j})$.

Since $\Phi_i(\overline{B}^{m_i})$ is compact and since we know by Proposition 26.11 that $\Psi_j(\overline{B}^{n_j}) \subset Y$ is Hausdorff it follows from Proposition 1.40 that $f\colon \Phi_i(\overline{B}^{m_i}) \to \Psi_j(\overline{B}^{n_j})$ is a homeomorphism. But this implies that $f^{-1} = g$ is continuous on $\Psi_j(\overline{B}^{n_j})$. But this implies that $g \circ \Psi_j$ is continuous. $\square$

Using Proposition 26.11 we can now prove the following very useful version of the Seifert–van Kampen Theorem 14.1. In view of Theorem 26.3 it can be viewed as a generalization of the Seifert-van Kampen Theorem 14.2 for manifolds.

Theorem 26.13. (Seifert–van Kampen for CW-complexes) *Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Let $x_0 \in A \cap B$. Then*

$$\pi_1(X, x_0) \;\cong\; \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

Examples.

(1) Let X be the wedge of two circles. We can view X as a CW-complex with one 0-cell x_0 and two 1-cells a and b. Then $A = \{x_0, a\}$ and $B = \{x_0, b\}$ are subcomplexes, each of which is homeomorphic to S^1. We refer to Figure 409 for an illustration. It follows, as on page 390, that

$$\pi_1(X, x_0) \;\underset{\substack{\uparrow \\ \text{Theorem 26.13}}}{\cong}\; \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \;\underset{\substack{\uparrow \\ \text{since } A \text{ and } B \text{ are homeomorphic to circles}}}{\cong}\; \langle a\rangle * \langle b\rangle \;=\; \langle a, b\rangle.$$

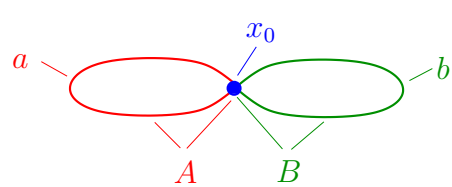


FIGURE 409.

(2) We consider

$$K \;:=\; \underbrace{S^1 \times S^1}_{=:A} \cup \underbrace{\overline{B}^2 \times \{1\}}_{=:B},$$

i.e. K is given by a torus where the "central hole" is filled in with a disk. We refer to Figure 410 for an illustration.[521] In the above definition of K we already wrote K as the union of two subsets. It is quite straightforward to see that K can be given

[521] Here the name K stands for "kiacherl".

a CW-structure such that A and B are subcomplexes.[522] We pick the base point $x_0 = (1,1)$. We obtain that

$$\pi_1(K,x_0) \underset{\substack{\uparrow \\ \text{Theorem 26.13}}}{\cong} \pi_1(A,x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B,x_0) \underset{\substack{\uparrow \\ \text{since } A\cap B = S^1\times\{1\}}}{\cong} \underbrace{\langle x,y|[x,y]\rangle *_{\langle s\rangle} \{e\}}_{\text{with } x \leftarrow\!\shortmid s} \underset{\substack{\uparrow \\ \text{Lemma 13.12}}}{\cong} \langle y\rangle.$$

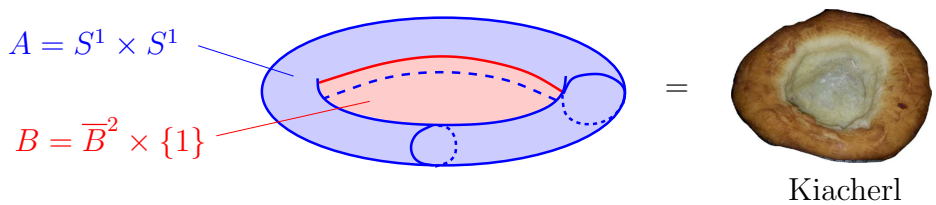


FIGURE 410.

Proof of Theorem 26.13 (∗). [523] Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Note that $A \cap B$ is also a subcomplex. Let $x_0 \in A \cap B$.

We use the notation of Proposition 26.11 (8). We write $U = \Phi(A)$ and $V = \Phi(B)$. Note that by Proposition 26.11 (8b) we have $U \cap V = \Phi(A \cap B)$. Since $A \cap B$ is path-connected and since $A \cap B$ is a deformation retract of $\Phi(A \cap B)$ we see that $U \cap V = \Phi(A \cap B)$ is also path-connected. Therefore we can apply the Seifert–van Kampen Theorem 14.1 to the decomposition $X = U \cup V$. We obtain the following commutative diagram

$$\begin{array}{ccc} \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) & \xrightarrow{\cong} & \pi_1(X,x_0) \\ \uparrow & & =\uparrow \\ \pi_1(A,x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B,x_0) & \longrightarrow & \pi_1(X,x_0), \end{array}$$

where the two vertical maps are induced by the inclusions. By Proposition 26.11 (8a) and Proposition 9.12 (1) the three inclusion induced maps $\pi_1(A,x_0) \to \pi_1(U,x_0)$, $\pi_1(B,x_0) \to \pi_1(V,x_0)$ and $\pi_1(A \cap B,x_0) \to \pi_1(U \cap V,x_0)$ are isomorphisms. The top horizontal map is an isomorphism by the Seifert–van Kampen Theorem 14.1. It follows that the bottom horizontal map is also an isomorphism. □

The following proposition is one of the key technical results about infinite CW-complexes.

Proposition 26.14. *Every compact* (*connected*) *subset of a CW-complex X is contained in a finite* (*connected*) *subcomplex.*

Remark. The statement of the proposition and also its proof have more than a passing resemblance with Lemma 26.5.

[522] How would you define the CW-structures?

[523] The proof is very similar to the proof of the Seifert-van Kampen Theorem for manifolds 14.2.

Before we start out with the proof of Proposition 26.14 we formulate the following pretty corollary.

Corollary 26.15. *A CW-complex is compact if and only if it is finite.*

Proof. As we remarked before, it is a consequence of Proposition 26.11 (3) that every finite CW-complex is compact. Now suppose that X is a compact CW-complex. It follows from Proposition 26.14 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, ie. X si a finite CW-complex. □

The proof of Proposition 26.14 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 26.16. *Let X be a CW-complex. Then the following statements hold:*

(1) *Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.*
(2) *If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.*
(3) *Every cell of X is contained in a finite subcomplex of X.*[524]

Proof of Lemma 26.16 $(*)$. We start out with the following claim.

Claim. Let Y be a CW-complex. Then the statements (1), (2) and (3) are equivalent.

Let Y be a CW-complex. First we show (1) $\Rightarrow$ (2). Let $C \subset Y$ be a compact subset. For each cell that intersects C we pick a single point. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of compact set we obtain from Lemma 1.18 (1) that A is compact. Since A is compact and discrete we obtain from Lemma 1.19 (1) that A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells as claimed.

Now we will prove (2) $\Rightarrow$ (3). So let $C_\sigma \subset Y$ be an n-cell. We denote by U_σ the corresponding open cell. By Proposition 26.11 (3) and by definition of a cell we know that $C_\sigma \setminus U_\sigma$ is a compact subset of the $(n-1)$-skeleton Y^{n-1}. It follows from (2) that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n-1$. By the same argument each of these lower-dimensional closed cells is again contained in a finite union of open cells of smaller dimension, except again for the cell itself. A straightforward induction argument now shows that we end up with finitely many cells which form a subcomplex of Y.

Next we will show (3) $\Rightarrow$ (1). Let $A \subset Y$ be a subset such that no two points lie in the same open cell. We consider the intersection of A with a closed cell. By (3) this intersection is contained in a finite subcomplex Y'. Since A has at most one point in common with any open cell this intersection $A \cap Y'$ must be finite and hence closed in $A \cap Y'$ by Proposition 26.11 (1). By Proposition 26.11 (3) the finite subcomplex Y' is a closed subset of X. Hence $A \cap Y'$ is a closed subset of X. For any point $x \in A$ the set

[524] This implies in particular that the *closure* of each open cell is contained in the union of *finitely* many open cells. This property is sometimes referred to as X being "closure finite" which explains the "C" in the name CW-complex.

$A \setminus \{x\}$ also satisfies the hypothesis for A, and so it must also be a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. This concludes the proof of the claim.

Now let X be a CW-complex. First note that statement (1) holds for X^0 by definition of the topology on X^0. Now suppose we know (1) for the n-skeleton X^n for some $n \in \mathbb{N}_0$. By claim we know that (2) and (3) also for X^n. But the proof of (2) $\Rightarrow$ (3) for a given n-cell only used (2) for subsets of X^{n-1}. Thus we actually have (3) for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all n. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. □

Now we can provide the proof of Proposition 26.14.

Proof of Proposition 26.14. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 26.16 (2) the subset C is contained in a union of finitely many open cells. By Lemma 26.16 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.
If C is connected, then we take the component of Z that contains C. By Lemma 26.10 (4) this is again a subcomplex. □

In many situations the following proposition makes it possible to reduce the study of arbitrary CW-complexes to the study of finite CW-complexes.

Proposition 26.17. *Let X be a CW-complex with countably many cells. Then the following hold:*

(a) *There exists a sequence $A_1 \subset A_2 \subset \dots$ of* finite *subcomplexes A_i of X such that $X = \bigcup_{i \in \mathbb{N}_0} A_i$.*
(b) *If X is connected, then we can arrange in* (a) *that every A_i is connected.*

Proof. Let X be a CW-complex with countably many cells. We number them with $e_1, e_2, \dots$. By Lemma 26.16 (3) and Proposition 26.14 there exists for each i a finite connected subcomplex B_i that contains e_i. We define $A_1 := B_1$ and iteratively we define $A_i := A_{i-1} \cup B_i$. By Lemma 26.10 (4) we know that each A_i is a finite subcomplex. Thus this sequence satisfies (a).

Now suppose that X is connected. We define the sequence by $A_1 := B_1$ and iteratively we define A_i as follows:

(1) We pick a path $p\colon [0,1] \to X$ from a point in A_{i-1} to a point in B_i.
(2) Then we denote by A_i any finite connected subcomplex that contains the compact connected set $A_{i-1} \cup B_i \cup p([0,1])$. Such a finite connected subcomplex exists by Lemma 26.16 (3) and Proposition 26.14.

This sequence clearly has the desired properties (a) and (b). □

Corollary 26.18. *Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Given any compact subset K of X there exists an $i \in \mathbb{N}$ with $K \subset X_i$.*

Proof. Let K be a compact subset of X. It follows from Proposition 26.17 that K is contained in a finite subcomplex Y of X. From $X = \bigcup_{i\in\mathbb{N}} X_i$ and from the fact that each X_i is a subcomplex of X we obtain that every cell is contained in some X_i. But this implies that the finite subcomplex Y is also contained in some X_i. $\square$

We conclude this section with the following proposition that is an immediate consequence of Proposition 16.4 and Corollary 26.18.

Proposition 26.19. *Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i\in\mathbb{N}} X_i$. Then for any $w \in X_1$ we have*

$$\pi_1(X, w) \;=\; \varinjlim \pi_1(X_i, w).$$

Examples.

(1) We have[525]

$$\pi_1(\mathbb{R}\mathrm{P}^\infty) \underset{\substack{\uparrow \\ \text{see page 655}}}{=} \pi_1\Big(\bigcup_{n\in\mathbb{N}} \mathbb{R}\mathrm{P}^n\Big) \underset{\substack{\uparrow \\ \text{Proposition 26.19}}}{=} \varinjlim \begin{array}{c} \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^3) \\ \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^2) \\ \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^1) \end{array} = \varinjlim \begin{array}{c} \uparrow\cong \\ \mathbb{Z}_2 \\ \uparrow\cong \\ \mathbb{Z}_2 \\ \uparrow \\ \mathbb{Z} \end{array} \cong \mathbb{Z}_2.$$

(2) In Exercise 26.4 we will use Proposition 26.19 to show that there exists a CW-complex X with $\pi_1(X) \cong (\mathbb{Q}, +)$.

26.6. **Constructions of more CW-complexes.** In this and the next section we will see how one can build new CW-complexes out of given CW-complexes. We start out with the following lemma.

Lemma 26.20.

(1) *Let $\{X_i\}_{i\in I}$ be a family of CW-complexes. The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure such that each X_i is a subcomplex.*
(2) *Let X be a CW-complex and let A be a subcomplex. The quotient X/A admits a natural CW-structure such that the projection map $X \to X/A$ is cellular and where the cells of X/A are in one-to-one correspondence to the cells of X that do not entirely lie in A together with one 0-cell given by the point $[A]$.*
(3) *Let $\{X_i\}_{i\in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $a_i \in X_i^0$. If we use these points to form the wedge $\bigvee_{i\in I} X_i$, then the wedge admits a natural CW-complex structure such that each X_i is a subcomplex.*

[525] By Corollary 7.19 we know that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$ for $n \geq 2$. But why is $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$?

(4) *Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. Then*

$$(X \sqcup Y)/\sim \qquad \text{where } a \sim f(a) \text{ for } a \in A$$

admits a natural CW-structure such that X and Y are subcomplexes.

Convention. When we form the wedge of a family of a CW-complexes, then we always use a point that lies in the 0-skeleton, so that we can view the wedge in a natural way as a CW-complex.

Proof.

(1) The first statement is basically obvious.

(2), (4) In both cases it should be pretty clear what the CW-structure should be. The problem that arises is that one needs to show that the topology of the new CW-complex agrees in fact with the intended topology. If X (and Y) are finite CW-complexes then this follows easily from Proposition 1.40. The general case gets more technical. We refer to [LW, p. 60 and Theorem 5.11] or alternatively to [FPi, p. 63] for complete proofs.

(3) This follows easily from (1) and (2). □

Later on we will make good use of the following lemma.

Lemma 26.21. *Let X be a CW-complex and let $n \in \mathbb{N}$. Let $\{\varphi_i\colon \overline{B}^{n+1} \to X^n\}_{i\in I}$ be the characteristic maps of the $(n+1)$-cells of X. For each $i \in I$ let $\overline{B}_i^{n+1}$ be a copy of $\overline{B}^{n+1}$ and let S_i^n be a copy of S^n. We write $S_i^n = \partial\overline{B}_i^{n+1}$.*

(1) *The map*

$$\begin{aligned} f\colon \bigvee_{i\in I} \overbrace{(\overline{B}_i^{n+1}/S_i^n)}^{\text{homeomorphic to } S^{n+1}\text{, see page 99}} &\to X^{n+1}/X^n \\ [x] &\mapsto [\varphi_i(x)] \quad \text{if } x \in \overline{B}_i^{n+1} \end{aligned}$$

is a homeomorphism.

(2) *If X has only one 0-cell $\{x_0\}$ and no cells in dimensions $1, \dots, n$, then the homeomorphism f, together with the homeomorphism $\overline{B}^{n+1}/S^n \to S^{n+1}$ from page 99 induces a homeomorphism*

$$\bigvee_{i\in I} S_i^{n+1} \to X^{n+1}.$$

Example. In Figure 411 we see the torus together with a CW-structure X that has two 2-dimensional cells. It follows from Lemma 26.21 (1) that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres.

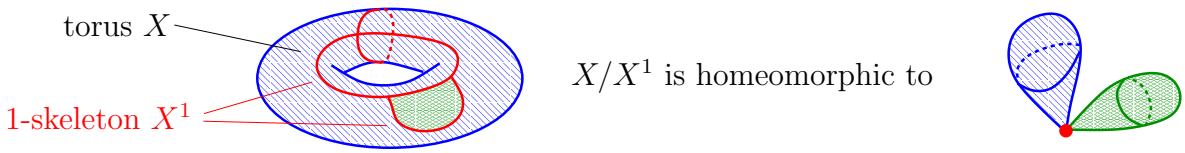


FIGURE 411.

Proof. Clearly (2) is just a neat special case of (1). Thus it remains to prove (1). First note that it is basically clear that the given map f is a bijection. Furthermore it follows immediately from Lemma 9.17 that f is continuous. We use Lemma 26.20 to view both sides of Lemma 26.21 as CW-complexes. The map f gives a bijection between the cells of the two sides. Thus it follows from Lemma 26.12 that f is a homeomorphism. □

Lemma 26.22. *Let $f_i\colon X_i \to X_{i+1}$, $i \in \mathbb{N}$ be a sequence of cellular inclusion maps between CW-complexes. Then the direct limit*[526] $\varinjlim X_i = \bigcup_{i\in I} X_i$ *admits a CW-structure such that each X_i is a subcomplex of* $\varinjlim X_i = \bigcup_{i\in I} X_i$.

Example. Lemma 26.22 allows us to view the line with infinitely many spheres attached as a CW-complex. We refer to Figure 412 for an illustration.

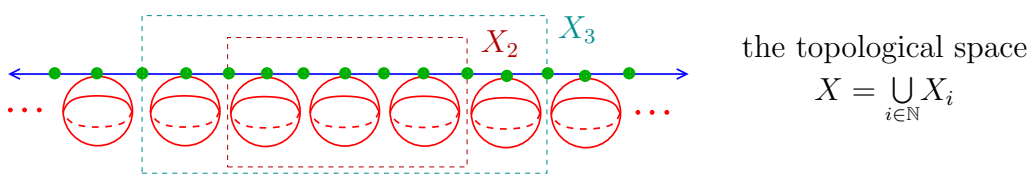


FIGURE 412.

Proof. We leave it to the reader to write down the, notationally messy but mathematically elementary, proof. □

Let X and Y be two CW-complexes. We can form the product topological space $X \times Y$. We would like to equip $X \times Y$ with a CW-structure. Using the homeomorphism from Proposition 1.48 (2) we identify in the following the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n = [0,1]^n$. For $p + q = n$ we evidently have $I^n = I^p \times I^q$. The idea is now to define a CW-structure on $X \times Y$ such that the product of a p-cell in X with a q-cell in Y corresponds to a $(p+q)$-cell in $X \times Y$.

[526] We write explicitly "direct limit" to remind the reader that, according to the discussion on page 485, the topology on $\varinjlim X_i = \bigcup_{i\in I} X_i$ is given by the condition that a subset U is open if and only if each intersection $U \cap X_i$ is an open subset of X_i.

In the following we want to execute this idea rigorously. We first make the observation that

$$\partial(I^n) \;=\; \partial I^p \times I^q \cup I^p \times \partial I^q.$$

We define a CW-structure on $X \times Y$ inductively as follows:

(1) We put $(X \times Y)^0 = X^0 \times Y^0$.
(2) Now we suppose that the $(n-1)$-skeleton $(X \times Y)^{(n-1)}$ has already been defined. We consider a p-cell e in X which is given by a characteristic map $\varphi\colon I^p \to X^p$ and an $(n-p)$-cell f in Y which is given by a characteristic map $\psi\colon I^{n-p} \to Y^{n-p}$. Then we denote by $e \times f$ the n-cell in $X \times Y$ which is given by the attaching map[527]

$$\begin{array}{rcl} \partial(I^n) = \partial I^p \times I^{n-p} \cup I^p \times \partial I^{n-p} & \to & (X \times Y)^{n-1} \\ (x, y) & \mapsto & (\varphi(x), \psi(y)). \end{array}$$

All these n-cells together define the n-skeleton of $X \times Y$.

We refer to this CW-structure on $X \times Y$ as the *product CW-structure*.

Examples.

(1) If we equip $S^1 = [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 = [0,1] \times [0,1]/\sim$ is precisely the CW-structure that we had already encountered on page 646. We refer to Figure 413 for an illustration.

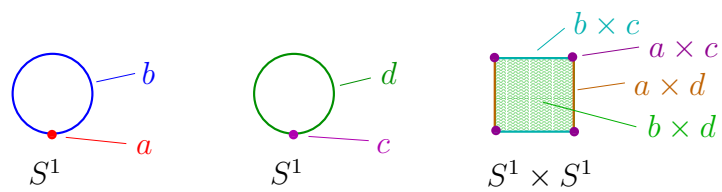


FIGURE 413.

(2) Given $a = s_0 < s_1 < \cdots < s_k = b$ and $c = t_0 < t_1 < \cdots < t_l = d$ we can view the intervals $[a, b]$ and $[c, d]$ as CW-complexes where the 1-cells are given precisely by the intervals $[s_i, s_{i+1}]$, $i = 0, \ldots, k-1$ and $[t_j, t_{j+1}]$, $j = 0, \ldots, l-1$. The product CW-complex is the topological space $[a, b] \times [c, d]$ where the 2-cells are given by the rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ with $i \in \{0, \ldots, k-1\}$ and $j \in \{0, \ldots, l-1\}$. The analogous statement holds also for higher-dimensional hyperrectangles. For example, given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we can equip the cube $[0,1]^n$ with a CW-structure where the n-cells are precisely given by the cubes $[\frac{s_1}{k}, \frac{s_1+1}{k}] \times \cdots \times [\frac{s_n}{k}, \frac{s_n+1}{k}]$ with $s_1, \ldots, s_n \in \{0, \ldots, k-1\}$.

[527] Why does the image of this attaching map lie in $(X \times Y)^{n-1}$?

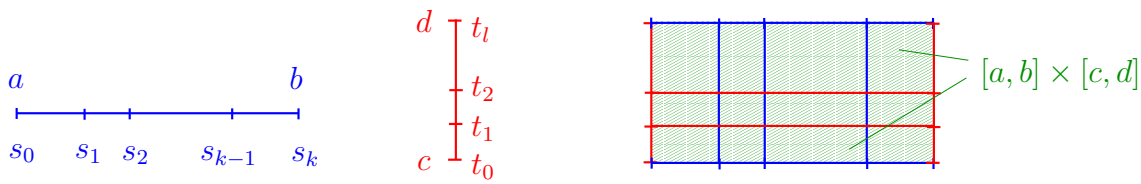


FIGURE 414.

Now comes a major subtlety. We just equipped the set $X \times Y$ with a CW-structure and thus with a topology, but does this topology equal the original topology?

We have the following proposition which deals with all the situations we will care about.

Proposition 26.23. *Let X and Y be CW-complexes. If at least one of the two CW-complexes has only countably many cells, then the product CW-structure is indeed a CW-structure for the topological space $X \times Y$.*

Example. One of the most relevant examples for us is the case $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell. It follows from Proposition 26.23 that given any CW-complex X the product CW-structure defines a CW-structure on $X \times [0,1]$. By definition of the product CW-structure $X \times \{0\}$ and $X \times \{1\}$ are subcomplexes.[528]

Proof. If X and Y are both finite CW-complexes, then the conclusion is again a consequence of Proposition 1.40. The general case is dealt with in [LW, Theorem II.5.2]. (If one of X or Y is a finite CW-complex, then the statement is also proved in [Hat, Theorem A.6]. If both CW-complexes have countably many cells, then the statement is also proved in [FPi, Proposition 2.2.3] or [Mi14, Lemma 2.1].) □

Remark. The formulation of Proposition 26.23 already suggests that the conclusion does not hold without some assumptions on X and Y. In fact, as discussed in [LW, Chapter II.5], the hypotheses of Proposition 26.23 can be loosened even further. But, as shown in [Dow] or [Hat, p. 524], there are indeed examples of CW-complexes X and Y such that the above product CW-structure on $X \times Y$ is not a CW-structure for the topological space $X \times Y$. Finally, [Broo, Theorem 1] gives a necessary and sufficient condition for the product of two CW-complexes to be a CW-complex.

We conclude this section with the following corollary that shows that several constructions that we are familiar with can also be performed in the category of CW-complexes.

Corollary 26.24. *Let X be a CW-complex.*[529]

[528] In fact this special case can be proved along the same lines as Lemma 27.1.

[529] Strictly speaking, or Bourbakily speaking if that's an adverb, we should really specify in each case what the "natural CW-structure" is. But this should hopefully be clear in each case, and we do not encumber the reader with unnecessary extra notation.

(1) *The cone*

$$\mathrm{Cone}(X) \;=\; (X \times [0,1])/\sim \qquad \textit{where } (x,0) \sim (y,0) \textit{ for every } x,y \in X$$

admits a natural CW-structure such that $X = X \times \{1\}$ *is a subcomplex.*

(2) *The suspension*

$$\Sigma(X) \;=\; (X \times [-1,1])/\sim \qquad \textit{where for every } x,y \in X \textit{ we have } (x,-1) \sim (y,-1) \textit{ and } (x,1) \sim (y,1)$$

admits a natural CW-structure such that $X = X \times \{0\}$ *is a subcomplex.*

(3) *If* A *is a CW-complex and if* $f\colon A \to X$ *is a cellular map, then the mapping cone*

$$\mathrm{Cone}(f\colon A \to X) \;=\; (X \sqcup \mathrm{Cone}(A))/\sim \;\; \textit{where } (a,1) \sim f(a) \textit{ for all } a \in A.$$

and the mapping cylinder

$$\mathrm{Cyl}(f\colon A \to X) \;=\; (A \times [0,1] \sqcup X)/\sim \;\; \textit{where } (a,1) \sim f(a) \textit{ for all } a \in A.$$

admit natural CW-structures. In both cases X *is naturally a subcomplex and in the latter case* $A = A \times \{0\}$ *is also a subcomplex.*

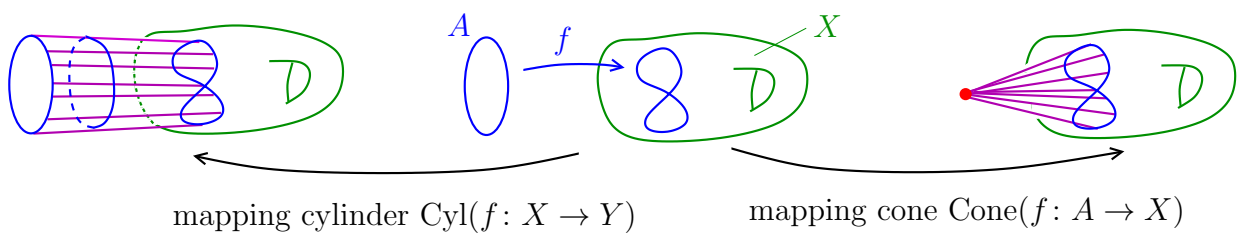


FIGURE 415.

Proof. All four statements can be proved easily using Proposition 26.23 and Lemma 26.20. We leave it to the reader to fill in the details. □

26.7. **Coverings of CW-complexes.**

Proposition 26.25. *Let* $p\colon \widetilde{X} \to X$ *be a covering of a connected CW-complex* X*. We can equip* $\widetilde{X}$ *with the structure of a CW-complex such that the following statements hold:*

(1) *The map* $p\colon \widetilde{X} \to X$ *is a cellular map.*

(2) *For any* $k \in \mathbb{N}_0$ *we have*[530]

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(3) *The dimension of* $\widetilde{X}$ *is the same as the dimension of* X*.*

[530]Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.

(4) *If X is a finite CW-complex and if p is a finite covering, then*

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Proof. Let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex X. We write $n := [\widetilde{X} : X]$. Let $\psi\colon \overline{B}^k \to X$ be a characteristic map of a k-cell of X. We write $x = \psi(0)$. We consider the following diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \overline{B}^k \arrow[ur, dashed, "\widetilde{\psi}"] \arrow[r, "\psi"'] & X. \end{tikzcd}$$

Since $\overline{B}^k$ is simply connected and o-locally path-connected we can apply Proposition 20.2 which says in this context that for each $\widetilde{x} \in p^{-1}(x)$ there exists a unique lift $\overline{B}^k \to \widetilde{X}$ with $\widetilde{\psi}(0) = \widetilde{x}$. By the definition on page 259, the degree $n = [\widetilde{X} : X]$ of the covering equals the cardinality of $p^{-1}(x)$. Thus we see that ψ gives rise to n maps $\overline{B}^k \to \widetilde{X}$. It is straightforward to verify that all these lifts of the characteristic maps for the cells of X define a CW-structure for $\widetilde{X}$. We leave the verification of the details as an exercise to the reader. Alternatively we refer to [Bre, Theorem IV.8.10] for details.

It follows immediately from the above construction and the definitions that the CW-structure we had just defined on $\widetilde{X}$ has the desired properties (1) to (4). □

Examples.

(1) We equip S^1 with the CW-structure consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ the universal covering of S^1. If we apply the construction of the CW-structure on $\mathbb{R}$ given in the proof of Proposition 26.25, then we end up equipping $\mathbb{R}$ with the CW-structure where the 0-cells correspond to $\mathbb{Z}$ and the 1-cells are precisely the intervals $[k, k+1]$, with $k \in \mathbb{Z}$. We illustrate the CW-structure of $\mathbb{R}$ in Figure 416.

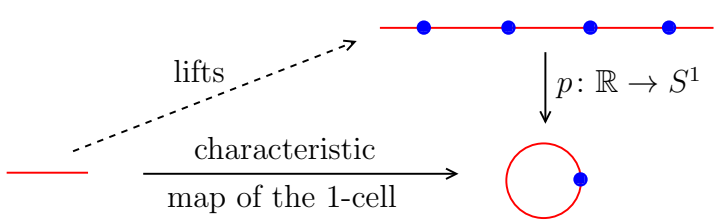


FIGURE 416.

(2) Almost the same approach as in (1), applied to the universal covering $\mathbb{R}^2 \to S^1 \times S^1$, endows $\mathbb{R}^2$ with a CW-structure that is illustrated in Figure 417.

(3) In the proof of Hadamard's Theorem 23.9 we constructed an explicit covering map $\mathbb{D} \to H_8/\sim \;\cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we

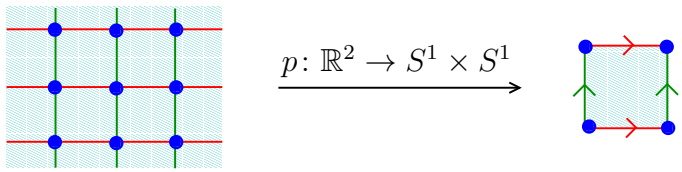


FIGURE 417.

obtain from Proposition 26.25 the CW-structure on $\mathbb{D}$ that is shown in Figure 418.

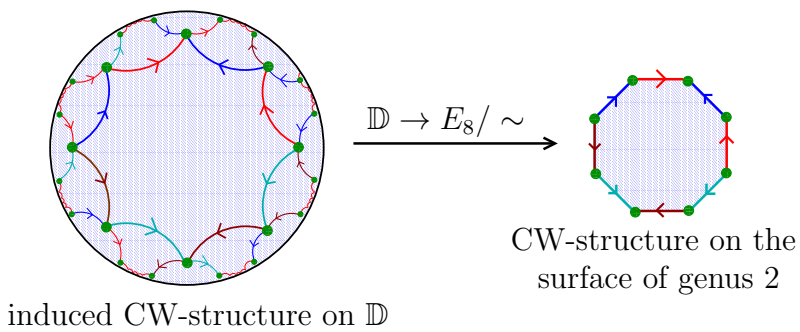


FIGURE 418.

We conclude this discussion of CW-complexes with an application to group theory. Given a free group F, i.e. given a group F that is isomorphic to the free group $\langle S\rangle$ generated by some set S we refer to the cardinality of S as the *rank* of F.[531]

Now we can formulate the following corollary to Proposition 26.25. The statement is purely group theoretic, but it is surprisingly hard to prove the statement without appealing to topological methods.[532]

Corollary 26.26. *Let F be a finitely generated free group.*

(1) *Every subgroup of F is again a free group.*[533]
(2) *If G is a finite-index subgroup of F, then*[534]

$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Examples.

[531]It follows from the discussions on the pages 417 and 368 that the rank of F is well-defined.

[532]Try to prove it using only group theoretic methods!

[533]This statement was first proved in 1927 by Otto Schreier [Schre].

[534]Does the equality hold if G is infinite-index and if we interpret $[F : G] \cdot (\operatorname{rank}(F) - 1)$, as usual, as $\infty \cdot (\operatorname{rank}(F) - 1) = \infty$ for $F \ncong \mathbb{Z}$?

(1) If $F = \mathbb{Z}$ then the formula from Corollary 26.26 does indeed show that every finite-index subgroup of $\mathbb{Z}$ is again a free group of rank one, i.e. every finite-index subgroup isomorphic to $\mathbb{Z}$.
(2) Let $\alpha\colon F := \langle a, b\rangle \to \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. It follows from Corollary 26.26 that the kernel $G := \ker(\alpha)$ is a free group of rank $[F : G] \cdot (\mathrm{rank}(F) - 1) + 1 = 3 \cdot 1 + 1 = 4$. In Exercise 26.7 we will determine four elements of $G = \ker(\alpha)$ that are generators for this free group.

Proof. Let F be a free group of rank m and let G be a subgroup of F. Let X be a 1-dimensional CW-complex with one 0-cell P and m 1-cells. By Proposition 27.9 we have $\pi_1(X, P) \cong F$.

By Proposition 26.11 (6) and Proposition 20.8 there exists a covering $p\colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$. By Corollary 7.15 the map p_* is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G.

By Proposition 26.25 we can equip Y with the structure of a 1-dimensional CW-structure. We deduce from Proposition 27.9 that $G \cong \pi_1(Y, Q)$ is a free group.

Finally, suppose that G is in fact a finite-index subgroup of F. It follows from Proposition 26.25 that $\chi(Y) = [F : G] \cdot \chi(X)$. It is now a consequence of Proposition 27.9 that $\mathrm{rank}(G) - 1 = [F : G] \cdot (\mathrm{rank}(F) - 1)$. □

Remark.

(1) It is not hard to modify the proof of Corollary 26.26 to show that the subgroup of any free group (not necessarily finitely generated) is again a free group.
(2) There are purely group theoretic proofs of Corollary 26.26, see e.g. [Schre], [HallM, Theorem 7.2.1] or [Bau, Chapter III.3], but the topological proof that we had just provided is arguably the clearest proof.
(3) In principle one could have tried to prove Corollary 26.26 using spatial graphs in the sense of page 134 instead of using CW-complexes. But then one would have had to show that coverings of spatial graphs are again spatial graphs. This would have been exceedingly painful[535], whereas, as we have just seen in Proposition 26.25, it is almost a triviality to show that coverings of 1-dimensional CW-complexes are again 1-dimensional CW-complexes. This shows that sometimes it pays to use more abstract language to prove statements even about "basic objects".
(4) Corollary 26.26 is a special case of the *Kurosh Subgroup Theorem*. This theorem says that if G is the product of groups $A_i, i \in I$ and if H is a subgroup of G, then H is a free product of a free group and groups $B_i, i \in I$ where each B_i is a (potentially trivial) group that is conjugate to a subgroup of A_i. We refer to [LS, Theorem IV.1.10] for details and a proof.

Example. Let G be a graph with one vertex v and two edges which we denote by s and t. By a slight abuse of notation we denote the corresponding elements in $\pi_1(G, v)$ by s and t

[535] You can try to do this by hand.

as well. By Lemma 12.5 we can identify $\pi_1(G,v)$ with the free group $\langle s,t\rangle$ on the generators s and t. Now let $\phi\colon \pi_1(G,v) = \langle s,t\rangle \to \mathbb{Z}$ be the homomorphism that is given by $\phi(s) = 0$ and $\phi(t) = 1$. By Proposition 20.8 there exists a covering $q\colon (\widehat{G},\widehat{v}) \to (G,v)$ with $q_*(\pi_1(\widehat{G},\widehat{v})) = \ker(\phi)$, furthermore by Proposition 20.9 this covering is unique up to equivalence, in the sense of the definition on page 563.

We consider the infinite covering illustrated in Figure 148, which is very similar to the finite covering studied on pages 627.[536] For $i \in \mathbb{Z}$ we denote by x_i the loop in $(\widetilde{G},\widetilde{v})$ as illustrated in Figure 420. We denote by x_i also the corresponding element in $\pi_1(\widetilde{G},\widetilde{v})$. It follows from Proposition 26.27 that $\pi_1(\widetilde{G},\widetilde{v})$ equals the free group $\langle \dots x_{-2}, x_{-1}, x_0, x_1, x_2 \dots\rangle$. It is straightforward to see that

$$p_*(x_i) \;=\; t^i s t^{-i}.$$

By Corollary 7.15 the homomorphism p_* is a monomorphism. Thus we see that $t^i s t^{-i}$, $i \in \mathbb{Z}$, generate an infinitely generated free subgroup of the finitely generated generated free group $\langle s,t\rangle$. Furthermore we see that $p_*(\pi_1(\widetilde{G},\widetilde{v})) \subset \ker(\phi\colon \pi_1(G,v) \to \mathbb{Z})$. It is easy to see that the deck transformation group is isomorphic to $\mathbb{Z}$.

Now we claim that the coverings $p\colon \widetilde{G} \to G$ and $q\colon \widehat{G} \to G$ are equivalent. By Proposition 20.9 it suffices to show that $p_*(\pi_1(\widetilde{G},\widetilde{v})) = q_*(\pi_1(\widehat{G},\widehat{v}))$. We consider the following sequence of maps

$$\begin{array}{ccccccc}
\mathbb{Z} & \cong & D(p\colon \widetilde{G} \to G) & \cong & \pi_1(G,v)/p_*(\pi_1(\widetilde{G},\widetilde{v})) & \twoheadrightarrow & \pi_1(G,v)/\ker(\phi\colon \pi_1(G,v) \to \mathbb{Z}) \xrightarrow[\cong]{\phi} \mathbb{Z} \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{see above} & & \text{isomorphism of Proposition 24.6} & & \text{since } p_*(\pi_1(\widetilde{G},\widetilde{v})) \subset \ker(\phi\colon \pi_1(G,v) \to \mathbb{Z}) &
\end{array}$$

Thus, reading from left to right, we get an epimorphism from $\mathbb{Z}$ onto itself. But any such epimorphism has to be an isomorphism. It follows that the third map is also an isomorphism, i.e. $p_*(\pi_1(\widetilde{G},\widetilde{v})) = \pi_1(G,v)/\ker(\phi\colon \pi_1(G,v) \to \mathbb{Z})$, but the latter group is precisely $q_*(\pi_1(\widehat{G},\widehat{v}))$. So we get the desired equality of subgroups.

26.8. **Spanning trees of graphs** $(*)$**.** In Proposition 27.9 we saw that the fundamental group of every connected 1-dimensional CW-complex is a free group. In this short section we will specify more precisely what are generators of this free group. To do so we need the following definition.

Definition.

(1) A *tree* is a contractible graph.

[536]This covering can also be described rigorously. More precisely, we view G as the union of two copies A and B of S^1 glued along the common point 1. We define

$$\widetilde{G} \;=\; (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/(n \sim (1,n)) \text{ for } n \in \mathbb{Z}.$$

Then the projection map

$$p\colon (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/\sim \;\to\; G = (S^1 \times \{A\} \sqcup S^1 \times \{B\})/(1,A) \sim (1,B)$$

is given by $p(x) = (e^{2\pi i x}, A)$ for $x \in \mathbb{R}$ and $p((z,n)) = (z,B)$ for $(z,n) \in S^1 \times \mathbb{Z}$.

deck transformation is given by translation by $n \in \mathbb{Z}$

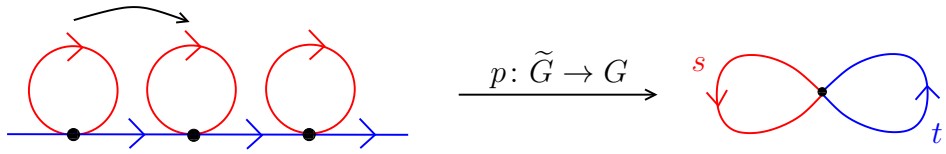


FIGURE 419.

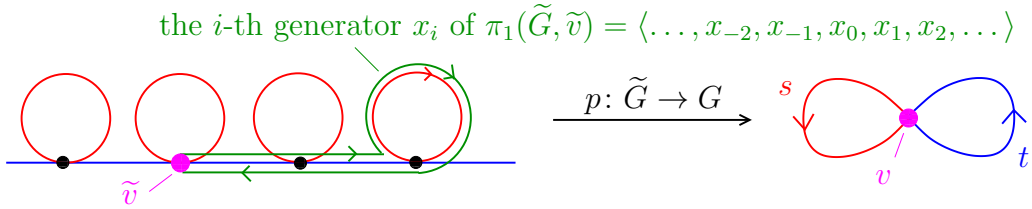


FIGURE 420.

(2) Let G be a graph. A *spanning tree* is a subcomplex T that is a tree and that contains all vertices of G.

Remark.

(1) It follows from Proposition 9.20 that the above definition of a tree is consistent with the definition that we had given on page 318.
(2) We leave it to the reader to show that for a finite group the above definition of a spanning tree is equivalent to the one we had given on page 320.

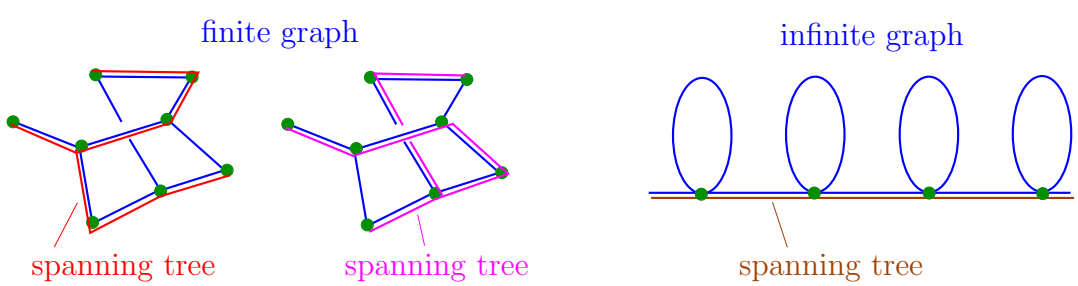


FIGURE 421.

In Figure 421 we give examples of spanning trees in graphs. We see in particular that in general there is no unique spanning tree. Now we can formulate the following generalization of Lemma 12.5.

Proposition 26.27. *Let G be a connected graph. Then the following holds:*

(1) *The graph G admits a spanning tree T. In fact, given any vertex P on G there exists a tree that admits a deformation retraction to P.*

(2) *The group $\pi_1(G)$ is isomorphic to a free group where the cardinality of the generating set equals the cardinality of the set of edges in $G \setminus T$.*

(3) *An explicit description of the isomorphism in (2) is given as follows. Pick a base point v in the spanning tree and for each cell e in $G \setminus T$ pick a path a_e from p to one of the endpoints and pick a path b_e from the other endpoint back to p*[537] *and let t_e be the loop given by first going along a_e, then going through e and then going back to the base point via b_e. Then the map*

$$\begin{array}{rcl} \left\langle \{x_e\}_{e \text{ edge in } G \setminus T} \right\rangle & \to & \pi_1(G, v) \\ x_e & \mapsto & [t_e] \end{array}$$

is an isomorphism.

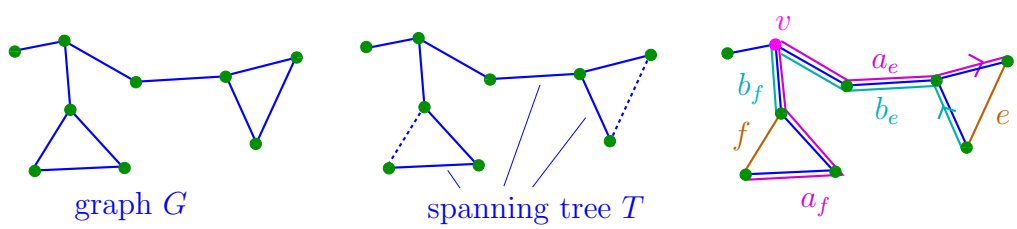


FIGURE 422.

Sketch of proof. We only provide an outline of the proof. The full details can be found in [Hat, p. 84] or [FPi, Lemma 2.6.3]. Let G be a connected graph.

(1) We pick a vertex P on G. We define $V_0 = \{P\}$ and iteratively we define for $i \geq 1$ the set V_i to be the set of all vertices that are not contained in V_{i-1} but that are connected to a vertex in V_{i-1} by an edge. Now we define $T_0 = \{P\}$. Once T_{i-1} is defined we pick for each vertex in V_i a single edge that connects the vertex to a vertex in V_{i-1}. Then it shown in [Hat, Proposition 1.A.1] that $\bigcup_{i \geq 0} T_i$ is a spanning tree.[538]

(2) If G is a finite graph, then the argument of the proof of Proposition 9.21 shows that the projection map $G \to G/T$ is a homotopy equivalence. The same statement also holds if G is an infinite graph, in this case the statement is proved in [Hat, Proposition 1.A.2]. The topological space G/T is again a graph with one vertex, given by the point $[T] \in G/T$ and an edge for each edge in $G \setminus T$. In particular G/T is homeomorphic to the wedge of circles, one for each edge in $G \setminus T$. It then follows from the straightforward generalization of Proposition 12.3 to the case of the wedge

[537] The two paths exist since a tree is in particular connected.

[538] Since G is connected, every vertex lies in some V_i, hence each vertex lies in some T_i. It is also straightforward to see that each T_i is the deformation retract of T_{i+1}. Now one just needs to work a little harder to show that $T_0 = \{P\}$ is a deformation retract of T, i.e. T is contractible.

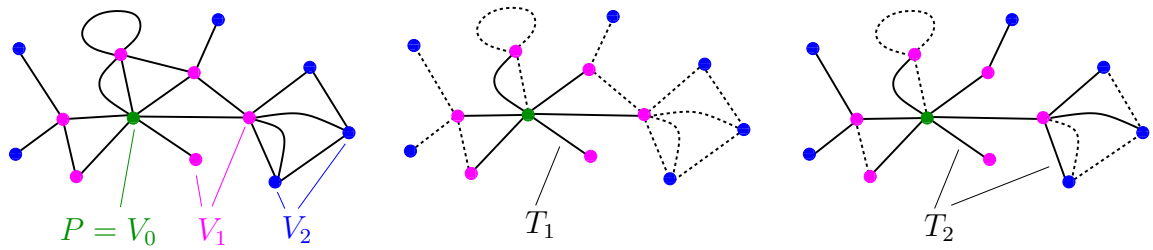


FIGURE 423.

of arbitrarily many topological spaces that the fundamental group of such a wedge of circles is the free group, where the generators correspond precisely to the circles, i.e. the edges in $G \setminus T$. □

Exercises for Chapter 26.

Exercise 26.1.

(a) Is every compact subset of $\mathbb{R}^2$ a CW-complex?
(b) Is the open ball $B_1(0)$ a CW-complex?
(c) Let X be a CW-complex and let $x \in X$. Show that $\{x\}$ is a closed subset of X.

Exercise 26.2.

(a) We equip $\mathbb{R}^\infty$ with the metric that is given by

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \ := \ \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

We consider the following two topologies on $\mathbb{R}^\infty$:

$$\mathcal{S} \ := \ \{U \subset \mathbb{R}^\infty \mid \text{ for every } P \in U \text{ there exists } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

and

$$\mathcal{T} \ := \ \{U \subset \mathbb{R}^\infty \mid \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Show that $\mathcal{S} \subset \mathcal{T}$.

(b) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{S}$. Does the Heine-Borel Theorem hold in $\mathbb{R}^\infty$? More precisely, is every bounded closed subset of $\mathbb{R}^\infty$ compact?
(c) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{T}$. Show that the projection map $p\colon \mathbb{R}^\infty \to \mathbb{R}^n$ is continuous and open.

Exercise 26.3. Show that the action

$$\begin{array}{rcl} \{\pm 1\} \times S^\infty & \to & S^\infty \\ (\epsilon, v) & \mapsto & \epsilon \cdot v \end{array}$$

of the group $\{\pm 1\}$ on the topological space S^∞ is continuous and discrete.

Exercise 26.4. Let $A = S^1 \times [0, 1]$ be an annulus.

(a) (i) We consider

$$X := \Big(\bigcup_{k\in\mathbb{N}_0} A\times\{k\}\Big)/(z,0,k)\sim(z^2,1,k-1) \quad \text{for } z\in S^1, k\in\mathbb{N}.$$

What is $\pi_1(X)$?

(ii) We consider

$$Y := \Big(\bigcup_{k\in\mathbb{N}_0} A\times\{-k\}\Big)/(z,0,-k)\sim(z^2,1,-k-1) \quad \text{for } z\in S^1, k\in\mathbb{N}.$$

What is $\pi_1(Y)$?

In both cases the group you obtain is an abelian group that is a subgroup of $(\mathbb{R},+)$.

(b) Give an example of topological space Z such that $\pi_1(Z)\cong(\mathbb{Q},+)$.

Exercise 26.5. We equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one cell in dimensions $0,1,2$. Let $p\colon S^2\to\mathbb{R}\mathrm{P}^2 = S^2/\pm 1$ be the usual 2-fold covering. The CW-structure on $\mathbb{R}\mathrm{P}^2$ induces a CW-structure on S^2. What does this CW-structure on S^2 look like?

Exercise 26.6. We consider the following subset of $\mathbb{R}^2$:

$$X := \bigcup_{n\in\mathbb{N}} \text{circle of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n},0) = \bigcup_{n\in\mathbb{N}} \{(x,y)\in\mathbb{R}^2 \mid (x-\tfrac{1}{n})^2+y^2=\tfrac{1}{n^2}\}.$$

This subset, equipped with the subspace topologically is colloquially referred to as the *Hawaiian earrings*, see Figure 424 for an illustration.

(a) Let $A=\{0\}\cup\{\frac{1}{n}\mid n\in\mathbb{N}\}$. Show that X is homeomorphic to $[0,1]/A$.
Remark: We encountered the topological space $[0,1]/A$ in Exercise 1.36.
(b) Show that X is compact.
(c) Is X homeomorphic to the wedge of countably infinitely many circles?
(d) Show that $\pi_1(X)$ is infinitely generated.
(e) Even better than (d): Show that $\pi_1(X)$ is uncountable generated.
(f) Can X be equipped with a CW-structure?

The combination of (b) and (d) implies, perhaps somewhat surprisingly, that there exist compact topological spaces whose fundamental groups are infinitely generated.
Remark: The precise determination of $\pi_1(X)$ is actually rather delicate, we refer to [MoM] and [Smi] for details. In particular, despite appearances, it is shown in [Smi] that $\pi_1(X)$ is not a free group.

Exercise 26.7. Let $\alpha\colon\langle a,b\rangle\to\mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a)=1$ and $\alpha(b)=0$. Furthermore let X be the wedge of two circles A and B. As usual we equip X with the CW-structure with one 0-cell P and two 1-cells a and b. Furthermore we make the identification $\pi_1(X)=\langle a,b\rangle$.

(a) "Construct" the covering $p\colon\widetilde{X}\to X$ corresponding to $\ker(\alpha)$, i.e. construct the covering p such that $p_*(\pi_1(\widetilde{X}))=\ker(\alpha)$. More precisely, give the CW-structure on $\widetilde{X}$ and indicate, e.g. using colors, the cellular map p.

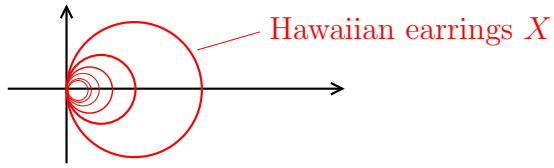


FIGURE 424.

(b) We know from Corollary 26.26 that $\ker(\alpha)$ is a free group on four generators. Determine four elements of $\ker(\alpha)$ that generate $\ker(\alpha)$ as a free group. You do not have to justify your answer.

27. CW-complexes II: Advanced properties

In this chapter we will prove several key technical theorems on CW-complexes, in particular we will prove the Homotopy Extension Theorem 27.4 and the Cellular Approximation Theorem 27.15. The reader who is in a hurry could just have a look at the Cellular Approximation Theorem 27.15 and then move on.

27.1. **Constructing homotopies on infinite-dimensional CW-complexes** $(*)$**.** On several occasions throughout these notes we will want to construct a homotopy on an infinite-dimensional CW-complex by working our way up through the skeleta. In the end one needs to combine infinitely many homotopies which obviously leads to continuity issues. This rather technical section deals with these issues. It is best to initially skip this section and to go back to it if later on one really wants to know how an argument works.

We start out with the following subtle lemma.

Lemma 27.1. *Let X be a CW-complex.*

(1) *The following two topologies on $X \times [0,1]$ agree:*
   (a) *The product topology on the product $X \times [0,1]$.*
   (b) *The topology on $X \times [0,1]$ viewed as the direct limit $\varinjlim (X^k \times [0,1])$. In other words, as discussed on page 488, this is the topology where a given subset W of $X \times [0,1] = \varinjlim (X^k \times [0,1])$ is open if and only each intersection $W \cap (X^k \times [0,1])$ is an open subset of $X^k \times [0,1]$.*

(2) *A map $\Phi\colon X \times [0,1] \to Y$ to a topological space Y is continuous if and only if all the restrictions to $X^k \times [0,1]$, $k \in \mathbb{N}_0$, are continuous.*

Proof. Let X be a CW-complex. First note that second statement is an immediate consequence of the of the first statement. Also, it follows immediately from the fact that $X = \varinjlim X_k$ that any set that is open with respect to (a) is also open with respect to (b). Thus it suffices to show that any set that is open with respect to (b) is also open with respect to (a).

Let $W \subset X \times [0,1]$ be a set that is open in $\varinjlim (X^k \times [0,1])$. We need to show that W is also open with respect to the product topology on $X \times [0,1]$. Let $(x_0, t_0) \in W$ be a point. By Lemma 1.59 it suffices to show that there exists a neighborhood V of $x_0 \in X$ and a neighborhood K of $t_0 \in [0,1]$ such that $V \times K$ is contained in W. We start out with the following claim.

Claim. There exists a compact neighborhood K of $t_0 \in [0,1]$ such that $\{x_0\} \times K \subset W$.

To prove the claim we pick $k \in \mathbb{N}_0$ with $x_0 \in X^k$. Since W is open in $\varinjlim (X^k \times [0,1])$ we know that the intersection $(X^k \times [0,1]) \cap W$ is an open neighborhood of $(x_0, t_0) \in X^k \times [0,1]$. By definition of the product topology on $X^k \times [0,1]$ there exists in particular an open neighborhood J of $t_0 \in [0,1]$ such that $\{x_0\} \times J \subset W \cap (X^k \times [0,1])$. In the topological

space $[0,1]$ any open neighborhood J of x_0 also contains a compact neighborhood K. This K has the desired properties. This concludes the proof of the claim.

Next we consider $V := \{x \in X \,|\, \{x\} \times K \subset W\}$. Note that $x_0 \in V$. Thus by the discussion in the beginning of the proof it remains to prove the following claim.

Claim. The subset V is open in X.

We use Lemma 26.8 to show that V is open in X. Thus let $\Phi_e \colon \overline{B}^n \to X$ be the characteristic map of an n-cell of X. We need to show that the preimage $\Phi_e^{-1}(V)$ is open in $\overline{B}^n$. We write $U := (\Phi_e \times \mathrm{id}_{[0,1]})^{-1}(W \cap (X^n \times [0,1]))$. Note that U is an open subset of $\overline{B}^n \times [0,1]$ since $W \cap (X^n \times [0,1])$ is open by our hypothesis on W and since the map $\Phi_e \times \mathrm{id}_{[0,1]} \colon \overline{B}^n \times [0,1] \to X^n \times [0,1]$ is continuous. Basically by definition we have the equality

$$\Phi_e^{-1}(V) \;=\; \{y \in \overline{B}^n \,|\, \{y\} \times K \subset U\}.$$

Now that we are dealing with $\overline{B}^n \times [0,1]$ it is an elementary topology exercise to show that the set on the right-hand side is indeed open. This concludes the proof of the second claim and thus of the lemma. □

In Lemma 9.2 we saw that one can combine finitely many homotopies. Later on we will want to combine infinitely many homotopies of CW-complexes. This slightly delicate procedure can be performed using the following lemma.

Lemma 27.2. *Let X be a CW-complex, let A be a subset of X and let Y be a topological space. Finally let $W_0 \subset W_1 \subset W_2 \subset \dots$ be a sequence of subcomplexes of X such that for each $k \in \mathbb{N}_0$ there exists an $l \in \mathbb{N}_0$ such that the k-skeleton X^k is contained in W_l. (In many applications we have $W_k = X^k$.) Let $f_0, f_1, \dots$ be maps from X to Y. Suppose that for each $k \in \mathbb{N}_0$ we are given a homotopy $H(k) \colon X \times [0,1] \to Y$ rel $A \cup W_k$ from f_k to f_{k+1}. (Since it is easy to overlook this bit of the hypothesis we stress that $H(k)$ is in particular a homotopy rel W_k). We define*[539]

$$\begin{array}{rcl} g \colon X & \to & Y \\ x & \mapsto & f_k(x) \quad \textit{where } k \in \mathbb{N}_0 \textit{ such that } x \in W_k. \end{array}$$

For $k = 0, 1, 2, \dots$ we write $s_k = \sum_{j=1}^{k} \frac{1}{2^j}$. Then the map

$$\begin{array}{rcl} H \colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} H(k)(x, 2^{k+1} \cdot (t - s_k)), & \textit{if } t \in [s_k, s_{k+1}), \\ g(x), & \textit{if } t = 1 \end{cases} \end{array}$$

that is illustrated in Figure 425 is a homotopy from f_0 to g. We refer to the homotopy H as the combination of $\{H(k)\}_{k \in \mathbb{N}_0}$.

[539] It follows easily from the setup that the map g is well-defined, i.e. given $x \in X$ the point $g(x)$ does not depend on the choice of k.

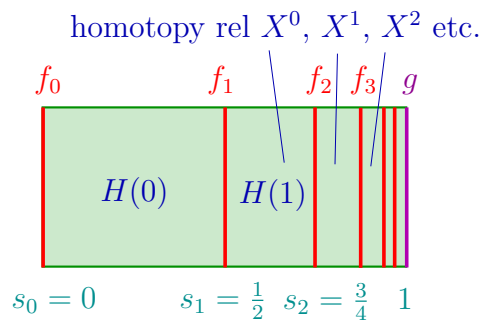


FIGURE 425. Illustration of Lemma 27.2.

Proof. We only need to show that H is continuous. Note that for each $l \in \mathbb{N}_0$ it follows from our choice of maps f_l and Lemma 5.3 that the restriction of H to $W_l \times [0,1]$ is continuous. Since each X^k is contained in some W_l we see that the restriction of H to each $X^k \times [0,1]$ is continuous. But now it follows from Lemma 27.1 (2) that $H\colon X \times [0,1] \to Y$ is also continuous. □

We conclude this technical section with an application that we will need once on a much later occasion. But the application doubles also a nice illustration for how to apply the previous lemma.

Definition. Let Z be a CW-complex. We equip $[0,\infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$. We equip $Z \times [0,\infty)$ with the product CW-structure. We refer to the subcomplex

$$T(Z) \;:=\; \bigcup_{i \in \mathbb{N}_0} Z^i \times [i, i+1] \quad \subset \; Z \times [0,\infty)$$

as the *telescope of* Z..

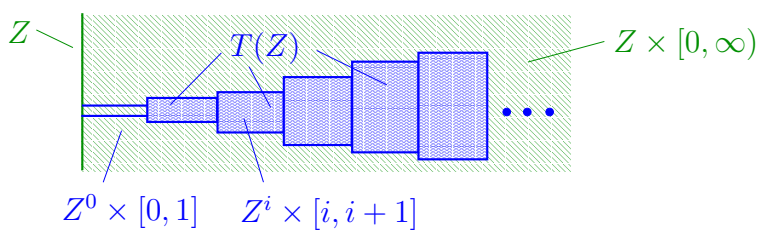


FIGURE 426.

The following lemma shows that the telescope $T(Z)$ is homotopy equivalent to Z.

Lemma 27.3. *Let Z be a CW-complex. Suppose we are in the following situation:*

($*$) *For each $k \in \mathbb{N}_0$ we are given a deformation retraction r_k from $Z \times [0,1]$ to the subset $(Z^k \times [0,1]) \cup (Z \times \{1\})$. (Later on in Lemma 27.8 we will see that such that a deformation retraction always exists.)*

The following statements hold:

(1) *$T(Z)$ is a deformation retract of $Z \times [0,\infty)$.*
(2) *The telescope $T(Z)$ is homotopy equivalent to Z.*

Proof. First note that $Z = Z \times \{0\}$ is clearly a deformation retract of $Z \times [0,\infty)$. Thus it follows from Lemmas 9.10 and 9.8 that (2) is a consequence of (1). Therefore it remains to prove (1).

As in the definition of the telescope $T(Z)$ we view $[0,\infty)$, $X := Y := Z \times [0,\infty)$ and $T(Z)$ as CW-complexes. For each $k \in \mathbb{N}_0$ we are given a deformation retraction

$$r_k \colon (Z \times [0,1]) \times [0,1] \;\to\; Z \times [0,1]$$

from $Z \times [0,1]$ to $(Z^k \times [0,1]) \cup (Z \times \{1\})$. We denote by

$$\begin{array}{rcl} \rho_k \colon Z \times [k,k+1] & \to & (Z^{k-1} \times [k,k+1]) \cup (Z^k \times \{k+1\}) \\ (z,t) & \mapsto & (r_k(z,t-k) + (0,k), 1) \end{array}$$

the corresponding retraction, with the intervals shifted by k. We make the following preparations:

(1) Given $k \in \mathbb{N}_0$ we define $W_k := Z^k \times [0,\infty)$. Note that it follows from the definition of the product CW-structure on $Z \times [0,\infty)$ that the k-skeleton of $X = Z \times [0,\infty)$ is contained in W_{k+1}.
(2) For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} f_k \colon X = Z \times [0,\infty) & \to & Y = Z \times [0,\infty) \\ (z,t) & \mapsto & \begin{cases} (z,t), & \text{if } x \in Z^k \text{ and } t \geq k+1, \\ (\rho_k \circ \cdots \circ \rho_1 \circ \rho_0)(x), & \text{if } z \in Z^k \text{ and } t < k+1. \end{cases} \end{array}$$

(3) We consider the map

$$\begin{array}{rcl} H(k) \colon X \times [0,1] & \to & Y = Z \times [0,\infty) \\ ((z,t),s) & \to & \begin{cases} (z,t), & \text{if } t \geq k+1, \\ r_{k+1}(\rho_k(z,t),s), & \text{if } t < k+1. \end{cases} \end{array}$$

This defines a homotopy from f_k to f_{k+1} rel W_k. Note that $H(0)_1 = \mathrm{id}_X$.

As in the statement of Lemma 27.2 we consider the map

$$\begin{array}{rcl} g \colon X & \to & Y \\ x & \mapsto & f_k(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

One can easily verify that this is a retraction of $X = Z \times [0,\infty)$ to the telescope $T(Z)$. It follows from Lemma 27.2 that there exists a homotopy from $H(0)_1 = \mathrm{id}$ to g, i.e. $T(Z)$ is in fact a deformation retract of $X = Z \times [0,\infty)$. $\square$

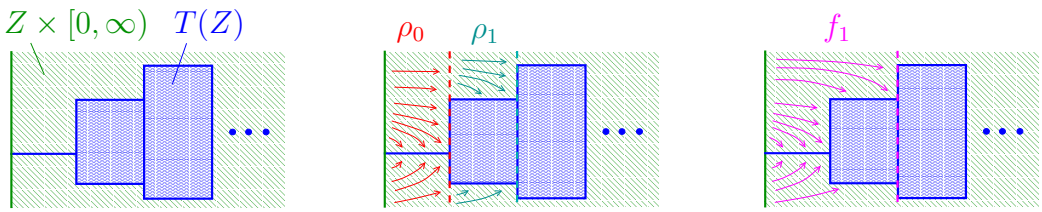


FIGURE 427.

27.2. **The Homotopy Extension Theorem.** On several occasions we will need the following somewhat technical theorem.

Theorem 27.4. (Homotopy Extension Theorem) *Let X be a CW-complex, let Y be a topological space and let $f_0\colon X \to Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G\colon A \times [0,1] \to Y$ from $g_0 := f_0|_A$ to some map $g_1\colon A \to Y$. Then there exists a homotopy $F\colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A\times[0,1]}$ equals G.*

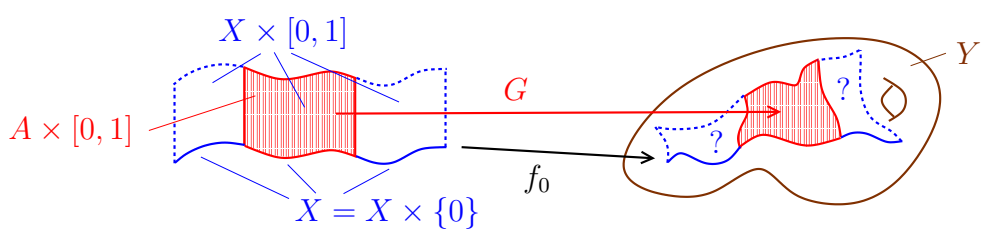


FIGURE 428. Illustration of the Homotopy Extension Theorem 27.4.

Remark. The statement of the Homotopy Extension Theorem does not hold for arbitrary subsets of X. For example, let $X = \mathbb{R}$ and $A = (0,1)$. We write $Y = \mathbb{R} \times \{0\} \cup (0,1) \times [0,1]$ and let $f\colon X \to Y$ be the obvious inclusion map and let $G\colon (0,1) \times [0,1] \to Y$ be the obvious inclusion map. (See Figure 429 for an illustration.) It is an amusing exercise to show that G cannot be extended to a homotopy on all of $\mathbb{R}$.[540] It is an equally amusing exercise to show that the homotopy can be extended if throughout we replace the open interval $A = (0,1)$ by the closed interval $A = [0,1]$.

Before we delve into the long and strenuous proof of the Homotopy Extension Theorem 27.4 let us prove the following neat corollary.

Corollary 27.5. *Let X be connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y. There exists a map $h\colon X \to X$ with $h(P) = Q$ that is homotopic to the identity, in particular h is a homotopy equivalence.*

[540]Here is a hint: if there was an extension of G to a homotopy $F\colon X \times [0,1] \to Y$, what would/should $F(1,1)$ be?

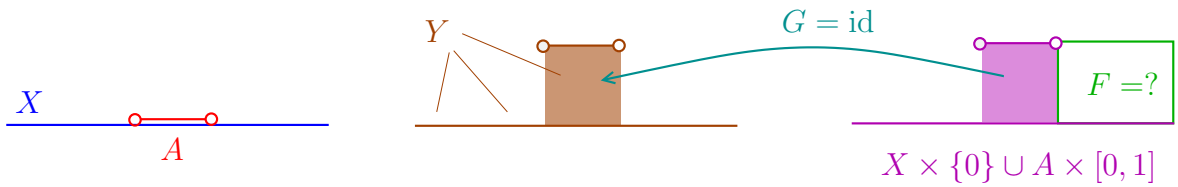


FIGURE 429.

Example. We consider the CW-complex X and the two points P and Q shown in Figure 430. It should be pretty clear that there is no homeomorphism f of X with $f(P) = Q$. But Corollary 27.5 gives us a decent replacement, namely a map $h\colon X \to X$ with $h(P) = Q$ which is homotopic to the identity.

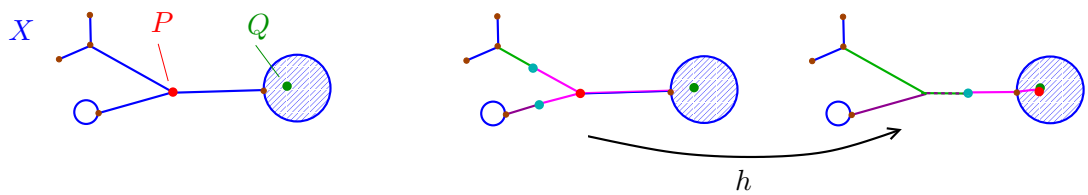


FIGURE 430.

Proof. Let X be connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y. We write $A := \{P0\}$. Since P is a point in the 0-skeleton of X we see that A is a subcomplex of A. Since X is connected we obtain from Proposition 26.11 that there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$. We apply the Homotopy Extension Theorem 27.4 to $f_0 = \mathrm{id}\colon X \to X$, $g_0(P) := P$, $g_1(P) = Q$ and $G(P,t) = \gamma(t)$. The resulting homotopy $F\colon X \times [0,1] \to X$ and $h := F_1$ give us the desired homotopy and the desired map. Since id is a homotopy equivalence and since $h = F_1$ is homotopic to $\mathrm{id} = F_0$ we obtain from Lemma 9.8 that h is also a homotopy equivalence. □

Before we can provide the proof of Theorem 27.4 we will need to formulate and prove several lemmas.

Lemma 27.6. *For any $n \in \mathbb{N}_0$ there exists a deformation retraction from the solid cylinder $\overline{B}^n \times [0,1]$ to the "cup" $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$.*

Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} r\colon \overline{B}^n \times [0,1] & \to & (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \\ P & \mapsto & \text{the unique point on } \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1] \text{ that} \\ & & \text{gets hit by the ray from } (0,2) \in \mathbb{R}^n \times \mathbb{R} \text{ through } P. \end{array}$$

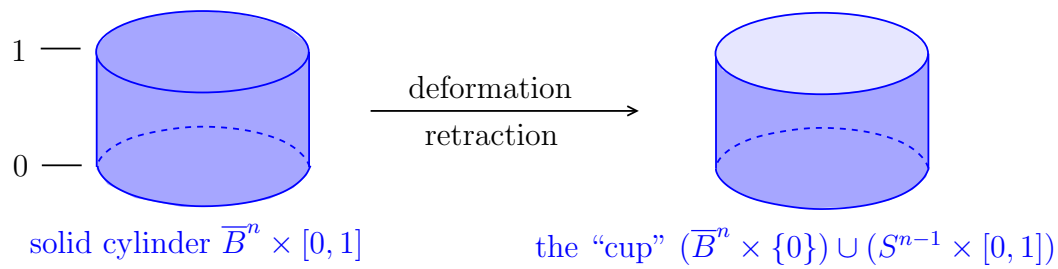


FIGURE 431.

is a retraction. We refer to Figure 432 for an illustration. Since $\overline{B}^n \times [0,1]$ is convex we can define the corresponding deformation retraction as follows:

$$\begin{array}{rcl} (\overline{B}^n \times [0,1]) \times [0,1] & \to & \overline{B}^n \times [0,1] \\ (P,t) & \mapsto & P \cdot (1-t) + r(P) \cdot t. \end{array}$$

□

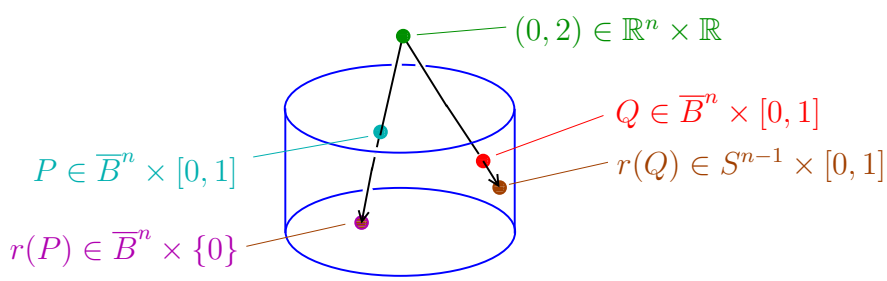


FIGURE 432.

Lemma 27.7. *If X is a CW-complex and if A is a subcomplex of X, then there exists a retraction from $X \times [0,1]$ to the subcomplex $(X \times \{0\}) \cup (A \times [0,1])$.*

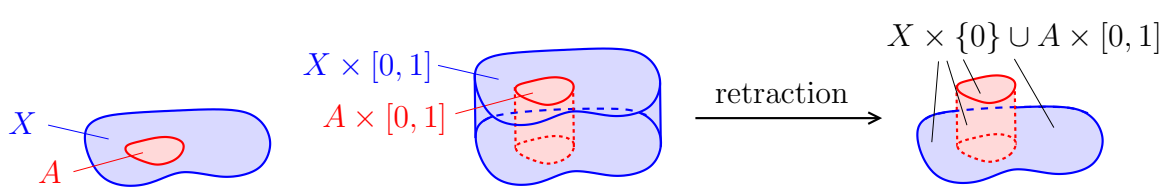


FIGURE 433. Illustration of Lemma 27.7.

Proof. Let X be a CW-complex and let A be a subcomplex. We start out with the following claim.

Claim. For every $n \in \mathbb{N}_0$ there exists a retraction

$$r_n \colon X \times \{0\} \ \cup \ (X^n \cup A) \times [0,1] \ \to \ X \times \{0\} \ \cup \ (X^{n-1} \cup A) \times [0,1].$$

Let $n \in \mathbb{N}_0$. We denote by $\varphi_i \colon \overline{B}^n \to X$, $i \in I$, the characteristic maps of all the n-cells of X that do not lie in A. By Lemma 27.6 there exists a retraction r from $\overline{B}^n \times [0,1]$ to $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$. The idea is to apply this retraction to all k-cells of X to obtain the desired retraction. More precisely we consider the map

$$\begin{aligned} r_n \colon X \times \{0\} \cup (X^n \cup A) \times [0,1] \ &\to \ X \times \{0\} \cup (X^{n-1} \cup A) \times [0,1] \\ (P,t) \ &\mapsto \ \begin{cases} \varphi_i(r(Q,t)), & \text{if } P = \varphi_i(Q) \text{ with } Q \in \overline{B}^n \text{ and } i \in I, \\ (P,t), & \text{otherwise.} \end{cases} \end{aligned}$$

This map is easily seen to be a retraction.[541] This concludes the proof of the claim.

Now we turn to the actual proof of the lemma. For clarity we first consider the case that X is a finite-dimensional CW-complex. So let $n \in \mathbb{N}_0$ be the dimension of the CW-complex X. We denote by $r_0, \dots, r_n$ the retractions from the claim. Then the map

$$s_n := r_0 \circ r_1 \circ \cdots \circ r_n \colon \underbrace{X \times \{0\} \cup (X^n \cup A) \times [0,1]}_{= X \times [0,1]} \ \to \ \underbrace{X \times \{0\} \cup (X^{-1} \cup A) \times [0,1]}_{= X \times \{0\} \cup A \times [0,1]}$$

is the desired retraction. We refer to Figure 434 for an illustration of the argument.

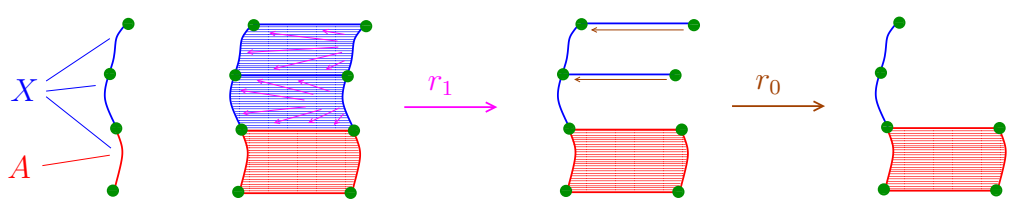


FIGURE 434.

Now we turn to the general case, i.e. X can now be infinite-dimensional. As before we denote by $r_0, r_1, \dots$ the retractions provided by the claim and given $k \in \mathbb{N}_0$ we again consider the retraction

$$s_k := r_0 \circ \cdots \circ r_k \colon X \times \{0\} \cup (X^k \cup A) \times [0,1] \ \to \ X \times \{0\} \cup (A \times [0,1]).$$

We consider the following map

$$\begin{aligned} H \colon X \times [0,1] \ &\to \ X \times \{0\} \cup A \times [0,1] \\ (x,t) \ &\mapsto \ s_k(x,t) \quad \text{if } x \in X^k. \end{aligned}$$

We make the following observations:

(1) It follows immediately from the fact that the r_k are appropriate retractions that the map H is well-defined, i.e. given $(x,t) \in X \times [0,1]$ the value of $s_k(x,t)$ is independent of the choice of k.

[541] It follows from Lemma 26.8 that the map is in fact continuous.

(2) Let $k \in \mathbb{N}_0$. By (1) the restriction of H to $X^k \times [0,1]$ agrees with the restriction of the map s_k to $X^k \times [0,1]$. The map s_k is the composition of finitely many continuous maps, thus continuous. It follows from Lemma 27.1 (2) that the map s on $X \times [0,1]$ is continuous with respect to the usual product topology.
(3) It follows immediately from the definition of H that it is the identity on the subset $(X \times \{0\}) \cup (A \times [0,1])$ of $X \times [0,1]$.

We have thus shown that H is indeed the desired retraction. □

In one situation later on we will need the following stronger version of Lemma 27.7. For most readers it is best to skip this lemma.

Lemma 27.8. $(*)$ *If X is a CW-complex and if A is a subcomplex of X, then there exists a* deformation *retraction from $X \times [0,1]$ to the subcomplex $(X \times \{0\}) \cup (A \times [0,1])$.*

Proof $(*)$. Let X be a CW-complex and let A be a subcomplex. Using the fact that the retraction in Lemma 27.6 comes actually from a deformation retraction one can show, using basically the same argument as in Lemma 27.8, that for every $n \in \mathbb{N}_0$ there exists a *deformation* retraction

$$R(n)\colon (X \times \{0\} \ \cup \ (X^n \cup A) \times [0,1]) \times [0,1] \ \to \ X \times \{0\} \ \cup \ (X^{n-1} \cup A) \times [0,1].$$

We denote by $r_n := R(n)_1$ the corresponding retraction. For each $k \in \mathbb{N}$ we consider the map

$$\begin{array}{rcl} f_k\colon X \times [0,1] & \to & (X^{k-1} \cup A) \times [0,1] \\ (x,t) & \mapsto & \begin{cases} (r_k \circ \cdots \circ r_l)(x,t), & \text{if } x \in X^l \text{ and } l \geq k, \\ (x,t), & \text{if } x \in X^{k-1}. \end{cases} \end{array}$$

It follows from Lemma 27.1 (2) that this map f_k is continuous. Now we consider the following map

$$\begin{array}{rcl} F\colon (X \times [0,1]) \times [0,1] & \to & X \times [0,1] \\ ((x,t),s) & \mapsto & \begin{cases} (x,t), & \text{if } x \in X^k \text{ and } t \in [0, \frac{1}{2^{k+1}}], \\ r_k\big((x,t), 2^{k+1} \cdot (s - \frac{1}{2^{k+1}})\big), & \text{if } x \in X^k \text{ and } t \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}], \\ f_k(x,t), & \text{if } x \in X^k \text{ and } t \in [\frac{1}{2^k}, 1]. \end{cases} \end{array}$$

We refer to Figure 435 for an illustration. It follows from Lemma 5.3 that the restriction of F to each $(X^k \times [0,1]) \times [0,1]] = (X \times [0,1])^{k+1} \times [0,1]$ is continuous. Thus it follows from Lemma 27.1 (2) that $F\colon X \times [0,1] \to X \times [0,1]$ is continuous. This map is the desired deformation retraction. □

Now we can easily provide a proof of Theorem 27.4.

Proof of Theorem 27.4. Let X be a CW-complex, let Y be a topological space and let $f_0\colon X \to Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G\colon A \times [0,1] \to Y$ from $g_0 := f_0|_A$ to some map $g_1\colon A \to Y$. By Lemma 27.7 there exists a retraction

$$H\colon X \times [0,1] \ \to \ (X \times \{0\}) \cup (A \times [0,1]).$$

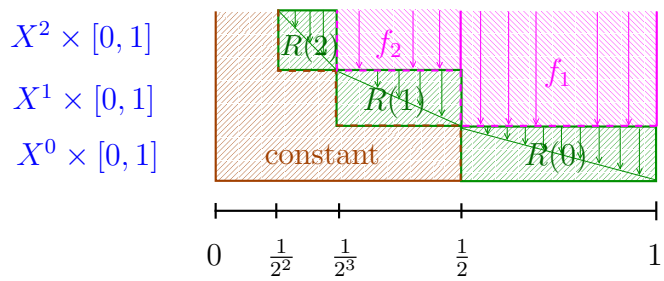


FIGURE 435. Illustration for the proof of Lemma 27.8.

The theorem follows from the following claim.

Claim. The map

$$\begin{array}{rcccl} F\colon X \times [0,1] \xrightarrow{H} (X \times \{0\}) \cup (A \times [0,1]) & \to & Y & & \\ (x,0) & \mapsto & f_0(x), & \text{for } x \in X, \\ (a,t) & \mapsto & G(a,t), & \text{for } (a,t) \in A \times [0,1]. \end{array}$$

has all the desired properties.

First we have to verify that the map is continuous. In fact, for continuity we do have to worry a little bit about the right-hand map. But note that Lemma 26.10 (2) says that the subcomplex A is a closed subset of X. Thus we see that the map on the right-hand side is continuous when restricted to the *closed* subsets $X \times \{0\}$ and $A \times [0,1]$. Thus it is continuous by Lemma 1.32 (2). Note it is clear that F has the desired properties, i.e. that it is a homotopy $F\colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A\times[0,1]}$ equals G. This concludes the proof of the claim. □

27.3. **Fundamental groups of CW-complexes.**

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;:=\; \sum_n (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite graph, then, according to the discussion on page 645, we can view G as a CW-complex. In this case the Euler characteristic agrees with the definition of the Euler characteristic of a graph that we gave on page 318.
(2) For the CW-structures provided by Lemma 26.2 we have $\chi(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1 + (-1)^n)$ and $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$.
(3) A short glance at the three examples in Figure 399 shows that these three different CW-structures for the torus have the same Euler characteristic. Later on we will find out whether or not that is a coincidence.

Proposition 27.9. *Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$. Then the following hold:*

(1) *The fundamental group $\pi_1(X, x_0)$ is a free group.*
(2) *If X is a finite CW-complex, then $\pi_1(X, x_0)$ is a free group with $1-\chi(X)$ generators.*

Proof. In Lemma 12.5 we showed that if G is a finite connected graph, then $\pi_1(G)$ is a free group with $1 - \chi(G)$ generators. Verbatim the same proof applies to finite 1-dimensional CW-complexes, which proves (2).

Now we turn to the proof of (1). We will only provide the proof for the case that X has countably many cells. The general case is proved in [Hat, Proposition 1.A.1]. Thus let X be a connected countable 1-dimensional CW-complex and let $x_0 \in X$. If X is finite, then we are already done by the above argument. So suppose that X is infinite. Since X is connected this implies that X has infinitely many 1-cells. We start out with the following claim.

By Proposition 26.17 there exists a sequence of finite connected subcomplexes A_i, $i \in \mathbb{N}_0$ with the following properties:

(a) each A_i is a subcomplex of A_{i+1},
(b) $X = \bigcup_{i \in \mathbb{N}_0} A_i$.

By Proposition 5.11 we can assume without loss of generality that x_0 is a point in A_1.

The proof of the following claim is a slight generalization of the proof of Lemma 12.5. We leave the details as an exercise to the reader.

Claim. Let C be a connected finite 1-dimensional CW-complex and let B be a connected subcomplex. Let $x \in B$. Suppose we are given a finite set b and an isomorphism $\phi\colon \pi_1(B, x) \cong \langle b \rangle$. Then there exists a finite set c, disjoint from b, and an isomorphism $\psi\colon \pi_1(C, x) \cong \langle b \sqcup c \rangle$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B, x) & \xrightarrow{\cong} & \langle b \rangle \\ \downarrow & & \downarrow \\ \pi_1(C, x) & \xrightarrow{\cong} & \langle b \sqcup c \rangle, \end{array}$$

where the left vertical map is the inclusion induced map and the right vertical map is the obvious map.

Iteratively we apply the claim to the above sequence $A_0, A_1, A_2, \dots$ of connected 1-dimensional CW-complexes and we obtain disjoint finite sets $a_0, a_1, a_2, \dots$ and identifications $\pi_1(A_i, x_0) = \langle a_0 \sqcup \cdots \sqcup a_i \rangle$. We obtain that

$$\pi_1(X, x_0) \underset{\substack{\uparrow \\ \text{Proposition 26.19}}}{\cong} \varinjlim \pi_1(A_i, x_0) = \varinjlim \langle a_0 \sqcup \cdots \sqcup a_i \rangle \underset{\substack{\uparrow \\ \text{as on page 484}}}{=} \Big\langle \bigsqcup_{i \in \mathbb{N}} a_i \Big\rangle.$$

We have thus shown that $\pi_1(X, x_0)$ is a free group. $\square$

The following definition is evidently inspired by the definition of a CW-complex.

Definition. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a map. We refer to $X \cup_\varphi \overline{B}^n$ as the *topological space X with an n-cell attached via the attaching map φ*. We refer to the image of $\overline{B}^n$ in $X \cup_\varphi \overline{B}^n$ as an n-cell.

The following proposition now tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 27.10. *Let X be a topological space, let $n \geq 2$, let $\varphi\colon S^{n-1} \to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. Then the following hold:*

(a) *If $n > 2$, then the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.*

(b) *If $n = 2$, then $\varphi\colon S^1 \to X$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$, and the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism*

$$\pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^n, x_0\big).$$

Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then[542]

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\substack{\uparrow \\ \text{by Proposition 27.10}}}{=} \pi_1(S^1, 1)/\langle[\varphi]\rangle \underset{\substack{\uparrow \\ \text{see Proposition 7.18}}}{=} \langle x\rangle/\langle x^n\rangle \cong \mathbb{Z}_n.$$

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\mathrm{id} = \varphi\colon S^1 \to S^1$ that is given by the identity. The resulting topological space is illustrated in Figure 436. This topological space is again simply connected by Proposition 27.10.

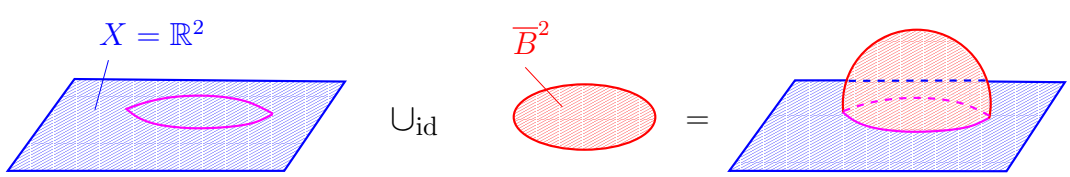


FIGURE 436.

(3) Using Propositions 27.9 and 27.10 we can now easily compute fundamental groups of finite CW-complexes. For example, as in Figure 437 we view the surface Σ of genus 2 as a CW-complex with one 0-cell P, four 1-cells x_1, y_1, x_2, y_2 and one 2-cell. We denote

[542] We leave it as an exercise to convince yourself that for $n = 2$ the topological space $S^1 \cup_\varphi \overline{B}^2$ is actually homeomorphic to $\mathbb{R}\mathrm{P}^2$.

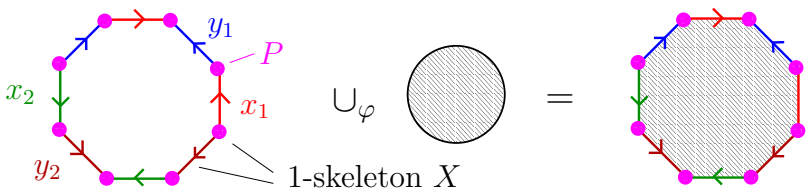


FIGURE 437.

by X the 1-skeleton of Σ and we denote by $\varphi\colon S^1 \to X$ the attaching map of the 2-cell. We have

$$\pi_1(\Sigma, P) \;=\; \pi_1\big(X \cup_\varphi \overline{B}^2, P\big) \;\underset{\substack{\uparrow \\ \text{by Proposition 27.10}}}{=}\; \pi_1(X,P)/\langle\!\langle[\varphi]\rangle\!\rangle \;\underset{\substack{\uparrow \\ \text{by Proposition 27.9 and since } [\varphi]=[x_1,y_1][x_2,y_2]}}{=}\; \langle x_1, y_1, x_2, y_2\rangle/\langle\!\langle[x_1,y_1][x_2,y_2]\rangle\!\rangle.$$

Of course we had obtained the same result much earlier. The advantage of CW-complexes is that Proposition 27.10 absorbs many of the technical arguments we had to deal with in the proof of Proposition 14.3, i.e. in our initial calculation of $\pi_1(\Sigma, P)$.

Proof $(*)$. Let X be a topological space, let $n \geq 2$, let $\varphi\colon S^{n-1} \to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. The key idea of the proof is to apply the Seifert–van Kampen Theorem 14.1 to the following two open[543] subsets of of $X \cup_\varphi \overline{B}^n$

$$U \;=\; X \cup_\varphi \big\{z \in \overline{B}^n \,\big|\, \|z\| > \tfrac{1}{4}\big\} \quad \text{and} \quad V \;=\; \big\{z \in \overline{B}^n \,\big|\, \|z\| < \tfrac{3}{4}\big\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path in U from x_0 to x_1. We refer to Figure 438 for an illustration. Note that V is a ball, hence $\pi_1(V, x_1)$ is the trivial group. We consider

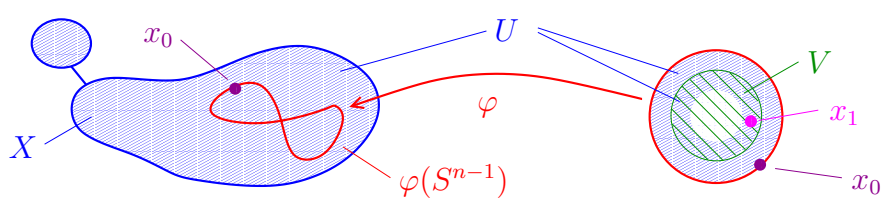


FIGURE 438.

[543]Why are they open subsets?

the following diagram of maps

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r] \arrow[d, "\cong"'] & \pi_1(X\cup_\varphi \overline{B}^n, x_0) \arrow[d, "="'] \\
\pi_1(U,x_0) \arrow[r] \arrow[d, "\cong"'] & \pi_1(X\cup_\varphi \overline{B}^n, x_0) \arrow[d, "\cong"] \\
\pi_1(U,x_1) \arrow[r] \arrow[d] & \pi_1(X\cup_\varphi \overline{B}^n, x_1) \arrow[d, "="] \\
\pi_1(U,x_1)/\langle\langle \pi_1(U\cap V,x_1)\rangle\rangle \arrow[r] \arrow[d, "\cong"', "\text{Lemma 13.12}"] & \pi_1(X\cup_\varphi \overline{B}^n, x_1) \arrow[d, "="] \\
\pi_1(U,x_1) *_{\pi_1(U\cap V,x_1)} \underbrace{\pi_1(V,x_1)}_{=\{e\}} \arrow[r, "\cong", "\text{Seifert–van Kampen}"'] & \pi_1(X\cup_\varphi \overline{B}^n, x_1).
\end{tikzcd}$$

We want to prove the following three statements:

(i) the diagram commutes,
(ii) all vertical maps except possibly the third vertical map on the left are isomorphisms,
(iii) the bottom horizontal map is an isomorphism.

The two vertical maps on the second level are the isomorphisms induced by the choice of base point coming from the chosen path from x_0 to x_1. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily that the diagram commutes.

Most of the maps are obviously isomorphisms. The top left vertical map is an isomorphism by Lemma 9.10 and Proposition 9.12 (1) since X is a deformation retract of U.[544] The bottom vertical map on the left is an isomorphism by Lemma 13.12.

Finally the bottom horizontal map is an isomorphism by the Seifert–van Kampen Theorem 14.1.

Now we consider the case $n \geq 3$. In this case $\pi_1(U\cap V, x_1)$ is trivial since $U\cap V$ is homotopy equivalent to S^{n-1} which has trivial fundamental group by Proposition 5.14. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are now isomorphisms and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U\cap V$ is homotopy equivalent to S^1 which implies that $\pi_1(U\cap V, x_1) \cong \mathbb{Z}$. Under the isomorphism $\Phi\colon \pi_1(X,x_0) \to \pi_1(U,x_1)$ this infinite cyclic subgroup corresponds to $\langle[\varphi]\rangle$. It is straightforward to see that we get a

[544] Why is it a deformation retract?

commutative diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_1\big(X\cup_\varphi \overline{B}^n, x_0\big) \arrow[dd] \\
& \pi_1(X,x_0)/\langle\langle[\varphi]\rangle\rangle \arrow[ur] \arrow[dl, "\cong"'] & \\
\pi_1(U,x_1)/\langle\langle\pi_1(U\cap V,x_1)\rangle\rangle \arrow[rr, "\cong"'] & & \pi_1\big(X\cup_\varphi \overline{B}^n, x_1\big).
\end{tikzcd}$$

This concludes the proof of the proposition. □

The following corollary says that every finitely presented group is the fundamental group of a finite CW-complex.

Corollary 27.11. *Given any finitely presented group π there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y)\cong\pi$.*

Proof. Let $\pi=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_m\rangle$ be a finitely presented group. We denote by X the wedge of k circles which we view as a CW-complex with one 0-cell x_0 and k 1-cells. Following the discussion on page 390 we can make the identification $\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$. Now given $i=1,\dots,m$ we pick a loop $\varphi_i\colon S^1\to X$ in the base point x_0 such that $[\varphi_i]$ represents $r_i\in\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$. We write $Y_0=X$ and for $i=1,\dots,m$ we iteratively define Y_i to be the 2-dimensional CW-complex that is given by attaching a 2-cell to Y_{i-1} with the attaching map φ_i. It follows immediately from Proposition 27.10, applied m times, that

$$\begin{aligned}\pi_1(Y_m,x_0) &\cong \langle g_1,\dots,g_k\rangle/\langle\langle[\varphi_1]\rangle\rangle\dots/\langle\langle[\varphi_k]\rangle\rangle\\ &= \langle g_1,\dots,g_k\rangle/\langle\langle r_1,\dots,r_m\rangle\rangle \;=\; \langle g_1,\dots,g_k\,|\,r_1,\dots,r_m\rangle.\end{aligned}$$ □

The following proposition says in particular that the fundamental group of a CW-complex is the same as the fundamental group of its 2-skeleton.

Proposition 27.12. *Let X be a connected CW-complex with countably many cells and let x_0 be a point in the 0-skeleton of X.*

1. *The inclusion induced map*[545] $\pi_1(X^1,x_0)\to\pi_1(X,x_0)$ *is an epimorphism.*
2. *For any $k\geq 2$ the inclusion induced map $\pi_1(X^k,x_0)\to\pi_1(X,x_0)$ is an isomorphism.*
3. *If X has only finitely many cells in dimensions 0 and 1, then $\pi_1(X,x_0)$ is finitely generated.*
4. *If X has only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X,x_0)$ is finitely presented.*

Example. Let $X=\mathbb{R}\mathrm{P}^\infty$ be the infinite real projective space with the CW-structure from page 655. For this CW-complex the 1-skeleton is $X^1=\mathbb{R}\mathrm{P}^1\cong S^1$ and the 2-skeleton is $X^2=\mathbb{R}\mathrm{P}^2$. It follows from Proposition 27.12 (1) that $\mathbb{Z}\cong\pi_1(S^1)\to\pi_1(\mathbb{R}\mathrm{P}^\infty)$ is an epimorphism and it follows from Proposition 27.12 (2) that $\mathbb{Z}_2\cong\pi_1(\mathbb{R}\mathrm{P}^2)\to\pi_1(\mathbb{R}\mathrm{P}^\infty)$ is

[545]Recall that for $k\in\mathbb{N}$ we denote by X^k the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.

an isomorphism. This calculation of $\pi_1(\mathbb{R}\mathrm{P}^\infty)$ is certainly easier than the argument from page 667.

Proof. We first suppose that X is a finite CW-complex. Then the first two statements follow immediately from iteratively applying Proposition 27.10 finitely many times. If X is an infinite CW-complex, but with countably many cells, then one can reduce the first two statements to the finite case using Propositions 26.17 and 26.19. We leave the details to the reader.

Now we turn to the proof of (3). By (1) it suffices to show that the fundamental group of a finite connected 1-dimensional CW-complex is finitely generated, but we showed this in Proposition 27.9.

Finally we turn to the proof of (4). By (2) it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. Let $x_0 \in X$. We denote by $\varphi_1, \dots, \varphi_k$ the attaching maps of the 2-cells. For $i = 1, \dots, k$ we pick a path p_i from x_0 to a point on $\varphi_i(S^1)$ and we write $\psi_i = p_i * \varphi_i * \overline{p}_i$. We then have

$$\begin{array}{ccccl} \pi_1(X) & \cong & \pi_1(X^1)/\langle\langle[\psi_1]\rangle\rangle \dots /\langle\langle[\psi_k]\rangle\rangle & = & \pi_1(X^1)/\langle\langle[\psi_1], \dots, [\psi_k]\rangle\rangle. \\ & \uparrow & & = & \{\text{finitely generated free group}\}/\langle\langle[\psi_1], \dots, [\psi_k]\rangle\rangle. \\ & \text{Propositions 5.11 and 27.10} & & \uparrow & \\ & & & \text{Statement (3)} & \end{array}$$

We have thus shown that $\pi_1(X)$ is a finitely presented group. □

The following proposition is an immediate consequence of Proposition 27.12 and Theorem 26.3.

Proposition 27.13. *The fundamental group of any connected, compact manifold M is finitely presented.*

The following corollary is an immediate but useful consequence of Proposition 27.12 (1).

Corollary 27.14. *Let X be a connected CW-complex with countably many cells. If X has no 1-cells, then X is simply connected, i.e. $\pi_1(X) = 0$.*

Examples.

(1) On page 648 we pointed out that for $n \geq 2$ the sphere S^n admits a CW-structure with one 0-cell and one n-cell. From Corollary 27.14 we now recover the result from Proposition 5.14 that S^n is simply connected.
(2) It follows from Lemma 26.2 and Corollary 27.14 that for any n we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. Similarly it follows, using Lemma 26.7, that $\pi_1(\mathbb{C}\mathrm{P}^\infty) = 0$.

27.4. **The Cellular Approximation Theorem.** When we study maps between CW-complexes it is reasonable to consider maps that respect the CW-structures. This leads us to the following definition.

Definition. A map $g\colon X \to Y$ between CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0,1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0,1] \times [0,1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t) \end{array},$$

which is illustrated in Figure 439 on the left, is not a cellular map.

Remark. It is clear that the composition of two cellular maps is again cellular. Thus we can form the *category of CW-complexes* where the objects are CW-complexes and where the morphisms are cellular maps.

The following theorem is the main result of this section.

Theorem 27.15. (Cellular Approximation Theorem) *Let $g\colon X \to Y$ be a map between CW-complexes and let A be a subcomplex of X such that the restriction of g to A is a cellular map.*[546] *The map g is homotopic rel A to a cellular map $\widetilde{g}\colon X \to Y$.*

Remark. Let $g\colon X \to Y$ be a *homeomorphism* between CW-complexes. By the Cellular Approximation Theorem 27.15 we know that g is homotopic to a cellular map h. But in general we cannot arrange that h is again a *homeomorphism*. For example consider the torus $T = ([0,1] \times [0,1])/\sim = \mathbb{R}^2/\mathbb{R}^2$ with the standard CW-structure from page 646 which has precisely one 0-cell $*$ that corresponds to the origin. As in Theorem 7.17 we make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^2, *)$ with $\mathbb{Z}^2$. Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ v & \mapsto & P \cdot v. \end{array}$$

In Lemma 3.28 we saw that this map is a homeomorphism. Furthermore, in Exercise 7.12 we saw that the induced map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ is given by multiplication by P. On the other hand the map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ induced by any *cellular homeomorphism* $\widetilde{g}$ on $\mathbb{R}^2/\mathbb{Z}^2$ is represented by a matrix of the form[547] $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. It follows from Exercise 9.20 (a) that g cannot be homotopic, rel $*$, to any such $\widetilde{g}$.

[546] For example $f\colon X \to Y$ could be any map between CW-complexes, then the hypothesis is satisfied for A the empty set.

[547] Why is that?

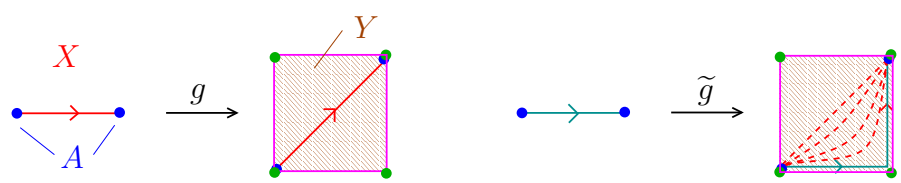
Example. We consider again the above example of a non-cellular map. More precisely, we consider again the above diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1]\times[0,1] \\ t & \mapsto & (t,t). \end{array}$$

We want to deform g into a cellular map, while keeping the endpoints fixed. Put differently, we want to find a homotopy rel $A = \{(0,0),(1,1)\}$ from g to a cellular map $\widetilde{g}$. Such a map $\widetilde{g}$ is given by

$$\begin{array}{rcl} \widetilde{g}\colon X = [0,1] & \to & Y = [0,1]\times[0,1] \\ t & \mapsto & \begin{cases} (2t,0), & \text{if } t\in[0,\frac{1}{2}] \\ (1,2t-1), & \text{if } t\in[\frac{1}{2},1] \end{cases} \end{array}$$

One way of stating this example is that we turn g into a cellular map $\widetilde{g}$ by "pushing g off



g is not a cellular map $\widetilde{g}$ is a cellular map

FIGURE 439.

the high-dimensional cell". The proof of Cellular Approximation Theorem 27.15 is built on the same concept. But somewhat similar to the proof of Proposition 5.14, we have to worry about space-filling curves that exist by Proposition 1.91.

Before we attack the proof of the Cellular Approximation Theorem 27.15 we want to formulate a relative version thereof. To formulate this relative version we need the following semi-obvious definition.

Definition. A *pair of CW-complexes* is a pair (X,A) where X is a CW-complex and A is a subcomplex of X.

Now we can formulate the promised theorem.

Theorem 27.16. (Cellular Approximation Theorem for Pairs) *Suppose we are given a map $g\colon (X,A)\to(Y,B)$ between pairs of CW-complexes. Let Z be a subcomplex of A such that the restriction of g to Z is a cellular map. Then g is homotopic rel Z to a cellular map $\widetilde{g}\colon (X,A)\to(Y,B)$.*[548]

Proof assuming the Cellular Approximation Theorem 27.15. The desired homotopy is constructed as follows:

[548]Here we refer to page 642 for the discussion on the notion of homotopies of pairs of topological spaces rel a subset.

(1) By the Cellular Approximation Theorem 27.15 applied to $g|_A\colon A \to B$ there exists a homotopy F rel Z from $g|_A\colon A \to B$ to a cellular map.
(2) By the Homotopy Extension Theorem 27.4 we can extend the homotopy F from (1) to a homotopy $G\colon X \times [0,1] \to Y$. Note that G is still a homotopy rel Z. We write $h = g_1\colon X \to Y$. Note that $h|_A$ is cellular and that $h(A) \subset B$.
(3) We apply the Cellular Approximation Theorem 27.15 to h and we obtain a homotopy H rel A from h to a cellular map.
(4) The combination of the homotopies G and H has all the desired properties. □

Now we turn to the proof of Cellular Approximation Theorem 27.15. The following proposition is the key step towards the desired proof.

Proposition 27.17. *Let V be a CW-complex and W be a topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$ be another map. Finally let*

$$h\colon V \cup_\psi \overline{B}^n \;\to\; W \cup_\varphi \overline{B}^k$$

be a map with $h(\psi(S^{n-1})) \subset W$. If $n < k$, then h is homotopic rel V to a map h' such that $h'(B^n)$ does not intersect B^k.

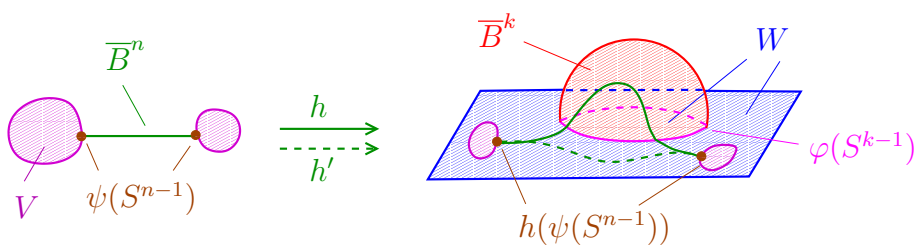


FIGURE 440. Illustration of Proposition 27.17.

The statement of Proposition 27.17 is fairly obvious, if there exists a point $P \in B^k$ that does not get hit by $h(B^n)$. Indeed, if such a point P exists, then we can use the deformation retraction $\overline{B}^n \setminus \{p\} \to S^{n-1}$ to push h on B^n off the ball B^k. Put in general such a point P does not exist, since even for $n < k$ there exist continuous surjective maps $B^n \to B^k$. To get around this problem requires a fairly hard, not particularly inspiring, argument which we will give later on in Section 27.5.

In the following we will use Proposition 27.17 to prove the Cellular Approximation Theorem 27.15.

Corollary 27.18. *Let Z be a finite n-dimensional CW-complex.*

(1) *Let $g\colon Z \to Y$ be a map to a k-dimensional CW-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. If $n < k$, then g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^{k-1}$.*

(2) *Let $g\colon Z \to Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. The map g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^n$.*

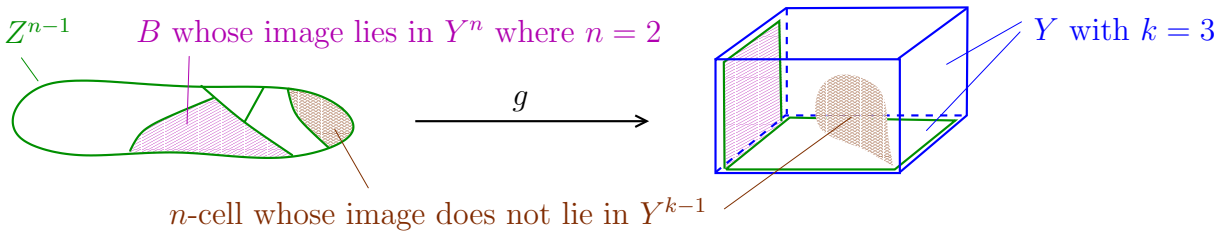


FIGURE 441. Illustration for Corollary 27.18 (1).

Proof. Let Z be a finite n-dimensional CW-complex. By Proposition 26.11 (3) the CW-complex Z is compact. It follows from Proposition 26.14 that the image of the compact set Z in a CW-complex is contained in a finite subcomplex. Therefore we can assume in both (1) and (2) that Y is in fact a finite CW-complex.

(1) Let $g\colon Z \to Y$ be a map to a finite k-dimensional CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. We suppose that $n < k$. We denote by $\widehat{Z}$ the union of Z^{n-1} with all the n-cells of Z whose image is contained in Y^{k-1}. In other words we can write $Z = \widehat{Z} \cup e_1 \cup \cdots \cup e_r$ where $e_1, \ldots, e_r$ are the n-cells such that the image is not contained in Y^{k-1}. We denote by $f_1, \ldots, f_s$ the k-cells of Y. Note that by hypothesis on B the interiors of the cells $e_1, \ldots, e_r$ are disjoint from B.

In the following we apply Proposition 27.17 precisely $r \cdot s$ times to get the desired homotopy. More precisely, we do the following:

(a) We apply Proposition 27.17 to the map

$$h\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-1}) \cup e_r \quad \to \quad (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s$$

to obtain a homotopy rel $\widehat{Z} \supset Z^{n-1} \cup B$ to a map h_{rs} such that the image of the interior of e_r under the map h_{rs} does not intersect the interior of f_s. Next we apply Proposition 27.17 to the map

$$h_{rs}\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-2} \cup e_r) \cup e_{r-1} \quad \to \quad (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s.$$

Iterating this process we obtain a homotopy rel $\widehat{Z}$ to a map such that the image of the interiors of the cells $e_1, \ldots, e_r$ does not intersect the interior of f_s.

(b) Again iterating the process above we obtain a homotopy rel $\widehat{Z}$ to a map such that image of the interiors of the cells $e_1, \ldots, e_r$ does not intersect the interior of any of the f_i. But that means that the image of Z lies in Y^{k-1}.

(2) Let $g\colon Z \to Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subset Y^n$. As we pointed out in the beginning of the proof, we can assume that Y is finite-dimensional, we denote the dimension by k. But then statement (2) is just an immediate consequence of applying (1) iteratively $k-n$ times. □

Now we turn to the proof of the Cellular Approximation Theorem 27.15.

Proof. Let X be a CW-complex and let A be a (possibly empty) subcomplex of X. Furthermore let Y be a CW-complex. We prove the Cellular Approximation Theorem 27.15 only in the special case that X has finitely many cells. The general case is dealt with in [Hat, Theorem 4.8].

Let $g\colon X \to Y$ be map.

(1) We say an n-cell e of X is *g-bad* if $g(e)$ is not contained in Y^n.
(2) If g is non-cellular, then we define

$$n(g) \ := \ \text{minimal dimension of a cell that is } g\text{-bad.}$$

If g is cellular, then we define $n(g) := \infty$.

Claim. Let $g\colon X \to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. Then there exists a homotopy from g to a map g' rel A such that $n(g') > n(g)$.

We write $n = n(g)$. We apply Corollary 27.18 (2) to $g\colon Z := X^n \to Y$ and $B = A^n$.[549] We obtain a homotopy G rel $X^{n-1} \cup A^n$ from $g\colon Z = X^n \to Y$ to a map $g'\colon Z = X^n \to Y$ with $g'(X^n) \subset Y^n$. We can extend this homotopy to a homotopy G on $(X^n \cup A) \times [0,1]$ via the trivial homotopy on $A \times [0,1]$. Now we apply the Homotopy Extension Theorem 27.4 to obtain a homotopy rel $X^n \cup A$ from $g\colon X \to Y$ to a map $g'\colon X \to Y$ such that $g'(X^n) \subset Y^n$. Note that by the choice of our homotopy we have $g'|_{X^{n-1}} = g|_{X^{n-1}}$. By definition of $n(g)$ the map $g'|_{X^{n-1}} = g|_{X^{n-1}}$ is cellular. Thus we see that $n(g') \geq n+1 = n(g)+1$. This concludes the proof of the claim.

Finally let $g\colon X \to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. We apply the claim iteratively as many times as possible to obtain a sequence of maps $g = g_1, g_2, g_3, \ldots : X \to Y$, each of which is homotopic to g rel A, such that $n(g) = n(g_1) < n(g_2) < \ldots$. Note that n can take on only finitely many values, namely $\{0, \ldots, \dim(X), \infty\}$. This means that the sequence is finite which implies that the last map is cellular. □

27.5. **Proof of Proposition 27.17** $(*)$. [550] Before we can even think of proving Proposition 27.17 we need to introduce a few more definitions and we need to prove a rather technical lemma.

Definition.

[549] Note that by definition of n we have $g(X^{n-1}) \subset Y^{n-1}$.

[550] Here the $(*)$ indicates once again that we do not provide the proof in the lecture. You are reading this section at your own risk.

(1) A *closed half-space in* $\mathbb{R}^n$ is a set of the form $\{v \in \mathbb{R}^n \mid \langle v, w\rangle \geq 0\}$, for some fixed non-zero vector $w \in \mathbb{R}^n$.
(2) A *convex polyhedron in* $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ that is the intersection of finitely many closed half-spaces.
(3) A *polyhedron in* $\mathbb{R}^n$ is the union of a finite number of convex polyhedra.[551]
(4) Given a polyhedron P a map $f\colon P \to \mathbb{R}^k$ is called *piecewise linear*if there exists a subdivision of P into finitely many convex polyhedra $Q_1, \ldots, Q_m$ such that the restriction of f to each Q_i is affine linear.[552]

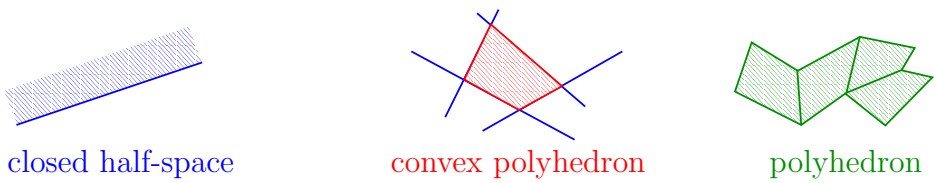


FIGURE 442.

The following, very technical lemma, will get us around the problem of space filling curves, i.e. it will get us around the problem that we saw in Proposition 1.91 that there exist surjective continuous maps $\overline{B}^n \to \overline{B}^k$ even if $n < k$.

Lemma 27.19. *Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be map. Furthermore let $g\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ be a map. Then there exists a (possibly empty) polyhedron $K \subset [0,1]^n$ and a map $\widetilde{g}\colon [0,1]^n \to Z$ with the following properties:*
(1) *The map $\widetilde{g}$ is homotopic to g rel $g^{-1}(W)$.*
(2) *We have $\widetilde{g}(K) \subset B^k$.*
(3) *The restriction $\widetilde{g}|_K\colon K \to B^k$ is piecewise linear.*
(4) *There exists a non-empty open subset $U \subset B^k$ such that $\widetilde{g}^{-1}(U) \subset K$.*

Example. Suppose that in the setting of Lemma 27.19 there exists an open subset $U \subset B^k$ such that $g([0,1]^n) \cap U = \varnothing$. Then the empty polyhedron $K = \varnothing$ and the map $\widetilde{g} = g$ already have the desired property.

Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be map. Furthermore let $g\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ be a map. We write $D_1 := \overline{B}_{\frac{1}{4}}(0) \subset B^k$ and $D_2 := \overline{B}_{\frac{1}{2}}(0) \subset B^k$. The set $g^{-1}(D_2) \subset [0,1]^n$ is closed and therefore compact by the Heine-Borel Theorem 1.21. We obtain from Proposition 1.107 that g is uniformly continuous on $g^{-1}(D_2)$.[553] This implies that we can find an $\epsilon > 0$ with the following two properties:

[551]Note that the union of zero convex polyhedra is the empty set.
[552]A map $f\colon V \to \mathbb{R}^k$ from a subset V of $\mathbb{R}^n$ is called *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$-matrix A such that $f(v) = P + Av$ for all $v \in V$.
[553]Why can't we conclude that g is uniformly continuous on all of $[0,1]^n$?

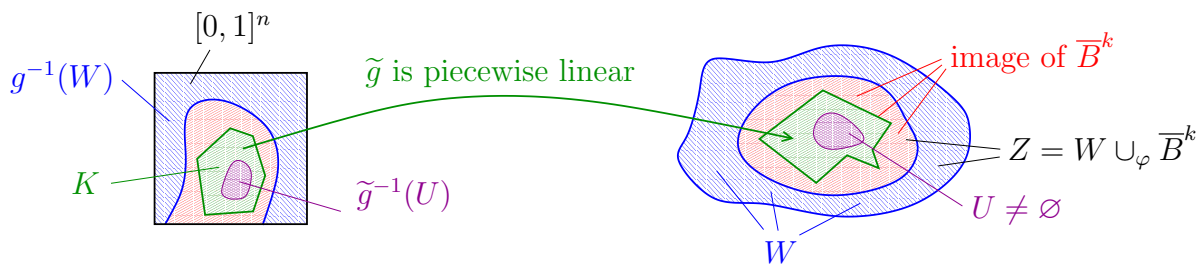


FIGURE 443. Illustration of Lemma 27.19.

(i) For any $x, y \in g^{-1}(D_2)$ with $\|x - y\| < \epsilon$ we have $\|g(x) - g(y)\| < \frac{1}{8}$.
(ii) We have[554]

$$\epsilon \ < \ \text{distance between } g^{-1}(D_1) \text{ and } [0,1]^n \setminus g^{-1}\big(\mathring{D}_2\big).$$

We divide $[0,1]^n$ into cubes such that the diameter of each cube is less than ϵ. We denote by K_1 the union of all the cubes that meet $g^{-1}(D_1)$ and we denote by K_2 the union of all the cubes meeting K_1. By the second property of ϵ we have $K_2 \subset g^{-1}(D_2)$. We refer to Figure 444 for an illustration.

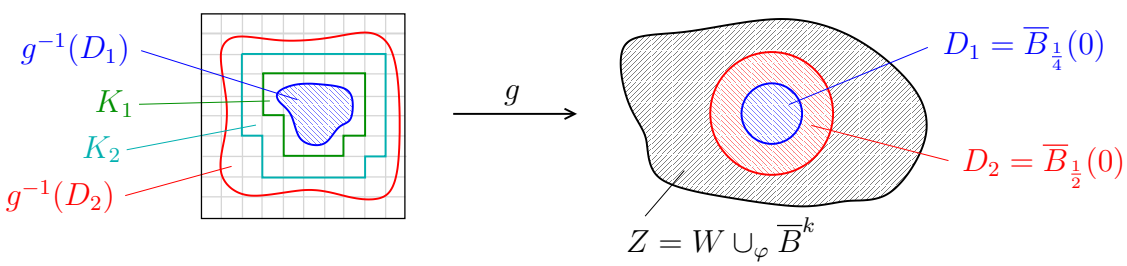


FIGURE 444.

We need one more definition: Suppose we are given points $P_1, \dots, P_{n+1} \in \mathbb{R}^n$ such that the vectors $P_2 - P_1, \dots, P_{n+1} - P_1$ are linearly independent. We refer to

$$\text{convex hull of } P_1, \dots, P_{n+1} \ = \ \left\{ \sum_{i=1}^{n+1} t_i P_i \,\middle|\, \sum_{i=1}^{n+1} t_i = 1 \right\}$$

[554]Here recall that, according to the definition on page 89, for two non-empty subsets A and B of $\mathbb{R}^n$ we refer to

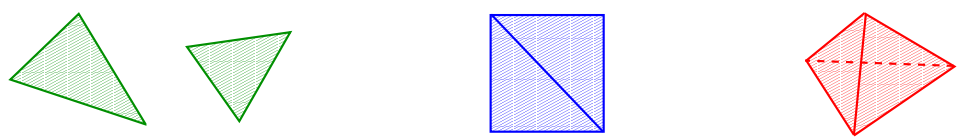
$$d(A,B) \ := \ \inf\{\|P - Q\| \mid P \in A \text{ and } Q \in B\}$$

as the distance between A and B. If A and B are compact and disjoint, then it follows from Corollary 1.56 that $d(A,B) > 0$.

as a *simplex* in $\mathbb{R}^n$.[555] We will make use of the following trivial observation: if we are given $w_1, \ldots, w_{n+1} \in \mathbb{R}^k$, then there exists a unique affine linear map Φ with $\Phi(P_i) = w_i$, namely the map that is given by

$$\Phi\Big(\sum_{i=1}^{n+1} t_i P_i\Big) \ := \ \sum_{i=1}^{n+1} t_i w_i.$$

We leave it as an exercise to show that each cube can be subdivided[556] into simplices. We also refer to Figure 445 for an illustration.



simplices in $\mathbb{R}^2$ | a square can be decomposed into two simplices | simplex in $\mathbb{R}^3$

FIGURE 445.

Now we subdivide all the cubes in K_2 into simplices. By the above discussion there exists a map $h\colon K_2 \to \mathbb{R}^k$ that equals g on all the vertices of the simplices in the subdivision of the cubes.[557] Furthermore we choose a map $\psi\colon K_2 \to [0,1]$ with $\psi|_{\partial K_2} \equiv 0$ and $\varphi|_{K_1} \equiv 1$.[558] Now we consider the homotopy[559]

$$\begin{array}{rcl} F\colon [0,1]^n \times [0,1] & \to & Z = W \cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} (1 - t\cdot\psi(P))\cdot g(P) + t\cdot\psi(P)\cdot h(P), & \text{if } P \in K_2 \\ g(P), & \text{otherwise.} \end{cases} \end{array}$$

This is a homotopy from g to a map $\widetilde{g}\colon [0,1] \to Z = W \cup_\varphi \overline{B}^k$ that restricts to the piecewise linear map h on K_1.

Claim. The point $0 \in D_1$ does not lie in $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$.

[555]Given a subset S of $\mathbb{R}^k$ we defined on page 78 the *convex hull* of S as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. As we pointed out, if S is a finite subset of $\mathbb{R}^k$, then the *convex hull* of S is given by

$$\Big\{ \sum_{s\in S} a_s \cdot s \,\Big|\, a_s \in [0,1] \text{ and } \sum_{s\in S} a_s = 1 \Big\}.$$

[556]Here, by "subdivide" we mean that given any cube K there exist simplices $S_1, \ldots, S_k$ such that the interiors of the S_i are disjoint and such that $K = S_1 \cup \cdots \cup S_k$.

[557]Note that it follows from the uniqueness statement above that the map g is in fact continuous.

[558]Such a map can be constructed as follows: we assign the value 1 to all vertices in K_1 and we assign the value 0 to all other vertices in K_2. By the above "trivial observation" this defines a map on K_2 with the desired property.

[559]Since $\varphi|_{\partial K_2} \equiv 0$ we see that F is indeed continuous.

Now we provide the proof of the claim.

(1) On $[0,1]^n \setminus K_2$ we have $\widetilde{g} = g$. By Property (ii) of ϵ we have $g^{-1}(D_1) \subset K_2$. It follows that the image of $[0,1]^n \setminus K_2$ lies outside of D_1.

(2) Now let σ be a simplex of K_2 that does not lie in K_1. Note that $K_2 \subset g^{-1}(D_2)$. It follows from Property (i) of ϵ that $g(\sigma)$ is contained in some ball B_σ of radius $\frac{1}{8}$. Since $g(\sigma) \subset B_\sigma$ and since B_σ is convex we also have $h(\sigma) \subset B_\sigma$. Again by the convexity of B_σ we have $\widetilde{g}(\sigma) \subset B_\sigma$. We know that B_σ is not contained in D_1 since σ contains points outside K_1, hence outside $g^{-1}(D_1)$. The radius of B_σ is half that of D_1, so it follows that 0 is not in B_σ, and hence 0 does not lie in $\widetilde{g}(\sigma)$.

This concludes the proof of the claim.

Since $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$ is compact it follows from the claim that there exists in fact an open neighborhood U of 0 that does not intersect the image of $[0,1]^n \setminus K_1$. Put differently, we have $\widetilde{g}^{-1}(U) \subset K_1$. It is now straightforward to verify that $K = K_1$, $\widetilde{g}$ and U have the desired properties. □

Lemma 27.20. *Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k.*

Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k.

By Proposition 1.47 there exists a homeomorphism $\Phi\colon [0,1]^n \to \overline{B}^n$. So after precomposing h with Φ we might as well assume that the domain of h is $[0,1]^n$, i.e. that h is a map $h\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ with $h(\partial[0,1]^n) \subset W$.

We pick a polyhedron K, a map $\widetilde{h}\colon [0,1]^n \to Z$ and a non-empty open subset $U \subset B^n$ with $\widetilde{h}^{-1}(U) \subset K$ as in the statement of Lemma 27.19. Since $n < k$ and since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear we deduce that $\widetilde{h}(K)$ cannot contain the open subset U of B^n.[560] We pick a point $P \in U \setminus \widetilde{h}(K)$. By statement (4) of Lemma 27.19 we know that P does not lie in the image of $\widetilde{h}\colon [0,1]^n \to Z$.

As we pointed out on page 309, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1}. We can extend this deformation retraction via the identity map to a deformation retraction from $Z = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W. If we compose $\widetilde{h}$ with this deformation retraction we obtain a homotopy, rel S^{n-1}, from $\widetilde{h}$ to a map h' that misses B^k. □

Now we can finally provide the proof of Proposition 27.17.

Proof of Proposition 27.17. Let V be a CW-complex and W be a topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$

[560] Indeed, since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear and since $n < k$ we know that the image $\widetilde{h}(K)$ is set of Lebesgue-measure zero in B^n whereas a non-empty open subset has positive Lebesgue-measure.

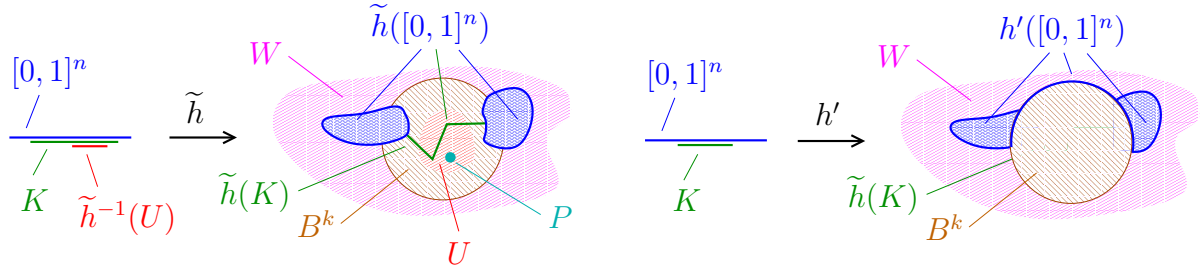


FIGURE 446. Illustration for the proof of Proposition 27.17.

be another map. Finally let

$$h\colon V \cup_\psi \overline{B}^n \;\to\; W \cup_\varphi \overline{B}^k$$

be a map with $h(\psi(S^{n-1})) \subset W$. We suppose that $n < k$.

We denote by θ the obvious map $\overline{B}^n \to V \cup_\psi \overline{B}^n$, furthermore we write $h = g \circ \theta$. By our hypothesis we have $h(S^{n-1}) \subset W$. By Lemma 27.20 there exists a homotopy G rel S^{n-1} from h to a map $h'\colon \overline{B}^n \to W$ that misses B^k. The desired homotopy is now given by[561][562]

$$\begin{array}{rcl} F\colon (V \cup_\psi \overline{B}^n) \times [0,1] & \to & W \cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} G(Q,t), & \text{if } P = \theta(Q), \text{ for some } Q \in \overline{B}^k, \\ g(P), & \text{otherwise.} \end{cases} \end{array}$$

□

Exercises for Chapter 27.

Exercise 27.1. Let X be a CW-complex and let $A \subset X$ be a subcomplex. Suppose that A admits a deformation retraction to a point. Show that X/A is homotopy equivalent to X.
Hint: Use the Homotopy Extension Theorem 27.4.

Exercise 27.2. We consider the map

$$\begin{array}{rcl} \varphi\colon S^1 & \to & S^1 \\ e^{it} & \mapsto & \begin{cases} e^{3it}, & \text{if } t \in [0, \frac{2\pi}{3}], \\ e^{-3it}, & \text{if } t \in [\frac{2\pi}{3}, \frac{4\pi}{3}], \\ e^{3it}, & \text{if } t \in [\frac{4\pi}{3}, 2\pi]. \end{cases} \end{array}$$

[561] We view $V \cup_\psi \overline{B}^n$ as a CW-complex where V is a subcomplex and the image of $\overline{B}^n$ is an n-cell. Furthermore we equip $V \cup_\psi \overline{B}^n$ with the product CW-structure. It follows from Lemma 27.1 and Lemma 26.9, applied to the subcomplexes given by $V \times [0,1]$ and $\theta(\overline{B}^n) \times [0,1]$, that the map F is continuous.

[562] One might be tempted to use a different approach to show that F is continuous: Namely one might want to write $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ and then argue using Lemma 1.83 (3) that the given map is continuous. But this approach is problematic since it is not clear that the two topologies on $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ agree.

Put differently, φ wraps S^1 first around S^1 clockwise, then counter clockwise and then once again clockwise. We write $Y = S^1$ and we consider

$$X \ := \ Y \cup_\varphi \overline{B^2},$$

where we identify $z \in \partial\overline{B}^2 = S^1$ with $\varphi(z) \in S^1 = Y$.

(a) Show that X is homotopy equivalent to $\overline{B}^2$.

(b) Is X homeomorphic to $\overline{B}^2$?

28. Higher homotopy groups

After this somewhat technical chapter on CW-complexes we now return to our original goal, namely trying to distinguish topological spaces. Using the fundamental group we had established several non-trivial results, for example we proved that

(1) S^1 and S^k with $k \geq 2$ are not homeomorphic,
(2) $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ are not homeomorphic,
(3) surfaces of different genus are not homeomorphic,
(4) we showed that S^1 is not a deformation retract of $\overline{B}^2$,
(5) the trefoil knot is not isotopic to the trivial knot.

On the other hand there are still many even rather elementary questions we cannot answer. For example at the moment we cannot show that for $k \neq l$ with $k, l \geq 2$ the spheres S^k and S^l are not homeomorphic and we cannot show that for $k \neq l$ with $k, l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. Also, the somewhat mysterious lens spaces $L(p, q)$ still evaded our attempts at distinguishing them for a given p.

The problem with fundamental groups is that we only probe a topological space X with loops. The natural idea is now to go from using maps $S^1 \to X$ to studying maps $S^n \to X$ for $n \geq 2$. As we will see, this idea leads us naturally to the definition of the "higher homotopy groups" of a topological space.

28.1. **Definition of the higher homotopy groups.**

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote, as usual, by ∂I^n its boundary as a subset of $\mathbb{R}^n$.

Now let X be a topological space and let $x_0 \in X$. With the definitions from page 641 we can write

$$\pi_1(X, x_0) \;=\; \text{homotopy classes of maps } (I, \partial I) \to (X, x_0).$$

This definition can be easily generalized to higher dimensions:

Definition. Given a pointed topological space (X, x_0) and $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \;=\; \text{homotopy classes of maps } (I^n, \partial I^n) \to (X, x_0).$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the *n-th homotopy group.*

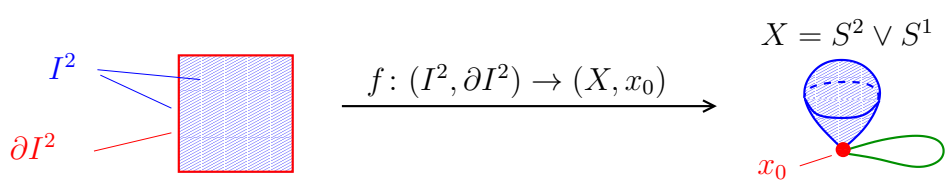


Figure 447. Illustration of an element in $\pi_2(S^2 \vee S^1, x_0)$.

Remark. This definition makes in particular sense for $n = 0$, we can thus consider $\pi_0(X, x_0)$. In Section 1.17 we had already defined $\pi_0(X)$ as the set of path-components of X. In fact it is straightforward to see that the map

$$\begin{array}{rcl} \pi_0(X, x_0) & \to & \pi_0(X) \\ [\gamma\colon I^0 = \{0\} \to X] & \to & \text{path-component of } X \text{ that contains } \gamma(0) \end{array}$$

is a bijection.

Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Here is what we do:

Definition. Let $((X, x_0)$ be a pointed topological space. Let $n \geq 1$ and let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \dots, x_n) & \mapsto & \begin{cases} f(2x_1, \dots, 2x_n), & \text{if } (x_1, \dots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \dots, 2x_n - 1), & \text{if } (x_1, \dots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

This map is illustrated in Figure 448.[563] Applying Lemma 5.3 several times one sees that

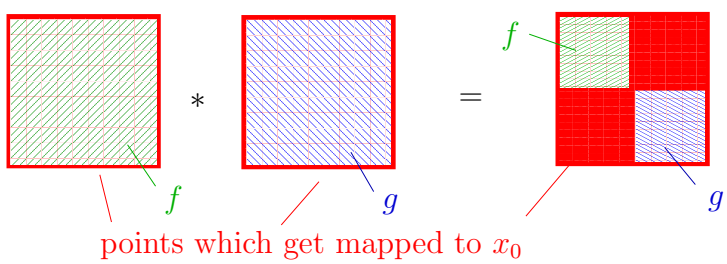


FIGURE 448.

this map is indeed continuous.

It is straightforward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g. This leads us to the following definition:

Definition. Let $((X, x_0)$ be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g]. \end{array}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we had we already introduced in Chapter 5.2.

[563]For mysterious reasons in all figures of this chapter the origin $(0, 0)$ is in the top left corner and the point $(1, 1)$ is the vertex at the bottom right.

Now we can formulate the following proposition:

Proposition 28.1. *Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:*

(1) *The map*

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) *Given a map $f\colon (I^n, \partial I^n) \to (X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map $f \circ \Phi\colon (I^n, \partial I^n) \to (X, x_0)$ where $\Phi\colon I^n \to I^n$ is the homeomorphism given by $\Phi(x_1, x_2, \ldots, x_n) = (1 - x_1, x_2, \ldots, x_n)$.*

As we will see, the proof of Proposition 28.1 is very similar to the proof of Proposition 5.7 and the proof of Proposition 5.6. To deal with the technical differences between the case $n = 1$ and the general case we need a few more definitions and one lemma.

Definition.

(1) A *cuboid* is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.
(2) We say a cuboid $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is *non-degenerate* if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a map.

(3) We say a cuboid $A \subset I^n$ is *boundary-trivial* if f sends all points on the boundary ∂A of A to x_0.
(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A. We define

$$\begin{array}{rcl} f(A, C)\colon I^n & \to & X \\ x = (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\Big(a_1 + (x_1 - c_1)\frac{b_1 - a_1}{d_1 - c_1}, \ldots, a_n + (x_n - c_n)\frac{b_n - a_n}{d_n - c_n}\Big) & \text{if } x \in C. \end{cases} \end{array}$$

By applying Lemma 5.3 several times one can easily verify that this map is continuous.

The definition of $f(A, C)$ is sketched in Figure 449.

Lemma 28.2. *Let (X, x_0) be a pointed topological space and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map. Furthermore let $A \subset I^n$ be a boundary-trivial cuboid and let C be a non-degenerate cuboid that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.*

Proof $(*)$. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A. For $t \in [0, 1]$ we write

$$C_t = \big[a_1(1 - t) + c_1 t, b_1(1 - t) + d_1 t\big] \times \cdots \times \big[a_n(1 - t) + c_n t, b_n(1 - t) + d_n t\big].$$

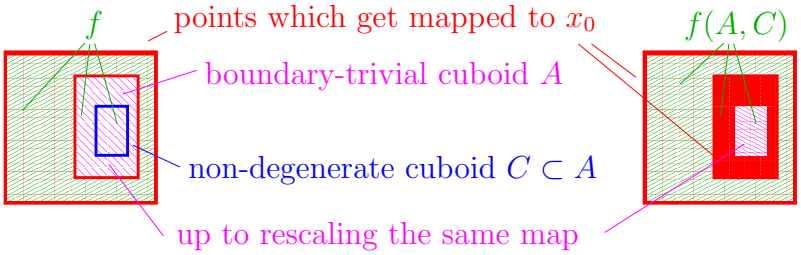


FIGURE 449.

Note that $C_0 = A$ and $C_1 = C$. It is now straightforward to see that the map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x,t) & \mapsto & F(A, C_t)(x) \end{array}$$

is a homotopy between f and $f(A,C)$. □

Now we can provide the proof of Proposition 28.1

Proof of Proposition 28.1. The neutral element of the product structure is given by the equivalence class of the constant map

$$\begin{array}{rcl} c_{x_0} \colon I^n & \to & X \\ x & \mapsto & x_0. \end{array}$$

Indeed, it follows immediately from Lemma 28.2, applied to $A = [0,1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any map $f \colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.

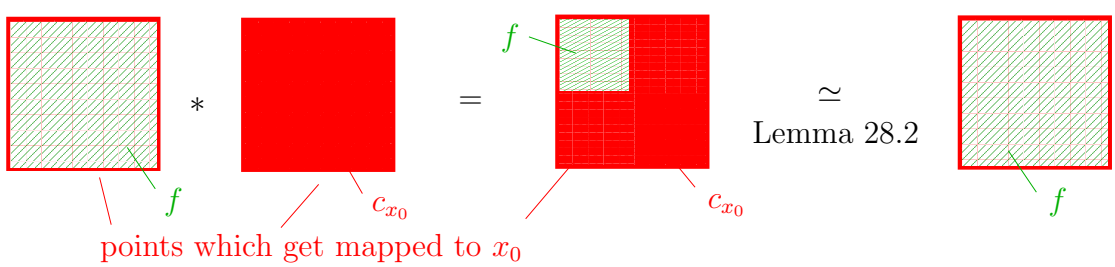


FIGURE 450.

Now consider three maps $f, g, h \colon (I^n, \partial I^n) \to (X, x_0)$. By applying Lemma 28.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. Figure 451. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.

Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a map $f \colon (I^n, \partial I^n) \to (X, x_0)$. We

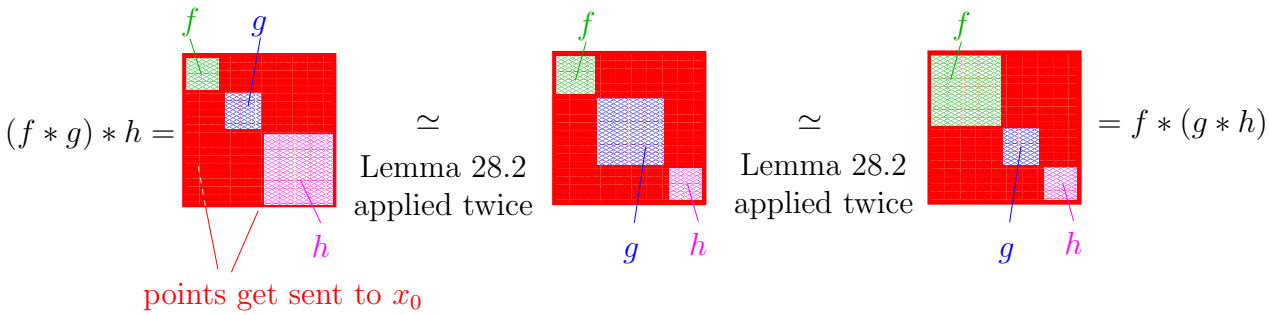


FIGURE 451.

consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1,\dots,x_n) & \mapsto & f(1-x_1,x_2,\dots,x_n). \end{array}$$

We claim that $[f]\cdot[\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from Lemma 28.2 that $f * \overline{f}$ is homotopic to the map

$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1,\dots,x_n) & \mapsto & \begin{cases} f(2x_1,x_2,\dots,x_n), & \text{if } x_1 \in [0,\frac{1}{2}], \\ \underbrace{\overline{f}(2x_1-1,x_2,\dots,x_n)}_{=f(2-2x_1,x_2,\dots,x_n)}, & \text{if } x_1 \in [\frac{1}{2},1]. \end{cases} \end{array}$$

Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[564][565]

$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1,\dots,x_n,t) & \mapsto & \begin{cases} f(2(x_1-\frac{t}{2}),x_2,\dots,x_n), & \text{if } x_1 \in [\frac{t}{2},\frac{1}{2}], \\ f(2-2(x_1+\frac{t}{2}),x_2,\dots,x_n), & \text{if } x_1 \in (\frac{1}{2},1-\frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

is a homotopy from h to c_{x_0}. We refer to Figure 452 for an illustration of the argument. Basically the same argument also shows that $[\overline{f}]\cdot[f] = [\overline{f} * f] = [c_{x_0}]$. We have thus verified both statements of the proposition. □

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}) = 0$ for any $n \geq 1$.

[564]This map is continuous by Lemma 5.3.

[565]Even for $n = 1$ this homotopy is different from the homotopy used in Proposition 5.6 (3). Why is that?

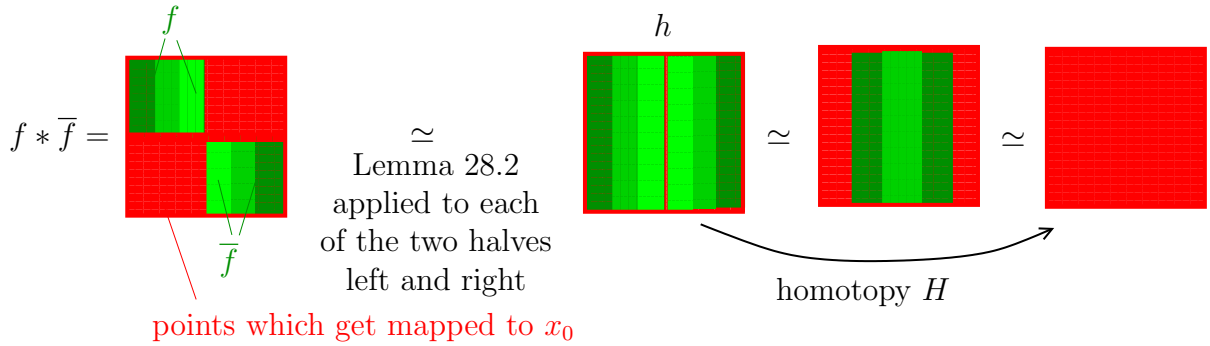


FIGURE 452.

The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler.

Proposition 28.3. *For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.*

Proof. Suppose we are given two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in Figure 453 for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by Lemma 28.2. We leave the details to the reader. □

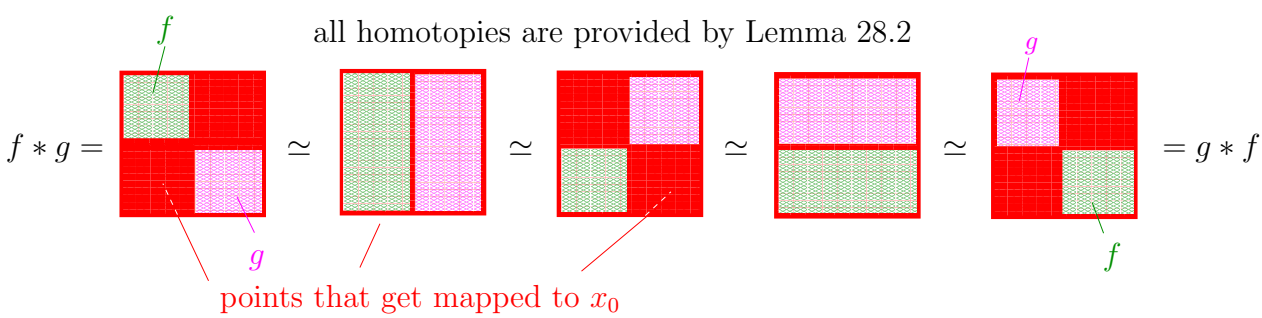


FIGURE 453. Illustration of the proof of Proposition 28.3.

28.2. **Alternative descriptions of homotopy groups.** In this short section we present several alternative points of views regarding elements in homotopy groups. In the following let $n \in \mathbb{N}$.

(1) We denote by $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from page 80.
(2) We write $* = (-1, 0, \dots, 0) \in S^n$. We denote by $\Psi\colon (\overline{B}^n / S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the homeomorphism from page 99.

Now let X be a topological space and let $x_0 \in X$. The above homeomorphisms Φ and Ψ, together with Lemma 1.68, define bijections

$$\begin{array}{ll} & \text{(homotopy classes) of maps } (I^n, \partial^n) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (S^n, *) \to (X, x_0). \end{array}$$

The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \to (X, x_0)$.

Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. Unfortunately with this definition it is a little less clear how the group structure on $\pi_n(X, x_0)$ is defined. To visualize the group structure we need to work a little:

(1) First note that the homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n \to \overline{B}^n/S^{n-1}$.
(2) We consider the homeomorphism $\Theta := \overline{\Phi}^{-1} \circ \Psi^{-1}\colon S^n \to I^n/\partial I^n$.
(3) We write $K := \{(x_1, \ldots, x_n) \in I^n \mid x_1 = \frac{1}{2}\}$.

In Figure 454 we show K and its journey through the maps Φ and Ψ.

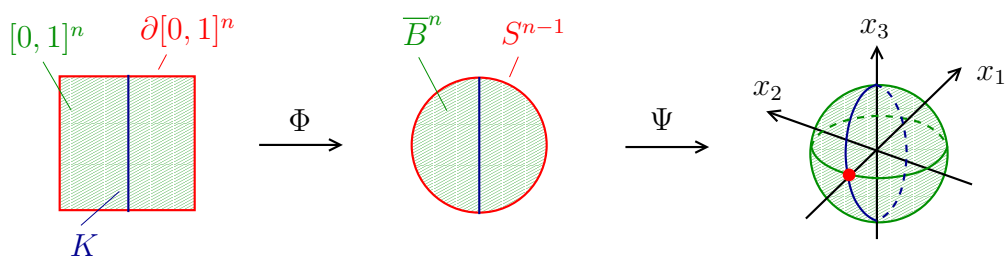


FIGURE 454.

(4) We denote by $S_1^n \vee S_2^n$ the wedge of two copies of S^n along the point $* \in S^n$.
(5) We refer to the map[566]

$$\begin{array}{rcl} S^n \xrightarrow{\Theta} I^n/\partial I^n & \to & I^n/(\partial I^n \cup K) \to S_1^n \vee S_2^n \\ [x_1, \ldots, x_n] & \mapsto & \begin{cases} \Theta^{-1}([2x_1, x_2, \ldots, x_n]) \in S_1^n, & \text{if } x_1 \in [0, \frac{1}{2}), \\ \Theta^{-1}([2x_1 - 1, x_2, \ldots, x_n]), & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

as the *pinching map*.

[566]It follows easily from Lemma 1.32 (2') that this map is continuous.

(1) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given as follows:

$$\begin{array}{rcl} \pi_n(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([f_1\colon S^n \to X],[f_2\colon S^n \to X]) & \mapsto & \big[S^n \ \to \ S^n_1 \vee S^n_2 \ \xrightarrow{f_1 \vee f_2} \ X\big]. \end{array}$$

In Figure 455 we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[567]

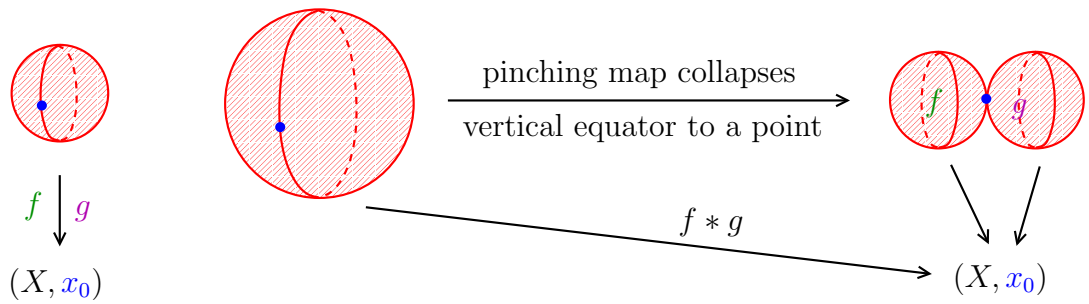


FIGURE 455.

We conclude this section with the following lemma that can be viewed as a generalization of Lemma 5.1.

Lemma 28.4. *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Furthermore let $f\colon (S^n, *) \to (X, x_0)$ be a map. Then the following statements are equivalent:*

(1) *the map $f\colon S^n \to X$ is homotopic to a constant map,*
(2) *there exists a map $F\colon \overline{B}^{n+1} \to X$ so that $F|_{S^n} = f$,*
(3) *f represents the trivial element in $\pi_n(X, x_0)$.*

Proof $(*)$. The equivalence of (2) and (3) is proved in a similar way as Lemma 5.1. We leave it to the reader to make the fairly straightforward modifications.

We turn to the proof that (3) implies (2). This is basically obvious. Indeed, the fact that f represents the trivial element in $\pi_n(X, x_0)$ means that there exists a homotopy of maps of pairs $(S^n, *) \to (X, x_0)$ from f to the constant map. But this means in particular that f is homotopic to the constant map c_{x_0}.

Finally suppose that (1) holds. This means that we assume that there exists a homotopy $H\colon S^n \times [0,1] \to X$ with $H_0 = f$ and $H_1 = c_{x_0}$. This time we do not assume that $H(*, t) = *$ for all $\in [0,1]$. We consider the map

$$\begin{array}{rcl} F\colon \overline{B}^{n+1} & \to & X \\ P & \mapsto & \begin{cases} H(Q, 2-2t), & \text{if } P = Q \cdot t \text{ with } Q \in S^n \text{ and } t \in [1, \frac{1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

[567]The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

It follows from Lemma 1.32 (2) that this map is indeed continuous. By construction we have $F|_{S^n} = f$. Thus we see that (2) is satisfied. This concludes the proof that all three statements are equivalent. $\square$

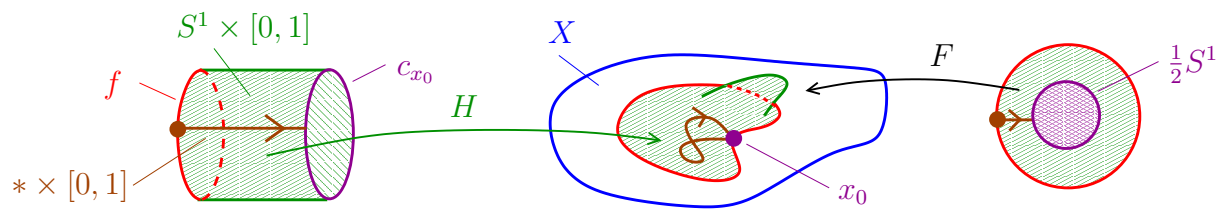


FIGURE 456. Illustration for the proof of Lemma 28.4.

28.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. Perhaps not surprisingly, many of the basic statements about the fundamental group can be easily generalized to the setup of higher homotopy groups.

For example the following proposition can be viewed as a generalization of Proposition 5.11.

Proposition 28.5. *Let X be a topological space. Furthermore let x_0 and x_1 be two points in X which lie in the same component of X. Then the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.*

Remark. If X is a path-connected topological space and if we are only interested in the isomorphism type of the n-th homotopy group of X, then sometimes we use this proposition as an excuse to just write $\pi_n(X)$, i.e. we suppress the base point from the notation.

Proof. Let X be a topological space and let x_0 and x_1 be two points which lie in the same component of X. By definition this means that x_0 and x_1 are connected through a path $\gamma\colon [0,1] \to X$. Let $f\colon (I^n, \partial I^n) \to (X, x_1)$ be a map. We consider the map[568][569]

$$\begin{aligned} f^\gamma\colon (I^n, \partial I^n) &\to (X, x_0) \\ (t_1, \dots, t_n) &\to \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1, \dots, t_n) \in \partial\big(\big[\frac{t}{4}, \frac{4-t}{4}\big]^n\big) \text{ for } t \in [0,1]. \end{cases} \end{aligned}$$

The definition of f^γ is illustrated in Figure 457. It remains to prove the following claim:

Claim.

(1) the map $\Phi\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $[f] \mapsto [f^\gamma]$ is well-defined,
(2) the map Φ is a bijection, and

[568] If we view elements in $\pi_1(X, x_i)$ as represented by maps $(S^1, *) \to (X, x_i)$, what does the corresponding map $\pi_1(X, x_1) \to \pi_1(X, x_0)$ look like?

[569] One can easily verify that the map is continuous.

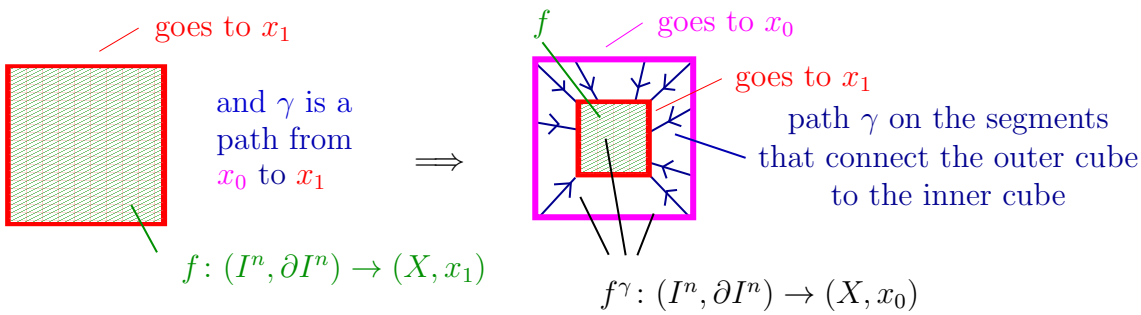


FIGURE 457. Illustration of the proof of Proposition 28.5.

(3) the map Φ is a group homomorphism.

We quickly discuss the three statements of the claim:

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t) = \gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ given by $[g] \mapsto [f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Now let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f * g)^\gamma$. (These two maps are illustrated in Figure 458.) It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. □

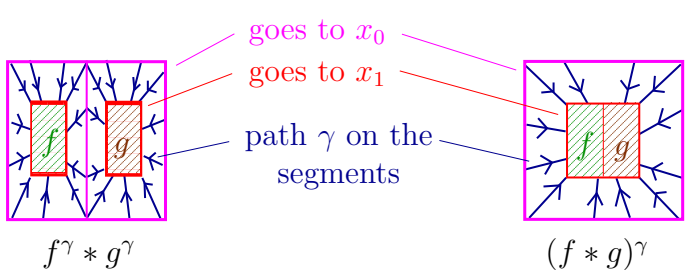


FIGURE 458. Second illustration of the proof of Proposition 28.5.

Definition. Let $f\colon X \to Y$ be a map between two topological spaces, let $n \in \mathbb{N}$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. The map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, x_0)] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, x_0) \xrightarrow{f} (Y, y_0)] \end{array}$$

is well-defined map and it is a group homomorphism for every $n \geq 1$.[570] As usual we say that f_* is the map induced by f.

The following proposition is straightforward generalization of Proposition 6.1.

Proposition 28.6. *Let $n \geq 1$, let $\mathcal{T}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. Then*

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{T}) & \to & \mathrm{Ob}(\mathcal{G}) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{T}) & \to & \mathrm{Mor}(\mathcal{G}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

is a covariant functor.

Proof. It follows immediately from the definitions that

$$\mathrm{id}_* = \mathrm{id} \qquad \text{for all pointed pairs } (X, x_0),$$

and that

$$(g \circ f)_* = g_* \circ f_* \quad \text{for all maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0).$$

But this is exactly what we need to show. □

We continue with the following fairly elementary but useful proposition.

Proposition 28.7. *Let $n \geq 1$.*

(1) *Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then*

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism.

(2) *If X is a contractible topological space, then the following two statements hold:*
   (a) *for any base point $x_0 \in X$ we have $\pi_n(X, x_0) = 0$,*
   (b) *any map $f\colon S^n \to X$ can be extended to a map $\overline{B}^{n+1} \to X$.*

(3) *Let (X, x_0) be a pointed topological space. If we denote by Y the component of X that contains x_0 and if we denote by $i\colon Y \to X$ the inclusion map, then the map*

$$i_*\colon \pi_n(Y, x_0) \to \pi_n(X, x_0)$$

is an isomorphism.

Proof.

(1) This statement is proved the same way as Proposition 9.12.
(2) Let X be a contractible topological space.

[570] Both statements follow basically immediately from the definitions.

(a) By definition of "contractible" the topological space X is homotopy equivalent to a topological space consisting of a single point $\{*\}$. Evidently $\pi_n(\{*\},*)=0$, which by (1) implies that $\pi_n(X,x_0)=0$ for some base point x_0. In Lemma 9.9 we saw that a contractible topological space is path-connected. Thus it follows from Proposition 28.5 that $\pi_n(X,x_0)=0$ for any base point x_0.

(b) This statement is proved basically the same way as Lemma 5.1. For completeness' sake we carry out the argument. So let $f\colon S^n\to X$ be a map. We pick a base point $*$ for S^n and we set $x_0:=f(*)$. By (a) we know that $\pi_n(X,x_0)=0$. We take the point of view from page 715. Thus we can view $[f]$ as an element in $\pi_n(X,x_0)=0$. By definition $[f]=0$ means that there exists a homotopy $F\colon S^n\times[0,1]\to X$ such that $F(p,0)=f(p)$ for all $p\in S^n$, that $F(p,1)=x_0$ for all $p\in S^n$ and that $F(*,t)=x_0$ for all $t\in[0,1]$. Then the desired extension is given by the map

$$\begin{array}{rcl}\overline{B}^{n+1} & \to & X\\ r\cdot v & \mapsto & F(v,(1-r))\end{array}$$

where we write a point in $\overline{B}^{n+1}$ as $r\cdot v$ with $r\in[0,1]$ and $v\in S^n$.

(c) This statement is an easy consequence of Lemma 1.101 and the fact that I^n is path-connected. □

Example. The topological space $\mathbb{R}^n$ is homotopy equivalent to a point P. Thus it follows from Proposition 28.7 and the discussion on page 713 that $\pi_k(\mathbb{R}^n)\cong\pi_k(P)\cong 0$. Exactly the same way, using the discussion on page 308, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^n$ vanish, e.g. we have $\pi_k\big(\overline{B}^n\big)=\pi_k(B^n)=0$.

The following proposition is the high-dimensional analogue of Proposition 7.21.

Proposition 28.8. *Let A and B be two topological spaces and let $a_0\in A$ and $b_0\in B$. We consider the inclusion maps*

$$\begin{array}{rcl} i\colon A & \to & A\times B\\ a & \mapsto & (a,b_0)\end{array}\quad \textit{and}\quad \begin{array}{rcl} j\colon B & \to & A\times B\\ b & \mapsto & (a_0,b)\end{array}$$

and we consider the projection maps

$$\begin{array}{rcl} p\colon A\times B & \to & A\\ (a,b) & \mapsto & a\end{array}\quad \textit{and}\quad \begin{array}{rcl} q\colon A\times B & \to & B\\ (a,b) & \mapsto & b.\end{array}$$

Then the maps

$$\begin{array}{rcl} \Phi\colon \pi_n(A,a_0)\times\pi_n(B,b_0) & \to & \pi_n(A\times B,(a_0,b_0))\\ (x,y) & \to & i_*(x)\cdot j_*(y)\end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \pi_n(A\times B,(a_0,b_0)) & \to & \pi_n(A,a_0)\times\pi_n(B,b_0)\\ z & \to & (p_*(z),q_*(z))\end{array}$$

are both homomorphisms[571] *and they are inverses of one another, in particular both are isomorphisms.*

Proof. The proof of this proposition is almost verbatim the same as the proof of Proposition 7.21. □

In Proposition 5.14 we saw that $\pi_1(S^n) = 0$ for $n \geq 2$. Now we can prove the analogous statement for higher homotopy groups. More precisely, we have the following proposition.

Proposition 28.9. *Let $n \in \mathbb{N}$. For any $k \in \{1, \ldots, n-1\}$ we have*

$$\pi_k(S^n) \;=\; 0.$$

The proof of Proposition 28.9 relies on the following more general and actually very convenient proposition.

Proposition 28.10. *Let X be a CW-complex with a single 0-cell x_0 and let $n \in \mathbb{N}$. If X has no cells in dimensions $1, \ldots, n$, then $\pi_n(X, x_0) = 0$ for $k = 1, \ldots, n$.*

Note that the above proposition is a generalization of Corollary 27.14.

Proof of Proposition 28.10. Let X be a CW-complex with a single 0-cell x_0 and let $n \in \mathbb{N}$. We suppose that X has no cells in dimensions $1, \ldots, n$. Let $k \in \{1, \ldots, n\}$. We want to show that $\pi_k(X, x_0) = 0$. We take the point of view from page 715, i.e. we view elements of $\pi_k(X, x_0)$ as homotopy classes of maps $(S^k, *) \to (X, x_0)$. As on page 648 we consider the k-sphere S^k as a CW-complex with one 0-cell $*$ and one k-cell. Let $g\colon (S^k, *) \to (X, x_0)$ be a map. By the Cellular Approximation Theorem 27.15, applied to $g\colon S^k \to X$ and the subcomplex $\{*\} \subset S^k$, there exists a homotopy F rel $\{*\}$ from the map g to a cellular map $\widetilde{g}$. Since F is a homotopy rel $\{*\}$ we see that F also defines a homotopy of maps $(S^k, *) \to (X, x_0)$ of pairs of topological spaces. Thus we see that $[g] = [\widetilde{g}] \in \pi_k(X, x_0)$.

Since $\widetilde{g}$ is cellular we know that $\widetilde{g}(S^k)$ lies in the k-skeleton of X. But by our hypothesis on X the k-skeleton of X consists only of the point x_0. Put differently, $\widetilde{g}$ is the constant map that sends all points in S^k to x_0. We have thus shown that $\pi_k(X, x_0) = 0$. □

Proof of Proposition 28.9. Again, as on page 648 we consider the n-sphere S^n as a CW-complex with one 0-cell and one n-cell. With this CW-structure it follows immediately from Proposition 28.10 that $\pi_k(S^n) = 0$ for $k = 1, \ldots, n-1$. □

The following proposition is also a straightforward generalization from the setting of fundamental groups to the setting of higher homotopy groups. The proof is almost verbatim the same as the proof of Proposition 26.19 and of Proposition 16.4.

Proposition 28.11. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that $X = \bigcup_{i \in \mathbb{N}} X_i$ and such that of one the following three conditions holds:*

[571] In fact in this case it follows immediately from Proposition 28.3 and Lemma 7.20 that Φ is a homomorphism.

(1) *every compact subset of X is already contained in one of the X_i, or*
(2) *each X_i is open in X, or*
(3) *X is a CW-complex and each X_i is a subcomplex.*

Then for any $w \in X_1$ and any $n \in \mathbb{N}$ the inclusion induced maps $\pi_n(X_i, w) \to \pi_n(X, w)$ induce an isomorphism

$$\pi_n(X, w) \;=\; \lim_{\longrightarrow} \pi_n(X_i, w).$$

From this proposition we obtain immediately the following, perhaps not overly surprising, corollary.

Corollary 28.12. *For every $n \in \mathbb{N}$ we have $\pi_n(S^\infty) = 0$.*

Proof. We equip S^∞ and all spheres S^k, $k \in \mathbb{N}$ with the CW-complex structure that we defined on page 655. We have

$$\pi_n(S^\infty) \;=\; \pi_n\Big(\bigcup_{k\in\mathbb{N}} S^k\Big) \;\overset{\substack{\text{by Proposition 28.11 and by definition}\\ \text{of the topology of } S^\infty}}{\overset{\downarrow}{=}}\; \lim_{\longrightarrow} \pi_n(S^k) \;\overset{\substack{\text{by Proposition 28.9 we have}\\ \pi_n(S^k)=0 \text{ for } k>n}}{\overset{\downarrow}{=}}\; 0.$$

□

28.4. **Covering spaces and higher homotopy groups.** One of the main tools in studying fundamental groups was to study the relationship between the fundamental group of a topological space and coverings of the given topological space. Perhaps somewhat surprisingly, for $n \geq 2$ we have a completely different relationship between the n-th homotopy group and coverings.

Proposition 28.13. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Let $n \geq 2$. Then the following two statements hold:*

(1) *The map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism.*
(2) *The map*[572]

$$\begin{array}{rcl} \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ [f\colon (S^n, *) \to (B, b_0)] & \to & [\textit{unique lift of } f \textit{ to a map } (S^n, *) \to (X, x_0)] \end{array}$$

is well-defined and it is the inverse to p_*.

The following is an immediate, and very useful, corollary to Proposition 28.13.

Corollary 28.14. *Let X be a path-connected and reasonable topological space. Given any $n \geq 2$ we have*[573]

$$\pi_n(X) \;\cong\; \pi_n(\textit{universal covering of } X).$$

[572]Here we take the point of view from page 715 where we saw that we view the elements of $\pi_n(B, b_0)$ as homotopy classes of maps $(S^n, *) \to (B, b_0)$.

[573]In Corollary 20.11 we showed that every connected, path-connected and reasonable topological space X admits a universal covering $p\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected.

Now we present some examples for Corollary 28.14. We will consider a few more examples after the proof of Proposition 28.13.

Examples.

(1) Let $k \in \mathbb{N}$ and $n \geq 2$. We have

$$\pi_n((S^1)^k) \;=\; \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \;\underset{\substack{\uparrow \\ \text{Proposition 28.13 page 720}}}{\cong}\; \pi_n(\mathbb{R}^k) \;\underset{\uparrow}{=}\; 0.$$

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1 \times S^1$ vanish.

(2) We consider $X = S^1 \vee S^2$. On page 568 we had determined its universal covering $\widetilde{X}$. For convenience we picture it again in Figure 459. It follows from Proposition 28.13 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.

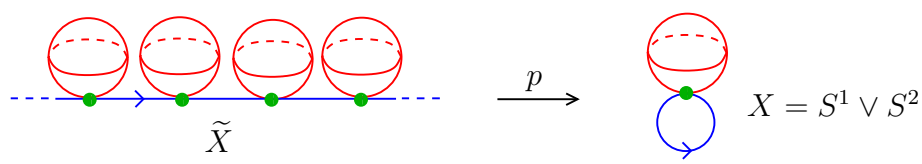


FIGURE 459.

(3) One of our goals is still to complete the classification of lens spaces, in particular we would like to answer Question 7.8. So let $L(p,q)$ and $L(p,r)$ be two lens spaces. Then for any $n \geq 2$ we have

$$\pi_n(L(p,q)) \;\underset{\substack{\uparrow \\ \text{Proposition 28.13}}}{=}\; \pi_n(S^3) \;\underset{\uparrow}{=}\; \pi_n(L(p,r)).$$

We do not know the higher homotopy groups of S^3, but this argument shows that higher homotopy groups are useless for distinguishing lens spaces.

Proof of Proposition 28.13. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and furthermore let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X,x_0) \to \pi_n(B,b_0)$.

We saw on page 715 that we can think of elements in $\pi_n(B,b_0)$ as homotopy classes of maps $(S^n,*) \to (B,b_0)$. Thus let $f\colon (S^n,*) \to (B,b_0)$ be given. We are in the following

setting[574]

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] & (B,b_0). \end{tikzcd} \quad \xrightarrow[\text{since } \pi_1(S^n)=0]{\text{by Proposition 20.2}} \quad \begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] \arrow[ur, dashed, "\text{exists unique } \widetilde{f}"] & (B,b_0). \end{tikzcd}$$

Furthermore it follows again from Proposition 20.2, applied to $Z = S^n \times [0,1]$, that the lifts of two homotopic maps $(S^n,*) \to (B,b_0)$ are two homotopic maps $(S^n,*) \to (X,x_0)$.[575] Summarizing we just constructed a map

$$\begin{array}{rcl} \lambda\colon \pi_n(B,b_0) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [\widetilde{f}]. \end{array}$$

Now we have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B,b_0)$ we have

$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; \big[p \circ \widetilde{f}\big] \;=\; [f].$$

Now let $[g] \in \pi_n(X,x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \;\underset{\substack{\uparrow \\ \text{since } g \text{ is the unique lift of } p\circ g \text{ to } x_0.}}{=}\; [g].$$

□

Now we give a few more applications of Proposition 28.13.

Examples.

(1) Let Σ be a surface of genus ≥ 2. In Theorem 23.9 we have seen that there exists a covering $p\colon \mathbb{D} \to \Sigma$ where $\mathbb{D} = B_1(0) \subset \mathbb{C}$. Since $\mathbb{D}$ is convex it follows from Proposition 28.13 and the discussion on page 720 that $\pi_n(\Sigma) \cong \pi_n(\mathbb{D}) = 0$ for all $n \geq 2$.

Together with the example on page 723 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

(2) Let Σ be a surface of genus ≥ 1 and let $f\colon \Sigma \to \Sigma$ be a self-homeomorphism. We denote by $M(\Sigma, f)$ the mapping torus that we introduced on page 469. For $n \geq 2$

[574] In Proposition 20.2 we need that the sphere S^n is of path-connected and o-locally path-connected, but there are no hypotheses on B or X.

[575] Here there is a little hidden subtlety: if $F\colon S^n \times [0,1] \to B$ is a homotopy between two maps $f, g\colon (S^n,*) \to (B,b_0)$, then we can lift f, g to maps $\widetilde{f}, \widetilde{g}\colon (S^n,*) \to (X,x_0)$. Furthermore we can apply Proposition 20.2 to lift F to a map $\widetilde{F}\colon S^n \times [0,1] \to X$ such that $\widetilde{F}(*,0) = x_0$. We still need to show that $\widetilde{F}(*,t) = x_0$ for all $t \in [0,1]$. We know that for every $t \in [0,1]$ we have $\widetilde{F}(*,t) \in p^{-1}(b_0)$. By Lemma 7.4 the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon [0,1] \to p^{-1}(b_0)$ is constant by Lemma 1.92. But since it takes the value x_0 for $t = 0$ it takes the value x_0 for all $t \in [0,1]$.

we have

$$\pi_n(M(\Sigma,f)) \underset{\substack{\uparrow \\ \text{by Lemma 15.23} \\ \text{and Proposition 28.13}}}{\cong} \pi_n(\Sigma\times\mathbb{R}) \underset{\substack{\uparrow \\ \text{Proposition 28.7}}}{\cong} \pi_n(\Sigma) \underset{\substack{\uparrow \\ \text{previous example}}}{=} 0.$$

Lemma 28.15. *Let X be a 1-dimensional CW-complex and let $x_0 \in X$ be base point. Then $\pi_n(X,x_0)=0$ for every $n\geq 2$.*

Sketch of proof. Let X be a 1-dimensional CW-complex, let $x_0\in X$ and let $n\geq 2$. We first assume that X is finite and connected. We write $k=1-\chi(X)$. We have

$$\pi_n(X) \underset{\substack{\uparrow \\ \text{Proposition 9.21 and} \\ \text{Proposition 28.7}}}{\cong} \pi_n\Big(\bigvee_{i=1}^{k} S^1\Big) \underset{\substack{\uparrow \\ \text{Proposition 28.13}}}{\cong} \pi_n\Big(\text{universal covering of } \bigvee_{i=1}^{k} S^1\Big) \underset{\substack{\uparrow \\ \text{Proposition 28.11 and Figure 460}}}{\cong} \varinjlim \pi_n(\text{trees } T_i) \underset{\substack{\uparrow \\ \text{Proposition 9.20} \\ \text{and Proposition 28.7}}}{=} \varinjlim 0 = 0.$$

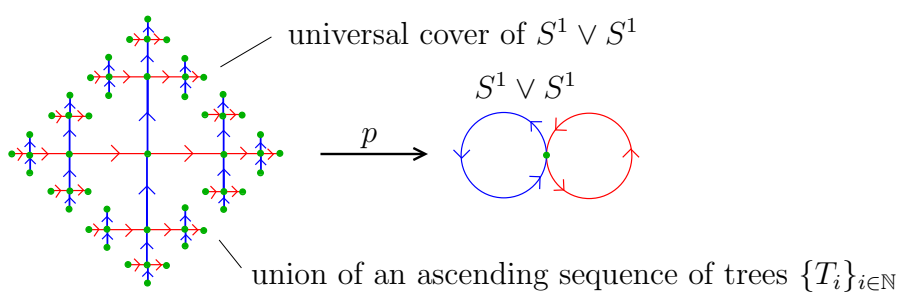


FIGURE 460.

Now suppose that X is an arbitrary 1-dimensional CW-complex. Let $[f]\in\pi_n(X,x_0)$. It follows from Proposition 26.14 that the image of f lies in a finite connected subcomplex Y of X. But by the above discussion $[f]=0\in\pi_n(Y,x_0)$, hence, by functoriality of π_n, we also have $[f]=0\in\pi_n(X,x_0)$. □

28.5. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.

So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 28.16. *Let $n\geq 2$. Does the identity map* $\mathrm{id}\colon S^n\to S^n$ *define a non-trivial element in the homotopy group $\pi_n(S^n,*)$?*

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k>n$? On page 723 we saw that all these groups are trivial for $n=1$. But what happens for $n>1$? Can the higher homotopy groups of spheres be non-trivial?

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 113 we gave an identification $\mathbb{C}\mathrm{P}^1 = S^2$. We consider the following map which is usually referred to as the *Hopf*[576] *map*:

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

The homotopy class of H defines an element in $\pi_3(S^2)$.

Question 28.17. *Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?*

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(P)$ is a circle. Furthermore, if we use the explicit homeomorphism from Lemma 1.41 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) the circle $S^1 \times \{0\}$ corresponds to the points $(z, 0) \in S^3$, in particular the circle gets sent to the point $[1 : 0] \in \mathbb{C}\mathrm{P}^1$,
(2) the circle that is given by the z-axis together with ∞ corresponds to the points $(0, z) \in S^3$, in particular the circle gets sent to the point $[0 : 1] \in \mathbb{C}\mathrm{P}^1$.

Next recall that Lemma 18.1 we gave an explicit identification[577]

$$\left(\overline{B}^2 \times S^1\right) \sqcup_{S^1 \times S^1} \left(S^1 \times \overline{B}^2\right) \xrightarrow{\cong} S^3.$$

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z, z) \,|\, z \in S^1\}$ is the preimage of the point $[1 : 1] \in \mathbb{C}\mathrm{P}^1$. In Figure 461 we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$.

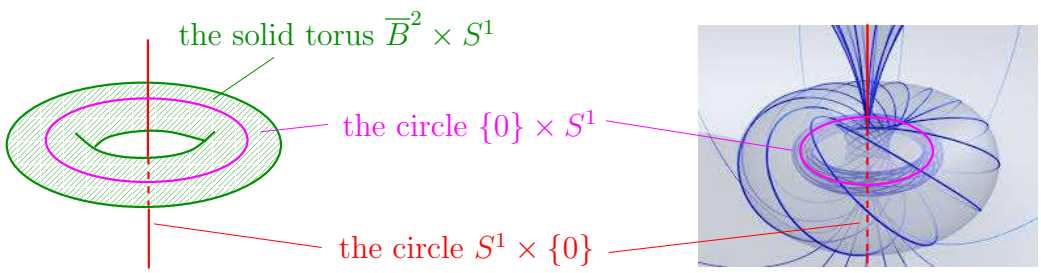


FIGURE 461.

We introduce one more construction of maps that represent potentially non-trivial elements in homotopy groups of spheres.

[576]Heinz Hopf (1894-1971) was a German mathematician.

[577]Note that in Lemma 18.2 we saw that the corresponding embedding $\overline{B}^2 \times S^1 \to S^3$ is essential given by the standard embedding.

Definition.

(1) Let X be a topological space. Recall that on page 452 we defined the *suspension of* X to be the topological space

$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the *North Pole* of $\Sigma(X)$.

(2) Given a map $f\colon X \to Y$ between topological space we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the *suspension of* f.

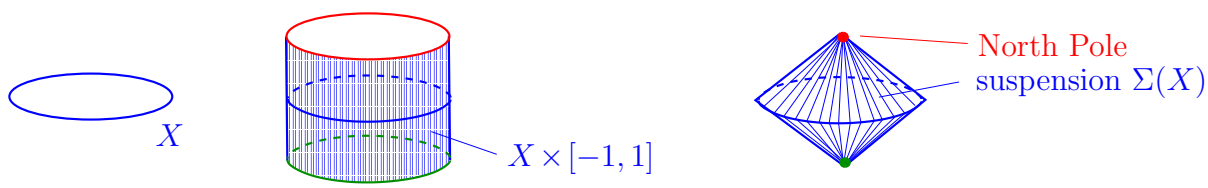


FIGURE 462.

Remark. Let $n \in \mathbb{N}_0$. Recall that on page 453 we wrote down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0,\dots,0,1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}.

Now we have the following elementary lemma.

Lemma 28.18. *Let $k \in \mathbb{N}_0$.*

(1) *Let (X,x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map*

$$\begin{array}{rcl} \pi_k(X,x_0) & \to & \pi_{k+1}(\Sigma(X),*) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

(2) *Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map*

$$\begin{array}{rcl} \pi_k(S^n,*) & \to & \pi_{k+1}(S^{n+1},*) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(S^n) = S^{n+1}\big] \end{array}$$

is a well-defined group homomorphism. We refer to it also as the suspension homomorphism.

Proof.

(1) It follows from Lemma 15.4 (4) that this map is well-defined.[578] We leave it as an exercise to show that the map is in fact a group homomorphism.[579]
(2) This statement follows immediately from (1). □

We conclude this chapter with the following question:

Question 28.19. *Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?*

Unfortunately at the moment we do not have the tools to answer any of the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert-van Kampen Theorem 14.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapter, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres, in fact the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the coming sections.

28.6. **The Poincaré Conjecture.** Before we bid a temporary farewell to the fundamental group and to higher homotopy groups we want to quickly discuss the Poincaré Conjecture.

Definition.

(1) Let $k \in \mathbb{N}_0$. We say that a topological space X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \dots, k$.[580][581]
(2) An n-dimensional topological manifold X that is closed and that is $\lfloor \frac{n}{2} \rfloor$-connected is called an *n-dimensional homotopy sphere*. If X is in fact a manifold, then we refer to it as a *smooth n-dimensional homotopy sphere*.

Remark.

[578] In an earlier version of these it was written that "it is straightforward to see that the map is well-defined". If one traces through the ancestry of Lemma 15.4 (4) one realizes that this lemma is, despite appearances, by no means obvious.

[579] This exercise might in fact be rather painful if one does not pick the correct point of view. It seems like most authors agree, since I could not find a single reference where it is shown that the suspension map is indeed a homomorphism.

[580] For example, X is 0-connected if and only if X is path-connected and *non-empty*. Furthermore, X is 1-connected if and only if it is simply connected.

[581] Since a k-connected topological space is in particularly 0-connected and thus path-connected we see that we can ignore base points.

(1) Since $\lfloor \frac{3}{2} \rfloor = 1$ we see that a 3-dimensional homotopy sphere is the same as a connected closed 3-dimensional topological manifold with $\pi_1(M) = 0$.
(2) We have seen in Proposition 28.9 that each S^n is in fact a homotopy sphere. A smooth n-dimensional homotopy sphere is thus a closed n-dimensional manifold that cannot be distinguished from the standard sphere S^n using the first $\lfloor \frac{n}{2} \rfloor$ homotopy groups.
(3) The definition using $\lfloor \frac{n}{2} \rfloor$-connected sounds slightly arbitrary, but we will see in a later chapter that an n-dimensional manifold that is $\lfloor \frac{n}{2} \rfloor$-connected is in fact $(n-1)$-connected. Furthermore we will see that a homotopy sphere cannot be distinguished from an ordinary sphere by any of the higher homotopy groups either.

One can now ask whether every (smooth) homotopy sphere is S^n. Here one needs to be a little careful what one means by "is". There are two reasonable interpretations. One could either interpret "is" as "is diffeomorphic to" or as "is homeomorphic to". So we really have two, a priori different, questions that we can ask:

Question 28.20. *Let $n \in \mathbb{N}$.*
(1) *Is every smooth n-dimensional homotopy sphere diffeomorphic to S^n?*
(2) *Is every n-dimensional homotopy sphere homeomorphic to S^n?*

For $n = 1$ we saw in Proposition 3.53 and Proposition 3.49 that the answer to both questions is "yes". Similarly we saw in Theorems 14.9 and 14.9 that the answer to questions is also "yes" for $n = 2$. For $n = 1$ and $n = 2$ it was basically known by the end of the 19th century that the answers are "yes".[582]

The above question was first raised in dimension $n = 3$ by Henri Poincaré in 1905. This special case became known as the Poincaré Conjecture in dimension three.

The first important result dealing with Question 28.20 was proved by John Milnor [Mi1] in 1956. In 1962 he obtained the Fields medal for proving the following theorem which shows that the above two questions raised in Question 28.20 are indeed two different questions.[583]

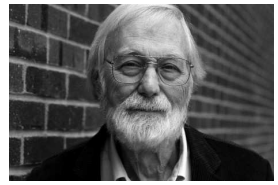
Theorem 28.21. (Milnor) *There exists a 7-dimensional manifold that is a homeomorphic to S^7 but that is not diffeomorphic to S^7.*

Remark. An n-dimensional manifold that is homeomorphic to S^n but not diffeomorphic to S^n is often called an *exotic n-sphere*. Milnor thus showed that in dimension 7 there exists an exotic 7-sphere. Michel Kervaire and John Milnor [KeM] showed that for $n \geq 5$ there exist at most finitely many, pairwise non-diffeomorphic, exotic n-spheres. Furthermore, in dimensions 5 and 6 there do not exist any exotic spheres. Somewhat earlier it had been shown in 1952 by Edwin Moise [Moi] that in dimension 3 there does not exist an exotic sphere. On the other hand, it follows from the work of Kervaire–Milnor [KeM] and

[582] Even though at that point the concept of a (topological) manifold was not fully developed yet.

[583] The Fields medal is the most prestigious award in mathematics. It is awarded every four years to usually 2-4 mathematicians. One of the conditions is that you can only be awarded a Fields medal if you are under 40.

Henri Poincaré (1854-1912)



John Milnor (∗1931)

FIGURE 463.

FIGURE 464. The Fields medal

much more recent work of Hill–Hopkins–Ravenel [HHR] and Wang–Xu [WX] that the only odd dimensions n for which S^n has a unique smooth structure are 1, 3, 5 and 61. It is conjectured, see e.g. [WX, Conjecture 1.17] that for dimensions greater than 4, the only spheres with a unique smooth structure are S^5, S^6, S^{12}, [56] and S^{61}. We refer to [BHHM] for some recent work regarding even-dimensional spheres.

In 1961 Stephen Smale [Sma] proved the following theorem, for which he obtained the Fields medal in 1966.

Theorem 28.22. (Smale) *In dimensions $n \geq 5$ every homotopy sphere is homeomorphic to S^n.*

The next major result was obtained by Michael Freedman [Fre] in 1981. He was also awarded a Fields medal for his work.

Theorem 28.23. (Freedman) *Every 4-dimensional homotopy sphere is homeomorphic to S^4.*

The proof by Freedman is so hard and specialized that only very few mathematicians have ever understood the proof.[584]

Now we turn to the dimension $n = 3$. Now is a good moment to state the following theorem which implies, in particular, that in the 3-dimensional setting the two questions asked Question 28.20 are equivalent.

[584] I was told at some point that the total number of mathematicians who understood the proof is six, it is not clear whether at this moment anybody understands the proof.

Theorem 28.24. *Let $n = 2$ or $n = 3$. Every topological n-dimensional manifold*[585] *admits a unique smooth structure, in particular two n-dimensional manifolds are diffeomorphic if and only if they are homeomorphic.*

Proof. The 2-dimensional case was proved by Tibor Radó [Rad], see also [AS], [Moi2, p. 60] and [Hat4] for more modern proofs.

We turn to the 3-dimensional case. By work of Edwin Moise [Moi] [Moi2, p. 252 and 253] any topological 3-manifold (not necessarily compact) admits a unique "piecewise-linear structure". (See also [Bin, Shal] for alternative proofs). By work of Munkres [Mun4, p. 333][Mun2, Theorems 6.2 and 6.3], and independently Whitehead [Whd4, Corollary 1.18], this implies that a topological 3-manifold admits a unique smooth structure. (See also [Kui], [Thu2, Chapter 3.10] and for a more detailed discussion.) □

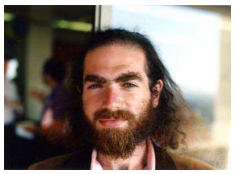
After Freedman's work it took another 20 years before finally the original Poincaré Conjecture was dealt with. More precisely, in 2002 Grigori Perelman [Per1, Per2, Per3, MTi] proved the following result.

Theorem 28.25. (Perelman) *Every 3-dimensional homotopy sphere is diffeomorphic to S^3.*

Perelman would have gotten the Fields medal in 2006 if he had not turned it down. Furthermore, the Poincaré Conjecture was one of the seven Millennium Problems[586] for which the Clay foundation awards a prize of one million dollars. This prize was also turned down by Perelman.

Stephen Smale (∗1930)

Michael Freedman (∗1951)



Grigory Perelman (∗1966)

Figure 465.

What remains to be done is the following problem which is often referred to as the *smooth Poincaré Conjecture in dimension 4*.

Question 28.26. *Does there exist a 4-dimensional manifold that is homeomorphic to S^4 but that is not diffeomorphic to S^4?*

[585] Note that here we do not add any extra hypothesis, so the topological manifold could be non-compact.
[586] https://en.wikipedia.org/wiki/Millennium_Prize_Problems

This question is considered to be one of the hardest problems in topology. Unfortunately there has been very little progress so far.

It might be surprising to see that the last open problem in this context is in dimension 4. But it turns out that the study of 4-dimensional manifolds is much harder than the study of manifolds in lower or higher dimensions.

The following theorem shows very clearly the special role played by 4-dimensional topology.

Theorem 28.27.

(1) *For $n \neq 4$ every n-dimensional manifold homeomorphic to $\mathbb{R}^n$ is also diffeomorphic to $\mathbb{R}^n$.*
(2) *In dimension 4 there exist uncountably many manifolds that are homeomorphic to $\mathbb{R}^4$ but that are pairwise non-diffeomorphic.*

The first statement was proved by John Stallings [Sta] in 1961. The second statement was proved by Cliff Taubes [Tau] in 1987. The latter result builds on the ground breaking work of Simon Donaldson [Don] which he proved as a second year PhD student and for which he obtained a Fields medal in 1986.

Exercises for Chapter 28.

Exercise 28.1. Let X be a topological space, let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map with $n \geq 1$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \dots, x_n) & \mapsto & f(1-x_1, 1-x_2, \dots, 1-x_n). \end{array}$$

Is $[\overline{f}] \in \pi_n(X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$?

Exercise 28.2. Let $k \leq l$ be natural numbers, let $f\colon S^k \to S^l$ be a map and let $x_0 \in S^k$. Show that for any $i \in \mathbb{N}$ the map $f_*\colon \pi_i(S^k, x_0) \to \pi_i(S^l, f(x_0))$ is the trivial map.
Hint: Use Proposition 28.9.

Exercise 28.3. In this exercise you can assume that we already know that $\pi_3(S^2) \cong \mathbb{Z}$. Show that there exists an epimorphism $\pi_3(S^1 \vee S^2) \to \mathbb{Z}^3 \oplus \mathbb{Z}_4$.

Exercise 28.4. Let $n \in \mathbb{N}$ and let K be a non-empty compact subset of $\mathbb{R}^n$. Let $r \in \mathbb{R}$ such that $r > \sup\{\|P\| \mid P \in K\}$. Show that for any $k \in \mathbb{N}$ the inclusion induced map $\pi_k(S_r^{n-1}(0)) \to \pi_k(\mathbb{R}^n \setminus K)$ is a monomorphism.

Exercise 28.5. We consider the pseudocircle, i.e. we consider the slightly weird topological space $X = \{A, B, C, D\}$ from page 66. Show that $\pi_n(X, A) = 0$ for any $n \geq 2$.
Hint: Find a convenient covering space of X.

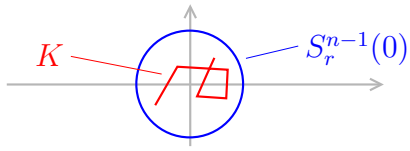


FIGURE 466. Illustration of Exercise 28.4.

Exercise 28.6. Let (X, x_0) be a topological space. We denote by $\Sigma(X)$ the suspension of X and we denote by $*$ the North Pole of $\Sigma(X)$. Is the suspension homomorphism

$$\pi_1(X_0, x_0) \ \to \ \pi_2(\Sigma(X), *),$$

that we introduced in Lemma 28.18, necessarily a monomorphism?

Exercise 28.7. Let M be a connected n-dimensional topological manifold with non-trivial fundamental group. Show that the suspension $\Sigma(M)$ is not an $(n+1)$-dimensional topological manifold.
Hint: Consider one of the two suspension points. Show that there is no neighborhood of such a point that is homeomorphic to an open subset of $\mathbb{R}^{n+1}$.
Remark: It is surprisingly tricky to write down a rigorous argument. Also note that the exercise is similar in spirit to Exercise 18.7.

29. The homology groups of a topological space

One of our main goals is to show that for $m \neq n$ the spheres S^m and S^n are not homeomorphic. We have seen that for most cases fundamental groups are of no use. Furthermore, it might be that in principle higher homotopy groups can distinguish such spheres, but since we cannot compute higher homotopy groups, those are of no use either. So we need a new approach to finding invariants that can distinguish spheres.

In this chapter we want to introduce a new functor from the category of topological spaces to an algebraic category, more precisely, we will introduce a functor from the category of topological spaces to the category of abelian groups.

29.1. **Singular chains.**

Definition.

(1) Given k points $P_1, \dots, P_k$ in $\mathbb{R}^n$ such that $P_2 - P_1, \dots, P_k - P_1$ are linearly independent we refer to the convex hull of $P_1, \dots, P_k$ as a *k-simplex*.

(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n \ := \ \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the *standard n-simplex*.

(3) Given the standard n-simplex Δ^n we define its boundary as[587]

$$\partial \Delta^n \ = \ \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

Example. The standard 0-simplex Δ^0 is just the point $\{1\} \in \mathbb{R}$, the standard 1-simplex Δ^1 equals the segment in $\mathbb{R}^2$ from $(1,0)$ to $(0,1)$ and the standard 2-simplex Δ^2 is the triangle in $\mathbb{R}^3$ spanned by the vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$. We refer to Figure 467 for an illustration of the standard simplices in dimension 0, 1 and 2 and for an illustration of the boundary of the standard 2-simplex.

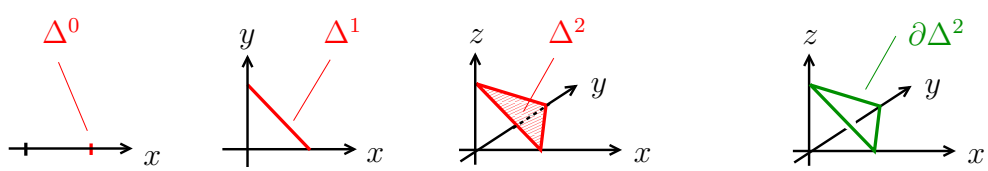


Figure 467.

For the record we state the following elementary lemma.

[587]The definition of the "boundary" of the standard n-simplex Δ^n is the "obvious one". Nonetheless note that if we had taken the original definition of the boundary of a subset A in some subspace X as introduced on page 50 and applied it to $\Delta^n \subset \mathbb{R}^{n+1}$, then the boundary of Δ^n in $\mathbb{R}^{n+1}$ would be all of Δ^n.

Lemma 29.1. *The map*

$$\begin{array}{rcl} \Delta^n & \overset{\cong}{\to} & \left\{(x_1,\dots,x_n)\in[0,1]^n \,\middle|\, \sum_{i=1}^n x_i \in [0,1]\right\} \xrightarrow[\text{to } Q=(\frac{1}{n+1},\dots,\frac{1}{n+1})]{\text{Proposition 1.48 (2) applied}} \overline{B}^n \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}). \end{array}$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.

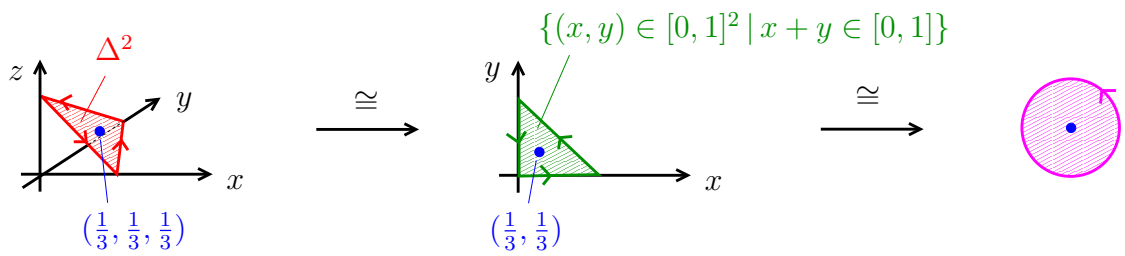


FIGURE 468. Illustration of the canonical homeomorphism from Lemma 29.1.

Definition. An *r-dimensional face* of the standard n-simplex is defined as a subset of the form

$$\{(t_0,\dots,t_n)\in\Delta^n \,|\, t_{d_1}=t_{d_2}=\dots=t_{d_{n-r}}=0\},$$

where $d_1<d_2<\dots<d_{n-r}\in\{0,\dots,n\}$ are fixed. We illustrate the definition in Figure 469 on the left.

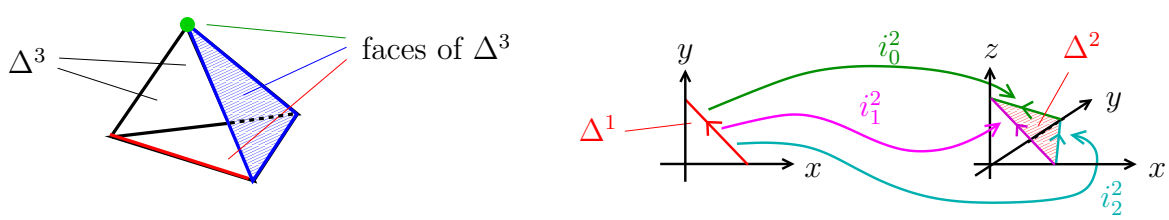


FIGURE 469.

Remark.

(1) The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.

(2) Every r-dimensional face of Δ^r is homeomorphic to the standard r-simplex. For later we fix homeomorphism in the case of $(n-1)$-dimensional faces. More precisely, for $j\in\{0,\dots,n\}$ we consider the map

$$\begin{array}{rcl} i_j^n\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{array}$$

This map is sometimes called the *j-th face map*. It is straightforward to see that these face maps are homeomorphisms. The face maps $i_0^2, i_1^2, i_2^2 \colon \Delta^1 \to \Delta^2$ are illustrated in Figure 469 on the right.

The first idea for the definition of homology groups is now the following. In the definition of the fundamental group $\pi_1(X, x_0)$ we had considered all maps $f \colon [0,1] \to X$ with the extra condition $f(0) = f(1) = x_0$. Using this extra condition we could concatenate loops to eventually end up with a group structure. Now given $n \in \mathbb{N}_0$ we want to consider all maps from Δ^n to X with no extra condition on the boundary. This time there is no reasonable way how two such maps can be composed to give a new map. Since we nonetheless want the notion of a group we turn the set of all maps $\Delta^n \to X$ into a group by "brute force", i.e. by considering the free abelian group generated by these maps.

This leads us to the following definition.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) A *singular n-simplex* is a map $\sigma \colon \Delta^n \to X$.
(2) We define

$$C_n(X) \ := \ \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $C_n(X) = 0$.
(3) We refer to an element in $C_n(X)$ as a *singular n-chain in X*.

Examples.

(1) For $n = 0$ a singular n-simplex is a map from Δ^0 to X. Such a map is uniquely determined by the corresponding point in X. Therefore frequently we will identify singular 0-simplices with points in X.
(2) For points $z_0, \dots, z_k \in \mathbb{R}^n$ we consider the map

$$\begin{aligned} [z_0, \dots, z_k] \colon \Delta^k &\to \mathbb{R}^n \\ (t_0, \dots, t_k) &\mapsto \sum_{j=0}^{k} t_j \cdot z_j. \end{aligned}$$

The map $[z_0, \dots, z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0, \dots, z_k$ in $\mathbb{R}^n$. Put differently, $[z_0, \dots, z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

Remark. Let X be a topological space. By the discussion on page 366 we can view every element in $C_n(X)$, i.e. every singular n-chain, as a linear combination $a_1\sigma_1 + \dots + a_k\sigma_k$ where $a_1, \dots, a_k \in \mathbb{Z}$ and where $\sigma_1, \dots, \sigma_k$ are singular n-simplices.

Now let $f \colon X \to Y$ be a map between topological spaces. For each n we consider

$$\begin{aligned} f_* \colon C_n(X) &\to C_n(Y) \\ \sum_{i=1}^{k} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i). \end{aligned}$$

It is easy to verify that $X \mapsto C_n(X)$ and $f \mapsto f_*$ defines a functor from the category of topological spaces to the category of abelian groups.

Have we made any progress towards our goal of distinguishing topological spaces? Not really. For example, if X is a manifold of dimension ≥ 1, then given any $n \in \mathbb{N}_0$ there exist uncountable many maps $\Delta^n \to X$, i.e. the group $C_n(X)$ is a free abelian group with uncountably many generators. Such groups are totally useless for distinguishing non-homeomorphic topological spaces.

29.2. **Definition of the homology groups of a topological space.** Before we give the definition of homology groups it is perhaps helpful to recall the definition of de Rham cohomology.

Let M be a manifold. We write

$$\Omega^n(M) \ := \ \{\text{smooth } n\text{-forms on } M\}.$$

The isomorphism type of this vector space is an invariant of the diffeomorphism type of the manifold M, but it is rather useless, since for $1 \leq n \leq \dim(M)$ this is a vector space of uncountable dimension. To cut $\Omega^n(M)$ into something much smaller we defined linear maps $d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)$ with $d_{n+1} \circ d_n = 0$. Finally we defined the n–th de Rham cohomology group of M as

$$H^n_{\mathrm{dR}}(M) \ := \ \ker\{d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)\}/\operatorname{im}\{d_{n-1} \colon \Omega^{n-1}(M) \to \Omega^n(M)\}.$$

We are now in a similar situation: to a topological space X we associated the free abelian group $C_n(X)$ which is "too big". The idea now is to introduce a "differential" as for the case of differential forms. But in this case there is no good way to associate to a map $\Delta^n \to X$ a new map $\Delta^{n+1} \to X$. But by restricting the map $\Delta^n \to X$ to each of the $(n-1)$-dimensional faces of Δ^n which are, as we have seen, homeomorphic to the standard $(n-1)$-simplex, we can associate to a singular n-simplex $\sigma \colon \Delta^n \to X$ a sum of singular $(n-1)$-simplices, i.e. a singular $(n-1)$-chain. To do this we will use the maps $i^n_k \colon \Delta^{n-1} \to \Delta^n$, $k = 0, \ldots, n$ from page 735 to identify the standard $(n-1)$-simplex with the $(n-1)$-dimensional faces of Δ^n. Now we can define

$$\widetilde{\partial}_n \sigma \ := \ \sum_{j=0}^{n} \sigma \circ i^n_j.$$

By Lemma 11.1 this defines a map $\widetilde{\partial}_n \colon C_n(X) \to C_{n-1}(X)$. But it is quite straightforward to see that for this definition the composition of the maps $\widetilde{\partial}_n \colon C_n(X) \to C_{n-1}(X)$ and $\widetilde{\partial}_{n-1} \colon C_{n-1}(X) \to C_{n-2}(X)$ is not zero[588]. But we can rectify this problem by cleverly introducing signs. This leads us to the following definition.

[588]Indeed, let S be a set and let $\mathbb{Z}^{(S)}$ be the free abelian group generated by S. We define $\epsilon \colon \mathbb{Z}^{(S)} \to \mathbb{Z}$ to be the linear map that assigns to each $s \in S$ the value 1. It follows easily from the definitions that for any $\sigma \colon \Delta^n \to X$ we have $\epsilon(\widetilde{\partial}_n \sigma) = n+1$ and $\epsilon((\widetilde{\partial}_{n-1} \circ \widetilde{\partial}_n)(\sigma)) = n \cdot (n+1)$.

Definition. Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. The *boundary* $\partial_n\sigma \in C_{n-1}(X)$ of σ is defined as

$$\partial_n\sigma \;:=\; \sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n.$$

By Lemma 11.1 the map $\sigma \mapsto \partial_n\sigma$ extends uniquely to a linear map

$$\begin{array}{rcl} \partial_n\colon C_n(X) & \to & C_{n-1}(X) \\ \sum_{i=1}^{k} a_i\sigma_i & \mapsto & \sum_{i=1}^{k} a_i \cdot \partial\sigma_i, \end{array}$$

which we refer to as the *n-th boundary map*. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have[589]

$$\partial\sigma \;=\; (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 \;=\; \sigma(0,1) - \sigma(1,0).$$

Put differently, a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$. Here we view points in X again as singular 0-simplices. We sketch the definition of a singular 1-simplex in Figure 470.

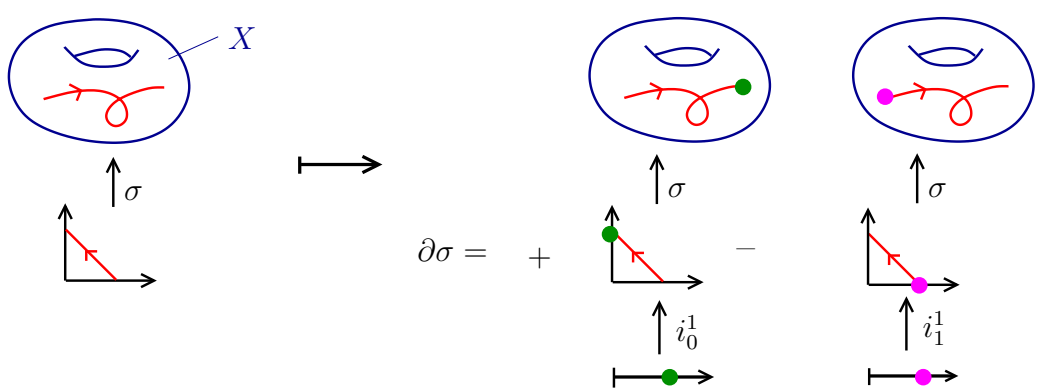


FIGURE 470. Boundary of a singular 1-simplex.

(2) In Figure 471 we sketch the boundary of a singular 2-simplex.

As promised, these boundary maps have the following fundamental property:

Proposition 29.2. *Let X be a topological space. For any $n \in \mathbb{N}_0$ the composition*

$$\partial_{n-1} \circ \partial_n\colon C_n(X) \;\to\; C_{n-2}(X)$$

[589]Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.

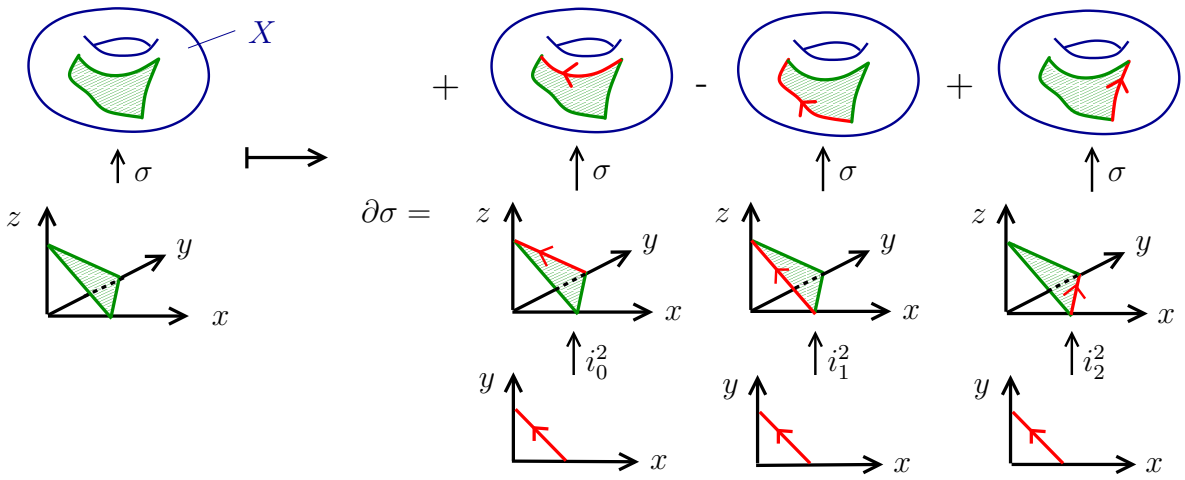


FIGURE 471. Boundary of a singular 2-simplex.

is the zero map.

Proof. It suffices to show that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for every singular n-simplex $\sigma\colon \Delta^n \to X$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then it follows immediately from the definitions that

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) = \sum_{k=0}^{n-1}(-1)^k\sum_{j=0}^{n}(-1)^j\cdot(\sigma\circ i_j^n)\circ i_k^{n-1}\\ &= \sum_{0=k<j\le n}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) + \sum_{0=j\le k\le n-1}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) =: (*).\end{aligned}$$

Now we want to show that each expression $\sigma\circ(i_j^n\circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j\le k$ we have[590]

$$i_j^n\circ i_k^{n-1} = i_{k+1}^n\circ i_j^{n-1}.$$

Now we continue with the above calculation:

$$\begin{aligned}(*) &\underset{\substack{\uparrow\\ \text{above calculation of } i_j^n\circ i_k^{n-1}}}{=} \sum_{0=k<j\le n}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) + \sum_{0=j\le k\le n-1}(-1)^{k+j}\cdot\sigma\circ(i_{k+1}^n\circ i_j^{n-1})\\ &\underset{\substack{\uparrow\\ \text{substitution } l=k+1}}{=} \sum_{0=k<j\le n}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) + \sum_{0=j<l\le n}(-1)^{l+j-1}\cdot\sigma\circ(i_l^n\circ i_j^{n-1}) \underset{\substack{\uparrow\\ \text{all summands left and right cancel}}}{=} 0.\end{aligned}$$

□

The following is arguably the most important definition of algebraic topology.

[590]Indeed, if $j\le k$, then the zero-entries of $(i_j^n\circ i_k^{n-1})(t_0,\dots,t_{n-1})$ are the entries j and $k+1$.

Definition. Let X be a topological space and $n \in \mathbb{N}_0$. We define the *n-th singular homology group of X* as the quotient group[591]

$$H_n(X) \ := \ \ker\{\partial_n \colon C_n(X) \to C_{n-1}(X)\} \,/\, \operatorname{im}\{\partial_{n+1} \colon C_{n+1}(X) \to C_n(X)\}.$$

For $n \in \mathbb{Z}_{<0}$ we extend this definition to $H_n(X) := 0$. Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".[592]

One of the key goals of this course is to develop tools for computing homology groups of topological spaces. In particular we want to compute the homology groups for all spheres. In the subsequent discussions it is helpful to introduce a few more definitions.

Definition. Let X be a topological space.

(1) We say $c \in C_n(X)$ is an *n-cycle*, if $\partial c = 0$.
(2) If c is an n-cycle, then we refer to $[c] \in H_n(X)$ as the *homology class of c*.
(3) We say two singular n-chains c and d are *homologous* if there exists a singular $(n+1)$-chain e with $\partial e = c - d$. In particular, if c and d are cycles, then they are homologous if they represent the same homology class.
(4) An n-cycle c is called *null-homologous*, if $[c] = 0 \in H_n(X)$.
(5) Given a singular 1-simplex $\sigma \colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$. Put differently, $\overline{\sigma}$ is given by reversing the orientation of σ.

The following lemma will be proved in Exercises 29.1 and 29.3.

Lemma 29.3. *Let X be a topological space and let $\sigma \colon \Delta^1 \to X$ be a singular 1-simplex.*

(1) *If σ is a constant map, then σ is null-homologous.*
(2) *The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.*
(3) *Let $s \in (0,1)$ and consider the singular 1-simplices*

$$\begin{array}{rcl} \alpha \colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \textit{and} \quad \begin{array}{rcl} \beta \colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s + t(1-s), 1 - (s + t(1-s))). \end{array}$$

Then the singular 1-chains σ and $\alpha + \beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.

[591] It follows from Proposition 29.2 that we have the following inclusion:

$$\operatorname{im}\{\partial_{n+1} \colon C_{n+1}(X) \to C_n(X)\} \ \subset \ \ker\{\partial_n \colon C_n(X) \to C_{n-1}(X)\}.$$

Furthermore the groups $C_*(X)$ are abelian, in particular any subgroup is also abelian and normal. Therefore it does indeed make sense to consider the quotient group.

[592] Later on we will also introduce "cellular homology". At that point it will be useful to remember that our present homology groups have a more precise name, namely singular homology groups.

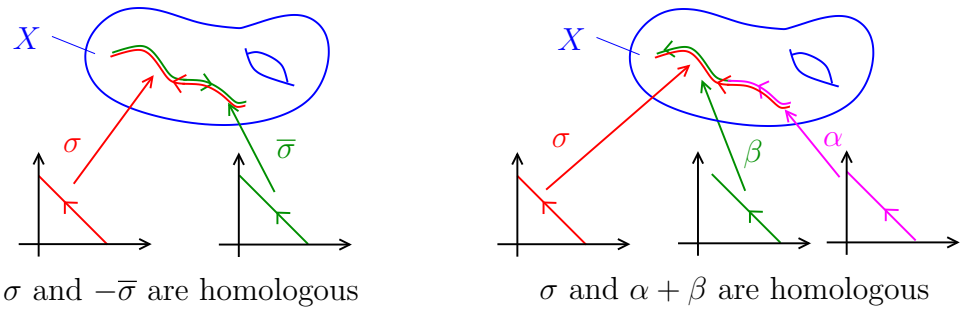


FIGURE 472. Illustration of Lemma 29.3.

Examples.

(1) We consider Figure 473. We see three singular chains on the torus, and it is straightforward to show that all three singular chains are cycles. Now the question arises,

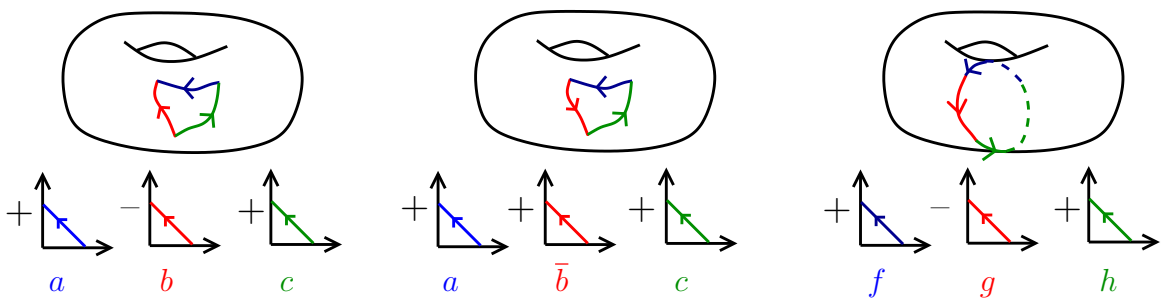


FIGURE 473. Three examples of 1-cycles on the torus.

which of these cycles are in fact null-homologous. In Figure 1 we see that the first cycle $a-b+c$ in Figure 473 is the boundary of a singular 2-simplex D. One can easily

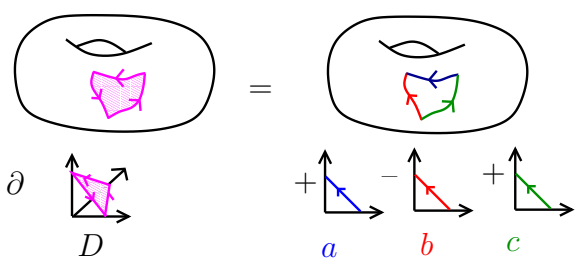


FIGURE 474. The first cycle in Figure 473 is the boundary of a 2-dimensional singular chain.

see that the second cycle, $a+\overline{b}+c$ in Figure 473 is *not* the boundary of a singular 2-simplex. On the other hand by Lemma 29.3 we know that $b+\overline{b}$ is a null-homologous

cycle, i.e. there exists an $E \in C_2(X)$ such that $b + \bar{b} = \partial E \in C_1(X)$. Thus it follows that

$$a + \bar{b} + c \;=\; a + \partial E - b + c \;=\; a - b + c + \partial E = \partial D + \partial E \;=\; \partial(D + E),$$

i.e. $a + \bar{b} + c$ is null-homologous.

It is not clear whether the third cycle in Figure 473 is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement.

(2) In Figure 475 we show the torus $T = [0,1] \times [0,1]/\sim$ and two singular simplices $\alpha\colon \Delta^2 \to T$ and $\beta\colon \Delta^2 \to T$ that are given by affine linear maps.[593] It is straightforward to verify that $\alpha - \beta$ is a cycle, hence it represents an element in $H_2(T)$. We will later see whether or not this element is zero in $H_2(T)$.

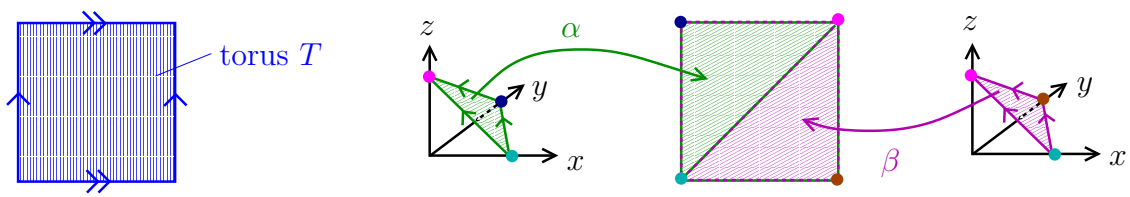


FIGURE 475.

(3) In Exercise 29.4 we will try to generalize the approach taken for the torus to other 2-dimensional manifolds, namely the sphere, the Klein bottle and the projective space $\mathbb{R}P^2$.

29.3. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will be infinitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the 0-th homology of a path-connected topological space. The key to the calculation is the following definition.

Definition. Given a topological space X we refer to the map

$$\begin{aligned} \epsilon_X\colon C_0(X) &\to \mathbb{Z} \\ \sum_{i=1}^{k} a_i\sigma_i &\mapsto \sum_{i=1}^{k} a_i \end{aligned}$$

as the *augmentation map*.

[593]More precisely, α is the unique affine linear map with $\alpha(1,0,0) = [(0,0)]$, $\alpha(0,1,0) = [(1,0)]$ and $\alpha(0,0,1) = [(1,1)]$ whereas β is the unique affine linear map with $\beta(1,0,0) = [(0,0)]$, $\beta(0,1,0) = [(0,1)]$ and $\beta(0,0,1) = [(1,1)]$.

The following lemma shows that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 29.4. *Let X be a topological space X.*

(1) *The map $\epsilon_X \circ \partial_1 \colon C_1(X) \to \mathbb{Z}$ is the zero map.*
(2) *The map*

$$\begin{array}{rcl} \epsilon_X \colon H_0(X) & \to & \mathbb{Z} \\ [\sigma] & \mapsto & [\epsilon_X(\sigma)] \end{array}$$

is well-defined. In the following we refer to this map also as the augmentation map.

Proof. Let X be a topological space.

(1) Let $c = \sum\limits_{i=1}^{n} b_i \sigma_i \in C_1(X)$. We calculate that

$$\epsilon(\partial_1(c)) \;=\; \epsilon\Big(\sum_{i=1}^{n} b_i \cdot \partial_1(\sigma_i)\Big) \;\underset{\substack{\uparrow \\ \text{for every singular 1-simplex } \sigma \text{ we} \\ \text{have } \partial\sigma = \sigma(0,1) - \sigma(1,0), \text{ see page 738}}}{=}\; \epsilon\Big(\sum_{i=1}^{n} b_i \cdot \sigma_i(0,1) - b_i \cdot \sigma_i(1,0)\Big) \;\underset{\substack{\uparrow \\ \text{by definition of } \epsilon}}{=}\; \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} b_i \;=\; 0.$$

(2) Note that

$$\begin{array}{rcl} H_0(X) & = & \ker\{\partial_0 \colon C_0(X) \to C_{-1}(X)\} \,/\, \operatorname{im}\{\partial_1 \colon C_1(X) \to C_0(X)\} \\ & = & C_0(X) \,/\, \operatorname{im}\{\partial_1 \colon C_1(X) \to C_0(X)\}. \\ & \underset{\substack{\uparrow \\ \text{since } C_{-1}(X) = 0}}{} & \end{array}$$

Thus we see that Statement (2) is an immediate consequence of Statement (1). □

Now we can formulate the following proposition.

Proposition 29.5. *Let X be a path-connected non-empty topological space and let P be a point in X. Then the augmentation map*

$$\epsilon = \epsilon_X \colon H_0(X) \;\to\; \mathbb{Z}$$

is an isomorphism where the inverse is given by the map

$$\begin{array}{rcl} \iota \colon \mathbb{Z} & \to & H_0(X) \\ n & \mapsto & n \cdot [P]. \end{array}$$

In particular we have $H_0(X) \cong \mathbb{Z}$.

Proof. Let X be a non-empty path-connected topological space. Let P be a point in X. It is clear that $\epsilon \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ is surjective. It remains to show that ϵ is injective.

Thus let $c \in C_0(X)$ with $\epsilon(c) = 0$. By definition of $C_0(X)$ we can find points $Q_1, \ldots, Q_k$ in X and $a_1, \ldots, a_k \in \mathbb{Z}$ such that $c = \sum\limits_{i=1}^{k} a_i \cdot Q_i$. We choose a point P in X. Since X is

path-connected we can find, for $i = 1, \dots, k$, a path from P to Q_i. Put differently, for each $i = 1, \dots, k$ we can find a singular 1-simplex σ_i with $\partial\sigma_i = Q_i - P$. Then we have

$$\partial\left(\sum_{i=1}^{k} a_i\sigma_i\right) \;=\; \sum_{i=1}^{k} a_i\partial\sigma_i \;=\; \sum_{i=1}^{k} a_i(Q_i - P) \;=\; \sum_{i=1}^{k} a_iQ_i - \underbrace{\left(\sum_{i=1}^{k} a_i\right)}_{=\epsilon(c)=0} \cdot P \;=\; \sum_{i=1}^{k} a_iQ_i.$$

This implies that $[c] = 0 \in H_0(X)$. Thus we have shown that ϵ is injective. □

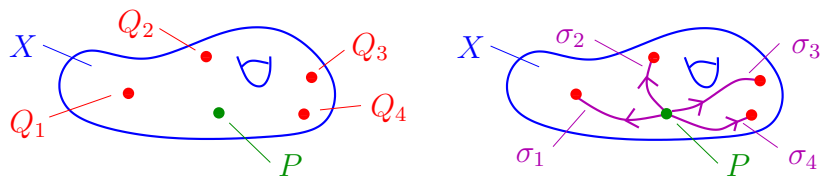


FIGURE 476. Illustration of the proof of Proposition 29.5.

The proof of the following lemma is Exercise 29.2.

Lemma 29.6. *Let X be a topological space which consists of a single point. Then the following holds*

$$H_n(X) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

The calculation of $H_0(X)$ in Proposition 29.5 and the calculation of the homology of a point in Lemma 29.6 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder.

29.4. **Algebraic chain complexes.** Before we continue with the discussion of homology groups it is convenient to introduce several algebraic definitions. The following definitions are just the obvious algebraic abstractions of the definitions that we had already introduced.

Definition.

(1) An (*algebraic*) *chain complex* (C_*, ∂_*) is a sequence

$$\dots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of maps between abelian groups[594] such that for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $C_n = 0$ for $n \in \mathbb{Z}_{<0}$.

(2) We define the *n-th homology group of the chain complex* precisely the same way as we defined the homology of a topological space, namely we set

$$H_n(C) \;:=\; \ker\{\partial_n \colon C_n \to C_{n-1}\} \,/\, \operatorname{im}\{\partial_{n+1} \colon C_{n+1} \to C_n\}.$$

[594]It is perhaps worth pointing out that we do not assume that the chain groups C_* are free abelian groups.

(3) As before we refer to elements in the kernel of $\partial_n \colon C_n \to C_{n-1}$ as *cycles*. Furthermore we say that a cycle $z \in C_n$ is *null-homologous* if $[z] = 0 \in H_n(C)$.

On some occasions we will need the following generalization:

(4) A *generalized* (*algebraic*) *chain complex* (C_*, ∂_*) is a sequence

$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$$

of maps between abelian groups such that for every $i \in \mathbb{Z}$ we have $\partial_{i-1} \circ \partial_i = 0$. Basically all present and future definitions and results for chain complexes also make sense for generalized chain complexes.

Now we introduce a new notion, namely the notion of a chain map between chain complexes.

Definition. A *chain map* $f \colon C_* \to D_*$ between two chain complexes[595] consists of a family $\{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ of maps such that for each n the following equality holds:[596]

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n.$$

Put differently the maps f_n have the property that the following diagram commutes:

$$\begin{tikzcd}
\ldots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \ldots \\
\ldots \arrow[r, "\partial"] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & \ldots
\end{tikzcd}$$

Remark. On many occasions we will use the elementary observation that the composition of two chain maps $f \colon C_* \to D_*$ and $g \colon D_* \to E_*$ defines a chain map $g \circ f \colon C_* \to E_*$.

Now we can prove the following lemma:

Lemma 29.7. *Let $f \colon C_* \to D_*$ be a chain map between chain complexes C_* and D_*. Then the map*

$$\begin{array}{rcl} f_* \colon H_n(C_*) & \to & H_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is well-defined.

In the following we refer to the map $f_* \colon H_n(C) \to H_n(D)$ as the *induced map on homology*.

---

[595] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let (C_*, ∂_*) be a chain complex" we say simply "let C_* be a chain complex".

[596] More precisely, for any $c \in C$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \to C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

Proof. Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} f_{n-1}(\partial_n c) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} f_n(0) = 0.$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $H_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] = [f_n(c-d)] \underset{\substack{\uparrow \\ \text{since } c,d \text{ are homologous there} \\ \text{exists an } e \in C_{n+1} \text{ with } \partial_{n+1} e = c-d}}{=} [f_n(\partial_{n+1}(e))] \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} [\partial_{n+1}(f_{n+1}(e))] = 0 \in H_n(D).$$

□

We can state the above results more formally. More precisely, we consider the category $\mathcal{C}$ of chain complexes which is given by the objects

$$\mathrm{Ob}(\mathcal{C}) := \text{all chain complexes},$$

and for $C_*, D_* \in \mathrm{Ob}(\mathcal{C})$ the morphisms are given by

$$\mathrm{Mor}(C_*, D_*) := \text{all chain maps from } C_* \text{ to } D_*.$$

The following lemma follows easily from Lemma 29.7 and from the definitions:

Lemma 29.8. *For each $n \in \mathbb{N}_0$ the maps*

$$C_* \mapsto H_n(C_*)$$

and

$$(f\colon C_* \to D_*) \mapsto (f_*\colon H_n(C_*) \to H_n(D_*))$$

define a functor from the category of chain complexes to the category of abelian groups.

29.5. **The functoriality of homology groups.** Let $f\colon X \to Y$ be a map between topological spaces. We already saw on page 736 that for each $n \in \mathbb{N}_0$ the map f induces a map

$$\begin{aligned} f_*\colon C_n(X) &\to C_n(Y) \\ \sum_{i=1}^k a_i \cdot \sigma_i &\mapsto \sum_{i=1}^k a_i \cdot f \circ \sigma_i. \end{aligned}$$

Now we can formulate the following lemma:

Lemma 29.9.

(1) *Let $f\colon X \to Y$ be a map between topological spaces and let $n \in \mathbb{N}_0$. For any $c \in C_n(X)$ we have*

$$f_*(\partial_n c) = \partial_n(f_*(c)) \in C_n(Y).$$

In particular $f_*\colon C_*(X) \to C_*(Y)$ is a chain map.

(2) *The maps*

$$\begin{array}{rcl} \textit{topological space } X & \mapsto & C_*(X) \end{array}$$

and

$$\begin{array}{rcl} (f\colon X \to Y) & \mapsto & (f_*\colon C_*(X) \to C_*(Y)) \end{array}$$

define a functor from the category of topological spaces to the category of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a map between topological spaces and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We have

$$f_*(\partial\sigma) = f_*\left(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\right) = \sum_{j=0}^{n}(-1)^j\cdot f\circ(\sigma\circ i_j^n) = \sum_{j=0}^{n}(-1)^j\cdot\underbrace{(f\circ\sigma)}_{=f_*\sigma}\circ i_j^n = \partial(f_*\sigma).$$

Since the singular n-simplices generate $C_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon C_*(X) \to C_*(Y)$ is a chain map.

(2) The second statement is an almost immediate consequence of the first statement. □

In particular, using Lemma 29.7 we immediately obtain the following lemma.

Lemma 29.10. *Let $f\colon X \to Y$ be a map between topological spaces. Then the map*

$$\begin{array}{rcl} f_*\colon H_n(X) & \to & H_n(Y) \\ \left[\sum_{i=1}^{k} a_i\cdot\sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i\cdot(f\circ\sigma_i)\right] \end{array}$$

is well-defined.

Example. Let $f\colon X \to Y$ be a map between two topological spaces. It follows immediately from the definitions that the following diagram commutes

$$\begin{tikzcd} H_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon_X"'] & & H_0(Y). \arrow[dl, "\epsilon_Y"] \\ & \mathbb{Z} & \end{tikzcd}$$

where the maps ϵ_X and ϵ_Y denote the augmentation maps. In particular, if X and Y are path-connected topological spaces, then it follows easily from Proposition 29.5 that the map $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism. Furthermore, if $X = Y$, i.e. if f is a self-map on a path-connected topological space X, then f_* is in fact the identity on $H_0(X)$.

The combination of Lemmas 29.9 and 29.8 now gives us the following lemma.

Lemma 29.11. *For each $n \in \mathbb{N}_0$ the map*

$$X \mapsto H_n(X)$$

together with the map

$$(f\colon X \to Y) \mapsto \begin{pmatrix} f_*\colon H_n(X) & \to & H_n(Y) \\ \left[\sum_{i=1}^k a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^k a_i \cdot (f \circ \sigma_i)\right] \end{pmatrix}$$

defines a functor from the category of topological spaces to the category of abelian groups.

29.6. **Direct products and direct sums** $(*)$**.** In this short section we recall the definition of the direct product and the direct sum of a family of abelian groups.

Let $G_i, i \in I$ be a family of abelian groups. We define the *direct product* as

$$\prod_{i\in I} G_i := \Big\{\text{all maps } g\colon I \to \bigcup_{i\in I} G_i \text{ such that for all } i \in I \text{ we have } g(i) \in G_i\Big\}.$$

Given such a map we usually write $g_i := g(i)$ for $i \in I$ and we denote such a map by $\{g_i\}_{i\in I}$. The direct product is again a group with respect to the group structure

$$\begin{array}{rcl} \prod_{i\in I} G_i \times \prod_{i\in I} G_i & \to & \prod_{i\in I} G_i \\ \big((a_i)_{i\in I}, (b_i)_{i\in I}\big) & \mapsto & (a_i + b_i)_{i\in I}. \end{array}$$

Furthermore we consider the *direct sum*

$$\bigoplus_{i\in I} G_i := \Big\{(g_i)_{i\in I} \in \prod_{i\in I} G_i \,\Big|\, \text{there exists only finitely many } i \in I \text{ with } g_i \neq e\Big\}.$$

The same way one can also define the direct product respectively the direct sum of rings, algebras, vector spaces, modules and so on.

Example. Let $I = \mathbb{N}$ and for each $i \in I = \mathbb{N}$ we pick $G_i = \mathbb{R}$. Then

$$\prod_{i\in\mathbb{N}} \mathbb{R} = \text{all real-valued sequences } i \mapsto a_i$$

and

$$\bigoplus_{i\in\mathbb{N}} \mathbb{R} = \text{all real-valued sequences with only finitely many non-zero terms.}$$

In general the direct product is "significantly larger" than the direct sum. For example the direct product $\prod_{i\in\mathbb{N}} \mathbb{Z}$ is uncountable while the direct sum $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}$ is countable.[597]

For some element $g_j \in G_j$, following the discussion on page 366, we sometimes denote by g_j also the element in $\bigoplus_{i\in I} G_i$ that is given by $i \mapsto e$ if $i \neq j$ and $j \mapsto g_j$. In particular we

[597] Why is that the case?

can view each G_j as a subgroup of $\bigoplus_{i\in I} G_i$. Furthermore for $i_1,\dots,i_l \in I$ and $g_{i_j} \in G_{i_j}$ we can consider the corresponding element

$$g_{i_1} + \cdots + g_{i_l} \in \bigoplus_{i\in I} G_i.$$

It follows immediately from the definitions that any element of $\bigoplus_{i\in I} G_i$ is of that form.

Finally let $G_i, i \in I$ be a family of abelian groups, let H be an abelian group and let $\varphi_i\colon G_i \to H, i \in I$ be a family of homomorphisms. We refer to

$$\begin{array}{rcl} \bigoplus_{i\in I}\varphi_i \quad : \bigoplus_{i\in I} G_i & \to & H \\ g_{i_1} + \cdots + g_{i_l} & \mapsto & \varphi_{i_1}(g_{i_1}) + \cdots + \varphi_{i_l}(g_{i_l}) \end{array}$$

as the direct sum of the homomorphisms φ_i, $i \in I$.

29.7. **The homology groups of a direct sum.** Now we return to the study of chain complexes. Let C_* and D_* be two chain complexes. The same way as we introduced the direct sum of abelian groups we can also define the direct sum of chain complexes. The following lemma says that homology commutes with direct sums and direct products. The proof is a standard exercise in homological algebra, which oddly enough nobody ever feels like doing, see [Weib, p. 5].

Lemma 29.12. *If $\{C_a\}_{a\in A}$ is a family of chain complexes, then the obvious inclusion maps $C_a \to \bigoplus_{a\in A} C_a$ and $C_a \to \prod_{a\in A} C_a$ induce isomorphisms*

$$\bigoplus_{a\in A} H_n(C_a) \xrightarrow{\cong} H_n\Big(\bigoplus_{a\in A} C_a\Big) \quad \text{and} \quad \prod_{a\in A} H_n(C_a) \xrightarrow{\cong} H_n\Big(\prod_{a\in A} C_a\Big).$$

Now we can formulate and prove the following lemma.

Lemma 29.13. *Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. Then the map*

$$\bigoplus_{a\in A} i_a\colon \quad \bigoplus_{a\in A} C_*(X_a) \to C_*(X)$$

is an isomorphism of chain complexes. In particular for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \quad \bigoplus_{a\in A} H_n(X_a) \to H_n(X)$$

is an isomorphism.

The lemma says in particular that the homology groups of a topological space X are the direct sum of the homology groups of the path-components of X. Put differently, this shows that in most cases it suffices to consider the homology groups of path-connected topological spaces.

Example. One of our goals is to calculate the homology groups of the spheres. Now we can calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, by Lemmas 29.6 and 29.13 we have

$$H_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

We will later reduce the computation of the homology groups of all other spheres to this calculation.

Proof. Since the standard simplex Δ^n is path-connected, the image of a map $\sigma\colon \Delta^n \to X$ lies in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{singular } n\text{-simplices in } X_a \quad = \quad \text{singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{aligned} \bigoplus_{a\in A} i_a \colon \bigoplus_{a\in A} C_n(X_a) &\to C_n(X) \\ \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) &\mapsto \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) \end{aligned}$$

is an isomorphism of free abelian groups. From the definitions it follows immediately that this a chain map. The second statement of the lemma now follows from Lemma 29.12. □

The following corollary is an immediate consequence of Proposition 29.5 and Lemma 29.12.

Corollary 29.14. *Let X be a topological space. As on page 127 we denote by $\pi_0(X)$ the set of path-components of X. Then*

$$H_0(X) \cong \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}$, then

$$H_0(X) \cong \mathbb{Z}^n.$$

Exercises for Chapter 29.

Exercise 29.1.

(a) Let X be a topological space and let $x \in X$. For each $n \in \mathbb{N}_0$ denote by σ_n the singular n-simplex $\sigma_n\colon \Delta^n \to X$ which is defined by $\sigma_n(P) = x$ for all $P \in \Delta^n$. Determine $\partial\sigma_n \in C_{n-1}(X)$.

(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. Show that if σ is a constant map, then σ is null-homologous.

Exercise 29.2. Prove Lemma 29.6.

Exercise 29.3.

(a) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplices

$$\begin{array}{rcl} \sigma_1\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(\frac{1}{2}t, 1-\frac{1}{2}t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \sigma_2\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & (\frac{1}{2}+\frac{1}{2}t, \frac{1}{2}-\frac{1}{2}t). \end{array}$$

Show that σ and $\sigma_1 + \sigma_2$ are homologous.

(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplex

$$\begin{array}{rcl} \overline{\sigma}\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(1-t, t). \end{array}$$

Show that $\sigma + \overline{\sigma}$ is a cycle and that this cycle is null-homologous.

We refer to Figure 472 for an illustration of the definitions.

Exercise 29.4.

(a) Try to find "interesting" elements in the following homology groups:
  (i) $H_2(S^2)$,
  (ii) H_2(surface Σ of genus 2),
  (iii) H_2(Klein bottle K) and
  (iv) $H_2(\mathbb{R}\mathrm{P}^2)$.
  More precisely, try to find singular 2-cycles such that the images of the singular 2-simplices cover the whole surface and such that the images of the singular 2-simplices overlap only in "1-dimensional" subsets.

(b) Make a guess for which surfaces you expect H_2 to be zero and for which it is non-zero.

Exercise 29.5. Provide an example of a chain complex (C_*, ∂_*) such that all C_* are finitely generated free abelian groups and such that $H_i(C_*) = 0$ for all even $i \in \mathbb{N}_0$ and such that $H_i(C_*) = \mathbb{Z}_2$ for all odd $i \in \mathbb{N}_0$.

Exercise 29.6.

(a) Determine the isomorphism types of the homology groups of the chain complex

$$0 \to C_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}} C_0 = \mathbb{Z}^2 \to 0.$$

(b) Determine the isomorphism types of the homology groups of the chain complex

$$0 \to C_2 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}} C_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -2 & 1 \end{pmatrix}} C_0 = \mathbb{Z} \to 0.$$

(Why is this a chain complex?)

30. Homology and homotopies

It was not overly difficult to introduce the homology groups of a topological space. We also proved several straightforward properties of homology groups. But so far we are unable to calculate any interesting homology group. In this section we will show in particular that homotopy equivalent topological spaces have isomorphic homology groups. That does not yet lead to the calculation of non-trivial homology groups, but this result will play an important role in the calculation of the homology groups of spheres later on.

30.1. **Chain homotopies.** In this section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups.

Definition. Let $f, g\colon C_* \to D_*$ be chain maps between chain complexes C_* and D_*. A *chain homotopy* $P = \{P_n\}_{n\in\mathbb{N}_0}$ *between* f *and* g consists of family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are *chain homotopic* and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:

$$\begin{tikzcd}
{} \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_*,g_*"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_*,g_*"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_*,g_*"] \arrow[dl, "P_{n-1}"'] & {} \\
{} \arrow[r, "\partial_{n+2}"] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & {}
\end{tikzcd}$$

The following basic lemma will be proved in Exercise 30.1

Lemma 30.1. *Let C_* and D_* be two chain complexes.*

(1) *Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.*
(2) *Let $f, g\colon C_* \to D_*$ be chain maps, and let $e\colon B_* \to C_*$ and $h\colon D_* \to E_*$ be two other chain maps. Then*

$$f \simeq g \quad\Longrightarrow\quad f \circ e \simeq g \circ e \quad \text{and} \quad h \circ f \simeq h \circ g.$$

Now we can formulate the following lemma.

Lemma 30.2. *Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. Then*

$$f \simeq g \quad\Longrightarrow\quad f_* = g_*\colon H_n(C) \to H_n(D) \text{ for all } n \in \mathbb{N}_0.$$

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$f(z) - g(z) \;=\; (f-g)(z) \;=\; (\partial_{n+1}P_n + P_{n-1}\partial_n)(z) \;=\; (\partial_{n+1}P_n)(z) + P_{n-1}\underbrace{\partial_n z}_{=0} \;=\; \partial P_n z,$$

i.e. $f(z)$ and $g(z)$ are homologous. □

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon C_* \to D_*$ is called a *chain homotopy equivalence between C_* and D_**, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g\circ f \simeq \mathrm{id}_{C_*}$ and $f\circ g \simeq \mathrm{id}_{D_*}$.
(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are *chain homotopy equivalent* and we write $C_* \simeq D_*$.

Remark. It is again straightforward to show that "homotopy equivalence" of chain complexes is indeed an equivalence relation.

For later we record the following straightforward corollary to Lemma 30.2.

Corollary 30.3. *If $f\colon C_* \to D_*$ is a chain homotopy equivalence between chain complexes, then the induced maps $f_*\colon H_n(C) \to H_n(D)$ are isomorphisms.*

We also record the following useful little lemma.

Lemma 30.4. $(*)$ *Let (C_*, ∂_*) be a chain complex. If there exist maps $P_n\colon C_n \to C_{n+1}$ such that for every $n \in \mathbb{N}_0$ we have*

$$\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n \;=\; \mathrm{id}_{C_n},$$

then all homology groups of (C_*, ∂_*) vanish.

Proof $(*)$. Let D_* be the zero chain complex and for each $n \in \mathbb{N}_0$ let $f_n\colon C_n \to D_n = 0$ be the trivial map. Since the homology groups of D_* it suffices, by Corollary 30.3, to show that f_* is a chain homotopy equivalence. For each $n \in \mathbb{N}_0$ let $g_n\colon D_n \to C_n$ be the trivial map. Evidently $f_n \circ g_n = \mathrm{id}_{D_*}$. Thus it remains to show that $g_*\circ f_* \simeq \mathrm{id}_{C_*}$. In other words we need to show that $0 \simeq \mathrm{id}_{C_*}$. But that is exactly what the maps P_n do. □

30.2. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. As we will see shortly, this is indeed the case. For the reader's convenience we recall that if $f, g\colon X \to Y$ are two maps between topological spaces, then a *homotopy between the maps f and g* is a map

$$F\colon X \times [0,1] \to Y,$$

such that

$$F(x,0) \;=\; f(x) \text{ and } F(x,1) \;=\; g(x) \text{ for all } x \in X.$$

If there exists a homotopy between f and g, then we say that f and g are *homotopic* and we write $f \simeq g$.

Proposition 30.5. *Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two maps. If f and g are homotopic, then the following hold:*

(1) *the induced maps f_* and g_* from $C_*(X)$ to $C_*(Y)$ are chain homotopic,*
(2) *for each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon H_n(X) \to H_n(Y)$.*

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g\colon X \to Y \quad \Longrightarrow \quad f_* \simeq g_*\colon C_*(X) \to C_*(Y) \quad \Longrightarrow \quad f_* = g_*\colon H_n(X) \to H_n(Y).$$

We start out with a discussion of the key idea behind the proof of Proposition 30.5. Afterwards we will fill in the details details. So let $f, g\colon X \to Y$ be two maps between topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in C_{n+1}(Y)$ and $P(\partial\sigma) \in C_n(Y)$ such that

$$\partial P(\sigma) \;=\; -P(\partial\sigma) \,+\, f_*(\sigma) \,-\, g_*(\sigma).$$

Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + \begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \underbrace{\Delta^n \times \{0\}}_{=\Delta^n}\end{matrix} - \begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \underbrace{\Delta^n \times \{1\}}_{=\Delta^n}\end{matrix}.$$

To get an idea we consider Figure 477. We make the following trivial observation: we have

$$\partial(\Delta^n \times [0,1]) \;=\; \partial\Delta^n \times [0,1] \;\cup\; \Delta^n \times \{0\} \;\cup\; \Delta^n \times \{1\}.$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k\colon C_k(X) \to C_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu\colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]})\colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In Figure 478 we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.

To generalize this observation we have to introduce some more notation.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \dots, n$ we write

$$v_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{0\} \;\in\; \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{1\} \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

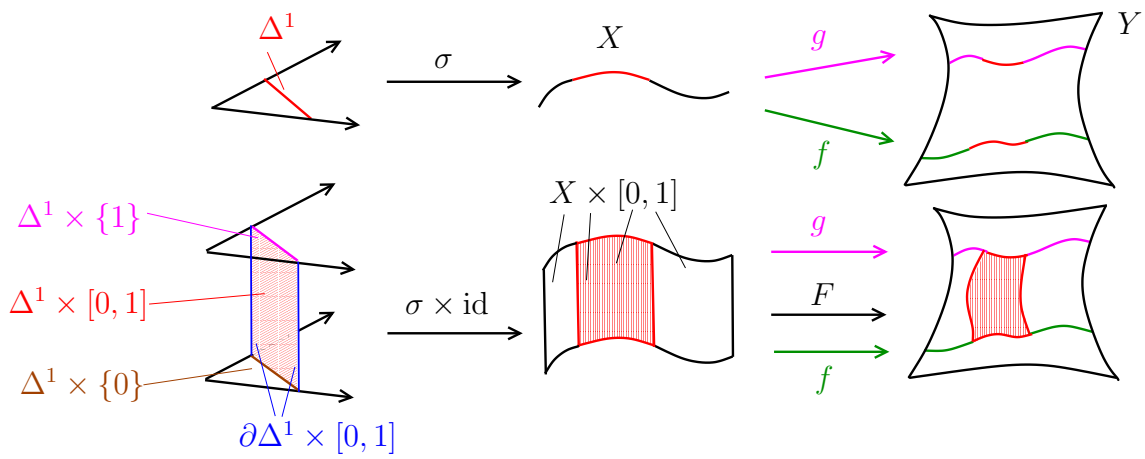


FIGURE 477.

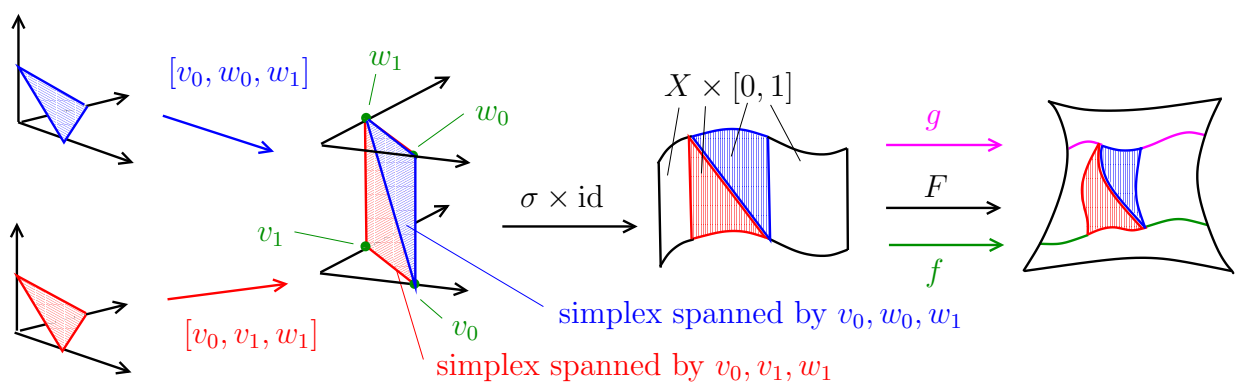


FIGURE 478.

The points $v_0, \ldots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $v_0, \ldots, v_n$ span the simplex $\Delta^n \times 1$. Furthermore for each i the points $v_0, \ldots, v_i, w_i, \ldots, w_n$ span an $(n+1)$-simplex that is a subset of $\Delta^n \times [0,1]$.[598]

(2) For $a_1, \ldots, a_{k+1} \in \Delta^n \times [0,1]$ we consider, as on page 736, the map

$$\begin{aligned} [a_0, \ldots, a_k] \colon \Delta^k &\to \Delta^n \times [0,1] \\ (\lambda_0, \ldots, \lambda_k) &\mapsto \sum_{j=0}^{k} \lambda_j a_j. \end{aligned}$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \ldots, a_k$ in $\Delta^n \times [0,1]$.

[598]It is a straightforward exercise to show that the union of these $(n+1)$-dimensional simplices is precisely $\Delta^n \times [0,1]$. We will not make use of this fact.

(3) We define

$$\Omega_n \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0,\dots,v_j,w_j,\dots,w_n]}_{\Delta^{n+1}\to\Delta^n\times[0,1]} \;\in\; C_{n+1}(\Delta^n\times[0,1]).$$

In the following lemma we compute the boundary of $\Omega_n \in C_{n+1}(\Delta^n \times [0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

Lemma 30.6. *Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write*

$$\begin{array}{rcl} \eta_t\colon \Delta^n & \to & \Delta^n\times[0,1] \\ x & \mapsto & (x,t). \end{array}$$

With this notation we have

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times \mathrm{id}_{[0,1]})_*\big(\underbrace{\Omega_{n-1}}_{\in C_n(\Delta^{n-1}\times[0,1])}\big) \;=\; \eta_1-\eta_0 \;\in\; C_n(\Delta^n\times[0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{aligned} \partial\Omega_n &= \partial\Big(\sum_{j=0}^{n}(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,w_n]\Big) \\ &= \sum_{k=0}^{n+1}(-1)^k\cdot\sum_{j=0}^{n}(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,w_n]\circ i_k^{n+1}. \end{aligned}$$

It is straightforward to see that for any $k \in \{0,\dots,n+1\}$ we have[599]

$$[v_0,\dots,v_j,w_j,\dots,w_n]\circ i_k^{n+1} \;=\; \begin{cases} [v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n], & \text{if } k\le j, \\ [v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n], & \text{if } k>j. \end{cases}$$

Thus it follows that[600]

$$\begin{aligned} \partial\Omega_n \;=\; & \sum_{0\le k\le j\le n} (-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n] \\ & + \sum_{n+1\ge k>j\ge 0}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n]. \end{aligned}$$

[599]The first case, $k \le j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_h. If $h \ge k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_{h+1}. If $k \le j$ this corresponds just to the map $[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]$.
The second case, $k > j$ is proved precisely the same way.

[600]Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Now we split the first sum into the cases $k = j$ and $k < j$ and we split the second sum into the cases $k = j + 1$ and $k > j + 1$. This leads us to the following equality:

$$\begin{aligned}\partial\Omega_n \quad = \quad & \sum_{k=0}^{n}(-1)^{2k}\cdot[v_0,\dots,v_{k-1},w_k,\dots,w_n] \\ & +\sum_{k=1}^{n+1}(-1)^{2k+1}\cdot[v_0,\dots,v_{k-1},w_k,\dots,w_n] \\ & + \sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n] \\ & + \sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n].\end{aligned}$$

Next we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n + 1$. Thus we obtain that

$$(*)\qquad \begin{aligned}\partial\Omega_n \quad = \quad & [w_0,\dots,w_n]-[v_0,\dots,v_n] \\ & + \sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n] \\ & + \sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_k},\dots,w_n].\end{aligned}$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[601]

$$\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \underset{\substack{\uparrow \\ \text{definition of } (i_j^n\times\mathrm{id}_{[0,1]})_* \text{ and substitution } r=j}}{=} \sum_{r=0}^{n}\sum_{s=0}^{n-1}(-1)^r\cdot(-1)^s\cdot(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\dots,v_s,w_s,\dots,w_{n-1}].$$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n \to \Delta^n \times [0, 1]$:

$$(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\dots,v_s,w_s,\dots,w_{n-1}] \quad = \quad \begin{cases}[v_0,\dots,\widehat{v_r},\dots,v_{s+1},w_{s+1},\dots,w_n], & \text{if } r\le s,\\ [v_0,\dots,v_s,w_s,\dots,\widehat{w_{r+1}},\dots,w_n], & \text{if } r>s.\end{cases}$$

Thus we see that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \quad = \quad & \sum_{0\le r\le s\le n-1}(-1)^s\cdot(-1)^r\cdot[v_0,\dots,\widehat{v_r},\dots,v_{s+1},w_{s+1},\dots,w_n] \\ & + \sum_{n\ge r>s\ge 0}(-1)^s\cdot(-1)^r\cdot[v_0,\dots,v_s,w_s,\dots,\widehat{w_{r+1}},\dots,w_n].\end{aligned}$$

[601] Here $v_0,\dots,v_{n-1},w_0,\dots,w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1}\times[0,1]$.

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;&=\; \sum_{0\le k\le j-1\le n-1}(-1)^{j-1}\cdot(-1)^k\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]\\ &\quad+\sum_{n\ge k-1>j\ge 0}(-1)^j\cdot(-1)^{k-1}\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_k},\dots,w_n].\end{aligned}$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. □

The following lemma implies in particular the first statement of Proposition 30.5. The second statement of Proposition 30.5 regarding the induced maps on homology follows immediately from the first statement together with Lemma 30.2. In other words, the following lemma completes the proof of Proposition 30.5.

Lemma 30.7. *Let X be a topological space and let $F\colon X\times[0,1]\to X$ be a homotopy from a map f to a map g. Given any $n\in\mathbb{N}_0$ we define*

$$P_n\colon C_n(X) \;\to\; C_{n+1}(Y)$$

to be the map that is induced by

$$(\sigma\colon\Delta^n\to X) \;\mapsto\; (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\Omega_n).$$

Then for any $n\in\mathbb{N}_0$ we have

$$\partial P_n + P_{n-1}\partial \;=\; g_* - f_*$$

as maps from $C_n(X)$ to $C_n(Y)$.

Proof. Let X be a topological space and let $F\colon X\times[0,1]\to X$ be a homotopy from a map f to a map g. Furthermore let $\sigma\colon\Delta^n\to X$ be a singular k-simplex. Then

$$\begin{aligned}(\partial P_n + P_{n-1}\partial)(\sigma) \;&=\; \partial(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) \;=\; \partial(P_n(\sigma)) + \sum_{j=0}^{n}(-1)^j\cdot P_{n-1}(\sigma\circ i_j^n)\\ &=\; \partial((F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n}(-1)^j\cdot(F\circ(\sigma\circ_j^n\times\mathrm{id}_{[0,1]}))_*(\Omega_{n-1})\\ &=\; (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*\Big(\partial(\Omega_n)) + \sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big)\\ &\quad\uparrow\\ &\quad\text{by Lemma 29.9 (1) and since } (\sigma\times\mathrm{id})\circ(i_j^n\times\mathrm{id}) = (\sigma\circ i_j^n)\times\mathrm{id}\\ &=\; (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\eta_1-\eta_0) \;=\; f\circ\sigma - g\circ\sigma \;=\; g_*(\sigma) - f_*(\sigma).\\ &\quad\uparrow \qquad\qquad\qquad\qquad \uparrow\\ &\quad\text{Lemma 30.6} \qquad\qquad\qquad \text{since } F \text{ is a homotopy from } f \text{ to } g\end{aligned}$$

□

We recall the following definition from page 307.

Definition. Let X and Y be topological spaces.

(1) A map $f\colon X \to Y$ is a *homotopy equivalence between X and Y*, if there exists a homotopy inverse to f, i.e. a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that the topological spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.
(3) We say X is *contractible*, if X is homotopy equivalent to $Y = \{*\}$, i.e. if X is homotopy equivalent to a topological space which consists of precisely one point.

Now we obtain the following corollary to Proposition 30.5. This corollary mirrors the fact, proved in Propositions 9.12 (1) and 28.7, that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 30.8. *Let X and be Y topological spaces and let $n \in \mathbb{N}_0$.*

(1) *If $f\colon X \to Y$ is a homotopy equivalence (e.g. a homeomorphism), then the induced map $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*
(2) *If X and Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$.*
(3) *If X is contractible, then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \geq 1$.*
(4) *If X is a deformation retract of Y, then the inclusion induced map $H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof.

(1) Let $f\colon X \to Y$ be a homotopy equivalence and let $g\colon Y \to X$ be a homotopy inverse of f, i.e. g is a map such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. It follows from Proposition 30.5 and the functoriality of homology that $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_{H_n(Y)}$ and similarly $g_* \circ f_* = \mathrm{id}_{H_n(X)}$. But that means precisely that $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.
(2) This statement follows immediately from (1).
(3) This statement follows immediately from (2) and Lemma 29.6.
(4) This statement follows immediately from (1) and Lemma 9.10. □

Examples.

(1) On page 308 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from Corollary 30.8, that

$$H_i(\text{convex subset of } \mathbb{R}^n) \;\cong\; H_i(B^n) \;\cong\; H_i(\overline{B}^n) \;\cong\; H_i(\mathbb{R}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

(2) In Exercise 30.2 we will use Corollary 30.8 to show that if A is a compact convex subset of $\mathbb{R}^n$ and if $x \in A$, then the inclusion $\mathbb{R}^n \backslash A \to \mathbb{R}^n \backslash \{x\}$ induces isomorphisms $H_i(\mathbb{R}^n \setminus A) \to H_i(\mathbb{R}^n \setminus \{x\})$ for all i.

Remark. Given a map $f\colon X \to Y$ between topological spaces we denote by $[f]$ the homotopy equivalence class of f, and we denote by $[X, Y]$ the set of equivalence classes of maps from X

to Y. As we pointed out on page 305,

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{H}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

together with the composition

$$\begin{array}{rcl} \mathrm{Mor}(X,Y)\times\mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g\circ f] \end{array}$$

defines a category $\mathcal{H}$, called the *homotopy category*. Proposition 30.5 thus says that for each n the maps

$$X \mapsto H_n(X)$$

and

$$[f] \mapsto f_*\colon H_n(X)\to H_n(Y)$$

define a functor from the category $\mathcal{H}$ to the category $\mathcal{A}$ of abelian groups. In particular we now have the following commutative diagram of categories and functors:

$$\begin{tikzcd} \text{category } \mathcal{T} \text{ of topological spaces} \arrow[rr, "X\mapsto X"] \arrow[dr, "X\mapsto H_n(X)"'] & & \text{homotopy category } \mathcal{H} \arrow[dl, "X\mapsto H_n(X)"] \\ & \text{category of abelian groups.} & \end{tikzcd}$$

Exercises for Chapter 30.

Exercise 30.1. Prove Lemma 30.1.

Exercise 30.2. Let A be a compact convex subset of $\mathbb{R}^n$ and let $x\in A$. Show that the inclusion $\mathbb{R}^n\setminus A\to\mathbb{R}^n\setminus\{x\}$ induces isomorphisms $H_i(\mathbb{R}^n\setminus A)\to H_i(\mathbb{R}^n\setminus\{x\})$ for all i.

31. Long exact sequences and the homology of quotient spaces

In this section, given a topological space X and a "good" subset A we will relate the homology groups of X, A and the quotient space X/A via a long exact sequence. We will use this long exact sequence to fulfill our dream of calculating the homology groups of all spheres. Unfortunately it will take a very long time to prove the existence of this long exact sequence.

31.1. **Reduced homology groups.** In this section we introduce the reduced homology groups of a topological space. These are a minute but at times surprisingly useful variation on the usual homology groups.

Definition. Let X be a a topological space. By Lemma 29.4 (1) we know that the following is a generalized[602] chain complex

$$\begin{array}{ccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 \\ & & & & & & \downarrow & & \downarrow \\ \dots & \xrightarrow{\partial_3} & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} \to 0. \\ & & & & & & \sum_{i=1}^{k} n_i\sigma_i & \mapsto & \sum_{i=1}^{k} n_i \end{array}$$

We introduce the following three definitions.

(1) We refer to this chain complex as the *augmented chain complex* $\widetilde{C}_*(X)$ *of* X.
(2) We define the *k-th reduced homology group* $\widetilde{H}_k(X)$ *of* X to be the k-th homology group of the augmented chain complex.
(3) If $f\colon X \to Y$ is a map between topological spaces we denote by $f_*\colon \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$ the map that is the usual map on $\widetilde{C}_k(X) = C_k(X)$ for $k \geq 0$ and that is the identity on the final $\mathbb{Z}$-term.

The following elementary lemma summarizes a few properties of reduced homology groups.

Lemma 31.1.

(1) *If $f\colon X \to Y$ is a map between two topological spaces, then the induced maps $f_*\colon \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$ are chain maps. In particular we see that we get well-defined induced maps $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$.*
(2) *For any $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} X & \mapsto & \widetilde{H}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)) \end{array}$$

define a functor from the category of topological spaces to the category of abelian groups.

[602] Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

(3) *For each $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} \widetilde{H}_n(X) & \to & H_n(X) \\ [\sigma] & \to & [\sigma] \end{array}$$

are well-defined. Furthermore they define a natural transformation from the functor $X \mapsto \widetilde{H}_n(X)$ to the functor $X \mapsto H_n(X)$.

(4) *For every non-empty topological space X the following statements hold:*
   (a) *Given any point $P \in X$ the map*

$$\begin{array}{rcl} \widetilde{H}_0(X) \oplus \mathbb{Z} & \to & H_0(X) \\ \sigma \oplus n & \mapsto & \sigma + [n \cdot P] \end{array}$$

   is an isomorphism.
   (b) *Given any $n \geq 1$ the natural map $\widetilde{H}_n(X) \to H_n(X)$ is an isomorphism.*

(5) *If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{H}_0(X) \cong \mathbb{Z}^{n-1}$.*

(6) *The statements of Corollary 30.8* (1), (2) *and* (4) *also hold the same way for reduced homology groups and the corresponding induced maps. For example, if X is a deformation retract of some topological space Y, then the inclusion induced maps $\widetilde{H}_k(X) \to \widetilde{H}_k(Y)$ are isomorphisms for all $k \in \mathbb{N}_0$.*

(7) *If X is a contractible topological space, then $\widetilde{H}_k(X) = 0$ for all $k \in \mathbb{N}_0$.*

The lemma shows in particular that reduced homology groups are almost the same as the usual homology groups. So it might sound slightly silly to introduce them. But we will see that on several occasions reduced homology groups make it easier to formulate some statements.

Remark.

(1) If X is a non-empty topological space, then it follows from Lemma 31.1 (4) that we have isomorphisms

$$H_n(X) \;\cong\; \begin{cases} \widetilde{H}_0(X) \oplus \mathbb{Z}, & \text{if } n = 0, \\ \widetilde{H}_n(X), & \text{if } n \neq 0. \end{cases}$$

Furthermore, by Lemma 31.1 (3), the isomorphism for $n \geq 1$ is natural. The isomorphism for $n = 0$ depends on the choice of a point.

(2) By Lemma 31.1 (7) we know for any $k \in \mathbb{N}_0$ and any $n \in \mathbb{N}_0$ we have

$$\widetilde{H}_n(\Delta^k) \;=\; \widetilde{H}_n(B^k) \;=\; \widetilde{H}_n(\overline{B}^k) \;=\; \widetilde{H}_n(\mathbb{R}^k) \;=\; 0.$$

(3) We will often use Lemma 31.1 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

Proof.

(1) Let $f\colon X \to Y$ be a map between two topological spaces. It is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{tikzcd} C_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon"'] & & C_0(Y) \arrow[dl, "\epsilon"] \\ & \mathbb{Z}. & \end{tikzcd}$$

It follows almost immediately from the fact that this diagram commutes that the induced maps $f_*\colon \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$ are chain maps. Furthermore we obtain from Lemma 29.7 that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions.

(4) Let X be a non-empty topological space. By the definition of $\widetilde{H}_n(X)$ we only have to deal with the case $n = 0$. We pick a point $P \in X$. We denote by $i\colon \mathbb{Z} \to H_0(X)$ the map that is given by $i(n) = [n \cdot P]$. Evidently we have $\epsilon \circ i = \mathrm{id}_{\mathbb{Z}}$. It follows easily from this observation and from $\widetilde{H}_0(X) = \ker(\epsilon)$ that the map[603]

$$\begin{array}{rcl} \widetilde{H}_0(X) \oplus \mathbb{Z} & \to & H_0(X) \\ c \oplus n & \mapsto & c + i(n) \end{array}$$

is an isomorphism.

(5) This statement follows immediately from (1), Corollary 29.14 and Lemma 11.8 (4).

(6) This statement is proved exactly the same way as Corollary 30.8.

(7) This final statement follows immediately from (5) and (6). □

31.2. **Long exact sequences.** We need two more definitions before we can formulate one of the key theorems of homology theory.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is *good* if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.

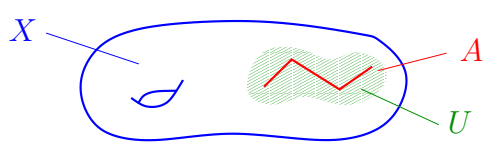


FIGURE 479. Illustration of a good subset.

Remark. Our definition of a good subset is the same as on [Hat, p. 114], except that we do not demand that a good subset is non-empty.

Examples.

[603] Why is this map an epimorphism?

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 387.
(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. It is clear that S^{n-1} is a closed subset. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction is given by the map
$$\begin{array}{rcl} (\overline{B}^n \setminus \{0\}) \times [0,1] & \to & \overline{B}^n \setminus \{0\} \\ (x,t) & \mapsto & \left(1 - t\frac{1-\|x\|}{\|x\|}\right) \cdot x. \end{array}$$
(3) We can generalize the first example: It follows easily from Proposition 3.21 (3) and the Collar Neighborhood Theorem 10.10 that every boundary component of a manifold is good.
(4) Finally it follows from Lemma 26.10 (2) and Proposition 26.11 (8a) that every subcomplex A of a CW-complex X is good.

The following definition might be familiar from earlier algebra courses.

Definition. A sequence of group homomorphisms
$$\dots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \dots$$
is called *exact*, if for each n we have
$$\ker(f_n) \;=\; \operatorname{im}(f_{n+1}).$$

Remark.
(1) An exact sequence satisfies in particular that $f_n \circ f_{n+1} = 0$ for all $n \in \mathbb{N}_0$.
(2) We have the following equivalences of statements:
$$\begin{array}{rcl} \text{the sequence } 0 \to A \xrightarrow{f} B \quad \text{is exact} & \Longleftrightarrow & f \text{ is injective} \\ \text{the sequence } \quad A \xrightarrow{f} B \to 0 \text{ is exact} & \Longleftrightarrow & f \text{ is surjective} \\ \text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} & \Longleftrightarrow & f \text{ is an isomorphism} \end{array}$$
(3) Sometimes we also use the following observations:
$$\begin{array}{rcl} \text{the sequence } 0 \to A \to B \xrightarrow{f} C \text{ is exact} & \Longrightarrow & A \cong \ker(f) \\ \text{the sequence } A \xrightarrow{f} B \to C \to 0 \text{ is exact} & \Longrightarrow & C \cong \operatorname{coker}(f) \end{array}$$

Now we can formulate the following theorem.

Theorem 31.2. *Let X be a topological space and let $A \subset X$ be a non-empty good subset. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection map. Then for each n there exists a homomorphism $\partial\colon \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A)$ such that the sequence*
$$\begin{array}{rcccccc} & & & \dots & \xrightarrow{p_*} & \widetilde{H}_{n+1}(X/A) & \xrightarrow{\partial} \\ \to & \widetilde{H}_n(A) & \xrightarrow{i_*} & \widetilde{H}_n(X) & \xrightarrow{p_*} & \widetilde{H}_n(X/A) & \xrightarrow{\partial} \\ \to & \widetilde{H}_{n-1}(A) & \xrightarrow{i_*} & \dots & & & \end{array}$$

is exact.

It will take a while to prove Theorem 31.2. Therefore we will first consider several applications which will convince us that proving Theorem 31.2 is worth the effort.

31.3. **The homology groups of spheres.** Using Theorem 31.2 we can now determine the homology groups of spheres and we get a nice and satisfactory result.

Proposition 31.3. *For each n we have*

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Put differently, the combination of Lemma 31.1 and Proposition 31.3 says that for any $n > 0$ we have

$$H_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

Furthermore we have

$$H_k(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

By Proposition 3.18 (2) we know that the for $k \neq l$ the spheres S^k and S^l are not diffeomorphic. The following corollary is a serious upgrade of this fact.

Corollary 31.4. *For $k \neq l$ the spheres S^k and S^l are not homeomorphic.*

So we have finally managed to distinguish spheres of different dimensions. Our initial attempt had been to use higher homotopy groups to prove that result. The definition of higher homotopy groups is arguably much more natural than the definition of homology groups, but the latter groups are significantly easier to compute.

Proof of Proposition 31.3 assuming Theorem 31.2. We prove the proposition by induction on n. If $n = 0$, then $S^n = S^0$ consists of precisely two points, and the proposition follows from Lemmas 29.6, 29.13 and 31.1.

Now we suppose that the statement of the proposition holds for $n - 1$. We want to determine the reduced homology of S^n. On page 99 we remarked that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. We had just seen on page 764 that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply Theorem 31.2 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{rcrcrcl} \dots & \longrightarrow & \widetilde{H}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{H}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} & \\ \rightarrow \quad \widetilde{H}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{H}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} & \\ \rightarrow \quad \widetilde{H}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & & \end{array}$$

From the induction hypothesis, from the homeomorphism $S^n \cong \overline{B}^n/S^{n-1}$ and from the fact that $\widetilde{H}_i(\overline{B}^n) = 0$ for all i (obtained on page 762) we now see that the above long exact

sequence is of the following form:

$$\begin{array}{rcccccc} & \dots & \longrightarrow & 0 & \xrightarrow{p_*} & \widetilde{H}_{k+1}(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{H}_k(S^{n-1}) & \xrightarrow{i_*} & 0 & \xrightarrow{p_*} & \widetilde{H}_k(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{H}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & \end{array}$$

In particular we obtain for each k an exact sequence

$$0 \;\to\; \widetilde{H}_{k+1}(S^n) \;\xrightarrow{\partial}\; \widetilde{H}_k(S^{n-1}) \;\to\; 0.$$

But this means that the maps ∂ are isomorphisms, and using our induction hypothesis we obtain the desired result that

$$\widetilde{H}_k(S^n) \;=\; \widetilde{H}_{k-1}(S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1, \text{ i.e. if } k = n \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

□

In Lemma 1.90 we had used elementary arguments to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Furthermore, in Lemma 9.13 we had used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 31.5. *For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.*

Proof. Clearly we only have to deal with the case $k, l \geq 1$. Let $k, l \in \mathbb{N}$ and suppose there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\begin{array}{ccccccccc} & & \text{Corollary 30.8} & & & & \text{Corollary 30.8} & & \\ & & \downarrow & & & & \downarrow & & \\ \mathbb{Z} & \cong & \widetilde{H}_{k-1}(S^{k-1}) & \cong & \widetilde{H}_{k-1}(\mathbb{R}^k \setminus \{0\}) & \cong & \widetilde{H}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) & \cong & \widetilde{H}_{k-1}(S^{l-1}). \\ & \uparrow & & & & \uparrow & & & \\ & \text{Proposition 31.3} & & & & \text{since } f \text{ is a homeomorphism} & & & \end{array}$$

So we have $\widetilde{H}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by Proposition 31.3 that is only possible if $k = l$. □

In Lemma 6.2 we had used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with Proposition 31.3, now gives us the following proposition.

Proposition 31.6. *For any $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.*

Remark. I found the following amusing description of Proposition 31.6 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stay there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep."

Proof. Let us suppose there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram

$$\begin{tikzcd} & \widetilde{H}_{n-1}(\overline{B}^n) \arrow[dr, "r_*"] & \\ \widetilde{H}_{n-1}(S^{n-1}) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*"] & & \widetilde{H}_{n-1}(S^{n-1}). \end{tikzcd}$$

This diagram commutes since it follows from the functoriality of the homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{H}_{n-1}(S^{n-1})$ which, by Proposition 31.3, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism, but the upper map factorizes through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. $\square$

Theorem 31.7. (Brouwer[604] Fixed Point Theorem) *Every map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.*[605]

Proof. Suppose there exists a map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. Since f admits no fixed points there exists for each x precisely one ray which starts at $f(x)$ and which goes through x. We consider the following map which is illustrated in Figure 480:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{array}{l}\text{the intersection point of } S^{n-1} \text{ with the} \\ \text{uniquely determined ray from } f(x) \text{ to } x.\end{array} \end{array}$$

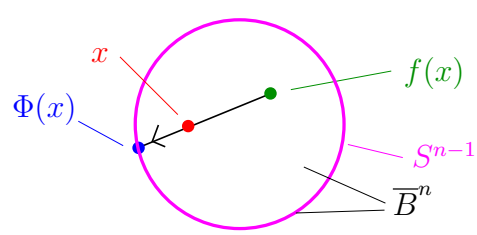


FIGURE 480.

It is elementary to show that Φ is continuous.[606] It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from

[604]Luitzen Brouwer (1881-1966) was a Dutch mathematician.

[605]How can one prove the statement for $n = 1$ using elementary methods from real analysis?

[606]Here is the precise argument: given $x \in \overline{B}^n$ the ray r_x emanating from x and through $f(x)$ is given by $r + x(t) = x + t(f(x) - x)$ for $t \in \mathbb{R}_{\geq 0}$. For any x there exists a unique $t_x \geq 0$ with $\|r_x(t_x)\| = 1$, and $\Phi(x) = r_x(t_x)$. It suffices to show that $x \mapsto t_x$ is continuous. One can determine t_x through the

$\overline{B}^n$ to S^{n-1}. But according to Proposition 31.6 this not possible. Thus we have obtained a contradiction. □

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 31.8. (Perron-Frobenius Theorem) *Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.*

Remark. There are more precise statements one can prove about matrices with positive entries we refer to [Meyr, Section 8.2] and also to

`https://en.wikipedia.org/wiki/Perron-Frobenius_theorem`

for details. Supposedly the Perron-Frobenius Theorem and some of the more refined results mentioned in [Meyr, Section 8.2] play a major role in Google's algorithm, see e.g.

`http://www.math.pku.edu.cn/teachers/yaoy/Fall2011/lecture07.pdf`

for more information.

Proof. We consider the convex set

$$T \ := \ \big\{(t_1, \dots, t_n) \in [0,1]^n \,\big|\, \sum_{i=1}^n t_i = 1\big\}.$$

This set is just the standard $(n-1)$-simplex endowed with a different name. We saw in Lemma 29.1 that it is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 31.7 that any self-map $f\colon T \to T$ has a fixed point.

Let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon T & \to & T \\ v = (v_1, \dots, v_n) & \mapsto & \frac{Av}{|Av|}. \end{array}$$

Note that this map is defined since our hypothesis that all a_{ij} are positive and the fact that all entries of v are non-negative and that at least one entry of v is positive imply that $Av \neq 0$. As we had just discussed above, the map $f\colon T \to T$ has a fixed point $v \in T$. But this means that $Av = |Av| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|Av|$. □

We conclude this discussion of the Brouwer Fixed Point Theorem with two applications outside of mathematics:

---

quadratic equation

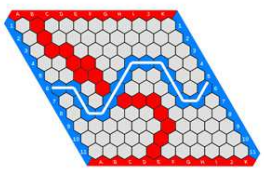
$$1 = \|r_x(t_x)\|^2 = \langle r_x(t_x), r_x(t_x)\rangle = 2\langle x, f(x) - x\rangle + \|f(x) - x\| = \langle x + t_x(f(x) - x), x + t_x(f(x) - x)\rangle$$

and the quadratic formula for polynomials of degree two which is continuous in the coefficients.

(1) The board game "Hex" was invented in 1942 by Piet Hein[607] and it was re-invented in 1948 by John Nash[608] The game is illustrated in Figure 481 and an online version can be found at

`http://www.lutanho.net/play/hex.html`

David Gale [Gal] showed in 1979 that the statement that the board game "Hex" cannot end in a draw, is equivalent to the Brouwer Fixed-Point Theorem. We also refer to [Gh, p. 102] for a discussion of Hex and the Brouwer Fixed-Point Theorem.



11×11 Hex gameboard showing a winning configuration for Blue

FIGURE 481.

(2) The Brouwer Fixed Point Theorem can also be used to show the existence of a "Nash equilibrium" in game theory. We refer to the following website for more information:

`https://en.wikipedia.org/wiki/Nash_equilibrium`.

31.4. **Basic homological algebra.** Now that we have made use of Theorem 31.2 we also need to provide a proof of Theorem 31.2. In this chapter we will first develop several algebraic methods which later on will be used in the proof of the existence of the exact sequence of Theorem 31.2.

Definition. A *short exact sequence* is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial groups. A *short exact sequence of chain complexes*

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to C_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$$

form a short exact sequence.

[607]Piet Hein (1905-1996) was a Danish mathematician.

[608]John Nash (1928-2015) was an American mathematician who was awarded the Nobel prize in economics in 1994. The movie "a beautiful mind" was based on his struggles with schizophrenia.

Remark. Throughout the subsequent discussion in the following pages it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "p"] \arrow[d, "\partial_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial_n"] & B_n \arrow[r, "p"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "p"] \arrow[d, "\partial_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i"] & B_{n-2} \arrow[r, "p"] & C_{n-2} \arrow[r] & 0.
\end{tikzcd}$$

Now suppose that we are given such a short exact sequence of chain complexes. We will define a map

$$\partial_n \colon H_n(C) \ \to \ H_{n-1}(A)$$

as follows:

(1) Let $z \in H_n(C)$.
(2) We pick $c \in C_n$ with $z = [c]$.
(3) By the surjectivity of p we have $c = p(b)$ for some $b \in B_n$.
(4) We have

$$\begin{array}{ccccccc} p(\partial b) & = & \partial p(b) & = & \partial c & = & 0. \\ & & \uparrow & & & \uparrow & \\ & & \text{since the diagram commutes} & & & \text{since } c \text{ is a cycle} & \end{array}$$

(5) Since the second horizontal row in the above diagram is exact it follows immediately from (4) that $\partial b = i(a)$ for some uniquely determined $a \in A_{n-1}$.
(6) We put $\partial_n(z) := [a]$.

We refer to this map $\partial_n \colon H_n(C) \to H_{n-1}(A)$ as the *connecting homomorphism.*

The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 31.9.

(1) *Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. The connecting homomorphism $\partial_n \colon H_n(C) \to H_{n-1}(A)$ is well-defined and it is indeed a homomorphism.*
(2) *If*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "a"] & B_* \arrow[r, "p"] \arrow[d, "b"] & C_* \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "p"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{array}{ccc} H_n(C) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle a_*} \\ H_n(C') & \xrightarrow{\partial_n} & H_{n-1}(A') \end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the commutative diagram on page 770 in mind and one has to use on several occasions that the vertical maps are chain maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary.

Proof.

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

(1) We start out with the following claim.

Claim. The map

$$\partial_n \colon H_n(C) \ \to \ H_{n-1}(A)$$

is well-defined.

Let $z \in H_n(C)$. We denote by c, b and a the elements as in the definition of ∂_n. We need to show that the element a is indeed a cycle and that the definition of $\partial_n([c])$ does not depend on the choices of c and b.

The proofs of these two statements is not particularly difficult and it is done via "diagram chasing". First we note that

$$i(\partial a) \ \underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=} \ \partial i(a) \ = \ \partial(\partial b) \ = \ 0.$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a]$ depends only on the homology class $[c]$.

(a) If we chose a different element in B_n, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial(b + i(a')) \ = \ \partial b + \partial(i(a')) \ \overset{\substack{\text{since the diagram commutes} \\ \downarrow}}{=} \ \underbrace{\partial b}_{=i(a)} + i(\partial a') \ = \ i(a + \partial a').$$

Since $[a] = [a + \partial a'] \in H_{n-1}(A)$ this does not change the homology class.

(b) If we picked an other representative for $[c]$, then this representative would be of the form $c + \partial c'$ for some $c' \in C_{n+1}$. Since the map $p\colon B_n \to C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial b') \;=\; p(b) + p(\partial b') \;\underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=}\; c + \partial p(b') \;=\; c + \partial c'.$$

Since $\partial(b + \partial b') = \partial b$ we have ended up with the same element in B_{n-1} as before.

This concludes the proof of the claim.

Claim. The map

$$\partial_n\colon H_n(C) \;\to\; H_{n-1}(A)$$

is a homomorphism.

Thus let $[c_1]$ and $[c_2]$ be given. For $j = 1, 2$ we choose b_j as in this definition of ∂_n, and we denote by a_j the uniquely determined element with $i(a_j) = \partial b_j$. Then we also have $p(b_1 + b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. It follows that

$$\partial_n([c_1 + c_2]) \;=\; [a_1 + a_2] \;=\; [a_1] + [a_2] \;=\; \partial_n([c_1]) + \partial_n([c_2]).$$

Thus we have proved the claim.

(2) The second statement follows easily from the definitions. We leave the verification of the details as an exercise to the reader. □

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 31.10. *Let*

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact.

Remark.

(1) In the following we refer to the exact sequence of Proposition 31.10 as the as *long exact sequence in homology* corresponding to the short exact sequence of chain complexes.

(2) Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{\;i\;} & B_* & \xrightarrow{\;p\;} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{\;i\;} & B'_* & \xrightarrow{\;p\;} & C'_* & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain maps where the horizontal sequences are exact. It follows from Lemma 31.9 that the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{p_*} & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow \dots \\
& \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow a_* & \\
\dots \longrightarrow & H_n(A') & \xrightarrow{i_*} & H_n(B') & \xrightarrow{p_*} & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \longrightarrow \dots
\end{array}$$

Put differently, we have defined a functor form the category of short exact sequences of chain complexes[609] to the category[610] of long exact sequences of abelian groups.

Proof of Proposition 31.10.

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact. We examine the exactness at the three groups $H_{n-1}(A), H_n(B)$ and $H_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

(1) the image of the map to the left is contained in the image of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,
(2) the kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) First let $[c] \in H_n(C)$. We want to show that $i(\partial[c]) = 0$. We choose b with $p(b) = c$ and denote by $a \in A_{n-1}$ the element with $i(a) = \partial b$. Then we have

$$i(\partial[c]) \;\underset{\substack{\uparrow \\ \text{definition of } \partial\colon H_n(C) \to H_{n-1}(A)}}{=}\; i([a]) \;=\; [\partial b] \;=\; 0 \in H_{n-1}(B).$$

Now let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_n$. It follows immediately from the definitions that $\partial[p(b)] = [a]$.

[609]Here the objects are short exact sequences of chain complexes and morphisms between two short exact sequences $0 \to A_* \to B_* \to C_* \to 0$ and $0 \to A'_* \to B'_* \to C'_* \to 0$ are given by a triple of chain maps $a\colon A_* \to A'_*$, $b\colon B_* \to B'_*$ and $c\colon C_* \to C'_*$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\
& \downarrow a & & \downarrow b & & \downarrow c & & \\
0 \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0.
\end{array}$$

[610]Here the objects are chain complexes of abelian groups and morphisms are chain maps.

(B) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subset \ker(p_*)$.
Conversely let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in C_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. From the commutativity it follows that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \mathrm{im}(i_*)$.

(C) Let $[b] \in H_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we choose some $b \in B$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$.
Finally let $[c] \in \ker(\partial\colon H_n(C) \to H_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Then there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in H_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. Then it follows that

$$\partial(b - i(a')) \;=\; \partial b - \partial i(a') \;=\; \partial b - i(\partial a') \;=\; \partial b - i(a) \;=\; 0,$$

i.e. $b - i(a')$ is a cycle. On the other hand we have $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. □

The proof of the following lemma is also done via "diagram chasing". We will provide the proof in Exercise 31.4.

Lemma 31.11. (Five Lemma) *Let*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, two heads, "a"] & B \arrow[r, "j"] \arrow[d, "b", "\cong"'] & C \arrow[r, "k"] \arrow[d, "c"] & D \arrow[r, "l"] \arrow[d, "d", "\cong"'] & E \arrow[d, hook, "e"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

be a diagram of groups (*here for once we do not suppose that the groups are abelian*) *such that the horizontal sequences are exact and such that every squares commute* (*if all the groups are abelian, then it suffices it the squares commute up to a sign*)*. If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.*[611]

31.5. **Relative homology groups.** In this section we will introduce the relative homology groups which will play an important role in the proof of Theorem 31.2 and also in the remainder of this course.

Definition. Let (X, A) be a pair of topological spaces.[612] We view each chain group $C_n(A)$ as a subgroup of $C_n(X)$. Thus we can define

$$C_n(X, A) \;:=\; C_n(X)/C_n(A)$$

[611] In many applications it suffices to remember the following slightly weaker statement: if the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

[612] Recall that according to the definition on page 243 this means that X is a topological space and $A \subset X$ is a subset.

and we consider the boundary map[613]

$$\begin{array}{rcl} \partial_n \colon C_n(X,A) & \to & C_{n-1}(X,A) \\ [c] & \mapsto & [\partial_n c]. \end{array}$$

We refer to

$$\ldots C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \ldots$$

as the chain complex of the pair (X,A).[614] Given any $n \in \mathbb{N}_0$ we define the *n-th relative homology group of* (X,A) as follows:

$$H_n(X,A) \ := \ H_n(C_*(X,A)).$$

Remark. Let (X,A) be a pair of topological spaces and let $n \in \mathbb{N}_0$. We write

$$S \ := \ \{\sigma \colon \Delta^n \to X \mid \operatorname{im}(\sigma) \cap (X \setminus A) \neq \varnothing\},$$

i.e. S denotes the set of all singular n-simplices of X that are not completely contained in A. Then it is straightforward to show that the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} & \to & C_n(X,A) \\ \sum\limits_{i=1}^{n} n_i\sigma_i & \mapsto & \left[\sum\limits_{i=1}^{n} n_i\sigma_i\right] \end{array}$$

is an isomorphism. In particular we see that $C_n(X,A)$ is a free-abelian group.

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $C_n(\varnothing) = 0$ and $C_n(X,\varnothing) = C_n(X)$, i.e. we have $H_n(X,\varnothing) = H_n(X)$. In the following we will freely go back and forth between the notations $H_n(X,\varnothing)$ and $H_n(X)$.

(2) If X is a path-connected topological space and if A is non-empty, then $H_0(X,A) = 0$. Indeed, let $P \in X$ be a point. We pick a point $Q \in A$. Since X is path-connected we can pick a map $\sigma \colon \Delta^1 \to X$ such that $\sigma(0,1) = P$ and $\sigma(1,0) = Q$. As discussed on page 738, in $C_*(X,A)$ we have $\partial\sigma = P - Q$, but since $Q = 0 \in C_0(X,A)$ we have $\partial\sigma = P$, which shows that $P = 0 \in H_0(X,A)$. But this clearly implies that $H_0(X,A) = 0$. We refer to Figure 482 for an illustration.

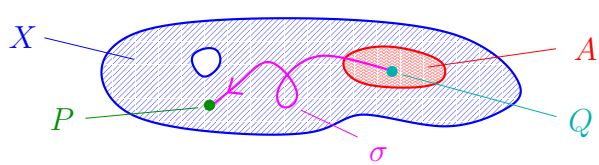


FIGURE 482.

[613]The map ∂_n is well-defined, since for any $c \in C_n(A) \subset C_n(X)$ the boundary ∂c lies obviously in $C_{n-1}(A) \subset C_{n-1}(X)$.

[614]Since $\partial_{n-1} \circ \partial_n \colon C_n(X) \to C_{n-2}(X)$ is the zero map it follows that $\partial_{n-1} \circ \partial_n \colon C_n(X,A) \to C_{n-2}(X,A)$ is also the zero map.

(3) We let $X = \mathbb{R}^2$ and $A = B_1(-2) \cup B_1(2)$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_1(2)$ and $\sigma(1,0) \in B_1(-2)$. (We refer to Figure 483 for an illustration.) Then we have

$$\partial\sigma \;=\; \underbrace{\sigma(0,1)}_{\in B_1(2)} - \underbrace{\sigma(1,0)}_{\in B_1(-2)}.$$

This shows that $\partial\sigma$ lies in $C_0(A)$. Put differently, the singular 1-simplex σ, viewed as an element in $C_1(X,A)$, is a cycle and therefore it represents an element in $H_1(X,A)$. Later on we will see whether or not $[\sigma] \in H_1(X,A)$ is non-trivial.

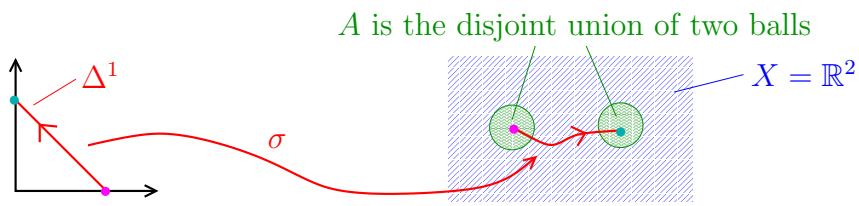


FIGURE 483.

Now let $f\colon (X,A) \to (Y,B)$ be a map of pairs of topological spaces, i.e. a map $f\colon X \to Y$ with $f(A) \subset B$. Then f induces maps

$$\begin{array}{rcl} f_*\colon H_n(X,A) & \to & H_n(Y,B) \\ \left[\sum\limits_{j=1}^n a_j\sigma_j\right] & \to & \left[\sum\limits_{j=1}^n a_j(f\circ\sigma_j)\right]. \end{array}$$

It is straightforward to show that these maps are well-defined.[615]

Remark. If (Y,B) is a pair of topological spaces and if $X \subset Y$ and $A \subset B$, then we have the obvious inclusion map $(X,A) \to (Y,B)$ and we therefore obtain the induced maps $H_n(X,A) \to H_n(Y,B)$ in relative homology. If it is clear from the context that we are dealing with such an inclusion induced map, then on many occasions we will not explicitly point out that we are working with the inclusion induced maps.

Now we consider the homology of a topological space relative to a point.

[615]We refer to the category that is given by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{P}) & := & \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) & := & \text{all maps } (X,A) \to (Y,B) \end{array}$$

with the usual composition of maps as the *category of topological pairs*. Then for each $n \in \mathbb{N}_0$ the maps

$$(X,A) \;\to\; H_n(X,A)$$

and

$$(f\colon (X,A) \to (Y,B)) \;\mapsto\; (f_*\colon H_n(X,A) \to H_n(Y,B))$$

define a functor from the category of topological pairs to the category of groups.

Lemma 31.12. *Given a topological space X, a point $x_0 \in X$ and $n \in \mathbb{N}_0$ we consider the map*

$$\begin{tikzcd} \widetilde{H}_n(X) \arrow[r] \arrow[rrr, bend right, "=:\Phi_{X,x_0}"'] & H_n(X) \arrow[r, "="] & H_n(X,\varnothing) \arrow[r] & H_n(X,\{x_0\}). \end{tikzcd}$$

Then the following holds:

(1) *The map*

$$\Phi_{X,x_0}\colon \widetilde{H}_n(X) \;\to\; H_n(X,\{x_0\})$$

is an isomorphism.

(2) *For any map $f\colon X \to Y$ of topological spaces and any $a \in X$ we have the following commutative diagram*

$$\begin{tikzcd} \widetilde{H}_n(X) \arrow[r, "f_*"] \arrow[d, "\Phi_{X,x_0}"'] & \widetilde{H}_n(Y) \arrow[d, "\Phi_{Y,f(x_0)}"] \\ H_n(X,\{x_0\}) \arrow[r, "f_*"] & H_n(Y,\{f(x_0)\}). \end{tikzcd}$$

In other words, if we denote by $\mathcal{P}$ the category of pointed topological spaces and we denote by $\mathcal{A}$ the category of abelian groups, then the isomorphisms from Lemma 31.12 (1) *give a natural transformation from the functor*

$$\begin{array}{rcl} \mathcal{P} & \to & \mathcal{A} \\ (X,x_0) & \mapsto & H_n(X,x_0) \end{array} \quad \textit{to the functor} \quad \begin{array}{rcl} \mathcal{P} & \to & \mathcal{A} \\ (X,x_0) & \mapsto & \widetilde{H}_n(X). \end{array}$$

In particular, the relative homology groups $H_n(X,x_0)$ and the reduced homology group $\widetilde{H}_n(X)$ are naturally isomorphic.

Proof. We will provide the proof of the first statement in Exercise 31.7. The second statement follows basically immediately from the definitions. □

Before we can state the next proposition we need to introduce the notion of a triple of topological spaces which is defined in almost the same fashion as a pair of topological spaces.

Definition.

(1) A *triple of topological spaces* is a triple (X,B,A) where X is a topological space and $A \subset B \subset X$ are subsets.
(2) A map $f\colon (X,B,A) \to (X',B',A')$ between triples of topological spaces is a continuous map $f\colon X \to X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 31.13. *Let X be a topological space and let $A \subset B \subset X$ be two subsets. We denote by $i\colon (B,A) \to (X,A)$ and by $p\colon (X,A) \to (X,B)$ the obvious maps. Then the following statements hold:*

(1) *For each $n \in \mathbb{N}_0$ the map*[616][617]

$$\begin{array}{rcl} \partial_n \colon H_n(X,B) & \to & H_{n-1}(B,A) \\ \Big[\sum\limits_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)\Big] & \mapsto & \Big[\sum\limits_{j=1}^{k} a_j \cdot \partial\sigma_j\Big] \end{array}$$

is well-defined.

(2) *Let $\mathcal{T}$ be the category of triples of topological spaces.*[618] *The maps ∂_n from* (1) *define a natural transformation from the functor $(X,B,A) \to H_n(X,B)$ to the functor $(X,B,A) \to H_{n-1}(B,A)$. In down-to-earth language this means that for a map $f\colon (X,A,B) \to (\widetilde{X},\widetilde{A},\widetilde{B})$ of triples of topological spaces the following diagram commutes:*

$$\begin{tikzcd} H_n(X,B) \arrow[r, "\partial_n"] \arrow[d, "f_*"'] & H_{n-1}(B,A) \arrow[d, "f_*"] \\ H_n(\widetilde{X},\widetilde{B}) \arrow[r, "\partial_n"] & H_{n-1}(\widetilde{B},\widetilde{A}). \end{tikzcd}$$

(3) *The maps*

$$\ldots \to H_n(B,A) \xrightarrow{i_*} H_n(X,A) \xrightarrow{p_*} H_n(X,B) \xrightarrow{\partial_n} H_{n-1}(B,A) \to \ldots$$

form an exact sequence.

(4) *The map in* (3) *which assigns to a triple (X,B,A) of topological spaces a long exact sequence of homology groups is functorial. More precisely, given a map $f\colon (X,B,A) \to (X',B',A')$ between triples of topological spaces the following diagram commutes*

$$\begin{tikzcd} \ldots \arrow[r] & H_n(B,A) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_n(X,B) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(B,A) \arrow[r] \arrow[d, "f_*"] & \ldots \\ \ldots \arrow[r] & H_n(B',A') \arrow[r] & H_n(X',A') \arrow[r] & H_n(X',B') \arrow[r, "\partial"] & H_{n-1}(B',A') \arrow[r] & \ldots. \end{tikzcd}$$

The long exact sequence of Proposition 31.13 (2) is sometimes called the *long exact sequence of the triple* (X,B,A) and we refer to ∂_n as the *connecting homomorphism* of the triple (X,B,A).

Proof. It follows immediately from the definitions and the third isomorphism theorem[619] that

$$0 \to C_*(B,A) \xrightarrow{i_*} C_*(X,A) \xrightarrow{p_*} C_*(X,B) \to 0$$

[616]The fact that here the symbol "∂_n" denotes the boundary map of singular chains and the desired map $H_n(X,B) \to H_{n-1}(B,A)$ is an unfortunate problem in our choice of notation.

[617]Here we use that every homology class in $H_n(X,B)$ admits by definition a representative in the quotient $C_n(X,B) = C_n(X)/C_n(B)$, so it can be represented by a singular n-chain in X.

[618]It should be clear from the definitions preceding the definition what the objects and morphisms of this category are.

[619]The third isomorphism theorem implies that $(C_*(X)/C_*(A))/(C_*(B)/C_*(A)) \cong C_*(X)/C_*(B)$.

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 770, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined.

The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(B,A) & \longrightarrow & C_n(X,A) & \longrightarrow & C_n(X,B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ 0 & \longrightarrow & C_{n-1}(B,A) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & C_{n-1}(X,B) & \longrightarrow & 0. \end{array}$$

Let $z \in H_n(X,A)$. Now we will determine the image of z under the connecting homomorphism $H_n(X,B) \to H_{n-1}(B,A)$ which during the proof of the claim we denote by ∂^c.

We pick a representative $\sum_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ in $C_n(X)$. Note that $\sum_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $C_n(X,A)$. We then take the boundary of the above singular n-chain under the boundary map $\partial_n \colon C_n(X,A) \to C_{n-1}(X,A)$ and we end up with the singular n-chain $\sum_{j=1}^{k} a_j \cdot \partial\sigma_j \in C_{n-1}(X,A)$. As we have seen in the proof of Lemma 31.9 this singular n-chain lies in fact in $C_{n-1}(B,A) \subset C_{n-1}(X,A)$. The homology class thereof is by definition precisely $\partial^c(z) \in H_{n-1}(B,A)$. Summarizing we have shown that

$$\partial^c(z) \;=\; \Big[\sum_{j=1}^{k} a_j \cdot \partial\sigma_j\Big].$$

This shows that the map defined in (1) agrees with ∂^c. This concludes the proof of the claim.

Statement (2) of the lemma is an easy consequence of Lemma 31.9. The fact that the stated sequence of homology groups is exact is now an immediate consequence of Proposition 31.10. Finally the functoriality of the long exact sequence is an immediate consequence of (2). □

For convenience we record two important special cases of Proposition 31.13 in the following corollary.

Corollary 31.14. *Let X be a topological space and let $B \subset X$ be a subset. Then there exists a long exact sequence*

$$\ldots \;\to\; H_n(B) \;\xrightarrow{i_*}\; H_n(X) \;\xrightarrow{p_*}\; H_n(X,B) \;\xrightarrow{\partial}\; H_{n-1}(B) \;\to\; \ldots$$

Furthermore, if B is non-empty then there exists also a long exact sequence

$$\ldots \;\to\; \widetilde{H}_n(B) \;\xrightarrow{i_*}\; \widetilde{H}_n(X) \;\xrightarrow{p_*}\; H_n(X,B) \;\xrightarrow{\partial}\; \widetilde{H}_{n-1}(B) \;\to\; \ldots$$

Both long exact sequences define a functor from the category of pairs of topological spaces that we defined on page 243, to the category of chain complexes.

The long exact sequence of Corollary 31.14 is sometimes called the *long exact sequence* (*in reduced homology*) *of the pair* (X, B).

Proof. The first statement follows from Proposition 31.13 applied to $A = \varnothing$ and the second statement follows from Proposition 31.13 applied to $A = \{a\}$ a point in B. $\square$

Remark. Let (X, B, A) be a triple of topological spaces. Then $(X, B, \varnothing) \to (X, B, A)$ is a map between triples of topological spaces and by the naturality of the long exact sequence we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow{=} & & \downarrow & \\
\dots \longrightarrow & H_n(B,A) & \longrightarrow & H_n(X,A) & \longrightarrow & H_n(X,B) & \xrightarrow{\partial} & H_{n-1}(B,A) & \longrightarrow \dots .
\end{array}$$

Example. We let $X = \mathbb{R}^2$ and $A = B_1(-2) \cup B_1(2)$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By Corollary 31.14 we have a long exact sequence

$$H_2(X) \to H_2(X,A) \to H_1(A) \xrightarrow{i_*} H_1(X) \to H_1(X,A) \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

By Corollary 30.8 (3) together with Lemma 29.13 we have $H_i(X) = H_i(\mathbb{R}^2) = 0$ and also $H_i(A) = H_i(B_1(-2)) \oplus H_i(B_1(2)) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $H_i(X, A) = 0$ for $i \geq 2$. We denote by $\epsilon\colon H_0(\mathbb{R}^2) \to \mathbb{Z}$ the augmentation map that is an isomorphism by Proposition 29.5. Similarly we denote by $\mu_1\colon H_0(B_1(-2)) \to \mathbb{Z}$ and $\mu_2\colon H_0(B_1(2)) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H_1(X,A) & \xrightarrow{\partial} & H_0(A) & \xrightarrow{i_*} & H_0(X) & \longrightarrow & H_0(X,A) & \longrightarrow 0 \\
& {\scriptstyle =}\downarrow & & {\scriptstyle \mu_1\oplus\mu_2}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle =} & \\
0 \longrightarrow & H_1(X,A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_0(X,A) & \longrightarrow 0.
\end{array}$$

We want to determine the matrix representing the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ at the bottom. We do so in the following claim.

Claim. The homomorphism $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$.

We pick a point $P \in B_1(-2)$ and we pick a point $Q \in B_1(2)$. It follows from Proposition 29.5 and Lemma 29.13 that the map

$$\begin{array}{rcl}
\Psi\colon \mathbb{Z}^2 & \to & H_0(A) = H_0(B_1(-2)) \oplus H_0(B_1(2)) \\
(m,n) & \mapsto & [m \cdot P + n \cdot Q]
\end{array}$$

is the inverse to $\mu_1 \oplus \mu_2$. Now it is easy to verify that $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1} = \epsilon \circ i_* \circ \Psi$ applied to $(1, 0)$ and $(0, 1)$ gives in both cases the value 1. So the map is described by the matrix $(1\ 1)$. This concludes the proof of the claim.

Since the top sequence in the above commutative diagram is exact and since the vertical maps are isomorphisms we see that the bottom sequence is also exact. It is now straightforward to see that

$$H_1(X,A) \;\cong\; \ker\left\{\mathbb{Z}^2 \xrightarrow{(1\;1)} \mathbb{Z}\right\} \;\cong\; \mathbb{Z} \quad \text{and} \quad H_0(X,A) \;\cong\; \operatorname{coker}\left\{\mathbb{Z}^2 \xrightarrow{(1\;1)} \mathbb{Z}\right\} \;\cong\; 0.$$

Now we return to the situation of the example on page 776. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_1(2)$ and $\sigma(1,0) \in B_1(-2)$. We have

$$((\mu_1 \oplus \mu_2) \circ \partial)[\sigma]) \;\underset{\substack{\uparrow \\ \text{description of } \partial \text{ in Proposition 31.13}}}{=}\; ((\mu_1 \oplus \mu_2)([\partial\sigma]) \;=\; (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0,1)}_{\in B_1(2))}] - [\underbrace{\sigma(1,0)}_{\in B_1(-2))}]) \;=\; (1,-1).$$

Since $(1,-1)$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\;1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $H_1(X,A) \cong \mathbb{Z}$. In particular it is non-zero.

Recall that, as defined on page 641, we say that two maps $f,g\colon (X,A) \to (Y,B)$ between pairs of topological spaces are homotopic, if there exists a map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, and $F(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$. The following proposition extends Proposition 30.5 to the relative case.

Proposition 31.15. *Let $f,g\colon (X,A) \to (Y,B)$ be two maps between pairs of topological spaces. If f and g are homotopic, then the maps $f_*, g_*\colon C_*(X,A) \to C_*(Y,B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have*

$$f_* \;=\; g_*\colon H_n(X,A) \;\to\; H_n(Y,B).$$

Proof. Let $f,g\colon (X,A) \to (Y,B)$ be two homotopic maps between pairs of topological spaces. To prove the proposition we have to look again at the proof of Proposition 30.5. There in the proof we had explicitly constructed a chain $P_n\colon C_n(X) \to C_{n+1}(Y)$, $n \in \mathbb{N}_0$ between the maps f_* and g_* from $C_*(X)$ to $C_*(Y)$. It follows immediately from the definition of the chain homotopy that it has the property that $C_n(A)$ gets sent to $C_{n+1}(B)$. In particular the chain homotopy descends to maps $Q_n\colon C_n(X,A) \to C_{n+1}(Y,B)$ which then form a chain homotopy between the maps f_* and g_* from $C_*(X,A)$ to $C_*(Y,B)$. It now follows immediately from Lemma 30.2 that $f_* = g_*\colon H_n(X,A) \to H_n(Y,B)$. $\square$

We conclude this section with the following corollary.

Corollary 31.16. *Let X be a topological space.*

(1) *Let $A \subset B$ be subsets of X. If A is a deformation retract of B, then for any $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(X,A) \;\to\; H_n(X,B)$$

is an isomorphism.

(2) *If A is a deformation retract of X, then $H_n(X,A) = 0$ for all $n \in \mathbb{N}_0$.*

(3) *Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then for any $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(Y, A) \ \to \ H_n(X, A)$$

is an isomorphism.

Proof. Let X be a topological space.

(1) Let $A \subset B$ be a deformation retract of B. We consider the following commutative diagram

$$\begin{array}{ccccccccc} H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(X), \end{array}$$

where the vertical maps are induced by the inclusions of topological spaces and the horizontal maps are given by the long exact sequences of the pairs (X, A) and (X, B) from[620] Corollary 31.14. The second and the fifth vertical map are both evidently isomorphisms. Since A is a deformation retract of B we obtain from Corollary 30.8 that the first and the fourth map are isomorphisms. We conclude with the Five Lemma 31.11 that the middle vertical map is also an isomorphism.

(2) The second statement follows from the first statement applied to $B = X$ and the trivial observation that $H_n(X, X) = 0$ for all $n \in \mathbb{N}_0$.

(3) The third statement is proved almost the same way as the first statement. □

The statement of Corollary 31.14 looks already very similar to the statement of Theorem 31.2 that we actually want to prove. Now we "only" have to find the connection between the groups $H_n(X, A)$ and $H_n(X/A)$. The notations already look very similar, and they do so for good reasons, since in both cases "one ignores what lies in A". Nonetheless, proving the precise relationship will still require some substantial piece of work.

31.6. **The Excision Theorem.** The key step in finding the connection between the groups $H_n(X, A)$ and $H_n(X/A)$ is given by the following Excision Theorem 31.17.

Theorem 31.17. (Excision Theorem) *Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism*

$$H_n(X \setminus Z, A \setminus Z) \ \stackrel{\cong}{\to} \ H_n(X, A).$$

The proposition thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise [621] a subset Z from the interior of A, then this does not affect the relative homology groups.

[620] Note that here we use not only the existence of the two long exact sequences in homology but we also use the functoriality of the long exact sequence.

[621] The verb "to excise" means to cut out, or in German "ausschneiden".

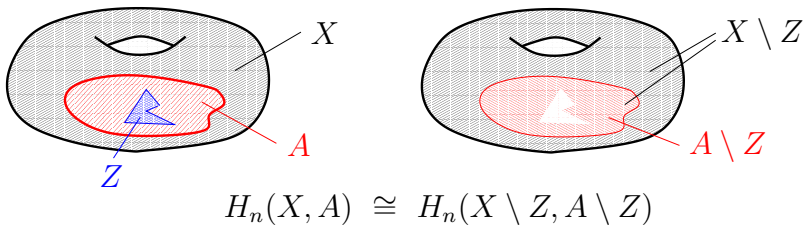


FIGURE 484. Illustration of the Excision Theorem 31.17.

Before we turn to the proof of the Excision Theorem 31.17 we first want to show that we can now relate the relative homology groups of a pair (X, A) to the homology groups of the quotient space X/A, assuming $A \subset X$ is non-empty and good. To do so we need the following definition.

Definition. Let X be a topological space and let $A \subset X$ be a non-empty[622] subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the obvious projection map. For each $n \in \mathbb{N}_0$ we refer to

$$\begin{tikzcd} H_n(X,A) \arrow[r, "p_*"] \arrow[rr, bend right] & H_n(X/A, A/A) & \widetilde{H}_n(X/A) \arrow[l, "\text{Lemma 31.12}"', "\cong"] \end{tikzcd}$$

as the *natural homomorphism* $H_n(X, A) \to \widetilde{H}_n(X/A)$.

Lemma 31.18. $(*)$ *We denote by $\mathcal{Q}'$ the category of non-trivial pairs*[623] *of topological spaces that we introduced on page 243. Furthermore denote by $\mathcal{A}$ the category of abelian groups. The above natural homomorphisms define a natural transformation from the functor*

$$\begin{array}{ccc} \mathcal{Q}' & \to & \mathcal{A} \\ (X,A) & \mapsto & H_n(X,A) \end{array} \quad \textit{to the functor} \quad \begin{array}{ccc} \mathcal{Q}' & \to & \mathcal{A} \\ (X,A) & \mapsto & \widetilde{H}_n(X/A). \end{array}$$

Proof. This lemma follows basically immediately from the definitions. □

The following proposition is much more interesting than the previous lemma.

Proposition 31.19. *Let X be a topological space and let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is non-empty and good, then the natural homomorphism $H_n(X, A) \to \widetilde{H}_n(X/A)$ is an isomorphism.*

Example. Let X be a topological space, let $A \subset X$ be a non-empty subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $H_n(X, A) \to \widetilde{H}_n(X/A)$ is in general not an isomorphism. In fact the situation is worse, in general the two groups $H_n(X, A)$ and $\widetilde{H}_n(X/A)$ are in general not even abstractly isomorphic.

[622] Readers who have too much time on their hands can start a discussion on whether the definition also makes sense if A is the empty set.

[623] A pair (X, A) of topological spaces is called non-trivial if $A \neq \varnothing$.

For example consider $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. In Exercise 26.6 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings. (See Figure 485 for an illustration.) In Exercise 31.11 we will prove the following two statements:

(1) The group $H_1(X, A)$ is countable.
(2) The group $H_1(X/A)$ is uncountable.

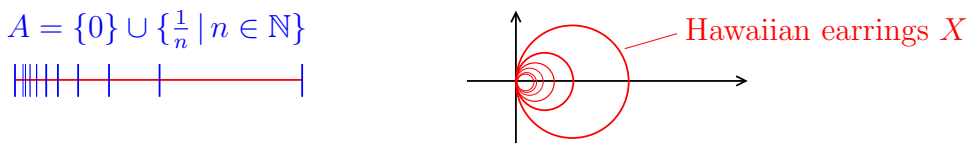


FIGURE 485.

Proof of Proposition 31.19 assuming the Excision Theorem 31.17. Let X be a topological space and let $A \subset X$ be a non-empty good subset. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.
(2) We denote by $p\colon X \to X/A$ the projection map.
(3) We denote by $q\colon X \setminus A \to X/A \setminus A/A$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from Lemma 1.30 that q is continuous.

Now we consider the following diagram

$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,U) & \longleftarrow & H_n(X \setminus A, U \setminus A) \\ \downarrow p_* & & \downarrow p_* & & \downarrow q_* \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, U/A) & \longleftarrow & H_n(X/A \setminus A/A, U/A \setminus A/A), \end{array}$$

here all the horizontal maps are induced by inclusions of pairs of topological spaces. It follows easily from the definitions that the diagram commutes.

We start out with the following claim.

Claim. The map

$$q\colon (X \setminus A, U \setminus A) \;\to\; (X/A \setminus A/A, U/A \setminus A/A)$$

is a homeomorphism of pairs of topological spaces.

It is clear that $q\colon X \setminus A \to X/A \setminus A/A$ is a bijection with $q(U \setminus A) = U/A \setminus A/A$. By Lemma 1.39 it remains to show that $q\colon X \setminus A \to X/A \setminus A/A$ is continuous and open. As we pointed out above, q is continuous. Thus it remains to show that q is open. Now let $V \subset X \setminus A$ be a subset that is open with respect to the subspace topology. We need to show that $q(V) \subset X/A \setminus A/A$ is open. We break the argument into four steps:

(1) It follows from Lemma 1.35 (1) and the fact that A is by hypothesis closed[624], that V is also an open subset of X.
(2) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (1) is an open subset of X.
(3) It follows from (2) and the definition of the quotient topology that $p(V)$ is an open subset of X/A.
(4) We have $q(V) = p(V) \cap (X/A \setminus A/A)$. It follows from (3) and the definition of the subspace topology that $q(V)$ is an open subset of $X/A \setminus A/A$.

This concludes the proof of the claim.

It follows from the claim that the vertical map on the right is an isomorphism. Our actual goal is to show that the vertical map on the left is an isomorphism. This follows from the commutativity of the above diagram and from the following claim:

Claim. All horizontal maps are isomorphisms.

We distinguish three different situations:

(1) It follows from the Excision Theorem 31.17 that both the top horizontal and the bottom horizontal map on the right-hand side are isomorphisms.
(2) Since A is deformation retract of U we obtain from Corollary 31.16 that the top left horizontal map is an isomorphism.
(3) Recall that A is a deformation retract of U. It follows from Corollary 9.15 that A/A is a deformation retract of U/A.[625] As in (2) we now see that the bottom-left horizontal map is an isomorphism.

Thus we have proved the claim. □

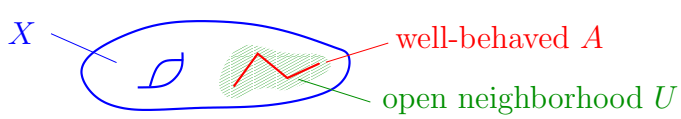


FIGURE 486. Illustration for the proof of Proposition 31.19.

Proof of Theorem 31.2 assuming the Excision Theorem 31.17. Let X be a topological space and let $A \subset X$ be a non-empty good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $q\colon X \to X/A$ the projection map. We consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "p_*"] \arrow[dr, "q_*"'] & H_n(X,A) \arrow[r, "\partial"] \arrow[d] & \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r, "i_*"] & \dots \\
& & & \widetilde{H}_n(X/A). & & &
\end{tikzcd}$$

[624]The property of being "closed" is the easily overlooked second condition for being good.

[625]The diligent reader who checks the statements we refer to will not fail to notice, that Corollary 9.15 rests eventually on the not-so-trivial Theorem 2.15. It is all too often overlooked (see see e.g. earlier version of these notes) that Corollary 9.15 is required and that it is non-trivial.

We make the following comments and observations:

(1) The horizontal sequence is the long exact coming from Corollary 31.14.
(2) The vertical map is the above natural homomorphism. It follows easily from the definition of the vertical map that the triangle commutes.
(3) Since A is good we obtain from Proposition 31.19 that the vertical map is an isomorphism.

We now define $\widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$ in the obvious way. By the above discussion we obtain the desired long exact sequence. □

31.7. **The proof of the Excision Theorem 31.17: the idea.** We are still not done with the proof of Theorem 31.2 since we still have to prove the Excision Theorem 31.17. In this section we want to come up with the idea for the proof of the Excision Theorem 31.17. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z \subset A \subset X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \ \to \ H_n(X, A).$$

Let us attempt to prove that the map $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ is surjective. We take an element in $H_n(X, A)$ and we choose a representative $a_1\sigma_1 + \cdots + a_r\sigma_r$ where $\sigma_i \colon \Delta^n \to X$, $i = 1, \dots, r$ are singular n-simplices in X.

We make the following two basic observations:

(1) If the singular n-simplices $\sigma_1, \dots, \sigma_r$ lie in $X \setminus Z$,[626] then $[a_1\sigma_1 + \cdots + a_r\sigma_r]$ evidently lies in the image of the inclusion induced map $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$.
(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $C_n(X, A) = C_n(X)/C_n(A)$.

The combination of these two observations proves the following lemma.

Lemma 31.20. *Let X be a topological space and let $Z \subset A$ be two subsets of X. Let $c = a_1\sigma_1 + \cdots + a_r\sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$.*

Unfortunately not every cycle in $C_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$.

31.8. **The proof of the Excision Theorem 31.17: the full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 31.17. This will take a surprisingly long time.

[626] Given a topological space X and a subset A we say that *a singular simplex $\sigma \colon \Delta^n \to X$ lies in A* if the image of σ lies in A.

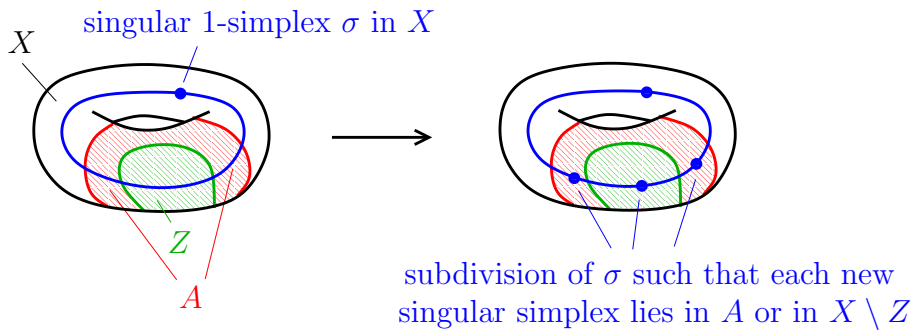


FIGURE 487. Illustration of the key idea of the proof of the Excision Theorem 31.17.

Thus let again X be a topological space. A *covering of* X is a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X such that

$$X \;=\; \bigcup_{i\in I} \mathring{U}_i.$$

Given a covering $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $C_n(X)$:

$$C_n^{\mathcal{U}}(X) \;:=\; \Big\{ \sum_{j=1}^{k} a_j\sigma_j \,\Big|\, \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \Big\}.$$

It follows quite easily from the definitions that the restriction of usual the boundary map $\partial_n\colon C_n(X) \xrightarrow{\partial} C_{n-1}(X)$ to $C_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n\colon C_n^{\mathcal{U}}(X) \xrightarrow{\partial} C_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1}\colon C_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} C_{n-1}^{\mathcal{U}}(X)$ is the zero map. We write

$$\begin{aligned} H_n^{\mathcal{U}}(X) \;&:=\; H_n\big(C_*^{\mathcal{U}}(X), \partial_*\big) \\ &=\; \ker\big\{\partial_n\colon C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)\big\} \,/\, \operatorname{im}\big\{\partial_{n+1}\colon C_{n+1}^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)\big\}. \end{aligned}$$

The homology groups $H_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later, this proposition is at the heart of many methods for computing homology groups.

Proposition 31.21. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X. Then the inclusion map*

$$C_*^{\mathcal{U}}(X) \;\to\; C_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$H_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; H_n(X).$$

Remark. In fact a stronger statement holds: the inclusion map $C_*^{\mathcal{U}}(X) \to C_*(X)$ is in fact a chain homotopy equivalence. In a later chapter we will show that this stronger statement follows in fact from Proposition 31.21 and a purely algebraic argument. Alternatively one

can fight one's way through the proof of Proposition 31.21 and upgrade the prove to this stronger statement. This approach is for example pursued in [Hat, Proposition 2.21]. The disadvantage is that it makes the proof even more complicated than it already is.

The idea of the proof of Proposition 31.21 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma\colon \Delta^n \to X$ by a singular n-chain of the form $\sum_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in Lemma 29.3 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use is given by "barycentric subdivision". The key idea is sketched in Figure 488 for the case $n = 2$.

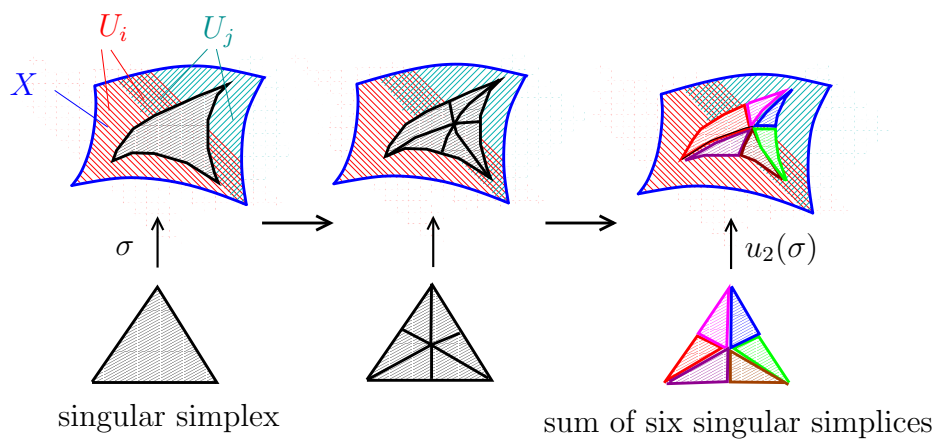


FIGURE 488. Barycentric decomposition of a singular 2-simplex.

Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[627] of the standard n-simplex Δ^n.

(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)$ which is defined as follows:[628]

$$\begin{aligned} c_{l+1}(\mu)\colon \Delta^{l+1} &\to \Delta^n \\ (t_0, \dots, t_{l+1}) &\mapsto \begin{cases} \mu\Big(\overbrace{\frac{1}{t_0+\dots+t_l} \cdot (t_0, \dots, t_l)}^{\in \Delta^l}\Big) \cdot (1 - t_{l+1}) + B_n \cdot t_{l+1}, & \text{if } t_{l+1} < 1, \\ B_n, & \text{if } t_{l+1} = 1. \end{cases} \end{aligned}$$

[627]The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.

[628]It is straightforward to verify that the image of $c_{l+1}(\mu)$ lies in Δ^n and that $c_{l+1}(\mu)$ is indeed continuous.

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. By Lemma 11.1 this map $\mu \to c_{l+1}(\mu)$ defines a linear map $C_l(\Delta^n) \to C_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-*st cone map*.

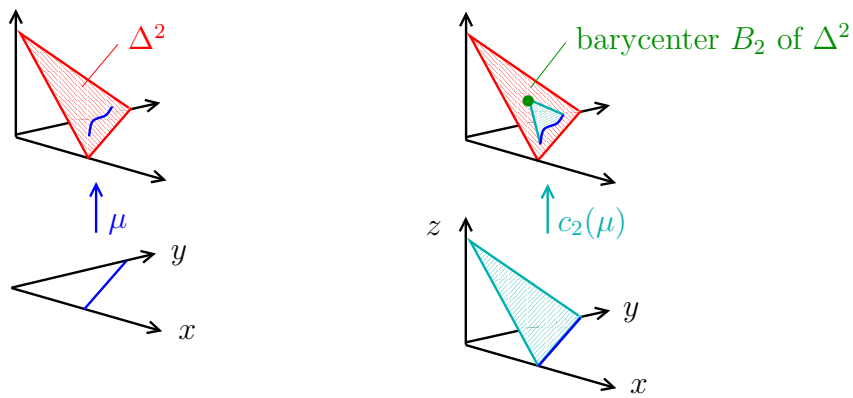


FIGURE 489. Illustration of the cone map.

(3) For $n = 0, 1, 2, 3, \ldots$ we now inductively do the following:
  (a) we define an element $\chi_n \in C_n(\Delta^n)$,
  (b) and furthermore for each topological space X we define a homomorphism $u_n \colon C_n(X) \to C_n(X)$.

We do this inductive procedure as follows:
  (i) We set $\chi_0 = \mathrm{id} \colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $C_0(X)$.
  (ii) Now suppose $\chi_{n-1} \in C_{n-1}(\Delta^{n-1})$ and that furthermore all the homomorphisms $u_{n-1} \colon C_{n-1}(X) \to C_{n-1}(X)$ are already defined. Then we define

$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\,\mathrm{id}_{\Delta^n}}_{\in C_{n-1}(\Delta^n)})) \;\in\; C_n(\Delta^n).$$

Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n \colon C_n(X) & \to & C_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l \colon \Delta^n \to X) & \mapsto & \underbrace{\sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{\chi_n}_{\in C_n(\Delta^n)})}_{\in C_n(X)}. \end{array}$$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in Figure 490. For $n = 2$ we already gave a more schematic sketch in Figure 488 without considering signs and orientations.

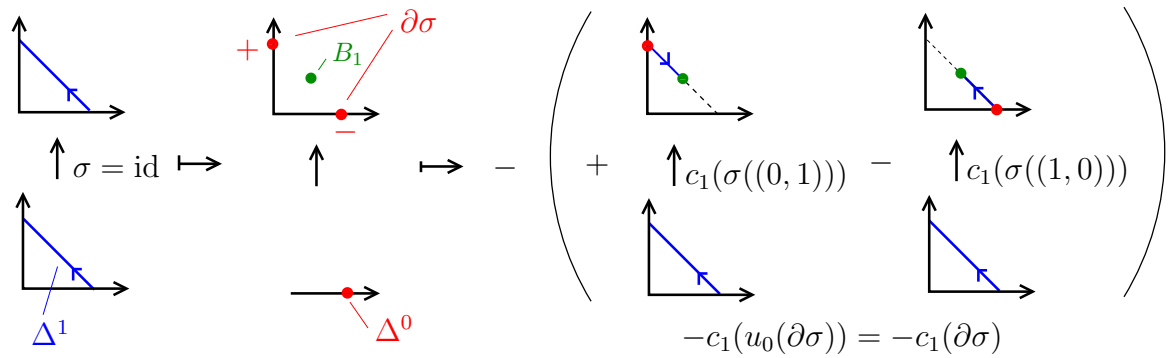


FIGURE 490. Illustration of the definition of first subdivision map.

The definition of χ_1 is illustrated in Figure 490.

Before we can continue we need to prove several technical lemmas about the subdivision maps.

Lemma 31.22. *For any map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have*

$$f_* \circ u_n \;=\; u_n \circ f_*\colon C_n(X) \;\to\; C_n(Y).$$

As we will see, Lemma 31.22 follows immediately from the definitions.

Proof. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{array}{ccccc} f_*(u_n(\sigma)) & = & f_*(\sigma_*(\chi_n)) & = & (f\circ\sigma)_*(\chi_n) & = & u_n(f_*(\sigma)). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by definition of } u_n(\sigma) & & \begin{array}{c}\text{since } (f\circ\sigma)_* = f_*\circ\sigma_* \\ \text{as maps } C_n(\Delta^n)\to C_n(Y)\end{array} & & \begin{array}{c}\text{by definition of } u_n(f_*(\sigma))\end{array} & \end{array}$$

Since the desired equality holds for all generators of $C_n(X)$ it also holds for all of $C_n(X)$. $\square$

Remark. For category-language aficionados Lemma 31.22 can be reformulated as follows. Given $n \in \mathbb{N}_0$ we consider the functor C_n from the category of topological spaces to the category of abelian groups which is given by

$$\text{topological space } X \;\mapsto\; C_n(X)$$

and

$$(f\colon X \to Y) \;\mapsto\; (f_*\colon C_n(X) \to C_n(Y)).$$

Given a topological space X we denote by $\Phi_X\colon C_n(X) \to C_n(X)$ the map that is given by the n-th subdivision map that we defined on page 789. It follows from Lemma 31.22 that these maps define a natural transformation from the functor C_n to itself.

Lemma 31.23. *Let X be a topological space. The maps $u_n\colon C_n(X) \to C_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(C_*(X), \partial)$ to itself.*

The statement of Lemma 31.23 is equivalent to the statement that for any singular n-simplex σ in X we have

$$u_{n-1}(\partial\sigma) \;=\; \partial(u_n(\sigma)).$$

Put differently, Lemma 31.23 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in Figure 491. The picture is quite convincing.

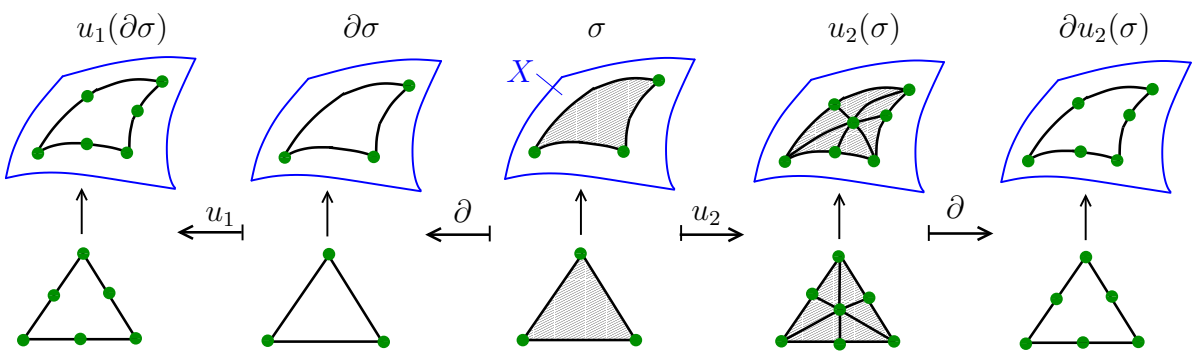


FIGURE 491. Illustration of Lemma 31.23.

Proof of Lemma 31.23 $(*)$. [629] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.
(2) As on page 736, given points $x_0, \ldots, x_k \in \Delta^n$ we denote by $[x_0, \ldots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by[630]

$$\begin{array}{rcl} \Delta^k & \to & \Delta^n \\ (t_0, \ldots, t_k) & \mapsto & \sum\limits_{i=0}^{k} t_i x_i, \end{array}$$

we refer to such a singular k-simplex as a *linear* singular simplex.
(3) We denote by $C_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $C_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon C_k(\Delta^n) \to C_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon C_k^{\mathrm{lin}}(\Delta^n) \to C_{k-1}^{\mathrm{lin}}(\Delta^n)$.
(4) For $x_0, \ldots, x_k \in \Delta^n$ we write

$$[x_0, \ldots, \widehat{x_i}, \ldots, x_k] \;:=\; [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k].$$

The proof of the lemma will be broken into three claims.

[629] We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

[630] It is straightforward to show that this map does indeed take values in Δ^n.

Claim. For any $\mu \in C_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu) + (-1)^{k+1}\cdot\mu \;\in\; C_k(\Delta^n).$$

We illustrate the claim in Figure 492. To prove the claim we first note that for any choice

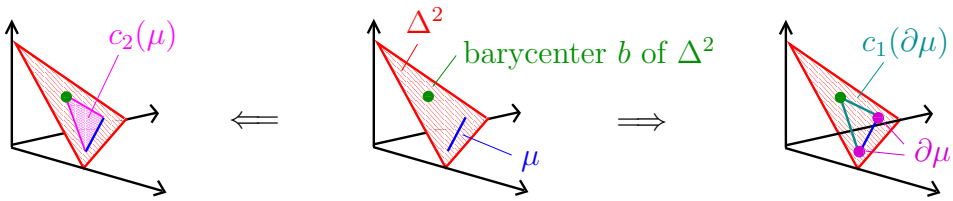


FIGURE 492.

of points $x_0,\dots,x_{l-1}\in\Delta^n$ we have by definition of c_l that

$$c_l([x_0,\dots,x_{l-1}]) \;=\; [x_0,\dots,x_{l-1},b].$$

It now follows that for any $w_0,\dots,w_k\in\Delta^n$ we have the following equality in $C_k(\Delta^n)$:

$$\begin{aligned}
\partial(c_{k+1}([w_0,\dots,w_k])) &\overset{\substack{\text{by the above equality}\\ \downarrow}}{=} \partial([w_0,\dots,w_k,b])\\
&\underset{\substack{\uparrow\\ \text{definition of }\partial\text{ and by the above equality}}}{=} \sum_{j=0}^{k}(-1)^j\cdot[w_0,\dots,\widehat{w_j},\dots,w_k,b] + (-1)^k\cdot[w_0,\dots,w_k]\\
&\overset{\downarrow}{=} c_k(\partial([w_0,\dots,w_k])) + (-1)^k\cdot[w_0,\dots,w_k].
\end{aligned}$$

The claim immediately follows from this equality.

Claim. For any $n\in\mathbb{N}_0$ and any $\nu\in C_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial u_n(\nu) \;=\; u_{n-1}(\partial\nu).$$

We will prove this claim by induction on n. The case $n=0$ is trivial. So now suppose we have already shown the case $n-1$. It suffices to show the claim for all linear singular

n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial u_n(\nu) &\overset{\substack{\text{definition of } u_n \text{ and } \chi_n\text{, see page 789}\\ \downarrow}}{=} \partial(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n \cdot \partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\substack{\text{since } \partial\circ\nu_* = \nu_*\circ\partial\\ \downarrow}}{=} (-1)^n \cdot \nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big)\\
&\underset{\substack{\uparrow\\ \text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in C^{\mathrm{lin}}_{n-1}(\Delta^n)}}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big)\\
&\underset{\substack{\uparrow\\ \text{by induction hypothesis we have } \partial u_{n-1}(\rho) = u_{n-2}(\partial\rho) \text{ for } \rho \in C^{\mathrm{lin}}_{n-1}(\Delta^n)\text{, applied here to } \rho = \partial\,\mathrm{id}_{\Delta^n}}}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \;+\; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}))\\
&\underset{\substack{\uparrow\\ \text{Lemma 31.22 applied to } f=\nu}}{=} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{by definition}}}{=} u_{n-1}(\partial\nu).
\end{aligned}$$

This concludes the proof of the claim.

The following claim is just a reformulation of the statement of the lemma.

Claim. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{aligned}
\partial(u_n(\sigma)) &= \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{Lemma 31.22 applied to } f=\sigma\\ \downarrow}}{=} \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{since } \sigma_* \text{ is a chain map}\\ \downarrow}}{=} \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n})))\\
&\underset{\substack{\uparrow\\ \text{by the previous claim}}}{=} \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{Lemma 31.22 applied to } f=\sigma}}{=} u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{since } \sigma_* \text{ is a chain map}}}{=} u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = u_{n-1}(\partial\sigma).
\end{aligned}$$

This concludes the proof of the claim. □

Lemma 31.24. *Let X be a topological space. The chain maps*

$$u_n\colon C_n(X) \;\to\; C_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon C_n(X) \to C_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n \;=\; -P_{n-1} \circ \partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $C_{-1}(X) = 0$, accordingly we define $P_{-1}\colon C_{-1}(X) \to C_0(X)$ to be the zero map. Let us suppose that we have have already found maps $P_{-1}, \dots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon C_n(X) \to C_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) \;=\; -(P_{n-1}\circ\partial_n)(\sigma) \,+\, (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in C_{n+1}(X)$ with

$$\partial E_\sigma \;=\; -(P_{n-1}\circ\partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0}\circ\partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) \;=\; 0 \in C_0(X),$$

and we put $E_\sigma := 0 \in C_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 759 that $H_n(X) = 0$. Therefore to show that an element $c \in C_n(X)$ is the boundary of some element in $C_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{aligned}
\partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) \;&\overset{\substack{\text{by Lemma 31.23 the subdivision maps form a chain map, i.e. } \partial\circ u_n = u_{n-1}\circ\partial \\ \downarrow}}{=}\; -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\
&=\; \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0.
\end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in C_{n+1}(X)$ with $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon C_n(X) \to C_{n+1}(X)$ the map which assigns to each singular n-simplex σ the element $E_\sigma \in C_{n+1}(X)$. Then it follows from the definition of E_σ that

$$\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n \;=\; u_n - \mathrm{id}$$

as maps $C_n(X) \to C_n(X)$. This concludes the proof of Lemma 31.24 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{aligned}
P_n\colon C_n(X) \;&\to\; C_{n+1}(X) \\
\sum_{l=1}^{r} a_l\cdot(\sigma_l\colon \Delta^n \to X) \;&\mapsto\; \sum_{l=1}^{r} a_l\cdot(\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\substack{\uparrow \\ \text{defined since } \Delta^n \text{ is convex}}}).
\end{aligned}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. □

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon C_n(X) \to C_n(X)$.

Lemma 31.25. *For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $C_*(X) \to C_*(X)$ that is chain homotopic to the identity.*

Proof. In Lemma 31.23 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $C_*(X) \to C_*(X)$. As we remarked on page 745, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By Lemma 31.24 we have $u_* \simeq \mathrm{id}$. By Lemma 30.1 (2) this implies $u_*^2 \simeq u_*$, and by Lemma 30.1 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. □

Definition. Given a singular chain c we can write it uniquely as $c = a_1\sigma_1 + \cdots + a_s\sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We then say $\sigma_1, \ldots, \sigma_s$ are the *pieces* of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the *diameter* is defined as
$$d(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a, b \in A\big\} \in \mathbb{R}_{\geq 0}.$$
(2) Given a singular simple $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Now we can formulate the following lemma.

Lemma 31.26. *For any $x_0, \ldots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \ldots, x_k])$ we have*[631]
$$d(|\tau|) \;\leq\; \tfrac{n}{n+1} \cdot d(|[x_0, \ldots, x_k]|).$$

FIGURE 493. Illustration of Lemma 31.26.

In the proof of Lemma 31.26 we will need the following elementary[632] geometric lemma.

[631] Recall the definition on page 736 of the map $[x_0, \ldots, x_k]\colon \Delta^k \to \mathbb{R}^n$.

[632] Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

Lemma 31.27. *Let $y_0, \dots, y_k \in \mathbb{R}^n$.*

(1) *We have the equality*

$$d(|[y_0, \dots, y_k]|) \;=\; \max\big\{\|y_i - y_j\| \;\big|\; i, j = 1, \dots, k\big\}.$$

(2) *We denote by $b = \frac{1}{k}\sum_{i=0}^{k} y_i$ the barycenter of $y_0, \dots, y_k$. Then for any $i = 0, \dots, k$ we have*

$$\|y_i - b\| \;\le\; d(|[y_0, \dots, y_k]|).$$

Proof of Lemma 31.27 $(*)$.

(1) The inequality "$\ge$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0, \dots, y_k]| \;=\; \Big\{\sum_{i=0}^{k} t_i \cdot y_i \;\Big|\; t_i \in [0,1] \text{ and } \sum_{i=0}^{k} t_i = 1\Big\}.$$

We turn to the proof of the reverse inequality "$\le$". So we pick $a, b \in |[y_0, \dots, y_k]|$. By definition we can write $a = \sum_{i=0}^{k} s_i y_i$ and $b = \sum_{i=0}^{k} t_i y_i$ with $s_i, t_i \in [0,1]$ and $\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then have

$$\begin{aligned}
\|a - b\| \;&\overset{\text{since } \sum_{i=0}^{k} t_i = 1}{\overset{\downarrow}{=}}\; \Big\|a - \sum_{i=0}^{k} t_i y_i\Big\| \;=\; \Big\|\sum_{i=0}^{k} t_i (a - t_i y_i)\Big\| \;\le\; \sum_{i=0}^{k} \|t_i (a - t_i y_i)\| \\
&\le\; \sum_{i=0}^{k} t_i \cdot \max_{i=0,\dots,k}\{\|a - y_i\|\} \;\underset{\text{since } \sum_{i=0}^{k} t_i = 1}{\overset{\uparrow}{=}}\; \max_{i=0,\dots,k}\{\|a - y_i\|\} \;\underset{\text{same argument applied to } a = \sum_{j=0}^{k} s_j y_j}{\overset{\uparrow}{\le}}\; \max_{i,j=0,\dots,k}\{\|y_j - y_i\|\}
\end{aligned}$$

(2) We denote by b_i the barycenter of $y_0, \dots, \widehat{y_i}, \dots, y_k$, i.e.

$$b_i \;=\; \tfrac{1}{k}\sum_{j \ne i} y_j.$$

We then have

$$b \;=\; \tfrac{1}{k+1} y_i + \tfrac{k}{k+1} b_i.$$

It follows that

$$\begin{aligned}
\|b - y_i\| \;=\; \Big\|\tfrac{1}{k+1} y_i + \tfrac{k}{k+1} b_i - y_i\Big\| \;&=\; \Big\|\tfrac{k}{k+1} b_i - \tfrac{k}{k+1} y_i\Big\| \\
&=\; \tfrac{k}{k+1}\|b_i - y_i\| \;\le\; \tfrac{k}{k+1} d(|[y_0, \dots, y_k]).
\end{aligned}$$

Thus we have obtained the desired inequality. □

Now we are ready to provide the proof of Lemma 31.26.

Proof of Lemma 31.26 (∗). We need to show that for any $x_0, \dots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \dots, x_k])$ we have

$$d(|\tau|) \;\leq\; \tfrac{n}{n+1} \cdot d(|[x_0, \dots, x_k]|).$$

We prove the claim by induction on k. For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0, \dots, x_k \in \Delta^n$ and denote by $b = \frac{1}{k}\sum_{i=0}^{k} x_i$ the barycenter of $x_0, \dots, x_k$. We denote by $\sigma_1, \dots, \sigma_r$ the pieces of $\partial[x_0, \dots, x_k]$.[633]

Now let τ be a piece of $u_k([x_0, \dots, x_k])$. By the definition of the subdivision map given on page 789 we see that $|\tau| = |[w_0, \dots, w_{k-1}, b]|$ where $[w_0, \dots, w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1, \dots, r\}$. By Lemma 31.27 (1) it now suffices to prove the following claim.

Claim.

(1) For any $i, j = 0, \dots, k-1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1} d(|[x_0, \dots, x_k]|)$.
(2) For any $i = 0, \dots, k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1} d(|[x_0, \dots, x_k]|)$.

Now we provide a proof of these two inequalities. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\|w_i - w_j\| \;\leq\; d([w_0, \dots, w_{k-1}]) \;\underset{\substack{\uparrow \\ \text{induction hypothesis} \\ \text{applied to } \sigma_s}}{\leq}\; \tfrac{k-1}{k} d(\sigma_s) \;\underset{\substack{\uparrow \\ \text{since } |\sigma_s| \subset |[x_0, \dots, x_k]| \text{ and } \frac{k-1}{k} \leq \frac{k}{k+1}}}{\leq}\; \tfrac{k}{k+1} d([x_0, \dots, x_k]).$$

Now let $i \in \{0, \dots, k-1\}$. We then have

$$\|w_i - b\| \;\underset{\substack{\uparrow \\ \text{Lemma 31.27 (2)}}}{\leq}\; \tfrac{k}{k+1} d(|[w_0, \dots, w_{k-1}]|) \;\underset{\substack{\uparrow \\ \text{since } |[w_0, \dots, w_{k-1}]| \subset |[x_0, \dots, x_k]|}}{\leq}\; \tfrac{k}{k+1} d(|[x_0, \dots, x_k]|)$$

This concludes the proof of the claim. □

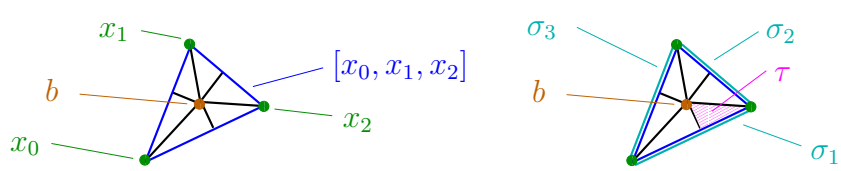


FIGURE 494. Illustration of the proof of Lemma 31.26.

Lemma 31.28. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X. For every singular n-simplex $\sigma \colon \Delta^n \to X$ there exists an m such that $u_n^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$.*

[633] Why does it not say that the number of pieces is $k+1$?

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By Lebesgue's Lemma 1.105 there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $d(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\mathring{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We have

$$\begin{aligned} u_n^m(\sigma) \;=\; u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) \;&\overset{\text{Lemma 31.22}}{=}\; \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \;\overset{\text{by Lemma 31.26 and } \left(\frac{n}{n+1}\right)^m < \delta}{=}\; \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{has diameter less than } \delta\end{pmatrix} \\ &\underset{\text{by definition of } \delta}{=}\; \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{of each piece lies in some } \sigma^{-1}(\mathring{U}_i)\end{pmatrix}. \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in C_n^{\mathcal{U}}(X)$. □

Now we can prove Proposition 31.21.

Proof of Proposition 31.21. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X. We want to show that the inclusion map $\iota\colon C_*^{\mathcal{U}}(X) \to C_*(X)$ induces for each n an isomorphism

$$\iota_*\colon H_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; H_n(X).$$

We first show that the map $\iota_*\colon H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is surjective. Let $z \in H_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j\sigma_j \in C_n(X)$ which represents $z \in H_n(X)$. It follows from Lemma 31.28, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}$ such that $u_n^m(c) \in C_n^{\mathcal{U}}(X)$. By Lemma 31.25 the maps $\{u_n^m\}_{n\in\mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon C_k(X) \to C_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z \;=\; [c] \;\underset{\text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \;=\; u_n^m - \mathrm{id}}{=}\; \big[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\big] \;=\; \big[u_n^m(c) - \partial_{n+1}P_n(c)\big] \;\underset{\text{equality in } H_n(X)}{=}\; \overbrace{[u_n^m(c)]}^{\in \iota_*(H_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is surjective.

The argument that the map $\iota_*\colon H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is injective is very similar. Indeed, let $[c] \in H_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in H_n(X)$. This means that there exists an $e \in C_{n+1}(X)$ with $\partial e = c$. From Lemma 31.28 it follows, as above, that there exists an $m \in \mathbb{N}$ with $u_{n+1}^m(e) \in C_{n+1}^{\mathcal{U}}(X)$. We again choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \;\underset{\text{since } \partial P_{n+1} + P_n\partial = u_{n+1}^m - \mathrm{id}}{=}\; \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial\underbrace{u_{n+1}^m(e)}_{\in C_{n+1}^{\mathcal{U}}(X)} \;-\; \partial\underbrace{P_n(c)}_{\in C_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain, in $C^{\mathcal{U}}_{n+1}(X)$ i.e. $[c] = 0 \in H^{\mathcal{U}}_n(X)$. Put differently, we have now proved that $\iota_*\colon H^{\mathcal{U}}_n(X) \xrightarrow{\cong} H_n(X)$ is injective. $\square$

Using Proposition 31.21 we can now prove the Excision Theorem 31.17.

Proof of the Excision Theorem 31.17. Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \ \xrightarrow{\cong}\ H_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B, put differently, the two sets $\{A, B\}$ is a covering of X.

For each n we consider the following commutative diagram of abelian groups:[634]

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "="] & C_n^{\{A,B\}}(X) \arrow[r] \arrow[d, hook] & C_n^{\{A,B\}}(X)/C_n(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X, A) \arrow[r] & 0.
\end{tikzcd}$$

Note that by definition the horizontal sequences are in fact short exact sequences of chain complexes. We first make the observation that it follows immediately from the definition of $C_n^{\{A,B\}}(X)$ that the inclusion

$$C_n(B) \ \to\ C_n^{\{A,B\}}(X)$$

induces a chain map

$$C_n(B, A \cap B) \ \to\ C_n^{\{A,B\}}(X)/C_n(A)$$

that is in fact an isomorphism. Thus in the above diagram we can replace the chain complex $C_n^{\{A,B\}}(X)/C_n(A)$ by $C_n(B, A \cap B) = C_n(X \setminus Z, A \setminus Z)$.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. From the remark on page 772 we obtain the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "="] & H_n^{\{A,B\}}(X) \arrow[r] \arrow[d] & \overbrace{H_n(B, A \cap B)}^{=H_n(X\setminus Z, A\setminus Z)} \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "="] & H_{n-1}^{\{A,B\}}(X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r, "\partial"] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

[634]Here we use the definition from page 787 which in this context says that

$$C_n^{\{A,B\}}(X) \ :=\ \{\sigma + \tau \,|\, \sigma \in C_n(A) \text{ and } \tau \in C_n(B)\}.$$

From Proposition 31.21 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 31.11 that the middle map is also an isomorphism. □

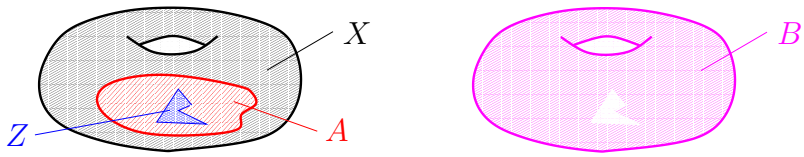


FIGURE 495. Illustration for the proof of the Excision Theorem 31.17.

Exercises for Chapter 31.

Exercise 31.1. Let

$$\mathbb{Z} \xrightarrow{f} B \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}_6 \to 0$$

be an exact sequence, i.e. we know that $g \circ f = 0$, we know that $\operatorname{im}(f) = \ker(g)$ and we know that g is an epimorphism. What are the possible isomorphism types of the group B?

Exercise 31.2. Let X be a topological space with $\widetilde{H}_0(X) \cong \mathbb{Z}$.

(a) Show that given any $\epsilon \in \{-1, 0, 1\}$ there exists a map $f\colon X \to X$ such that the induced map f_* on $\widetilde{H}_0(X)$ is given by multiplication by ϵ.

(b) Show that given any map $f\colon X \to X$ the induced map f_* on $\widetilde{H}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

Exercise 31.3. Determine the homology groups of $S^2 \vee S^1$.

Exercise 31.4.

(a) Let

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

be a commutative diagram such that the horizontal sequences are exact. Show that if a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.

(b) Does the statement in (a) also hold, if a injective and if e is surjective ?

Exercise 31.5. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{\varphi} & B_* & \xrightarrow{\psi} & C_* & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{\varphi'} & B'_* & \xrightarrow{\psi'} & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of homomorphisms between abelian groups. Show that there exists an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(a) \arrow[r, "\varphi"] & \ker(b) \arrow[r, "\psi"] & \ker(c) \arrow[dll, rounded corners] \\
& \operatorname{coker}(a) \arrow[r, "\varphi'"] & \operatorname{coker}(b) \arrow[r, "\psi'"] & \operatorname{coker}(c) \arrow[r] & 0.
\end{tikzcd}$$

This statement is sometimes referred to as the "Snake Lemma".
Hint: Try to prove the statement with as little effort as possible.

Exercise 31.6. For readers who really really enjoy diagram chases we have the following problem on offer. Suppose we are given the following commutative diagram of

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_* \arrow[r] \arrow[d] & B_* \arrow[r, "h_B"] \arrow[d, "v_B"] & C_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_* \arrow[r, "h_D"] \arrow[d, "v_D"] & E_* \arrow[r] \arrow[d] & F_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_* \arrow[r] \arrow[d] & H_* \arrow[r] \arrow[d] & J_* \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical and horizontal sequences are exact. We suppose that the chain complex H_* is acyclic, i.e. we suppose that homology groups of H_* vanish. Show that for any $k \in \mathbb{N}_0$ the following diagram commutes up to a minus sign:

$$\begin{tikzcd}
H_k(D_*) \arrow[r, "v_{D*}"] \arrow[d] & H_k(G_*) & H_{k+1}(K_*) \arrow[l, "\partial", "\cong"'] \arrow[d, "\partial"', "\cong"] \\
H_k(E) \arrow[r, "v_{B*}^{-1}", "\cong"'] & H_k(B) \arrow[r, "h_{B*}"] & H_k(C).
\end{tikzcd}$$

Exercise 31.7. Let X be a topological space, let $x_0 \in X$ be a point and let $n \in \mathbb{N}_0$. Show that the map

$$\begin{tikzcd}
\widetilde{H}_n(X) \arrow[r] \arrow[rrr, bend right, "=:\Phi_{X,x_0}"'] & H_n(X) \arrow[r, "="] & H_n(X, \varnothing) \arrow[r] & H_n(X, \{x_0\})
\end{tikzcd}$$

is an isomorphism.

Exercise 31.8. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that any map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint: use the Brouwer Fixed Point Theorem.

Exercise 31.9. Let (X, A) be a pair of topological spaces. Suppose there exists a deformation retraction $r\colon X \times [0,1] \to X$ from X to some subset $Y = r_1(X)$. We write $B = r_1(A)$. Show that the inclusion induced maps $H_k(Y, B) \to H_k(X, A)$ are isomorphisms.

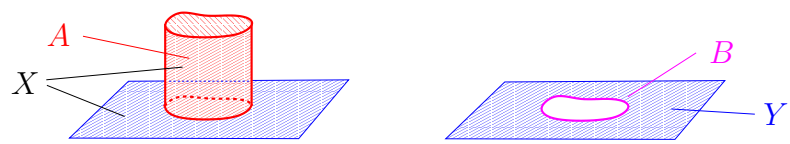


FIGURE 496. Illustration of Exercise 31.9.

Exercise 31.10. Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m$ be a nested sequence of topological spaces.

(a) Suppose that for each $k = 0, \dots, m-1$ we have $H_n(X_{k+1}, X_k) = 0$ for all $n \in \mathbb{N}_0$. Show that $H_n(X_m, X_0) = 0$ for all $n \in \mathbb{N}_0$.
(b) Let $n \in \mathbb{N}_0$. Suppose that for each $k = 0, \dots, m-1$ we have $H_n(X_{k+1}, X_k) = 0$. Does it also follow that $H_n(X_m, X_0) = 0$?

Exercise 31.11. We consider $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Prove the following two statements:

(a) The group $H_1(X, A)$ is countable.
(b) The group $H_1(X/A)$ is uncountable. For psychological reasons it might be convenient to recall that in Exercise 26.6 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings.

Exercise 31.12. Let X be a non-empty bounded subset of $\mathbb{R}^n$. We pick an $r \in \mathbb{R}_{>0}$ such that X is contained in the bounded n-ball $\overline{B}^n_r$ of radius r. We consider the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & \mathbb{R}^n \setminus X \\ x & \mapsto & 2r \cdot x. \end{array}$$

Show that the induced map $f_*\colon H_{n-1}(S^{n-1}) \to H_{n-1}(\mathbb{R}^n \setminus X)$ is a monomorphism.

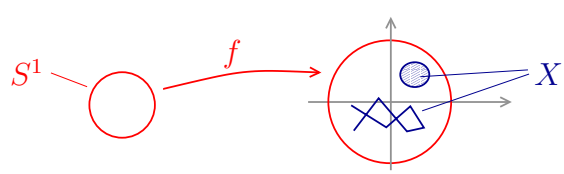


FIGURE 497. Illustration of Exercise 31.12.

Exercise 31.13. Show that for $k \geq 1$ the wedge $S^k \vee S^k$ is not a manifold.

Exercise 31.14. Compute the homology groups of the quotient of S^2 obtained by identifying the North and South Pole to a point. We refer to Figure 498 for an illustration.

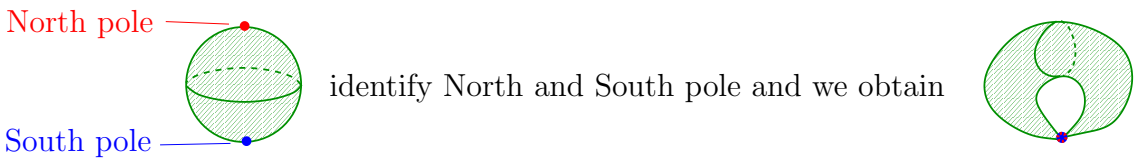

North pole
South pole
identify North and South pole and we obtain

Figure 498.

32. Applications of the long exact sequence in homology

32.1. **Applications to topological manifolds** (∗)**.** In this section we will apply some of the results that we have obtained in the previous chapter to the study of topological manifolds. For the most part these results will generate a sense of déjà vu. In fact we had proved many of these results for manifolds using different methods. The generalization of these results to topological manifolds is non-trivial, but most readers will have a hard time building up enthusiasm for these results. Thus it might be best to have a quick glimpse at the results and then move on to the next section.

First of all we can now generalize Proposition 14.11 from the case of 2-dimensional topological manifolds to the higher-dimensional case.

Proposition 32.1. *Let X be an n-dimensional topological manifold. The following statements hold:*

(1) *Every point on X admits either a chart of type (i) or it admits a chart of type (ii).*
(2) *The boundary $Y := \partial X$ is an $(n-1)$-dimensional topological manifold with $\partial Y = \varnothing$.*
(3) *If X is a closed topological manifold, then ∂X is also a closed $(n-1)$-dimensional topological manifold.*
(4) *If Z is a proper submanifold of X, then $\partial Z = X \cap Y$.*
(5) *Any open subset Y of X is an n-dimensional topological manifold with $\partial Y = Y \cap \partial X$.*
(6) *Let M be an m-topological manifold with empty boundary and let N be an n-dimensional topological manifold with possibly non-empty boundary. Then $M \times N$ is an $(m+n)$-dimensional topological manifold with $\partial(M \times N) = M \times \partial N$.*

This proposition can be viewed as the analogue of Proposition 3.21 for topological manifolds.

Example. It follows from Proposition 32.1 and the examples of charts for $\overline{B}^n$ given on page 173 that the boundary of the n-dimensional topological manifold is indeed S^{n-1}. We had suspected this for a long time, but only do we have the tools for providing a proof.

Remark.

(1) Given a topological manifold X we had on page 171 defined its boundary as the set of all points that do not admit a chart of type (i). If X is equipped with a smooth structure, i.e. if we view X as a manifold, then on page 182 we defined the boundary of the manifold X as the set of all points that do not admit a chart of type (i) *from the given smooth atlas*. A priori we thus had two different definitions of the boundary. Fortunately Proposition 32.1 (1) tells us that these two definitions coincide.
(2) Let M and N be two n-dimensional topological manifolds with empty boundary. If $M \times [0,1]$ is homeomorphic to $N \times [0,1]$, then it follows from Proposition 32.1 (6) that M and N are homeomorphic. Even though this sounds quite reasonable, very similar sounding statements can easily be wrong. For example there exists a 3-dimensional topological manifold W (called the *Whitehead manifold*) which is

non-compact and has no boundary, which is not homeomorphic to $\mathbb{R}^3$ but such that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. We refer to [Kir4, p. 87] and [AR] for details.

Proof. For $n = 2$ we proved (1) and (2) in Proposition 14.11. In the general case the proof of (1) and (2) is formally exactly the same as the proof of Proposition 14.11 where we proved the statement for $n = 2$. We just need to replace the fundamental group by the $(n-1)$-st reduced homology groups and we need to replace the properties of the fundamental group we used in the proof of Proposition 14.11 by the corresponding properties of the reduced homology groups. In particular we need to replace the fact that $\pi_1(S^1) \neq 0$ by the corresponding statement, given in Proposition 31.3, that $\widetilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$ and we need to replace the fact that the fundamental group of a star-shaped subset of $\mathbb{R}^2$ has trivial fundamental group by the fact, obtained in Corollary 30.8, that the $(n-1)$-st reduced homology group of a star-shaped subset of $\mathbb{R}^n$ vanishes. We leave the details to the reader.

We turn to the proofs of the remaining statements:

(3) We assume that X is closed, i.e. it is compact and it has no boundary. By (2) we know that ∂X also has no boundary. Furthermore, by Lemma 3.6 (2) we know that ∂X is compact. Thus ∂X is also a closed topological manifold.
(4) Statement (4) is proved in a fashion similar to (1). We leave the details as an exercise to the reader.
(5) This statement follows easily from (1).
(6) In Proposition 3.4 we already showed that $M \times N$ is an $(m+n)$-dimensional topological manifold. As in the proof of Proposition 3.42 this statement follows easily from (1). □

Let X be a topological manifold and let Y be a submanifold. It follows almost immediately from the definitions that Y is topological manifold in its own right. In particular it makes sense to consider its boundary ∂Y and in particular it makes sense to introduce the following definition.

Definition. Let X be a topological manifold and let Y be a submanifold. We write

$$\partial_0 Y \;:=\; \underset{\substack{\uparrow \\ \text{here } \partial Y \text{ denotes the boundary of } Y \text{ viewed as a topological manifold}}}{\partial Y} \cap \overline{X \setminus Y} \quad \text{and } \partial_1 Y \;:=\; Y \cap \partial X.$$

The definition is illustrated in Figure 499.

The next proposition gives us some results on the boundary of submanifolds. It can be viewed as an analogue of Proposition 3.24.

Proposition 32.2. *Let X be an m-dimensional topological manifold and let Y be an n-dimensional submanifold.*

(1) *$\partial_0 Y$ and $\partial_1 Y$ are disjoint and each of them is a union of components of ∂Y.*
(2) *If Y is a proper submanifold, then $\partial Y = \partial_1 Y = Y \cap \partial X$.*

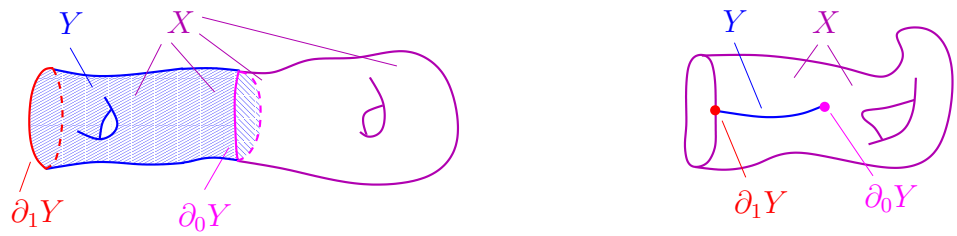


FIGURE 499.

(3) *If Y is of codimension zero, then the interior $\mathring{Y}$ of Y as a subset of X agrees with $Y \setminus \partial_0 Y$. In particular $Y \setminus \partial_0 Y$ is an open subset of X.*
(4) *If Y is of codimension zero and if Y is compact, then the set $Z := X \setminus \mathring{Y}$ is also an n-dimensional submanifold of X and we have $Y \cap Z = \partial_0 Y = \partial_0 Z$.*

Proof. All four statements can be deduced from Proposition 32.1 and the methods of the proof thereof. We leave the details of the proof as an exercise to the reader. □

Lemma 32.3. *Let M be a k-dimensional topological manifold. Then for any $P \in M$ and any $l \in \mathbb{N}_0$ we have*

$$H_l(M, M \setminus \{P\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and } l = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let P be a point on a k-dimensional topological manifold M. First consider the case that $P \in M \setminus \partial M$. It follows from the definition of ∂M, given on page 171, that we can pick a chart $\Phi\colon U \to B^k$ for P with $\Phi(P) = 0$. We then have

$$\begin{array}{ccccccc}
 & & \text{isomorphism since } \Phi \text{ is a homeomorphism} & & & & \\
 & & \downarrow & & & & \\
H_l(M, M \setminus \{P\}) & \leftarrow & H_l(U, U \setminus \{P\}) & \xrightarrow{\Phi_*} & H_l(B^k, B^k \setminus \{0\}) & \xrightarrow{\partial} & \widetilde{H}_{l-1}(B^k \setminus \{0\}) \leftarrow \widetilde{H}_{l-1}(S^{k-1}). \\
 & \uparrow & & & \uparrow & & \uparrow \\
 & \begin{array}{c}\text{isomorphism by the} \\ \text{Excision Theorem 31.17 applied} \\ \text{to } X = M, A = M \setminus \{P\}, Z = M \setminus U\end{array} & & & \begin{array}{c}\text{isomorphism by the long exact} \\ \text{sequence in reduced homology} \\ \text{of the pair } (B^k, B^k \setminus \{0\}) \text{ and the} \\ \text{fact that } \widetilde{H}_*(B^k) = 0 \text{ by page 762}\end{array} & & \begin{array}{c}\text{isomorphism since} \\ S^{k-1} \text{ is homotopy} \\ \text{equivalent to } B^k \setminus \{0\}\end{array}
\end{array}$$

For $P \in M \setminus \partial M$ the statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 31.3.

The case that $P \in \partial M$ is dealt with in a very similar way. The proof of this case is dealt with in Exercise 32.3. □

Examples. Our goal now is to give explicit generators for the relative homology groups $H_k(M, M \setminus \{x\}) \cong \mathbb{Z}$. We first do so for open subsets of $\mathbb{R}^k$ and then we "transport" those generators to topological manifolds using charts.

(1) Let $U \subset \mathbb{R}^k$ be an open subset that contains 0. Let $\sigma\colon \Delta^k \to U$ be an injective affine linear map. We suppose that 0 lies in the image of the interior $\overset{\circ}{\Delta}{}^k := \Delta^k \setminus \partial\Delta^k$ of Δ^k. We claim that σ represents a generator of $H_k(U, U \setminus \{0\})$. To prove this claim we consider the commutative diagram

$$\begin{tikzcd}
H_k(\Delta, \partial\Delta^k) \arrow[rr, "\sigma_*"] \arrow[d, "\sigma_*"] & & H_k(U, U \setminus \{0\}) \arrow[d, "\cong"] \\
H_k(\sigma(\Delta^k), \sigma(\partial\Delta^k)) \arrow[r] & H_k(\mathbb{R}^k, \sigma(\partial\Delta^k)) \arrow[r] & H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})
\end{tikzcd}$$

where the maps that are not induced by σ are induced by the obvious inclusions. The left vertical map is an isomorphism since $\sigma\colon \Delta^k \to \sigma(\Delta^k)$ is a homeomorphism. The horizontal maps are isomorphisms by Corollary 31.16 since $\sigma(\Delta^k)$ is a deformation retract of $\mathbb{R}^k$ and since $\sigma(\partial\Delta^k)$ is deformation retract of $\mathbb{R}^k \setminus \{0\}$. By Lemma 32.9 we know that id_{Δ^n} represents a generator of $H_n(\Delta^n, \partial\Delta^n)$. It follows from the above that $\sigma \circ \mathrm{id}_{\Delta^n} = \sigma$ represents a generator of $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$. But the right vertical map is an isomorphism by the Excision Theorem 31.17, hence σ also represents a generator of $H_k(U, U \setminus \{0\})$.

(2) Let M be a k-dimensional topological manifold and let $P \in M \setminus \partial M$. In the following we want to give an example of an explicit singular k-simplex that represents a generator for $H_k(M, M \setminus \{P\}) \cong \mathbb{Z}$. We pick a chart $\Phi\colon U \to B^k$ around P with $\Phi(P) = 0$ and we pick an affine linear map $\sigma\colon \Delta^k \to B^k$ as in (1). It follows from (1) and the proof of Lemma 32.3 together that $\Phi^{-1} \circ \sigma\colon \Delta^k \to M$ represents a generator of $H_k(M, M \setminus \{P\})$. We refer to Figure 500 for an illustration.

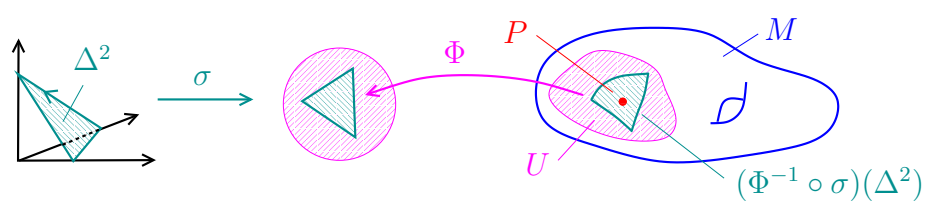


FIGURE 500.

We will prove the following corollary in Exercise 32.2.[635]

Corollary 32.4. *If M is a non-empty topological manifold, then $M \neq \partial M$, i.e. there exists a point in $M \setminus \partial M$.*

The following proposition is the generalization of Proposition 3.18 from manifolds to topological manifolds.

[635]One might think that the statement of the corollary is obvious, but it does not follow from the definition of a topological manifold. One really does need to use Lemma 32.3.

Proposition 32.5. *Let M be an m-dimensional non-empty topological manifold and let N be an n-dimensional topological manifold. If M is homeomorphic to an open subset of N (e.g. if M is homeomorphic to N), then $m = n$.*

It follows in particular from Proposition 32.5 that the dimension of a non-empty topological manifold is well-defined.

Proof. By Proposition 32.1 (5) it suffices to consider the case that we are given a homeomorphism $f\colon M \to N$ between a non-empty m-dimensional topological manifold M and a topological manifold N. By Corollary 32.4 we can find a point $P \in M \setminus \partial M$. Then

$$\begin{array}{ccccc} \mathbb{Z} & \cong & H_m(M, M \setminus \{P\}) & \xrightarrow{\cong} & H_m(N, N \setminus \{f(P)\}). \\ & \uparrow & & \uparrow & \\ & \text{Lemma 32.3} & & \text{since } f \text{ is a homeomorphism} & \end{array}$$

It now follows from Lemma 32.3 that N is also m-dimensional. □

We conclude this section with the following variation of the Excision Theorem 31.17.

Theorem 32.6. *Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that following statements hold:*

(1) *$X \setminus Z$ is a deformation retract of $X \setminus U$*
(2) *$A \setminus Z$ is a deformation retract of $A \setminus U$.*

Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A).$$

The hypotheses of Theorem 32.6 might sound complicated, but as we will see, in practice these conditions are often satisfied.

Example. Let $n \in \mathbb{N}$. We denote by $\overline{B}^{\pm}_n$ the upper and lower hemisphere of S^n and we view S^{n-1} as a subset of S^n in the obvious way. It follows from Theorem 32.6 and the choices of X, A, Z and U illustrated in Figure 501 that the inclusion induced map

$$H_n(\overline{B}^n_+, S^{n-1}) \to H_n(S^n, \overline{B}^n_-)$$

is an isomorphism.

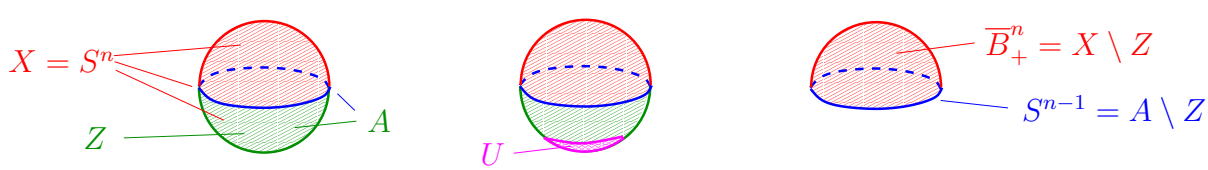


FIGURE 501. excision-lower-hemisphere

We can easily generalize the above example. More precisely, Theorem 32.6 also allows us to prove the following variation of the Excision Theorem 31.17. It says, under mild technical hypothesis, that in the relative homology of an m-dimensional manifold we can excise the interior of an m-dimensional submanifold.

Theorem 32.7. (Excision Theorem for Manifolds) *Let X be an m-dimensional manifold and let A be a subset of X. Furthermore let B be a compact m-dimensional submanifold of X that is contained in A. We write*[636] $\mathring{B} := B \setminus \partial_0 B$. *If $\mathring{B} \subset A$, then for every $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n\big(X \setminus \mathring{B}, A \setminus \mathring{B}\big) \xrightarrow{\cong} H_n(X, A)$$

is an isomorphism.

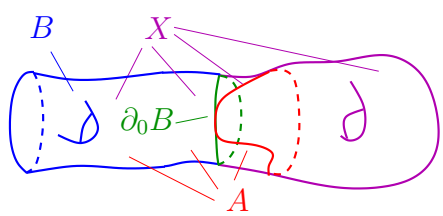


FIGURE 502. Illustration of Theorem 32.7.

Proof of Theorem 32.7 using Theorem 32.6. By Proposition 32.2 we know that $\partial_0 B$ is a union of boundary components of B. Thus we can apply the Collar Neighborhood Theorem 10.10 to $\partial_0 B$. We obtain an embedding $\varphi\colon \partial_0 B \times [-1,0] \to B$ that is the identity map on $\partial_0 B \times \{0\} = \partial_0 B$. To simplify the notation we identify $\partial_0 B \times [-1,0]$ with its image in B. We set $Z = \mathring{B} = B \setminus \partial_0 B \times \{0\}$ and we set $U = B \setminus (\partial_0 B \times (-1,0])$.

Claim. There exists a deformation retract from $X \setminus U$ to $X \setminus Z$.

First note that we can write $X \setminus U = (X \setminus Z) \cup (\partial_0 B \times [-1,0])$. Furthermore note that $(X \setminus Z) \cap (\partial_0 B \times [-1,0]) = \partial_0 B \times \{0\}$. The idea is to use Lemma 12.10 to conclude that the obvious deformation retraction from $\partial_0 B \times [-1,0]$ to $\partial_0 B \times \{0\}$ gives rise to the desired deformation retraction.

Unfortunately we have to go to the laborious task of verifying the hypothesis of Lemma 12.10, i.e. we need to argue that $X \setminus Z$ and $\partial_0 B \times [-1,0]$ are closed subsets of $X \setminus U$. To do so we note that the hypothesis that B is compact implies by Proposition 3.21 that $\partial_0 B$ is compact and thus that $(\partial_0 B \times [-1,0]$ is compact. It follows from Lemma 1.18 that $\partial_0 B \times [-1,0]$ is a closed subset of $X \setminus U$. Furthermore, since $Z = \mathring{B}$ we see that $X \setminus Z$ is a closed subset of X. It follows from Lemma 1.6 that $X \setminus Z$ is also a closed subset of $X \setminus U$. Thus we have now verified that the hypothesis of Lemma 1.6 hold. This concludes the proof of the claim.

[636]The amnesic reader is reminded of the fact that we defined $\partial_0 B$ on page 805.

Claim. There exists a deformation retract from $A \setminus U$ to $A \setminus Z$.

Since $\mathring{B} \subset A$ we have $A \setminus U = (A \setminus Z) \cup (\partial_0 B \times [-1,0])$. Furthermore note that $(A \setminus Z) \cap (\partial_0 B \times [-1,0]) = \partial_0 B \times \{0\}$. Thus basically the same argument as in the proof of the previous claim shows that such a deformation retraction exists. This concludes the proof of the claim.

The statement of Theorem 32.7 now follows immediately from Theorem 32.6. □

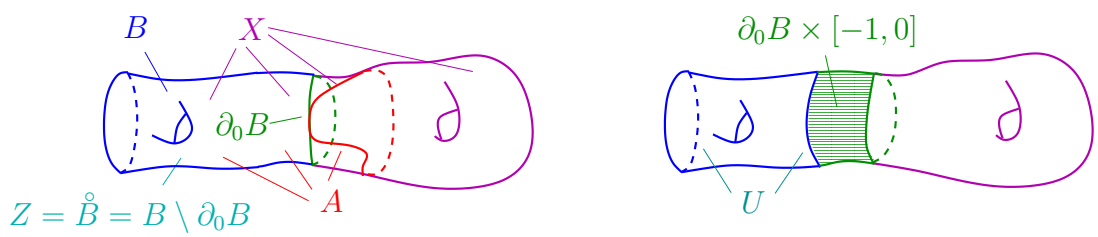


FIGURE 503. Illustration of the proof of Theorem 32.7.

Proof of Theorem 32.6. We consider the following commutative diagram of inclusion induced maps

$$\begin{tikzcd} H_n(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d] & H_n(X, A) \\ H_n(X \setminus U, A \setminus U). \arrow[ur, "\cong"'] & \end{tikzcd}$$

The diagonal map is an isomorphism by the Excision Theorem 31.17. Thus it suffices to show that the vertical map is an isomorphism.

We consider the following commutative diagram:

$$\begin{tikzcd} H_n(A \setminus Z) \arrow[r] \arrow[d, "="] & H_n(X \setminus Z) \arrow[r] \arrow[d] & H_n(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d] & H_{n-1}(A \setminus Z) \arrow[r] \arrow[d, "="] & H_{n-1}(X \setminus Z) \arrow[d] \\ H_n(A \setminus Z) \arrow[r] \arrow[d] & H_n(X \setminus U) \arrow[r] \arrow[d, "="] & H_n(X \setminus U, A \setminus Z) \arrow[r] \arrow[d] & H_{n-1}(A \setminus Z) \arrow[r] \arrow[d] & H_{n-1}(X \setminus U) \arrow[d, "="] \\ H_n(A \setminus U) \arrow[r] & H_n(X \setminus U) \arrow[r] & H_n(X \setminus U, A \setminus U) \arrow[r] & H_{n-1}(A \setminus U) \arrow[r] & H_{n-1}(X \setminus U). \end{tikzcd}$$

Here the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by Corollary 31.14. Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X \setminus Z$ is a deformation retract of $X \setminus U$ and from Corollary 31.16 that the second and the fifth vertical map are also isomorphisms. It follows from the Five Lemma 31.11 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A \setminus Z$ is a deformation retract of $A \setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. □

32.2. **Explicit generators of homology groups.** By Proposition 31.3 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{H}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the homology groups of all finite-dimensional spheres. In this section, given $n \in \mathbb{N}_0$, we want to determine an explicit singular n-chain which is a generator of $\widetilde{H}_n(S^n)$.

We start out with the following elementary lemma that we will use explicitly and implicitly on many occasions.

Lemma 32.8. *Let $n \in \mathbb{N}_0$. We have*

$$\widetilde{H}_k(\overline{B}^n, S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } k = n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$:

$$\ldots \;\to\; \underbrace{\widetilde{H}_k(\overline{B}^n)}_{= 0,\ \text{see page } 762} \;\to\; H_k(\overline{B}^n, S^{n-1}) \;\xrightarrow{\partial}\; \widetilde{H}_{k-1}(S^{n-1}) \;\to\; \underbrace{\widetilde{H}_{k-1}(\overline{B}^n)}_{= 0,\ \text{see page } 762} \;\to\; \ldots$$

The statement of the lemma is now an immediate consequence of Proposition 31.3. □

We continue with the following lemma.

Lemma 32.9. *For any $n \in \mathbb{N}_0$ the following two statements hold:*

(1) $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
(2) *The identity map* $\mathrm{id}\colon \Delta^n \to \Delta^n$ *represents a generator of* $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. The first statement is an immediate consequence of Lemmas 32.8 and 29.1. Now we turn to the proof of the second statement. We prove the second statement by induction on n. It follows easily from Proposition 29.5 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n - 1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $H_n(\Delta^n, \partial\Delta^n)$ and $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. We introduce the following notation:

(1) We consider

$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0\},$$

i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. It is elementary[637] to show that Λ is a deformation retract of Δ^n. We refer to Figure 504 for an illustration.

(2) We write

$$\mathring{\Lambda} := \bigcup_{i=0}^{n-1} \{(t_0, \dots, t_n) \in \Lambda \,|\, t_i = 0 \text{ and } t_n > 0\},$$

the set $\mathring{\Lambda}$ is thus the "interior"[638] of Λ.

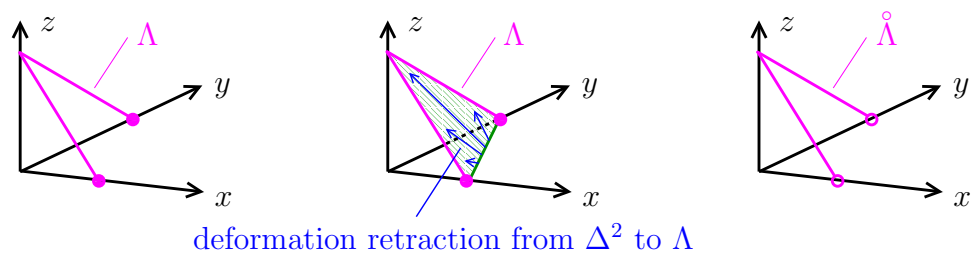


FIGURE 504.

In the following we view $\mathbb{R}^n$ as the subset of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subset of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.

Claim. We consider the maps

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{i_*} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Here the map on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right-hand map is induced by the inclusion. Then the following holds:

(1) Both maps are isomorphisms.
(2) $\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot i_*([\mathrm{id}_{\Delta^{n-1}}])$.

It is clear that a proof of the claim together with the induction hypothesis implies that $[\mathrm{id}_{\Delta^n}]$ is a generator of $H_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies Lemma 32.9.

Now we turn to the proof of the claim. Of the two maps in the claim we first consider the map ∂ on the left. It follows from the exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ that

[637]When we say that it is "elementary to show something" this usually means that one can show the statement using basic methods without the need to use advanced results. In this case one can write down a deformation retraction "by hand", but it is not entirely easy to do so. We leave it as an exercise to write down such a deformation retraction.

[638]Why is "interior" put in quotes?

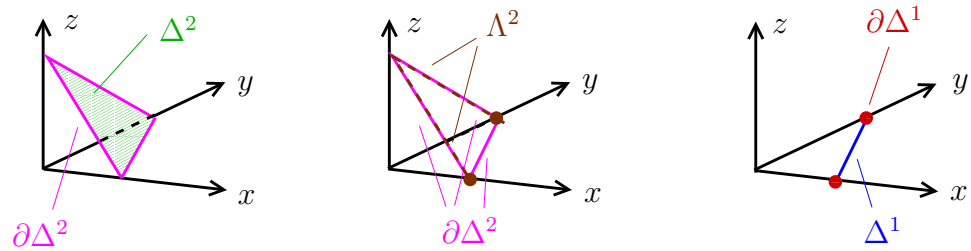


FIGURE 505. Illustration of the pairs $(\Delta^n, \partial\Delta^n)$, $(\partial\Delta^n, \Lambda)$ and $(\Delta^{n-1}, \partial\Delta^{n-1})$.

the following sequence is exact:

$$\underbrace{H_n(\Delta^n, \Lambda)}_{\substack{= 0 \text{ by Corollary 31.16 since} \\ \Lambda \text{ is a def. retract of } \Delta^n}} \xrightarrow{p} H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{i_*} \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{\substack{= 0 \text{ by Corollary 31.16 since} \\ \Lambda \text{ is a def. retract of } \Delta^n}}.$$

This shows that $H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore we have

$$\partial([\mathrm{id}_{\Delta^n}]) \underset{\substack{\uparrow \\ \text{Proposition 31.13}}}{=} [\partial\, \mathrm{id}_{\Delta^n}] = \Big[\underbrace{\sum_{j=0}^{n-1} (-1)^j\, \mathrm{id} \circ i_j^n}_{\text{lies in } C_{n-1}(\Lambda)} + (-1)^n \cdot \underbrace{\mathrm{id} \circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}}\Big] \underset{\substack{\uparrow \\ \text{since we work in } H_{n-1}(\partial\Delta^n, \Lambda)}}{=} (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}].$$

Now we turn our attention to the right-hand map in the statement of the claim. It follows from Theorem 32.6, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0, \ldots, 0, 1)$[639] that the inclusion induced map

$$H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = H_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Lambda}}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} H_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism. Furthermore we evidently have $i_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) = (-1)^{\cdot}[\mathrm{id}_{\Delta^{n-1}}] = (-1)^n \cdot i_*([\mathrm{id}_{\Delta^{n-1}}]).$$

This concludes the proof of the claim and thus also of the lemma. □

Now we want to find explicit generators of $H_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$. We consider the map $\alpha\colon \Delta^n \to S^n$ that is given by the concatenation of the following two maps

$$\begin{array}{rcl} \Delta^n \xrightarrow{\text{Lemma 29.1}} \overline{B}^n & \to & S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ x & \mapsto & \big(x, \sqrt{1 - \|x\|^2}\big). \end{array}$$

This map restricts to a homeomorphism $\Delta^n \to S^n_{\geq 0} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\}$ and it sends $\partial\Delta^n$ to $S^{n-1} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}$. The definition of α is also sketched in Figure 506.

[639] One can easily show that U has the property required in Theorem 32.6.

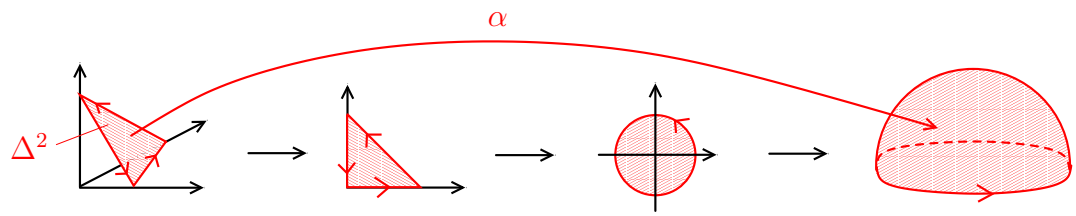


FIGURE 506.

Now we denote by $\beta\colon \Delta^n \to S^n$ the map which is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$. The maps $\alpha\colon \Delta^n \to S^n$ and $\beta\colon \Delta^n \to S^n$ both define singular n-chains in S^n. Note that α and β agree on $\partial\Delta^n$. Both maps are sketched in Figure 507.

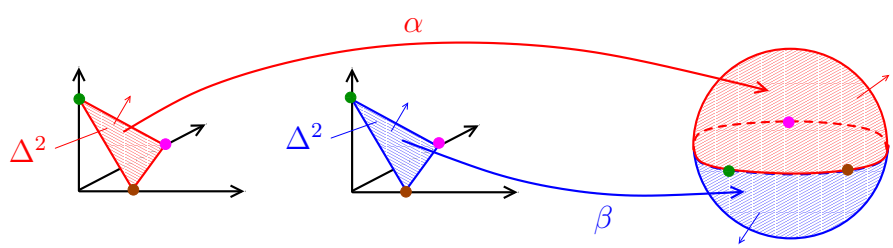


FIGURE 507.

Lemma 32.10. *Let $n \geq 1$. The above singular n-chain $\alpha - \beta$ is a cycle and it represents a generator of $\widetilde{H}_n(S^n) = H_n(S^n)$.*

Definition.

(1) We refer to the generator from Lemma 32.10 as the *standard generator of $H_n(S^n)$*.[640] With great foresight we denote it by $[S^n]$.
(2) Slightly more generally, if I is an interval and x is a point in I, then the inclusion $S^n \times \{x\} \to S^n \times I$ is a homotopy equivalence which implies by Corollary 30.8 that $H_n(S^n \times \{x\}) \to H_n(S^n \times I)$ is an isomorphism. We refer to the image of the standard generator of $H_n(S^n \times \{x\}) = H_n(S^n)$ as the *standard generator* of $H_n(S^n \times I)$.[641]

Proof. We start out with the following claim.

Claim. The singular n-chain $\alpha - \beta$ is a cycle.

[640]It follows easily from Lemma 29.3 and the definitions that the two definitions of standard generator of $H_1(S^1)$ agree.

[641]It follows from Corollary 30.8 that the definition of the standard generator of $H_n(S^n \times I)$ does not depend on the choice of x.

We denote by ρ the reflection in the hyperplane defined by $x_{n+1} = 0$. We have

$$\begin{array}{ccccccc} & & & & \text{since } \alpha(\partial\Delta^n) \subset \mathbb{R}^n \times 0 \text{ we have } \partial\alpha \in C_{n-1}(\mathbb{R}^n \times 0) & & \\ & & & & \downarrow & & \\ \partial(\alpha - \beta) & = & \partial(\alpha - \rho_*(\alpha)) & = & (\mathrm{id} - \rho_*)(\partial\alpha) & = & 0. \\ & & & \uparrow & & \uparrow & \\ & & & \text{since } \rho_* \text{ is a chain map} & & \text{since } \rho = \mathrm{id} \text{ on } \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1} & \end{array}$$

This concludes the proof of the claim.

We denote by

$$S^n_{\geq 0} \quad := \quad \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\}$$

and

$$S^n_{\leq 0} \quad := \quad \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \leq 0\}$$

the two "hemispheres". By the discussioin on page 73 both hemispheres are homeomorphic to the closed ball $\overline{B}^n$, in particular they are contractible which implies by the discussion on page 762 that their reduced homology groups vanish. We also consider the "equator"

$$S^n_{=0} \quad := \quad \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}.$$

Next we consider the following sequence of maps

$$\begin{array}{ccccccc} \text{isomorphism by the long exact} & & \text{isomorphism by the} & & & & \\ \text{sequence in reduced homology} & & \text{Excision Theorem} & & \text{isomorphism since} & & \\ \text{of the pair } (S^n, S^n_{\leq 0}) & & \text{for Manifolds 32.7} & & \alpha \text{ is a homeomorphism} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \widetilde{H}_n(S^n) & \to & H_n(S^n, S^n_{\leq 0}) & \xleftarrow{i_*} & H_n(S^n_{\geq 0}, S^n_{=0}) & \xleftarrow{\alpha_*} & H_n(\Delta^n, \partial\Delta^n) \\ [\alpha - \beta] & \mapsto & [\alpha - \beta] = [\alpha] & \leftarrow\!\shortmid & [\alpha] & \leftarrow\!\shortmid & [\mathrm{id}_{\Delta^n}]. \\ & & \uparrow & & & & \\ & & \text{since } \beta = 0 \text{ in } C_n(S^n, S^n_{\leq 0}) & & & & \end{array}$$

So we have seen that all maps are isomorphisms. Furthermore, by Lemma 32.9 we know that $[\mathrm{id}_{\Delta^n}]$ is a generator of $H_n(\Delta^n, \partial\Delta^n)$. But $[\mathrm{id}_{\Delta^n}]$ and $[\alpha - \beta]$ get sent to the same element in $H_n(S^n, S^n_{\leq 0})$, therefore $[\alpha - \beta]$ must also be generator of $\widetilde{H}_n(S^n)$. $\square$

In the following lemma we give an alternative description of the standard generator of $H_1(S^1)$.

Lemma 32.11. *The singular 1-simplex*

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & S^1 \\ (1-t, t) & \mapsto & e^{2\pi i t} \end{array}$$

is a cycle that represents the standard generator $[S^1]$ *of* $H_1(S^1) \cong \mathbb{Z}$.

FIGURE 508. Two representatives of the standard generator of $H_1(S^1)$.

Proof. We denote by $\alpha, \beta\colon \Delta^1 \to S^1$ the singular 1-simplices in the definition of the standard generator of $H_1(S^1)$. We have

$$\begin{array}{ccccccc} [\alpha - \beta] & = & [\alpha + \overline{\beta}] & = & [\sigma] & \in & H_1(S^1). \\ & \uparrow & & \uparrow & & & \\ & \text{by Lemma 29.3 (2) and (3)} & & & & & \end{array}$$

We refer to Figure 508 for an illustration. □

Remark. By Lemma 32.11 we know that there exists a singular 1-simplex σ that represents a generator of $H_1(S^1)$. In Exercise 32.1 we will discuss the question, for which $n \in \mathbb{N}$ does there exist a singular n-simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $H_n(S^n)$.

We conclude this section with a short discussion of several related homology groups and their "standard generators".

Definition. Let $n \in \mathbb{N}$. The *standard generator of* $H_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element that gets sent to the standard generator of $\widetilde{H}_{n-1}(S^{n-1})$ under the connecting homomorphism $H_n(\overline{B}^n, S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$. We denote the standard generator by $[\overline{B}^n]$.

We continue with the following lemma that relates several of the generators of homology groups that we have encountered so far.

Lemma 32.12. *Let $n \in \mathbb{N}$. We denote by $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from Lemma 29.1 and we denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from page 99. We consider the following diagram of isomorphisms*

$$\begin{array}{ccccccccc} H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{f_*} & H_n(\overline{B}^n, S^{n-1}) & \longrightarrow & H_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) & \xrightarrow{g_*} & H_n(S^n, *) & \longleftarrow & H_n(S^n). \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & & & & & \\ \widetilde{H}_{n-1}(\partial\Delta^{n-1}) & \xrightarrow{f_*} & \widetilde{H}_{n-1}(S^{n-1}) & & & & & & \end{array}$$

Then the following statements hold:

(1) *The standard generator $[S^n]$ of $H_n(\overline{B}^n, S^{n-1})$ has the same image in $H_n(S^n, *)$ as the standard generator $[\overline{B}^n]$ of $H_n(S^n)$.*

(2) *We have*

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \textit{standard generator } [\overline{B}^n] \textit{ of } H_n(\overline{B}^n, S^{n-1}).$$

(3) *The image of the standard generator* $[\overline{B}^n]$ *in* $H_{n-1}(S^{n-1})$ *equals the standard generator* $[S^{n-1}]$ *of* $H_{n-1}(S^{n-1})$.

(4) *We have*

$$f_*(\partial[\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \textit{standard generator } [S^{n-1}] \textit{ of } H_{n-1}(S^{n-1}).$$

Proof. The proof of the first two statements is an annoying exercise which mischievously we leave to the reader. The third statement is just a repetition of the definition of the standard generator $[\overline{B}^n]$. Finally the last statement is an immediate consequence of (2) and (3) together with the fact, shown in Proposition 31.13, that the square in the above diagram commutes. □

We conclude this section with the following lemma.

Lemma 32.13. *Let* $n \in \mathbb{N}$. *We consider the following sequence of maps*

$$H_n(S^n) \;\to\; H_n(S^n, \overline{B}^n_+) \;\underset{\substack{\uparrow \\ \textit{isomorphism by the discussion on page 808}}}{\overset{\cong}{\longleftarrow}}\; H_n(\overline{B}^n_+, S^{n-1}) \;\xrightarrow{\partial}\; \widetilde{H}_{n-1}(S^{n-1}).$$

The image of the standard generator $[S^n] \in H_n(S^n)$ *under this map is* $(-1)^n \cdot [S^{n-1}]$.

Proof. Let $\alpha, \beta \in C_n(S^n)$ be the n-chains in the definition of the standard generator $[S^n]$. It follows easily from the definitions of the above maps that $[S^n] = [\alpha - \beta]$ gets sent to $[\partial\alpha] \in H_{n-1}(S^{n-1})$. But by Lemma 32.12 (4) this equals $(-1)^n \cdot [S^{n-1}]$. □

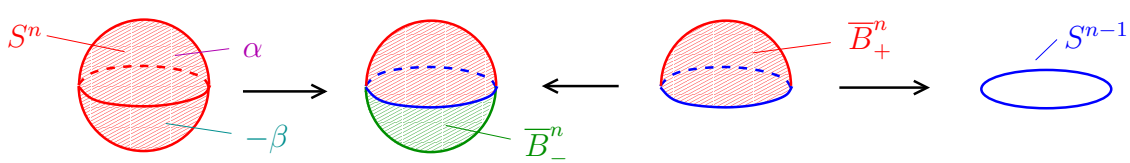


FIGURE 509. Illustration for the proof of Lemma 32.13.

32.3. **Suspensions and homology groups.** In the next section we will study self-maps of spheres. In that section it will be useful to consider the effect of taking suspensions on homology. Since this is going to be interesting for larger classes of topological spaces we dedicate this short section to the study of homology groups of suspensions of topological spaces.

The following definition follows the definition on page 452 and the convention given by Lemma 15.2 (3).

Definition. Let X be a topological space.

(1) We define the *suspension of* X to be the topological space
$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$
where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.

(2) We denote by $p\colon X \times [-1,1] \to \Sigma(X)$ be the obvious projection. We write
$$C_-X \;:=\; p(X \times [-1,1)) \quad \text{and} \quad C_+X \;:=\; p(X \times (-1,1]).$$

As shown in Lemma 15.2 (3) we can view $X = X \times \{0\}$ as a subspace of $C_\pm(X)$ and of $\Sigma(X)$.

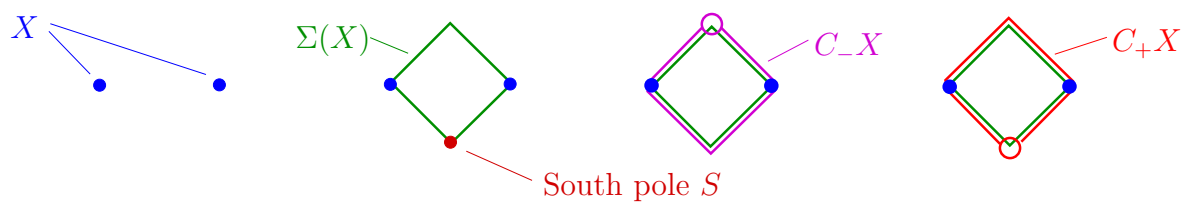


FIGURE 510.

The following proposition is the main result of this section.

Proposition 32.14. *Let X be a non-empty topological space and let $n \in \mathbb{Z}$.*[642]

(1) *The following maps are isomorphisms:*[643]
$$\widetilde{H}_n(X) \;\xleftarrow{\partial}\; H_{n+1}(C_+X, X) \;\to\; H_{n+1}(\Sigma(X), C_-X) \;\leftarrow\; \widetilde{H}_{n+1}(\Sigma(X)).$$
The first horizontal map is the connecting homomorphism of the pair (C_+X, X) from Corollary 31.14. The second horizontal map is induced by the inclusion $(C_+X, C_-X \cap C_+X) \to (\Sigma(X), C_-X)$. Finally the third horizontal map is the natural map given by Lemma 31.1 (3).

(2) *The resulting isomorphism*
$$\Sigma_X\colon \widetilde{H}_n(X) \;\xrightarrow{\cong}\; \widetilde{H}_{n+1}(\Sigma(X))$$
is natural. In other words, given any map $f\colon X \to Y$ between topological spaces the following diagram commutes:
$$\begin{array}{ccc} \widetilde{H}_n(X) & \xrightarrow{\Sigma_X} & \widetilde{H}_{n+1}(\Sigma(X)) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma(f)_*} \\ \widetilde{H}_n(Y) & \xrightarrow{\Sigma_Y} & \widetilde{H}_{n+1}(\Sigma(Y)). \end{array}$$

Proof. Let $n \in \mathbb{N}$.

[642]Note that the proposition has non-trivial content for $n = -1$.

[643]We use Lemma 15.2 (3) to make the identification $X = C_-X \cap C_+$.

(a) By the same argument as in the proof of Lemma 15.1 we see that both C_-X and C_+X admit a deformation retraction to a point. It follows from Lemma 9.10 that both are contractible, therefore by Lemma 31.1 (7) the reduced homologies of C_-X and C_+X vanish. It follows in particular that the connecting homomorphism $\partial\colon H_{n+1}(C_+(X), X) \to \widetilde{H}_n(X)$ is an isomorphism.
(b) It is straightforward to show that the obvious map $X \times (-1,1) \to C_-X \cap C_+X$ is a homeomorphism. It follows easily that X is a deformation retract of $C_-X \cap C_+X$. Thus we obtain from Corollary 31.16 that the first horizontal map is an isomorphism.
(c) We denote by $S := [X \times \{-1\}] \in \Sigma(X)$ the "South Pole" of the suspension. Note that $\Sigma(X) \setminus S = C_+X$ and $C_-X \setminus S = C_-X \cap C_+X$. Using this observation we now see that the second horizontal map is an isomorphism by the Excision Theorem 31.17.[644]
(d) Similar to (a) we obtain from Corollary 31.16 that the third horizontal map is an isomorphism.
(d) First note that the connecting homomorphism is natural by Proposition 31.13 (2) and the fact that $\Sigma(f)|_X = f$. It is clear that all the horizontal maps in the definition of the isomorphism Σ_X are natural. Thus Σ_X itself is natural. □

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 453. It follows quite easily from the definition of the suspension isomorphism $\Sigma_{S^n}\colon H_n(S^n) \to H_{n+1}(S^{n+1})$ and Lemma 32.13 that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \;\in\; H_{n+1}(S^{n+1}).$$

32.4. **The degree of a self-map of a sphere.** Now that we know the homology groups of spheres, we want to study the maps on homology induced by maps $f\colon S^n \to S^n$. This is interesting in its own right, but it will also play a major role later on when we will try to determine the homology groups of CW-complexes.

Definition. Let $n > 0$ and let $f\colon S^n \to S^n$ be map. Since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$ we know that the induced map $f_*\colon H_n(S^n) \to H_n(S^n)$ is given by multiplication by a uniquely determined integer $\deg(f) \in \mathbb{Z}$ which we call the *degree of* f.

Remark. Let $n > 0$ and let $f\colon S^n \to S^n$ be a map. If $a \in H_n(S^n)$ is non-trivial, then the degree $\deg(f)$ is uniquely determined by the equality $f_*(a) = \deg(f) \cdot a$.

Example. We consider the map $f\colon S^1 \to S^1$ that is illustrated in Figure 511 on the left.[645] We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t, t) = e^{2\pi i t}$. By Lemma 32.11 we know that $[\tau] \in H_1(S^1)$ is a generator.

[644] One can easily verify that the hypotheses for the Excision Theorem 31.17 are satisfied.

[645] More precisely, given an "interval" on S^1 going from $e^{i\alpha}$ to $e^{i\beta}$ we mean the singular 1-simplex that is given by $(1-t, t) \mapsto e^{i(\alpha\cdot(1-t)+\beta\cdot t)}$.

We pick singular 1-simplices $\sigma_1, \dots, \sigma_5$ as in the middle of Figure 511 and we denote by $\gamma\colon \Delta^1 \to S^1$ the constant singular 1-simplex that sends all points to 1. Then

$$\begin{aligned} f_*([\tau]) \overset{\text{by Lemma 29.3 (3)}}{\overset{\downarrow}{=}} f_*([\sigma_1 + \dots + \sigma_5]) &= [\overbrace{\sigma_1 \circ f}^{=\gamma} + \overbrace{\sigma_2 \circ f}^{=\tau} + \overbrace{\sigma_3 \circ f}^{=\gamma} + \overbrace{\sigma_4 \circ f}^{=\overline{\tau}} + \overbrace{\sigma_5 \circ f}^{=\tau}] \\ &= [2 \cdot \tau + \overline{\tau} + 2 \cdot \gamma] \underset{\substack{\uparrow \\ \text{Lemma 29.3 (2)}}}{=} [\tau] = \underbrace{(+1)}_{=:\deg(f)} \cdot [\tau]. \end{aligned}$$

Summarizing we have shown that $\deg(f) = +1$. With almost the same approach we can

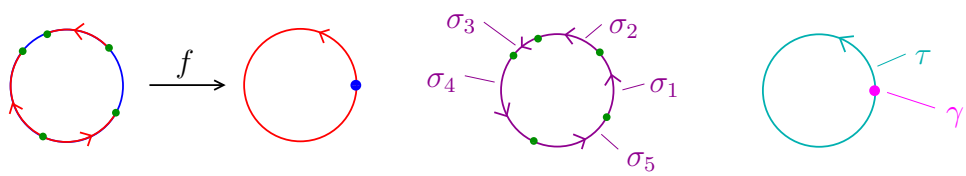
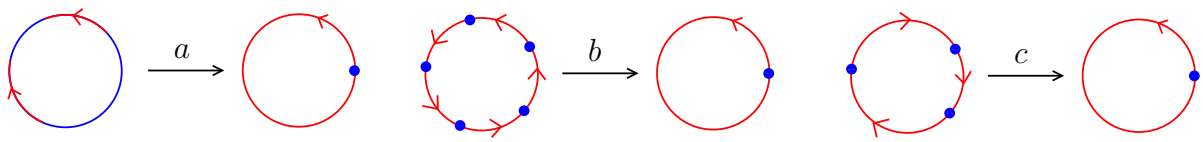


FIGURE 511.

calculate the degrees of the maps that are illustrated in Figure 512.



$\deg(a) = 0$ $\deg(b) = +5$ $\deg(c) = -3$

FIGURE 512.

The same arguments that we used in discussion of the examples gives us the following lemma. We leave the details as an exercise to the reader.

Lemma 32.15. *For any $k \in \mathbb{Z}$ the degree of the map*

$$\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{array}$$

equals k.

Proof. We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t, t) = e^{2\pi i t}$. By Lemma 32.11 we know that $[\tau] \in H_1(S^1)$ is a generator.

Now we first consider the case $k \geq 1$. For $j = 0, \dots, k-1$ we denote by $\sigma_j \colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t, t) \mapsto e^{2\pi i(\frac{t+j}{k})}$. Then

$$f_*([\tau]) \underset{\substack{\uparrow \\ \text{by Lemma 29.3 (3)}}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = \big[\overbrace{f \circ \sigma_0}^{=\tau} + \cdots + \overbrace{f \circ \sigma_{k-1}}^{=\tau}\big] = k \cdot [\tau].$$

The case $k < 0$ is proved is proved in a similar fashion using Lemma 29.3 (2). Finally the case $k = 0$ follows from the fact that the constant map $S^1 \to S^1$, $z \mapsto 1$ factors through a point which by Lemma 29.6 has trivial first homology. $\square$

In the following lemma we summarize several properties of the degree of the self-map of a sphere.

Lemma 32.16. *Let $f, g \colon S^n \to S^n$ be two maps. Then the following hold:*

(1) $\deg(\mathrm{id}_{S^n}) = 1$,
(2) *if f is not surjective, then* $\deg(f) = 0$,
(3) *if f is homotopic to g, then* $\deg(f) = \deg(g)$,
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$,
(5) *if f is a self-homeomorphism of S^n, then* $\deg(f^{-1}) = \deg(f)$,
(6) *if f is the reflection in a hyperplane, then* $\deg(f) = -1$,
(7) *if f is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then we have* $\deg(f) = (-1)^{n+1}$,

Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{H_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f \colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram

$$\begin{tikzcd} & S^n \setminus \{P\} \arrow[dr, hook, "i"] & \\ S^n \arrow[ur, "f"] \arrow[rr, "f"'] & & S^n \end{tikzcd}$$

which gives rise to the commutative diagram

$$\begin{tikzcd} & H_n(S^n \setminus \{P\}) \arrow[dr, "i_*"] & \\ H_n(S^n) \arrow[ur, "f_*"] \arrow[rr, "f_*"'] & & H_n(S^n). \end{tikzcd}$$

Since $S^n \setminus \{P\}$ is homeomorphic to B^n we know that it is contractible which implies by Corollary 30.8 that $H_n(S^n \setminus \{P\}) = 0$. But this means that $f_* \colon H_n(S^n) \to H_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.
(3) If f is homotopic to g, then it follows from Proposition 30.5 that $\deg(f) = \deg(g)$.
(4) We consider the following diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*", "\cdot \deg(f)"'] \arrow[rr, bend right, "(f \circ g)_*"'] & H_n(S^n) \arrow[r, "g_*", "\cdot \deg(g)"'] & H_n(S^n). \end{tikzcd}$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

(5) This statement follows immediately from (1) and (4).

(6) We consider again the singular chain

$$\begin{array}{rcl} \alpha\colon \Delta^n & \overset{\cong}{\to} & \Big\{(x_1,\dots,x_n) \in [0,1]^n \,\Big|\, \sum\limits_{i=1}^n x_i \in [0,1]\Big\} \overset{\cong}{\to} \overline{B}^n \to S^n. \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}) \qquad\qquad\qquad\qquad x \mapsto \big(x, \sqrt{1-\|x\|^2}\big). \end{array}$$

that we had already considered on page 813. Let ρ be the reflection in the hyperplane V defined by $x_{n+1} = 0$. We write $\beta = \rho \circ \alpha$. In Lemma 32.10 we saw that the singular n-chain $\alpha - \beta$ is a generator of $H_n(S^n)$.

We then have

$$\rho_*([\alpha - \beta]) \;=\; [\rho \circ \alpha] - [\rho \circ \beta] \;=\; [\beta] - [\underbrace{\rho \circ \rho}_{=\mathrm{id}} \circ \alpha] \;=\; [\beta] - [\alpha] \;=\; -[\alpha - \beta].$$

The map ρ_* is therefore given by multiplication with -1. This shows that reflection in V has degree -1.

Now let f be the reflection in some other hyperplane W of $\mathbb{R}^{n+1}$. It follows from Lemma 9.4 that ρ and f are homotopic maps of S^{n-1} to itself. By Proposition 30.5 this implies that τ and ρ induce the same map on $H_n(S^n)$, thus we see that $\deg(\tau) = \deg(\rho) = -1$.

(7) The reflection in the origin is the composition of the reflections in the $(n+1)$ hyperplanes $H_i := \{(t_1,\dots,t_{n+1}) \in \mathbb{R}^{n+1} \,|\, t_i = 0\}$, $i = 1,\dots,n+1$. The desired statement thus follows from (4) and (6). □

Lemma 32.17. *Let $A \in \mathrm{O}(n+1)$ be an orthogonal matrix. Then*

$$\deg\begin{pmatrix} S^n & \to & S^n \\ P & \mapsto & A \cdot P \end{pmatrix} \;=\; \det(A).$$

Proof. Let $A \in \mathrm{O}(n+1)$ be an orthogonal matrix. We give two slightly different proofs that $\deg(A) = \det(A)$.

(1) It is a standard fact proved in Linear Algebra, see e.g. [Cu, Theorem 33.1], that the map $P \mapsto A \cdot P$ is the composition of reflections in finitely many hyperplanes $H_1,\dots,H_k$.[646] We see that

$$\begin{array}{lllll} \deg(A) & = \deg(H_1) \cdot \dots \cdot \deg(H_k) & = (-1)^k & = \det(H_1) \cdot \dots \cdot \det(H_k) & = \det(A). \\ & \uparrow & \uparrow & \uparrow & \\ & \text{Lemma 32.16 (4)} & \text{Lemma 32.16 (6)} & \text{determinant of reflection} = -1 & \end{array}$$

(2) For the second proof we first assume that $\det(A) = 1$, i.e. that $A \in \mathrm{SO}(n+1)$. By Lemma 1.96 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a map

[646]For example, a matrix in $\mathrm{O}(2)$ corresponds to rotation by an angle α. But rotation by α is the same as the composition of the reflection in the line defined by the angle $\varphi = \frac{\alpha}{2}$ and the reflection in the line defined by the angle $\varphi = \alpha$.

$h\colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from id to multiplication by A. It follows from Lemma 32.16 (1) and (3) that $\deg(A) = 1$.
Now suppose that $\det(A) \neq 1$. Then $\det(A) = -1$ and if we compose A with the map h that is given by reflection in a hyperplane, then multiplicativity of determinants shows that $\det(h \circ A) = \det(h) \cdot \det(A) = (-1)^2 = 1$. From the above we see that $\deg(h \circ A) = 1$. From Lemma 32.16 (4) and (6) we easily obtain that $\deg(A) = -1$. □

We have now seen that the circle admits self-maps of arbitrary degrees and that every sphere admits self-maps of degree $-1, 0, 1$. It is natural to ask whether there are also maps on higher-dimensional spheres that have other degrees. The combination of Lemma 32.15 and the following lemma shows that in fact every $k \in \mathbb{Z}$ is realized as the degree of every high-dimensional sphere.

Lemma 32.18. *Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map. Following the discussion on page 820, we denote by $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ the suspension of f and we make the identification $\Sigma(S^n) = S^{n+1}$. Then*

$$\deg(\Sigma(f)\colon S^{n+1} \to S^{n+1}) \;=\; \deg(f\colon S^n \to S^n).$$

Example. Let $k \in \mathbb{Z}$. We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^2 \\ (re^{i\varphi}, w) & \mapsto & (re^{i\varphi k}, w). \end{array}$$

Under the identification $S^2 = \Sigma(S^1)$ from page 820 the map g equals $\Sigma(z \mapsto z^k)$. Thus it follows from Lemmas 32.15 and 32.18 that $\deg(g) = \deg(\Sigma(z \mapsto z^k)) = k$.

Proof. We consider the following diagram

$$\begin{array}{ccc} \widetilde{H}_n(S^n) & \xrightarrow{\Sigma_{S^{n+1}}} & \widetilde{H}_{n+1}(S^{n+1}) \\ {\scriptstyle \cdot \deg(f)} \big\downarrow {\scriptstyle f_*} & & {\scriptstyle \cdot \deg(\Sigma(f))} \big\downarrow {\scriptstyle \Sigma(f)_*} \\ \widetilde{H}_n(S^n) & \xrightarrow{\Sigma_{S^{n+1}}} & \widetilde{H}_{n+1}(S^{n+1}). \end{array}$$

By Proposition 32.14 the diagram commutes and the horizontal maps are isomorphisms. Thus we see that $\deg(\Sigma(f)) = \deg(f)$. □

Theorem 32.19. *Let $f\colon S^n \to S^n$ be a map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed-point.*

Example. Theorem 32.19 says in particular that every map $f\colon S^2 \to S^2$ of degree one has a fixed-point. Together with Lemma 32.16 this implies that every map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed-point. This means that we cannot "deform" the identity to a map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed-point free map.

Proof. Let $f\colon S^n \to S^n$ be a map which admits no fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\operatorname{id}$.

Our hypothesis implies that $(1-t)f(x) \neq tx$ for all $t \in [0,1]$ and all $x \in S^n$.[647] Therefore we can consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}. \end{array}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\begin{array}{ccccc} \deg(f) & = & \deg(-\operatorname{id}) & = & (-1)^{n+1}. \\ & \uparrow & & \uparrow & \\ & \text{Lemma 32.16 (3)} & & \text{Lemma 32.16 (7)} & \end{array}$$

□

In the following we recall the definition of the "naive" tangent space of a submanifold of $\mathbb{R}^n$.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold and let $P \in M \setminus \partial M$.

(1) A vector $v \in \mathbb{R}^n$ is called a *tangent vector to M in the point P* if there exists a smooth path

$$\begin{array}{rcl} \gamma\colon (-\epsilon,\epsilon) & \to & M \\ t & \mapsto & \gamma(t) \end{array}$$

such that $\gamma(0) = P$ and $\gamma'(0) = v$.

(2) The set of all tangent vectors at the point P is called the *tangent space* T_PM.[648]

(3) A *vector field* on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in T_PM$.

Examples.

(1) Some fairly elementary considerations show that the tangent space of the sphere S^n at a point P is given as follows:

$$T_PS^n = \{v \in \mathbb{R}^{n+1} \mid P \cdot v = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[649]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \dots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}). \end{array}$$

[647]This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)f(x)\| = 1-t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = tx$.

[648]For the sake of exposition we use Proposition 3.32 to blur the distinction between the "official" tangent space T_PM that we introduced on page 195 and the visual tangent space V_PM that we introduced on page 196.

[649]Indeed, for any $P \in S^{2k-1}$ we have $P \cdot v(P) = 0$, i.e. $v(P)$ does indeed lie in the tangent space T_PS^{2k-1}.

This vector field has the property that it is nowhere vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

(3) For even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$

This vector field vanishes at the "north pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "south pole" $(0, \ldots, 0, -1)$. Both types of vector fields are sketched in Figure 513.

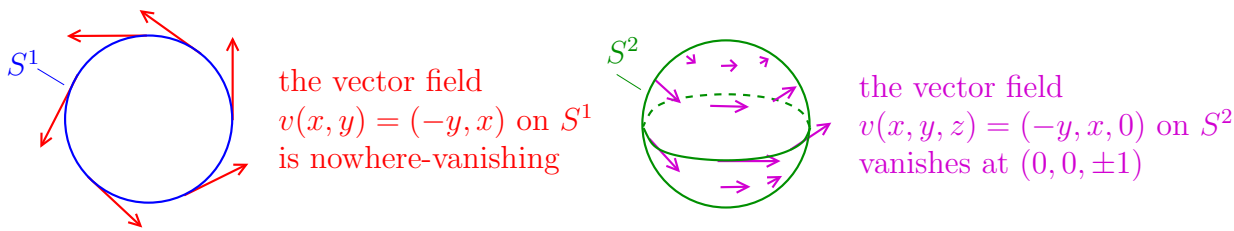


FIGURE 513.

One can now ask whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following proposition says that this is not possible.

Theorem 32.20. *Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.*[650]

Remark.

(1) For $n = 2$ we get the following "real life application": on the surface of earth there always exists at least one point where the wind does not blow at all. This fact is illustrated at the following website

`https://earth.nullschool.net/`

(2) In the literature Theorem 32.20 is known as the "hairy ball theorem" since the theorem says that "you can't comb a hairy ball flat without creating a cowlick".

(3) In German we refer to Theorem 32.20 as the "Satz vom (gekämmten) Igel", for the same reason as we refer to it as the "hairy ball theorem" in English.

Proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. which is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} v(x). \end{array}$$

[650] On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 32.4.

Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \mapsto & S^n \\ (x,t) & \mapsto & \cos(\pi t)x + \sin(\pi t)w(x). \end{array}$$

We refer to Figure 514 for an illustration. The map does indeed take values in S^n, since

$$(\cos(\pi t)x + \sin(\pi t)w(x)) \cdot (\cos(\pi t)x + \sin(\pi t)w(x)) \underset{\substack{\uparrow \\ \text{since } w(x)\cdot x = 0}}{=} \cos(\pi t)^2\|x\|^2 + \sin(\pi t)^2\|w(x)\|^2 \underset{\substack{\uparrow \\ \text{since } \|x\| = \|w(x)\| = 1}}{=} 1.$$

The map $F_0 = F(-,0)$ is the identity and the map $F_1 = F(-,1)$ equals F is $-\operatorname{id}$. Put differently, the map F is a homotopy between the two maps id and $-\operatorname{id}$. It follows that

$$1 \underset{\substack{\uparrow \\ \text{Lemma 32.16 (1)}}}{=} \deg(\operatorname{id}) \underset{\substack{\uparrow \\ \text{Lemma 32.16 (3)}}}{=} \deg(-\operatorname{id}) \underset{\substack{\uparrow \\ \text{Lemma 32.16 (7)}}}{=} (-1)^{n+1}.$$

But this means that n is odd. □

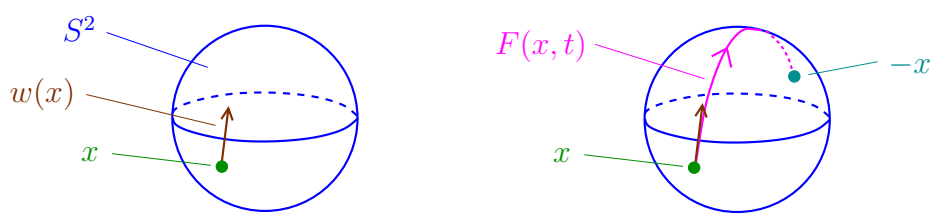


FIGURE 514.

32.5. **The local degree.** As the reader will have noticed, it is not entirely trivial to compute the isomorphism types of the homology groups of a given topological space. But there is one problem which is often even harder, namely given a map $f\colon X \to Y$ between topological spaces, how can we determine the induced map on homology? For example, let us consider the following question.

Question 32.21. *Let f be a complex polynomial of degree ≥ 1. By Exercise 1.11 we know that the map*

$$\begin{array}{rcl} \Theta(f)\colon S^2 = \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} = S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{array}$$

is continuous. What is the induced map on homology, i.e. what is the degree of $\Theta(f)$?

In this section we will develop a method for determining the degree of a self-map of a sphere which can be applied in many "reasonable" cases. In particular we will be able to answer Question 32.21. The answer will give us a new proof of the Fundamental Theorem of Algebra.

We start out with the following convention.

Convention. Let $n \geq 1$. We use the standard generator of $H_n(S^n) \cong \mathbb{Z}$, as defined on page 814, to make the identification $H_n(S^n) = \mathbb{Z}$. Now let $x \in S^n$ be a point and let U be a neighborhood of x in S^n. We have isomorphisms

$$\begin{array}{ccccccc} \mathbb{Z} & = & H_n(S^n) & \overset{\cong}{\rightarrow} & H_n(S^n, S^n \setminus \{x\}) & \overset{\cong}{\leftarrow} & H_n(U, U \setminus \{x\}). \\ & & & \uparrow & & \uparrow & \\ & & & \text{isomorphism by the long exact} & & \text{isomorphism by the} & \\ & & & \text{of the pair } (S^n, S^n \setminus \{x\}) & & \text{Excision Theorem 31.17} & \end{array}$$

Throughout this section will use these isomorphisms to identify the above homology groups with $\mathbb{Z}$.

Definition.

(1) Let $\Phi\colon \mathbb{Z} \to \mathbb{Z}$ be a homomorphism. It is given by multiplication by a unique integer which we refer to as the *degree* $\deg(\phi)$.
(2) We say a map $f\colon X \to Y$ between two topological spaces is *nice at a point* $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a *nice neighborhood of* x.
(3) Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x and let V be a neighborhood of $f(x)$ such that $f(U) \subset V$. The *local degree of* f *at the point* x is defined as[651]

$$\deg f|_x \ := \ \deg\big(f_*\colon \underbrace{H_n(U, U \setminus \{x\})}_{=\mathbb{Z}} \ \to \ \underbrace{H_n(V, V \setminus \{f(x)\})}_{=\mathbb{Z}}\big).$$

Example. We consider the functions

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t\sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the function f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the function g the point $x = 0$ is not nice.

Lemma 32.22. *The local degree* $\deg f|_x$ *of a map* $f\colon S^n \to S^n$ *at a nice point* $x \in S^n$ *is well-defined.*

Proof. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x and let V, V' be neighborhoods of $f(x)$ such that $f(U) \subset V$ and such that $f(U') \subset V'$. After replacing U' by $U \cap U'$ and replacing V' by $V \cap V'$ we can without loss of generality assume that $U' \subset U$ and $V' \subset V$. We consider the following

[651]Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (V, V \setminus \{f(x)\})$ since U is nice.

commutative diagram

$$\begin{array}{ccc} H_n(U', U' \setminus \{x\}) & \xrightarrow{f_*} & H_n(V', V' \setminus \{f(x)\}) \\ \downarrow\cong & & \downarrow\cong \\ H_n(U, U \setminus \{x\}) & \xrightarrow{f_*} & H_n(V, V \setminus \{f(x)\}). \end{array}$$

The vertical maps are isomorphisms by the Excision Theorem 31.17. It follows immediately that the degree of the top map agrees with the degree of the bottom map. □

Lemma 32.23. *Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$. If f is a homeomorphism, then for every nice point $x \in S^n$ we have*

$$\deg f|_x \;=\; \deg f.$$

Proof. From the hypothesis that f is a homeomorphism we obtain that $U = S^n$ is a nice neighborhood for x. We take $V = S^n$ and we consider the following diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \downarrow\cong & & \downarrow\cong \\ H_n(S^n, S^n \setminus \{x\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{f(x)\}) \end{array}$$

The statement now follows immediately from the definitions. □

The following lemma can often be used to determine the local degree of a map.

Lemma 32.24. *Let $f\colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.*

(1) *If f is a local homeomorphism around x, then x is nice and*

$$\deg f|_x \;=\; \pm 1.$$

(2) *If f is a local diffeomorphism around x, then x is nice and*

$$\deg f|_x \;=\; \begin{cases} 1, & \text{if } f \text{ is o-locally orientation-preserving at } x, \\ -1, & \text{if } f \text{ is o-locally orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, |y|)$. It follows from Lemma 32.24 (2) that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1. In Exercise 35.3 we will prove that the local degree at the points ± 1 is in both cases zero.

In the proof of Lemma 32.24 we will make use of the following lemma which we will also use on several other occasions later on.

Lemma 32.25. *Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by*

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & (D_P f)(x) + f(P) \end{array}$$

the linearization of P.

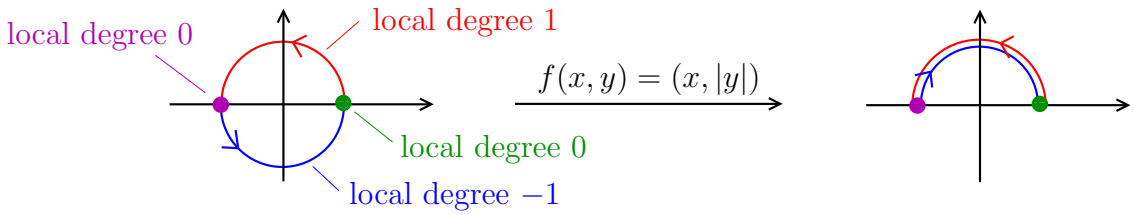


FIGURE 515.

(1) *There exists an open neighborhood $U' = B_r(P)$ of P that is contained in U such that for any $t \in [0,1]$ the map*

$$\begin{array}{rcl} g_t \colon U' & \to & \mathbb{R}^n \\ x & \mapsto & (1-t)\cdot f(x) \,+\, t\cdot \lambda(x) \end{array}$$

is a diffeomorphism onto its image and such that the image $g_t(U') = g_t(B_r(P))$ is convex.

(2) *The restrictions of f and φ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ are homotopic, in particular we have for any $k \in \mathbb{N}_0$ that*

$$f_* \,=\, \varphi_* \colon H_k(U', U' \setminus \{P\}) \;\to\; H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\}).$$

Proof.

(1) The existence of $U' = B_r(P)$ is an amusing exercise that requires a good understanding of what the differential actually means. We will fill in the details in Exercise 35.5.
(2) Since each g_t is a diffeomorphism onto its image with $g_t(P) = f(P)$ we see that each g_t maps $U' \setminus \{P\}$ to $\mathbb{R}^n \setminus \{f(P)\}$. Thus we see that the maps f and φ define homotopic maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ of pairs of topological spaces. The final statement regarding the induced maps on relative homology groups is an immediate consequence of Proposition 31.15. □

Now we can provide the proof of Lemma 32.24.

Proof of Lemma 32.24. Let $f \colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.

(1) Suppose that f is a local homeomorphism around the point x. Recall that this means that there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f \colon U \to V$ is a homeomorphism. It is clear that U is nice. Since the map f restricts to a homeomorphism $(U, U \setminus \{x\}) \to (V, V \setminus \{f(x)\})$ of pairs we see that the induced map $H_n(U, U \setminus \{x\}) \to H_n(V, V \setminus \{f(x)\})$ is an isomorphism, which implies immediately that $\deg f|_x = \pm 1$.
(2) We only sketch the proof of (2). We leave it as an exercise to fill in the details. First of all we introduce the following conventions and definitions:

(a) Let U of $\mathbb{R}^n$ be an open subset that contains 0. We use the obvious isomorphisms

$$\begin{array}{ccccccc} H_{n-1}(S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n \setminus 0) & \xrightarrow{\cong} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) & \xleftarrow{\cong} & H_n(U, U \setminus 0) \\ & & & \uparrow & & \uparrow & \\ & & & \text{long exact sequence of the pair } (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & & \text{Excision Theorem 31.17} & \end{array}$$

and the standard generator of $H_{n-1}(S^{n-1})$ to identify $H_{n-1}(U, U \setminus 0)$ with $\mathbb{Z}$.

(b) Given an open subset U of $\mathbb{R}^n$ and a map $f\colon U \to \mathbb{R}^n$ with $f(0) = 0$ we define

$$\deg(f) \;=\; \deg\big(f_*\colon \underbrace{H_n(U, U \setminus 0)}_{=\mathbb{Z}} \to \underbrace{H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)}_{=\mathbb{Z}}\big).$$

(c) Given a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ we write $\deg(A)$ for the degree of the map $\mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ that is given by multiplication by A.

Using orientation-preserving charts one sees easily that it suffices to prove the following claim.

Claim. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ such that $f(0) = 0$. Then

$$\deg(f) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving,} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

We start out with the following four observations:

(a) Since f is a diffeomorphism we know that the differential $D_0 f$ is an invertible matrix.[652] Furthermore, basically by definition, we know that f is orientation-preserving if $\det(D_0 f) > 0$ and we know that f is orientation-reversing if $\det(D_0 f) < 0$.

(b) It follows immediately from Lemma 32.25 applied to $f\colon U \to V$ that f and D_0 have the same degree.

(c) If A and B are two matrices in $\mathrm{GL}(n, \mathbb{R})$ that lie in the same path-component of $\mathrm{GL}(n, \mathbb{R})$, then the corresponding maps $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ are homotopic.[653] In particular they have the same degree by Proposition 31.15.

(d) We denote by R the matrix corresponding to the reflection in the hyperplane defined by $x_n = 0$. By Lemma 1.96 (1) a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ lies in the same path-component of $\mathrm{GL}(n, \mathbb{R})$ as the identity matrix id if $\det(A) > 0$ and it lies in the same path-component of $\mathrm{GL}(n, \mathbb{R})$ as R if $\det(A) < 0$.

[652] Indeed, by the chain rule for differentials the inverse of the $D_0 f$ is given by the differential of f^{-1} at the point $f(0) = 0$.

[653] Why is that?

Summarizing we have

$$\deg(f) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} \deg(D_0 f) \underset{\substack{\uparrow \\ \text{by (a), (c) and (d)}}}{=} \begin{cases} \deg(\mathrm{id}), & \text{if } f \text{ or.-pres.} \\ \deg(R), & \text{if } f \text{ or.-rev.} \end{cases} \underset{\substack{\uparrow \\ \text{similar argument as in Lemma 32.16 (6)}}}{=} \begin{cases} 1, & \text{if } f \text{ or.-pres.} \\ -1, & \text{if } f \text{ or.-rev.} \end{cases}$$

This concludes the proof of the claim. □

Using the following proposition we can now determine the degree of a map from local degrees.

Proposition 32.26. *Let $f\colon S^n \to S^n$ be a map where $n > 0$ and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.*[654] *Then*[655]

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

Proof (∗). We choose disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$ and we choose an open neighborhood V of y such that $f(U_i) \subset V$ for $i = 1, \dots, m$. We consider the following diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "f_*"] \arrow[d] \arrow[dr, "j"] & & H_n(S^n) \arrow[d, "\cong"] \\
\bigoplus_{i=1}^{m} H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "\oplus p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
\bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i) \arrow[u, "\cong"] \arrow[ur, "\oplus k_i"'] \arrow[rr, "f_*"] & & H_n(V, V \setminus y). \arrow[u, "\cong"', "q"]
\end{tikzcd}$$

Here the maps k_i and p_i are induced by inclusions. One sees easily that this diagram is commutative.

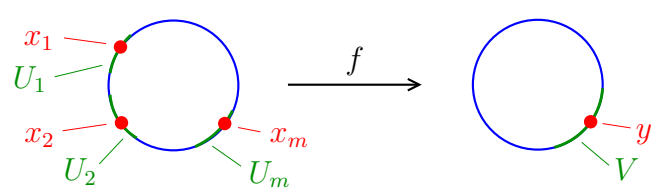


FIGURE 516. Illustration for the proof of Proposition 32.26.

[654] Does there always exist such a y?

[655] The hypothesis implies immediately that all the points $x_1, \dots, x_m$ are nice, hence the local degrees are defined.

It follows from the Excision Theorem 31.17 that the map[656]

$$\bigoplus_{i=1}^{m} k_i \,:\, \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i) \;\xrightarrow{\cong}\; H_n(S^n, S^n \setminus f^{-1}(y))$$

is an isomorphism. From the commutativity of the lower triangle it follows immediately that the map

$$\bigoplus_{i=1}^{m} p_i \,:\, H_n(S^n, S^n \setminus f^{-1}(y)) \;\to\; \bigoplus_{i=1}^{m} H_n(S^n, S^n \setminus x_i)$$

is also an isomorphism. Recall that we make the identifications

$$\mathbb{Z} \;=\; H_n(S^n, S^n \setminus x_i) \;=\; H_n(U_i, U_i \setminus x_i) \;=\; H_n(V, V \setminus y).$$

The commutativity of the triangles and the choice of the identifications with $\mathbb{Z}$ implies that $(p_i \circ j)(1) = (p_i \circ k_i)(1)$ for $i = 1, \dots, m$. Thus we see that

$$\sum_{i=1}^{m} (p_i \circ j)(1) \;=\; \sum_{i=1}^{m} (k_i \circ j)(1).$$

In particular it follows that

$$((p_1 \circ j)(1), \dots, (p_m \circ j)(1)) \;=\; ((p_1 \circ k_1)(1), \dots, (p_m \circ k_m)(1)).$$

Since $\bigoplus_{i=1}^{m} p_i$ is an isomorphism, we now obtain that

$$j(1) \;=\; \sum_{i=1}^{m} k_i(1).$$

Now we see that[657]

$$\begin{aligned}
\deg f \;&\overset{\substack{\text{by the definition of } \deg f \text{ and} \\ \text{by the chosen identifications with } \mathbb{Z}}}{=}\; (q^{-1} \circ f_*)(j(1_{H_n(S^n)})) \;=\; (q^{-1} \circ f_*)\Big(\sum_{i=1}^{m} k_i(1)\Big) \\
&= \sum_{i=1}^{m} (q^{-1} \circ f_* \circ k_i)(1_{H_n(U_i, U_i \setminus x_i)}) \;\underset{\substack{\text{by the commutativity of the} \\ \text{lower rectangle}}}{=}\; \sum_{i=1}^{m} f_*(1_{H_n(U_i, U_i \setminus x_i)}) \;\underset{\substack{\text{by the definition of } \deg f|_{x_i} \\ \text{and the identifications with } \mathbb{Z}}}{=}\; \sum_{i=1}^{m} \deg f|_{x_i}.
\end{aligned}$$

□

Examples.

[656]More precisely, it follows from the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i) \arrow[rr, "\cong"] \arrow[dr] & & H_n\Big(\bigcup_i U_i, \bigcup_i U_i \setminus x_i\Big) \arrow[dl, "\cong", "\text{Excision Theorem 31.17}"'] \\
& H_n(S^n, S^n \setminus f^{-1}(y)). &
\end{tikzcd}$$

that the left diagonal map is an isomorphism.

[657]Here we write $1_{H_n(S^n)}$ to indicate $1 \in H_n(S^n)$ and we use the same notation for the other spaces.

(1) The map $f\colon S^1 \to S^1$ given by $f(x,y) = (x,|y|)$ that is illustrated in Figure 515 has degree zero by Lemma 32.16. Taking preimages of points in S^1 one sees that in each case the sum of the local degrees of the preimages is zero.
(2) In Figure 517 we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$. We

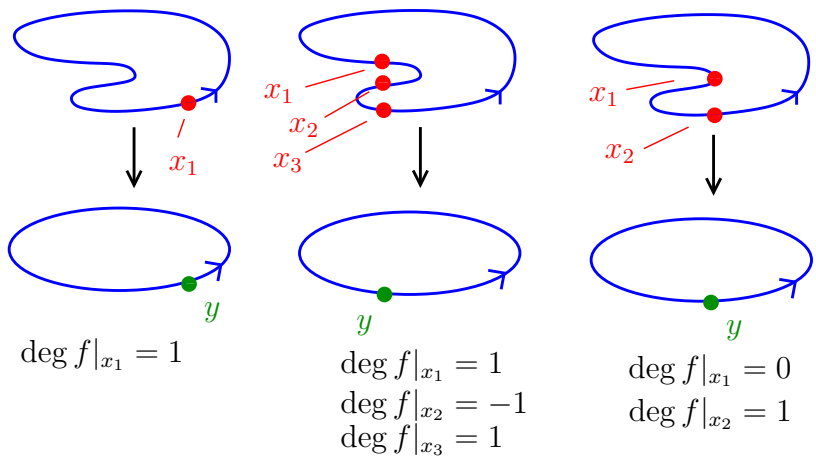


FIGURE 517.

use Lemma 32.24 to determine the various degrees at the given points.[658] We see that the number of preimages and the degrees at the points in the preimages vary, but in all three cases the sum of the degrees at the points in the preimage is the same.
(3) We consider the map

$$\begin{array}{rcl} f\colon S^1 & \to & S^1 \\ z & \mapsto & z^m \end{array}$$

We had already determined in Lemma 32.15 that $\deg(f) = m$. Now we will show that the degree can also easily be determined using Proposition 32.26.

First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = e^{2\pi i j/m}$ where $j = 1, \dots, m$. It follows from Lemma 32.24 (2) that for each i we have

$$\deg f|_{x_i} = \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

Thus it follows from Proposition 32.26 that $\deg f = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then $\deg f = 0$. The easiest way to see this is to take $y = -1$, since in this case the preimage is the empty set.

As promised, we can now provide an answer to Question 32.21:

[658] Why is in the third example the degree $\deg f|_{x_1}$ equal to zero?

Proposition 32.27. *Let f be a complex polynomial of degree ≥ 1. The degree of the map*

$$\begin{array}{rcl} \Theta(f)\colon S^2 = \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} = S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

equals the degree of the polynomial f.

Sketch of proof. Let f be a complex polynomial of degree $n \geq 1$. We also consider the polynomial $g(z) = z^n$. We calculate that

$$\deg(\Theta(f)) \overset{\substack{\text{by Lemma 32.16 (3), since by Exercise 9.4 we know that } \Theta(f) \text{ and } \Theta(g) \text{ are homotopic} \\ \downarrow}}{=} \deg(\Theta(g(z)=z^n)) \underset{\substack{\uparrow \\ \text{follows from Proposition 32.26 and Lemma 32.24, see Exercise 32.5}}}{=} n.$$

□

More excitingly, we can provide a new proof for the Fundamental Theorem of Algebra.

Theorem 7.22. (Fundamental Theorem of Algebra) *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

Proof. Let f be a nonconstant polynomial with complex coefficients. In other words, f is a complex polynomial of degree $n \geq 1$. By Proposition 32.27 we know that the degree of the self-map $\Theta(f)$ of $S^2 = \mathbb{C}\cup\{\infty\}$ equals $n \geq 1$. By Lemma 32.16 (2) this implies that $\Theta(f)\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is surjective, in particular 0 lies in the image of $\Theta(f)$. Since $\Theta(f)(\infty) = \infty$ we see that 0 lies in the image of $\mathbb{C}$. In other words, there exists a $z \in \mathbb{C}$ with $\underbrace{\Theta(f)(z)}_{=f(z)} = 0$. □

Exercises for Chapter 32.

Exercise 32.1. For which $n \in \mathbb{N}$ does there exist a singular simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $H_n(S^n)$?

Exercise 32.2. Let M be a non-empty topological manifold. Show that $M \neq \partial M$, i.e. show that there exists a point in $M \setminus \partial M$.

Exercise 32.3. Let M be an m-dimensional topological manifold and let $P \in \partial M$. Show that $H_k(M, M\setminus\{P\}) = 0$ for all k.

Exercise 32.4. On S^{2k} we had found a vector field which vanishes at *precisely two* points. Does there exist a vector field on S^2 which vanishes on only *one* point?

Exercise 32.5. Let $n \geq 1$. We consider the complex polynomial $g(z) = z^n$. Show that the degree of the map $\Theta(g)\colon S^2 \to S^2$ equals n.
Hint: Use Proposition 32.26 and Lemma 32.24.

Exercise 32.6. Let $f = f(z)$ be a polynomial with complex coefficients of degree ≥ 1. Let $\Theta(f)$ be the corresponding self-map of $S^2 = \mathbb{C} \cup \{\infty\}$. Show that the local degree of $\Theta(f)$ at a root of $f(z)$ equals the multiplicity of the root.

33. The Mayer–Vietoris sequence and its applications

In this chapter we will state and prove the Mayer–Vietoris sequence for homology groups. It makes it possible to compute the homology groups of a topological space from the homology groups of appropriate subspaces. The role of the Mayer–Vietoris sequence in homology theory is similar to the role of the Seifert-van Kampen Theorem 14.1 for fundamental groups. The Mayer–Vietoris sequence will give us the tool to compute the homology groups of many topological spaces that we are interested in. In particular we will determine the homology groups of $\mathbb{R}\mathrm{P}^2$, the torus and the Klein bottle.

33.1. **Split exact sequences.** We already saw in Theorem 31.2 that homology groups of different topological spaces are sometimes related by a long exact sequence of homology groups. The Mayer–Vietoris sequence that we will introduce later on in this chapter is another instance of such a long exact sequence. Very often one ends up in the situation that one knows the homology groups of two out of three topological spaces and we would like to determine the homology groups of the third topological space from the long exact sequence.

Algebraically the question is as follows: suppose we are given an exact sequence

$$\ldots \to A_n \to B_n \to C_n \to A_{n-1} \to \ldots$$

of abelian groups where we know the groups A_* and C_*, can we determine the groups B_* from this long exact sequence? The same question of course also arises when the roles of A, B and C are switched.

In this section we will study the simplest case, namely the case of a short exact sequence

$$0 \to A \to B \to C \to 0.$$

If we know A and C, can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \to 0$$

could be of the form

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto(x,0)} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(x,y)\mapsto y} \mathbb{Z}_2 \to 0$$

but it could also be of the form[659]

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto 2x} \mathbb{Z}_4 \xrightarrow{x\mapsto x} \mathbb{Z}_2 \to 0.$$

In this short section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of (not necessarily abelian) groups *splits*, if there exists a *splitting* $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

[659] Are there any other possibilities for the isomorphism type of B?

For example the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k \mapsto (k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l) \mapsto l} \mathbb{Z}_n \to 0$$

splits, in fact a splitting is given by the map $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k \mapsto nk} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

does not split.[660]

Lemma 33.1. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. If C is a free abelian group, then the short exact sequence splits.

Proof. Let $\{c_i\}_{i \in I}$ be a basis of the free abelian group C. For each $c_i \in C$ we choose a $b_i \in B$ with $p(b_i) = c_i$. By Lemma 11.1 there exists a unique homomorphism $s\colon C \to B$ with $s(c_i) = b_i$ for all i. We have $(p \circ s)(c_i) = c_i$ for each c_i. But then it follows from the uniqueness part of Lemma 11.1 that $p \circ s = \mathrm{id}_C$. The homomorphism s is thus indeed a splitting. □

The following lemma gives two useful criteria for a short exact sequence to split.

Lemma 33.2. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. Then the following statements are equivalent:

(1) *there exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$,*
(2) *there exists an isomorphism $\Phi\colon B \to A \oplus C$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & & B \arrow[dd, "\Phi", "\cong"'] \arrow[dr, "p"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the obvious projection map,
(3) *the short exact sequence splits.*

Proof.
(3)⇒(1) Let $s\colon C \to B$ be a splitting, i.e. let s be a homomorphism such that $p \circ s = \mathrm{id}_C$. We need to find an appropriate homomorphism $t\colon B \to A$. Since the sequence

[660] Why does it not split?

is exact, our task is basically the same as finding an appropriate homomorphism $B \to \ker(p)$. Let $b \in B$. We have

$$p(b - s(p(b))) \;=\; p(b) - (\underbrace{p \circ s}_{=\mathrm{id}})(p(b)) \;=\; p(b) - p(b) \;= 0.$$

By the exactness of the short exact sequence there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. We define $t(b) := a$. It follows easily from the fact that p and s are homomorphisms that the map $t\colon B \to A$ is a homomorphism. It remains to show that $t \circ i = \mathrm{id}_A$. Let $a \in A$. We have to show that $t(i(a)) - a = 0$. Since i is injective it suffices to show that $i(t(i(a))) - i(a) = 0$. We have

$$i(t(i(a))) - i(a) \;\underset{\substack{\uparrow \\ \text{since } t(y) \text{ is the unique element} \\ \text{with } i(t(y)) = y - s(p(y))}}{=}\; (i(a) - s(p(i(a)))) - i(a) \;\underset{\substack{\uparrow \\ \text{since } p \circ i = 0}}{=}\; i(a) - i(a) \;=\; 0.$$

(1)⇒(2) Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{aligned} \Phi\colon B &\to A \oplus C \\ b &\mapsto t(b) \oplus p(b). \end{aligned}$$

It follows easily from the exactness of the initial sequence that this map is in fact an isomorphism.[661] It is straightforward to verify that Φ has all the other desired properties.

(2)⇒(3) Now suppose we are given an isomorphism $\Phi\colon B \to A \oplus C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the map

$$C \xrightarrow{c \mapsto (0,c)} A \oplus C \xrightarrow{\Phi^{-1}} B$$

is a splitting. □

Remark. We quickly want to discuss the more general case of short exact sequences of groups that are not necessarily abelian. The three statements of Lemma 33.2 (with minor modifications) still make sense. But it is no longer true that (3) implies (1).[662] But it is true that (1) and (2) are equivalent, in fact one can use almost verbatim the same proof. For the reader's convenience we spell out the statement, namely given a short exact sequence

$$\{e\} \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; \{e\}$$

of groups the following two statements are equivalent:

(1) there exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$,

[661] In fact the argument is basically the same as in the proof of Lemma 31.1 (4a).

[662] In the above proof of (3) ⇒ (1) we did use that B is abelian. In fact one can also give easy counterexamples, e.g. consider $0 \to \mathbb{Z}_3 \to S_3 \to \mathbb{Z}_2 \to 0$ where $S_3 \to \mathbb{Z}_2$ is the unique epimorphism from the permutation group S_3 onto $\mathbb{Z}_2$.

(2) there exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & B \arrow[dr, "p"] \arrow[dd, "\Phi", "\cong"'] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0. \\
 & & A \times C \arrow[ur] & &
\end{tikzcd}$$

In many applications we only need the following corollary that is an immediate consequence of Lemmas 33.1 and 33.2.

Corollary 33.3. *Let*

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of abelian groups. Then the following hold:

(1) *If C is a free abelian group, then $B \cong A \oplus C$.*
(2) *Suppose A and C are free abelian groups. If X is a basis for A and if $Y \subset B$ is a subset such that $p(Y)$ is a basis for C, then $X \cup Y$ is a basis for the free abelian group B.*

Proof. The first statement is an immediate consequence of Lemmas 33.1 and 33.2. The second statement follows easily from Lemma 33.2 (1) $\Rightarrow$ (3). We leave the details as an exercise to the reader. □

The above discussion deals with the case of a short exact sequence. The following lemma allows us to break a long exact sequence into shorter exact sequences.

Lemma 33.4. *Let*

$$\ldots \;\to\; A_{n+2} \;\xrightarrow{f_{n+2}}\; A_{n+1} \;\xrightarrow{f_{n+1}}\; A_n \;\xrightarrow{f_n}\; A_{n-1} \;\xrightarrow{f_{n-1}}\; A_{n-2} \;\to\; \ldots$$

be an exact sequence of abelian groups. Then for any n the sequences

$$0 \;\to\; \operatorname{coker}\left\{A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right\} \;\xrightarrow{f_{n+1}}\; A_n \;\xrightarrow{f_n}\; A_{n-1} \;\xrightarrow{f_{n-1}}\; A_{n-2} \;\to\; \ldots$$

and

$$\ldots \;\to\; A_{n+2} \;\xrightarrow{f_{n+2}}\; A_{n+1} \;\xrightarrow{f_{n+1}}\; A_n \;\xrightarrow{f_n}\; \ker\left\{A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right\} \;\to\; 0$$

are also exact. In particular

$$0 \;\to\; \operatorname{coker}\left\{A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right\} \;\xrightarrow{f_{n+1}}\; A_n \;\xrightarrow{f_n}\; \ker\left\{A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right\} \;\to\; 0$$

is a short exact sequence.

Proof. The statements follow immediately from the definitions. □

Example. A typical application is that if we are given an exact sequence

$$\dots\ A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n=0} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3}\dots$$

of abelian groups such that the map f_n is the zero map, then the exact sequence can be split "into the two halves" along that zero map, i.e.

$$0 \quad \to \quad A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3}\dots$$

and

$$\dots\ A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \quad \to 0$$

are also exact sequences. In the remainder of this course we will frequently use this fact without explicitly citing Lemma 33.4.

33.2. **The Mayer–Vietoris sequence.** In the following theorem and throughout the subsequent discussion we will use the following notation: given a topological space D and a subset $C \subset D$, we denote the corresponding inclusion map $C \to D$ by i_C. To avoid pandemonium with subscripts we often drop denote the inclusion induced maps on homology by i_C instead of the more accurate $(i_C)_*$.

Theorem 33.5. (Mayer–Vietoris) [663] *Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Then the following two statements hold:*

(1) *For each $n \in \mathbb{N}$ there exists a unique homomorphism*

$$\partial_n \colon H_n(X) \ \to\ H_{n-1}(A \cap B)$$

which has the following property: for every $a \in C_n(A)$ and $b \in C_n(B)$ such that $a + b \in C_n(X)$ is a cycle we have[664]

$$\partial_n([a+b]) \ = \ [\,\partial a\,] \ = \ -\,[\,\partial b\,] \ \in\ H_{n-1}(A \cap B).$$

We refer to $\partial_n H_n(X) \to H_{n-1}(A \cap B)$ as the connecting homomorphisms *corresponding to (X, A, B).*

(2) *The following sequence is exact:*

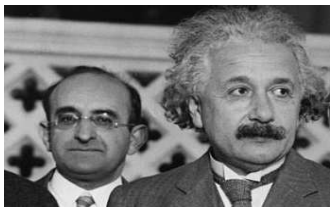
$$\cdots \to H_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \to \dots$$

Totally analogous statements also hold if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.

The long exact sequence in Theorem 33.5 is usually called the Mayer–Vietoris sequence of the decomposition $X = A \cup B$.

[663] Walther Mayer (1887-1948) and Leopold Vietoris (1891-2002) were two Austrian mathematicians. Vietoris was the longest living Austrian man ever.

[664] Here in the expression ∂a we mean by ∂ the boundary map $\partial \colon C_n(A) \to C_{n-1}(A)$. In our case we will see that $\partial a \in C_{n-1}(A \cap B) \subset C_{n-1}(A)$. A similar statement holds for ∂b.



Walther Mayer (1887-1948) with some other person

Leopold Vietoris (1891-2002) at 110

FIGURE 518.

Example. In Figure 519 we consider the annulus X together with two closed subsets A and B with $X = \mathring{A} \cup \mathring{B}$. Furthermore we show chains $a \in C_1(A)$ and $b \in C_1(B)$ such that $a + b$ form a cycle in $C_1(X)$. The connecting homomorphism of the Mayer–Vietoris sequence sends $[a + b] \in H_1(X)$ to $[\partial a] = -[\partial b] \in H_0(A \cap B)$.

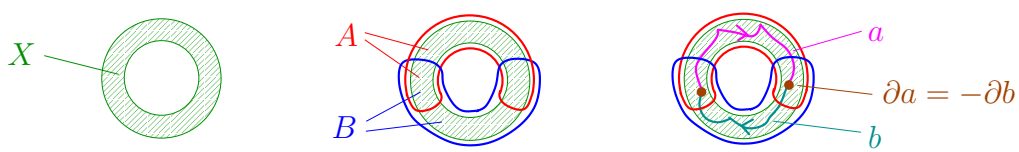


FIGURE 519.

Proof. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. As a reminder, following the notation from page 787 we write

$$C_n^{\{A,B\}}(X) \ := \ \{a + b \,|\, a \in C_n(A) \text{ and } b \in C_n(B)\}.$$

The inclusion maps induce the following sequence of chain maps:

$$0 \to C_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A,B\}}(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

We prove the claim in the following four steps:

(1) It is clear that each of the maps $C_*(A \cap B) \to C_*(A)$ and $C_*(A \cap B) \to C_*(B)$ is already injective, in particular the map $i_{A\cap B} \oplus -i_{A\cap B}$ is injective.
(2) It is an immediate consequence of the definition of $C_*^{\{A,B\}}(X)$ that the map $i_A + i_B$ is surjective.
(3) It follows immediately from the definitions that the composition of both maps is the zero map,[665] i.e. $\mathrm{im}(i_{A\cap B} \oplus -i_{A\cap B}) \subset \ker(i_A + i_B)$.

[665]Note that here we use the minus-sign that appears in the first map.

(4) It remains to show that $\ker(i_A + i_B) \subset \operatorname{im}(i_{A\cap B} \oplus -i_{A\cap B})$. Thus let $x \in C_n(A)$ and $y \in C_n(B)$ such that $(i_A + i_B)(x + y) = 0$. By definition this means that x and y, viewed as elements in $C_n(X)$, satisfy $x = -y$. This means that x lies in $C_n(A)$ and also in $C_n(B)$. Put differently, $x \in C_n(A) \cap C_n(B) = C_n(A \cap B)$. But then we have $(i_{A\cap B} \oplus -i_{A\cap B})(x) = x + y$.

Thus we have proved the claim.

For each $n \in \mathbb{N}$ we denote by

$$p_n \colon H_n^{\{A,B\}}(X) \;\to\; H_{n-1}(A \cap B)$$

the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 770. We deduce from Proposition 31.10 that

$$\cdots \to H_n(A\cap B) \xrightarrow{i_{A\cap B}\oplus -i_{A\cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A+i_B} H_n^{\{A,B\}}(X) \xrightarrow{p_n} H_{n-1}(A\cap B) \to \dots$$

is an exact sequence. It follows from Proposition 31.21, together with our hypothesis that $X = \mathring{A} \cup \mathring{B}$, that for every $n \in \mathbb{N}_0$ the inclusion map

$$C_*^{\{A,B\}}(X) \;\to\; C_*(X)$$

induces an isomorphism

$$\Phi_n \colon H_n^{\{A,B\}}(X) \;\xrightarrow{\cong}\; H_n(X).$$

For each $n \in \mathbb{N}_0$ we now write $\partial_n := p_n \circ \Phi_n^{-1}$.

It remains to show that these maps have the two desired properties stated in the theorem:

(2) It is clear from the above discussion that the sequence given in (2) is indeed exact.

(1) We leave it as an exercise to verify that it follows from the definition of the connecting homomorphism p_n on page 770 that for every $a \in C_n(A)$ and $b \in C_n(B)$ such that $a + b \in C_n(X)$ is a cycle we have

$$\partial_n([a+b]) \;=\; p_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; H_{n-1}(A\cap B).$$

Furthermore, the isomorphism $\Phi_n \colon H_n^{\{A,B\}}(X) \xrightarrow{\cong} H_n(X)$ shows that every element in $H_n(X)$ is of the form $[a + b]$ for some $a \in C_n(A)$ and $b \in C_n(B)$, which implies that ∂_n is uniquely determined by the above rule.

We leave it as an exercise to verify that the analogous statements also hold if we replace the homology groups of $A\cap B, A, B$ and X by the corresponding reduced homology groups. □

In some applications it is useful to know that the connecting homomorphisms from the Mayer–Vietoris Theorem 33.5 behaves well under induced maps, i.e. that it is a natural map. This is formulated more precisely in the following lemma.

Lemma 33.6. $(*)$ *Let X and Y be two topological spaces and let $A, B \subset X$ and $C, D \subset Y$ be subsets with $X = \mathring{A} \cup \mathring{B}$ and $Y = \mathring{C} \cup \mathring{D}$. If $f \colon X \to Y$ is a map with $f(A) \subset C$ and*

$f(B) \subset D$, then for every $n \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} H_n(X) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A \cap B) \arrow[d, "f_*"] \\ H_n(Y) \arrow[r, "\partial"] & H_{n-1}(C \cap D) \end{tikzcd}$$

where the horizontal maps are the connecting homomorphisms introduced in the Mayer–Vietoris Theorem 33.5 (1).

Proof. The statement follows immediately from the definition of the connected homomorphisms. □

We conclude this section with the following variation on the Mayer–Vietoris Theorem 33.5. This theorem might sound rather unnatural, but it will play an important role in a later chapter.

Theorem 33.7. (The inverted Mayer–Vietoris Theorem) $(*)$ *Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon H_n(X, U \cup V) \to H_{n-1}(X, U \cap V)$ such that*

$$\ldots \to H_n(X, U \cap V) \xrightarrow{i \oplus -i} H_n(X, U) \oplus H_n(X, V) \xrightarrow{i+i} H_n(X, U \cup V) \xrightarrow{\partial} H_{n-1}(X, U \cap V) \to \ldots$$

is an exact sequence.

Proof $(*)$. As in the proof of Theorem 33.5 we have a short exact sequence

$$0 \;\to\; C_n(X, U \cap V) \;\to\; C_n(X, U) \oplus C_n(X, V) \;\to\; C_n(X)/C_n^{\{U,V\}}(U \cup V) \;\to\; 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to show that the homology groups of the chain complex on the right are precisely the homology groups of the pair $(X, U \cup V)$. To do so we consider the following commutative pair of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C_*^{\{U,V\}}(U \cup V) \arrow[r] \arrow[d] & C_*(X) \arrow[r] \arrow[d] & C_*(X)/C_*^{\{U,V\}}(U \cup V) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_*(U \cup V) \arrow[r] & C_*(X) \arrow[r] & C_*(X, U \cup V) \arrow[r] & 0. \end{tikzcd}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{tikzcd} \ldots \arrow[r] & H_n^{\{U,V\}}(U \cup V) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n\big(C_*(X)/C_*^{\{U,V\}}(U \cup V)\big) \arrow[r] \arrow[d] & H_{n-1}^{\{U,V\}}(U \cup V) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & H_n(U \cup V) \arrow[r] & H_n(X) \arrow[r] & H_n(X, U \cup V) \arrow[r] & H_{n-1}(U \cup V) \arrow[r] & \ldots \end{tikzcd}$$

It follows from Proposition 31.21 that the first and fourth vertical map are an isomorphism. The second map and the fifth map (which for space reasons is not shown) are the identity. Hence it follows from the Five Lemma 31.11 that the third vertical map is also an isomorphism. □

33.3. **Applications of the Mayer–Vietoris sequence.** As a warm-up we first consider the homology groups of the suspension $\Sigma(X)$ of a topological space X. More precisely, we have the following lemma which basically gives a new proof for Proposition 32.14.

Lemma 33.8. *Let $k \in \mathbb{Z}$. Given any topological space X we have a natural isomorphism*

$$\Sigma_X \colon \widetilde{H}_k(X) \xrightarrow{\cong} \widetilde{H}_{k+1}(\Sigma(X)).$$

Remark. Let X be a topological space. In Exercise 33.4 we will see that the two natural isomorphisms $\widetilde{H}_k(\Sigma(X)) \xrightarrow{\cong} \widetilde{H}_{k-1}(X)$ that we constructed in Proposition 32.14 and that we constructed in the proof of Lemma 33.8 are actually the same on the nose.

Proof. Let $k \in \mathbb{Z}$.

(1) Let X be a topological space. We denote by $p \colon X \times [-1,1] \to \Sigma(X)$ the obvious projection. As on page 818 we consider the subspaces $C_+X = p(X \times (-1,1]))$ and $C_-X = p(X \times [-1,1))$. Note that C_+X and C_-X are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+X \cup C_-X$. By the same argument as in the proof of Lemma 15.1 we see that both C_+X and C_-X admit a deformation retraction to a point. It follows from Lemma 9.10 that both are contractible, therefore by Lemma 31.1 (7) the reduced homologies of C_+X and C_-X vanish. It follows immediately from the long exact sequence in reduced homology from the Mayer–Vietoris Theorem 33.5 that the connecting homomorphism

$$\partial \colon \widetilde{H}_{k+1}(\Sigma(X)) \to \widetilde{H}_k(C_+X \cap C_-X)$$

is in fact an isomorphism. But $X = X \times \{0\}$ is evidently a deformation retract of $C_+X \cap C_-X = X \times (-1,1)$. Thus we obtain from Lemma 31.1 (6) that the inclusion induced map $i_* \colon \widetilde{H}_k(X \times \{0\}) \to \widetilde{H}_k(C_+X \cap C_-X)$ is an isomorphism. We define Σ_X to be the map

$$\widetilde{H}_k(X) = \widetilde{H}_k(X \times \{0\}) \xrightarrow{i_*} \widetilde{H}_k(C_+X \cap C_-X) \xrightarrow{\partial^{-1}} \widetilde{H}_{k+1}(\Sigma(X)).$$

(2) It follows easily from the definition of Σ_X and Lemma 33.6 that the isomorphism Σ_X is natural. □

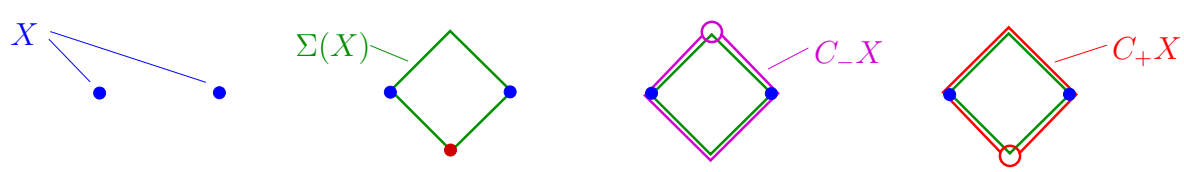


FIGURE 520. Illustration for the proof of Lemma 33.8.

As our next example we want to determine the homology groups of the 2-dimensional projective space $\mathbb{R}\mathrm{P}^2$.

Lemma 33.9. *We have*

$$H_n(\mathbb{R}\mathrm{P}^2) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. In the proof of Lemma 33.9 we use the description from page 113 of $\mathbb{R}\mathrm{P}^2$ as

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\sim \quad \text{where } x \sim y \text{ if } x, y \in \partial\overline{B}^2 \text{ and } x = -y.$$

In the following we denote by p the projection map $\overline{B}^2 \to \overline{B}^2/\sim$. We consider the open subsets

$$A \;:=\; p\big(\big\{z \in \overline{B}^2 \,\big|\, |z| > \tfrac{1}{4}\big\}\big) \quad \text{and} \quad B \;:=\; p\big(\big\{z \in \overline{B}^2 \,\big|\, |z| < \tfrac{3}{4}\big\}\big).$$

Evidently we have $\mathring{A} \cup \mathring{B} = A \cup B = \mathbb{R}\mathrm{P}^2$. From Mayer–Vietoris Theorem 33.5 we obtain the following long exact sequence of homology groups:

$$\cdots \to \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \;\to\; \widetilde{H}_2(\mathbb{R}\mathrm{P}^2) \;\xrightarrow{\partial}\; \widetilde{H}_1(A \cap B) \;\to\; \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \;\to\; \widetilde{H}_1(\mathbb{R}\mathrm{P}^2) \;\to \ldots$$

Now we have to collect all the available information on the reduced homology groups of

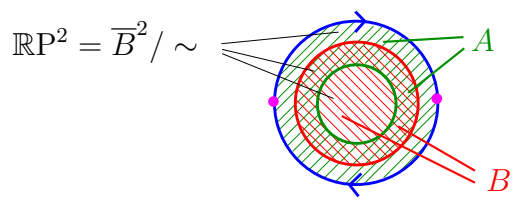


FIGURE 521. Illustration for the proof of Lemma 33.9.

A, B and $A \cap B$. We make the following observations:

(1) We remarked on page 762 that the reduced homology of a disk vanishes, i.e. we have $\widetilde{H}_n(B) = 0$ for all $n \in \mathbb{N}_0$.

(2) The map $j\colon S^1 \to A \cap B = p(\{z \in \overline{B}^2 \,|\, |z| \in (\frac{1}{4}, \frac{3}{4})\}$ given by $z \mapsto \frac{1}{2}z$ is a homotopy equivalence, since $p(\frac{1}{2}S^1)$ is a deformation retract of $A \cap B$. It follows from Corollary 30.8 (2) and Proposition 31.3 that $\widetilde{H}_n(A \cap B) = 0$ for $n \neq 1$ and that $j_*\colon \widetilde{H}_1(S^1) \to \widetilde{H}_1(A \cap B)$ is an isomorphism.

(3) We consider the retraction

$$\begin{array}{rcl} r\colon A & \to & \{z \in \partial\overline{B}^2\}/z \sim -z \\ p(se^{i\varphi}) & \mapsto & e^{i\varphi} \end{array}$$

which is easily seen to be a homotopy equivalence and we also consider the homeomorphism

$$\begin{array}{rcl} f\colon \{z \in \partial\overline{B}^2\}/z \sim -z & \to & S^1 \\ z & \mapsto & z^2. \end{array}$$

As in (2) it follows that $\widetilde{H}_n(A) = 0$ for $n \neq 1$ and that $(f \circ r)_* \colon \widetilde{H}_1(A) \to \widetilde{H}_1(S^1)$ is an isomorphism.

The above Mayer–Vietoris sequence is thus in our case of the following form:

$$\begin{array}{ccccccccccc}
\ldots \to 0 \to \widetilde{H}_3(\mathbb{R}\mathrm{P}^2) \to 0 \to 0 \to \widetilde{H}_2(\mathbb{R}\mathrm{P}^2) & \xrightarrow{\partial} & \widetilde{H}_1(A \cap B) & \xrightarrow{\quad i_* \quad} & \widetilde{H}_1(A) & \xrightarrow{i_*} & \widetilde{H}_1(\mathbb{R}\mathrm{P}^2) \to 0 \\
 & & {\scriptstyle j_*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (f\circ r)_*} & & \\
 & & H_1(S^1) & \xrightarrow[\cdot \deg(f\circ r\circ i\circ j)]{(f\circ r\circ i\circ j)_*} & H_1(S^1). & &
\end{array}$$

We obtain immediately that $\widetilde{H}_n(\mathbb{R}\mathrm{P}^2) = 0$ for $n \geq 3$. Furthermore it follows immediately from the definitions that the map $f \circ r \circ i \circ j \colon S^1 \to S^1$ is given by $z \mapsto z^2$ which implies by Lemma 32.15 that $\deg(f \circ r \circ i \circ j) = 2$. Thus we obtain the following exact sequence

$$0 \;\to\; \widetilde{H}_2(\mathbb{R}\mathrm{P}^2) \;\to\; \mathbb{Z} \;\xrightarrow{\cdot 2}\; \mathbb{Z} \;\to\; \widetilde{H}_1(\mathbb{R}\mathrm{P}^2) \;\to\; 0.$$

It follows from the discussion on page 764 that

$$\widetilde{H}_2(\mathbb{R}\mathrm{P}^2) \;\cong\; \ker\left\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\right\} \;=\; 0 \quad \text{and} \quad \widetilde{H}_1(\mathbb{R}\mathrm{P}^2) \;\cong\; \operatorname{coker}\left\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\right\} \;=\; \mathbb{Z}_2. \qquad \square$$

33.4. **The Mayer–Vietoris Theorem for manifolds and CW-complexes.** When we studied fundamental groups we first proved the Seifert-van Kampen Theorem 14.1 for the decomposition of a topological space X into two *open* subsets. Later on we showed in Theorem 14.2 and Theorem 26.13 that an analogous statement holds if we decompose a manifold into suitable submanifolds and if decompose a CW-complex into subcomplexes.

In a similar fashion we now state and prove the "manifold version" and the "CW-complex version" of the Mayer–Vietoris Theorem 33.5.

Theorem 33.10. (Mayer–Vietoris Theorem for manifolds) *Let X be an m-dimensional manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that the following statements hold:*

(1) $X = A \cup B$,
(2) $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B,
(3) A and B are closed subsets of X.

The following two statements hold:

(1) *For each $n \in \mathbb{N}$ there exists a unique homomorphism $\partial_n \colon H_n(X) \to H_{n-1}(A \cap B)$ which has the following property: If $a \in C_n(A)$ and $b \in C_n(B)$ are such that $a + b \in C_n(X)$ is a cycle, then*

$$\partial_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; H_{n-1}(A \cap B).$$

(2) *The following sequence is exact:*

$$\cdots \to H_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A \cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \to \ldots$$

A totally analogous statement also holds if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.

Proof $(*)$. Let X be an m-dimensional manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that $X = A \cup B$ and such that $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.

We pick maps $f\colon \partial A \times [0,1) \to A$ and $g\colon \partial B \times [0,1) \to B$ that are provided by the Collar Neighborhood Theorem 10.10. We set

$$U \;:=\; A \cup \underbrace{g((A \cap B) \times [0,1))}_{\subset B \times [0,1)} \quad \text{and} \quad V \;:=\; B \cup \underbrace{f((A \cap B) \times [0,1))}_{\subset A \times [0,1)}.$$

As in the proof of Theorem 14.2 we note that the following statements hold:

(1) A is a deformation retract of U,
(2) B is a deformation retract of V, and
(3) $A \cap B$ is a deformation retract of $U \cap V$.

Now we consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r, dotted] & H_n(A \cap B) \arrow[r, "i \oplus -i"] \arrow[d] & H_n(A) \oplus H_n(B) \arrow[r, "i+i"] \arrow[d] & H_n(X) \arrow[r, dotted, "\partial_n"] \arrow[d, "="] & H_{n-1}(A \cap B) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(U \cap V) \arrow[r, "i \oplus -i"] & H_n(U) \oplus H_n(V) \arrow[r, "i+i"] & H_n(X) \arrow[r, "\partial_n"] & H_{n-1}(U \cap V) \arrow[r] & \dots
\end{tikzcd}$$

where the bottom sequence is given by the Mayer–Vietoris Theorem 33.5. We make the following observations:

(1) The two squares to the left commute since all the relevant maps are given by inclusions.
(2) The vertical maps are isomorphisms by Corollary 30.8.
(3) By (2) we can now define the dotted map $H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B)$ such that the third square also commutes.

The theorem is now an immediate consequence of the Mayer–Vietoris Theorem 33.5. $\square$

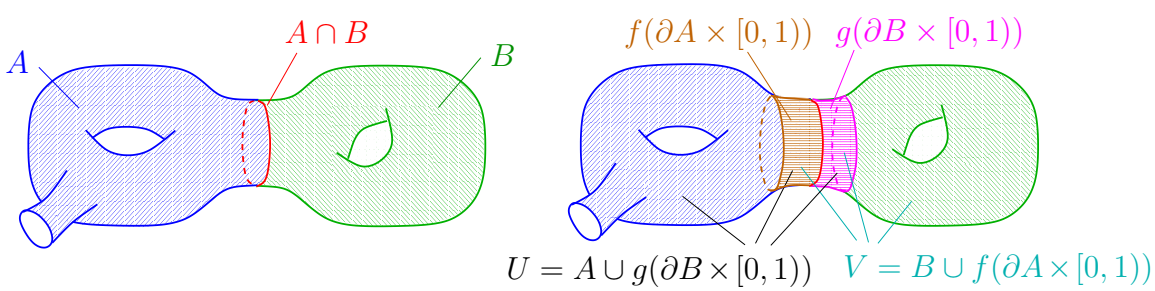


FIGURE 522. Illustration of the proof of Theorem 33.10.

Now we turn to CW-complexes.

Theorem 33.11. (Mayer–Vietoris Theorem for CW-complexes) *Let X be a CW-complex. If A and B are two subcomplexes of X such that $X = A \cup B$, then exactly the same conclusion as in the Mayer–Vietoris Theorem 33.10 for manifolds holds.*

Proof $(*)$. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. By Proposition 26.11 (8) there exist open neighborhoods U of A and V of B such that the following statements hold:

(1) A is a deformation retract of U,
(2) B is a deformation retract of V, and
(3) $A \cap B$ is a deformation retract of $U \cap V$.

The proof is now verbatim the same as the proof of the Mayer–Vietoris Theorem 33.10 for manifolds. □

Later on we will generalize the above Mayer–Vietoris theorems to "excisive triads". But for the time being we will be able to handle most situations with the Mayer–Vietoris theorems that we had stated.

33.5. **The homology groups of the torus and the Klein bottle.** In this section we will determine the homology groups of the torus and the Klein bottle.

Lemma 33.12. *Let $T = S^1 \times S^1$ be the torus, then*

$$H_n(T) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Throughout the proof we identify the torus T with $S^1 \times \mathbb{R}/4\mathbb{Z}$ in the obvious way. As usual we denote by $p\colon S^1 \times \mathbb{R} \to T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection map. In the following we write $A := p(S^1 \times ([0,1] \cup [3,4]))$ and $B := p(S^1 \times [1,3])$. Evidently we have $T = A \cup B$. Furthermore we write $C_1 := p(S^1 \times \{1\})$ and $C_2 := p(S^1 \times \{3\})$. Evidently we have $A \cap B = C_1 \cup C_2$.

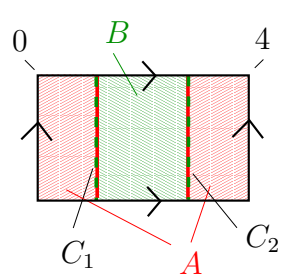


FIGURE 523.

Clearly the hypotheses for the Mayer–Vietoris Theorem 33.10 for manifolds are satisfied for $T = A \cup B$ and we obtain the following long exact sequence in homology[666]

$$\begin{array}{ccccccc}
 & & & \dots & \to & H_2(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} H_1(C_1) \\ H_1(C_2) \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} H_1(A) \\ H_1(B) \end{matrix} & \xrightarrow{i_A+i_B} & H_1(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} H_0(C_1) \\ H_0(C_2) \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} H_0(A) \\ H_0(B) \end{matrix} & \xrightarrow{i_A+i_B} & H_0(T) & \longrightarrow \quad 0.
\end{array}$$

By Lemma 32.11, for each interval I and each point $x \in I$ the map $(1-t, t) \to (e^{2\pi i t}, x)$ is a singular chain that represents a generator of $H_1(S^1 \times I) \cong \mathbb{Z}$. Now we use this generator to identify the groups $H_1(A)$, $H_1(B)$, $H_1(C_1)$ and $H_1(C_2)$ with $\mathbb{Z}$. The topological spaces A, B, C_1 and C_2 are path-connected, so we can identify their 0-th homology groups with $\mathbb{Z}$ using the isomorphism from Proposition 29.5. Thus we obtain the following Mayer–Vietoris sequence

$$\begin{array}{ccccccc}
 & & & 0 & \to & H_2(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{i_A+i_B} & H_1(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{i_A+i_B} & H_0(T) & \longrightarrow \quad 0.
\end{array}$$

To determine the homology groups $H_2(T)$ and $H_1(T)$ we now need to understand the maps in the above long exact sequence. It follows immediately from the choice of the identifications of the 0-th and first homology groups that all the inclusion maps induce the identity. Put differently, we obtain the following long exact sequence:

$$\begin{array}{ccccccc}
 & & & 0 & \to & H_2(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{(1\ 1)} & H_1(T) & \xrightarrow{\partial} \\
\to & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} & \xrightarrow{(1\ 1)} & \mathbb{Z} & \longrightarrow \quad H_0(T).
\end{array}$$

[666]Here we write

$$\begin{matrix} H_i(C_1) \\ H_i(C_2) \end{matrix} \text{ instead of } H_i(C_1) \oplus H_i(C_2) \quad \text{and} \quad \begin{matrix} H_i(A) \\ H_i(B) \end{matrix} \text{ instead of } H_i(A) \oplus H_i(B).$$

The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

It follows from the discussion on page 764 that

$$H_2(T) \;\cong\; \ker\left(\begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \right) \;=\; \mathbb{Z}\cdot\begin{pmatrix}1\\ -1\end{pmatrix} \;\cong\; \mathbb{Z}.$$

Using Lemma 33.4 we obtain the following short exact sequence:

$$0 \;\to\; \underbrace{\operatorname{coker}\left(\begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \right)}_{\cong \begin{matrix}\mathbb{Z}\\ 0\end{matrix}} \;\to\; H_1(T) \;\to\; \underbrace{\ker\left(\begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \right)}_{\cong \mathbb{Z}\cdot\left(\begin{smallmatrix}1\\ -1\end{smallmatrix}\right)} \;\to\; 0.$$

Thus we obtain a short exact sequence

$$0 \;\to\; \begin{matrix}\mathbb{Z}\\ 0\end{matrix} \;\to\; H_1(T) \;\to\; \mathbb{Z}\cdot\begin{pmatrix}1\\ -1\end{pmatrix} \;\to\; 0.$$

It follows from Corollary 33.3 that $H_1(T) \cong \mathbb{Z}^2$. Finally it follows from the above long exact sequence and the discussion on page 764 that[667]

$$H_0(T) \;\cong\; \operatorname{coker}\left(\begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \right) \;\cong\; \mathbb{Z}.$$

This concludes the proof of the lemma. □

Now we also want to determine explicit generators for $H_1(T) \cong \mathbb{Z}^2$ and $H_2(T) \cong \mathbb{Z}$. We will do so in the following lemma.

Lemma 33.13. *Let $T = S^1 \times S^1$ be the torus, then the following hold:*[668]

(1) *The singular 1-simplices c and d shown in Figure 524 on the left are generators for $H_1(T) \cong \mathbb{Z}^2$.*
(2) *The singular 2-chain $\alpha - \beta$ that is given by the singular 2-simplices α and β shown in Figure 524 on the right is a generator for $H_2(T) \cong \mathbb{Z}$.*[669]

Proof. We continue with the notation that we introduced in the proof of Lemma 33.12. Recall that this means that we identify the torus T with the product $S^1 \times \mathbb{R}/4\mathbb{Z}$, we denote by $p\colon S^1 \times \mathbb{R} \to T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection map, we write $A := p(S^1 \times ([0,1] \cup [3,4]))$, $B := p(S^1 \times [1,3])$ and we write $C_1 := p(S^1 \times \{1\})$ and $C_2 := p(S^1 \times \{3\})$.

(1) Let c be the standard generator of $H_1(B)$ and let a and b be the singular simplices in A and B that are indicated in Figure 525. We consider the following short exact

[667] Of course we already knew by Proposition 29.5 that $H_0(T) \cong \mathbb{Z}$.

[668] Here the singular simplices in Figure 524 are understood to be given by the obvious affine maps $\Delta^n \to [0,1] \times [0,1]/\sim$.

[669] This singular 2-chain is precisely the one that we had already considered in Figure 475. In particular this answers our question from page 742, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $H_2(T)$.

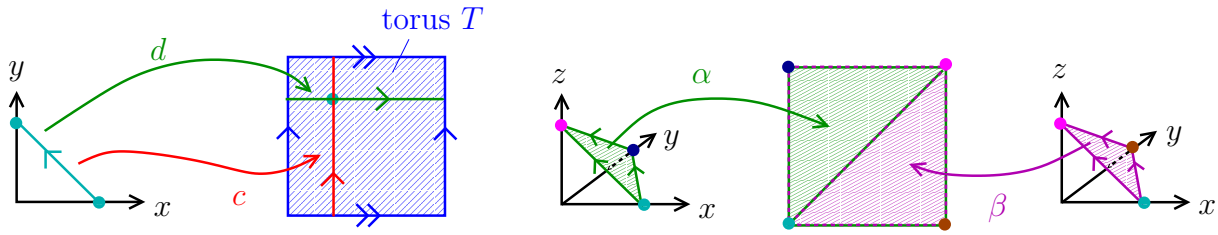


FIGURE 524.

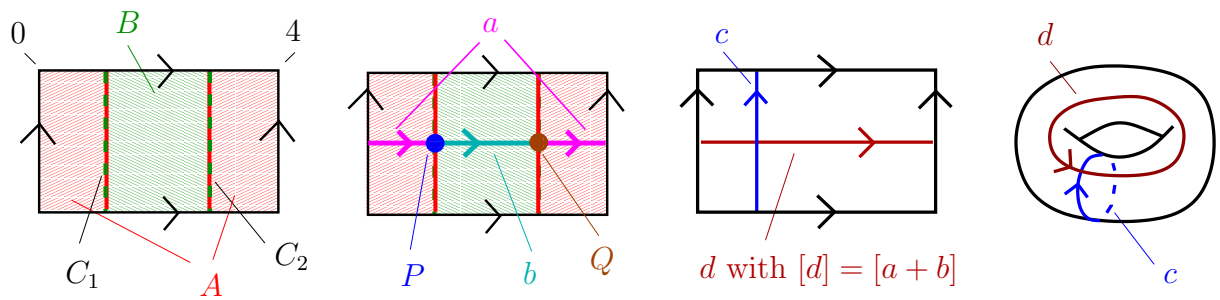


FIGURE 525.

sequence that we had obtained in the proof of Lemma 33.12:

$$0 \to \operatorname{coker}\begin{pmatrix} H_1(C_1)=\mathbb{Z} \\ H_1(C_2)=\mathbb{Z} \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \begin{matrix} H_1(A)=\mathbb{Z} \\ H_1(B)=\mathbb{Z} \end{matrix}\Bigg) \to H_1(T) \xrightarrow{\partial_1} \ker\begin{pmatrix} H_0(C_1)=\mathbb{Z} \\ H_0(C_2)=\mathbb{Z} \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \begin{matrix} H_0(A)=\mathbb{Z} \\ H_0(B)=\mathbb{Z} \end{matrix}\Bigg) \to 0.$$

It is clear that $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{matrix} H_1(C_1)=\mathbb{Z} \\ H_1(C_2)=\mathbb{Z} \end{matrix}$ represents a generator of the group on the left-hand side. Furthermore note that $a+b$ is a cycle in T with $a \in C_1(A)$ and $b \in C_1(B)$, thus we have

$$\partial_1([a+b]) \underset{\substack{\uparrow \\ \text{see the definition of } \partial_1 \text{ in Theorem 33.10 (1)}}}{=} [\partial a] = [P-Q] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \begin{matrix} H_1(C_1)=\mathbb{Z} \\ H_1(C_2)=\mathbb{Z} \end{matrix}$$

Thus we see that $\partial_1([a+b])$ is a generator of the group on the right-hand side of the above short exact sequence. It follows from Corollary 33.3 (2) that $[c]$ and $[a+b]$ generate the free abelian group $H_1(T) \cong \mathbb{Z}^2$. The desired statement now follows from the observation that Lemma 29.3 (3) implies that $[a+b] = [d]$.

(2) Let α and β be the two singular 2-simplices that are shown in Figure 524 on the right. As we will see, the proof that $\alpha - \beta$ represents a generator of $H_2(T)$ is quite similar to the proof of Lemma 32.10.

First note that it is straightforward to verify that the singular 2-chain $\alpha - \beta$ is indeed a cycle. We denote by A and B the images of $\alpha\colon \Delta^2 \to T$ and $\beta\colon \Delta^2 \to T$. (We refer

to Figure 526 for an illustration.) We consider the following maps

$$\begin{tikzcd}
H_2(T) \arrow[d] \arrow[dr, "{[\alpha-\beta]\mapsto[\alpha-\beta]}"] & & & H_2(\Delta^2,\partial\Delta^2) \arrow[dlll, "{[\mathrm{id}_{\Delta^2}]\mapsto[\alpha]}"'] \arrow[d, "\cong"'] \\
H_2(T/B) \arrow[r] & \underbrace{H_2(T/B,B/B)}_{\text{here } [\alpha-\beta]=[\alpha]} & H_2(A/\partial A,\partial A/\partial A) \arrow[l, "\cong"] & H_2(\Delta^2/\partial\Delta^2,\partial\Delta^2/\partial\Delta^2). \arrow[l, "\alpha_*"', "\cong"]
\end{tikzcd}$$

We make the following observations:

(a) By Lemma 32.9 we know that $[\mathrm{id}_{\Delta^2}] \in H_2(\Delta^2,\partial\Delta^2) \cong \mathbb{Z}$ is a generator.
(b) The two horizontal maps on the right are induced by homeomorphisms of pairs, hence they are isomorphisms.
(c) The vertical homomorphism on the right is an isomorphism by Proposition 31.19 and Lemma 31.12.
(d) The statements (a), (b) and (c) imply that $[\alpha]$ represents a generator of the group $H_2(T/B,B/B) \cong \mathbb{Z}$.
(e) The singular 2-simplex $\beta\colon \Delta^2 \to B$ is zero in $C_2(T/B,B/B)$. Therefore we have $[\alpha-\beta]=[\alpha] \in H_2(T/B,B/B)$.
(f) The statements (d) and (e) imply that the image of $[\alpha-\beta] \in H_2(T) \cong \mathbb{Z}$ under the map $H_2(T) \to H_2(T/B,B/B)$ is a generator of $H_2(T/B,B/B) \cong \mathbb{Z}$. But this implies that $[\alpha-\beta]$ is a generator of $H_2(T) \cong \mathbb{Z}$. □

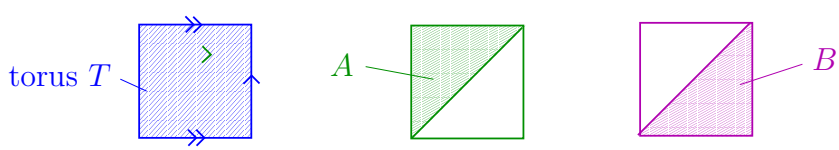


FIGURE 526. Illustration for the proof of Lemma 33.13 (2).

In Exercise 33.1 we will determine the homology groups of the Klein bottle. More precisely, using the Mayer–Vietoris Theorem 33.10 for manifolds we will prove the following lemma.

Lemma 33.14. *We denote by K the Klein bottle. Then*

$$H_n(K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}\oplus\mathbb{Z}_2, & \text{if } n=1,\\ 0, & \text{otherwise.}\end{cases}$$

Furthermore we can determine explicitly two generators of $H_1(K)$. More precisely, we consider Figure 527. There the singular 1-simplex d generates the $\mathbb{Z}_2$-summand and the

singular 1-simplex c generates a $\mathbb{Z}$-summand of $H_1(K)$.[670] We leave the verification of this statement as an exercise to the reader.

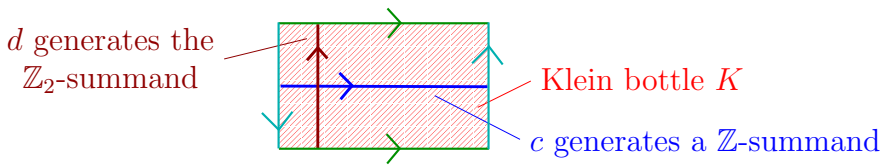


FIGURE 527.

Remark. With some effort one could now use the Mayer–Vietoris Theorem 33.10 to compute the homology groups of the surfaces of genus ≥ 2. We will not do so, since we will soon get to know an alternative approach to calculating homology groups, with which we can easily compute these homology groups.[671]

33.6. **The homology groups of a knot complement.** Finally we want to determine the first homology group of the complement of a knot. We recall the relevant definitions from pages 360 and 362.

(1) A *knot* is a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1.
(2) Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem 10.19 there exists an orientation-preserving embedding $F\colon \overline{B}^2 \times S^1 \to S^3$ with $\Phi(\{0\} \times S^1) = K$. For any $* \in S^1$ we refer to the oriented submanifold $F(S^1 \times \{*\})$ as a *meridian* of the knot K.

We refer to Figure 528 for an illustration of an oriented knot and a meridian. We also introduce one new definition:

(3) Let M be a manifold and let C be a closed oriented curve in M, see page 306 for the definition. Sometimes, by a slight abuse of notation, we denote by C also the element in $H_1(M)$ that is given by the image of the standard generator $[S^1]$ of $H_1(S^1)$ under the map $H_1(S^1) \xrightarrow{\Phi_*} H_1(C) \to H_1(M)$.[672]

Lemma 33.15. *Let $K \subset S^3$ be a knot. Then*[673]

$$H_0(S^3 \setminus K) \;\cong\; \mathbb{Z} \quad \textit{and} \quad H_1(S^3 \setminus K) \;\cong\; \mathbb{Z}.$$

Furthermore every meridian is a generator of $H_1(S^3 \setminus K)$.

[670] It is straightforward to see that c generates an element of infinite order since the "vertical projection" $K \to [0,1]/\sim$ sends c to a generator of $H_1([0,1]/\sim) = H_1(S^1)$.

[671] Of course it is a great exercise to determine the homology groups of the surfaces of genus ≥ 2 with our present knowledge.

[672] It follows from Proposition 20.5 and Proposition 30.5 that, up to a sign, the homology class does not depend on the choice of the orientation-preserving diffeomorphism Φ.

[673] By Proposition 29.5 the first statement just says that $S^3 \setminus K$ is path-connected. This seems obvious from the pictures, but it is nonetheless non-trivial to prove.

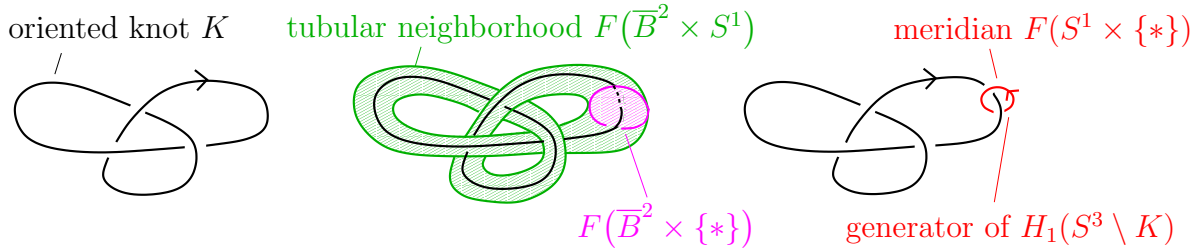


FIGURE 528.

Later on we will compute the other homology groups of $S^3 \setminus K$.

Remark. The result of Lemma 33.15 is easy to state but also slightly disappointing since it shows that the first homology groups cannot be used to distinguish different knots. In contrast to this fact we saw in Proposition 18.5 and Lemma 18.6 that the fundamental group can be used to distinguish the trivial knot from the trefoil.

Proof. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 10.19 there exists an embedding $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ with $\Phi(\{0\} \times S^1) = K$. In the following we write $\overline{B}^2 \times S^1$ instead of $\Phi(\overline{B}^2 \times S^1)$. It is clear that the interiors of $S^3 \setminus K$ and $\overline{B}^2 \times K$ cover S^3. Therefore we can apply the Mayer–Vietoris Theorem 33.5 to $S^3 = (S^3 \setminus K) \cup (\overline{B}^2 \times K)$ and we obtain the following exact sequence

$$\begin{array}{ccccccc}
\overbrace{H_2(S^3)}^{=0} & \longrightarrow & H_1\big((\overline{B}^2 \setminus 0) \times K\big) & \longrightarrow & H_1(S^3 \setminus K) \oplus H_1(\overline{B}^2 \times K) & \longrightarrow & \overbrace{H_1(S^3)}^{=0} \longrightarrow \\
 & \longrightarrow & \widetilde{H}_0((\overline{B}^2 \setminus 0) \times K) & \longrightarrow & \widetilde{H}_0(S^3 \setminus K) \oplus \widetilde{H}_0\big(\overline{B}^2 \times K\big) & \longrightarrow & \underbrace{\widetilde{H}_0(S^3)}_{=0} \longrightarrow 0.
\end{array}$$

Now we have to study the homology groups of the various spaces in more detail.

(1) We denote by $\mu = \Phi(S^1 \times \{*\})$ the meridian and we define $\lambda = \Phi(\{*\} \times S^1)$. It follows from Lemma 33.13 that μ and λ is a basis for $H_1(\Phi(S^1 \times S^1))$. The inclusion $S^1 \times S^1 \to (\overline{B}^2 \setminus 0) \times K$ is a homotopy equivalence. It follows from the above, Proposition 29.5 and Corollary 30.8 that $H_1\big((\overline{B}^2 \setminus 0) \times K\big) \cong \mathbb{Z} \cdot \mu \oplus \mathbb{Z} \cdot \lambda$ and that $\widetilde{H}_0((\overline{B}^2 \setminus 0) \times K) = 0$.

(2) We write $\gamma = \{0\} \times K$. The inclusion $\{0\} \times K \to \overline{B}^2 \times K$ is a homotopy equivalence. Similar to (1) we see that $H_1(\overline{B}^2 \times K) \cong \mathbb{Z} \cdot \gamma$ and that $\widetilde{H}_0(\overline{B}^2 \times K) = 0$.

(3) Under the inclusion induced map $H_1\widetilde{((\overline{B}^2 \setminus 0) \times K)} = \mathbb{Z} \cdot \mu \oplus \mathbb{Z} \cdot \lambda \to H_1(\overline{B}^2 \times K) = \mathbb{Z} \cdot \gamma$ we have $\mu \mapsto 0$[674] and we have $\lambda \mapsto \gamma$.[675].

[674]This can be seen as follows: the map $H_1(* \times S^1) \to H_1(\overline{B}^2 \times K)$ factors through $H_1(* \times \overline{B}^2) = 0$.
[675]Why is that?

Thus we obtain an exact sequence of the following form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}\cdot\mu\oplus\mathbb{Z}\cdot\lambda & \xrightarrow[\cong]{\substack{\mu\mapsto(\mu,0)\\ \lambda\mapsto(\lambda,\gamma)}} & H_1(S^3\setminus K)\oplus\mathbb{Z}\cdot\gamma & \longrightarrow & 0 & \longrightarrow & \\ & \longrightarrow & 0 & \longrightarrow & \widetilde{H}_0(S^3\setminus K)\oplus 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Now it follows immediately from Corollary 33.3 and Lemma 11.8 (1) that $H_1(S^3\setminus K)\cong\mathbb{Z}$ and that $\widetilde{H}_0(S^3\setminus K)\cong 0$.[676] Furthermore it follows easily that $\mu\in H_1(S^3\setminus K)=H_1(S^3\setminus K)$ is a generator.[677] $\square$

Remark. In the proof of Lemma 33.15 we used the Mayer–Vietoris Theorem 33.5 in a different way than in the previous examples. More precisely, instead of decomposing $S^3\setminus K$ into smaller pieces, we used that for a decomposition $X=U\cup V$ the Mayer–Vietoris sequence not only relates the homology groups of U,V and $U\cap V$ to the homology groups of X, but also that it relates the homology groups of U to the homology groups of $V,U\cap V$ and X.

The following lemma will be proved in Exercise 33.5.

Lemma 33.16. *Let $X\subset\mathbb{R}^n$ be a non-empty compact subset. We denote by $S^n=\mathbb{R}^n\cup\{\infty\}$ the topological space $\mathbb{R}^n$ with a "point at infinity". Then for any $k\in\mathbb{N}_0$ we have*

$$H_k(\mathbb{R}^n\setminus X)\ \cong\ \begin{cases} H_k((\mathbb{R}^n\cup\{\infty\})\setminus X)\oplus\mathbb{Z}, & \text{if } k=n-1,\\ H_k((\mathbb{R}^n\cup\{\infty\})\setminus X), & \text{otherwise.}\end{cases}$$

33.7. **The homology groups of a mapping cone** $(*)$**.** In this section we will discuss the homology groups of a mapping cone. Before we start the discussion we recall the relevant definitions.

(1) On page 450, given a topological space A we defined the *cone on A* as

$$\mathrm{Cone}(A)\ :=\ (A\times[0,1])/\sim \qquad \text{where } (a,0)\sim(a',0) \text{ for all } a,a'\in A.$$

(2) Given a map $f\colon A\to X$ between topological spaces we defined on page 458 the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A\to X)\ :=\ (X\sqcup\mathrm{Cone}(A))/\sim \ \text{ where } (a,1)\sim f(a) \text{ for all } a\in A.$$

First we intend to understand the homology groups of the mapping cone of an inclusion $i\colon A\to X$ where A is a non-empty subset of some topological space X. We refer to $P=[A\times 0]$ as the cone point of $\mathrm{Cone}(A)$ and of $\mathrm{Cone}(i\colon A\to X)$. We have the following

[676] Can we use the same approach to determine $H_2(S^3\setminus K)$?

[677] Why does this follow?

isomorphisms

$$\begin{array}{ccccc}
 & \text{by Corollary 31.14 since we know by Lemma 15.1 that the Cone}(A)\text{ is contractible} & & & \\
 & \downarrow & & & \\
\underbrace{\widetilde{H}_n((X \sqcup \mathrm{Cone}(A))/\sim)}_{=\widetilde{H}_n(\mathrm{Cone}(i\colon A\to X))} & \overset{\cong}{\to} & H_n((X \sqcup \mathrm{Cone}(A))/\sim, \mathrm{Cone}(A)) & & \\
 & \overset{\cong}{\leftarrow} & H_n((X \sqcup \mathrm{Cone}(A))/\sim \backslash P, \mathrm{Cone}(A)\backslash P) & \overset{\cong}{\leftarrow} & H_n(X,A). \\
 & \uparrow & & \uparrow & \\
 & \text{Excision Theorem 31.17} & \text{from a slight variation on Lemma 15.7 (1) we obtain a deformation retraction, thus the inclusion induces an isomorphism by Exercise 31.9} & &
\end{array}$$

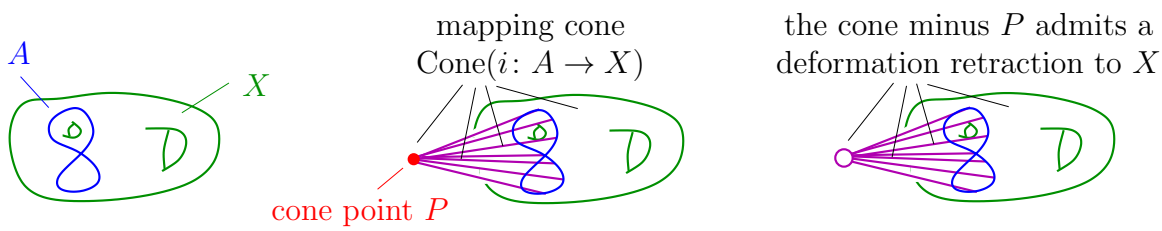


FIGURE 529.

The following lemma summarizes the discussion.

Lemma 33.17. *We denote by $\mathcal{P}$ the category of pairs of topological spaces and we denote by $\mathcal{A}$ the category of abelian groups. The above isomorphisms define a natural isomorphism from the functor*[678]

$$\begin{array}{rcl} \mathcal{P} & \to & \mathcal{A} \\ (X,A) & \mapsto & \widetilde{H}_n(\mathrm{Cone}(i\colon A\to X)) \end{array} \quad \textit{to the functor} \quad \begin{array}{rcl} \mathcal{P} & \to & \mathcal{A} \\ (X,A) & \mapsto & H_n(X,A). \end{array}$$

Remark.

(1) It is worth comparing Lemma 31.12 to Lemma 33.17.
(2) In Proposition 31.19 we showed that, under favorable circumstances, we can replace the relative homology group $H_n(X,A)$ by the homology group $\widetilde{H}_n(X/A)$. Lemma 33.17 says that we can replace any relative homology group by the homology group of the mapping cone of the inclusion.
(3) Loosely speaking, Lemma 33.17 says that for all intents and purposes we can always replace relative the homology groups of a pair (X,A) of topological spaces by the relative homology of the mapping cone of the inclusion $A \to X$.

Proof. We had just seen that the map is an isomorphism. It is straightforward to verify that the maps define a natural transformation. □

[678] As in Lemma 15.9 (3) we see that the left-hand side does indeed define a functor.

Given an inclusion map $i\colon A \to X$ of (non-empty) topological spaces we just saw in Lemma 33.17 that the reduced homology of the mapping cone is naturally isomorphic to the relative homology groups of (X, A). One of the key features of relative homology groups is that they fit into a long exact sequence of homology groups.

The following lemma proves the neat fact that the homology groups of any mapping cone fit into a long exact sequence.

Lemma 33.18.

(1) *Let $f\colon A \to X$ be a map between topological spaces.*
   (a) *We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$ the obvious inclusion map.*
   (b) *We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also Figure 530.) We consider the following diagram*

$$\begin{tikzcd}
\dots \arrow[r] & H_k(U) \oplus H_k(V) \arrow[r] & H_k(\mathrm{Cone}(f)) \arrow[r, "\partial"] \arrow[dr, "=:\partial^{k-1}(f)"'] & H_{k-1}(U \cap V) \arrow[r] \arrow[d, "="] & \dots \\
 & & & H_{k-1}(A). &
\end{tikzcd}$$

Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 33.5[679]*. Furthermore the vertical map comes from the obvious identifications $H_{k-1}(U \cap V) = H_{k-1}(A \times \frac{1}{2}) = H_{k-1}(A)$. From the above diagram we obtain a map*

$$\partial_k(f)\colon H_k(\mathrm{Cone}(f)) \ \to \ H_{k-1}(A).$$

With these definitions the following sequence is exact

$$\dots \longrightarrow H_k(A) \xrightarrow{f_*} H_k(X) \xrightarrow{j(f)_*} \widetilde{H}_k(\mathrm{Cone}(f)) \xrightarrow{\partial_k(f)} H_{k-1}(A) \longrightarrow \dots$$

(2) *The connecting homomorphism in* (1) *is natural. In particular given any commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\
\widetilde{A} \arrow[r, "\widetilde{f}"] & \widetilde{X}
\end{tikzcd}$$

of maps between topological spaces the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & H_k(A) \arrow[r, "f_*"] \arrow[d, "\varphi_*"] & H_k(X) \arrow[r, "j(f)_*"] \arrow[d, "\psi_*"] & \widetilde{H}_k(\mathrm{Cone}(f)) \arrow[r, "\partial_k(f)"] \arrow[d] & H_{k-1}(A) \arrow[r] \arrow[d, "\varphi_*"] & \dots \\
\dots \arrow[r] & H_k(\widetilde{A}) \arrow[r, "\widetilde{f}_*"] & H_k(\widetilde{X}) \arrow[r, "j(\widetilde{f})_*"] & \widetilde{H}_k(\mathrm{Cone}(\widetilde{f})) \arrow[r, "\partial_k(\widetilde{f})"] & H_{k-1}(\widetilde{A}) \arrow[r] & \dots.
\end{tikzcd}$$

[679]Note that U and V are open subsets of $\mathrm{Cone}(f)$.

Here the vertical maps between the homology groups of the mapping cones are induced by the maps from Lemma 15.9 (3).

(3) *Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i\colon A \to X$ the inclusion map. The following diagram*

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{H}_k(A) \arrow[r, "i"] \arrow[d, "="] & \widetilde{H}_k(X) \arrow[r] \arrow[d, "="] & H_k(X,A) \arrow[r, "\partial"] \arrow[d, "\cong"] & \widetilde{H}_{k-1}(A) \arrow[r] \arrow[d, "="] & \widetilde{H}_{k-1}(X) \arrow[r] \arrow[d, "="] & \dots \\
\dots \arrow[r] & \widetilde{H}_k(A) \arrow[r, "i"] & \widetilde{H}_k(X) \arrow[r] & \widetilde{H}_k(\mathrm{Cone}(i)) \arrow[r, "\partial"] & \widetilde{H}_{k-1}(A) \arrow[r] & \widetilde{H}_{k-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

commutes. Here the middle vertical map is the natural isomorphism from Lemma 33.17, the top sequence is the long exact sequence arising from Corollary 31.14 the bottom sequence is the exact sequence from (1).

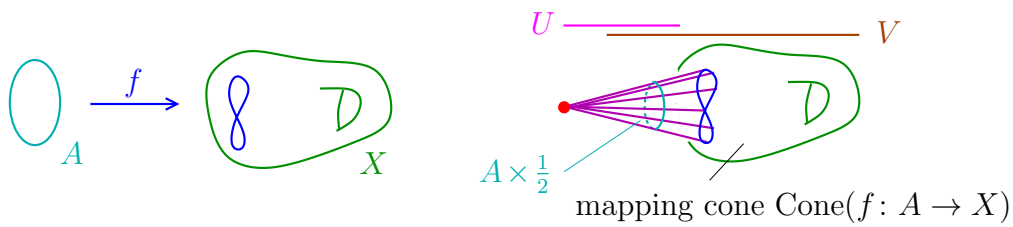


FIGURE 530. Illustration of Lemma 33.18.

Proof.

(1) The statement follows from the Mayer–Vietoris Theorem 33.5 applied to the open covering $\mathrm{Cone}(f) = U \cup V$ and the following two observations:
   (a) As in Lemma 15.1 we see that the topological space U is homotopy equivalent to a point, in particular it only contributes to the exact sequence in dimension 0, which then gets swallowed up by replacing $H_0(\mathrm{Cone}(f);\mathbb{Z})$ by the reduced version.
   (b) The subset $X \subset V$ is a deformation retract.

   We leave it as a highly instructive exercise to fill in the remaining details.

(2) This statement follows from Lemma 33.6.

(3) We will prove this statement in Exercise 33.9. □

The last lemma of this section gives a neat reinterpretation of the connecting homomorphism in the long exact sequence of Lemma 33.18.

Lemma 33.19. $(*)$ *Let $f\colon A \to X$ be a map between topological spaces. For any $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{H}_k(\mathrm{Cone}(f\colon A \to X)) & \xrightarrow{\quad\partial\quad} & \widetilde{H}_{k-1}(A) \\ \downarrow & & \cong\downarrow \Sigma_A \\ \widetilde{H}_k(\mathrm{Cone}(f\colon A \to X)/X) & \xrightarrow[\cong]{\varphi} & \widetilde{H}_k(\Sigma(A)). \end{array}$$

Here we used the following notation:

1. *The top horizontal map is the connecting homomorphism of the long exact sequence given in Lemma 33.18.*
2. *The left vertical map is induced by the obvious projection.*
3. *The map $\varphi\colon \mathrm{Cone}(f\colon A \to X)/X \to \Sigma(A)$ is the natural homeomorphism from Lemma 15.8 (2).*
4. *The map $\Sigma_A\colon \widetilde{H}_{k-1}(A) \to \widetilde{H}_k(A)$ is the natural isomorphism from Lemma 33.8.*

Proof $(*)$. The statement follows immediately from the explicit definition of the maps ∂ and Φ_A via Mayer-Vietoris sequences, from the explicit definition of the map φ, and from the naturality of the connecting homomorphism of the Mayer-Vietoris sequence, see Lemma 33.6. We leave it to the reader to fill in the details. □

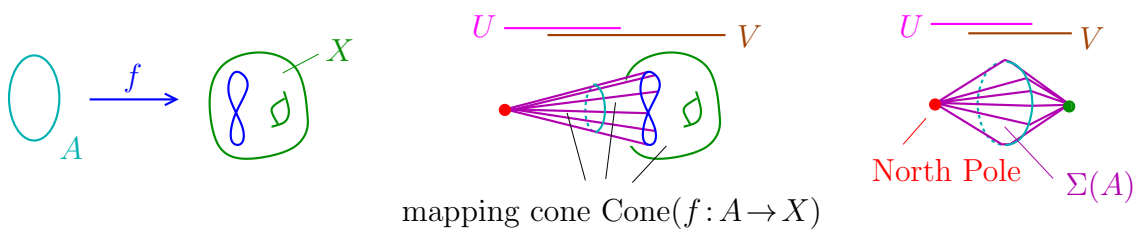


FIGURE 531. Illustration for the proof of Lemma 33.19.

33.8. **The homology groups of a mapping torus** $(*)$**.** On page 469 we defined the mapping torus of a self-homeomorphism and in Proposition 15.25 we had determined the fundamental group of the mapping torus. In this section we will determine the homology groups of a mapping torus.

For convenience we recall from page 469 the definition of the mapping torus of a self-homeomorphism. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We refer to

$$M(X,f) \;:=\; X \times [0,1]/(x,0) \sim (f(x),1)$$

as the *mapping torus of* (X,f).

Proposition 33.20. *Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Then for any $n \in \mathbb{N}_0$ we have a short exact sequence*[680]

$$0 \to \operatorname{coker}\big(H_n(X) \xrightarrow{f_*-\mathrm{id}} H_n(X)\big) \xrightarrow{i} H_n(M(X,f)) \to \ker\big(H_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} H_{n-1}(X)\big) \to 0,$$

where the map i is induced by the inclusion $X = X \times \{0\} \to M(X,f)$.

Example. First let $X = S^1$ and let $f\colon S^1 \to S^1$ be the self-homeomorphism that is given by $f(z) = \overline{z} = z^{-1}$. We saw on page 470 that the mapping torus $M(S^1, f)$ is homeomorphic to the Klein bottle. We make the standard identifications $H_0(S^1) = \mathbb{Z}$ and $H_1(S^1) = \mathbb{Z}$. It follows from the discussion on page 747 the induced map $f_*\colon H_0(S^1) \to H_0(S^1)$ is the identity map. Furthermore we showed in Lemma 32.15 that the induced homomorphism $f_*\colon \mathbb{Z} = H_1(S^1) \to H_1(S^1) = \mathbb{Z}$ is given by multiplication by -1. The short exact sequences of Proposition 33.20 are therefore of the form

$$\begin{array}{lccccccccc}
n=2\colon & & & & 0 & \to & H_2(M(S^1,f)) & \to & \ker\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} & \to & 0 \\
n=1\colon & 0 & \to & \operatorname{coker}\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} & \to & & H_1(M(S^1,f)) & \to & \ker\{\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\} & \to & 0
\end{array}$$

and

$$n=0\colon \quad 0 \to \operatorname{coker}\{\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\} \to H_0(M(S^1,f)) \to 0.$$

Using Corollary 33.3 we now deduce that

$$H_2(M(S^1,f)) = 0, \quad H_1(M(S^1,f)) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \quad \text{and} \quad H_0(M(S^1,f)) = 0.$$

This confirms, as it should, the calculation that we had made in Lemma 33.14.

Proof. The proof of the proposition is quite similar to the calculation of the homology groups of the torus.[681] Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We compute the homology of the following space that is clearly homeomorphic to the mapping torus:

$$M := X \times [0,10]/(x,0) \sim (f(x),10).$$

We denote by $p\colon X \times [0,10] \to M$ the obvious projection map. Furthermore we write $A := p(X \times ([0,4) \cup (6,10])) \subset M$ and $B := p(X \times (2,8)) \subset M$. Evidently the interiors of A and B cover all of M. Finally we write $C_1 := p(X \times (2,4))$ and $C_2 := p(X \times (6,8))$. Thus we have $A \cap B = C_1 \cup C_2$. Finally, given $t \in [0,10]$ we denote by

$$\begin{array}{rcl} i_t\colon X & \to & p(X \times \{t\}) \\ x & \mapsto & p(x,t) \end{array}$$

the obvious inclusion map. We write $i_{3,A}$ for the composition of i_3 with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation.

The following claim is a straightforward consequence of Proposition 30.5. We leave the details as an exercise to the reader.

[680] Recall that $H_{-1}(X)$ is defined to be the trivial group.

[681] In fact if one reads the subsequent proof with $X = S^1$ and $f = \mathrm{id}$ in mind, then one obtains almost the same argument as for the torus.

Claim. For any $n \in \mathbb{N}_0$ the following two diagrams commute

$$\begin{tikzcd} H_n(X) \arrow[rr, "\mathrm{id}"] \arrow[dr, "i_{3,B}"'] & & H_n(X) \arrow[dl, "i_{7,B}"] \\ & H_n(B) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_n(X) \arrow[rr, "f_*^{-1}"] \arrow[dr, "i_{3,A}"'] & & H_n(X) \arrow[dl, "i_{7,A}"] \\ & H_n(B) & \end{tikzcd} .$$

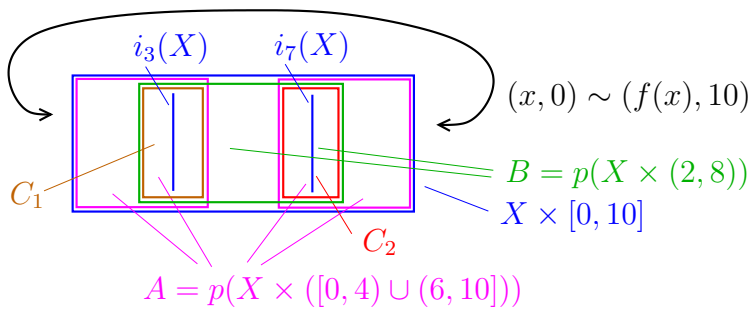


FIGURE 532.

We consider the following diagram:[682]

$$\begin{tikzcd}[column sep=small] \arrow[r] & \begin{matrix} H_n(C_1) \\ H_n(C_2) \end{matrix} \arrow[r, "\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}"] & \begin{matrix} H_n(A) \\ H_n(B) \end{matrix} \arrow[r, "(i_A\ i_B)"] & H_n(M) \arrow[r, "\partial"] & \begin{matrix} H_{n-1}(C_1) \\ H_{n-1}(C_2) \end{matrix} \arrow[r, "\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}"] & \begin{matrix} H_{n-1}(A) \\ H_{n-1}(B) \end{matrix} \arrow[r] & {} \\ \arrow[r] & \begin{matrix} H_n(X) \\ H_n(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \mathrm{id} & f_* \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}"] & \begin{matrix} H_n(X) \\ H_n(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}"] \arrow[r, "(i_A\ i_B)"] & H_n(M) \arrow[u, "="] \arrow[r, "\partial"] & \begin{matrix} H_{n-1}(X) \\ H_{n-1}(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \mathrm{id} & f_* \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}"] & \begin{matrix} H_{n-1}(X) \\ H_{n-1}(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}"] \arrow[r] & {} \\ \arrow[r] & H_n(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ -\mathrm{id} \end{pmatrix}"] \arrow[r, "\mathrm{id} - f_*"] & H_n(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] \arrow[r, "i"] & H_n(M) \arrow[u, "="] \arrow[r, "\partial"] & H_{n-1}(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ -\mathrm{id} \end{pmatrix}"] \arrow[r, "\mathrm{id} - f_*"] & H_{n-1}(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] \arrow[r] & 0. \end{tikzcd}$$

The above diagram arose as follows:

(1) On top we wrote down the Mayer–Vietoris sequence from Theorem 33.5 for the decomposition $M = A \cup B$, which is of course exact.

(2) We observed that the topological spaces A, B, C_1, C_2 are homotopy equivalent to X, more precisely, the maps $i_3 \colon X \to A, i_3 \colon X \to C_1$, $i_7 \colon X \to C_2$ and $i_7 \colon X \to B$ are homotopy equivalences. It follows from Corollary 30.8 that the induced maps on homology are isomorphisms. In particular the vertical maps between the first and the second horizontal sequence are isomorphisms.

[682]Here we use the same notation as on page 849.

(3) We wanted the top part of the diagram to commute. Since the top horizontal maps and the vertical maps were given we had to find the appropriate maps for the second horizontal sequence. For the first square to commute we needed to find maps $a, b, c, d\colon H_n(X) \to H_n(X)$ such that

$$\underbrace{\begin{pmatrix} i_{C_1,A} & i_{C_2,A} \\ -i_{C_1,B} & -i_{C_2,B} \end{pmatrix}\begin{pmatrix} i_{3,C_1} & 0 \\ 0 & i_{7,C_2} \end{pmatrix}}_{=\begin{pmatrix} i_{3,A} & i_{7,A} \\ -i_{3,B} & -i_{7,B} \end{pmatrix}} = \underbrace{\begin{pmatrix} i_{3,A} & 0 \\ 0 & i_{7,B} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=\begin{pmatrix} i_{3,A}\circ a & i_{3,A}\circ b \\ i_{7,B}\circ c & i_{7,B}\circ d \end{pmatrix}}.$$

It follows from the above claim that a solution is given by $a = \mathrm{id}, b = f_*, c = -\mathrm{id}$ and $d = -\mathrm{id}$.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We leave it as an exercise to verify that the bottom sequence is also exact.

As we had just pointed out, the bottom horizontal sequence is exact. The proposition is thus an immediate consequence of Lemma 33.4. □

Now we can also easily prove the following lemma.

Lemma 33.21. *Let X be a topological space. Then for any $n \in \mathbb{N}_0$ we have an isomorphism*

$$H_n(X \times S^1) \;\cong\; H_n(X) \oplus H_{n-1}(X).$$

Example. Applying Lemma 33.21 iteratively one can show that for $k = 0, \dots, n$ we have

$$H_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}$$

and that all other homology groups vanish.

Proof. Let $n \in \mathbb{N}_0$. We identify $X \times S^1$ with $M(X, f := \mathrm{id})$. By Proposition 33.20 we have a short exact sequence

$$0 \to \operatorname{coker}\big(H_n(X) \xrightarrow{f_*-\mathrm{id}} H_n(X)\big) \xrightarrow{i} H_n(M(X,f)) \to \ker\big(H_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} H_{n-1}(X)\big) \to 0,$$

where the map i is induced by the inclusion $X = X \times \{0\} \to M(X, f)$. In our case $f_* - \mathrm{id}_* = 0$, so we obtain the short exact sequence

$$0 \;\to\; H_n(X) \;\xrightarrow{i}\; H_n(X \times S^1) \;\to\; H_{n-1}(X) \;\to\; 0$$

where the map i is induced by the inclusion $X = X \times \{1\} \to X \times S^1$. The inclusion map $X \to X \times S^1$ admits the splitting $p\colon X \times S^1 \to X$ that is given by projection onto the first factor. In particular $p_* \circ i_* = (p \circ i)_* = \mathrm{id}$. It follows from Lemma 33.2 that the above short exact sequence splits and we obtain the desired result. □

Remark. A careful reading of the proof of Lemma 33.21 shows that we can write down explicitly two homomorphisms $p\colon H_n(X \times S^1) \to H_n(X)$ and $q\colon H_n(X \times S^1) \to H_{n-1}(X)$ such that $p \oplus q\colon H_n(X \times S^1) \to H_{n-1}(X) \oplus H_{n-1}(X)$ is an isomorphism. In fact two such homomorphisms are given as follows:

(1) First we consider the homomorphism $p\colon H_n(X \times S^1) \to H_n(X)$ that is induced by the projection $X \times S^1 \to X$.
(2) Secondly we consider the homomorphism that is given by

$$\begin{array}{ccccc} H_n(X \times S^1) & \to & H_n(X \times S^1, X \times S^1_{\leq 0}) & \underset{\substack{\uparrow \\ \text{Excision Theorem 31.17}}}{\overset{\cong}{\leftarrow}} & H_n(X \times S^1_{\geq 0}, X \times \{\pm 1\}) \\ & \underset{\substack{\uparrow \\ \text{connecting homomorphism}}}{\to} & H_{n-1}(X \times \{\pm 1\}) & \underset{\substack{\uparrow \\ \text{projection induced homomorphism}}}{\to} & H_{n-1}(X \times \{1\}) \;=\; H_{n-1}(X). \end{array}$$

We conclude this long chapter with the following lemma.

Lemma 33.22. *Let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. Then there exists a short exact sequence*

$$0 \;\to\; \operatorname{coker}\big(H_1(X) \xrightarrow{f_* - \mathrm{id}} H_1(X)\big) \;\xrightarrow{i}\; H_1(M(X,f)) \;\to\; \mathbb{Z} \;\to\; 0$$

and we have an isomorphism

$$H_1(M(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(H_n(X) \xrightarrow{f_* - \mathrm{id}} H_n(X)\big).$$

Proof. It follows from the discussion on page 747 that in this case $f_* - \mathrm{id}\colon H_0(X) \to H_0(X)$ is the zero map. Thus we obtain from Proposition 33.20 the short exact sequence

$$0 \;\to\; \operatorname{coker}\big(H_1(X) \xrightarrow{f_* - \mathrm{id}} H_1(X)\big) \;\xrightarrow{i}\; H_1(M(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

Since $\mathbb{Z}$ is free abelian it follows from Corollary 33.3 that

$$H_1(M(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(H_n(X) \xrightarrow{f_* - \mathrm{id}} H_n(X)\big).$$

□

Exercises for Chapter 33.

Exercise 33.1.

(a) We denote by

$$K \;=\; [0,1] \times [0,1]/\sim \;\text{ with } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1, 1-y)$$

the Klein bottle. Show that

$$H_n(K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Give an example of a map $f\colon S^1 \to K$ such that the inclusion induced map $f_*\colon H_1(S^1) \to H_1(K)$ is a monomorphism. Prove that the map f that you provide does indeed have this property.

Exercise 33.2. Compute the isomorphism types of the homology groups of the real line with two zeros.

Exercise 33.3. Determine the isomorphism types of the homology groups of the topologist's sine curve that we introduced on page 123.

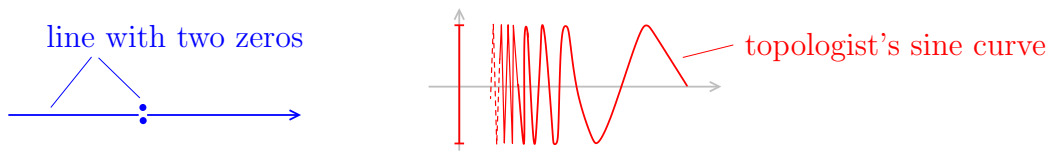


FIGURE 533. Illustration for Exercises 33.2 and 33.3.

Exercise 33.4. Let X be a topological space and let $k \in \mathbb{Z}$. Show that the two natural isomorphisms $\widetilde{H}_k(X) \xrightarrow{\cong} \widetilde{H}_{k-1}(\Sigma(X))$ that we constructed in Proposition 32.14 and that we constructed in the proof of Lemma 33.8 are actually the same.

Exercise 33.5. Let $X \subset \mathbb{R}^n$ be a non-empty compact subset. We denote by $S^n = \mathbb{R}^n \cup \{\infty\}$ the space $\mathbb{R}^n$ with a "point at infinity".

(a) Show that for $k \neq n-1$ we have
$$H_k(\mathbb{R}^n \setminus X) \;\cong\; H_k(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=S^n} \setminus X).$$

(b) Show that there exists a short exact sequence
$$0 \;\to\; \mathbb{Z} \;\to\; H_{n-1}(\mathbb{R}^n \setminus X) \;\to\; H_{n-1}((\mathbb{R}^n \cup \{\infty\}) \setminus X) \;\to\; 0.$$

(c) Show that the short exact sequence in (b) splits.

Exercise 33.6. Let X be a topological space and let $n \in \mathbb{N}$. What is the isomorphism type of $H_n(X \times S^1)$ in terms of the homology groups of X?

Exercise 33.7. Let $k \in \mathbb{N}$. We consider the topological space
$$X_k \;:=\; ([-k,k] \cup S^2 \times \{-k,\dots,k\})/\sim$$
where given $i \in \{-k,\dots,k\}$ we identify $i \in [-k,k]$ with $((0,0,-1),i) \in S^2 \times \{-k,\dots,k\}$. For $i = -k,\dots,k$ we denote by
$$\begin{array}{rcl} f_i\colon S^2 & \to & X_k \\ P & \mapsto & (P,i) \end{array}$$
the various "obvious" inclusion maps. (We refer to Figure 534 for an illustration.)

(a) Show that $H_0(X_k) \cong \mathbb{Z}$.
(b) Show that $H_i(X_k) = 0$ for $i \neq 0,2$.
(c) Let $[S^2] \in H_2(S^2)$ be the standard generator. Show that the map
$$\begin{array}{rcl} \mathbb{Z}^{2k+1} & \to & H_2(X_k) \\ (a_{-k},\dots,a_k) & \to & \sum\limits_{i=-k}^{k} a_i \cdot f_{i*}([S^2]) \end{array}$$
is an isomorphism.

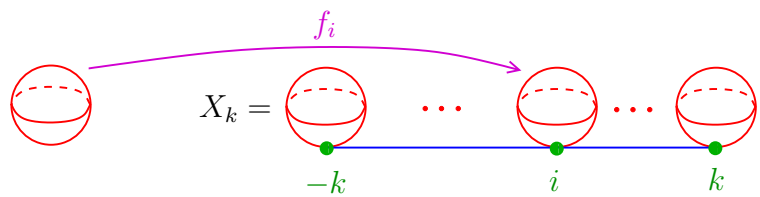


FIGURE 534.

Exercise 33.8. Let X be a topological space. We consider the corresponding suspension $\Sigma(X) = (X \times [-1,1])/\sim$ which is given by identifying all points in $X \times \{-1\}$ and by identifying all points in $X \times \{1\}$.

(a) Given a singular k-simplex $\sigma\colon \Delta^k \to X$ we define the two maps

$$\begin{array}{rcl} c_\pm(\sigma)\colon \Delta^{k+1} & \to & \Sigma(X) \\ (t_0, t_1, \ldots, t_{k+1}) & \mapsto & \begin{cases} \left(\left[\left(\sigma\left(\frac{1}{1-t_0}(t_1, \ldots, t_{k+1})\right), \pm t_0\right)\right], & \text{if } t_0 < 1, \\ [X \times \{\pm 1\}], & \text{if } t_0 = 1. \end{cases} \end{array}$$

Show that the maps $c_-(\sigma)$ and $c_+(\sigma)$ are continuous. (We refer to Figure 535 for an illustration of the definition of $c_\pm$.)
Hint: Consider $\Delta^k \times [0,1]$ and $(\Delta^k \times [0,1])/(\Delta^k \times \{1\})$.

(b) We extend $c_\pm$ in an obvious way to maps $c_\pm\colon C_k(X) \to C_k(\Sigma(X))$ and we define $s = (-1)^{k+1} \cdot (c_+ - c_-)$. Show that for any $\sigma\colon \Delta^k \to X$ we have

$$\partial s(\sigma) \;=\; \begin{cases} s(\partial\sigma), & \text{if } k > 0, \\ [X \times \{1\}] - [X \times \{-1\}], & \text{if } k = 0. \end{cases}$$

(c) Let $k \in \mathbb{N}_0$. Show that the map

$$\begin{array}{rcl} \widetilde{H}_k(X) & \to & H_{k+1}(\Sigma(X)) \\ \sigma & \mapsto & s(\sigma) \end{array}$$

is well-defined and that it is an isomorphism.

Remark: We refer to c_+, c_- as *cone operators* and we refer to s as the *suspension operator*..

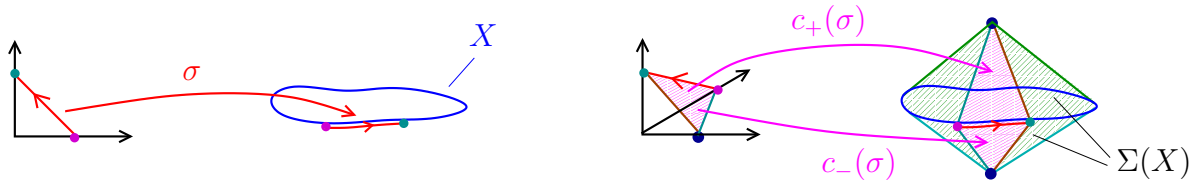


FIGURE 535.

Exercise 33.9. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i\colon A \to X$ the inclusion map. We consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i"] \arrow[d, "="] & \widetilde{H}_n(X) \arrow[r] \arrow[d, "="] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "\cong"] & \widetilde{H}_{n-1}(A) \arrow[r] \arrow[d, "="] & \widetilde{H}_{n-1}(X) \arrow[r] \arrow[d, "="] & \dots \\
\dots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i"] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(\mathrm{Cone}(i)) \arrow[r, "\partial"] & \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

where the middle vertical map is the natural isomorphism from Lemma 33.17, where the top sequence is the long exact sequence arising from Corollary 31.14. and where the bottom sequence is the exact sequence from Lemma 33.18. Show that the diagram commutes.

Exercise 33.10. Let $X = \mathbb{Z}_n$ be the topological space whose underlying set is $\mathbb{Z}_n$ and whose topology is given by the discrete topology. Let $f\colon X = \mathbb{Z}_n \to X = \mathbb{Z}_n$ be the homeomorphism given by $f(x) = x + 1 \in \mathbb{Z}_n$. Compute the homology groups of the mapping torus $M(X, f)$.

34. Homology groups and direct limits

In this chapter we will discuss the relationship between homology groups and direct limits. This will allow us to determine the homology groups of suitable nested sequences of topological spaces. In a similar vein we will determine the homology groups of the wedge of arbitrarily many topological spaces. We concludes this chapter with the introduction of the Moore spaces. We will in particular see that all abelian group can appear as homology groups of topological spaces.

For the reader who is in a hurry only few of the results of this chapter are really important. In particular, very soon we will make use of Proposition 34.9. For this proposition we give two proofs, one building on the theme of this chapter. The other proof can be read independently of the rest of the chapter.

34.1. **Direct limits and induced maps.** Given a directed set $(I, \leq)$ and a direct system $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ of abelian groups we write, as on page 487

$$\varinjlim K_i \ := \ \Big(\bigsqcup_{i\in I} K_i\Big)/\sim \quad \text{where } k_i \sim \chi_{ij}(k_i) \text{ for all } i \leq j,\ k_i \in K_i$$

We adopt the following notation, given $k_i \in K_i$ we denote by $[k_i] \in \varinjlim K_i$ the element represented by k_i. We make the following observation that follows immediately from the above definition of the direct limit.

Observation 34.1. *Any element in* $\varinjlim K_i$ *is of the form* $[k_i]$ *for some* $k_i \in K_i$.

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ij}\}_{i\leq j})$ be direct systems of abelian groups.

(1) We define a *homomorphism between the above two direct systems* to be a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of homomorphisms that has the property that for any $i \leq j$ the following diagram commutes:

$$\begin{tikzcd} K_i \arrow[r, "f_i"] \arrow[d, "\chi_{ij}"'] & L_i \arrow[d, "\lambda_{ij}"] \\ K_j \arrow[r, "f_j"] & L_j. \end{tikzcd}$$

(2) Given such a homomorphism we refer to

$$\begin{array}{ccc} \varinjlim K_i & \to & \varinjlim L_i \\ [k_i] & \mapsto & [f_i(k_i)] \end{array}$$

as the induced map. Sometimes we denote it by $\varinjlim f_i$.[683][684] It is straightforward to verify that the induced maps are functorial.

Proposition 34.2. *Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{m_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given homomorphisms $f_i\colon K_i \to L_i$ and $g_i\colon L_i \to M_i$ of the direct systems. If for all $i \in I$ the sequence*

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. We write $f = \varinjlim f_i$ and $g = \varinjlim g_i$. We have to prove that $g \circ f = 0$ and that $\ker(g) \subset \operatorname{im}(f)$. Both statements statements follow our hypothesis together with Observation 34.1 and the fact that in the definition of the induced map between direct limits we can take any representative. We leave the few remaining details to the reader. □

The following corollary says that direct limits commute with kernels and cokernels.

Corollary 34.3. *Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a homomorphism $f_i\colon K_i \to L_i$ of the direct systems. Then there exist canonical isomorphisms*

$$\varinjlim \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\big(\varinjlim K_i \to \varinjlim L_i\big)$$

and

$$\varinjlim \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\big(\varinjlim K_i \to \varinjlim L_i\big).$$

Proof. For each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. Evidently these homomorphisms form a homomorphism between the various direct systems. It follows from Proposition 34.2 (applied four times!) that the sequence

$$0 \to \varinjlim \ker(f_i) \to \varinjlim K_i \to \varinjlim L_i \to \varinjlim \operatorname{coker}(f_i) \to 0$$

is also exact. But this observation already implies the desired statements. □

[683] It follows immediately from the definitions that this map is well-defined, i.e. the map does not depend on the choice $[k_i]$ of a representative for an element in the direct limit.

[684] More formally one could say that the maps $K_i \to L_i \to \varinjlim L_i$ induce, by the universal property of a direct limit, a map $\varinjlim K_i \to \varinjlim L_i$. It is straightforward to see that this map is exactly the one we wrote down.

34.2. **The homology groups of a nested sequence of topological spaces.** In Propositions 16.4 and 28.11 we saw that in favorable situations the homotopy groups of a union of a nested sequence of topological spaces is the direct limit of the corresponding fundamental groups. Now we will see that a similar statement holds for homology groups.

Proposition 34.4. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:*

(1) *every compact subset of X is already contained in one of the X_i, or*
(2) *each X_i is open in X, or*
(3) *X is a CW-complex and each X_i is a subcomplex.*

Then for any $n \in \mathbb{N}_0$ the inclusion induced maps $H_n(X_i) \to H_n(X)$ induce an isomorphism

$$\varinjlim H_n(X_i) \xrightarrow{\cong} H_n(X).$$

We remark that by Corollary 26.18 hypothesis (3) implies hypothesis (1) and that by Lemma 16.5 hypothesis (2) implies hypothesis (1). Thus it suffices to prove Proposition 34.4 under the hypothesis (1). But in that case the proposition is an immediate consequence of the next two lemmas.

Lemma 34.5. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that every compact subset of X is already contained in one of the X_i. Then the chain complexes $C_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $C_*(X_i) \to C_*(X_j)$ for $i \leq j$ form a direct system of chain complexes.*[685] *Furthermore the inclusion induced maps $C_*(X_i) \to C_*(X)$ induce an isomorphism*[686]

$$\varinjlim C_*(X_i) \xrightarrow{\cong} C_*(X)$$

of chain complexes.

Proof. Given a topological space Y and $n \in \mathbb{N}_0$ we denote by $S_n(Y)$ the set of singular n-simplices in Y. It follows immediately from our hypothesis on the X_i, the fact that Δ^n is compact and Lemma 1.37 that

$$S_n(X) = \bigcup_{i\in\mathbb{N}} S_n(X_i).$$

As in the example on page 484 we now see that

$$\varinjlim C_n(X_i) = \varinjlim \mathbb{Z}^{(S_n(X_i))} = \mathbb{Z}^{(S_n(X))} = C_n(X).$$

We leave it as an exercise to verify that the inclusion maps also induce an isomorphism of chain complexes. □

The second lemma required for the proof of Proposition 34.4 is purely algebraic.

[685] We refer to page 482 for the definition of a direct system.

[686] The argument of Proposition 16.1 or respectively of page 487 shows that the direct limit of a direct system of chain complexes always exists.

Lemma 34.6. *Let $(I, \leq)$ be a directed set and furthermore let $(\{C_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{H_n(C_i)\}_{i\in I}, \{f_{ij*}\}_{i\leq j})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $H_n(C_i) \to H_n\big(\varinjlim C_i\big)$ induce an isomorphism*

$$\varinjlim H_n(C_i) \xrightarrow{\cong} H_n\big(\varinjlim C_i\big).$$

Proof. We have

$$\begin{aligned} \varinjlim H_n(C_i) &= \varinjlim \operatorname{coker}\{C_{i+1} \to \ker\{C_i \to C_{i-1}\}\} \\ &\underset{\substack{\uparrow \\ \text{Corollary 34.3 applied twice}}}{=} \operatorname{coker}\big\{\varinjlim C_{i+1} \to \ker\big\{\varinjlim C_i \to \varinjlim C_{i-1}\big\}\big\} \;=\; H_n\big(\varinjlim C_i\big). \end{aligned}$$

The fact that this isomorphism is induced by the maps $H_n(C_i) \to H_n(\varinjlim C_i)$ follows easily from the various definitions. □

We conclude this short section with a few examples:

Example. For any $n \in \mathbb{N}_0$ we have

$$H_n(S^\infty) \underset{\substack{\uparrow \\ \text{by Proposition 34.4,} \\ \text{where we view } S^\infty \text{ as a} \\ \text{CW-complex as on page 655}}}{=} \varinjlim_{k\in\mathbb{N}} H_n(S^k) \underset{\substack{\uparrow \\ \text{Proposition 31.3}}}{=} \varinjlim_{k\in\mathbb{N}} \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} \underset{\substack{\uparrow \\ \text{Lemma 16.2}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

The next lemma is basically also just an example. But since we will need it later on we formulate it as a lemma.

Lemma 34.7. *We consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider*

$$X \;:=\; (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim \qquad \textit{where } n \sim ((0,0,-1), n) \textit{ for } n \in \mathbb{Z}.$$

Then[687]

$$H_n(X) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } n = 0, \\ \mathbb{Z}^\infty, & \textit{if } n = 2, \\ 0, & \textit{else.} \end{cases}$$

[687]Recall that

$$\mathbb{Z}^\infty \;:=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \,|\, x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\}.$$

In fact we can specify an isomorphism on $H_2(X)$. Given $i \in \mathbb{Z}$ we denote by $f_i \colon S^2 \to X$ the "obvious" i-th inclusion map. Then an isomorphism is given by

$$\begin{array}{rcl} \mathbb{Z}^\infty \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} & \to & H_2(X) \\ \bigoplus_{i\in\mathbb{Z}} a_i & \mapsto & \sum_{i\in\mathbb{Z}} a_i \cdot f_i([S^2]). \end{array}$$

Proof. We view X as a CW-complex in the obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k \ := \ ([-k,k] \cup S^2 \times \{-k,\dots,k\})/\sim .$$

Note that X_k is the interval $[-k,k]$ with $2k+1$ spheres attached. (We refer to Figure 536 for an illustration.) We have

$$H_n(X) \underset{\substack{\uparrow \\ \text{Proposition 34.4}}}{=} \varinjlim H_n(X_k) \underset{\substack{\uparrow \\ \text{see Exercise 33.7}}}{=} \varinjlim \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{2k+1}, & \text{if } n=2, \\ 0, & \text{else} \end{cases} \underset{\substack{\uparrow \\ \text{see page 484 and Lemma 16.2}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^\infty, & \text{if } n=2, \\ 0, & \text{else.} \end{cases}$$

The statement about the precise isomorphism of $H_2(X)$ also follows from the explicit description of the isomorphism Exercise 33.7 and the discussion on page 484. □

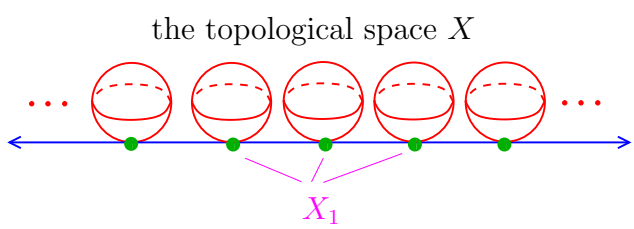


FIGURE 536.

34.3. **The homology groups of the wedge of topological spaces.** The following proposition says that under a mild hypothesis "homology groups are additive" under the operation of taking the wedge of two topological spaces. The proposition is very similar in vein to Proposition 12.3.

Proposition 34.8. *Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be good points. We denote by $A \vee B$ the wedge defined using a and b. Then the natural inclusion maps $i \colon A \to A \vee B$ and $j \colon B \to A \vee B$ that we introduced on page 317 induce for every*

$n \in \mathbb{N}_0$ an isomorphism

$$i_* \oplus j_* \colon \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \quad \to \quad \widetilde{H}_n(A \vee B).$$

Furthermore, if we denote by $p \colon A \vee B \to A$ and $q \colon A \vee B \to A$ the natural projection maps, then an inverse to the above isomorphism is given by

$$p_* + q_* \colon \widetilde{H}_n(A \vee B) \quad \to \quad \widetilde{H}_n(A) \oplus \widetilde{H}_n(B).$$

Proof $(*)$. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. We pick an open neighborhood C in A of a such that $\{a\}$ is a deformation retract of C. Furthermore we pick an open neighborhood D of b in B such that b is a deformation retract of D. We consider $U := A \vee D$

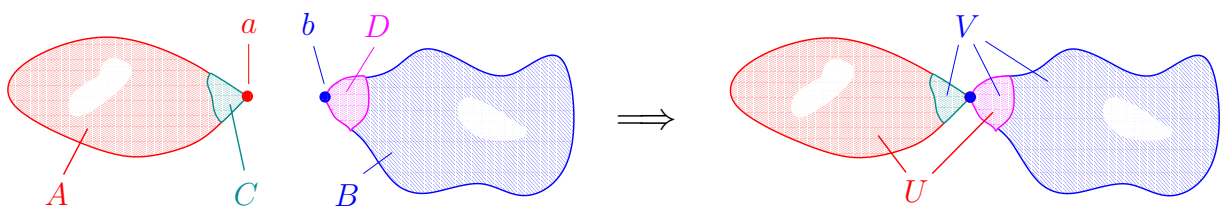


FIGURE 537. Schematic image for the proof of Proposition 34.8.

and $V := B \vee C$. We consider the following diagram of maps

$$\begin{array}{ccccccccc} & & & & \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) & & & & \\ & & & & \cong\downarrow \qquad \downarrow\cong & & & & \\ \dots & \longrightarrow & \widetilde{H}_n(U \cap V) & \longrightarrow & \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) & \longrightarrow & \widetilde{H}_n(X) & \longrightarrow & \widetilde{H}_{n-1}(U \cap V) \longrightarrow \dots \end{array}$$

We make the following observations:

(1) The bottom sequence is exact by the Mayer–Vietoris Theorem 33.5.
(2) As in the proof of Proposition 12.3 we see that the intersection $U \cap V$ deformation retracts to x_0, in particular we obtain from Corollary 30.8 that we have the equality $\widetilde{H}_n(U \cap V) = \widetilde{H}_{n-1}(U \cap V) = 0$.
(3) By the proof of Proposition 12.3 we know that A is a deformation retract of U and that B is a deformation retract of V. Therefore the vertical maps are isomorphisms by Corollary 30.8.

The combination of these three observations shows that

$$i_* \oplus j_* \colon \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \quad \to \quad \widetilde{H}_n(A \vee B)$$

is an isomorphism. It is clear that $(p_* + q_*) \circ (i_* + j_*)$ is the identity on $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B)$, so we also get the second isomorphism. $\square$

Proposition 34.8 has nothing to do with the content of the present chapter, we could easily have proved it in the previous chapter. But odd as it may sound, soon we will be quite

interested in the homology groups of the wedge $\bigvee_{i\in I} X_i$ of an arbitrary family of topological spaces. If I is a finite set, i.e. if we take the wedge of finitely many topological spaces, then we can use Proposition 34.8 and an elementary induction argument to determine the homology groups.

It is a priori less clear how one can deal with the case that I is an infinite set. Fortunately the next proposition allows us to handle the general situation. Proposition 34.8 to the wedge of an arbitrary family of topological spaces.

Proposition 34.9. *Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$.*[688] *Given $j \in K$ we denote by*

$$i_j\colon A_j \to \bigvee_{k\in K} A_k \quad \textit{respectively} \quad p_j\colon \bigvee_{k\in K} A_k \to A_j$$

the natural inclusion map respectively the natural projection map that we introduced on page 317. Then for every $n \in \mathbb{N}_0$ the map

$$\bigoplus_{k\in K} p_{k*}\colon \widetilde{H}_n\Big(\bigvee_{k\in K} A_k\Big) \to \bigoplus_{k\in K} \widetilde{H}_n(A_k)$$

is an isomorphism where the inverse is given by

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \widetilde{H}_n(A_k) \to \widetilde{H}_n\Big(\bigvee_{k\in K} A_k\Big)$$

We like Proposition 34.9 so much that we provide two different proofs. The first proof is quick and slick, but depending on your taste the first proof might have feel slightly unsatisfactory. The second proof is somewhat longer, but it fits in better with the discussion of this chapter.

First proof of Proposition 34.9. We have the following isomorphisms:

$$\widetilde{H}_n\Big(\bigvee_{k\in K} A_k\Big) \underset{\substack{\uparrow \\ \text{definition of the} \\ \text{wedge operation}}}{=} \widetilde{H}_n\Big(\bigsqcup_{k\in K} A_k \Big/ \bigsqcup_{k\in K}\{a_k\}\Big) \underset{\substack{\uparrow \\ \text{since each } a_k\in A_k \text{ is simple we see} \\ \text{that } \bigsqcup_{k\in K}\{a_k\} \text{ is a good subset of } \bigsqcup_{k\in K} A_k, \\ \text{thus we can apply Proposition 31.19}}}{\overset{\cong}{\longleftarrow}} H_n\Big(\bigsqcup_{k\in K} A_k, \bigsqcup_{k\in K}\{a_k\}\Big) \underset{\substack{\uparrow \\ \text{by the obvious generalization} \\ \text{of Lemma 29.13}}}{\overset{\cong}{\longleftarrow}} \bigsqcup_{k\in K} H_k(A_k, \{a_k\})$$

$$\underset{\substack{\uparrow \\ \text{Lemma 31.12}}}{\overset{\cong}{\longleftarrow}} \bigsqcup_{k\in K} \widetilde{H}_k(A_k).$$

We leave it as an exercise to verify that the isomorphism is indeed given by $\bigoplus_{k\in K} p_{k*}$ and that the inverse is the map $\bigoplus_{k\in K} i_{k*}$. □

[688] Recall that by the discussion on page 387 any point on a topological manifold is good and that by Proposition 26.11 (5) any point on a CW-complex is good.

Second proof of Proposition 34.9. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. We will show that the map

$$\bigoplus_{k\in K} i_k \colon \bigoplus_{k\in K} \widetilde{H}_n(A_k) \;\to\; \widetilde{H}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is an isomorphism. Once we will have established this fact it is straightforward to show that the inverse is given by $\bigoplus_{k\in K} p_{k*}$.

The idea of the proof is to show that the map is a monomorphism and an epimorphism, in a way which is reminiscent to the approach taken in the proofs of Proposition 34.4 and 34.4. Given a subset $J \subset K$ we write $B_J := \bigvee_{j\in J} A_j$ and we write $B := B_K$. Given any two finite subsets $J \subset \widetilde{J} \subset K$ we have the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{j\in J} \widetilde{H}_n(A_j) \arrow[rr, "\cong"] \arrow[dr, hook] \arrow[ddr, hook] & & \widetilde{H}_n(B_J) \arrow[dr] \arrow[ddr] & \\
& \bigoplus_{j\in \widetilde{J}} \widetilde{H}_n(A_j) \arrow[rr, "\cong"] \arrow[dl, hook] \arrow[ddr] & & \widetilde{H}_n(B_{\widetilde{J}}) \arrow[ddl] \\
\bigoplus_{k\in K} \widetilde{H}_n(A_k) \arrow[rr] & & \widetilde{H}_n(B). &
\end{tikzcd}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces.
(2) Since J and $\widetilde{J}$ are finite we obtain, as pointed out above, from Proposition 34.8 that the top two horizontal maps are isomorphisms.
(3) By Exercise 9.18 we know that given any compact subset $L \subset B$ there exists a finite subset $J \subset K$ with $L \subset B_J$.

Now we show that the above map $\bigoplus_{k\in K} i_{k*}$ is an epimorphism. So suppose we are given a homology class $[\sigma] \in \widetilde{H}_n(B)$. As in the proof of Lemma 34.5 we see that it follows from (3) there exists a finite subset $J \subset K$ such that $\sigma \in C_n(B_J)$. But considering the above blue diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\sigma]$ actually lies in the image of $\bigoplus_{k\in K} i_k$.

In Exercise 34.1 we will use the above diagram in its full glory to show that the map $\bigoplus_{k\in K} i_{k*}$ is also a monomorphism. $\square$

34.4. **Moore spaces** $(*)$**.** Now that we have played with homology groups the following question arises: which sequences of abelian groups can appear as the sequence of homology groups $H_*(X)$ of a topological space? It turns out that anything goes, there are basically no restrictions, except that we know by Corollary 29.14 that $H_0(X)$ is necessarily a free abelian group.

In other words, we have the following proposition.

Proposition 34.10. *If $\{A_n\}_{n\in\mathbb{N}_0}$ is a sequence of abelian groups. If A_0 is a free abelian group, then there exists a CW-complex X such that for any $n \in \mathbb{N}_0$ we have $H_n(X) \cong A_n$.*

This proposition is hardly the most exciting result of these notes. But the following definition, which we will need in the proof of Proposition 34.10, will in fact turn out to be convenient.

Definition. Let $n \in \mathbb{N}$ and let π be an abelian group. We say that a topological space X is a *Moore space of type $M(\pi, n)$* if the following conditions are satisfied:

(1) $H_n(X) \cong \pi$,
(2) for any $i \neq n$ we have $\widetilde{H}_i(X) = 0$.
(3) If $n \geq 2$, then we also demand that X is simply connected.

Example. Let $n \in \mathbb{N}$. It follows from Proposition 5.14 and Proposition 31.3 that S^n is a Moore space of type $(\mathbb{Z}, n)$.

Proposition 34.11. *Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional CW-complex that has no cells in dimensions $1, \ldots, n-1$ and that is a Moore space of type $M(\pi, n)$.*

Proof of Proposition 34.10 assuming Proposition 34.11. Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of abelian groups. By Proposition 34.11 we can find for each $n \in \mathbb{N}$ a CW-complex X_n that is a Moore space of type $M(\pi, n)$. We consider the wedge $Y := \bigvee_{n\in\mathbb{N}} X_n$. By Lemma 26.20 (3) we can equip Y with a CW-structure. It follows from Proposition 34.9 that for every $n \in \mathbb{N}$ we have $H_n(X) \cong H_n(X_n) \cong A_n$. □

In the proof of Proposition 34.11 we will also need the following little calculation which will be of independent use.

Lemma 34.12. *Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$, let $\omega \in H_n(X, x_0)$ and let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two maps. Then*

$$[f_*(\omega)] + [g_*(\omega)] \underset{\substack{\uparrow \\ \text{product of } f \text{ and } g \text{ as defined on page 710}}}{=} [(f * g)_*(\omega)] \in H_n(X, x_0) = H_n(X).$$

Proof $(*)$. We write $J = I^n$, $J_1 = [0, \frac{1}{2}] \times I^{n-1}$ and $J_2 = [\frac{1}{2}, 1] \times I^{n-1}$. Furthermore we write $K = \partial J \cup (\{\frac{1}{2}\} \times I^{n-1})$. We refer to Figure 538 for an illustration. For $k = 1, 2$ we denote by $i_k\colon (J_k, \partial J_k) \to (J, K)$ the inclusion map and we denote by $\varphi_k\colon (J_k, \partial J_k) \to (J, \partial J)$ the obvious homeomorphism given by stretching the last coordinate.

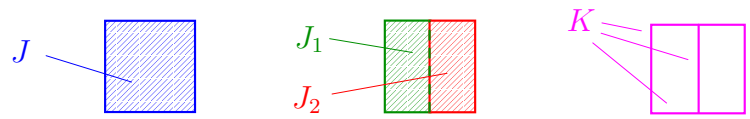


FIGURE 538. Illustration for the proof of Lemma 34.12.

The map $f * g\colon (J, \partial J) \to (X, x_0)$ factors through a map $(J, K) \to (X, x_0)$ that we also denote by $f * g$. Thus we have have the following commutative diagram

$$\begin{tikzcd}
 & (J_1, \partial J_1) \arrow[d, "i_1"] \arrow[dr, "f\circ\varphi_1^{-1}"] & \\
(J, \partial J) \arrow[ur, "\varphi_1"] \arrow[r, "i"] \arrow[dr, "\varphi_2"'] & (J, K) \arrow[r, "f*g"] & (X, x_0) \\
 & (J_2, \partial J_2). \arrow[u, "i_2"'] \arrow[ur, "g\circ\varphi_2^{-1}"'] &
\end{tikzcd}$$

For $k = 1, 2$ we write $\omega_k = (\varphi_k)_*(\omega)$. Thus it suffices to show that $i_{1*}(\omega_1) + i_{2*}(\omega_2) = i_*(\omega)$. To prove this equality we consider the following commutative diagram[689]

$$\begin{tikzcd}
 & H_n(J_1, \partial J_1) \arrow[r, "k_1"] \arrow[d, "i_1"] & H_n(J, \partial J_1 \cup J_2) \\
H_n(J, \partial J) \arrow[ur, "\varphi_1"] \arrow[r, "i"] \arrow[dr, "\varphi_2"'] & H_n(J, K) \arrow[ur, "j_1"'] \arrow[dr, "j_2"] & \\
 & H_n(J_2, \partial J_2) \arrow[u, "i_2"'] \arrow[r, "k_2"'] & H_n(J, J_1 \cup \partial J_2),
\end{tikzcd}$$

where all maps, except for φ_1 and φ_2, are induced by the obvious inclusion maps. We have

$$(j_1)(i(\omega)) \;=\; (k_1 \circ \varphi_1)(\omega) \;=\; j_1(i_1(\omega_1)) \;\underset{\substack{\uparrow \\ \text{since } j_1 \circ i_2 = 0}}{=}\; j_1(i_1(\omega_1)) + j_1(i_2(\omega_2))$$

and similarly we show that $(j_2)(i(\omega)) = j_2(i_1(\omega_1)) + j_2(i_2(\omega_2))$.

It remains to show that $\ker(j_1) \cap \ker(j_2) = 0$. We consider the following diagram

$$\begin{tikzcd}
 & & H_n(J, \partial J_1 \cup J_2) \\
 & H_n(J, K) \arrow[ur, "j_1"'] \arrow[dr, "j_2"] & \\
H_n(J, \partial J_1 \cup J_2) \arrow[ur] & H_n(J_2, \partial J_2) \arrow[l, "\cong"] \arrow[u, "i_2"'] \arrow[r, "k_2", "\cong"'] & H_n(J, J_1 \cup \partial J_2).
\end{tikzcd}$$

[689]To simplify the notation we denote the induced map by the same symbol as the map between pairs of topological spaces, i.e. we drop the subscript $*$ from the notation.

The bottom two horizontal maps are isomorphisms by the Excision Theorem 32.6. The diagonal sequence is exact by Proposition 31.13. It now follows from some mild diagram chasing that $\ker(j_1) \cap \ker(j_2) = 0$. □

Lemma 34.13.

(1) *Let $f\colon S^n \to S^n_1 \vee S^n_2$ be the pinching map that we introduced on page 715. Let $i \in \{1,2\}$. If we denote by $S^n_1 \vee S^n_2 \to S^n_i = S^n$ the obvious projection, then the induced map $(p_i \circ f)_*\colon H_n(S^n) \to H_n(S^n)$ is the identity.*

(2) *Given any $k \in \mathbb{Z}$ there exists a map $g_k\colon S^n \to S^n$ with $g_k(*) = *$ such that $g_{k*}\colon H_n(S^n) \to H_n(S^n)$ is given by multiplication by k.*

Proof.

(1) The first statement follows easily from Lemma 34.12.

(2) The second statement follows almost immediately from Lemma 32.15 together with Proposition 32.14. □

Now we turn to the proof of Proposition 34.11.

Proof of Proposition 34.11. Let $n \in \mathbb{N}$ and and let π be an abelian group. Since π is abelian it follows immediately from Lemma 42.16 (1)[690] that there exist sets A and B, a homomorphism $\varphi\colon \mathbb{Z}^{(A)} \to \mathbb{Z}^{(B)}$ and a homomorphism $\rho\colon G \to \pi$ such that the following sequence is exact:

$$0 \to \mathbb{Z}^{(A)} \xrightarrow{\varphi} \mathbb{Z}^{(B)} \xrightarrow{\rho} \pi \to 0.$$

We let $X := \bigvee_{a\in A} S^n_a$ and we let $Y := \bigvee_{b\in B} S^n_b$. By Lemma 26.20 (3) we can equip X with the obvious CW-structure which has precisely one 0-cell and which has precisely one n-cell for each $a \in A$. Similarly we equip Y with the obvious CW-structure. Using Proposition 34.9 and using the standard generator of $H_n(S^n)$ we can make the obvious identifications $H_n(X) = \mathbb{Z}^{(S)}$ and $H_n(Y) = \mathbb{Z}^{(T)}$.

Claim. There exists a map $h\colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{h_*} & H_n(Y) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{Z}^{(A)} & \xrightarrow{\varphi} & \mathbb{Z}^{(B)}. \end{array}$$

This map φ can be constructed explicitly using the following ingredients:

(1) the pinching map $f\colon S^n \to S^n_1 \vee S^n_2$,

(2) the maps $g_k\colon S^n \to S^n$ from Lemma 34.13,

(3) the natural inclusions and projections from page 317,

(4) and the wedge of maps as introduced on page 316.

[690] In this case the fact that we refer to a later lemma need not worry us. As the reader can see, the proof of Lemma 42.16 (1) relies only on information that we already know.

The notation becomes somewhat cumbersome, thus we leave it as an exercise to the reader to fill in the details. A baby example is the content of Exercise 34.2. This concludes the proof of the claim.

Now let $Z := \mathrm{Cone}(h\colon X \to Y)$ be the mapping cone of the map $h\colon X \to Y$. It follows from the above claim and Lemma 33.18 that we have the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc}
\dots \longrightarrow & \overbrace{H_{n+1}(Y)}^{=0} & \longrightarrow & H_{n+1}(Z) & \longrightarrow & H_n(X) & \xrightarrow{h_*} & H_n(Y) & \longrightarrow & H_n(Z) & \longrightarrow & \overbrace{H_{n-1}(X)}^{=0} & \longrightarrow \dots \\
 & \downarrow & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & & & \downarrow & \\
\dots \longrightarrow & 0 & \longrightarrow & H_{n+1}(Z) & \longrightarrow & \mathbb{Z}^{(A)} & \xrightarrow{\varphi} & \mathbb{Z}^{(B)} & \longrightarrow & H_n(Z) & \longrightarrow & 0 & \longrightarrow \dots .
\end{array}$$

It follows that $H_n(Z) \cong \mathrm{coker}(\varphi) \cong \pi$. Furthermore, since φ is injective we see that $H_{n+1}(Z) = 0$. Furthermore we see that all other reduced homology groups of Z are also zero.

Next we equip Z with the natural CW-structure coming from Corollary 26.24. There is one small fly in the ointment. The CW-structure has two 0-cells (namely one corresponding to the tip of the cone and the 0-cell of Y), one 1-cells and otherwise indeed only cells in dimensions n and $n+1$. Let A be the subcomplex that consists of the 1-cell that connects the two 0-cells. By Lemma 26.20 we can view X/A as a CW-complex which now has only one 0-cell and otherwise only cells in dimension n and $n+1$. By the discussion on page 764 we know that A is a good subset of X. It follows from Theorem 31.2 that the reduced homology groups of X/A coincide with the reduced homology groups of X. Thus we see that X/A is the CW-complex of our dreams. □

Exercises for Chapter 34.

Exercise 34.1. In the second proof of Proposition 34.9, given on page 874, show that the map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \widetilde{H}_n(A_k) \;\to\; \widetilde{H}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.
Hint: Consider the commutative diagram shown on page 874.

Exercise 34.2. Let $n \in \mathbb{N}$ and let $X = S^n \vee S^n$ and let $Y = S^n \vee S^n \vee S^n$. As in the proof of Proposition 34.11 we can make the identification $H_n(X) = \mathbb{Z}^2$ and $H_n(Y) = \mathbb{Z}^3$. Show that there exists a map $\varphi\colon X \to Y$ such that the map $\varphi_*\colon H_n(X) \to H_n(Y)$ is represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ -5 & 1 \\ 0 & 7 \end{pmatrix}.$$

Exercise 34.3. Let X and Y be topological spaces and let $x_0 \in X$ and $y_0 \in Y$ be good points. We write $\widetilde{x}_0 = [(x_0, 0)] \in \Sigma(X)$ and $\widetilde{y}_0 = [(y_0, 0)] \in \Sigma(Y)$.

(a) Show that $\widetilde{x}_0 \in \Sigma(X)$ and $\widetilde{y}_0 \in \Sigma(Y)$ are good points. We use these points to define the wedge $\Sigma(X) \vee \Sigma(Y)$.

(b) Let $i\colon X \to X \vee Y$ and $j\colon Y \to X \vee Y$ be the obvious maps. Show that the map

$$(\Sigma(i) \vee \Sigma(j))_*\colon H_k(\Sigma(X) \vee \Sigma(Y)) \;\to\; H_k(\Sigma(X \vee Y))$$

is an isomorphism for each k.

Remark: You could make use of the *natural* isomorphism of Lemma 33.8.

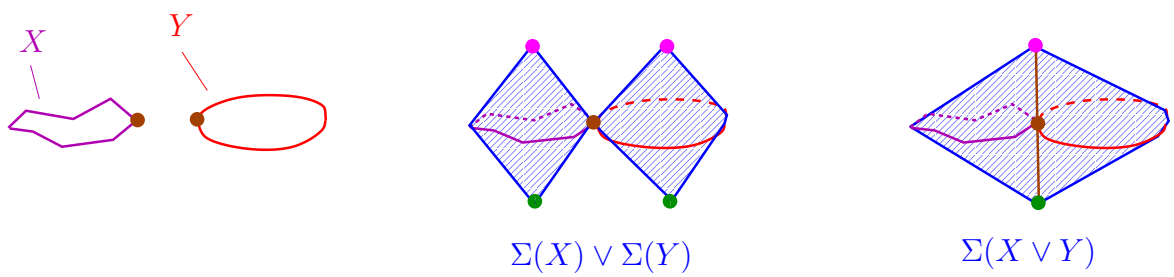


FIGURE 539. Illustration of Exercise 34.3.

35. Cellular homology

We saw in Propositions 27.9 and 27.10 that it is quite easy to compute the fundamental group of a finite CW-complex. In this section we will develop a relatively straightforward method for computing the homology groups of a finite CW-complex. We will use this method for finally determining the homology groups of the surfaces of genus ≥ 2 and all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

35.1. **The definition of cellular homology.** Now we turn, as promised, to the study of homology groups of CW-complexes. We need to recall two definitions:

(1) First recall that in Lemma 29.1 we gave a homeomorphism $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, \partial\overline{B}^n)$. This homeomorphism together with Lemma 32.9 gives us the following isomorphism:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cong} & H_n(\Delta^n, \partial\Delta^n) & \xrightarrow[\cong]{f_*} & H_n(\overline{B}^n, \partial\overline{B}^n) \\ 1 & \mapsto & [\mathrm{id}_{\Delta^n}] & & \end{array}$$

In the following we will usually identify $H_n(\overline{B}^n, \partial\overline{B}^n)$ with $\mathbb{Z}$ using the above isomorphism.

(2) Let X be a topological space and let $A \subset X$ be a non-empty subset. Following the discussion on page 783 we refer to

$$\begin{tikzcd} H_n(X,A) \arrow[r] \arrow[rr, bend right] & H_n(X/A, A/A) & \widetilde{H}_n(X/A) \arrow[l, "\text{Lemma 31.12}"', "\cong"] \end{tikzcd}$$

as the natural homomorphism $H_n(X,A) \to \widetilde{H}_n(X/A)$. We will often write down this map without explicitly referring to it as being the natural homomorphism.

Now we can formulate one of the key lemmas regarding homology groups of CW-complexes.

Lemma 35.1. *For every CW-complex X the following statements hold:*

(1) *Denote by I_n the set of n-cells and given $i \in I_n$ denote by $\Phi_i\colon \overline{B}^n_i \to X^n$ the corresponding characteristic map. Then for every $k \in \mathbb{N}_0$ all the maps*[691]

$$\bigoplus_{i\in I_n} H_k(\overline{B}^n_i, \partial\overline{B}^n_i) \;\to\; \bigoplus_{i\in I_n} \widetilde{H}_k\big(\overline{B}^n_i/\partial\overline{B}^n_i\big) \;\xrightarrow{\oplus\Phi_i}\; \widetilde{H}_k(X^n/X^{n-1}) \;\leftarrow\; H_k(X^n, X^{n-1})$$

are isomorphisms. In particular the following holds:

(a) *For $k \neq n$ we have $H_k(X^n, X^{n-1}) = 0$.*

(b) *The group $H_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set of n-cells in the CW-structure.*

(2) *For $k > n$ we have $H_k(X^n) = 0$.*

(3) *For $k < n$ the inclusion $i\colon X^n \to X$ induces an isomorphism $i_*\colon H_k(X^n) \xrightarrow{\cong} H_k(X)$.*

(4) *The inclusion $i\colon X^n \to X$ induces an epimorphism $i_*\colon H_n(X^n) \twoheadrightarrow H_n(X)$.*

Remark. The statements (3) and (4) of Lemma 35.1 can be viewed as "homology analogues" of Proposition 27.12.

[691] Here the maps to the left and right are examples of the aforementioned natural homomorphism

Proof. Let X be a CW-complex.

(1) We have

$$\begin{array}{ccccccc}
 & \text{isomorphism by Proposition 31.19} & & \text{isomorphism by Proposition 34.9} & & & \\
 & \downarrow & & \downarrow & & & \\
\bigoplus_{i\in I_n} H_k(\overline{B}_i^n, \partial\overline{B}_i^n) & \to & \bigoplus_{i\in I_n} \widetilde{H}_k(\overline{B}_i^n/\partial\overline{B}_i^n) & \to & \widetilde{H}_k\Big(\bigvee_{i\in I_n} \overline{B}_i^n/\partial\overline{B}_i^n\Big) & & \\
 & & & \to & \widetilde{H}_k(X^n/X^{n-1}) & \leftarrow & H_k(X^n, X^{n-1}). \\
 & & & \uparrow & & \uparrow & \\
 & & & \text{by Lemma 26.21 (1) we know that the characteristic maps induce a homeomorphism } \bigvee_{i\in I_n} \overline{B}_i^n/\partial\overline{B}_i^n \to X^n/X^{n-1} & & \text{isomorphism by Proposition 31.19 and Proposition 26.11 (8)} &
\end{array}$$

The remaining statements are now an immediate consequence of Lemma 32.8.

(2) We fix $k \in \mathbb{N}$. We want to show that $H_k(X^n) = 0$ for $n = 0, \dots, k-1$. First of all we have $H_k(X^0) = 0$ by Proposition 29.5 and Lemma 29.13.[692] Now assume that we already know that $H_k(X^n) = 0$ for some $n < k-1$. We want to show that $H_k(X^{n+1}) = 0$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to H_{k+1}(X^{n+1}, X^n) \to \underbrace{H_k(X^n)}_{=0} \to H_k(X^{n+1}) \to \underbrace{H_k(X^{n+1}, X^n)}_{\substack{= 0 \text{ by (1) since } n<k\text{-1} \\ \text{i.e. since } k \neq n+1}} \to \dots$$

Thus we obtain that indeed $H_k(X^{n+1}) = 0$.

(3) We fix $k \in \mathbb{N}_0$. Let $m > k$. This time we consider the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{H_{k+1}(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k<m} \to H_k(X^m) \to H_k(X^{m+1}) \to \underbrace{H_k(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k<m} \to \dots$$

This implies in particular that for each $k < n \leq l$ we have inclusion induced isomorphisms $H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} H_k(X^l)$. In particular, if X is finite-dimensional, then claim (3) follows immediately from this observation. The infinite-dimensional case follows from the finite-dimensional case together with Proposition 34.4.

(4) In (3) we showed that $H_n(X^{n+1}) \to H_n(X)$ is an isomorphism. Hence it suffices to show that $H_n(X^n) \to H_n(X^{n+1})$ is an epimorphism. To show this we consider the following excerpt from long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n) \to H_n(X^{n+1}) \to \underbrace{H_n(X^{n+1}, X^n)}_{= 0 \text{ by (1)}} \to \dots$$

We see that the inclusion induced map $H_n(X^n) \to H_n(X^{n+1})$ is indeed an epimorphism. □

[692] Here we use that X^0 has the discrete topology.

Definition. Given a CW-complex X, we denote by $d = d_n$ the map

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "=:d=d_n"'] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X^{n-1}, X^{n-2}) \end{tikzcd}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the map induced by the projection $C_{n-1}(X^{n-1}) \to C_{n-1}(X^{n-1}, X^{n-2})$. We refer to d_n as the *cellular boundary map*.[693]

The name "cellular boundary map" already suggests that the maps d_n have the following important property:

Lemma 35.2. *For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.*

Proof. We consider the following commutative diagram of maps

$$\begin{tikzcd} & H_n(X^n) \arrow[dr, "j_n"] & & & \\ H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\ & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \end{tikzcd}$$

The map $\partial_n \circ j_n \colon H_n(X^n) \to H_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}), i.e. the map $\partial_n \circ j_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. □

Definition. Let X be a CW-complex.

(1) We write $C_n^{\mathrm{CW}}(X) := H_n(X^n, X^{n-1})$ and we refer to $(C_*^{\mathrm{CW}}(X), d_n)$ as the *cellular chain complex* of the CW-complex X.[694]
(2) We denote the homology groups of the cellular chain complex by $H_n^{\mathrm{CW}}(X)$ and we refer to these groups as the *cellular homology groups of the CW-complex X*.

Example. We consider the n-sphere S^n for $n \geq 2$. On page 648 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. The cellular chain complex $C_*^{\mathrm{CW}}(X)$ is thus of the form

$$0 \to \underbrace{\mathbb{Z}}_{=C_n^{\mathrm{CW}}(X)} \to 0 \to \ldots \to 0 \to \underbrace{\mathbb{Z}}_{=C_0^{\mathrm{CW}}(X)} \to 0.$$

Since $n \geq 2$ both the $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps and we obtain that

$$H_i^{\mathrm{CW}}(X) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

[693]It follows easily from the definitions that d_n is precisely the connecting homomorphism of the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}).

[694]It follows from Lemma 35.2 that $(H_n(X^n, X^{n-1}), d_n)$ is indeed a chain complex.

These are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence.

The "usual" homology groups define functors from the category of topological spaces to the category of abelian groups. We will now see that in a more restricted setting cellular homology groups also define a functor.

Definition.

(1) A map $f\colon X \to Y$ between CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.
(2) We denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps.

We conclude this section with the following lemma.

Lemma 35.3.

(1) *Every cellular map $f\colon X \to Y$ induces a chain map*
$$f_*\colon C_*^{\mathrm{CW}}(X) \;\to\; C_*^{\mathrm{CW}}(Y)$$
and hence also maps
$$f_*\colon H_*^{\mathrm{CW}}(X) \;\to\; H_*^{\mathrm{CW}}(Y)$$
on cellular homology.
(2) *For each $n \in \mathbb{N}_0$ the maps*
$$\begin{array}{rcl} X & \mapsto & H_n^{\mathrm{CW}}(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon H_n^{\mathrm{CW}}(X) \to H_n^{\mathrm{CW}}(Y)) \end{array}$$
define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{A}$ of abelian groups.

Proof.

(1) Let $f\colon X \to Y$ be a cellular map. Since f is cellular we get for every $n \in \mathbb{N}_0$ an induced map $H_n(X^n, X^{n-1}) \to H_n(Y^n, Y^{n-1})$, i.e. we get an induced map $f_*\colon C_n^{\mathrm{CW}}(X) \to C_n^{\mathrm{CW}}(Y)$. It follows almost immediately from the naturality of the connecting homomorphism, see Proposition 31.13 (2), that the above maps $f_*\colon C_n^{\mathrm{CW}}(X) \to C_n^{\mathrm{CW}}(Y)$, $n \in \mathbb{N}_0$ actually form a chain map.
(2) This statement follows easily from the fact that the usual chain complexes and homology groups are covariantly functorial. □

35.2. **The relationship between cellular and singular homology.** Now we can formulate and prove the following proposition.

Proposition 35.4. *Let $n \in \mathbb{N}_0$.*

(1) *For each CW-complex X there exists a uniquely determined isomorphism*
$$\Phi_X\colon H_n(X) \;\xrightarrow{\cong}\; H_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[695]

$$\begin{tikzcd} & H_n(X) \arrow[dr, "\Phi_X"] & \\ H_n(X^n) \arrow[ur] \arrow[dr, "j_n"'] & & H_n^{\mathrm{CW}}(X). \\ & \ker(d_n) \arrow[ur, two heads] & \end{tikzcd}$$

(2) *We denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps. The homomorphisms from* (1) *define a natural isomorphism between the functors $X \mapsto H_n(X)$ and the functor $X \mapsto H_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f\colon X \to Y$ and any $n \in \mathbb{N}_0$ the following diagram commutes*

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"] \arrow[d, "\Phi_X"', "\cong"] & H_n(Y) \arrow[d, "\Phi_Y", "\cong"'] \\ H_n^{\mathrm{CW}}(X) \arrow[r, "f_*"] & H_n^{\mathrm{CW}}(Y) \end{tikzcd} .$$

Example. Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious we can already use the proposition to do some calculations. As an example, we return to the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n$ which he defined on page 111 as

$$\mathbb{C}\mathrm{P}^n \;:=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

In Lemma 26.2 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no other cells. In particular the cellular chain complex is of the form

$$0 \to \underbrace{C_{2n}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{C_{2n-1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \cdots \to \underbrace{C_{2}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{C_{1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{C_{0}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are also zero. In particular the homology groups equal the cellular chain groups. Thus we have shown that

$$H_k(\mathbb{C}\mathrm{P}^n) \;\underset{\substack{\uparrow \\ \text{Proposition 35.4}}}{\cong}\; H_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; C_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

[695]Here we again denote by $j_n\colon H_n(X^n) \to H_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $H_n(X^n)$ in $H_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n\colon H_n(X^n) \to H_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

It follows immediately from inspecting the cellular chain complex that for any $k \leq l$ and any $i \in \{0, \ldots, 2k\}$ the inclusion induced map

$$H_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) \xrightarrow{\cong} H_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^l; \mathbb{Z})$$

is an isomorphism. By the naturality of the isomorphism of Proposition 35.4 the same conclusion also holds for the inclusion induced map

$$H_i(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) \xrightarrow{\cong} H_i(\mathbb{C}\mathrm{P}^l; \mathbb{Z})$$

on singular homology. It follows from Lemma 26.7 (2) that we can view the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. From the arguments above it now follows that

$$H_k(\mathbb{C}\mathrm{P}^\infty) \;\cong\; H_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; C_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore, as above we see that the inclusion induced map

$$H_i(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) \xrightarrow{\cong} H_i(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$$

is an isomorphism for $i \leq k+1$.

Proof of Proposition 35.4. We add a couple of maps to the commutative diagram that we had already encountered in the proof of Lemma 35.2:

$$\begin{tikzcd}
 & & & 0 \\
0 \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] & \\
 & H_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "j_n"] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}). \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, hook, "j_{n-1}"'] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd}$$

Claim. All the diagonal sequences in the above commutative diagram are exact.

From Lemma 35.1 (1) and (2) we know that

$$H_n(X^{n+1}, X^n) \;=\; 0, \quad H_n(X^{n-1}) \;=\; 0 \quad \text{and} \quad H_{n-1}(X^{n-2}) \;=\; 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. This concludes the proof of the claim.

Now we consider the inclusion induced maps

$$\begin{array}{ccccc} H_n(X^n)/\operatorname{im}(\partial_{n+1}) & \xrightarrow{\cong} & H_n(X^{n+1}) & \xrightarrow{\cong} & H_n(X). \\ & \uparrow & & \uparrow & \\ & \text{isomorphism since the left} & & \text{isomorphism by} & \\ & \text{diagonal sequence is exact} & & \text{Lemma 35.1 (3)} & \end{array}$$

We us this map to make the identification $H_n(X) = H_n(X^n)/\operatorname{im}(\partial_{n+1})$. Next we prove the following claim.

Claim. The map $j_n\colon H_n(X^n) \to H_n(X^n, X^{n-1})$ induces an isomorphism

$$\Phi\colon H_n(X) = H_n(X^n)/\operatorname{im}(\partial_{n+1}) \ \to\ \ker(d_n)/\operatorname{im}(d_{n+1}) = H_n^{\mathrm{CW}}(X).$$

We will prove this claim once again by a "diagram chase".[696]

(1) Since $\partial_n \circ j_n = 0$ it follows that $j_n(H_n(X^n)) \subset \ker(d_n)$.
(2) The map j_{n-1} is injective, i.e. $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(j_n)$, i.e. the map j_n induces an epimorphism

$$H_n(X^n) \ \to\ \ker(d_n).$$

Since the left triangle commutes it also induces an epimorphism

$$H_n(X^n)/\operatorname{im}(\partial_{n+1}) \ \to\ \ker(d_n)/\operatorname{im}(d_{n+1}).$$

(3) If an element $j_n(c) \in H_n(X^n, X^{n-1})$ lies in the image of $d_{n+1} = j_n \circ \partial_{n+1}$, then $j_n(c) = j_n(\partial_{n+1}(d))$ for some $d \in H_{n+1}(X^{n+1}, X^n)$. Since j_n is injective it follows that $c = \partial_{n+1}(d)$. Thus we have now shown that the map

$$H_n(X^n)/\operatorname{im}(\partial_{n+1}) \ \to\ \ker(d_n)/\operatorname{im}(d_{n+1})$$

is injective.

This concludes the proof of the claim.

By construction the isomorphism Φ has the property that the following diagram commutes:

$$\begin{tikzcd} & H_n(X) \arrow[dr, "\Phi"] & \\ H_n(X^n) \arrow[ur, two heads] \arrow[dr, "j_n"'] & & H_n^{\mathrm{CW}}(X). \\ & \ker(d_n) \arrow[ur, two heads] & \end{tikzcd}$$

Note that the top-left diagonal map is an epimorphism by Lemma 35.1 (4). This implies immediately that Φ is uniquely determined.

Finally we leave it as a straightforward exercise to the reader that the maps Φ define a natural isomorphism. $\square$

[696]It is arguably best not to read the remainder of the proof but to go on the chase on one's own.

On page 884 we already saw that Proposition 35.4 can be useful, even if we cannot determine the boundary maps d_n. Another, more theoretical application is, that the cellular chain complex is "much smaller" than the singular chain complex. For example, if X is a CW-complex with finitely many cells, then the chain group $C_*^{\mathrm{CW}}(X)$ in the cellular chain complex are *finitely generated* free abelian groups. This is in contract to the singular chain groups $C_*(X)$ which are in almost all cases free abelian groups of uncountable rank.[697]

Using this observation we can now easily prove the following proposition. It can be viewed as an analogue to Proposition 27.12.

Proposition 35.5. *Let X be a CW-complex. The following statements hold:*

(1) *If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then $H_n(X)$ is generated by at most d elements.*
(2) *If X consists only of finitely many cells, then all homology groups of X are finitely generated groups.*
(3) *If X is finite-dimensional, then*

$$H_k(X) \;=\; 0 \;\; \textit{for all } k > \dim(X).$$

(4) *If X is an n-dimensional CW-complex, then $H_n(X)$ is a free abelian group.*

Example. It follows from Proposition 35.5 that for any graph G we have $H_i(G) = 0$ for every $i \geq 2$ and that $H_1(G)$ is a free abelian group.

Proof. Let X be a CW-complex X.

(1) If X contains precisely d cells of dimension n, then it follows from Lemma 35.1 that $C_n^{\mathrm{CW}}(X) \cong \mathbb{Z}^d$. By Lemma 11.8 any subgroup U of $\mathbb{Z}^d$ is a free abelian group with rank $\operatorname{rank}(U) \leq d$. Thus it follows that

$$H_n(X) \;\cong\; \underbrace{\ker\left\{d_n\colon C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)\right\}}_{\text{free abelian group of rank} \leq d} \Big/ \operatorname{im}\left\{d_{n+1}\colon C_{n+1}^{\mathrm{CW}}(X) \to C_n^{\mathrm{CW}}(X)\right\}$$

is generated by at most d elements.
(2) This statement follows immediately from (1).
(3) This statement is in fact already a consequence of Lemma 35.1 (2).
(4) Now suppose that X is an n-dimensional CW-complex X. We have

$$\begin{array}{ccccc} H_n(X) & \cong & H_n^{\mathrm{CW}}(X) & = & \ker\left\{d_n\colon C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)\right\}, \\ & \uparrow & & \uparrow & \\ & \text{Proposition 35.4} & & \text{we have } C_{n+1}^{\mathrm{CW}}(X) = 0 \text{ since } X \text{ is } n\text{-dimensional} & \end{array}$$

i.e. $H_n(X)$ is a subgroup of the free abelian group $C_n^{\mathrm{CW}}(X)$. It follows from Lemma 11.2 that the group $H_n(X)$ itself is a free abelian group. □

[697] For which topological spaces and which n is the singular chain complex $C_n(X)$ finitely generated? Can you give a complete answer?

In Theorem 26.3 we saw that every compact n-dimensional manifold admits a CW-structure with finitely many cells and such that the maximal dimension of a cell is n. If we combine this result with Proposition 35.5 we obtain immediately the following useful result.

Proposition 35.6. *Let M be a compact n-dimensional manifold. Then the following holds:*

(1) *All homology groups are finitely generated abelian groups.*
(2) *For any $k > n$ we have $H_k(M) = 0$.*

Remark.

(1) Note that the statement of Proposition 35.6 is in contrast to the situation for higher homotopy groups. As formulated in Question 28.17 we have the strong suspicion that $\pi_3(S^2) \neq 0$.
(2) Thus we have now shown that for a CW-complex X the singular homology groups $H_n(X)$ are isomorphic to the cellular homology groups $H_n^{\mathrm{CW}}(X)$. One might now ask, whether one can generalize this statement, namely whether the singular chain complex $(C_*(X), \partial_*)$ and the cellular chain complex (C_*^{CW}, d_*) are in fact chain homotopic. This is indeed the case. But the proof requires more homological algebra than we have at the moment at our disposal. Therefore we postpone the proof to a later chapter.

35.3. **The cellular boundary maps.** If we want to use Proposition 35.4 to calculate homology groups of CW-complexes we need to develop tools for computing the cellular boundary maps d_n in the cellular chain complex. Let X be a CW-complex. Throughout this section we will use the following notation:

(1) For $n \in \mathbb{N}_0$ we denote by I_n the set of n-cells of X.
(2) Given an n-cell α we denote by $\varphi_\alpha \colon S_\alpha^{n-1} \to X^{n-1}$ the corresponding attaching map and we denote by $\Phi_\alpha \colon \overline{B}_\alpha^n \to X^n$ the corresponding characteristic map.
(3) Given $n \in \mathbb{N}_0$ we denote by
$$\Psi_n \colon \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \;\to\; X^n / X^{n-1}$$
the map that is given by
$$x \;\mapsto\; \Phi_\alpha(x), \qquad \text{if } x \in \overline{B}_\alpha^n.$$
It is straightforward to verify that Ψ_n is a homeomorphism.
(4) Given $\beta \in I_n$ we denote by
$$i_\beta \colon \overline{B}_\beta^n / \partial \overline{B}_\beta^n \to \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \quad \text{and} \quad p_\beta \colon \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \to \overline{B}_\beta^n / \partial \overline{B}_\beta^n,$$
the obvious inclusion map respectively the obvious projection map. By Proposition 34.9 the maps
$$\bigoplus_{\beta \in I} H_n\big(\overline{B}_\beta^n / \partial \overline{B}_\beta^n\big) \underset{\oplus i_\beta}{\overset{\oplus p_\beta}{\rightleftarrows}} H_n\Big(\bigvee_{\beta \in I_n} \overline{B}_\beta^n / \partial \overline{B}_\beta^n\Big).$$
are inverses of one-another.

(5) Given $\alpha \in I_n$ we denote, by a slight abuse of notation, the standard generator of $H_n(\overline{B}^n_\alpha, \partial\overline{B}^n_\alpha) \cong \mathbb{Z}$, as defined on page 816, by α. Put differently, we make the identification $H_n(\overline{B}^n_\alpha, \partial\overline{B}^n_\alpha) = \mathbb{Z}\cdot\alpha$.

It follows from the discussion above and from Lemma 35.1 and its proof that all the maps in the following sequence are isomorphisms:

$$\begin{tikzcd}
\bigoplus_{\alpha\in I_n} \overbrace{H_n\big(\overline{B}^n_\alpha, \partial\overline{B}^n_\alpha\big)}^{=\mathbb{Z}\cdot\alpha} \arrow[r, "\cong"] \arrow[rrr, bend right, "=:\Omega_n"'] & \bigoplus_{\alpha\in I_n} \widetilde{H}_n(\overline{B}^n_\alpha/\partial\overline{B}^n_\alpha) \arrow[r, "\cong"] & \widetilde{H}_n\big(\bigvee_\alpha \overline{B}^n_\alpha/\partial\overline{B}^n_\alpha\big) \arrow[r, "\cong", "\Psi_n"'] & \widetilde{H}_n(X^n/X^{n-1}) & H_n(X^n, X^{n-1}). \arrow[l, "\cong"']
\end{tikzcd}$$

By another abuse of notation we often denote $\Omega_n(\alpha)$ also by α. Put differently, we identify $C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1})$ with the free abelian group generated by the set I_n of all n-cells.

Our goal now is to determine the matrix of the cellular boundary map

$$d_n\colon \quad \overbrace{\underbrace{H_n(X^n, X^{n-1})}_{\substack{\text{free abelian group}\\ \text{generated by } I_n}}}^{=C_n^{\mathrm{CW}}(X)} \quad \to \quad \overbrace{\underbrace{H_{n-1}(X^{n-1}, X^{n-2})}_{\substack{\text{free abelian group}\\ \text{generated by } I_{n-1}}}}^{=C_{n-1}^{\mathrm{CW}}(X)}.$$

corresponding to the bases given by the cells. As a warm-up with start out with the case $n = 1$.

Lemma 35.7. *Let X be a CW-complex. We adopt all the notations introduced above.*

(1) *For a 1-cell α with attaching map $\varphi_\alpha\colon \overline{B}^1 \to X^0$ we have*

$$d_1(\alpha) \;=\; \varphi_\alpha(1) - \varphi_\alpha(-1).$$

(2) *If X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.*

Proof. The second statement of the lemma is evidently a consequence of the first statement. Therefore it suffices to prove the first statement. So let α be a 1-cell with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^0$. We have to determine the map

$$\begin{tikzcd}
\mathbb{Z}\cdot\alpha \arrow[r] & \bigoplus_{j\in I_1} H_1(\overline{B}^1_j, \partial\overline{B}^1_j) \arrow[r, "\Omega_1"] & H_1(X^1, X^0) \arrow[d, "d_1"] & & \\
& & H_0(X^0, \underbrace{X^{-1}}_{=\varnothing}) \arrow[r, "\Omega_0^{-1}"] & \bigoplus_{i\in I_0} H_0(\overline{B}^0_i, \underbrace{\partial\overline{B}^0_i}_{=\varnothing}). &
\end{tikzcd}$$

Since $X^{-1} = \varnothing$ we see that in the present case we have $d_1 = \partial_1$, i.e. the cellular boundary map d_1 is the connecting homomorphism $H_1(X^1, X^0) \to H_0(X^0)$ of the long exact sequence of the pair (X^1, X^0). It follows immediately from the definitions of the connecting homomorphism, as defined in Proposition 31.13 (1), that $d_1(\alpha) = \varphi_\alpha(1) - \varphi_\alpha(-1)$. $\square$

Example. We consider the graph G that is given in Figure 540 with two vertices β_1, β_2 and four edges $\alpha_1, \dots, \alpha_4$. We view it as a CW-complex in the obvious way. According to

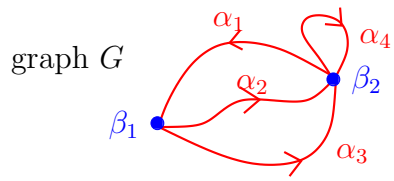


FIGURE 540.

Lemma 35.7 the cellular chain complex of X is of the following form

$$0 \;\to\; C_1^{\mathrm{CW}}(G) = \mathbb{Z}^4 \xrightarrow{\begin{array}{c|cccc} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \beta_1 & 1 & -1 & -1 & 0 \\ \beta_2 & -1 & 1 & 1 & 0 \end{array}} C_0^{\mathrm{CW}}(G) = \mathbb{Z}^2 \;\to\; 0.$$

Thus we obtain from Proposition 35.4 that

$$H_1(G) \cong \ker\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}^3 \quad \text{and} \quad H_0(G) \cong \operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}.$$

After this example we return to the discussion of the cellular boundary maps. Now let $n > 1$. We want to determine the cellular boundary map

$$d_n \colon H_n(X^n, X^{n-1}) \;\to\; H_{n-1}(X^{n-1}, X^{n-2}).$$

Given an n-cell α and an $(n-1)$-cell β we consider the map

$$S_\alpha^{n-1} = \partial \overline{B}_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow[\cong]{\Psi_{n-1}^{-1}} \bigvee_{j \in I_{n-1}} \overline{B}_j^{n-1}/\partial \overline{B}_j^{n-1} \xrightarrow{p_\beta} \overline{B}_\beta^{n-1}/\partial \overline{B}_\beta^{n-1} \xrightarrow[\cong]{\Theta} S_\beta^{n-1}$$

where $q \colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map and where $\Theta \colon \overline{B}^n/\partial \overline{B}^n \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from page 99. Since S_α^{n-1} and S_β^{n-1} are copies of S^{n-1} we can consider the degree of the map as defined on page 819.

Now we will see that this degree is just the $\beta \times \alpha$–entry of the matrix that represents the cellular boundary map d_n with respect to the bases given by all n-cells and all $(n-1)$-cells. More precisely we have the following proposition:

Proposition 35.8. *Let X be a CW-complex. We use all the notation that we introduced above. Then for each n-cell α we have*

$$d_n(\alpha) \;=\; \sum_{\beta \in I_{n-1}} d_{\alpha,\beta} \cdot \beta$$

where

$$d_{\alpha\beta} \;:=\; \textit{degree of the map } S_\alpha^{n-1} \xrightarrow{\Theta \circ p_\beta \circ \Psi_{n-1}^{-1} \circ q \circ \varphi_\alpha} S_\beta^{n-1}.$$

Proof. We consider the following commutative[698][699] diagram

$$\begin{tikzcd}
\bigoplus_{i\in I_n} H_n(\overline{B}_i^n,\partial\overline{B}_i^n) \arrow[r, "\cong"'] \arrow[rrr, bend left, "\Omega_n"] & \bigoplus_{i\in I_n} \widetilde{H}_n(\overline{B}_i^n/\partial\overline{B}_i^n) \arrow[dr, "\cong"'] & & \widetilde{H}_n(X^n/X^{n-1}) \arrow[r, "\cong"'] & H_n(X^n,X^{n-1}) \arrow[d, "\partial_n"'] \arrow[ddd, bend left, "d_n"] \\
H_n(\overline{B}_\alpha^n,\partial\overline{B}_\alpha^n) \arrow[u] \arrow[d, "\cong"', "\partial"] & & \widetilde{H}_n\Big(\bigvee_{i\in I_n} \overline{B}_i^n/\partial\overline{B}_i^n\Big) \arrow[ur, "\Psi_n"', "\cong"] & & \\
\widetilde{H}_{n-1}(S_\alpha^{n-1}) \arrow[rrr] \arrow[d, "\cdot d_{\alpha\beta}"] & & & \widetilde{H}_{n-1}(X^{n-1}) \arrow[dl, "q"] \arrow[d, "j_{n-1}"'] \\
\widetilde{H}_{n-1}(S_\beta^{n-1}) & & \widetilde{H}_{n-1}\Big(\bigvee_{j\in I_{n-1}} \overline{B}_j^{n-1}/\partial\overline{B}_j^{n-1}\Big) \arrow[dl, "p_\beta"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[l, "\Psi_{n-1}^{-1}"', "\cong"] & H_{n-1}(X^{n-1},X^{n-2}). \arrow[l, "\cong"'] \arrow[lll, bend left, "\Omega_{n-1}^{-1}"] \\
H_{n-1}(\overline{B}_\beta^{n-1},\partial\overline{B}_\beta^{n-1}) \arrow[u, "\cong"', "\partial"] \arrow[r, "\cong"'] & \widetilde{H}_{n-1}(\overline{B}_\beta^{n-1}/\partial\overline{B}_\beta^{n-1}) \arrow[ul, "\Theta_*"'] & & &
\end{tikzcd}$$

The proposition now follows directly from the definitions and the commutativity of the above diagram. We leave it as an exercise to the reader to unravel all the definitions and to digest the diagram. □

Example.

(1) We consider the surface Σ of genus 2. As usual we view Σ as a CW-complex with one 0-cell P, four 1-cells a, b, c, d and one 2-cell F. We denote by $\varphi\colon S^1 \to \Sigma^1$ the attaching map of the 2-cell F. (We refer to Figure 541 for an illustration.) It follows from the discussion on page 819 that the degree of the map $p_a \circ \Psi_1^{-1} \circ q \circ \varphi\colon S_F^1 \to S_a^1$ is zero. The same conclusion holds for all other 1-cells. It follows from Proposition 35.8 that the cellular boundary map d_2 in the cellular chain complex of Σ is zero. Since Σ has only one 0-cell we know from Lemma 35.7 that the cellular boundary map d_1 is also zero. Thus we see that

$$\begin{array}{ccccccc} H_n(\Sigma) & \cong & H_n^{\mathrm{CW}}(\Sigma) & = & C_n^{\mathrm{CW}}(\Sigma) & \cong & \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ & \uparrow & & \uparrow & & & \\ & \text{Proposition 35.4} & & \begin{array}{c}\text{since all the cellular} \\ \text{boundary maps are zero}\end{array} & & & \end{array}$$

In Figure 542 on the left-hand side we show closed curves a, b, c, d. It follows fairly easily from the above calculation and the explicit isomorphism of Proposition 35.4

[698] Why is the diagram commutative?

[699] The strange shape of the diagram is due to space constraints.

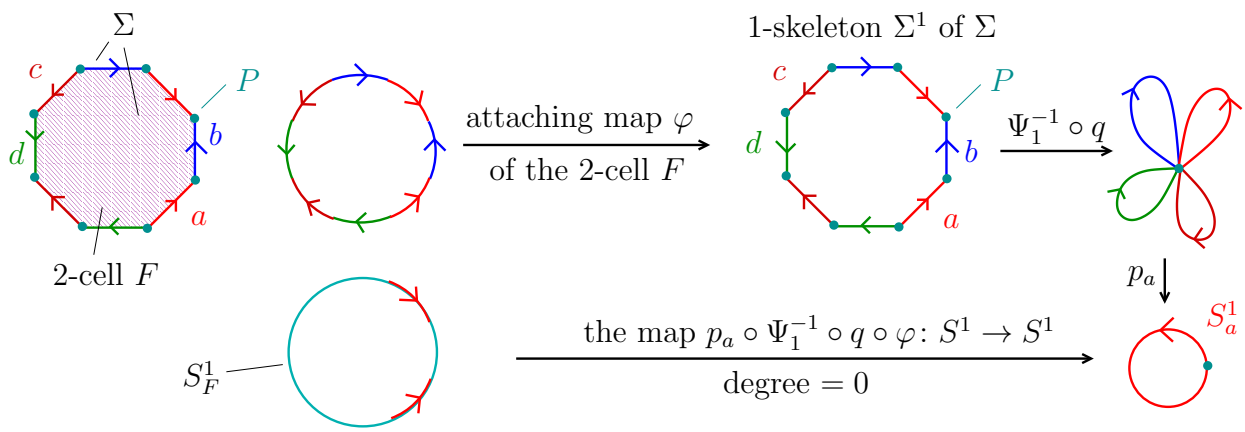


FIGURE 541.

that these curves represent a basis for $H_1(\Sigma) \cong \mathbb{Z}^4$. These curves in turn are isotopic to the curves a', b', c', d' on the right-hand side, which therefore, by Proposition 30.5, also represent a basis for $H_1(\Sigma) \cong \mathbb{Z}^4$.

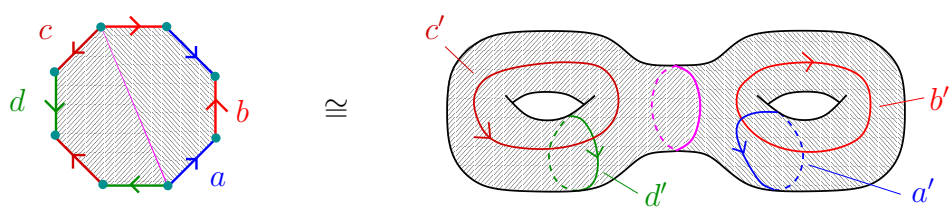


FIGURE 542. A basis for $H_1(\Sigma)$.

(2) As a reality check we also consider once again the Klein bottle K. We view K as a CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F. (We refer to Figure 543 for an illustration.) It follows from Proposition 35.8, the discussion on page 819 and Lemma 35.7 that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \to 0.$$

Therefore we see that $H_2(K) = 0$, $H_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $H_0(K) \cong \mathbb{Z}$. This is of course the same result that we had already obtained in Lemma 33.14.

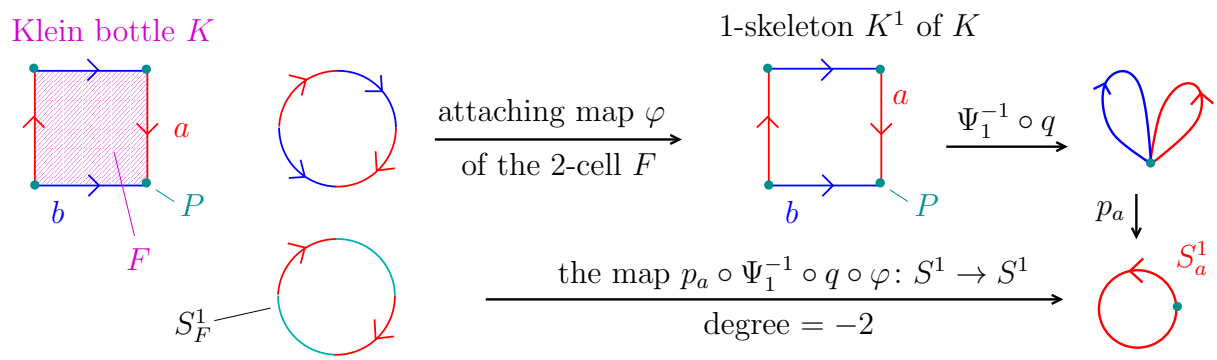


FIGURE 543.

35.4. **The homology groups of 2-dimensional manifolds.** Recall that given $g, m \in \mathbb{N}_0$ we write[700]

$$\Sigma_{g,m} \quad := \quad \text{the surface of genus } g \text{ minus } m \text{ open disks}$$

and for $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we write

$$R_{k,m} \quad := \quad \text{the manifold } k \cdot \mathbb{R}\mathrm{P}^2 \text{ minus } m \text{ open disks.}$$

In Theorem 14.14 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $R_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Our goal in this section will be to determine the homology groups of $\Sigma_{g,n}$ and $R_{k,n}$. Before we consider the general case we will consider the special case $\Sigma_{2,1}$ in great detail.

Example. We consider $\Sigma_{2,1}$ and we equip it with the CW-structure that is illustrated in Figure 544. This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and it has one 2-cell F. Given $i = 1, 2$ we denote by A_i the matrix that represents the cellular boundary

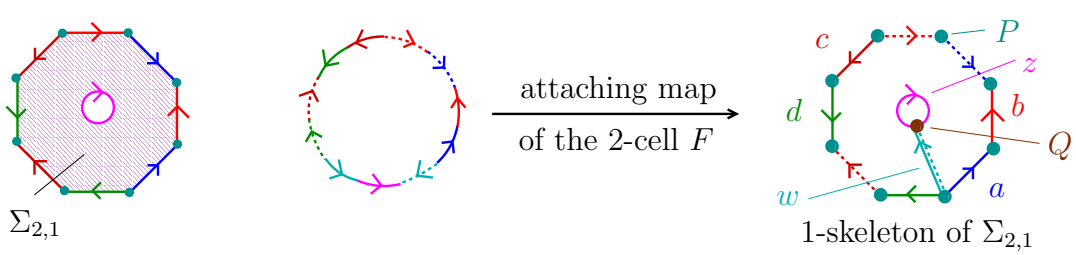


FIGURE 544.

[700]Also recall the definition on page 440 what we mean by a "surface minus m open disks".

map $d_i\colon C_i^{\mathrm{CW}}(\Sigma_{2,1}) \to C_{i-1}^{\mathrm{CW}}(\Sigma_{2,1})$ with respect to the given bases.

The argument on page 891 shows that $A_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ w \\ z \end{array} \begin{array}{c} F \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$ It follows from Lemma 35.7 that we have $A_1 = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & w & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$.

Some elementary linear algebra shows that

$$H_n(\Sigma_{2,1}) \underset{\substack{\uparrow \\ \text{Proposition 35.4}}}{\cong} H_n^{\mathrm{CW}}(\Sigma_{2,1}) = H_n\Big(0 \to \mathbb{Z} \xrightarrow{A_2} \mathbb{Z}^6 \xrightarrow{A_1} \mathbb{Z}^2 \to 0\Big) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

From this discussion we get in fact a more precise statement. Namely, by the choice of the CW-structure on $\Sigma_{2,1}$ we have that the map inclusion map $\partial\Sigma_{2,1} \to \Sigma_{2,1}$ is in fact a cellular map. Therefore it follows from Proposition 35.4 (2) that the following diagram commutes

$$\begin{tikzcd} H_1(\partial\Sigma_{2,1}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_1(\Sigma_{2,1}) \arrow[d, "\cong"] \\ \mathbb{Z} = H_1^{\mathrm{CW}}(\partial\Sigma_{2,1}) \arrow[r, "f_*", "{[z]\mapsto[z]}"'] & H_1^{\mathrm{CW}}(\Sigma_{2,1}) = \mathbb{Z}^4 \end{tikzcd}$$

where the vertical maps denote the isomorphisms from Proposition 35.4. The 1-cell z is null-homologous in $C_*^{\mathrm{CW}}(\Sigma_{2,1})$ since we have just seen that it is the boundary of the 2-cell F. Thus we see that the bottom horizontal map is the zero map, but that implies that the top horizontal map is also the zero map.

After the calculations that we just did it we can now easily prove the following lemma.

Lemma 35.9.

(1) *Let $g \in \mathbb{N}_0$. Then the inclusion induced map*

$$H_1(\partial\Sigma_{g,1}) \to H_1(\Sigma_{g,1})$$

is the zero map.

(2) *Let M be a manifold and let C be an oriented curve in M. If there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$, then $C = 0 \in H_1(M)$.*[701]

[701] Following the convention of page 853 the curve C defines a homology class in $H_1(M)$ that we also denote by C.

Example. Loosely speaking Lemma 35.9 (2) says that if a curve C is the boundary of an compact orientable surface in M, then C is null-homologous. For example in Figure 545 we see a curve C on a surface M. Since C is the boundary of a compact orientable subsurface contained in M we see that $C = 0 \in H_1(M)$.

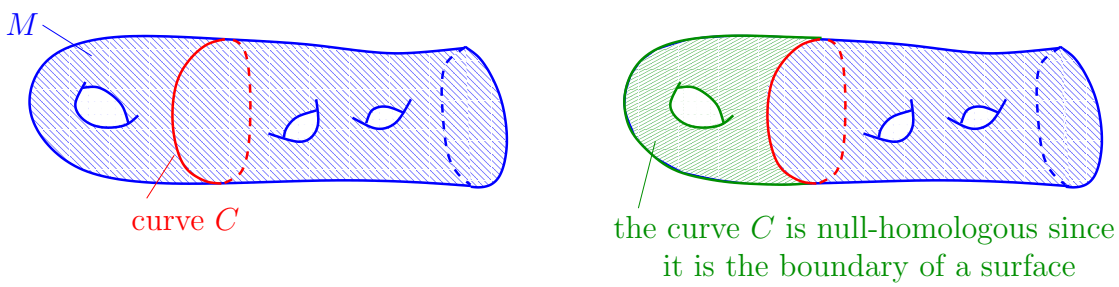


FIGURE 545.

Proof.

(1) For $g = 1$ we proved the statement in the previous example. The general case is proved entirely the same way.
(2) Now let M be a manifold and let C be an oriented curve in M. Suppose there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$. We get the commutative diagram

$$\begin{tikzcd}
 & & H_1(C) \arrow[dr] & \\
H_1(S^1) \arrow[urr] \arrow[dr] & & & H_1(M). \\
 & H_1(\partial\Sigma_{g,1}) \arrow[r] & H_1(\Sigma_{g,1}) \arrow[ur, "\psi_*"'] &
\end{tikzcd}$$

By the first statement the bottom horizontal map is the zero map. But that shows that the homology class represented by C is also zero. $\square$

In the following proposition we give the calculation for the homology groups of all connected, compact 2-dimensional topological manifolds.

Proposition 35.10. *For any $g \in \mathbb{N}_0$ we have*

$$H_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \qquad \begin{array}{c}\textit{Furthermore} \\ \textit{for } m \geq 1 \\ \textit{we have}\end{array} \qquad H_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$H_n(k \cdot \mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \qquad \begin{array}{c}\textit{Furthermore} \\ \textit{for } m \geq 1 \\ \textit{we have}\end{array} \qquad H_n(R_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups $\Sigma_{g,m}$ is a slight generalization of the arguments that we provided above and on page 891. We leave the details to the reader.

We will deal with the homology of $R_{k,m}$ in Exercise 35.1. The calculation is evidently just a variation of the arguments we provided for Σ_2 and $\Sigma_{2,2}$. □

Remark. A careful reading of the Proof of Proposition 35.10 gives us some extra information. First of all we endow $\Sigma_{g,m}$ with an orientation. The boundary components of $\Sigma_{g,m}$ then inherit an orientation. By the discussion on page 853 the boundary components, viewed as oriented curves, then define elements $c_1, \dots, c_m \in H_1(\Sigma_{g,m})$. Then there exists an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto \sum c_i} \bigoplus_{i=1}^{m} \mathbb{Z} \cdot c_i \to H_1(\Sigma_{g,m}) \underset{\substack{\uparrow \\ \text{inclusion induced map}}}{\to} H_1(\Sigma_g) \to 0.$$

As we saw in Figure 542, a basis for $H_1(\Sigma_g)$ is given by a pair of curves for each "hole". The above exact sequence shows that a basis for the homology group $H_1(\Sigma_{g,m}) \cong \mathbb{Z}^{2g+m-1}$ is obtained by a pair of curves for each "hole" together with boundary curves for all except one boundary component. This statement is illustrated in Figure 546.

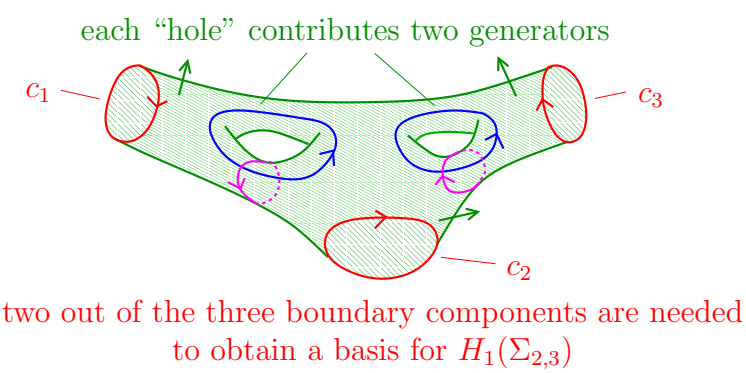
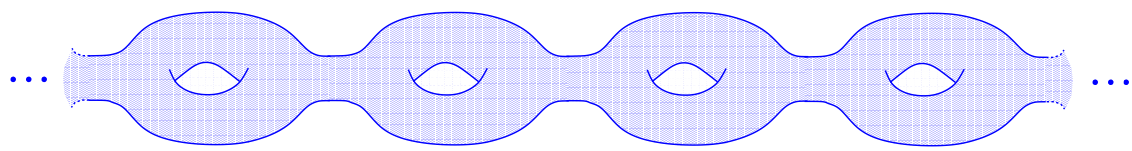


FIGURE 546.

We conclude this section with the surface Σ_∞ of infinite genus that we had already encountered on page 492. Using a variation of the proof of Lemma 16.6 one can show, using the above calculations and Proposition 34.4, that

$$H_n(\Sigma_\infty) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

We leave the details as an exercise to the reader.



surface of "infinite genus"

FIGURE 547.

35.5. **The homology groups of real projective spaces.** In this section we will compute the homology groups of real projective spaces $\mathbb{R}\mathrm{P}^n$. This calculation is significantly trickier than the previous ones, since the calculation of the cellular boundary maps is not as straightforward as in our previous examples. This time we will need the theory of local degrees that we developed in Section 32.5.

Proposition 35.11. *For any $n \in \mathbb{N}_0$ we have*[702]

$$H_k(\mathbb{R}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } k = 0, \\ \mathbb{Z}_2, & \textit{if } k \textit{ is odd and } k < n, \\ 0, & \textit{if } k \textit{ is even and } 0 < k \neq 0, \\ 0, & \textit{if } k > n, \\ \mathbb{Z}, & \textit{if } k = n \textit{ and } n \textit{ is odd.} \end{cases}$$

Proof $(*)$. As a reminder we defined

$$\mathbb{R}\mathrm{P}^n \;=\; S^n/\sim \qquad \text{where } x \sim -x \text{ for all } x \in S^n.$$

In Lemma 26.2 we already saw that we can view $\mathbb{R}\mathrm{P}^n$ as a CW-complex with the following two properties:

(1) the CW-structure has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells,
(2) for each $k \leq n$ the k-skeleton of $\mathbb{R}\mathrm{P}^n$ is given by $\mathbb{R}\mathrm{P}^k$.

[702]For $n = 2$ we thus obtain, as we should, the same result as in Section 33.3.

Now we recall the definition of this CW-structure. For each $k \in \mathbb{N}_0$ we define

$$\begin{array}{rcl} \varphi_k \colon S^{k-1} = \partial \overline{B}^k & \to & \mathbb{R}\mathrm{P}^{k-1} \\ (x_1, \dots, x_k) & \mapsto & [(x_1, \dots, x_k)]. \end{array}$$

Then we obtain a homeomorphism

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{k-1} \cup_{\varphi_k} \overline{B}^k & \xrightarrow{\cong} & \mathbb{R}\mathrm{P}^k \\ x & \mapsto & \begin{cases} (x, 0), & \text{if } x \in \mathbb{R}\mathrm{P}^{k-1} \\ \big[\big(x, \sqrt{1 - \|x\|^2}\big)\big], & \text{if } x \in \overline{B}^k. \end{cases} \end{array}$$

Thus we see inductively that we can view $\mathbb{R}\mathrm{P}^n$ as a CW-complex with a single cell in the dimensions $0, \dots, n$ where the attaching maps are given precisely by the maps $\varphi_1, \dots, \varphi_n$. As usual we denote the characteristic map of the k-cell, which in this case is the obvious map $\overline{B}^k \to \mathbb{R}\mathrm{P}^k$, by Φ_k.

Claim. For each $k = 1, \dots, n$ the cellular boundary map

$$d_k \colon \mathbb{Z} = C_k^{\mathrm{CW}}(X) \ \to \ C_{k-1}^{\mathrm{CW}}(X) = \mathbb{Z}$$

is given by multiplication by $1 + (-1)^k$.

Let $k \in \{1, \dots, n\}$. We denote by $\Theta \colon \overline{B}^{k-1} / \partial \overline{B}^{k-1} \to S^{k-1}$ the homeomorphism from page 99 and we denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1} / \mathbb{R}\mathrm{P}^{k-2}$ the projection map. By Proposition 35.8 it suffices to show that the degree of the map

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi_k"] \arrow[rrr, bend right, "=:\Omega"'] & \mathbb{R}\mathrm{P}^{k-1} \arrow[r, "q"] & \mathbb{R}\mathrm{P}^{k-1} / \mathbb{R}\mathrm{P}^{k-2} \arrow[r, "\Phi_{k-1}^{-1}"] & \overline{B}^{k-1} / \partial \overline{B}^{k-1} \arrow[r, "\Theta"] & S^{k-1} \end{tikzcd}$$

equals $1 + (-1)^k$. (We refer to Figure 548 for an illustration of the map.) We pick the point $y = (0, \dots, 0, 1) \in S^{k-1}$. The preimages of y under the map Ω are $x_1 = (0, \dots, 0, 1)$ and $x_2 = (0, \dots, 0, -1)$. We leave it as an exercise to verify that[703] $\deg \Omega|_{x_1} = 1$ and that $\deg \Omega|_{x_2} = (-1)^n$. The desired statement now follows from Proposition 32.26. This concludes the proof of the claim.

It follows from the claim that the cellular chain complex of X is of the form

$$0 \ \to \ \mathbb{Z} \ \xrightarrow{1+(-1)^n} \ \mathbb{Z} \ \dots \ \mathbb{Z} \ \xrightarrow{\cdot 2} \ \mathbb{Z} \ \xrightarrow{0} \ \mathbb{Z} \ \to \ 0.$$

It is now straightforward to determine the cellular homology groups which, by Proposition 35.4, are isomorphic to the singular homology groups of $\mathbb{R}\mathrm{P}^n$. $\square$

[703] It is not entirely easy to struggle through all the orientation conventions and all the maps, but it can indeed be done.

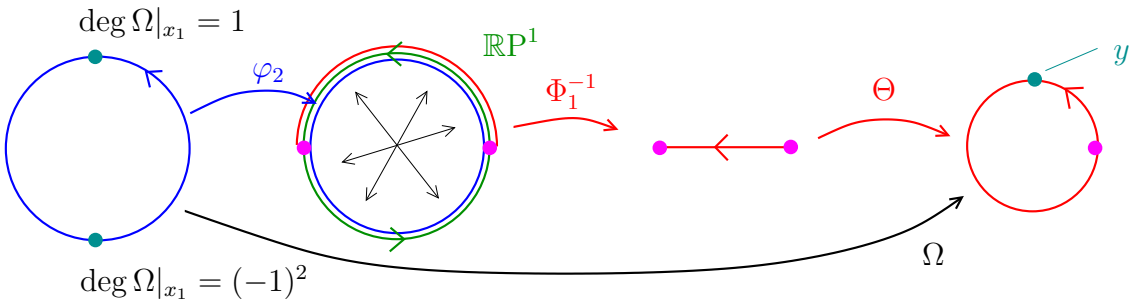


FIGURE 548. Illustration from the proof of Proposition 35.11.

Remark. Using the approach of the proof of Proposition 35.11, applied to the infinite CW-complex $\mathbb{R}\mathrm{P}^\infty$, or alternatively using $\mathbb{R}\mathrm{P}^\infty = \bigcup_n \mathbb{R}\mathrm{P}^n$ and Proposition 34.4, we see that

$$H_k(\mathbb{R}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes. There are two related answers:

(1) we also want to study topological spaces that are not CW-complexes,
(2) it is much harder to prove some statements using cellular chain complexes, for example it is very difficult, if not impossible, to show that homeomorphic CW-complexes have isomorphic homology groups, just using the definition of cellular chain complexes.

A general slogan in topology is that for calculations it is usually best to work with "small objects", whereas for proving general properties, it is often best to work with "big objects".

Exercises for Chapter 35.

Exercise 35.1.

(a) Determine the homology groups of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$.
(b) Determine the homology groups of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ minus an open disk.

Note: In both parts you can make use of the CW-structure for $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ illustrated in Figure 549.
Remark: The homology groups are stated as Proposition 35.10. But it is a very good exercise to try to determine the homology groups without looking at the proposition.

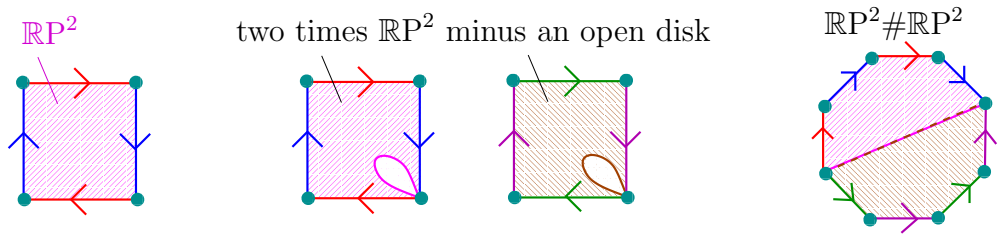


FIGURE 549.

Exercise 35.2. Compute the homology groups of the quotient of S^2 obtained by identifying north and south poles to a point.

Exercise 35.3.

(a) We consider the map $f\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, |y|)$. Give a careful proof that the local degree at the point -1 is zero.

(b) Let $f\colon S^1 \to S^1$ be a map and let $z \in S^1$ be a nice point. Suppose that $\deg f|_z = 1$. Does it follow that f is a local homeomorphism at z? It suffices to give a short justification for your answer.

Exercise 35.4. Show that given any finitely generated abelian group A there exists a topological space X with $H_0(X) = \mathbb{Z}$ and $H_1(X) \cong A$ and with $H_i(X) = 0$ for $i \geq 2$.

Exercise 35.5. Let U be an open subset of $\mathbb{R}^n$ with $0 \in U$. Let $f\colon U \to \mathbb{R}^n$ be a differentiable map with $f(0) = 0$ such that $A := Df_0$ is invertible. We consider the linearization

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & A \cdot x \end{array} \quad \text{and the homotopy} \quad \begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot f(x) + t \cdot \lambda(x). \end{array}$$

Show that there exists an open neighborhood $V = B_r(0)$ of 0, contained in U, with the following properties:

(1) for each $t \in [0, 1]$ the map $F_t\colon V \to \mathbb{R}^n$ is a diffeomorphism onto its image,
(2) for each $t \in [0, 1]$ the image $F_t(V) = F_t(B_r(0))$ is a convex subset of $\mathbb{R}^n$.

Remarks:

(1) Recall that the hypothesis on f implies that
$$\lim_{v \to 0} \frac{f(v) - Av}{\|v\|} = 0.$$
(2) Since A is invertible we know from Exercise 1.18 that there exists a $\lambda > 0$ such that $\|Av\| \geq \lambda v$ for all v.
(3) Hint: it is not as hard as it sounds.

Exercise 35.6.

(a) Let X be the quotient space of S^2 under the identification $x \sim -x$ for all $x \in S^1$. Determine the homology groups of $X = S^2/\sim$.

(b) Let $m, n \in \mathbb{N}$ with $m < n$. As usual we view $\mathbb{R}\mathrm{P}^m$ as a subset of $\mathbb{R}\mathrm{P}^n$. Compute the homology groups of the quotient space $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$.

Exercise 35.7. Let C be a closed curve in a manifold M. Suppose that C is the boundary of an embedded compact orientable surface in M. Show that the inclusion induced map $H_1(C) \to H_1(M)$ is the zero map.
More concisely, if a closed curve bounds an compact orientable surface, then the closed curve represents the trivial element in homology.

Exercise 35.8. Let A be a finitely generated abelian group. Does there exist a topological space X with $H_0(X) = \mathbb{Z}$, $H_3(X) \cong A$ and $H_i(X) = 0$ for $i \neq 0, 3$?

Exercise 35.9. Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose X is compact. Does it follow that $H_n(X)$ is a finitely generated group?

36. The Jordan Curve Theorem

We start out with the following proposition.

Proposition 36.1.

(1) *For every injective map* $h\colon \overline{B}^k \to S^n$ *we have*

$$\widetilde{H}_i\big(S^n \setminus h(\overline{B}^k)\big) \;=\; 0 \quad \textit{for all } i \in \mathbb{N}_0.$$

(2) *For every injective map* $h\colon S^k \to S^n$ *we have*

$$\widetilde{H}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } i = n-k-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Example.

(1) Let $K \subset S^3$ be a knot, i.e. let K be a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1. This means in particular that there exists an embedding $h\colon S^1 \to S^3$ with $h(S^1) = K$. In Lemma 33.15 we had already determined that $H_1(S^3 \setminus K) \cong \mathbb{Z}$. Now we can compute the higher homology groups as well. More precisely, the combination of Proposition 36.1 (2) and Lemma 31.1 says that

$$H_i(S^3 \setminus K) \;=\; H_i(S^3 \setminus h(S^1)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) In Proposition 36.1 we do not assume that h is in any way smooth. For example there are injective maps $h\colon S^1 \to S^2$ of infinite length, see e.g.

`https://en.wikipedia.org/wiki/Koch_snowflake`,

to which the proposition nonetheless applies. Reduced homology is only defined for non-empty topological spaces. It is implicit in the statement of Proposition 36.1 that $S^n \setminus h(\overline{B}^k)$ is not the empty set. Even this statement is not entirely obvious since, as we have pointed out on several occasions, there are non-injective maps $h\colon \overline{B}^1 \to S^n$ that are in fact surjective.

(3) In Figure 550 we see the image of an injective map $\varphi\colon [0,1] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a closed curve $C \subset S^3 \setminus \varphi([0,1])$. It follows from Proposition 36.1 (1) that $H_1(S^3 \setminus \varphi([0,1])) = 0$, in particular C is null-homologous in $H_1(S^3 \setminus \varphi([0,1])) = 0$.[704] But on [Sti, p. 152] it is shown that the curve C represents in fact a non-trivial element in $\pi_1(S^3 \setminus \varphi([0,1]))$.

To make the exposition more readable we prove the two parts of Proposition 36.1 separately.

[704] In Lemma 35.9 we saw that the boundary curve of an orientable surface with one boundary component is null-homologous. In Exercise 36.1 we will see that the closed curve C is in fact the boundary of an orientable surface that is embedded in $S^3 \setminus \varphi([0,1])$.

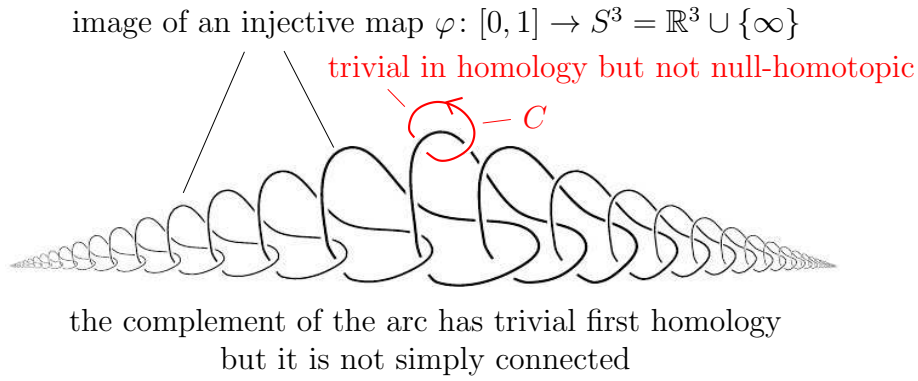


FIGURE 550.

Proof of Proposition 36.1 (1). We write $I = [-1,1]$. By Proposition 1.47 (2) there exists a homeomorphism $I^k \xrightarrow{\cong} \overline{B}^k$. Therefore in the statement of Proposition 36.1 (1) we can replace $\overline{B}^k$ by I^k. So we now want to show that for any injective map $h\colon I^k \to S^n$ we have

$$\widetilde{H}_i(S^n \setminus h(I^k)) \;=\; 0 \qquad \text{for all } i \in \mathbb{N}_0.$$

We will prove this statement by induction on k. We start out with $k = 0$. So let $h\colon I^0 = \{0\} \to S^n$ be an injective map. It follows from the stereographic projection defined in Lemma 1.41 that $S^n \setminus h(\{0\})$ is homeomorphic to $\mathbb{R}^n$. Hence we obtain from Corollary 30.8 (3) that $\widetilde{H}_*(S^n \setminus h(\{0\})) = 0$.

Now we do the induction step. We assume we know the desired statement for all injective maps $I^{k-1} \to S^n$. Now let $h\colon I^k \to S^n$ be an injective map. We suppose that the desired conclusion does not hold, i.e. we suppose that $\widetilde{H}_j(S^n \setminus h(I^k)) \neq 0$ for some j. We pick a non-trivial element $g_0 \in \widetilde{H}_j(S^n \setminus h(I^k))$.

We write $Q = I^{k-1}$, $J_0 = I$ and we write

$$A_1 \;=\; S^n \setminus h(Q \times [0, \tfrac{1}{2}]) \quad \text{and} \qquad A_2 \;=\; S^n \setminus h(Q \times [\tfrac{1}{2}, 1]).$$

Since h is injective we have

$$A_1 \cap A_2 \;=\; S^n \setminus h(I^k) \qquad \text{and} \quad A_1 \cup A_2 \;=\; S^n \setminus h(Q \times \{\tfrac{1}{2}\}).$$

From Lemma 1.37 (3) we know that A_1 and A_2 are open subsets of $A_1 \cup A_2$. Therefore we can apply the Mayer-Vietoris Theorem 33.5 and we obtain the long exact sequence

$$\ldots \to \underbrace{\widetilde{H}_{j+1}(A_1 \cup A_2)}_{=0 \text{ by induction}} \to \widetilde{H}_j(A_1 \cap A_2) \;\to\; \widetilde{H}_j(A_1) \oplus \widetilde{H}_j(A_2) \;\to\; \underbrace{\widetilde{H}_i(A_1 \cup A_2)}_{=0 \text{ by induction}} \to \ldots$$

So we see that the middle map is an isomorphism. Thus there exists an $i \in \{1,2\}$ such that the image of g_0 is non-trivial in $\widetilde{H}_j(A_i)$. If $i = 1$, then we set $J_1 := [0, \frac{1}{2}]$, otherwise

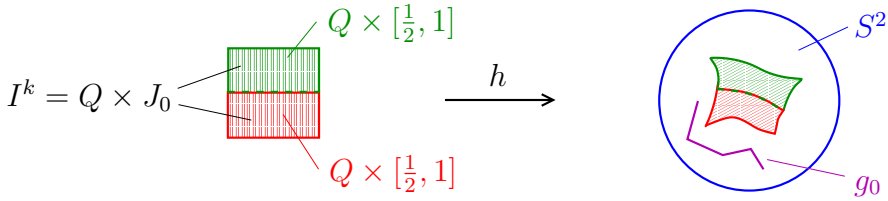


FIGURE 551. Illustration for the proof of Proposition 36.1 (1).

we set $J_1 := [\frac{1}{2}, 1]$. We denote by

$$\varphi_1 \colon \widetilde{H}_j(S^n \setminus h(Q \times J_0)) \;\to\; \widetilde{H}_j(S^n \setminus h(Q \times J_1))$$

the inclusion induced map. By the above exact sequence φ_1 is an epimorphism. We denote by ρ_1 the inverse of the induced isomorphism

$$\widetilde{H}_j(S^n \setminus h(Q \times J_0))/\ker(\varphi_1) \;\to\; \widetilde{H}_j(S^n \setminus h(Q \times J_1)).$$

Now we iterate this process by running the same argument with J_0 replaced by J_1 and so on. For each $m \in \mathbb{N}$ we can find an interval J_m of length $\frac{1}{2^m}$ that is contained in J_{m-1} such that for the inclusion induced homomorphism

$$\varphi_m \colon \widetilde{H}_j(S^n \setminus h(Q \times J_{m-1})) \;\to\; \widetilde{H}_j(S^n \setminus h(Q \times J_m))$$

we have that $g_m := \varphi_m(g_{m-1})$ is non-zero. We define ρ_m the same way from φ_m as we had obtained ρ_1 from φ_1.

By the definition of the direct limit we have the following commutative diagram:[705]

$$\begin{tikzcd}
\vdots & & \\
\widetilde{H}_j(S^n \setminus h(Q \times J_2)) \arrow[u, two heads, "\varphi_3"] \arrow[rr] \arrow[drr, "\rho_2 \circ \rho_1"] & & \varinjlim \widetilde{H}_j(S^n \setminus h(Q \times J_m)) \arrow[d, dashed] \\
\widetilde{H}_j(S^n \setminus h(Q \times J_1)) \arrow[u, two heads, "\varphi_2"] \arrow[urr] \arrow[drr, "\rho_1"] & & \\
\widetilde{H}_j(S^n \setminus h(Q \times J_0)) \arrow[u, two heads, "\varphi_1"] \arrow[urr] \arrow[rr, "\text{image of } g_0 \text{ is non-zero}"'] & & \dfrac{\widetilde{H}_j(S^n \setminus h(Q \times I))}{\bigcup_{m \in \mathbb{N}} \ker(\varphi_m \circ \cdots \circ \varphi_1)}
\end{tikzcd}$$

[705]Note that we have

$$\ker(\varphi_1) \subset \ker(\varphi_2 \circ \varphi_1) \subset \ker(\varphi_3 \circ \varphi_2 \circ \varphi_1) \subset \ldots$$

It follows that the union of these subgroups is again a subgroup.

Since each J_m has half the length of J_{m-1} we conclude that the intersection $\bigcap_{m\in\mathbb{N}} J_m$ contains a unique point $t \in [0,1]$.[706] We have

$$\lim_{\longrightarrow} \widetilde{H}_j(S^n \setminus h(Q \times J_m)) \underset{\substack{\uparrow \\ \text{by Proposition 34.4 since the} \\ \text{subsets } S^n \setminus h(Q\times J_m) \text{ are open}}}{\cong} \widetilde{H}_j\Big(\bigcap_{m\in\mathbb{N}} S^n \setminus h(Q \times J_m)\Big) \underset{\substack{\uparrow \\ \text{since } h \text{ is injective}}}{=} \widetilde{H}_j(S^n \setminus h(Q \times \{t\})) \underset{\substack{\uparrow \\ \text{by induction} \\ \text{since } Q\times\{t\} \cong I^{k-1}}}{=} 0.$$

But we have now obtained a contradiction, since the bottom horizontal map of the above commutative diagram has a non-trivial image but it factors through the trivial group. □

Proof of Proposition 36.1 (2). We have to show that for every injective map $h\colon S^k \to S^n$ we have

$$\widetilde{H}_i\big(S^n \setminus h(S^k)\big) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

We prove this statement by induction on k. For $k = 0$ we have

$$\widetilde{H}_i(S^n \setminus h(S^0)) \underset{\substack{\uparrow \\ \text{since } S^n\setminus h(S^0) \text{ is homotopy} \\ \text{equivalent to } S^{n-1}}}{=} \widetilde{H}_i(S^{n-1}) \underset{\substack{\uparrow \\ \text{by Proposition 31.3}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

[707]

Now suppose we know the statement for some $k \in \mathbb{N}$. Suppose we are given an injective map $h\colon S^{k+1} \to S^n$. We denote by $S^{k+1}_{\le 0}$ the lower hemisphere of S^{k+1} and we denote by $S^{k+1}_{\ge 0}$ the upper hemisphere of S^{k+1}. Note that $S^{k+1}_{\le 0} \cap S^{k+1}_{\ge 0} = S^k$. We apply the Mayer-Vietoris Theorem 33.5 to the decomposition of $S^n \setminus h(S^{k+1})$ into the open sets $S^n \setminus h(S^{k+1}_{\le 0})$ and $S^n \setminus h(S^{k+1}_{\ge 0})$. We obtain a long exact sequence[708]

$$\cdots \to \widetilde{H}_i(S^n \setminus h(S^k)) \to \underbrace{\widetilde{H}_i((S^n \setminus h(S^{k+1}_{\le 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\le 0} \cong \overline{B}^{k+1}}} \oplus \underbrace{\widetilde{H}_i(S^n \setminus h(S^{k+1}_{\ge 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\ge 0} \cong \overline{B}^{k+1}}} \to \widetilde{H}_i(S^n \setminus h(S^{k+1})) \to \dots.$$

Thus we see that

$$\widetilde{H}_i(S^n \setminus h(S^{k+1})) \cong \widetilde{H}_{i-1}(S^n \setminus h(S^k)) \underset{\substack{\uparrow \\ \text{by induction}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i-1 = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

□

The following immediate corollary to Proposition 36.1 says that every closed curve in $\mathbb{R}^2$, i.e. the image of an injective map $S^1 \to \mathbb{R}^2$, separates S^2 into two path-components. This

[706] Why does this follow? What results from real analysis do we need to draw that conclusion?

[707] Can you give a rigorous proof for the statement that the complement of two points in S^n is homotopy equivalent to S^{n-1}?

[708] Here we use, as in the proof of Proposition 36.1 (1), implicitly the injectivity of h.

might be intuitively obvious, but it is actually non-trivial.[709] The theorem was first proved in 1887 by Camille Jordan.[710]

Corollary 36.2. (Jordan Curve Theorem) *If C is a subset of $\mathbb{R}^2$ that is homeomorphic to S^1, then $\mathbb{R}^2 \setminus C$ consists of two path-connected components.*

Proof. Let C be a subset of $\mathbb{R}^2$ that is homeomorphic to S^1. We pick an injective map $h\colon S^1 \to S^2 = \mathbb{R}^2 \cup \{\infty\}$ with $h(S^1) = C$. Now we see that

$$\begin{array}{ccccccc} \widetilde{H}_0(\mathbb{R}^2 \setminus C) & = & \widetilde{H}_0(\mathbb{R}^2 \setminus h(S^1)) & \cong & \widetilde{H}_0(S^2 \setminus h(S^1)) & \cong & \mathbb{Z} \\ & & & \uparrow & & \uparrow & \\ & & & \text{see Exercise 33.5} & & \text{Proposition 36.1 (2)} & \end{array}$$

It follows from Lemma 31.1 (5) that $\mathbb{R}^2 \setminus C$ has precisely two path-components. □

Proposition 36.1, together with Lemma 31.1, says in particular, that if $h\colon S^{n-1} \to S^n$ is an injective map, then

$$H_i(S^n \setminus h(S^{n-1})) \cong \begin{cases} \mathbb{Z}^2, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, homologically the topological space $S^n \setminus h(S^{n-1})$ looks like the disjoint union of two open n-dimensional balls. One might now suspect that $S^n \setminus h(S^{n-1})$ is in fact homeomorphic to the union of two open n-dimensional balls.

This is in fact true in dimension $n = 2$. More precisely Arthur Schönflies[711] proved in 1908 the following theorem. We refer to [Tn, Section 3] for a proof of the theorem.[712]

Theorem 36.3. (Schönflies Theorem) *Let $h\colon S^1 \to S^2$ be an injective map. Then h extends to a homeomorphism of S^2, i.e. there exists a homeomorphism $H\colon S^2 \to S^2$ such that $H|_{S^1} = h$. In particular both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks.*

Remark. If $h\colon S^1 \to S^2$ is an embedding, then the conclusion of the Schönflies Theorem 36.3 also follows from the much easier Proposition 40.16.

It is natural to ask whether the conclusion of the Schönflies Theorem also holds in higher dimensions. But this hope breaks down already one dimension higher. More precisely, James Alexander [713] showed in 1924 that there exists an injective map $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected. The

[709] Try to give an elementary proof for the statement.

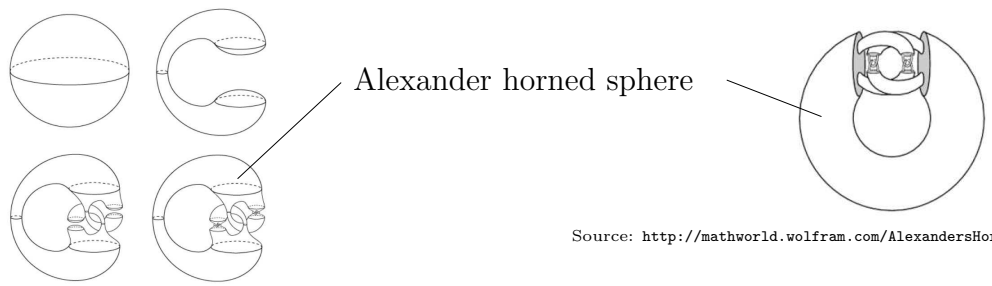
[710] Camille Jordan (1838-1922) was a French mathematician. It is the same Jordan whom you encountered in linear algebra.

[711] Arthur Schönflies (1853–1928) was a German mathematician.

[712] More precisely, in [Tn, Section 3] a proof for the following statement is given:
Let $g\colon S^1 \to \mathbb{R}^2$ be an injective map. Then g extends to a homeomorphism of $\mathbb{R}^2$.
We leave it as an exercise to deduce Theorem 36.3 from that statement.

[713] James Alexander (1888-1971) was an American mathematician. He also introduced the Alexander polynomial of a knot.

image $h(S^2)$ is called the *Alexander horned sphere*. It is illustrated in Figure 552. A more detailed description of the Alexander horned sphere and the proof that the complement is not simply connected is given in [Al] and also [Hat, p. 170].



Source: https://conan777.wordpress.com/2010/04/18/

Source: http://mathworld.wolfram.com/AlexandersHornedSphere.html

FIGURE 552.

We have now seen that injective maps $S^2 \to S^3$ can be pretty "wild". The map that defines the Alexander horned sphere is continuous, but it is certainly not smooth. So what happens if we restrict ourselves to smooth maps? In fact in the smooth setting we have the following theorem.

Theorem 36.4. (Generalized smooth Schönflies Theorem) *Let $h\colon S^{n-1} \to S^n$ be an embedding. If $n \neq 4$, then there exists a diffeomorphism $H\colon S^n \to S^n$ with $H|_{S^{n-1}} = h$.*

For $n = 2$ the statement follows fairly easily from the classification of smooth 2-dimensional manifolds that we gave in Theorem 14.14, see e.g. [Hirs, p. 207]. For $n = 3$ this statement was proved by James Alexander [Al2], see also [Hat3, Chapter 1] for an exposition of the proof. For $n \geq 5$ this statement was independently proved by Barry Mazur [Maz] in 1959 and by Morton Brown [BrownM] in 1960.

The case $n = 4$ is missing in the Generalized Schönflies Theorem. In fact it is still unknown whether the statement holds for $n = 4$.

Question 36.5. *Is it true that given any embedding $h\colon S^3 \to S^4$ there exists a diffeomorphism $H\colon S^4 \to S^4$ with $H|_{S^3} = h$?*

This question is very much open and there is not even a consensus whether one expects the answer to be yes or no. If you can answer this question you would almost certainly obtain a Fields medal.[714]

We conclude this section with two more consequences of Proposition 36.1. The following theorem was first proved by Luitzen Brouwer [Brou] in 1911. It can be viewed as a significant strengthening of Proposition 3.20 (1).

[714]But hurry up, you can only get a Fields medal if you are under 40.

Theorem 36.6. (Invariance of Domain Theorem) *Let U be an open subset of $\mathbb{R}^n$ and let $h\colon U \to \mathbb{R}^n$ be an injective map. Then $h(U)$ is also an open subset of $\mathbb{R}^n$.*[715]

Remark. The Open Mapping Theorem from complex analysis, see [Lan, Theorem II.6.2], says that if $G \subset \mathbb{C}$ is an open subset and if $f\colon G \to \mathbb{C}$ is a non-constant *holomorphic* function, then $f(G)$ is also open. In fact, one can deduce the "Open Mapping Theorem" from the Invariance of Domain Theorem 36.6 since by [Jä2, Satz 10] every non-constant holomorphic function on an open subset of $\mathbb{C}$ is o-locally the power of an injective function.

In the proof of Theorem 36.6 we need the following elementary lemma.

Lemma 36.7. *Let $U \subset \mathbb{R}^n$ be a subset. If U is open in $\mathbb{R}^n$, then every path-component of U is also an open subset of $\mathbb{R}^n$.*

Proof. Let C be a path-component of the subset subset $U \subset \mathbb{R}^n$. Let $P \in C$. Since U is open there exists an $r > 0$ such that $B_r(P) \subset U$. But all points in $B_r(P)$ are path-equivalent to P, hence $B_r(P) \subset C$. □

Now we can provide the proof of Theorem 36.6.

Proof of Theorem 36.6. By the homeomorphism from Lemma 1.41 we can view $\mathbb{R}^n$ as an open subset of $\mathbb{R}^n \cup \{\infty\} = S^n$. So we can view h as a map from U to S^n and we need to show that $h(U)$ is an open subset of S^n.

Let $x \in U$. We pick a closed n-ball B in U that contains x in the interior. It suffices to show that $h(B \setminus \partial B) \subset h(U)$ is open in S^n. Note that $S^n \setminus h(\partial B)$ is open by Lemma 1.37 (3). By Lemma 36.7 it therefore suffices to show that $h(B \setminus \partial B)$ is a path-component of $S^n \setminus h(\partial B)$. We will do so in the following claim.

Claim. The topological space $S^n \setminus h(\partial B)$ has precisely two path-components namely the subsets $h(B \setminus \partial B)$ and $S^n \setminus h(B)$.

By Proposition 36.1 (2) and Lemma 31.1 we know that

$$S^n \setminus h(\partial B) \;=\; h(B \setminus \partial B) \sqcup (S^n \setminus h(B))$$

has two path-components. It suffices to show that the two sets on the right-hand side are path-connected. The first set $h(B \setminus \partial B)$ is path-connected since it is the image of the path-connected topological space $B \setminus \partial B$, while the second set $S^n \setminus h(B)$ is path-connected by Proposition 36.1 (1) and Lemma 31.1. This concludes the proof of the claim. □

Corollary 36.8. *Let M be a non-empty closed n-dimensional topological manifold and let N be a connected n-dimensional topological manifold without boundary. Let $h\colon M \to N$ be a map.*

(1) *If $h\colon M \to N$ is o-locally injective, then h is also surjective.*
(2) *If h is injective, then h is already a homeomorphism.*

[715] How can one prove the theorem for $n = 1$ using only the methods from real analysis?

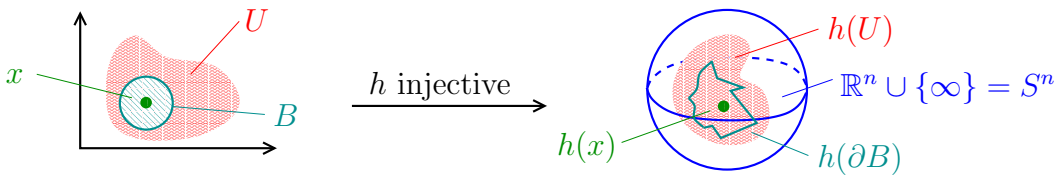


FIGURE 553. Illustration for the proof of Theorem 36.6.

Example. Corollary 36.8 implies in particular that there exists no injective map from the closed manifold S^n into the manifold $\mathbb{R}^n$.[716]

Proof. The second statement of the corollary is an immediate consequence of the first statement and Proposition 1.40. So it suffices to prove the first statement.

Let $h\colon M \to N$ be a strongly locally injective map from a non-empty closed n-dimensional topological manifold M to a connected n-dimensional topological manifold N without boundary.

The image $h(M)$ is closed in N since M is compact and since N is Hausdorff. Since N is connected and since M is non-empty it suffices to show that $h(M)$ is also open in N.

So let $Q \in h(M)$. We pick $P \in M$ with $h(P) = Q$. We pick a chart $\Phi\colon U \to V$ around P and we pick a chart $\Psi\colon W \to X$ around Q. After possibly replacing U by $U \cap h^{-1}(W)$ we can assume that $h(U) \subset W$. Since h is o-locally injective we can, after possibly shrinking U even more, assume that $h|_U$ is injective.

From Theorem 36.6 we know that $(\Psi \circ h \circ \Phi^{-1})(V)$ is an open subset of $\mathbb{R}^n$. Since Ψ is a homeomorphism we see that $h(\Phi^{-1}(V)) = h(U)$ is an open neighborhood of Q that is contained in $h(M)$. $\square$

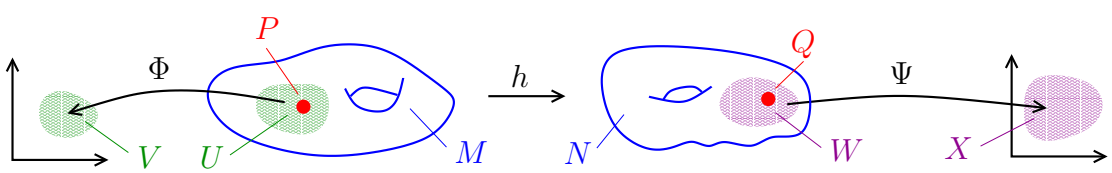


FIGURE 554. Illustration for the proof of Corollary 36.8.

Exercises for Chapter 36.

Exercise 36.1. In Figure 555 we see the image of an injective map $\varphi\colon I \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a curve $C \subset S^3 \setminus \varphi(I)$. We know from Proposition 36.1 (1) that x is null-homologous. Show that x is the boundary of an orientable surface Σ that is embedded in $S^3 \setminus \varphi(I)$. (In this context this can be seen as the converse to Exercise 35.7.)

[716] How can you show this for $n = 1$, just using methods from real analysis?

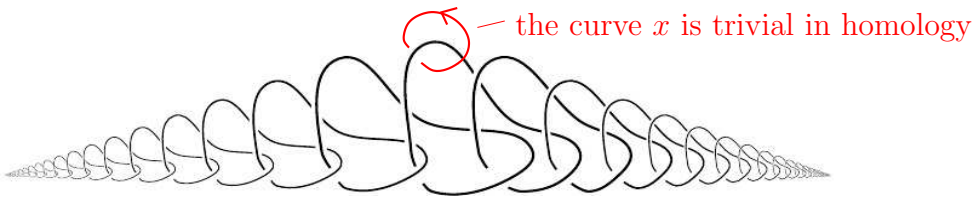


FIGURE 555.

37. Topological robotics

In this chapter we will first study the matrix groups SO(3) and SU(2), viewed as topological spaces. In particular we will determine $\pi_1(\mathrm{SO}(3))$ and we will use this result to give an application to topological robotics.

37.1. **The matrix groups** SO(3) **and** SU(2)**.** We recall that Lemma 3.45 implies in particular that we can view SO(3) and SU(2) as closed 3-dimensional manifolds. In this section we will see that the manifolds SO(3) and SU(2) are diffeomorphic to manifolds we are already very familiar with.

We start out our discussion of SO(3) and SU(2) with some definitions. We consider the set of matrices[717]

$$T \;:=\; \big\{B \in \mathrm{SL}(2,\mathbb{C}) \,\big|\, \mathrm{tr}(B) = 0 \text{ and } B = \overline{B}^T\big\}.$$

Note that T is a 3-dimensional real vector space. We equip it with the basis given by the so-called "Pauli matrices"[718]

$$E_1 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and we equip T with the inner product given by

$$\begin{array}{rcl} T \times T & \mapsto & \mathbb{R} \\ (A,B) & \mapsto & g(A,B) := \frac{1}{2}\,\mathrm{tr}\left(A\overline{B}^T\right). \end{array}$$

We consider the isomorphism

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^3 & \to & T \\ e_i & \mapsto & E_i. \end{array}$$

It follows from a straightforward calculation that Φ defines an isometry from $(\mathbb{R}^3, \langle\,,\,\rangle)$ to (T,g).[719] It follows immediately from this observation that Φ induces an isomorphism $\mathrm{SO}(3) \to \mathrm{SIsom}(T,g)$, where $\mathrm{SIsom}(T,g)$ denotes the group of isometries of (T,g) of determinant $+1$. We use this isomorphism to identify these two groups.

Lemma 37.1. *The map*

$$\begin{array}{rcl} q\colon \mathrm{SU}(2) & \to & \mathrm{SIsom}(T,g) = \mathrm{SO}(3) \\ A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a smooth map between manifolds and it is a well-defined group homomorphism. Furthermore for $A, A' \in \mathrm{SU}(2)$ we have

$$q(A) \;=\; q(A') \quad \Longleftrightarrow \quad A \;=\; \pm A'.$$

[717]Recall that the trace $\mathrm{tr}(A)$ of a matrix is defined as the sum of all the diagonal entries.

[718]Wolfgang Pauli (1900–1958) was Swiss-American theoretical physicist and one of the pioneers of quantum physics. These matrices were also independently introduced by Paul Dirac. Wikipedia write: "In 1928, building on 2×2-spin matrices which he [Paul Dirac] discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, [...] he proposed the Dirac equation as a relativistic equation of motion for the wave function of the electron. This work led Dirac to predict the existence of the positron, the electron's antiparticle [...]."

[719]In the calculation we only have to show that $g(E_i, E_j) = \frac{1}{2}\,\mathrm{tr}(E_i\overline{E}_j^T) = \delta_{ij} = \langle e_i, e_j\rangle$ for all choices of $i, j \in \{1,2,3\}$. But this can be verified easily.

Example. Given $t \in \mathbb{R}$ we consider the matrix

$$A = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \in \mathrm{SU}(2).$$

We want to determine $q(A) \in \mathrm{SO}(3)$. This means that we have to express the linear map $T \to T$ that is given by $B \mapsto ABA^{-1}$ with respect to the ordered basis $\{E_1, E_2, E_3\}$. A straightforward calculation shows that

$$\begin{aligned} AE_1A^{-1} &= E_1, \\ AE_2A^{-1} &= \begin{pmatrix} 0 & e^{-2it} \\ e^{2it} & 0 \end{pmatrix} \underset{\substack{\uparrow \\ \text{since } e^{is} = \cos(s) + i\sin(s) \\ \downarrow}}{=} \cos(2t) \cdot E_2 + \sin(2t) \cdot E_3, \\ AE_3A^{-1} &= \begin{pmatrix} 0 & -ie^{-2it} \\ ie^{2it} & 0 \end{pmatrix} = -\sin(2t) \cdot E_2 + \cos(2t) \cdot E_3. \end{aligned}$$

But this implies that the homomorphism $T \to T$ given by conjugation with A, with respect to the ordered basis $\{E_1, E_2, E_3\}$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}, \text{ put differently, we have } q\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

Proof of Lemma 37.1 $(*)$. First we have to show that for $A \in \mathrm{SU}(2)$ and $B \in T$ we also have $ABA^{-1} \in T$. This is indeed the case since

$$\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B) = 0 \quad \text{and} \quad \left(\overline{ABA^{-1}}\right)^T = (\overline{A}^{-1})^T \overline{B}^T \overline{A}^T \underset{\substack{\uparrow \\ \text{since } B = \overline{B}^T \text{ and } A^{-1} = \overline{A}^T}}{=} A\overline{B}^T A^{-1}.$$

It follows easily from the definition that q is a group homomorphism. We also have to show that for $A \in \mathrm{SU}(2)$ we get indeed an isometry of (T, g). So let $B, C \in T$. Then we have indeed

$$\begin{aligned} g(q(A)(B), q(A)(C)) &= \tfrac{1}{2}\,\mathrm{tr}\left(ABA^{-1} \cdot \overline{ACA^{-1}}^T\right) = \tfrac{1}{2}\,\mathrm{tr}\left(AB\overbrace{A^{-1} \cdot (\overline{A}^{-1})^T}^{=\mathrm{id}}\overline{C}^T\overline{A}^T\right) \\ &= \tfrac{1}{2}\,\mathrm{tr}\left(AB\overline{C}^T\overline{A}^T\right) = \tfrac{1}{2}\,\mathrm{tr}\left(B\overline{C}^T\underbrace{\overline{A}^T A}_{=\mathrm{id}}\right) = \tfrac{1}{2}\,\mathrm{tr}\left(B \cdot \overline{C}^T\right). \end{aligned}$$

This shows that q defines group homomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$. By Lemma 1.96 we know that $\mathrm{SU}(2)$ is path-connected. The image $q(\mathrm{SU}(2))$ is a path-connected subset of $\mathrm{O}(3)$ that contains $q(\mathrm{id}) = \mathrm{id}$. It follows again Lemma 1.96 that $q(\mathrm{SU}(2))$ is contained in $\mathrm{SO}(3)$.

Finally let $A, A' \in \mathrm{SU}(2)$. If $A = -A'$ then we clearly have $q(A) = q(A')$. Now suppose that conversely we have $q(A) = q(A')$. We then have in particular $q(A)(E_i) = q(A')(E_i)$ for $i = 1, 2, 3$, i.e. we have $AE_iA^{-1} = A'E_i(A')^{-1}$ for $i = 1, 2, 3$. These equalities give us equations relating the four entries of the matrix A to the four entries of the matrix A'. An elementary calculation then shows that the only solutions are $A = \pm A'$. We leave the details to the reader.

This discussion shows that q is a well-defined map and that it is a group homomorphism. After one has unraveled the definitions it will also have become apparent that the map g is indeed a smooth map between manifolds. $\square$

The following theorem is the main result of this section.

Theorem 37.2. *We denote by* $q\colon \mathrm{SU}(2) \to \mathrm{SIsom}(T, g) = \mathrm{SO}(3)$ *the map from Lemma 37.1. Then the following two statements hold:*

(1) *the map q is a 2-fold covering map,*
(2) *there exist diffeomorphisms $f\colon S^3 \to \mathrm{SU}(2)$ and $g\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ such that the following diagram commutes:*

$$\begin{tikzcd} S^3 \arrow[r, "f", "\cong"'] \arrow[d, "p"'] & \mathrm{SU}(2) \arrow[d, "q"] \\ \mathbb{R}\mathrm{P}^3 \arrow[r, "g", "\cong"'] & \mathrm{SO}(3), \end{tikzcd}$$

where we denote by $p\colon S^3 \to \mathbb{R}\mathrm{P}^3 = S^3/\pm 1$ the usual projection map.

In the proof of Theorem 37.2 we will need the following technical lemma.

Lemma 37.3. *Let $q\colon M \to N$ be a map between two n-dimensional topological manifolds. Suppose the following conditions are satisfied:*

(1) *the map q is surjective,*
(2) *the map q is o-locally injective, i.e. for each $P \in M$ there exists a neighborhood U such that $q|_U$ is injective,*
(3) *for each point $Q \in N$ the preimage $q^{-1}(Q)$ is a finite set.*

Then q is a covering map.

Proof $(*)$. Since q is surjective we only have to show that each $Q \in N$ admits an open neighborhood that is uniformly covered. So let $Q \in N$. We denote its preimages by $P_1, \ldots, P_r$. Let $i \in \{1, \ldots, m\}$. It follows easily from (2) and the hypothesis that M is a topological manifold that we can pick a closed ball B_i which is a neighborhood around P_i and such that $q\colon B_i \to N$ is injective. Since the topological manifold N is in particular Hausdorff it follows from Proposition 1.40 that $q\colon B_i \to q(B_i)$ is a homeomorphism. It follows from the proof of Corollary 36.8 that $q(\overset{\circ}{B}_i)$ is an open neighborhood of Q. It is now straightforward to see that $U := \bigcap_{i=1}^{m} q(\overset{\circ}{B}_i)$ is a uniformly covered open neighborhood of Q. $\square$

Now we can provide the proof of Theorem 37.2.

Proof of Theorem 37.2. Somewhat confusingly the proof of Theorem 37.2 is given in three steps (a), (b) and (c).

(a) One can easily show that the map

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w) & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{f}\colon \mathrm{SU}(2) & \to & S^3 \\ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} & \mapsto & (z,w) \end{array}$$

are inverses to one another and that both maps are smooth. This implies that both maps, in particular f, is a diffeomorphism.

(b) Now we want to show that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering map. By Lemma 37.1 we know that q is o-locally injective. But then it follows from Corollary 36.8 and the fact that SU(2) and SO(3) are closed 3-dimensional manifolds and the fact that SO(3) is connected (which follows from the Lemma 1.96 (1) together with Lemma 1.102) that q is in fact also a surjection. Since the preimage of each $A \in \mathrm{SO}(3)$ consists of precisely two points in SU(2) we can apply Lemma 37.3 to deduce that q is indeed a 2-fold covering map.

(c) For any $(z,w) \in S^3$ we have $q(f(-z,-w)) = q(-f(z,w)) = q(f(z,w))$. Thus we see that the map

$$\begin{array}{rcl} g\colon \mathbb{R}\mathrm{P}^3 & \to & \mathrm{SO}(3) \\ [(z,w)] & \mapsto & q(f(z,w)) \end{array}$$

is well-defined. It follows immediately from the fact, obtained in (b), that q is surjective, that g is also surjective. Similarly it is straightforward to show, using (b), that the map g is injective. Since q, f and p are local diffeomorphisms and since locally g is given by a composition of q, f, p and their local inverses we see that g is a local diffeomorphism. But by Lemma 3.15 (3) map between manifolds that is a local diffeomorphism and a bijection is in fact a diffeomorphism.

Finally we note that it follows easily from the definitions that $g \circ p = q \circ f$, i.e. the diagram given in the theorem commutes. □

Remark. An elegant direct proof that $\mathbb{R}\mathrm{P}^3 = \overline{B}^3/\sim$ is homeomorphic to SO(3) is given at wikipedia:

`https://en.wikipedia.org/wiki/Rotation_group_SO(3)`

under "topology".

From Theorem 37.2 and Corollary 7.19 we obtain immediately the following, perhaps initially somewhat surprising, corollary.

Corollary 37.4. *We have*

$$\pi_1(\mathrm{SU}(2)) \;=\; 0 \quad \textit{and} \quad \pi_1(\mathrm{SO}(3)) \;\cong\; \mathbb{Z}_2.$$

In particular SU(2) *is the universal covering of* SO(3).[720]

[720]The fact SU(2) is the universal covering of SO(3) plays an important role in theoretical physics and it is essential in understanding the notion of "spin" of an electron. At least that is a slogan that I remember from the days when I studied physics.

Remark. We can actually give an explicit description of the non-trivial element in the fundamental group $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. More precisely, we consider the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$. The calculation on page 912 shows that a lift of the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ to a path $[0,1] \to \mathrm{SU}(2)$ to the starting point $\mathrm{id} \in \mathrm{SU}(2)$ is given by

$$\begin{array}{rcl} \widetilde{\gamma}\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} e^{-i\pi t} & 0 \\ 0 & e^{i\pi t} \end{pmatrix}. \end{array}$$

Note that $\widetilde{\gamma}(1) = -\mathrm{id}$, i.e. $\widetilde{\gamma}$ is *not* a loop. Since $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering and since $\mathrm{SU}(2)$ is simply connected it follows from Lemma 7.16 that $[\gamma]$ is non-trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

Corollary 37.4 also raises the following question.

Question 37.5. *Let $n \geq 4$. What is $\pi_1(\mathrm{SO}(n))$?*[721][722]

We will return to this question later on in a later chapter.

37.2. **The belt trick.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ using a belt. We proceed as follows.

(1) First of all note that a matrix in $\mathrm{SO}(3)$ is precisely given by an ordered orthonormal basis, i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$.
(2) We equip $\mathrm{SO}(3)$ with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.
(3) Now suppose we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.
(4) Any point on a twisted belt defines a triple of orthogonal vectors:
    (a) the first vector lies in the plane defined by the belt and points away from the buckle,
    (b) the second vector also lies in the plane defined by the belt and points towards the red edge,
    (c) the third is orthogonal to the belt and it is attached to the top side of the belt.
(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.

[721] Recall that Lemma 1.96 (1) says that $\mathrm{SO}(n)$ is path-connected.
[722] What is $\pi_1(\mathrm{SO}(2))$?

(6) We say that the end of the belt is in standard position if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to it is the identity matrix.

(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0,1] \to \mathrm{SO}(3)$. In particular, if the end of the belt is in standard position, then the path defined by the belt is in fact a loop in $\mathrm{SO}(3)$ at the base point id.

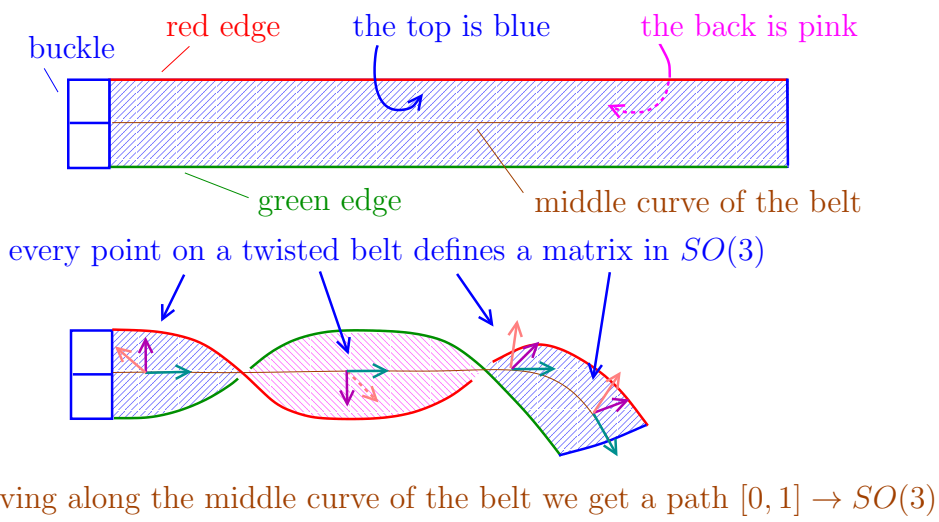


FIGURE 556.

(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.

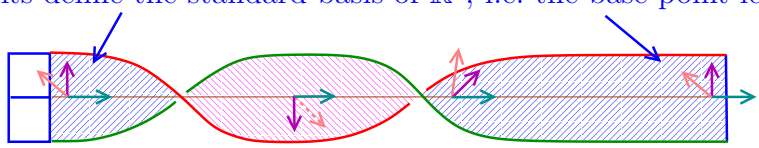
(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0,1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely the loop that we had considered on page 915. In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}_2$.

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from Corollary 37.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$, i.e. $\gamma * \gamma$ is null-homotopic.

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in Figure 559.

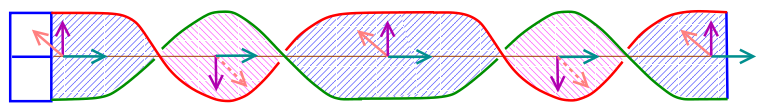
The fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ lies at the heart of some other tricks, in particular the "plate trick" and "Dirac's string trick". More information and some videos can be found

both points define the standard basis of $\mathbb{R}^3$, i.e. the base point $\mathrm{id} \in SO(3)$



the path $\gamma\colon [0,1] \to SO(3)$ defined by the matrices along the middle curve is a loop and thus defines an element $[\gamma] \in \pi_1(SO(3), \mathrm{id})$

FIGURE 557.



the double full twist defines the loop $\gamma * \gamma$ which represents $[\gamma]^2 \in \pi_1(SO(3), \mathrm{id})$

FIGURE 558.

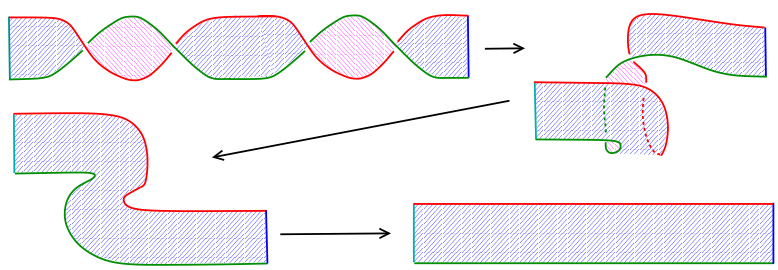


FIGURE 559.

here:

https://en.wikipedia.org/wiki/Plate_trick
https://www.youtube.com/watch?v=JaIR-cWk_-o
https://www.youtube.com/watch?v=JDJKfs3HqRg
https://www.youtube.com/watch?v=ICEIgznuHmg

37.3. **Topological robotics.** In this section we want to discuss a "real-life" application of Corollary 37.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally

(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.

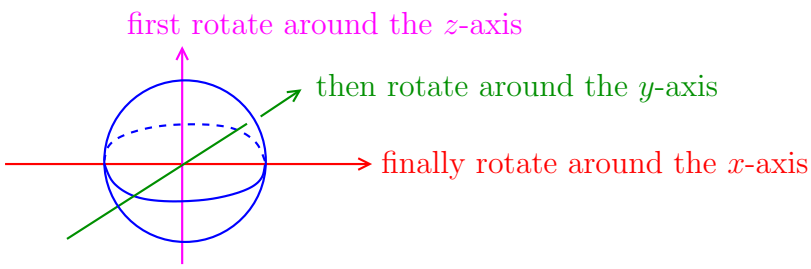


FIGURE 560.

Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:

Lemma 37.6. *The map*

$$\begin{array}{rcl} \Phi\colon S^1 \times S^1 \times S^1 & \to & \mathrm{SO}(3) \\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\textit{rotation by } \alpha \\ \textit{around the } x\textit{-axis}\end{array} \circ \begin{array}{c}\textit{rotation by } \beta \\ \textit{around the } y\textit{-axis}\end{array} \circ \begin{array}{c}\textit{rotation by } \gamma \\ \textit{around the } z\textit{-axis}\end{array} \end{array}$$

is a surjection.[723]

Remark. One can use Lemma 37.6 and a variation of the argument on page 912 to give an alternative proof for the statement that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective.

Sketch of proof. Let $A \in \mathrm{SO}(3)$ be matrix with columns v_1, v_2, v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1 = e_1$. Since $v_1 = e_1, v_2, v_3$ is an orthonormal basis we see that v_2, v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2 = e_2$ and $v_3 = e_3$. $\square$

This lemma gives rise to the following practical problem: given a matrix $P \in \mathrm{SO}(3)$, can we continuously find parameters $\alpha(P), \beta(P), \gamma(P) \in S^1$ such that these parameters give back the matrix P? More precisely, we have the following question.

Question 37.7. *Does there exist a continuous map* $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ *such that* $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$*?*

It turns out that the answer is negative. More precisely, we have the following proposition.

Proposition 37.8. *There is no continuous map* $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ *such that* $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$.

[723]The map is also continuous. Why is that?

This proposition falls into the area of "topological robotics", more information on this example and on topological robotics can be found in [Gh, p. 76]. Other aspects of topological robotics are discussed in the monograph [Far].

Proof. Let us suppose there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram

$$\begin{tikzcd}
 & \overbrace{\pi_1(S^1 \times S^1 \times S^1)}^{\cong \mathbb{Z}^3} \arrow[dr, "\Phi_*"] & \\
\underbrace{\pi_1(\mathrm{SO}(3))}_{\cong \mathbb{Z}_2} \arrow[ur, "\Psi_*"] \arrow[rr, "\mathrm{id}_*"] & & \underbrace{\pi_1(\mathrm{SO}(3))}_{\cong \mathbb{Z}_2}.
\end{tikzcd}$$

By Corollary 37.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ and from Corollary 7.19 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. $\square$

37.4. **Linkages.** We conclude this chapter with a somewhat unrelated but fun topic that also appears in topological robotics.

(1) We recall that an abstract graph G is a triple (V, E, λ) where V is a non-empty set, E is a set and λ is a map

$$\lambda\colon E \ \to \ \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of* G and the elements of E are called the *edges of* G. Furthermore, given $e \in E$ the points in $\lambda(e)$ are called the *endpoints of* e.

(2) Given $n \in \mathbb{N}$ we define the *linear graph of length* n to be the graph $G(n)$ which is given by the vertex set $V = \{0, \dots, n\}$ and the edge set $\{e_0, \dots, e_{n-1}\}$ with $\lambda(e_i) = \{i, i+1\}$ for $i = 0, \dots, n-1$. This graph is illustrated in Figure 561.

the linear graph $G(4)$

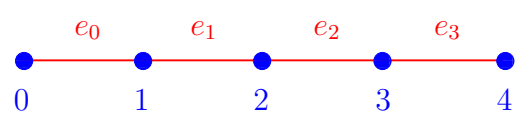


FIGURE 561.

Definition.

(1) A *planar linkage* is a triple $(G = (V, E), \lambda, \psi)$ consisting of a finite abstract graph $G = (V, E)$, a function $\lambda\colon E \to \mathbb{R}_{>0}$ and a map $\psi\colon W \to \mathbb{C} = \mathbb{R}^2$ where W is a subset of V. Put differently, λ assigns a length to each edge and ψ represents a subset of the vertices by points in $\mathbb{C}$.

(2) A *configuration* of a planar linkage $(G = (V, E), \lambda, \psi)$ is a map $f\colon V \to \mathbb{R}^2 = \mathbb{C}$ such that $f|_W = \psi$ and such that for every edge e the distance between the endpoints of e is $\lambda(e)$.[724]

(3) The set $C(G = (V, E), \lambda, \psi)$ of all configurations is called the *configuration space* of the linkage. We have an injective map

$$\begin{array}{rcl} C(G = (V, E), \lambda, \psi) & \to & \mathbb{C}^{|V|} := \text{maps}(V \to \mathbb{C}) \\ f & \mapsto & (v \mapsto f(v)) \end{array}$$

thus we can view the configuration space $C(G = (V, E), \lambda, \psi)$ as a topological space.

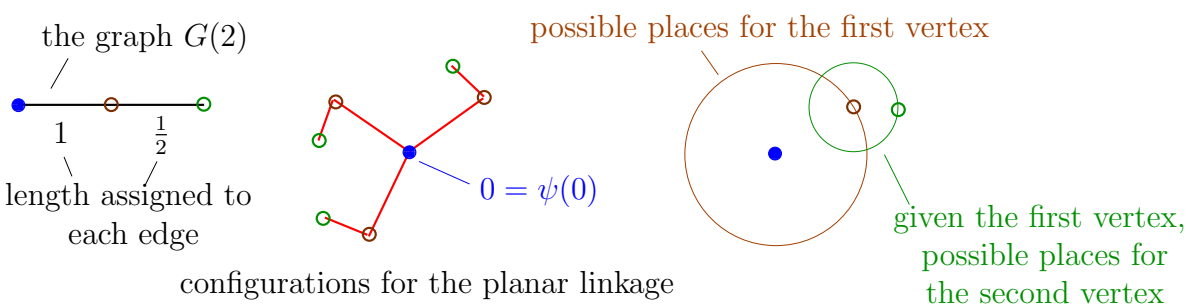


FIGURE 562.

In some cases the topological shape of the configuration space of a planar linkage is easy to describe. More precisely, we have the following lemma which is a straightforward consequence of the definitions and Proposition 1.40. We leave the proof as an exercise.

Lemma 37.9. *Let $G(m)$ be the linear graph with m edges. We consider a planar linkage of the form $(G(m), \lambda, \psi(0) = 0)$. For $i = 1, \dots, m$ we write $r_i = \lambda(e_i)$. Then the map*

$$\begin{array}{rcl} (S^1)^m = S^1 \times \cdots \times S^1 & \to & C(G(m), \lambda, \psi) \\ (z_1, \dots, z_m) & \mapsto & \left(\begin{array}{rcl} \{0, \dots, m\} & \to & \mathbb{C} \\ k & \mapsto & \sum_{i=1}^{k} r_i z_i \end{array} \right) \end{array}$$

is a homeomorphism.

For general linkages the topology of the configuration spaces are much harder to describe than for free planar linkages.

(1) Given $r_1, \dots, r_m > 0$ we denote by $C(r_1, \dots, r_m, P, Q)$ the configuration space corresponding to the planar linkage defined by $G(m)$, $\lambda(e_i) = r_i$ and by the conditions that $\psi(0) = P$ and $\psi(m) = Q$.

[724] In particular if e has only one endpoint, then we need $\lambda(e) = 0$.

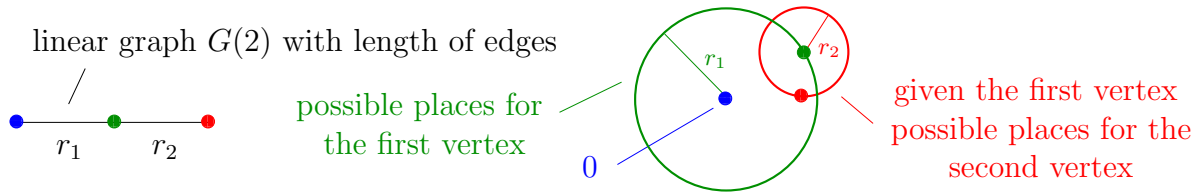


FIGURE 563. Sketch for the proof of Lemma 37.9.

(a) Using elementary arguments one can easily show that

$$C(r_1, r_2, P, Q) \;=\; \begin{cases} \varnothing, & \text{if } r_1 + r_2 < d(P,Q), \\ \text{one point}, & \text{if } r_1 + r_2 = d(P,Q), \\ \text{two points}, & \text{if } r_1 + r_2 > d(P,Q). \end{cases}$$

(b) Kevin Walker [Walkb, p. 35], see also [Gh, p. 12], showed in 1985 in his bachelor thesis that the surfaces of genus $0, 1, \ldots, 4$ are homeomorphic to configuration spaces of the type $(r_1, r_2, r_3, r_4, P, Q)$ for suitable values of $r_1, \ldots, r_4$ and points P and Q.

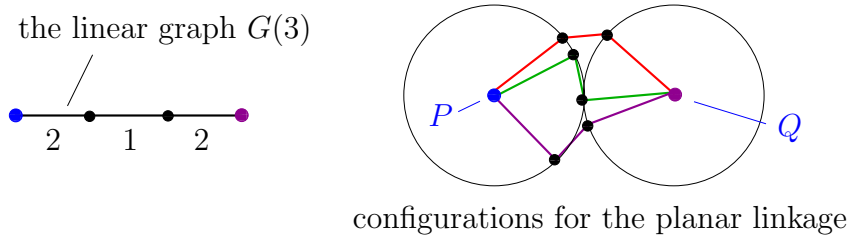


FIGURE 564.

(2) Michael Kapovich and John Millson [KaM] showed that given any compact manifold M there exists a planar linkage such that each component of the configuration space is homeomorphic to M. In particular this shows that any compact manifold can be represented by a 2-dimensional physical object.

38. The first homology group and the fundamental group

In this section we want to relate the first homology group to the fundamental group of a path-connected topological space X. Both are defined by mapping 1-dimensional manifolds into X, hence it is reasonable to expect that there is a relationship. But in general these two groups associated to a topological space cannot be the same, since the first homology group is by definition abelian, whereas we saw on many occasions that fundamental groups can be non-abelian.

38.1. **The Hurewicz homomorphism.** Let X be a topological space. Throughout this section we denote by Φ the following bijection:

$$\begin{array}{rcl} \Phi\colon \{\text{paths } [0,1]\to X\} & \to & \{\text{singular 1-simplices in } X\} \\ (\gamma\colon [0,1]\to X) & \mapsto & \begin{pmatrix} \Delta^1 & \to & X \\ (1-t,t) & \mapsto & \gamma(t) \end{pmatrix}. \end{array}$$

On many occasions we will use the trivial observation that for any path $\gamma\colon [0,1]\to X$ we have $\Phi(\overline{\gamma}) = \overline{\Phi(\gamma)}$.

We start out with the following lemma.

Lemma 38.1. *Let X be a topological space. If f and g are two paths in X such that the endpoint of f agrees with the starting point of g, then the singular 1-chains $\Phi(f)+\Phi(g)$ and $\Phi(f*g)$ are homologous.*

Proof. The lemma follows quite easily from Lemma 29.3. We refer to Figure 565 for a sketch of the proof. We leave it as an exercise to fill in the details. □

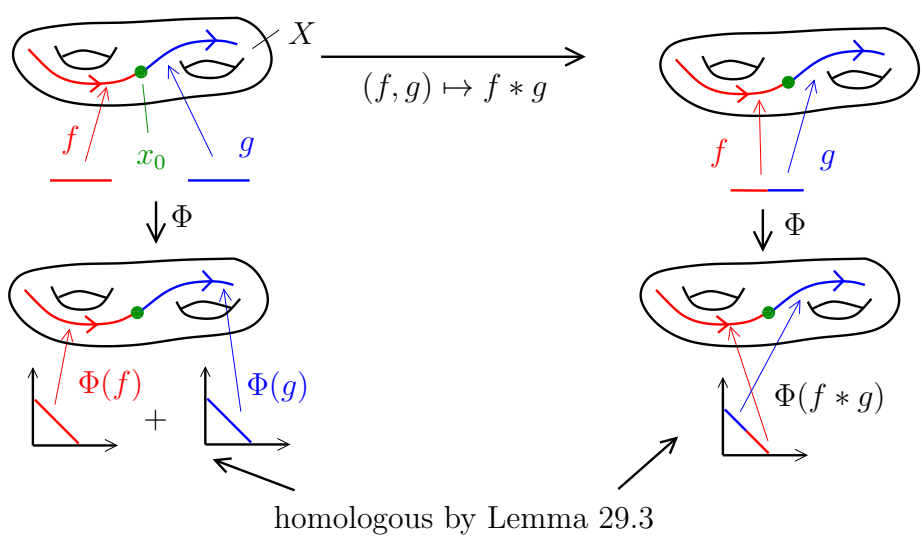


Figure 565. Illustration of the proof of Lemma 38.1.

Now we can easily prove the following proposition.

Proposition 38.2. *Let (X, x_0) be a pointed topological space.*

(1) *If γ is a loop in (X, x_0), then $\Phi(\gamma) \in C_1(X)$ is a cycle.*
(2) *The map Φ induces a well-defined map*

$$\begin{array}{rcl} \Phi_{(X,x_0)} \colon \pi_1(X, x_0) & \to & H_1(X) \\ [\gamma] & \mapsto & [\Phi(\gamma)]. \end{array}$$

(3) *The map $\Phi_{(X,x_0)} \colon \pi_1(X, x_0) \to H_1(X)$ is a group homomorphism.*
(4) *Let $\mathcal{P}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. The Hurewicz homomorphisms*

$$\Phi_{(X,x_0)} \colon \pi_1(X, x_0) \ \to \ H_1(X)$$

define a natural transformation from the functor

$$\begin{array}{rcl} \mathcal{P} & \to & \mathcal{G} \\ (X, x_0) & \mapsto & \pi_1(X, x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{P} & \to & \mathcal{G} \\ (X, x_0) & \mapsto & H_1(X). \end{array}$$

In plain English this means the following, given a map $f \colon X \to Y$ between topological spaces and given $x_0 \in X$ the following diagram commutes:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\Phi_{(X,x_0)}"'] & \pi_1(Y, f(x_0)) \arrow[d, "\Phi_{(Y,f(x_0))}"] \\ H_1(X) \arrow[r, "f_*"] & H_1(Y). \end{tikzcd}$$

Proof.

(1) The first statements follows immediately from the definitions and the description of $\partial_1 \colon C_1(X) \to C_0(X)$ given on page 738.
(2) We denote by $\sim$ the equivalence relation on $[0, 1]$ that is generated by $0 \sim 1$. We denote by $\alpha \colon \Delta^1 \to [0, 1]/\sim$ the map that is given by $\alpha(1 - t, t) = [t]$. Note that α defines an element in $H_1([0, 1]/\sim)$. Now suppose that $f, g \colon [0, 1] \to X$ are two path-homotopic loops in x_0. We denote by $\widetilde{f}, \widetilde{g} \colon [0, 1]/\sim \to X$ the induced maps. Then

$$\begin{array}{ccccccccc} [\Phi(f)] & = & \big[\widetilde{f} \circ \alpha\big] & = & \widetilde{f}_*([\alpha]) & = & \widetilde{g}_*([\alpha]) & = & \big[\widetilde{g} \circ \alpha\big] & = & [\Phi(g)]. \\ & \uparrow & & & & \uparrow & & & & & \\ & \text{since } \Phi(f) \text{ and } \widetilde{f} \circ \alpha \text{ are the same map} & & & & \text{by Proposition 30.5, since } \widetilde{f} \text{ and } \widetilde{g} \text{ are homotopic maps } [0,1]/\sim \to X & & & & & \end{array}$$

(3) The third statement is an immediate consequence of Lemma 38.1.
(4) The last statement follows easily from the definitions. □

[724]Recall that we denote by $\mathcal{P}$ the category of pointed topological spaces and that we denote by $\mathcal{G}$ the category of groups.

Definition. Given a pointed topological space (X, x_0) we refer to the homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & H_1(X) \\ [\gamma] & \mapsto & [\Phi(\gamma)] \end{array}$$

as the *Hurewicz homomorphism.*[725]

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

(1) The Hurewicz homomorphism is in general not injective since $\pi_1(X, x_0)$ is in many cases a non-abelian group whereas $H_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.
(2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X, x_0)$ depends only on the path-component of x_0, while $H_1(X)$ "sees" all path-components of X. But as we will see in the next lemma, this turns out to be the only obstacle to Φ being surjective.

Proposition 38.3. *Let X be a topological space and let $x_0 \in X$. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \ \to \ H_1(X)$$

is surjective.

In the proof of of Proposition 38.3 we will need the following lemma.

Lemma 38.4. *Let X be a set and let $x_1, \dots, x_k$ and $y_1, \dots, y_l$ be elements in X. If*

$$\sum_{i=1}^{k} x_i \ - \ \sum_{i=1}^{l} y_i \ = \ 0 \ \in \ \mathbb{Z}^{(X)},$$

then $k = l$ and there exists a permutation $s \in S_k$ such that $y_i = x_{s(i)}$ for $i = 1, \dots, k$.

Proof $(*)$. Let X be a set and let $x_1, \dots, x_k$ and $y_1, \dots, y_l$ be elements in X such that

$$\underbrace{\sum_{i=1}^{k} x_i}_{=:a} \ = \ \underbrace{\sum_{i=1}^{l} y_i}_{=:b} \ \in \ \mathbb{Z}^{(X)}.$$

It follows from Lemma 11.1 that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x) = 1$ for all $x \in X$. By definition we have $\psi(a) = k$ and $\psi(b) = l$. So from $a = b$ we deduce that $k = l$.

It follows again from Lemma 11.1 that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x_k) = 1$ and $\psi(x) = 0$ for all $x \in X \setminus \{x_k\}$. By definition we have $\psi(a) > 0$. From $\psi(b) = \psi(a) > 0$ it follows that there exists an $n \in \{1, \dots, k\}$ such that $y_n = x_k$. We define $s(k) = n$. We obtain an equality

$$\sum_{i=1}^{k-1} x_i \ = \ \sum_{i \in \{1,\dots,k\} \setminus \{s(k)\}} y_i \ \in \ \mathbb{Z}^{(X)}$$

[725]Witold Hurewicz (1904-1956) was a Polish mathematician.

and we iterate this procedure to define $s(k-1), \dots, s(1)$. By construction the map $s\colon \{1,\dots,k\} \to \{1,\dots,k\}$ is a bijection $\square$

Now we can provide the proof of Proposition 38.3.

Proof of Proposition 38.3. We denote by Ψ the inverse of the above bijection

$$\Phi\colon \{\text{paths } [0,1] \to X\} \;\to\; \{\text{singular 1-simplices in } X\}$$

We pick a homology class $z \in H_1(X)$. Let $c = \sum_{i=1}^{k} a_i\sigma_i$ be a cycle in $C_1(X)$ that represents z. It follows from Lemma 29.3 that, after possibly replacing a singular chain σ_i by $\overline{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that the σ_i's are different, we can in fact suppose that $a_1 = \dots = a_k = 1$. Since c is a cycle we have

$$0 \;=\; \partial_1(c) \;=\; \partial_1\Big(\sum_{i=1}^{k}\sigma_i\Big) \;\underset{\substack{\uparrow \\ \text{see discussion on page 738}}}{=}\; \sum_{i=1}^{k}\sigma_i(0,1) - \sigma_i(1,0)$$

It follows from Lemma 38.4 that there exists a permutation $s \in S_k$ such that for each i the "endpoint" $\sigma_i(0,1)$ of σ_i is just the "starting point" $\sigma_{s(i)}(1,0)$ of $\sigma_{s(i)}$.

Since X is path-connected we can choose for each $i \in \{1,\dots,k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1,0)$ of σ_i.

We obtain the following equality in $H_1(X)$:

$$\begin{aligned}
c \;=\; \sum_{i=1}^{k}\sigma_i \;&\overset{\substack{\text{by Lemma 29.3 the singular chain} \\ \Phi(p_i)+\Phi(\overline{p_i}) = \Phi(p_i)+\overline{\Phi(p_i)} \\ \text{is a null-homologous cycle} \\ \downarrow}}{=}\; \sum_{i=1}^{k}\sigma_i + \sum_{i=1}^{k}\Phi(p_i) + \sum_{i=1}^{k}\Phi(\overline{p}_i) \;\overset{\substack{\text{since } \Phi\circ\Psi = \mathrm{id} \text{ and since permutation of the} \\ \text{summands does not change the sum} \\ \downarrow}}{=}\; \sum_{i=1}^{k}\big(\Phi(p_i) + \Phi(\Psi(\sigma_i)) + \Phi(\overline{p}_{s(i)})\big) \\
&\underset{\substack{\uparrow \\ \text{by Lemma 38.1}}}{=}\; \sum_{i=1}^{k}\Phi\big(\underbrace{p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\substack{\text{loop in } (X,x_0) \\ \text{by choice of } s}}\big) \;\underset{\substack{\uparrow \\ \text{by Lemma 38.1}}}{=}\; \Phi\Bigg(\underbrace{\prod_{i=1}^{k} p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\text{loop in } (X,x_0)}\Bigg).
\end{aligned}$$

Thus we have shown that $z = [c] \in \Phi(\pi_1(X,x_0))$. This concludes the proof that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to H_1(X)$ is surjective. $\square$

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so we need to recall some definitions and results from Section 13.3.

Definition. Let π be a group.

(1) For $x, y \in \pi$ we denote by $[x,y] := xyx^{-1}y^{-1}$ the *commutator of x and y*.
(2) We refer to

$$[\pi,\pi] \;:=\; \langle\!\langle \{[x,y] \mid x,y \in \pi\} \rangle\!\rangle \;=\; \Big\{ \prod_{i=1}^{n}[x_i,y_i] \;\Big|\; x_1,y_1,\dots,x_n,y_n \in \pi \Big\}$$

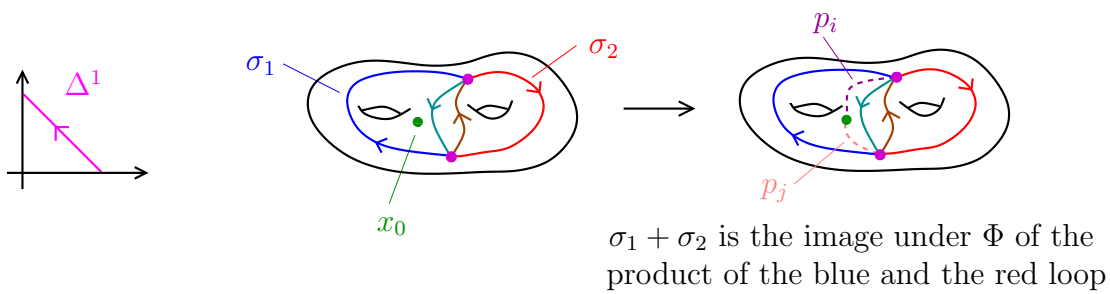


FIGURE 566. Sketch for the proof of Proposition 38.3.

as the *commutator subgroup* of π.

(3) We refer to $\pi_{\mathrm{ab}} := \pi/[\pi,\pi]$ as the *abelianization* of π.

In Section 13.3 we saw that the abelianization is "the largest abelian quotient" of π. More precisely, if $\alpha\colon \pi \to H$ is a homomorphism from π to an abelian group H, then Proposition 13.10 says that there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:

$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, "\alpha"'] & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[d, "\exists!\,\beta"] \\ & H. \end{tikzcd}$$

In particular, given a pointed topological space (X, x_0) the corresponding Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to H_1(X)$ induces a homomorphism $\pi_1(X,x_0)_{\mathrm{ab}} \to H_1(X)$ that we also denote by $\Phi_{(X,x_0)}$ and that we also refer to as the Hurewicz homomorphism. Now we can formulate the following proposition.

Theorem 38.5. (Hurewicz Theorem) *Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \;\to\; H_1(X)$$

is a natural[726] *isomorphism.*

Examples.

(1) If we compare our calculations of fundamental groups with our calculations of homology groups, then we see that we had encountered this isomorphism many times.
   (a) In Corollary 7.19 we showed that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$, whereas on page 862 we saw that $H_1((S^1)^n) \cong \mathbb{Z}^n$.

[726] Whenever we write "natural" one needs to work out what the categories and functors are. In this case we consider the category of path-connected topological spaces, the category of groups and the functors $(X,x_0) \to \pi_1(X,x_0)_{\mathrm{ab}}$ and $(X,x_0) \mapsto H_1(X)$.

(b) For $n \geq 2$ we saw in Corollary 7.19 that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$ and in Lemma 33.9 we had calculated that $H_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(c) In Proposition 14.3 we showed that

$$\pi_1(\text{Klein bottle}) \;\cong\; \langle x, y \,|\, yxyx^{-1}\rangle.$$

The abelianization of that group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ which is isomorphic to the first homology of the Klein bottle that we had determined in Lemma 33.14.

(d) In Proposition 14.3 we showed that for any $g \in \mathbb{N}$ we have

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g]\rangle.$$

The abelianization of this group is isomorphic to $\mathbb{Z}^{2g}$ which is precisely the first homology group that we had determined in Proposition 35.10.

(e) Let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. On page 863 we saw that there exists a short exact sequence

$$0 \;\to\; \operatorname{coker}\big(H_1(X) \xrightarrow{f_* - \mathrm{id}} H_1(X)\big) \;\xrightarrow{i}\; H_1(M(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

On the other hand it follows from Proposition 15.25 and the remark on page 478 that there exists an isomorphism $\psi\colon \pi_1(M(X,f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ where the action of $1 \in \mathbb{Z}$ on $\pi_1(X)$ is induced by f. Therefore we can put $\pi_1(M(X,f))$ into a short exact sequence

$$0 \;\to\; \pi_1(X) \;\to\; \pi_1(M(X,f)) \;\to\; \mathbb{Z} \;\to\; 0$$

Now we apply the abelianization functor to these groups and homomorphisms. We obtain the exact sequence[727]

$$\pi_1(X)_{\mathrm{ab}} \;\to\; \pi_1(M(X,f))_{\mathrm{ab}} \;\to\; \mathbb{Z} \;\to\; 0$$

which, using the Hurewicz Theorem 38.5, we can rewrite as

$$H_1(X) \;\to\; H_1(M(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

This is, fortunately, consistent with the above short exact sequence.

(2) Note that we now have a new, and depending on one's taste, more satisfying proof of Corollary 18.15: The Hurewicz Theorem 38.5 together with Lemma 33.15 implies that given any knot $K \subset S^3$ the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Proof of the Hurewicz Theorem 38.5 $(*)$. Let (X, x_0) be a pointed path-connected topological space. We write $\pi := \pi_1(X, x_0)$. By Proposition 38.2 we know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to H_1(X)$ is natural. Furthermore it follows from Proposition 38.3 that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to H_1(X)$ is surjective. It remains to show that $\Phi_{(X,x_0)}$ is injective.

[727] In Exercise 13.3 we saw that if $\varphi\colon A \to B$ is an epimorphism, then the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism. We had also seen that there exist monomorphisms $\varphi\colon A \to B$ such that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is not a monomorphism.

In the following we will make the identification $[0,1] = \Delta^1$ that is given by the homeomorphism $t \mapsto (1-t, t)$. In particular we will identify paths $[0,1] \to X$ with singular 1-simplices $\Delta^1 \to X$.

So let $f\colon [0,1] = \Delta^1 \to X$ be a loop in (X, x_0) such that $[f] = 0 \in H_1(X)$. By definition this means that there exists a 2-chain $\sum_{j=1}^{s} n_j \sigma_j$ in X whose boundary is the singular 1-simplex f. Furthermore, since we do not demand that the σ_j are distinct, we can assume that all n_j are either -1 or 1. Given $j \in \{1, \dots, s\}$ and $k \in \{0,1,2\}$ we write[728]

$$\mu_{(j,k)} \;:=\; \sigma_j \circ i_k^2 \colon [0,1] = \Delta^1 \to X \quad \text{ and } n_{(j,k)} \;:=\; (-1)^k \cdot n_j.$$

We write $A = \{1, \dots, s\} \times \{0,1,2\}$. We have the following equality in $C_1(X)$:

$$f \;=\; \partial_1\Big(\sum_{j=1}^{s} n_j \sigma_j\Big) \;\underset{\substack{\uparrow \\ \text{definition of } \partial_1}}{=}\; \sum_{j=1}^{s}\sum_{k=0}^{2} n_{(j,k)} \cdot \mu_{j,k} \;=\; \sum_{a \in A} n_a \cdot \mu_a.$$

By comparing the left and the right-hand side of this equality we see that there exists a $b \in A$ with $n_b \cdot \mu_b = 1 \cdot f$. We write $A_\pm = \{a \in A \setminus \{b\} \,|\, n_a = \pm 1\}$. Now we obtain from the above equality that

$$\sum_{a \in A_+} \mu_a \;+\; \sum_{a \in A_-} (-1) \cdot \mu_a \;=\; 0.$$

We make the following choices and introduce the following definitions:

(1) For each point P in X that is a starting or an endpoint of some μ_a we pick a path β_P from x_0 to P. For $P = x_0$ we just pick the constant path.
(2) Given $\mu_{(i,j)}\colon [0,1] = \Delta^1 \to X$ we write

$$\widehat{\mu}_{(i,j)} \;:=\; \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

Note that $\widehat{\mu}_{(i,j)}$ is a loop in (X, x_0), in particular it defines an element in the group $\pi = \pi_1(X, x_0)$ which, by a slight abuse of notation, we denote again by $\widehat{\mu}_{(i,j)}$.

We make the following elementary observation: for any $(i,j) \in \{1, \dots, s\} \times \{0,1,2\}$ we have the following equality in $\pi = \pi_1(X, x_0)$:

$$(*) \qquad \widehat{\mu}_{(i,j)}^{-1} \;:=\; \beta_{\text{starting point of } \overline{\mu}_{(i,j)}} * \overline{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \overline{\mu}_{(i,j)}}.$$

[728]Recall that for $j \in \{0, \dots, n\}$ we denote by i_j^n the map

$$\begin{array}{rcl} i_j^n\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{array}$$

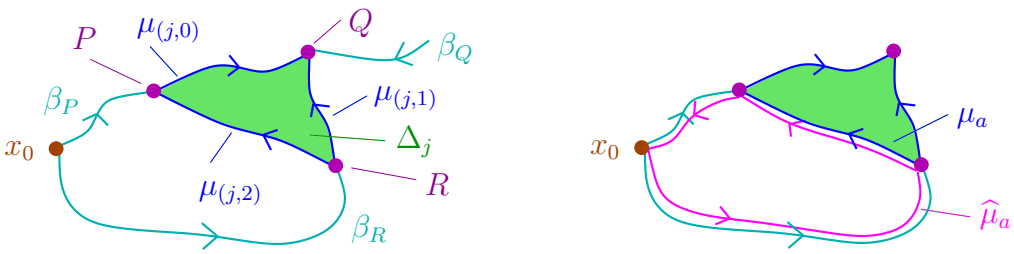


FIGURE 567. Illustration for the proof of the Hurewicz Theorem 38.5.

In π_{ab} we have the following equalities:[729]

$$e \underset{\uparrow}{=} \prod_{j=1}^{s} \beta_{\text{starting point of } \mu_{(j,0)}} * \big(\overbrace{\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}}^{\text{loop in the starting point of } \mu_{(j,0)}}\big)^{n_j} * \overline{\beta}_{\text{starting point of } \mu_{(j,0)}}$$

since $\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}$ is null-homotopic

$$\underset{\uparrow}{=} \prod_{j=1}^{s} \widehat{\mu}_{(j,0)}^{n_{(j,0)}} \cdot \widehat{\mu}_{(j,1)}^{n_{(j,1)}} \cdot \widehat{\mu}_{(j,2)}^{n_{(j,2)}} \qquad = \prod_{a \in \{1,\dots,k\} \times \{0,1,2\}} \widehat{\mu}_a^{n_a}$$

follows from inserting null-homotopic paths $\overline{\beta}_P * \beta_P$ together with the above equality $(*)$

$$\underset{\uparrow}{=} f \cdot \prod_{a \in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_{\psi(a)}^{n_{\psi(a)}} \qquad \underset{\uparrow}{=} f \cdot \prod_{a \in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_a^{-n_a} = f.$$

since $\{1,\dots,k\} \times \{0,1,2\} = (1,0) \sqcup A_- \sqcup \psi(A_-)$ since $\mu_{(1,0)} = f$ and since $\beta_{x_0} = c_{x_0}$ for $a \in A_-$ we have $\mu_{\psi(a)} = \mu_a$ and $n_a = -n_a$.

This completes the proof of the proposition.[730] □

For most topological spaces for which we had computed the fundamental group we had also determined the homology groups. But there is one rather interesting exception, namely the lens spaces. From Corollary 7.19 and the Hurewicz Theorem 38.5 we obtain immediately the following corollary.

Corollary 38.6. *For any coprime $p, q \in \mathbb{N}$ we have*

$$H_1(\textit{lens space } L(p,q)) \cong \mathbb{Z}_p.$$

Remark. The Hurewicz Theorem 38.5 is perhaps a mixed blessing. On the one hand it allows us to use the techniques that we had developed in earlier chapters to determine first homology groups. On the other hand it tells us that all the information that is contained in the first homology group is already contained in the fundamental group. So we do not get any new information. In particular, as we saw in Corollary 38.6, the first homology

[729] In the notation we do not distinguish between loops and elements they represent in π_{ab}.

[730] Where did we actually use that we work with π_{ab} instead of π?

group cannot help us in our long-standing goal of distinguishing lens spaces with the same fundamental group.

Calculating (isomorphism types) of homology groups can get quite involved. But what is often even harder is to determine the maps on homology groups induced by maps between topological spaces. Using Proposition 38.2 (4) we can transfer calculations of induced maps on fundamental groups to obtain the corresponding induced maps on homology groups.

For example we have the following lemma, which later on will be used on many occasions.

Lemma 38.7. *Let $n \in \mathbb{N}$. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$ the n-dimensional torus.*

(1) *We denote by $\sigma \in H_1(S^1)$ the standard generator introduced on page 814. Given $i \in \{1, \dots, n\}$ we denote by*

$$\begin{array}{rcl} p_i \colon S^1 & \to & (S^1)^n \\ z & \mapsto & (1, \dots, 1, z, 1, \dots, 1) \end{array}$$

the obvious i-th inclusion map and we write $z_i = p_{i*}(\sigma)$. Then $e_1, \dots, e_n$ is a basis of $H_1(T)$.

(2) *Let $A \in \mathrm{SL}(n, \mathbb{Z})$ be a matrix. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. By Exercise 7.12, the map*

$$\begin{array}{rcl} f(A) \colon T = \mathbb{R}^n/\mathbb{Z}^n & \to & T = \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [Av] \end{array}$$

is a homeomorphism. With respect to the above basis $e_1, \dots, e_n$ of $H_1(T)$ the induced map $f(A)_* \colon H_1(T) \to H_1(T)$ is given by multiplication by the matrix A.

Proof.

(1) This statement follows easily from the definitions, Theorem 7.17 and the Hurewicz Theorem 38.5.

(2) We use the isomorphism from Theorem 7.17 to make the identification $\pi_1(T, 0) = \mathbb{Z}^n$. By Proposition 38.2 (4), Theorem 7.17 and Exercise 7.12 we obtain the following commutative diagram

$$\begin{tikzcd} H_1(T) \arrow[d, "f(A)"'] & \pi_1(T,0) \arrow[l, "\Phi_{(T,0)}"'] \arrow[r, "\cong"] \arrow[d, "f(A)_*"] & \mathbb{Z}^n \arrow[d, "v \mapsto A \cdot v"] \\ H_1(T) & \pi_1(T,0) \arrow[l, "\Phi_{(T,0)}"'] \arrow[r, "\cong"] & \mathbb{Z}^n. \end{tikzcd}$$

The desired statement follows from the observation that the horizontal identifications of $H_1(T)$ with $\mathbb{Z}^n$ are precisely the ones corresponding to the basis $e_1, \dots, e_n$. □

As another illustration of Proposition 38.2 (4) we also prove the following corollary.

Corollary 38.8. $(*)$ *Let $p\colon Y \to X$ be a finite-index covering of path-connected topological spaces. Then the cokernel of the map*

$$p_*\colon H_1(Y) \;\to\; H_1(X)$$

is finite.

Remark. Let $p\colon Y \to X$ be a finite-index covering of path-connected topological spaces. Let $y_0 \in Y$. By Corollary 7.15 the induced map $p_*\colon \pi_1(Y, y_0) \to \pi_1(X, p(y_0))$ is a monomorphism. On the other hand it is in general not true that that the induced homomorphism $p_*\colon H_1(Y) \to H_1(X)$ is a monomorphism. For example on page 260 we saw that there exists a 2-fold covering map $p\colon T \to K$ from the torus T to the Klein bottle K. By Lemmas 33.12 and 33.14 we know that $H_1(T) \cong \mathbb{Z}^2$ and that $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. By Lemma 11.8 (2) there is no monomorphism from the group $\mathbb{Z}^2$ to $\mathbb{Z} \oplus \mathbb{Z}_2$.

Proof $(*)$. We pick a base point y_0 of Y and we write $x_0 = p(y_0)$. We consider the following diagram of maps:

$$\begin{tikzcd}
\pi_1(Y, y_0) \arrow[rr, "p_*"] \arrow[dd, "\Phi_Y"'] \arrow[dr] & & \pi_1(X, x_0) \arrow[dd, "\Phi_X"] \arrow[dr] & \\
& \pi_1(Y, y_0)_{\mathrm{ab}} \arrow[rr, "p_*"] \arrow[dl, "\cong"'] & & \pi_1(X, x_0)_{\mathrm{ab}} \arrow[dl, "\cong"'] \\
H_1(Y) \arrow[rr, "p_*"] & & H_1(X). &
\end{tikzcd}$$

Here it follows from the Hurewicz Theorem 38.5 that the lower diagonal maps are isomorphisms. The "big rectangle in the back" commutes by Proposition 38.2 (4). The upper parallelogram commutes by Proposition 13.10 (7). Since the lower diagonal maps are isomorphisms it follows that the lower parallelogram also commutes.

By Lemma 7.16 and by our hypothesis that p is a finite covering we know that $p_*(\pi_1(Y, y_0))$ is a finite index subgroup of $\pi_1(X, x_0)$, i.e. $p_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ has a finite cokernel. It follows[731] that $p_*\colon \pi_1(Y, y_0)_{\mathrm{ab}} \to \pi_1(X, x_0)_{\mathrm{ab}}$ also has a finite cokernel. From the commutativity of the lower parallelogram it follows that the map $p_*\colon H_1(Y) \to H_1(X)$ has a finite cokernel as well. $\square$

38.2. **The Hurewicz homomorphism in higher dimensions.** The Hurewicz Theorem 38.5 also raises the question whether there is a connection between higher homotopy groups and higher homology groups.

[731]This can be seen as follows: First we write $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(Y, y_0))$. Let $g_1, \ldots, g_k$ be coset representatives for G/H. One can easily verify that the images of $g_1, \ldots, g_k$ in G_{ab} have the property that $G_{\mathrm{ab}} = g_1 H_{\mathrm{ab}} \cup \cdots \cup g_k H_{\mathrm{ab}}$.

Definition. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. We denote by $\omega \in H_n(S^n) \cong \mathbb{Z}$ the standard generator as defined on page 814. We refer to the map

$$\begin{array}{rcl} \Phi_{(X,x_0)} \colon \pi_n(X,x_0) & \to & H_n(X) \\ [f \colon S^n \to X] & \mapsto & f_*(\omega) \end{array}$$

as the *Hurewicz homomorphism.*[732]

Remark. Using the bijection given on page 228 and using Lemma 32.11 one can easily show that for $n = 1$ this definition of the Hurewicz homomorphism agrees with the definition given on page 924.

In the next lemma we show in particular that the Hurewicz homomorphism is in fact a homomorphism.

Lemma 38.9.

(1) *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. The Hurewicz homomorphism*

$$\Phi_{(X,x_0)} \colon \pi_n(X,x_0) \ \to \ H_n(X)$$

is a homomorphism.

(2) *Let $\mathcal{P}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. The Hurewicz homomorphisms define a natural transformation from the functor*

$$\begin{array}{rcl} \mathcal{P} & \to & \mathcal{G} \\ (X,x_0) & \mapsto & \pi_n(X,x_0) \end{array} \quad \textit{to the functor} \quad \begin{array}{rcl} \mathcal{P} & \to & \mathcal{G} \\ (X,x_0) & \mapsto & H_n(X). \end{array}$$

Proof $(*)$. As usual we write $I = [0,1]$. By Proposition 1.47 (2) and Lemma 29.1 there exists a homeomorphism $\varphi \colon (\Delta^n, \partial\Delta^n) \to (I^n, \partial I^n)$. It follows from Lemma 32.9 that $\varphi \colon \Delta^n \to I^n$ represents a generator of $H_n(I^n, \partial I^n)$. Now we write $\omega := [\varphi] \in H_n(I^n, \partial I^n)$. Using the bijection from page 715 we see that it suffices show that for any two maps $f, g \colon (I^n, \partial I^n) \to (X, x_0)$ we have

$$[f_*(\omega)] + [g_*(\omega)] \ = \ [(f * g)_*(\omega)] \ \in \ H_n(X,x_0) = H_n(X).$$

But we showed precisely this equality in Lemma 34.12. □

The following lemma gives in particular an affirmative answer to Question 28.16.

Lemma 38.10. *For every $n \in \mathbb{N}$ the Hurewicz homomorphism*

$$\Phi \colon \pi_n(S^n, *) \ \to \ H_n(S^n) \ \cong \ \mathbb{Z}$$

sends $[\mathrm{id}_{S^n} \colon S^n \to S^n] \in \pi_n(S^n)$ to the standard generator of $H_n(S^n)$, in particular the Hurewicz homomorphism is an epimorphism.

[732]It follows from Proposition 30.5 that the Hurewicz homomorphism is indeed well-defined.

Proof. We denote by $\omega \in H_n(S^n)$ the standard generator. By definition of the Hurewicz homomorphism we have

$$\Phi(\underbrace{[\mathrm{id}_{S^n}\colon S^n \to S^n]}_{\in \pi_n(S^n,*)}) \;=\; (\mathrm{id}_{S^n})_*([\omega]) \;=\; [\omega].$$

Thus we see that the standard generator of $H_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi\colon \pi_n(S^n,*) \to H_n(S^n)$ is an epimorphism. □

It is natural to ask whether, given a path-connected topological space X and $n \geq 2$, the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \;\to\; H_n(X)$$

is necessarily an isomorphism. But browsing through our examples shows that the Hurewicz homomorphism is in general neither a monomorphism nor an isomorphism:

(1) In Lemma 33.12 we showed that $H_2(T) \cong \mathbb{Z}$ is non-zero, whereas on page 723 we showed that $\pi_2(T) = 0$. This shows that the Hurewicz homomorphism $\pi_2(T) \to H_2(T)$ is not an epimorphism.
(2) Here are two examples that show that in general the Hurewicz homomorphism is not a monomorphism:
   (a) We consider $X = \mathbb{R}\mathrm{P}^2$. By Proposition 28.13 and Lemma 38.10 we know that $\pi_2(\mathbb{R}\mathrm{P}^2) \cong \pi_2(S^2)$ is non-trivial. On the other hand we saw in Lemma 33.9 that $H_2(\mathbb{R}\mathrm{P}^2) = 0$.
   (b) We consider $X = S^1 \vee S^2$. It follows immediately from Propositions 31.3 and 34.8 that $H_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by Proposition 28.13 and the discussion on page 568 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$ where $\widetilde{X}$ is the real line "with infinitely many 2-spheres attached", see e.g. Figure 459 for an illustration. Using the projections onto the various spheres and using Lemma 38.10 one can now show that $\pi_2(\widetilde{X})$ admits an epimorphism onto $\mathbb{Z}^\infty$.[733] It follows from Lemma 11.8 (2) that there is no monomorphism from $\pi_2(X) = \pi_2(\widetilde{X})$ into $H_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$, whatever it is precisely, cannot be a monomorphism.

Now we recall that for $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$ and $k \leq l$ we can view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^l$. Now we can formulate the following application of Lemma 38.10.

Lemma 38.11. *For $k < l$ the submanifold $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$.*

Proof. We have

$$\underset{\substack{\uparrow \\ \text{Proposition 28.13}}}{\pi_k(\mathbb{R}\mathrm{P}^k) \;\cong\; \pi_k(S^k)} \underset{\substack{\uparrow \\ \text{Lemma 38.10}}}{\;\neq\; 0} \qquad \text{and} \qquad \underset{\substack{\uparrow \\ \text{Proposition 28.13}}}{\pi_k(\mathbb{R}\mathrm{P}^l) \;\cong\; \pi_k(S^l)} \underset{\substack{\uparrow \\ \text{Proposition 28.9}}}{\;=\; 0.}$$

[733] The proof of this statement is a slight variation of the proof of Exercise 28.3.

With this calculation, and the fact that π_n is functorial, the argument of Lemma 6.2 can be easily modified to show that $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$. $\square$

We conclude this chapter with the following question.

Question 38.12. *Let $k < l$. Is the complex projective space $\mathbb{C}\mathrm{P}^k$ a retract of $\mathbb{C}\mathrm{P}^l$?*

Exercises for Chapter 38.

Exercise 38.1. Let $T = S^1 \times S^1$ be the torus. Let $\varphi, \psi \colon S^1 \to T$ be two embeddings such that $\varphi_*([S^1]) = \psi_*([S^1]) \in H_1(T)$. Show that φ and ψ are homotopic maps.

39. Simplicial complexes and homology groups of manifolds

In this section we introduce the notion of a simplicial complex. Simplicial complexes can be viewed as special classes of CW-complexes with particularly nice properties. We will see that every manifold admits the structure of a simplicial complex and we will use this fact to obtain several results on homology groups of manifolds.

Remark. The eternal problem in topology is to strike the right balance between rigor and geometric intuition. In this chapter we will at times be slightly less rigorous than in the other chapters.

Later on, in a later chapter we will reprove most of the results from this section more generally for topological manifolds. These proofs will then be very rigorous but much less intuitive.

39.1. **Simplicial complexes.** In the following we recall several definitions from page 734.

Definition.

(1) Given $n \in \mathbb{N}_0$ we refer to

$$\Delta^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1 + \cdots + x_{n+1} = 1 \text{ and } x_i \geq 0 \text{ for all } i = 1, \dots, n+1\}$$

as the *standard n-simplex*.

(2) A *face* of codimension r of the standard n-simplex is defined as a subset of the form

$$\{(x_1, \dots, x_{n+1}) \in \Delta^n \,|\, x_{d_1} = x_{d_2} = \cdots = x_{d_r} = 0\},$$

where $d_1 < d_2 < \cdots < d_r \in \{1, \dots, n+1\}$ are fixed. We refer to Figure 469 for an illustration.

(3) The boundary $\partial\Delta^n$ of the standard n-simplex is defined as the union of all faces of Δ^n of codimension 1.

We also introduce two new definitions:

(4) We refer to

$$\mathring{\Delta}^n \;:=\; \Delta^n \setminus \partial\Delta^n$$

as the *interior of* Δ^n. This is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ and we equip it with the orientation which is given by the rule that a basis $\{v_1, \dots, v_n\}$ for $T_P\mathring{\Delta}^n$ is positive, if the basis $\{(1, \dots, 1), v_1, \dots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$.[734]

(5) Given $i \in \{1, \dots, n+1\}$ let

$$F_i \;=\; \{(x_1, \dots, x_{n+1}) \in \Delta^n \,|\, x_i = 0\}$$

be the face of Δ^n that is given by setting $x_i = 0$. We equip the interior of F_i with the orientation which is given by the rule that a basis $\{v_1, \dots, v_{n-1}\}$ for $T_P\mathring{F}_i$ is positive, if the basis $\{(1, \dots, 1), v_1, \dots, v_{n-1}, e_i\}$ is a positive basis for $\mathbb{R}^{n+1}$.

[734]Here, as always, we equip $\mathbb{R}^k$ with the orientation that is defined by the standard basis $e_1, \dots, e_k$.

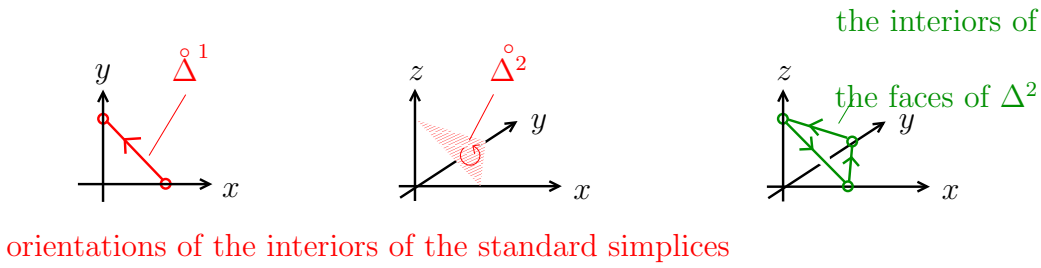


FIGURE 568.

Remark. The homeomorphism $\Delta^n \to S^n_{\geq 0} = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ that is given in page 813 and that is illustrated in Figure 506 restricts to an orientation-preserving diffeomorphism $\mathring{\Delta}^n \to S^n_{>0}$.

Next we introduce the notion of a simplicial precomplex.

Definition.

(1) A *simplicial precomplex* is a topological space X that is Hausdorff together with a family of injective maps $\varphi_i \colon \Delta^{n_i} \to X$, $i \in I$.[735]
(2) We refer to the image of a map $\varphi_i \colon \Delta^{n_i} \to X$ as an *n_i-dimensional simplex of X*.[736] The image of a face of Δ^{n_i} is called a *face of the simplex*, the image of $\partial\Delta^{n_i}$ is called the *boundary of the simplex*, and the image of the interior $\mathring{\Delta}^{n_i}$ is called the *interior of the simplex*.
(3) The *dimension* $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty simplicial precomplex X is defined as the maximum dimension of a simplex.
(4) A *subcomplex* of a simplicial precomplex X is defined as a union of simplices of X.

Now we can introduce the notion of a simplicial complex, which is what we are really interested in.

Definition. A *simplicial complex* is a simplicial precomplex $(X, \{\varphi_i \colon \Delta^{n_i} \to X\}_{i \in I})$ such that the following conditions are satisfied:

(1) For each $x \in X$ there exists a unique simplex such that x lies in the interior of the simplex.
(2) Every face of every simplex of X is again a simplex of X.
(3) If the images of φ_i and φ_j intersect, then there exist faces F_i of Δ^{n_i} and F_j of Δ^{n_j} such that the following two conditions are satisfied:
   (a) we have $\varphi_i(\Delta^{n_i}) \cap \varphi_j(\Delta^{n_j}) = \varphi_i(F_i) = \varphi_j(F_j)$,

[735]Note that it follows from Proposition 1.40 that each map $\varphi_i \colon \Delta^{n_i} \to X$ is a homeomorphism onto its image.

[736]It follows from Footnote 735 together with Proposition 32.5 that the definition of the dimension of a simplex is well-defined. More precisely, it follows that for two injective maps $\varphi \colon \Delta^k \to X$ and $\psi \colon \Delta^l \to X$ with $k \neq l$ the images have to be different.

(b) the map $\varphi_j^{-1} \circ \varphi_i \colon F_i \to F_j$ is an affine homeomorphism.[737]

(4) A subset $U \subset X$ is open if and only if for every $i \in I$ the preimage $\varphi_i^{-1}(U)$ is open in Δ^{n_i}.[738]

Remark. The above definition of a simplicial complex is highly non-standard. We use it since it allows us to quickly state and prove the desired results on homology groups of manifolds. We refer to [Mun] for the standard definition of a simplicial complex.

Put slightly more naively, a simplicial complex is a topological space which is "built out of simplices glued along faces". We refer to Figure 569 for an illustration of examples and non-examples of simplicial complexes.

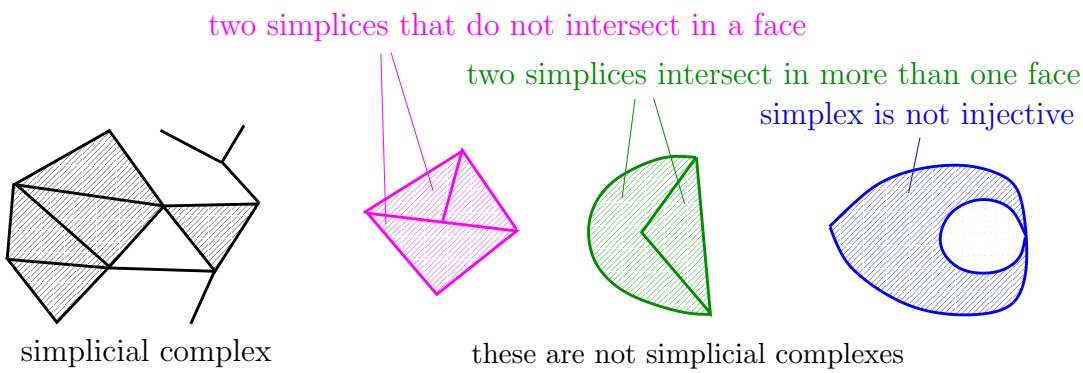


FIGURE 569.

We leave the straightforward proof of the following lemma as an exercise to the reader.

Lemma 39.1. *Let X be a simplicial complex.*

(1) *The boundary of every n-simplex is contained in a subcomplex of dimension $\leq n-1$.*

(2) *If A is a subcomplex and if c is a simplex of X such that the interior of c intersects A, then c already lies in A.*

Let $n \in \mathbb{N}_0$. Recall that in Lemma 29.1 we had written down a specific homeomorphism $(\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n)$. If we now precompose the maps $\Delta^n \to X$ of a simplicial complex with these homeomorphisms we easily obtain the following lemma from Lemma 39.1 (2). We leave it as an exercise to the reader to fill in all the details.

Lemma 39.2. *Any simplicial complex can also be viewed as CW-complex where given any $n \in \mathbb{N}$ the n-simplices of the simplicial complex correspond precisely to the n-cells of the CW-complex.*

[737]Note that F_i respectively F_j are subsets of $\mathbb{R}^{n_i}$ respectively $\mathbb{R}^{n_j}$. An *affine homeomorphism* between F_i and F_j is an affine, linear bijective map from F_i to F_j, i.e. it is a map of form $v \mapsto A \cdot v + P$ for some invertible matrix A and vector P that restricts to a bijection from F_i to F_j.

[738]From the continuity of the maps $\varphi_i, i \in I$ it follows that if $U \subset X$ is open, then for all $i \in I$ the preimage $\varphi_i^{-1}(U)$ is open in Δ^{n_i}. The statement of (4) is thus that the converse also holds.

Remark. It is straightforward to see that every 1-dimensional CW-complex is also a 1-dimensional simplicial complex. On the other hand not every CW-complex is necessarily a simplicial complex. For example, if $n \geq 2$ and if we consider the CW-complex with one 0-cell and one n-cell, then this CW-complex is not a simplicial complex.

39.2. **The top-dimensional homology group of a manifold.**

Definition. Let X be a Hausdorff space. A *simplicial structure* for X consists of a family of injective maps $\varphi_i \colon \Delta^{n_i} \to X$, $i \in I$ such that $(X, \{\varphi_i\}_{i\in I})$ is a simplicial complex.[739]

The following proposition is one of the keys for getting a grip on the invariants of manifolds.

Proposition 39.3. *Let M be an n-dimensional manifold. Then M admits an n-dimensional simplicial structure $\{\varphi_i \colon \Delta^{n_i} \to M\}_{i\in I}$ with the following properties:*

(1) *for each $i \in I$ the restriction of φ_i to $\varphi_i \colon \mathring{\Delta}^{n_i} \to M$ is an embedding,*[740]
(2) $(*)$ *for each $i \in I$ the map*[741]

$$\begin{aligned} \Big\{(t_1, \dots, t_{n_i}) \in [0,1]^{n_i} \;\Big|\; \sum_{j=1}^{n_i} t_j \in [0,1]\Big\} &\to M \\ (t_1, \dots, t_{n_i}) &\to \varphi_i\Big(t_1, \dots, t_{n_i}, 1 - \sum_{j=1}^{n_i} t_j\Big) \end{aligned}$$

is smooth,
(3) *the boundary ∂M is a subcomplex.*

Furthermore, if N is a closed submanifold of M, then one can arrange that the following extra statement holds:

(4) *The submanifold N is a subcomplex.*

Remark. As we had just pointed out in Lemma 39.2, every simplicial complex is also a CW-complex. Thus we can view Proposition 39.3 as a strengthening of Theorem 26.3.

Proof. The fact that one can find a simplicial structure that satisfied (1), (2) and (3) is precisely is [Mun2, Theorem 10.6]. Given a submanifold it follows from solving [Mun2, Problem 10.14] that we can also arrange (4) to hold. □

Examples.

(1) In Figure 570 we show the torus $T = [0,1] \times [0,1]/\sim$ with three different sets of maps $\Delta^2 \to T$. Only the maps on the right define a simplicial structure for the torus.
(2) In Figure 571 we show simplicial structures for the annulus, the 2-dimensional sphere and for the torus minus an open disk.

[739] In the literature a simplicial structure is also often called a *triangulation*.

[740] Note that given $n \geq 2$ the standard-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is not a submanifold of $\mathbb{R}^{n+1}$ (why not?), but the interior $\mathring{\Delta}^n$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

[741] The left-hand side is a subset of $\mathbb{R}^{n_i}$ and the right-hand side is a manifold. Thus by the definition on page 184 it makes sense to say whether nor not this map is smooth.

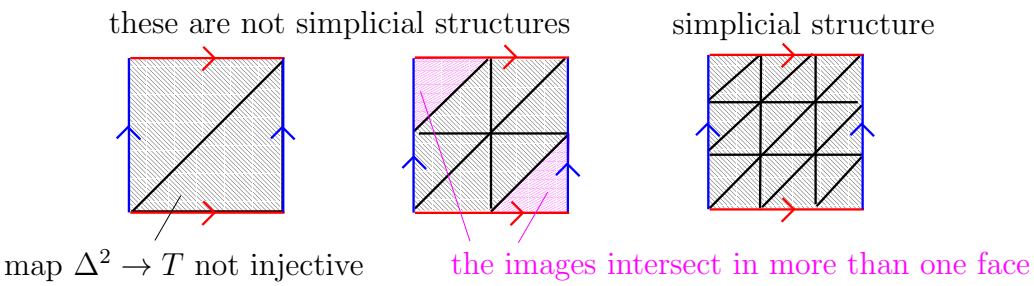


FIGURE 570. The torus as a simplicial complex.

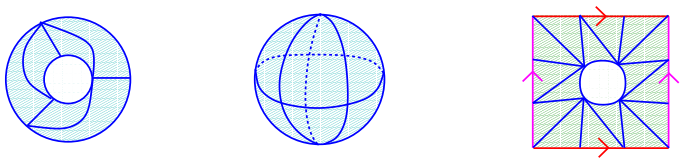


FIGURE 571.

(3) A simplicial structure for a surface can be viewed as an approximation by triangles. Such simplicial structures play an important role in computer visualizations. In this context one usually talks of triangulations. We refer to Figure 572 for more illustrations.

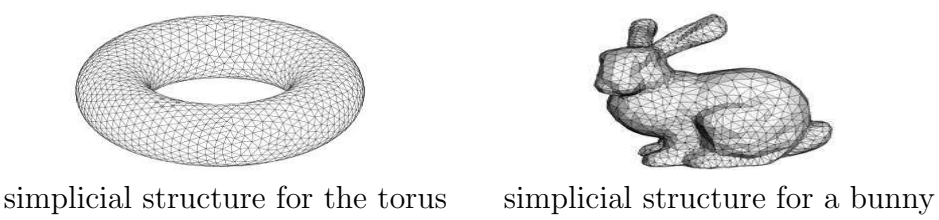


FIGURE 572.

Definition. Let X be a simplicial complex and let c be a k-simplex of X. We say c has *order* r, if there exist precisely r simplices of dimension $k+1$ which contain c as a face.

Lemma 39.4. *Let M be an n-dimensional manifold that is equipped with a simplicial structure.*

(1) *If $k \leq n-1$, then every k-dimensional simplex of M has order at least one.*
(2) *The simplicial structure has finitely many simplices if and only if it has finitely many n-simplices.*

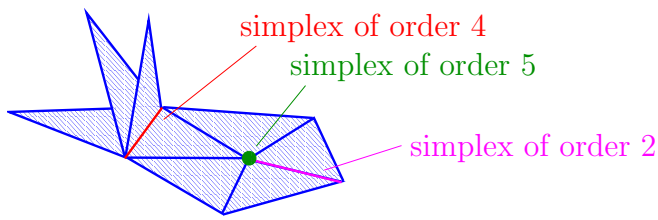


FIGURE 573.

(3) *Let c be an $(n-1)$-dimensional simplex of M. Then one of the following holds:*
   (a) *Either c lies in the boundary of M, in this case the order of c equals one,*
   (b) *or the interior of c lies in the interior of M, in this case the order is equal to two.*

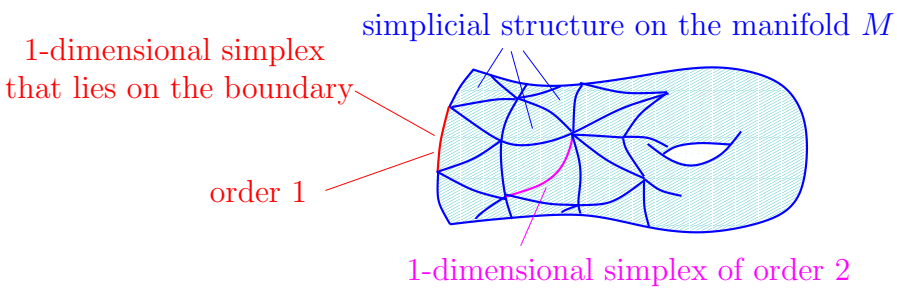


FIGURE 574. Illustration of Lemma 39.4.

Sketch of a proof for Lemma 39.4 $(*)$. Let M be an n-dimensional manifold that is equipped with a simplicial structure.

(1) Let $k \leq n-1$. Suppose there exists a k-dimensional simplex c of M of order 0. Let P be a point in the interior of c. Since c has order 0 the interior $U = \mathring{c}$ of the simplex c is an open neighborhood of P in M.[742] But then we have

$$H_k(M, M \setminus \{P\}) \underset{\substack{\uparrow \\ \text{Excision Theorem 31.17}}}{\cong} H_k(U, U \setminus \{P\}) \underset{\substack{\uparrow \\ \text{by Lemma 32.3 since } U \text{ is a} \\ k\text{-dimensional manifold} \\ \text{without boundary}}}{\cong} \mathbb{Z}.$$

But since $k \leq n-1$ this contradicts Lemma 32.3 applied to the n-dimensional manifold M.

(2) We only have to show that if M has finitely many n-simplices, then for $k = n, \dots, 0$ the number of k-simplices is finite. This clearly holds for $k = n$. Suppose we know the statement for some $k \in \{1, \dots, n\}$. Each k-simplex has only finitely many faces, so

[742] This statement requires a little bit of thought.

there exist only finitely many $(k-1)$-simplices that are the face of a k-simplex. But by (1) every $(k-1)$-simplex is the face of a k-simplex.

(3) Let c be an $(n-1)$-dimensional simplex of M. We start out with the following claim.

Claim. Let P be a point in the interior of c.
(i) If P lies on the boundary of M, then c has order one.
(ii) If P lies in the interior of M, then c has order two.

Suppose that c has order k. By (1) we know that $k \geq 1$. We write

$$X_k := \big\{(z,w) \,\big|\, z = re^{2\pi ij/k}, r \in [0,1), j \in \{0,\dots,k-1\} \text{ and } w \in \mathbb{R}^{n-1} \text{ with } \|w\| < 1\big\}.$$

We sketch X_k for $k = 1,2,3$ in Figure 575. Since P lies in the interior of c it follows[743]

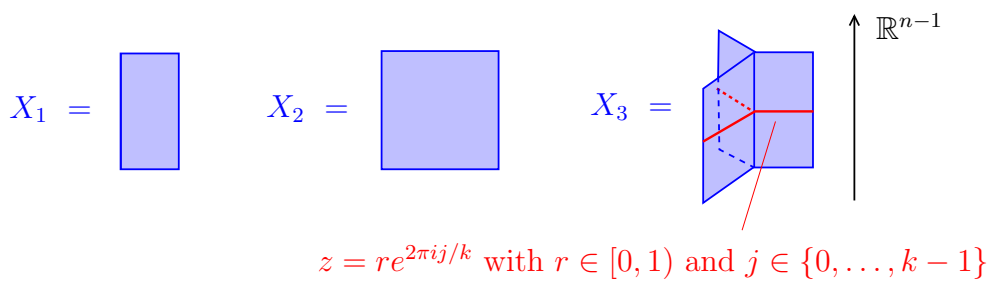


FIGURE 575.

that there exists an open neighborhood U of P and a homeomorphism $\Phi\colon U \to X_k$ with $\Phi(P) = 0$. Now we see that

$$\begin{array}{ccccccc} H_n(M, M\setminus\{P\}) & \cong & H_n(U, U\setminus\{P\}) & \xrightarrow{\Phi} & H_n(X_k, X_k\setminus\{0\}) & \cong & \mathbb{Z}^{k-1}. \\ & \uparrow & & & & \uparrow & \\ & \text{Excision Theorem 31.17} & & & & \text{left as an exercise} & \end{array}$$

The desired statement now follows from Lemma 32.3. This concludes the proof of the claim.

Finally we turn to the actual proof of statement (3) of the lemma.

(a) First suppose that the interior of c contains a point on the boundary of M. Then it follows from the claim that the order of c equals one. Since ∂M is a subcomplex of M it follows from Lemma 39.1 (3) that all of c is contained in the boundary of M.

(b) Now suppose that the interior of c does not contain a point that lies on the boundary of M, i.e. we suppose that the interior c lies in the interior of M. It follows from the claim that the order of c equals two. □

[743]This step requires some thought.

In Proposition 35.6 we already saw that for a given n-dimensional compact manifold M all homology groups are finitely generated abelian groups and that for $k > n$ we have $H_k(M) = 0$. Now we can prove the following proposition which gives us more information on homology groups of manifolds.[744]

Proposition 39.5. *Let M be a connected non-empty n-dimensional manifold. Then the following holds*[745]

$$H_n(M) \cong \begin{cases} \mathbb{Z}, & \textit{if } M \textit{ is orientable and closed,} \\ 0, & \textit{otherwise.} \end{cases}$$

Examples.

(1) In Question 3.40 we had asked whether $\mathbb{R}\mathrm{P}^{2n}$ is orientable. Now we can answer this question in the negative. More precisely, it follows from Proposition 39.5 and the calculation of the homology of real projective spaces in Proposition 35.11 that $\mathbb{R}\mathrm{P}^{2n}$ is non-orientable.
(2) On page 266 we pointed out that every lens space $L(p, q)$ is an orientable closed compact 3-dimensional manifold. It follows from Proposition 39.5 that $H_3(L(p, q)) \cong \mathbb{Z}$. Unfortunately the calculation does not help us in our goal of answering Question 7.8.

Sketch of a proof of Proposition 39.5. [746] Let M be a connected non-empty n-dimensional manifold. We pick an n-dimensional simplicial structure $\{\varphi_k \colon \Delta^{n_k} \to M\}_{k\in K}$ for M as in Proposition 39.3. Now we introduce some notation and conventions.

(1) We denote by $I \subset K$ the subset corresponding to the $(n-1)$-simplices. Given $i \in I$ we write $X_i = \varphi_i(\Delta^{n-1})$.
(2) Similarly we denote by $J \subset K$ the subset corresponding to the n-simplices of M. Given $j \in J$ we write $Y_j = \varphi_j(\Delta^n)$.
(3) By Lemma 39.2 we can view M as a CW-complex, where the cells correspond to the simplices.
(4) It follows from Proposition 26.14, statement (3) and Lemma 39.4 (2) that M is compact if and only if J is finite.
(5) Using the orientations for the interiors of the standard simplices that we introduced on page 935 and using the embeddings $\varphi_k, k \in K$ we view the interiors of $X_i, i \in I$ as oriented $(n-1)$-dimensional submanifolds of M and we view the interiors of Y_j, $j \in J$, as oriented n-dimensional submanifolds of M.[747]
(6) We use the convention introduced on page 889 to view the $X_i, i \in I$ as a basis for $C^{\mathrm{CW}}_{n-1}(M)$ and to view the $Y_j, j \in J$ as a basis for $C^{\mathrm{CW}}_n(M)$.

[744] Note that Proposition 39.5 is consistent with the calculation of H_2 of 2-dimensional manifolds that we gave in Proposition 35.10.

[745] Recall that a manifold is called closed if it is compact and if it has no boundary.

[746] A careful reading of the proof shows that the argument has several little gaps. We leave it as an exercise to find all gaps and to fill them. Alternatively you can wait for a later chapter.

[747] Here we use that the restrictions of the maps φ_i to $\overset{\circ}{\Delta}{}^{n_i}$ are embeddings.

(7) Given $i \in I$ and $j \in J$ we denote by a_{ij} the coefficient of X_i in $d_n(Y_j)$ where $d_n \colon C_n^{\mathrm{CW}}(M) \to C_{n-1}^{\mathrm{CW}}(M)$ denotes the cellular boundary map.

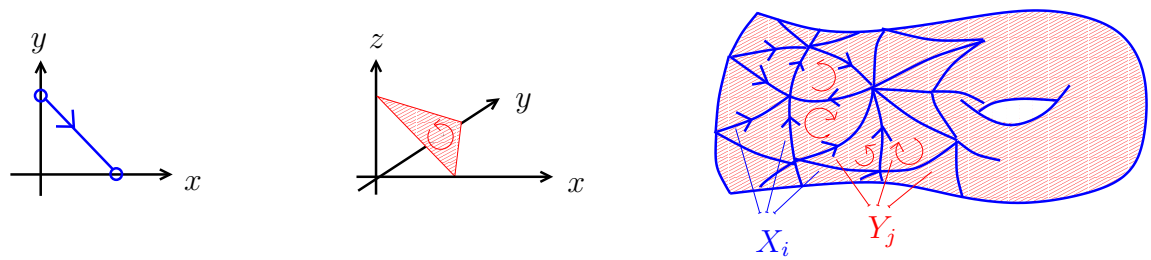


FIGURE 576. Illustration for the proof of Proposition 39.5.

We start out with the following observation:

$$(*) \quad H_n(M) \underset{\substack{\uparrow \\ \text{Proposition 35.4}}}{\cong} H_n^{\mathrm{CW}}(M) \underset{\substack{\uparrow \\ M \text{ has no simplices of dimension } n+1\text{, hence } C_{n+1}^{\mathrm{CW}}(M)=0}}{=} \ker\left\{C_n^{\mathrm{CW}}(M) \xrightarrow{d_n} C_{n-1}^{\mathrm{CW}}(M)\right\} = \{\text{cycles in } C_n^{\mathrm{CW}}(M)\}.$$

We continue with the following claim which is the key ingredient to the proof of Proposition 39.5.

Claim 1. Let $i \in I$. Then the following hold:

(1) If the $(n-1)$-simplex X_i lies on the boundary of M, then there exists precisely one $j \in J$ with $a_{ij} \neq 0$ and we have $a_{ij} = \pm 1$.
(2) If the $(n-1)$-simplex X_i does not lie the boundary of M, then there exist precisely two $j_1, j_2 \in J$ with $a_{i,j_1} \neq 0$ and $a_{i,j_2} \neq 0$. Furthermore we have $a_{i,j_1} = \pm 1$ and $a_{i,j_2} = \pm 1$. Finally, we have $a_{i,j_1} = -a_{i,j_2}$ if and only if the orientations of Y_{j_1} and Y_{j_2} agree along X_i.

Most of the statements of the claim follow immediately from Lemma 39.4, from the calculation of the cellular boundary map in Proposition 35.8 and from the definitions. It only remains to prove the statement regarding the orientations.

So suppose that an $(n-1)$-simplex X_i lies on the face of an n-simplex Y_j. Recall that we denote by $\varphi_i \colon \Delta^{n-1} \to M$ and $\varphi_j \colon \Delta^n \to M$ the corresponding maps from the simplicial structure. We denote by $F \subset \Delta^n$ the corresponding face of Δ^n. Using Proposition 35.8 and Lemma 32.24 one can provide an elementary, albeit lengthy, argument to show that, with the sign conventions from page 935, the following equalities:

$$a_{ij} = \begin{cases} +1, & \text{if } \varphi_i^{-1} \circ \varphi_j \colon F \to \Delta^{n-1} \text{ is orientation-preserving,} \\ -1, & \text{if } \varphi_i^{-1} \circ \varphi_j \colon F \to \Delta^{n-1} \text{ is orientation-reversing.} \end{cases}$$

Now suppose we have two simplices Y_{j_1} and Y_{j_2} that meet along X_i. Then it is straightforward to see that the orientations of Y_{j_1} and Y_{j_2} agree along X_i if and only if they induce

the opposite orientations on X_i. We refer to Figure 577 for an illustration. This concludes the proof of Claim 1.

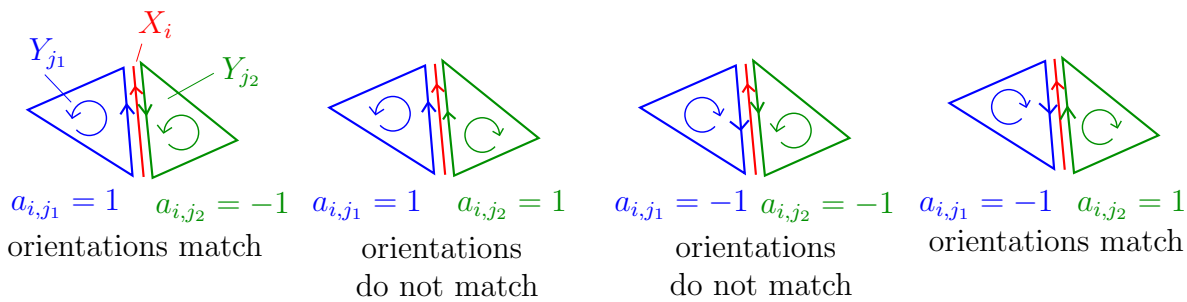


FIGURE 577.

Now we will prove the following claim that clearly implies the proposition:

Claim 2.

(1) If M is orientable and closed, then $H_n(M) \cong \mathbb{Z}$.
(2) If $H_n(M) \neq 0$, then M is orientable and closed.

First we suppose that M is orientable and closed. As we had observed above, the hypothesis that M is compact implies that I is finite. We choose an orientation for M. Given $i \in I$ we define

$$\epsilon_i \ := \ \begin{cases} +1, & \text{if the orientations of } M \text{ and } Y_i \text{ match,} \\ -1, & \text{otherwise.} \end{cases}$$

It follows immediately from Claim 1 that

$$c \ := \ \sum_{i \in I} \epsilon_i \cdot Y_i \ \in \ C_n^{\mathrm{CW}}(M)$$

is an n-cycle in $C_n^{\mathrm{CW}}(M)$. Now suppose that we are given another cycle $d = \sum_{i \in I} d_i \cdot Y_i$. We pick $i_0 \in I$ and we define $m \in \mathbb{Z}$ via $d_{i_0} = m \cdot \epsilon_{i_0}$. We set $I' := \{i \in I \,|\, d_i = m \cdot \epsilon_i\}$. It follows from Claim 1 and the hypothesis that M is connected that $I' = I$. Thus we see that $d = m \cdot c$. Together with the above observation $(*)$ we now see that $H_n(M)$ is isomorphic to $\mathbb{Z}$. This concludes the proof if M is orientable and closed.

Now suppose that $H_n(M) \neq 0$. We need to show that M is orientable and closed. By the observation $(*)$ there exists a non-zero cycle $d = \sum_{i \in I} d_i \cdot Y_i$. We pick $i_0 \in I$ with $d_{i_0} \neq 0$. We set $I' = \{i \in I \,|\, |d_i| = |d_{i_0}|\}$. We write $M' = \bigcup_{i \in I'} Y_i$. This is a submanifold[748] of M which, by the Claim 1 is closed. Since M is connected we obtain from Corollary 36.8 that $M' = M$. But this implies that M is closed.

[748]This statement is frankly not so clear.

Given $k = 0, \dots, n$ we denote by M^k the union of all k-simplices, i.e. M^k is the k-skeleton of M, viewed as a CW-complex. Now we can orient M as follows:

(i) Given a point $P \in \overset{\circ}{Y}_i$ we now endow it with the orientation of Y_i if $a_i > 0$, otherwise we endow it with the opposite orientation. This defines an orientation on $M^n \setminus M^{n-1}$.

(ii) It follows again from Claim 1 that the orientations from (1) extend to orientations on the interiors of $(n-1)$-simplices. This defines an orientation on $M^n \setminus M^{n-2}$.

(iii) Finally let P be a point on M^{n-2}. We pick a chart $\Phi\colon U \to V$ around P such that V is an open ball in $\mathbb{R}^n$. Since $M^{n-2} \cap U$ is of codimension 2 we see that $U \setminus (U \cap M^{n-2})$ is connected.[749]. It follows that Φ is either orientation-preserving for all $Q \in U \setminus (U \cap M^{n-2})$ or Φ is orientation-reversing for all $Q \in U \setminus (U \cap M^{n-2})$. Now we can use Φ to define an orientation for all points in U that agrees with the given orientation on $U \setminus (U \cap M^{n-2})$.

This concludes the proof of Claim 2. □

In a very similar fashion to Proposition 39.5 we can also prove the following proposition. We leave the details to the reader.

Proposition 39.6. *Let M be a connected non-empty n-dimensional manifold. Then the following holds:*

$$H_n(M, \partial M) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

39.3. **The fundamental class of a compact orientable manifold.** In the previous section we had just seen that for a connected, orientable, compact non-empty n-dimensional manifold M we have $H_n(M, \partial M) \cong \mathbb{Z}$. In the following we will see that a choice of an orientation determines a generator $[M]$ of $H_n(M, \partial M)$.

We recall that in Lemma 29.1 we gave a homeomorphism $\Theta\colon \Delta^n \to \overline{B}^n$ that restricts to a homeomorphism $\partial\Delta^n \to \partial\overline{B}^n$. Furthermore we recall that in Lemma 32.9 we showed that id_{Δ^n} represents a generator of $H_n(\Delta^n, \partial\Delta^n)$. Now we write[750]

$$\left[\,\overline{B}^n\right] \;:=\; (-1)^n \cdot \Theta_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot [\Theta] \;\in\; H_n(\overline{B}^n, \partial\overline{B}^n).$$

Now we can formulate the following proposition.

Proposition 39.7. *Let M be a compact oriented connected non-empty n-dimensional manifold. Then there exists a unique generator $[M]$ of $H_n(M, \partial M) \cong \mathbb{Z}$ with the following property: For any orientation-preserving embedding*[751] *$\Phi\colon \overline{B}^n \to M$ with $\Phi\big(\overline{B}^n\big) \subset M \setminus \partial M$ we have*

$$\Phi_*\big(\big[\overline{B}^n\big]\big) \;=\; \textit{the image of } [M] \textit{ under the map } H_n(M, \partial M) \to H_n(M, M \setminus \Phi(B^n)).$$

[749]This statement requires an argument that we are skipping

[750]The sign $(-1)^n$ is only necessary to make sure that the various sign conventions are consistent.

[751]We view $\overline{B}^n$ as a manifold with boundary and we equip it with the usual orientation where the standard basis $e_1, \dots, e_n$ is a positive basis for each tangent space.

Proof $(*)$. Let M be a compact oriented connected non-empty n-dimensional manifold. To simplify the notation we prove the proposition only in the closed case.

We pick a simplicial structure for the manifold M. We can and will assume that all the maps $\varphi_i\colon \Delta^n \to M$, restricted to $\mathring{\Delta}^n$, are orientation-preserving. We denote by $Y_i = \varphi_i(\Delta^n)$, $i = 1, \dots, k$ the corresponding n-simplices. In the proof of Proposition 39.5 we saw that

$$c \;:=\; \sum_{i\in I} Y_i \;\in\; C_n^{\mathrm{CW}}(M)$$

is a cycle. In the following we denote by $[M]$ the image of $[c]$ under the inverse of the isomorphism $H_n(M) \to H_n^{\mathrm{CW}}(M)$ from Proposition 35.4. We claim that $[M]$ has the desired properties. More precisely, we will prove the following claim.

Claim.

(a) There exists an orientation-preserving embedding $\Psi\colon \overline{B}^n \to M$ such that

$$\Psi\big([\overline{B}^n]\big) \;=\; \text{the image of } [M] \text{ under the map } H_n(M) \to H_n(M, M\setminus \Psi(B^n)).$$

(b) For any orientation-preserving embedding $\Phi\colon \overline{B}^n \to M$ we have

$$\Phi\big([\overline{B}^n]\big) \;=\; \text{the image of } [M] \text{ under the map } H_n(M) \to H_n(M, M\setminus \Phi(B^n)).$$

We start out with the proof of (a). We pick an orientation-preserving embedding map $\Psi\colon \overline{B}^n \to M$ such that $\Psi(\overline{B}^n)$ lies in the interior of an n-simplex Y_i of M and such that ∂Y_i is a deformation retract of $Y_i \setminus \Psi(B^n)$. (We refer to Figure 578 for an illustration.) We consider the following commutative diagram of homomorphisms:

$$\begin{tikzcd}
H_n(\Psi(\overline{B}^n), \Psi(\partial\overline{B}^n)) \arrow[r, "\cong", "\text{exc.}"'] & H_n(Y_i, Y_i\setminus\Psi(B^n)) \arrow[r, "\cong", "\text{exc.}"'] & H_n(M, M\setminus\Psi(B^n)) & H_n(M) \arrow[l] \\
H_n(\overline{B}^n, \partial\overline{B}^n) \arrow[u, "\Psi_*", "\cong"'] & H_n(Y_i, \partial Y_i) \arrow[u, "(*)", "\cong"'] \arrow[r, "\cong", "\text{exc.}"'] & H_n(M, M\setminus\mathring{Y}_i) \arrow[u, "\cong"'] & H_n(M) \arrow[l] \arrow[u, "\text{id}", "\cong"'] \\
H_n(\Delta^n, \partial\Delta^n) \arrow[u, "\Theta_*", "\cong"'] \arrow[r, "\varphi_{i*}"] & H_n^{\mathrm{CW}}(Y_i, \partial Y_i) \arrow[u, "\cong"'] \arrow[r, "\cong"] & H_n^{\mathrm{CW}}(M, M\setminus\mathring{Y}_i) \arrow[u, "\cong"'] & H_n^{\mathrm{CW}}(M). \arrow[l] \arrow[u, "\cong"', "{[M]}", "{[c]}"'] \arrow[lll, bend left, "{[\varphi_i]\mapsto[c]}"]
\end{tikzcd}$$

Here are the justification for some of the arrows and the claim that the diagram commutes:

(1) The horizontal maps in the first and second row and the vertical maps from the second to the first row are the obvious inclusion maps, which implies that the two top squares commute.
(2) The maps marked by "exc." are isomorphisms by the Excision Theorem 31.17.
(3) The bottom vertical maps are the isomorphisms from Proposition 35.4.
(4) The lower two squares commute by Proposition 35.4.[752]

[752]To be precise, one really needs a natural isomorphism between relative singular and cellular homology. That can be proved the same way as we proved Proposition 35.4.

(5) It follows from the long exact sequence in homology of the triple $(Y_i, Y_i \setminus \Psi(B^n), \partial Y_i)$, from the fact that we chose Ψ such that ∂Y_i is a deformation retract of $Y_i \setminus \Psi(B^n)$, and from Corollary 30.8 (4) that the homomorphism $(*)$ is an isomorphism.
(6) We denote by $[\varphi_i] \in H_n^{\mathrm{CW}}(Y_i, \partial Y_i)$ the generator which is defined by the homeomorphism $\varphi_i \colon (\Delta^n, \partial\Delta^n) \to (Y, \partial Y)$.
(7) It follows immediately from the definition of c that the image of $[c]$ under the map $H_n^{\mathrm{CW}}(M) \to H_n^{\mathrm{CW}}(Y_i, \partial Y_i)$ is precisely $[\varphi_i]$. Using this observation one obtains easily that the rectangle to the left commutes.

The homeomorphism $\Theta \colon \Delta^n \to \overline{B}^n$ restricts to a diffeomorphism $\mathring{\Delta}^n \to B^n$. It follows easily from the definitions that this diffeomorphism is orientation-preserving if and only if $(-1)^n$ is even. Since φ_i and Ψ are orientation-preserving we see, using a slight variation of Lemma 32.24, that the images of $[\overline{B}^n] = (-1)^n \cdot [\Theta]$ and $[\varphi_i]$ in $H_n(Y_i, Y_i \setminus \Psi(B^n))$ agree. From the commutative diagram it follows that $[\overline{B}^n]$ and $[M]$ get sent to the same element of $H_n(M, M \setminus \Psi(B^n))$. This concludes the proof of part (a) of the claim.

Now we turn to the proof of part (b). Thus let $\Phi \colon \overline{B}^n \to M$ be another orientation-preserving embedding. By Theorem 10.28 there exists a diffeomorphism f of M with the following two properties:

(1) the map f is homotopic to the identity,
(2) $f \circ \Psi = \Phi$.

It follows from (1) and from Proposition 30.5 that the map $f_* \colon H_n(M) \to H_n(M)$ is the identity. We consider the following commutative diagram

$$\begin{tikzcd}
 & H_n(M, M \setminus \Psi(\overline{B}^n)) \arrow[dd, "f_*"'] & H_n(M) \arrow[l] \arrow[dd, "f_*=\mathrm{id}"'] \\
H_n(\overline{B}^n, \partial\overline{B}^n) \arrow[ur, "\Psi_*"] \arrow[dr, "\Phi_*"'] & \Psi_*([\overline{B}^n]) \downarrow \Phi_*([\overline{B}^n]) & [M] \downarrow [M] \\
 & H_n(M, M \setminus \Phi(\overline{B}^n)) & H_n(M). \arrow[l]
\end{tikzcd}$$

It now follows immediately from this diagram that $[M]$ also has the desired property for Φ. This concludes the proof of part (b) of the claim. $\square$

Definition.

(1) Let M be a compact, oriented, connected, non-empty n-dimensional manifold. We refer to the uniquely determined element $[M] \in H_n(M, \partial M)$ from Proposition 39.7 as the *fundamental class of M*.
(2) If M is a compact oriented non-empty n-dimensional manifold with components $N_1, \ldots, N_k$, then we define

$$[M] \;:=\; \sum_{i=1}^{k} [N_i] \;\in\; H_n(M, \partial M) \;=\; \bigoplus_{i=1}^{k} H_n(N_i, \partial N_i).$$

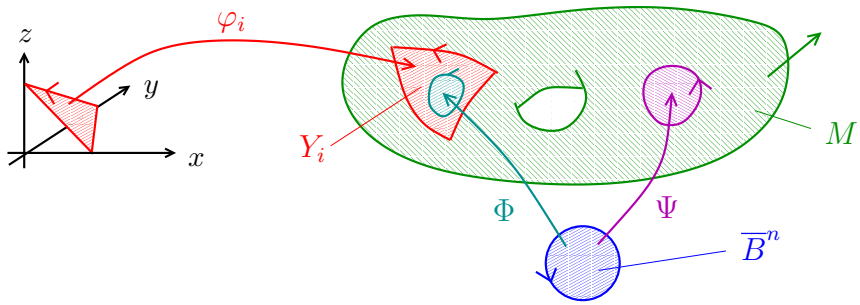


FIGURE 578. Illustration for the proof of Proposition 39.7.

Examples.

(1) If we apply Proposition 39.7 to $\mathrm{id}\colon \overline{B}^n \to \overline{B}^n$ we see that the homology class $\left[\overline{B}^n\right]$ that we introduced on page 945 is indeed the fundamental class of the oriented manifold $\overline{B}^n$.
(2) We endow S^n with the usual orientation, i.e given $P \in S^n$ and a basis $v_1, \dots, v_n$ of $T_P M$ we say it is a positive basis if $P, v_1, \dots, v_n$ is a positive basis for $\mathbb{R}^{n+1}$. It follows easily from the definitions that the corresponding fundamental class $[S^n] \in H_n(S^n)$ is precisely the standard generator that we introduced on page 814.
(3) If we make the usual identification $T = [0,1] \times [0,1]/\sim$ and if we give $[0,1] \times [0,1]$ the usual orientation, then the generator of $H_2(T)$ given in Lemma 33.13 equals *minus* the fundamental class of T.

Given an oriented manifold M we denote, as usual, by $-M$ the same manifold but with opposite orientation. Now we can formulate the following lemma.

Lemma 39.8. *For any compact, oriented, non-empty n-dimensional manifold M we have*

$$[-M] \;=\; -[M] \;\in\; H_n(M).$$

Proof $(*)$. Let M be a compact, oriented, non-empty n-dimensional manifold. We claim that $-[M]$ is the fundamental class of $-M$. We pick an embedding $\Phi\colon \overline{B}^n \to M \setminus \partial M$ that is orientation-preserving with respect to the standard orientation of $\overline{B}^n$ and the orientation $-M$. We denote by $r\colon H_n(M, \partial M) \to H_n(M, M \setminus \Phi(B^n))$ the obvious map. Furthermore we denote by $\rho\colon \overline{B}^n \to \overline{B}^n$ the map that is given reflection in the x-axis. Then

$$\Phi_*([\overline{B}^n]) \;\underset{\substack{\uparrow \\ \text{since } \rho^2 = \mathrm{id}}}{=}\; (\Phi \circ \rho)_*(\rho_*([\overline{B}^n])) \;\underset{\substack{\uparrow \\ \text{as in Lemma 32.16}}}{=}\; -(\Phi \circ \rho)_*([\overline{B}^n]) \;\underset{\substack{\uparrow \\ \text{since } \rho \circ \Phi\colon \overline{B}^n \to M \text{ is} \\ \text{orientation-preserving}}}{=}\; -r([M]) \;=\; r(-[M]).$$

□

On 205 we saw that the boundary of an oriented manifold inherits an orientation. For compact manifolds the following proposition gives the relationship between the fundamental class of the manifold and of its boundary.

Proposition 39.9. *If M is a compact, oriented n-dimensional manifold with non-empty boundary, then the connecting homomorphism*

$$\partial\colon H_n(M,\partial M) \;\to\; H_{n-1}(\partial M)$$

of the long exact sequence in homology of the pair $(M,\partial M)$ has the property that

$$[M] \;\mapsto\; [\partial M].$$

Proof $(*)$. Let M be a compact, oriented n-dimensional manifold with boundary components $N_1,\dots,N_k$. By Proposition 39.3 the manifold M admits a simplicial structure such that ∂M is a subcomplex. We can and will assume that all the maps for the n-simplices in the simplicial structure are orientation-preserving. Now we use this simplicial structure to define the cellular n-cycle $c \in C_n^{\mathrm{CW}}(M)$ as in the proof of Proposition 39.7. We also use the restriction of the simplicial structure to the various boundary components to define cellular cycles $d_i \in C_{n-1}^{\mathrm{CW}}(N_i)$, $i=1,\dots,k$. Now we consider the following diagram

$$\begin{tikzcd}
H_n^{\mathrm{CW}}(M) \arrow[r, "{[c]\mapsto[d_1]+\cdots+[d_k]}"] \arrow[d, "{[c]\mapsto[M]}"', "\cong"] & \bigoplus_{i=1}^{k} H_{n-1}^{\mathrm{CW}}(N_i) \arrow[d, "\cong"', "{[d_i]\mapsto[N_i]}"] \\
H_n(M) \arrow[r] & \bigoplus_{i=1}^{k} H_{n-1}(N_i)
\end{tikzcd}$$

where the vertical isomorphisms are given by Proposition 35.4. The proposition is an immediate consequence of the following observations:

(1) As in the proof of Propositions 39.5 we see that in the cellular chain complex of M we have $d_n(c) = d_1 + \cdots + d_k$.[753] This observation explains why on the top we have the assignment $[c] \mapsto [d_1] + \cdots + [d_k]$.[754]
(2) By definition of the fundamental classes the vertical maps send $[c]$ to $[M]$ and $[d_i]$ to $[N_i]$.
(3) The diagram commutes by Proposition 35.4. □

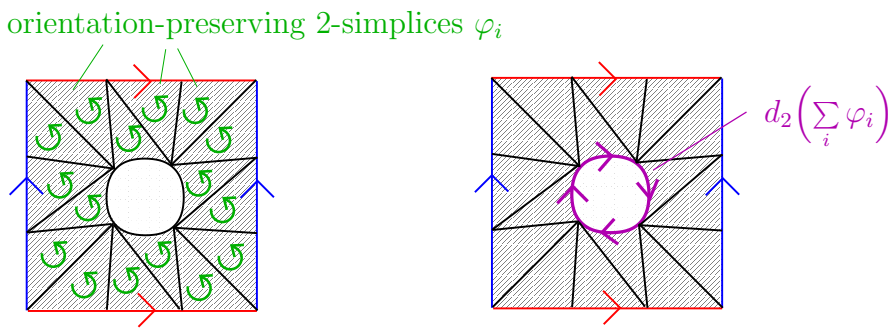
Example. In Figure 579 we consider the surface F that is given removing an open disk from a torus. We illustrate a simplicial structure with oriented 2-simplices. The sum of the oriented 2-simplices is a generator of $H_2^{\mathrm{CW}}(F,\partial F) = H_2(F,\partial F)$, and the boundary of the sum is a generator of $H_1^{\mathrm{CW}}(\partial F) = H_1(\partial F)$.

For the record we state the following elementary corollary to Proposition 39.9.

Corollary 39.10. *Let M be a compact, oriented n-dimensional manifold with non-empty boundary. We denote by $i\colon H_{n-1}(\partial M) \to H_{n-1}(M)$ the inclusion map.*

[753] A careful proof would require checking that all the various sign conventions work out to give the desired result.

[754] We skip the evidently painful discussion of the signs that arise.



torus, minus one open disk, viewed as a simplicial complex

FIGURE 579.

(1) *We have*[755]

$$\ker\left\{i_*\colon H_{n-1}(\partial M) \;\to\; H_{n-1}(M)\right\} \;=\; \mathbb{Z}\cdot[\partial M] \;\in\; H_{n-1}(\partial M).$$

(2) *If M has one boundary component, then $i_*\colon H_{n-1}(\partial M) \to H_{n-1}(M)$ is the zero map.*

(3) *The boundary ∂M is not a retract of M.*

Remark.

(1) In Proposition 31.6 we showed that the sphere $S^{n-1} = \partial\overline{B}^{n-1}$ is not a retract of the closed ball $\overline{B}^n$. Corollary 39.10 (3) can thus be viewed as a generalization of Proposition 31.6 from $\overline{B}^n$ to the case of compact oriented connected manifolds with non-trivial boundary.

(2) The question now arises, whether the condition that M is oriented is necessary in the statement of Corollary 39.10. We will discuss the case of the Klein bottle minus one open disk in Exercise 39.2.

Proof. Let M be a compact, oriented n-dimensional manifold with non-empty boundary.

(1) Using Proposition 39.9 and the long exact sequence in homology of the pair $(M, \partial M)$ we obtain that the following sequence is exact:

$$H_n(M,\partial M) \xrightarrow{[M]\mapsto[\partial M]} H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M).$$

Statement (1) is now an immediate consequence.

[755]Put differently, if M has boundary components $N_1,\dots,N_k$, then

$$\ker\left\{H_{n-1}(\partial M) \;\to\; H_{n-1}(M)\right\} \;=\; \sum_{i=1}^{k}[N_i] \;\in\; H_{n-1}(\partial M) \;=\; \bigoplus_{j=1}^{k} H_{n-1}(N_j).$$

(2) This statement is an immediate consequence of (1) and the fact that $[\partial M]$ is a generator of $H_{n-1}(\partial M) \cong \mathbb{Z}$.
(3) Using (1) one can prove the last statement almost the same way as Proposition 31.6. We just need to replace π_n by H_n. We leave the details to the reader. □

The following proposition gives us some extra information about homology groups of manifolds with boundary.

Proposition 39.11. *Let M be a connected n-dimensional manifold. If M is non-compact or if M has non-empty boundary, then the following statements hold:*
(1) *M is homotopy equivalent to an $(n-1)$-dimensional CW-complex,*
(2) *$H_{n-1}(M)$ is a free abelian group.*

Examples.
(1) The Möbius band and the annulus are both homotopy equivalent to S^1.
(2) Given $g \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we denote as usual by $\Sigma_{g,k}$ the surface of genus g with k open disks removed. In Figure 580 we sketch a deformation retraction from $\Sigma_{2,2}$ to the graph of Euler characteristic -4.
More generally, almost the same argument shows that $\Sigma_{g,k}$ with $k \geq 1$ admits a deformation retraction to a graph with k vertices and $2g+2(k-1)$ edges, i.e. a graph of Euler characteristic $k-(2g+2(k-1)) = -2g-k+2$. Using Proposition 9.21 we also see that $\Sigma_{g,k}$ is homotopy equivalent to the wedge of $2g+k-1$ circles.

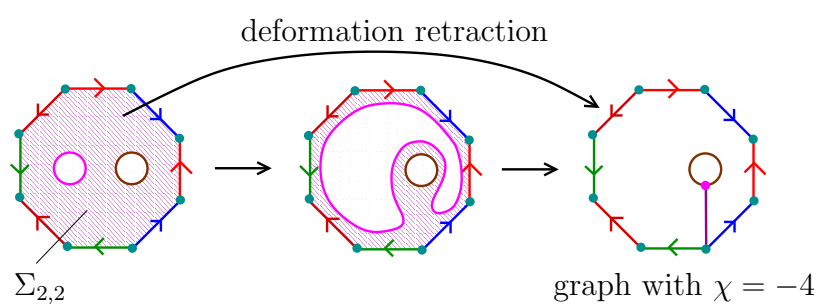


FIGURE 580.

(3) The non-compact manifold that is given by removing a point from the torus is easily seen to be homotopy equivalent to the wedge of two circles.
(4) In Figure 581 we show the surface M of infinite genus, which is non-compact, and a 1-dimensional CW-complex that is deformation retract of M. It is admittedly not overly obvious why the 1-dimensional CW-complex is indeed a deformation retract of M.

Proof. Let M be a connected n-dimensional manifold. We first consider the case that M is non-compact. In this situation it follows from [NR, Theorem 2.2], or alternatively [Whd5, Lemma 2.1], that M is homotopy equivalent to an $(n-1)$-dimensional CW-complex. It

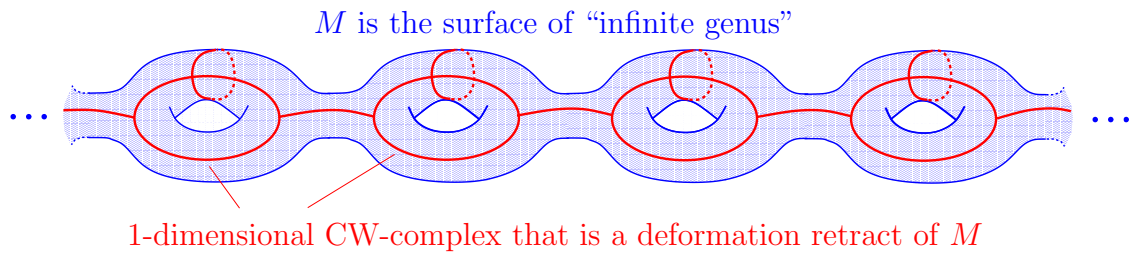


FIGURE 581.

is then a consequence of Corollary 30.8 (4) and Proposition 35.5 that $H_{n-1}(M)$ is a free abelian group.

Now suppose that M is compact, but that it has a non-empty boundary. By the Collar Neighborhood Theorem 10.10 there exists a collar neighborhood $\partial M \times [0,1)$. We write $N = M \setminus \partial M \times [0, \frac{1}{2})$. It is straightforward to see that N is a deformation retract of both M and $M \setminus \partial M$. In particular M is homotopy equivalent to the non-compact manifold $M \setminus \partial M$, which, by the above, is in turn homotopy equivalent to an $(n-1)$-dimensional CW-complex. □

Remark. Let M be a compact connected n-dimensional manifold with non-empty boundary. In this remark we give an intuitive argument why M is homotopy equivalent to a finite $(n-1)$-dimensional CW-complex.

First of all, by Proposition 39.3 we know that M admits a simplicial structure such that ∂M is a subcomplex. From Lemma 39.4 it follows that M contains an $(n-1)$-simplex c of order one. Let d be the n-dimensional simplex of W with $c \subset \partial d$. We denote by c' the union of all faces of d except for c. As on page 812 one can easily show that c' is a deformation retract of d. We denote by V_1 the union of all simplices of M except for d. Since c' is a deformation retract of d we see that V_1 is a deformation retract of M. We refer to Figure 582 for an illustration.

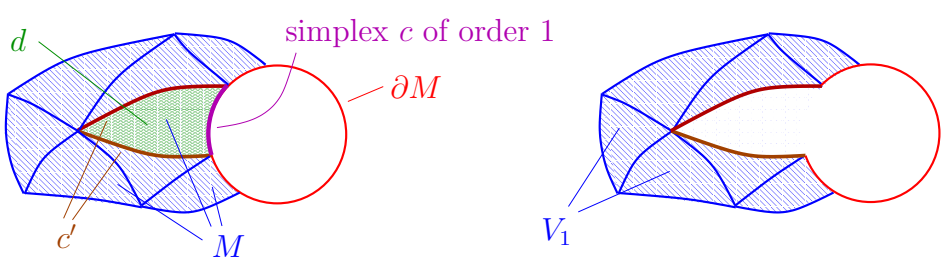
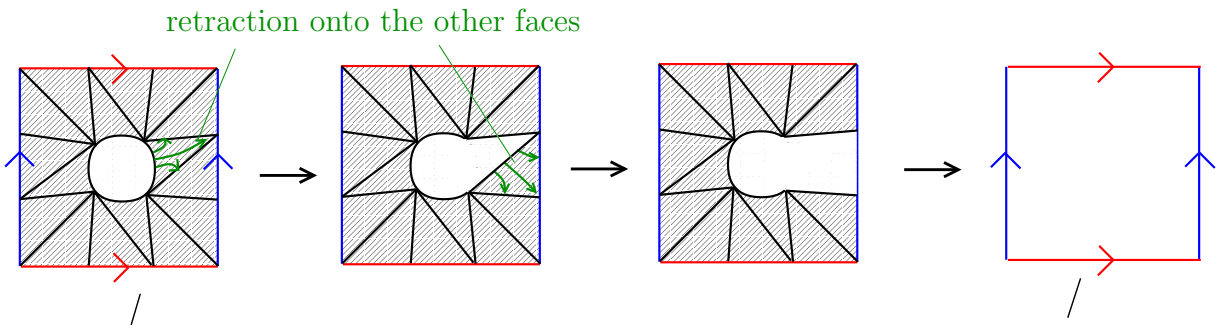


FIGURE 582.

Now we iterate this procedure[756] and we obtain a sequence $M =: V_0 \supset V_1 \supset \cdots \supset V_k$ of subcomplexes such that V_k has no n-simplices and where each V_{i+1} is a deformation retract of V_i. This implies in particular that the $(n-1)$-dimensional CW-complex V_k is also a deformation retract of M. This procedure is illustrated in Figure 583 for M the torus minus one open disk.



simplicial structure for the surface of genus one with one boundary component

1-dimensional simplicial complex

FIGURE 583.

We conclude this section with the following corollary.

Corollary 39.12. *Let M be a connected 2-dimensional manifold. If M is non-compact, then $\pi_1(M)$ is a free group.*

Remark. The statement of Corollary 39.12 generalizes Lemma 16.6.

Proof. Let M be a connected 2-dimensional manifold that is non-compact. By Proposition 39.11 we know that M is homotopy equivalent to a one-dimensional CW-complex. It follows from Proposition 9.12 (1) and Proposition 27.9 (1) that $\pi_1(M)$ is a free group. □

We conclude this section with a short discussion of fundamental groups of 2-dimensional manifolds.

Proposition 39.13.

(1) *If M is a connected 2-dimensional manifold that is non-compact, then $\pi_1(M)$ is a free group.*
(2) *If M is a connected 2-dimensional manifold, then every subgroup of $\pi_1(M)$ of infinite index is free.*
(3) *If M is a surface of genus ≥ 2 and if $\Gamma \subset \pi_1(M)$ is a* normal *non-trivial subgroup of infinite-index, then $\pi_1(M)$ is a free group of infinite index.*

[756]Here is the snag: how can one prove that we can iterate this procedure, i.e. why can we always find an $(n-1)$-simplex of degree one?

Example. The 2-dimensional manifolds shown in Figure 584 are non-compact and thus their fundamental groups are free. For the example on the left we saw this already in Lemma 16.6.

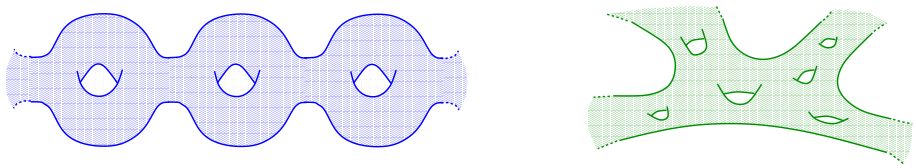


FIGURE 584.

Proof.

(1) Let M be a connected 2-dimensional manifold that is non-compact. By Proposition 39.11 we know that M is homotopy equivalent to a one-dimensional CW-complex. It follows from Proposition 9.12 (1) and Proposition 27.9 (1) that $\pi_1(M)$ is a free group.

(2) Let M be a connected 2-dimensional manifold and let $\Gamma \subset \pi_1(M)$ be a subgroup of infinite index. By Proposition 20.8 there exists a connected covering $p\colon \widetilde{M} \to M$ of infinite degree with $\pi_1(\widetilde{M}) \cong \Gamma$. By Lemma 7.4 (5) we know that $\widetilde{M}$ is non-compact so it follows from Proposition 8.1 (2) and from statement (1) above that $\Gamma \cong \pi_1(\widetilde{M})$ is a free group.

(3) By (2) it remains to show that Γ is infinitely generated. A proof for this statement is given in [Cat, Lemma 3.4], alternatively one can prove the second statement using [Lü4, Theorem 0.7]. A significantly stronger statement is proved in [BH, Theorem 1]. □

We will prove the following corollary in Exercise 39.1.

Corollary 39.14. *For any $g \in \mathbb{N}$ the fundamental group of the surface of genus g is torsion-free.*

39.4. **The homology groups of the connected sum of two manifolds.** Let M and N be oriented two closed oriented connected n-dimensional manifolds. Now we recall the definition of the connected sum $M\#N$ that we gave on page 355. First we pick an orientation-preserving embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and we pick an orientation-reversing embedding $\psi\colon \overline{B}^n \to N \setminus \partial N$. Then we set

$$M\#N \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

In Proposition 12.11 we showed that $\pi_1(M\#N)$ is the free product of $\pi_1(M)$ and $\pi_1(N)$. Now we have a very similar result for the homology groups of $M\#N$.

Proposition 39.15. *Let M and N be two closed oriented connected n-dimensional non-empty manifolds. Then for any $k \in \mathbb{N}_0$ we have*

$$H_k(M\#N) \;\cong\; \begin{cases} H_k(M) \oplus H_k(N), & \text{if } k \neq 0, n, \\ \mathbb{Z}, & \text{if } k = 0, n. \end{cases}$$

Moreover, for $k \neq 0, n$ the inclusion maps induce isomorphisms

$$H_k(M) \oplus H_k(N) \;\xleftarrow{\cong}\; H_k(M \setminus B^n) \oplus H_k(N \setminus B^n) \;\xrightarrow{\cong}\; H_k(M\#N)$$

where $B^n \subset M$ and $B^n \subset N$ denote the open balls used in the definition of the connected sum of M and N.

Remark. In Exercise 39.6 we will consider the homology groups of $M\#N$ if we drop the condition that the manifolds are closed.

The key to the proof of Proposition 39.15 is the following lemma.

Lemma 39.16. *Let X and Y be two compact orientable connected n-dimensional manifolds such that $\partial X \cong S^{n-1}$ and $\partial Y \cong S^{n-1}$. We pick a diffeomorphism $\varphi\colon \partial X \to \partial Y$.*

(1) *There exists an isomorphism*

$$H_n(X \cup_\varphi Y) \;\cong \mathbb{Z}$$

and for any $k < n$ the inclusion maps induce isomorphisms

$$\widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) \;\xrightarrow{\cong}\; \widetilde{H}_k(X \cup_\varphi Y).$$

(2) *In particular for $Y = \overline{B}^n$ we obtain that for any $k < n$ the inclusion induces an isomorphism*

$$\widetilde{H}_k(X) \;\xrightarrow{\cong}\; \widetilde{H}_k(X \cup_\varphi \overline{B}^n).$$

Proof. The second statement is an immediate consequence of the first statement and the fact that $\widetilde{H}_k(\overline{B}^n) = 0$. So it suffices to prove the first statement. To simplify the notation we write $S^{n-1} = \partial X = \partial Y$. We also write $Z = X \cup_\varphi Y$. It follows the Mayer-Vietoris Theorem 33.10 that there exists a long exact sequence

$$\to \widetilde{H}_k(S^{n-1}) \xrightarrow{i \oplus -i} \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) \xrightarrow{i+i} \widetilde{H}_k(Z) \xrightarrow{\partial_k} \widetilde{H}_{k-1}(S^{n-1}) \to \dots$$

where i always stands for the corresponding inclusion induced map. By Proposition 31.3 we have $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_k(S^n) = 0$ for $k \neq n$. We see immediately that for any $k < n$ the inclusion maps induce an isomorphism $\widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) \to \widetilde{H}_k(Z)$.

Now we turn to the n-th homology of Z. First note that by Proposition 39.6 we have $\widetilde{H}_n(X) = \widetilde{H}_n(Y) = 0$. Therefore the above long exact sequence gives rise to an exact sequence

$$0 \to \widetilde{H}_n(Z) \xrightarrow{\partial_{n-1}} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{i \oplus -i} \widetilde{H}_{n-1}(X) \oplus \widetilde{H}_{n-1}(Y) \to \widetilde{H}_{n-1}(Z) \to 0.$$

Since S^{n-1} is the unique boundary component of X we deduce from Corollary 39.10 that the inclusion induced map $\widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(X)$ is the zero map. The same way we

see that the inclusion induced map $\widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(Y)$ is also the zero map. It now follows from the above long exact sequence that $\widetilde{H}_n(Z) \cong \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. $\square$

Now we can easily provide the proof of Proposition 39.15.

Proof of Proposition 39.15. Let M and N be oriented two closed oriented connected non-empty n-dimensional manifolds. As required in the definition of the connected sum of M and N we pick an orientation-preserving embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and we pick an orientation-reversing embedding $\psi\colon \overline{B}^n \to N \setminus \partial N$. Recall that

$$M\#N \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

We write $X = M \setminus \varphi(B^n)$ and $Y = N \setminus \psi(B^n)$. It follows immediately from Lemma 39.16 (1) that $H_n(M\#N) = H_n(X \cup_{S^{n-1}} Y) \cong \mathbb{Z}$.

Now let $k < n$. In this case we have

$$\begin{array}{ccccc} H_k(M) \oplus H_k(N) & \xleftarrow{\cong} & H_k(\underbrace{M \setminus \varphi(B^n)}_{=X}) \oplus H_k(\underbrace{N \setminus \psi(B^n)}_{=Y}) & \xrightarrow{\cong} & H_k(M\#N) \\ & \uparrow & & \uparrow & \\ & \text{Lemma 39.16 (1) applied to} & & \text{Lemma 39.16 (1) applied to } M\#N = X \cup Y & \\ & M = X \cup \overline{B}^n \text{ and } N = Y \cup \overline{B}^n & & & \end{array}$$

$\square$

Examples.

(1) On page 884 we showed that

$$H_k(\mathbb{C}\mathrm{P}^2) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Proposition 39.15 that

$$H_k(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Since the homology groups do not depend on the orientation we have the same homology groups for $\overline{\mathbb{C}\mathrm{P}}^2$ and the calculation shows that the homology groups of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ agree.

(2) Let $T = S^1 \times S^1$ be the torus. In Figure 224 we saw that the connected sum $T\#T$ is the surface Σ_2 of genus 2. More generally, the same argument shows that the connected sum of g copies of the torus T is the surface Σ_g of genus g. Together with the calculation of the homology groups of the torus in Lemma 33.12 we now see that

$$H_k(\Sigma_g) \;\cong\; H_k(\underbrace{T\#\ldots\#T}_{g \text{ copies}}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g}, & \text{if } k = 1, \\ 0, & \text{if } k > 2. \end{cases}$$

This agrees of course with the result obtained in Proposition 35.10.

Proposition 39.17. *If M is a closed, orientable, connected n-dimensional manifold, then the homology group $H_{n-1}(M)$ is a free abelian group.*

Remark. Note the example of $M = \mathbb{R}\mathrm{P}^2$ shows that the statement of Proposition 39.17 does not hold if we drop the condition that M has to be orientable.

Proof. Let M be a closed, orientable, connected n-dimensional manifold. We pick an embedding $\Phi\colon \overline{B}^n \to M$. Since M is closed, orientable and connected we can apply Lemma 39.16 which says that $H_{n-1}(M) \cong H_{n-1}(M \setminus \Phi(B^n))$. But $M \setminus \Phi(B^n)$ is a manifold with boundary, hence it follows from Proposition 39.11 that $H_{n-1}(M \setminus \Phi(B^n))$ is a free abelian group. □

Remark. In this section we have obtained many results on the homology groups of manifolds. It is natural to ask whether the statements also hold for topological manifolds. In the above arguments we made use of the fact that manifolds have a simplicial structure. This is in general not the case for topological manifolds. In a later chapter we will reprove the above statements for topological manifolds using a different approach. This new approach will be more general, but also more abstract and less intuitive.

39.5. **Representing homology classes by manifolds** $(*)$.

Definition. Let M be a manifold.

(1) If N is a closed, oriented, non-empty n-dimensional submanifold of M, then, by a slight abuse of notation, we denote by $[N]$ the image of the fundamental class $[N]$ in $H_n(M)$ under the inclusion induced map $H_n(N) \to H_n(M)$.
(2) We say that a homology class $z \in H_n(M)$ *is represented by a submanifold* if there exists an n-dimensional closed, oriented, non-empty submanifold N such that $[N] = z \in H_n(M)$.

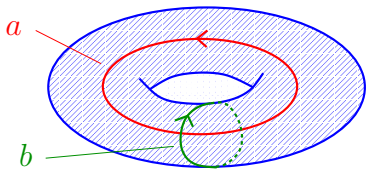
Examples.

(1) We first consider the 2-dimensional torus $T = S^1 \times S^1$. Under the usual isomorphism $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$ the basis element $(1, 0)$ is represented by the oriented submanifold $S^1 \times \{1\}$ and the basis element $(0, 1)$ is represented by the oriented submanifold $\{1\} \times S^1$. This situation is illustrated in Figure 586 on the left.

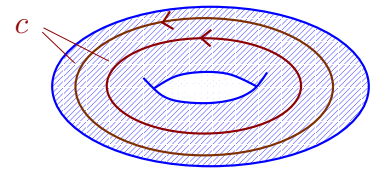
    Next we consider the homology class $(2, 0)$. As shown in Figure 586 on the right, we can represent it by a disconnected submanifold. But can we also represent it by a connected submanifold?
(2) We consider the surface Σ_2 of genus 2. As in Figure 224 we view Σ_2 as the connected sum $T\#T$ of two tori. If we use the calculation on page 956 to determine an isomorphism $H_1(\Sigma_2) \cong \mathbb{Z}^4$, then the basis elements are represented by the submanifolds a_1, b_1, a_2, b_2 that are illustrated in Figure 586.[757] [758]

[757] Another point of view is that the submanifolds a_1, b_1, a_2, b_2 are given by deforming the curves x_1, y_1, x_2, y_2 that are shown in Figure 259.

[758] What do these submanifolds look like in the formal picture of the surface of genus 2 as the quotient $E_8/\sim$ of the Euclidean octagon?

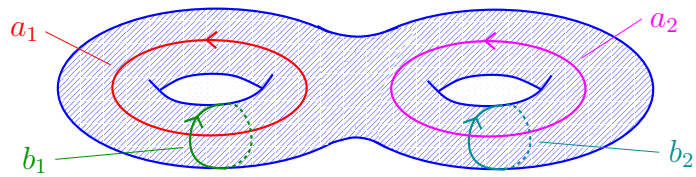


basis for homology of H_1(torus)



homology class represented
by a disconnected submanifold

FIGURE 585.



basis for homology of H_1(surface of genus 2)

FIGURE 586.

(3) We consider again the complex projective space $\mathbb{C}\mathrm{P}^n$. We saw on page 649 that for $k \leq n$ the map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

is an embedding. We use this map to view $\mathbb{C}\mathrm{P}^k$ as a submanifold of $\mathbb{C}\mathrm{P}^n$. It follows from the discussion on page 884 and the natural isomorphism of Proposition 35.4 that

$$H_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^i], & \text{if } k = 2i \text{ for some } i \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, the generators of the non-trivial homology groups of the complex projective space $\mathbb{C}\mathrm{P}^n$ are represented by the complex projective spaces $\mathbb{C}\mathrm{P}^k$ with $k = 0, \ldots, n$.

If X is a topological space that is not a manifold, then it does not make sense to ask whether a homology class is represented by a submanifold. But given a homology class $c \in H_n(X)$ it still makes sense to ask whether there exists a map from a closed oriented manifold M such that the image of the fundamental class $[M]$ equals c. This leads to the following definition.

Definition. Let X be a topological space and let $z \in H_n(X)$ be a homology class. We say *z can be realized by a manifold* if there exists an n-dimensional closed oriented connected, non-empty manifold M and a map $f\colon M \to X$ such that $f_*([M]) = z$.

Example. If X is a path-connected topological space, then it follows from Proposition 38.3 that any class in $H_1(X)$ is realized by a manifold. Furthermore a relatively straightforward argument, see [Cal, p. 3] shows that also every class in $H_2(X)$ is realized by a manifold.

The following theorem was first proved by René Thom [Tho, Tho2] in 1954[759] (See also [Rudy, Theorem IV.7.37]). The proof of the theorem uses methods that go well beyond our present abilities. We will not make use of the theorem.

Theorem 39.18.

(1) *If X is a topological space, then every homology class of dimension ≤ 6 can be realized by a manifold.*
(2) *Given any $n \geq 7$ there exists a topological space X and a homology class in $H_n(X)$ that is not realized by a manifold.*
(3) *Given any $n \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}$ such that for any topological space X and any homology class $z \in H_n(X)$ the class $k \cdot z$ can be realized by a manifold.*

39.6. **The degree of a map between oriented manifolds.**

Definition. Let $f\colon M \to N$ be a map between two closed oriented connected, non-empty n-dimensional manifolds. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$f_*([M]) \;=\; \deg(f) \cdot [N].$$

We refer to $\deg(f)$ as the *degree of f*.

Example. For a map $f\colon S^n \to S^n$ the above degree is exactly the same as the degree that we defined on page 819.

In the following lemma we summarize two key properties of the degree.

Lemma 39.19.

(1) *Let $f\colon L \to M$ and $g\colon M \to N$ be two maps between two closed oriented connected, non-empty n-dimensional manifolds. Then*

$$\deg(g \circ f) \;=\; \deg(g) \cdot \deg(f).$$

(2) *Let $f\colon M \to N$ be a map between two closed oriented connected, non-empty n-dimensional manifolds. Let $y \in N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$ and such that f is a local diffeomorphism around each x_i. Then*

$$\deg f \;=\; \sum_{i=1}^{m} \begin{array}{rl} 1, & \text{if } f \text{ is strongly locally orientation-preserving at } x_i \\ -1, & \text{if } f \text{ is o-locally orientation-reversing at } x_i. \end{array}$$

Proof.

(1) Exactly as in the proof of Lemma 32.16 (4) we have

$$(g \circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f) \cdot [L]) \;=\; \deg(g) \cdot \deg(f) \cdot [L].$$

This shows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

[759] René Thom (1923-2002) was a French mathematician who received the Fields Medal in 1958.

(2) The second statement is proved in a very similar fashion to Proposition 32.26 and Lemma 32.24. We leave it as an exercise to make the not overly difficult modifications of the proofs. □

The following proposition gives us many examples of degrees of maps between manifolds.

Proposition 39.20. *Let $p\colon \widetilde{N} \to N$ be a finite covering of closed oriented connected, non-empty manifolds. If p is orientation-preserving, then*

$$\deg(p) \;=\; [\widetilde{N} : N].$$

Proof. This proposition follows immediately from Lemma 39.19 (2). □

Proposition 39.21. *Let M be a closed, oriented, non-empty n-dimensional manifold. Then for any $k \in \mathbb{Z}$ there exists a map $f\colon M \to S^n$ of degree k.*

Proof. Let M be a closed, oriented, non-empty n-dimensional manifold and let $k \in \mathbb{Z}$. By Lemmas 32.15 and 32.18 there exists a map $g\colon S^n \to S^n$ of degree k. It follows from this fact and Lemma 39.19 that it suffices to show that there exists a map $f\colon M \to S^n$ of degree ± 1. We will show the existence in the remainder of the proof.

First we pick an orientation-preserving embedding $\Phi\colon \overline{B}^n \to M$. This map descends to a homeomorphism $\overline{B}^n/S^{n-1} \to M/(M \setminus \Phi(B^n))$ that we denote by $\overline{\Phi}$. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from page 99. We denote by f the map given by

$$f\colon M \;\to\; M/(M \setminus \Phi(B^n)) \;\xrightarrow{\overline{\Phi}^{-1}}\; \overline{B}/S^{n-1} \;\xrightarrow{\Omega}\; S^n.$$

We claim that f is a map of degree ± 1. This follows quite easily from Lemma 39.19 (2), but since we did not prove that lemma we will give a direct proof.

It suffices to show that f induces an isomorphism $\widetilde{H}_n(M) \to \widetilde{H}_n(S^n)$. Since $\overline{\Phi}$ and Ω are homeomorphisms it suffices to show that the first map induces an isomorphism on homology. We consider the following diagram of homomorphisms

$$\begin{tikzcd}
\underbrace{\widetilde{H}_n(M \setminus \Phi(B^n))}_{=0} \arrow[r] & \widetilde{H}_n(M) \arrow[r] \arrow[dr] & H_n(M, M \setminus \Phi(B^n)) \arrow[r] \arrow[d, "\cong"'] & \widetilde{H}_{n-1}(M \setminus \Phi(B^n)) \arrow[r] & \widetilde{H}_{n-1}(M) \\
& & \widetilde{H}_n(M/(M \setminus \Phi(B^n))). & &
\end{tikzcd}$$

We make the following remarks:

(1) The horizontal sequence is the long exact sequence in reduced homology of the pair $(M, M \setminus \Phi(B^n))$,
(2) the group on the left is zero by Proposition 39.5,
(3) the map on the right is an isomorphism by Lemma 39.16.
(4) the vertical map is the natural isomorphism of Proposition 31.19.

It follows that the diagonal map is an isomorphism. □

The following theorem says that the degree of a map is a complete invariant up to homotopy. We will not make use of this theorem.

Theorem 39.22. (Hopf Theorem) *Let M be a closed oriented connected, non-empty n-dimensional manifold and let $f, g\colon M \to S^n$ be two maps. Then*

$$\deg(f) = \deg(g) \quad \Longleftrightarrow \quad \textit{the maps } f \textit{ and } g \textit{ are homotopic.}$$

Proof. The "$\Leftarrow$"-direction is an immediate consequence of Proposition 30.5. The "$\Rightarrow$"-direction is much harder, it was initially proved by Heinz Hopf whom we had already encountered on page 726. We refer to [OR, p. 191] or [Mi2, p. 51] for a proof. We will prove a generalization later on in Theorem **??** □

For the record we also state the following open question, see [Kir2, Problem 5.26], which was initially posed by Heinz Hopf.

Question 39.23. (Hopf) *Given a closed, orientable manifold M, is every map $f\colon M \to M$ of degree ± 1 a homotopy equivalence?*

In the following discussion, given $g \in \mathbb{N}_0$ we denote as usual by Σ_g the surface of genus g.

Lemma 39.24. *For any $g \geq k$ there exists a degree-one map from Σ_g to Σ_k*

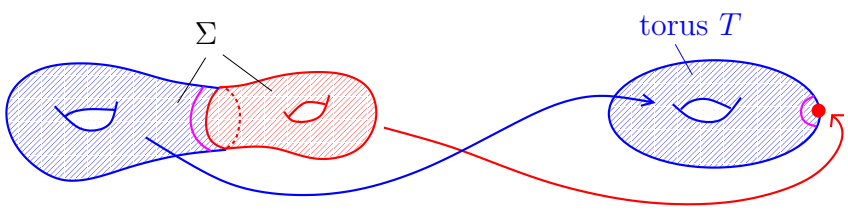


FIGURE 587. A degree-one map from the surface of genus 2 to the torus.

Proof. The case $k = 0$ follows immediately from Proposition 39.21. In the following we will cover the case $g = 2$ and $k = 1$. All other cases will be a variation of the subsequent argument and we leave them as an exercise to the reader. We denote by Σ the surface of genus 2 and we denote by T the torus.

We consider the maps $\Phi\colon \overline{B}^2 \to \Sigma$, $\Psi\colon \overline{B}^2 \to T$ and $\mu\colon \Sigma \to T$ that are illustrated in Figure 588. These maps can be chosen such that $\Psi = \mu \circ \Phi$. We had already encountered the map μ on page 250. We claim that μ is a degree-one map. This follows fairly easily from Lemma 39.19 (2), but since we did not prove that lemma we will give a direct proof.

We will follow the construction of the proof of Proposition 39.21 to obtain a degree-one map $f\colon \Sigma \to S^2$ from Φ and to obtain a degree-one map $g\colon T \to S^2$ from Ψ. It follows from $\Psi = \mu \circ \Phi$ and the explicit constructions that $f = g \circ \mu$. We obtain the following commutative diagram

$$\begin{tikzcd} H_2(\Sigma) \arrow[rr, "\mu_*"] \arrow[dr, "f_*", "{[\Sigma]\to[S^2]}"'] & & H_2(T) \arrow[dl, "g_*"', "{[T]\mapsto[S^2]}"] \\ & H_2(S^2). & \end{tikzcd}$$

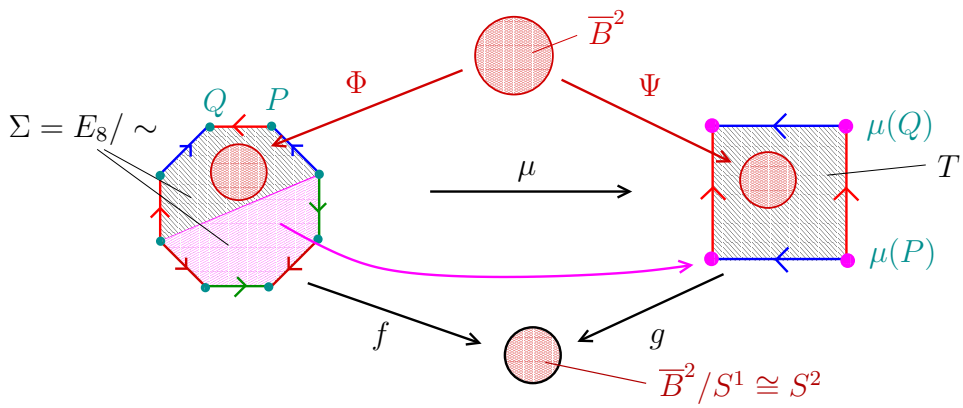


FIGURE 588.

Since f and g are by degree-one maps we see that the diagonal maps are isomorphisms with $f_*([\Sigma]) = [S^2]$ and $g_*([T]) = [S^2]$. We have thus shown that $\mu\colon \Sigma \to T$ is a degree-one map. $\square$

The following question arises naturally.

Question 39.25. *Suppose that $g < k$. Does there exist a degree-one map from Σ_g to Σ_k?*

The following proposition is the key to answering Question 39.25.

Proposition 39.26. *Let $f\colon M \to N$ be a map between closed oriented connected, non-empty n-dimensional manifolds. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.*

The statement of Proposition 39.26 is somewhat surprising, in so far as it takes as input fairly simple information, namely the induced map on n-th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f\colon M \to N$ be a map between closed oriented connected n-dimensional manifolds. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N, Q)$ and $\Gamma = f_*(\pi_1(M, P))$.

By Proposition 20.8[760] there exists a path-connected covering $p\colon (\widetilde{N}, \widetilde{Q}) \to (N, Q)$ such that $p_*(\pi_1(\widetilde{N}, \widetilde{Q})) = \Gamma$ and such that $[\widetilde{N} : N] = [\pi : \Gamma]$. By Proposition 8.1 the covering space $\widetilde{N}$ is an oriented manifold without boundary. By Proposition 20.2 the map $f\colon (M, P) \to (N, Q)$ lifts to a map $\widetilde{f}\colon (M, P) \to (\widetilde{N}, \widetilde{Q})$. We can summarize the situation

[760]Here we use that according to the discussion on pages 549 any manifold is reasonable.

in the following simple commutative diagram

$$\begin{tikzcd} & \widetilde{N} \arrow[d, "p", "\text{covering of degree } [\pi\colon\Gamma]"'] \\ M \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & N. \end{tikzcd}$$

First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N} : N] = [\pi : \Gamma]$ and Lemma 7.4 (5) that $\widetilde{N}$ is not compact. By Proposition 39.5 we therefore have $H_n(\widetilde{N}) = 0$. But this shows that $f_* = p_* \circ \widetilde{f}_* \colon H_n(M) \to H_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is in fact a finite-index subgroup. It follows from $[\widetilde{N} : N] = [\pi : \Gamma]$ and Lemma 7.4 (4) that $\widetilde{N}$ is compact. Therefore, by the above we now know that $\widetilde{N}$ is a closed oriented connected n-dimensional manifold. We deduce that

$$\pm 1 \;=\; \deg(f) \;\underset{\substack{\uparrow \\ \text{Lemma 39.19}}}{=}\; \deg(\widetilde{f}) \cdot \deg(p) \;\underset{\substack{\uparrow \\ \text{Proposition 39.20}}}{=}\; \deg(\widetilde{f}) \cdot [\widetilde{N} : N] \;=\; \deg(\widetilde{f}) \cdot [\Gamma\colon \pi].$$

This shows that $[\Gamma\colon \pi] = 1$, i.e. $\Gamma = \pi$. □

Now we can give a negative answer to Question 39.25.

Corollary 39.27. *If there exists a degree-one map from Σ_g to Σ_k, then $g \geq k$.*

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k. By Proposition 39.26 we obtain an epimorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By Proposition 14.3 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from Lemma 11.6 that $g \geq k$. □

We conclude this chapter with several questions. Recall that it follows from Lemma 39.24 that there exists a degree-one map $S^2 \times S^2 \to S^4$. It is natural to ask whether the converse holds.

Question 39.28. *Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?*

Later on we will also address the following question.

Question 39.29. *Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$?*

Finally it is also interesting to study degrees of self-maps of manifolds. It follows from Lemmas 32.15 and 32.18 that given any $k \in \mathbb{Z}$ and given any $n \in \mathbb{N}$ there exists a map $f\colon S^n \to S^n$ of degree k. Furthermore it follows easily from Proposition 39.20 that the torus $T = S^1 \times S^1$ admits a self-map of any degree.

We conclude with the following questions.

Question 39.30.

(1) *Let $g \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?*
(2) *Let $g \geq 2$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?*
(3) *Does there exist a self-map $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ of degree -1?*

We will address the first question in Exercise 39.5 and we will address the last two questions in a later chapter.

Exercises for Chapter 39.

Exercise 39.1. Let $g \in \mathbb{N}_{\geq 2}$. Use Proposition 39.13 to show that the fundamental group of the surface of genus g is torsion-free.

Exercise 39.2. Let K be the Klein bottle minus one open disk. Is ∂K a retract of K?

Exercise 39.3. Let $K \subset S^3$ be a knot and let $\Phi\colon S^1 \times \overline{B}^2 \to S^3$ be an embedding such that $\Phi(S^1 \times 0) = K$, i.e. $\Phi(S^1 \times \overline{B}^2)$ is a tubular neighborhood of K. We refer to $X = S^3 \setminus \Phi(S^1 \times B^2)$ as the exterior of the knot K. This is a compact 3-manifold with boundary that is given by the torus $\Phi(S^1 \times S^1)$. We denote by $\mu = \Phi(* \times S^1)$ the meridian of K. In Lemma 33.15 we saw that μ, viewed as an element in $H_1(X)$, is a generator of $H_1(X)$.

(a) Show that the kernel of $H_1(\Phi(S^1 \times S^1)) \to H_1(X)$ is isomorphic to $\mathbb{Z}$. We refer to either of the two generators of the kernel as the longitude λ of K.
(b) Sketch longitudes for the three knots that are shown in Figure 589.
(c) Now suppose that we are given two knots K and J. We define X_K, X_J, μ_K, μ_J and λ_K and λ_J as above. We pick a homeomorphism $\varphi\colon \partial X_K \to \partial X_J$ with $\varphi_*(\mu_K) = \lambda_J$ and $\varphi_*(\lambda_K) = \mu_K$. We denote by $Y = X_K \cup_\varphi X_J$ the result of gluing X_K to X_J along the boundary. Compute the homology groups of Y.

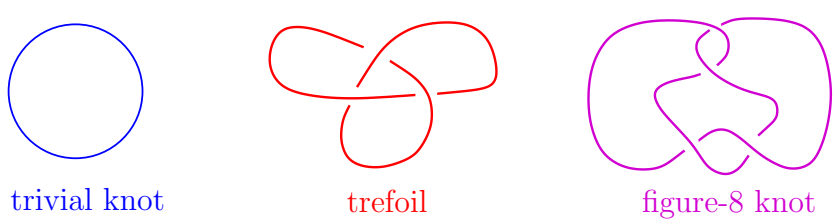


FIGURE 589.

Exercise 39.4.

(a) Let $L \subset S^3$ be a m-component link, i.e. L is a union of m disjoint 1-dimensional submanifolds $L_1, \dots, L_m$ of S^3, each of which is diffeomorphic to S^1. We pick disjoint tubular neighborhoods $L_i \times \overline{B}^2$, $i = 1, \dots, m$. We write $X_L := S^3 \setminus \bigcup_{i=1}^m L_i \times B^2$.

Determine the homology groups of X_L.
Hint: Use the Mayer-Vietoris sequence for manifolds.

(b) In Figure 590 we see two 2-component links, namely the unlink and the Hopf link. Are these two links isotopic?

2-component trivial link Hopf link

FIGURE 590.

Exercise 39.5. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g.

(a) Let $g = 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(b) Let $g \geq 1$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?

Exercise 39.6. Let M and N be two oriented, connected n-dimensional manifolds. We suppose that M is closed and that N has *non-empty* boundary. Express the homology groups of $M\#N$ in terms of the homology groups of M and N.

40. The Euler characteristic

40.1. **The Euler characteristic and homology groups.** Let X be a finite CW-complex, i.e. let X be a CW-complex with finitely many cells. Recall that on page 691 we defined the Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;=\; \sum_n (-1)^n \cdot \text{number of } n\text{-cells}.$$

A short glance on the examples in Figure 399 shows that the three different CW-structures for the torus provided in that figure have the same Euler characteristic.

The following proposition explains why this is not a coincidence.

Proposition 40.1. *Let X be a finite CW-complex, then*[761]

$$\chi(X) \;=\; \sum_n (-1)^n \cdot \operatorname{rank}(H_n(X)).$$

Examples.

(1) It follows either from the CW-structure of the n-dimensional sphere S^n with one 0-cell and one n-cell given on page 648 or from the calculation of the homology groups in Proposition 31.3 that

$$\chi(S^n) \;=\; 1 + (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Let $n \in \mathbb{N}_0$. We want to determine the Euler characteristic of $\overline{B}^n$. It is actually somewhat tricky to write down a CW-structure for $\overline{B}^n$. But it follows immediately from Proposition 40.1 and Corollary 30.8 (3) that $\chi(\overline{B}^n) = 1$.

(3) Given any $k, n \in \mathbb{N}_0$ we deduce from Corollary 30.8, Proposition 31.3 and Proposition 40.1 that $\chi(S^k \times \overline{B}^n) = \chi(S^k) = 1 + (-1)^k$.

(4) It follows either from the CW-structure provided on page 646 or alternatively from Proposition 35.10 that for a surface Σ_g of genus g we have

$$\chi(\Sigma_g) \;=\; 2 - 2g.$$

(5) It follows immediately from the CW-structures provided in Lemma 26.2 that for any $n \in \mathbb{N}_0$ we have

$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even} \end{cases} \qquad \text{and} \qquad \chi(\mathbb{C}\mathrm{P}^n) \;=\; n+1.$$

Before we turn to the proof of Proposition 40.1 we draw the following corollary.

[761] As a reminder, if G is a finitely generated abelian group, then it follows from the classification of finitely generated abelian groups, see Theorem 11.4, that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We then call r the *rank* of G. Put differently, let $T \subset G$ be the torsion subgroup, then G/T is isomorphic to $\mathbb{Z}^r$ for an r, and this r is just the rank of G.

Corollary 40.2. *Let X and Y be finite CW-complexes. Then*

$$X \text{ and } Y \text{ are homotopy equivalent} \quad \Longrightarrow \quad \chi(X) = \chi(Y).$$

Proof. The corollary is an immediate consequence of Proposition 40.1 together with Corollary 30.8 (2). □

Example. On page 437 we introduced two examples of closed orientable 2-dimensional manifolds for which we found it difficult to determine the genus. We show the same surfaces also in Figure 591. Let us first consider the surface on the left: it is given identifying opposite sides of the hexagon E_6. The obvious CW-structures on the hexagon H_6 descends to a CW-structure on $E_6/\sim$. We make the following observations:

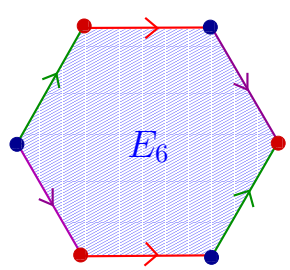
(1) the single 2-cell on E_6 defines a 2-cell for $E_6/\sim$,
(2) we identify every 1-cell of E_6 with the opposite 1-cell, thus $E_6/\sim$ has three 1-cells,
(3) we identify all 0-cells that differ by an even number of vertices, thus $E_6/\sim$ has precisely two 0-cells.

Summarizing we see that $E_6/\sim$ has Euler characteristic $2-3+1=0$. We had just seen on page 966 that the surface of genus g has Euler characteristic $2-2g$. Thus we obtain from Theorem 14.10 that $E_6/\sim$ is diffeomorphic to the surface of genus zero, i.e. the torus.

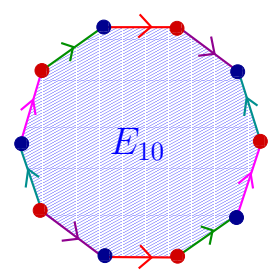
The same argument shows that $E_{10}/\sim$ has Euler characteristic $2-5+1=-2$ and that it is diffeomorphic to the surface of genus one. Evidently the same argument shows that if we take a regular $(4g+2)$-gon and if we identify opposite sides, then we obtain the surface of genus g.

It is a great mental exercise to visualize that the surface on the left is a torus and that the surface on the right is a surface of genus one.

hexagon with opposite sides identified



dodecagon with opposite sides identified



in each case there are two equivalence classes of vertices: red and blue

FIGURE 591.

The key step in the proof of Proposition 40.1 is the following purely algebraic lemma.

Lemma 40.3. *Let*

$$0 \to C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of finitely generated abelian groups. Then the following equality holds:

$$\sum_{n=0}^{k}(-1)^n \operatorname{rank} C_n \;=\; \sum_{n=0}^{k}(-1)^n \operatorname{rank} H_n(C_*).$$

The same statement holds if the C_i are vector spaces over a field and if we replace ranks by dimensions.

Proof. For the above chain complex (C_n, ∂_n) we write for each $n \in \mathbb{N}_0$[762]

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

These groups form the following two types of short exact sequences

(a) $$0 \hookrightarrow Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

and

(b) $$0 \hookrightarrow B_n \to Z_n \to H_n \to 0.$$

Now we see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n \operatorname{rank} C_n &\overset{\substack{\text{by (a) and Lemma 11.6 (2) we know that } \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n(\operatorname{rank} Z_n + \operatorname{rank} B_{n-1})\\
&\underset{\substack{\uparrow\\ \text{by (b) and Lemma 11.6 (2) we know that } \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n}}{=} \sum_{n=0}^{k}(-1)^n(\operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1})\\
&= \sum_{n=0}^{k}(-1)^n \operatorname{rank} H_n + \sum_{n=0}^{k}(-1)^n \operatorname{rank} B_n + \sum_{n=0}^{k}(-1)^n \operatorname{rank} B_{n-1}\\
&\underset{\substack{\uparrow\\ \text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0}}{=} \sum_{n=0}^{k}(-1)^n \operatorname{rank} H_n + \sum_{n=0}^{k-1}(-1)^n \operatorname{rank} B_n - \sum_{m=0}^{k-1}(-1)^m \operatorname{rank} B_m\\
&= \sum_{n=0}^{k}(-1)^n \operatorname{rank} H_n.
\end{aligned}$$

Basically the same proof also applies if the C_i are vector spaces over a field and if we replace ranks by dimensions. □

Now we can easily provide the proof of Proposition 40.1.

[762]It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

Proof of Proposition 40.1. Let X be a finite k-dimensional CW-complex. We have

$$\begin{array}{rcl} \chi(X) & = & \sum\limits_{n=0}^{k}(-1)^n \cdot \text{number of } n\text{-cells} \overset{\substack{\text{Lemma 35.1}\\ \downarrow}}{=} \sum\limits_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(C_n^{\mathrm{CW}}(X)) \\ & \underset{\substack{\uparrow\\ \text{Lemma 40.3}}}{=} & \sum\limits_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(H_n^{\mathrm{CW}}(X)) \underset{\substack{\uparrow\\ \text{Proposition 35.4}}}{=} \sum\limits_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(H_n(X)). \end{array}$$

□

Using Lemma 40.3 we can also easily proof the following lemma which be will useful on several occasions.

Lemma 40.4. *Let*

$$0 \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to 0$$

be an exact sequence of finitely generated abelian groups. Then

$$\sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} A_n = 0.$$

Proof. We can view this exact sequence of finitely generated abelian groups as a chain complex whose homology groups vanish. The lemma is therefore an immediate consequence of Lemma 40.3. □

Definition.

(1) Given a topological space X and $n \in \mathbb{N}_0$ such that $H_n(X)$ is a finitely generated abelian group we refer to

$$b_n(X) := \operatorname{rank}(H_n(X))$$

as the *n-th Betti number of X*.[763]

(2) Let X be a topological space. If all homology groups are finitely generated and if $\sum\limits_n b_n(X)$ is finite, then we define the *Euler characteristic of X* to be

$$\chi(X) := \sum_n (-1)^n \cdot b_n(X).$$

It follows from Proposition 40.1 that this new definition of the Euler characteristic of a finite CW-complex agrees with the earlier definition.

Example. Using Proposition 33.20 and using Lemma 40.4 several times we will prove the following statement in Exercise 40.3. Let X be a topological space that is homotopy equivalent to a finite CW-complex. Then for any self-homeomorphism f of X we have

$$\chi(M(X,f)) = 0.$$

[763]These numbers are named after the Italian mathematician Enrico Betti (1823-1892)

`http://de.wikipedia.org/wiki/Enrico_Betti`

who had studied an early version of homology groups.

40.2. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will now see that it behaves better than the homology groups under several natural operations.

The following lemma makes it possible to determine the Euler characteristic of a CW-complex from suitable decompositions.

Lemma 40.5. *Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. Then the following equality holds*

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

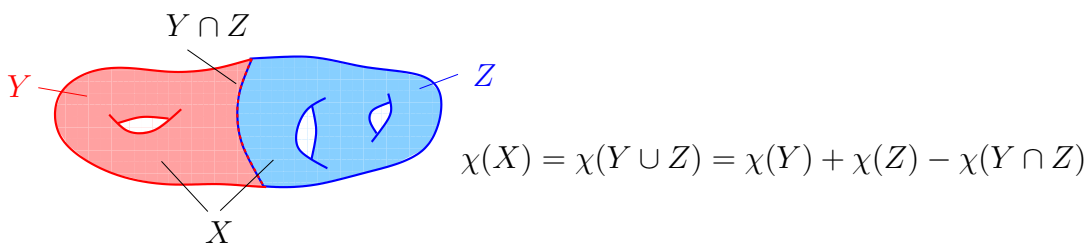


FIGURE 592.

Proof. We give two proofs for the lemma:

(1) Given $n \in \mathbb{N}_0$ and given a subset W we denote by $n(W)$ the number of open n-cells that are contained in W. In our case we have

$$n(X) \;\underset{\uparrow}{=}\; n(Y) + n(Z) - n(Y \cap Z).$$

since every open n-cell of X lies in Y or Z, the n-cells that lie in $Y \cap Z$ get counted twice in $n(Y) + n(Z)$, so we have to subtract $n(Y \cap Z)$

By performing the alternating sum over all n we obtain immediately that

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

We have thus proved the desired equality.

(2) The lemma also follows from applying Lemma 40.4 and Proposition 40.1 to the long exact sequence of homology groups that we obtain by applying the Mayer-Vietoris Theorem for CW-complexes 33.11 to the decomposition $X = Y \cup Z$. □

Example. As usual, given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks. Now we want to determine the Euler characteristic $\chi(\Sigma_{g,n})$. We could do so by using Propositions 40.1 and 35.10. But here we use an alternative approach. By Theorem 26.3 there exists a CW-structure on Σ_g such that $\Sigma_{g,n}$ and the closed disks that we remove are subcomplexes. It now follows that

$$2 - 2g \;\underset{\substack{\uparrow \\ \text{see page 966}}}{=}\; \chi(\Sigma_g) \;=\; \chi\Big(\Sigma_{g,n} \cup \bigcup_{i=1}^{n} \overline{B}^2\Big) \;\underset{\substack{\uparrow \\ \text{Lemma 40.5}}}{=}\; \chi(\Sigma_{g,n}) + \sum_{i=1}^{n} \chi(\overline{B}^2) - \sum_{i=1}^{n} \chi(\partial \overline{B}^2) \;\underset{\substack{\uparrow \\ \text{see page 966}}}{=}\; \chi(\Sigma_{g,n}) + n.$$

Thus we see that $\chi(\Sigma_{g,n}) = 2 - 2g - n$.

We record the above example in a lemma.

Lemma 40.6.

(1) *Let $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. If we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks, then $\chi(\Sigma_{g,n}) = 2 - 2g - n$.*

(2) *Let Σ and Σ' be two compact orientable connected 2-dimensional manifolds. If Σ and Σ' have the same number of boundary components and if $\chi(\Sigma) = \chi(\Sigma')$, then Σ and Σ' are diffeomorphic.*

Proof.

(1) We proved this statement in the example preceding the lemma.

(2) This statement follows immediately from (1) and the classification of compact orientable connected 2-dimensional manifolds, see Theorem 14.15. □

We return to the ever-popular study of knots in S^3.

Definition. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 10.19 there exists an embedding $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ with $\Phi(\{0\} \times S^1) = K$. We refer to

$$X_K \;:=\; S^3 \setminus \Phi(B^2 \times S^1)$$

as the *exterior of K*.

Lemma 40.7. *For any knot $K \subset S^3$ the following statements hold:*

(1) *The exterior X_K of the knot K is a compact orientable connected 3-dimensional manifold and the boundary ∂X_K is diffeomorphic to the torus.*

(2) *The diffeomorphism type of X_K as an oriented manifold does not depend on the choice of Φ.*

(3) *The exterior X_K is a deformation retract of the knot complement $S^3 \setminus K$.*

(4) *We have $\chi(X_K) = 0$.*

(5) *For every $n \in \mathbb{N}_0$ we have*[764]

$$H_n(S^3 \setminus K) \underset{\substack{\uparrow \\ \text{induced by the inclusion} \\ X_K \to S^3 \setminus K}}{\overset{\cong}{\leftarrow}} H_n(X_K) \;=\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \cdot [\mu_K], & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

where μ_K denotes a meridian of K.

Proof $(*)$. Let $K \subset S^3$ be a knot. We pick an embedding $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ such that $\Phi(\{0\} \times S^1) = K$.

(1) This statement is basically a consequence of Proposition 10.1 and 3.24.

(2) This statement is an immediate consequence of Proposition 10.20.

[764] We gave an alternative calculation of these homology groups in Exercise 39.4.

(3) A deformation retraction from $S^3 \setminus K$ to X_K is given by the map

$$\begin{array}{rcl} (S^3 \setminus K) \times [0,1] & \to & X_K = S^3 \setminus \Phi(\overline{B}^2 \times S^1) \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in Y = S^3 \setminus \Phi(\overline{B}^2 \times S^1), \\ \Phi(((1-t)r+t)e^{i\alpha}, z), & \text{if } x = \Phi(re^{i\alpha}, z) \in \Phi(\overline{B}^2 \times S^1). \end{cases} \end{array}$$

It follows easily from Lemma 1.32 (2) and Proposition 10.2 that this map is indeed continuous.

(4) By Theorem 26.3 there exists a CW-structure for S^3 such that X_K and $\Phi(\overline{B}^2 \times S^1)$ are subcomplexes. We have

$$\underset{\substack{\uparrow \\ \text{see page 966}}}{0 =} \chi(S^3) = \chi\big(X_K \cup \Phi(\overline{B}^2 \times S^1)\big) \underset{\substack{\uparrow \\ \text{Lemma 40.5}}}{=} \chi(X_K) + \chi\big(\overline{B}^2 \times S^1\big) - \chi\big(\partial\overline{B}^2 \times S^1\big) \underset{\substack{\uparrow \\ \text{see page 966}}}{=} \chi(X_K).$$

(5) By Corollary 30.8, Lemma 9.10 and by (3) we know that the inclusion $X_K \to S^3 \setminus K$ induces an isomorphism on all homology groups. Furthermore, together with Lemma 33.15 we already know that $H_0(X_K) = \mathbb{Z}$ and that $H_1(X_K) = \mathbb{Z} \cdot [\mu_K]$. Since X_K is a compact, oriented n-dimensional manifold with non-empty boundary it follows from Proposition 39.11 that $H_2(X_K)$ is a free abelian group and that $H_i(X_K) = 0$ for $i \geq 3$. Thus it now suffices to show that $\operatorname{rank}(H_2(X_K)) = 0$. But this follows immediately from $\chi(X_K) = 0$ and from the calculation of all the other homology groups of X_K. □

Lemma 40.8. *If M and N are two compact oriented connected n-dimensional manifolds, then*

$$\chi(M\#N) \;=\; \chi(M) + \chi(N) + \begin{cases} -2, & \textit{if } n \textit{ is even}, \\ 0, & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof (∗). In this case we again have two slightly different proofs. First of all, the lemma follows quite easily from Proposition 40.1 together with the calculation of the homology groups of connected sum that we gave in Proposition 39.15. Alternatively we can argue directly using Lemma 40.5.

More precisely, we have

$$\chi(M \setminus B^n) \;\underset{\substack{\uparrow \\ \text{Lemma 40.5}}}{=}\; \chi(M) - \chi(\overline{B}^n) + \chi(S^{n-1}) \;\underset{\substack{\uparrow \\ \text{since } \chi(\overline{B}^n) = 1 \text{ and } \chi(S^{n-1}) = 1 + (-1)^{n-1}}}{=}\; \chi(M) + (-1)^{n-1}.$$

The same way we also see that $\chi(N \setminus B^n) = \chi(N) + (-1)^{n-1}$. It then follows, using the same type of arguments, that

$$\begin{array}{rcl} \chi(M\#N) & = & \chi(M \setminus B^n) + \chi(N \setminus B^n) - \chi(S^{n-1}) \\ & = & \chi(M) + (-1)^{n-1} + \chi(N) + (-1)^{n-1} - (1 + (-1)^{n-1}) \\ & = & \chi(M) + \chi(N) - 1 + (-1)^{n-1}. \end{array}$$

□

Examples.

(1) In Figure 224 we saw that the connected sum of two tori is the surface of genus two. By Lemma 40.8 we obtain that[765]

$$\chi(\text{surface of genus two}) \;=\; \underbrace{\chi(\text{torus})}_{=0} + \underbrace{\chi(\text{torus})}_{=0} - 2 \;=\; -2.$$

We had obtained the same result on page 966 using a different method.

(2) From the CW-structure for $\mathbb{R}\mathrm{P}^2$ that we gave in Figure 400 we read off that $\chi(\mathbb{R}\mathrm{P}^2) = 1$. It follows from Lemma 40.8 that for any $k \in \mathbb{N}$ we have

$$\chi(k \cdot \mathbb{R}\mathrm{P}^2) \;=\; 1 - k.$$

Using the same argument as in Lemma 40.6 one can also show that

$$\chi(k \cdot \mathbb{R}\mathrm{P}^2 \text{ minus } n \text{ open disks}) \;=\; 1 - k - n.$$

Let X and Y be two finite CW-complexes. On page 670 we introduced the product CW-structure which by Proposition 26.23 is indeed a CW-structure for the topological space $X \times Y$. The following lemma relates the Euler characteristics of X, Y and $X \times Y$ in an appealing way.

Lemma 40.9. *Let X and Y be two finite CW-complexes, then*

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Example. For any finite CW-complex X we have $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

Proof. Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y.

We equip $X \times Y$ with the product CW-structure from page 670. By construction we have

$$\#n\text{-cells of } X \times Y \;=\; \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y \;=\; \sum_{k+l=n} c_k \cdot d_l.$$

It follows that

$$\chi(X) \cdot \chi(Y) \;=\; \Big(\sum_k (-1)^k \cdot c_k\Big) \cdot \Big(\sum_l (-1)^l \cdot d_l\Big) \;=\; \sum_n (-1)^n \sum_{k+l=n} c_k \cdot d_l \;=\; \chi(X \times Y). \;\square$$

Now let $p\colon \widetilde{X} \to X$ be a k-fold covering of path-connected topological spaces. We saw in Corollary 7.15 that the induced map $p_*\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism and we saw in Lemma 7.16 (3) that $p_*(\pi_1(\widetilde{X}))$ is a subgroup of index k of $\pi_1(X)$. There is no analogous statement for homology groups. For example the projection map $p\colon S^2 \to S^2/\sim = \mathbb{R}\mathrm{P}^2$ is a 2-fold covering, but $H_2(S^2) = \mathbb{Z}$ and $H_2(\mathbb{R}\mathrm{P}^2) = 0$, which shows that the induced map on homology is in general not a monomorphism.

It is therefore perhaps surprising that we already saw in Proposition 26.25 that the Euler characteristic behaves well under finite coverings. For convenience we recall, a slightly

[765] How many arguments can you find in the lecture notes that show that the Euler characteristic of the torus is zero?

simplified version, of the statement of Proposition 26.25.

Proposition 26.25. *Let $p\colon \widetilde{Y} \to Y$ be a finite covering of a finite connected CW-complex Y. Then $\widetilde{Y}$ is also a finite CW-complex with*

$$\chi(\widetilde{Y}) \;=\; [\widetilde{Y} : Y] \cdot \chi(Y).$$

Example. We consider again the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$. In this case we have by the calculation on page 846 or alternatively in Proposition 35.11 that

$$\begin{array}{rcl} H_0(S^2) & \cong & \mathbb{Z}, \\ H_1(S^2) & \cong & 0, \\ H_2(S^2) & \cong & \mathbb{Z}, \\ H_i(S^2) & \cong & 0, \text{ for } i \geq 3, \end{array} \quad \text{and} \quad \begin{array}{rcl} H_0(\mathbb{R}\mathrm{P}^2) & \cong & \mathbb{Z}, \\ H_1(\mathbb{R}\mathrm{P}^2) & \cong & \mathbb{Z}_2, \\ H_2(\mathbb{R}\mathrm{P}^2) & \cong & 0, \\ H_i(\mathbb{R}\mathrm{P}^2) & \cong & 0, \text{ for } i \geq 3. \end{array}$$

Thus we see that the homology groups are quite different. Nonetheless we observe that $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{R}\mathrm{P}^2) = 1$, as we had determined on page 966.

Now we can finally give an answer to Question 20.10, i.e. we can determine how the genus of a surface changes if we take finite covers. More precisely we have the following lemma.

Lemma 40.10. *Let Σ be the surface of genus g and let $p\colon \widetilde{\Sigma} \to \Sigma$ be a k-fold connected covering. Then*[766]

$$\textit{genus}(\widetilde{\Sigma}) \;=\; k \cdot (g-1) + 1.$$

Example. In Figure 347 we illustrate a 3-fold covering of the surface of genus 4 over the surface of genus 2.

Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the remark on page 966 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.
(2) By Proposition 26.25 we have $\chi(\widetilde{\Sigma}) = k \cdot \chi(\Sigma)$. □

Remark. In a similar vein one can now also answer Question 20.14. Recall that in that question we had considered a 6-fold covering $X \to B$ where B is the 2-dimensional sphere minus three open disks. On page 573 we saw that X has six boundary components. We leave it as an exercise to adapt the above arguments to determine the genus of X. (Hint: it is non-zero!)

In Question 14.5 we had asked whether the fundamental group of the surface of genus g is isomorphic to the free group F_{2g} on $2g$ generators. Now we can answer this question in the negative.

[766]Recall that by Theorem 14.10 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional manifold $\widetilde{\Sigma}$ is diffeomorphic to precisely the surface of genus $\widetilde{g}$.

Proposition 40.11. *If $g \geq 1$, then the group*

$$\pi_1(\textit{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.

Proof. Let Σ be a surface of genus $g \geq 1$. We write $\pi = \pi_1(\Sigma)$. Suppose that there exists an isomorphism $\varphi\colon \pi \to F$ where F is the free group on $2g$ generators. Let $\alpha\colon \pi \to \mathbb{Z}_k$ be an epimorphism for some $k \geq 2$.[767] We denote by $p\colon \widetilde{\Sigma} \to \Sigma$ the covering which satisfies $p_*(\pi_1(\widetilde{\Sigma})) = \ker(\alpha)$ which is given by Proposition 20.8. By Lemma 7.16 (3) this is a k-fold covering. We get the following commutative diagram:

$$\begin{tikzcd}
\pi \arrow[d, "\varphi", "\cong"'] & \ker(\alpha) \arrow[l] \arrow[r, "\cong", "\text{Lemma 40.10}"'] \arrow[d, "\varphi", "\cong"'] & \pi_1(\Sigma_{k\cdot(g-1)+1}) \arrow[r, "\text{abelianization}"] \arrow[d, "\cong"] \arrow[dr] & \mathbb{Z}^{2k(g-1)+2} \arrow[d, "\cong"] \\
F & \ker(\alpha \circ \varphi^{-1}) \arrow[l] \arrow[r, "\cong", "\text{Corollary 26.26}"'] & F_{k(2g-1)+1} \arrow[r, "\text{abelianization}"] & \mathbb{Z}^{k(2g-1)+1} \\
 & & & H_1(\Sigma_{k\cdot(g-1)+1}) \arrow[u, "\cong"']
\end{tikzcd}.$$

It follows from $k \geq 2$ that $2k(g-1)+2 \neq k(2g-1)+1$, hence the two groups on the right cannot by isomorphic. We have thus obtained a contradiction. $\square$

Remark. One can more generally ask, if X is a finite set, and if r is a non-trivial element in the free group $\langle X \rangle$ generated by X, can the group $\langle X \mid r \rangle$ be isomorphic to $\langle X \rangle$? It turns out that the answer is no, one approach to proving this purely group theoretic statement is as follows:

(1) A group π is called *Hopfian*[768] if every epimorphism $\varphi\colon \pi \to \pi$ is in fact a monomorphism.[769]
(2) Every finitely generated free group is Hopfian. This follows from the fact that free groups are residually finite and that finitely generated residually finite groups are Hopfian. We refer to [Cohe2, p. 11] and [Bog, Chapter 2.29] for details. See also [MKS, p. 109] for an alternative proof that free groups are Hopfian.
(3) Now suppose there does exist an isomorphism $\varphi\colon \langle X \,|\, r \rangle \to \langle X \rangle$. We denote by $\psi\colon \langle X \rangle \to \langle X \,|\, r \rangle$ the obvious epimorphism. Then $\varphi \circ \psi\colon \langle X \rangle \to \langle X \rangle$ is an epimorphism. By (2) it is an isomorphism, which implies in particular that ψ is a monomorphism. But r lies in the kernel of ψ, this is only possible if $r = e \in \langle X \rangle$.

At the end of this section we return to the study of lens spaces. Now we are finally in a position where we can calculate all the homology groups.

[767] Why does such an epimorphism exist?
[768] This notion is once again named after Heinz Hopf.
[769] For example Lemma 11.8 (5) says that every finitely generated free abelian group is Hopfian.

Lemma 40.12. *Let p, q be coprime natural numbers. For the corresponding lens space $L(p,q)$ we have*

$$H_n(L(p,q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0,3 \\ 0, & \text{if } n = 2 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

The calculation of this lemma is rather disappointing, it shows that homology groups cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have made no progress at all on Question 7.8.

Proof. It follows from Proposition 7.10 and the definition of a lens space that $L(p,q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with Proposition 26.25 that $\chi(L(p,q)) = 0$. We already know from Proposition 29.5 that $H_0(L(p,q)) \cong \mathbb{Z}$, from Corollary 38.6 that $H_1(L(p,q)) \cong \mathbb{Z}_p$ and from page 942 that $H_3(L(p,q)) \cong \mathbb{Z}$. Also we know from Proposition 35.6 that $H_n(L(p,q)) = 0$ for $n \geq 4$. So at the moment we are still missing $H_2(L(p,q))$. It now follows from Proposition 40.1 that

$$\operatorname{rank} H_2(L(p,q)) = \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank} H_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank} H_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank} H_3(L(p,q))}_{=1} = 0.$$

Furthermore we know from Proposition 39.17 that $H_2(L(p,q))$ is a free abelian group. Therefore we see that $H_2(L(p,q)) = 0$. □

40.3. **Groups acting on spheres.** We recall that by definition a group G acts *freely* on a topological space X if for any $g \neq e$ and any $x \in X$ we have $g \cdot x \neq x$. For example for each $k \in \mathbb{N}$ the action

$$\begin{array}{rcl} \mathbb{Z}_k \times S^1 & \to & S^1 \\ (l, z) & \mapsto & e^{2\pi i l/k} \cdot z \end{array}$$

is free. We can easily generalize this example. More precisely, let S^{2n-1} be an odd-dimensional sphere. We view S^{2n-1} as subset of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then it is straightforward to show that

$$\begin{array}{rcl} \mathbb{Z}_k \times S^{2n-1} & \to & S^{2n-1} \\ (l, (z_1, \dots, z_n)) & \mapsto & (e^{2\pi i l/k} z_1, \dots, e^{2\pi i l/k} z_n) \end{array}$$

is a free and continuous action of $\mathbb{Z}_k$ on S^{2n-1}.

It is less clear what finite groups can act freely and continuously on even-dimensional spheres. The group $\mathbb{Z}_2 \cong \{\pm 1\}$ acts of course freely and continuously on any sphere. The following proposition says that $\mathbb{Z}_2$ is in fact the only finite non-trivial group that can act freely and continuously on an even-dimensional sphere. This proposition and the above example give a complete answer to Question 7.9.

Proposition 40.13. *If a finite group G acts freely and continuously on S^{2n}, then G is either trivial or $G \cong \mathbb{Z}_2$.*

Proof. Let G be a finite group that acts freely and continuously on S^{2n}. By Lemma 7.6 the action is also discrete. By Proposition 7.10 the projection map $S^{2n} \to S^{2n}/G$ is a covering of degree $|G|$. We obtain the equalities

$$\begin{array}{ccccc} 2 & = & \chi(S^{2n}) & = & |G| \cdot \chi(S^{2n}/G). \\ & \uparrow & & \uparrow & \\ & \text{see page 765} & & \text{Proposition 26.25} & \end{array}$$

If follows that $|G|$ divides two, i.e. $|G| = 1$ or $|G| = 2$. In first case the group G is the trivial group and in the second case we have $G \cong \mathbb{Z}_2$.[770] □

Example. Let Σ be the surface of genus four. By the remark on page 966 we know that $\chi(\Sigma) = -6$. The proof of Proposition 40.13 shows that the order of any finite group acting freely and continuously on Σ has to have an order that divides 6.

In Figure 593 we sketch a discrete and continuous action of the group $\mathbb{Z}_3$ on Σ. We leave it as an exercise to give a rigorous description of this action and to determine whether the groups $\mathbb{Z}_2$ and $\mathbb{Z}_6$ can act continuously and freely on Σ.

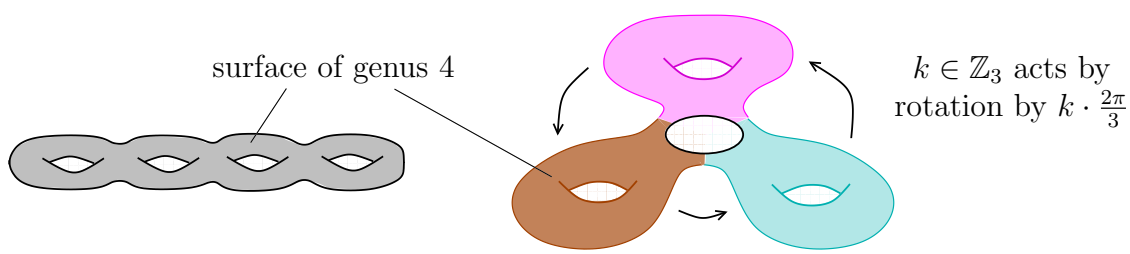


FIGURE 593.

40.4. **Curves on surfaces.** We start out with the following definition.

Definition. Let Σ be an oriented 2-dimensional manifold and let C and D be two closed oriented curves in Σ. We say C and D are *equivalent* if there exists an orientation-preserving self-diffeomorphism f of Σ which restricts to an orientation-preserving diffeomorphism $f\colon C \to D$.

In this section we want to give an easy and useful criterion for showing that two curves on a given 2-dimensional manifold are equivalent. The key to formulating the criterion is the following definition.

Definition. Let M be a compact orientable connected n-dimensional manifold and furthermore let $N \subset M$ be a proper submanifold of codimension one.

(1) We say N is *separating* if $M \setminus N$ is disconnected, otherwise we see that N is *non-separating*. (This definition is illustrated in Figure 594.)

[770] Why is every group with two elements isomorphic to $\mathbb{Z}_2$?

(2) If N is closed and separating, then we pick a tubular neighborhood $[-1,1]\times N$ as provided by the Tubular Neighborhood Theorem 10.19. We refer to $M\setminus((-1,1)\times N)$ as M *cut along* N. By Proposition 3.24 we know that M cut along N is again an n-dimensional manifold.[771] (This definition is illustrated in Figure 595.)

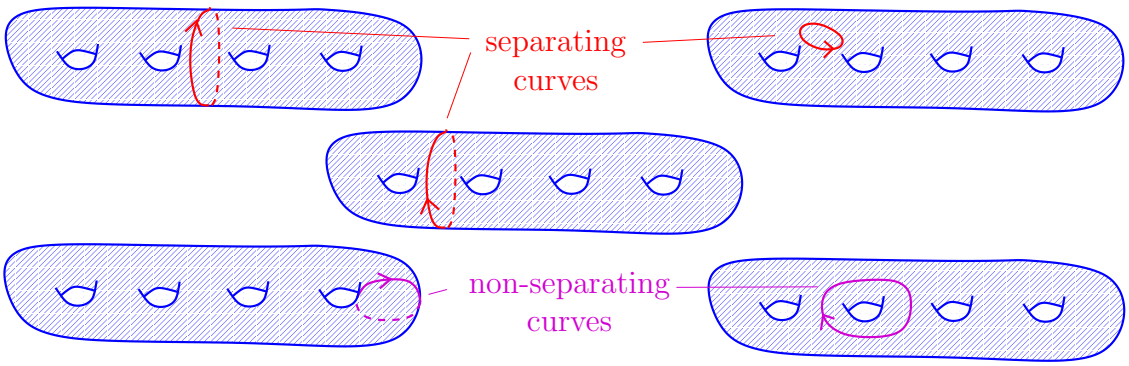


FIGURE 594.

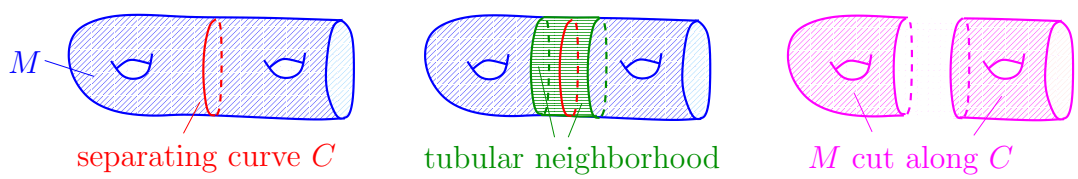


FIGURE 595.

It is clear that if two curves are equivalent, then either both are separating or both are non-separating. Furthermore if they are both separating, then the results of cutting the 2-dimensional manifold along the curves are diffeomorphic. The following proposition gives us the converse to this observation.

Proposition 40.14. *Let Σ a compact oriented connected 2-dimensional manifold and let C and D be two closed oriented curves in $\Sigma\setminus\partial\Sigma$.*

(1) *If C and D are non-separating, then C and D are equivalent.*
(2) *Suppose C and D are separating. If Σ cut along C is diffeomorphic to Σ cut along D, then C and D are equivalent.*

Example. It follows easily from Proposition 40.14 and Exercise 40.5 that any separating curve on the surface of genus four is equivalent to one of the separating curves shown in Figure 594. Furthermore, it follows from Proposition 40.14 that the two non-separating curves shown in Figure 594 are actually equivalent

[771] It follows from Proposition 10.20 that the definition of the diffeomorphism type of "M cut along N" does not depend on the choice of the tubular neighborhood.

Proof. We only prove statement (1). We leave it to the reader to make the minor adjustments that are needed to handle statement (2). Thus let Σ a compact oriented connected 2-dimensional manifold and let C and D be two closed oriented non-separating curves on Σ. We pick two tubular neighborhoods $[-1,1] \times C$ and $[-1,1] \times D$. We write $M := \Sigma \setminus (-1,1) \times C$ and we write $N := \Sigma \setminus (-1,1) \times D$. It follows from Proposition 3.24 that M and N are compact oriented connected 2-dimensional manifolds. Note that M and N have the same number of boundary components. Furthermore note that it follows easily from Lemma 40.5 that $\chi(M) = \chi(\Sigma) = \chi(N)$. Therefore we conclude from Lemma 40.6 that there exists an orientation-preserving diffeomorphism $\Phi\colon M \to N$. It follows from Corollary 14.16 that in fact we can find such an orientation-preserving diffeomorphism that has the extra property that $\Phi(\{\epsilon\} \times C) = \{\epsilon\} \times D$ for $\epsilon \in \{-1,1\}$. In fact, if one takes the orientation conventions into account one sees that Φ restricts to orientation-preserving diffeomorphisms $\Phi\colon \{\epsilon\} \times C \to \{\epsilon\} \times D$ for $\epsilon \in \{-1,1\}$.

Using Proposition 20.5 it is now straightforward to see that Φ extends to an orientation-preserving diffeomorphism $\Psi\colon \Sigma \to \Sigma$ that in turn restricts to an orientation-preserving diffeomorphism $C \to D$. □

We conclude this section with two corollaries to Proposition 40.14.

Corollary 40.15. *Let Σ be a compact 2-dimensional manifold and let C be a closed oriented curve on Σ. If C is non-separating, then C is not null-homotopic.*

Proof. Let Σ be a compact 2-dimensional manifold and let C be a closed oriented non-separating curve on Σ. By Theorem 14.14 we can assume that Σ is the surface of genus g, for some $g \in \mathbb{N}_0$, with some open disks removed. Since adding the disks does not change connectivity we can in fact assume that Σ is the surface of genus g. It follows from Exercise 40.6 that $g \geq 1$. It follows from the calculation of the fundamental group of the surface Σ of genus g, see e.g. Proposition 14.3, that Σ admits closed oriented non-separating curves that are not null-homotopic. By Proposition 40.14 this means that C is also not null-homotopic. □

Remark. What is the "right" proof for a statement is often a matter of taste. In this instance one could make the case that the results of Section 17 give the more natural proof of Corollary 40.15.

The following proposition might sound self-evident, but as we will see, the proof requires some serious input from our earlier work.

Proposition 40.16. *Let Σ be a compact orientable 2-dimensional manifold. Furthermore let $f\colon S^1 \to \Sigma$ be an embedding. If f is null-homotopic, then there exists an embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.*

Sketch of proof. Let Σ be a compact orientable 2-dimensional manifold. Furthermore let $f\colon S^1 \to \Sigma$ be an embedding that is null-homotopic. We write $C = f(S^1)$. By Corollary 40.15 we know that C is separating. Let A and B be the two components of Σ cut

along C. The argument from page 433 shows that if neither A nor B is a disk, then C would represent a non-trivial element in $\pi_1(\Sigma)$. Thus by our hypothesis one of A or B is a disk. But this observation, together with Proposition 20.5, easily implies that there exists such an embedding F. □

40.5. **Graphs** (∗)**.** Recall that on page 645 we defined a topological graph to be a 1-dimensional CW-complex. Furthermore recall that we refer to the 0-cells as the vertices of the graph and we refer to the 1-cells as the edges of the graph. The Euler characteristic of a finite graph G is thus given by

$$\chi(G) \;=\; \#\text{vertices} - \#\text{edges}.$$

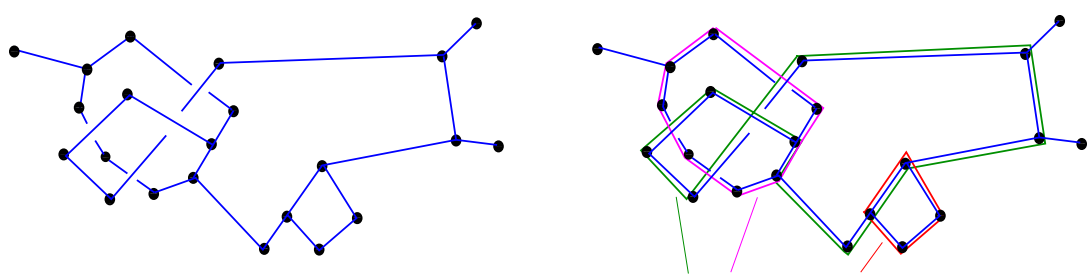
In particular this matches the original definition that we gave on page 318.

Using the results of this chapter we can now easily prove the following lemma:

Lemma 40.17. *Let G be a connected graph with v vertices and e edges. Then*

$$H_0(G) \cong \mathbb{Z}, \quad H_1(G) \cong \mathbb{Z}^{e-v+1} \quad \text{and} \quad H_i(G) = 0 \text{ for } i \geq 2.$$

Example. In Figure 596 we see a connected graph G with 22 vertices and 24 edges. It follows from Lemma 40.17 that its first homology group is isomorphic to $\mathbb{Z}^3$. In Figure 596 we sketched three cycles which represent a generating set for $H_1(G) \cong \mathbb{Z}^3$. But there are infinitely many other choices of generating sets.[772]



graph G with 22 vertices and 24 edges generating set for $H_1(G) \cong \mathbb{Z}^3$

FIGURE 596.

Proof. Let G be a connected graph with v vertices and e edges. Now that we have developed so many techniques for computing homology groups it is perhaps not surprising that we can provide three different proofs for the lemma.

(1) First of all, from Proposition 29.5 together with Proposition 26.11 (7) it follows immediately that $H_0(G) \cong \mathbb{Z}$. Furthermore, from Proposition 35.5 it follows that $H_i(G) = 0$ for $i \geq 2$ and that $H_1(G)$ is torsion-free.

[772] What are other examples of generating sets for $H_1(G)$?

From Proposition 40.1 we obtain that

$$\chi(X) \;=\; \sum_i (-1)^i \operatorname{rank}(H_i(X)).$$

Putting everything together we see that

$$1-\operatorname{rank}(H_1(G)) \;=\; \chi(X) \;=\; \#0\text{-cells} - \#1\text{-cells} \;=\; v-e.$$

Since $H_1(G)$ is torsion-free we obtain that $H_1(G) \cong \mathbb{Z}^{e-k+1}$.

(2) We can also modify the argument of (1) as follows: instead of calculating $H_1(G)$ using the Euler characteristic, we could also have determined that $H_1(G) \cong \mathbb{Z}^{e-k+1}$ by combining the calculation of the fundamental group in Lemma 12.5 with the isomorphism of the Hurewicz Theorem 38.5.

(3) An alternative proof for Lemma 40.17 is given as follows: we have

$$\begin{array}{ccccccc} \widetilde{H}_j(G) & \cong & \widetilde{H}_j(\text{wedge of } 1-\chi(G) \text{ circles}) & \cong & \bigoplus_{i=1}^{1-\chi(G)} \widetilde{H}_j(S^1) & \cong & \begin{cases} \mathbb{Z}^{1-\chi(G)}, & \text{if } j=1, \\ 0, & \text{otherwise.} \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 9.21 (3) and Corollary 30.8} & & \text{Proposition34.8} & & \text{Proposition 31.3} & \end{array}$$

It is a matter of taste which of the three proof is the nicest. □

40.6. **The Lefschetz Fixed Point Theorem** (∗)**.** The following theorem was proved by Solomon Lefschetz in the 1920s.[773]

Theorem 40.18. (Lefschetz Fixed Point Theorem) *Let X be a finite CW-complex (e.g. a compact manifold) and let $f\colon X \to X$ be a map. If*[774]

$$\tau(f) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(f_*\colon H_n(X) \to H_n(X)) \;\neq\; 0,$$

then f has a fixed point, i.e. there exists an $x \in X$ with $f(x) = x$.

A proof for the Lefschetz Fixed Point Theorem 40.18 is given in [Hat, Theorem 2.C.3] or in [BrownR]. In fact the book by Brown [BrownR] is completely dedicated to the proof of the Lefschetz Fixed Point Theorem 40.18 and its numerous applications and generalizations.

Corollary 40.19. *Let X be a finite connected CW-complex such that $H_n(X)$ is torsion for all $n \geq 1$. Then any map $f\colon X \to X$ has a fixed point.*

---

[773] Solomon Lefschetz (1884-1972) was an American mathematician.

[774] Let H be a finitely generated abelian group and let $\varphi\colon H \to H$ be a homomorphism. We write $\overline{H} := H/\text{torsion}$. The homomorphism φ defines a homomorphism $\overline{\varphi}\colon \overline{H} \to \overline{H}$. By the classification of finitely generated abelian groups, see Theorem 11.4, there exists an isomorphism $\psi\colon \overline{H} \xrightarrow{\cong} \mathbb{Z}^k$. We define $\operatorname{tr}(\varphi)$ to be the trace of the homomorphism $\mathbb{Z}^k \xrightarrow{\psi^{-1}} \overline{H} \xrightarrow{\overline{\varphi}} \overline{H} \xrightarrow{\psi} \mathbb{Z}^k$. As in linear algebra one can easily show that this definition does not depend on the choice of ψ.

Proof. Let X be a finite connected CW-complex such that $H_n(X)$ is torsion for all $i \geq n$. We calculate that

$$\tau(f) \;:=\; \mathrm{tr}(\underbrace{f_*\colon H_0(X) \to H_0(X)}_{\substack{f_* = \mathrm{id} \text{ by the discussion} \\ \text{on page 747}}}) \;+\; \sum_{n\in\mathbb{N}} (-1)^n \cdot \underbrace{\mathrm{tr}(f_*\colon H_n(X) \to H_n(X))}_{= 0, \text{ since } H_n(X) \text{ torsion}} \;=\; 1.$$

It follows from the Lefschetz Fixed Point Theorem 40.18 that f has a fixed point. □

Examples.

(1) If we apply Corollary 40.19 to $X = \overline{B}^n$, then we recover the Brouwer Fixed Point Theorem 31.7.
(2) In Proposition 35.11 we saw that the hypothesis of Corollary 40.19 are satisfied for even-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2n}$. Thus we see that every self-map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point.[775]
(3) We consider the CW-complex S^n. Let $f\colon S^n \to S^n$ be a map. By the discussion on page 747 and the definition of the degree of f we have

$$\tau(f) \;=\; \tau(\mathrm{id}) = 1 + (-1)^n \cdot \deg(f).$$

The Lefschetz Fixed Point Theorem 40.18 thus says that if $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point. Thus we see that Theorem 32.19 is a special case of the Lefschetz Fixed Point Theorem 40.18.

Exercises for Chapter 40.

Exercise 40.1. Show that every CW-structure for the torus with one 0-cell has precisely one 2-cell.

Exercise 40.2. Let $g \in \mathbb{N}_0$. By Proposition 14.6 the connected sum $F = \Sigma_g \# \mathbb{R}\mathrm{P}^2$ of the surface of genus g with the real projective plane is non-orientable. So according to Proposition 8.3 it admits a 2-fold covering $\widetilde{F} \to F$ that is orientable. Determine the genus of $\widetilde{F}$?
Note: we had raised this as a question on page 434.

Exercise 40.3. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. In Proposition 33.20 we showed that for any $n \in \mathbb{N}_0$ there exists a short exact sequence

$$0 \to \mathrm{coker}\big(H_n(X) \xrightarrow{f_*-\mathrm{id}} H_n(X)\big) \to H_n(M(X,f)) \to \ker\big(H_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} H_{n-1}(X)\big) \to 0.$$

(a) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Given $A \in \mathrm{SL}(2,\mathbb{Z})$ we consider the homeomorphism

$$\begin{aligned} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 &\to \mathbb{R}^2/\mathbb{Z}^2 \\ [v] &\mapsto [Av]. \end{aligned}$$

[775] What happens for odd-dimensional real projective spaces? For example, does there exists a self-map $\mathbb{R}\mathrm{P}^3 \to \mathbb{R}\mathrm{P}^3$ without fixed-points?

Determine the homology groups of $M(T, f(-\mathrm{id}))$.
Remark: You can use the fact that under the usual identification $H_1(T) = \mathbb{Z}^2$ the induced map $f(A)_*$ is given by multiplication by A.[776]

(b) Does there exist a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $H_1(M(T, f(A))) \cong \mathbb{Z}$?

(c) Let X be a topological space that is homotopy equivalent to a finite CW-complex. Show that for any self-homeomorphism f of X we have

$$\chi(M(X, f)) \;=\; 0.$$

Exercise 40.4. For a topological space X and $n \in \mathbb{N}_0$ we denote by $b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$ the rank of $H_n(X)$. Does there exist a topological space X such that $b_1(X) < \infty$ but such that for a *finite* cover $\widetilde{X}$ of X we have $b_1(\widetilde{X}) = \infty$?

Exercise 40.5. Given $g, n, \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. Let $g, k, m, n \in \mathbb{N}_0$. Suppose there exists an embedding $\Sigma_{g,m} \to \Sigma_{k,n}$. Show that $g \leq k$.
Hint: use Proposition 3.24.

Exercise 40.6. Show that $\overline{B}^2$ and S^2 do not admit closed non-separating curves.
Hint: Given a closed curve C on $X = \overline{B}^2$ or $X = S^2$ consider the Euler characteristic of X cut along C.

Exercise 40.7. Let Σ be a 2-dimensional manifold. A *curve system* is a collection of disjoint curves (i.e. closed connected 1-dimensional manifolds) $C_1, \ldots, C_k$ on Σ with the following two properties:

(1) None of the C_i is the boundary of a disk on F.
(2) There is no annulus on F such that the boundary is the union of two curves.

In Figure 597 on the right we see a curve system on the surface of genus two consisting of three curves.

(a) Is there a curve system consisting of four curves on the surface of genus two?
(b) Given $g \in \mathbb{N}_0$, what is the maximal number of curves in a curve system on the surface of genus g?

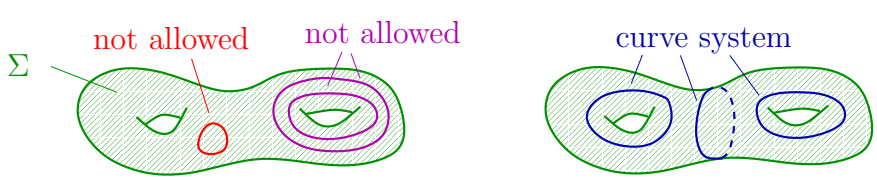


FIGURE 597.

[776] Why does this hold?

Exercise 40.8. Let $\varphi\colon A \to B$ be a homomorphism between free groups of the same finite rank. Suppose that $\varphi_*\colon H_1(A) \to H_1(B)$ is an isomorphism. Show that φ is a monomorphism.
Hint: Use that finitely generated free groups are Hopfian. I.e. given any finitely generated free group F any epimorphism $\varphi\colon \pi \to \pi$ is in fact a monomorphism.

41. Applications of the Euler characteristic (∗)

In this chapter we show that the fairly simple concept of the Euler characteristic can be used to give answers to several questions about 3-dimensional objects.

41.1. **Building a leather football.** In this section we will discuss the shapes of footballs and the platonic solids. The results of this section will not be used later on and we will present some of the material in a slightly informal, somewhat less rigorous style.

First we recall that an n-gon on the 2-dimensional sphere S^2 is called *regular* if the isometry group of the sphere acts transitively on the set of vertices and the set of faces of the n-gon. In particular all interior angles are the same. For example the classical leather football, see Figure 598, is made out of regular pentagons and hexagons.

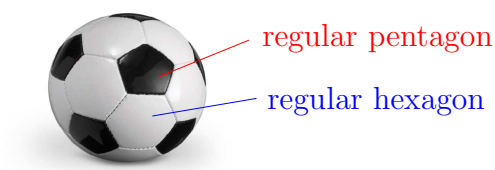


Figure 598.

Now we want to determine how many pentagons and hexagons are required to make a football. So suppose a football is made out of p spherical regular pentagons and h spherical regular hexagons where the vertices of the pentagons are precisely the vertices of hexagons. This endows S^2 with a CW-structure. We first note that each 1-cell bounds two 2-cells. Furthermore, elementary spherical geometry, see e.g. [Bon, p. 53] or [Fen, p. 260], says that the interior angle of a spherical pentagon and a spherical hexagon are larger than the corresponding Euclidean angles. Therefore we see that that interior angles are larger than[777] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. So at any vertex there are not more than three hexagons and pentagons that meet, which of course implies that there are precisely three hexagons and pentagons that meet. Therefore we see that

$$\begin{aligned} \#\text{2-cells} &= p + h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p + 6h) \\ \#\text{0-cells} &= \tfrac{1}{3}(5p + 6h). \end{aligned}$$

By Proposition 40.1 and the above calculation of the Euler characteristic of S^2 we now have

$$2 \;=\; \chi(S^2) \;=\; \sum_{n=0}^{2}(-1)^n \cdot \#n\text{-cells} \;=\; \tfrac{1}{3}(5p+6h) - \tfrac{1}{2}(5p+6h) + (p+h) \;=\; \tfrac{1}{6}p.$$

Thus we see that a football has to have precisely 12 pentagons. On the other hand, the number of hexagons is not fixed. For example, the standard football has 20 hexagons, as

[777] Where does that formula come from?

pictured in Figure 599 on the left. But in principle one could also build a football with zero hexagons, see Figure 600 one the right, and one obtains a spherical dodecahedron. One could also build with many hexagons, but they would be much less round since they would have bigger flat parts.

The 1-skeleton of this CW-structure were in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football. More information can be found at

`https://en.wikipedia.org/wiki/Buckminsterfullerene`

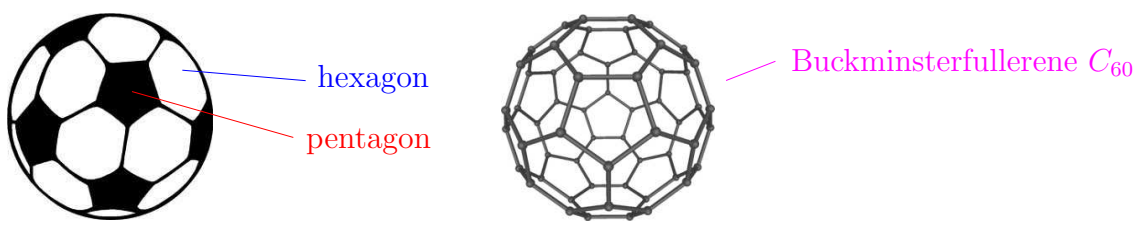


FIGURE 599.

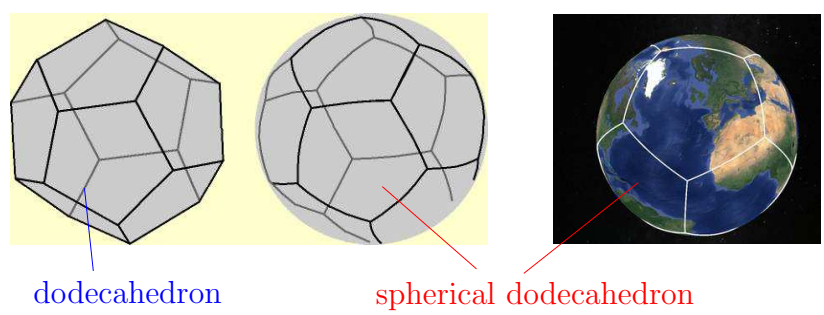


FIGURE 600.

41.2. **Platonic solids.** Recall that on page 78 we defined the *convex hull* of a subset $S \subset \mathbb{R}^n$ as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^k$.

Definition.

(1) A *convex polyhedron* is the convex hull of finitely many points in $\mathbb{R}^k$.[778] [779][780]
(2) The *dimension of a convex polyhedron* is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

We leave the proof of the following lemma as an exercise to the reader.

Lemma 41.1. *If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set V such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.*

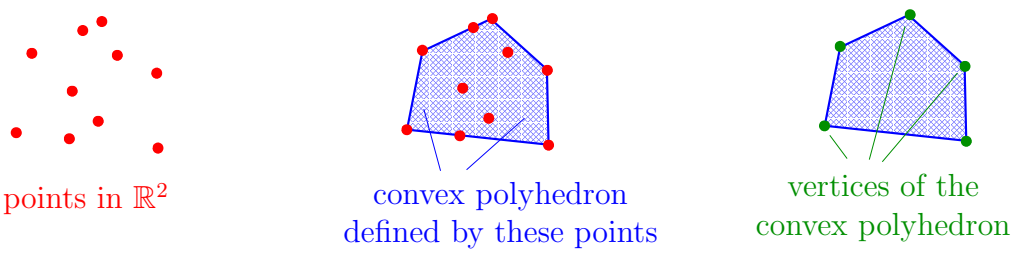


FIGURE 601. Illustration of Lemma 41.1.

Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.

(1) We refer to the finite set of Lemma 41.1 as the *vertices of the convex polyhedron* P.
(2) If P is a k-dimensional convex polyhedron, then an *edge* of the convex polyhedron is the convex hull of two vertices that has the property that it lies on ∂P.
(3) If P is a k-dimensional convex polyhedron, then an *m-dimensional face* of the convex polyhedron is the convex hull of m vertices that has the property that it lies on ∂P. (In particular, by definition a 1-dimensional face is exactly the same as an edge.)
(4) An edge is a 1-dimensional face and sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.
(5) The *valence* of a vertex is defined as the number of edges that contain the vertex.
(6) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called *equivalent* if there exists an $r > 0$, a point $v \in \mathbb{R}^k$, and an orthogonal matrix $A \in \mathrm{O}(k)$ with $P = A \cdot (r \cdot Q) + v$.

Example. The notion of a vertex, an edge, a face and the valence of a vertex are illustrated in Figure 602.

[778] It is an amusing little exercise to show that this notion of a convex polyhedron agrees with the notion that we had originally defined on page 703.

[779] In the literature a convex polyhedron is often also called a polytope.

[780] Recall that we saw on page 79 that the convex hull of points $P_1, \dots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{ \sum_{i=1}^n t_i P_i \,\Big|\, t_1, \dots, t_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_n = 1 \Big\}.$$

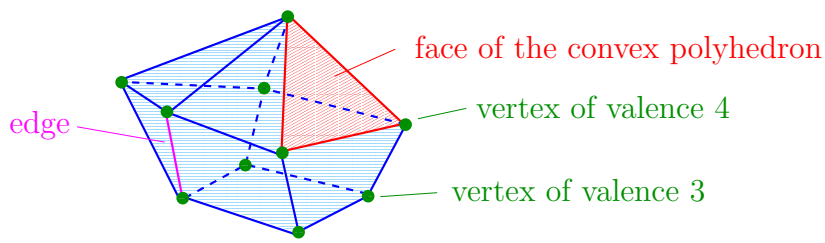
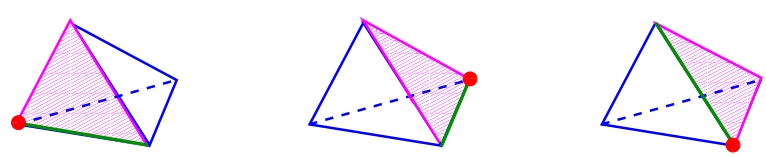


FIGURE 602.

We move on to the next definition.

Definition. Let P be a convex m-dimensional polyhedron in $\mathbb{R}^k$.

(1) A *rotational symmetry* of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{O}(k)$ such that $A \cdot P = P$. The *rotational symmetry group* $S(P) \subset \mathrm{O}(k)$ is defined as the group of all rotational symmetries of P.
(2) A *flag* of P is a sequence $F_0 \subset F_1 \dots F_{m-1}$ of faces of P such that each F_i is i-dimensional.
(3) A convex polyhedron P is called *regular* if the rotational symmetry group $S(P)$ acts transitively on the set of flags of P.



three flags in the regular tetrahedron T

FIGURE 603.

Example.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. We show T together with three flags in Figure 603. Some elementary geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as the *regular tetrahedron*.
(2) The standard n-simplex Δ^n is an n-dimensional convex polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$. Each vertex has valence n. In this case the above definition of a face agrees with the definition that we gave on page 935. It is straightforward to see that Δ^n is regular.
(3) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the points $r, re^{2\pi i/n}, \dots, re^{2\pi i(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, re^{2\pi i/n}, \dots, re^{2\pi i(n-1)/n}$, where each vertex has valence 2. We leave it as an exercise to show that, up to

equivalence, these are all regular 2-dimensional convex polyhedra in $\mathbb{R}^2$. We refer to Figure 604 for an illustration.

(4) The convex polyhedron spanned by $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is a regular 3-dimensional convex polyhedron where each vertex has valence 4. This convex polyhedron is usually called the *regular octahedron*. We refer to Figure 604 for an illustration.

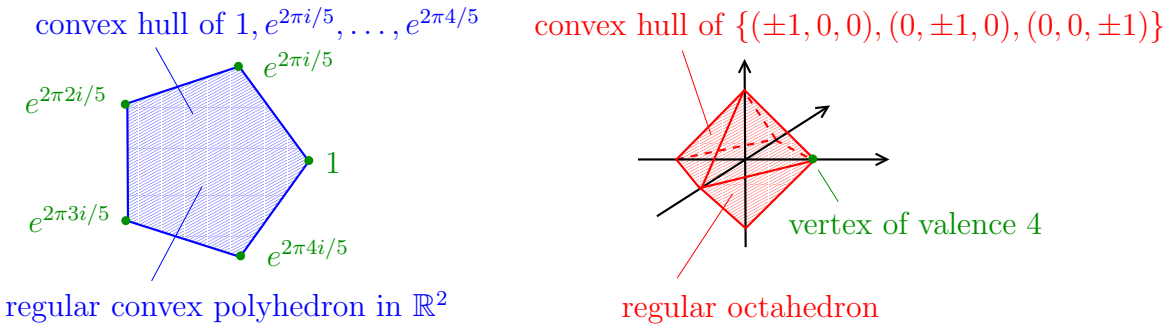


FIGURE 604.

(5) We consider the cube $[-1, 1]^3 \subset \mathbb{R}^3$. In [Arm, p. 38] it is shown that $S([-1, 1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements. We refer to Figure 605 for an illustration. Then the map

$$\begin{array}{rcl} S([-1,1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \left(\begin{array}{ccc} D & \to & D \\ d & \mapsto & A \cdot d \end{array} \right) \end{array}$$

is shown in [Arm, p. 38] to be an isomorphism.[781] It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *regular hexahedron*.

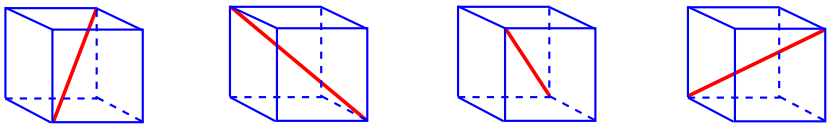


FIGURE 605. The four diagonals in a cube.

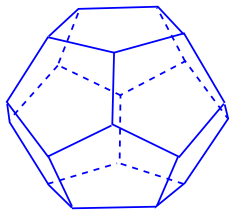
(6) Finally we consider the regular dodecahedron D shown in Figure 606.[782] We can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. Then the map

$$\begin{array}{rcl} S(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

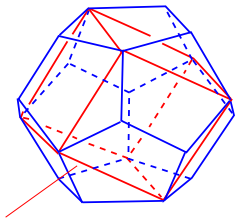
[781] It is a good exercise to verify this statement.

[782] The fact that a regular dodecahedron really exists is shown rigorously in [Cox, Section 6.7].

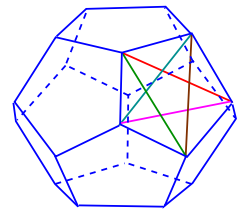
is shown in [Arm, p. 40] to be an isomorphism.



regular dodecahedron



cube inscribed into the regular dodecahedron



the five diagonals of a face correspond precisely to one inscribed cube

FIGURE 606.

The following definition is one of the oldest in the history of mathematics.

Definition. A *Platonic solid* is a regular 3-dimensional convex polyhedron in $\mathbb{R}^3$.

Theorem 41.2. *Up to rotation and stretching the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids illustrated in Figure 607.*[783] *They have the following types of faces and valences and they have the following rotational symmetry groups:*

	tetrahedron	*hexahedron*	*octahedron*	*dodecahedron*	*icosahedron*
number of faces	4	6	8	12	20
shape of face	*3-gon*	*4-gon*	*3-gon*	*5-gon*	*3-gon*
valence of vertex	4	3	4	3	5
rotational symmetry group	A_4	S_4	S_4	A_5	A_5

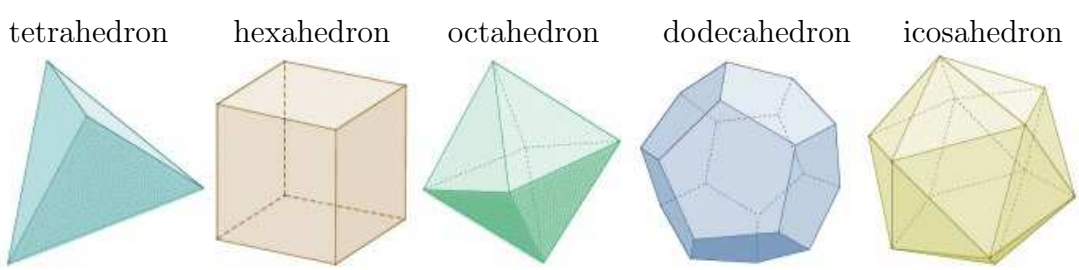


FIGURE 607. The five platonic solids.

[783] A precise definition of the platonic solids in Figure 607 is given in [Cox, Section 1.2].

Sketch of proof. We will not provide a full proof for the theorem. We refer to [Cox, Section 6.7] for a full proof of the first three rows and we refer to [Arm, p. 40] or alternatively to [Aa, Chapter 5.5] for a proof of the statement regarding the rotational symmetry groups.[784] But we will show that any platonic solid fits into the table of the theorem.

So let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

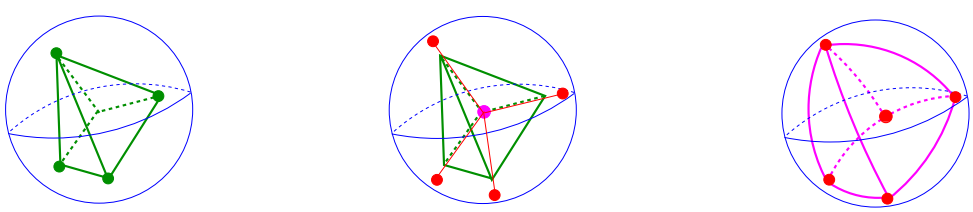
(1) It follows easily from the definition of a regular polyhedron that there exists an m such that each face of P is a regular m-gon.
(2) The number of faces that meet a vertex is the same for every vertex and it is at least three.

The following claim tells us the possible values for m and the number of faces that meet at a vertex.

Claim.

possible values for m	3	4	5
faces that meet at a vertex	$3, 4, 5$	4	3

First note that after a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere, see Figure 608. Recall that the spherical interior angle of a regular m-gon is greater than the Euclidean angle of a regular m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $3(\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. This concludes the proof of the claim.



embed polyhedron in the 3-ball | projection of vertices onto the sphere | projection of edges onto the sphere

FIGURE 608.

We now treat the cases $m = 3, 4, 5$ separately. First we first consider the case $m = 3$, i.e. all the faces of the convex polyhedron are equilateral triangles. Using Proposition 1.47 (2)

[784]We had sketched the proofs for the regular hexahedron and the regular dodecahedron above.

one can easily show that P admits a CW-structure such that

$$\begin{array}{rcl} \#\text{ of 0-cells} &=& \#\text{ of vertices,} \\ \#\text{ of 1-cells} &=& \#\text{ of edges,} \\ \#\text{ of 2-cells} &=& \#\text{ of faces,} \\ \#\text{ of 3-cells} &=& 1. \end{array}$$

Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#2\text{-cells} = k, \quad \#1\text{-cells} = \tfrac{1}{2}\cdot 3k \quad \text{and} \quad \#0\text{-cells} = \tfrac{1}{3}\cdot 3k.$$

Since the Euler characteristic of S^2 equals two we get

$$2 = \chi(S^2) = \sum_{n=0}^{2}(-1)^n\cdot \#n\text{-cells} = k - \tfrac{3}{2}k + k = \tfrac{1}{2}k.$$

It follows that $k = 4$. Almost the same calculation shows that if four triangles meet at a vertex, then $k - \frac{3}{2}k + \frac{3}{4}k = 2$, i.e. $k = 8$. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $k - \frac{3}{2}k + \frac{3}{5}k = 2$, i.e. $k = 20$.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. □

On page 989 we had just seen that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in Theorem 41.2 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html`

41.3. **Homology spheres.**

Definition. We say that an n-dimensional topological manifold X is a *homology n-sphere* if for any $k \in \mathbb{N}_0$ we have $H_k(X) \cong H_k(S^n)$.

It follows from the classification of 1-dimensional topological manifolds that we had stated in Proposition 3.49 and Theorem 14.9 and the calculation of homology groups in Proposition 35.10 that for $n = 1, 2$ every homology n-sphere is homeomorphic to the standard sphere S^n. In 1900 Henri Poincaré[785] claimed that the same also holds for $n = 3$, i.e he claimed that every homology 3-sphere is in fact homeomorphic to S^3. Shortly

[785]Henri Poincaré (1854-1912) was a French mathematician. He can be viewed as the "father of algebraic topology", he first introduced the fundamental group and the homology groups of topological spaces. We

afterwards, in 1904, Henri Poincaré himself found a counterexample to this claim. More precisely, he proved the following proposition.

Proposition 41.3. *There exists a* 3*-dimensional manifold M that is a homology* 3*-sphere, but such that $\pi_1(M)$ is a non-trivial group.*

After proving Proposition 41.3 Henri Poincaré formulated the following conjecture.

Conjecture 41.4. (Poincaré Conjecture) *If M is a topological homology 3- sphere that is simply connected, then M is homeomorphic to S^3.*

As we had discussed in Section 28.6, this conjecture was solved almost a 100 years later by Grigori Perelman [Per1, Per2, Per3, MTi].

Nowadays there are many ways to prove Proposition 41.3. In the following we will give Poincaré's example of such a manifold.[786]

In Theorem 41.2 we saw that we can view A_5 as a subgroup of SO(3). In Lemma 37.1 we gave an explicit group homomorphism $q\colon S^3 = \mathrm{SU}(2) \to \mathrm{SO}(3)$. It is a consequence of Theorem 37.2 (1) that q is an epimorphism and that the kernel of q is given by $\pm\,\mathrm{id}$. We write $\Gamma := q^{-1}(A_5)$. Since A_5 has $\frac{1}{2}5! = 60$ elements and since q has a kernel with two elements we see that Γ is a group with 120 elements.

Proposition 41.5. *The topological space $M := S^3/\Gamma$ is a homology 3-sphere with non-trivial fundamental group.*

The manifold M is called the *Poincaré homology sphere*. It was initially discovered by Henri Poincaré in 1904 by a different construction. The Poincaré homology sphere is ubiquitous in topology and can be described in many different ways. We refer to [KSc] for altogether eight different ways to construct the Poincaré homology sphere.

In the proof of Proposition 41.5 we will make use of the following lemma.

Lemma 41.6. *Let M be an orientable closed 3-dimensional manifold that is finitely covered by S^3. If $H_1(M) = 0$, then M is a homology 3-sphere.*

As we will see, the proof of Lemma 41.6 is almost entirely the same as the proof of Lemma 40.12.

Proof. Let M be an orientable closed 3-dimensional manifold that admits a k-fold covering $p\colon S^3 \to M$. Suppose that $H_1(M) = 0$. By Proposition 39.5 it remains to show that

had encountered Henri Poincaré before, the disk model $\mathbb{D}$ for hyperbolic space goes back to Poincaré and it is often called the "Poincaré model of hyperbolic space".

[786]In Exercise 39.3 we saw that if we glue two knot complements to one another in an appropriate way, then we obtain a homology 3-sphere. To obtain a proof of Proposition 41.3 one still needs to show that with that construction one gets a manifold that is not diffeomorphic to S^3. For example one can show that if one glues two non-trivial knots to one another, then the fundamental group is always non-trivial. This statement is in turn a consequence of the "Loop Theorem" [Pap] in 3-manifold topology, together with basic facts in group theory [Ser, Theorem 1, p.3].

$H_2(M) = 0$. By Proposition 39.17 we know that $H_2(M)$ is a free abelian group. So it suffices to show that $b_2(M) = 0$. We have

$$\begin{array}{ccccccc} b_2(M) & = & \chi(M) & = & \frac{1}{k} \cdot \chi(S^3) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } b_0(M) = b_3(M) = 1 & & \text{Proposition 26.25} & & \text{page 966} & \\ & \text{and } b_1(M) = 0 & & & & & \end{array}$$

□

Now we can provide the proof of Proposition 41.5.

Proof of Proposition 41.5. It follows from Proposition 3.25 that $M := S^3/\Gamma$ is a 3-dimensional manifold and it follows from Proposition 5.14 and Theorem 7.17 that $\pi_1(M) \cong \Gamma$.

Now we will show the slightly weaker statement that M or a 2-fold cover of M is a homology sphere. The fact that M is indeed a homology sphere is proved in [Bre, p. 354] or alternatively in [Toen, p. 133]. By Lemma 41.6 it suffices to prove the following claim.

Claim. We either have $H_1(M) = 0$ or there exists a 2-fold cover M' of M with $H_1(M') = 0$.

By the Hurewicz Theorem 38.5 it suffices to show that the abelianization of Γ is trivial. Since Γ is a finitely generated group this is equivalent to showing that Γ does not admit an epimorphism onto a non-trivial abelian group. By construction the group Γ fits into a short exact sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{\rho} \Gamma \to A_5 \to 0.$$

By Proposition 13.10 we get an induced exact sequence

$$\mathbb{Z}_2 \xrightarrow{\rho} \Gamma_{\text{ab}} \to (A_5)_{\text{ab}} \to 0.$$

From the algebra course we know that the abelianization of A_5 is trivial. This means there are now two possibilities:

(1) Either $\Gamma_{\text{ab}} = 0$, in which case we are done.
(2) Or the map $\mathbb{Z}_2 \xrightarrow{\rho} \Gamma_{\text{ab}}$ is an isomorphism.[787] In this case the map $\Gamma \to \Gamma_{\text{ab}} \xrightarrow{\rho^{-1}} \mathbb{Z}_2$ is a splitting of $\rho\colon \mathbb{Z}_2 \to \Gamma$. The argument of Lemma 33.2 shows that this implies that $\Gamma \cong \mathbb{Z}_2 \times A_5$. In particular A_5 is a subgroup of $\pi_1(M) = \Gamma$ of index two. By Proposition 20.8 there exists a 2-fold covering M' of M with fundamental group A_5. But then $H_1(M') = (A_5)_{\text{ab}} = 0$.

This concludes the proof of the claim. □

41.4. **Planar graphs.** We recall several definitions that we had made along the way:

(1) On page 133 we defined an *abstract graph* G as a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

[787] In fact this case does not occur, see e.g. [KSc] for details. But I don't know a quick argument which proves that $\Gamma_{\text{ab}} = 0$.

The elements of V are called *vertices of* G and the elements of E are called the *edges of* G.

(2) On page 645 we defined a topological graph to be a one-dimensional CW-complex.
   (a) We refer to the 0-cells as the *vertices* and the 1-cells as the *edges* of the topological graph.
   (b) Given a topological the *underlying abstract graph* is defined as the abstract graph $(V, E, \varphi\colon V \to E)$ given by setting V to the be set vertices and E the set of edges and given $e \in E$ we define $\varphi(e)$ to be the vertices that lie on e.

Examples.

(1) For $n \in \mathbb{N}$ we define the *complete graph* K_n as the abstract graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \ldots, n\} \text{ with } i \neq j\}$. Furthermore we define $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge. In Figure 610 we show topological realizations of the complete graphs K_1, K_2, K_3, K_4 and K_5.

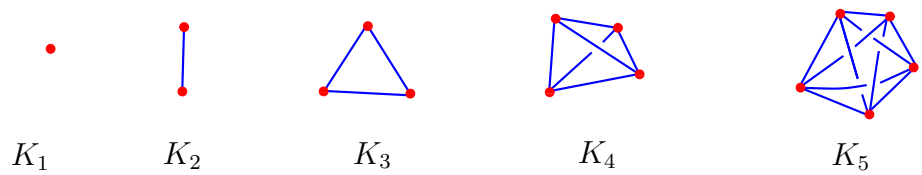


FIGURE 609.

(2) For $m, n \in \mathbb{N}$ we define the *complete bipartite graph* $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$, the edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}\}$ and the map $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge. In Figure 610 we show topological realizations of the complete bipartite graphs $K_{2,3}$, $K_{1,5}$ and $K_{3,3}$.

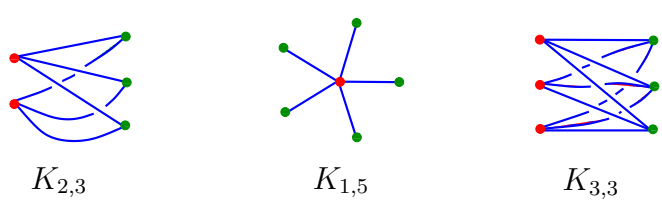


FIGURE 610.

On page 135 we had already mentioned that every finite graph can be realized as a spatial graph in $\mathbb{R}^3$. It is natural to ask whether we can reduce this dimension from 3 to 2. This leads us to the following dimension.

Definition.

(1) We say a finite topological graph G is *planar* if there exists a spatial graph in $\mathbb{R}^2$ that is homeomorphic to G.
(2) We say a finite abstract graph is *planar* if it admits a planar topological realization.

Examples.

(1) The above graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. The above topological realizations of K_4 and $K_{2,3}$ are not planar. But as we see in Figure 611 both graphs admit planar realizations.

topological realizations of K_4 topological realizations of $K_{2,3}$

FIGURE 611.

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar.

Now the following question arises.

Question 41.7. *Are the graphs K_5 and $K_{3,3}$ also planar?*

If this question sounds to abstract and bipartite graphs sound too otherworldly let me reformulate the question regarding $K_{3,3}$.

Question 41.8. *Suppose we are in charge of finding efficient bus routes in a given city. Furthermore suppose the city has three main employers A, B and C. After work the employees, naturally, either want to go to the train station α, or the swimming pool β or the shopping mall γ. Is it possible to find bus routes which connect each of the three employers with each of the three destinations such that the bus routes do not intersect?*[788]

To answer that question we need a few more definitions.

Definition. Let G be a graph.

(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

$$V(X_n) = \{1, \dots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{n,1\}\}.$$

[788]It follows easily from Proposition 10.23 that the answer to this question does not depend on the precise location of A, B, C and α, β, γ, as long as they are distinct.

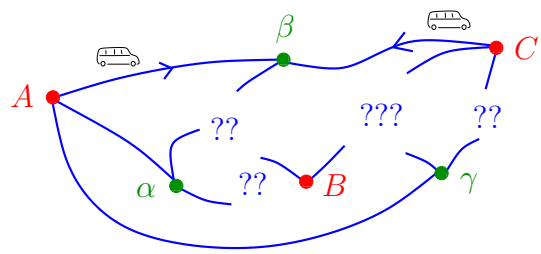


FIGURE 612.

(2) A *cycle of length* n in G is an injective map $\varphi\colon X_n \to G$ of graphs.[789]
(3) The *girth* of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.

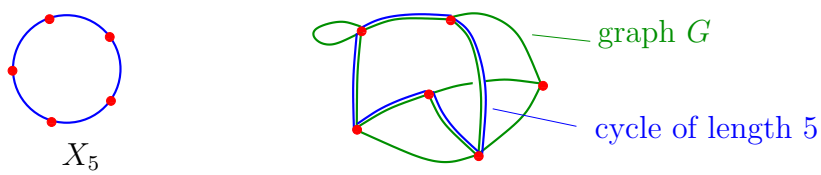


FIGURE 613.

Examples.

(1) A topological realization of X_n is given by S^1 equipped with the CW-structure that has n zero-cells and n one-cells.
(2) The girth of the graph shown in Figure 613 to the right is one, since it has an edge where the two endpoints agree.
(3) For any $n \geq 3$ the girth of the complete graphs K_n equals three.
(4) For any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four.

Proposition 41.9. *Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then*

$$e \;\leq\; \tfrac{g}{g-2} \cdot (v-2).$$

Sketch of proof. Let G be a with v vertices and e edges which admits a planar realization and for which the girth g is finite. By abuse of notation we denote the planar realization by G as well. Without loss of generality we can assume that G lies in $\overline{B}^2$.

Now we view G as a subset of the sphere $S^2 = \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus G$ as *faces* and we denote by f the number of faces. We refer to Figure 614 for an illustration. We make the following observation:

[789]We refer to page 133 for the definition of a map between abstract graphs.

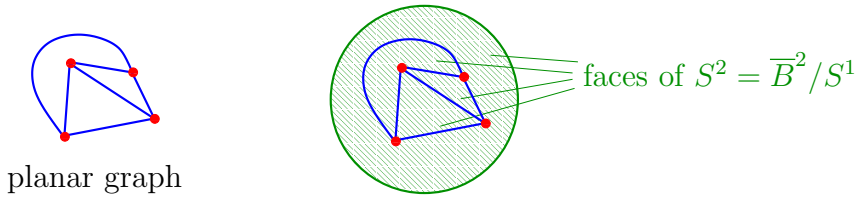


FIGURE 614.

(1) The vertices, edges and faces define a CW-structure for $S^2 = \overline{B}^2/S^1$.[790]
(2) We make the following two observations:
   (a) The boundary of each face is a cycle. By the definition of the girth each face is bounded by at least g edges.
   (b) Each edge lies in the boundary of at most two faces.

   Now we deduce that

$$\begin{aligned} g \cdot f &\leq \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f,e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary } \leq 2e. \end{aligned}$$

   Here the first inequality follows from (a) and the last inequality follows from (b). Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$\begin{array}{ccccccccc} 2 & = & \chi(S^2) & = & v - e + f & \leq & v - e + \frac{2}{g}e & = & v - \frac{g-2}{g}e. \\ & \uparrow & & & & \uparrow & & & \\ & \text{see page 966} & & & & \text{since } g \cdot f \leq 2e & & & \end{array}$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v-2)$. □

The following proposition gives in particular a negative answer to Question 41.7.

Proposition 41.10. *If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.*

Proof.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore the girth is three. It is straightforward to verify that the inequality of Proposition 41.9 is only satisfied for $m = 3$ and $m = 4$.

[790] Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is "clear from the picture" and it can be proved using the Schönflies Theorem 36.3. But we will not attempt to give a rigorous proof.

(2) Let $m, n \in \mathbb{N}_{\geq 2}$. Again it easy to verify that the complete bipartite graph $K_{m,n}$ has $m + n$ vertices and $m \cdot n$ edges. Furthermore,as we pointed out above, the girth is four. It is straightforward to verify that the inequality of Proposition 41.9 is only satisfied if $m = 2$ or $n = 2$.
(3) The Proposition now follows from (1) and (2) and the observation that if a graph is planar, then any subgraph is also planar. □

Remark. Let G be a graph that contains the complete graph K_7. By Proposition 41.10 the graph G cannot be planar. John Conway and Cameron Gordon [CG] showed in 1984 a stronger statement: any spatial embedding of G in $\mathbb{R}^3$ contains a non-trivial knot. More precisely, for any topological realization of G as a topological graph in $\mathbb{R}^3$ there exists a C in $G \subset S^3$ that is knotted in the sense that $\pi_1(\mathbb{R}^3 \setminus C)$ is not isomorphic to $\mathbb{Z}$.

Now we will consider the question, to what degree the converse of Proposition 41.10 holds. We start out with the following definition.

Definition. We say a graph is *simple* if its girth is at least three.

Put differently, a graph G is simple if the following two conditions are satisfied:

(1) the graph G has no loops, i.e. if there is no edge where the two endpoints agree,
(2) the graph G does not have multiple edges, i.e. there are no two edges with the same endpoints.

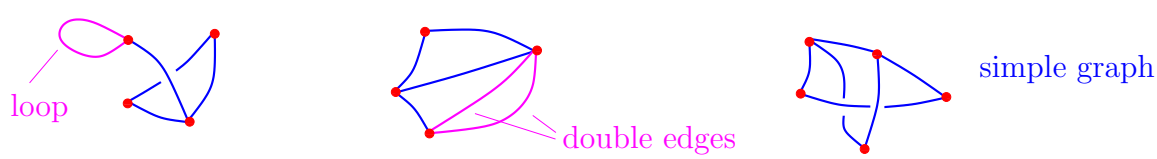


FIGURE 615.

In the remainder of this section we restrict ourselves to simple graphs. We start out with the following observations.

(1) If a simple graph G admits a vertex v of valence[791] one, then v is the endpoint of a single edge e. If we delete v and e we obtain a new graph G' such that the topological realization of G' is homotopy equivalent to the topological realization of G. It is straightforward to see that G is planar if and only if G' is planar.
(2) If a simple graph admits a vertex v of valence two, then v is the endpoint of two edges e_1 and e_2. We can combine e_1 and e_2 into one edge and eliminate v. This way we obtain a new graph with a homeomorphic topological realization.

This shows that for the most part we can restrict ourselves to simple graphs where every vertex has valence at least three.

[791]Recall that the *valence* of a vertex is defined as the number of edges that contain the vertex.

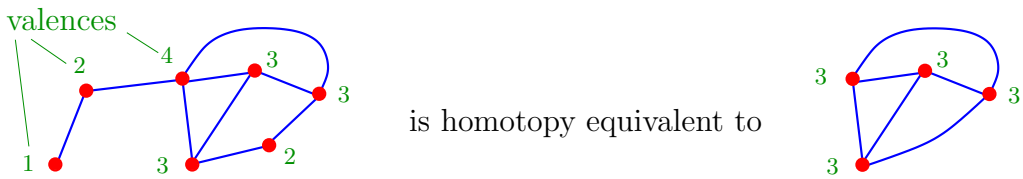


FIGURE 616.

If we restrict ourselves to simple graphs such that every vertex has valence at least three, then the graphs K_5 and $K_{3,3}$ turn out to be the only obstruction to being planar. More precisely the following theorem was proved by Kuratowski[792] in 1930. We refer to [BoM, Chapter 10.5] for a proof.

Theorem 41.11. (Kuratowski's Reduction Theorem) *Let G be a finite simple graph such that every vertex has valence at least three. Then the following holds:*

$$G \text{ is planar} \quad \Longleftrightarrow \quad G \text{ contains neither } K_5 \text{ nor } K_{3,3}.$$

[792]Kazimierz Kuratowski (1896-1980) was a Polish mathematician.

42. Homology with coefficients

In this chapter we will introduce the homology groups of a topological space X with coefficients in a ring R. These homology groups will have many applications, for example they will be essential in relating the homology groups of a manifold M to the de Rham cohomology groups of M. They will also be essential in proving theoretic statements, for example we will need to work with $\mathbb{F}_2$-coefficients to prove the higher-dimensional analogue of the Borsuk-Ulam Theorem 7.23. Before we can introduce the homology groups with coefficients we will need to introduce or recall some basic algebraic concepts.

42.1. **The tensor product of abelian groups.** The key to defining homology groups with coefficients is the notion of the tensor product of two abelian groups.

Definition. Let A and B be abelian groups. We define the *tensor product* $A \otimes B$ as follows:

$$A \otimes B \ := \left\{ \begin{array}{l} \text{free abelian group which is generated by all} \\ \text{symbols of the form } a \otimes b \text{ with } a \in A,\, b \in B \end{array} \right\} / \, N(A,B)$$

where $N(A,B)$ is the subgroup generated by the elements of the form

$$(a+a') \otimes b - (a \otimes b + a' \otimes b) \qquad \text{and} \qquad a \otimes (b+b') - (a \otimes b + a \otimes b')$$

with $a, a' \in A$ and $b, b' \in B$. For $a \in A$ and $b \in B$ we denote, by a slight abuse of notation, the image of $a \otimes b$ in $A \otimes B$ again by $a \otimes b$.

We summarize some elementary properties of the tensor product of two abelian groups in the following lemma.

Lemma 42.1. *Let A and B be two abelian groups.*

(1) *For any $a, a' \in A$ and $b, b' \in B$ we have*

$$\begin{array}{rcl} (a+a') \otimes b & = & a \otimes b + a' \otimes b \ \in \ A \otimes B \\ a \otimes (b+b') & = & a \otimes b + a \otimes b' \ \in \ A \otimes B. \end{array}$$

(2) *For any $a \in A, b \in B$ and $m \in \mathbb{Z}$ we have*

$$(m \cdot a) \otimes b \ = \ a \otimes (m \cdot b) \ = \ m \cdot (a \otimes b).$$

(3) *Any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} a_i \otimes b_i$.*

(4) *Let $n \in \mathbb{N}$. For any $c \in A \otimes \mathbb{Z}_n$ we have $n \cdot c = 0 \in A \otimes \mathbb{Z}_n$.*

Proof. Let A and B be two abelian groups.

(1) This statement is an immediate consequence of the definition of the tensor product.

(2) Let $a \in A$ and $b \in B$. We first consider the case that $m > 0$. We have

$$(m \cdot a) \otimes b \ = \ \left(\sum_{i=1}^{m} a\right) \otimes b \ \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \ \sum_{i=1}^{m} a \otimes b \ = \ m \cdot (a \otimes b) \ \underset{\substack{\uparrow \\ \text{same argument backwards}}}{=} \ \ldots \ = \ a \otimes (m \cdot b).$$

It follows easily from (1) that $0\otimes b = a\otimes 0 = 0$ and that $(-a)\otimes b = a\otimes(-b) = -a\otimes b$. Using these observations we get statement (2) for $m = 0$ and we can easily reduce the case $m < 0$ of (2) to the case $m > 0$ of (2).

(3) It follows immediately from the definition of the tensor product, together with the notational convention of "formal linear combinations" that we introduced on page 366 that any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$. But by (2) we know that $r_i \cdot (a_i \otimes b_i) = (r_i \cdot a_i) \otimes b_i$ thus we get the desired result.

(4) Let $n \in \mathbb{N}$ and let $c \in A\otimes\mathbb{Z}_n$. By (3) we can write $c = \sum_{i=1}^{m} a_i \otimes r_i$ with $a_1, \dots, a_m \in A$ and $r_1, \dots, r_m \in \mathbb{Z}_n$. Then

$$n \cdot c \;=\; n \cdot \Big(\sum_{i=1}^{m} a_i \otimes r_i\Big) \;\underset{\substack{\uparrow \\ \text{by statement (2)}}}{=}\; \sum_{i=1}^{m} a_i \otimes \underbrace{n \cdot r_i}_{=0} \;=\; 0.$$

□

Definition. Let A, B and C be abelian groups. We say that a map $f\colon A\times B \to C$ is *bilinear*, if for all $a, a' \in A$, $b, b' \in B$ the following equalities hold:

$$f(a + a', b) \;=\; f(a, b) + f(a', b) \quad \text{and} \quad f(a, b + b') \;=\; f(a, b) + f(a, b').$$

We leave the elementary proof of the following lemma as an exercise to the reader.

Lemma 42.2. *Let A and B be abelian groups.*

(1) *The map*

$$\begin{array}{rcl} f\colon A \times B & \to & A \otimes B \\ (a, b) & \mapsto & a \otimes b \end{array}$$

is bilinear.

(2) *Given any abelian group G and any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[dr, "g"'] & A \otimes B \arrow[d, "\varphi \text{ unique}"] \\ & G. \end{tikzcd}$$

(3) *Given any abelian group G the map*

$$\begin{array}{rcl} \mathrm{Hom}(A \otimes B, G) & \to & \textit{set of bilinear maps } A \times B \to G \\ (\varphi\colon A \otimes B \to G) & \mapsto & (A \times B \xrightarrow{f} A \otimes B \xrightarrow{\varphi} C) \end{array}$$

is a bijection.

Remark. One can show quite easily that if a homomorphism $g\colon A \times B \to H$ is bilinear and if this homomorphism satisfies the universal property of Lemma 42.2 (2), then there exists an isomorphism $H \xrightarrow{\cong} A \otimes B$. We will not make use of this observation.

On many occasions we will also make use of the following lemma.

Lemma 42.3. *Let A, B, C and $A_i, i \in I$ be abelian groups.*

(1) *The maps*

$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} b_i \otimes a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \left(\bigoplus_{i\in I} A_i\right) \otimes B & \to & \bigoplus_{i\in I} A_i \otimes B \\ \left(\sum_{i=1}^{n} a_i\right) \otimes b & \mapsto & \sum_{i=1}^{n} a_i \otimes b \end{array}$$

are well-defined and they are isomorphisms.

(2) *The map*

$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ \sum_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$

is well-defined and it is an isomorphism.

(3) *Let $k \in \mathbb{N}$. The maps*[793]

$$\begin{array}{rcl} \mathbb{Z}_k \otimes A & \to & A/kA \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i a_i. \end{array}$$

are well-defined and they are isomorphisms.

(4) *Suppose $(I, \leq)$ is an ordered set and $(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j})$ is a direct system of abelian groups. Then*

$$\lim_{\longrightarrow} (A_i \otimes B) \;=\; \left(\lim_{\longrightarrow} A_i\right) \otimes B.$$

Proof. We refer to [Lan2, Corollary XVI.2.2] for a proof for the distributivity in (1). The other statements in (1) to (3) follow fairly easily from the definitions and Lemma 42.1. We refer to [Mun, Chapter 50] for details. A proof of the last statement is given in Bourbaki[794] [Bou, Chapter II.6.3] or alternatively in [Matb, Theorem A1]. □

Remark.

(1) The isomorphisms of Lemma 42.3 are natural, in the sense that they define natural transformations between appropriate functors. For example if we denote by $\mathcal{A}$ the category of abelian groups, then the isomorphisms of Lemma 42.3 (3) define a natural transformation between the two functors

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{A} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{A} & \to & \mathcal{A} \\ A & \mapsto & \mathbb{Z} \otimes A. \end{array}$$

[793]Note that for an abelian group A and $k \in \mathbb{Z}$ the set $kA := \{k \cdot a \,|\, a \in A\}$ is a subgroup of A and we can thus form the quotient A/kA.

[794]More information on the books written by "Nicolas Bourbaki" can be found here

`https://en.wikipedia.org/wiki/Nicolas_Bourbaki`

As an example, for every $k \in \mathbb{N}$ and any abelian group A we get a commutative diagram

$$\begin{tikzcd} \mathbb{Z} \otimes A \arrow[r, "\cong", "n\otimes a \mapsto n\cdot a"'] \arrow[d, "\cdot k \otimes \mathrm{id}"'] & A \arrow[d, "\cdot k"] \\ \mathbb{Z} \otimes A \arrow[r, "\cong", "n\otimes a \mapsto n\cdot a"'] & A. \end{tikzcd}$$

We use these natural isomorphisms from Lemma 42.3 to identify the various groups.

(2) Note that Lemma 42.3 (2) allows us to write the tensor product $A \otimes B \otimes C$ unambiguously without having to worry about parentheses.

(3) In Lemma 42.3 we saw that the tensor product commutes with forming direct sums. The analogous statement is not true if we replace direct sums by direct products. More precisely, in Exercise 42.6 we will see that

$$\Big(\prod_{n\in\mathbb{N}} \mathbb{Z}_n\Big) \otimes \mathbb{Q} \neq 0 \quad \text{but} \quad \prod_{n\in\mathbb{N}} \underbrace{(\mathbb{Z}_n \otimes \mathbb{Q})}_{=0,\text{ by Lemma 42.3 (3)}} = 0.$$

Examples.

(1) Let $m, n \in \mathbb{N}$. We write $r = \gcd(m,n)$ and $s = m/r$. Then

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \underset{\substack{\uparrow \\ \text{Lemma 42.3 (3)}}}{\cong} \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_n/rs\mathbb{Z}_n = \mathbb{Z}_n/r\cdot(s\mathbb{Z}_n) \underset{\substack{\uparrow \\ \text{since } s \text{ is coprime to } n}}{=} \mathbb{Z}_n/r\mathbb{Z}_n \xleftarrow[\cong]{\cdot n/r} Z_r = \mathbb{Z}_{\gcd(m,n)}.$$

(2) Let A be a finitely generated abelian group of rank r. By the classification of finitely generated abelian groups, see Theorem 11.4, we have $A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$ where $a_1, \dots, a_n$ are non-zero natural numbers. For $S = \mathbb{Q}, S = \mathbb{R}$ or $S = \mathbb{C}$ we have

$$A \otimes S \cong \Big(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}\Big) \otimes S \underset{\substack{\uparrow \\ \text{Lemma 42.3 (1)}}}{=} (\mathbb{Z}\otimes S)^r \oplus \Big(\bigoplus_{i=1}^n \mathbb{Z}_{a_i} \otimes S\Big) \underset{\substack{\uparrow \\ \text{Lemma 42.3 (3)}}}{\cong} S^r \oplus \bigoplus_{i=1}^n S/a_iS \underset{\substack{\uparrow \\ \text{since } a_i \neq 0 \text{ in } S = \mathbb{Q},\mathbb{R},\mathbb{C}}}{=} S^r.$$

(3) Let A be an abelian group, then

$$\ker\begin{pmatrix} A & \to & A\otimes\mathbb{Q} \\ a & \mapsto & a\otimes 1 \end{pmatrix} = \text{all torsion elements of } A.$$

If A is finitely generated, then this follows easily from the classification of finitely generated abelian groups together with Lemma 42.3. If A is infinitely generated, then we can view it as the direct limit of finitely generated abelian groups and using Lemma 42.3 the statement can be reduced to the finitely generated case. We leave the details as an exercise to the reader.

(4) Recall that given a set W we denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W. Let A and B be two sets. We then have

$$\mathbb{Z}^{(A)} \otimes \mathbb{Z}^{(B)} = \Big(\bigoplus_{a\in A} \mathbb{Z}\cdot a\Big) \otimes \Big(\bigoplus_{b\in B} \mathbb{Z}\cdot b\Big) \overset{\text{Lemma 42.3 (1)}}{\overset{\downarrow}{=}} \bigoplus_{a\in A, b\in B} \mathbb{Z}\cdot a \otimes \mathbb{Z}\cdot b \overset{\text{Lemma 42.3 (3)}}{\overset{\downarrow}{=}} \bigoplus_{a\in A, b\in B} \mathbb{Z}\cdot (a,b) = \mathbb{Z}^{(A\times B)}.$$

(5) Let A and B be two finitely generated abelian groups. Using examples (1) and (2), the classification of finitely generated abelian groups and Lemma 42.3 (1) and (4) one can now easily show that

$$\operatorname{rank}(A\otimes B) \;=\; \operatorname{rank}(A)\cdot \operatorname{rank}(B).$$

Definition. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. We define $f\otimes g$ to be the homomorphism that is given by[795]

$$\begin{array}{rcl} f\otimes g\colon A\otimes B & \to & A'\otimes B' \\ \sum\limits_{i=1}^{n} a_i\otimes b_i & \mapsto & \sum\limits_{i=1}^{n} f(a_i)\otimes g(b_i) \end{array}$$

For future reference we record the following lemma:

Lemma 42.4. *Let $f\colon A\to A'$ and $f'\colon A'\to A''$ be homomorphisms and let $g\colon B\to B'$ and $g'\colon B'\to B''$ be homomorphisms between abelian groups. Then we have*

$$(f'\otimes g')\circ(f\otimes g) \;=\; (f'\circ f)\otimes(g'\otimes g).$$

Proof. The statement follows immediately from the definitions. □

The following elementary lemma will play a major role.

Lemma 42.5. *Let R be a commutative ring.*

(1) *Given any abelian group A the map*

$$\begin{array}{rcl} (A\otimes R)\times R & \mapsto & A\otimes R \\ \Big(\sum\limits_{i=1}^{n} a_i\otimes b_i, r\Big) & \mapsto & \sum\limits_{i=1}^{n} a_i\otimes b_i r \end{array}$$

defines an R-module structure on $A\otimes R$.

(2) *If $f\colon A\to B$ is a homomorphism of abelian groups, then the map*

$$\begin{array}{rcl} f\otimes \mathrm{id}\colon A\otimes R & \to & B\otimes R \\ \sum\limits_{i} a_i\otimes r_i & \mapsto & \sum\limits_{i} f(a_i)\otimes r_i \end{array}$$

is a homomorphism of R-modules.

(3) *The map*

$$\begin{array}{rcl} A & \mapsto & A\otimes R \\ (f\colon A\to B) & \mapsto & (f\otimes \mathrm{id}\colon A\otimes R\to B\otimes R) \end{array}$$

defines a functor from the category of abelian groups to the category of R-modules.

[795] It is straightforward to verify that the map $f\otimes g$ is indeed well-defined and that it is a homomorphism.

Proof. Once again the statements follow immediately from the definitions. □

Lemma 42.5 shows that tensoring with a commutative ring "turns" an abelian group into a module. In particular if we tensor with a field, we obtain a vector space. This is very convenient since sometimes it allows us to reduce problems about abelian groups into problems about vector spaces that we can handle with techniques from linear algebra.

For example we can now give a meaningful definition of the rank of any abelian group.

Definition. The rank of an abelian group A is defined as

$$\operatorname{rank}(A) \ := \ \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}).$$

Remark. Previously, on page 371 we introduced the rank of finitely generated abelian groups. It follows from the discussion on page 1004 that for finitely generated abelian groups the two definitions of rank agree. Put differently, the above definition is an extension of the earlier notion of rank to all abelian groups.

We conclude this section with the following lemma.

Lemma 42.6. *Let*

$$0 \ \to \ A \ \xrightarrow{i} \ B \ \xrightarrow{p} \ C \ \to \ 0$$

be a short exact sequence and let G be an abelian group. If the short exact sequence splits, then

$$0 \ \to \ A \otimes G \ \xrightarrow{i \otimes \mathrm{id}} \ B \otimes G \ \xrightarrow{p \otimes \mathrm{id}} \ C \otimes G \ \to \ 0$$

is also a short exact sequence and it also splits.

Remark. In most applications we will consider a short exact sequence

$$0 \ \to \ A \ \xrightarrow{i} \ B \ \xrightarrow{p} \ C \ \to \ 0$$

where C is a free abelian group. It then follows from Lemma 33.1 that the short exact sequence splits, which then implies by Lemma 42.6 that for any abelian group G the sequence

$$0 \ \to \ A \otimes G \ \xrightarrow{i \otimes \mathrm{id}} \ B \otimes G \ \xrightarrow{p \otimes \mathrm{id}} \ C \otimes G \ \to \ 0$$

is also a short exact sequence which furthermore splits.

Proof. According to Lemma 33.2 we can without loss of generality suppose that the short exact sequence is of the form

$$0 \ \to \ A \ \xrightarrow{a \mapsto (a,0)} \ A \oplus C \ \xrightarrow{(a,c) \mapsto c} \ C \ \to \ 0.$$

If we tensor this short exact sequence with G, then using $(A \oplus C) \otimes G = A \otimes G \oplus C \otimes G$ we obtain the sequence

$$0 \to A \otimes G \xrightarrow{\sum_i a_i \otimes g_i \mapsto \left(\sum_i a_i \otimes g_i, 0\right)} A \otimes G \oplus C \otimes G \xrightarrow{\left(\sum_i a_i \otimes g_i, \sum_j c_j \otimes g_j\right) \mapsto \sum_j c_j \otimes g_j} C \otimes G \to 0,$$

which is evidently exact. It is also clear that the short exact sequence splits. □

42.2. **The homology groups of a topological space with coefficients.** Let

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be an algebraic chain complex and let G be an abelian group. We tensor the chain complex with G, i.e. we consider the following sequence of homomorphisms:[796]

$$\ldots \to C_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \mathrm{id}} \ldots \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1 \otimes G \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0 \otimes G \to 0.$$

For all i we have

$$(\partial_{i-1} \otimes \mathrm{id}) \circ (\partial_i \otimes \mathrm{id}) \underset{\substack{\uparrow \\ \text{Lemma 42.4}}}{=} (\partial_{i-1} \circ \partial_i) \otimes \mathrm{id} = 0 \otimes \mathrm{id} = 0,$$

i.e. the above sequence of maps is again a chain complex. We denote the corresponding homology groups by $H_n(C;G) := H_n(C_* \otimes G, \partial_* \otimes \mathrm{id})$. If we tensor the chain complex C_* with a commutative ring R, then it follows from the discussion in the previous section that we obtain a chain complex of R-modules, i.e. all chain groups are R-modules and all chain maps are R-homomorphisms. In particular also all homology groups are R-modules.

Remark. Most of the results which we have proved for chain complexes of abelian groups hold, with the obvious modifications, also for chain complexes of R-modules. In particular one can easily convince oneself that Lemma 29.7, Lemma 30.2, Corollary 30.3 and Propositions 31.10 and Lemma 31.9 also hold in in this more general context. In the remainder of the course we will use these generalizations without further comment.

Now we are ready to define the homology groups of a topological space with coefficients in an abelian group G.

Definition. Let X be a topological space. For an abelian group G we define

$$C_*(X;G) := C_*(X) \otimes G.$$

As discussed above, we obtain a chain complex and we denote by

$$H_k(X;G) := H_k(C_*(X;G), \partial \otimes \mathrm{id})$$

the corresponding homology groups. We call $H_k(X;G)$ the *k-th homology group of X with G-coefficients*. We define the notions of "cycle", "homologous" and "null-homologous" in exactly the same way as on page 740.

Example. We consider an electric circuit as a finite graph X where at the vertices we have resistors or capacitors. Kirchhoff's current rule, see e.g.

`https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws`

states that

[796]Note that if G has also the structure of a commutative ring, then this becomes a chain complex of G-modules.

at any node (vertex) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node,

or equivalently

the algebraic sum of currents in a network meeting at a point is zero.

Put differently, a current defines a 1-cycle in $C_1(X;\mathbb{R})$.[797]

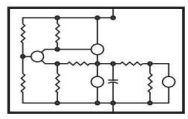
electric circuit 

FIGURE 617.

We start out with the following three observations. Let X be a topological space.

(1) If the coefficients are given by a commutative ring R, e.g. if $R = \mathbb{Z}_n$ or $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, then, as we pointed out above, each $C_n(X;R) = C_n(X) \otimes R$ is an R-module, each boundary map is an R-module homomorphism, and each homology group $H_n(X;R)$ is also an R-module.

(2) From Lemma 42.3 (3) it follows immediately that we have natural isomorphisms $C_*(X;\mathbb{Z}) \cong C_*(X)$ and $H_*(X;\mathbb{Z}) \cong H_*(X)$. In the following we will identify these groups and we will go back and forth between these two notations, depending on which one is more convenient.

(3) Let G be an abelian group. Given any singular k-simplex σ, given any $a \in \mathbb{Z}$ and any $g \in G$ we have by definition of the tensor product that $a\sigma \otimes g = \sigma \otimes ng$. From this observation it follows that any element in $C_k(X;G)$ can be written as

$$\sum_{i=1}^{n} \sigma_i \otimes g_i \quad \text{where } \sigma_1, \dots, \sigma_n \text{ are singular } k\text{-simplices and } g_1, \dots, g_n \in G.$$

Most results for homology groups of topological spaces carry over, with the obvious modifications, to the theory of homology groups with G-coefficients. For example, given an abelian group G we have the following statements:

(1) A map $f\colon X \to Y$ between two topological spaces X and Y induces a chain map

$$f_* \otimes \mathrm{id}\colon C_*(X;G) \to C_*(Y;G)$$

which in turn, for each $k \in \mathbb{N}_0$, induces a homomorphism

$$(f_* \otimes \mathrm{id})_*\colon H_k(X;G) \to H_k(Y;G).$$

To simplify the notation we usually denote this induced map on homology groups also by f_*.

(2) The k-th homology group with G-coefficients defines a functor from the category of topological spaces to the category of abelian groups.

[797]This observation and the figure are due to Ghrist's excellent book on applied topology [Gh].

(3) If R is a commutative ring, then a map $f\colon X \to Y$ between topological spaces induces a chain map of R-homomorphisms $C_*(X;R) \to C_*(Y;R)$ and R-homomorphisms $f_*\colon H_k(X;R) \to H_k(Y;R)$. Furthermore the k-th homology group with R-coefficients defines a functor from the category of topological spaces to the category of R-modules.

(4) The augmentation map $\epsilon\colon H_0(X;G) \to G$ is defined almost the same way as in Lemma 29.4. More precisely, it is given by the map

$$\begin{aligned} \epsilon_X\colon H_0(X;G) &\to G \\ \Big[\sum_{i=1}^k \sigma_i \otimes g_i\Big] &\mapsto \sum_{i=1}^k g_i. \end{aligned}$$

If X is path-connected, then the argument of Proposition 29.5 implies immediately that the augmentation map $\epsilon\colon H_0(X;G) \to G$ is an isomorphism. At times we will use this natural isomorphism to make the identification $H_0(X;G) = G$.

(5) Precisely as on page 761 we can use the augmentation map $\epsilon\colon H_0(X;G) \to G$ to define reduced homology groups $\widetilde{H}_k(X;G)$ with G-coefficients.

(6) We saw in Proposition 30.5 that if $f, g\colon X \to Y$ are two homotopic maps between topological spaces, then the induced maps $f_*, g_*\colon C_*(X) \to C_*(Y)$ are chain homotopic. By tensoring the chain homotopies we see that the corresponding induced maps $C_*(X) \otimes G \to C_*(Y) \otimes G$ are also chain homotopic, and we see that for any $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon H_n(X;G) \to H_n(Y;G).$$

In particular, if X and Y are homotopy equivalent topological spaces, then for any $n \in \mathbb{N}_0$ we have an isomorphism

$$H_n(X;G) \xrightarrow{\cong} H_n(Y;G).$$

(7) Given a pair (X, A) of topological spaces we can tensor the relative chain complex $C_*(X, A)$ with G. We obtain the *relative chain complex* $C_*(X, A; G) := C_*(X, A) \otimes G$. The corresponding homology groups are the *relative homology groups* $H_k(X, A; G)$.

(8) Let (X, B, A) be a triple of topological spaces. As pointed out on page 775, the relative chain groups $C_n(X, B; G)$ are free abelian groups. Thus it follows from the discussion on page 1006 that we obtain a short exact sequence of chain complexes $0 \to C_*(B, A; G) \to C_*(X, A; G) \to C_*(X, B; G) \to 0$. By applying the obvious generalization of Proposition 31.10 to this short exact sequence we obtain the following long exact sequence of homology groups with G-coefficients:

$$\ldots \to H_n(B, A; G) \xrightarrow{i_*} H_n(X, A; R) \xrightarrow{p_*} H_n(X, B; G) \xrightarrow{\partial} H_{n-1}(B, A; G) \to \ldots$$

(9) We have an obvious version of the Mayer-Vietoris Theorem 33.5 for homology groups with G-coefficients. The proof is basically identical to the proof of Theorem 33.5, we only have to note that when we consider the exact sequence

$$0 \to C_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A,B\}}(X) \to 0,$$

then it is a consequence of the discussion on page 1006 that the corresponding sequence with G-coefficients is also exact.

(10) We also have an obvious version of the Excision Theorem 31.17, the proof for G-coefficients is almost verbatim the same as before.

(11) For a CW-complex X we can also consider the corresponding cellular chain complex $C_*^{\mathrm{CW}}(X;G) := C_*^{\mathrm{CW}}(X) \otimes G$ and the corresponding cellular homology groups $H_*^{\mathrm{CW}}(X;G)$. Then, as in Proposition 35.4, one can show that there exists a natural isomorphism $H_*(X;G) \xrightarrow{\cong} H_*^{\mathrm{CW}}(X;G)$.

(12) If X is a finite CW-complex and if $\mathbb{F}$ is a field, then the argument of Proposition 40.1 shows that

$$\chi(X) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(X;\mathbb{F})).$$

Examples.

(1) Let $n \in \mathbb{N}$. As on page 648 we equip S^n with the CW-structure with one 0-cell and one n-cell. For any abelian group G it follows from (10), and similar to the discussion on page 882, that

$$H_k(S^n;G) \;=\; H_k^{\mathrm{CW}}(S^n;G) \;\cong\; \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

The same result holds for reduced homology with G-coefficients except that in this case we have $\widetilde{H}_0(S^n;G) = 0$.

(2) It follows from (10) and the same argument as in the proof of Proposition 35.6 that for any compact n-dimensional manifold M and any abelian group G we have $H_k(M;G) = 0$ for $k > n$.

(3) We want to determine homology groups of the real projective plane $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients. As usual we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell that we introduced on page 898. In the proof of Proposition 35.11 we saw that the cellular chain complex is given by

$$0 \;\to\; \mathbb{Z} \;\xrightarrow{\cdot 2}\; \mathbb{Z} \;\xrightarrow{0}\; \mathbb{Z} \;\to\; 0,$$

i.e. $H_2(\mathbb{R}\mathrm{P}^2) = 0$, $H_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$ and $H_0(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}$. On the other hand, if we tensor the above cellular chain complex with $\mathbb{F}_2$, then using Lemma 42.3 (3) we obtain the chain complex

$$0 \;\to\; \mathbb{F}_2 \;\xrightarrow{\cdot 2 = 0}\; \mathbb{F}_2 \;\xrightarrow{0}\; \mathbb{F}_2 \;\to\; 0.$$

Using the isomorphism between cellular and singular homology with $\mathbb{F}_2$-coefficients we obtain that $H_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$, $H_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$ and $H_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$.

Note that even though the homology groups of $\mathbb{R}\mathrm{P}^2$ with $\mathbb{Z}$-coefficients and $\mathbb{F}_2$-coefficients are quite different, in both cases the alternating sums of the dimensions and ranks give, as promised in Proposition 40.1 and in (10) above, the Euler characteristic.

Indeed, we have

$$\operatorname{rank}(H_0(X)) - \operatorname{rank}(H_1(X)) + \operatorname{rank}(H_2(X)) \;=\; 1-0+0 \;=\; 1,$$

and

$$\dim_{\mathbb{F}_2}(H_0(X;\mathbb{F}_2)) - \dim_{\mathbb{F}_2}(H_1(X;\mathbb{F}_2)) + \dim_{\mathbb{F}_2}(H_2(X;\mathbb{F}_2)) \;=\; 1-1+1 \;=\; 1.$$

(4) Basically the same argument as in (3) shows that for any $n \in \mathbb{N}$ we have

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. Why should we study the homology groups of a topological space with G-coefficients? There are several answers one can give:

(1) At times it is easier to work over a field instead of working over the ring $\mathbb{Z}$. For example, if we only want to determine the Euler characteristic of a CW-complex we can now use a field, even a very simple one like $\mathbb{F}_2$, instead of working with the integers.
(2) If we work over the field $\mathbb{F}_2$ then many calculations become particularly simple, since we do not need to worry about signs. Even in the previous example we already saw that working with $\mathbb{F}_2$-coefficients can simplify calculations.
(3) Recall that given a manifold M we denote its de Rham cohomology groups $H^k_{\mathrm{dR}}(M)$, which are in fact real vector space. For each k we can now also study the homology groups $H_k(M;\mathbb{R})$ which are also real vector spaces. The notations $H^k_{\mathrm{dR}}(M)$ and $H_k(M;\mathbb{R})$ look similar and the names "de Rham cohomology" and "homology" also sound similar. We will study the relationship between these two objects later on in a later chapter.

Throughout this chapter we will use the following notation:

Notation. We denote by $\mathcal{A}$ the category of abelian groups.

We conclude this section with the following elementary lemma which follows immediately from the definitions.

Lemma 42.7.

(1) *Let X be a topological space and let $\varphi\colon A \to B$ be a homomorphism of abelian groups. Then the maps*

$$\begin{aligned} \varphi_*\colon C_*(X;A) &\to C_*(X;B) \\ \Big[\sum_{i=1}^k \sigma_i \otimes a_i\Big] &\mapsto \Big[\sum_{i=1}^k \sigma_i \otimes \varphi(a_i)\Big] \end{aligned}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$\varphi_*\colon H_n(X;A) \;\to\; H_n(X;B).$$

(2) *Given a topological space X and $n \in \mathbb{N}_0$ the maps $A \mapsto H_n(X;A)$ from (1) define a covariant functor from the category of abelian groups $\mathcal{A}$ to itself.*

(3) *The maps from* (1) *define a natural transformation from "homology with A-coefficients" to "homology with B-coefficients", i.e. given a map $f\colon X \to Y$ of topological spaces we get a commutative diagram*

$$\begin{array}{ccc} H_n(X;A) & \xrightarrow{f_*} & H_n(Y;A) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ H_n(X;B) & \xrightarrow{f_*} & H_n(Y;B). \end{array}$$

42.3. **Exact functors.** The following question naturally arises from the discussion in the previous section:

Question 42.8. *Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?*

The example on page 1010 shows that the naive guess that perhaps $H_k(X;G) \cong H_k(X) \otimes G$ is in general not correct. More precisely, we saw that

$$\mathbb{F}_2 = H_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \ \neq \ H_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2 = 0.$$

In the following section we will see that, despite this sobering example, one can always deduce the homology groups $H_*(X;G)$ of a topological space X from the usual homology groups $H_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.

Before we can hope to answer Question 42.8 we have to introduce many more definitions and algebraic tools. We start out with the following definition from category theory.

Definition. Let $F\colon \mathcal{A} \to \mathcal{A}$ be a covariant functor from the category $\mathcal{A}$ of abelian groups to itself.

(1) We say F is *left-exact*, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C)$$

is also exact.

(2) We say F is *right-exact*, if for every exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

(3) We say F is *exact* if it is left-exact and right-exact, put differently, the functor F is exact, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

Remark. Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups and let $F\colon \mathcal{A} \to \mathcal{A}$ be a covariant functor from the category $\mathcal{A}$ of abelian groups to itself.

(1) If F is a left-exact functor, then "taking F" commutes with "taking kernels", more precisely, if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $0 \to \ker(\varphi) \to A \to B$ we see that $F(\ker(\varphi)) = \ker(\varphi_*\colon F(A) \to F(B))$.
(2) Similarly, if F is right-exact, then "taking F" commutes with "taking cokernels", i.e. if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $A \to B \to \operatorname{coker}(\varphi) \to 0$ we see that $F(\operatorname{coker}(\varphi)) = \operatorname{coker}(\varphi_*\colon F(A) \to F(B))$.

Lemma 42.9. *Let G be an abelian group. If G is free abelian, then tensoring with G is exact.*

Proof. Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence and let $G \cong \mathbb{Z}^{(S)}$ be a free abelian group. It follows from Lemma 42.3 (1) and (3) that tensoring the above short exact sequence with $G \cong \mathbb{Z}^{(S)}$ results in an exact sequence that is isomorphic to the direct sum of S copies of the original sequence, hence it is again exact. □

Example. In contrast to the previous lemma we will now see that tensoring with an abelian group G is in general not a *left-exact* functor. For example for any $k \in \mathbb{N}_{\geq 2}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}_k$$

is exact, but after tensoring with $\mathbb{Z}_k$ we obtain, using Lemma 42.3 (3), the sequence

$$0 \longrightarrow \mathbb{Z}_k \xrightarrow[=0]{\cdot k} \mathbb{Z}_k \longrightarrow \mathbb{Z}_k$$

that is not exact.

In contrast to the previous example we will now see that tensoring with an abelian group G is a *right-exact* functor.

Lemma 42.10. *We denote by $\mathcal{A}$ the category of abelian groups. For every abelian group G the functor*

$$\begin{array}{ccc} \mathcal{A} & \to & \mathcal{A} \\ A & \mapsto & A \otimes G \end{array}$$

is right-exact.

Proof. Let

$$A \quad \xrightarrow{i} \quad B \quad \xrightarrow{j} \quad C \quad \to \quad 0$$

be an exact sequence of abelian groups and let G be an abelian group. We have to show that the sequence

$$A \otimes G \quad \xrightarrow{i\otimes \mathrm{id}} \quad B \otimes G \quad \xrightarrow{j\otimes \mathrm{id}} \quad C \otimes G \quad \to \quad 0$$

is also exact.

(a) Since $j\colon B \to C$ is an epimorphism it follows immediately from Lemma 42.1 (3) that the homomorphism

$$\begin{aligned} j \otimes \mathrm{id}\colon B \otimes G &\to C \otimes G \\ \sum_{i=1}^{n} b_i \otimes g_i &\mapsto \sum_{i=1}^{n} j(b_i) \otimes g_i \end{aligned}$$

is also an epimorphism.

(b) By Lemma 42.4 we have $(j \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) = (j \circ i) \otimes \mathrm{id} = 0$, i.e. we have

$$\mathrm{im}\{i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G\} \ \subset \ \ker\{j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G\}.$$

(c) By (b) it remains to show that

$$\ker\{j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G\} \ \subset \ \mathrm{im}\{i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G\}.$$

Put differently it remains to show that the homomorphism

$$\Psi\colon B \otimes G/\,\mathrm{im}\{i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G\} \quad \to \quad C \otimes G$$

that is induced by $j \otimes \mathrm{id}$, i.e. that is given by[798]

$$[b \otimes g] \quad \mapsto \quad j(b) \otimes g,$$

is a monomorphism. We will do so by providing a left-inverse for Ψ, i.e. we will provide a map Φ with $\Phi \circ \Psi = \mathrm{id}$.

Claim. The map

$$\begin{aligned} \varphi\colon C \times G &\to B \otimes G/\,\mathrm{im}\{i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G\} \\ (c, g) &\mapsto b \otimes g \quad \text{ where } b \in B \text{ is an element with } j(b) = c. \end{aligned}$$

is well-defined.

Let $(c, g) \in C \times G$. Since j is by hypothesis surjective we know that there exists a $b \in B$ with $j(b) = c$. We have to show that $\varphi(c, g)$ is independent of the choice of b. So let $b, b' \in B$ with $j(b) = j(b') = c$, i.e. with $j(b - b') = 0$. From the exactness of the original short exact sequence at B it follows that $b - b' = i(a)$ for an $a \in A$. Then we obtain that

$$b \otimes g - b' \otimes g \ = \ (b - b') \otimes g \ = \ i(a) \otimes g \in \mathrm{im}\{i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G\}.$$

Thus we have proved that φ is well-defined. This concludes the proof of the claim.

[798] Note that it is a consequence of (b) that Ψ is well-defined.

The map φ is easily seen to bilinear. It follows from Lemma 42.2 (2) that φ induces a well-defined map

$$\begin{array}{rcl} \Phi\colon C\otimes G & \to & B\otimes G/\operatorname{im}\{i\otimes \mathrm{id}\colon A\otimes G\to B\otimes G\} \\ \sum\limits_{i=1}^{k} c_i\otimes g_i & \mapsto & \sum\limits_{i=1}^{k}\varphi(c_i,g_i). \end{array}$$

By construction we have $(\Phi\circ\Psi)([b\otimes g]) = [b\otimes g]$ for all $b\in B$ and $g\in G$. But the elements $b\otimes g$ generate $B\otimes G$, hence $\Phi\circ\Psi = \mathrm{id}$, i.e. the map Φ is indeed a right-inverse to the map Ψ. □

42.4. **The G-torsion of an abelian group.** Our final goal is to relate the homology groups $H_*(C)$ of a chain complex to the homology groups $H_*(C;G)$ of the same chain complex with G-coefficients. To state the theorem relating these groups we need to introduce the G-torsion of an abelian group.

We start our discussion with the definition of a free abelian group that we already gave on page 366, we also recall some basic facts regarding free abelian groups, and we introduce one new definition.

(1) Let S be a non-empty set. We refer to

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s\in S$$

as the free abelian group generated by S.

(2) A group G is called *free abelian* if there exists a set S and also an isomorphism $G\xrightarrow{\cong}\mathbb{Z}^{(S)}$.

(3) It follows easily from the definitions that a group G is free abelian if and only if there exists a subset $\mathcal{B}=\{b_i\}_{i\in I}\subset G$ with the following property: For any abelian group A and each choice of elements $a_i\in A, i\in I$ there exists a unique homomorphism $\varphi\colon G\to A$ such that $\varphi(b_i)=a_i$ for all $i\in I$. As on page 369 we call such a subset $\mathcal{B}$ a *basis* of G.

(4) A *free resolution* of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are free abelian groups. If there exists an $n\in\mathbb{N}_0$ with $F_n\neq 0$ but $F_m=0$ for $m>n$, then we refer to n as the *length* of the resolution. If no such n exists, then we say that the length of the resolution is infinite.

Examples.

(1) Let $m\in\mathbb{N}$, then a free resolution for $H=\mathbb{Z}_m$ is given by

$$0\to\mathbb{Z}\xrightarrow{\cdot m}\mathbb{Z}\to\mathbb{Z}_m\to 0.$$

(2) Let G be a *finitely generated* abelian group. By the classification of finitely generated abelian groups, see Theorem 11.4, there exists an $r\in\mathbb{N}_0$, non-zero natural numbers

$a_1, \dots, a_k$ and an isomorphism

$$\Phi\colon G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Thus we obtain a free resolution[799]

$$0 \to \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \to \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}}_{\cong G} \to 0.$$

In most cases, when we want to work with explicit examples, this type of free resolution will turn out to be the most useful free resolution.

(3) We recall the following basic fact: if M is an abelian group, then by Lemma 11.1 there exists a unique homomorphism $\alpha(M)\colon \mathbb{Z}^{(M)} \to M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism. Now let H be an abelian group. We consider

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We set $H_1 := \ker\left(\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H\right)$. Then

$$H_1 \hookrightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

is exact. Now we consider

$$\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(H_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. Now we set $H_2 := \ker\left(\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H)} H\right)$ and we iterate this procedure. This way we obtain a free resolution of H.[800] Note that this free resolution is unambiguously defined, we did not make any choices along the way. Therefore we refer to this free resolution of H as the *canonical free resolution of H*.

The last example provides the proof for the following lemma.

Lemma 42.11. *Every abelian group (not necessarily finitely generated) admits a canonical free resolution.*

The following definition is one of the key definitions in homological algebra.

[799] Note that the homomorphism to the left is a monomorphism since $a_1, \dots, a_k$ are non-zero.

[800] Note that this procedure does not stop, i.e. it produces an infinite free resolution.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the canonical free resolution from page 1016. We define the *n-th G-torsion group of H* as

$$\mathrm{Tor}_n(H,G) := H_n(F_* \otimes G).$$

The following proposition is the key to calculating torsion-groups.

Proposition 42.12. *Let G and H be abelian groups and let*

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. Then there exists a canonical isomorphism

$$\mathrm{Tor}_n(H,G) \xrightarrow{\cong} H_n(F_* \otimes G).$$

Example. In many situations we are only interested in determining the isomorphism type of $\mathrm{Tor}_n(H,G)$ where H is a given finitely generated abelian group. In such situations it is usually much more convenient to work with the free resolution given on page 1016. For example, let $m,n \in \mathbb{N}$. We want to determine the isomorphism type of $\mathrm{Tor}_i(\mathbb{Z}_m,\mathbb{Z}_n)$ for $i \in \mathbb{N}_0$. We will do so by doing the following three step process:

(1) We pick the free resolution $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0$,
(2) we tensor this free resolution of $\mathbb{Z}_m$ with $\mathbb{Z}_n$ and from Lemma 42.3 (3) we obtain the sequence $0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to \mathbb{Z}_m \otimes \mathbb{Z}_n \to 0$,
(3) we drop the term to the right and we compute the homology groups.

Now using an argument as in the calculation of the tensor product $\mathbb{Z}_m \otimes \mathbb{Z}_n$ on page 1004 we see that

$$\mathrm{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n) = H_1\big(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\big) = \ker\{\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n\} \cong \mathbb{Z}_{\gcd(m,n)}$$

and

$$\mathrm{Tor}_0(\mathbb{Z}_m,\mathbb{Z}_n) = H_0\big(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\big) = \mathrm{coker}\{\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n\} \cong \mathbb{Z}_{\gcd(m,n)}.$$

We will fill in the details in Exercise 42.5.

The proof of Proposition 42.12 will require some effort. We start out with the following definition.

Definition. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a free resolution of H'. An *extension of α to the free resolutions F_* and F'_** is a sequence of homomorphisms $\alpha_i\colon F_i \to F'_i$, $i \in \mathbb{N}_0$ such that the following diagram commutes:

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\ldots \arrow[r] & F'_2 \arrow[r, "f'_2"] & F'_1 \arrow[r, "f'_1"] & F'_0 \arrow[r, "f'_0"] & H' \arrow[r] & 0.
\end{tikzcd}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and the maps α_* define a chain map from one complex to the other.

The following lemma shows in particular that extensions of homomorphisms between abelian groups always exist.

Lemma 42.13. *Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a free resolution of H'. Then the following statements hold:*

(1) *There exists an extension of α to the free resolutions F_* and F'_*.*
(2) *Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.*
(3) *Let $\alpha_n\colon F_n \to F'_n$, $n \in \mathbb{N}_0$ be an extension of $\alpha\colon H \to H$ and let G be an abelian group. The map*

$$\begin{array}{rcl} \alpha_n \otimes \mathrm{id}\colon H_n(F_* \otimes G) & \to & H_n(F'_* \otimes G) \\ \left[\sum\limits_{i=1}^{k} c_i \otimes g_i\right] & \mapsto & \left[\sum\limits_{i=1}^{k} \alpha_n(c_i) \otimes g_i\right] \end{array}$$

is well-defined and it is independent of the choice of the extension.

Proof $(*)$. Let F_* be a free resolution of H and let F'_* be a free resolution of H'. Statements (1) and (2) are both a straightforward consequence of some mild diagram chasing and the flexibility which Lemma 11.1 provides for defining homomorphisms starting from a free abelian group. In the following we fill in all the details.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} \dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\ \dots \arrow[r] & F'_2 \arrow[r, "f'_2"] & F'_1 \arrow[r, "f'_1"] & F'_0 \arrow[r, "f'_0"] & H' \arrow[r] & 0. \end{tikzcd}$$

First we define the homomorphism α_0. We choose a basis $\mathcal{B}_0$ of the free abelian group F_0. For each $x_i \in \mathcal{B}_0$ we choose an $x'_i \in F'_0$ with $f'_0(x'_i) = \alpha(f_0(x_i))$. This is possible, since f'_0 is surjective. By Lemma 11.1 there exists a unique homomorphism $\alpha_0\colon F_0 \to F'_0$ with $\alpha_0(x_i) = x'_i$ for all $i \in I$. It is now evident that the right square commutes.

Now we turn to the definition of the map α_1. We choose a basis $\mathcal{B}_1$ of the free abelian group F_1. Let $x_i \in \mathcal{B}_1$. It follows from the commutativity of the square on the right that

$$f'_0((\alpha_0 \circ f_1)(x_i)) \;=\; \alpha(\underbrace{(f_0 \circ f_1)}_{=0}(x_i)) \;=\; 0.$$

Since the lower sequence is exact there exists an $x'_i \in F'_1$ with $f'_1(x'_i) = (\alpha_0 \circ f_1)(x_i)$. Again by Lemma 11.1 there exists a unique homomorphism $\alpha_1\colon F_1 \to F'_1$ such that $\alpha_1(x_i) = x'_i$ for all $i \in I$. It is now evident that the "second square from the right"

also commutes. It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \ldots$.

(2) Now let $\alpha_i, \alpha'_i \colon F_i \to F'_i, i \in \mathbb{N}_0$ be two extensions of α to the free resolutions F_* and F'_*. We have to find a chain homotopy from the chain map α_i to the chain map α'_i. The maps $\beta_i = \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = 0 \colon H \to H'$. Therefore it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. The construction of the chain homotopy is very similar to the construction of the maps α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for any $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\beta_2"] \arrow[dl, "\lambda_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1"] \arrow[dl, "\lambda_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0"] \arrow[dl, "\lambda_0"] & H \arrow[r] \arrow[d, "\beta=0"] \arrow[dl, "\lambda_{-1}"] & 0 \\
\ldots \arrow[r] & F'_2 \arrow[r, "f'_2"'] & F'_1 \arrow[r, "f'_1"'] & F'_0 \arrow[r, "f'_0"'] & H' \arrow[r] & 0.
\end{tikzcd}$$

We put $\lambda_{-1} = 0$. Next we need to find a map $\lambda_0 \colon F_0 \to F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. For each $x \in \mathcal{B}_0$ we choose an $x' \in F'_1$ with $f'_1(x') = \beta_0(x)$. The existence of such x' follows from the observation that $f'_0(\beta_0(x)) = \beta(f_0(x)) = 0$ and from the exactness of the lower sequence.

Now we suppose that we have already found maps $\lambda_0, \ldots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \ldots, i.$$

We choose again a basis $\mathcal{B}_i$ of the free abelian group F_i. We have to assign to x an element $x' \in F'_{i+1}$ which satisfies the equation

$$f'_{i+1}(x') \;=\; \beta_i(x) - \lambda_{i-1}(f_i(x)).$$

From the exactness of the lower sequence it follows that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) = 0$. It turns out that indeed we have

$$\begin{aligned}
f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) \;&=\; f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i \;\underset{\substack{\uparrow \\ \text{since the } \beta_i\text{'s are a chain map, i.e. } f'_i \circ \beta_i = \beta_{i-1} \circ f_i}}{=}\; \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\
&=\; (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i \;\underset{\substack{\uparrow \\ \text{by } (*)}}{=}\; \lambda_{i-2} \circ f_{i-1} \circ f_i \;\underset{\substack{\uparrow \\ \text{since } f_{i-1} \circ f_i = 0}}{=}\; 0.
\end{aligned}$$

(3) It is straightforward to verify that the map is well-defined. It follows from (2) and Lemma 30.2 that any two extensions induce the same map on the homology groups. □

Using Lemma 42.13 we can now prove the following lemma.

Lemma 42.14. *Let*

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

and

$$\ldots \to F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H \to 0$$

be two free resolutions of an abelian group H. We pick an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$ *to the free resolutions F_* and F'_*.*

(1) *The maps α_* form a chain homotopy equivalence between the two chain complexes F_* and F'_*.*
(2) *Let G be an abelian group. The*

$$H_n(F_* \otimes G) \xrightarrow{\cong} H_n(F'_* \otimes G)$$

from Lemma 42.13 (3) *is an isomorphism.*

Remark. The statement of Lemma 42.14 (1) is sometimes called the *fundamental theorem of homological algebra*, see e.g. [Lü, Satz 6.3].

Proof. We pick an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$.

(1) We apply Lemma 42.13 to the free resolutions F'_* and F_* (i.e. we swap the roles of the free resolutions) and we obtain maps $\{\beta_i\}_{i\in\mathbb{N}_0}$ that give us the lower half of the following commutative diagram.

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F'_2 \arrow[r, "f'_2"] \arrow[d, "\beta_2"] & F'_1 \arrow[r, "f'_1"] \arrow[d, "\beta_1"] & F'_0 \arrow[r, "f'_0"] \arrow[d, "\beta_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0.
\end{tikzcd}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\beta_2\circ\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1\circ\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0\circ\alpha_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0.
\end{tikzcd}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id}\colon H \to H$ to the free resolutions F_i and F_i. But the identity maps also define an extension. By Lemma 42.13 the extensions $\{\beta_i \circ \alpha_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic.

(2) We had just shown that $\alpha_*\colon F_* \to F'_*$ is a chain homotopy equivalence. One easily verifies that $\alpha_* \otimes \mathrm{id}\colon F_* \otimes G \to F'_* \otimes G$ is also a chain homotopy equivalence. The statement now follows from Corollary 30.3. □

We are now also in a position to prove Proposition 42.12.

Proof. Let G and H be abelian groups and let (F_*, f_*) be a free resolution of H. We denote by (F'_*, f'_*) the canonical free resolution of H. Then by Lemma 42.14 we have a canonical isomorphism

$$\mathrm{Tor}_n(H, G) \;=\; H_n(F'_* \otimes G) \;\xrightarrow{\cong}\; H_n(F_* \otimes G).$$ □

Remark. Using Lemma 42.13 we can now show that the torsion groups $\mathrm{Tor}_n(G, H)$ are "covariantly functorial in both entries". More precisely the following two statements hold:

(1) We fix a group G. Let $\beta\colon H \to H'$ be a homomorphism between two abelian groups. It follows from Lemma 42.13 (3) applied to the canonical free resolutions of H and H' that β induces for each n a canonical homomorphism

$$\beta_*\colon\ \mathrm{Tor}_n(H, G) \;\to\; \mathrm{Tor}_n(H', G).$$

One can now easily show that the maps given by

$$\begin{array}{rcl} H & \mapsto & \mathrm{Tor}_n(H, G) \\ (\beta\colon H \to H') & \mapsto & (\beta_*\colon\ \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H', G)) \end{array}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

(2) We fix a group H. Let $\alpha\colon G \to G'$ be a homomorphism between two abelian groups. Let F_* be the canonical free resolution of H. Then $\mathrm{id} \otimes \alpha$ defines a chain map from $F_* \otimes G$ to $F_* \otimes G'$ and we get an induced map

$$\alpha_*\colon\ \mathrm{Tor}_n(H, G) \;\to\; \mathrm{Tor}_n(H, G').$$

One can now easily show that the maps given by

$$\begin{array}{rcl} G & \mapsto & \mathrm{Tor}_n(H, G) \\ (\alpha\colon G \to G') & \mapsto & (\alpha_*\colon\ \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H, G')) \end{array}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

The following lemma shows that the 0-th torsion groups are objects that we are already familiar with.

Lemma 42.15. *Let G and H be abelian groups, then there exists a natural isomorphism* [801]

$$\mathrm{Tor}_0(H, G) \;\xrightarrow{\cong}\; H \otimes G.$$

[801] Note that "natural isomorphism" means in this context that given an abelian group G we have natural isomorphisms from the functor $H \mapsto \mathrm{Tor}_0(H, G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H, G)$ to the functor $G \mapsto H \otimes G$.

In the remainder of this course we will use the isomorphism from Lemma 42.15 to identify the groups $\mathrm{Tor}_0(H,G)$ and $H \otimes G$.

Proof. Let G and H be abelian groups. Let

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be the canonical free resolution of H. We know from Lemma 42.10 that the sequence

$$(*) \qquad F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G \xrightarrow{f_0 \otimes \mathrm{id}} H \otimes G \to 0$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned} \mathrm{Tor}_0(H,G) &\overset{\text{by definition}}{=} H_0\big(\cdots \to F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G \to 0\big) \\ &= \mathrm{coker}\,\big(F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G\big) \underset{\text{isomorphism since } (*) \text{ is exact}}{\xrightarrow{f_0 \otimes \mathrm{id}}} H \otimes G. \end{aligned}$$

It is straightforward to verify that these isomorphisms define in fact a natural isomorphism from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$. □

The following lemma shows that we do not need to worry about higher torsion groups for abelian groups.

Lemma 42.16.

(1) *Every abelian group H admits a free resolution of length* 1.
(2) *For any two abelian groups H and G we have* $\mathrm{Tor}_n(H,G) = 0$ *for* $n \geq 2$.

Proof.

(1) Let H be an abelian group. If H is finitely generated, then we already saw on page 1016 that H admits a free resolution of length 1. Now suppose that H is any abelian group. We have the exact sequence

$$0 \to \ker\big(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\big) \to \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

where $\alpha(H)$ is defined as in the proof of Lemma 42.11. It follows from Lemma 11.2 that the group $\ker\big(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\big)$, as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$, is itself free abelian. So in this general case we also found a free resolution of length 1.

(2) Let G and H be abelian groups. By (1) there exists a free resolution of length 1, i.e. of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

It follows immediately from Proposition 42.12 that $\mathrm{Tor}_n(H,G) \cong H_n(F_* \otimes G) = 0$ for $n \geq 2$. □

Definition. Given two abelian groups H and G we now write $\mathrm{Tor}(H,G) := \mathrm{Tor}_1(H,G)$ and we refer to this group as as the *G-torsion of H*.

Remark. For many purposes the long discussion of G-torsion groups can be summarized in the following statement: it follows from Proposition 42.12 that for a free resolution $0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{j} H \to 0$ of an abelian group H we have

$$\mathrm{Tor}(H,G) \;\cong\; \ker\big\{F_1 \otimes G \xrightarrow{i\otimes \mathrm{id}} F_0 \otimes G\big\}.$$

Now we summarize several properties of the torsion groups $\mathrm{Tor}(H,G)$ in the following lemma.

Lemma 42.17. *Let G, H, $\{H_i\}_{i\in I}$ and $\{G_j\}_{j\in J}$ be abelian groups. Then the following holds:*

(1) *there exists a natural isomorphism* $\mathrm{Tor}\big(\bigoplus_i H_i, G\big) \;\cong\; \bigoplus_i \mathrm{Tor}(H_i,G)$,
(2) *there exists a natural isomorphism* $\mathrm{Tor}\big(H, \bigoplus_j G_j\big) \;\cong\; \bigoplus_j \mathrm{Tor}(H,G_j)$,
(3) *if H is a torsion-free abelian group, then* $\mathrm{Tor}(H,G) = 0$,
(4) *if G is a torsion-free abelian group, then* $\mathrm{Tor}(H,G) = 0$,
(5) *if T denotes the torsion subgroup*[802] *of H, then* $\mathrm{Tor}(H,G) = \mathrm{Tor}(T,G)$,
(6) *for all $n \in \mathbb{N}$ we have a natural isomorphism* $\mathrm{Tor}(\mathbb{Z}_n, G) \;\cong\; \ker\big\{G \xrightarrow{\cdot n} G\big\}$,
(7) *for all $n, m \in \mathbb{N}$ we have* $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \;\cong\; \mathbb{Z}_{\gcd(n,m)}$.

Remark.

(1) Using this lemma one can usually determine the G-torsion of an abelian group H without any problems. For example the groups $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are torsion-free abelian groups[803] Thus it follows immediately from Lemma 42.17 (4) that for every abelian group H we have
$$\mathrm{Tor}(H,\mathbb{Q}) \;=\; \mathrm{Tor}(H,\mathbb{R}) \;=\; \mathrm{Tor}(H,\mathbb{C}) \;=\; 0.$$
(2) It is a straightforward consequence of the classification of finitely generated abelian groups, see Theorem 11.4, together with Lemma 42.17 that for any two finitely generated abelian groups G and H the corresponding torsion group $\mathrm{Tor}(H,G)$ is finite.[804]

Proof.

(1) For each $i \in I$ we choose a free resolution F^i_* of H_i. Then $\bigoplus_i F^i_*$ is a free resolution of $\bigoplus_i H_i$. The statement follows easily from the natural isomorphisms
$$\Big(\bigoplus_i F^i_j\Big) \otimes G \;\cong\; \bigoplus_i (F^i_j \otimes G)$$

[802]The torsion subgroup of an abelian group is defined as the subgroup of all elements of finite order.
[803]Note though that in Lemma 11.9 we saw that the abelian group $(\mathbb{Q},+)$ is not free abelian.
[804]This fact explains the name "torsion group".

of Lemma 42.3 together with Lemma 29.12 and Proposition 42.12.

(2) We pick a free resolution F_* for H. As in (1) the statement follows easily from the natural isomorphisms

$$F_i \otimes \bigoplus_j G_j \;\cong\; \bigoplus_j F_i \otimes G_j$$

of Lemma 42.3 together with Lemma 29.12 and Proposition 42.12.

(3) If H is a free abelian group, then $0 \to F_0 := H \xrightarrow{\text{id}} H \to 0$ is already a free resolution, and it follows immediately from Proposition 42.12 that $\operatorname{Tor}(H,G) = 0$. Later on we will only use the special case that H is a free abelian group. Therefore we do not provide the proof of the general case of a torsion-free abelian group which is not free. We refer to [Hat, p. 265] for the proof.

(4) By Lemma 42.16 (1) we can pick a free resolution $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H$ of length 1 for H. First we consider the case that G is a free abelian group. In this case we have $G \cong \bigoplus_I \mathbb{Z}$ for an index set I. It follows that

$$\operatorname{Tor}(H,G) = \operatorname{Tor}\big(H, \bigoplus_I \mathbb{Z}\big) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \bigoplus_I \operatorname{Tor}(H,\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 42.12}}}{\cong} \bigoplus_I H_1(0 \to \underbrace{F_1 \otimes \mathbb{Z}}_{=F_1} \to \underbrace{F_0 \otimes \mathbb{Z}}_{=F_0} \to 0) \;= 0.$$

Now we no longer suppose that G is free abelian, we just suppose that G is an abelian group that is torsion-free. We denote by $\{G_i\}_{i\in I}$ the set of finitely generated subgroups of G. It follows from Theorem 11.4 that each G_i is a free abelian group. We write $i \le j$ if and only if $G_i \subset G_j$. Together with the inclusion maps the groups $\{G_i\}_{i\in I}$ form a direct system. We leave it as an exercise to verify that

$$G \;=\; \varinjlim G_i.$$

We have

$$\begin{aligned}
\operatorname{Tor}(H,G) \;=\; \operatorname{Tor}(H, \varinjlim G_i) \;&=\; H_1\big(0 \to F_1 \otimes \varinjlim G_i \to F_0 \otimes \varinjlim G_i\big) \\
&\underset{\substack{\uparrow \\ \text{by Lemma 42.3 (4)}}}{=} H_1\big(\varinjlim (0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)\big) \\
&\underset{\substack{\uparrow \\ \text{Lemma 34.6}}}{=} \varinjlim \underbrace{H_1(0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)}_{\substack{= 0\text{, by the above, since } G_i \text{ is a} \\ \text{free abelian group}}} \;=\; 0.
\end{aligned}$$

(5) If H is a finitely generated abelian group, then it follows from Theorem 11.4 that we can write H as a direct sum $F \oplus T$, where is T the torsion subgroup and where F is a free abelian group. It follows from (1) and (2) that

$$\operatorname{Tor}(H,G) \;\cong\; \operatorname{Tor}(F \oplus T, G) \underset{\substack{\uparrow \\ \text{by (1)}}}{\cong} \operatorname{Tor}(F,G) \oplus \operatorname{Tor}(T,G) \underset{\substack{\uparrow \\ \text{by (3)}}}{=} \operatorname{Tor}(T,G).$$

For the proof in the case that H is not finitely generated we once again refer to [Hat, p. 265] for a proof.

(6) We consider the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

for $H = \mathbb{Z}_n$. After tensoring with the abelian group G, and using Lemma 42.3 we obtain the sequence

$$0 \to G \xrightarrow{\cdot n} G \to \mathbb{Z}_n \to 0.$$

It follows from Proposition 42.12 that $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker\left\{G \xrightarrow{\cdot n} G\right\}$.[805]

(7) We showed this isomorphism on page 1017. Alternatively one can deduce the statement using (6). □

42.5. **The Universal Coefficient Theorem.** Before we can state the Algebraic Universal Coefficient Theorem we need to introduce the following definition.

Definition. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. Throughout this chapter, given $n \in \mathbb{N}_0$, we denote by μ the natural[806] homomorphism[807]

$$\begin{array}{rcl} \mu\colon H_n(C_*) \otimes G & \to & H_n(C_* \otimes G) \\ \sum\limits_{i=1}^{m} [c_i] \otimes g_i & \mapsto & \left[\sum\limits_{i=1}^{m} c_i \otimes g_i\right]. \end{array}$$

Now we can formulate and prove the following theorem which is the key result of the whole chapter.

Theorem 42.18. (The Algebraic Universal Coefficient Theorem) *Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}_0$*

[805]The same argument, together with Lemma 42.15 can be used to give a proof of Lemma 42.3 (3), which back then was perhaps shoved under the carpet a little too quickly.

[806]Here "natural" means that for a chain map $f_*\colon C_* \to D_*$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} H_n(C_*) \otimes G & \xrightarrow{\mu} & H_n(C_* \otimes G) \\ \downarrow & & \downarrow \\ H_n(D_*) \otimes G' & \xrightarrow{\mu} & H_n(D_* \otimes G') \end{array}$$

where the vertical maps are induced by f and φ in the, hopefully, obvious way.

[807]It is straightforward to show that the map is well-defined. We will discuss in Exercise 42.1 whether the "obvious converse" is also well-defined.

there exists a natural[808] *short exact sequence of the form*

$$0 \;\to\; H_n(C) \otimes G \;\xrightarrow{\mu}\; H_n(C;G) \;\xrightarrow{p}\; \mathrm{Tor}(H_{n-1}(C), G) \;\to\; 0.$$

Proof. Let (C_n, ∂_n) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. Note that it follows from Lemma 11.2 that B_n and Z_n are free abelian groups.

We consider the following diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes. Recall that the groups B_n are free abelian groups, thus it follows from Lemmas 33.1 that each of the horizontal sequences splits.

[808]Here "natural" means that if we are given a chain map $f\colon C \to D$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups, then the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_n(C) \otimes G \arrow[r, "\mu"] \arrow[d] & H_n(C;G) \arrow[r, "p"] \arrow[d] & \mathrm{Tor}(H_{n-1}(C), G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_n(D) \otimes G' \arrow[r, "\mu"] & H_n(D;G') \arrow[r, "p"] & \mathrm{Tor}(H_{n-1}(D), G') \arrow[r] & 0
\end{tikzcd}$$

commutes. Here the vertical maps are all induced by the chain map $f\colon C \to D$ and the homomorphism $\varphi\colon G \to G'$. (Wait, in which way do f and φ induce the three vertical maps?) We had already pointed out above that the left-hand square commutes.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G and we obtain the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "0"] & C_n \otimes G \arrow[r, "\partial_n \otimes \mathrm{id}"] \arrow[d, "\partial_n \otimes \mathrm{id}"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] \arrow[d] & C_{n-1} \otimes G \arrow[r, "\partial_{n-1} \otimes \mathrm{id}"] \arrow[d] & B_{n-2} \otimes G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This is again a sequence of chain complexes.

Since the original short exact sequences split it follows from Lemmas 42.6 that the horizontal sequences obtained from tensoring with G are still exact. Therefore we can apply Proposition 31.10 to obtain a long exact sequence of homology groups. Since the vertical boundary maps on the left and on the right are the zero maps, we obtain the following long exact sequence

$$\ldots \to B_n \otimes G \xrightarrow{d_n} Z_n \otimes G \to H_n(C;G) \xrightarrow{\partial_n} B_{n-1} \otimes G \xrightarrow{d_{n-1}} Z_{n-1} \otimes G \to \ldots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. It follows easily from the definition that the connecting homomorphism $d_n \colon B_n \otimes G \to Z_n \otimes G$ is the map that is given by $i_n \otimes \mathrm{id} \colon B_n \otimes G \to Z_n \otimes G$ where $i_n \colon B_n \to Z_n$ denotes the obvious inclusion map.[809]

Using Lemma 33.4 the above discussion implies that we obtain the following short exact sequence

$$0 \to \operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) \to H_n(C;G) \to \ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}} Z_{n-1} \otimes G\right) \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions. We have

$$\begin{array}{ccccc}
\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) & \cong & \operatorname{Tor}_0(H_n(C), G) & \cong & H_n(C) \otimes G \\
 & \uparrow & & \uparrow & \\
 & \text{by Proposition 42.12 since } B_n \xrightarrow{i_n} Z_n \to H_n(C) & & \text{Lemma 42.15} & \\
 & \text{is a free resolution of } H_n(C) & & &
\end{array}$$

Similarly we have

$$\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}} Z_{n-1} \otimes G\right) \cong \operatorname{Tor}(H_{n-1}(C), G).$$

We have thus shown the existence of the desired short exact sequences. The statement that the short exact sequence is natural follows easily from the definitions and Lemma 42.13 (2). We leave the verification of the details as an exercise to the reader. □

[809] Why is this the case?

The following theorem is for us the main application of the Universal Coefficient Theorem 42.18.

Theorem 42.19. (Universal Coefficient Theorem) *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a natural*[810] *short exact sequence*

$$0 \;\to\; H_n(X,A) \otimes G \;\xrightarrow{\mu}\; H_n(X,A;G) \;\to\; \mathrm{Tor}(H_{n-1}(X,A), G) \;\to\; 0.$$

Proof. Let (X, A) be a pair of topological spaces and let G be an abelian group. As we pointed out on page 775, the chain groups $C_n(X, A)$, $n \in \mathbb{N}_0$, are free abelian groups. Thus we can apply the Universal Coefficient Theorem 42.18 and we immediately obtain the desired short exact sequence. □

Examples.

(1) We consider again the real projective space $\mathbb{R}\mathrm{P}^n$. In Proposition 35.11 we saw that

$$H_k(\mathbb{R}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \le n, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

It follows from the Universal Coefficient Theorem 42.19, the discussion on page 1004 and Lemma 42.17 (7) that for any even number $2k \in \{0, \dots, n\}$ we have a short exact sequence

$$0 \;\to\; \underbrace{H_{2k}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\cong 0 \otimes \mathbb{F}_2 = 0} \;\xrightarrow{\mu}\; H_{2k}(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(H_{2k-1}(\mathbb{R}\mathrm{P}^n), \mathbb{F}_2)}_{\cong \mathrm{Tor}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2} \;\to\; 0$$

whereas for $2k + 1 \in \{0, \dots, n\}$ an odd number we have a short exact sequence

$$0 \;\to\; \underbrace{H_{2k+1}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\cong \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \text{ or} \cong \mathbb{Z} \otimes \mathbb{F}_2 \cong \mathbb{F}_2} \;\xrightarrow{\mu}\; H_{2k+1}(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(H_{2k}(\mathbb{R}\mathrm{P}^n), \mathbb{F}_2)}_{\cong \mathrm{Tor}(0, \mathbb{F}_2) = 0 \text{ or } \cong \mathrm{Tor}(\mathbb{Z}, \mathbb{F}_2) = 0} \;\to\; 0.$$

Also it follows easily that $H_k(\mathbb{R}\mathrm{P}^n) = 0$ for $k > n$. Summarizing we obtain the following result

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

[810] Here "natural" means that for every map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces and any homomorphism $\varphi\colon G \to G'$ between abelian groups we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(X,A) \otimes G & \xrightarrow{\mu} & H_n(X,A;G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(X,A), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_n(Y,B) \otimes G' & \xrightarrow{\mu} & H_n(Y,B;G') & \longrightarrow & \mathrm{Tor}(H_{n-1}(Y,B), G') & \longrightarrow & 0, \end{array}$$

where the vertical maps are the obvious (?) ones induced by f and φ.

Fortunately this calculation agrees with the calculation on page 1011.

(2) As on pages 99 and 113 we make the identifications $S^2 = \overline{B}^2/S^1$ and $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z$ where $z \in S^1$. We denote by f the obvious projection map[811]

$$f\colon \mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z \;\to\; S^2 = \overline{B}^2/S^1.$$

As in Figure 400 we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell and as on page 648 we equip S^2 with the CW-structure with one 0-cell and one 2-cell. With these CW-structures the above map f is given by collapsing the 1-cell to the unique 0-cell of S^2, in particular f is a cellular map. By Proposition 35.4 and the discussion on page 1010 we obtain the following commutative diagram

$$\begin{tikzcd}
H_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \arrow[r, "f_*"] \arrow[d, "\Phi_{\mathbb{R}\mathrm{P}^2}", "\cong"'] & H_2(S^2;\mathbb{F}_2) \arrow[d, "\cong"', "\Phi_{S^2}"] \\
H_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \arrow[r, "f_*"] & H_2^{\mathrm{CW}}(S^2;\mathbb{F}_2).
\end{tikzcd}$$

As we saw on page 1010, the 2-cell is a generator of $H_2^{\mathrm{CW}}(S^2;\mathbb{F}_2)$ and it gets sent to the 2-cell in S^2 which is, as we saw on page 882, a generator of $H_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. Thus we see that the bottom map is an isomorphism, it follows that the top map is also an isomorphism. Summarizing the short exact sequences of the Universal Coefficient Theorem 42.19 give us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overbrace{H_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{F}_2}^{=0} \arrow[r, "\mu"] \arrow[d, "f_*", "=0"'] & \overbrace{H_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)}^{\cong\mathbb{F}_2} \arrow[r] \arrow[d, "f_*", "\cong"'] & \overbrace{\mathrm{Tor}(H_1(\mathbb{R}\mathrm{P}^2),\mathbb{F}_2)}^{\cong\mathbb{F}_2} \arrow[r] \arrow[d, "f_*", "=0"'] & 0 \\
0 \arrow[r] & \underbrace{H_2(S^2)\otimes\mathbb{F}_2}_{\cong\mathbb{F}_2} \arrow[r, "\mu"] & \underbrace{H_2(S^2;\mathbb{F}_2)}_{\cong\mathbb{F}_2} \arrow[r] & \underbrace{\mathrm{Tor}(H_1(S^2),\mathbb{F}_2)}_{=0} \arrow[r] & 0.
\end{tikzcd}$$

In the following corollary we give two instances where the tor-terms in the Universal Coefficient Theorem 42.19 are zero, i.e. two instances where we can "pull out" the coefficients.

Corollary 42.20.

(1) *Let (X,A) be a pair of topological spaces and let G be a subgroup of $(\mathbb{C},+)$. Then for each $n \in \mathbb{N}_0$ the map*

$$\mu\colon H_n(X,A;\mathbb{Z})\otimes G \;\to\; H_n(X,A;G)$$

is an isomorphism.

(2) *Let M be a closed, orientable, connected, non-empty n-dimensional manifold, then for any abelian group G the map*

$$\mu\colon H_n(M;\mathbb{Z})\otimes G \;\to\; H_n(M;G)$$

[811]Note that this is not a covering map and that this map has nothing to do with the more familiar projection map $S^2 \to \mathbb{R}\mathrm{P}^2$.

is an isomorphism.

Example. It follows immediately from Proposition 35.11 together with Corollary 42.20 and Lemma 42.3 (3) that

$$H_k(\mathbb{R}\mathrm{P}^{2n+1};\mathbb{R}) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k=0,2n+1,\\ 0, & \text{otherwise.}\end{cases} \qquad \text{and} \qquad H_k(\mathbb{R}\mathrm{P}^{2n};\mathbb{R}) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k=0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The first statement is an immediate consequence of the Universal Coefficient Theorem 42.19 and the fact, obtained in Lemma 42.17 (4), that the torsion-groups are zero for any subgroup of $(\mathbb{C},+)$. The second statement follows similarly, we just need to replace Lemma 42.17 (4) by Lemma 42.17 (3) together with Proposition 39.17. □

Remark.

(1) Let (X,A) be a pair of topological spaces and let R be a subring of $\mathbb{C}$. Since the map $\mu\colon H_n(X,A;\mathbb{Z})\otimes R \to H_n(X,A;R)$ is functorial we can and we will in the following identify the groups $H_n(X,A;\mathbb{Z})\otimes R$ and $H_n(X,A;R)$ using the isomorphism from Corollary 42.20.
(2) Let R be a subring of $\mathbb{C}$. Lemma 42.17 (4) and the Universal Coefficient Theorem 42.18 show that taking the tensor product with R is an exact functor in the sense of the definition on page 1013.
(3) If M is a closed, *oriented*, connected n-dimensional manifold, then the fundamental class allows us to make the identification $H_n(M;\mathbb{Z}) = \mathbb{Z}$ and the corollary now allows us to make the identification $H_n(M;G) = G$ for any abelian group G.

We conclude this section with the following lemma which we will need later on in the generalization of the Borsuk-Ulam Theorem 7.23 to higher dimensions.

Lemma 42.21. *Let $f\colon S^n \to S^n$ be a map and let $k\in\mathbb{N}$. Then the induced map*

$$f_*\colon H_n(S^n;\mathbb{Z}_k) \;\to\; H_n(S^n;\mathbb{Z}_k)$$

is given by multiplication by $\deg(f)$.

Proof $(*)$. Let $f\colon S^n \to S^n$ be a map and let $k\in\mathbb{N}$. Since the short exact sequence of the Universal Coefficient Theorem 42.19 is natural we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(S^n)\otimes\mathbb{Z}_k & \xrightarrow{\mu} & H_n(S^n;\mathbb{Z}_k) & \longrightarrow & \mathrm{Tor}(H_{n-1}(S^n),\mathbb{Z}_k) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f_*\otimes\mathrm{id}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
0 & \longrightarrow & H_n(S^n)\otimes\mathbb{Z}_k & \xrightarrow{\mu} & H_n(S^n;\mathbb{Z}_k) & \longrightarrow & \mathrm{Tor}(H_{n-1}(S^n),\mathbb{Z}_k) & \longrightarrow & 0.
\end{array}$$

Note that $H_{n-1}(S^n)$ is torsion-free, hence $\mathrm{Tor}(H_{n-1}(S^n),\mathbb{Z}_k) = 0$ by Lemma 42.17. It follows from the exactness of the horizontal sequences that the horizontal maps μ on the left are isomorphisms. By definition the map $f_*\colon H_n(S^n)\to H_n(S^n)$ is given by multiplication by $\deg(f)$, hence $f_*\otimes\mathrm{id}$ is also given by multiplication by $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. □

42.6. **Splittings of the Universal Coefficient Theorem.** Our goal had been to determine the homology groups $H_n(C;G)$ only in terms of the original groups $H_k(C)$ and the abelian group G. The Algebraic Universal Coefficient Theorem 42.18 gave us a short exact sequence

$$0 \;\to\; H_n(C)\otimes G \;\xrightarrow{\mu}\; H_n(C;G) \;\xrightarrow{p}\; \mathrm{Tor}(H_{n-1}(C),G) \;\to\; 0.$$

But in general, as we already saw on page 836, given a short exact sequence, it is not possible to determine the middle group of a short exact sequence from the groups to the left and the right. But we saw in Lemma 33.2 that if a short exact sequence of abelian groups

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

splits, then the middle group B is the direct sum of the "outer" groups A and C. The question arises, whether the short exact sequence in the Universal Coefficient Theorem 42.18 splits. This is fortunately the case:

Proposition 42.22. *Let (C_n,∂_n) be a free chain complex*[812] *and let G be an abelian group. Then for any $n\in\mathbb{N}_0$ the short exact sequence*

$$0 \;\to\; H_n(C)\otimes G \;\xrightarrow{\mu}\; H_n(C;G) \;\to\; \mathrm{Tor}(H_{n-1}(C),G) \;\to\; 0$$

of the Algebraic Universal Coefficient Theorem 42.18 splits. In particular there exists an isomorphism

$$H_n(C;G) \;\cong\; H_n(C)\otimes G \oplus \mathrm{Tor}(H_{n-1}(C),G).$$

Proof. Let (C_n,∂_n) be a chain complex of free abelian groups and furthermore let G be an abelian group. According to Lemma 33.2 it suffices to construct a homomorphism $\eta\colon H_n(C;G)\to H_n(C)\otimes G$ such that $\eta\circ\mu$ is the identity on $H_n(C)\otimes G$.

For each $n\in\mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We consider the short exact sequence

$$0 \;\to\; Z_n \;\hookrightarrow\; C_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0.$$

The group $B_{n-1}\subset C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by Lemma 11.2. Therefore we can deduce from Lemma 33.1 that the above short exact sequence splits. In particular, according to Lemma 33.2, for each $n\in\mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n\to Z_n$ such that p_n is the identity on $Z_n\subset C_n$.

[812] A chain complex (C_*,∂_*) is called *free*, if all the groups C_n are free abelian groups.

Claim. The following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift left, "p_{n+1}"] \arrow[dr, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift left, "p_n"] \arrow[dr, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, shift left, "p_{n-1}"] & \dots \\
& Z_{n+1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n+1}"] & Z_n \arrow[u, hook, shift left] \arrow[d, two heads, "q_n"] & Z_{n-1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n-1}"] & \\
\dots \arrow[r] & \underbrace{H_{n+1}(C)}_{=Z_{n+1}/B_{n+1}} \arrow[r, "0"] & \underbrace{H_n(C)}_{=Z_n/B_n} \arrow[r, "0"] & \underbrace{H_{n-1}(C)}_{=Z_{n-1}/B_{n-1}} \arrow[r, "0"] & \dots
\end{tikzcd}$$

where the maps q_n are just the obvious projection maps.

We start at C_{n+1}. If we go down and then right, then we end up with the zero map to $H_n(C)$. On the other hand, if we first go right, then we end up in $B_n \subset Z_n$, but then the projection q_n to $H_n(C) = Z_n/B_n$ is also the zero map. This concludes the proof of the claim.

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

Now we tensor the top and the bottom horizontal sequences with G and we obtain the following diagram of chain complexes.

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \otimes G \arrow[r, "\partial_{n+1}\otimes\mathrm{id}"] \arrow[d, "(q_{n+1}\circ p_{n+1})\otimes\mathrm{id}"] & C_n \otimes G \arrow[r, "\partial_n\otimes\mathrm{id}"] \arrow[d, "(q_n\circ p_n)\otimes\mathrm{id}"] & C_{n-1} \otimes G \arrow[r, "\partial_{n-1}\otimes\mathrm{id}"] \arrow[d, "(q_{n-1}\circ p_{n-1})\otimes\mathrm{id}"] & \dots \\
\dots \arrow[r] & H_{n+1}(C) \otimes G \arrow[r, "0"] & H_n(C) \otimes G \arrow[r, "0"] & H_{n-1}(C) \otimes G \arrow[r, "0"] & \dots
\end{tikzcd}$$

The induced maps on the homology of these two chain complexes give us a homomorphism

$$\eta\colon H_n(C;G) \;=\; H_n(C_* \otimes G) \;\to\; H_n(\text{lower sequence}) = H_n(C) \otimes G.$$

It remains to prove the following claim.

Claim. The composition $\eta \circ \mu$ is the identity on $H_n(C) \otimes G$.

Let $[\sigma] \otimes g \in H_n(C) \otimes G$ where $\sigma \in Z_n$. Then

$$(\eta \circ \mu)([\sigma] \otimes g) \;=\; \eta([\sigma \otimes g]) \;\underset{\substack{\uparrow \\ \text{definition of } \eta}}{=}\; (q_n(p_n(\sigma))) \otimes g \;\underset{\substack{\uparrow \\ \text{since } p_n = \mathrm{id} \text{ on } Z_n}}{=}\; q_n(\sigma) \otimes g \;=\; [\sigma] \otimes g.$$

This concludes the proof of the claim. □

The following immediate corollary to Proposition 42.22 shows in particular that the homology groups with G-coefficients of a topological space X are completely determined by the usual homology groups of X and the abelian group G.

Theorem 42.23. *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ the short exact sequence*

$$0 \;\to\; H_n(X, A) \otimes G \;\xrightarrow{\mu}\; H_n(X, A; G) \;\to\; \mathrm{Tor}(H_{n-1}(X, A), G) \;\to\; 0$$

from the Universal Coefficient Theorem 42.19 splits. In particular there exists an isomorphism

$$H_n(X, A; G) \;\cong\; H_n(X, A) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X, A), G).$$

The question now arises, whether given an abelian group G there exists in fact a *natural* isomorphism

$$H_n(X; G) \;\cong\; H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G).$$

The following proposition says that such a natural isomorphism does not exist for $G = \mathbb{F}_2$. More precisely, we have the following proposition.

Proposition 42.24. *It is not possible assign to each topology space X an isomorphism*

$$\Phi_X \colon H_n(X; \mathbb{F}_2) \;\to\; H_n(X) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(H_{n-1}(X), \mathbb{F}_2)$$

such that for any map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{tikzcd}
H_n(X; \mathbb{F}_2) \arrow[r, "\Phi_X", "\cong"'] \arrow[d, "f_*"] & H_n(X) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(H_{n-1}(X), \mathbb{F}_2) \arrow[d, "f_*"] \arrow[d, phantom, shift right=6ex, "f_*"] \\
H_n(Y; \mathbb{F}_2) \arrow[r, "\Phi_Y", "\cong"'] & H_n(Y) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(H_{n-1}(Y), \mathbb{F}_2).
\end{tikzcd}$$

Proof. So suppose instead that such a map Φ_X exists for every topological space X. We consider the map

$$f \colon \mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z \;\to\; S^2 = \overline{B}^2/S^1$$

from page 1029. We obtain the following diagram:

$$\begin{tikzcd}
\overbrace{H_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)}^{\cong \mathbb{F}_2} \arrow[r, "\Phi_{\mathbb{R}\mathrm{P}^2}", "\cong"'] \arrow[d, "f_*"'] & \overbrace{H_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2}^{=0} \oplus \overbrace{\mathrm{Tor}(H_1(\mathbb{R}\mathrm{P}^2), \mathbb{F}_2)}^{\cong \mathbb{F}_2} \arrow[d, "f_*", "=0"'] \arrow[d, phantom, shift right=6ex, "f_*", "=0"'] \\
\underbrace{H_2(S^2; \mathbb{F}_2)}_{\cong \mathbb{F}_2} \arrow[r, "\Phi_{\mathbb{R}\mathrm{P}^2}", "\cong"'] & \underbrace{H_2(S^2) \otimes \mathbb{F}_2}_{\cong \mathbb{F}_2} \oplus \underbrace{\mathrm{Tor}(H_1(S^2), \mathbb{F}_2)}_{=0}.
\end{tikzcd}$$

The horizontal maps are by hypothesis isomorphisms. We saw on page 1029 that the vertical map on the left is an isomorphism. On the other hand the two vertical maps on the right are zero since either the domain or the target is zero. Thus this diagram cannot possibly commute. We have thus found a contradiction to our assumption. $\square$

42.7. **Isomorphisms on homology groups induce chain homotopies** $(*)$**.** We have now all the tools to prove the following purely algebraic proposition.

Proposition 42.25. *Let C_* and C'_* be two free chain complexes. Suppose that for each $n \in \mathbb{N}_0$ we are given an isomorphism $\varphi_n \colon H_n(C_*) \to H_n(C'_*)$, then there exists a chain map $C_* \to C'_*$, which induces the maps φ_n, and which is a chain homotopy equivalence.*

Remark. Let C_* be a chain complex and let G be a group. The combination of the Universal Coefficient Theorem 42.18 with Theorem 42.23 shows that the homology groups of the tensor product $C_* \otimes G$ do not contain more formation than was already contained in the homology groups of C_*. Proposition 42.25 gives another explanation for this fact since it shows that the chain complex C_*, is up to chain homotopy, determined by its homology groups. But the homology groups of $C_* \otimes G$ depend only on the chain homotopy type of C_*.

The statement of Proposition 42.25 is going to be an immediate consequence of the next two lemmas. Before we can state the first lemma we need to introduce one more definition.

Definition. We say that a chain complex is *basic* if it is of the form $0 \to E_n \xrightarrow{e_n} E_{n-1} \to 0$ where E_n and E_{n-1} are free abelian groups and where e_n is injective.

The first lemma implies Proposition 42.25 in the special case that both chain complexes are basic.

Lemma 42.26. *Suppose that we have two basic chain complexes $0 \to E_n \overset{e_n}{\hookrightarrow} E_{n-1} \to 0$ and $0 \to F_n \overset{f_n}{\hookrightarrow} F_{n-1} \to 0$. Then for every isomorphism*

$$\varphi \colon H_{n-1}(E_*) = \operatorname{coker}(e_n) \ \to \ \operatorname{coker}(f_n) = H_{n-1}(F_*)$$

there exists a chain map $E_* \to F_*$ that induces that isomorphism and that is a homotopy equivalence.

Proof. By Lemma 42.13, applied twice, there exist maps $\alpha_i \colon E_i \to F_i$, $i = n-1, n$ and there exist maps $\beta_i \colon F_i \to E_i$, $i = n-1, n$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & E_n \arrow[r, "e_n"] \arrow[d, "\alpha_n"] & E_{n-1} \arrow[r, "p"] \arrow[d, "\alpha_{n-1}"] & \operatorname{coker}(e_n) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & F_n \arrow[r, "f_n"] \arrow[d, "\beta_n"] & F_{n-1} \arrow[r, "q"] \arrow[d, "\beta_{n-1}"] & \operatorname{coker}(f_n) \arrow[r] \arrow[d, "\varphi^{-1}"] & 0 \\
0 \arrow[r] & E_n \arrow[r, "e_n"] & E_{n-1} \arrow[r, "p"] & \operatorname{coker}(e_n) \arrow[r] & 0
\end{tikzcd}$$

commutes. From the second part of Lemma 42.13 it follows that $\beta_* \circ \alpha_* \simeq \mathrm{id}_{E_*}$ and that $\alpha_* \circ \beta_* \simeq \mathrm{id}_{F_*}$. Put differently, α_* is a chain homotopy equivalence. $\square$

The next lemma says that we can decompose every free chain complex into a direct sum of basic chain complexes. Proposition 42.25 is an immediate consequence of Lemma 42.26, Lemma 29.12 and the following lemma.

Lemma 42.27. *Let (C_*, ∂_*) be a free chain complex. Then there exist basic chain complexes (F_*^n, ∂_*), $n \in \mathbb{N}_0$ such that*

$$C_* \;\cong\; \bigoplus_{n\in\mathbb{N}_0} F_*^n.$$

Proof. Let (C_*, ∂_*) be a chain complex. For $n \in \mathbb{N}_0$ we write $Z_n = \ker(\partial_n \colon C_n \to C_{n-1})$ and $B_n = \operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n)$. Note that we have a short exact sequence

$$0 \;\to\; Z_n \;\to\; C_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0.$$

Since C_{n-1} is free abelian it follows from Lemma 11.2 that the subgroup B_{n-1} is also free abelian. It follows immediately from Lemmas 33.1 and 33.2 that there exists an isomorphism $\Phi_n \colon C_n \to Z_n \oplus B_{n-1}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & C_n \arrow[dr, "\partial_n"] \arrow[dd, "\cong", "\Phi_n"'] & & \\
0 \arrow[r] & Z_n \arrow[ur, "i"] \arrow[dr] & & B_{n-1} \arrow[r] & 0 \\
 & & Z_n \oplus B_{n-1} \arrow[ur] & &
\end{tikzcd}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the obvious projection map. Given $n \in \mathbb{N}_0$ we now denote by F_*^n the chain complex that is given by

$$0 \;\to\; B_{n-1} \;\hookrightarrow\; Z_{n-1} \;\to\; 0.$$

As before it follows from Lemma 11.2 that this is a basic chain complex. It is straightforward to verify that

$$\bigoplus_{n\in\mathbb{N}_0} \Phi_n \colon C_* \;\to\; \bigoplus_{n\in\mathbb{N}_0} F_*^n$$

is an isomorphism of chain complexes. □

At the end of this section we return to the relationship between cellular and singular chain complexes of a CW-complex. More precisely, let X be a CW-complex. In Proposition 35.4 we showed that for any $n \in \mathbb{N}_0$ there exists a natural isomorphism $H_n(X) \to H_n^{\mathrm{CW}}(X)$. Picking up the discussion from page 888 we can now generalize this statement. More precisely from Proposition 35.4 and 42.25 we immediately obtain the following corollary.

Corollary 42.28. *Let X be a CW-complex. Then the singular chain complex $C_*(X)$ is chain homotopic to the cellular chain complex $C_*^{\mathrm{CW}}(X)$.*

42.8. **Homological algebra over an arbitrary commutative ring** $(*)$**.** A lot of the algebra that we had studied in the previous sections can be generalized from the case of abelian groups, i.e. the case of $\mathbb{Z}$-modules, to modules over a commutative ring. We will not make use of the generalizations that we discuss in this short section.

Throughout this section let R be a commutative ring. Let V and W be two R-modules. We say that an R-homomorphism $f\colon V \times W \to X$ to an R-module X is *R-bilinear*, if for all $v, v' \in V$, $w, w' \in W$ and $r \in R$ the following holds:

$$\begin{array}{rcl} f(v+v',w) & = & f(v,w)+f(v',w), \\ f(rv,w) & = & rf(v,w) \end{array} \quad \text{and} \quad \begin{array}{rcl} f(v,w+w') & = & f(v,w)+f(v,w') \\ f(v,rw) & = & rf(v,w). \end{array}$$

We say an R-bilinear map $f\colon V \times W \to X$ has the *universal R-bilinear property*, if for every R-bilinear map $g\colon V \times W \to Y$ there exists a *unique* R-homomorphism $\varphi\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, "\varphi \text{ unique}"] \\ & Y. \end{tikzcd}$$

Using a variation of the definition of the tensor product of abelian groups that we gave on page 1001 one can now fairly easily prove that there exists an R-bilinear map $f\colon V \times W \to X$ which has the universal R-bilinear property.

A variation of the usual "universal property" argument shows that if $f\colon V \times W \to X$ and $f'\colon V \times W \to X'$ are maps that both have the universal R-bilinear property, then there exists a unique isomorphism $\varphi\colon X \to X'$ of R-modules such that $\varphi(f(v,w)) = f'(v,w)$ for all $v \in V$ and $w \in W$.

We denote by $V \otimes_R W$ the R-module that satisfies satisfies the above property. (By the universal property we see that $V \otimes_R W$ is uniquely determined up to isomorphism.) Note that $V \otimes_R W$ comes with an R-bilinear map $V \times W \to V \otimes_R W$. Given $v \in V$ and $w \in W$ we denote the image of (v,w) by $v \otimes w$. If $R = \mathbb{Z}$, then it follows from the remark after Lemma 42.2 that $V \otimes_{\mathbb{Z}} W$ is isomorphic to $V \otimes W$.

As before we see that given an R-module W the map $V \mapsto V \otimes_R W$ defines a functor from the category of R-modules to the category of R-modules. The proof of Lemma 42.10 shows that this functor is right-exact.

We define a *free R-module* precisely the same way as we defined a free abelian group. A *free resolution* of an R-module M is an exact sequence

$$\cdots \to \ F_2 \ \xrightarrow{f_2} \ F_1 \ \xrightarrow{f_1} \ F_0 \ \xrightarrow{f_0} \ M \ \to \ 0$$

where each F_i is a free R-module. We can construct the canonical free resolution of an R-module M the same was as we did on page 1016.

Given two R-modules M and N we can define the torsion R-modules $\operatorname{Tor}_n^R(M,N)$ as before. More precisely, we take the canonical free resolution

$$\cdots \to \ F_2 \ \xrightarrow{f_2} \ F_1 \ \xrightarrow{f_1} \ F_0 \ \xrightarrow{f_0} \ M \ \to \ 0$$

and we define

$$\operatorname{Tor}_n^R(M,N) \ := \ H_n(F_* \otimes_R N).$$

Here the following statements hold:

(1) As in Proposition 42.12 one can show that the isomorphism type of $\mathrm{Tor}_n^R(M,N)$ can be determined using any free resolution for M.
(2) The argument of page 1021 shows that for each $n \in \mathbb{N}_0$ the R-torsion module $\mathrm{Tor}_n^R(M,N)$ is covariantly functorial in both entries.
(3) The proof of Lemma 42.15 can also be generalized to show that given any two R-modules M and N we have a natural isomorphism $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$.
(4) The obvious analogues of Lemma 42.17 (1) and (2) also hold in the more general context. Furthermore the obvious analogues of Lemma 42.17 (3) and (4) hold if we replace "torsion-free abelian group" by "free R-module".[813] In general there are no analogues of Lemma 42.17 (5), (6) and (7).
(5) For general commutative rings it is no longer true though that $\mathrm{Tor}_i^R(M,N) = 0$ for $i \geq 2$. For example for $R = \mathbb{Q}[t_1, \ldots, t_k]$ the R-torsion modules can be non-zero for $i = 1, \ldots, k$ and they are only zero for $i > k$. We refer to [Mac, Chapter VII.4] for details.

Now let (C_*, ∂_*) be a chain complex of R-modules. Furthermore let M be another R-module. We consider the chain complex $C_* \otimes_R M$ of R-modules. It is natural to ask whether one can again determine the homology groups of $C_* \otimes_R M$ from the torsion groups $\mathrm{Tor}_k^R(H_l(C), M)$ with $k \in \mathbb{N}_0$. If R is a principal ideal domain (usually abbreviated by PID), then almost the same statement as in the Universal Coefficient Theorem 42.18 holds. On the other hand, if R is not a principal ideal, then as we pointed out above, the higher torsion groups are in general non-zero and the problem becomes much more subtle. The torsion-terms and the homology with M-coefficients are then related by a "spectral sequence", see e.g. [Rot2, Theorem 10.90] for details.

42.9. **The Eilenberg-Steenrod axioms for homology** $(*)$**.** In this section we give a very short introduction to the concept of (co-) homology theories which was introduced in the seminal book of Eilenberg–Steenrod [ES].

Before we can do so we introduce the following definition.

Definition. Let R be a commutative ring and let S be a set.

(1) An *S-graded R-module* is defined as an R-module M together with a direct sum decomposition $M = \bigoplus_{s \in S} M_s$ into R-modules M_s, $s \in S$. If $x \in M$ lies in some M_s, then we call x *homogeneous* and we refer to $\deg(x) := s$ as the *degree of* x.
(2) A *morphism $f\colon M \to N$ between S-graded R-modules M and N* is an R-module homomorphism such that for each $s \in S$ we have $f(M_s) \subset N_s$. Equivalently, it is a family of R-module homomorphisms $\{f_s\colon M_s \to N_s\}_{s \in S}$.
(3) The *category of S-graded R-modules* is the category where the objects are S-graded R-modules and the morphisms are the ones described in (2).

[813] Note that in general "R-torsion free module" is not going to be strong enough.

Examples. Given a pair of topological spaces (X, Y) and an abelian group G we write

$$H_*(X, Y; G) \ := \ \bigoplus_{n \in \mathbb{N}_0} H_n(X, Y; G)$$

and we view $H_*(X, Y; G)$ as an $\mathbb{N}_0$-graded $\mathbb{Z}$-module.

The following definition goes back to [ES] and [Mi10].

Definition. A *homology theory* is a covariant functor

$$\mathcal{H}\colon \text{category of pairs of topological spaces} \ \to \ \text{category of } \mathbb{N}_0\text{-graded abelian groups}$$

that satisfies the following axioms:[814]

(1) (Homotopy invariant) If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps between pairs of topological spaces, then $f_* = g_*$.
(2) (Connecting homomorphism) Given any pair (X, A) of topological spaces and any $n \in \mathbb{N}$ there exists a natural connecting homomorphism $\partial\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$[815] such that the sequence
$$\ldots \ \to \ \mathcal{H}_{n+1}(X, A) \ \xrightarrow{\partial} \ \mathcal{H}_n(A) \ \to \ \mathcal{H}_n(X) \ \to \ \mathcal{H}_n(X, A) \ \xrightarrow{\partial} \ \mathcal{H}_{n-1}(A) \ \to \ \ldots$$
is exact.
(3) (Excision) If U and A are subsets of a topological space X with $\overline{U} \subset \mathring{A}$, then for each $n \in \mathbb{N}_0$ the map $\mathcal{H}_n(X \setminus U, A \setminus U) \to \mathcal{H}_n(X, A)$ is an isomorphism.
(4) (Additivity) Let $\{(X_j, A_j)\}_{j \in J}$ be a family of pairs of topological spaces. We write $X = \bigsqcup_{j \in J} X_j$ and $A = \bigsqcup_{j \in J} A_j$. If given $j \in J$ we denote by $i_j\colon X_j \to X$ the obvious inclusion map, then for any $k \in \mathbb{N}_0$ the induced map
$$\bigoplus_{j \in J} i_{j*}\colon \bigoplus_{j \in J} \mathcal{H}_k(X_j, A_j) \ \to \ \mathcal{H}_k(X, A)$$
is an isomorphism.
(5) (Dimension axiom) We denote by $\star$ the topological space that consists of a single point. For $n \neq 0$ we have $H_n(\star) = 0$.

Example. It follows from Proposition 31.15, Corollary 31.14, the Excision Theorem 31.17, the obvious generalization of Lemma 29.13 and Lemma 29.6 that singular homology is indeed a homology theory.

The following theorem gives a uniqueness statement for homology theories.

Theorem 42.29. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two homology theories and let $\varphi\colon \mathcal{H}_0(\star) \to \mathcal{H}'_0(\star)$ be an isomorphism. Then there exists a natural isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ viewed as functors*

$$\textit{category of CW-complexes} \ \to \ \textit{category of } \mathcal{N}_0\textit{-graded groups.}$$

[814]The axioms are usually called Eilenberg-Steenrod axioms.

[815]Here we use the obvious notation that for a topological space Y we write $\mathcal{H}_n(Y) := \mathcal{H}_n(Y, \varnothing)$.

Sketch of a proof. Let G be an abelian group. We start our proof with the following observation: Given a pair (X, A) of CW-complexes we saw in Corollary 26.24 (3) that we can equip the mapping cone $\mathrm{Cone}(i\colon A \to X)$ of the inclusion map with a natural CW-structure. Given $n \in \mathbb{N}_0$ we define the n-th cellular homology $H_n^{\mathrm{CW}}(X, A; G)$ as the reduced cellular homology with G-coefficients of the mapping cone.

Proposition 35.4, together with Lemma 33.17, Proposition 35.4 and the discussion on page 1010 shows that for pairs of CW-complexes there exists a natural isomorphism between singular homology with G-coefficients and cellular homology with G-coefficients. Closer inspection of the proof shows that we can prove this statement only using the above Eilenberg-Steenrod axioms and the fact that $H_0(\star; G) \cong G$.

Now let $\alpha\colon \mathcal{H}_0(\star) \to G$ be an isomorphism. By the above one can show that there exists a natural isomorphism from $\mathcal{H}$ to $H_n^{\mathrm{CW}}(-; G)$. The same applies to $\mathcal{H}'$ and we obtain the theorem as long as φ is an isomorphism. This approach to the proof is developed in more detail in [tD2, Chapter 12.2]. □

Remark.

(1) Closely related results are also proved in [ES, p. 100] and [Hu, p. 52], see also [GH, p. 138].
(2) The axioms of a cohomology theory are defined basically the same way with the obvious modifications and the analogue of Theorem 42.29 also holds in this context, see [DK, Theorem 1.31].
(3) If one drops the dimension axiom, then there are other (co-) homology theories, for example "topological K-theory". We refer to [Karo] for details.

Exercises for Chapter 42.

Exercise 42.1. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. We consider the following map:

$$\begin{array}{ccc} H_n(C_* \otimes G) & \to & H_n(C_*) \otimes G \\ \left[\sum\limits_{i=1}^{n} c_i \otimes g_i\right] & \mapsto & \sum\limits_{i=1}^{n} [c_i] \otimes g_i \end{array}$$

Is this map well-defined?

Exercise 42.2. What are the homology groups of $\mathbb{R}\mathrm{P}^5$ with $\mathbb{F}_3$-coefficients?

Exercise 42.3. Let $A \to B \to C$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there exists an exact sequence of the form

$$0 \to \mathrm{Tor}(A, G) \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0.$$

Exercise 42.4. Determine $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z})$.

Exercise 42.5. Let $m, n \in \mathbb{N}$. Show that

$$\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}.$$

Exercise 42.6. Show that

$$\Big(\prod_{n\in\mathbb{N}} \mathbb{Z}_n\Big) \otimes \mathbb{Q} \neq 0.$$

43. The Künneth Theorem

In this chapter we want to study the following question:

Question 43.1. *Let X and Y be two topological spaces. What is the connection between the homology groups of X and Y on the one hand and the homology groups of the product topological space $X \times Y$ on the other hand?*

43.1. **The tensor product of chain complexes.** First we want to find some connection between the chain complexes of X, Y and $X \times Y$. To do so we need the notion of the tensor product of chain complexes. Therefore we start out with a short excursion into homological algebra, i.e. into the study of chain complexes.

Lemma 43.2. *Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we define*

$$(\mathcal{C} \otimes \mathcal{C}')_n \ := \bigoplus_{p+q=n} C_p \otimes C'_q.$$

The consider the maps

$$d\colon (\mathcal{C} \otimes \mathcal{C}')_n \ \to \ (\mathcal{C} \otimes \mathcal{C}')_{n-1}$$

which are uniquely determined by

$$c_p \otimes c'_q \ \mapsto \ \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q$$

for $c_p \in C_p$ and $c'_q \in C'_q$. The maps turn $\mathcal{C} \otimes \mathcal{C}'$ into a chain complex, called the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

We illustrate the definition of the tensor product of chain complexes in Figure 618.

$$\begin{tikzcd}
C_0 \otimes C'_2 \arrow[d, "\mathrm{id} \otimes \partial'_2"] & C_1 \otimes C'_2 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_2"] & C_2 \otimes C'_2 \arrow[l, "\partial_2 \otimes \mathrm{id}"'] \arrow[d, "\mathrm{id} \otimes \partial'_2"] \\
C_0 \otimes C'_1 \arrow[d, "\mathrm{id} \otimes \partial'_1"] & C_1 \otimes C'_1 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_1"] & C_2 \otimes C'_1 \arrow[l, "\partial_2 \otimes \mathrm{id}"'] \arrow[d, "\mathrm{id} \otimes \partial'_1"] \\
C_0 \otimes C'_0 & C_1 \otimes C'_0 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] & C_2 \otimes C'_0 \arrow[l, "\partial_2 \otimes \mathrm{id}"']
\end{tikzcd}$$

Figure 618. The tensor product of chain complexes.

Proof. We still have to show that the maps d are indeed boundary maps of a chain complex, i.e. we have to verify that $d \circ d = 0$. This is the case since for any $c_p \in C_p$ and $c'_q \in C'_q$ we

have

$$\begin{aligned} d(d(c_p \otimes c'_q)) &= d(\overbrace{\partial c_p}^{\in C_{p-1}} \otimes c'_q) + d((-1)^p \cdot c_p \otimes \overbrace{\partial' c'_q}^{\in C'_{q-1}}) \\ &= \underbrace{\partial\partial c_p}_{=0} \otimes c'_q + (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot c_p \otimes \underbrace{\partial'\partial' c'_q}_{=0} \\ &= (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q - (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q \;=\; 0. \end{aligned}$$

Thus we have shown that $(\mathcal{C} \otimes \mathcal{C}', d)$ is indeed a chain complex.[816] □

Example. Let

$$\mathcal{C} \;=\; \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \dots \qquad \text{and} \qquad \mathcal{C}' \;=\; 0 \;\to\; \underset{\substack{\uparrow \\ \text{in degree } k}}{G} \;\to\; 0$$

be two chain complexes where the second consists of a single abelian group. In this case we have

$$(\mathcal{C}_* \otimes \mathcal{C}'_*)_n \;=\; \mathcal{C}_{n-k} \otimes G,$$

i.e. the tensor product is just given by tensoring the chain groups of $\mathcal{C}$ by G and shifting the degrees by $-k$.

We conclude this section with the following lemma.

Lemma 43.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes.*

(1) *Given two chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the induced map*

$$\begin{aligned} f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' &\to \mathcal{D} \otimes \mathcal{D}' \\ \sum_{i=1}^n c_i \otimes c'_i &\mapsto \sum_{i=1}^n f(c_i) \otimes f'(c'_i) \end{aligned}$$

is also a chain map.

(2) *If $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopic maps, then the induced maps*

$$f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

are also chain homotopic.

(3) *If $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are homotopy equivalences, then the induced map*

$$f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

is also a homotopy equivalence.

[816] Note that in the calculation we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we defined the boundary map "naively" as $d(c_p \otimes c'_q) = \partial c_p \otimes c'_q + c_p \otimes \partial' c'_q$, then the result would not have been a chain complex.

Proof. The proof of the lemma is elementary. For completeness' sake and for a good conscience we provide the proof of the first statement. So let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes and let $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps. Let $c \in C_k$ and $c' \in C'_l$, then

$$\begin{aligned}\partial(f(c)\otimes f'(c')) &\underset{\substack{\uparrow \\ \text{since } \deg(f(c))=\deg(c)=k}}{=} \partial f(c)\otimes f'(c') + (-1)^k\cdot f(c)\otimes \partial f'(c') \underset{\substack{\uparrow \\ \text{since } f \text{ and } f' \text{ are chain maps}}}{=} f(\partial c)\otimes f'(c') + (-1)^k\cdot f(c)\otimes f'(\partial c') \\ &= (f\otimes f')\big(\partial c\otimes c' + (-1)^k\cdot c\otimes \partial c'\big) \qquad = (f\otimes f')(\partial(c\otimes c')).\end{aligned}$$

□

43.2. **The Eilenberg-Zilber Theorem.** The following theorem, which was first proved in 1953, relates the singular chain complex $C_*(X\times Y)$ of the product of two topological spaces X and Y to the tensor product $C_*(X)\otimes C_*(Y)$ of the two individual chain complexes.

Theorem 43.4. (Eilenberg-Zilber) *Let X and Y be topological spaces. Then there exist natural*[817] *chain maps*

$$\Upsilon\colon C_*(X)\otimes C_*(Y) \;\to\; C_*(X\times Y)$$

and

$$\Theta\colon C_*(X\times Y) \;\to\; C_*(X)\otimes C_*(Y),$$

which are naturally chain homotopy-inverses of one another. In particular for each $n\in\mathbb{N}_0$ we have a natural isomorphism

$$H_n(X\times Y) \xrightarrow[\cong]{\Theta_*} H_n(C_*(X)\otimes C_*(Y)).$$

Remark. The proof of the Eilenberg-Zilber Theorem is technically rather intricate. We postpone the proof to algebraic topology III. For the time being we refer to [Mun, Theorem 59.2] or alternatively to [Mas2, Chapter XI.6] or to [Bre, Chapter VI.1.4] for a proof. What we will do instead is that we will prove an analogous statement for CW-complexes which will be sufficient for the calculations of homology groups that we want to do in this chapter.

Proposition 43.5. *Let X and Y be two CW-complexes with finitely many cells in each dimension. Then there exists a natural*[818] *isomorphism*

$$C_*^{\mathrm{CW}}(X\times Y) \xrightarrow{\cong} C_*^{\mathrm{CW}}(X)\otimes C_*^{\mathrm{CW}}(Y)$$

of chain complexes.

[817]Here "natural" means that for maps $f\colon X\to X'$ and $g\colon Y\to Y'$ we obtain commutative diagrams

$$\begin{tikzcd} C_*(X)\otimes C_*(Y) \arrow[r, "\Upsilon"] \arrow[d, "f_*\otimes f'_*"] & C_*(X\times Y) \arrow[d, "(f\times g)_*"] \\ C_*(X')\otimes C_*(Y') \arrow[r, "\Upsilon"] & C_*(X'\times Y') \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} C_*(X\times Y) \arrow[r, "\Theta"] \arrow[d, "(f\times g)_*"] & C_*(X)\otimes C_*(Y) \arrow[d, "f_*\otimes f'_*"] \\ C_*(X'\times Y') \arrow[r, "\Theta"] & C_*(X')\otimes C_*(Y'). \end{tikzcd}$$

[818]Here the notion "natural" means "natural" in the category of CW-complexes, i.e. the isomorphisms Φ_{XY} are only natural with respect to cellular maps $X\to X'$ and $Y\to Y'$, i.e. the commutative diagram from Footnote 817 only exists for such cellular maps.

Remark. Recall that, following the convention of page 752, given two chain complexes C_* and D_* we write $C_* \simeq D_*$ if they are homotopy equivalent. Suppose that X and Y are CW-complexes which have only finitely many cells in each dimension. Then we have

$$\begin{array}{ccccccc} C_*(X\times Y) & \simeq & C_*^{\mathrm{CW}}(X\times Y) & \xrightarrow{\cong} & C_*^{\mathrm{CW}}(X)\otimes C_*^{\mathrm{CW}}(Y) & \simeq & C_*(X)\otimes C_*(Y). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Corollary 42.28 and Lemma 43.3} & & \text{Proposition 43.5} & & \text{Corollary 42.28 and Lemma 43.3} & \end{array}$$

For such X and Y this argument proves the existence of the chain homotopies in the statement of the Eilenberg-Zilber Theorem 43.4. But note that the chain homotopy equivalences of Lemma 43.3 are not natural, hence the above chain equivalences are not natural.

Proof. Let X and Y be two CW-complexes with finitely many cells in each dimension. On page 669 we introduced a CW-structure on $X\times Y$ where the n-cells of $X\times Y$ are precisely of the form $e\times f$ where e is a p-cell of X and f is an $(n-p)$-cell of Y. We quickly recall the definition of the product CW-structure on $X\times Y$:

(1) Using the homeomorphism from Proposition 1.48 (2) we identify the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n=[0,1]^n$.
(2) We put $(X\times Y)^0 = X^0\times Y^0$.
(3) Now we suppose that the $(n-1)$-skeleton $(X\times Y)^{(n-1)}$ has already been defined. We consider a p-cell e in X which is given by a characteristic map $\varphi\colon I^p\to X^p$ and an $(n-p)$-cell f in Y which is given by a characteristic map $\psi\colon I^{n-p}\to Y^{n-p}$. Then we denote by $e\times f$ the n-cell in $X\times Y$ which is given by the attaching map

$$\begin{array}{rcl} \partial(I^n)=\partial I^p\times I^{n-p}\cup I^p\times\partial I^{n-p} & \to & (X\times Y)^{n-1} \\ (x,y) & \mapsto & (\varphi(x),\psi(y)). \end{array}$$

All these n-cells together define the n-skeleton of $X\times Y$.

Since X and Y have only finitely many cells in each dimension the above construction does indeed define a CW-structure on $X\times Y$.

Given $n\in\mathbb{N}_0$ we consider the map

$$C_n^{\mathrm{CW}}(X\times Y) \;\to\; C_p^{\mathrm{CW}}(X)\otimes C_{n-p}^{\mathrm{CW}}(Y)$$

which is induced by the property that for any p-cell e of X and any $(n-p)$-cell f of Y we have[819]

$$[e\times f] \;\mapsto\; [e]\otimes[f].$$

As on page 1005 one can show that this map is an isomorphism. It remains to prove that the above maps form a chain map. This statement in turn is equivalent to the following claim.

[819]We follow the convention of page 889 where saw that cells naturally give rise to a basis of the cellular chain complex.

Claim. For any p-cell e of X and any q-cell f of Y we have

$$\underbrace{\partial(e\times f)}_{\in C^{\mathrm{CW}}_{p+q-1}(X\times Y)} \;=\; \underbrace{\partial e\otimes f}_{\in C^{\mathrm{CW}}_{p-1}(X)\otimes C^{\mathrm{CW}}_{q}(Y)} \;+\; (-1)^p\cdot \underbrace{e\otimes\partial f}_{\in C^{\mathrm{CW}}_{p}(X)\otimes C^{\mathrm{CW}}_{q-1}(Y)}.$$

It is possible to prove this formula using the description of the cellular boundary map in Proposition 35.8 and by doing a very careful sign-analysis. We illustrate this argument in Figure 619.

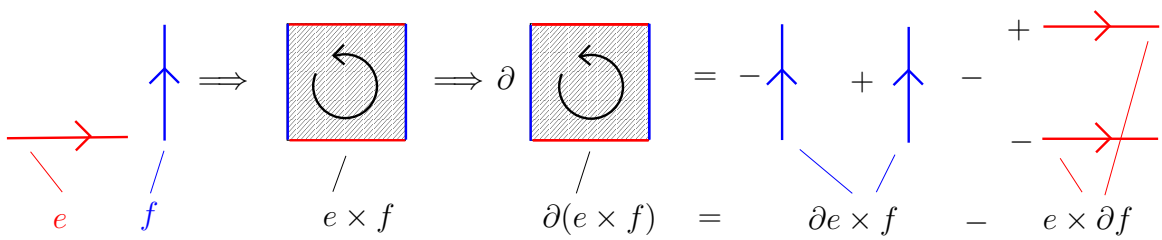


FIGURE 619. The boundary of the product of two 1-cells.

Following [Bre, Theorem 12.2] we will give an alternative proof instead. Given an k-cell σ and an $(k-1)$-cell μ we denote by $[\mu:\sigma]$ the coefficient of μ in $\partial\sigma$. It follows quite easily from Proposition 35.8 that for $p,q\in\mathbb{N}_0$ there exists a unique sign $\epsilon_{p,q}\in\{-1,1\}$ such that for any CW-complexes K and L, any p-cell e of K, any q-cell f of L and any $(q-1)$-cell g of L we have

$$[e\times g : e\times f] \;=\; \epsilon_{p,q}\cdot[g:f].$$

It follows immediately from the definitions that $\epsilon_{0,q}=1$ for all q.

Now let e be a p-cell of a CW-complex K and let f be a q-cell of a CW-complex L. We denote by τ the set of $(p-1)$-cells of K and we denote by ϕ the set of $(q-1)$-cells of L. We compute that

$$\begin{aligned}\partial(e\times f) \;&=\; \sum_\tau[\tau\times f : e\times f]\cdot(\tau\times f) \;+\; \sum_\phi[e\times\phi : e\times f]\cdot(e\times\phi)\\ &=\; \sum_\tau[\tau:e]\cdot(\tau\times f) \;+\; \sum_\phi\epsilon_{p,q}\cdot[\phi:f]\cdot(e\times\phi) \;=\; (\partial e)\times f \;+\; \epsilon_{p,q}\cdot e\times(\partial f).\end{aligned}$$

It remains to show that $\epsilon_{p,q}=(-1)^p$. We will prove this indirectly. More precisely, since we know that $\partial^2=0$ in the cellular chain complex we obtain that

$$\begin{aligned}0 \;=\; \partial\partial(e\times f) \;&=\; \overbrace{(\partial\partial e)\times f}^{=0} + \epsilon_{p-1,q}\cdot(\partial e)\times(\partial f) + \epsilon_{p,q}\cdot(\partial e)\times(\partial f) + \epsilon_{p,q-1}\epsilon_{p,q}\cdot e\times\overbrace{(\partial\partial f)}^{=0}\\ &=\; \epsilon_{p-1,q}\cdot(\partial e)\times(\partial f) + \epsilon_{p,q}\cdot(\partial e)\times(\partial f) \;=\; (\epsilon_{p-1,q}+\epsilon_{p,q})\cdot(\partial e)\times(\partial f).\end{aligned}$$

But this equality can hold for all CW-complexes K and L only if $\epsilon_{p,q}=-\epsilon_{p-1,q}$. Since $\epsilon_{0,q}=1$ we obtain iteratively that $\epsilon_{p,q}=(-1)^p$. $\square$

43.3. **The Künneth Theorem for chain complexes.** Our goal is to determine the homology groups of the product $X \times Y$ of two topological spaces in terms of the homology groups of X and Y. In light of the Eilenberg-Zilber Theorem 43.4, or alternatively, considering Proposition 43.5, we have to find a connection between the homology groups of two chain complexes $\mathcal{C}$, $\mathcal{C}'$ and the corresponding tensor product $\mathcal{C} \otimes \mathcal{C}'$.

We start out with the following elementary lemma.

Lemma 43.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. The map*

$$\begin{array}{rcl} \Theta\colon H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') & \to & H_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\ \sum_{i=1}^{r} [c_i] \otimes [c_i'] & \mapsto & \sum_{i=1}^{r} [c_i \otimes c_i'] \end{array}$$

is well-defined and it is natural with respect to the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (C_n', \partial_n')$ be two chain complexes. First note that if $c_i \in C_p$ and $c_i' \in C_q$ are cycles, then

$$d(c_i \otimes c_i') \;=\; \underbrace{\partial c_i}_{=0} \otimes c_i' + (-1)^p c_i \otimes \underbrace{\partial c_i'}_{=0} \;=\; 0$$

i.e. $c_i \otimes c_i' \in \mathcal{C} \otimes \mathcal{C}'$ is indeed a cycle. Second, if $c_i + \partial e_i$ is an other representative of $[c_i]$ then

$$[(c_i + \partial e_i) \otimes c_i'] \;\underset{\substack{\uparrow \\ \text{since } \partial' c' = 0}}{=}\; [c_i \otimes c_i' + d(e_i \otimes c_i')] \;=\; [c_i \otimes c_i'].$$

The same way one can show that the value of Θ does not depend on the choice of representative of the class in $H_q(\mathcal{C}')$. It is straightforward to verify that Θ is in fact a natural map. □

Example. In some instances the homomorphism Θ from Lemma 43.6 is an isomorphism. For example let $\mathcal{C} = (C_n, \partial_n)$ be a chain complex of free abelian groups such that all boundary maps $C_n \to C_{n-1}$ in $\mathcal{C}$ are the zero map. In this case we have for any chain complex $\mathcal{C}'$ that

$$\bigoplus_{n \in \mathbb{N}_0} H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}_*') \;\underset{\substack{\uparrow \\ \text{since all boundary maps} \\ \text{in } \mathcal{C} \text{ are zero}}}{=}\; \bigoplus_{n \in \mathbb{N}_0} C_p \otimes H_n(\mathcal{C}_{*-p}') \;\underset{\substack{\uparrow \\ \text{by Theorem 42.18} \\ \text{and Lemma 42.17 (3)} \\ \text{since all } C_p \text{ are free abelian}}}{=}\; \bigoplus_{n \in \mathbb{N}_0} H_n(C_p \otimes \mathcal{C}_{*-p}') \;\underset{\substack{\uparrow \\ \text{by the example on page 1042} \\ \text{since } \mathcal{C} \text{ is a direct sum} \\ \text{of trivial chain complexes}}}{=}\; H_n(\mathcal{C}_* \otimes \mathcal{C}_*').$$

Now we can formulate the following proposition which is the key algebraic result of this chapter.

Theorem 43.7. (Künneth Theorem for Chain Complexes)[820] *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If $\mathcal{C}$ is free, i.e. if all chain groups of the chain complex $\mathcal{C}$ are free abelian*

[820] Hermann Künneth (1892-1975) was a German mathematician.

groups, then there exists a natural short exact sequence

$$0 \;\to\; \bigoplus_{p+q=n} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') \;\xrightarrow{\Theta}\; H_n(\mathcal{C} \otimes \mathcal{C}') \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(\mathcal{C}), H_q(\mathcal{C}')) \;\to\; 0.$$

If $\mathcal{C}'$ is also a free chain complex, then this short exact sequence splits.

Remark.

(1) The adjective "natural" means in this context that if $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=n} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') \arrow[r, "\Theta"] \arrow[d, "f_* \otimes f'_*"] & H_n(\mathcal{C} \otimes \mathcal{C}') \arrow[r] \arrow[d, "(f \otimes f')_*"] & \bigoplus_{p+q=n} \mathrm{Tor}(H_p(\mathcal{C}), H_q(\mathcal{C}')) \arrow[r] \arrow[d, "\mathrm{Tor}(f_*, f'_*)"] & 0 \\
0 \arrow[r] & \bigoplus_{p+q=n} H_p(\mathcal{D}) \otimes H_q(\mathcal{D}') \arrow[r, "\Theta"] & H_n(\mathcal{D} \otimes \mathcal{D}') \arrow[r] & \bigoplus_{p+q=n} \mathrm{Tor}(H_p(\mathcal{D}), H_q(\mathcal{D}')) \arrow[r] & 0.
\end{tikzcd}$$

Here the middle map is given by Lemma 43.3 (1) and the map on the right is given by the functoriality of the torsion-groups that we had discussed on page 1021.

(2) If $\mathcal{C}'$ is a chain complex which consists of a single abelian group G, i.e. if it is of the form $0 \to G \to 0$ with G in degree 0, then $H_0(\mathcal{C}') = G$ and all other homology groups are zero. The Künneth Theorem together with the example on page 1042 thus gives us the short exact sequence

$$0 \;\to\; H_n(\mathcal{C}) \otimes G \;\to H_n(\underbrace{\mathcal{C} \otimes G}_{=\mathcal{C}\otimes\mathcal{C}'}) \;\to\; \mathrm{Tor}(H_{n-1}(\mathcal{C}), G) \;\to\; 0,$$

i.e. we recover the Universal Coefficient Theorem 42.18 as a special case of the Künneth Theorem. In fact, as we will see, the proof of the Künneth Theorem is quite similar to the proof of the Universal Coefficient Theorem 42.18.

(3) The fact that the short exact sequence of the Künneth Theorem splits implies that the middle group is isomorphic to the direct sum of the two groups to the left and right. But, as we saw in the discussion of the Universal Coefficient Theorem 42.18, see Proposition 42.24, this isomorphism is not natural.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (C'_n, \partial'_n)$ be two chain complexes such that all C_n are free abelian groups. As we will see, the proof of the theorem is quite similar to the proof of the Universal Coefficient Theorem 42.18.

For $p \in \mathbb{N}_0$ we consider again

$$Z_p \;:=\; \ker\{\partial_p\colon C_p \to C_{p-1}\} \quad \text{and} \quad B_p \;:=\; \mathrm{im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}.$$

We denote by $i\colon Z_p \to C_p$ the inclusion map. Furthermore we denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_p where all the boundary maps are the zero map. The same way we obtain from the groups B_p a chain complex $\mathcal{B}$.

Claim. The sequence

$$0 \;\to\; \mathcal{Z}\otimes\mathcal{C}' \;\xrightarrow{i\otimes\mathrm{id}}\; \mathcal{C}\otimes\mathcal{C}' \;\xrightarrow{\partial\otimes\mathrm{id}}\; \mathcal{B}\otimes\mathcal{C}' \;\to\; 0$$

is a sequence of chain maps and it is exact at each degree.

It follows easily from the definitions of the boundary maps in the various chain complexes, see also the diagram on page 1027, that the homomorphisms $i\otimes\mathrm{id}\colon \mathcal{Z}\otimes\mathcal{C}'\to\mathcal{C}\otimes\mathcal{C}'$ and $\partial\otimes\mathrm{id}\colon \mathcal{C}\otimes\mathcal{C}'\to\mathcal{B}\otimes\mathcal{C}$ are chain maps. It remains to show that the sequence is exact at each degree. The argument is exactly the same as on page 1027. For the reader's convenience we recall the argument. First note that by definition the sequences

$$0 \;\to\; Z_p \;\xrightarrow{i}\; C_p \;\xrightarrow{\partial}\; B_{p-1} \;\to\; 0$$

are exact. Also recall that by hypothesis each C_n is a free abelian group. It follows from Lemma 11.2 that each B_n is also free abelian. By Lemma 33.1 this means that the above short exact sequence splits. But then it follows from Lemma 42.6 that the sequence stays exact even after tensoring it with any C'_q. The claim is an immediate consequence of these observations.

From Proposition 31.10 and the claim we thus obtain the following long exact sequence[821]

$$\cdots\to H_n(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_n} H_n(\mathcal{Z}\otimes\mathcal{C}') \to H_n(\mathcal{C}\otimes\mathcal{C}') \to H_{n-1}(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_{n-1}} H_{n-1}(\mathcal{Z}\otimes\mathcal{C}')\to\dots$$

The connecting homomorphism $d_n\colon H_n(\mathcal{B}\otimes\mathcal{C}')\to H_n(\mathcal{Z}\otimes\mathcal{C}')$ from Proposition 31.10 is hereby, as in the proof of the Universal Coefficient Theorem 42.18, induced by the map $i\otimes\mathrm{id}$ where i denotes the inclusion $\mathcal{B}\to\mathcal{Z}$.

Since the boundary maps in the chain complexes $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the example on page 1046 that we obtain the following long exact sequence:

$$\dots\xrightarrow{i\otimes\mathrm{id}}\bigoplus_{p+q=n} Z_p\otimes H_q(\mathcal{C}')\to H_n(\mathcal{C}\otimes\mathcal{C}')\to\bigoplus_{p+q=n-1} B_p\otimes H_q(\mathcal{C}')\xrightarrow{i\otimes\mathrm{id}}\bigoplus_{p+q=n-1} Z_p\otimes H_q(\mathcal{C}')\to\dots$$

From Lemma 33.4 we obtain a short exact sequence

$$0 \;\to\; \bigoplus_{p+q=n}\overbrace{\operatorname{coker}\left\{B_p\otimes H_q(\mathcal{C}')\xrightarrow{i\otimes\mathrm{id}} Z_p\otimes H_q(\mathcal{C}')\right\}}^{\substack{=H_p(\mathcal{C})\otimes H_q(\mathcal{C}\text{'}),\text{ since}\\ 0\to B_p\to Z_p\to H_p(\mathcal{C})\to 0\text{ is a free resolution}}} \;\to\; H_n(\mathcal{C}\otimes\mathcal{C}') \;\to$$
$$\to\; \bigoplus_{p+q=n-1}\underbrace{\ker\left\{B_p\otimes H_q(\mathcal{C}')\xrightarrow{i\otimes\mathrm{id}} Z_p\otimes H_q(\mathcal{C}')\right\}}_{\substack{=\operatorname{Tor}(H_p(\mathcal{C}),H_q(\mathcal{C}\text{'})),\text{ since}\\ 0\to B_p\to Z_p\to H_p(\mathcal{C})\to 0\text{ is a free resolution}}} \to 0.$$

This proves the existence of the short exact sequence. Using Lemma 42.13 one can show that this short exact sequence is natural. We leave the verification of the details as an exercise to the reader.

[821]The index goes down by one in the map $H_n(\mathcal{C}\otimes\mathcal{C}')\to H_{n-1}(\mathcal{B}\otimes\mathcal{C}')$, since the map is induced by the short exact sequence $0\to Z_p\to C_p\to B_{p-1}\to 0$.

Finally we suppose that $\mathcal{C}'_*$ is also a free chain complex. We leave it as an exercise to adapt the argument of the proof of Proposition 42.22 to show that the above short exact sequence splits. Alternatively we refer to [Mun, Chapter 58] for details. $\square$

43.4. **The Künneth Theorem for topological spaces.** Now we return to our initial Question 43.1 of the chapter, namely, what is the relationship between the homology groups of the topological spaces X, Y and $X \times Y$?

The Eilenberg-Zilber Theorem 43.4 together with the Künneth Theorem for chain complexes from Theorem 43.7 now gives us the following proposition:

Theorem 43.8. (The Künneth Theorem for topological spaces) *Let X and Y be topological spaces. Then there exists a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{\Theta} H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0,$$

and this short exact sequence is natural with respect to the topological spaces X and Y. Furthermore the above short exact sequence splits.[822]

Examples.

(1) Let X be a topological space. We want to determine the homology groups of $S^1 \times X$. The homology groups of S^1 are of course $\mathbb{Z}$ in the dimensions 0 and 1. For any abelian group A we have by Lemma 42.3 that $\mathbb{Z} \otimes A = A$ and by Lemma 42.17 we have $\mathrm{Tor}(\mathbb{Z}, A) = 0$. It follows from the Künneth Theorem 43.8 for topological spaces that there exists an isomorphism

$$H_n(S^1 \times X) \xrightarrow{\cong} H_{n-1}(X) \oplus H_n(X).$$

This is of course precisely the same result as in Lemma 33.21.

The map $H_n(X) \to H_n(S^1 \times X)$ is fairly easily seen to be the map induced by the projection $S^1 \times X \to X$. It is less clear what the map $H_n(S^1 \times X) \to H_{n-1}(X)$ should be. Some more effort shows that this map is precisely the map that we had described in detail on page 863.

(2) Let X and Y be two path-connected topological spaces. Then we have isomorphisms

$$\begin{array}{ccccccc} H_1(X \times Y) & \cong & H_1(X) \otimes H_0(Y) \oplus H_0(X) \otimes H_1(Y) & = & H_1(X) \oplus H_1(Y). \\ & \uparrow & & \uparrow & \\ & \text{Künneth Theorem 43.8 for topological spaces} & & \text{Proposition 29.5 and Lemma 42.3} & \end{array}$$

This is consistent with the product formula for fundamental groups proved in Proposition 7.21 and with the isomorphism of the Hurewicz Theorem 38.5.

[822] But as in the case of the Universal Coefficient Theorem 42.18, see Proposition 42.24, the splitting is not natural.

Now we discuss the product $S^2 \times S^4$ in slightly more detail. It follows easily from the Künneth Theorem 43.8 for topological spaces, Proposition 31.3, Proposition 29.5, Lemma 42.3 and Lemma 42.17 that

$$H_k(S^2 \times S^4) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 884 these are precisely the homology groups of the complex projective space $\mathbb{C}\mathrm{P}^3$. Furthermore it follows from Propositions 5.14 and 7.21 that $S^2 \times S^4$ is simply connected and it follows from the discussion on page 697 that $\mathbb{C}\mathrm{P}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute[823], agree for $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$. This raises the following question which we will answer in a later chapter.

Question 43.9. *Are the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ homeomorphic?*

In a very similar fashion to the above calculation of the homology groups of $S^2 \times S^4$ one can show that

$$H_k(S^2 \times S^2) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 956 these are precisely the homology groups of the connected sums $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$. As above, all three topological spaces are furthermore simply connected. So this raises the following question.

Question 43.10. *Are the topological spaces $S^2 \times S^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ homeomorphic?*

The Künneth Theorem 43.8 for topological spaces can also be used to prove more theoretical results. For example we have the following proposition.

Proposition 43.11. *Let M and N be two closed manifolds. Then the following holds:*

$$M \times N \text{ is orientable} \quad \Longleftrightarrow \quad M \text{ and } N \text{ are orientable.}$$

Example. It follows from the proposition and the fact that the Klein bottle K is not orientable, see Lemma 3.39, that the product $K \times K$ is also not orientable.

Proof. Let M be an m-dimensional closed manifold and let N be an n-dimensional closed manifold. By restricting to the various components of M and N we can without loss of generality assume that M and N are connected.

By the Künneth Theorem 43.8 for topological spaces we have a short exact sequence

$$0 \to \bigoplus_{p+q=m+n} H_p(M) \otimes H_q(N) \xrightarrow{\Theta} H_{m+n}(M \times N) \to \bigoplus_{p+q=m+n-1} \mathrm{Tor}(H_p(M), H_q(N)) \to 0.$$

We consider the two terms to the left and right of the short exact sequence:

[823]As of right now we do not know the higher homotopy groups of these two manifolds, but these will not turn out to be very useful.

(1) By Proposition 35.6 we know that all homology groups of a manifold beyond its dimension vanish. It follows that there is only one potentially non-zero summand left, namely $H_m(M) \otimes H_n(N)$.
(2) Again using Proposition 35.6 we see that the only potentially non-zero summands on the right are terms involving $H_m(M)$ and $H_n(N)$, but these are torsion-free by Proposition 39.5. Therefore the torsion-groups are zero by Lemma 42.17.

We have thus shown that the above short exact sequences reduces to an isomorphism

$$H_m(M) \otimes H_n(N) \xrightarrow{\cong} H_{m+n}(M \times N).$$

The proposition now follows immediately from Proposition 39.5. □

We conclude this chapter with the following proposition.

Proposition 43.12. $(*)$ *Let M and N be manifolds and let $\varphi \in H_k(M)$ and $\psi \in H_l(N)$. If φ is realized by a closed, oriented k-dimensional submanifold A and if ψ is realized by a closed, oriented l-dimensional submanifold B, then $\Theta(\varphi \times \psi) \in H_{k+l}(M \times N)$ is realized by the closed $(k+l)$-dimensional submanifold $A \times B$, equipped with an appropriate orientation.*

Example. If we use the isomorphism from Künneth Theorem 43.8 for topological spaces and the standard generators of $H_2(S^2)$ to make the identification

$$H_2(S^2 \times S^2) \;=\; H_2(S^2) \otimes H_0(S^2) \;\oplus\; H_0(S^2) \otimes H_2(S^2) \;=\; \mathbb{Z} \oplus \mathbb{Z},$$

then the element $(1,0)$ is represented by the submanifold $S^2 \times \{*\}$ and the element $(0,1)$ is represented by the submanifold $\{*\} \times S^2$.

Proof. Let M and N be manifolds. Suppose $\varphi \in H_k(M)$ is realized by a closed, oriented k-dimensional submanifold A and that $\psi \in H_l(N)$ is realized by a closed, oriented l-dimensional submanifold B. By Proposition 43.11 we know that $A \times B$ is a closed, orientable $(k+l)$-dimensional submanifold of $M \times N$.

Claim. With an appropriate orientation the submanifold $A \times B$ represents $\Theta(\varphi, \psi)$.

We denote by $i\colon A \to M$ and by $j\colon B \to N$ the inclusion maps. By the naturality of the Künneth Theorem 43.8 for topological spaces we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=k+l} H_p(A) \otimes H_q(B) \arrow[r, "\Theta"] \arrow[d, "i_* \otimes j_*"] & H_{k+l}(A \times B) \arrow[r] \arrow[d, "(i \times j)_*"] & \bigoplus_{p+q=k+l-1} \mathrm{Tor}(H_p(A), H_q(B)) \arrow[r] \arrow[d, "\mathrm{Tor}(i_*, j_*)"] & 0 \\
0 \arrow[r] & \bigoplus_{p+q=k+l} H_p(M) \otimes H_q(N) \arrow[r, "\Theta"] & H_{k+l}(M \times N) \arrow[r] & \bigoplus_{p+q=k+l-1} \mathrm{Tor}(H_p(M), H_q(N)) \arrow[r] & 0.
\end{tikzcd}$$

By considering only the terms corresponding to $H_k(A) = \mathbb{Z} \cdot [A]$ and $H_l(B) = \mathbb{Z} \cdot [B]$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z} \cdot [A] \otimes \mathbb{Z} \cdot [B] \arrow[r, "\Theta"] \arrow[d, "{[A] \otimes [B] \mapsto \varphi \otimes \psi}"] & H_{k+l}(A \times B) = \mathbb{Z} \cdot [A \times B] \arrow[d, "(i \times j)_*"] \\
H_k(M) \otimes H_l(N) \arrow[r, "\Theta"] & H_{k+l}(M \times N).
\end{tikzcd}$$

The claim now follows from the fact, pointed out in the proof of Proposition 43.11, that the top horizontal map is an isomorphism. $\square$

Exercises for Chapter 43.

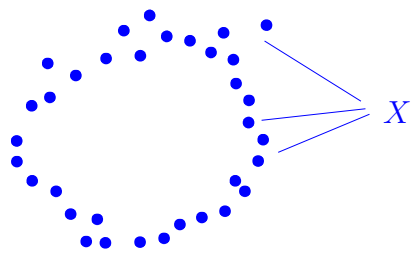
Exercise 43.1. Let $n \in \mathbb{N}$. Show that for any $k \in \mathbb{N}_0$ there exists an isomorphism

$$H_k((S^1)^n) \;\cong\; \mathbb{Z}^{\binom{n}{k}} \quad \text{where } \binom{n}{k} := \begin{cases} \frac{n!}{k! \cdot (n-k)!}, & \text{if } k \in \{0, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

44. Applications of homology groups

44.1. **Persistent homology** (∗)**.** In this section we outline the idea behind "persistent homology". Persistent homology was introduced by Gunnar Carlsson [Carl] about ten years ago. More information and a much more careful treatment of persistent homology can for example be found in [EH]. Some extra information can also be found in [Gh].

We consider Figure 620. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.



dots that appear to the eye to form a loop

Figure 620.

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0. So how can we use the mathematical machine that we have developed over the last two semesters to "see the loop"?

Given $r \in [0, \infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x. The homeomorphism type of $X(r)$ changes with the radius r. For example in Figure 621 we see that at $r = 1$ all but one ball are joined up and we see that $H_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $H_1(X(1))$. Two of the generators, namely c_1, c_2 are formed "by accident", namely they arise from balls which are close by. This gets captured by the fact that they become null-homologous in $H_1(X(2))$ whereas c_3 represents a non-trivial element for "much longer", i.e. for much larger values of r. Put differently, c_3 is a persistent element in homology, hence it captures global information.

Summarizing, given a subset X of $\mathbb{R}^2$ we can study the topological spaces $X(r)$ for any $r \in [0, \infty)$ and furthermore for any $r < s$ we can study the inclusion induced map $H_n(X(r)) \to H_n(X(s))$. If there is a non-zero element in $H_n(X(r))$ which stays non-zero for a "long time", then we say it is persistent and we hope that it captures something interesting.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much any data one has in practice, e.g. blue pixels of a picture, measurements in an experiment, statistical data and so on, can be viewed as a finite subset of some $\mathbb{R}^n$, so in principle one can study its persistent homology. The real question is, whether one captures something

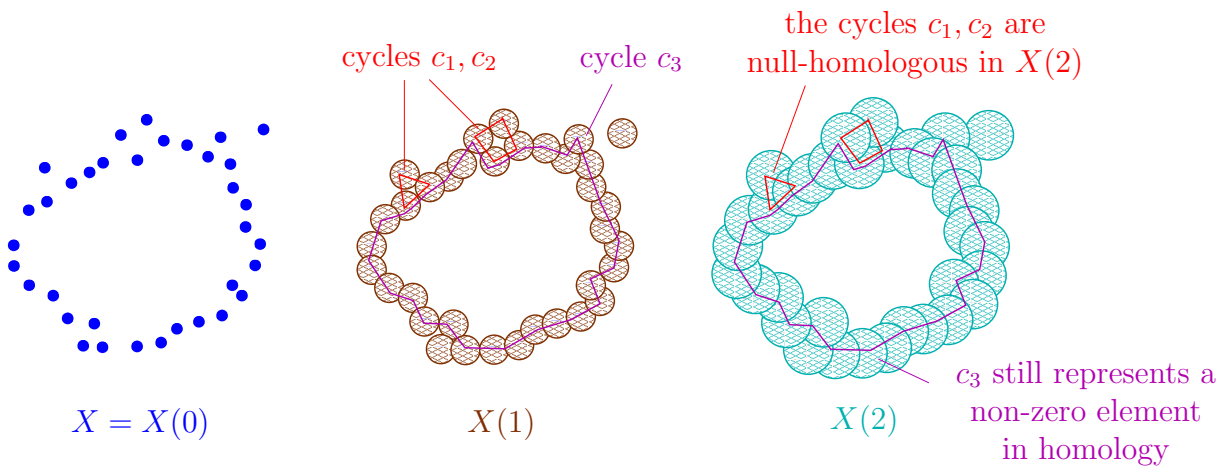


FIGURE 621.

significant or not. For example persistent homology was applied in [SHP][824] to study the "The topological shape of Brexit". It is perhaps debatable whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

44.2. **Division algebras.** We start out with the following purely algebraic definition.

Definition. An *algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{ccc} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is *commutative* if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) We say that the algebra is *associative* if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(3) We say that the algebra *has an identity* if there exists an element $1 \in A$ with $1 \neq 0$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.
(4) A *division algebra* is an algebra such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Examples.

(1) The complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot i$ with the usual multiplication are an algebra over $\mathbb{R}$ that is commutative and associative, it has an identity, and it is a division algebra.
(2) The set of $(n \times n)$-matrices $M(n \times n, \mathbb{F})$ over a field $\mathbb{F}$, where the multiplication is given by the usual matrix multiplication, is an associative algebra over $\mathbb{F}$ with an identity. If $n \geq 2$ this algebra is clearly neither commutative nor is it a division algebra.

[824]Here is an easy way to tell that that paper was not written by mathematicians: In a mathematics paper the authors are invariably listed according to the alphabetic order of their last names. In most other sciences this is not the case.

(3) The real vector space $\mathbb{R}^3$ together with the cross product

$$\begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} & \mapsto & \begin{pmatrix} v_2w_3 - w_2v_3 \\ -(v_1w_3 - w_1v_3) \\ v_1w_2 - w_1v_2 \end{pmatrix} \end{array}$$

is an algebra over $\mathbb{R}$. We recall that the geometric interpretation of the cross product is given as follows: for $v, w \in \mathbb{R}^3$ the cross product $v \times w$ is the unique vector in $\mathbb{R}^3$ that satisfies the following three properties:

(a) the vector is orthogonal to v and w,
(b) the length equals the area of the parallelogram spanned by v and w,
(c) we have $\det(v\ w\ v \times w) \geq 0$.[825]

We make the following observations:

(a) Clearly we have $v \times w = -w \times v$, hence the algebra is non-commutative.
(b) We have

$$(e_1 \times e_1) \times e_2 \ = \ 0 \times e_2 \ = \ 0 \ \neq \ -e_2 \ = \ e_1 \times e_3 \ = \ e_1 \times (e_1 \times e_2),$$

which shows that this algebra is non-associative.
(c) It follows easily from the definition or the geometric interpretation that for $v \neq 0$ there is no vector w with $v \times w = v$, hence the algebra does not have an identity element.
(d) Finally it is straightforward to see that $(\mathbb{R}^3, \times)$ is not a division algebra. More precisely, one can easily show, using the geometric interpretation of the cross product, that given $a, b \in \mathbb{R}^3$ there exists a vector $x \in \mathbb{R}^3$ with $a \times x = b$ if and only if b is orthogonal to a.

We consider the next example in greater detail.

Definition. The *quaternions* $\mathbb{H}$ are defined as the 4-dimensional real vector space $\mathbb{R}^4$ spanned by $1, i, j, k$ and the multiplication which is determined by

(1) $i^2 = j^2 = k^2 = -1$,
(2) $1 \cdot i = i \cdot 1 = i, \quad 1 \cdot j = j \cdot 1 = j, \ \ 1 \cdot k = k \cdot 1 = k$, and
(3) $i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$ and $j \cdot i = -k$, $k \cdot j = -i$, $i \cdot k = -j$.[826]

For a quaternion

$$z \ := \ a \cdot 1 + b \cdot i + c \cdot j + d \cdot k \in \mathbb{H}$$

we define the *conjugate*

$$\overline{z} \ := \ a \cdot 1 - b \cdot i - c \cdot j - d \cdot k$$

and the *norm*

$$|z| \ := \ \sqrt{a^2 + b^2 + c^2 + d^2}$$

[825] If v and w are non-zero this just means that v, w and $v \times w$ form a positive basis for $\mathbb{R}^3$.

[826] Put differently, we have $i \cdot j = k$ and $j \cdot i = -k$, and both of the equalities hold for *cyclic* permutations of $\{i, j, k\}$.

the same way as we did for complex numbers.

The following lemma summarizes a few key properties of the quaternions. In particular we see that the conjugate and the norm of the quaternions behave in a quite similar fashion as the conjugate and norm of complex numbers.

Lemma 44.1.

(1) *The quaternions are a non-commutative algebra with identity.*
(2) *The quaternions are associative.*
(3) *For all $z, w \in \mathbb{H}$ we have*

$$\overline{\overline{z}} \;=\; z, \quad \overline{z+w} \;=\; \overline{z}+\overline{w} \quad \text{and} \quad \overline{z\cdot w} \;=\; \overline{w}\cdot\overline{z}.$$

(4) *If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.*
(5) *For all $z, w \in \mathbb{H}$ we have $|z \cdot w| = |z| \cdot |w|$.*
(6) *For every $z \in \mathbb{H}$ we have $z \cdot \overline{z} = |z|^2 \in \mathbb{R} = \mathbb{R} \cdot 1$.*
(7) *For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} := \frac{\overline{z}}{|z|^2}$.*
(8) *The quaternions are a division algebra.*

FIGURE 622. Brougham Bridge in Dublin

Proof. The first statement is clear. We turn to the proof of associativity. Since the quaternions are an algebraic we only have to check associativity for products of three of the generators $\{1, i, j, k\}$ and that can be done easily by hand.

Statements (3), (4), (5), (6) and (7) follow from elementary calculations. Finally note that it follows from (1) and (7) that the quaternions are in fact a division algebra. □

In Theorem 37.2 we saw that SU(2) is diffeomorphic to S^3. We can now give a refined version of this statement.

Proposition 44.2.

(1) *The set $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ forms a group under multiplication of quaternions.*

(2) *The map*[827]

$$\begin{aligned} q\colon S^3 &\to \mathrm{SU}(2) \\ (z,w) = z + w\cdot j &\mapsto \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{aligned}$$

from Lemma 37.1 is a diffeomorphism and it defines an isomorphism of groups.

Proof.

(1) This statement follows immediately from Lemma 44.1 (5).
(2) We already saw that in Theorem 37.2 that this map is a diffeomorphism. An elementary, albeit slightly tedious calculation, shows that the map

$$\begin{aligned} q\colon \mathbb{H} &\to M(2\times 2,\mathbb{C}) \\ (z,w) = z + w\cdot j &\mapsto \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{aligned}$$

is a ring homomorphism, in particular it preserves the ring structure. This implies that the given map $q\colon S^3 \to \mathrm{SU}(2)$ is a group homomorphism. □

Before we move on to the octonions it is helpful to give an alternative description of the quaternions. Namely we define the Hamiltonians as elements in $\mathbb{C}\oplus\mathbb{C}$ with the multiplication given by

$$(z_1,z_2)\cdot(w_1,w_2) \;:=\; (z_1w_1 - \overline{w}_2z_2, z_2\overline{w}_1 + w_2z_1)$$

the conjugate given by

$$\overline{(w,z)} \;:=\; (\overline{w},-z)$$

and the norm given by

$$|(w,z)| \;:=\; \sqrt{(w,z)\cdot\overline{(w,z)}}.$$

One easily verifies that this norm is multiplicative, i.e. given $a,b\in\mathbb{H}=\mathbb{C}\oplus\mathbb{C}$ one has $|a\cdot b| = |a|\cdot|b|$. An elementary calculation shows that the map $(w,z)\mapsto w+zj$ defines an isomorphism between the two descriptions of $\mathbb{H}$, furthermore this isomorphism preserves the conjugate and the norm.

Considering the above description of the quaternions one might hope to be lucky and just try to recycle the definitions:

Definition. The *Cayley octonions* $\mathbb{O}$ (or usually just *octonions*) are defined as $\mathbb{O}=\mathbb{H}\oplus\mathbb{H}$ with the multiplication multiplication given by[828]

$$(z_1,z_2)\cdot(w_1,w_2) \;:=\; (z_1w_1 - \overline{w}_2z_2, z_2\overline{w}_1 + w_2z_1)$$

the *conjugate* given by

$$\overline{(w,z)} \;:=\; (\overline{w},-z)$$

[827]Note that it follows from $i\cdot j = k$ that any quaternion $a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$ can be uniquely written as $(a\cdot 1 + b\cdot i) + (c\cdot 1 + d\cdot i)\cdot j$ where $a\cdot 1 + b\cdot i$ and $c\cdot 1 + d\cdot i$ are complex numbers.

[828]In fact these formulas also work for defining $\mathbb{C}$ as $\mathbb{R}\oplus\mathbb{R}$ where we equip $\mathbb{R}$ with the trivial involution.

and the *norm* given by

$$|(w,z)| \ := \ \sqrt{(w,z)\cdot\overline{(w,z)}}.$$

The same argument as in the proof of Lemma 44.1 also gives us the following "lemma".

Lemma 44.3. *The statements of Lemma 44.1 also hold for octonions, except that the octonions are not associative.*

In particular we see that $\mathbb{O}$ is again a division algebra. Evidently this algebra is not commutative, but even worse, it is not even associative.[829]

Remark. The octonions were discovered[830] by Arthur Cayley in 1845, shortly after Hamilton found the quaternions.

Even though the octonions are not associative they have a faint shadow of associativity which will actually turn out to be useful:

Lemma 44.4. *Given any two elements $x, y \in \mathbb{O}$ the subalgebra generated by x and y is associative.*[831]

Proof. This lemma is proved on [Bae, p. 149].. □

The discussion shows, that starting from $\mathbb{R}$ one can construct the complex numbers, which give rise to the quaternions and finally we obtain the octonions. When we move from the complex numbers to the quaternions we loose commutativity and when we go from the quaternions to the octonions we also loose associativity. As is explained in [Bae, Section 2.2] there is one underlying principle behind these constructions, and in fact one can continue to obtain a 16-dimensional algebra $\mathcal{S}$ with an identity, this algebra is evidently neither commutative nor associative, it has multiplicative inverses, i.e. for each $a \in \mathcal{S}\setminus\{0\}$ there exists a $b \in \mathcal{S}$ with $a\cdot b = b\cdot a = 1$, but it is not a division algebra.[832]

After the discovery of quaternions and the octonions the question arose whether one can also find division algebras on $\mathbb{R}^n$ for some $n \neq 1, 2, 4, 8$. We will discuss this question in the remainder of this section.

The following theorem is the best theorem that we can prove in this direction with our present knowledge.

Theorem 44.5. *Any finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.*

Remark.

[829] To quote from [Frie3, p. 5], "here, associativity is obviously a concern".

[830] Or were they invented by Arthur Cayley? This question whether mathematics is "discovered" or "invented" is a source for endless philosophical discussion.

[831] An algebra which has such a property is sometimes called *alternative*.

[832] How can it be that an algebra has multiplicative inverses, but it is not a division algebra?

(1) Note that in Theorem 44.5 we do not assume that the division algebra is associative. It is a nice exercise in algebra and real analysis to show, using the Fundamental Theorem of Algebra, the weaker statement that if V is a finite-dimensional commutative and *associative* division algebra over $\mathbb{R}$, then V is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
(2) The statement of Theorem 44.5 does not hold if we drop the "finite dimension" condition. For example the field $\mathbb{R}(t)$ of rational functions over $\mathbb{R}$ is a division algebra, but it is an infinite-dimensional real vector space.

In the proof of Theorem 44.5 we will need the following basic lemma which might already be familiar from earlier algebra courses.

Lemma 44.6. *Let A be a* finite-dimensional *algebra over a field $\mathbb{F}$. The following two statements are equivalent:*

(1) *A is a division algebra,*
(2) *A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.*

Remark. Let A be a division algebra over a field $\mathbb{F}$. If A is associative, then it is clear that A has no zero-divisors. Indeed, if $a \cdot b = 0$ with $a \neq 0$, then we pick x with $x \cdot a = 1$ and we see that $0 = x \cdot 0 = x \cdot (a \cdot b) = (x \cdot a) \cdot b = 1 \cdot b = b$. The point about Lemma 44.6 is that it applies to non-associative algebras, as long as they are finite-dimensional.

Proof. Let A be a finite-dimensional algebra over a field $\mathbb{F}$. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned}
\text{for any } y \in A \text{ there exists } x \in A \text{ with } ax = y \quad &\Leftrightarrow \text{ the map } A \to A,\ x \mapsto ax \text{ is surjective} \\
&\Leftrightarrow \text{ the map } A \to A,\ x \mapsto ax \text{ is injective} \\
&\underset{\substack{\text{since } x \mapsto ax \text{ is a homomorphism of finite-dimensional } \mathbb{F}\text{-vector spaces}}}{\uparrow} \\
&\Leftrightarrow \text{ for any } b \neq 0 \text{ we have } ab \neq 0.
\end{aligned}$$

The same statement holds for left-multiplication. The equivalence of (1) and (2) now follows easily from this observation and the definitions. □

Now we turn to the proof of Theorem 44.5.

Proof of Theorem 44.5. Let $(A, *)$ be a finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity. Since A is a finite-dimensional vector space we can, after applying an isomorphism, assume that $A = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

We first consider the case $n = 1$. We define $a := 1 * 1$. Since "$*$" is bilinear we have for all $x, y \in \mathbb{R}$ that $x * y = (x \cdot 1) * (y \cdot 1) = x \cdot (1 * 1) \cdot y = xay$. Since the algebra is a division algebra we obtain from Lemma 44.6 that $a \neq 0$. Then the map $\varphi(x) = ax$ defines an isomorphism from the algebra $(\mathbb{R}, *)$ to the algebra $(\mathbb{R}, \cdot)$.[833]

[833]Indeed, for $x, y \in \mathbb{R}$ we have $\varphi(x * y) = \varphi(x \cdot a \cdot y) = a \cdot x \cdot a \cdot y = (a \cdot x) \cdot (a \cdot y) = \varphi(x) \cdot \varphi(y)$.

Now suppose that $n \geq 2$. We first want to show that $n = 2$. To do this we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x) := \dfrac{x * x}{\|x * x\|}. \end{array}$$

This is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from Lemma 44.6 that $x * x \neq 0$. The map f is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[834]

Since "$*$" is bilinear we have $f(-x) = f(x)$ for all $x \in \mathbb{R}^n$. It follows from Lemma 1.66 that the map f descends to a continuous map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} & \to & S^{n-1} \\ [x] & \mapsto & \overline{f}([x]) := f(x). \end{array}$$

Claim. The map $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is injective.

So let $[x], [y] \in \mathbb{R}\mathrm{P}^{n-1} = S^{n-1}/\sim$. We have

$$\begin{aligned} \underbrace{\overline{f}([x])}_{=f(x)} = \underbrace{\overline{f}([y])}_{=f(y)} \implies f(x) = f(y) \implies & x * x = \alpha^2 \cdot (y * y) \text{ for } \alpha := \sqrt{\|x * x\| \cdot \|y * y\|^{-1}} \\ \implies & 0 = x * x - \alpha^2 \cdot (y * y) \underset{\substack{\uparrow \\ \text{since "}*\text{" is commutative and bilinear}}}{=} (x + \alpha y) * (x - \alpha y) \\ \underset{\substack{\uparrow \\ \text{Lemma 44.6}}}{\implies} & x = \pm \alpha y \underset{\substack{\uparrow \\ \text{since } x, y \in S^{n-1}}}{\implies} x = \pm y \implies [x] = [y]. \end{aligned}$$

This concludes the proof of the claim.

Since S^{n-1} is closed and connected (here we use that $n \geq 2$) and since $\mathbb{R}\mathrm{P}^{n-1}$ is a closed manifold it follows from Corollary 36.8 that $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is in fact a homeomorphism. In particular the fundamental groups of $\mathbb{R}\mathrm{P}^{n-1}$ and S^{n-1} are isomorphic. But it follows from Proposition 5.14 and Corollary 7.19 that this is only possible if $n = 2$.

It remains to show that the algebra $(A, *) = (\mathbb{R}^2, *)$ is isomorphic to the algebra $(\mathbb{C}, \cdot)$. We start out with the following claim.

Claim. There exists a $k \in A \setminus \mathbb{R} \cdot 1$ with $k * k = c \cdot 1$ for some $c \in \mathbb{R}_{<0}$.

We first pick a random element $j \in A \setminus \mathbb{R} \cdot 1$. Then 1 and j are a basis for $A = \mathbb{R}^2$ and therefore we can write $j * j = a \cdot 1 + b \cdot j$ for some $a, b \in \mathbb{R}$. Then multiplying out and using this equality we see that

$$\underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} * \underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} = \underbrace{\left(a + \tfrac{1}{4} b^2\right)}_{=:c} \cdot 1.$$

[834]Indeed, the fact that the map is bilinear implies that there exists a real $n \times n$-matrix P such for any $v, w \in \mathbb{R}^n$ we have $v \cdot w = v^T P w$, hence the map is continuous.

We set $k := j - \frac{b}{2} \cdot 1$ and $c := a + \frac{1}{4}b^2$. We still need to show that $c < 0$. Suppose that $c \geq 0$. Then $k * k = c \cdot 1$ implies that $(k + \sqrt{c} \cdot 1) * (k - \sqrt{c} \cdot 1) = k * k - c \cdot 1 = 0$. From Lemma 44.6 we conclude that $k = \pm\sqrt{c} \cdot 1$. But this is not possible since $k = j - \frac{b}{2} \cdot 1$ is linearly independent from 1. This concludes the proof of the claim.

One can now easily verify that

$$\begin{array}{rcl} A & \to & \mathbb{C} \\ x \cdot 1 + y \cdot k & \mapsto & x + y \cdot \frac{i}{\sqrt{-c}} \quad \text{with } x, y \in \mathbb{R} \end{array}$$

is an isomorphism from the algebra $(A, *)$ to the algebra $(\mathbb{C}, \cdot)$. □

It is natural to ask what other division algebras exist. The following theorem classifies associative division algebras.

Theorem 44.7. *Any finite-dimensional division algebra over $\mathbb{R}$ which is associative and which has an identity is isomorphic to $\mathbb{R}, \mathbb{C}$ or to $\mathbb{H}$.*

Proof. The statement was first proved by Frobenius in 1878. A short self-contained proof is for example given in [Pal2]. □

Later on, once we have introduced cohomology groups, we will address the following question.

Question 44.8. *For which n, besides $n = 1, 2, 4, 8$ does $\mathbb{R}^n$ admit the structure of a division algebra?*

We conclude this section with a short discussion of quaternionic projective spaces and an interesting open question.

On page 648, given $n \in \mathbb{N}_0$, we introduced the real projective space $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In the following we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Since the quaternions $\mathbb{H}$ are non-commutative we do the constructions carefully.

Lemma 44.9. *Given $v, w \in \mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$ we write $v \sim w$ if there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. This defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$.*

Proof.

(1) It is clear that $\sim$ is reflexive.
(2) Since $\mathbb{H}$ is a skew field we see that $\sim$ is symmetric.
(3) Finally suppose that $u \sim v$ and $v \sim w$. Thus there exist $g, h \in \mathbb{H} \setminus \{0\}$ with $gu = v$ and $hv = w$. Then

$$(hg)u \underset{\substack{\uparrow \\ \text{since the quaternions are associative!}}}{=} h(gu) = hv = w.$$

□

Now we can define the quaternionic projective spaces.

Definition. Let $n \in \mathbb{N}_0$.

(1) For $(h_0, \dots, h_n) \in \mathbb{H}^{n+1} \setminus \{0\}$ we denote by $[h_0 : \dots : h_n]$ the corresponding equivalence class.
(2) Given $n \in \mathbb{N}$ we define the *quaternionic projective space*

$$\mathbb{H}\mathrm{P}^n \ := \ (\mathbb{H}^{n+1} \setminus \{(0, \dots, 0)\})/\sim .$$

The following lemma summarizes some key properties of quaternionic projective spaces. $\mathbb{H}\mathrm{P}^n$. We will make use of some them later on.

Lemma 44.10. *Let $n \in \mathbb{N}_0$.*
(1) *$\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable manifold.*
(2) *The map*

$$\begin{array}{rcl} \mathbb{H}\mathrm{P}^1 & \to & \mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\} \overset{\text{by Lemma 1.41}}{\overset{\downarrow}{=}} S^4 \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

is a homeomorphism.
(3) *$\mathbb{H}\mathrm{P}^n$ admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \dots, 4n$.*
(4) *We have $\chi(\mathbb{H}\mathrm{P}^n) = n + 1$.*
(5) *$\mathbb{H}\mathrm{P}^n$ is 3-connected, i.e. we have $\pi_i(\mathbb{H}\mathrm{P}^3) = 0$ for $i = 1, 2, 3$.*

Proof.
(1) This statement is proved in a very similar way as Lemma 21.6.
(2) The proof of this statement is basically the same as the proof of Lemma 1.82 (2).
(3) The proof of this statement is an easy modification of the proof of Lemma 26.2.
(4) This statement follows immediately from (3).
(5) This follows from (3) and Proposition 28.10. □

Since the octonions are not associative the definition on page 648 cannot be applied to the octonions. But there is some hope in low dimensions. More precisely, as we will now see, one can use the octonions to construct manifolds that have all the properties that one would naively expect $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ to have.

Definition.
(1) We equip[835]

$$T_2 \ := \ \{(x, y) \in \mathbb{O}^2 \,|\, |x|^2 + |y|^2 = 1\}$$

with the equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \quad \Longleftrightarrow \quad x_1\overline{x_1} \ = \ x_2\overline{x_2}, \quad x_1\overline{y_1} \ = \ x_2\overline{y_2}, \quad x_1\overline{y_1} \ = \ x_2\overline{y_2}.$$

We set

$$\mathbb{O}\mathrm{P}^1 \ := \ T_2/\sim$$

and we refer to $\mathbb{O}\mathrm{P}^1$ as the *octonion projective line*.

[835] What happens if we use these definitions of $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$, but if we replace the octonions by our more familiar friends $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$?

(2) We equip

$$T_3 \ := \ \left\{ (x,y,z) \in \mathbb{O}^3 \,\middle|\, \begin{array}{c} |x|^2+|y|^2+|z|^2 = 1 \text{ and } x,y,z \\ \text{generate an associative algebra} \end{array} \right\}$$

with the equivalence relation

$$(x_1,y_1,z_1) \sim (x_2,y_2,z_2) \quad \Longleftrightarrow \quad \begin{array}{ccc} x_1\overline{x_1} = x_2\overline{x_2}, & x_1\overline{y_1} = x_2\overline{y_2}, & x_1\overline{z_1} = x_2\overline{z_2}, \\ y_1\overline{y_1} = y_2\overline{y_2}, & y_1\overline{z_1} = y_2\overline{z_2}, & z_1\overline{z_1} = z_2\overline{z_2}. \end{array}$$

We set

$$\mathbb{O}\mathrm{P}^2 \ := \ T_3/\sim$$

and we refer to $\mathbb{O}\mathrm{P}^2$ as the *octonion projective plane*. In the literature the octonion projective plane is often also called the *Cayley projective plane*.

(3) We denote the equivalence class of a point $(x,y) \in T_2$ by $[x:y]$ and we denote the equivalence class of a point $(x,y,z) \in T_3$ by $[x:y:z]$.

The following proposition hopefully convinces the reader that the octonion line $\mathbb{O}\mathrm{P}^1$ and the octonion plane $\mathbb{O}\mathrm{P}^2$ deserve their names.

Proposition 44.11.

(1) *The map*

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\cup\{\infty\} = \mathbb{R}^8\cup\{\infty\} \overset{\text{by Lemma 1.41}}{=} S^8 \\ [x:y] & \mapsto & \begin{cases} x\cdot y^{-1}, & \text{if } y \neq 0, \\ \infty, & \text{if } [x:y] = [1:0] \end{cases} \end{array}$$

is a homeomorphism.

(2) *The octonion projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional manifold.*

(3) *The map*[836]

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x_1:y_1] & \mapsto & [x_1:y_1:0] \end{array}$$

is a homeomorphism onto its image. We use it to view $\mathbb{O}\mathrm{P}^1$ as a subspace of $\mathbb{O}\mathrm{P}^1$.

(4) *The octonion projective plane $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Furthermore this CW-structure can be chosen such that the 8-skeleton is precisely $\mathbb{O}\mathrm{P}^1$.*

(5) *We have $\chi(\mathbb{O}\mathrm{P}^2) = 3$.*

(6) *$\mathbb{O}\mathrm{P}^2$ is 7-connected, i.e. we have $\pi_i(\mathbb{O}\mathrm{P}^2) = 0$ for $i = 1,2,\ldots,7$.*

Proof. The proof of the first statement is basically identical to the proof of Lemma 1.82 (2). Statements are (3), (4) and (5) are shown in [Lackm].[837] Statement (5) is an immediate consequence of (4) and Statement (6) follows from (4) and Proposition 28.10. □

[836] It follows from Lemma 44.4 that this map is well-defined, i.e. given any point $(x_1,y_1) \in T_2$ we have indeed $(x_1,y_1,0) \in T_3$.

[837] Many of the statements can also be found in [Bae].

It is natural to ask whether there exists a manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ for $n \geq 3$. This slightly vague question can be formulated in different ways. Here is one possible interpretation of this question.

Question 44.12. *Let $n \geq 3$. Does there exist a closed orientable $8n$-dimensional manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \dots, 8n$?*

We will return to this question at a later stage.

Proposition 44.11 says in particular that the octonion plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional manifold that is 7-connected and that has odd Euler characteristic. The following question is, to the best of my knowledge, open:

Question 44.13. *Does there exist an 8-connected, closed, orientable manifold with odd Euler characteristic?*

In [Hoe, Theorem 1.2] it is shown that the dimension of an 8-connected, closed, orientable manifold with odd Euler characteristic, if it exists, has to be divisible by 16.

44.3. **The transfer map.** Before we can give the next application of homology theory, namely the Borsuk-Ulam Theorem in higher dimensions, we need to introduce and study the transfer map in homology corresponding to a finite covering.

Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected[838] it follows from Proposition 20.2 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We write

$$p^*(\sigma) \ := \ \sum_{i=1}^{k} \widetilde{\sigma}_i.$$

Given any commutative ring R we refer to the map

$$\begin{array}{rcl} p^*\colon C_*(X;R) & \to & C_*(\widetilde{X};R) \\ \sum_{i=1}^{n} \sigma_i \otimes r_i & \mapsto & \sum_{i=1}^{n} p^*(\sigma_i) \otimes r_i \end{array}$$

as the *transfer map* of the covering. It is straightforward to verify that the transfer map is in fact a chain map. In particular it induces a map

$$p^*\colon H_*(X;R) \ \to \ H_*(\widetilde{X};R).$$

Example. Let $p\colon \widetilde{M} \to M$ be a finite covering of a compact, oriented, non-empty n-dimensional manifold. We equip M with a simplicial structure. Similar to the proof of Proposition 26.25 we can lift this simplicial structure of M to a simplicial structure on $\widetilde{M}$. It follows now easily from the proof of Proposition 39.7 that the transfer map applied to the fundamental class of M equals the fundamental class of the compact oriented manifold $\widetilde{M}$, i.e. that $p^*([M]) = [\widetilde{M}]$.

[838] To be precise, we also use that Δ^n is o-locally path-connected.

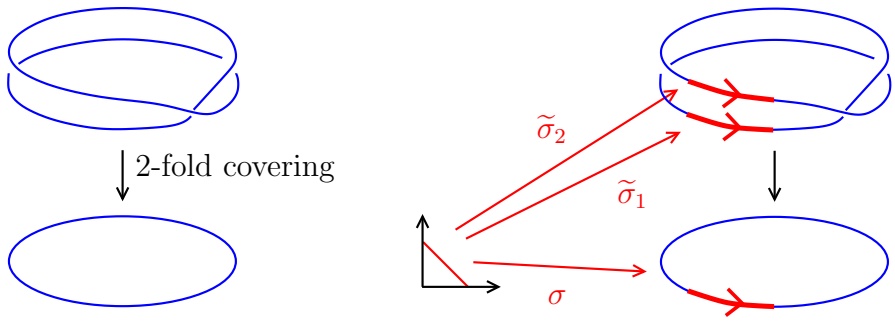


FIGURE 623.

Proposition 44.14. *Let $p\colon \widetilde{X} \to X$ be a covering of degree $k \in \mathbb{N}$ of a topological space X and let $n \in \mathbb{N}_0$. Then the following hold:*

(1) *For any commutative ring R the transfer map $p^*\colon C_n(X;R) \to C_n(\widetilde{X};R)$ is a monomorphism.*
(2) *For any commutative ring R the map $p_* \circ p^*\colon H_n(X;R) \to H_n(X;R)$ is given by multiplication by k.*
(3) *The projection map $p_*\colon H_n(\widetilde{X};\mathbb{Q}) \to H_n(X;\mathbb{Q})$ is an epimorphism.*

Remark.

(1) If $p\colon \widetilde{X} \to X$ is a finite covering of path-connected topological spaces, then we saw in Corollary 38.8 that the map $p_*\colon H_1(\widetilde{X}) \to H_1(X)$ has finite cokernel. This implies in particular[839] that $p_*\colon H_1(\widetilde{X};\mathbb{Q}) \to H_1(X;\mathbb{Q})$ is an epimorphism. Therefore we can view Proposition 44.14 (3) as a generalization of Corollary 38.8.
(2) The conclusion of Proposition 44.14 (3) does not hold if we drop the condition that the covering is finite. For example, we have an infinite covering $p\colon \mathbb{R} \to S^1$ but $0 = H_1(\mathbb{R};\mathbb{Q}) \xrightarrow{p_*} H_1(S^1;\mathbb{Q}) = \mathbb{Q}$ is not an epimorphism.

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X.

(1) Let $c \in C_n(X;R)$ be non-trivial. We write $c = \sum_{i=1}^{n} \sigma_i \otimes r_i$ where $\sigma_1, \dots, \sigma_n$ are pairwise different singular simplices in X and where $r_1, \dots, r_n$ are all non-zero. Since each σ_i lifts to k different singular simplices and since for $i \neq j$ the lifts of σ_i and σ_j differ we see immediately from the definition that $p^*(\sigma)$ is also non-zero.
(2) Let R be a commutative ring. We denote by $p^*\colon C_n(X;R) \to C_n(\widetilde{X};R)$ the transfer map. Let $\sigma\colon \Delta^m \to X$ be a singular simplex. We denote the corresponding lifts by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k$. Then
$$(p_* \circ p^*)(\sigma) \;=\; p_*(\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_k) \;=\; \underbrace{p \circ \widetilde{\sigma}_1 + \cdots + p \circ \widetilde{\sigma}_k}_{k \text{ summands}} \;=\; \underbrace{\sigma + \cdots + \sigma}_{k \text{ summands}} \;=\; k \cdot \sigma.$$

[839] Why does this follow?

We thus see that $p_* \circ p^* \colon C_m(X) \to C_m(X)$ is multiplication by k for any m. It follows easily that the map

$$H_n(X;R) \xrightarrow{p^*} H_n(\widetilde{X};R) \xrightarrow{p_*} H_n(X;R)$$

is also given by multiplication by k.

(3) Since multiplication by $k \neq 0$ is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^* \colon H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})$ is an isomorphism. In particular the second map, namely p_*, is an epimorphism. □

Proposition 44.15. *Let $p \colon \widetilde{X} \to X$ be a 2-fold covering. Then there exists a long exact sequence of the form*

$$\ldots \to H_n(X;\mathbb{F}_2) \xrightarrow{p^*} H_n(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} H_n(X;\mathbb{F}_2) \xrightarrow{\partial} H_{n-1}(X;\mathbb{F}_2) \to \ldots.$$

Example. Now we will see that we can use Proposition 44.15 to calculate the homology groups $H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following basic facts:

(1) By the discussion on page 1010 we know that

$$H_k(S^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k=0 \text{ or } k=n, \\ 0, & \text{otherwise} \end{cases}$$

(2) By the obvious generalization of Proposition 29.5 we know that $H_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \mathbb{F}_2$.

(3) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional manifold we know from the discussion on page 1010 that $H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying Proposition 44.15 to the 2-fold covering $p \colon S^n \to \mathbb{R}\mathrm{P}^n$ we obtain the following long exact sequence:

$$\to \underbrace{H_{n+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=0} \xrightarrow{\partial} H_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{H_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} H_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} H_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{H_{n-1}(S^n;\mathbb{F})}_{=0} \to$$

which continues all the way to

$$\ldots \to H_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{H_1(S^n;\mathbb{F})}_{=0} \xrightarrow{p_*} H_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} H_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{H_0(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \underbrace{H_0(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=\mathbb{F}} \to 0.$$

Now we can make the following observations:

(1) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms

$$H_{n-1}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} H_{n-2}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \ldots \xrightarrow[\cong]{\partial} H_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} H_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}.$$

(2) Now that we know that $H_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$ we obtain from the upper end of the long exact sequence that[840]

$$H_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} H_n(S^n;\mathbb{F}) \quad \text{and} \quad H_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} H_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})$$

[840] Why does this follow, why are there no other possibilities?

are also isomorphisms.

Summarizing we have shown that

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \;\cong\; \begin{cases} \mathbb{F}, & \text{if } k=0,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

This agrees of course with the result that we had obtained on page 1011.

Proof of Proposition 44.15. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. By Proposition 31.10 it suffices to show that the following sequence of chain maps

$$0 \;\to\; C_*(X;\mathbb{F}) \;\xrightarrow{p^*}\; C_*(\widetilde{X};\mathbb{F}) \;\xrightarrow{p_*}\; C_*(X;\mathbb{F}) \;\to\; 0$$

is in fact exact. As in Lemma 7.5 we consider the continuous map

$$\begin{aligned} f\colon \widetilde{X} &\to \widetilde{X}\\ x &\mapsto \text{the unique other element of } p^{-1}(p(x)).\end{aligned}$$

Using Proposition 20.2 we pick for each singular cell σ of X a lift $\widetilde{\sigma}$ to $\widetilde{X}$. We make the following observations which follow easily from the definitions and Proposition 20.2:

(a) given any singular n-simplex σ we have $p_*(\widetilde{\sigma}) = \sigma$,
(b) given any singular n-simplex σ we have $p^*(\sigma) = \widetilde{\sigma} + f_*(\widetilde{\sigma})$,
(c) given any singular simplex μ of $\widetilde{X}$ we have $p_*(f_*(\mu)) = p_*(\mu)$,
(d) any singular singular n-simplex μ of $\widetilde{X}$ is of the form $\mu = \widetilde{\sigma}$ or $\mu = f_*(\widetilde{\sigma})$ for some singular n-simplex σ of X.

Now we turn to the proof of the exactness of the above sequence.

(1) We first show that the map $p_*\colon C_n(\widetilde{X};\mathbb{F}) \to C_n(X;\mathbb{F}) = C_n(X)\otimes\mathbb{F}$ is surjective. So let $c = \sigma_1\otimes b_1 + \cdots + \sigma_k \otimes b_k \in C_n(X;\mathbb{F})$. It follows from (a) that

$$p_*\big(\widetilde{\sigma}_1\otimes b_1 + \cdots + \widetilde{\sigma}_k\otimes b_k\big) \;=\; \sigma_1\otimes b_1 + \cdots + \sigma_k\otimes b_k \;=\; c.$$

(2) By Proposition 44.14 we know that the transfer map p^* is a monomorphism.
(3) Next we show that $\mathrm{im}(p^*) \subset \ker(p_*)$. For any singular n-simplex σ of $\widetilde{X}$ we have by (c) above that[841] $(p_*\circ p^*)(\sigma) = p_*(\widetilde{\sigma} + f_*(\widetilde{\sigma})) = 2\sigma$, but it follows from Lemma 42.1 (4) that any multiple of two is zero in $C_*(X;\mathbb{F}) = C_*(X)\otimes\mathbb{F}$.

[841]That is of course basically the same argument as in the proof of Proposition 44.14.

(4) Finally we have to show that $\ker(p_*) \subset \operatorname{im}(p^*)$. So let $c \in \ker(p_*)$. By (d) we can write

$$c = (\widetilde{\sigma}_1 \otimes a_1 + f_*(\widetilde{\sigma}_1) \otimes b_1) + \cdots + (\widetilde{\sigma}_k \otimes a_k + f_*(\widetilde{\sigma}_k) \otimes b_k)$$

where $\sigma_1, \dots, \sigma_k$ are pairwise different and where $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{F}$. By (c) we have

$$p(c) = \sigma_1 \otimes (a_1 + b_1) + \cdots + \sigma_k(a_k + b_k).$$

But this implies that $a_i = -b_i$ for $i = 1, \dots, k$. But since we work over $\mathbb{F}$ we actually have $a_i = b_i$ for $i = 1, \dots, k$. It follows from (b) that

$$c = p^*(\sigma_1 \otimes a_1 + \cdots + \sigma_k \otimes a_k),$$

as desired. □

44.4. **The Borsuk-Ulam Theorem and the Ham-Sandwich Theorem.** Our goal in this section is to prove the following generalization of Theorem 7.23 from the 2-dimensional case to the higher-dimensional situation.

Theorem 44.16. (Borsuk-Ulam) *For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.*

Example. If our universe happens to be diffeomorphic to S^3, then there are two antipodal points where the temperature, distance to the Kneitinger and strength of the magnetic field agree.

The key ingredient in the proof of Theorem 44.16 is the following proposition.

Proposition 44.17. *Let $f\colon S^n \to S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.*

Proof. Let $f\colon S^n \to S^n$ be a map such that $f(-x) = -f(x)$ for all $x \in S^n$. By Lemma 42.21 it suffices to show that $f_*\colon H_n(S^n;\mathbb{Z}_2) \to H_n(S^n;\mathbb{Z}_2)$ is an isomorphism.

We start out with the observation that the map $f\colon S^n \to S^n$ evidently descends to a well-defined map

$$\begin{aligned} \overline{f}\colon \mathbb{R}\mathrm{P}^n &\to \mathbb{R}\mathrm{P}^n \\ [x] &\mapsto [f(x)]. \end{aligned}$$

Claim. The diagram

$$\begin{array}{ccccccccccc}
\to H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & H_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & H_{k-1}(S^n;\mathbb{Z}_2) & \to \\
\downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} & \\
\to H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & H_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & H_{k-1}(S^n;\mathbb{Z}_2) & \to
\end{array}$$

where the horizontal sequences are the exact sequences given by Proposition 44.15, commutes.

By the remark on page 772 it suffices to show that the following diagram commutes for any $k \in \mathbb{N}_0$:

$$\begin{tikzcd}
0 \arrow[r] & C_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "p^*"] \arrow[d, "\overline{f}_*"] & C_k(S^n;\mathbb{Z}_2) \arrow[r, "p_*"] \arrow[d, "f_*"] & C_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & 0 \\
0 \arrow[r] & C_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "p^*"] & C_k(S^n;\mathbb{Z}_2) \arrow[r, "p_*"] & C_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r] & 0.
\end{tikzcd}$$

Here the right-hand square commutes since by definition we have $p \circ f = \overline{f} \circ p$. Now we argue that the left-hand square commutes. So let $\sigma\colon \Delta^k \to \mathbb{R}\mathrm{P}^n$ be a singular k-simplex. We pick a point $P \in \Delta^k$. Furthermore we pick $Q \in S^n$ with $[Q] = \sigma(1, 0, \ldots, 0) \in \mathbb{R}\mathrm{P}^n$. We denote by $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ the two lifts of σ to S^n. Now we consider the singular simplex $\overline{f} \circ \sigma$. It remains to show that $p^*(\overline{f} \circ \sigma) = f \circ \widetilde{\sigma}_1 + f \circ \widetilde{\sigma}_2$. But this is indeed the case for the following reason:

(1) by construction we have $p \circ f = \overline{f} \circ p$ which implies that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are both lifts of $\overline{f} \circ \sigma$,
(2) it follows from $f(Q) = -f(Q)$ that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are two different lifts.

This concludes the proof of the claim.

Now we consider the following diagram:

$$\begin{tikzcd}
H_n(S^n;\mathbb{Z}_2) \arrow[d, "f_*"] & H_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[l, "p^*"'] \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"] & H_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"] & \ldots \arrow[r, "\partial"] & H_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"] & H_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[d, "\overline{f}_*"] \\
H_n(S^n;\mathbb{Z}_2) & H_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[l, "p^*"'] \arrow[r, "\partial"] & H_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "\partial"] & \ldots \arrow[r, "\partial"] & H_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \arrow[r, "\partial"] & H_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2).
\end{tikzcd}$$

We make the following observations:

(1) the diagram commutes by the previous claim,
(2) all the horizontal maps are isomorphisms by the discussion on page 1066,
(3) by the discussion on page 747 the right vertical map is an isomorphism.

It follows that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. □

Now we turn to the proof of the Borsuk-Ulam Theorem 44.16.

Proof of Theorem 44.16. Let $g\colon S^n \to \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rcccl}
\Phi\colon S^{n-1} \times [0,1] & \to & S^n & \to & S^{n-1}. \\
(y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\
 & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}}_{\text{defined since } g(x) \neq g(-x)}
\end{array}$$

The map $\Phi\colon S^{n-1}\times[0,1]\to S^{n-1}$ defines a homotopy from the map $f_0(x):=\Phi(x,0)$ to the map $f_1(x):=\Phi(x,1)$.

(1) By definition we have $f_0(-x)=\Phi(-x,0)=-\Phi(x,0)=-f_0(x)$. It follows from Proposition 44.17 that f_0 has odd degree.
(2) The map f_1 is a constant map, hence it has has degree zero.

But homotopic maps have the same degree by Lemma 32.16 (3). We have thus obtained a contradiction. □

The Borsuk-Ulam Theorem has an almost endless string of consequences, many of which lie in combinatorics and graph theory, we refer to [Mato] for an extensive discussion of the Borsuk-Ulam theorem and its corollaries. In the following we will state and prove two particularly interesting consequences of the Borsuk-Ulam Theorem 44.16. The first theorem was first proved by the Soviet mathematicians Lazar Lusternik and Lev Schnirelmann in 1930.

Theorem 44.18. (Lusternik-Schnirelmann) *Let $A_1,\dots,A_{n+1}$ be closed subsets of S^n with $A_1\cup\dots\cup A_{n+1}=S^n$. Then there exists an $i\in\{1,\dots,n+1\}$ such that A_i contains antipodal points, i.e. there exists a point x on S^n such that both x and $-x$ lie in A_i.*

Examples.

(1) For $n=1$ we obtain that if we have a decomposition $S^1=A\cup B$ with A and B two closed subsets (which do not need to be connected), then the subset A or the subset B contains antipodal points. How can one prove this statement "naively"?
(2) We consider the surface of our earth, we take A_1 to be the closure of the land mass, we take A_2 to be the closure of the Pacific ocean and we take A_3 to be the closure of all seas except for the Pacific. Then at least one of A_1,A_2 or A_3 contains a pair of antipodal points. Which of A_1,A_2 and A_3 is it?
(3) The proposition is optimal in the following sense: it is easy to find three closed subsets A_1,A_2,A_3 that cover all of S^1, but such that none of the A_i contains antipodal points.

Proof. Let $A_1,\dots,A_{n+1}$ be closed subsets of S^n with $A_1\cup\dots\cup A_{n+1}=S^n$. Without loss of generality we can assume that all A_i are non-empty. (Indeed, at least one of the A_i is evidently non-empty, if one of them is empty we just replace it with the non-empty A_i.) Given a non-empty subset A of S^n and $x\in S^n$ we write

$$d(x,A)\ :=\ \inf\{d(x,a)\mid a\in A\},$$

where $d(x,a)$ denotes the distance function on the Riemannian manifold S^n that we introduced on page 594.[842] We make the following two observations:

(1) If A is closed, then $d(x,A)=0$ if and only if $x\in A$.
(2) It follows easily from Proposition 22.8 that $d(x,A)$ is continuous in x.

[842]Put differently, $d(x,a)$ is the infimum of all lengths of piecewise smooth paths on S^n from x to a.

Now we consider the function

$$\begin{aligned} f\colon S^n &\to \mathbb{R}^n \\ x &\mapsto (d(x, A_1), \ldots, d(x, A_n)). \end{aligned}$$

By the Borsuk-Ulam Theorem 44.16 there exists an $x \in S^n$ with $f(x) = f(-x)$. We first suppose that there exists an $i \in \{1, \ldots, n\}$ with $d(x, A_i) = 0$. From $f(x) = f(-x)$ it follows that $d(-x, A_i) = 0$ as well. Since A_i is closed it follows that both x and $-x$ lie in A_i.

Now suppose that $d(-x, A_i) = d(x, A_i) \neq 0$ for $i = 1, \ldots, n$. Since $A_1 \cup \cdots \cup A_{n+1} = S^n$ it follows that both x and $-x$ lie in A_{n+1}. □

Before we can state the next consequence of the Borsuk-Ulam Theorem 44.16 we need to introduce one more definition.

Definition. Let $A \subset \mathbb{R}^n$ be a measurable subset[843] of finite volume. We say a hyperplane $H \subset \mathbb{R}^n$ *bisects* A if for the two half-spaces V, V' determined by H we have the equality $\mathrm{vol}(A \cap V) = \mathrm{vol}(A \cap V')$.

We conclude this chapter with the following pretty theorem.

Theorem 44.19. (Ham-Sandwich Theorem) *Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Then there exists an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ that bisects $A_1, \ldots, A_n$.*

Remark.

(1) The Ham-Sandwich Theorem gets its name from the following "application". Suppose we are given a sandwich with an upper piece of bread A_1, some ham A_2 and a lower piece of bread A_3. Then using a knife one can cut the sandwich, along a hyperplane, into two pieces such that each piece of bread and the ham are bisected. This situation is, more or less convincingly, illustrated in Figure 624.

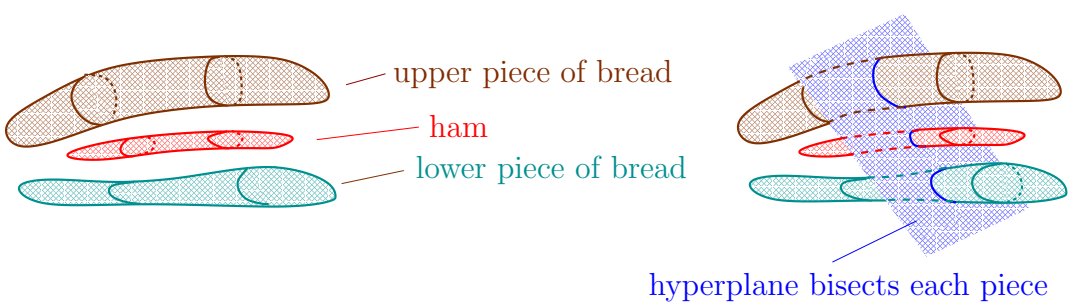


FIGURE 624. Illustration of the "Ham-Sandwich Theorem".

(2) The 2-dimensional version of the Ham-Sandwich Theorem is sometimes referred to as the "Pancake Theorem", since one can bisect the two pancakes along a line into two equal sized halves.

[843]Here "measurable" means Lebesgue-measurable in the sense of say [Frie4, Caro] or [Rudi2, Chapter 2].

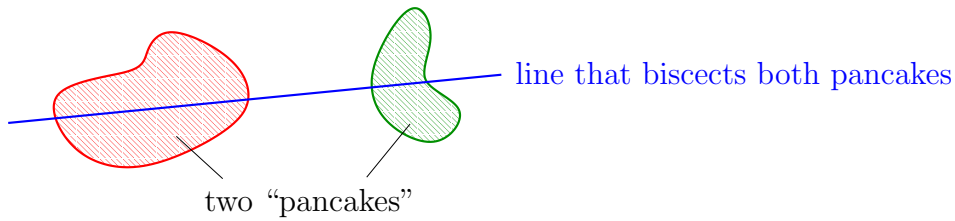


FIGURE 625. Illustration of the "Pancake Theorem".

In the proof of the Ham-Sandwich Theorem we will need the following result from measure theory.

Proposition 44.20. *Suppose that for each $t \in \mathbb{R}$ we have a measurable subset X_t of $\mathbb{R}^n$ such that for each $s \leq t \in \mathbb{R}$ we have $X_s \subset X_t$. Then $\bigcup_{t\in\mathbb{R}} X_t$ is also measurable and*

$$\lim_{t\to\infty} \operatorname{vol}(X_t) \;=\; \operatorname{vol}\Big(\bigcup_{t\in\mathbb{R}} X_t\Big).$$

Proof of Proposition 44.20. The proposition can be deduced easily from basic facts of the Lebesgue measure, see [Caro, Chapter 16] or [Frie3, Proposition 6.2 (6)]. □

Now we can provide the proof of the Ham-Sandwich Theorem.

Proof of the Ham-Sandwich Theorem. Let $A_1, \dots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Without loss of generality we can assume that the volume of A_n is positive. Given a point $P \in S^{n-1}$ and $t \in \mathbb{R}$ we write

$$H_{P,t} \;:=\; \{t \cdot P + w \mid P \cdot w = 0\}$$

and

$$\begin{aligned} V^+_{P,t} &:= \{t \cdot P + w \mid P \cdot w \geq 0\} \\ V^-_{P,t} &:= \{t \cdot P + w \mid P \cdot w \leq 0\}. \end{aligned}$$

(We refer to Figure 626 on the left for an illustration.) Put differently, $H_{P,t}$ is the hyperplane through $t \cdot P$ orthogonal to P, and $V^\pm_{P,t}$ are the two half-spaces determined by $H_{P,t}$. We make the following two observations:

(i) by definition we have $V^\pm_{-P,-t} = V^\mp_{P,t}$,
(ii) for a subset $A \subset \mathbb{R}^n$ of finite volume we have the equality

$$\operatorname{vol}(A \cap V^\pm_{P,t}) \;\underset{\substack{\uparrow \\ \text{since } H_{P,t} = V^+_{P,t} \cap V^-_{P,t} \text{ is a set of measure zero}}}{=}\; \operatorname{vol}(A \cap (\mathbb{R}^n \setminus V^\mp_{P,t})) \;=\; \operatorname{vol}(A) - \operatorname{vol}(V^\mp_{P,t}).$$

We continue with the following claim.

Claim. There exists a function

$$t\colon S^{n-1} \;\to\; \mathbb{R}$$

with the following two properties:

(1) for any $P \in S^{n-1}$ we have $\mathrm{vol}(A_n \cap V^+_{P,t(P)}) = \frac{1}{2}\,\mathrm{vol}(A_n)$, i.e. the hyperplane $H_{P,t(P)}$ bisects A_n,
(2) for any $P \in S^{n-1}$ we have $t(-P) = -t(P)$,
(3) the function t is continuous.

To prove the claim we first fix $P \in S^{n-1}$. We consider the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ t & \mapsto & \mathrm{vol}(A_n \cap V^+_{P,t}). \end{array}$$

This function is continuous[844] and we have

$$\lim_{t\to\infty} \mathrm{vol}(A_n \cap V^+_{P,t}) \underset{\substack{\uparrow \\ \text{Proposition 44.20}}}{=} \mathrm{vol}\Big(\bigcup_{t\in\mathbb{R}} (A_n \cap V^+_{P,t})\Big) = \mathrm{vol}(A_n).$$

and

$$\lim_{t\to-\infty} \mathrm{vol}(A_n \cap V^+_{P,t}) \underset{\substack{\uparrow \\ \text{by (ii)}}}{=} \mathrm{vol}(A_n) - \underbrace{\lim_{t\to-\infty} \mathrm{vol}(A_n \cap V^-_{P,t})}_{=\,\mathrm{vol}(A_n)\text{ by Proposition 44.20}} = 0.$$

We deduce from the Intermediate Value Theorem that

$$T := \big\{ t \in \mathbb{R} \,\big|\, \mathrm{vol}(A_n \cap V^+_{P,t}) = \tfrac{1}{2}\,\mathrm{vol}(A_n) \big\}$$

is a non-empty set.[845] It is straightforward to see that it is a connected subset of $\mathbb{R}$, i.e. it is an interval. Since $\mathrm{vol}(A_n) > 0$ we know that T is a finite interval. We define

$$t(P) := \tfrac{1}{2}(\inf(T) + \sup(T)).$$

Note that it follows immediately from the definitions that $t(-P) = -t(P)$. Furthermore with some elementary measure theory one can show that the function $S^{n-1} \to \mathbb{R}$ given by $P \mapsto t(P)$ is continuous. We leave the verification of the details as an exercise to the reader. This concludes the proof of the claim.

Now we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & \mathbb{R}^{n-1} \\ P & \mapsto & \big(\mathrm{vol}\big(A_1 \cap V^+_{P,t(P)}\big), \ldots, \mathrm{vol}\big(A_{n-1} \cap V^+_{P,t(P)}\big)\big). \end{array}$$

From the continuity of $t\colon S^{n-1} \to \mathbb{R}$ it follows that f itself is continuous. By the Borsuk-Ulam Theorem 44.16, there exists a point $P \in S^{n-1}$ with $f(P) = f(-P)$. We claim that the hyperplane $H_{P,t(P)}$ has the desired property.

[844] This requires at least a little bit of thought since we do not assume that A is bounded, we only assume it has finite volume. We leave it as an exercise to prove this statement using basics facts of Lebesgue measures.

[845] One might think that T consists of just a point, but for example if A_n is disconnected, then it can happen that T is in fact a interval that is not just a point.

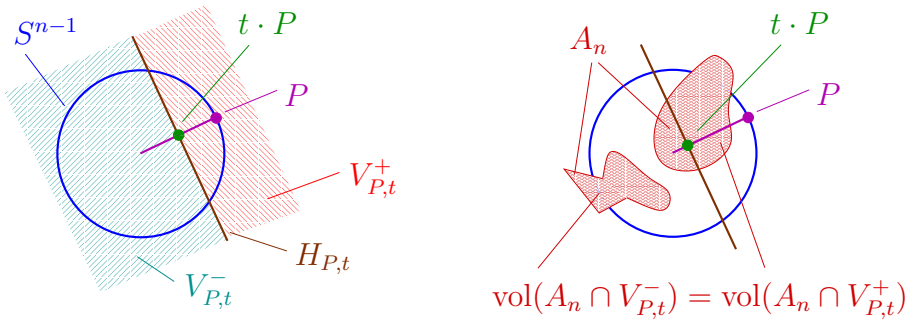


FIGURE 626. Illustration for the proof of the Ham-Sandwich Theorem 44.19

By definition of $t(P)$ the hyperplane $H_{P,t(P)}$ bisects A_n. Furthermore, for $i = 1, \dots, n-1$ we have

$$\mathrm{vol}\left(A_i \cap V^+_{P,t(P)}\right) \underset{\substack{\uparrow \\ \text{since } f(P)=f(-P)}}{=} \mathrm{vol}\left(A_i \cap V^+_{-P,t(-P)}\right) \underset{\substack{\uparrow \\ \text{since } t(-P)=-t(P)}}{=} \mathrm{vol}\left(A_i \cap V^+_{-P,-t(P)}\right) \underset{\substack{\uparrow \\ \text{since } V^+_{P,s}=V^-_{-P,-s}}}{=} \mathrm{vol}\left(A_i \cap V^-_{P,t(P)}\right).$$

This shows that the hyperplane $H_{P,t(P)}$ also bisects $A_1, \dots, A_{n-1}$. □

Remark. The proof for the existence of the hyperplane that bisects the given sets $A_1, \dots, A_n$ is non-constructive, in the sense that even with some flexibility in mathematical rigor one cannot turn it into an algorithm that given $A_1, \dots, A_n$ finds the hyperplane that bisects all of the subsets.

Exercises for Chapter 44.

Exercise 44.1. Let $q_1, q_2 \in S^3 = \{h \in \mathbb{H} \mid |h| = 1\}$. We consider the map

$$\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1}. \end{array}$$

Show that the map is linear and that it is represented by a matrix in SO(4).

45. Topological manifolds

In this chapter we will discuss topological manifolds in more detail. In particular we will generalize some results that we had initially obtained for manifolds to topological manifolds. We will also introduce the notion of orientability for topological manifolds which extends the notion of orientability that we introduced in Section 3.8 for manifolds.

45.1. **Topological manifolds and CW-structures.** In Chapters 35 and 39 we had discussed the homology groups of manifolds at length. The key tool hereby was that compact manifolds are finite CW-complexes and that every manifold can be viewed as a simplicial complex. The situation is more complicated for topological manifolds. The following theorem can be viewed as a weak version of Theorem 26.3 for topological manifolds.

Theorem 45.1.

(1) *Every topological manifold is homotopy equivalent to a CW-complex.*
(2) *Every compact topological manifold is homotopy equivalent to a finite CW-complex.*

Proof. Olof Hanner [Han, Theorem 3.3] showed in 1950 that every topological manifold is an "absolute neighborhood retract". By [Mi9, Theorem 1] this implies that M is homotopy equivalent to a CW-complex.[846]

James West [Wes] showed in 1977 that every compact "absolute neighborhood retract" is homotopy equivalent to a compact CW-complex. But by Corollary 26.15, a compact CW-complex is the same as a finite CW-complex. An alternative proof for the theorem of James West is given by Rob Kirby and Larry Siebenmann [KSi, KSi2]. □

Of course it would be much nicer if every compact topological manifold was in fact a simplicial complex or at least a CW-complex. In the following we summarize what is known about the existence of simplicial structures and of CW-structures on topological manifolds. We will not make use of these more general statements, in the lecture notes we will only use Theorem 45.1.

(1) Tibor Radó [Rad] (see also [MTh, Theorem 3.1.1]) showed in 1926 that every compact 2-dimensional topological manifold has a simplicial structure, in particular it has a CW-structure. In 1952 Edwin Moise [Moi] proved the analogous result for 3-dimensional topological manifolds.
(2) In the 1970s Rob Kirby and Larry Siebenmann [KSi2, Essay III.2] showed that every *closed* topological manifold of dimension ≥ 6 has a CW-structure. In 1982 Frank Quinn [Qu] extended this result to the case of 5-dimensional manifolds.
(3) Andrew Casson, see [AM, page xvi] showed in the 1980s that there exist closed topological 4-dimensional manifolds that do not have a simplicial structure.
(4) Ciprian Manolescu [Man] showed in 2013 that for every $n \geq 5$ there exists a closed n-dimensional topological manifold that does not admit a simplicial structure.

[846] In fact in [Mi9, Proposition 1] it is shown that every compact topological manifold is "dominated" by a finite CW-complex which is good enough for many conclusions.

(5) It seems to be unknown whether every closed 4-dimensional topological manifold has a CW-structure and it seems to be unknown whether every compact n-dimensional topological manifold with non-empty boundary of dimension $n \geq 4$ has a CW-structure.
(6) Using [Wall, Corollary 5.1] and the computations of homology groups of topological manifolds one can prove the following refinement of Theorem 45.1:
   (a) Every closed n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional CW-complex.
   (b) Every compact, connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional CW-complex.

   The proof of these statements (a) and (b) actually relies on the Poincaré Duality Theorem that we will prove later on.

We can summarize most of the above statements in the following table:

	$n=1,2,3$	$n=4$	$n \geq 5$
each closed top. n-mfd. is homotopy equiv. to a finite CW-cplx.	✓	✓	✓
each closed top. n-mfd. admits a CW-structure	✓	?	✓
each compact top. n-mfd. is homotopy equ. to a finite CW-cplx.	✓	✓	✓
each compact top. n-manifold admits a CW-structure	✓	?	?

In this and the following chapter we will see that many of the results that we had stated and proved for manifolds also hold for topological manifolds. In particular we will extend the results from Chapter 39 on homology groups of manifolds to the case of topological manifolds. The proofs of this chapter are also more rigorous than the proofs in Chapter 39, but they are arguably at times less intuitive than the arguments using simplicial structures.

We start out with the following proposition which is the generalization of Propositions 27.13 and 35.6 (1) to the case of topological manifolds.

Proposition 45.2. *Let M be a compact topological manifold and let $\mathbb{F}$ be a field. Then the following hold:*

(1) *The group $\pi_1(M)$ is finitely presented.*
(2) *All homology groups $H_k(M)$ are finitely generated abelian groups and the homology groups $H_k(M;\mathbb{F})$ are finite-dimensional. Furthermore there exists an l such that all homology groups in dimension $\geq l$ vanish.*[847]
(3) *Let X be a subset of M that is homeomorphic to a compact manifold. (For example, X could be a compact submanifold, or by Lemma 3.6 (2), we could take*[848] *$X = \partial M$.) Then all relative homology groups $H_k(M, X)$ are finitely generated abelian groups and the homology groups $H_k(M, X;\mathbb{F})$ are finite-dimensional.*

[847]These statements can also be proved without the full power of Theorem 45.1, see e.g. [Bre, E.5.Corollary] or [Hat, p. 527].

[848]Note that the boundary of a topological manifold is in general *not* a submanifold.

Proof. The first two statements of the proposition are an immediate consequence of Theorem 45.1 together with Proposition 27.12, Corollary 30.8, Proposition 35.5 and the discussion on page 1010.

For the final statement we consider the long exact sequence of the pair (M, X), given by Corollary 31.14. By (2) the homology groups of X and M are finitely generated respectively finite-dimensional. It follows from Lemma 11.6 that the relative homology groups are also finitely generated respectively finite-dimensional. □

45.2. **The Topological Collar Neighborhood Theorem.** The following theorem is the analogue of the Collar Neighborhood Theorem 10.10 for topological manifolds.

Theorem 45.3. (Topological Collar Neighborhood Theorem) *Let M be a topological manifold. Then there exists a map $\Phi\colon \partial M \times [0,1] \to M$ with the following three properties:*

(1) *the map Φ is a homeomorphism onto its image,*
(2) *the image of $\partial M \times [0,1)$ is an open subset of M,*
(3) *for all $P \in \partial M$ we have $\Phi(P, 0) = P$.*

Proof. If M is compact, then the proof is fairly elementary, see e.g. [Vic, Theorem Appendix II.7] or [Hat, p. 253]. If M is non-compact, then the proof is harder, see e.g. [BrownM2, Theorem 3], where we actually need that topological manifolds are "metrizable", see Theorem 3.10. We also refer to [Arm2] for a proof of the theorem and some more refined statements. □

The following uniqueness statements for collar neighborhoods is almost verbatim the same as the statement of Proposition 10.11.

Proposition 45.4. $(*)$ *Let M be a topological manifold and let $f_0, f_1\colon \partial M \times [0,1] \to M$ be two embeddings that are the identity map on $\partial M \times \{0\} = \partial M$. If ∂M is compact, then there exists an isotopy*

$$H\colon M \times [0,1] \;\to\; M$$

rel ∂M from $H_0 = \mathrm{id}_M$ to a homeomorphism $\Phi = H_1\colon M \to M$ with $\Phi_1 \circ f_0 = f_1$. In particular Φ restricts to a homeomorphism

$$\Phi\colon f_0(\partial M \times [0,1])) \;\to\; f_1(\partial M \times [0,1]).$$

Proof $(*)$. The proof is sketched in [KSi, Essay I, Theorem A.2]. □

Convention. Let M be a compact topological manifold. We pick a map $\Phi\colon \partial M \times [0,1] \to M$ as in the conclusion of the Topological Collar Neighborhood Theorem 45.3. Usually we identify $\partial M \times [0,1]$ with the image in M under the map Φ and we just refer to $\partial M \times [0,1]$ as a "collar neighborhood of ∂M". Sometimes we also find it convenient to use other intervals in our notation, e.g. we might write "let $\partial M \times [-3,0]$ be a collar neighborhood". It is then understood that $\partial M \times \{0\} = \partial M$.

For later we record the following corollary to the Topological Collar Neighborhood Theorem 45.3.

Corollary 45.5. *Let M be an n-dimensional topological manifold. If ∂M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional topological manifold without boundary.*

Proof. By the Topological Collar Neighborhood Theorem 45.3 we can pick a collar neighborhood $\partial M \times [0,2]$. We write $W := M \setminus (\partial M \times [0,1))$ and we set $M' := M \setminus \partial M$. We consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q,t) & \mapsto & \begin{cases} (R, s(1-t)+t), & \text{if } Q = (R,s) \in \partial M \times [0,1], \\ Q, & \text{otherwise.} \end{cases} \end{array}$$

Using Lemma 1.32 (2) and Proposition 10.2 one can show that this map is continuous. This map defines a deformation retraction from M to W and also from $M \setminus \partial M$ to W. It follows from Lemmas 9.10 and 9.8 that M and M' are homotopy equivalent. In Exercise 45.5 we will see that M' is non-compact. Finally note that it follows from Proposition 32.1 that M' does not have any boundary. □

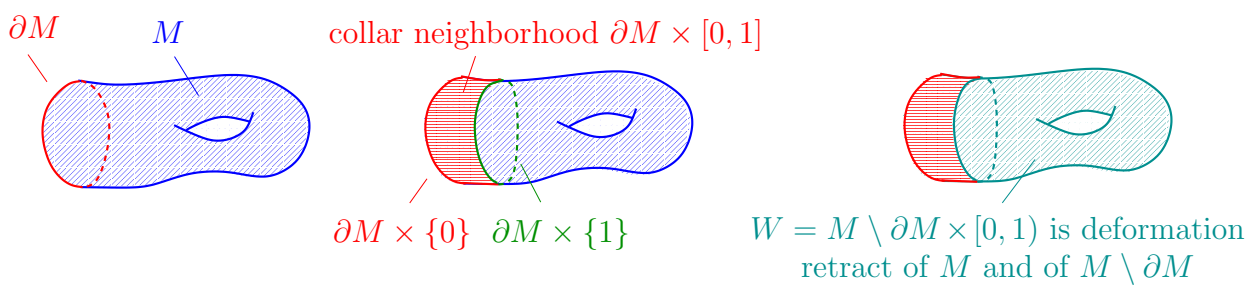


FIGURE 627. Illustration for the proof of Corollary 45.5.

We conclude this discussion of the Topological Collar Neighborhood Theorem 45.3 with the following proposition that looks uncannily similar to Proposition 10.12.

Proposition 45.6. *Let M be an n-dimensional topological manifold (we do not assume that M is connected). Let A and B be two distinct boundary components of M. Furthermore let $f\colon A \to B$ be a homeomorphism. Then the following two statements hold:*

(1) *The topological space*

$$M/a \sim f(a) \quad \text{for } a \in A$$

is an n-dimensional topological manifold.

(2) *If M consists of two components X and Y and if $A \subset X$ and $B \subset Y$, then the images of X and Y in $M/a \sim f(a)$ are codimension zero submanifolds.*

(3) *If M is compact, then $M/a \sim f(a)$ is also compact.*

(4) *The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.*

Proof. The proof of Proposition 45.6 is almost the same as the proof of Proposition 10.12. We just need to make the following replacements:

(1) We need to replace Proposition 3.21 (3a) by Lemma 3.6.
(2) We need to replace the Collar Neighborhood Theorem 10.10 by the Topological Collar Neighborhood Theorem 45.3.
(3) We need to replace Proposition 3.21 (1) by Proposition 32.1 (1). □

45.3. **The Euler characteristic of compact topological manifolds.** The following definition is a slight generalization of the definition from page 969.

Definition.

(1) Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to[849]
$$b_n(X, A) \ := \ \mathrm{rank}(H_n(X, A))$$
as the *n-th Betti number of* (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$.
(2) Let (X, A) be a pair of topological spaces. If all $b_n(X, A)$ are finite and if the sum $\sum\limits_{n\in\mathbb{N}_0} b_n(X, A)$ is finite, then we define the Euler characteristic of (X, A) to be
$$\chi(X, A) \ := \ \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X, A).$$
For $A = \varnothing$ we write $\chi(X) := \chi(X, \varnothing)$.

Remark.

(1) It follows from Proposition 40.1 that for a finite CW-complex the above definition agrees with the original definition of the Euler characteristic of a finite CW-complex.
(2) Let X be a compact manifold and let A be a compact submanifold. It follows from Proposition 45.2 that the Euler characteristic $\chi(X, A)$ is defined.

It turns out that the formulas we proved before for the Euler characteristic of CW-complexes also hold for topological manifolds. More precisely, we have the following proposition, of which the first three statements are the analogue of Lemmas 40.9 and 40.5 and of Proposition 26.25 in our new context.

Proposition 45.7.

(1) *Let X and Y be homotopy equivalent topological spaces. If the Euler characteristic is defined for one of the two topological spaces, then it is defined for the other topological space and we have*
$$\chi(X) \ = \ \chi(Y).$$
(2) *Let X be and Y be two compact topological manifolds, then*
$$\chi(X \times Y) \ = \ \chi(X) \cdot \chi(Y).$$

[849] We refer to page 1006 for the definition of the rank of an arbitrary abelian group.

(3) *Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $Y \cap Z$ is also a compact submanifold of X. Then the following equality holds*

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

(4) *If X is a compact topological manifold and A is a compact submanifold, then*

$$\chi(X, A) \;=\; \chi(X) \;-\; \chi(A).$$

Furthermore, for any field $\mathbb{F}$ we have

$$\chi(X, A) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(X, A; \mathbb{F})).$$

(5) *Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Then*

$$\chi(\widetilde{X}) \;=\; \left[\widetilde{X} : X\right] \cdot \chi(X).$$

Sketch of proof.

(1) This statement follows immediately from Corollary 30.8.

(2) This equality can be proved using the Künneth Theorem 43.7 for Chain Complexes. We will fill in the details in Exercise 45.1.

(3) We will not make use of this statement. Therefore we hide, with a good conscience, the proof in Exercise 45.2.

(4) Let X be a compact topological manifold and let A be a compact submanifold.

(a) It follows immediately from considering the long exact sequence in homology of the pair (X, A) together with Lemma 40.4 that $\chi(X, A) = \chi(X) - \chi(A)$.

(b) Let $\mathbb{F}$ be a field. Let Y be a compact topological manifold. From Theorem 45.1 we know that Y is homotopy equivalent to a finite CW-complex Z. We observe that

$$\chi(Y) \;\underset{\substack{\uparrow \\ \text{Corollary 30.8}}}{=}\; \chi(Z) \;\underset{\substack{\uparrow \\ \text{see page 1010}}}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(Z; \mathbb{F})) \;\underset{\substack{\uparrow \\ \text{by the generalization of Corollary 30.8} \\ \text{to } \mathbb{F}\text{-coefficients, see page 1009}}}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(Y; \mathbb{F})).$$

Using this observation we can now prove that

$$\chi(X, A) \;\underset{\substack{\uparrow \\ \text{by (a)}}}{=}\; \chi(X) - \chi(A) \;\underset{\substack{\uparrow \\ \text{by the above observation}}}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(X; \mathbb{F})) - \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(A; \mathbb{F}))$$

$$\underset{\substack{\uparrow \\ \text{by the long exact sequence of the pair } (X, A) \text{ with } \mathbb{F}\text{-coefficients and Lemma 40.3}}}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(H_i(X, A; \mathbb{F})).$$

(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Without loss of generality we can assume that X and $\widetilde{X}$ are connected. By Theorem 45.1 there exists a homotopy equivalence $f\colon X \to Y$, where Y is a finite

CW-complex. By Proposition 20.8 there exists a connected covering $q\colon \widetilde{Y} \to Y$ such that $q_*(\pi_1(\widetilde{Y})) = f_*(p_*(\pi_1(\widetilde{X})))$. Using Proposition 20.2 one can now easily show that $f\colon X \to Y$ lifts to map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ and that this map is again a homotopy equivalence[850]. Now we obtain that

$$\begin{array}{ccccccc}
 & & & \text{we have } \big[\pi_1(\widetilde{Y}) : \pi_1(Y)\big] = \big[\pi_1(\widetilde{X}) : \pi_1(X)\big] \text{ since } f_* \text{ is an isomorphism} & & & \\
 & & & \downarrow & & & \\
\chi\big(\widetilde{X}\big) & = & \chi\big(\widetilde{Y}\big) & = \big[\pi_1(Y) : q_*(\pi_1(\widetilde{Y}))\big]\cdot\chi(Y) & = & \big[\pi_1(X) : p_*(\pi_1(\widetilde{X}))\big]\cdot\chi(X) & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{by (1) since } \widetilde{X} \text{ and } \widetilde{Y} & & \text{Proposition 26.25} & & \text{by (1) we have } \chi(Y) = \chi(X) \text{ since } X & \\
 & \text{are homotopy equivalent} & & \text{and Lemma 7.16} & & \text{and } Y \text{ are homotopy equivalent} & \\
 & & & & & = \big[\widetilde{X} : X\big]\cdot\chi(X). & \\
 & & & & & \uparrow & \\
 & & & & & \text{Lemma 7.16} &
\end{array}$$

□

45.4. **Variations of the excision theorem.** One of the key tools throughout this course will be the Excision Theorem 31.17 and its many different incarnations. Recall that given an abelian group G the Excision Theorem 31.17 says that given a topological space X and subsets $Z \subset A \subset X$ the inclusion induces an isomorphism

$$H_n(X \setminus Z, A \setminus Z; G) \xrightarrow{\cong} H_n(X, A; G)$$

under the technical, and often quite annoying, condition, that $\overline{Z} \subset \mathring{A}$. In this short section we state two theorems that deal with this technical condition in favorable situations. The first of the two theorems is basically just a reformulation.

Theorem 45.8. (Excision Theorem)

(1) *Let X be a topological space, let $U \subset X$ be an open subset and let $K \subset X$ be a subset such that $\overline{K} \subset U$. Then for any abelian group G and any $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(U, U \setminus K; G) \to H_n(X, X \setminus K; G)$$

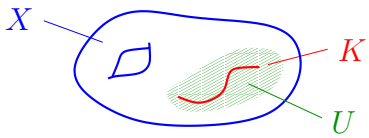
is an isomorphism.

(2) *The hypotheses of* (1) *are satisfied if X is Hausdorff* (*e.g. if X is a topological manifold*) *and if K is a compact subset that is contained in an open set U.*

Proof.

(1) Let X be a topological space, let $U \subset X$ be an open subset and let $K \subset X$ be a subset such that $\overline{K} \subset U$. We set $Z := X \setminus U$ and $A := X \setminus K$. By our hypothesis

[850]Here the point is that not only the homotopy equivalences but also the homotopies lift to the covering spaces. We leave the details to the reader.



here we have $H_n(U, U \setminus K; G) \xrightarrow{\cong} H_n(X, X \setminus K; G)$

FIGURE 628. Illustration of the Excision Theorem 45.8.

on K we have $Z \subset A$. We then have

$$\begin{array}{ccccc} H_n(U, U \setminus K; G) & \xrightarrow{=} & H_n(X \setminus Z, A \setminus Z; G) & \xrightarrow{\cong} & H_n(X, X \setminus K; G) \\ & & \uparrow & & \uparrow \end{array}$$

since $X \setminus Z = U$ and $A \setminus Z = U \setminus K$ — we have $\overline{Z} = Z = X \setminus U \subset X \setminus \overline{K} \subset$ interior of $X \setminus K = \mathring{A}$ thus we can apply the Excision Theorem 31.17

(2) If X is a topological manifold and if K is a compact subset that is contained in an open set U, then it follows from Lemma 1.18 (2)[851] that K is also a closed subset, hence we have $\overline{K} \subset U$. □

The next lemma is at first glance a little dull, but later on it will be used on several occasions. In the proof we will also require the Excision Theorem 45.8.

Lemma 45.9. *Let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. Then for any abelian group G and any $i \in \mathbb{N}_0$ the following inclusion induced maps are isomorphisms:*

$$\begin{array}{rrcl} (1) & H_i(\mathbb{R}^n \setminus A; G) & \to & H_i(\mathbb{R}^n \setminus \{x\}; G) \\ (2) & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A; G) & \to & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \\ (3) & H_i(U, U \setminus A; G) & \to & H_i(U, U \setminus \{x\}; G). \end{array}$$

If $\mathring{A}$ is non-empty and if $x \in \mathring{A}$ is a point, then also the following inclusion induced maps are isomorphisms

$$\begin{array}{rrcl} (4) & H_i(\partial\overline{A}; G) & \to & H_i(\mathbb{R}^n \setminus \{x\}; G) \\ (5) & H_i(\mathbb{R}^n, \partial\overline{A}; G) & \to & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \end{array}$$

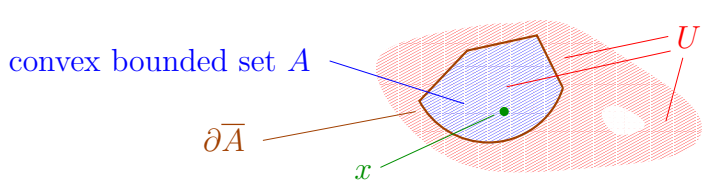


FIGURE 629. Illustration of Lemma 45.9.

[851] Note that we use that topological manifolds are by definition Hausdorff.

Proof. Let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. To simplify the notation we drop the coefficients G from the notation and without loss of generality we assume that $x = 0$. Now let $i \in \mathbb{N}_0$.

(1) We have to show that the inclusion induced map $H_i(\mathbb{R}^n \setminus A) \to H_i(\mathbb{R}^n \setminus \{0\})$ is an isomorphism. We had already stated this fact in Exercise 30.2. Now that we actually need the statement we give the full details.

Since A is bounded we can pick an $r > 0$ such that $A \subset B^n_{r/2}(0)$. Note that if $y \notin A$, then it follows from the convexity of A and the fact that $0 \in A$, that for any $s \geq 1$ the point sy cannot lie in A either. It is now straightforward to verify that the map

$$\begin{array}{rcl} F\colon (\mathbb{R}^n \setminus A) \times [0,1] & \to & \mathbb{R}^n \setminus A \\ x & \mapsto & \begin{cases} (x,t), & \text{if } x \in \mathbb{R}^n \setminus B^n_r(0), \\ x \cdot \underbrace{\left((1-t) + t \cdot \frac{r}{\|x\|}\right)}_{\geq 1 \text{ since } r > \|x\|}, & \text{if } x \in B^n_r(0) \setminus A. \end{cases} \end{array}$$

is a deformation retraction from $\mathbb{R}^n \setminus A$ to $\mathbb{R}^n \setminus B^n_r(0)$.[852] Clearly $\mathbb{R}^n \setminus B^n_r(0)$ is also a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Now we consider the following commutative diagram

$$\begin{tikzcd} H_i(\mathbb{R}^n \setminus A) \arrow[rr] & & H_i(\mathbb{R}^n \setminus \{0\}) \\ & H_i(\mathbb{R}^n \setminus B^n_r(0)). \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It follows from Corollary 30.8 and the above discussion that the two diagonal maps are isomorphisms. It follows that the horizontal map is also an isomorphism.

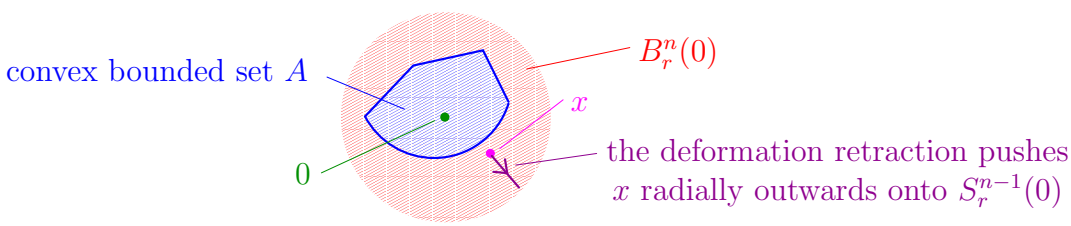


FIGURE 630.

(2) Now we consider the following commutative diagram of long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & H_i(\mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & H_i(\mathbb{R}^n) \arrow[r] \arrow[d, "\cong"] & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & H_{i-1}(\mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & H_{i-1}(\mathbb{R}^n) \arrow[r] \arrow[d, "="] & \dots \\ \dots \arrow[r] & H_i(\mathbb{R}^n \setminus \{x\}) \arrow[r] & H_i(\mathbb{R}^n) \arrow[r] & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \arrow[r] & H_{i-1}(\mathbb{R}^n \setminus \{x\}) \arrow[r] & H_{i-1}(\mathbb{R}^n) \arrow[r] & \dots \end{tikzcd}$$

[852] As always, checking that a map is a deformation retraction also requires showing that the map is continuous. In this case the map is continuous by Lemma 1.104 and the fact we pick r such that $A \subset B^n_{r/2}(0)$.

By (1) the first and the fourth vertical maps are isomorphisms. The second and the fifth vertical map are the identity. It follows from the Five Lemma 31.11 that the middle vertical map is also an isomorphism.

(3) We consider the following commutative diagram

$$\begin{array}{ccc} H_i(U, U \setminus A) & \longrightarrow & H_i(U, U \setminus \{x\}) \\ \downarrow & & \downarrow \\ H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \longrightarrow & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \end{array}$$

of inclusion induced homomorphisms. Since $\overline{A} \subset U$ we deduce from Theorem 45.8 that the two vertical maps are isomorphisms. By (2) we know that the bottom horizontal map is an isomorphism. Hence the top horizontal map is also an isomorphism.

Now we suppose that $\mathring{A}$ is non-empty and we suppose that x is a point in $\mathring{A}$. Without loss of generality we can assume that $x = 0$ and that $A = \overline{A}$.

(4) By Lemma 9.1 we know that there exists an isotopy

$$F\colon A \times [0,1] \ \to \ \mathbb{R}^n$$

from the inclusion map $i\colon A \to \mathbb{R}^n$ to a homeomorphism $\varphi\colon A \to \overline{B}^n$ that has the following two properties:

(a) $F_1 = \varphi$ restricts to a homeomorphism $\partial A \to S^{n-1}$, and

(b) F has the property that $F(0,t) = 0$ for all $t \in [0,1]$.

Now we have the following diagram

$$\begin{array}{ccccc} & & H_k(\partial A) & & \\ & \swarrow^{\cong}_{\varphi_*} & \downarrow \varphi_* & \searrow^{(i|_{\partial A})_*} & \\ H_k(S^{n-1}) & \xrightarrow[\cong]{} & H_k(\mathbb{R}^n \setminus \{0\}) & \xrightarrow[=]{} & H_k(\mathbb{R}^n \setminus \{0\}). \end{array}$$

Here all the unmarked maps are induced by inclusions. The right-hand triangle commutes by Corollary 30.8 since, by the above property (b), the isotopy F restricts to an isotopy $\partial A \times [0,1] \to \mathbb{R}^n \setminus \{0\}$. The left horizontal map is an isomorphism since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. It is now clear that the right-hand diagonal map is also an isomorphism.

(5) With the same argument as in (2) we can reduce the proof of this statement to (4). □

We conclude this section with another variation on the Excision Theorem. It can be summarized as saying that in relative homology one can excise the interior of a codimension 0 submanifold.

Theorem 45.10. (Excision Theorem for topological Manifolds) *Let X be an m-dimensional topological manifold and let A be a subset of X. Furthermore let B be an m-dimensional submanifold of X. We write $\overset{\circ}{B} := B \setminus \partial_0 B$.*[853] *If $\overset{\circ}{B} \subset A$, then for every $n \in \mathbb{N}_0$ and any abelian group G the inclusion induced map*

$$H_n\big(X \setminus \overset{\circ}{B}, A \setminus \overset{\circ}{B}; G\big) \xrightarrow{\cong} H_n(X, A; G)$$

is an isomorphism.

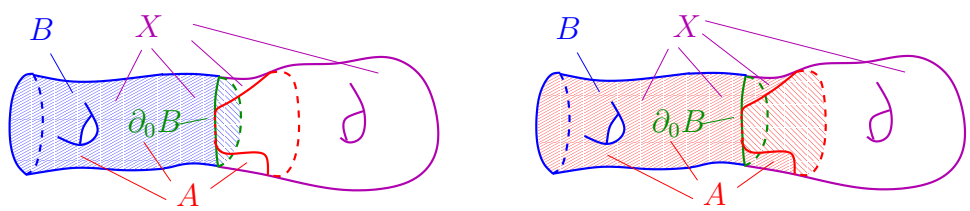


FIGURE 631. Illustration of the Excision Theorem 45.10.

The proof of Theorem 45.10 is verbatim the same as the proof of Theorem 32.7, except that in the proof we need to replace the Collar Neighborhood Theorem 10.10 by the Topological Collar Neighborhood Theorem 45.3 and we need to replace Proposition 3.21 (3) by Lemma 3.6.

45.5. **The double of a (topological) manifold.** In this short section we introduce the double of a topological manifold.

Definition. Given a topological manifold M we refer to

$$DM \;:=\; (M \times \{1\}) \cup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$$

as the *double of* M. We view M as a subset of DM via the inclusion $P \mapsto [(P, 1)]$.

Example. The double of M is given by taking two copies of M and identifying the boundary points "left and right" in the obvious way.

(1) The double of the n-ball $\overline{B}^n$ is homeomorphic to the n-dimensional sphere S^n. We refer to Figure 632 for an illustration.
(2) The double of the torus minus two open disks is the surface of genus 3. We refer to Figure 633 for an illustration.
(3) In Exercise 45.6 we will determine the double of the Möbius band and the Klein bottle minus an open disk.

The following lemma says in particular that the double of a compact (topological) manifold is a closed (topological) manifold.

[853] We refer to page 805 for the definition of $\partial_0 B$.

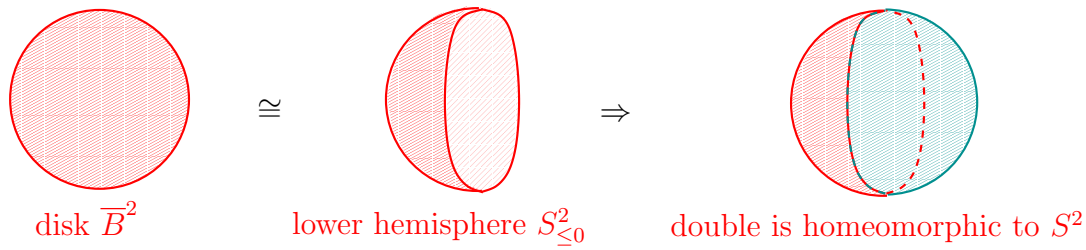


FIGURE 632.

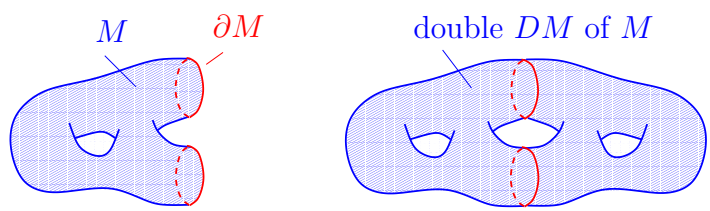


FIGURE 633.

Lemma 45.11. *Let M be a (topological) n-dimensional manifold, then the following statements hold:*

(1) *The double DM is also a (topological) n-dimensional manifold such that $M \subset DM$ is a submanifold.*
(2) *The boundary of the double DM is empty.*
(3) *If M is compact, then the double DM is also compact.*

Proof. For a topological manifold the statements follow immediately from Proposition 45.6 and for a manifold the statements follow immediately from Proposition 10.12. □

Lemma 45.11 says in particular that we can turn any (topological) manifold into a (topological) manifold without boundary by doubling. In particular, if the original (topological) manifold is compact, then the double is a closed (topological) manifold. The doubling trick is often useful when one tries to extend results for closed (topological) manifolds to results for compact (topological) manifolds.

The following definition allows us to relate a topological manifold to its double.

Definition. Let M be a topological manifold. We refer to the map

$$\begin{array}{rcl} r\colon DM = (M \times \{1\} \cup M \times \{2\})/\sim & \to & M \\ [(P,i)] & \mapsto & P \end{array}$$

as the *folding map*. We refer to Figure 634 for an illustration. (Note that it follows easily from Lemma 1.83 (3) that the folding map is continuous.)

The following lemma relates some of the algebraic invariants of a manifold M to the invariants of its double DM.

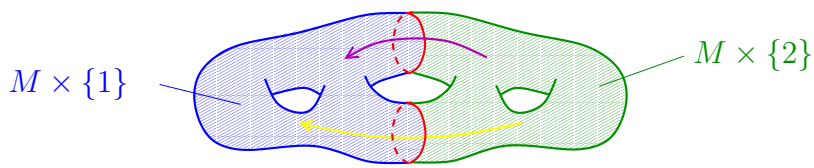


FIGURE 634. Illustration of the folding map $DM \to M$.

Lemma 45.12. *Let M be a topological manifold.*

(1) *The folding map $DM \to M$ is retraction, in particular the submanifold M is a retract of the double DM.*
(2) *Let G be an abelian group. Then for every $i \in \mathbb{N}_0$ the inclusion induced map $H_i(M;G) \to H_i(DM;G)$ is a monomorphism.*
(3) *If ∂M is non-empty and if x_0 is a point in ∂M, then the inclusion induced map $\pi_1(M,x_0) \to \pi_1(DM,x_0)$ is a monomorphism.*

Proof. We denote by $i\colon M = M \times \{1\} \to DM$ the obvious inclusion map. Clearly we have $r \circ i = \mathrm{id}_M$, i.e. r is a retraction and M is a retract of the double DM. The other two statements follow as usual from the functoriality of homology groups and fundamental groups. More precisely, in both cases we have $r_* \circ i_* = \mathrm{id}_*$, i.e. i_* needs to be a monomorphism. $\square$

We conclude this introduction to the double of a topological manifold with the following harmless but at times useful lemma. The lemma follows almost immediately from the definition, hence we leave it to the conscientious reader to fill in the details.

Lemma 45.13. *Let M be a topological manifold. For $i = 1,2$ we view $M \times \{i\}$ as a subset of DM and we denote by $j_i\colon M \to M \times \{i\}$ be the obvious map.*

(1) *For any $k \in \mathbb{N}_0$ and any commutative ring R the map*
$$\begin{array}{rcl} D\colon C_k(M,\partial M;R) & \to & C_k(DM;R) \\ [\underbrace{\sigma}_{\in C_k(M;R)}] & \mapsto & j_{1*}(\sigma) - j_{2*}(\sigma) \end{array}$$
is well-defined.
(2) *The maps from (1) define a chain map, in particular for any $k \in \mathbb{N}_0$ these maps induce a map*
$$D\colon H_k(M,\partial M;R) \to H_k(DM;R)$$
which is called the doubling homomorphism.
(3) *We make the identification $M = j_1(M \times \{1\})$ and we write $M' = j_2(M \times \{2\})$. The doubling homomorphism has the property that the following diagram commutes*
$$\begin{tikzcd} H_k(M,\partial M;R) \arrow[rr, "D"] \arrow[dr] & & H_k(DM;R) \arrow[dl] \\ & H_k(DM,M';R), & \end{tikzcd}$$

where the diagonal maps are the obvious maps.

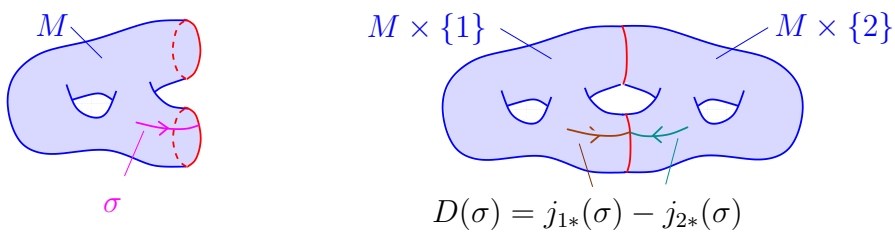


FIGURE 635.

45.6. **The local homology of topological manifolds.** Given a topological space X and a point $x_0 \in X$ the homology groups $H_n(X, X \setminus \{x_0\}; R)$ are often called the *local homology* of X at the point x_0. In this section we will study the local homology of topological manifolds, in particular we want to gain a good understanding of generators of these groups.

Some of the following definitions and conventions we had already introduced earlier.

Definition and notation. Let $n \in \mathbb{N}$.

(1) We equip $\mathbb{R}^n$ with the orientation given by the standard basis $e_1, \dots, e_n$. With this orientation we can view any open subset of $\mathbb{R}^n$ as an oriented manifold.
(2) We write
$$\partial\Delta^n \;=\; \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$
and we refer to $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ as the *interior* of Δ^n.
(3) As on page 935 we say that a basis $\{v_1, \dots, v_n\}$ for $T_P\mathring{\Delta}^n$ is *positive*, if the ordered set of vectors $\{(1, \dots, 1), v_1, \dots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$. This equips $\mathring{\Delta}^n$ with the structure of an oriented manifold.
(4) Given a subset V of $\mathbb{R}^n$ we say that a map $f\colon V \to \mathbb{R}^k$ is *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$-matrix A such that $f(v) = P + Av$ for all $v \in V$.

FIGURE 636.

Example. The projection map

$$\begin{array}{rcl} p\colon \Delta^n & \to & \mathbb{R}^n \\ (x_1,\dots,x_{n+1}) & \mapsto & (x_1,\dots,x_n) \end{array}$$

is an injective affine linear map. One can easily verify that the restriction of this map to the interior $\mathring{\Delta}^n$ is orientation-preserving if and only if n is even. We refer to Figure 637 for an illustration.

projection is orientation-reversing projection is orientation-preserving

FIGURE 637.

We continue with the following lengthy lemma.

Lemma 45.14. *Let R be a commutative ring.*

(1) *For any $k \in \mathbb{N}_0$ we have*

$$H_k(\Delta^n, \partial\Delta^n; R) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ R \cdot [\mathrm{id}_{\Delta^n} \otimes 1], & \text{if } k = n. \end{cases}$$

(2) *Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Then for any $k \in \mathbb{N}_0$ the induced map*

$$\Phi_*\colon H_k(\Delta^n, \partial\Delta^n; R) \;\to\; H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$$

is an isomorphism and this map does not depend on the choice of Φ.

(3) *Let $P \in \mathbb{R}^n$. We pick $r > 0$ such that $B_r^n(0)$ contains P. Then for any $k \in \mathbb{N}_0$ the inclusion induced maps*

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\leftarrow\; H_k\big(\mathbb{R}^n, \mathbb{R}^n \setminus B_r^n(0); R\big) \;\leftarrow\; H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

are isomorphisms and the resulting isomorphism

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\xrightarrow{\cong}\; H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

does not depend on the choice of r.

(4) *Let $P \in \mathbb{R}^n$ be a point. If $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in an affine hyperplane through P, then the induced map*

$$\rho_*\colon H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is given by multiplication by minus -1. The same conclusion also holds if we replace $\mathbb{R}^n$ by any open ball centered at P.

(5) *Let $U \subset \mathbb{R}^n$ be an open set and let $\overline{B}^n_r(Q) \subset U$ be a closed ball. Then for any $P \in B^n_r(Q)$ the inclusion induced maps*

$$\begin{array}{ccc} H_k(U, U \setminus B^n_r(Q); R) & \longrightarrow & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_r(Q); R) \\ \downarrow & & \downarrow \\ H_k(U, U \setminus \{P\}; R) & \longrightarrow & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \end{array}$$

are isomorphisms and they form a commutative diagram.

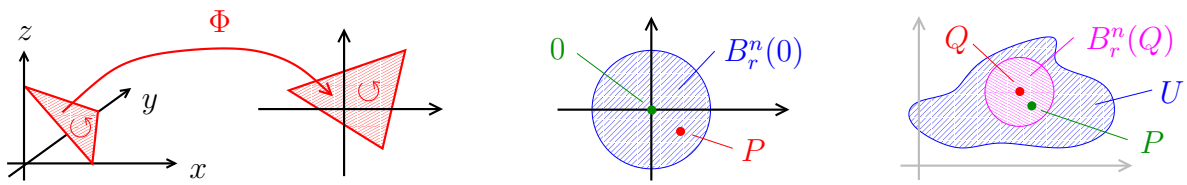


FIGURE 638. Illustration for Lemma 45.14 (2), (3) and (5)

Proof (∗).

(1) For $k \neq n$ the statement is proved the same way as Lemma 32.8. In Lemma 32.9 (2) we saw that the identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $H_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \cong \mathbb{Z}$. Basically the same proof shows that for any commutative ring R the element $\mathrm{id}_{\Delta^n} \otimes 1 \in C_n(\Delta^n, \partial\Delta^n; R)$ represents a generator of the homology group $H_n(\Delta^n, \partial\Delta^n; R) \cong R$.

(2) Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Note that it follows from Proposition 1.40 that Φ is a homeomorphism onto its image.

Then we have the following commutative diagram

$$\begin{array}{ccc} H_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\Phi_*} & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ {\scriptstyle \Phi_*}\downarrow & & \uparrow \\ H_k(\Phi(\Delta^n), \Phi(\partial\Delta^n); R) & \longrightarrow & H_k(\mathbb{R}^n, \Phi(\partial\Delta^n); R) \end{array}$$

where the bottom and the right vertical map are induced by inclusions. Here the left vertical map is an isomorphism since Φ is a homeomorphism onto its image.

Since Φ is an injective affine linear map we see that $\Phi(\Delta^n)$ is a convex subset of $\mathbb{R}^n$ with non-empty interior. By our hypothesis the point 0 lies in the interior of $\Phi(\Delta^n)$. It follows from Lemma 45.9 (5) that the right vertical map is an isomorphism. Clearly $\Phi(\Delta^n) \to \mathbb{R}^n$ is a homotopy equivalence, hence it follows from Corollary 31.16 that the bottom horizontal map is an isomorphism. Since the diagram commutes we have thus now shown, as desired, that the top horizontal map is also an isomorphism.

If $\Psi\colon \Delta^n \to \mathbb{R}^n$ is another such injective affine linear orientation-preserving map, then it follows from elementary linear algebra that there exists some $P \in \mathbb{R}^n$ and a matrix A with *positive* determinant such that

$$\Phi \;=\; (v \mapsto P + A \cdot v) \circ \Psi\colon \Delta^n \to \mathbb{R}^n.$$

We write $\Omega = A \circ \Psi$. it suffices to show that Ψ and Ω and also Ω and Φ induce the same maps $H_k(\Delta^n, \partial\Delta^n; R) \to H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$.

(a) By Lemma 1.96 (1) we know that $\{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) > 0\}$ is path-connected, in particular there exists a path $\gamma\colon [0,1] \to \mathrm{GL}(n, \mathbb{R})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We consider the map

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (Q,t) & \mapsto & \gamma(t) \cdot \Psi(Q). \end{array}$$

Since by hypothesis we have $0 \notin \Phi(\partial\Delta^n)$ and since $\gamma(t)(0) = 0$ we see that $\gamma(t)(\partial\Delta^n) \subset \mathbb{R}^n \setminus \{0\}$ for all t. In particular we deduce that F defines a homotopy between Ψ and Ω as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But by Proposition 31.15 this means that $\Psi_* = \Omega_*$ as maps on relative homology.

(b) Now we compare Ω and $\Phi = (w \mapsto P + w) \circ \Omega$. We consider the map

$$\begin{array}{rcl} G\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (Q,t) & \mapsto & (P \cdot t + \Omega(Q)). \end{array}$$

Note that $0 \in \Omega(\mathring{\Delta}^n) = A(\Psi(\mathring{\Delta}^n))$ and that $0 \in \Phi(\mathring{\Delta}^n)$ which implies by $\Phi = P + \Omega$ that $-P \in \Omega(\mathring{\Delta}^n)$. Since $\Omega(\mathring{\Delta}^n)$ is convex we see that G defines a homotopy between Ω and Φ as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But again by Proposition 31.15 this means that $\Omega_* = \Phi_*$ as maps on relative homology.

(3) The fact that both inclusion induced maps are isomorphisms is an immediate consequence of Lemma 45.9 (2). Now suppose we are given another such $s > 0$. Without loss of generality we can assume that $r \geq s$. We obtain the following commutative diagram:

$$\begin{tikzcd} & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_r(0); R) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[dd] & \\ H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) & & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_s(0); R) \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

All maps are induced by inclusions, so the diagram commutes. By Lemma 45.9 the diagonal maps are isomorphisms. It is now obvious that the composition of the top maps and the composition of the bottom maps agree.

(4) Let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane V through a point P. We denote by $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by addition by P and furthermore we write $W = f^{-1}(V)$. Note that W goes through the origin. Finally we denote by

$\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the hyperplane W. We consider the following diagram

$$\begin{array}{ccccccc}
H_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & H_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & H_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \\
\downarrow{\scriptstyle \sigma_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\
H_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & H_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & H_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R).
\end{array}$$

where μ is the natural map from page 1025 and where the right-hand horizontal maps are the connecting homomorphisms from Corollary 31.14.

It follows from Lemma 32.16 that the vertical map on the left is given by multiplication by -1. It is straightforward to verify that the diagram commutes and that all horizontal maps are isomorphisms. Furthermore the horizontal maps on the top and bottom are identical. It thus follows that the vertical map on the right is also given by multiplication by -1.

(5) It is clear that the diagram commutes since all maps are induced by inclusions. The horizontal maps are isomorphisms by the Excision Theorem 45.8 and the vertical maps are isomorphisms by Lemma 45.9. □

Convention. We use the isomorphisms from Lemma 45.14 to identify the various relative homology modules. In particular, for any open set U of $\mathbb{R}^n$, any open ball $B_r(P)$ with $\overline{B}_r(P) \subset U$, any point $Q \in U$ and any commutative ring R we make the identifications

$$R = H_n(\Delta^n, \partial\Delta^n; R) = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) = H_n(U, U \setminus B_r(P); R) = H_n(U, U \setminus \{Q\}; R).$$

Furthermore, we refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1] \in H_n(\Delta^n, \partial\Delta^n; R)$ as the *standard generator* of the various relative homology modules.

Example. We refer to Figure 639 for an illustration of representatives of the standard generators.[854] Here, for simplicity, given a singular 2-simplex $\Delta^2 \to \mathbb{R}^2$ we only sketch the image of $\partial\Delta^2$ with the given orientation. Finally note that it follows from Lemma 45.14 (4) that the singular simplex illustrated in the middle of Figure 639 represents *minus* the standard generator.

Lemma 45.15. *Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$, let $P \in U$ and let $\sigma\colon \Delta^n \to U$ be a singular n-simplex with the following properties:*

(1) *the point P lies in the image of $\mathring{\Delta}^n$,*
(2) *the restriction of σ to a map $\mathring{\Delta}^n \to U$ is a diffeomorphism onto its image.*

Then $\sigma \otimes 1 \in C_n(U) \otimes R$ represents a generator of $H_n(U, U \setminus \{P\}; R)$.[855] *Furthermore, if*

(3) *the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving,*

[854] At this occasion it is good to remember that if $\sigma\colon \Delta^k \to X$ is a map with $\sigma(\partial\Delta^n) \subset A$, then σ is a cycle in $C_k(X, A)$, i.e. it represents a class in $H_k(X, A)$.

[855] Note that we do not make any assumptions on the behavior of σ restricted to $\partial\Delta^n$. If one looks very carefully, then one realizes that one has to convince oneself that $\sigma(\partial\Delta^n)$ is a subset of $U \setminus \{P\}$, i.e. one has to show that it follows from (1) and (2), that $P \notin \sigma(\partial\Delta^n)$. How can one prove this statement?

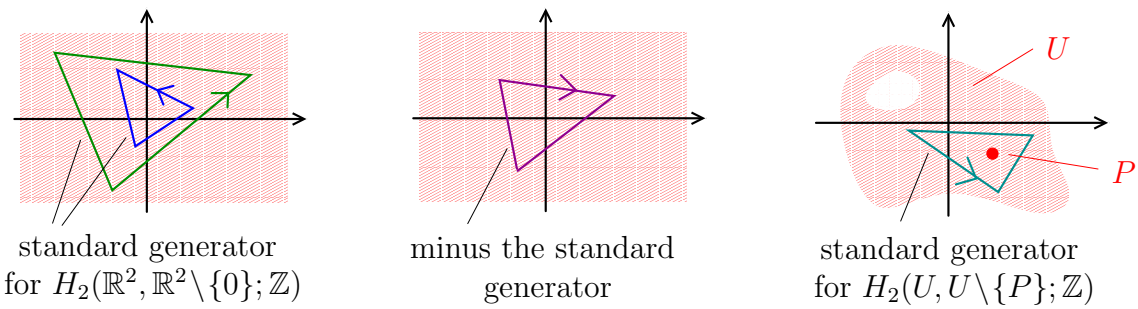


FIGURE 639.

then $\sigma \otimes 1$ represents the standard generator of $H_n(U, U \setminus \{P\}; R)$.

We refer to Figure 640 for an illustration of Lemma 45.15.

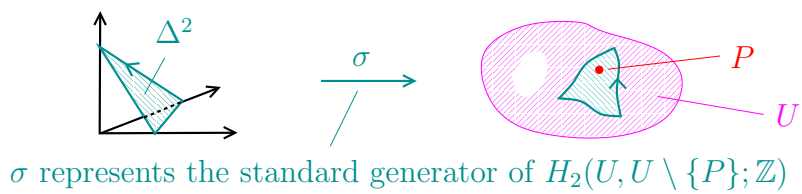


FIGURE 640.

Proof. We denote by $Q \in \mathring{\Delta}^n$ the point with $\sigma(Q) = P$. To simplify the notation we work throughout the proof with the coefficients $R = \mathbb{Z}$. Using the fact that the inclusion induced map

$$H_n(U, U \setminus \{P\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

is an isomorphism by the Excision Theorem 45.8 we can without loss of generality assume that $U = \mathbb{R}^n$. As always we denote by $D_Q\sigma \colon T_Q\mathring{\Delta}^n \to T_P U = \mathbb{R}^n$ the differential of σ at the point Q. We denote by

$$\begin{array}{rcl} f \colon \Delta^n & \to & \mathbb{R}^n \\ x & \mapsto & (D_Q\sigma)\underbrace{(x - Q)}_{\in T_Q\Delta^n} + P \end{array}$$

the linear approximation of σ at the point Q. We make the following two observations:

(1) Since $\sigma \colon \mathring{\Delta}^n \to U$ is a diffeomorphism onto its image we obtain that $D_Q\sigma$ is an isomorphism of vector spaces, in particular f is an injective affine linear map.
(2) Basically the same argument as in the proof of Lemma 32.25 shows that there exists an open neighborhood V of Q in $\mathring{\Delta}^n$ such that the restrictions of σ and f to maps

$$\sigma, f \colon (V, V \setminus \{Q\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

are homotopic.

Now consider the following diagram

$$\begin{array}{ccccccccc}
H_n(\Delta^n,\partial\Delta^n) & \longrightarrow & H_n(\Delta^n,\Delta^n\backslash\{Q\}) & \leftarrow & H_n(V,V\backslash\{Q\}) & \xrightarrow{=} & H_n(V,V\backslash\{Q\}) & \rightarrow & H_n(\Delta^n,\Delta^n\backslash\{Q\}) \\
\downarrow{\sigma_*} & & \downarrow{\sigma_*} & & \downarrow{\sigma_*} & & \downarrow{f_*} & & \downarrow{f_*} \\
H_n(\mathbb{R}^n,\mathbb{R}^n\backslash\{P\}) & \xrightarrow{=} & H_n(\mathbb{R}^n,\mathbb{R}^n\backslash\{P\}) & \xrightarrow{=} & H_n(\mathbb{R}^n,\mathbb{R}^n\backslash\{P\}) & \xrightarrow{=} & H_n(\mathbb{R}^n,\mathbb{R}^n\backslash\{P\}) & \xrightarrow{=} & H_n(\mathbb{R}^n,\mathbb{R}^n\backslash\{P\}).
\end{array}$$

It is clear that the first two squares and the last square commute. Furthermore the third square commutes by the above observation (2) together with Proposition 31.15. As in the proof of Lemma 45.14 (2) we see that the last vertical map is an isomorphism. It now follows easily that $\sigma \in C_n(\mathbb{R}^n)$ represents a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$.

Finally suppose that the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving. In this case, by definition of orientation-preserving, the map f is also orientation-preserving, therefore by the above commutative diagram the singular n-simplex σ does indeed represent the standard generator of $H_n(U, U \setminus \{P\})$. □

We continue with the following lemma which is almost identical to Lemma 32.3.

Lemma 45.16. *Let M be an n-dimensional topological manifold and let R be a commutative ring. Then for every $x \in M \setminus \partial M$ and any $k \in \mathbb{N}_0$ we have*

$$H_k(M, M \setminus \{x\}; R) \;\cong\; \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\Phi\colon U \to V$ is a chart around x and if $\sigma\colon \Delta^n \to V$ represents a generator of $H_n(V, V \setminus \{\Phi(x)\}; R)$, then $\Phi^{-1} \circ \sigma$ represents a generator of $H_n(M, M \setminus \{x\}; R)$.

We refer to Figure 641 for an illustration of Lemma 45.16.

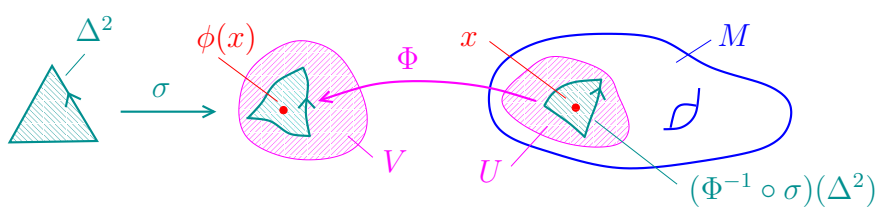


FIGURE 641.

Remark. Let $n \in \mathbb{N}_0$. A topological space X that has the property that for each $x \in X$ we have

$$H_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

is often called a *homology manifold*. By Lemma 45.16 such topological spaces can be viewed as generalizations of closed topological manifolds.

Let Y be an n-dimensional homology sphere.[856] By Proposition 32.14 we know that the suspension $\Sigma(Y)$ of Y is a homology manifold. On the other hand we saw in Exercise 28.7 that the suspension $\Sigma(Y)$ is *not* a topological manifold.[857] The famous Double Suspension Theorem of Jim Cannon [Can] and Richard Edwards [Edw] says that the double suspension $\Sigma(\Sigma(Y))$ of any n-dimensional homology sphere is homeomorphic to S^{n+2}. We will not make use of this deep theorem.

The proof of Lemma 45.16 is basically identical to the proof of Lemma 32.3. We recall the argument just in case some readers might have forgotten.

Proof. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $x \in M \setminus \partial M$. Since $x \notin \partial M$ we can pick a chart $\Phi\colon U \to V$ such that V is an open subset of $\mathbb{R}^n$. We write $y = \Phi(x)$. For any $k \in \mathbb{N}_0$ we then have

$$\begin{array}{ccccc} H_k(M, M\setminus\{x\}; R) & \overset{\cong}{\longleftarrow} & H_k(U, U\setminus\{x\}; R) & \xrightarrow[\cong]{\Phi_*} & H_k(V, V\setminus\{y\}; R) & \xrightarrow{\cong} & \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\ & \uparrow & \uparrow & & \uparrow & & \\ & \text{Excision Theorem 45.8} & \text{since } \Phi \text{ is a homeomorphism} & & \text{see Lemma 45.14 (5)} & & \end{array}$$

The last statement regarding generators now follows immediately from the above isomorphisms. □

45.7. **R-Sections on topological manifolds.** In Section 3.8 we defined the notion of an oriented manifold via orientations of the tangent spaces. The notion of a tangent space requires the smooth structure on a manifold, put differently, topological manifolds do not come with tangent spaces. In the next section we will see that we can still define the notion of an orientation of a topological manifold. Furthermore we will see that for manifolds this new notion of an orientation will be equivalent to the previous notion of an orientation.

The key to defining the new notion of orientation is the following, initially rather obscure definition.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An *R-section for M at $x \in M \setminus \partial M$* is an element $\mu_x \in H_n(M, M\setminus\{x\}; R) \cong R$.
(2) An *R-section for M* is a choice for each $x \in M \setminus \partial M$ of an R-section μ_x such that the following "continuity condition" is satisfied: for each $x \in M$ there exists an open neighborhood U of x in $M \setminus \partial M$ and an element $\mu_U \in H_n(M, M\setminus U; R)$ such that for each $y \in U$ the image of μ_U under the map

$$H_n(M, M\setminus U; R) \;\to\; H_n(M, M\setminus\{y\}; R)$$

[856]Recall that by the definition on page 992 an n-dimensional homology sphere is an n-dimensional topological manifold X such that for any $k \in \mathbb{N}_0$ we have $H_k(X;\mathbb{Z}) \cong H_k(S^n;\mathbb{Z})$. On page 993 we saw that there exists a 3-dimensional homology sphere with non-trivial fundamental group, namely the Poincaré homology sphere.

[857]To be super-precise we showed that it is not an $(n+1)$-dimensional topological manifold, but it follows easily from Proposition 32.5 that if $\Sigma(Y)$ was a topological manifold it would need to be $(n+1)$-dimensional.

equals μ_y.

(3) We denote by $\Gamma_R(M)$ the set of all R-sections of M.

Remark. Let M be an n-dimensional topological manifold and let R be a commutative ring. An example of an R-section for M is given by assigning to each $x \in M \setminus \partial M$ the zero element in $H_n(M, M \setminus \{x\}; R) \cong R$. It is clear that R-sections can be added and that one can multiply them by an element in R. This shows that the set $\Gamma_R(M)$ is an R-module in an obvious way.

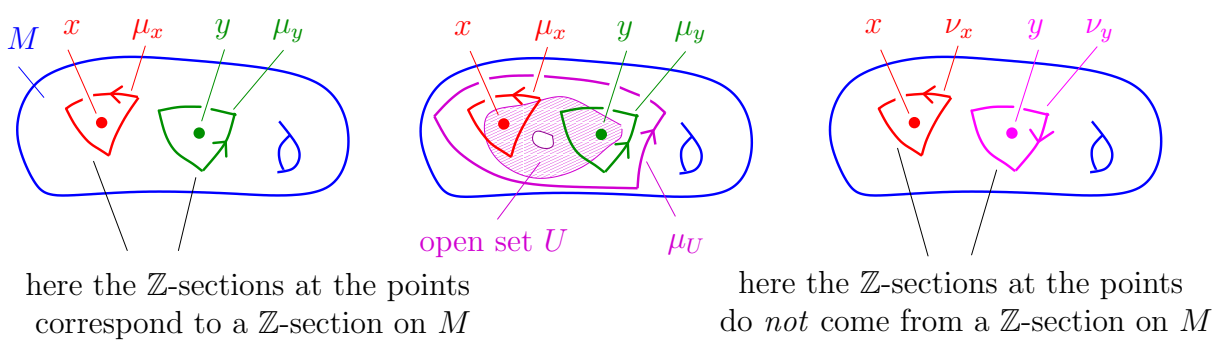
Example. In Figure 642 we attempt to illustrate the definition of a $\mathbb{Z}$-section on a surface M. Here, for simplicity, given a singular 2-simplex $\Delta^2 \to M$ we only sketch the image of $\partial\Delta^2$ with the given orientation.

(1) On the left of Figure 642 we show two points x and y with singular 2-simplices representing $\mu_x \in H_2(M, M \setminus \{x\}; \mathbb{Z})$ and $\mu_y \in H_2(M, M \setminus \{y\}; \mathbb{Z})$. Furthermore we sketch an open set U that contains x and y and an element $\mu_U \in H_2(M, M \setminus U; \mathbb{Z})$ that gets sent to μ_x and μ_y via the obvious maps[858]

$$H_n(M, M\setminus U; \mathbb{Z}) \to H_n(M, M\setminus\{x\}; \mathbb{Z}) \text{ and } H_n(M, M\setminus U; \mathbb{Z}) \to H_n(M, M\setminus\{y\}; \mathbb{Z}).$$

In fact one can show that these $\mathbb{Z}$-sections at x and y form part of a $\mathbb{Z}$-section of M.

(2) To the right of Figure 642 we show two $\mathbb{Z}$-sections ν_x and ν_y that cannot be part of a $\mathbb{Z}$-section of M.



here the $\mathbb{Z}$-sections at the points correspond to a $\mathbb{Z}$-section on M

here the $\mathbb{Z}$-sections at the points do *not* come from a $\mathbb{Z}$-section on M

FIGURE 642.

The following proposition will be used on many occasions.

Proposition 45.17. *Let M be a connected topological manifold.*

(1) *Let R be a commutative ring and let $\{\alpha_x\}_{x\in M\setminus\partial M}$ and $\{\beta_x\}_{x\in M\setminus\partial M}$ be two R-sections for M. If there exists an $x \in M \setminus \partial M$ with $\alpha_x = \beta_x$, then the two R-sections agree.*

[858]Note that in this example the group $H_2(M, M \setminus U; \mathbb{Z})$ is not isomorphic to $\mathbb{Z}$, i.e. the two maps are not isomorphisms.

(2) *Let* $\{\alpha_x\}_{x\in M\setminus\partial M}$ *be a* $\mathbb{Z}$*-section for* M*. Suppose there exists a* $y \in M \setminus \partial M$*, an* $m \in \mathbb{Z}$ *and a* $\beta_y \in H_n(M, M \setminus \{y\}; \mathbb{Z})$ *such that* $\alpha_y = m \cdot \beta_y$*, then there exists a* $\mathbb{Z}$*-section* $\{\beta_x\}_{x\in M\setminus\partial M}$ *such that* $\alpha_x = m \cdot \beta_x$ *for every* $x \in M \setminus \partial M$*.*

(3) *Let* $\{\alpha_x\}_{x\in M\setminus\partial M}$ *be a* $\mathbb{Z}$*-section for* M*. If there exists a* $y \in M\setminus\partial M$ *such that* α_y *is a generator of* $H_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$*, then* α_x *is a generator of* $H_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$ *for every* $x \in M \setminus \partial M$*.*

Before we provide the proof of Proposition 45.17 we need to prove one little lemma which in turn requires the following definition.

Definition. Let M be an n-dimensional topological manifold. We say that $Y \subset M$ is a *small ball* in M if there exists a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$. We illustrate the definition of a small ball in Figure 643.

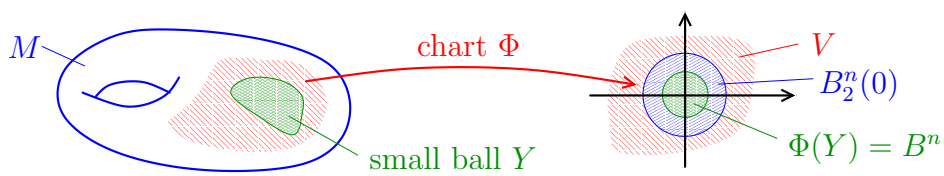


FIGURE 643.

Lemma 45.18. *Let* M *be an* n*-dimensional topological manifold.*

(1) *Given any point* $x \in M \setminus \partial M$ *and any neighborhood* U *of* x *there exists a small ball contained in* U*.*

(2) *Let* R *be a commutative ring. For any point* x *in a small ball* Y *and any* $k \in \mathbb{N}_0$ *the map*

$$H_k(M, M \setminus Y; R) \;\to\; H_k(M, M \setminus \{x\}; R)$$

is an isomorphism.

Proof. Let M be an n-dimensional topological manifold.

(1) The proof of the first statement is elementary.

(2) Now let x be a point that is contained in a small ball Y. To simplify the notation we drop the commutative ring R from the notation. We consider the long exact sequence of the triple $(M, M \setminus \{x\}, M \setminus Y)$ given by Corollary 31.14:

$$\to H_k(M\setminus\{x\}, M\setminus Y) \to H_k(M, M\setminus Y) \to H_k(M, M\setminus\{x\}) \to H_{k-1}(M\setminus\{x\}, M\setminus Y) \to$$

We see immediately that it suffices to show that all the relative homology groups $H_k(M \setminus \{x\}, M \setminus Y)$ vanish. Since Y is a small ball we can pick a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$. We pick $s \in (1, 2)$ and we write

$Z = \Phi^{-1}(B^n_s(0))$. Then

$$\underset{\substack{\uparrow \\ \text{Excision Theorem 45.8}}}{H_k(M\setminus\{x\}, M\setminus Y) \xleftarrow{\cong} H_k(Z\setminus\{x\}, Z\setminus Y)} \xrightarrow[\cong]{\Phi_*} \underset{\substack{\uparrow \\ \text{by Corollary 31.16, since } B^n_s(0)\setminus B^n \text{ is a} \\ \text{deformation retract of } B^n_s(0)\setminus\{\Phi(x)\}}}{H_k(B^n_s(0)\setminus\{\Phi(x)\}, B^n_s(0)\setminus B^n) = 0.}$$

We refer to Figure 644 for an illustration of the proof. □

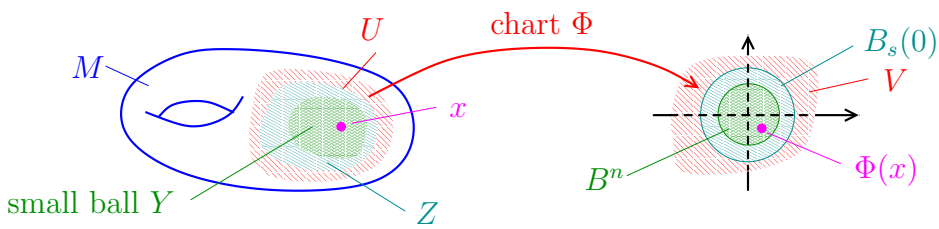


FIGURE 644. Illustration for the proof of Lemma 45.18.

Now we can provide the proof of Proposition 45.17.

Proof of Proposition 45.17. Let M be a connected n-dimensional topological manifold and let R be a commutative ring.

(1) Let $\{\alpha_x\}_{x\in M\setminus\partial M}$ and $\{\beta_x\}_{x\in M\setminus\partial M}$ be two R-sections for M such that there exists an $x \in M \setminus \partial M$ with $\alpha_x = \beta_x$. We set

$$V \;:=\; \{y \in M \setminus \partial M \,|\, \alpha_y = \beta_y\} \quad \text{and} \quad W \;:=\; \{y \in M \setminus \partial M \,|\, \alpha_y \neq \beta_y\}.$$

Since M is connected we also know that $M \setminus \partial M$ is connected.[859]. Since V is non-empty it suffices to show that V and W are both open. It suffices to prove the following claim.

Claim. Let $x \in M\setminus\partial M$. There exists an open neighborhood U of x such that either α_z and β_z agree for every $z \in U$ or disagree for every $z \in U$.

Let $x \in M \setminus \partial M$. It follows easily from the definition of an R-section and Lemma 45.18 (1) that there exists a small ball U in $M \setminus \partial M$ that contains x and elements $\alpha_U, \beta_U \in H_n(M, M \setminus U; R)$ such that for each $z \in U$ the image of α_U respectively β_U under the map

$$H_n(M, M \setminus U; R) \;\to\; H_n(M, M \setminus \{z\}; R)$$

equals α_z respectively β_z. By Lemma 45.18 this map is an isomorphism. So either α_z and β_z agree on all of U or they disagree on all of U. This concludes the proof of the claim.

[859]We will prove this statement in Exercise 45.4.

(2) Let $n \in \mathbb{N}$. We start out with the following definition: Given an element g of a group G with $G \cong \mathbb{Z}$ we say that g is divisible by n if there exists an $h \in G$ with $n \cdot h = g$. In this case we write $g/n := h$. Note that h is well-defined since $G \cong \mathbb{Z}$.

Let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be a $\mathbb{Z}$-section for M. Given $n \in \mathbb{N}$ we consider the two sets

$$V \;:=\; \{y \in M \setminus \partial M \,|\, \alpha_y \text{ is divisible by } n\}$$

and

$$W \;:=\; \{y \in M \setminus \partial M \,|\, \alpha_y \text{ is not divisible by } n\}.$$

As in (1) one shows that V and W are open.

If V is non-empty, then $V = M \setminus \partial M$ and it is straightforward to verify that $\beta_y := \alpha_y/n$, $y \in M \setminus \partial M$, is in fact a $\mathbb{Z}$-section. We leave the details as an exercise to the reader.

(3) Let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be a $\mathbb{Z}$-section for M. Let $y \in M \setminus \partial M$ be a point such that α_y is a generator of $H_n(M, M \setminus \{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Now let $z \in M \setminus \partial M$. If α_z is not a generator, then there exists a $\beta_z \in H_n(M, M \setminus \{z\}; \mathbb{Z}) \cong \mathbb{Z}$ and some $n \geq 2$ such that $\alpha_z = n \cdot \beta_z$. By (2) we can extend β_z to a $\mathbb{Z}$-section $\{\beta_x\}_{x \in M \setminus \partial M}$ such that $\alpha_x = n \cdot \beta_x$ for every $x \in M \setminus \partial M$. But this implies that α_y is not a generator. So we have obtained a contradiction. □

45.8. **Orientations of topological manifolds.** After the technical preparations from the last section we can now finally introduce the notion of an orientation of a topological manifold.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at a point $x \in M \setminus \partial M$ that is in fact a generator of the R-module $H_n(M, M \setminus \{x\}; R) \cong R$ is called an *R-orientation for M at $x \in M$*.
(2) An *R-section for M* that is an R-orientation at each point $x \in M \setminus \partial M$ is called an *R-orientation of M*.
(3) The topological manifold M is called *R-orientable* if it admits an R-orientation.
(4) Let M and N be two n-dimensional topological manifolds that are equipped with R-orientations $\{\alpha_x\}_{x \in M \setminus \partial M}$ and $\{\beta_x\}_{x \in N \setminus \partial N}$. Let $f \colon M \to N$ be a map that is a local homeomorphism. We say the map f is *orientation-preserving* if the following holds: for any open set $U \subset M$, such that $f \colon U \to f(U)$ is a homeomorphism and such that $f(U)$ is open, and any $x \in U \setminus \partial M$ the image of α_x under the map[860] [861] [862]

$$\begin{array}{ccccc} & \text{these maps are isomorphisms by the Excision Theorem 45.8} & & & \\ & \downarrow & & & \downarrow \\ H_n(M, M \setminus \{x\}) & \xleftarrow{\cong} & H_n(U, U \setminus \{x\}) & \xrightarrow[\cong]{f_*} & H_n(f(U), f(U) \setminus \{f(x)\}) \xrightarrow{\cong} H_n(N, N \setminus \{f(x)\}). \end{array}$$

[860] Since $f \colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$.

[861] It follows from the fact that $f \colon U \to f(U)$ is a homeomorphism, the fact that $x \in U \setminus \partial U$ and Propositionprop:boundary-of-n-dimensional-manifold that $f(x) \notin \partial N$.

[862] We suppress the R-coefficients from the notation for space reasons.

equals $\beta_{f(x)}$. Otherwise we call f *orientation-reversing.*

We leave the proof of the following lemma as an amusing exercise to the reader.

Lemma 45.19. *Let M and N be two n-dimensional topological manifolds and furthermore let $f, g\colon M \to N$ be two homeomorphisms. If f and g are isotopic, then f is orientation-preserving if and only if g is orientation-preserving.*

Before we start with the discussions of orientations in detail we get the following technical lemma out of the way.

Lemma 45.20. *Let M be a connected n-dimensional topological manifold.*

(1) *Let R be a commutative ring. If M is R-orientable, then for any $x \in M \setminus \partial M$ the map*
$$\begin{array}{rcl} \Phi\colon \Gamma_R(M) & \to & \overbrace{H_n(M, M \setminus \{x\}; R)}^{\cong R} \\ \{\alpha_y\}_{y \in M \setminus \partial M} & \mapsto & \alpha_x \end{array}$$
is an isomorphism of R-modules.

(2) *If M is non-orientable, then $\Gamma_{\mathbb{Z}}(M) = 0$.*

Proof. Let M be a connected n-dimensional topological manifold.

(1) Let R be a commutative ring such that M is R-orientable. We pick an R-orientation $\{\mu_y\}_{y \in M \setminus \partial M}$ for M. Now let $x \in M \setminus \partial M$. By definition μ_x is a generator of the R-module $H_n(M, M \setminus \{x\}; R) \cong R$, i.e. every element in $H_n(M, M \setminus \{x\}; R)$ is of the form $r \cdot \mu_x$ for some unique $r \in R$. We consider the map
$$\begin{array}{rcl} \Psi\colon H_n(M, M \setminus \{x\}; R) & \to & \Gamma_R(M) \\ r \cdot \mu_x & \mapsto & \{r \cdot \mu_y\}_{y \in M \setminus \partial M}. \end{array}$$
Evidently we have $\Phi \circ \Psi = \mathrm{id}$. It follows from our hypothesis that M is connected and from Proposition 45.17 (1) that $\Psi \circ \Phi$ is also the identity.

(2) Now we suppose that M admits a non-zero $\mathbb{Z}$-section $\{\alpha_x\}_{x \in M \setminus \partial M}$. We need to show that M is orientable. We pick $y \in M \setminus \partial M$ such that $\alpha_y \neq 0$. We pick a generator $\beta_y \in H_n(M, M \setminus \{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Then $\alpha_y = n \cdot \beta_y$ for some $n \in \mathbb{Z}$. It follows again from our hypothesis that M is connected and from Proposition 45.17 (2) that there exists a $\mathbb{Z}$-section $\{\beta_x\}_{x \in M \setminus \partial M}$ such that $\alpha_x = n \cdot \beta_x$ for all $x \in M \setminus \partial M$.

It follows immediately from Proposition 45.17 (3) that $\{\beta_x\}_{x \in M \setminus \partial M}$ is an orientation for M. □

In the following lemma we collect some basic facts about $\mathbb{Z}$-orientations.

Lemma 45.21. *Let M be a topological manifold of dimension ≥ 1.*

(1) *For any point $x \in M \setminus \partial M$ there exist precisely two $\mathbb{Z}$-orientations for M at x.*

(2) *If M is connected and orientable, then M admits precisely two $\mathbb{Z}$-orientations.*

Proof. The first statement is an immediate consequence of the fact, proved in Lemma 45.16, that for any $x \in M \setminus \partial M$ we have $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$. The second statement follows easily from Proposition 45.17 and the definitions. □

Examples.

(1) Let V be a connected open subset of $\mathbb{R}^n$. For any $x \in V$ we define

$$\mu_x \ := \ \text{the standard generator of } H_n(V, V \setminus \{x\}; \mathbb{Z}) \text{ as defined on page 1092.}$$

We claim that the μ_x define a $\mathbb{Z}$-orientation for V. Indeed, let $x \in V$. We pick an $r > 0$ such that $\overline{B}_r(x) \subset V$. We set $U := B_r(x)$. We denote by μ_U the standard generator of $H_n(V, V \setminus U; \mathbb{Z})$ as defined on page 1092. It follows from Lemma 45.14 (5) that μ_U has the desired property. We refer to this $\mathbb{Z}$-orientation as the *standard $\mathbb{Z}$-orientation* of V.

By Lemma 45.21 (2) the topological manifold V admits precisely two different $\mathbb{Z}$-orientations. We have already given one orientation above. By Lemma 45.14 the other one is given by a reflection of the standard generators in hyperplanes. In Figure 645 we sketch the two different orientations of V.

standard generators for $H_2(V, V \setminus \{x\}; \mathbb{Z})$ — the negative of the standard generators

FIGURE 645.

(2) In Figure 646 we sketch the two different orientations of the torus.

FIGURE 646.

(3) Let R be a commutative ring. Recall that for $n \geq 1$ we denote by $[S^n] \in H_n(S^n; \mathbb{Z})$ the standard generator that we introduced on page 814. It follows from the Universal Coefficient Theorem 42.19 together with Lemma 42.17 (3) that the natural map

$$H_n(S^n) \otimes R \ \to \ H_n(S^n; R)$$

is an isomorphism. By a slight abuse of notation we denote by $[S^n]$ also the image of $[S^n] \otimes 1 \in H_n(S^n; R)$ under the above map. Note that $[S^n] \in H_n(S^n; R) \cong R$ is a generator. We refer to it as the *standard generator of* $H_n(S^n; R)$. For any $x \in S^n$ it

follows easily from the long exact sequence of homology groups with R-coefficients of the pair $(S^n, S^n \setminus \{x\})$ that the natural map

$$H_n(S^n; R) \;\to\; H_n(S^n, S^n \setminus \{x\}; R)$$

is an isomorphism. We denote by $\mu_x \in H_n(S^n, S^n \setminus \{x\}; R)$ the image of $[S^n]$. By the above μ_x is an R-orientation at x. In fact all these R-orientations $\{\mu_x\}_{x \in S^n}$ define an R-orientation of S^n. Indeed, in the definition on page 1099 we just need to set $U = S^n$ and $\mu_U = [S^n] \in H_n(S^n, \underbrace{S^n \setminus S^n}_{=\varnothing}; \mathbb{Z})$. We refer to this orientation of S^n as the *standard R-orientation of S^n*.

The following proposition shows in particular that for manifolds the notion of orientability from Section 3.8 agrees with the above notion of orientability.

Proposition 45.22.

(1) *Give any manifold M there exists a bijection*

$$\left\{ \begin{array}{c} \textit{orientations on } M \textit{ in} \\ \textit{the sense of Section 3.8} \end{array} \right\} \xrightarrow{\cong} \{\, \mathbb{Z}\textit{-orientations on } M \,\}$$

such that the following two statements hold:

(a) *For any open subsets of some $\mathbb{R}^n$ the above bijections sends the standard orientation from page 199 to the above standard $\mathbb{Z}$-orientation.*

(b) *If $f\colon M \to N$ is a local diffeomorphism between oriented connected manifolds, then the following two statements are equivalent:*

(i) *f preserves the orientations on M and N in the sense of Section 3.8,*

(ii) *f preserves the $\mathbb{Z}$-orientations on M and N.*

(2) *Given an n-dimensional manifold M we have*

$$M \textit{ is orientable in the sense of Section 3.8} \iff M \textit{ is } \mathbb{Z}\textit{-orientable.}$$

In the proof of Proposition 45.22 we will use the following lemma which is basically just a slight reformulation of Lemma 32.25.

Lemma 45.23. *Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then for any $x \in V$ we have*

$$\begin{array}{c} \Psi \textit{ preserves the standard} \\ \mathbb{Z}\textit{-orientation of } V \textit{ at the point } x \end{array} \iff \begin{array}{c} \Psi \textit{ preserves the orientation of } V \textit{ in the} \\ \textit{sense of Section 3.8 at the point } x. \end{array}$$

Now we can provide the proof of Proposition 45.22.

Sketch of the proof of Proposition 45.22. First note that statement (2) is just a reader-friendly condensation of statement (1). Thus we need to prove statement (1). We define the promised bijections. Thus let M be an n-dimensional manifold. We first suppose that M is oriented in the sense of Section 3.8. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving in the sense of Section 3.8. We define

$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } H_n(V, V \setminus \{\Phi(x)\}; \mathbb{Z}) \text{ as defined on page 1092}\big).$$

It follows from Lemma 45.23 that this definition does not depend on the choice of the orientation-preserving chart Φ. We claim that the classes $\{\mu_x\}_{x\in M\setminus\partial M}$ form a $\mathbb{Z}$-orientation for M. We only have to show that these classes satisfy the "continuity condition". We claim that for any $x \in M\setminus\partial$ a sufficiently small ball U containing x has the desired property. The proof of this fact is very similar to the arguments used in the proof of Proposition 45.17. We leave the details as an exercise to the reader.

Now we define the inverse map. So suppose we are given a $\mathbb{Z}$-orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ for M. By Lemma 3.37 we only need to find an orientation for $M \setminus \partial M$. Thus let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x such that V is connected. It follows from Lemma 45.14 that, after possibly reflecting V in a hyperplane, we can assume that Φ preserves the $\mathbb{Z}$-orientation at x, i.e. it sends the given $\mathbb{Z}$-orientation μ_x to the standard orientation of the open subset V of $\mathbb{R}^n$ that we introduced on page 1101. We define

$$\text{orientation of } T_xM := \Phi^*(\text{standard orientation of } T_{\Phi(x)}V = \mathbb{R}^n).$$

It follows from Lemma 45.23 that this definition does not depend on the choice of Φ. As before, using small balls, one can show that this defines an orientation of M in the sense of Section 3.8. We leave the verification of the details once again as an exercise to the reader.

It is basically clear, that by construction, the bijections have the desired properties (a) and (b). This concludes the proof of statement (1). □

Convention. Proposition 45.22 says in particular that for manifolds the notion of orientability that we introduced in Section 3.8 agrees with the notion of $\mathbb{Z}$-orientability that we had just introduced. For $R = \mathbb{Z}$ we will therefore often drop the "R" from the definitions on page 1099, i.e. often we just say *orientation at a point*, *orientation* and *orientable*.

Remark. It follows basically from the definition that this definitions that under the equivalence of Proposition 45.22 the orientations of S^n introduced on page 200 and page 1102 agree.

The following proposition shows that the notion of R-orientability is more flexible than the previous notion of orientability.

Proposition 45.24. *Every topological manifold is $\mathbb{F}_2$-orientable.*

Example. Proposition 45.24 says in particular that the real projective plane, the Klein bottle and the Möbius band are $\mathbb{F}_2$-orientable.

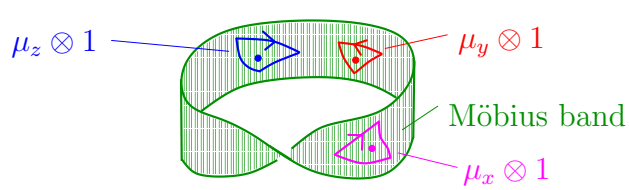


FIGURE 647.

Proof. Let M be an n-dimensional topological manifold. Let $x \in M \setminus \partial M$. By Lemma 45.16 we know that $H_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. We define μ_x to be the unique non-trivial element in $H_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. Using Lemma 45.18 it is straightforward to show that the continuity condition is satisfied, in particular we have defined an $\mathbb{F}_2$-orientation on M. $\square$

Lemma 45.25. *Let M be a topological manifold. If M is orientable, then it is R-orientable for any commutative ring R.*

Proof. Let M be an n-dimensional orientable topological manifold and let R be a commutative ring. Let $x \in M \setminus \partial M$. It follows from the Universal Coefficient Theorem 42.19, together with the fact that $H_{n-1}(M, M \setminus \{x\}; \mathbb{Z}) = 0$ obtained in Lemma 45.16, that the natural map

$$\begin{array}{rcl} \mu\colon H_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes R & \to & H_n(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{k} [\sigma_i] \otimes r_i & \mapsto & \left[\sum_{i=1}^{k} \sigma_i \otimes r_i\right] \end{array}$$

is an isomorphism.

Now we pick an orientation $\{\alpha_x\}_{x \in M \setminus \partial M}$ for M. For each $x \in M \setminus \partial M$ we denote by $\widetilde{\alpha}_x \in H_n(M, M \setminus \{x\}; R)$ the image of $\alpha_x \otimes 1$ under the above isomorphism. It is straightforward to verify that these generators $\{\widetilde{\alpha}_x\}_{x \in M \setminus \partial M}$ define an R-orientation for the topological manifold M. $\square$

We continue with the following proposition which is a variation of Proposition 8.3.

Proposition 45.26. *Let M be a connected non-orientable topological manifold without boundary. The following two statements hold:*

1. *There exists a connected 2-fold covering $p\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable.*
2. *If $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ are two connected 2-fold coverings such that $\widetilde{M}$ and $\widehat{M}$ are orientable, then there exists a homeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \widehat{M} \arrow[dl, "q"] \\ & M. & \end{tikzcd}$$

Remark. The statement of Proposition 45.26 also holds for connected non-orientable topological manifolds *with boundary*. How would you prove this more general statement? We will only need Proposition 45.26 for closed topological manifolds, so we will not attempt to prove the case of topological manifolds with boundary.

Definition. Let M be a connected non-orientable topological manifold without boundary. We refer to the connected 2-fold covering $p\colon \widetilde{M} \to M$ from Proposition 45.26, which is unique up to equivalence, as the *orientation covering* of M.

Sketch of proof. The proof is almost identical to the proof of Proposition 8.3. First let W be a topological manifold with empty boundary. We consider the set

$$\widetilde{W} := \{\mu_x \mid x \in W \text{ and } \mu_x \text{ is a } \mathbb{Z}\text{-orientation for } W \text{ at the point } x\}$$

and the projection map $p\colon \widetilde{W} \to W$ given by $\mu_x \mapsto x$. By hypothesis W has no boundary. It thus follows from Lemma 45.21 (1) that for every $x \in W$ the preimage $p^{-1}(x)$ has precisely two elements.

We equip $\widetilde{W}$ with a topology as in the proof of Proposition 8.3. It is then straightforward to see that $p\colon \widetilde{W} \to W$ is a 2-fold covering. Similar to the proof of Proposition 8.3 one can show that $\widetilde{W}$ is orientable.

Furthermore, if W is connected, then almost the same argument as in the proof of Proposition 8.3 shows that $\widetilde{W}$ is connected if and only if W is non-orientable. The conclusion of the proof of the proposition is, not surprisingly, almost verbatim the same as the denouement in the proof of Proposition 8.3. We leave it to the reader to flesh out all the details. □

Finally we conclude this section with the following corollary which is closely related to Corollary 8.4.

Corollary 45.27. *Let M be a connected topological manifold without boundary. If M is non-orientable, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.*

Proof. The proof of the corollary is basically identical to the proof of Corollary 8.4, we just need to replace Proposition 8.3 by Proposition 45.26. □

45.9. **Induced orientations.** (∗) In this final section we will show that various constructions of new topological manifolds out of a given oriented topological manifold are naturally equipped with orientations. For the most part the statements sound reasonable and familiar from the setting of manifolds. Nonetheless, the constructions of the induced representations for topological manifolds are at times quite different from the constructions of induced orientations for manifolds. Truth be told, the reader will not miss out on much by just moving on to the next chapter.

Lemma 45.28. *Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation.*

(1) *Let N be a topological manifold and let $f\colon N \to M$ be a map. If f is a local homeomorphism, then there exists a unique R-orientation on N such that f is orientation-preserving.*
(2) *If N is a submanifold of codimension zero (for example N could be an open subset of M), then N admits an R-orientation such that the inclusion map $N \to M$ is orientation-preserving.*
(3) *If $p\colon \widetilde{M} \to M$ is a covering, then there exists a unique orientation on $\widetilde{M}$ that turns p into an orientation-preserving map.*

Proof. To simplify the notation we suppress the ring R from the notation. Thus let M be an n-dimensional topological manifold that is equipped with an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$.

(1) Let N be a topological manifold and let $f\colon N\to M$ be a local homeomorphism. Let $x\in N\setminus\partial N$. We pick an open neighborhood U of x such that $f\colon U\to f(U)$ is a homeomorphism and such that the image $f(U)$ is an open subset of M. We define $\beta_x\in H_n(N,N\setminus\{x\};R)$ to be the image of α_x under the map[863]

$$\begin{array}{ccccccc} & & \text{these maps are isomorphisms by the Excision Theorem 45.8} & & & & \\ & & \downarrow & & & & \downarrow \\ H_n(M,M\setminus\{f(x)\}) & \xleftarrow{\cong} & H_n(f(U),f(U)\setminus\{f(x)\}) & \xrightarrow[\cong]{f_*^{-1}} & H_n(U,U\setminus\{x\}) & \xrightarrow{\cong} & H_n(N,N\setminus\{x\}). \end{array}$$

We leave it to the reader to verify that these classes β_x define an orientation on N. By definition the map f is orientation-preserving with respect to the orientations on M and N. It is elementary to verify that the orientation on N is uniquely determined.

(2) This statement follows from (1) since the inclusion map of a codimension zero submanifold is, basically by definition, a local homeomorphism.

(3) This statement also follows from (1) since a covering map is a local homeomorphism. □

The following lemma says in particular that the double of an R-orientable manifold is again R-orientable.

Lemma 45.29. *Let M be a connected topological manifold with non-empty boundary and let R be a commutative ring. Then*

$$M \text{ is } R\text{-orientable} \iff \text{the double } DM \text{ is } R\text{-orientable.}$$

Furthermore, if M is equipped with an R-orientation, then DM admits a unique R-orientation such that the inclusion map $M\to DM$ is orientation-preserving.

Sketch of proof $(*)$. Let M be a connected topological manifold. In the following we write $DM=(M\sqcup M')/\sim$ where M' is another copy of M. We apply the Topological Collar Neighborhood Theorem 45.3 to obtain a collar neighborhood $\partial M\times[-2,0]$ of ∂M in M. To simplify the notation we henceforth drop the coefficient ring R from the notation.

If DM is orientable, then it follows immediately from Lemma 45.28 that the submanifold M is also orientable.

Now suppose that M is equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$. We define orientations for the points of DM in $M\setminus\partial M$, $M'\setminus\partial M'$ and in $\partial M=\partial M'$ separately.

(1) Given $x\in M\setminus\partial M$ we denote by α_x the image of μ_x under the inclusion induced isomorphism $H_n(M,M\setminus\{x\})\to H_n(DM,DM\setminus\{x\})$ given by the Excision Theorem 45.10.

(2) We denote by $r\colon DM\to DM$ the obvious homeomorphism given by interchanging the two copies of M in DM. Given $x\in M'\setminus\partial M'$ we define $\alpha_x:=-r_*^{-1}(\alpha_{r(x)})$.

[863]Since $f\colon U\to f(U)$ is a homeomorphism we have $f(U\setminus\{x\})=f(U)\setminus\{f(x)\}$. Furthermore it follows from the fact that $x\in U\setminus\partial U$ and Proposition 32.1 that $f(x)\notin\partial N$.

(3) Finally let $x \in \partial M$. We define $\alpha_x \in H_n(DM, DM \setminus \{x\})$ to be the image of the class $\alpha_{x\times\{-1\}} \in H_n(DM, DM \setminus x \times \{-1\})$, which we defined in (1), under the isomorphism

$$H_n(DM, DM \setminus x\times\{-1\}) \underset{\uparrow}{\overset{\cong}{\Longleftarrow}} H_n(DM, DM \setminus x\times[-1,0]) \underset{\uparrow}{\overset{\cong}{\Longrightarrow}} H_n(DM, DM \setminus \overbrace{x\times\{0\}}^{=x}).$$

these maps are isomorphisms by an excision argument together with Lemma 45.9

We leave it as a slightly annoying exercise to the indefatigable reader to show that $\{\alpha_x\}_{x\in DM}$ does indeed define an orientation on DM. By construction of the orientation on DM the inclusion $M \to DM$ is orientation-preserving. The uniqueness of the orientation on DM is a consequence of Proposition 45.17 (1) and of our hypothesis that M is connected. $\square$

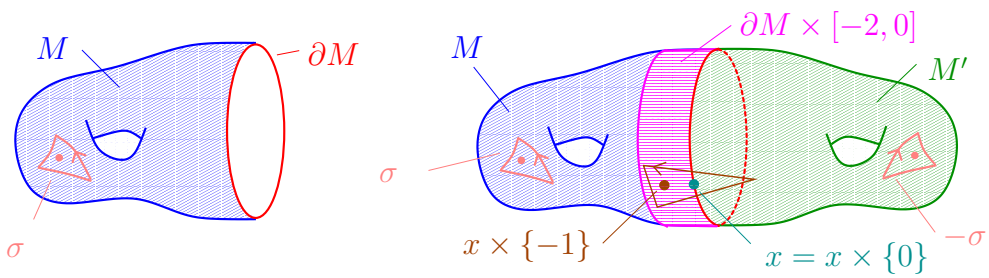


FIGURE 648.

Finally let M be an n-dimensional topological manifold equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$. Now we will see that the given orientation induces an orientation on ∂M.

Proposition 45.30. *Given any topological manifold M of dimension ≥ 2 exists a natural map*

$$\{\textit{orientations of } M\} \;\to\; \{\textit{orientations of } \partial M\}.$$

Here natural means that if $f\colon M \to N$ is a homeomorphism, then the following diagram commutes:

$$\begin{array}{ccc} \{\textit{orientations of } M\} & \longrightarrow & \{\textit{orientations of } \partial M\} \\ \downarrow f_* & & \downarrow f_* \\ \{\textit{orientations of } N\} & \longrightarrow & \{\textit{orientations of } \partial N\}. \end{array}$$

Convention. If M is an oriented topological manifold of dimension ≥ 2, then we always equip ∂M with the orientation provided by the construction in the proof of Proposition 45.30.

Proof $(*)$. We suppose we are given an orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. In the following we will define a natural orientation for ∂M.

We set $n := \dim(M)$ and we write $A = [0,1] \times [-1,1]^n$, we write $A_0 := \{0\} \times [-1,1]^{n-1}$ and we write $A' = [0,1) \times (-1,1)^n$. Now let $x \in \partial M$. Since M is a topological manifold

and since $x \in \partial M$ we can pick a chart $\Phi\colon U \to [0,2) \times (-2,2)^{n-1}$. We write $\Psi := \Phi^{-1}$. Furthermore we write $V = \Psi(A)$, $V_0 = \Psi(A_0)$ and we write $V' = \Psi(A')$. Finally we pick a point $y \in V'$. (We refer to Figure 649 for an illustration.)

We consider the following sequence of maps

$$\begin{array}{ccccc} H_n(M, M\setminus\{y\}) & \xleftarrow{(1)} & H_n\big(M, M\setminus V'\big) & \xrightarrow[(2)]{\partial} & H_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \\ & & & & \uparrow{\scriptstyle (3)} \\ & & & & H_{n-1}(V_0, V_0\setminus\{x\}) \xrightarrow[(4)]{} H_{n-1}(\partial M, \partial M\setminus\{x\}). \end{array}$$

Here all the maps are induced by the obvious inclusions of pairs of topological spaces, except for the second map which is the connecting homomorphism of the long exact sequence of the triple $(M, M \setminus V', (M\setminus V')\setminus\{x\})$ that is provided by Proposition 31.13.

Claim. All the maps in the above diagram are isomorphisms.

In the following we treat the four maps separately.

(1) It is elementary to see that $A \setminus \Psi^{-1}(y)$ is a deformation retract of ∂A. It follows that $M \setminus V'$ is a deformation retract of $M \setminus \{y\}$.[864] We obtain from Corollary 31.16 that (1) an isomorphism.

(2) We consider the long exact sequence of the triple $(M, M \setminus V', (M\setminus V')\setminus\{x\})$ that is provided by Proposition 31.13:

$$H_n(M, (M\setminus V')\setminus\{x\}) \to H_n(M, M\setminus V') \xrightarrow{\partial} H_{n-1}(M\setminus V', M\setminus V) \to H_{n-1}(M, (M\setminus V')\setminus\{x\}).$$

Similarly to (1) one sees that $(M \setminus V') \setminus \{x\}$ is a deformation retract of M. Therefore it follows from Corollary 31.16 that the groups to the left and right are zero. In other words, the connecting homomorphism is an isomorphism.

(3) It follows quite easily from the Excision Theorem 31.17 that (3) is an isomorphism.

(4) Replace (3) by (4) in the previous sentence.

This concludes the proof of the claim.

Given $x \in \partial M$ we now define $\nu_x \in H_{n-1}(\partial M, \partial M \setminus \{x\})$ to be the image of the generator $\mu_y \in H_n(M, M\setminus\{y\})$ under the above sequence of isomorphisms. We continue with our next claim.

Claim.

(1) The definition of ν_x is independent of the choice of y.

[864]Looking at Figure 649 this statement sounds reasonable. But the statement nonetheless requires a proof. So let r be a deformation retraction from $A \setminus \Psi^{-1}(y)$ to ∂A. We consider the map

$$\begin{array}{rcl} s\colon (M\setminus\{y\}) \times [0,1] & \to & M\setminus\{y\} \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin V\setminus\{y\}, \\ \Psi(r(\Psi^{-1}(P), t)), & \text{if } P \in V\setminus\{y\}. \end{cases} \end{array}$$

Note that $V \setminus \{y\}$ and $M \setminus \Psi(A')$ are closed subsets of $M \setminus \{y\}$. (Hmm, why is that?). It follows from Lemma 1.32 (2) that the map s is continuous.

(2) The definition of ν_x is independent of the choice of Φ.

We turn to the proof of the two statements.

(1) We need to show that for any $z \in \Psi(V')$ we have

$$\begin{array}{c}\text{preimage of } \mu_y \text{ under the map} \\ H_n(M, M\backslash V') \xrightarrow{\cong} H_n(M, M\backslash\{y\})\end{array} = \begin{array}{c}\text{preimage of } \mu_z \text{ under the map} \\ H_n(M, M\backslash V') \xrightarrow{\cong} H_n(M, M\backslash\{z\}).\end{array}$$

But this statement can be shown in a similar fashion as Proposition 45.17 (1). We leave it to the reader to fill in the details.

(2) Suppose we are given a different chart than Φ. We define W, W_0, W' in the obvious way. By possibly taking a third chart whose image lies in the intersection of V and W we can assume, without loss of generality, that $W \subset V$. (We refer to Figure 650 for an illustration.) Furthermore by (1) we can work with a point $y \in W$ for both charts. We consider the following diagram

$$\begin{tikzcd}
H_n(M, M\backslash\{y\}) & & H_n\big(M, M\backslash V'\big) \arrow[ll, "\cong", "(1)"'] \arrow[dl, dashed] \arrow[d, "(2)"', "\partial"] \\
H_n\big(M, M\backslash W'\big) \arrow[u, "\cong"', "(1)"] \arrow[d, "(2)"', "\partial"] & & H_{n-1}(M\backslash V', (M\backslash V')\backslash\{x\}) \arrow[dl, dashed] \\
H_{n-1}(M\backslash W', (M\backslash W')\backslash\{x\}) & & H_{n-1}(V_0, V_0\backslash\{x\}) \arrow[u, "(3)", "\cong"'] \arrow[d, "(4)"'] \\
H_{n-1}(W_0, W_0\backslash\{x\}) \arrow[u, "(3)"', "\cong"] \arrow[urr, dashed] \arrow[rr, "(4)"'] & & H_{n-1}(\partial M, \partial M\backslash\{x\}).
\end{tikzcd}$$

Here the two "outer paths" from the top left to the lower right correspond to the two "competing" definitions of ν_x. The dashed arrows are given by inclusions. We make the following observations:

(a) The triangle on top commutes since all maps are induced by inclusions.
(b) The upper parallelogram commutes since the connecting homomorphism is natural, see Proposition 31.13 (3) for details.
(c) The lower parallelogram commutes since all the maps are induced by inclusions. (Note that the reversal of the direction of the arrow is not a typo.)
(d) The triangle on the bottom commutes since all maps are induced by inclusions.

We saw above that all maps (1), (2), (3) and (4) are isomorphisms. By going from the top to the bottom we see that all the dashed arrows are isomorphisms.

Since all pieces of the diagram commute and since all maps are isomorphisms we see that the total diagram commutes. This shows that the two "'competing" definitions of ν_x agree.

This concludes the proof of the claim.

Using the previous claim it is now straightforward to show that the $\{\nu_x\}_{x\in\partial M}$ form an orientation for ∂M and that this orientation is natural. $\square$

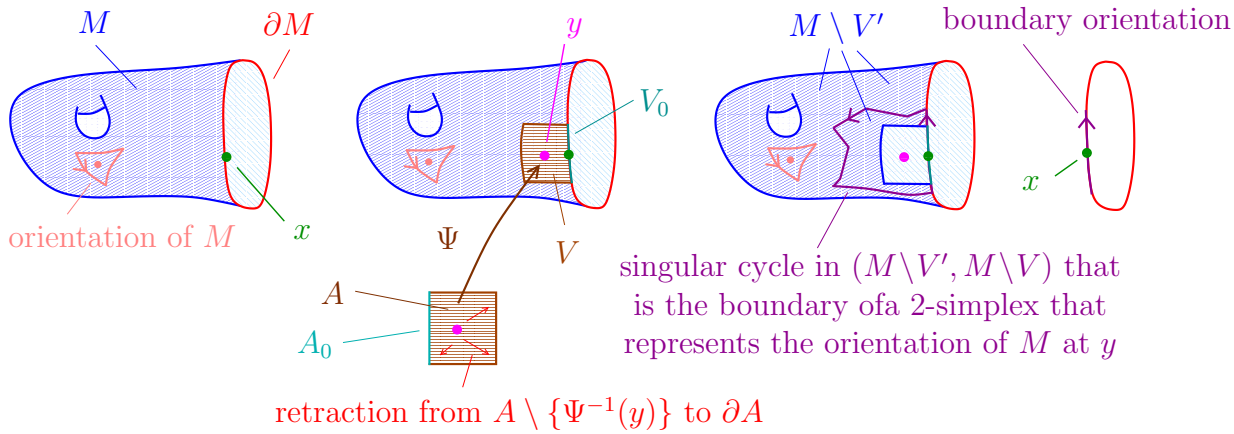


FIGURE 649.

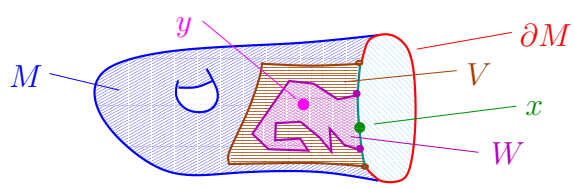


FIGURE 650.

Remark. Let M be an n-dimensional manifold that is equipped with an orientation in the sense of Section 3.8. On page 205 we had equipped the $(n-1)$-dimensional manifold ∂M with an orientation in the sense of Section 3.8. If one takes the strenuous journey through all conventions and definitions one sees that the boundary orientations from page 205 and the proof of Proposition 45.30 are compatible. More precisely, for any oriented n-dimensional M we have the following commutative "diagram"

$$\begin{tikzcd}
\text{oriented manifold } M \arrow[r, mapsto, "\text{take boundary}"] \arrow[d, mapsto, "\text{Proposition 45.22}"] & \text{oriented manifold } \partial M \arrow[d, mapsto, "\text{Proposition 45.22}"] \\
\mathbb{Z}\text{-oriented manifold } M \arrow[r, mapsto, "\text{take boundary}"] & \mathbb{Z}\text{-oriented manifold } \partial M.
\end{tikzcd}$$

Exercises for Chapter 45.

Exercise 45.1. Let X be and Y be two compact topological manifolds. Show that

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Exercise 45.2. Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $\partial Y = \partial Z$ is a submanifold of X. Show that

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

Exercise 45.3. Let X be a compact topological space. Is $H_1(X)$ necessarily finitely generated?

Exercise 45.4. Let M be a connected topological manifold. Show that $M \setminus \partial M$ is still connected.

Exercise 45.5. Let M be a topological manifold with non-empty boundary. Show that the topological space $M \setminus \partial M$ is non-compact.

Exercise 45.6. What is the double of the Möbius band and what is the double of the Klein bottle minus an open disk?

Exercise 45.7. We recall the following definitions:

(1) The suspension of a topological space X is defined to be the topological space
$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$
where we identify all points in $X \times \{-1\}$ and we identify all points in $X \times \{1\}$.

(2) A topological space X that has the property that for each $x \in X$ we have
$$H_k(X, X \setminus \{x\}; \mathbb{Z}) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$
is called an *n-dimensional homology manifold*.

(3) We say that an n-dimensional topological manifold X is a *homology n-sphere* if for any $k \in \mathbb{N}_0$ we have $H_k(X;\mathbb{Z}) \cong H_k(S^n;\mathbb{Z})$.

Let Y be an n-dimensional homology sphere.

(a) Show that $Y \times \mathbb{R}$ is an $(n+1)$-dimensional homology manifold.
(b) Show that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold.
(c) Show that if if $\pi_1(Y)$ is non-trivial, then the suspension $\Sigma(Y)$ is *not* a topological manifold.

Exercise 45.8. We view $\mathbb{R}$ as a topological manifold and we equip it with the standard $\mathbb{Z}$-orientation that we had introduced on page 1101. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map that is a local homeomorphism and that is orientation-preserving. Show that f is strictly monotonously increasing.

46. The fundamental class

46.1. The fundamental class of an oriented topological manifold.

Notation. Throughout this chapter we adopt the following notation. Let M be a topological manifold, let R be a commutative ring and let $x \in M \setminus \partial M$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the obvious map

$$\rho_x \colon H_k(M, \partial M; R) \quad \to \quad H_k(M, M \setminus \{x\}; R).$$

Given $\alpha \in H_k(M, \partial M; R)$ we often write $\alpha_x := \rho_x(\alpha)$, i.e. the above map is also written as

$$\alpha \quad \mapsto \quad \alpha_x.$$

With this notation we can now formulate the following elementary lemma.

Lemma 46.1. *Let M be an n-dimensional topological manifold and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} H_n(M, \partial M; R) & \to & \Gamma_R(M) \\ \alpha & \mapsto & \{\alpha_x\}_{x \in M \setminus \partial M} \end{array}$$

is a well-defined homomorphism of R-modules.

Proof. We only need to show that $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section, but this follows immediately from setting $U = M \setminus \partial M$ and $\alpha_U = \alpha$ in the definition given on page 1095. (Note that here we used that we showed in Lemma 3.6 (1) that $U = M \setminus \partial M$ is an open subset of M.) □

The following three theorems contain most of the main results of this chapter.

Theorem 46.2. *Let M be a compact, connected, non-empty n-dimensional topological manifold. Then the following statements are equivalent:*

(1) *the topological manifold M is orientable,*
(2) $H_n(M, \partial M; \mathbb{Z}) \neq 0$,
(3) $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$,
(4) *for every $x \in M \setminus \partial M$ the map*

$$\begin{array}{rcl} \rho_x \colon H_n(M, \partial M; \mathbb{Z}) & \to & H_n(M, M \setminus \{x\}; \mathbb{Z}) \\ \alpha & \mapsto & \alpha_x \end{array}$$

is an isomorphism.

Theorem 46.3. *Let M be a compact, connected, non-empty n-dimensional topological manifold M. Then the following two statements hold:*

(1) $H_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$, *and*
(2) *for every $x \in M \setminus \partial M$ the map*

$$\begin{array}{rcl} \rho_x \colon H_n(M, \partial M; \mathbb{F}_2) & \to & H_n(M, M \setminus \{x\}; \mathbb{F}_2) \\ \alpha & \mapsto & \alpha_x \end{array}$$

is an isomorphism.

Theorem 46.4. *Let M be a connected n-dimensional topological manifold and let R be a commutative ring. Then the following two statements hold:*

(1) *If M is not closed, then $H_i(M;R) = 0$ for $i \geq n$.*
(2) *If M is closed, then we have $H_i(M;R) = 0$ for $i > n$.*

Remark.

(1) Let M be a connected n-dimensional manifold. In Proposition 35.6 (2) we already saw, using Theorem 26.3, that $H_i(M) = 0$ for $i > n$. Furthermore, in Proposition 39.5 we showed that

$$H_n(M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and closed,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorems 46.2 and 46.4 thus generalize these earlier results to topological manifolds. The new proofs are also more self-contained since they do not make use of the existence of CW-structures and simplicial structures on manifolds.

(2) One might be tempted to consider Theorem 46.4 (2) as "intuitively clear". After all, how could an n-dimensional object possibly have homology groups in higher dimensions? To shatter this belief, fix some $n \geq 2$. We consider the following subset of $\mathbb{R}^n$:[865]

$$\begin{aligned} X &:= \bigcup_{n\in\mathbb{N}} n\text{-dimensional sphere of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n}, 0, \ldots, 0) \\ &= \bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R}^n \mid \|x - (\tfrac{1}{n}, 0, \ldots, 0)\| = \tfrac{1}{n^2}\}. \end{aligned}$$

In [BaM] it is shown, rather shockingly, that for every q with $q \equiv 1 \bmod (r-1)$ the group $H_q(X;\mathbb{Q})$ is not only non-zero, it is even uncountable.

We will not immediately prove these three theorems, but we will first draw many conclusions, before we head towards the somewhat intricate proofs of the above three theorems.

Definition. Let M be a compact n-dimensional topological manifold and let R be a commutative ring. An *R-fundamental class of M* is an element $\beta \in H_n(M, \partial M; R)$ with the property that for any $x \in M \setminus \partial M$ the element $\beta_x \in H_n(M, M \setminus \{x\}; R)$ is a generator. For $R = \mathbb{Z}$ we just say *fundamental class*.

Remark. Let M be a compact n-dimensional topological manifold and let R be a commutative ring.

(1) Let $[M]$ be a $\mathbb{Z}$-fundamental class for M. The image of $[M]\otimes 1$ under the natural map $\mu\colon H_n(M;\mathbb{Z}) \otimes R \to H_n(M;R)$ is an R-fundamental class. Indeed, let $x \in M \setminus \partial M$.

[865] As the reader will surely have noticed, this example can be viewed as a high-dimensional analogue of the Hawaiian Earings that we introduced in Exercise 26.6.

From Lemma 42.7 (3) we obtain a commutative diagram

$$\begin{array}{ccc} H_n(M;\mathbb{Z})\otimes R & \xrightarrow{\quad\mu\quad} & H_n(M;R) \\ \downarrow & & \downarrow \\ H_n(M,M\setminus\{x\};\mathbb{Z})\otimes R & \xrightarrow{\mu} & H_n(M,M\setminus\{x\};R) \end{array}$$

By Lemma 45.16 and the Universal Coefficient Theorem 42.19 the bottom horizontal map is an isomorphism. Since the image of the $\mathbb{Z}$-fundamental class $[M]$ under the obvious map $H_n(M;\mathbb{Z}) \to H_n(M, M\setminus\{x\};\mathbb{Z})$ is a generator it follows from the commutative diagram that the image of $\mu([M]\otimes 1)$ under the obvious map $H_n(M;R) \to H_n(M, M\setminus\{x\};R)$ is indeed a generator.

(2) If M has connected components $M_1,\dots,M_k$ and if for $i = 1,\dots,k$ we are given R-fundamental classes $[M_i]_R \in H_n(M_i,\partial M_i;R)$, then

$$\sum_{i=1}^{k}[M_i]_R \in H_n(M,\partial M;R) = \bigoplus_{i=1}^{k} H_n(M_i,\partial M_i;R)$$

is easily seen to be a R-fundamental class of M. The fundamental classes for the M_i are unique if and only if the fundamental class for M is unique.

The following corollary to Theorem 46.2 says in particular that an orientation determines and is in turn uniquely determined by a fundamental class.

Corollary 46.5. *Let M be a compact n-dimensional topological manifold.*

(1) *If $[M]$ is a fundamental class, then $\{[M]_x\}_{x\in M\setminus\partial M}$ defines an orientation for M.*
(2) *If $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an orientation for M, then there exists a unique fundamental class $[M]\in H_n(M,\partial M;\mathbb{Z})$ with $[M]_x = \alpha_x$ for all $x\in M\setminus\partial M$.*

Furthermore, if M is connected, non-empty and orientable, then the following two statements hold:

(3) *Every fundamental class $[M]\in H_n(M,\partial M;\mathbb{Z})$ is a generator of $H_n(M,\partial M;\mathbb{Z})\cong\mathbb{Z}$.*
(4) *Every generator of $H_n(M,\partial M;\mathbb{Z})\cong\mathbb{Z}$[866] is a fundamental class.*

Proof. Let M be a compact n-dimensional topological manifold. By considering the components of M separately we see that without loss of generality we can assume that M is connected. It is also clear that we only have to study the case that M is non-empty.

(1) Let $[M]$ be a fundamental class for the topological manifold M. It follows from Lemma 46.1 that the homology classes $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ form a section on M. By definition of a fundamental class each $\rho_x([M])$ is a generator. Therefore the section $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ is indeed an orientation.

[866]Here we use that in Theorem 46.2 we saw that for a compact orientable, non-empty n-dimensional topological manifold we have $H_n(M,\partial M;\mathbb{Z})\cong\mathbb{Z}$.

(2) Suppose we are given an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$ for M. By Theorem 46.2 (1) $\Rightarrow$ (4) we know that for every $x \in M \setminus \partial M$ the map

$$\rho_x\colon H_n(M, \partial M; \mathbb{Z}) \;\to\; H_n(M, M \setminus \{x\}; \mathbb{Z})$$

is an isomorphism. Now pick a $y \in M$. We write $[M] := \rho_y^{-1}(\alpha_y)$. Since ρ_y is an isomorphism the class $[M]$ is uniquely determined. So it remains to show that $[M]_x = \alpha_x$ for all $x \in M \setminus \partial M$. By Lemma 46.1 the classes $\{[M]_x\}_{x\in M\setminus\partial M}$ form a $\mathbb{Z}$-section. It follows from Proposition 45.17 that $[M]_x = \alpha_x$ for all $x \in M \setminus \partial M$.

(3) This statement follows immediately from the definition of a fundamental class and Theorem 46.2 (1) $\Rightarrow$ (4).

(4) The last statement is an immediate consequence of Lemma 46.1 and Theorem 46.2 (1) $\Rightarrow$ (4). □

We also have the following corollary to Theorem 46.3 which deals with $\mathbb{F}_2$-coefficients.

Corollary 46.6. *Let M be a compact, non-empty n-dimensional topological manifold. Then the following statements hold:*

(1) *If M is connected, then*[867]

$$[M]_{\mathbb{F}_2} \;:=\; \textit{the unique non-zero element of } H_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$$

is the unique $\mathbb{F}_2$-fundamental class of M.

(2) *If M has connected components $M_1, \dots, M_k$, then*

$$[M]_{\mathbb{F}_2} \;:=\; \sum_{i=1}^{k} [M_i]_{\mathbb{F}_2} \;\in\; H_n(M, \partial M; \mathbb{F}_2) = \bigoplus_{i=1}^{k} H_n(M_i, \partial M_i; \mathbb{F}_2)$$

is the unique $\mathbb{F}_2$-fundamental class of M.

Proof. It follows immediately from Theorem 46.3 (2) that the class $[M]_{\mathbb{F}_2}$ that we had just defined in (1) is indeed an $\mathbb{F}_2$-fundamental class for M and that it is unique. As we pointed out on page 1114, the second statement is an immediate consequence of the first statement. □

Remark. We have just shown in Corollary 46.6 that for every closed, connected n-dimensional topological manifold M there exists a uniquely determined $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2}$. Given a topological space X and a homology class $z \in H_n(X; \mathbb{F}_2)$ it was shown by René Thom [Tho, Tho2][868] in about 1954 (see also [Rudy, Theorem IV.7.33]) that there exists an n-dimensional closed, connected manifold M and a map $f\colon M \to X$ such that $f_*([M]_{\mathbb{F}_2}) = z$. This can be viewed as the $\mathbb{F}_2$-analogue of Theorem 39.18. We will not make use of this fact.

Let M be a compact n-dimensional manifold that is oriented in the sense of Section 3.8. In Proposition 39.7 we defined the fundamental class $[M]$ in $H_n(M, \partial M; \mathbb{Z})$. On the other hand, in Proposition 45.22 we saw that the orientation of M also determines a $\mathbb{Z}$-orientation

[867] Here in the formulation we already use Theorem 46.3 (1) to guarantee that $H_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.

[868] We had already encountered the Fields medalist René Thom on page 959.

and thus we obtain a fundamental class $[M] \in H_n(M, \partial M; \mathbb{Z})$ by Corollary 46.5. Not surprisingly we will now see that these two definitions agree.

Lemma 46.7. *The two definitions of the fundamental class of a compact oriented manifold agree.*

Proof $(*)$. Let M be a compact n-dimensional manifold that is oriented in the sense of Section 3.8. To simplify the notation we assume that M is closed and connected. We denote by $\alpha \in H_n(M; \mathbb{Z})$ the fundamental class of Proposition 39.7 and we denote by $\beta \in H_n(M; \mathbb{Z})$ the fundamental class of Corollary 46.5.

Let $x \in M$. By Theorem 46.2 (1) $\Rightarrow$ (4) it suffices to show that $\alpha_x = \beta_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$. We make the following choices:

(1) We pick an orientation-preserving chart $\Phi\colon U \to V$ around x with $\Phi(x) = 0$ and such that $\overline{B}^n \subset V$.
(2) We denote by $\Theta\colon \Delta^n \to \overline{B}^n$ the homeomorphism from Lemma 29.1. Note that with the orientation of $\mathring{\Delta}^n$ given on page 935 and the usual orientation of B^n this map Θ restricts to an orientation-preserving diffeomorphism $\mathring{\Delta}^n \to B^n$.

Now we consider the following diagram of maps

$$\begin{tikzcd}
H_n(M;\mathbb{Z}) \arrow[r] \arrow[dd] & H_n(M, M\setminus\Phi^{-1}(B^n);\mathbb{Z}) \arrow[ddl] & H_n(\overline{B}^n, \partial\overline{B}^n;\mathbb{Z}) \arrow[l, "\Phi_*^{-1}"'] \\
 & & H_n(\Delta^n, \partial\Delta^n;\mathbb{Z}) \arrow[u, "\Theta_*"'] \arrow[d, "\Theta_*"] \\
H_n(M, M\setminus\{x\};\mathbb{Z}) & H_n(U, U\setminus\{x\};\mathbb{Z}) \arrow[l, "\cong"] & H_n(V, V\setminus\{0\};\mathbb{Z}) \arrow[l, "\Phi_*^{-1}"']
\end{tikzcd}$$

where all maps without a name are induced by the obvious inclusions of pairs of topological spaces. It is straightforward to verify that the diagram commutes.

By definition of α the class $\alpha_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$ is given by starting with the class $[\mathrm{id}^n_\Delta] \in H_n(\Delta^n, \partial\Delta^n; \mathbb{Z})$ and traveling "counterclockwise" through the diagram. On the other hand, it follows from Lemma 45.15 that $\Theta_*([\mathrm{id}_{\Delta^n}])$ represents a standard generator of $H_n(V, V \setminus \{0\}; \mathbb{Z})$. From this it follows that the class $\beta_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$ is given by starting with $[\mathrm{id}^n_\Delta] \in H_n(\Delta^n, \partial\Delta^n; \mathbb{Z})$ and traveling "clockwise" through the diagram. Since the diagram commutes we have now shown that $\alpha_x = \beta_x$. $\square$

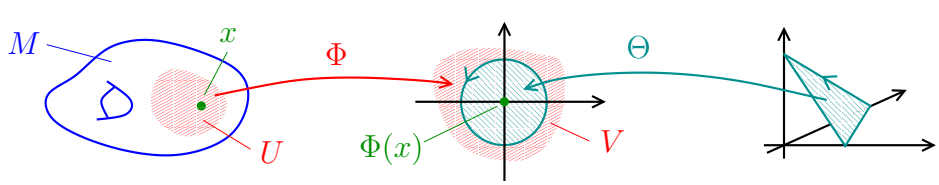


FIGURE 651. Illustration for the proof of Lemma 46.7.

The following proposition gives a useful criterion for showing that a given singular chain represents the fundamental class of a manifold.

Proposition 46.8. *Let M be a compact, connected n-dimensional manifold.*

(1) *Suppose that M is an oriented manifold. Let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in C_n(M, \partial M; \mathbb{Z})$ be a cycle. Suppose there exists a $j \in \{1, \dots, m\}$ such that the following four conditions are satisfied:*
   (a) *the map $\sigma_j \colon \mathring{\Delta}^n \to M$ is a diffeomorphism onto its image,*[869]
   (b) *for $i \neq j$ we have $\sigma_i(\Delta^n) \cap \sigma_j(\mathring{\Delta}^n) = \varnothing$,*
   (c) *$r_j = 1$,*
   (d) *$\sigma_j \colon \mathring{\Delta}^n \to M$ is orientation-preserving.*
   Then the cycle represents the fundamental class of the oriented manifold, i.e. we have

$$\left[\sum_{i=1}^{m} r_i \cdot \sigma_i\right] \;=\; [M] \;\in\; H_n(M, \partial M; \mathbb{Z}) \;\cong\; \mathbb{Z}.$$

(2) *Let $\sum_{i=1}^{m} \sigma_i \otimes r_i \in C_n(M, \partial M; \mathbb{F}_2)$ be a cycle for which there exists again a $j \in \{1, \dots, m\}$ that satisfies the above conditions* (a), (b) *and* (c), *then the cycle represents the $\mathbb{F}_2$-fundamental class of M, i.e. we have*

$$\left[\sum_{i=1}^{m} \sigma_i \otimes r_i\right] \;=\; [M]_{\mathbb{F}_2} \;\in\; H_n(M, \partial M; \mathbb{F}_2) \;\cong\; \mathbb{F}_2.$$

Notation. We will give many examples of fundamental classes of 2-dimensional manifolds. Since we want to be very careful what singular simplex we mean by a given picture we adopt the following convention:

A triangle T with a dot in a corner and a directional arrow in the center means the unique affine linear map $\Delta^2 \to T$ which sends $(1, 0, 0)$ to the vertex decorated with the dot and which respects the orientation of the vertices. We illustrate this notation in Figure 652. We also take some artistic liberties and allow triangles where the boundary is not given by segments and where the boundary consists of curves. In this case we expect that the boundary curves are parametrized with constant speed. Since the precise definition of the

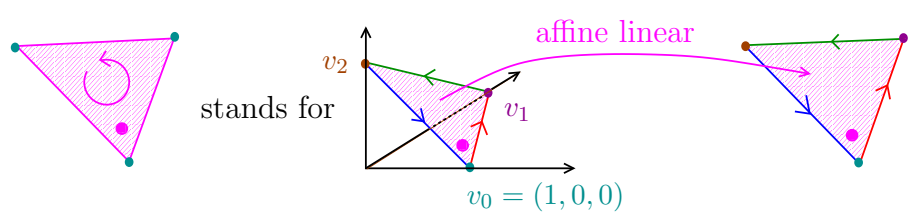


FIGURE 652.

[869]Note that we do *not* place any conditions on the restriction of σ_j to $\partial\Delta^n$.

boundary of a singular 2-simplex is often a source of confusion we recall the definition in Figure 653.

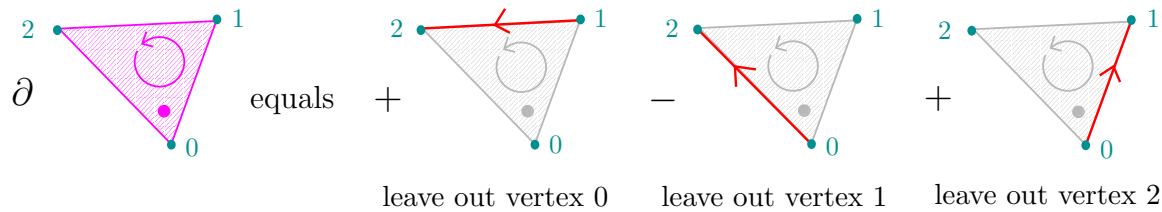


FIGURE 653.

Example. In Figure 654 we show the torus $T = [0,1] \times [0,1]/\sim$. We equip T with the orientation coming from the standard orientation on $(0,1) \times (0,1) \subset T$, which we can also view as an open subset of $\mathbb{R}^2$. In Figure 654 we also show two singular 2-simplices σ_1 and σ_2. It is straightforward to verify that $-\sigma_1 + \sigma_2$ is a cycle. (In the verification it is very helpful to keep Figure 653 in mind.) The summand σ_2 satisfies the conditions imposed in Proposition 46.8, thus we see that $-\sigma_1 + \sigma_2$ represents a fundamental class of T, in particular it represents a generator of $H_2(T)$. This is of course the same cycle that we had already initially encountered on page 742. We already saw in Lemma 33.13 in a somewhat ad hoc argument that it represents a generator of $H_2(T)$.

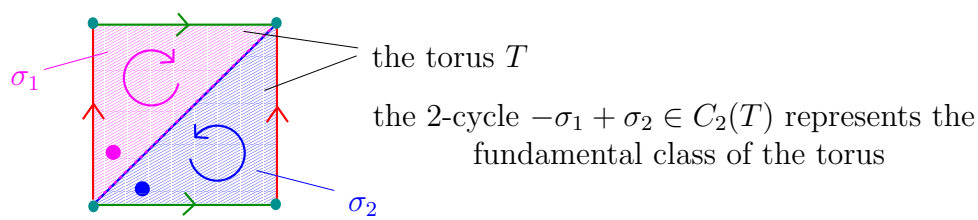


FIGURE 654.

Example. It follows easily from Proposition 46.8 and our orientation conventions that the standard generator of $H_n(S^n)$ as defined on page 814 represents the fundamental class of the manifold S^n with the usual orientation. In particular it follows from Lemma 32.11 that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ given by $\mu(1-t,t) = e^{2\pi i t}$ represents the fundamental class of the oriented manifold S^1.

Proof of Proposition 46.8 $(*)$. Let M be a compact n-dimensional manifold. To simplify the notation we only consider the case that M is closed. In the following we prove the proposition for $R = \mathbb{Z}$. The proof for $R = \mathbb{F}_2$ is almost identical, one just needs to replace Theorem 46.2 by Theorem 46.3

Thus let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in C_n(M;\mathbb{Z})$ be a cycle and $j \in \{1,\dots,m\}$ such that the following four conditions are satisfied:

(a) the map $\sigma_j\colon \mathring{\Delta}^n \to M$ is a diffeomorphism onto its image,
(b) for $i \neq j$ we have $\sigma_i(\Delta^n) \cap \sigma_j\big(\mathring{\Delta}^n\big) = \varnothing$,
(c) $r_j = 1$,
(d) $\sigma_j\colon \mathring{\Delta}^n \to M$ is orientation-preserving.

We first consider the special case that there exists a chart $\Phi\colon U \to V$ with $\sigma_j(\Delta^n) \subset U$. After possibly composing Φ with a reflection in $\mathbb{R}^n$ we can assume that Φ is in fact orientation-preserving. We pick $x \in \sigma_j(\mathring{\Delta}^n)$ and we write $y = \Phi(x)$. (We refer to Figure 655 for an illustration.) We consider the maps

$$\begin{array}{ccccccc}
 & \text{isomorphism by Theorem 46.2} & & & \text{isomorphism by the Excision Theorem 45.8} & & \text{isomorphism since } \Phi \text{ is a homeomorphism} \\
 & \downarrow & & & \downarrow & & \downarrow \\
H_n(M;\mathbb{Z}) & \to & H_n(M, M\setminus\{x\};\mathbb{Z}) & & \leftarrow H_n(U, U\setminus\{x\};\mathbb{Z}) & \xrightarrow{\Phi_*} & H_n(V, V\setminus\{y\};\mathbb{Z}) \\
\Big[\sum\limits_{i=1}^m r_i\cdot\sigma_i\Big] & \mapsto & \Big[\sum\limits_{i=1}^m r_i\cdot\sigma_i\Big] = [\underbrace{r_j\cdot\sigma_j}_{=\sigma_j}] & & \leftarrow\!\shortmid\ [\sigma_j] & & \mapsto\ [\Phi\circ\sigma_j].
\end{array}$$

since (b) implies that for $i \neq j$ we have $\sigma_i(\Delta^n) \subset M \setminus \{x\}$
and since by (c) we have $r_j = 1$

It follows immediately from our hypothesis (a) together with Lemmas 45.16 and 45.15 that $[\Phi \circ \sigma_j]$ is a generator of $H_n(V, V \setminus \{y\};\mathbb{Z})$. Furthermore, since $\sigma_j\colon \mathring{\Delta}^n \to M$ is orientation-preserving and since Φ is orientation-preserving, the desired equality follows easily from Lemma 45.23.

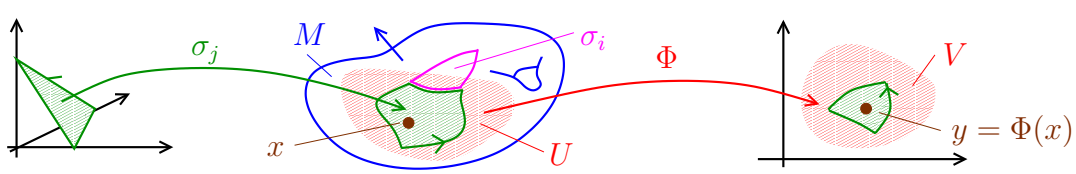


FIGURE 655. Illustration for the first part of the proof of Proposition 46.8.

Now we no longer assume that such a chart exists. We sketch the argument how to deal with this case. First we recall that on page 789 we introduced the subdivision map $u_k\colon C_k(M) \to C_k(M)$ with $k \in \mathbb{N}_0$. In Lemma 31.23 and 31.24 we saw that these maps form a chain map that is homotopy equivalent to the identity, in particular for any cycle $c \in C_n(M;\mathbb{Z})$ we have $[u_n(c)] = [c] \in H_n(M;\mathbb{Z})$.

Now we apply Lemma 31.28 to the open covering of M that is given by the domains of all charts in the atlas of the smooth structure on M. Then we obtain an $m \in \mathbb{N}$ and a singular chain $u_n^m(\sigma) = (u_n \circ \cdots \circ u_n)(\sigma)$ such that the image of each singular simplex appearing in $u_n^m(\sigma)$ lies in a chart and such that $u_n^m(\sigma)$ is homologous to σ. It follows easily from the definition of the subdivision map that $u_n^m(\sigma)$ still satisfies properties (a) to (d) for

an appropriate choice of summand. It follows from the previous observations that $u_n^m(\sigma)$ represents the fundamental class, but then so does σ. $\square$

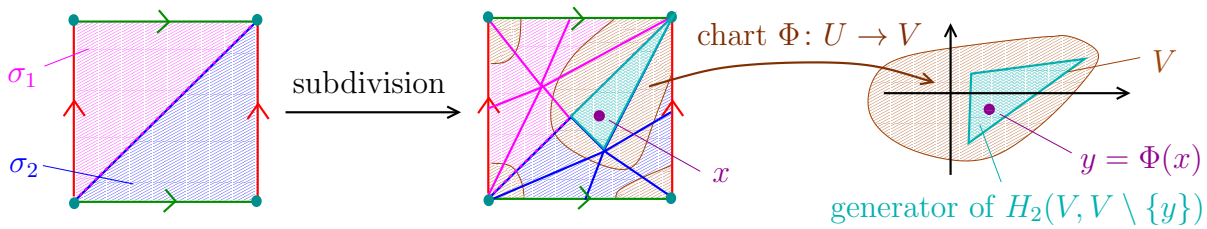


FIGURE 656. Illustration for the second part of the proof of Proposition 46.8.

Examples. In Figure 657 we show the surface of genus 2 with eight singular 2-simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\tau_1, \tau_2, \tau_3, \tau_4$. The interior of the octagon is an open subset of $\mathbb{R}^2$ and we equip the surface of genus 2 with the orientation coming from the standard orientation of $\mathbb{R}^2$. It is a fun exercise to verify that the singular 2-chain

$$\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 \; + \; \tau_1 - \tau_2 + \tau_3 - \tau_4$$

is indeed a cycle. It follows easily from Proposition 46.8 that the singular 2-chain represents a fundamental class of the surface of genus 2.

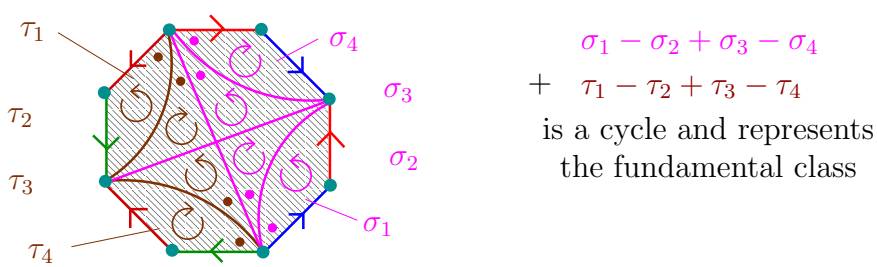


FIGURE 657.

Now we consider $\mathbb{F}_2$-fundamental classes of non-orientable 2-dimensional manifolds. Before we consider these examples we introduce one new piece of notation.

Notation. A bigon T with a cross in a corner and a directional arrow in the center means a map $\sigma\colon \Delta^2 \to T$ with the following three properties:

(1) the map σ collapses the face from v_0 to v_2 to the vertex decorated with the cross,
(2) on the two remaining faces the map σ is of constant speed,
(3) the restriction of σ to the interior is a diffeomorphism onto its image and it is orientation-preserving with the orientation specified.

We illustrate this notation in Figure 658.

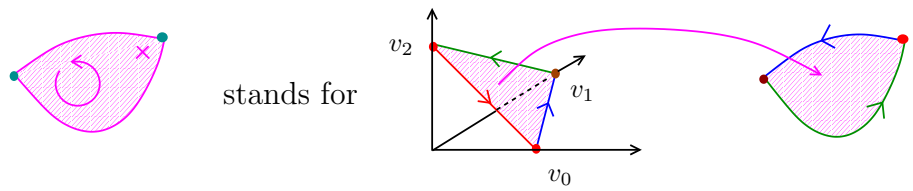


FIGURE 658.

Examples.

(1) Now we will give a representative for the $\mathbb{F}_2$-fundamental class for the real projective plane $\mathbb{R}\mathrm{P}^2$. More precisely, we consider the four singular simplices $\sigma_1, \sigma_2, \tau_1$ and τ_2 of $\mathbb{R}\mathrm{P}^2$ shown in Figure 659. It is straightforward to see that the singular 2-chain $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in C_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ is a cycle. Using Proposition 46.8 we see that it represents the $\mathbb{F}_2$-fundamental class of the real projective plane.

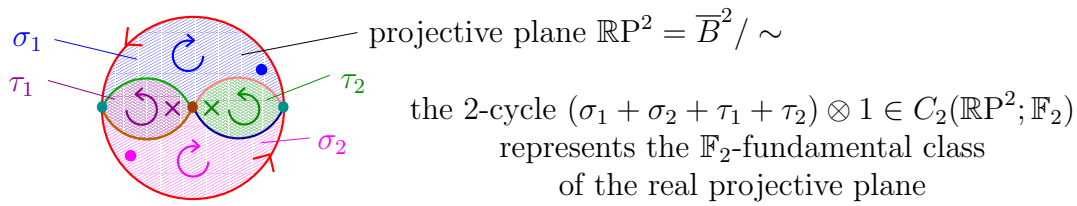


FIGURE 659.

(2) In Exercise 46.1 we will write down a representative for the $\mathbb{F}_2$-fundamental class of the Klein bottle.

46.2. **Lifts of sections on manifolds.** In this section we will prove the fundamental theorem of topological manifolds which lies at the heart of the proofs of Theorems 46.2, 46.3 and 46.4. Before we can state the lemma we introduce the following helpful notation.

Notation. Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset M$ be a subset.

(1) We write $H_n(M|A; R) := H_n(M, M \setminus A; R)$. Given a point $x \in M$ we use the obvious abbreviation $H_n(M|x; R) := H_n(M|\{x\}; R)$.
(2) Given a homology class $\alpha \in H_n(M|A; R)$ and a point $x \in A$ we denote by α_x the image of α under the obvious map $H_n(M|A; R) \to H_n(M|x; R)$.
(3) Let $\{\alpha_x\}_{x \in M}$ be an R-section on M and let $A \subset M$ be a subset. We call a class $\beta \in H_n(M|A; R)$ a *lift to* A if for all $x \in A$ we have $\beta_x = \alpha_x$.

The following lemma follows immediately from the definitions. But since we will refer to it on several occasions it is good to write it down.

Lemma 46.9. *Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset B \subset M$ be subsets:*

(1) *Since* $M \setminus B \subset M \setminus A$ *we have an inclusion induced map* $H_n(M|B;R) \to H_n(M|A;R)$.
(2) *Let* $\{\alpha_x\}_{x\in M}$ *be an* R*-section on* M*. If* $\mu_B \in H_n(M|B;R)$ *is a lift to* B*, then the image of* μ_B *under the above map* $H_n(M|B;R) \to H_n(M|A;R)$ *is a lift of the* R*-section to the subset* A*.*

The following theorem is the main technical result of this chapter.

Theorem 46.10. *Let* M *be an* n*-dimensional topological manifold with empty boundary and let* $A \subset M$ *be a compact subset. Let* R *be a commutative ring. Then the following hold:*

(i) *If* $\{\alpha_x\}_{x\in M}$ *is an* R*-section for* M*, then there exists a unique lift* $\alpha_A^M = \alpha_A$ *to* A*, i.e. there exists a unique class* $\alpha_A^M = \alpha_A \in H_n(M|A;R)$ *such that for each* $x \in A$ *we have* $\alpha_{A,x} = \alpha_x$*.*
(ii) *For* $i > n$ *we have* $H_i(M|A;R) = 0$*.*

Remark. To avoid any nervousness later on we remark that the statements (i) and (ii) hold trivially for $A = \varnothing$ since $H_n(M|\varnothing;R) = H_n(M,M;R) = 0$.

For later on we record the following lemma.

Lemma 46.11. *Let* M *be an oriented* n*-dimensional topological manifold with empty boundary and let* R *be a commutative ring. We suppose that* M *is equipped with an* R*-orientation. We use the notation from Theorem 46.10.*

(1) *Let* $K \subset L \subset M$ *be compact subsets. We denote by* $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ *the inclusion map of pairs. The map* $f_{KL*}\colon H_n(M, M\setminus L) \to H_n(M, M\setminus K)$ *sends* α_L^M *to* α_K^M*.*
(2) *Let* U *be an open subset of* M *and let* $A \subset U$ *be a compact subset. We use Lemma 45.28 to view* U *as an oriented topological manifold. If* $i\colon U \to M$ *denotes the inclusion map, then* $i_*(\alpha_A^U) = \alpha_A^M$*.*

Proof. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of Theorem 46.10. □

The very rough idea behind the proof of the fundamental theorem of topological manifolds can be summarized as follows:

(1) Using Mayer–Vietoris sequences we want to show that if the compact subsets A, B and $A \cap B$ have the desired properties, then so does $A \cup B$.
(2) Using charts we want to reduce the proof of the theorem to the case that $M = \mathbb{R}^n$.
(3) Finally for $M = \mathbb{R}^n$ we want to reduce the proof of the theorem to compact convex subsets A of $\mathbb{R}^n$.

The following lemmas give us the tools to put this idea into practice.

Lemma 46.12. *Let* M *be an* n*-dimensional topological manifold, let* R *be a commutative ring and let* A *and* B *be two compact subsets of* M*. If the statements* (i) *and* (ii) *from Theorem 46.10 hold for* A*,* B *and* $A \cap B$*, then they also hold for* $A \cup B$*.*

Proof. Let M be an n-dimensional topological manifold, let R be a commutative ring and let A and B be two compact subsets of M such that A, B and $A \cap B$ satisfy statements (i) and (ii). To simplify the notation we drop R from the notation. We consider the inverted Mayer–Vietoris Sequence from Theorem 33.7

$$\cdots \to \overbrace{H_{n+1}(M|A \cap B)}^{=0,\text{ by hypothesis on } A\cap B} \to H_n(M|A \cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A \cap B) \to \dots$$
$$\alpha \mapsto (\alpha, -\alpha) \qquad (\alpha, \beta) \mapsto \alpha + \beta$$

Now we can prove that $A \cup B$ also satisfies statements (i) and (ii).

(ii) For $i > n$ the group $H_i(M|A \cup B)$ is stuck, by our hypothesis on A, B and $A \cap B$, between groups that are zero, thus we see that $H_i(M|A \cup B) = 0$ for $i > n$.

(i) Now suppose we are given a section $\{\alpha_x\}_{x \in M}$. We denote by α_A, α_B and $\alpha_{A\cap B}$ the unique lifts to A, B and $A \cap B$ given by our hypothesis. By Lemma 46.9 the images of α_A and α_B in $H_n(M|A \cap B)$ are lifts to $A \cap B$. By the uniqueness of the lift $\alpha_{A\cap B}$ these images thus equal $\alpha_{A\cap B}$. By the exactness of the above sequence we see that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A\cup B})$ for some $\alpha_{A\cup B} \in H_n(M|A \cup B)$. We claim that $\alpha_{A\cup B}$ is indeed a lift to $A \cup B$. So let $x \in A \cup B$. First we consider the case that $x \in A$. We have the following commutative diagram

$$\begin{tikzcd} H_n(M|A \cup B) \arrow[r, "{[\alpha_{A\cup B} \mapsto \alpha_A]}"] \arrow[dr] & H_n(M|A) \arrow[d] \\ & H_n(M|x). \end{tikzcd}$$

We see that the image of $\alpha_{A\cup B}$ in $H_n(M|x)$ is the image of α_A in $H_n(M|x)$, hence it equals α_x. The same argument applies to the case that $x \in B$.

It remains to prove the uniqueness of the lift $\alpha_{A\cup B}$. If $\beta \in H_n(M|A \cup B)$ is another lift, then it follows immediately from Lemma 46.9 and the uniqueness of α_A and α_B that $\Phi(\beta) = (\alpha_A, -\alpha_B)$. By the exactness of the sequence we see that we already have the equality $\alpha_{A\cup B} = \beta$ in $H_n(M|A \cup B)$. □

Later on we will need the following variation of Lemma 46.12.

Lemma 46.13. *Let M be an n-dimensional topological manifold, let R be a commutative ring. Let $\mathcal{A}$ be a family of compact subsets of M with the following two properties:*

(1) *Statements* (i) *and* (ii) *from Theorem 46.10 hold for each $A \in \mathcal{A}$.*
(2) *The family $\mathcal{A}$ is closed under intersections, i.e. whenever $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.*

Then the statements hold for all finite unions of elements in $\mathcal{A}$.

Proof. Given $k \in \mathbb{N}$ we write

$$\mathcal{U}_k := \{\text{all subsets of } M \text{ that are the union of at most } k \text{ subsets in } \mathcal{A}\}.$$

We claim that for every $k \in \mathbb{N}$ statements (i) and (ii) hold for all elements in $\mathcal{U}_k$. We prove this claim by induction on k.

(1) If $k = 1$, then the claim holds by hypothesis (1).
(2) Suppose that the claim holds for some k. Let $A_1, \dots, A_k, A_{k+1} \in \mathcal{A}$. We write $B = A_2 \cup \dots \cup A_{k+1}$. Note that

$$A_1 \cap B \;=\; A_1 \cap (A_2 \cup \dots \cup A_{k+1}) \;=\; \bigcup_{i=2}^{k+1} \underbrace{(A_1 \cap A_i)}_{\in \mathcal{A} \text{ by (2)}} \;\in \mathcal{U}_k.$$

Thus by induction hypothesis we see that the statements hold for A_1, B and also for $A_1 \cap B$. So the statements also hold for $A_1 \cup B = A_1 \cup \dots \cup A_{k+1}$ by Lemma 46.12. □

Our next goal is to prove Theorem 46.10 for $M = \mathbb{R}^n$. We will proceed in several steps.

Lemma 46.14. *Let R be a commutative ring and let $\{\mu_x\}_{x \in \mathbb{R}^n}$ be an R-section on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ be a convex bounded subset and let $\beta \in H_n(\mathbb{R}^n | A; R)$. If $\mu_y = \beta_y$ for some $y \in A$, then $\mu_x = \beta_x$ for all $x \in A$, i.e. β is a lift to A.*

Proof. The proof is very similar to the proof of Proposition 45.17. The following are the two key observations:

(1) a convex set is path-connected and thus connected,
(2) since A is convex, for any $x \in A$ and any $\epsilon > 0$ the intersection $A \cap B_\epsilon(x)$ is the intersection of two convex bounded sets, hence convex and bounded.

In the proof of Proposition 45.17 we now need to replace Lemma 45.18 by Lemma 45.9. We leave it as an exercise for the reader to modify the proof of Proposition 45.17 to get a proof of Lemma 46.14. □

Lemma 46.15. *Statements* (i) *and* (ii) *from Theorem 46.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any convex compact subset A of $\mathbb{R}^n$.*

Proof. Let A be a convex compact subset of $\mathbb{R}^n$ and let R be a commutative ring. As pointed out above, if $A = \varnothing$, then there is nothing to prove. Thus we can assume that $A \neq \varnothing$.

Now we can prove that A satisfies the statements (i) and (ii).

(i) Let $\{\alpha_x\}_{x \in \mathbb{R}^n}$ be a section for $\mathbb{R}^n$. Since $A \neq \varnothing$ we can pick a $y \in A$. We denote by $\alpha_A \in H_n(\mathbb{R}^n | A)$ the preimage of α_y under the isomorphism

$$H_n(\mathbb{R}^n | A; R) \;\to\; H_n(\mathbb{R}^n | \{y\}; R)$$

given by Lemma 45.9. It follows from Lemma 46.14 that $\alpha_{A,x} = \alpha_x$ for all $x \in A$, i.e. we have shown that α_A is a lift to A. The uniqueness of the lift α_A follows again from the above isomorphism.

(ii) It follows Lemma 45.9 and 32.3 that $H_i(\mathbb{R}^n | A; R) = H_i(\mathbb{R}^n | \{y\}; R) = 0$ for every $i > n$. □

Lemma 46.16. *Every subset of $\mathbb{R}^n$ that is the union of finitely many convex compact sets satisfies the statements* (i) *and* (ii) *from Theorem 46.10.*

Proof. We consider

$$\mathcal{A} := \{\text{all convex compact subsets of } \mathbb{R}^n\}.$$

This family of subsets of $\mathbb{R}^n$ is closed under intersections.[870] By Lemma 46.15 the two statements hold for each $A \in \mathcal{A}$. The lemma is now an immediate consequence of Lemma 46.13. □

The following lemma is arguably the hardest step towards proving the fundamental theorem of topological manifolds.

Lemma 46.17. *Statements* (i) *and* (ii) *from Theorem 46.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any compact subset A of $\mathbb{R}^n$.*

Proof. To simplify the notation we again drop the R from the notation. Let $\{\alpha_x\}_{x\in\mathbb{R}^n}$ be a section for $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be an arbitrary compact set.

We first prove statement (i). We pick a closed ball B that contains A. We denote by $\alpha_B \in H_n(\mathbb{R}^n|B)$ the lift constructed in Lemma 46.15. By Lemma 46.9 the image α_A of the class α_B under the map $H_n(\mathbb{R}^n|B) \to H_n(\mathbb{R}^n|A)$ is a lift to A. To complete the proof of statement (i) we still need to prove the uniqueness of α_A.

This uniqueness statement is equivalent to showing that if $\beta \in H_n(\mathbb{R}^n|A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in H_n(\mathbb{R}^n|A)$. We pick a singular n-chain $z \in C_n(\mathbb{R}^n)$ that represents β. We denote by $C \subset \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in $\partial z \subset C_{n-1}(\mathbb{R}^n \setminus A)$. Note that C is the union of finitely many compact sets, hence C itself is compact.

Claim. There exist finitely many closed balls $B_1, \dots, B_m$ with the following three properties:

(1) each B_i is contained in $\mathbb{R}^n \setminus C$,
(2) each B_i contains a point $x_i \in A$,
(3) we have $A \subset B_1 \cup \cdots \cup B_m$.

Since C is compact and since $A \cap C = \varnothing$ we can find for each $x \in A$ an $r_x > 0$ with $B_{2r_x}(x) \subset \mathbb{R}^n \setminus C$. Since A is compact there exist finitely many points $x_1, \dots, x_m$ in A such that A is contained in $B_{r_{x_1}}(x_1) \cup \cdots \cup B_{r_{x_m}}(x_m)$. It is now clear that $B_i := \overline{B}_{r_{x_i}}(x_i)$, $i = 1, \dots, m$ have the desired property. We refer to Figure 660 for an illustration. This concludes the proof of the claim.

We write $K := B_1 \cup \cdots \cup B_m$. By (1) the singular n-chain $z \in C_n(\mathbb{R}^n)$ also represents an element $\gamma \in H_n(\mathbb{R}^n|K)$. Note that for each $x \in A$ we have a commutative diagram

$$\begin{tikzcd} H_n(\mathbb{R}^n|K) \arrow[rr, "\gamma=[z]\mapsto\beta=[z]"] \arrow[dr, "\gamma\mapsto\gamma_x"'] & & H_n(\mathbb{R}^n|A) \arrow[dl, "\beta\mapsto\beta_x"] \\ & H_n(\mathbb{R}^n|x). & \end{tikzcd}$$

[870]Here we use one of the most convenient properties of convex sets: the intersection of two convex sets is again convex.

It follows in particular from the commutative diagram that it suffices to show that we have $\gamma = 0 \in H_n(\mathbb{R}^n|K)$. We do so as follows:

(1) By our hypothesis on β and by the above commutative diagram we know that for each $i \in \{1, \dots, m\}$ we have $\gamma_{x_i} = \beta_{x_i} = 0$.
(2) Let $i \in \{1, \dots, m\}$. Since $\gamma_{x_i} = 0$ and since B_i is convex and bounded it follows from Lemma 46.14 (applied to the zero section and to the image of $\gamma \in H_n(\mathbb{R}^n|B_i)$) that $\gamma_x = 0$ for all $x \in B_i$.
(3) By (2) we know that $\gamma_x = 0$ for all $x \in K = B_1 \cup \dots \cup B_m$.
(4) But now it follows from Lemma 46.16 that $\gamma = 0$.

This concludes the proof of statement (i).

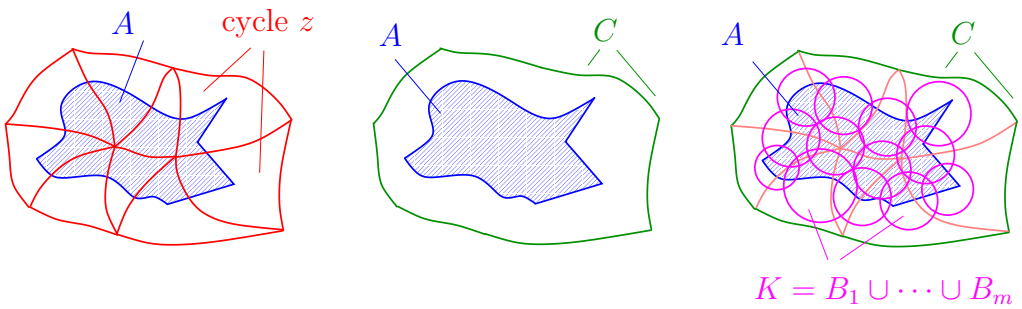


FIGURE 660. Illustration for the proof of Theorem 46.10.

Finally we need to show statement (ii) for A, i.e. we need to show that $H_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in H_i(\mathbb{R}^n|A)$. As above we pick a singular n-chain $z \in C_n(\mathbb{R}^n)$ that represents β. We define C, K and γ as above. By Lemma 46.16 we have $\gamma = 0 \in H_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in H_i(\mathbb{R}^n|A)$. □

Lemma 46.18. *Let M be an n-dimensional topological manifold with empty boundary and let $A \subset M$ be a compact subset. If there exists a chart $\Phi\colon U \to \mathbb{R}^n$ such that A is contained in U, then statements* (i) *and* (ii) *hold for A.*

Proof. The lemma is a fairly straightforward consequence of Lemma 46.17 and the fact that we have isomorphisms

$$\begin{array}{ccccc} H_n(M, M \setminus A; R) & \xleftarrow{\cong} & H_n(U, U \setminus A; R) & \xrightarrow{\Phi} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Phi(A); R) \\ & \uparrow & & \uparrow & \\ & \text{Excision Theorem 45.8} & & \text{isomorphism since } \Phi \text{ is a homeomorphism} & \end{array}$$

We leave it to the reader to fill in the details. □

Now we can finally complete the proof of Theorem 46.10.

Proof of Theorem 46.10. Let M be an n-dimensional topological manifold with empty boundary. We consider

$\mathcal{A}$:= all compact subsets A of M for which there exists a chart $\Phi\colon U \to \mathbb{R}^n$ with $A \subset U$.

By Lemma 46.18 statements (i) and (ii) hold for all subsets in $\mathcal{A}$. It is clear that $\mathcal{A}$ is closed under intersections. Thus it follows from Lemma 46.13 that the statements (i) and (ii) hold for all finite unions of sets in $\mathcal{A}$. Therefore it suffices to prove the following claim.

Claim. Every compact subset A of M is the union of finitely many sets in $\mathcal{A}$.

Let A be a compact subset of M. It is clear that given any point $x \in A$ there exists a chart $\Phi_x\colon U_x \to V_x$ around x such that the image is some open ball in $\mathbb{R}^n$. But since each open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ we can find around each $x \in A$ a chart around x such that the image V_x equals $\mathbb{R}^n$. Now we write $A_x := A \cap \Phi_x^{-1}(\overline{B}_1^n(0))$ and $U'_x := \Phi_x^{-1}(B_1^n(0))$. Since A is compact we can find finitely many $x_1, \dots, x_m$ in A such that A is contained in $U'_{x_1} \cup \cdots \cup U'_{x_m}$. By construction the corresponding charts $\Phi_i := \Phi_{x_i}\colon U_{x_i} \to \mathbb{R}^n$ and the sets $A_i := A_{x_i}$ have the desired properties. This concludes the proof of the claim.

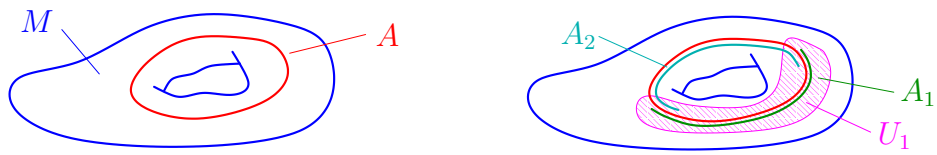


FIGURE 661. Illustration for the proof of Lemma 46.18.

It follows from Lemma 46.18 that statements (i) and (ii) hold for all the A_i and for all arbitrary intersections of the A_i. But then Lemma 46.12 implies that they also hold for $A = A_1 \cup \cdots \cup A_m$. □

46.3. **Proof of Theorem 46.4.** Now we are in a position that we can provide the long-delayed proofs of Theorems 46.2, 46.3 and 46.4. We start out with the proof of Theorem 46.4 since we will use that theorem in the proof of Theorem 46.2.

The key extra ingredient in the proof of Theorem 46.4 is the following proposition.

Proposition 46.19. *Let M be a connected n-dimensional topological manifold and let R be a commutative ring. If M is non-compact and if $\partial M = \varnothing$, then $H_k(M; R) = 0$ for every $k \geq n$.*

Proof. Let M be a connected non-compact n-dimensional topological manifold such that $\partial M = \varnothing$, let R be a commutative ring and let $k \geq n$. Let $\alpha \in H_k(M; R)$. We need to show that $\alpha = 0$. We represent α by a cycle $z = \sum_{i=1}^{m} \sigma_i \otimes r_i \in C_k(M; R) = C_k(M) \otimes R$. The union of the images of $\sigma_1, \dots, \sigma_m$ is a compact subset K of M. Since K is compact it follows from Exercise 46.3 that there exists an open subset $U \subset M$ that contains K and such that the

closure $\overline{U}$ is compact. We consider the open subset $V = M \setminus \overline{U}$ and we consider the long exact sequence of the triple $(M, U \cup V, V)$, see Corollary 31.14. It fits into the following commutative diagram

$$\begin{array}{ccccccc} \dots \longrightarrow H_{k+1}(M, U\cup V; R) & \xrightarrow{\partial} & H_k(U\cup V, V; R) & \longrightarrow & H_k(M, V; R) & \longrightarrow & \dots \\ & & \cong\uparrow & & \uparrow & & \\ & & H_k(U; R) & \xrightarrow{[z]\mapsto[z]=\alpha} & H_k(M; R). & & \end{array}$$

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open sets U and V.[871]

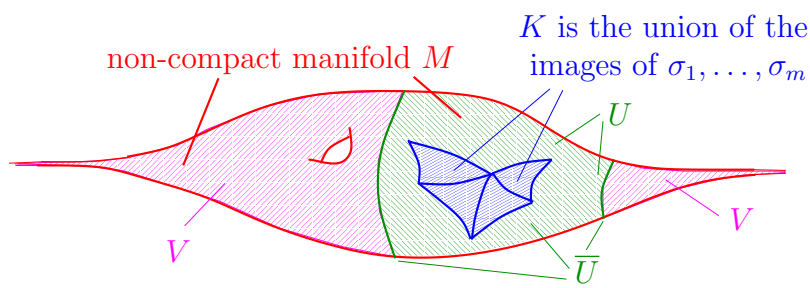


FIGURE 662. Illustration for the proof of Proposition 46.19.

Now we consider the modules to the top-left and top-right of the diagram. We have

$$\begin{array}{rcl} H_{k+1}(M, U\cup V; R) & = & H_{k+1}(M|\overbrace{M\setminus(U\cup V)}^{=\overline{U}\setminus U}; R) \;=\; 0 \quad \text{if } k > n-1 \\ & & \uparrow \\ & & \text{by Theorem 46.10 (ii) since } M \text{ has no boundary and since } \overline{U} \text{ and } \overline{U}\setminus U \text{ are compact} \\ & & \downarrow \\ H_k(M, V; R) & = & H_k(M|\underbrace{M\setminus V}_{=\overline{U}}; R) \;=\; 0 \quad \text{if } k > n. \end{array}$$

After these preparations we come to the actual proof that $\alpha = [z] = 0 \in H_k(M; R)$. We first consider the case $k > n$. We deduce from the above exact sequence and the above calculations that $H_k(U; R) = 0$. But z represents an element in $H_k(U; R)$. It follows that $[z] = 0$ in $H_k(M; R)$.

[871] One could at this point refer to the Excision Theorem 31.17 but this isomorphism follows already from the observation that $C_*(U \cup V) = C_*(U) \oplus C_*(V)$ as chain complexes, since U and V are both open subsets of $U \cup V$.

Now we consider the case $k = n$. It follows from the above discussion that we have a commutative diagram

$$\begin{array}{ccc} H_n(U \cup V, V; R) & \hookrightarrow & H_n(M, V; R) \\ \cong\uparrow & & \uparrow \\ H_n(U; R) & \xrightarrow{[z]\mapsto[z]=\alpha} & H_n(M; R). \end{array}$$

Thus it suffices to show that the image of α in $H_n(M, V; R)$ is zero. We showed in Lemma 46.1 that $\{\alpha_x\}_{x\in M}$ is an R-section of M.[872] Since M is non-compact we can find a point x outside of the compact set K. The class $\alpha_x \in H_n(M, M \setminus \{x\}; R)$ is again represented by z which lies in $K \subset M \setminus \{x\}$. It is thus clear that $\alpha_x = 0 \in H_n(M, M \setminus \{x\}; R)$. But since M is connected it follows from Proposition 45.17 (1) that $\{\alpha_x\}_{x\in M}$ is the zero section. Both the image of α in $H_n(M, V; R)$ and the zero-class are lifts of the zero section to $H_n(M, V; R)$. By Theorem 46.10 (i) both lifts agree, i.e. the image of α in $H_n(M, V; R)$ is zero. □

Now we can easily provide the proof of Theorem 46.4.

Proof of Theorem 46.4. Let M be a connected n-dimensional topological manifold and let R be a commutative ring. Recall that we need to prove the following two statements.

(1) If M is not closed, then $H_i(M; R) = 0$ for $i \geq n$.
(2) If M is closed, then we have $H_i(M; R) = 0$ for $i > n$.

First suppose that M is not closed. The hypothesis that M is not closed means that M is not compact or that it has a non-empty boundary. If M is not compact without boundary, then we had just proved in Proposition 46.19 that $H_i(M; R) = 0$ for $i \geq n$. If M has non-empty boundary, then we saw in Corollary 45.5 that M is homotopy equivalent to a non-compact n-dimensional topological manifold N without boundary. It follows from Corollary 30.8 and Proposition 46.19 that for any $i \geq n$ we have $H_i(M; R) \cong H_i(N; R) = 0$.

Now suppose that M is closed. Let $i > n$. We have

$$H_i(M; R) \;=\; H_i(M, \overbrace{M \setminus M}^{=\varnothing}; R) \;=\; H_i(M|M; R) \;\underset{\substack{\uparrow \\ \text{by Theorem 46.10 (ii) since } M \text{ is compact and since } M \text{ has no boundary}}}{=}\; 0.$$

□

46.4. **Proof of Theorems 46.2 and 46.3.** The key to proving Theorems 46.2 and 46.3 is the following proposition which later on we will apply to $R = \mathbb{Z}$ and $R = \mathbb{F}_2$.

Proposition 46.20. *Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring. Then the following hold:*

[872] Here we used a second time that M has no boundary.

(1) *If M is R-orientable, then for any $x \in M \setminus \partial M$ the map*

$$\begin{array}{ccccc} H_n(M, \partial M; R) & \to & H_n(M, M \setminus \{x\}; R) & \cong & R \\ & & & \uparrow & \\ & & & \textit{isomorphism by Lemma 45.16} & \end{array}$$

is an isomorphism.

(2) *If M is not orientable, then $H_n(M, \partial M; \mathbb{Z}) = 0$.*

Proof. Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring.

We first consider the case that M is actually closed. Recall that we denote by $\Gamma_R(M)$ the set of all R-sections of M. By Lemma 46.1 we have a well-defined R-module homomorphism

$$\begin{array}{rcl} H_n(M; R) & \mapsto & \Gamma_R(M) \\ \alpha & \mapsto & \{\alpha_x\}_{x \in M}. \end{array}$$

Since M is closed we can apply Theorem 46.10 (i) to the special case of the compact subset $A = M$ and we obtain that the above map is an isomorphism. The desired statements (1) and (2) now both follow from Lemma 45.20.

Now we consider the case that M is not closed, i.e. we suppose that $\partial M \neq \varnothing$. As usual we denote by DM the double of M. The idea now is to prove the desired statements for M by reducing them to the equivalent statements for the closed topological manifold DM that we had just proved.

As before we think of the double as $DM = (M \sqcup M')/\sim$ where M' is another copy of M. We consider the maps

$$\begin{tikzcd} \underbrace{H_n(M'; R)}_{= 0 \text{ by Thm. 46.4}} \arrow[r] & H_n(DM; R) \arrow[r] \arrow[dr, "\Phi"'] & H_n(DM, M'; R) \arrow[r] & H_{n-1}(M'; R) \arrow[r, "\text{injective}", "\text{by 45.12}"'] & H_{n-1}(DM; R) \\ & & H_n(M, \partial M; R), \arrow[u, "\cong"'] & & \end{tikzcd}$$

where on the top we have written down the long exact sequence of the pair (DM, M'). Note that the vertical map is an isomorphism by the Excision Theorem 45.10. The map Φ is defined to be the unique homomorphism that makes the triangle commute.

It follows immediately from the information provided in the above diagram that the map $\Phi\colon H_n(DM; R) \to H_n(M, \partial M; R)$ is an isomorphism. Now we turn to the proof of the two statements of the proposition.

(1) Now suppose that M is R-orientable. Let $x \in M \setminus \partial M$. We consider the following diagram

$$\begin{tikzcd}
H_n(DM;R) \arrow[rr, "{[DM]\mapsto\sigma_x}", "\text{isomorphism since } DM \text{ closed}"'] \arrow[dr, "\cong"'] & & H_n(DM, DM\setminus\{x\};R) \\
& H_n(DM, M';R) \arrow[ur] & \\
H_n(M,\partial M;R) \arrow[uur, "\cong", "\text{Excision Thm. 45.10}"'] \arrow[rr] & & H_n(M, M\setminus\{x\};R). \arrow[uu, "\mu_x\mapsto\sigma_x"', "\text{Excision Thm. 45.10}"]
\end{tikzcd}$$

Here all maps are the obvious inclusion induced maps. This implies that the diagram commutes. Since DM is closed and since it is R-orientable by Lemma 45.29 we know from the first part of the proof that the top horizontal map is an isomorphism. Furthermore by the above discussion we know that the top-left diagonal map is an isomorphism. Finally two of the maps are isomorphisms by the Excision Theorem 45.10. It now follows immediately that the bottom horizontal map is also an isomorphism. But that is exactly what we were supposed to show.

(2) If M is not orientable, then it follows from Lemma 45.29 that the double DM is not orientable either. We obtain that

$$\begin{array}{ccccc}
H_n(M,\partial M;\mathbb{Z}) & \cong & H_n(DM;\mathbb{Z}) & = & 0. \\
& \uparrow & & \uparrow & \\
& \text{see above} & \multicolumn{2}{l}{\text{since } DM \text{ is closed and non-orientable and}} & \\
& & \multicolumn{2}{l}{\text{since we have proved (1) in the closed case}} &
\end{array}$$

□

Remark. Note that the second statement of Proposition 46.20 only deals with integer coefficients. A more general statement is given by [Hat, Theorem 3.26]. This theorem says that if M is a closed connected n-dimensional topological manifold that is *not* R-orientable, then for any $x \in M$ the map $\rho_x \colon H_n(M;R) \to H_n(M, M\setminus\{x\};R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$. For example for $R = \mathbb{Z}$ we have $\{r \in \mathbb{Z} \mid 2r = 0\} = \{0\}$, i.e. the map ρ_x is the zero map and by injectivity we have $H_n(M;\mathbb{Z}) = 0$. In the case of closed topological manifolds we can thus view [Hat, Theorem 3.26] as a generalization of Proposition 46.20 (2). We will not make use of [Hat, Theorem 3.26].

Now we can finally provide the proofs of Theorems 46.2 and 46.3.

Proof of Theorem 46.2. Let M be a compact, connected, non-empty n-dimensional topological manifold. We have to show that the following statements are equivalent:

(1) the topological manifold M is orientable,
(2) $H_n(M,\partial M;\mathbb{Z}) \neq 0$,
(3) $H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$,
(4) for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon H_n(M,\partial M;\mathbb{Z}) \;\to\; H_n(M, M\setminus\{x\};\mathbb{Z})$$

is an isomorphism.

We have (1) $\Rightarrow$ (4) by Proposition 46.20 (1) applied with $R = \mathbb{Z}$. Since M is non-empty we obtain from Corollary 32.4 that there exists a point $x \in M \setminus \partial M$. Therefore we have (4) $\Rightarrow$ (3) by Lemma 45.16. Evidently we have (3) $\Rightarrow$ (2). Furthermore we have (2) $\Rightarrow$ (1) by Proposition 46.20 (2). $\square$

Proof of Theorem 46.3. Let M be a compact, connected, non-empty n-dimensional topological manifold. We have to show the following two statements:

(1) $H_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$, and
(2) for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon H_n(M, \partial M; \mathbb{F}_2) \quad \to \quad H_n(M, M \setminus \{x\}; \mathbb{F}_2)$$

is an isomorphism.

By Proposition 45.24 we know that M admits an $\mathbb{F}_2$-orientation. Statement (2) follows immediately from this fact and Proposition 46.20 (1) applied to $R = \mathbb{F}_2$. As in the proof of Theorem 46.2, the first statement is an immediate consequence of the second statement, together with the fact that there exists a point in $M \setminus \partial M$ and Lemma 45.16. $\square$

46.5. **Further properties of topological manifolds.** Let M be a compact n-dimensional topological manifold. With Theorem 46.2 we now completely understand the n-dimensional homology of M. In this section we will obtain some results on the $(n-1)$-dimensional homology of M.

The first part of the following proposition generalizes the result of Proposition 39.17 from manifolds to topological manifolds. The second part is new even for manifolds.

Proposition 46.21. *Let M be a connected, compact n-dimensional topological manifold.*

(1) *If M is orientable or if M is not closed, then $H_{n-1}(M; \mathbb{Z}) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.*
(2) *If M is non-orientable and closed, then $H_{n-1}(M; \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.*

Remark.

(1) As a reality check it is good to recall the calculation of the homology groups of closed topological 2-dimensional manifolds that we gave in Proposition 35.10. There we showed for any $g \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have

$$H_n(\Sigma_g) \;\cong\; \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0 \end{cases} \qquad \text{and} \qquad H_n(k \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0, \end{cases}$$

where, as always, we denote by Σ_g the surface of genus g. Fortunately these calculations are coherent with Proposition 46.21.

(2) It follows immediately from the Universal Coefficient Theorem 42.19, Theorem 46.2, Proposition 46.21 (1) and Lemma 42.17 that for any closed orientable n-dimensional topological manifold and any abelian group G we have $H_n(M; G) \cong G$.

In the proof of Proposition 46.21 we will use the following lemma which is an immediate consequence of the classification of finitely generated abelian groups, see Theorem 11.4, and of Lemma 42.17.

Lemma 46.22. *Let A be a finitely generated abelian group. Then the following holds:*

$$A \text{ is a free abelian group} \quad \Longleftrightarrow \quad \operatorname{Tor}(A, \mathbb{Z}_p) = 0 \text{ for all primes } p.$$

Now we can provide the proof of Proposition 46.21. We consider the two cases that the topological manifold is closed and that it has non-empty boundary separately.

Proof of Proposition 46.21 for closed topological manifolds. Let M be a closed, connected n-dimensional topological manifold. By Proposition 45.2 we know that $H_{n-1}(M;\mathbb{Z})$ is a finitely generated abelian group.

We first consider the case that M is orientable. We want to show that $H_{n-1}(M;\mathbb{Z})$ is a free abelian group. By Lemma 46.22 it suffices to show that $\operatorname{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) = 0$ for all primes p.

So let p be a prime. We have

$$\begin{array}{ccccc}
\mathbb{F}_p & \cong H_n(M;\mathbb{F}_p) & \cong H_n(M;\mathbb{Z}) \otimes \mathbb{F}_p \oplus \operatorname{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) & \cong \mathbb{F}_p \oplus \operatorname{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_p). \\
 & \uparrow & \uparrow & \uparrow \\
 & \begin{array}{c}\text{by Proposition 46.20 (1),} \\ \text{since by Lemma 45.25} \\ M \text{ is also } \mathbb{F}_p\text{-orientable}\end{array} & \text{Universal Coefficient Theorem 42.23} & \begin{array}{c}\text{by Theorem 46.2 (3) we} \\ \text{have } H_n(M;\mathbb{Z}) \cong \mathbb{Z}\end{array}
\end{array}$$

It follows from Lemma 11.8 that $\operatorname{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) = 0$.

Now we turn to the case that M is non-orientable. By Proposition 45.26 there exists a connected 2-fold covering $q\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\begin{tikzcd}
H_{n-1}(M;\mathbb{Z}) \arrow[r, "q^*"] \arrow[rr, bend right, "\cdot 2"'] & H_{n-1}(\widetilde{M};\mathbb{Z}) \arrow[r, "q_*"] & H_{n-1}(M;\mathbb{Z}).
\end{tikzcd}$$

Here q^* denotes the transfer map that we introduced on page 1064 and q_* denotes of course the map induced by the projection q. By Proposition 44.14 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case that we had just proved that $H_{n-1}(\widetilde{M};\mathbb{Z})$ is torsion-free. Now we see that

$$\begin{array}{ccccc}
\text{torsion subgroup of } H_{n-1}(M;\mathbb{Z}) & \subset \ker(q^*) & \subset \ker(q_* \circ q^*) & \subset \{a \in H_{n-1}(M;\mathbb{Z}) \mid 2a = 0\}. \\
 & \uparrow & & \uparrow \\
 & \text{since } H_{n-1}(\widetilde{M};\mathbb{Z}) \text{ is torsion-free} & & \text{by the above commutative diagram}
\end{array}$$

From these inclusions and the classification of finitely generated abelian groups we obtain easily that $H_{n-1}(M;\mathbb{Z}) \cong (\mathbb{Z}_2)^r \oplus \mathbb{Z}^k$ for some $r \in \mathbb{N}_0$ and some $k \in \mathbb{N}_0$.

It remains to show that $r = 1$. In light of Lemma 42.17 it suffices to show that we have $\operatorname{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2$. The argument is very similar to the argument in the orientable

case. More precisely, we have

$$\begin{array}{ccccccc} \mathbb{F}_2 & \cong & H_n(M;\mathbb{F}_2) & \cong & H_n(M;\mathbb{Z})\otimes\mathbb{F}_2 \oplus \operatorname{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{F}_2) & \cong & 0 \oplus \operatorname{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{F}_2). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by Proposition 45.24} & & \text{Universal Coefficient Theorem 42.23} & & \text{by Theorem 46.2 (3)} & \\ & \text{and Theorem 46.3} & & & & \text{we have } H_n(M;\mathbb{Z})=0 & \end{array}$$

□

Proof of Proposition 46.21 for topological manifolds with non-empty boundary (∗). Let M be a connected, compact n-dimensional topological manifold with non-empty boundary. We need to show that $H_{n-1}(M;\mathbb{Z})$ is a free abelian group. There are two approaches to the proof of this statement.

(1) We can imitate the proof of the closed case, or
(2) we reduce the proof of the proposition to the case of closed manifolds that we had just dealt with.

We will pursue the second approach. First recall that by Lemma 45.29 the double DM is orientable if and only if M is orientable.

We first consider the case that M is orientable. By Proposition 46.21 we know that $H_{n-1}(DM;\mathbb{Z})$ is a finitely generated free abelian group. It follows from Lemma 45.12 that the group $H_{n-1}(M;\mathbb{Z})$ is isomorphic to a subgroup of $H_{n-1}(DM;\mathbb{Z})$, hence by Lemma 11.8 (3) it is also a finitely generated free abelian group.

Now we consider the case that M is non-orientable. We denote by M' a copy of M and as usual we denote by $DM = M \cup_{\partial M=\partial M'} M'$ the double. We consider the Mayer–Vietoris sequence[873]

$$\cdots \to H_n(DM;\mathbb{Z}) \to H_{n-1}(\partial M;\mathbb{Z}) \to H_{n-1}(M;\mathbb{Z}) \oplus H_{n-1}(M';\mathbb{Z}) \to H_{n-1}(DM;\mathbb{Z}) \to \dots$$

We make the following observations:

(1) By Proposition 32.1 (3) we know that ∂M is a closed $(n-1)$-dimensional topological manifold. Thus we obtain from Theorem 46.2 that $H_{n-1}(\partial M;\mathbb{Z})$ is isomorphic to a free abelian group $\mathbb{Z}^r$ for some $r \in \mathbb{N}_0$. (We do not have any control over the orientability of ∂M and the number of components of ∂M but this does not matter.)
(2) Since DM is non-orientable we know from Theorem 46.2 that $H_n(DM;\mathbb{Z}) = 0$ and we know from Proposition 46.21 that $H_{n-1}(DM;\mathbb{Z}) \cong \mathbb{Z}^s \oplus \mathbb{Z}_2$ for some $s \in \mathbb{N}_0$.
(3) Since $M = M'$ we have $H_{n-1}(M;\mathbb{Z}) = H_{n-1}(M';\mathbb{Z})$. By the classification of finitely generated abelian groups we can pick an isomorphism $H_{n-1}(M;\mathbb{Z}) \cong F \oplus T$ where F is a free abelian group and T is a torsion group.

We can thus rewrite the above sequence as follows:

$$0 \;\to\; \mathbb{Z}^r \;\xrightarrow{\varphi}\; F \oplus F \oplus T \oplus T \;\to\; \mathbb{Z}^s \oplus \mathbb{Z}_2 \;\to\; \dots$$

The map $\varphi\colon \mathbb{Z}^r \to F \oplus F \oplus T \oplus T$ is a monomorphism, but since $\mathbb{Z}^r$ is torsion-free and since T is finite we see that $\operatorname{im}(\varphi) \cap (T \oplus T)$ is trivial. But this implies that $T \oplus T$ is isomorphic

[873]The existence of such a Mayer–Vietoris sequence is proved in a very similar fashion to the Mayer–Vietoris Theorem 33.10 for manifolds, we just need to replace the Collar Neighborhood Theorem 10.10 by the Topological Collar Neighborhood Theorem 45.3.

to a subgroup of $\mathbb{Z}^s \oplus \mathbb{Z}_2$. But this is only possible if $T = 0$. This concludes the proof that $H_{n-1}(M;\mathbb{Z})$ is a free abelian group. $\square$

46.6. **The fundamental class and induced orientations.** In Section 45.9 we saw that an orientation on a topological manifold M induces orientations on codimension-zero submanifolds of M, on the boundary ∂M and on covering spaces of M. If M is compact, then M admits a fundamental class and we will see in this section how it is related to the fundamental classes of the various corresponding oriented topological manifolds.

First of all, in Lemma 45.28 we saw that a codimension zero submanifold W of an oriented topological manifold M inherits an orientation. For compact topological manifolds the following lemma shows the relationship between the corresponding fundamental classes.

Lemma 46.23. *Let M be a compact n-dimensional topological manifold and let $W \subset M$ be a compact n-dimensional submanifold. We write $\mathring{W} = W \setminus \partial W$. Then the following two statements hold:*

(1) *The inclusion induced map $j_*\colon H_n(W, \partial W;\mathbb{Z}) \to H_n(M, M \setminus \mathring{W};\mathbb{Z})$ is an isomorphism.*
(2) *If M is oriented and if $[M]$ is the corresponding fundamental class, then the image of $[M]$ under the map*

$$H_n(M, \partial M;\mathbb{Z}) \;\to\; H_n(M, M \setminus \mathring{W};\mathbb{Z}) \;\xrightarrow{j_*^{-1}}\; H_n(W, \partial W;\mathbb{Z})$$

is the fundamental class for the oriented topological manifold W.

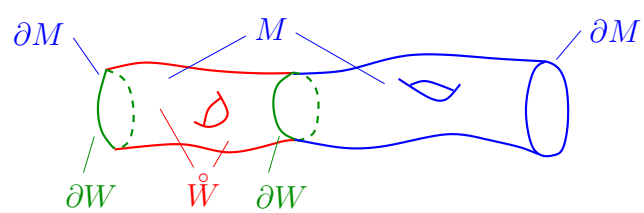


FIGURE 663. Illustration for the proof of Lemma 46.23.

Proof $(*)$.

(1) The first statement is an immediate consequence of the Excision Theorem 45.10 for topological manifolds.
(2) We denote by $\Phi\colon H_n(M, \partial M;\mathbb{Z}) \to H_n(W, \partial W;\mathbb{Z})$ the map given in statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. But this follows immediately from Corollary 46.5 and from considering for each $x \in W \setminus \partial W$ the

following commutative diagram

$$\begin{tikzcd}
H_n(M,\partial M;\mathbb{Z}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\Phi"] & H_n(M, M\setminus \mathring{W};\mathbb{Z}) \arrow[dl] & H_n(W,\partial W;\mathbb{Z}) \arrow[l, "\cong", "j_*"'] \arrow[d] \\
H_n(M, M\setminus\{x\};\mathbb{Z}) & & H_n(W, W\setminus\{x\};\mathbb{Z}). \arrow[ll, "\cong"']
\end{tikzcd}$$

We leave the few remaining details to the reader. □

In Lemma 45.28 (3) we showed that a covering space $\widetilde{M}$ of an oriented topological manifold M inherits an orientation from M. The following proposition can now be viewed as a generalization of Proposition 39.20.

Proposition 46.24. *Let M be a compact oriented n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. We denote by $[M]$ the fundamental class of M. Then*[874] *$p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$ of the oriented topological manifold $\widetilde{M}$ and we have $p_*\big(\big[\widetilde{M}\big]\big) = \big[\widetilde{M} : M\big] \cdot [M]$.*

Proof $(*)$. Let M be a compact n-dimensional topological manifold. To simplify the notation we assume that M is closed. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering and let $\{\mu_x\}_{x\in M}$ be an orientation for M. We denote by $\{\nu_x\}_{x\in\widetilde{M}}$ the orientation on $\widetilde{M}$ that we defined in Lemma 45.28 (3).

We want to show that $p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$. By Corollary 46.5 it suffices to show that for all $y \in \widetilde{M}$ we have $p^*([M])_y = \nu_y \in H_n(\widetilde{M}, \widetilde{M}\setminus\{y\};\mathbb{Z})$. So let $y \in \widetilde{M}$. We pick an open neighborhood V around y such that $U := p(V)$ is uniformly covered. We write $x = p(y)$ and we denote by $q\colon U \to V$ the inverse of the homeomorphism $p\colon V \to U$.

We pick a representative $z = a_1 \cdot \sigma_1 + \cdots + a_m \cdot \sigma_m \in C_n(M)$ for $[M]$. By applying Proposition 31.21 to the *open* covering of M given by $\{U, M\setminus\{x\}\}$ we can arrange that the image of each σ_i lies in U or in $M\setminus\{x\}$. We denote by z' the sum of all summands in z such that the image of σ_i lies in U.

A moment's thought shows that in $H_n(M, M\setminus\{x\};\mathbb{Z})$ we have $[z] = [z']$ and that in $H_n(\widetilde{M}, \widetilde{M}\setminus\{y\})$ we have $[p^*(z)] = [q_*(z')]$. From these observations we deduce that the following diagram commutes:

$$\begin{tikzcd}
H_n(M;\mathbb{Z}) \arrow[rrr, "p^*"] \arrow[d, "{[M]=[z]\mapsto[z]=[z']=[\mu_x]}"] & & & H_n(\widetilde{M};\mathbb{Z}) \arrow[d, "{[p^*z]\mapsto[p^*z]=[q_*(z')]}"'] \\
H_n(M, M\setminus\{x\};\mathbb{Z}) \arrow[rrr, bend right, "\mu_x\mapsto\nu_y"'] & H_n(U, U\setminus\{x\};\mathbb{Z}) \arrow[l, "\cong"'] \arrow[r, "\cong", "q_*"'] & H_n(V, V\setminus\{y\};\mathbb{Z}) \arrow[r, "\cong"] & H_n(\widetilde{M}, \widetilde{M}\setminus\{y\};\mathbb{Z}).
\end{tikzcd}$$

[874]Here $p^*\colon H_k(M;\mathbb{Z}) \to H_k(\widetilde{M};\mathbb{Z})$ denotes the transfer map that we introduced on page 1064.

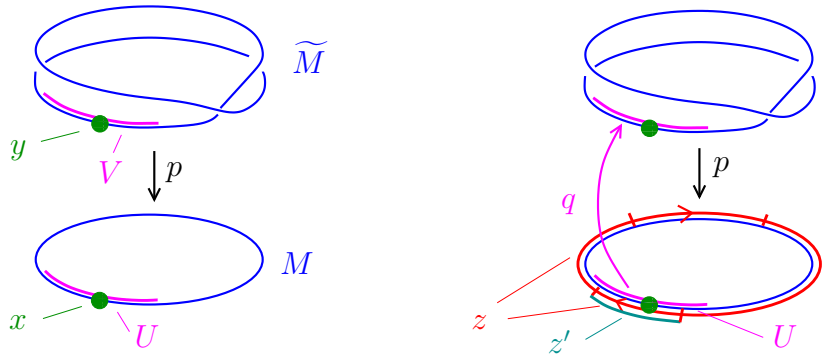


FIGURE 664.

Note that the bottom horizontal maps to the left and right are isomorphisms by the Excision Theorem 45.8 and the middle horizontal map is an isomorphism since q is a homeomorphism. It follows from this diagram that we indeed have $p^*([M])_y = \nu_y$.

Finally we have

$$p_*\big([\widetilde{M}]\big) \underset{\substack{\uparrow \\ \text{by the above}}}{=} p_*(p^*([M])) \underset{\substack{\uparrow \\ \text{Proposition 44.14}}}{=} [\widetilde{M} : M] \cdot [M].$$

□

In Lemma 45.29 we saw that the double of an oriented topological manifold with non-empty boundary comes with a natural orientation. The following lemma following relates the various fundamental classes in this construction.

Lemma 46.25. *Let M be a compact oriented n-dimensional topological manifold with non-empty boundary. If we equip DM with the orientation coming from Lemma 45.29, then*

$$[DM] \;=\; D([M]) \;\in\; H_n(DM;\mathbb{Z}),$$

where $D\colon H_n(M,\partial M;\mathbb{Z}) \to H_n(DM;\mathbb{Z})$ denotes the doubling homomorphism that we introduced in Lemma 45.13.

Proof. Let $\{\mu_x\}_{x\in M\setminus\partial M}$ be an orientation for M. We denote by $\{\alpha_x\}_{x\in DM}$ the corresponding orientation for DM that we had constructed in Lemma 45.29. To simplify the notation we write $DM = (M \sqcup M')/\sim$ where M' is a copy of M and we view M as a submanifold of the double. By Corollary 32.4 there exists an $x \in M \setminus \partial M$. We consider the following

diagram

$$\begin{tikzcd}
H_k(M,\partial M;\mathbb{Z}) \arrow[rr, "D"] \arrow[dr] \arrow[dd] & & H_k(DM;\mathbb{Z}) \arrow[dl] \arrow[ddl, "\cong"] \\
& H_k(DM,M';\mathbb{Z}) \arrow[d] & \\
H_n(M,M\setminus\{x\};\mathbb{Z}) \arrow[r, "\cong", "\mu_x\mapsto\alpha_x"'] & H_n(DM,DM\setminus\{x\};\mathbb{Z}). &
\end{tikzcd}$$

Here all maps are the obvious maps for pairs of topological spaces. The lemma follows from the following observations:

(1) The top triangle commutes by Lemma 45.13 (3). The parallelogram on the left and the triangle on the right commute by the functoriality of homology.
(2) The homomorphism to the bottom left is an isomorphism by the Excision Theorem 45.10 for topological Manifolds.
(3) The right diagonal map is an isomorphism by Theorem 46.2.
(4) By definition $[M]$ gets sent to $\mu_x \in H_n(M, M \setminus \{x\};\mathbb{Z})$ and by definition μ_x gets sent to α_x. It follows from the commutativity of the diagram that $D([M])$ gets sent to α_x. But since the right diagonal map is an isomorphism it follows, from the definition of $[DM]$, that $[DM] = D([M])$. □

In Proposition 45.30 we saw that the boundary of an oriented topological manifold inherits an orientation. The following proposition can now be seen as a generalization of Proposition 39.6 from manifolds to topological manifolds.

Proposition 46.26. *Let M be a compact n-dimensional topological manifold.*

(1) *We denote by $\partial\colon H_n(M,\partial M;\mathbb{Z}) \to H_{n-1}(\partial M;\mathbb{Z})$ the connecting homomorphism of the long exact sequence of the pair $(M,\partial M)$. If M is oriented, then*[875]
$$\partial[M] \;=\; [\partial M] \;\in\; H_{n-1}(\partial M;\mathbb{Z}).$$
(2) *Similarly, for the connecting homomorphism $\partial\colon H_n(M,\partial M;\mathbb{F}_2) \to H_{n-1}(\partial M;\mathbb{F}_2)$ we have*
$$\partial[M]_{\mathbb{F}_2} \;=\; [\partial M]_{\mathbb{F}_2} \;\in\; H_{n-1}(\partial M;\mathbb{F}_2).$$

Proof. The proof of the first part of the proposition follows from the definition of the orientation of the boundary ∂M and a judicious use of the Excision Theorem 31.17. We leave it as an exercise to the sedulous reader to fill in the details. Alternatively a proof can be found in [Bre, Lemma 9.1].

The second part is proved in a similar fashion. Not surprisingly for this part we also decline to provide the details of the argument. □

[875]Note that ∂M is a closed $(n-1)$-dimensional topological manifold by Proposition 32.1 (3).

Remark. Given a compact oriented n-dimensional topological manifold the conclusion of Proposition 46.26 can also be summarized in the following commutative "diagram"

$$\begin{tikzcd}
\mathbb{Z}\text{-oriented manifold } M \arrow[r, mapsto, "\text{take boundary}"] \arrow[d, mapsto, "\text{take fundamental class}"] & \mathbb{Z}\text{-oriented manifold } \partial M \arrow[d, mapsto, "\text{take fundamental class}"] \\
{[M]} \arrow[r, mapsto, "\text{apply connecting homomorphism}", "H_n(M,\partial M;\mathbb{Z})\to H_{n-1}(\partial M;\mathbb{Z})"'] & \partial[M] = [\partial M].
\end{tikzcd}$$

Example. We will illustrate Proposition 46.26 with the Möbius band M. By Proposition 45.24 and Theorem 46.3 it has an $\mathbb{F}_2$-fundamental class $[M] \in H_2(M, \partial M; \mathbb{F}_2)$. In Figure 665 we consider the Möbius band with two singular 1-simplices α_1, α_2 and two singular 2-simplices σ_1, σ_2. In this case we have

$$\partial\big(\underbrace{\sigma_1 + \sigma_2}_{=:\mu}\big) \;=\; \underbrace{\alpha_1 + \alpha_2}_{\in C_1(\partial M)}.$$

In particular, $\mu \otimes 1$ is a cycle in $C_2(M, \partial M; \mathbb{F}_2)$. It easily follows from Proposition 46.8 that $\mu \otimes 1$ represents the $\mathbb{F}_2$-fundamental class in $H_2(M, \partial M; \mathbb{F}_2)$. Furthermore, if we denote by $\partial\colon H_2(M, \partial M; \mathbb{F}_2) \to H_1(\partial M; \mathbb{F}_2)$ the connecting homomorphism of the long exact sequence in $\mathbb{F}_2$-homology of the pair $(M, \partial M)$, then

$$\partial[M] \;=\; \partial[\mu \otimes 1] \;\underset{\substack{\uparrow \\ \text{Proposition 31.13}}}{=}\; [\partial\mu \otimes 1] \;\underset{\substack{\uparrow \\ \text{by the above}}}{=}\; \big[(\alpha_1 + \alpha_2) \otimes 1\big],$$

but the last class is clearly[876] an $\mathbb{F}_2$-fundamental class for the circle ∂M.

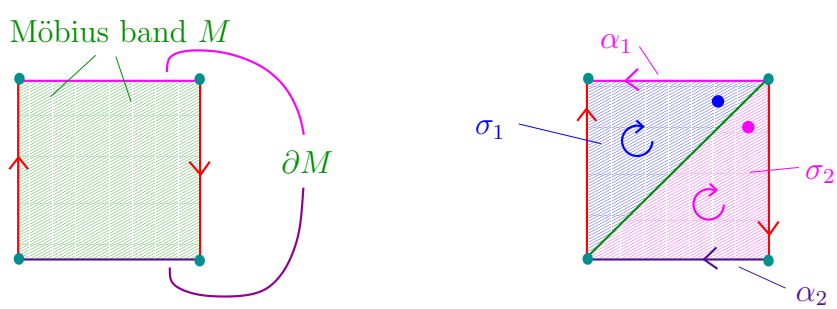


FIGURE 665.

The following corollary is the generalization of Corollary 39.10 to the case of topological manifolds.

[876]Why is this clear?

Corollary 46.27. *Let M be a compact n-dimensional topological manifold. Then the following two statements hold:*

(1) *We have*

$$\ker\left\{H_{n-1}(\partial M;\mathbb{F}_2) \to H_{n-1}(M;\mathbb{F}_2)\right\} = \mathbb{F}_2\cdot[\partial M]_{\mathbb{F}_2} \in H_{n-1}(\partial M;\mathbb{F}_2).$$

Furthermore, if M is oriented, then

$$\ker\left\{H_{n-1}(\partial M;\mathbb{Z}) \to H_{n-1}(M;\mathbb{Z})\right\} = \mathbb{Z}\cdot[\partial M] \in H_{n-1}(\partial M;\mathbb{Z}).$$

(2) *Let M be a compact n-dimensional topological manifold with non-trivial boundary. Then the boundary ∂M is not a retract of M.*

Proof. The proof is verbatim the same as the proof of Corollary 39.10, the only difference in the proof is that we need to replace Proposition 39.9 by Proposition 46.26.

As a reminder, the first statement is proved using the long exact sequence in homology of the pair $(M,\partial M)$ together with Proposition 46.26. The second statement follows from the first statement and the usual functoriality argument, this time we employ the functoriality of homology with $\mathbb{F}_2$-coefficients and the fact that $[\partial M]_{\mathbb{F}_2}$ is non-zero. □

Example. It follows from Corollary 46.27 that the boundary of the Möbius band M is not a retract of the Möbius band. A more down-to-earth proof is given as follows: we pick a base point $x_0\in\partial M$ and we denote by α and β the loops shown in Figure 666. Note that α represents a generator of $\pi_1(\partial M,x_0)\cong\mathbb{Z}$ and that β represents a generator of $\pi_1(M,x_0)\cong\mathbb{Z}$. We denote by $i\colon\partial M\to M$ the inclusion map. Note that $i_*([\alpha])=2\cdot[\beta]$. If there was a retraction $r\colon M\to\partial M$, then we would obtain a commutative diagram

$$\begin{tikzcd} & \pi_1(M,x_0) \arrow[dr, "r_*"] & \\ \pi_1(\partial M,x_0) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*"'] & & \pi_1(\partial M,x_0) \end{tikzcd}$$

which by the above gives rise to the commutative diagram

$$\begin{tikzcd} & \mathbb{Z} \arrow[dr, "??"] & \\ \mathbb{Z} \arrow[ur, "\cdot 2"] \arrow[rr, "\mathrm{id}"'] & & \mathbb{Z}, \end{tikzcd}$$

but such a commutative diagram cannot exist. Looking more carefully one sees that the two proofs that ∂M is not a retract of the Möbius band M are in some sense the same proof. More precisely, we have the following commutative diagram

$$\begin{tikzcd} \mathbb{Z}=\pi_1(\partial M,x_0) \arrow[r, "i_*"] \arrow[d] & \pi_1(M,x_0)=\mathbb{Z} \arrow[d] \\ \mathbb{Z}=H_1(\partial M;\mathbb{Z}) \arrow[r, "i_*"] \arrow[d] & H_1(M;\mathbb{Z})=\mathbb{Z} \arrow[d] \\ \mathbb{F}_2=H_1(\partial M;\mathbb{F}_2) \arrow[r, "i_*"] & H_1(M;\mathbb{F}_2)=\mathbb{F}_2. \end{tikzcd}$$

Here the top vertical maps are given by the Hurewicz homomorphism from page 924 and the bottom vertical maps are given by the map $[\alpha]\mapsto[\alpha\otimes 1]$ from page 1025. Since both maps are natural, see Proposition 38.2 (4) and page 1025, we see that the diagram commutes. The above observation that the top horizontal map is given by multiplication by 2 translates into the fact that the bottom horizontal map is the zero map.

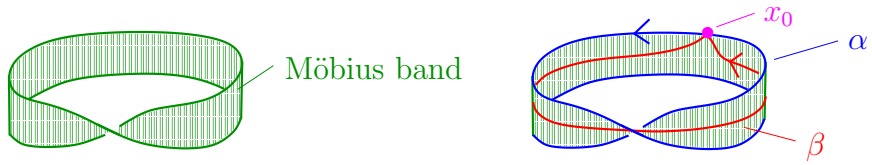


FIGURE 666.

46.7. **Lifts of homology orientations for topological manifolds with boundary** $(*)$**.** In this very technical section we generalize Theorem 46.10 (i) to the case of topological manifolds with boundary. Unless the reader is blessed with a surplus of energy it is best to skip this section.

Definition. Let M be a topological manifold. We say a compact subset $A \subset M$ is a *product close to the boundary* if there exists a collar neighborhood $\partial M \times [0,1]$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap (\partial M \times [0,1]) = K \times [0,1]$.

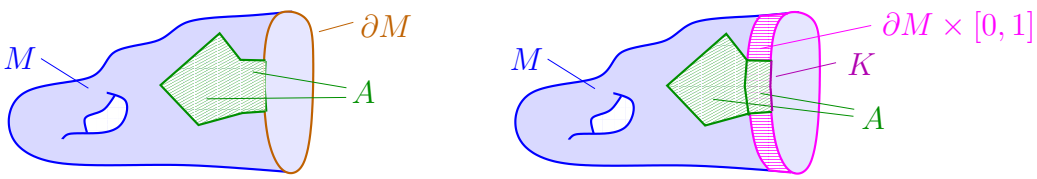


FIGURE 667.

Remark. If M is a topological manifold with *empty boundary*, then every compact subset is in fact a product close to the boundary.

We leave the proof of the following elementary lemma as an exercise to the reader.

Lemma 46.28. *Let M be a topological manifold and let $\partial M \times [0,1]$ be a collar neighborhood (which always exists by the topological Collar Neighborhood Theorem 45.3). Given any compact subset K of M there exists a compact subset A that contains K and that is a product close to the boundary.*

Throughout this section we will need the following notation.

Notation. Given a subset A of a topological manifold M we write $A_\partial := A \cap (M \setminus \partial M)$. In other words, we have $A_\partial = A \setminus \partial M$. Note that $(M \setminus A) \cup \partial M = M \setminus A_\partial$.

Now we can formulate the main result of this section.

Theorem 46.29. *Let M be an n-dimensional topological manifold and let $A \subset M$ be a compact subset that is a product close to the boundary. Let R be a commutative ring. If $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section for M, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A_∂, i.e. there exists a unique class $\alpha_A^M = \alpha_A \in H_n(M, M \setminus A_\partial; R)$ such that for each $x \in A_\partial$ we have $\alpha_{A,x} = \alpha_x$.*

Proof. Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $A \subset M$ be a compact subset that is a product close to the boundary. We introduce the following notation and conventions:

(1) As usual we denote by DM the double of M. As before we think of the double as $DM = (M \sqcup M')/\sim$ where M' is another copy of M. We recall that by Lemma 45.11 we know that DM is an n-dimensional topological manifold with empty boundary such that $M \subset DM$ is a submanifold.
(2) By definition there exists a collar neighborhood $\partial M \times [0,1]$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap (\partial M \times [0,1]) = K \times [0,1]$.
(3) We denote by $DA = A \cup A' \subset DM$ the double of A. As DA is the union of two compact sets we see that DA itself is compact.
(4) Given $t \in (0,1]$ we write $A_t := A \setminus (K \times [0,t)) = A \cap (M \setminus (\partial M \times [0,t)))$. It follows easily from Lemma 1.18 that A_t is again compact.
(5) It follows from the Excision Theorem 45.10 for topological manifolds that for any $P \in M \setminus \partial M$ the inclusion induced map

$$H_n(M, M \setminus \{P\}; R) \;\to\; H_n(DM, DM \setminus \{P\}; R)$$

is an isomorphism. We will use this isomorphism to identify these two homology groups.

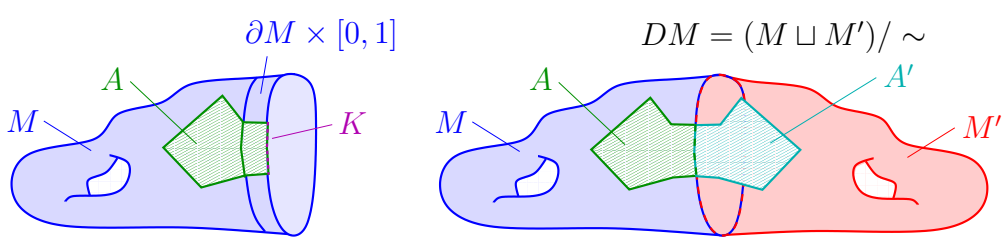


FIGURE 668. Illustration for the proof of Theorem 46.29.

The idea now is the same as in the proof of Proposition 46.20. Namely we want to prove the desired statement for M by reducing it to a corresponding statement for the topological manifold DM that has no boundary.

Claim. The inclusion induced map

$$\beta\colon H_n(M, M \setminus A_\partial; R) \;\to\; H_n(DM, (M \setminus A) \cup M'; R)$$

is an isomorphism.

Recall that A is a product close to the boundary. Using this fact and using a gentle generalization of the Excision Theorem 45.10 for topological manifolds one can easily show that the inclusion induced map is an isomorphism. We leave it to the reader to fill in the details. This concludes the proof of the claim.

We denote by Φ the composition of the following two maps

$$H_n(DM, DM \setminus DA; R) \;\to\; H_n(DM, (M \setminus A) \cup M'; R) \xrightarrow{\beta^{-1}} H_n(M, M \setminus A_\partial; R).$$

After all these preparations we now turn to the actual proof of the theorem. Thus let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-section for M. As in the proof of Lemma 45.29 we see that there exists an R-section $\{\alpha_x\}_{x \in DM}$ for DM that restricts to the given R-section on M. By Theorem 46.10 (i) there exists a lift $\mu_{DA} \in H_n(DM, DM \setminus DA; R)$ of the R-section on DM. We claim that $\Phi(\mu_{DA})$ is a lift for M. So let $P \in A_\partial$. We have the following commutative diagram

$$\begin{tikzcd}
H_n(DM, DM \setminus DA; R) \arrow[r] \arrow[d] \arrow[rr, bend left, "\Phi"] & H_n(DM, (M \setminus A) \cup M'; R) \arrow[d] & H_n(M, M \setminus A_\partial; R) \arrow[l, "\beta", "\cong"'] \arrow[d] \\
H_n(DM, DM \setminus \{P\}; R) \arrow[r, "="] & H_n(DM, DM \setminus \{P\}; R) & H_n(M, M \setminus \{P\}; R). \arrow[l, "\cong"']
\end{tikzcd}$$

Since all maps are induced by inclusions of pairs of topological spaces we see that the diagram commutes. By definition of μ_{DA} the left-hand vertical map sends μ_{DA} to α_P. But this implies that the right-hand vertical map sends $\Phi(\mu_{DA})$ to $\alpha_P \in H_n(M, M \setminus \{P\}; R)$.

Now we turn to the proof of the uniqueness of a lift. The following claim is the key to proving uniqueness.

Claim. Let $t \in (0, 1)$. Note that $A_t \subset A_\partial$. The two inclusion induced maps

$$\begin{aligned}
\gamma\colon H_n(M, M \setminus A_\partial; R) &\;\to\; H_n(M, M \setminus A_t; R) \\
\delta\colon H_n(M, M \setminus A_t; R) &\;\to\; H_n(DM, DM \setminus A_t; R)
\end{aligned}$$

are isomorphisms.

It follows immediately from the Excision Theorem 45.10 for topological manifolds that the map δ is an isomorphism. Regarding the map γ, note that it follows from the long exact sequence of the triple $(M, M \setminus A_t, M \setminus A_\partial)$, see Proposition 31.13, that it suffices to show that $H_i(M \setminus A_t, M \setminus A_\partial; R) = 0$ for all i. We pick $s \in (t, 1)$. We see that indeed we have

$$\begin{array}{c}
\text{by the Excision Theorem 31.17 where we excise everything in the complement of } (\partial M \times [0, s)) \cap A \\
\downarrow \\
H_i(M \setminus A_t, M \setminus A_\partial; R) \xleftarrow{\cong} H_i((\partial M \times [0, s)) \setminus (K \times [t, s)), (\partial M \times [0, s)) \setminus (K \times (0, s)); R) = 0. \\
\uparrow \\
\text{follows from Corollary 31.16 since } (\partial M \times [0, s)) \setminus (K \times (0, s)) \text{ is a} \\
\text{deformation retract of } (\partial M \times [0, s)) \setminus (K \times [t, s))
\end{array}$$

This completes the proof of the claim.

Now let $\mu, \nu \in H_n(M, M \setminus A_\partial; R)$ be two lifts of the R-section $\{\alpha_x\}_{x \in M \setminus \partial M}$. With the same arguments as before one can convince oneself easily of the fact that $\delta^{-1}(\gamma(\mu))$

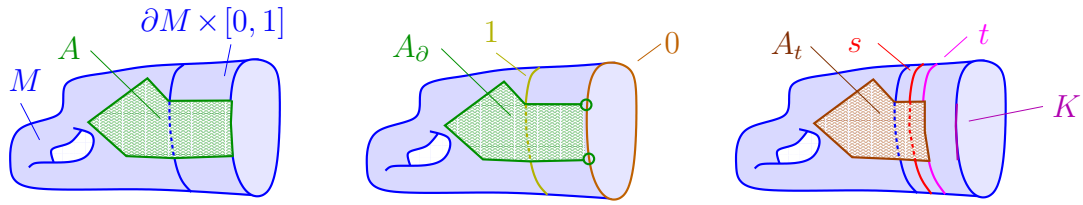


FIGURE 669. Second illustration for the proof of Theorem 46.29.

and $\delta^{-1}(\gamma(\nu))$ are lifts of the R-section $\{\alpha_x\}_{x\in DM}$. By Theorem 46.10 these two lifts are identical. But then it follows from the claim that also $\mu = \nu$. □

We conclude this long and tiring chapter with the following lemma that is a generalization of Lemma 46.11.

Lemma 46.30. *Let M be an oriented n-dimensional topological manifold and let R be a commutative ring. We suppose that M is equipped with an R-orientation. We use the notation from Theorem 46.29.*

(1) *Let $K \subset L \subset M$ be compact subsets that are products close to the boundary. We denote by $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ the inclusion map of pairs. The map $f_{KL*}\colon H_n(M, M\setminus L) \to H_n(M, M\setminus K)$ sends α_L^M to α_K^M.*
(2) *Let U be an open subset of M and let $A \subset U$ be a compact subset that is a product close to the boundary. We use Lemma 45.28 to view U as an oriented topological manifold. If $i\colon U \to M$ denotes the inclusion map, then $i_*(\alpha_A^U) = \alpha_A^M$.*

Proof. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of Theorem 46.29. □

Exercises for Chapter 46.

Exercise 46.1. Give an example of a representative for the $\mathbb{F}_2$-fundamental class for the Klein bottle.

Exercise 46.2. Can you find a representative for the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$ with two or three summands?

Exercise 46.3. Let M be a connected n-dimensional topological manifold without boundary and let K be a compact subset. Show that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact.

Exercise 46.4. Let M be a closed orientable connected non-empty topological manifold of dimension $n \geq 3$. We now know that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and that $H_{n-1}(M;\mathbb{Z})$ is torsion-free. Furthermore we know that $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Can we also say something about the other homology groups? More precisely, are there any restrictions on $H_{n-2}(M;\mathbb{Z})$ or can any finitely generated abelian group be isomorphic to $H_{n-2}(M;\mathbb{Z})$ for some such manifold?

Exercise 46.5.

(a) Let M be a compact, connected n-dimensional topological manifold with non-empty boundary. Show that $H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.
Hint: You can use that we already obtained some results for closed manifolds, in particular we have a good understanding of $H_{n-1}(DM;\mathbb{Z})$. You could start with the case that M is orientable and afterwards you could consider the case that M is non-orientable.

(b) Let M be a connected n-dimensional topological manifold. Suppose that M is non-compact. Does it follow that $H_{n-1}(M;\mathbb{Z})$ is a free abelian group? It suffices to give a short justification for your answer.

Exercise 46.6. Let $X = S^1 \cup \{*\}$ be the topological space that is given by the circle "with two 1's", where the topology is defined in the same way as we defined the topology on the "line with two 0's" on page 53. It is straightforward to see that X is o-locally homeomorphic to open subsets of $\mathbb{R}$, in fact one can easily find a finite atlas for X. Furthermore we know by Lemma 3.1 (4) that X is second-countable. But X is not a topological manifold since it is not Hausdorff. The argument of Exercise 33.2 shows that $H_1(X;\mathbb{F}_2) \cong \mathbb{F}_2^2$. This shows that in the proof of Theorem 46.3 at some point we must have used that topological manifolds are Hausdorff. Where did we make use of this fact?

47. Chirality

In this section we will take a short break from overly abstract concepts and we study chirality of objects in $\mathbb{R}^3$, knots and manifolds. We will run into several questions which will motivate us to develop new tools in the study of topological spaces.

47.1. **Chiral objects.** In physics or chemistry an object X in $\mathbb{R}^3$ is called *amphichiral* (or often short *achiral*) if it agrees with its mirror image, otherwise it is called *chiral*. To make this definition more precise we have to explain what we mean by "mirror image" and what we mean by "agree".

First of all, given a hyperplane $H \subset \mathbb{R}^3$ through the origin we denote by $\rho_H \colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the hyperplane H.[877] Sometimes in this chapter we will extend ρ_H to a map from $S^3 = \mathbb{R}^3 \cup \{\infty\}$ to itself by setting $\rho_H(\infty) = \infty$.

The following lemma says that any two reflections in a hyperplane through the origin differ by multiplication by a matrix in SO(3).

Lemma 47.1. *If H and G are two hyperplanes in $\mathbb{R}^3$ through the origin, then there exists a matrix $A \in \mathrm{SO}(3)$ such that for all $v \in \mathbb{R}^3$ we have*

$$\rho_G(v) \;=\; A \cdot \rho_H(v).$$

Sketch of proof. With respect to the standard basis of $\mathbb{R}^3$ the maps ρ_G and ρ_H are represented by orthogonal matrices P and Q with determinant -1. The matrix $A = PQ^{-1}$ then lies in SO(3) and for any $v \in \mathbb{R}^3$ we have $\rho_G(v) = Pv = A \cdot Qv = A \cdot \rho_H(v)$. □

The above discussion leads us to the following definitions.

Definition. Let X be a subset of $\mathbb{R}^3$.

(1) We say another subset Y of $\mathbb{R}^3$ *agrees with* X if there exists a matrix $A \in \mathrm{SO}(3)$ and a point $P \in \mathbb{R}^3$ such that $X = A \cdot Y + P = \{A \cdot y + P \,|\, y \in Y\}$.
(2) For every hyperplane $H \subset \mathbb{R}^3$ we refer to $\rho_H(X)$ as the *mirror image* of X.[878]
(3) We say X is *amphichiral* if its mirror image agrees with X, otherwise we say X is *chiral*.

Examples.

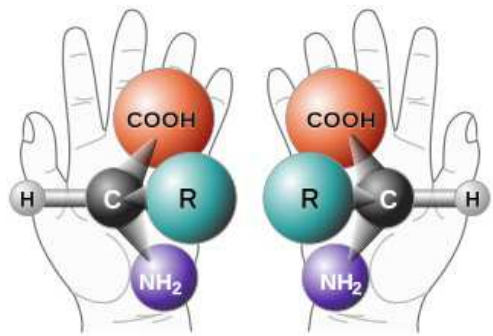
(1) Human hands are chiral, indeed, the left-hand does not agree with the right-hand. In fact the term chirality is derived from the Greek word for hand, $\chi\epsilon\iota\rho$.[879]
(2) Many molecules are chiral, see e.g. Figure 670 for an example of a chiral molecule.[880]

[877] How is the reflection in a hyperplane defined?

[878] To be precise, the definition of the mirror image depends on the choice of the hyperplane. By Lemma 47.1 reflections in two different hyperplanes differ by multiplication by a matrix in SO(3), i.e. the mirror images with respect to different hyperplanes agree in the above sense.

[879] The same word is also the etymological root for the German word "Chirurg".

[880] In fact, how can you show that the molecule is chiral?



source: `https://en.wikipedia.org/wiki/Chirality`

FIGURE 670.

47.2. **Chiral knots.** Now we turn our focus to knots. We recall some of the relevant definitions from Section 10.8.

Definition.

(1) A *knot* is a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.
(2) Given a knot $K \subset \mathbb{R}^3$ the reflection of K in any hyperplane is called *mirror* of K and denoted by K^*.[881]

Example. In Figure 47.2 we show the trefoil and its mirror and we also show the figure-8 knot and its mirror.

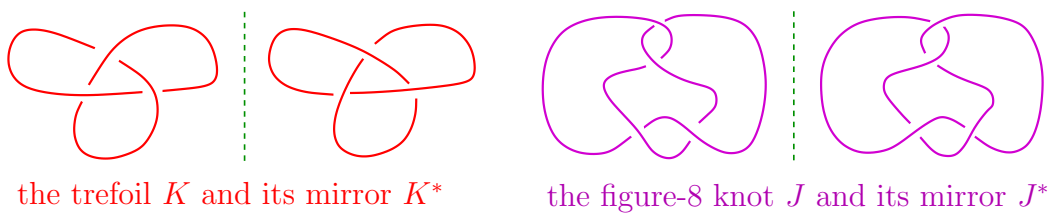


FIGURE 671.

Usually we are only interested in knots "up to deformation", i.e. we say that two knots are equivalent if one knot can be deformed into the other. This leads us to the following definition.

Definition. We say that *two knots K and J are equivalent* if there exists a smooth isotopy from K to J, i.e. a smooth map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

[881] Usually in pictures we consider the reflection in the xz-plane.

such that the following hold:

(1) we have $F(S^1 \times \{0\}) = K$ and $F(S^1 \times \{1\}) = J$, and
(2) for each $t \in [0,1]$ the map $S^1 \to S^3$ given by $z \mapsto F(z,t)$ is an embedding.

Examples.

(1) The knots shown in Figure 672 are equivalent.

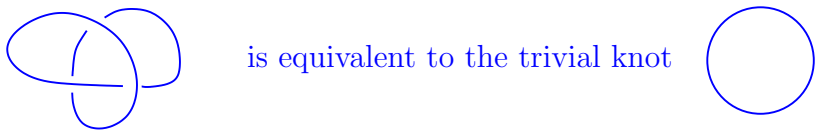


FIGURE 672.

(2) Let K and J be knots in $\mathbb{R}^3$. We claim that if K and J agree, then they are also equivalent. We pick $A \in \mathrm{SO}(3)$ and $P \in \mathbb{R}^3$ with $J = A \cdot K + P$. By Lemma 1.96 (1) we know that $\mathrm{SO}(3)$ is path-connected, in particular there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We pick an embedding $f\colon S^1 \to S^3$ with $f(S^1) = K$. The desired isotopy from K to $J = A \cdot K + P$ is then given by $F(z,t) := \gamma(t) \cdot f(z) + t \cdot P$.

We are now naturally led to the following definition.

Definition. We say that a knot K in $\mathbb{R}^3$ is *amphichiral* if it is equivalent to its mirror image, otherwise we call the knot K *chiral*.

Examples.

(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.
(2) In Figure 673 we show that the figure-8 knot J is equivalent to its mirror image J^*, i.e. the figure-8 knot is also amphichiral.

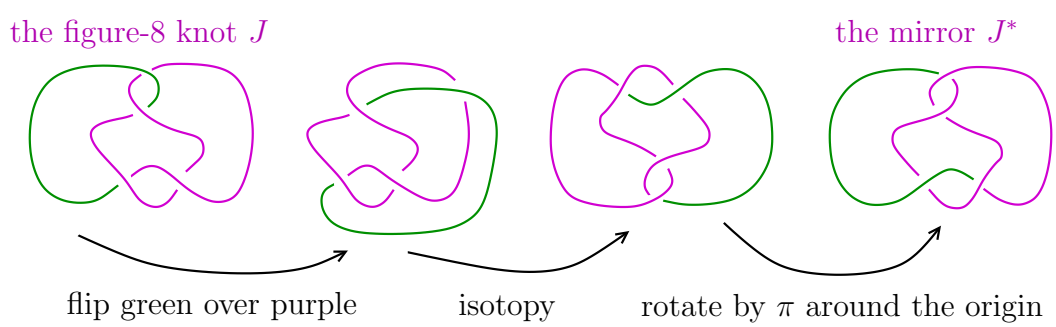


FIGURE 673.

It is now natural to ask whether the trefoil is amphichiral. But endless hours of playing around with the trefoil do not lead to any success. So the suspicion arises that the following question should be answered in the negative.

Question 47.2. *Is the trefoil chiral?*

The question that naturally arises is, what does this discussion have to do with homology groups? How can we address this question using the tools of algebraic topology? In the following section we will turn Question 47.2 into a question about homology groups.

47.3. **Chiral topological manifolds.**

Definition. Let M be a compact oriented topological manifold. We say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*.

In the next proposition we will give several criteria for a self-homeomorphism to be orientation-preserving.

Proposition 47.3. *Let M be a compact oriented connected n-dimensional topological manifold and let $f\colon M \to M$ be a homeomorphism. Then the following three statements are equivalent:*

(1) *the map f is orientation-preserving in the sense of the definition on page 1099,*
(2) *there exists a point $x \in M \setminus \partial M$ and an open neighborhood U of x, e.g. $U = M$, such that*[882]

$$f_*(\underbrace{\textit{orientation at } x}_{\in H_n(U,U\setminus\{x\};\mathbb{Z})}) \;=\; \underbrace{\textit{orientation at } f(x)}_{\in H_n(f(U),f(U)\setminus\{f(x)\};\mathbb{Z})},$$

(3) *we have $f_*([M]) = [M]$.*

Suppose M is in fact a manifold and that f is a diffeomorphism. We equip M with the orientation in the sense of Section 3.8 coming from Proposition 45.22. Then the above are equivalent to the statement

(4) *the map f is orientation-preserving in the sense of Section 3.8.*

Analogously the above statements also hold if in (1) *and* (4) *we replace "orientation-preserving" by "orientation-reversing" and if in* (2) *and* (3) *we add a minus sign.*

For the most part the proof of Proposition 47.3 boils down to the following lemma which is also useful in its own right.

Lemma 47.4. *Let $f\colon M \to N$ be a map between compact oriented connected n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$. Let $x \in M \setminus \partial M$ and let U be an open*

[882]Recall that by the construction of Lemma 45.28 the orientation on M induces an orientation on any open subset.

neighborhood of x with the property that f restricts to a homeomorphism $f\colon U \to f(U)$ and such that $f^{-1}(f(x)) = \{x\}$. Then

$$\begin{aligned}\deg(f\colon M \to N) &= \deg(f_*\colon H_n(M, M \setminus \{x\}; \mathbb{Z}) \to H_n(N, N \setminus \{f(x)\}; \mathbb{Z}) \\ &= \deg(f_*\colon H_n(U, U \setminus \{x\}; \mathbb{Z}) \to H_n(f(U), f(U) \setminus \{f(x)\}; \mathbb{Z}).\end{aligned}$$

Proof. Let M and N be compact connected n-dimensional topological manifolds that are equipped with orientations $\{\mu_x\}_{x \in M \setminus \partial M}$ and $\{\nu_x\}_{x \in N \setminus \partial N}$. Furthermore let f, $x \in M \setminus \partial M$ and U as above. We consider the following commutative diagram

$$\begin{tikzcd}
H_n(M, \partial M; \mathbb{Z}) \arrow[r, "f_*"] \arrow[d] & H_n(N, \partial N; \mathbb{Z}) \arrow[d] \\
H_n(M, M \setminus \{x\}; \mathbb{Z}) \arrow[r, "f_*"] & H_n(N, N \setminus \{f(x)\}; \mathbb{Z}) \\
H_n(U, U \setminus \{x\}; \mathbb{Z}) \arrow[r, "f_*"] \arrow[u] & H_n(f(U), f(U) \setminus \{f(x)\}; \mathbb{Z}). \arrow[u]
\end{tikzcd}$$

We make the following observations:

(1) the top vertical maps are isomorphisms by Theorem 46.2,
(2) the bottom vertical maps are isomorphisms by the Excision Theorem 45.8,
(3) it follows from Corollary 46.5 that the vertical maps send the fundamental classes $[M]$ and $[N]$ to the orientations μ_x respectively $\nu_{f(x)}$.

The desired equalities follow easily from these observations.[883] □

Now we can also provide a proof for Proposition 47.3.

Sketch of the proof $(*)$. The equivalence of (1), (2) and (3) follows easily from Lemma 47.4. Now suppose M that is in fact a manifold and that f is a diffeomorphism. The equivalence of (2) and (4) is a straightforward consequence of Lemma 45.23 and the Excision Theorem 45.8. Once again we leave it to the reader to fill in the details. □

Examples. Let $n \in \mathbb{N}_0$.

(1) It follows from Lemma 32.16 (6) that the map

$$\begin{aligned}\rho\colon S^n &\to S^n \\ (x_0, \dots, x_{n-1}, x_n) &\mapsto (x_0, \dots, x_{n-1}, -x_n),\end{aligned}$$

i.e. the map that is given by reflection in the $x_n = 0$ hyperplane, is orientation-reversing. This shows that the sphere S^n is amphichiral.

(2) Let M be a closed, oriented topological manifold. It follows from (1) and the naturality of the Künneth Theorem 43.8 for topological spaces that the map

$$\begin{aligned}S^n \times M &\to S^n \times M \\ (x, P) &\mapsto (\rho(x), P),\end{aligned}$$

[883] Where on earth did we use the hypothesis that $f^{-1}(f(x)) = \{x\}$?

is orientation-reversing. This shows that the product of the sphere S^n with any closed, oriented topological manifold is amphichiral.

Now we want to relate the notion of amphichiral knots to the notion of amphichiral manifolds. Before we can do so we recall the following definition.

Definition. Let $K \subset \mathbb{R}^3$ be a knot. We pick a trivial tubular neighborhood $\overline{B}^2 \times K$ as provided by the Tubular Neighborhood Theorem 10.19. We refer to $X_K := S^3 \setminus (B^2 \times K)$ as the *exterior of* K.[884]

FIGURE 674.

The following proposition together with Proposition 47.3 now relates Question 47.2 to homology groups.

Proposition 47.5. *Let $K \subset \mathbb{R}^3$ be a knot. If K is amphichiral, then the knot exterior X_K is also amphichiral.*

Remark. We had just seen in Proposition 47.5 that the exterior of an amphichiral knot is amphichiral. The converse of that statement is also true, i.e. if the exterior of a knot is amphichiral, then the knot is also amphichiral. But this direction is much harder to prove. To the best of my knowledge it follows from a very difficult theorem by Cameron Gordon and John Luecke [GL, p. 371] from 1989.

Proof. Let $K \subset \mathbb{R}^3$ be an amphichiral knot. We pick a trivial tubular neighborhood $\nu K = \overline{B}^2 \times K$ as provided by the Tubular Neighborhood Theorem 10.19 and we use it to define the knot exterior X_K. By Proposition 47.3 it suffices to show that there exists a homeomorphism $f\colon X_K \to X_K$ with $f_*([X_K]) = -[X_K]$.

We denote by ρ the reflection in a hyperplane of $\mathbb{R}^3$. As usual we extend ρ to a self-homeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by setting $\rho(\infty) = \infty$. We write $K^* = \rho(K)$ and $\nu K^* = \rho(\nu K)$. By hypothesis there exists a smooth isotopy from K^* to K. By the Isotopy Extension Theorem 10.21 this smooth isotopy extends to a smooth diffeotopy of S^3. This means that we have a smooth map

$$F\colon S^3 \times [0,1] \;\to\; S^3$$

[884]As we pointed out in Lemma 40.7, the exterior X_K is a compact orientable manifold and the diffeomorphism type of X_K as an oriented manifold is well-defined, i.e. it is independent of the choice of the tubular neighborhood.

such that $F_0 = \mathrm{id}$, such that each F_t is a diffeomorphism and such that $F_1(K^*) = K$. We write $\Phi := F_1 \colon S^3 \to S^3$. Then $\Phi(\nu K^*)$ is also a closed tubular neighborhood for K and by Proposition 10.20 there exists an orientation-preserving diffeomorphism $\Psi \colon S^3 \to S^3$ with $\Psi(\Phi(\nu K^*)) = \nu K$. We write $f := \Psi \circ \Phi \circ \rho$.

Now we consider the following commutative diagram

$$\begin{tikzcd}
H_3(S^3) \arrow[r, "\rho_*"] \arrow[d, "\cong"] & H_3(S^3) \arrow[r, "\Phi_*"] \arrow[d, "\cong"] & H_3(S^3) \arrow[r, "\Psi_*"] \arrow[d, "\cong"] & H_3(S^3) \arrow[d, "\cong"] \\
H_3(S^3, \nu K) \arrow[r, "\rho_*"] & H_3(S^3, \nu K^*) \arrow[r, "\Phi_*"] & H_3(S^3, \Phi(\nu K^*)) \arrow[r, "\Psi_*"] & H_3(S^3, \nu K) \\
H_3(X_K, \partial X_K) \arrow[u, "\cong"] \arrow[r, "\rho_*"] \arrow[rrr, bend right, "f_* = (\Psi \circ \Phi \circ \rho)_*"'] & H_3(X_{K^*}, \partial X_{K^*}) \arrow[u, "\cong"] \arrow[r, "\Phi_*"] & H_3(\Phi(X_{K^*}), \Phi(\partial X_{K^*})) \arrow[u, "\cong"] \arrow[r, "\Psi_*"] & H_3(X_K, \partial X_K). \arrow[u, "\cong"]
\end{tikzcd}$$

Here the upper left vertical map is an isomorphism by the long exact sequence of the pair $(S^3, \nu K)$ and the fact that $H_i(\nu K) = 0$ for $i = 2, 3$ since $\nu K = K \times B^2$ is homotopy equivalent to S^1. Exactly the same argument shows that in fact all upper vertical maps are isomorphisms. The lower vertical maps are induced by the inclusion of pairs of topological spaces. All of them are isomorphisms by the Excision Theorem for Manifolds 45.10.

By Proposition 47.3 we are done once we have shown that the composition of three bottom horizontal maps is given by multiplication by -1. By the commutative diagram it suffices to prove the analogous statement for the composition of the top three horizontal maps. It follows from Lemma 32.16 (6) that the top-left horizontal map is given by multiplication by -1. It is a consequence of Proposition 30.5 that the top-middle horizontal map is the identity, since Φ is homotopic to the identity. Finally using Proposition 47.3 we see that the top-right horizontal map is the identity since Ψ is orientation-preserving. $\square$

Now we continue with our discussion of examples of amphichiral manifolds.

Examples.

(1) We had just seen on page 1150 that topological manifolds of the type $S^n \times M$ are amphichiral. It follows that the torus $S^1 \times S^1$ is amphichiral.
(2) The surface of genus 2 is amphichiral. Pictorially an orientation-reversing self-homeomorphism is given by "reflection in the yz-plane", see Figure 675 on the left. In the "octagon-picture" an orientation-reversing self-homeomorphism is given by the reflection f in the line illustrated in Figure 675 on the left. It follows easily from Proposition 47.3 applied to the point $x = 0$ and U an open disk around x together with Lemma 45.14 that the map f is indeed orientation-reversing.
(3) All other compact oriented 2-dimensional manifolds are also amphichiral. We show some of the orientation-reversing maps in Figure 676. In Exercise 47.2 we will discuss how these maps can be described rigorously.

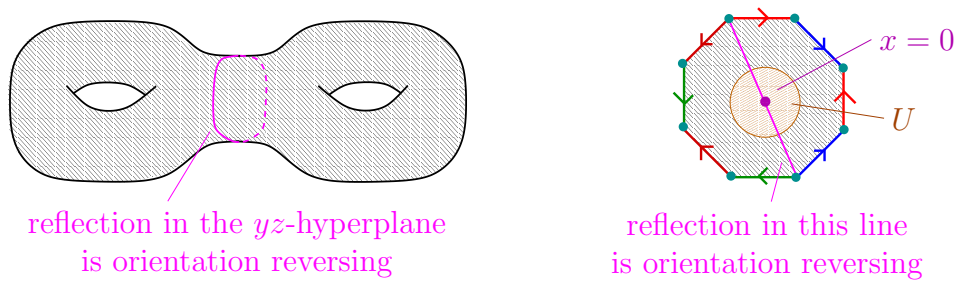


FIGURE 675.

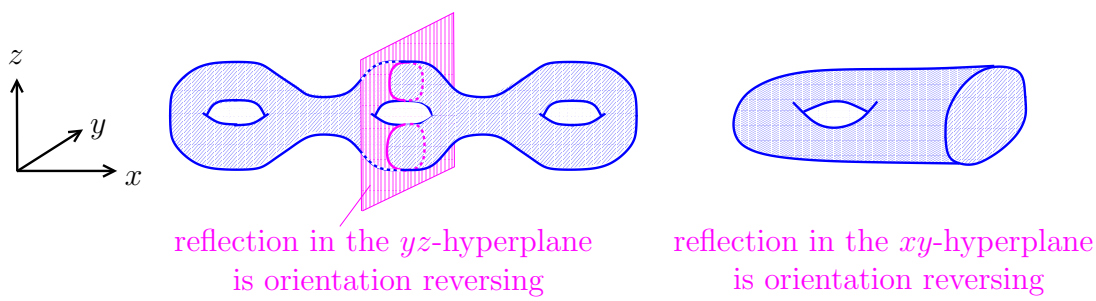


FIGURE 676.

(4) Example (2) can also be generalized as follows: if M is a compact oriented topological manifold, then its double $DM = (M \sqcup M')/\sim$ is amphichiral. Indeed, the map $f\colon DM \to DM$ that swaps the two copies of M is orientation-reversing. This follows immediately from the definition of the orientation of DM that we gave in Lemma 45.29.

The long list of examples of amphichiral manifolds raises the following question.

Question 47.6. *Are there compact oriented topological manifolds that are chiral?*

We end this chapter with a short discussion of lens spaces. Let $p \in \mathbb{N}$ and let $q \in \mathbb{N}$ be coprime to p. On page 266 we defined the lens space

$$L(p,q) \;=\; S^3/\mathbb{Z}_p \;=\; \big\{(z_1,z_2) \in \mathbb{C}^2 \,\big|\, |z_1|^2 + |z_2|^2 = 1\big\}/(z_1,z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$

Furthermore, on page 266 we had equipped $L(p,q)$ with the orientation which turns the obvious projection map $S^3 \to L(p,q)$ into an orientation-preserving map.

Now let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. In Lemma 7.7 we showed that if $q \equiv \pm r^{\pm 1} \bmod p$, then $L(p,q)$ and $L(p,r)$ are diffeomorphic.[885] The following lemma is now a slight refinement of Lemma 7.7.

Lemma 47.7. *Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$.*

[885] We still don't know whether the converse holds.

(1) *In the case that* $q \equiv r^{\pm 1} \bmod p$ *there exists an orientation-preserving diffeomorphism* $L(p,q) \to L(p,r)$.
(2) *On the other hand, if* $q \equiv -r^{\pm 1} \bmod p$, *then there exists an orientation-reversing diffeomorphism* $L(p,q) \to L(p,r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. As in the proof of Lemma 7.7 we define a diffeomorphism

$$\Phi\colon S^3 \to S^3$$

as follows:

$$\begin{array}{llll}
\text{Case 1: } q \equiv r \bmod p. & (z_1, z_2) & \mapsto & (z_1, z_2) \\
\text{Case 2: } q \equiv -r \bmod p. & (z_1, z_2) & \mapsto & (z_1, \overline{z_2}) \\
\text{Case 3: } q \equiv r^{-1} \bmod p. & (z_1, z_2) & \mapsto & (z_2, z_1) \\
\text{Case 4: } q \equiv -r^{-1} \bmod p. & (z_1, z_2) & \mapsto & (z_2, \overline{z_1})
\end{array}$$

In all four cases it is straightforward to verify that the map descends to a diffeomorphism $L(p,q) \to L(p,r)$. It follows fairly easily from Lemma 32.16 and Proposition 47.3 that the diffeomorphism Φ is orientation-preserving in cases 1 and 3 and that it is orientation-reversing in cases 2 and 4.[886] The lemma now follows from the fact that the projection maps $S^3 \to L(p,q)$ and $S^3 \to L(p,r)$ are orientation-preserving. □

Example. Note that $-2^{-1} \equiv -3 \equiv 2 \bmod 5$. Thus we obtain from Lemma 47.7 that there exists an orientation-reversing diffeomorphism $L(5,2) \to L(5,2)$, i.e. the lens space $L(5,2)$ is amphichiral.

The argument of the previous example does not work for the lens space $L(3,1)$. So the following question arises:

Question 47.8. *Is the lens space* $L(3,1)$ *chiral?*

With our present knowledge we are not able to answer Questions 47.2, 47.6 and 47.8. The key to answering these three questions will be the cohomology groups that we introduce in the next chapter.

Exercises for Chapter 47.

Exercise 47.1. Why is the molecule shown in Figure 677 chiral? How can you show rigorously that the two molecules left and right cannot be turned into one another using a translation and a matrix in SO(3)?

Exercise 47.2.

[886] For example in case 3 the map is given by $(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2)$ which is the composition of two reflections in hyperplanes, namely in the $x_1 = x_3$ and the $x_2 = x_4$ hyperplane.

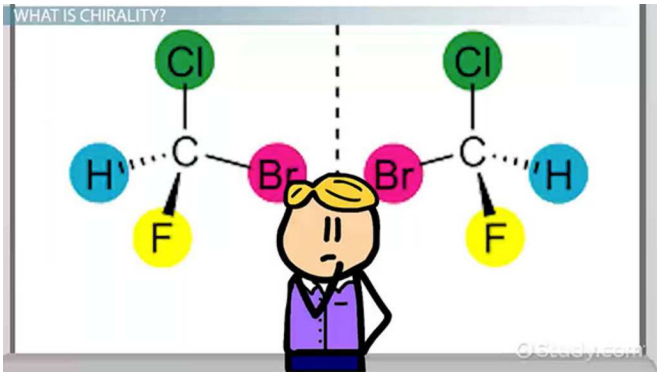


Figure 677.

(a) Give a rigorous proof for the statement that the surface of genus 3 is amphichiral. More precisely, give a precise description of the surface of genus 3 (e.g. as a dodecagon with appropriate identifications) and indicate carefully an orientation-reversing self-homeomorphism.
(b) Do the same exercise for the genus 2 surface with two open disks removed.

48. The cohomology groups of a topological space

48.1. **The Hom-functor.** We start out with the following obvious notation.

Notation. Given a group A and an abelian group G we write

$$\mathrm{Hom}(A,G) \;=\; \{\text{all group homomorphisms } A \to G\}.$$

The set of homomorphisms $\mathrm{Hom}(A,G)$ has the structure of an abelian group with the sum of two homomorphisms $\varphi,\psi\colon A \to G$ given by the group homomorphism that is defined by $(\varphi+\psi)(a) := \varphi(a)+\psi(a)$.

Example. We leave it as an exercise to show that for any $n \in \mathbb{N}$ and any abelian group G we have a natural isomorphism $\mathrm{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{\cdot n} G)$. In particular for any $m \in \mathbb{N}$ we have $\mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

On several occasions we will implicitly and explicitly use the following lemma which is an immediate consequence of Lemma 11.1.

Lemma 48.1. $(*)$ *Let G be an abelian group. If S is a non-empty set, then the homomorphism*[887] [888] [889]

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z}^{(S)},G) & \to & \{\textit{all maps from the set } S \textit{ to } G\} \\ (\varphi\colon \mathbb{Z}^{(S)} \to G) & \mapsto & \left(\begin{array}{ccc} S & \to & G \\ s & \mapsto & \varphi(s) \end{array}\right) \end{array}$$

is an isomorphism. In particular the map

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z},G) & \to & G \\ \varphi & \mapsto & \varphi(1) \end{array}$$

is an isomorphism.

Let G be an abelian group and let $f\colon A \to B$ be a group homomorphism. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}(B,G) & \to & \mathrm{Hom}(A,G) \\ (\varphi\colon B \to G) & \mapsto & \left(\begin{array}{ccc} A & \to & G \\ a & \mapsto & \varphi(f(a)) \end{array}\right). \end{array}$$

Example. Let G be an abelian group.

---

[887] Recall that on page 366, given a non-empty set S we defined

$$\mathbb{Z}^S \;:=\; \text{all maps from } S \text{ to } \mathbb{Z}.$$

and we defined the free abelian group generated by S to be the subgroup

$$\mathbb{Z}^{(S)} \;:=\; \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$

Following the notation introduced on page 366 we view each $s \in S$ as an element in $\mathbb{Z}^{(S)}$.

[888] Why do we write $\mathbb{Z}^{(S)}$ in the lemma? Should it not be $\mathbb{Z}^S$?

[889] Note that the group structure on G turns both sides of the lemma into a group, so the given map clearly respects this group structure, i.e. the given map is a homomorphism.

(1) If A is a subgroup of a group B and if $i\colon A \to B$ is the inclusion map, then for each homomorphism $\varphi\colon B \to G$ the homomorphism $i^*\varphi\colon A \to G$ is just the restriction of φ to $A \subset B$.
(2) If S is a subset of a set T, then we have a natural inclusion $i\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ and it follows from Lemma 48.1 that the map $i^*\colon \operatorname{Hom}(\mathbb{Z}^{(T)}, G) \to \operatorname{Hom}(\mathbb{Z}^{(S)}, G)$ is an epimorphism. Indeed, we just extend the map $S \to G$ to a map $T \to G$ by assigning the trivial element in G to all elements in $T \setminus S$.

It follows immediately from the definitions that

$$(\mathrm{id}_A)_* = \mathrm{id}_{\operatorname{Hom}(A,G)}, \quad \text{for all groups } A$$

and

$$(g \circ f)^* = f^* \circ g^* \qquad \text{for all homomorphisms } f\colon A \to B \text{ and } g\colon B \to C.$$

Put differently, given an abelian group G the maps

$$\begin{array}{rcl} A & \mapsto & \operatorname{Hom}(A, G) \\ (f\colon A \to B) & \mapsto & (f^*\colon \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G)) \end{array}$$

define a *contravariant* functor from the category of groups to the category of abelian groups.

Remark. If R is a commutative ring, for example if $R = \mathbb{Z}_n, R = \mathbb{Z}, R = \mathbb{Q}$ or $R = \mathbb{R}$, then for any group C the set of homomorphisms $\operatorname{Hom}(C, R)$ forms an R-module in an obvious way. More precisely, given a homomorphism $\varphi\colon C \to R$ and $r \in R$ we define $r\varphi$ to be the homomorphism given by $c \mapsto r \cdot \varphi(c)$. Now we see easily that the above maps $A \mapsto \operatorname{Hom}(A, R)$ and $f \mapsto f^*$ define a contravariant functor from the category of groups to the category of R-modules.

The following lemma might be familiar from linear algebra. We refer to [Rot2, Theorem 2.32(i)] for more details.

Lemma 48.2. *Let $\{A_i\}_{i\in I}$ be a family of groups and let G be an abelian group. Then the map*

$$\begin{array}{rcl} \operatorname{Hom}\left(\bigoplus_{i\in I} A_i, G\right) & \to & \prod_{i\in I} \operatorname{Hom}(A_i, G) \\ f & \mapsto & \prod_{i\in I} f|_{A_i} \end{array}$$

is an isomorphism.

The following rather technical lemma will be used later on.

Lemma 48.3. *Let $\{\{A_i\}_{i\in\mathbb{N}}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\le j}\}$ be a direct system of groups and let G be an abelian group. Then*

$$\{\{\operatorname{Hom}(A_i, G)\}_{i\in\mathbb{N}}, \{\varphi_{ij}^*\colon \operatorname{Hom}(A_j, G) \to \operatorname{Hom}(A_i, G)\}_{i\le j}\}$$

is an inverse system of abelian groups (or modules if G is a commutative ring), and we have a natural isomorphism

$$\operatorname{Hom}\left(\varinjlim A_i, G\right) \xrightarrow{\cong} \varprojlim \operatorname{Hom}(A_i, G)$$

of abelian groups (or modules if G is a commutative ring).

Proof. As so often for arguments regarding limits, the proof follows mostly from unraveling the definitions. We refer to [Rot2, Proposition 5.26] for more details. □

We continue with the following lemma which we will use on many occasions.

Lemma 48.4. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G be an abelian group. If C is free abelian, then the dual sequence

$$0 \to \operatorname{Hom}(C,G) \xrightarrow{p^*} \operatorname{Hom}(B,G) \xrightarrow{i^*} \operatorname{Hom}(A,G) \to 0$$

is also exact.

Proof. It follows from the hypothesis that C is a free abelian and from Lemma 33.1 that the initial short exact sequence splits. But this means that, according to Lemma 33.2, that there exists an isomorphism $\Phi\colon B \to A \oplus C$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "="] & B \arrow[r, "p"] \arrow[d, "\Phi", "\cong"'] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "a\mapsto(a,0)"] & A \oplus C \arrow[r, "(a,c)\mapsto c"] & C \arrow[r] & 0.
\end{tikzcd}$$

By applying the contravariant $\operatorname{Hom}(-,G)$ functor we obtain the upper half of the following commutative diagram:

$$\begin{tikzcd}
0 & \operatorname{Hom}(A,G) \arrow[l] & \operatorname{Hom}(B,G) \cong \arrow[l, "i^*"'] & \operatorname{Hom}(C,G) \arrow[l, "p^*"'] & 0 \arrow[l] \\
0 & \operatorname{Hom}(A,G) \arrow[l] \arrow[u, "="] & \operatorname{Hom}(A\oplus C,G) \arrow[l, "(a\mapsto(a,0))^*"'] \arrow[u, "\Phi^*"] & \operatorname{Hom}(C,G) \arrow[l, "((a,c)\mapsto c)^*"'] \arrow[u, "="] & 0 \arrow[l] \\
0 & \operatorname{Hom}(A,G) \arrow[l] \arrow[u, "="] & \operatorname{Hom}(A,G)\oplus\operatorname{Hom}(C,G) \arrow[l, "f\oplus g\mapsto f"'] \arrow[u, "(f\oplus)\mapsto(a\oplus c\mapsto f(a)+g(c))"'] & \operatorname{Hom}(C,G) \arrow[l, "g\mapsto 0\oplus g"'] \arrow[u, "="] & 0. \arrow[l]
\end{tikzcd}$$

One can easily verify that the lower half of the diagram commutes. Furthermore it is evident that the bottom sequence is exact. Finally it is straightforward to see that the vertical maps are isomorphisms we obtain that the upper sequence is also exact. But that is exactly what we wanted to prove. □

48.2. **Cochain complexes.** Now we introduce the notion of a cochain complex and of its cohomology groups. A cochain complex is exactly the same as a chain complex except that the boundary map now increases the degree by one instead of lowering it by one.

Definition.

(1) A *cochain complex* is a sequence

$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$

of maps between abelian groups such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as *coboundary maps*.

(2) We define the *n-th cohomology group* of a cochain complex (D^*, δ_*) to be

$$H^n(D) \;:=\; \frac{\ker\{\delta_n \colon D^n \to D^{n+1}\}}{\operatorname{im}\{\delta_{n-1} \colon D^{n-1} \to D^n\}}.$$

Example. It follows from [Le2, Theorem 14.24] that the groups

$$\Omega_n(M) \;:=\; \{\text{all smooth } n\text{-forms on } M\}, \quad n \in \mathbb{N}_0$$

together with the differentials $d_n \colon \Omega_n(M) \to \Omega_{n+1}(M)$ define a cochain complex. The corresponding cohomology groups are called the de Rham cohomology groups of M. We will return to this example in more detail at a later stage.

Now we will see that most of the definitions and algebraic results for chain complexes can be generalized almost verbatim to cochain complexes.

Definition.

(1) A *cochain map* $f \colon C^* \to D^*$ between two cochain complexes consists of a family $\{f_n \colon C^n \to D^n\}_{n \in \mathbb{N}_0}$ of maps that commute with the coboundary maps, i.e. such that for all $n \in \mathbb{N}_0$ we have

$$\underset{\substack{\uparrow \\ \delta_n \colon D^n \to D^{n+1}}}{\delta_n} \circ f_n \;=\; f_{n+1} \circ \underset{\substack{\uparrow \\ \delta_n \colon C^n \to C^{n+1}}}{\delta_n} \colon C^n \;\to\; D^{n+1}.$$

(2) Let $f, g \colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^*. A *cochain homotopy* $P = \{P_n\}_{n \in \mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n \colon C^n \to D^{n-1}\}_{n \in \mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \;=\; f_n - g_n.$$

If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are *cochain homotopic*.

Remark. For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:

$$\begin{tikzcd}
{} & C^{n+1} \arrow[l, "\delta_{n+1}"'] \arrow[d, "f_{n+1}, g_{n+1}"] \arrow[dr, "P_{n+1}"] & C^n \arrow[l, "\delta_n"'] \arrow[d, "f_n, g_n"] \arrow[dr, "P_n"] & C^{n-1} \arrow[l, "\delta_{n-1}"'] \arrow[d, "f_{n-1}, g_{n-1}"] & {} \arrow[l, "\delta_{n-2}"'] \\
{} & D^{n+1} \arrow[l, "\delta_{n+1}"] & D^n \arrow[l, "\delta_n"] & D^{n-1} \arrow[l, "\delta_{n-1}"] & {} \arrow[l, "\delta_{n-2}"]
\end{tikzcd}$$

The statements of the next lemma surely do not come as much of a surprise.

Lemma 48.5.

(1) *If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map*
$$\begin{array}{rcl} f_*\colon H^n(C) & \to & H^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$
is well-defined.

(2) *Two cochain maps that are cochain homotopic induce the same map on cohomology groups.*

(3) *Cochain complexes together with the cochain maps form a category, called the* category of cochain complexes.

Proof. The first statement is proved the same way as Lemma 29.7 and the second statement is proved the same way as Lemma 30.1. The last statement follows immediately from the definitions. □

One of the most important algebraic constructions in Algebraic Topology II was that we could associate to a short exact sequence of chain complexes a long exact sequence of homology groups. We have an almost identical construction for short exact sequences of cochain complexes and their cohomology groups. We will discuss this construction in detail. So let
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
be a short exact sequence of cochain complexes. In particular for each $n \in \mathbb{N}_0$ we have a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & C^{n+1} \arrow[r, "f_{n+1}"] & B^{n+1} \arrow[r, "g_{n+1}"] & A^{n+1} \arrow[r] & 0 \\ 0 \arrow[r] & C^n \arrow[r, "f_n"] \arrow[u, "\delta_n"] & B^n \arrow[r, "g_n"] \arrow[u, "\delta_n"] & A^n \arrow[r] \arrow[u, "\delta_n"] & 0 \end{tikzcd}$$
where the horizontal sequences are exact. Now we will define a *connecting homomorphism*
$$\delta\colon H^n(A) \to H^{n+1}(C).$$
The definition is almost verbatim the same as the definition on page 770 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in H^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) := [\gamma] \in H^{n+1}(C)$.

The following proposition is now the obvious analogue of Lemma 31.9 and Proposition 31.10. We leave the proof as an exercise to the reader.

Proposition 48.6.

(1) *The connecting homomorphisms are well-defined and they are indeed homomorphisms.*[890]

(2) *The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.*

(3) *If*

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

is a short exact sequence of cochain complexes, then

$$\cdots \to H^n(C) \xrightarrow{f_*} H^n(B) \xrightarrow{g_*} H^n(A) \xrightarrow{\delta} H^{n+1}(C) \xrightarrow{f_*} H_{n+1}(B) \xrightarrow{g_*} \dots$$

forms an exact sequence.

Finally the following lemma says that cohomology commutes with direct sums and direct products. The proof of the lemma is identical to the proof of Lemma 29.12 that we did not give.

Lemma 48.7. *If $\{C_a\}_{a\in A}$ is a family of cochain complexes, then the family of inclusion maps $C_a \to \bigoplus_{a\in A} C_a$, $a \in A$, and $C_a \to \prod_{a\in A} C_a$, $a \in A$, induce isomorphisms*

$$\bigoplus_{a\in A} H^n(C_a) \xrightarrow{\cong} H^n\Big(\bigoplus_{a\in A} C_a\Big) \quad \textit{and} \quad \prod_{a\in A} H^n(C_a) \xrightarrow{\cong} H^n\Big(\prod_{a\in A} C_a\Big).$$

48.3. **Dual cochain complexes.** Now we will present the arguably most important source of cochain complex.

Definition. Let

$$(C_*, \partial_*)\colon \ \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex and let G be an abelian group. For each $n \in \mathbb{N}_0$ we write

$$\delta_n := \partial_{n+1}^*\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G).$$

We obtain the following cochain complex[891]

$$\leftarrow \operatorname{Hom}(C_{n+1}, G) \xleftarrow{\delta_n} \operatorname{Hom}(C_n, G) \xleftarrow{\delta_{n-1}} \dots \xleftarrow{\delta_1} \operatorname{Hom}(C_1, G) \xleftarrow{\delta_0} \operatorname{Hom}(C_0, G) \leftarrow 0.$$

We refer to it as *cochain complex dual to* (C_*, ∂_*) and we denote it by $\operatorname{Hom}(C_*, G)$. We define the *$n$-th cohomology group of the chain complex (C_*, ∂_*) with G-coefficients* as the cohomology of the cochain complex $(\operatorname{Hom}(C_*, G), \delta_*)$, i.e. we define

$$H^n(C; G) := \frac{\ker\{\delta_n\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G)\}}{\operatorname{im}\{\delta_{n-1}\colon \operatorname{Hom}(C_{n-1}, G) \to \operatorname{Hom}(C_n, G)\}}.$$

[890]Furthermore, if the cochain complexes are cochain complexes over a commutative ring R, then the connecting homomorphism is in fact an R-module homomorphism.

[891]It follows the functoriality property of $\operatorname{Hom}(-, G)$ that $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed a cochain complex.

Remark.

(1) We had just defined $\delta_n := \partial^*_{n+1}$, i.e. the coboundary map in the cochain complex is the dual of the boundary map in the chain complex. This is the same definition that is used in most books on algebraic topology, see e.g. [Mun, p. 251] or [Hat, p. 189]. In the excellent book by Bredon [Bre, p. 321] the coboundary is defined as $\delta_n := (-1)^{n+1} \cdot \partial^*_{n+1}$.

The different conventions do not affect the definition of the cohomology groups, but they will affect the sign conventions in some formulas later on. In particular if one cites results from other books one needs to be careful about the sign conventions used.

(2) If R is a commutative ring and if (C_*, ∂_*) is a chain complex, then it follows immediately from the remark on page 1157 that the corresponding cohomology groups $H^n(C; R)$ come naturally with the structure of R-modules.

Many of the statements about chain complexes and their homology groups hold in a slightly modified version also for cohomology groups. For example we have the following purely algebraic lemma.

Lemma 48.8. *Let G be an abelian group.*

(1) *Let $f = \{f_n \colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map between chain complexes C_* and D_*, then for each $n \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} f^* \colon \operatorname{Hom}(D_n, G) & \to & \operatorname{Hom}(C_n, G) \\ (\varphi \colon D_n \to G) & \mapsto & \big(\underbrace{C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} G}_{=\varphi\circ f_n = f_n^*\varphi}\big) \end{array}$$

is a cochain map, in particular it induces a well-defined map

$$f^* \colon H^n(D; G) \to H^n(C; G).$$

(2) *For each $n \in \mathbb{N}_0$ the maps*

$$C_* \mapsto H^n(C; G) \quad \text{and} \quad f \mapsto f^*$$

define a contravariant *functor from the category of chain complexes to the category of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)*

(3) *Let $f, g \colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^* \colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G)$ are cochain homotopic and therefore the induced maps $f^*, g^* \colon H^n(D; G) \to H^n(C, G)$ are identical.*

(4) *If $f_* \colon C_* \to D_*$ is a chain homotopy equivalence, then for every $n \in \mathbb{N}_0$ the induced maps $f^* \colon H^n(D; G) \to H^n(C; G)$ are isomorphisms.*

Proof.

(1) Let $f = \{f_n \colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have
$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n$$
which implies by the fact that $A \mapsto \mathrm{Hom}(A, G)$ is a *contravariant* functor that
$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* \;=\; f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$
We have thus shown that f^* is a cochain map. The second statement follows immediately from Lemma 48.5.

(2) This statement follows easily from the definitions.

(3) Let $\{P_n \colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have
$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$
We set $Q_n := P_{n-1}^* \colon \mathrm{Hom}(D_n, G) \to \mathrm{Hom}(C_{n-1}, G)$. It follows again from the fact that $A \mapsto \mathrm{Hom}(A, G)$ is a *contravariant* functor that
$$\underbrace{P_n^*}_{=Q_{n+1}} \circ \underbrace{\partial_{n+1}^*}_{=\delta_n} + \underbrace{\partial_n^*}_{=\delta_{n-1}} \circ \underbrace{P_{n-1}^*}_{=Q_n} \;=\; f_n^* - g_n^*.$$
This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy. It follows from Lemma 48.5 that the induced maps on cohomology groups agree.

(4) The last statement follows immediately from (2) and (3). □

In Lemma 42.7 we basically saw that for any chain complex C_* and any homomorphism $f \colon A \to B$ of abelian groups the maps
$$\begin{array}{rcl} f_* \colon C_* \otimes A & \to & C_* \otimes B \\ \Big[\sum\limits_{i=1}^k \sigma_i \otimes a_i\Big] & \mapsto & \Big[\sum\limits_{i=1}^k \sigma_i \otimes f(a_i)\Big] \end{array}$$
form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map
$$f_* \colon H_n(C_* \otimes A) \;\to\; H_n(C_* \otimes B).$$
Similarly, it is straightforward to check that the maps
$$\begin{array}{rcl} f_* \colon \mathrm{Hom}(C_n, A) & \to & \mathrm{Hom}(C_n, B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$
define a cochain map, in particular we get for each $n \in \mathbb{N}_0$ an induced map
$$f_* \colon H^n(C_*; A) \;\to\; H^n(C_*; B)$$
The following lemma basically says that these maps have all the properties one could possibly hope for. The lemma can be viewed as a cohomological analogue of Lemma 42.7.

Lemma 48.9. $(*)$

(1) *Given a chain complex C_* and $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} A & \mapsto & H^n(C_*; A) \\ (f\colon A \to B) & \mapsto & (f_*\colon H^n(C_*; A) \to H^n(C_*; B)) \end{array}$$

form a covariant functor from the category of abelian groups to the category of abelian groups. In other words, "cohomology is covariant in the coefficients".

(2) *Given a group homomorphism $f\colon A \to B$ the maps f_* define a natural transformation from the contravariant functor $C_* \mapsto H^n(C_*; A)$ to the contravariant functor $C_* \mapsto H^n(C_*; B)$.*

Proof. The two statements follow basically immediately from the definitions. □

48.4. **The singular cohomology groups of topological spaces.** Now we apply the somewhat dry algebra from the last section to our favorite chain complex, namely the singular chain complex of a pair of topological spaces.

Definition. Let (X, A) be a pair of topological spaces and let G be an abelian group.

(1) For each $n \in \mathbb{N}_0$ we write

$$C^n(X, A; G) := \operatorname{Hom}(C_n(X, A), G) \quad \text{and} \quad \delta_n := \partial_{n+1}^*\colon C^n(X, A; G) \to C^{n+1}(X, A; G).$$

We use the following notation:

(a) We refer to the elements of $C^n(X, A; G)$ as *singular n-cochains*,

(b) we refer to the elements in the kernel of $\delta_n\colon C^n(X, A; G) \to C^{n+1}(X, A; G)$ as *singular n-cocycles* and

(c) we refer to the elements in the image of $\delta_{n-1}\colon C^{n-1}(X, A; G) \to C^n(X, A; G)$ as *singular n-coboundaries*.

When $n \in \mathbb{N}_0$ is understood then sometimes we just write δ instead of δ_n.

(2) We refer to the cochain complex

$$\cdots \leftarrow C^{n+1}(X, A; G) \xleftarrow{\delta_n} C^n(X, A; G) \xleftarrow{\delta_{n-1}} C^{n-1}(X, A; G) \leftarrow \ldots \xleftarrow{\delta_0} C^0(X, A; G) \leftarrow 0$$

as the *singular cochain complex of (X, A) with G-coefficients*. We define the *n-th singular cohomology group of (X, A) with G-coefficients*[892] as follows

$$H^n(X, A; G) := H^n(C_*(X, A); G) = \frac{\ker\{\delta_n\colon C^n(X, A; G) \to C^{n+1}(X, A; G)\}}{\operatorname{im}\{\delta_{n-1}\colon C^{n-1}(X, A; G) \to C^n(X, A; G)\}}.$$

(3) If $A = \varnothing$ then we just write $C^n(X; G) = C^n(X, \varnothing; G)$ and similar for cohomology groups.

(4) If $G = \mathbb{Z}$ then often we drop the coefficients from the notation, e.g. often we write $H^n(X, A) = H^n(X, A; \mathbb{Z})$.

[892] In most cases we drop the adjective "singular" and we just talk of the n-th cohomology group of (X, A) with coefficients in G. We only refer to it as singular cohomology when we want to distinguish it from de Rham cohomology.

Remark. As we had already pointed out on page 1162, if R is a commutative ring, e.g. if $R = \mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$, then the cohomology groups $H^n(X, A; R)$ can in fact be viewed as R-modules and the coboundary maps are R-module homomorphisms.

In the following we first discuss the definition and examples of the absolute cohomology groups $H^*(X; G)$. A little later we will discuss the arguably even more confusing relative cohomology groups $H^*(X, A; G)$. We start this discussion of absolute cohomology groups with the following remark.

Remark. Let G be an abelian group.

(1) Let X be a topological space and let $n \in \mathbb{N}_0$. We denote by $S_n(X)$ the set of singular n-simplices. We then have natural identifications

$$C^n(X; G) \;=\; \mathrm{Hom}(C_n(X), G) \;\underset{\substack{\uparrow \\ \text{by definition } C_n(X) = \mathbb{Z}^{(S_n(X))}}}{=}\; \mathrm{Hom}\left(\mathbb{Z}^{(S_n(X))}, G\right) \;\underset{\substack{\uparrow \\ \text{identification given by Lemma 48.1}}}{=}\; \text{maps from } S_n(X) \text{ to } G.$$

Put differently, a singular n-cochain is precisely the same as assigning to each singular n-simplex an element in G. We will use this bijection throughout the course without explicitly referring to it.

(2) Let X be a topological space. We recall that, following the discussion on page 736, we can identify the singular 0-simplices in X with the points in X. According to the discussion in (1) we can now make the identification

$$C^0(X; G) \;=\; \{\text{all maps from } X \text{ to } G\}.$$

Before we develop the theory of cohomology groups in detail we first want to consider some explicit examples. As a starter we consider cochains and cocycles on the topological spaces $\mathbb{R}$ and S^1.

Examples.

(1) We start out with the topological space $X = \mathbb{R}$. We consider the two functions

$$\begin{array}{rcl} \alpha_{\mathbb{R}} \colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_{\mathbb{Z}} \colon \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \leq x\}. \end{array}$$

As pointed out above, these two functions define singular 0-cochains. We denote these 0-cochains again by $\alpha_{\mathbb{R}}$ and $\alpha_{\mathbb{Z}}$. For any singular 1-simplex $\mu \colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta\alpha_{\mathbb{R}})(\mu) \;\underset{\substack{\uparrow \\ \text{definition of } \delta}}{=}\; \alpha_{\mathbb{R}}(\partial\mu) \;\underset{\substack{\uparrow \\ \text{see page 738}}}{=}\; \alpha_{\mathbb{R}}(\mu(0,1) - \mu(1,0)) \;\underset{\substack{\uparrow \\ \text{definition of } \alpha_{\mathbb{R}}}}{=}\; \mu(0,1) - \mu(1,0)$$

and, slightly more interesting, it follows immediately from the above discussion and the definition of $\alpha_{\mathbb{Z}}$ that

$$(\delta\alpha_{\mathbb{Z}})(\mu) \;=\; \alpha_{\mathbb{Z}}(\partial\mu) \;=\; \begin{cases} \#\{n \in \mathbb{Z} \,|\, \mu(1,0) < n \leq \mu(0,1)\}, & \text{if } \mu(0,1) \geq \mu(1,0) \\ -\#\{n \in \mathbb{Z} \,|\, \mu(0,1) < n \leq \mu(1,0)\}, & \text{if } \mu(1,0) \geq \mu(0,1). \end{cases}$$

In a somewhat informal way we can say that the cochain $\delta\alpha_{\mathbb{Z}}\colon C_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex μ measures how many integers the simplex μ "crosses from left to right". We refer to Figure 678 for an illustration of $\delta\alpha_{\mathbb{Z}}$.

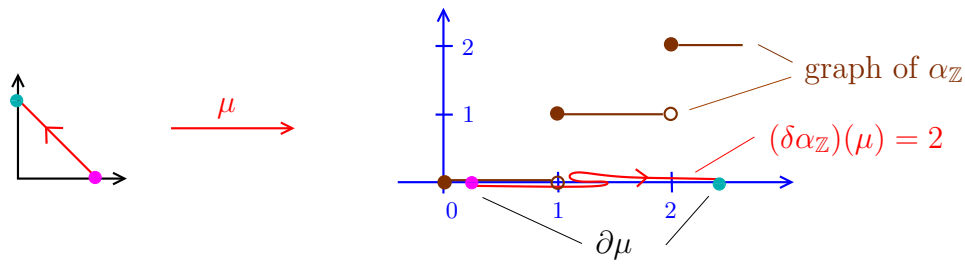


FIGURE 678.

(2) Now we consider the topological space $X = S^1$. As usual we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ the universal covering of S^1. Let $\sigma\colon \Delta^1 \to S^1$ be a singular 1-simplex. Since Δ^1 is simply connected it follows from Proposition 20.8 that σ lifts to a map $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$. The lift is not unique, but it is straightforward to see that any two lifts differ by addition by an integer. For a singular 1-simplex $\sigma\colon \Delta^1 \to S^1$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ of σ and we define

$$\theta_{\mathbb{R}}(\sigma) \;:=\; \underset{\substack{\uparrow \\ \in C_1(\mathbb{R})}}{(\delta\alpha_{\mathbb{R}})(\widetilde{\sigma})} \;=\; \widetilde{\sigma}(0,1) - \widetilde{\sigma}(1,0)$$

and we also define

$$\theta_{\mathbb{Z}}(\sigma) \;:=\; (\delta\alpha_{\mathbb{Z}})(\widetilde{\sigma}) \;=\; \begin{cases} \#\{n \in \mathbb{Z} \,|\, \widetilde{\sigma}(1,0) < n \le \widetilde{\sigma}(0,1)\}, & \text{if } \widetilde{\sigma}(0,1) \ge \widetilde{\sigma}(1,0) \\ -\#\{n \in \mathbb{Z} \,|\, \widetilde{\sigma}(0,1) < n \le \widetilde{\sigma}(1,0)\}, & \text{if } \widetilde{\sigma}(1,0) \ge \widetilde{\sigma}(0,1). \end{cases}$$

Since any two lifts of σ differ by addition by an integer we see that $\theta_{\mathbb{R}}(\sigma)$ and $\theta_{\mathbb{Z}}(\sigma)$ are both well-defined, i.e. independent of the choice of the lift $\widetilde{\sigma}$. The definitions of $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are illustrated in Figure 679.[893]

As we had explained on page 1165, these two maps $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ actually define singular 1-cochains in $C^1(S^1, \mathbb{R})$ and $C^1(S^1; \mathbb{Z})$.

[893]Here is an alternative description of the singular 1-cochain θ: We denote by $\Phi\colon [0,1] \to \Delta^1$ the homeomorphism that is given by $\Phi(t) = (1-t, t)$. Then for any singular 1-simplex $\sigma\colon \Delta^1 \to S^1$ we have

$$\theta_{\mathbb{R}}(\sigma) \;=\; \frac{1}{2\pi i} \int\limits_{\sigma\circ\Phi} \frac{1}{z}\, dz,$$

where the right-hand side denotes the path integral of the holomorphic function $\frac{1}{z}$ over the continuous path $\sigma \circ \Phi\colon [0,1] \to S^1$. We leave it as an amusing exercise to verify this equality using complex analysis.

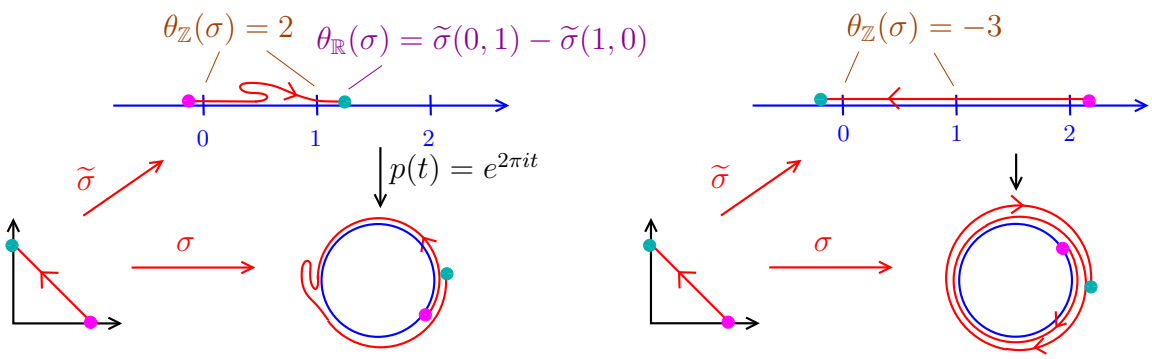


FIGURE 679.

As an example we consider the singular 1-simplex

$$\mu\colon \Delta^1 \;\to\; S^1 \quad \text{that is defined by} \quad \mu(1-t,t) \;=\; e^{2\pi i t}.$$

A lift of μ to the above universal covering $\mathbb{R} \to S^1$ is given by the singular 1-simplex

$$\widetilde{\mu}\colon \Delta^1 \;\to\; \mathbb{R} \quad \text{that is defined by} \quad \widetilde{\mu}(1-t,t) \;=\; t.$$

It follows immediately from the definitions that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$.

The following lemma shows that $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are actually quite interesting singular 1-cochains.

Lemma 48.10. *We continue with the notation from the previous example.*

(1) *The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are cocycles, in particular $\theta_{\mathbb{R}}$ defines an element in $H^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}}$ defines an element in $H^1(S^1;\mathbb{Z})$.*

(2) *The singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain, i.e. the cohomology classes $[\theta_{\mathbb{R}}] \in H^1(S^1;\mathbb{R})$ and $[\theta_{\mathbb{Z}}] \in H^1(S^1;\mathbb{Z})$ are non-trivial.*

Proof.

(1) We will first prove that the singular 1-cochain $\theta_{\mathbb{R}}$ is a cocycle. Thus we have to show that $\delta_1\theta_{\mathbb{R}} = 0 \in C^2(S^1,\mathbb{R}) = \operatorname{Hom}(C_2(S^1),\mathbb{R})$. So let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. We have to show that $(\delta_1\theta_{\mathbb{R}})(\sigma) = \theta_{\mathbb{R}}(\partial\sigma) = 0$.

As before we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ the universal covering of S^1. Since Δ^2 is simply connected we can apply Proposition 20.2 to obtain a lift of $\sigma\colon \Delta^2 \to S^1$ to a map $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$. We refer to Figure 680 for an illustration. Then

$$\theta_{\mathbb{R}}(\partial\sigma) \underset{\substack{\uparrow\\ \text{definition of } \partial}}{=} \theta_{\mathbb{R}}\Big(\sum_{j=0}^{2}(-1)^j\cdot\sigma\circ i_j^2\Big) \underset{\substack{\uparrow\\ \text{by definition of } \theta_{\mathbb{R}} \text{ and}\\ \text{since } \widetilde{\sigma}\circ i_j^2 \text{ is a lift of } \sigma\circ i_j^2}}{=} (\delta\alpha_{\mathbb{R}})\Big(\overbrace{\sum_{j=0}^{2}(-1)^j\cdot\widetilde{\sigma}\circ i_j^2}^{=\partial\widetilde{\sigma}}\Big) \underset{\substack{\uparrow\\ \text{by definition of } \delta\alpha_{\mathbb{R}} \text{ and } \partial\widetilde{\sigma}}}{=} \alpha_{\mathbb{R}}(\partial(\partial\widetilde{\sigma})) \underset{\substack{\uparrow\\ \text{since } \partial\circ\partial=0}}{=} 0.$$

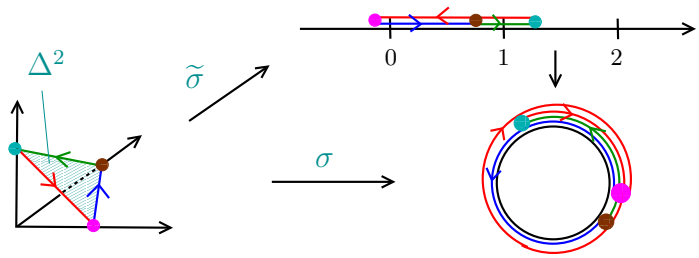


FIGURE 680.

Exactly the same argument applies to $\theta_{\mathbb{Z}}$.

(2) We still need to show that the singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain. So let β be a singular 0-cochain. As usual we view β as a function $S^1 \to \mathbb{R}$. We consider again the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ that is defined by $\mu(1-t,t) = e^{2\pi i t}$. Then

$$(\delta_0\beta)(\mu) \;=\; \beta(\,\partial(\mu)) \;=\; \beta\big(\underbrace{\mu(0,1)}_{=1\in S^1} - \underbrace{\mu(1,0)}_{=1\in S^1}\big) \;=\; 0.$$

But in the previous example we saw that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$. □

Remark. In Lemma 48.10 we have just given a mysteriously simple proof that $H^1(S^1;\mathbb{R}) \neq 0$ and also that $H^1(S^1;\mathbb{Z}) \neq 0$.

We continue with another example of cochains.

Example. We return to the topological space $X = \mathbb{R}$. We fix a point $y \in \mathbb{R}$. We consider the cochain

$$\varphi_y\colon C_0(\mathbb{R}) \;\to\; \mathbb{Z}$$

that is determined by the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \le y, \\ 1, & \text{else.} \end{cases} \end{array}$$

For any singular 1-simplex $d\colon \Delta^1 \to X = \mathbb{R}$ we have

$$(\delta_0\varphi_y)(d) \;=\; \varphi_y(\partial d) \;\underset{\substack{\uparrow \\ \text{since } \partial d = 1\cdot d(0,1) + (-1)\cdot d(1,0) \\ \text{and since } \varphi_y\colon C_0(\mathbb{R}) \to \mathbb{Z} \text{ is linear}}}{=}\; \varphi_y(d(0,1)) - \varphi_y(d(1,0)) \;=\; \begin{cases} 0, & \text{if } d(0,1), d(1,0) \le y, \\ 0, & \text{if } d(0,1), d(1,0) \ge y, \\ 1, & \text{if } d(1,0) \le y < d(0,1) \\ -1, & \text{if } d(0,1) \le y < d(1,0). \end{cases}$$

In a slightly informal way we can say that the homomorphism $\delta_0\varphi_y\colon C_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex d equals $+1$ if the simplex "crosses y from right to left" and it equals

-1 if the simplex "crosses y from left to right". We will make use of this singular 1-cochain on several occasions, hence we give it its own notation, namely we write

$$\gamma_y \ := \ \delta_0\varphi_y \ \in \ C^1(\mathbb{R};\mathbb{Z}).$$

We illustrate the singular 1-cochain $\gamma_y = \delta_0\varphi_y \in C^1(\mathbb{R};\mathbb{Z})$ in Figure 681.

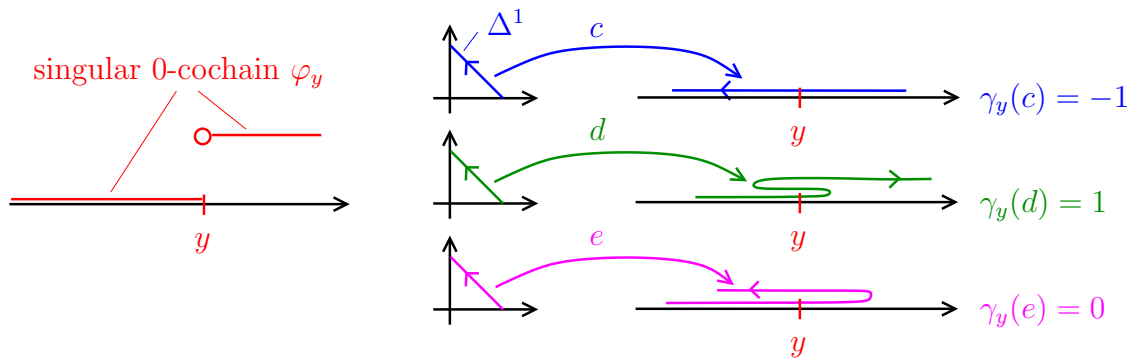


FIGURE 681.

After this long discussion of explicit cochains and cohomology classes we want to turn to more general results. The only two instances where we could compute homology groups without any problems was the 0-th homology group of a path-connected space, see Proposition 29.5, and the homology groups of a point, see Lemma 29.6. Not surprisingly we can also deal with the same situations with cohomology groups.

First of all we have the following analogue of Proposition 29.5.

Proposition 48.11. *Let X be a non-empty path-connected topological space and let G be an abelian group. Then the map*

$$\begin{array}{rcl} G & \to & H^0(X;G) \\ g & \mapsto & [\textit{0-cochain that assigns to each } x \in X \textit{ the value } g] \end{array}$$

is a well-defined map that is in fact an isomorphism. For any point $P \in X$ an inverse is given by the map

$$\begin{array}{rcl} H^0(X;G) & \to & G \\ [f] & \mapsto & f(P) \end{array}$$

Remark. At times we will use the isomorphism from Proposition 48.11 to identify the 0-th cohomology $H^0(X;G)$ of a path-connected topological space X with G.

Proof. We consider the "right-hand end" of the chain complex of $C_*(X)$ and of the cochain complex $\mathrm{Hom}(C_*(X),G)$:

$$C_1(X) \ \xrightarrow{\partial_1} \ C_0(X) \ \to \ 0 \quad \text{and} \quad \mathrm{Hom}(C_1(X),G) \ \xleftarrow{\delta_0=\partial_1^*} \ \mathrm{Hom}(C_0(X),G) \ \leftarrow \ 0.$$

Now we calculate that

$$\begin{aligned}
H^0(X;G) &= \ker\left\{\begin{array}{ccc} \mathrm{Hom}(C_0(X),G) & \to & \mathrm{Hom}(C_1(X),G) \\ (f\colon C_0(X)\to G) & \mapsto & (f\circ\partial_1\colon C_1(X)\to G)\end{array}\right\} \\
&\underset{\substack{\text{since the singular 1-simplices form a basis of } C_1(X)\\ \uparrow}}{=} \{\text{maps } f\colon X\to G \mid \underbrace{f(\partial_1(\sigma))}_{=f(\sigma(0,1))-f(\sigma(1,0))} = 0 \text{ for all singular 1-simplices } \sigma\colon \Delta^1\to X\} \\
&\underset{\substack{\text{by Lemma 48.1 we have the natural identification}\\ \mathrm{Hom}(C_0(X),G)=\text{maps from } X \text{ to } G\\ \uparrow}}{=} \text{all constant maps } X\to G \underset{\substack{\text{since } X \text{ is non-empty}\\ \uparrow}}{=} G.
\end{aligned}$$

since X is path-connected and since $\Delta^1\cong[0,1]$

The remaining statements of the proposition can be verified easily. □

In Exercise 48.1 we will prove the following lemma which can be viewed as an analogue of Lemma 29.6.

Lemma 48.12. *Let X be a topological space $X=\{x\}$ that consists of a single point x. For any abelian group G we have*

$$H^n(X;G) = \begin{cases} 0, & \text{if } n\geq 1,\\ G, & \text{if } n=0.\end{cases}$$

We want to conclude this section with a short discussion of relative cohomology classes. Before we do so we make the following observations:

Observation.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D,G) & \xrightarrow{\cong} & \text{all homomorphisms } C\to G \text{ that vanish on } D \\ (\varphi\colon C/D\to G) & \mapsto & (C\to C/D \xrightarrow{\varphi} G).\end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) If S is a set and T is a subset, then for any abelian group G the identification

$$\mathrm{Hom}(\mathbb{Z}^{(S)},G) = \text{all maps from } S \text{ to } G$$

from Lemma 48.1 restricts to an isomorphism

$$\left\{f\in\mathrm{Hom}(\mathbb{Z}^{(S)},G)\,\middle|\, f \text{ vanishes on } \mathbb{Z}^{(T)}\right\} = \text{all maps } S\to G \text{ that are trivial on } T.$$

(3) It follows immediately from Lemma 48.1 and from (1) and (2) that given a pair of topological spaces (X,A) and an abelian group G we have for every $n\in\mathbb{N}_0$ a natural

isomorphism

$$\underbrace{\mathrm{Hom}(C_n(X,A),G)}_{=C^n(X,A;G)=\mathrm{Hom}(C_n(X)/C_n(A),G)} \;=\; \text{all maps } S_n(X)\to G \text{ that are trivial on } S_n(A).$$

We will often use this natural isomorphism to identify the left-hand side with the right-hand side.

Examples.

(1) In Exercise 48.2 we will modify the argument of Proposition 48.11 to show that for any path-connected topological space with non-empty subset A we have

$$H^0(X,A;G) \;=\; 0 \quad \text{for any abelian group } G.$$

(2) We write $I=[0,1]$. In this example we want to construct a non-trivial element in $H^1(I,\partial I)$. We consider the function

$$\begin{aligned} f\colon I &\to \mathbb{Z} \\ x &\mapsto \begin{cases} 0, & \text{if } x\in[0,\frac12) \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

which we use to define the singular 1-cochain

$$\begin{aligned} \rho\colon C_1(I) &\to \mathbb{Z} \\ (\sigma\colon \Delta^1\to I) &\mapsto f(\sigma(0,1))-f(\sigma(1,0)). \end{aligned}$$

Loosely speaking, similar to the singular 1-cochain φ_y from page 1169 the singular 1-cochain $\rho(\sigma)$ measures whether σ crosses $\frac12$, and if yes, in what direction. We refer to Figure 682 for an illustration. Since Δ^1 is connected and since $\{0,1\}$ is discrete it follows from Lemma 1.92 that any singular 1-simplex $\sigma\colon\Delta^1\to\partial I=\{0,1\}$ is constant. In particular in this case we have $\rho(\sigma)=0$. Thus we see, using the above observation, that ρ defines a map

$$\rho\colon C_1(I,\partial I) \;\to\; \mathbb{Z}.$$

In Exercise 48.5 we will see that ρ is a singular 1-cocycle and that it defines a non-trivial element in $H^1(I,\partial I;\mathbb{Z})$.

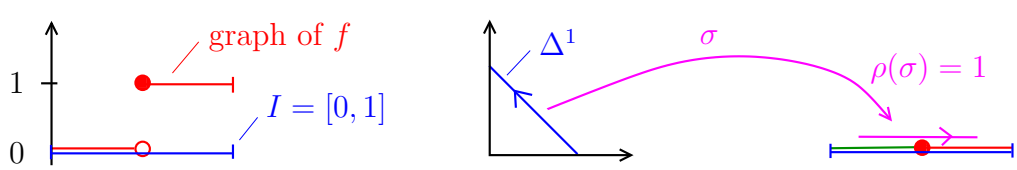


FIGURE 682.

48.5. **Basic properties of cohomology groups.** Now we will see that maps between pairs of topological spaces induce maps on cohomology groups.

Definition. Let G be an abelian group and let $f\colon (X,A) \to (Y,B)$ be a map between pairs of topological spaces. Recall that f induces a chain map $f_*\colon C_*(X,A) \to C_*(Y,B)$. For each $n \in \mathbb{N}_0$ we denote by

$$f^*\colon H^n(Y,B;G) \;\to\; H^n(X,A;G)$$

the induced map of cohomology groups that we obtain from Lemma 48.8 (1).

Example. We will see in Exercise 48.3 that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon H^0(Y;G) \to H^0(X;G)$. A very similar argument shows, that if Y is path-connected and X is non-empty, then $f^*\colon H^0(Y;G) \to H^0(X;G)$ is a monomorphism.

Notation. Let X be a topological space and let A be a subset. Furthermore let G be an abelian group. We denote by $i\colon A \to X$ the inclusion map. Sometimes we refer to $i^*\colon H^n(X;G) \to H^n(A;G)$ as the *restriction map.*

We summarize some basic properties of the induced maps in the following lemma.

Lemma 48.13. *Let G be an abelian group and let $n \in \mathbb{N}_0$.*

(1) *The maps*

$$\begin{array}{rcl} (X,A) & \mapsto & H^n(X,A;G) \\ (f\colon (X,A) \to (Y,B)) & \mapsto & (f^*\colon H^n(Y,B;G) \to H^n(X,A;G)) \end{array}$$

define a contravariant functor from the category of pairs of topological spaces to the category of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)

(2) *Let $f,g\colon (X,A) \to (Y,B)$ be maps between pairs of topological spaces. If f and g are homotopic, then they induce the same maps*

$$f^* \;=\; g^*\colon H^n(Y,B;G) \;\to\; H^n(X,A;G).$$

(3) *If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces, then the induced map*

$$f^*\colon H^n(Y;G) \;\xrightarrow{\cong}\; H^n(X;G)$$

is an isomorphism. In particular the cohomology groups of homotopy equivalent topological spaces are isomorphic.

(4) *Let X be a topological space, let $A \subset B$ be subsets of X and let $n \in \mathbb{N}_0$. If A is a deformation retract of B, then the inclusion induced map*

$$H^n(X,B;G) \;\to\; H^n(X,A;G)$$

is an isomorphism. Similarly, if B is a deformation retract of X, then the inclusion induced map

$$H^n(X,A;G) \;\to\; H^n(B,A;G)$$

is an isomorphism.

Example. Lemma 48.13 (4), together with Lemma 48.12 and the discussion on page 308 implies that for any $n \geq 1$, any abelian group G and any $k \in \mathbb{N}_0$ we have

$$H^n(\mathbb{R}^k; G) \;=\; H^n\big(\overline{B}^k; G\big) \;=\; H^n\big(B^k; G\big) \;=\; H^n\big([0,1]^k; G\big) \;=\; H^n\big(\Delta^k; G\big) \;=\; 0.$$

Proof.

(1) This statement follows immediately from Lemma 48.8 (2) together with the fact that the map $(X, A) \mapsto C_*(X, A)$ defines a covariant functor from the category of pairs of topological spaces to the category of chain complexes.
(2) This statement is an immediate consequence of Proposition 31.15 together with Lemma 48.8 (3).
(3) This statement follows from (1) and (2).
(4) This statement is proved almost verbatim the same way as Corollary 31.16 (1) and (3). □

Many of the statements about homology groups can be modified to give corresponding statements about cohomology groups. For example we have the following lemma.

Lemma 48.14. *Let X be a topological space and let G be an abelian group.*

(1) *We denote by $\{X_j\}_{j\in J}$ the path-components of X. The inclusion maps $\iota_j \colon X_j \to X$, $j \in J$, induce an isomorphism*

$$\prod_{j\in J} \iota_j^* \colon H^n(X; G) \;\xrightarrow{\cong}\; \prod_{j\in J} H^n(X_j; G).$$

(2) *Suppose that X has finitely many path-components $X_1, \ldots, X_k$. If for $j = 1, \ldots, k$ we pick a point $P_j \in X_j$, then the map*

$$\begin{array}{rcl} H^0(X; G) & \to & G^k = \overbrace{G \times \cdots \times G}^{k \text{ times}} \\ [f] & \mapsto & (f(P_1), \ldots, f(P_k)) \end{array}$$

is an isomorphism.

Proof. Let X be a topological space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We have the following isomorphisms of cochain complexes:

$$\begin{array}{ccccc} \mathrm{Hom}(C_*(X), G) & \xrightarrow{\left(\bigoplus_{j\in J} \iota_{j*}\right)^*} & \mathrm{Hom}\Big(\bigoplus_{j\in J} C_*(X_j), G\Big) & \xrightarrow{\cong} & \prod_{j\in J} \mathrm{Hom}(C_*(X_j), G). \\ & \uparrow & & \uparrow & \\ & \text{Lemma 29.13} & & \text{Lemma 48.2} & \end{array}$$

The first statement now follows easily from this observation together with the fact that cohomology commutes with direct products, see Lemma 48.7.

The second statement follows from the first statement together with the calculation of 0-th cohomology groups of path-connected topological spaces in Proposition 48.11. □

The following lemma is the analogue of Proposition 31.13.

Lemma 48.15. *Given a triple (X, B, A) of topological spaces and an abelian group G we obtain a long exact sequence*

$$\dots \xrightarrow{\delta_{n-1}} H^n(X, B; G) \xrightarrow{p^*} H^n(X, A; G) \xrightarrow{i^*} H^n(B, A; G) \xrightarrow{\delta_n} H^{n+1}(X, B; G) \xrightarrow{p^*} \dots$$

hereby the maps are given as follows:

(1) *p^* is the map induced by the map $p\colon (X, A) \to (X, B)$,*
(2) *i^* is the map induced by the map $i\colon (B, A) \to (X, A)$,*
(3) *the connecting homomorphism $\delta_n\colon H^n(B, A; G) \to H^{n+1}(X, B; G)$ is natural and it is given by the map*

$$\begin{array}{rcl} H^n(B, A; G) & \to & H^{n+1}(X, B; G) \\ [f\colon C_n(B, A) \to G] & \mapsto & \left[\begin{array}{rcl} C_{n+1}(X, B) & \to & G \\ \sigma & \mapsto & f(q(\partial_n(\sigma))) \\ \uparrow & & \\ \text{with } \sigma \in C_{n+1}(X) & & \end{array}\right] \end{array}$$

here $q\colon C_n(X) \to C_n(B)$ is the map given by sending all singular simplices that do not lie in B to zero.

Remark.

(1) An equivalent approach to expressing the connecting homomorphism is given as follows:
   (a) take a cocycle $f \in C^n(B, A; G) = \mathrm{Hom}(C_n(B, A), G)$,
   (b) denote by $\pi\colon C_n(B) \to C_n(B, A)$ the projection map,
   (c) denote by $\widetilde{f}$ the result of extending $f \circ \pi\colon C_n(B) \to G$ to a map $C_n(X) \to G$ by sending all singular simplices that do not lie in B to zero (sometimes we call $\widetilde{f}$ the *trivial extension* of $f \circ \pi$),
   (d) then consider the coboundary $\delta\widetilde{f} \in C^{n+1}(X; G)$, since f is a cocycle it follows that $\delta\widetilde{f}$ actually defines an element in $C^{n+1}(X, B; G)$ that we also denote by $\delta\widetilde{f}$, and which turns out to be a cocycle,
   (e) then $\delta[f] = [\delta\widetilde{f}] \in H^{n+1}(X, B; G)$.
(2) One of the difficulties with the long exact sequence of cohomology groups is that all maps seem to point the wrong way. You should use the Lucky Luke approach to cohomology: if Rantanplan (i.e. your intuition) points one way, you should take the opposite direction.
(3) The most frequent application of Lemma 48.15 is that given a pair (X, B) of topological spaces, i.e. given the triple $(X, B, \varnothing)$ and given an abelian group G we obtain a natural long exact sequence

$$\dots \xrightarrow{\delta_{n-1}} H^n(X, B; G) \xrightarrow{p^*} H^n(X; G) \xrightarrow{i^*} H^n(B; G) \xrightarrow{\delta_n} H^{n+1}(X, B; G) \xrightarrow{p^*} \dots$$

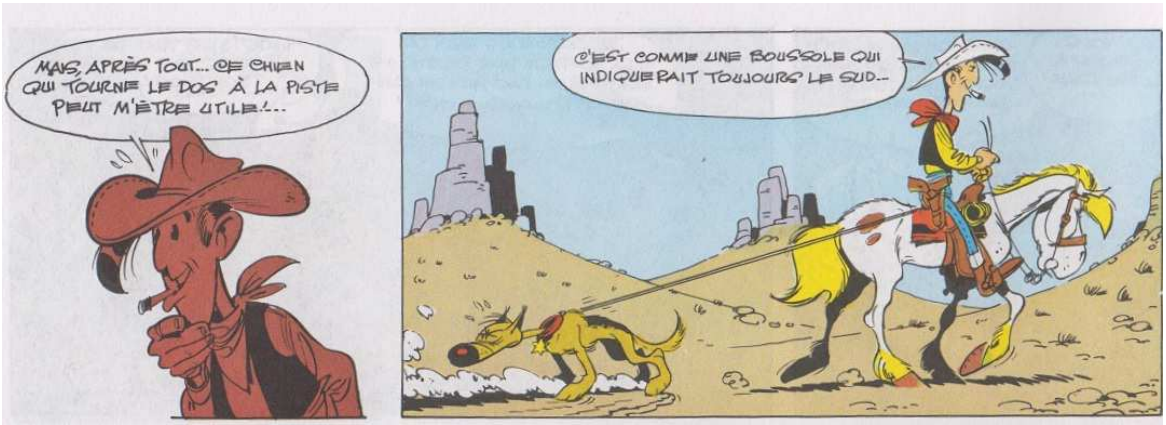


FIGURE 683. The Rantanplan rule.

Proof. Let (X, B, A) be a triple of topological spaces. As in the proof of Proposition 31.13 we note that

$$0 \;\to\; C_*(B, A) \;\xrightarrow{i_*}\; C_*(X, A) \;\xrightarrow{p_*}\; C_*(X, B) \;\to\; 0$$

is a short exact sequence of chain complexes. Now let G be an abelian group. Recall that we pointed out on page 775 that the chain groups $C_*(X, B)$ are free abelian groups. Thus we deduce from Lemma 48.4 that the following induced sequence of cochain complexes is also exact:

$$0 \;\to\; \mathrm{Hom}(C_*(X, B), G) \;\xrightarrow{p^*}\; \mathrm{Hom}(C_*(X, A), G) \;\xrightarrow{i^*}\; \mathrm{Hom}(C_*(B, A), G) \;\to\; 0.$$

The lemma now follows easily from Proposition 48.6 and the construction of the connecting homomorphism. We leave the details to the reader. □

Examples.

(1) Let X be a path-connected topological space and let A be any *non-empty* subset A of X. We denote by $i\colon A \to X$ the inclusion map. By the above remark we have a long exact sequence

$$0 \;\to\; H^0(X, A; G) \;\to\; H^0(X; G) \;\underset{\substack{\uparrow \\ \text{monomorphism, see page 1172}}}{\xrightarrow{i^*}}\; H^0(A; G) \;\to\; \dots$$

Thus it follows that $H^0(X, A; G) = 0$. In Exercise 48.2 we will provide an alternative proof for this fact using only the definitions of cohomology groups.

(2) Let $b \in \mathbb{R}_{>0}$. We consider the long exact sequence in cohomology with $\mathbb{Z}$-coefficients of the pair $(\mathbb{R}, \mathbb{R}\backslash[-b, b])$. Together with (1) and Lemma 48.14 we obtain the following

commutative diagram where the top horizontal sequence is exact:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{R};\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "{[f]\mapsto f(-b-1)}"] & H^0(\mathbb{R}\setminus[-b,b];\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "{[f]\mapsto (f(-b-1),f(b+1))}"] & H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z}) \arrow[r] & H^1(\mathbb{R};\mathbb{Z}) \arrow[d, "="] \\
& \mathbb{Z} \arrow[r, "m\mapsto(m,m)"'] & \mathbb{Z}^2 & & 0.
\end{tikzcd}$$

The cokernel of the bottom horizontal map is easily seen to be isomorphic to $\mathbb{Z}$ and it is generated by the element $(0,1)\in\mathbb{Z}^2$. Put differently, the cokernel of the map $i^*\colon H^0(\mathbb{R};\mathbb{Z})\to H^0(\mathbb{R}\setminus[-b,b];\mathbb{Z})$ is represented by the singular 0-cochain given by

$$\begin{array}{rcl} \psi\colon \mathbb{R}\setminus[-b,b] & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x\le -b, \\ 1, & \text{if } x\ge b. \end{cases} \end{array}$$

Now we want to determine $\delta\psi\in H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z})$. It is perhaps helpful to consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C_1(\mathbb{R},\mathbb{R}\setminus[-b,b]),\mathbb{Z}) \arrow[r, "p^*"] & \mathrm{Hom}(C_1(\mathbb{R}),\mathbb{Z}) \arrow[r, "i^*"] & \mathrm{Hom}(C_1(\mathbb{R}\setminus[-b,b]),\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}(C_0(\mathbb{R},\mathbb{R}\setminus[-b,b],\mathbb{Z}) \arrow[r, "p^*"] \arrow[u, "\delta_0"'] & \mathrm{Hom}(C_0(\mathbb{R}),\mathbb{Z}) \arrow[r, "i^*"] \arrow[u, "\delta_0"'] & \underbrace{\mathrm{Hom}(C_0(\mathbb{R}\setminus[-b,b],\mathbb{Z})}_{\text{contains } \psi} \arrow[r] \arrow[u, "\delta_0"'] & 0.
\end{tikzcd}$$

We start with the map $\psi\colon\mathbb{R}\setminus[-b,b]\to\mathbb{Z}$. We need to pick an extension to a map $\mathbb{R}\to\mathbb{Z}$. We might as well take the map $\varphi_0\colon\mathbb{R}\to\mathbb{Z}$ from page 1169. Then it follows from the discussion of the connecting homomorphism that the coboundary $\gamma_0:=\delta\varphi_0$ represents an element in $H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z})$, and that $\delta\psi=[\gamma_0]$ is a generator of $H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z})\cong\mathbb{Z}$.

48.6. **Reduced cohomology groups.** We saw on several occasions that the reduced homology groups $\widetilde{H}_k(X)$ of a a non-empty topological space can be convenient for expressing calculations and results. In this last section we quickly introduce the reduced cohomology groups.

Definition. Let X be a topological space. By Lemma 29.4 (1) we know that the following is a chain complex

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial_3} & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^k n_i\sigma_i & \mapsto & \sum_{i=1}^k n_i. & & \end{array}$$

Let G be an abelian group. In a fashion similar to reduced homology we now define the *k-th reduced cohomology group $\widetilde{H}^k(X;G)$ of X* to be the k-th cohomology group of the above chain complex.

The following lemma is the analogue of Lemma 31.1, and Lemma 31.12 (1) and Lemma 31.12 (2). We leave the proof as an exercise to the reader.

Lemma 48.16. *Let G be an abelian group.*

(1) *Let $f\colon X \to Y$ be a map between two topological spaces. For every $n \in \mathbb{N}_0$ the map f induces a map $f^*\colon \widetilde{H}^n(Y;G) \to \widetilde{H}^n(X;G)$ in the obvious way. For every $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} X & \mapsto & \widetilde{H}^n(X;G) \\ (f\colon X \to Y) & \mapsto & (f^*\colon \widetilde{H}^n(Y;G) \to \widetilde{H}^n(X;G)) \end{array}$$

define a functor from the category of topological spaces to the category of abelian groups.

(2) *For each $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} H^n(X;G) & \to & \widetilde{H}^n(X;G) \\ [\varphi] & \mapsto & [\varphi] \end{array}$$

define a natural transformation from the functor $X \mapsto H^n(X;G)$ to the functor $X \mapsto \widetilde{H}^n(X;G)$.

(3) *For every non-empty topological space X we have*

$$H^n(X;G) \;\cong\; \begin{cases} \widetilde{H}^0(X;G) \oplus G, & \text{if } n = 0, \\ \widetilde{H}^n(X;G), & \text{if } n \neq 0. \end{cases}$$

The isomorphism is natural for $n \geq 1$.

(4) *If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{H}^0(X;G) \cong G^{n-1}$.*

(5) *Given a topological space X, a point $x_0 \in X$ and $n \in \mathbb{N}_0$ we consider the map*

$$H^n(X,\{x_0\};G) \longrightarrow H^n(X,\varnothing;G) \xrightarrow{=} H^n(X;G) \longrightarrow \widetilde{H}^n(X;G).$$

For any $n \in \mathbb{N}_0$ this map forms a natural isomorphism from the contravariant functor $(X,x_0) \mapsto \widetilde{H}_n(X;G)$ to the contravariant functor $(X,x_0) \mapsto H^n(X,x_0;G)$.

(6) *Any homotopy equivalence between two topological spaces induces isomorphisms of reduced cohomology.*

Finally the following lemma plays the role of Corollary 31.14.

Lemma 48.17. *Let X be a topological space and let $B \subset X$ be a non-empty subset. Then given any abelian group G there exists a natural long exact sequence*

$$\ldots \xrightarrow{\delta_{n-1}} H^n(X,B;G) \xrightarrow{p^*} \widetilde{H}^n(X;G) \xrightarrow{i^*} \widetilde{H}^n(B;G) \xrightarrow{\delta_n} H^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$

Proof. The lemma follows immediately from Lemma 48.15 applied to the triple $(X,B,\{P\})$ where P is a point in B, together with Lemma 48.16 (5). □

Exercises for Chapter 48.

Exercise 48.1. Let X be a topological space $X = \{x\}$ that consists of a single point x. Show that for any abelian group G we have

$$H^n(X;G) \;\cong\; \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Exercise 48.2. Let X be a path-connected topological space and let A be a non-empty subset of X. Show that

$$H^0(X,A;G) \;=\; 0,$$

using only the definitions of cohomology groups.

Exercise 48.3. Show that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon H^0(Y;G) \to H^0(X;G)$.

Exercise 48.4. Let X be a topological space. Is the 0-th homology group $H_0(X;\mathbb{Z})$ necessarily isomorphic to the 0-th cohomology group $H^0(X;\mathbb{Z})$?

Exercise 48.5. We write $I = [0,1]$. As on page 1171 we consider the function

$$\begin{array}{rcl} f\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise.} \end{cases} \end{array}$$

This map gives rise to the following singular 1-cochain:

$$\begin{array}{rcl} \rho\colon C_1(I) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to I) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

As we pointed out on page 1171, the map ρ actually defines a cochain $\rho\colon C_1(I,\partial I) \to \mathbb{Z}$.

(a) Show that $\rho \in C_1(I,\partial I)$ is a singular 1-cocycle.
(b) Show that ρ defines a non-trivial element in $H^1(I,\partial I;\mathbb{Z})$.

Exercise 48.6. Let $X = \mathbb{R}^2$. We consider the singular 2-cochain $\varphi\colon C_2(\mathbb{R}^2;\mathbb{F}_2)$ that is given by

$$\varphi(\sigma) \;:=\; \begin{cases} 1, & \text{if } (0,0) \text{ lies in the interior of the convex hull} \\ & \text{of the three points } \sigma(1,0,0), \sigma(0,1,0), \sigma(0,0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Is φ a cocycle?

Exercise 48.7. Let $f\colon S^2 \to \mathbb{F}_2$ be a function such that $f(-x) = 1 - f(x)$ for all $x \in S^2$. Given a singular simplex $\tau\colon \Delta^1 \to S^2$ we write $\mu(\tau) = 0$ if f is constant on $\tau(\Delta^1)$ and otherwise we write $\mu(\tau) = 1$. We denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ the projection map. We consider the following cochain

$$\begin{array}{rcl} \varphi\colon C^2(\mathbb{R}\mathrm{P}^2) & \to & \mathbb{F}_2 \\ (\sigma\colon \Delta^2 \to \mathbb{R}\mathrm{P}^2) & \mapsto & \begin{array}{c} \text{pick a lift } \widetilde{\sigma}\colon \Delta^2 \to S^2 \text{ of } \sigma \text{ and calculate} \\ \mu(\widetilde{\sigma} \circ i_0^2) + \mu(\widetilde{\sigma} \circ i_1^2) + \mu(\widetilde{\sigma} \circ i_2^2). \end{array} \end{array}$$

(a) Show that φ is well-defined, i.e. show that $\varphi(\sigma)$ does not depend on the choice of the lift $\widetilde{\sigma}$.
Remark: Note that if $\widetilde{\sigma}\colon \Delta^2 \to S^2$ is a lift of σ, then the only other one is given by $\nu \circ \widetilde{\sigma}$ where $\nu\colon S^2 \to S^2$ is the map defined by $\nu(x) = -x$ for all $x \in S^2$.

(b) Show that φ is a cocycle in $C^2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.

(c) Does φ represent a non-zero cohomology class?

49. Chain complexes with isomorphic homology groups

In the following chapter we want to extend some of the deeper theorems and theories from the homological setup to the cohomological setting. Our goal will be to do so with as little effort as possible. To achieve this goal we will state and prove in this short chapter two purely algebraic propositions.

49.1. **Isomorphisms of homology groups are homotopy equivalences.** The goal of this chapter is to prove the following two propositions.

Proposition 49.1. *Suppose (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes of free abelian groups. If $\gamma_n \colon H_n(C_*) \to H_n(C'_*)$, $n \in \mathbb{N}_0$, is a family of homomorphisms, then there exists a chain map $f \colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have*

$$f_* = \gamma_n \colon H_n(C_*) \ \to \ H_n(C'_*).$$

Remark. We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "n \mapsto (0,n)"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(m,n) \mapsto m+2n"'] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f. It is straightforward to see that f induces the zero map on homology but that it is not chain homotopic to the zero chain map. This shows that the chain map from Proposition 49.1 is in general *not* unique up to chain homotopy.

The second proposition can be viewed as a converse to Lemma 30.2.

Proposition 49.2. *Let $f \colon C_* \to C'_*$ be a chain map between chain complexes of free abelian groups. If f induces an isomorphism of all homology groups, then f is in fact a chain homotopy equivalence.*

Remark. Note that we had already promised the previous proposition in the remark after Proposition 31.21.

Before we give the proofs of the two propositions we point out several consequences.

Corollary 49.3. *Suppose (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes of free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $H_n(C_*)$ and $H_n(C'_*)$ are isomorphic. Then for each $n \in \mathbb{N}_0$ and any abelian group G the following statements hold:*

1. *the homology groups $H_n(C_* \otimes G)$ and $H_n(C'_* \otimes G)$ are isomorphic and*
2. *the cohomology groups $H^n(C; G)$ and $H^n(C'; G)$ are also isomorphic.*

Remark. The first statement also follows from the fact that the short exact sequence in the Algebraic Universal Coefficient Theorem 42.18 splits, see Proposition 42.22.

Proof. Suppose (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes of free abelian groups. By our hypothesis there exist isomorphisms $\gamma_n \colon H_n(C_*) \to H_n(C'_*)$, $n \in \mathbb{N}_0$. By Proposition 49.1 these isomorphisms can be realized by a chain map $f \colon C_* \to C'_*$. But by Proposition 49.2 this chain map is a chain homotopy equivalence. Now we turn to the proofs of the two statements of the corollary.

(1) It is straightforward to see that $f \otimes \mathrm{id} \colon C_* \otimes G \to C'_* \otimes G$ is also a chain homotopy equivalence. Now the first statement follows from Corollary 30.3.
(2) The second statement follows immediately from Lemma 48.8. □

This corollary in turn gives us another, rather sobering corollary. It says that if our only goal is to distinguish (pairs of) topological spaces, then cohomology groups do not contain more information than homology groups.

Corollary 49.4. *Let (X, A) and (Y, B) be topological spaces and let G be an abelian group. Then*

$H_n(X, A)$ and $H_n(Y, B)$ are isomorphic for all $n \in \mathbb{N}_0$	$\Longrightarrow$	*the two groups $H_n(Y, B; G)$ and $H_n(X, A; G)$ and the two groups $H^n(Y, B; G)$ and $H^n(X, A; G)$ are isomorphic for all $n \in \mathbb{N}_0$.*

Remark. We will see later on in Exercise 51.7 that the analogous statement with the roles of homology and cohomology swapped does not hold. In particular there exist topological spaces X and Y such that $H^n(X; \mathbb{Z}) \cong H^n(Y; \mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $H_1(X; \mathbb{Z})$ is not isomorphic to $H_1(Y; \mathbb{Z})$.

Proof. This corollary follows immediately from Corollary 49.3 and the observation from page 775 that relative singular chain groups are free abelian groups. □

The previous corollary does not specify how the isomorphism of the cohomology groups is given. If we already start out with a map $f \colon (X, A) \to (Y, B)$ that induces an isomorphism of homology groups, then we can be more precise.

Corollary 49.5. *Let $f \colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let G be an abelian group. Then*

the induced map $f_* \colon H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all $n \in \mathbb{N}_0$	$\Longrightarrow$	*the induced maps $f_* \colon H_n(X, A; G) \to H_n(Y, B; G)$ and $f^* \colon H^n(Y, B; G) \to H^n(X, A; G)$ are isomorphisms for all $n \in \mathbb{N}_0$.*

Proof. Let $f \colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let G be an abelian group. If $f_* \colon H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all $n \in \mathbb{N}_0$, then it follows from Proposition 49.2 that $f_* \colon C_*(X, A) \to C_*(Y, B)$ is already a chain homotopy equivalence. (Here, as in the previous proof we use that relative singular chain groups are free abelian groups.) Now we get the desired conclusion from the discussion on page 1009 and Lemma 48.8. □

Later on we will also use the following purely algebraic corollary to Proposition 49.2.

Corollary 49.6. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. By Proposition 31.10 we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & H_{n+1}(C_*) & \xrightarrow{\partial} & H_n(A_*) & \longrightarrow & H_n(B_*) & \longrightarrow & H_n(C_*) & \xrightarrow{\partial} & H_{n-1}(A_*) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H_{n+1}(C'_*) & \xrightarrow{\partial} & H_n(A'_*) & \longrightarrow & H_n(B'_*) & \longrightarrow & H_n(C'_*) & \xrightarrow{\partial} & H_{n-1}(A'_*) & \longrightarrow & \dots \end{array}$$

By our hypothesis, together with Corollary 30.3, we know that always two out of three vertical maps are isomorphisms. But then it follows from the Five Lemma 31.11 that the third vertical map is also an isomorphism. Now Proposition 49.2 implies that the corresponding map of chain complexes is also a chain homotopy equivalence. □

As we pointed out above, the first time we had mentioned Proposition 49.2 was in the remark after Proposition 31.21. In fact Proposition 49.2 implies immediately that we can strengthen Proposition 31.21 to obtain the following result.

Proposition 49.7. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X. Then the inclusion map*[894]

$$C_*^{\mathcal{U}}(X) \;\to\; C_*(X)$$

is a chain homotopy equivalence.

We conclude this discussion of the consequences of Proposition 49.1 and 49.2 with the following lemma which we will use on several occasions.

Lemma 49.8. *Let (C_*, c_*) be a free chain complex such that for each $k \in \mathbb{N}_0$ the homology group $H_k(C)$ is finitely generated. Then there exists a chain complex (D_*, ∂_*) with the following two properties:*

(1) *The chain complexes (C_*, ∂_*) and (D_*, ∂_*) are chain homotopy equivalent.*
(2) *Each chain group D_k is a free abelian group of* finite rank.

[894]Recall that a *covering of X* is a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X such that the union of the interiors of the U_i equals X. On page 787 we defined

$$C_n^{\mathcal{U}}(X) \;=\; \Big\{ \sum_{j=1}^{k} a_j\sigma_j \;\Big|\; \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \Big\}$$

and we had observed that these groups form a subcomplex of $C_*(X)$. Note that these chain groups are free abelian groups generated by the union of the sets of singular n-simplices that lie in some U_i.

Proof. Let $k \in \mathbb{N}_0$. Recall that we assume that $H_k(C_*)$ is finitely generated. It follows immediately from the classification of finitely generated abelian groups, see Theorem 11.4, that there exists a chain complex D^k_* and an isomorphism $\varphi_k\colon H_k(D^k_*) \to H_k(C_*)$ such that only the chain groups D^k_k and D^k_{k+1} are non-zero and such that both chain groups are finitely generated free abelian groups.

We define $D_* = \bigoplus_{k\in\mathbb{N}_0} D^k_*$. It follows immediately from Proposition 49.1 and 49.2 that the chain complex D_* is chain homotopy equivalent to (C_*, c_*). □

49.2. **Proof of Propositions 49.1 and 49.2.** We start out with the proof of Proposition 49.1.

Proof of Proposition 49.1. Let (C_*, ∂_*) and (C'_*, ∂'_*) be two chain complexes of free abelian groups. Furthermore let $\gamma_n\colon H_n(C_*) \to H_n(C'_*)$, $n \in \mathbb{N}_0$ be a family of homomorphisms. We have to show that there exists a chain map $f\colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have

$$f_* = \gamma_n\colon H_n(C_*) \;\to\; H_n(C'_*).$$

We start out by recalling some of the usual notation and basic observations regarding chain complexes. More precisely, for each $n \in \mathbb{N}_0$ we write

$$Z_n \;:=\; \ker(\partial_n\colon C_n \to C_{n-1}), \quad B_n \;:=\; \operatorname{im}(\partial_{n+1}\colon C_{n+1} \to C_n) \quad \text{and} \quad H_n \;=\; Z_n/B_n.$$

Recall that these groups form the following two types of short exact sequences

$$\text{(a)} \qquad 0 \;\to\; Z_n \;\hookrightarrow\; C_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0,$$

and

$$\text{(b)} \qquad 0 \;\to\; B_n \;\hookrightarrow\; Z_n \;\to\; H_n \;\to\; 0.$$

Similarly we define Z'_n, B'_n and H'_n for the chain complex C'_*. Evidently these groups also form short exact sequences that we denote by (a′) and (b′).

Since all the chain groups are free abelian it follows from Lemma 11.2 that the subgroups Z_n, B_n, Z'_n and B'_n are also free abelian. In particular the sequences (b) and (b′) are free resolutions of H_n respectively H'_n. It follows from Lemma 42.13 that for each $n \in \mathbb{N}_0$ there exist homomorphisms β_n and ζ_n such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r] \arrow[d, "\beta_n"] & Z_n \arrow[r] \arrow[d, "\zeta_n"] & H_n \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & B'_n \arrow[r] & Z'_n \arrow[r] & H'_n \arrow[r] & 0.
\end{tikzcd}$$

Our goal is to extend the maps $\zeta_n\colon Z_n \to Z'_n$ to a chain map $f\colon C_n \to C'_n$. To do so we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[dr, hook] \arrow[dd, "\zeta_n"] & C_n \arrow[r, two heads, "\partial"] \arrow[d, dashed, "\psi_n"] & B_{n-1} \arrow[r] \arrow[dd, "\beta_{n-1}"] & 0 \\
& & Z_n \oplus B_{n-1} \arrow[ur, two heads] \arrow[d, "\zeta_n \oplus \beta_{n-1}"] & & \\
& & Z'_n \oplus B'_{n-1} \arrow[dr, two heads] & & \\
0 \arrow[r] & Z'_n \arrow[r, hook] \arrow[ur, hook] & C'_n \arrow[r, two heads, "\partial'"'] \arrow[u, dashed, "\psi'_n"'] & B'_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

Here the top sequence is precisely the short exact sequence (a) and the bottom sequence is the short exact sequence (a′). Here the unlabeled maps are the obvious inclusion and projection maps. It follows immediately from Lemmas 33.1 and 33.2 that there exist isomorphisms $\psi_n\colon C_n \xrightarrow{\cong} Z_n \oplus B_{n-1}$ and $\psi'_n\colon C'_n \xrightarrow{\cong} Z'_n \oplus B'_{n-1}$ which make the diagram commute. Now we define $f_n\colon C_n \to C'_n$ to be vertical homomorphism given by the diagram, i.e. $f_n = (\psi'_n)^{-1} \circ (\zeta_n \oplus \beta_{n-1}) \circ \psi_n$.

Now we need to show that the maps f_n form a chain map. We consider the diagram

$$\begin{tikzcd}
C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & B_{n-1} \arrow[r, hook] \arrow[d, "\beta_{n-1}"] & Z_{n-1} \arrow[r, hook] \arrow[d, "\zeta_{n-1}"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_n \arrow[r, "\partial'"'] & B'_{n-1} \arrow[r, hook] & Z'_{n-1} \arrow[r, hook] & C'_{n-1}
\end{tikzcd}$$

where all unlabeled maps are the obvious inclusion maps. The middle square commutes by the choice of β_{n-1} and ζ_{n-1}. We had arranged the maps f_n, f_{n-1} in such a way that the squares to the left and right commute. This shows that the rectangle commutes. But that was exactly what we needed to do to show that the maps f_n, $n \in \mathbb{N}_0$ form a chain map.

Finally note that it follows easily from the choice of ζ_n that the chain map f_n induces the given maps γ_n on homology. □

The remainder of this section will be occupied with the proof of Proposition 49.2. As we will see the proof is significantly trickier than the proof of Proposition 49.1. We start out with the following definition.

Definition. Let (C_n, c_n) and (D_n, d_n) be two chain complexes and let $\varphi\colon C_* \to D_*$ be a chain map. The *algebraic mapping cone of* φ is the chain complex $(M(\varphi), \partial_*)$ that is given by the chain groups

$$M(\varphi)_n \;:=\; \begin{matrix} C_{n-1} \\ D_n \end{matrix} \;:=\; C_{n-1} \oplus D_n$$

and the boundary maps[895]

$$\partial\colon \begin{array}{ccc} \overbrace{C_{n-1}}^{M(\varphi)_n} & \to & \overbrace{C_{n-2}}^{M(\varphi)_{n-1}} \\ D_n & & D_{n-1} \\ x & \mapsto & -c_{n-1}(x) \\ y & & \varphi_{n-1}(x)+d_n(y) \end{array} \qquad \text{which we write as block matrices} \qquad \begin{array}{cc} C_{n-1} & D_n \\ \end{array} \begin{array}{c} \\ \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} \end{array} \begin{array}{c} C_{n-2} \\ D_{n-1} \end{array}$$

Remark. Let $f\colon A \to X$ be a map between topological spaces. Recall that on page 458 we defined the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (A \times [0,1] \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a,b \in A$. As an application we showed on page 458 that for an inclusion map $i\colon A \to X$ there exists a natural isomorphism

$$\widetilde{H}_n(\mathrm{Cone}(i\colon A \to X)) \;\xrightarrow{\cong}\; H_n(X,A),$$

i.e. we can reinterpret relative homology groups as reduced homology groups of the corresponding mapping cone. Now suppose that A and X are finite CW-complexes and that f is a cellular map. By Corollary 26.24 (3) we can equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with a natural CW-structure. In Exercise 49.4 we will show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon C_*^{\mathrm{CW}}(A) \to C_*^{\mathrm{CW}}(X)$.

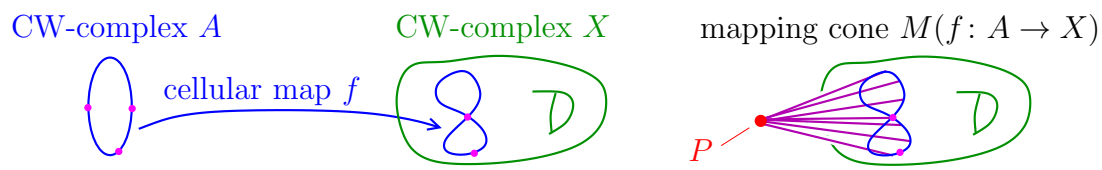


FIGURE 684.

Lemma 49.9. *Let $\varphi\colon C_* \to D_*$ be a chain map between two chain complexes (C_n, c_n) and (D_n, d_n). We denote by $i\colon D_n \to M(\varphi)_n = C_{n-1} \oplus D_n$ the obvious inclusion map and we denote by $p\colon M(\varphi)_n = C_{n-1} \oplus D_n \to C_{n-1}$ the obvious projection map. Then the following sequence is exact*

$$\ldots \to H_n(C_*) \xrightarrow{\varphi_*} H_n(D_*) \xrightarrow{i_*} H_n(M(\varphi)) \xrightarrow{p_*} H_{n-1}(C_*) \xrightarrow{\varphi_*} H_{n-1}(D_*) \to \ldots$$

[895]Note that

$$\partial_{n-1} \circ \partial_n \;=\; \begin{pmatrix} -c_{n-2} & 0 \\ \varphi_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} \;\underset{\uparrow}{=}\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since φ is a chain map and $c_{n-2} \circ c_{n-1} = 0$ and $d_{n-1} \circ d_n = 0$

This shows that we have indeed defined a chain complex.

Proof. For each $n \in \mathbb{N}_0$ we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & D_n \arrow[r, "i_n"] \arrow[d, "d_n"] & \begin{matrix} C_{n-1} \\ D_n \end{matrix} \arrow[r, "p_n"] \arrow[d, "\partial_n"', "\begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix}"] & C_{n-1} \arrow[r] \arrow[d, "c_{n-1}"] & 0 \\
0 \arrow[r] & D_{n-1} \arrow[r, "i_{n-1}"'] & \begin{matrix} C_{n-2} \\ D_{n-1} \end{matrix} \arrow[r, "p_{n-1}"'] & C_{n-2} \arrow[r] & 0.
\end{tikzcd}$$

We make the following observations:

(1) the horizontal sequences are exact,
(2) the left-hand square commutes, i.e. the maps i_* define a chain map,
(3) the right-hand square commutes up to a minus sign, i.e. we have

$$c_{n-1} \circ p_n \;=\; -p_{n-1} \circ \partial_n.$$

Fortunately this minus sign in the above equality does not change the fact that the maps p_n induce a map between homology groups.

Despite the minus sign in (3) we can use verbatim the same construction as on page 770 to obtain connecting homomorphisms $\partial_n \colon H_n(D_*) \to H_n(C_*)$, $n \in \mathbb{N}_0$, and verbatim the same proof as in Proposition 31.10 shows that these connecting homomorphisms give rise to a long exact sequence of the form

$$\ldots \;\to\; H_n(C_*) \;\xrightarrow{\partial_n}\; H_n(D_*) \;\xrightarrow{i_*} H_n(M(\varphi)) \;\xrightarrow{p_*} H_{n-1}(C_*) \;\xrightarrow{\partial_{n-1}}\; H_{n-1}(D_*) \;\to\; \ldots$$

It follows easily from the definition of the connecting homomorphism on page 770 that for each $n \in \mathbb{N}_0$ we have the equality

$$\partial_n \;=\; \varphi_* \colon H_n(C_*) \;\to\; H_n(D_*).$$

We have thus now shown that the sequence in the statement of the lemma is exact. □

The following lemma will be used in the proof of Proposition 49.2.

Lemma 49.10. *Let (C_*, ∂_*) be a chain complex of free abelian groups such that $H_n(C_*) = 0$ for all $n \in \mathbb{N}_0$. Then the identity map and the zero map of C_* to itself are chain homotopic.*

Proof. For each $n \in \mathbb{N}_0$ we write as usual

$$Z_n \;=\; \ker\{\partial_n \colon C_n \to C_{n-1}\} \quad \text{and} \quad B_n \;=\; \operatorname{im}\{\partial_{n+1} \colon C_{n+1} \to C_n\}.$$

By our hypothesis on the homology groups of C_* we have $B_n = Z_n$ for all $n \in \mathbb{N}_0$. Thus for each $n \in \mathbb{N}_0$ we have the short exact sequence

$$0 \;\to\; Z_n \;\to\; C_n \;\xrightarrow{\partial_n}\; Z_{n-1} \;\to\; 0.$$

The group $Z_{n-1} \subset C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by Lemma 11.2. It follows immediately from Lemmas 33.1 and Lemma 33.2 that for each $n \in \mathbb{N}_0$ we can find a direct sum decomposition $C_n = Z_n \oplus A_n$ such that

$\partial_n\colon A_n \to B_{n-1} = Z_{n-1}$ is an isomorphism. We denote by $\varphi_n\colon Z_{n-1} \to A_n$ its inverse. Now we define

$$\begin{array}{rcl} P_n\colon C_n = Z_n \oplus A_n & \to & C_{n+1} = Z_{n+1} \oplus A_{n+1} \\ z \oplus a & \mapsto & 0 \oplus \varphi_{n+1}(z). \end{array}$$

Using the definitions one can now easily verify that for each $n \in \mathbb{N}_0$ we have

$$P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}\colon \underbrace{Z_n \oplus A_n}_{=C_n} \;\to\; \underbrace{Z_n \oplus A_n}_{=C_n},$$

i.e. the maps $\{P_n\}_{n\geq 0}$ define the desired chain homotopy from the identity to the zero map. For orientation it might be convenient to keep in mind the following diagram:

$$\begin{tikzcd}
C_{n+1} = & Z_{n+1} \oplus A_{n+1} \arrow[d, shift right, "\partial_{n+1}"'] \\
C_n = & Z_n \oplus A_n \arrow[u, shift right, "P_n"'] \arrow[d, shift right, "\partial_n"'] \\
C_{n-1} = & Z_{n-1} \oplus A_{n-1} \arrow[u, shift right, "P_{n-1}"']
\end{tikzcd}$$

This concludes the proof of the lemma. □

Now we are finally fully prepared to provide the proof of Proposition 49.2.

Proof of Proposition 49.2. Let $\varphi\colon C_* \to D_*$ be a chain map between chain complexes (C_*, c_*) and (D_*, d_*) of free abelian groups which induces an isomorphism on all homology groups. We need to show that φ is in fact a chain homotopy equivalence.

As above we denote by $(M(\varphi)_*, \partial_*)$ the mapping cone of φ. We consider the long exact sequence from Lemma 49.9:

$$\ldots \to H_n(C_*) \xrightarrow{\varphi_*} H_n(D_*) \xrightarrow{i_*} H_n(M(\varphi)) \xrightarrow{p_*} H_{n-1}(C_*) \xrightarrow{\varphi_*} H_{n-1}(D_*) \to \ldots$$

By our hypothesis the maps φ induce isomorphisms of homology groups, thus we see that $H_n(M(\varphi)) = 0$ for all $n \geq 0$. By Lemma 49.10 this means that we have a chain homotopy $\{P_n\}_{n\geq 0}$ from the identity map to the zero map. (Note that here we use that by our hypothesis on C_* and D_* each $M(\varphi)_n = C_{n-1} \oplus D_n$ is a free abelian group.)

Recall that this means that for each $n \in \mathbb{N}_0$ we have the equality

$$(*) \qquad P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}_{C_{n-1}\oplus D_n}\,.$$

Using the obvious matrix notation we write

$$P_n \;=\; \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} : \underbrace{\begin{matrix} C_{n-1} \\ D_n \end{matrix}}_{=M(\varphi)_n} \;\to\; \underbrace{\begin{matrix} C_n \\ D_{n+1}. \end{matrix}}_{=M(\varphi)_{n+1}}$$

Now we can rewrite the above equality $(*)$ and we obtain that

$$\begin{pmatrix} P^{11}_{n-1} & P^{12}_{n-1} \\ P^{21}_{n-1} & P^{22}_{n-1} \end{pmatrix} \circ \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} + \begin{pmatrix} -c_n & 0 \\ \varphi_n & d_{n+1} \end{pmatrix} \circ \begin{pmatrix} P^{11}_{n} & P^{12}_{n} \\ P^{21}_{n} & P^{22}_{n} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{C_{n-1}} & 0 \\ 0 & \mathrm{id}_{D_n} \end{pmatrix}.$$

Looking at the four entries separately we obtain the following four equalities:

$$\begin{array}{lrcl}
\text{(a)} & -P^{11}_{n-1} \circ c_{n-1} + P^{12}_{n-1} \circ \varphi_{n-1} - c_n \circ P^{11}_n & = & \mathrm{id}_{C_{n-1}}, \\
\text{(b)} & P^{12}_{n-1} \circ d_n - c_n \circ P^{12}_n & = & 0, \\
\text{(c)} & -P^{21}_{n-1} \circ c_{n-1} + P^{22}_{n-1} \circ \varphi_{n-1} + \varphi_n \circ P^{11}_n + d_{n+1} \circ P^{21}_n & = & 0, \\
\text{(d)} & P^{22}_{n-1} \circ d_n + \varphi_n \circ P^{12}_n + d_{n+1} \circ P^{22}_n & = & \mathrm{id}_{D_n}\,.
\end{array}$$

We make the following observation:

(1) equation (b) implies that the homomorphisms $\{P^{12}_n \colon D_n \to C_n\}_{n\in\mathbb{N}_0}$ define a chain map $D \to C$.

Now we claim that the chain map $\{P^{12}_n\}_{n\in\mathbb{N}_0}$ is a chain homotopy inverse to the chain map φ. We do so with the following two observations:

(2) equation (a) implies that $P^{12} \circ \varphi \simeq \mathrm{id}_{C_*}$ where the chain homotopy is given by the maps $P^{11}_n \colon C_{n-1} \to C_n$,
(3) whereas equation (d) implies that $\varphi \circ P^{12} \simeq \mathrm{id}_{D_*}$, where the chain homotopy is given by the maps $P^{22}_n \colon D_n \to D_{n+1}$,

The last two statements together imply that φ is indeed a chain homotopy equivalence. □

49.3. **Singular and cellular chain complexes.** In Chapter 35 we introduced the cellular chain complex and the cellular homology groups of a CW-complex. We saw that cellular homology groups are naturally isomorphic to singular homology groups. At that point we did not say anything about the relationship between the cellular chain complex and the singular chain complex. We will rectify this in this section. The extra information that we gain in this section will stand us in good stead in the next chapter when we explore the relationship between cellular cohomology and singular cohomology.

First we recall some definitions and some notation. Let X be a CW-complex.

(1) Recall that given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X, i.e. the union of all cells of dimension $\leq n$.
(2) We write $C^{\mathrm{CW}}_n(X) := H_n(X^n, X^{n-1})$ and we denote by $d_n \colon C^{\mathrm{CW}}_n(X) \to C^{\mathrm{CW}}_{n-1}(X)$ the cellular boundary map from page 882.
(3) We refer to

$$\ldots \xrightarrow{d_{n+2}} C^{\mathrm{CW}}_{n+1}(X) \xrightarrow{d_{n+1}} C^{\mathrm{CW}}_n(X) \xrightarrow{d_n} C^{\mathrm{CW}}_{n-1}(X) \xrightarrow{d_{n-1}} \ldots$$

as the *cellular chain complex*. The corresponding homology groups are denoted by $H^{\mathrm{CW}}_n(X)$.

In Proposition 35.4 we had seen that, given a CW-complex X the cellular homology groups $H^{\mathrm{CW}}_k(X)$ and the singular homology groups $H_k(X)$ are naturally isomorphic.

In the following we want to relate the cellular chain complex $C_*^{\mathrm{CW}}(X)$ to the singular chain complex $C_*(X)$. By the aforementioned Proposition 35.4, together with the newly acquired Propositions 49.1 and 49.2 we know that there exists a chain homotopy equivalence $C_*(X) \to C_*^{\mathrm{CW}}(X)$. But there is no obvious chain homotopy equivalence, in fact there is not even an obvious "interesting" chain map between these two chain complexes. But it turns out that there is a nice "'intermediate" chain complex which maps into both of the above chain complexes:

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we define the[896] *intermediate cellular chain complex*

$$C_n^{\mathrm{int}}(X) \;:=\; \ker\big\{C_n(X^n) \xrightarrow{\partial} C_{n-1}(X^n) \to C_{n-1}(X^n, X^{n-1})\big\}.$$

The following proposition shows that the intermediate cellular chain complex sits comfortably between the singular and the cellular chain complex.

Proposition 49.11. *Let X be a CW-complex. The two natural chain maps*

$$i\colon C_*^{\mathrm{int}}(X) \hookrightarrow C_*(X) \quad \textit{and} \quad p\colon C_*^{\mathrm{int}}(X) \twoheadrightarrow C_*^{\mathrm{CW}}(X),$$

which are given by obvious inclusion and the obvious projection, are chain homotopy equivalences.

Remark. The proof of the proposition is modelled on the proofs provided in [Schub2, p. 303] or alternatively [Lü3, Lemma 4.2] and [Lü6, p. 263].

Proof. By Proposition 49.2 it suffices to show that for each $n \in \mathbb{N}_0$ the maps

$$(1)\;\; i_*\colon H_n(C_*^{\mathrm{int}}(X)) \to \underbrace{H_n(C_*(X))}_{=H_n(X)} \quad \text{and} \quad (2)\;\; p_*\colon H_n(C_*^{\mathrm{int}}(X)) \to \underbrace{H_n(C_*^{\mathrm{CW}}(X))}_{=H_n^{\mathrm{CW}}(X)}$$

are isomorphisms. So let $n \in \mathbb{N}_0$. Before we start with showing that the two maps above are isomorphisms we recall that in Lemma 35.1 we proved the following two convenient facts:

(a) We have $H_{n-1}(X^{n-2}) = 0$ and $H_n(X^{n-1}) = 0$.
(b) The inclusion $k\colon X^n \to X$ induces an isomorphism $k_*\colon H_n(X^{n+1}) \xrightarrow{\cong} H_n(X)$.
(c) The inclusion $k\colon X^n \to X$ induces an epimorphism $k_*\colon H_n(X^n) \twoheadrightarrow H_n(X)$.

Claim 1. Let $n \in \mathbb{N}_0$.

(i) The map $i_*\colon H_n(C_*^{\mathrm{int}}) \to H_n(C_*(X)) = H_n(X)$ is a monomorphism.
(ii) The map $i_*\colon H_n(C_*^{\mathrm{int}}) \to H_n(C_*(X)) = H_n(X)$ is an epimorphism.

[896]It follows almost immediately from the definitions that $C_*^{\mathrm{int}}(X)$ is a subcomplex of $C_*(X)$, in particular it is a chain complex in its own right.

We consider the following two commutative diagrams

$$\begin{tikzcd} C_{n+1}^{\mathrm{int}}(X) \arrow[r, hook, "j"] \arrow[d, "\partial"] & C_{n+1}(X^{n+1}) \arrow[r, hook, "k"] \arrow[d, "\partial"] & C_{n+1}(X) \arrow[d, "\partial"] \\ C_n^{\mathrm{int}}(X) \arrow[r, hook, "j"] & C_n(X^{n+1}) \arrow[r, hook, "k"] & C_n(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_n^{\mathrm{int}}(X) \arrow[r, hook, "j"] \arrow[d, "\partial"] & C_n(X^n) \arrow[r, hook, "k"] \arrow[d, "\partial"] & C_n(X) \arrow[d, "\partial"] \\ C_{n-1}^{\mathrm{int}}(X) \arrow[r, hook, "j"] & C_{n-1}(X^n) \arrow[r, hook, "k"] & C_{n-1}(X). \end{tikzcd}$$

Both statements follow easily from the diagrams together with (b) and (c) above:

(i) Let $\sigma \in C_n^{\mathrm{int}}(X)$ be a cycle with the property that $i(\sigma)$ is null-homologous. It follows from (b) that $j(\sigma)$ is already null-homologous, i.e. there exists a $\tau \in C_{n+1}(X^{n+1})$ with $\partial\tau = j(\sigma)$. Note that $j(\sigma) = \sigma \in C_n(X^n)$. Thus we see that τ actually lies in $C_{n+1}^{\mathrm{int}}(X)$. But this implies that σ is null-homologous in $C_*^{\mathrm{int}}(X)$.

(ii) Let $[\sigma] \in H_n(X)$. By (c) we know that there exists a cycle $\tau \in C_n(X^n)$ with $[k(\tau)] = [\sigma]$. Since $\partial\tau = 0$ we see that $\tau \in C_n^{\mathrm{int}}(X)$. Since the bottom-left horizontal map is a monomorphism we see that $\tau \in C_n^{\mathrm{int}}(X)$ is also a cycle. But this shows that $[\sigma] = [i(\tau)]$.

This concludes the proof of the claim.

Claim 2. Let $n \in \mathbb{N}_0$.

(i) The map $p_*\colon H_n(C_*^{\mathrm{int}}) \to H_n(C_*^{\mathrm{CW}}(X))$ is a monomorphism.

(ii) The map $p_*\colon H_n(C_*^{\mathrm{int}}) \to H_n(C_*^{\mathrm{CW}}(X))$ is an epimorphism.

For orientation purposes we consider the following commutative diagram:

$$\begin{tikzcd} C_{n+1}^{\mathrm{int}}(X) \arrow[r, hook] \arrow[d, "\partial"] & C_{n+1}(X^{n+1}) \arrow[r, two heads] \arrow[d, "\partial"] & C_{n+1}(X^{n+1}, X^n) \arrow[d, "d"] \\ C_n^{\mathrm{int}}(X) \arrow[r, two heads, "p"] \arrow[d, "\partial"] & C_n(X^n) \arrow[r] \arrow[d, "\partial"] & C_n(X^n, X^{n-1}) \arrow[d, "d"] \\ C_{n-1}^{\mathrm{int}}(X) \arrow[r, two heads, "p"] & C_{n-1}(X^n) \arrow[r] & C_{n-1}(X^{n-1}, X^{n-2}). \end{tikzcd}$$

As we will see, the claim follows from (a) and some chasing around in the above diagram:

(i) Let $\sigma \in C_n^{\mathrm{int}}(X)$ be a cycle. Suppose that $p(\sigma) = 0 \in H_n^{\mathrm{CW}}(X)$. This means that there exists a $\mu \in C_{n+1}(X^{n+1}, X^n)$ with $\partial\mu = p(\sigma)$. We pick a $\tau \in C_{n+1}(X^{n+1})$ that represents σ. Note that $\partial\tau = \sigma + \alpha$ with $\alpha \in C_n(X^{n-1})$. Since $\partial(\partial\tau) = 0$ and since $\partial\sigma = 0$ we see that $\partial\alpha = 0$. From (a) we obtain that there exists a $\beta \in C_{n+1}(X^{n-1})$ with $\partial\beta = \alpha$. Note that $\partial(\tau - \beta) = \partial\tau - \partial\beta = \sigma$. Since $\sigma \in C_n(X^n)$ we conclude that $\tau - \beta \in C_{n+1}^{\mathrm{int}}(X)$. Thus we have shown that $[\sigma] = 0 \in H_n(C_*^{\mathrm{int}}(X))$.

(ii) Given a class in $H_n(C_*^{\mathrm{CW}}(X))$ we pick a representative in $C_n^{\mathrm{CW}}(X)) = C_n(X^n, X^{n-1})$ for which in turn we pick a lift $\sigma \in C_n(X^n)$. Note that $\partial\sigma \in C_{n-1}(X^{n-2})$. But by (a) we know that $H_{n-1}(X^{n-2}) = 0$. Hence there exists a $\mu \in C_n(X^{n-2})$ with

$\partial\mu = \partial\sigma$. Note that $\partial(\sigma - \mu) = 0$, in particular $\sigma - \mu$ is a cycle in $C_n^{\mathrm{int}}(X)$. Furthermore note that σ and $\sigma - \mu$ represent the same element in $C_n(X^n, X^{n-1})$.

This concludes the proof of the claim. □

Exercises for Chapter 49.

Exercise 49.1. We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "n \mapsto (0,n)"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(m,n) \mapsto m+2n"'] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f.

(a) Convince yourself that f induces the zero map on homology.
(b) Is the chain map f chain homotopic to the zero chain map?

Exercise 49.2. Suppose (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes. We do *not* assume that the chain groups are necessarily free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $H_n(C_*)$ and $H_n(C'_*)$ are isomorphic. Does it follow that for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $H^n(C_*; G)$ and $H^n(C'_*; G)$ are also isomorphic?

Exercise 49.3. Let X and Y are topological spaces. Suppose that $H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z})$ for some $n \in \mathbb{N}_0$. Does it follow that $H^n(X; \mathbb{Z})$ and $H^n(Y; \mathbb{Z})$ are also isomorphic?

Exercise 49.4. Let $f\colon A \to X$ be a map between topological spaces. We defined the corresponding *mapping cone* as

$$M(f\colon A \to X) \;:=\; ((A \times [0,1]) \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a, b \in A$. Now suppose that A and X are finite CW-complexes and that f is a cellular map.

We equip the mapping cone $M(f\colon A \to X)$ with the natural CW-structure of Corollary 26.24. Show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon C_*^{\mathrm{CW}}(A) \to C_*^{\mathrm{CW}}(X)$.

50. Properties of cohomology groups

After the disappointment of Corollary 49.4 one might be tempted to drop cohomology groups and move on to a different subject. But later on we will see that cohomology groups can be very interesting after all. The key, as we will see, is that cohomology is a contravariant functor whereas homology is a covariant functor.

Therefore we now persevere with the study of cohomology groups. In this chapter we want to extend the following deep theorems and theories from the homological setup to the cohomological setting:

(1) The Excision Theorem 31.17,
(2) the isomorphism between singular homology and cellular homology from Proposition 35.4,
(3) the Mayer-Vietoris Theorem 33.5.

These theorems will allow us in particular to give many more calculations of cohomology groups.

50.1. **Excision theorems for cohomology groups.** We start out with the analogue of the Excision Theorem 31.17.

Theorem 50.1. (Excision Theorem for Cohomology Groups) *Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion*

$$i\colon (X \setminus Z, A \setminus Z) \quad \to \quad (X, A)$$

induces for each $n \in \mathbb{N}_0$ and every abelian group G an isomorphism

$$i^*\colon H^n(X, A; G) \quad \xrightarrow{\cong} \quad H^n(X \setminus Z, A \setminus Z; G).$$

Proof. In the original Excision Theorem 31.17 we showed that the inclusion induced map

$$i_*\colon H_n(X \setminus Z, A \setminus Z) \quad \to \quad H_n(X, A)$$

is an isomorphism for every $n \in \mathbb{N}_0$. But then it follows immediately from Corollary 49.5 that the inclusion induced maps

$$i^*\colon H^n(X, A; G) \quad \xrightarrow{\cong} \quad H^n(X \setminus Z, A \setminus Z; G).$$

are also isomorphisms. □

Remark. Using Corollary 49.5 we also obtain the "cohomological" versions of the various other excision theorems, e.g. Theorem 45.8 and Theorem 45.10. For example, let X be an m-dimensional topological manifold and let B be a compact m-dimensional submanifold of X. We write $\mathring{B} := B \setminus \partial_0 B$.[897] For every $n \in \mathbb{N}_0$ and any abelian group G the inclusion induced map

$$H^n(X, B; G) \xrightarrow{\cong} H^n\big(X \setminus \mathring{B}, \partial_0 B; G\big)$$

is an isomorphism.

[897] We refer to page 805 for the definition of $\partial_0 B$.

Psychologically it is often helpful to replace relative homology groups by the homology groups of a quotient space. In Proposition 31.19 we had seen that we can perform this switch for reasonable subsets. The following proposition does the same for cohomology groups.

Proposition 50.2. *Let G be an abelian group. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $p\colon (X,A) \to (X/A, A/A)$ the obvious projection map. We consider the following map:*

$$\begin{tikzcd} \widetilde{H}^n(X/A;G) \arrow[rr, bend right] & H^n(X/A, A/A; G) \arrow[l, "\text{Lemma 48.16 (5)}", "\cong"'] \arrow[r, "p^*"] & H^n(X,A;G) \end{tikzcd}$$

The following two statements hold:

1. *The above maps define a natural transformation from the contravariant functor $(X,A) \mapsto \widetilde{H}^n(X,A)$ to the contravariant functor $(X,A) \mapsto H^n(X,A;G)$.*
2. *If A is a non-empty good subset*[898]*, then the above map $\widetilde{H}^n(X/A;G) \to H^n(X,A;G)$ is an isomorphism.*

Proof. At this stage we have all the tools to rewrite the proof of Proposition 31.19. In particular we can use the above Excision Theorem 50.1 for Cohomology Groups to replace the Excision Theorem 31.17. We leave it to the reader to verify that nothing goes wrong. □

Example. Let G be an abelian group. From the calculation of the cohomology groups of finitely many points, see Lemmas 48.12 and 48.14, we obtain that

$$H^k(S^0;G) \quad \cong \quad \begin{cases} G^2, & \text{if } k=0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

Similar to the proof of Proposition 31.3 we can use Proposition 50.2 to calculate, via an induction argument, that for $n \geq 1$ we have

$$H^k(S^n;G) \quad \cong \quad \begin{cases} G, & \text{if } k=0 \text{ or } k=n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

We will give an alternative calculation of the cohomology groups of spheres in the next section using cellular cohomology.

50.2. **Cellular cohomology.** In Chapter 35 we introduced the cellular homology of a CW-complex. We saw that it is naturally isomorphic to singular homology and we saw that in many examples it is straightforward to compute cellular homology. Now we will repeat the trick and introduce cellular cohomology.

First we recall some definitions and some notation. Let X be a CW-complex.

1. Recall that given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X.
2. We write $C_n^{\mathrm{CW}}(X) := H_n(X^n, X^{n-1})$ and we denote by $d_n\colon C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 882.

[898]Recall according to the definition on page 763 this means that A is closed and that there exists an open neighborhood U of A in X such that A is a deformation retract of U.

(3) We refer to

$$\dots \xrightarrow{d_{n+1}} C_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} C_n^{\mathrm{CW}}(X) \xrightarrow{d_n} C_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \dots$$

as the *cellular chain complex*.

(4) Let G be an abelian group. We remark that for a finite CW-complex we have

$$\mathrm{Hom}\left(C_n^{\mathrm{CW}}(X), G\right) \underset{\substack{\uparrow \\ \text{Lemmas 35.1 and 48.1}}}{\cong} \text{maps from the set of } n\text{-cells to } G \;=\; G^{\#\, n\text{-cells}}.$$

We refer to elements in $\mathrm{Hom}\left(C_n^{\mathrm{CW}}(X), G\right)$ as *cellular n-cochains*.

Example. Given a CW-complex X an integral cellular 1-cochain in $\mathrm{Hom}\left(C_1^{\mathrm{CW}}(X), \mathbb{Z}\right)$ assigns to *every* 1-cell an integer. We refer to Figure 685 for an illustration. In contrast an element in $C_1^{\mathrm{CW}}(X)$ can be viewed as a formal linear combination of 1-cells which is the same as saying that an element in $C_1^{\mathrm{CW}}(X)$ assigns a non-zero integer to only *finitely many* 1-cells.

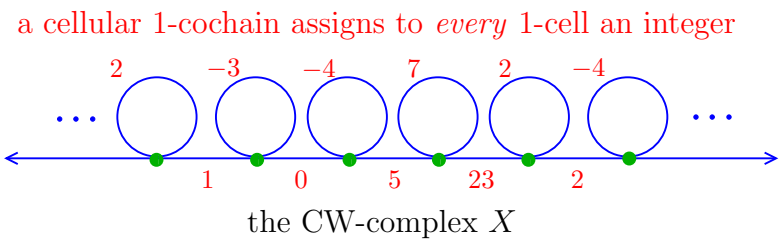


FIGURE 685.

Definition. Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) Let X be a CW-complex. We write $f_n := d_{n+1}^*$. We refer to

$$\dots \leftarrow \mathrm{Hom}\left(C_{n+1}^{\mathrm{CW}}(X), G\right) \xleftarrow{f_n} \mathrm{Hom}\left(C_n^{\mathrm{CW}}(X), G\right) \leftarrow \mathrm{Hom}\left(C_{n-1}^{\mathrm{CW}}(X), G\right) \xleftarrow{f_{n-1}} \dots$$

as the *cellular cochain complex*. We define the *n-th cellular cohomology group of X with G-coefficients* as

$$H_{\mathrm{CW}}^n(X;G) := H^n\left(C_*^{\mathrm{CW}}(X), G\right) = \frac{\ker\left\{f_n\colon \mathrm{Hom}(C_n^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(C_{n+1}^{\mathrm{CW}}(X), G)\right\}}{\mathrm{im}\left\{f_{n-1}\colon \mathrm{Hom}(C_{n-1}^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(C_n^{\mathrm{CW}}(X), G)\right\}}.$$

(2) If $f\colon X \to Y$ is a cellular map, then by Lemma 35.3 we get an induced chain map $f_*\colon C_*^{\mathrm{CW}}(X) \to C_*^{\mathrm{CW}}(Y)$ and thus we obtain from Lemma 48.8 (1) an induced map $f^*\colon H_{\mathrm{CW}}^n(Y;G) \to H_{\mathrm{CW}}^n(X;G)$.

For completeness' sake we jot down the following fairly self-evident lemma.

Lemma 50.3. $(*)$ *Let G be an abelian group and let $n \in \mathbb{N}_0$. The maps*

$$X \mapsto H^n_{\mathrm{CW}}(X) \quad \text{and} \quad (f\colon X \to Y) \mapsto (f^*\colon H^n_{\mathrm{CW}}(Y;G) \to H^n_{\mathrm{CW}}(X;G))$$

define a contravariant functor from the category of CW-complexes to the category of abelian groups.

Proof. This lemma follows immediately from Lemmas 35.3 and 48.8 (2). □

Perhaps not surprisingly we have the following theorem which plays a similar role as Proposition 35.4.

Proposition 50.4. *For every CW-complex X, any abelian group G and any $n \in \mathbb{N}_0$ there exists a natural isomorphism*[899]

$$H^n(X;G) \xrightarrow{\cong} H^n_{\mathrm{CW}}(X;G).$$

Remark. An alternative proof of this theorem is given in [Hat, p. 203].

First proof of Proposition 50.4. Let X be a CW-complex. Recall that on page 1189 we introduced the intermediate cellular chain complex that is given by

$$C^{\mathrm{int}}_n(X) := \ker\left\{C_n(X^n) \xrightarrow{\partial} C_{n-1}(X^n) \to C_{n-1}(X^n, X^{n-1})\right\} \quad \text{for } n \in \mathbb{N}_0.$$

We consider the two *natural* chain maps

$$i\colon C^{\mathrm{int}}_*(X) \hookrightarrow C_*(X) \quad \text{and} \quad p\colon C^{\mathrm{int}}_*(X) \twoheadrightarrow C^{\mathrm{CW}}_*(X),$$

which are given by the obvious inclusion and the obvious projection. In Proposition 49.11 we had seen that these two chain maps are chain homotopy equivalences. Thus it follows from Lemma 48.8 (4) that we get isomorphisms

$$H^n(X;G) = H^n(C_*(X);G) \xrightarrow[\cong]{i^*} H^n(C^{\mathrm{int}}_*(X);G) \xleftarrow[\cong]{p^*} H^n(C^{\mathrm{CW}}_*(X);G) = H^n_{\mathrm{CW}}(X;G).$$

As pointed out above, the two chain maps i and p are natural. Therefore we see that the above isomorphism $H^n(X;G) \to H^n_{\mathrm{CW}}(X;G)$ is in fact natural. □

Alternative proof of Proposition 50.4 without "natural". Let X be a CW-complex. We denote by $C_* = C_*(X)$ the singular chain complex and we denote by $D_* = C^{\mathrm{CW}}_*(X)$ the cellular chain complex. In Proposition 35.4 we showed that all the homology groups of the chain complexes C_* and D_* are isomorphic. It follows from Corollary 49.3 that for any abelian group G and any $n \in \mathbb{N}_0$ the cohomology groups $H^n(X;G) = H^n(C_*;G)$ and $H^n_{\mathrm{CW}}(X;G) = H^n(D_*;G)$ are isomorphic. This approach to proving Proposition 50.4 circumvents the use of Proposition 49.11. But it is not clear how this approach can be used to show that there exists a *natural* isomorphism. Of course for many applications this is not an issue. □

[899] Here "natural" means that there exists a natural transformation between the contravariant functors $X \mapsto H^n(X;G)$ and $X \mapsto H^n_{\mathrm{CW}}(X;G)$ from the category of CW-complexes to the category of abelian groups.

Examples. Let G be an abelian group.

(1) We use cellular cohomology to once again compute the cohomology groups of spheres. As we saw on page 648 we can view S^n, $n \geq 1$, as a CW-complex with one 0-cell and one other cell of dimension n. Furthermore the cellular boundary maps are the zero maps.[900] Thus we see that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0.$$

The dual cochain complex is then given by

$$0 \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z}, G)}_{=G} \xleftarrow{0} \cdots \xleftarrow{0} \underbrace{\mathrm{Hom}(\mathbb{Z}, G)}_{=G} \leftarrow 0.$$

Summarizing, precisely as on page 1193 we obtain that

$$H^k(S^n; G) \cong \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) If Γ is a connected, finite graph, then it follows from Proposition 9.21 that Γ is homotopy equivalent to a CW-complex X with one 0-cell and $\chi(\Gamma) + 1$ 1-cells, see Figure 686. The unique cellular boundary map in the cellular chain complex of X is zero by Lemma 35.7. Summarizing we see that for any $n \in \mathbb{N}_0$ we have

$$H^n(\Gamma; G) \underset{\substack{\uparrow \\ \text{Lemma 48.13}}}{\cong} H^n(X; G) \underset{\substack{\uparrow \\ \text{Proposition 50.4}}}{\cong} H^n_{\mathrm{CW}}(X; G) \underset{\substack{\uparrow \\ \text{since the coboundary map is zero}}}{\cong} \mathrm{Hom}(C_n^{\mathrm{CW}}(X); G) \cong \begin{cases} G, & \text{if } n = 0, \\ G^{\chi(\Gamma)+1}, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

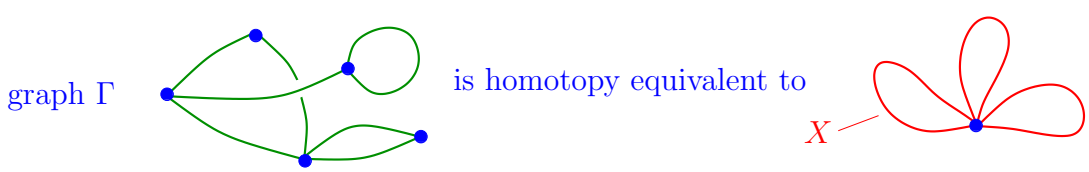


FIGURE 686.

(3) Given $g \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks. On page 951 we had argued that for $m \geq 1$ the topological space $\Sigma_{g,m}$ is homotopy equivalent to the wedge of $2g + m - 1$ circles. The previous example, together with Lemma 48.13, thus gives us the calculation of the cohomology groups of $\Sigma_{g,m}$ for $m \geq 1$.

(4) Let Σ_g be the surface of genus g. In Figure 400 we showed that we can view Σ_g as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. On page 891 we saw that the

[900] For $n \geq 2$ this follows by definition, for $n = 1$ this is a consequence of Lemma 35.7.

cellular boundary maps are all zero. Hence for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$H^n(\Sigma_g; G) \underset{\substack{\uparrow \\ \text{Proposition 50.4}}}{\cong} H^n_{\mathrm{CW}}(\Sigma_g; G) = \mathrm{Hom}\left(C^{\mathrm{CW}}_n(\Sigma_g); G\right) \cong \begin{cases} G, & \text{if } n = 0 \text{ or } n = 2, \\ G^{2g}, & \text{if } n = 1, \text{ and} \\ 0, & \text{if } n \neq 0, 1, 2. \end{cases}$$

In all the examples so far the cohomology groups are isomorphic to the corresponding homology groups that we had calculated earlier. Now we will see an example where the homology and cohomology groups differ.

Examples.

(1) We consider the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. In the proof of Proposition 35.11 we saw that there exists a CW-structure on $\mathbb{R}\mathrm{P}^n$ such that the corresponding cellular chain complex is of the following form:

$$0 \to \underset{\substack{\uparrow \\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \to \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

which allowed us to compute the homology groups of $\mathbb{R}\mathrm{P}^n$. Now let G be an abelian group. We dualize this chain complex, i.e. we apply the functor $\mathrm{Hom}(-, G)$. Using the isomorphism $\mathrm{Hom}(\mathbb{Z}, G) \to G$ from Lemma 48.1 we see that the cochain complex is isomorphic to the following cochain complex:

$$0 \leftarrow G \xleftarrow{\cdot(1+(-1)^n)} G \leftarrow \ldots \xleftarrow{0} G \xleftarrow{\cdot 2} G \xleftarrow{0} G \leftarrow 0.$$

For $G = \mathbb{Z}$ we thus obtain from Proposition 50.4 that for any $k \in \{0, \ldots, n\}$ we have

$$H^k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is even and } k \neq 0, \\ 0, & \text{if } k \text{ is odd and } k \neq n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases} \quad \text{but } H_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases}$$

where on the right-hand side we recall the calculation of the homology groups using the above cellular chain complex, see also Proposition 35.11. Therefore we see that for $n \geq 2$ the isomorphism types of the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients do *not* coincide.

(2) To add to the confusion we also compute the cohomology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. The corresponding dual cellular cochain complex is then given by

$$0 \to \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow \ldots \leftarrow \mathbb{F}_2 \xleftarrow{\cdot 2 = 0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \to 0$$

and we see that

$$H^k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

If we compare this result with the calculation on page 1011 of the homology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients we see that we get the same result. More precisely, the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients are isomorphic.

One can easily modify the above argument and apply it to $\mathbb{R}\mathrm{P}^\infty$, which we can view as a CW-complex with precisely one cell in each dimension, see 655. We then obtain easily that for each $k \in \mathbb{N}_0$ we have

$$H^k(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2) \;\cong\; \mathbb{F}_2$$

(3) We recall that in Lemma 26.2 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no odd-dimensional cells. It follows easily from Proposition 50.4 that

$$H^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 26.7 (2) we can view the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. Using Proposition 50.4 we obtain that

$$H^k(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ even,} \\ 0, & \text{if } k \text{ odd.} \end{cases}$$

Given $k < l \in \mathbb{N} \cup \{\infty\}$ we have an inclusion $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^l$. Since the isomorphism of Proposition 50.4 is natural and since the cellular chain complexes of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^\infty$ agree up to the dimension $2n+1$ we see that the inclusion induced maps $H^k(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \to H^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ are actually isomorphisms up to dimension $2n$.

50.3. **Explicit generators of cohomology groups.** In the previous section we had used cellular cohomology to compute the isomorphism types of the cohomology groups of many different topological spaces. But this calculation of the isomorphism types does not give us any explicit generators of the cohomology groups. For example on page 1166 we introduced a singular 1-cochain $\theta_\mathbb{Z} \colon C_1(S^1) \to \mathbb{Z}$ and we showed in Lemma 48.10 that $\theta_\mathbb{Z}$ defines a non-zero cohomology class $[\theta_\mathbb{Z}] \in H^1(S^1; \mathbb{Z})$. By the discussion on page 1196 we now also know that $H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$, but we do not know whether $[\theta_\mathbb{Z}] \in H^1(S^1; \mathbb{Z})$ is a generator.

On the other hand in earlier chapters we had obtained a pretty good understanding of generators of homology groups. The idea now is to translate information on homology groups to information on cohomology groups. The key tool relating homology and cohomology is given by the Kronecker pairing that we now introduce.

Lemma 50.5. *Let C_* be a chain complex.*

(1) *Let G be an abelian group. The map*

$$\begin{array}{rcl} \langle\ ,\ \rangle \colon H^n(C; G) \times H_n(C) & \to & G \\ ([\varphi], [c]) & \mapsto & \langle [\varphi], [c] \rangle := \varphi(c) \\ & & \uparrow \end{array}$$

here $\varphi \in \mathrm{Hom}(C_n, G)$ and $c \in C_n$, so $\varphi(c) \in G$ is defined

is well-defined and bilinear.[901] *In particular, if (X, A) is a pair of topological spaces, then the map*

$$\begin{array}{rcl} \langle\ ,\ \rangle_X \colon H^n(X,A;G) \times H_n(X,A) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle[\varphi],[c]\rangle_X := \varphi(c) \end{array}$$

is well-defined and bilinear. We refer to both pairings as the Kronecker pairing.[902][903]

(2) $(*)$ *If $\varphi\colon G_1 \to G_2$ is a group homomorphism then the following diagram commutes:*

$$\begin{tikzcd} H^n(C;G_1) \times H_n(C) \arrow[r, "\langle -,- \rangle"] \arrow[d, "\varphi_* \times \mathrm{id}"] & G_1 \arrow[d, "\varphi"] \\ H^n(C;G_2) \times H_n(C) \arrow[r, "\langle -,- \rangle"] & G_2. \end{tikzcd}$$

Proof. Let C_* be a chain complex.

(1) Let G be an abelian group. Let $\varphi \in \mathrm{Hom}(C_n, G)$ be a cocycle and let $c \in C_n$ be a cycle. Furthermore let $\alpha \in \mathrm{Hom}(C_{n-1}, G)$ and let $d \in C_{n+1}$. Then the following equalities hold in G:

$$\begin{array}{ccccccccc} (\varphi + \delta\alpha)(c) & = & \varphi(c) + \alpha(\partial c) & = & \varphi(c) & \text{and} & \varphi(c + \partial d) & = & \varphi(c) + (\delta\varphi)(d) & = & \varphi(c). \\ & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\ & \text{definition of } \delta & & \text{since } c \text{ is a cycle} & & & & \text{definition of } \delta & & \text{since } \varphi \text{ is a cocycle} & \end{array}$$

This calculation shows that the definition of the Kronecker pairing does not depend on the choices of the representatives of the cohomology and the homology classes. Finally note that it is clear that the Kronecker pairing is bilinear.

(2) This statement follows immediately from the definitions. □

Even though we do not need the following lemma in the near future, we record it already for later on.

Lemma 50.6. *Let $f\colon C_* \to D_*$ be a chain map between two chain complexes and let G be an abelian group. Then for any $c \in H_n(C)$ and $\varphi \in H^n(D;G)$ we have*

$$\langle f^*(\varphi), c\rangle \ = \ \langle \varphi, f_*(c)\rangle.$$

In particular, suppose that $f\colon (X,A) \to (Y,B)$ is a map between two pairs of topological spaces. Let $c \in H_n(X,A)$ and let $\varphi \in H^n(Y,B;G)$. Then

$$\langle f^*(\varphi), c\rangle_X \ = \ \langle \varphi, f_*(c)\rangle_Y.$$

[901] In general, given a commutative ring R a map $\Theta\colon U \times V \to W$ between R-modules is called *bilinear*, if for all $u, u' \in U$, $v, v' \in V$ and $r \in R$ we have

$$\begin{array}{rclcrcl} \Theta(u+u',v) & = & \Theta(u,v) + \Theta(u',v) & \text{and} & \Theta(ru,v) & = & r\Theta(u,v), \quad \text{and} \\ \Theta(u,v+v') & = & \Theta(u,v) + \Theta(u,v') & \text{and} & \Theta(u,rv) & = & r\Theta(u,v). \end{array}$$

In our special case we consider the case of $\mathbb{Z}$-modules.

[902] Often we drop the subscript X in the notation.

[903] Leopold Kronecker (1823–1891) was a German mathematician who worked on number theory. I don't know how his name got attached to this pairing.

Proof. The statement follows immediately from the definitions. Indeed, if $[c] \in H_n(C)$ and $[\varphi] \in H^n(D;G)$, then

$$\langle f^*([\varphi]),[c]\rangle \;=\; \langle [f^*\varphi],[c]\rangle \;=\; (f^*\varphi)(c) \;=\; \varphi(f_*(c)) \;=\; \langle [\varphi],[f_*(c)]\rangle \;=\; \langle [\varphi], f_*([c])\rangle.$$

The statement regarding maps between pairs of topological spaces is an immediate consequence of the purely algebraic statement. □

Now we state the lemma which will allow us in many situations to show that given cocycles form a basis for cohomology groups.

Lemma 50.7. *Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose that $H^n(X;\mathbb{Z})$ and $H_n(X;\mathbb{Z})$ are free abelian groups of rank k and suppose we are given $\varphi_1,\dots,\varphi_k \in H^n(X;\mathbb{Z})$ and $c_1,\dots,c_k \in H_n(X;\mathbb{Z})$. If*

$$\det\Big(\underbrace{\langle \varphi_i, c_j\rangle_{i,j=1,\dots,k}}_{=\,k\times k\text{-matrix}}\Big) \;=\; \pm 1,$$

then $\varphi_1,\dots,\varphi_k \in H^n(X;\mathbb{Z})$ and $c_1,\dots,c_k \in H_n(X;\mathbb{Z})$ are both bases.

Example. Let $X = S^1$. We denote by $\theta_{\mathbb{Z}}\colon C_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 1166. We consider the singular cycle $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto e^{2\pi i t}$. As on page 1167 we see that

$$\langle [\theta_{\mathbb{Z}}],[\mu]\rangle \;=\; \theta_{\mathbb{Z}}(\mu) \;=\; \alpha_{\mathbb{Z}}\left(\text{boundary of lift } \begin{array}{rcl} \widetilde{\mu}\colon \Delta^1 & \to & \mathbb{R} \\ (1-t,t) & \mapsto & t \end{array} \text{ of } \mu\right) \;=\; 1.$$

From Proposition 31.3 we know that $H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ and by the discussion on page 1193 we know that $H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore it follows from Lemma 50.7 that $[\theta_{\mathbb{Z}}] \in H^1(S^1;\mathbb{Z})$ is in fact a generator.[904] In particular we have an equality $H^1(S^1;\mathbb{Z}) = \mathbb{Z}\cdot[\theta_{\mathbb{Z}}]$.

Remark. Lemma 50.7 is really quite generous, it gives us as output that both sets are bases. In particular to apply the lemma we do not even have to verify that the c_i form a basis.

Lemma 50.7 is an immediate consequence of the following purely algebraic lemma.

Lemma 50.8. *Let A and B be two free abelian groups of rank k, let $a_1,\dots,a_k \in A$ and let $b_1,\dots,b_k \in B$. Furthermore let $\langle\ ,\ \rangle\colon A\times B \to \mathbb{Z}$ be a bilinear pairing. If*

$$\det(\langle a_i, b_j\rangle_{i,j=1,\dots,k}) \;=\; \pm 1,$$

then both $a_1,\dots,a_k$ and $b_1,\dots,b_k$ are bases.

Proof. We denote by S the free abelian group generated by the set $\{a_1,\dots,a_k\}$ and we denote by T the free abelian group generated by the set $\{b_1,\dots,b_k\}$. We denote by $i\colon S \to A$ and by $j\colon T \to B$ the obvious maps given by $a_i \mapsto a_i$ and $b_i \mapsto b_i$. We denote by

[904] It also follows from Lemma 50.7 that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ represents a generator for $H_1(S^1) \cong \mathbb{Z}$. We had obtained that result in Lemma 32.11.

$\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ the homomorphism given by $a \mapsto (b \mapsto \langle a,b\rangle)$.[905] The claim follows from the following three observations:

(1) Let $\gamma\colon G \to H$ be a homomorphism between two finitely generated free abelian groups of the same rank. Lemma 11.8 (5) says that if γ is an epimorphism, then γ it is already an isomorphism.

(2) We consider the following sequence of homomorphisms

$$S \xrightarrow{i} A \xrightarrow{\Phi} \mathrm{Hom}(B,\mathbb{Z}) \xrightarrow{j^*} \mathrm{Hom}(T,\mathbb{Z}).$$

With respect to the basis $a_1,\dots,a_k$ of S and the dual basis $b_1^*,\dots,b_k^*$ of $\mathrm{Hom}(T,\mathbb{Z})$ this linear map is represented by the matrix $\langle a_i,b_j\rangle_{i,j=1,\dots,k}$. Thus by our hypotheses this map is an isomorphism. Furthermore by our hypotheses each of the individual groups is a free abelian group of rank k.

(3) Since the map in (2) is an isomorphism it is in particular an epimorphism. Starting from the right and using (1) three times in a row we see that all maps are in fact isomorphisms. In particular we have $S = A$. The same argument turned around shows that $T = B$. □

Now we will use Lemma 50.7 to determine explicit bases for the first cohomology groups of surfaces. We start out with the torus.

Lemma 50.9. *We consider the two projection maps*

$$\begin{array}{rcl} p\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & v \end{array} \quad \textit{and} \quad \begin{array}{rcl} q\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & w. \end{array}$$

Then $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $H^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. We denote as usual by $\mu\colon \Delta^1 \to S^1$ our favorite singular 1-cycle of S^1 that is given by $\mu(1-t,t) = e^{2\pi i t}$. For simplicity we write $\theta = \theta_{\mathbb{Z}}$. We denote by

$$\begin{array}{rcl} i\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (z,1) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (1,z) \end{array}$$

the two obvious inclusion maps. We calculate that

$$\begin{pmatrix} \langle [p^*\theta],[i_*(\mu)]\rangle & \langle [p^*\theta],[j_*(\mu)]\rangle \\ \langle [q^*\theta],[i_*(\mu)]\rangle & \langle [q^*\theta],[j_*(\mu)]\rangle \end{pmatrix} \overset{\substack{\text{since } \langle [p^*\theta],[i_*(\mu)]\rangle = (p^*\theta)(i_*(\mu)) = \theta(p_*(i_*(\mu))) \text{ and similarly for the other terms} \\ \downarrow}}{=} \begin{pmatrix} \theta(p_*(i_*(\mu))) & \theta(p_*(j_*(\mu))) \\ \theta(q_*(i_*(\mu))) & \theta(q_*(j_*(\mu))) \end{pmatrix} \underset{\substack{\uparrow \\ \text{since } p\circ i = \mathrm{id} \text{ and } q\circ j = \mathrm{id} \text{ and since } p\circ j \equiv 1 \in S^1 \text{ and } q\circ i \equiv 1 \in S^1 \\ \text{here we denote by 1 also the constant singular 1-simplex given by } (1-t,t)\mapsto 1}}{=} \begin{pmatrix} \theta(\mu) & \theta(1) \\ \theta(1) & \theta(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We know from Lemma 33.12 that $H_1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$ and we saw on page 1197 that $H^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$. The above calculation, together with Lemma 50.7 shows that $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $H^1(S^1\times S^1;\mathbb{Z})$. □

[905] The fact that $\langle\,,\,\rangle$ is bilinear implies that $b \mapsto \langle a,b\rangle$ is a homomorphism and that $\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ is a homomorphism.

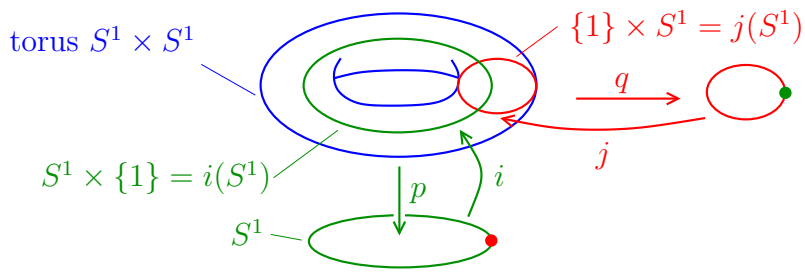


FIGURE 687.

Together with Proposition 48.11 we have now found explicit generators for the cohomology groups of the torus in dimensions 0 and 1. The following question arises:

Question 50.10. *Can we find an explicit generator of $H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$?*

Now we turn to surfaces of higher genus. As usual, given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. In Figure 688 we show projection maps $p_i \colon \Sigma_g \to S^1 \times S^1$, $i = 1, \dots, g$.[906] The following lemma is a generalization of Lemma 50.9.

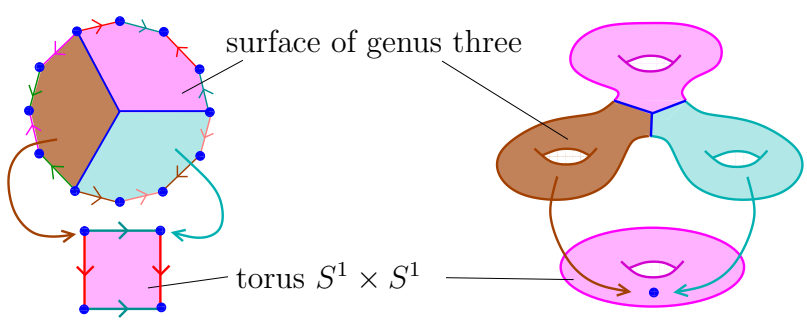


FIGURE 688.

Lemma 50.11. *We denote by $\alpha, \beta \in H^1(S^1 \times S^1; \mathbb{Z})$ the basis given by Lemma 50.9. A basis for $H^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is given by $p_i^*(\alpha), p_i^*(\beta)$, $i = 1, \dots, g$.*

Proof. The proof of the lemma is very similar to the proof of Lemma 50.9. For $i = 1, \dots, g$ we write $\varphi_i = p_i^*(\alpha)$ and $\psi_i := p_i^*(\beta)$. We consider the singular 1-cycles $x_1, \dots, x_g$ and $y_1, \dots, y_g$ that are indicated in Figure 689. As in the proof of Lemma 50.9 it follows easily from the definitions that (for $g = 2$) we have

$$\begin{pmatrix} \langle \varphi_1, x_1 \rangle & \langle \varphi_1, y_1 \rangle & \langle \varphi_1, x_2 \rangle & \langle \varphi_1, y_2 \rangle \\ \langle \psi_1, x_1 \rangle & \langle \psi_1, y_1 \rangle & \langle \psi_1, x_2 \rangle & \langle \psi_1, y_2 \rangle \\ \langle \varphi_2, x_1 \rangle & \langle \varphi_2, y_1 \rangle & \langle \varphi_2, x_2 \rangle & \langle \varphi_2, y_2 \rangle \\ \langle \psi_2, x_1 \rangle & \langle \psi_2, y_1 \rangle & \langle \psi_2, x_2 \rangle & \langle \psi_2, y_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[906] In fact for $g = 2$ we already saw this map in Figure 142.

The lemma now follows immediately from Lemma 50.7. □

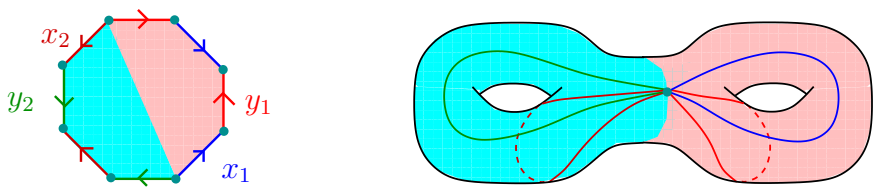


FIGURE 689. Illustration for the proof of Lemma 50.11.

50.4. **Excisive triads and the Mayer–Vietoris Theorem for cohomology groups.** Before we state and prove the Mayer–Vietoris Theorem for cohomology groups it is helpful to introduce the following definition.

Definition.

(1) A *triad of topological spaces* is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.[907]
(2) A map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ of triads is a map $f\colon X \to Y$ with $f(A_1) \subset B_1$ and $f(A_2) \subset B_2$.
(3) Given a triad (X, A_1, A_2) of topological spaces we consider the following subgroup of $C_n(A_1 \cup A_2)$:[908]

$$C_n^{\{A_1,A_2\}}(A_1 \cup A_2) := \left\{ \sum_{j=1}^{k} a_j \sigma_j \in C_n(A_1 \cup A_2) \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

The usual boundary map on the groups $C_*(A_1 \cup A_2)$ restricts to a boundary map on $C_*^{\{A_1,A_2\}}(A_1 \cup A_2)$, so we can view $C_*^{\{A_1,A_2\}}(A_1 \cup A_2)$ as a chain complex. We denote the corresponding homology groups by $H_*^{\{A_1,A_2\}}(A_1 \cup A_2)$.
(4) We say a triad (X, A_1, A_2) is *excisive* if the inclusion map

$$C_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; C_*(A_1 \cup A_2)$$

is a chain homotopy equivalence of chain complexes.

Example. In Exercise 50.1 we will see that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

After this negative example we want to collect some positive examples of excisive triads.

[907]This definition should not be confused with the definition of a *triple* of topological spaces, which was defined as a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets. In a triad we do not demand that the third set is contained in the second set.

[908]The definition is basically the same as on page 787 except that this time we do not demand that the interiors of A_1 and A_2 cover all of X.

Proposition 50.12. *Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:*

(1) $A_1 \subset A_2$ *or* $A_2 \subset A_1$,
(2) $A_1 = \varnothing$ *or* $A_2 = \varnothing$ *or* $A_1 = A_2$,
(3) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ *where we denote by* $\mathring{A}_1$ *and* $\mathring{A}_2$ *the interiors of* A_1 *and* A_2 *in* $A_1 \cup A_2$,
(4) A_1 *and* A_2 *are both open subsets of* $A_1 \cup A_2$,
(5) $A_1 \cup A_2$ *is a topological manifold and* A_1 *and* A_2 *are submanifolds such that each component of* $A_1 \cap A_2$ *is a boundary component of* A_1 *and a boundary component of* A_2, *and such that* A_1 *and* A_2 *are closed subsets,*
(6) X *is a CW-complex and* A_1 *and* A_2 *are both subcomplexes.*

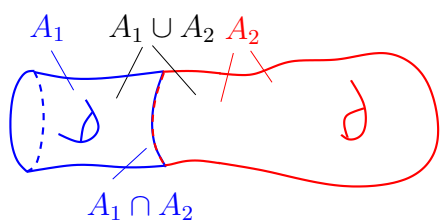


FIGURE 690. Illustration of Proposition 50.12 (5).

In the proof of Proposition 50.12 we will need the following lemma which gives us several criteria for a triad to be excisive.

Lemma 50.13. *Let (X, A_1, A_2) be a triad of topological spaces. The following statements are equivalent:*

(1) *the triad is excisive,*
(2) *for every $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n^{\{A_1, A_2\}}(A_1 \cup A_2) \;\to\; H_n(A_1 \cup A_2)$$

is an isomorphism,
(3) *for every $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(A_1, A_1 \cap A_2) \;\to\; H_n(A_1 \cup A_2, A_2)$$

is an isomorphism.

Furthermore, if G is an abelian group and if the triad is excisive, then statements (2) *and* (3) *also hold for G-coefficients.*

Proof. Let (X, A_1, A_2) be a triad of topological spaces. The equivalence of (1) and (2) is an immediate consequence of Corollary 30.3 and Proposition 49.2.

Now we turn to the proof that (2) and (3) are equivalent. We write $Y = A_1 \cup A_2$. We consider the following sequences of chain complexes:

$$\begin{array}{ll} \text{(a)} & 0 \;\to\; C_*^{\{A_1,A_2\}}(Y) \;\to\; C_*(Y) \;\to\; C_*(Y)/C_*^{\{A_1,A_2\}}(Y) \;\to\; 0 \\ \text{(b)} & 0 \;\to\; \underbrace{C_*(A_1)/C_*(A_1\cap A_2)}_{=C_*(A_1,A_1\cap A_2)} \;\to\; \underbrace{C_*(Y)/C_*(A_2)}_{=C_*(Y,A_2)} \;\to\; C_*(Y)/C_*^{\{A_1,A_2\}}(Y) \;\to\; 0. \end{array}$$

The top sequence is exact by definition. It is easy to see that the bottom sequence is also exact. We can thus apply Proposition 31.10 to get the corresponding long exact sequences of homology groups:

$$\begin{array}{l} \text{(a)} \to H_n\big(C_*^{\{A_1,A_2\}}(Y)\big) \to H_n(Y) \to H_n\big(C_*(Y)/C_*^{\{A_1,A_2\}}(Y)\big) \xrightarrow{\partial} H_{n-1}\big(C_*^{\{A_1,A_2\}}(Y)\big) \to \\ \text{(b)} \to H_n(A_1, A_1\cap A_2) \to H_n(Y, A_2) \to H_n\big(C_*(Y)/C_*^{\{A_1,A_2\}}(Y)\big) \xrightarrow{\partial} H_{n-1}(A_1, A_1\cap A_2) \to \end{array}$$

But using these long exact sequences we see that

$$\text{(2) holds} \;\underset{\substack{\uparrow \\ \text{long exact sequence (a)}}}{\Longleftrightarrow}\; H_n\big(C_*(Y)/C_*^{\{A_1,A_2\}}(Y)\big) = 0 \text{ for all } n \in \mathbb{N}_0 \;\underset{\substack{\uparrow \\ \text{long exact sequence (b)}}}{\Longleftrightarrow}\; \text{(3) holds.}$$

Finally suppose that (X, A_1, A_2) is excisive and that G is an abelian group. The fact that the map in (2) is also an isomorphism with G-coefficients follows from basically the same argument as above. Furthermore, if (2) holds for G-coefficients, then again the same argument as above shows that also (3) holds with G-coefficients. □

Now we turn to the proof of Proposition 50.12.

Proof of Proposition 50.12. Let (X, A_1, A_2) be a triad of topological spaces.

(1) If $A_1 \subset A_2$, then $C_*^{\{A_1,A_2\}}(A_1 \cup A_2) = C_*(A_2) = C_*(A_1 \cup A_2)$, so the triad is excisive. The same argument applies if $A_2 \subset A_1$.
(2) This case is just a special case of (1).
(3) If $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$, then the desired statement follows immediately from Proposition 49.7.
(4) The case that A_1 and A_2 are both open subsets of $A_1 \cup A_2$ is a special case of (3).
(5) Suppose that $A_1 \cup A_2$ is a compact topological manifold and that A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2. The Excision Theorem 45.10 for topological manifolds implies that for every $n \in \mathbb{N}_0$ the inclusion induced map

$$H_n(A_1, A_1 \cap A_2) \;\to\; H_n(A_1 \cup A_2, A_2)$$

is an isomorphism. By Lemma 50.13 (3) ⇒ (1) this implies that the triad (X, A_1, A_2) is excisive.
(6) Suppose that X is a CW-complex and that A_1 and A_2 are both subcomplexes. The proof of the Excision Theorem 32.7 for Manifolds can easily be modified to give the corresponding statement for CW-complexes. More precisely, in the proof of the

Excision Theorem 32.7 for Manifolds we need to replace the Collar Neighborhood Theorem 10.10 by Proposition 26.11 (8). Furthermore we need to replace Proposition 3.21 (3) by Lemma 26.10 (3). We then proceed as in (5). We leave it to the reader to fill in the details. □

Now we can formulate the following theorem, which in view of Proposition 50.12 is a generalization of the various Mayer–Vietoris Theorems 33.5, 33.10, and 33.11 that we had formulated earlier.

Theorem 50.14. (Mayer–Vietoris Theorem) *Given any excisive triad (X, A, B) with $X = A \cup B$ there exists a natural*[909] *long exact sequence*

$$\cdots \to H_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \to \dots$$

where the various maps denoted by i are the obvious inclusion induced maps. The same long exact sequence also exists for homology with coefficients in any abelian group G.

Proof. Let (X, A, B) be a triad with $X = A \cup B$. In the proof of the original Mayer–Vietoris Theorem 33.5 we showed that the sequence

$$0 \to C_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A,B\}}(X) \to 0$$

of chain maps is exact. We deduce from Proposition 31.10 that we get a long exact sequence of the form

$$\cdots \to H_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n^{\{A,B\}}(X) \xrightarrow{p_n} H_{n-1}(A \cap B) \to \dots$$

This long exact sequence is natural by the remark on page 773. Now we suppose that (X, A, B) is in fact excisive. By definition, and since $X = A \cup B$, this means that the inclusion map

$$C_*^{\{A,B\}}(X) \to C_*(X)$$

is a chain homotopy equivalence, hence for every $n \in \mathbb{N}_0$ it induces a natural isomorphism

$$\Phi_n \colon H_n^{\{A,B\}}(X) \xrightarrow{\cong} H_n(X).$$

For each $n \in \mathbb{N}_0$ we now replace $H_n^{\{A,B\}}(X)$ by $H_n(X)$ and we write $\partial_n := p_n \circ \Phi_n^{-1}$ and we finally get the desired natural long exact sequence.

The argument for homology with G-coefficients is very similar, see also page 1010. We leave the details to the dedicated reader. □

[909] Here "natural" means that given a map $f \colon (X, A, B) \to (Y, C, D)$ of excisive triads with $X = A \cup B$ and $Y = C \cup D$ the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_n(A \cap B) & \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} & H_n(A) \oplus H_n(B) & \xrightarrow{i_A + i_B} & H_n(X) & \xrightarrow{\partial_n} & H_{n-1}(A \cap B) & \longrightarrow \dots \\
 & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
\dots \longrightarrow & H_n(C \cap D) & \xrightarrow{i_{C\cap D} \oplus -i_{C\cap D}} & H_n(C) \oplus H_n(D) & \xrightarrow{i_C + i_D} & H_n(Y) & \xrightarrow{\partial_n} & H_{n-1}(C \cap D) & \longrightarrow \dots
\end{array}$$

We also have the following analogue for cohomology groups.

Theorem 50.15. (Mayer–Vietoris Theorem for Cohomology Groups) *Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. Then there exists a natural*[910] *long exact sequence*

$$\cdots \to H^n(X;G) \xrightarrow{i^* \oplus i^*} H^n(A;G) \oplus H^n(B;G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B;G) \to H^{n+1}(X;G) \to \dots$$

where the maps i^* are the restrictions to the various subspaces.

Proof. Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. We start out with the following claim.

Claim. The inclusion map

$$C_*^{\{A,B\}}(X) \quad \to \quad C_*(X)$$

induces for each $n \in \mathbb{N}_0$ a natural isomorphism

$$H^n(X;G) \quad \to \quad H^n\big(C_*^{\{A,B\}}(X);G\big).$$

The claim follows immediately from the definition of an excisive triad together with Lemma 48.8 (3).

Now we turn to the actual proof of the Mayer–Vietoris Theorem for Cohomology Groups. Exactly as in the proof of the Mayer–Vietoris Theorem 50.14 we start out with the fact that the following sequence is exact:

$$0 \to C_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A \cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A,B\}}(X) \to 0.$$

The groups $C_*^{\{A,B\}}(X)$ are clearly free abelian groups. Thus we deduce from Lemma 48.4 and Proposition 48.6 that there exists a natural long exact sequence

$$\to \underbrace{H^n\big(C_*^{\{A,B\}}(X);G\big)}_{=H^n(X;G) \text{ by the claim}} \to H^n(A;G) \oplus H^n(B;G) \to H^n(A \cap B;G) \to \underbrace{H^{n+1}\big(C_*^{\{A,B\}}(X);G\big)}_{=H^{n+1}(X;G) \text{ by the claim}} \to$$

□

As an application of the Mayer–Vietoris Theorem 50.15 for Cohomology Groups we will determine the cohomology groups of the wedge of two topological spaces. We will not make use of this example.

Lemma 50.16. $(*)$ *Let X and Y be topological spaces and let $x \in X$ and $y \in Y$ be good points.*[911] *We use these points to define the wedge $X \vee Y$. Then $(X \vee Y, X, Y)$ is an excisive triad.*

---

[910] The meaning of "natural" is evidently totally analogous to the interpretation given in Footnote 909.

[911] Recall that according to the definition on page 387 we say that a point x in a topological space X is *good*, if $\{x\}$ is a closed subset and if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U.

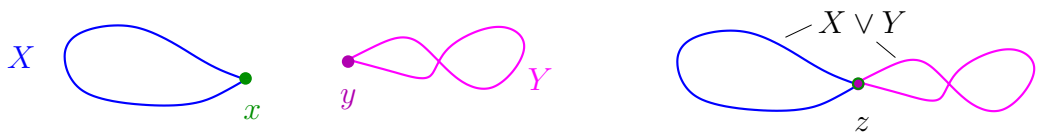


FIGURE 691.

Proof $(*)$. We denote by z the point in $X \vee Y$ that is given by identifying x with y. For each $k \in \mathbb{N}_0$ we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & H_k(X,\{x\};\mathbb{Z}) \oplus H_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & H_k(X,\{x\};\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & H_k(X \vee Y,\{z\};\mathbb{Z}) & \longrightarrow & H_k(X \vee Y, Y;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where all the maps are the obvious maps. The top horizontal sequence is evidently exact. The bottom horizontal sequence is an excerpt from the long exact sequence of the triple $(\{y\} = \{z\}, Y, X \vee Y)$, see Proposition 31.13. The excerpt shown is also exact since $(Y, \{y\})$ is a retract of $(X \vee Y, \{z\})$, which implies that for any $l \in \mathbb{N}_0$ the inclusion induced map $H_l(Y,\{y\};\mathbb{Z}) \to H_l(X \vee Y,\{z\};\mathbb{Z})$ is a monomorphism.

The middle vertical map in the above diagram is an isomorphism by Proposition 34.8 and the left vertical map is the identity. It follows from the Five Lemma 31.11 that the right vertical hand map is an isomorphism. We obtain from Lemma 50.13 (3) $\Rightarrow$ (1) that $(X \vee Y, X, Y)$ is an excisive triad. $\square$

We can now prove the following proposition that is the analogue of Proposition 34.9.

Proposition 50.17. $(*)$ *Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. Let G be an abelian group. Given $j \in K$ we denote by*

$$i_j \colon A_j \to \bigvee_{k\in K} A_k \quad \textit{respectively} \quad p_j \colon \bigvee_{k\in K} A_k \to A_j$$

the natural inclusion map respectively the natural projection map that we introduced on page 317. Then for every $n \in \mathbb{N}_0$ the map

$$\prod_{k\in K} i_k^* \colon \widetilde{H}^n\Big(\bigvee_{k\in K} A_k; G\Big) \to \prod_{k\in K} \widetilde{H}^n(A_k;G)$$

is an isomorphism where the inverse is given by

$$\prod_{k\in K} p_k^* \colon \prod_{k\in K} \widetilde{H}^n(A_k;G) \to \widetilde{H}^n\Big(\bigvee_{k\in K} A_k; G\Big)$$

Proof $(*)$.

(1) First we consider the case we are given just two topological spaces. By Lemma 50.16 we can use the Mayer–Vietoris Theorem 50.15 for Cohomology Groups to prove, completely analogously to Proposition 34.8, the desired statement.

(2) The case that we are dealing with finitely many topological spaces follows from (1) by induction.
(3) The case that K is infinite is more tricky. There are two possible approaches:
   (a) In the proof of Proposition 34.9, that deals with the corresponding statement for homology groups, we gave two different proofs. The first proof generalizes to the present setting. It is not clear to me how the second proof can be generalized. But fortunately providing one proof is enough.
   (b) Alternatively we can consider the map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \widetilde{C}_n(A_k) \;\to\; \widetilde{C}_n\Big(\bigvee_{k\in K} A_k\Big).$$

   By Proposition 34.9 this chain map induces an isomorphism of chain complexes. It follows from Proposition 49.2 that this map is in fact a chain homotopy equivalence. It follows from Lemma 48.8 that the above map induces an isomorphism of dual cochain complexes. The desired statement now follows from throwing Lemmas 48.2 and 48.7 into the bargain. The same logic applies to the projections. □

Recall that in Lemma 33.8 we showed that for any $k \in \mathbb{Z}$ and any topological space X we have a natural isomorphism

$$\Sigma_X\colon \widetilde{H}_k(X) \;\xrightarrow{\cong}\; \widetilde{H}_{k+1}(\Sigma(X)).$$

Not surprisingly we have a totally analogous statement in cohomology:

Lemma 50.18. $(*)$ *Let $k \in \mathbb{Z}$ and let G be an abelian group. Given any topological space X we have a natural isomorphism*

$$\Sigma_X\colon \widetilde{H}^k(\Sigma(X);G) \;\xrightarrow{\cong}\; \widetilde{H}^{k+1}(X;G)$$

which has the property that for any $\varphi \in H^k(X;G)$ and any $\sigma \in H_k(X)$ we have

$$\langle \Sigma_X(\varphi), \Sigma_X(\sigma)\rangle_{\Sigma(X)} \;=\; \langle \varphi, \sigma\rangle_X \;\in\; G.$$

Proof $(*)$. The construction of the isomorphism Σ_X is basically the same as the construction of the corresponding isomorphism in cohomology. In the Lemma 33.8 we just need to replace the Mayer–Vietoris Theorem 33.5 by the Mayer–Vietoris Theorem 50.15 for Cohomology Groups. We leave it to the reader to fill in the details. □

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 453. On page 819 we showed that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1}\cdot[S^{n+1}] \;\in\; H_{n+1}(S^{n+1};\mathbb{Z}).$$

From the equality in Lemma 50.18 we obtain the corresponding equality

$$\Sigma_{S^n}([S^n]^*) \;=\; (-1)^{n+1}\cdot[S^{n+1}]^* \;\in\; H^{n+1}(S^{n+1};\mathbb{Z}).$$

Exercises for Chapter 50.

Exercise 50.1. Show that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

Exercise 50.2. Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 50.3. Let $\{A_k\}_{k\in K}$ be a family of CW-complexes. Suppose that for each $k \in K$ suppose that we are given a point in the 0-skeleton of A_k. Provide a proof for Proposition 50.17 using cellular cohomology. More precisely, provide a proof along the following lines:

(a) Equip $X := \bigvee_{k\in K} A_k$ with the CW-structure provided by Lemma 26.20.
(b) Determine the relationship between the reduced cellular chain complexes of X and of the A_i.
(c) Make use of Proposition 49.11.
(d) Make use of Lemmas 48.2 and 48.7.

51. The Universal Coefficient Theorems

51.1. **The Ext group.** One of the questions that arises from the last chapter is, how can we explain the erratic relationship between homology groups and cohomology groups that we saw on page 1197. Furthermore it would be interesting to know whether given a chain complex one can determine the cohomology groups with G-coefficients from the usual homology group.

These questions are purely in the realm of algebra. More precisely, let (C_*, ∂_*) be a chain complex of free abelian groups and let G an abelian group. We want to find a connection between the cohomology groups $H^*(C; G)$ and the homology groups $H_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 42 where we showed that we can determine homology groups of a chain complex with G-coefficients from the usual homology group of the chain complex.

As usual we denote by $\mathcal{A}$ the category of abelian groups. On page 1013 we defined what it means for a covariant functor $F\colon \mathcal{A} \to \mathcal{A}$ to be exact, left-exact and right-exact. The same way we can introduce these notions for contravariant functors $F\colon \mathcal{A} \to \mathcal{A}$.

Definition. A contravariant functor $F\colon \mathcal{A} \to \mathcal{A}$ is called *left-exact*[912] if for all exact sequences

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact. Furthermore a contravariant functor $F\colon \mathcal{A} \to \mathcal{A}$ is called *right-exact*, if for all exact sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

In Lemma 42.10 we saw that the covariant functor $-\otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $\mathrm{Hom}(-, G)$ the situation is precisely the converse. More precisely, in the next lemma we will show that this functor is left-exact and immediately afterwards we will give an example that shows that the contravariant functor $\mathrm{Hom}(-, G)$ is in general not right-exact.

Lemma 51.1. *For any abelian group G the contravariant functor* $\mathrm{Hom}(-, G)$ *is left-exact.*

[912] Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

Proof $(*)$. Let G be an abelian group and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of abelian groups. We need to show that the sequence

$$0 \longrightarrow \mathrm{Hom}(C,G) \xrightarrow{\beta^*} \mathrm{Hom}(B,G) \xrightarrow{\alpha^*} \mathrm{Hom}(A,G)$$

is also exact. It is clear that β^* is injective and that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\mathrm{im}(\beta^*) \subset \ker(\alpha^*)$. It remains to show that $\ker(\alpha^*) \subset \mathrm{im}(\beta^*)$. So let $f\colon B \to G$ be a homomorphism such that $\alpha^*(f) = 0$. This means that f vanishes on $\alpha(A)$, i.e. $f\colon B \to G$ can be written as $f = \overline{f} \circ \pi$ where $\pi\colon B \to B/\alpha(A)$ is the projection map and where $\overline{f}\colon B/\alpha(A) \to G$ is some homomorphism. But by the exactness of the original sequence the homomorphism β descends to an isomorphism $\psi\colon B/\alpha(A) \to C$. Now we see that $f = \overline{f} \circ \psi^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \psi^{-1})$. $\square$

Now we give the example, promised above, that shows that the contravariant functor $\mathrm{Hom}(-,G)$ is in general not right-exact.

Example. We consider the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2$$

where $p\colon \mathbb{Z} \to \mathbb{Z}_2$ denotes the obvious projection map. We apply the contravariant functor $\mathrm{Hom}(-,\mathbb{F}_2)$. The resulting cochain complex is of the form[913]

$$\mathbb{F}_2 \xrightarrow[\cong]{p^*} \mathbb{F}_2 \xrightarrow[=0]{(\cdot 2)^*} \mathbb{F}_2 \to 0.$$

Since this cochain complex is not exact it follows that the functor $\mathrm{Hom}(-,\mathbb{F}_2)$ is not right-exact.

In Chapter 42.4 we introduced the torsion groups $\mathrm{Tor}(H,G)$ which "measure" to what degree the functor $-\otimes G$ fails to be a left-exact functor. The same way we now introduce the so called Ext-groups $\mathrm{Ext}(H,G)$ which "measure" to what degree the functor $\mathrm{Hom}(-,G)$ fails to be right-exact.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the canonical free resolution from page 1016. We define the *n-th G-Ext-group of H* as

$$\mathrm{Ext}_n(H,G) := H^n(F_*;G).$$

Remark. Let $n \in \mathbb{N}_0$. On page 1021 we saw that $\mathrm{Tor}_n(H,G)$ is covariantly functorial in both G and H. Now we will see that the groups $\mathrm{Ext}_n(H,G)$ are contravariantly functorial in H and covariantly functorial in G:

[913]Here we use the identification $\mathrm{Hom}(\mathbb{Z},G) = G$ from Lemma 48.1 and we use that multiplication by 2 is the zero endomorphism of $\mathbb{F}_2$.

(1) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups and let G be an abelian group. Then it follows from Lemmas 42.13 and 48.8 together with the definition of the G-Ext groups that α induces a homomorphism

$$\alpha^*\colon \operatorname{Ext}_n(H',G) \;\to\; \operatorname{Ext}_n(H,G).$$

It is now straightforward to show that

$$\begin{array}{rcl} H & \mapsto & \operatorname{Ext}_n(H,G) \\ (\alpha\colon H \to H') & \mapsto & (\alpha^*\colon \operatorname{Ext}_n(H',G) \to \operatorname{Ext}_n(H,G)) \end{array}$$

is a *contravariant* functor from the category of abelian groups to the category of abelian groups.

(2) On the other hand, if $\beta\colon G \to G'$ is a homomorphism between abelian groups then one sees easily that β induces a homomorphism

$$\beta_*\colon \operatorname{Ext}_n(H,G) \;\to\; \operatorname{Ext}_n(H,G')$$

Furthermore one can easily prove that

$$\begin{array}{rcl} G & \mapsto & \operatorname{Ext}_n(H,G) \\ (\beta\colon G \to G') & \mapsto & (\beta_*\colon \operatorname{Ext}_n(H,G) \to \operatorname{Ext}_n(H,G')) \end{array}$$

is a *covariant* functor from the category of abelian groups to the category of abelian groups.

The following proposition is the analogue of Proposition 42.12. The proof is almost the same as the proof of Proposition 42.12. We leave it to the reader to make the necessary changes.

Proposition 51.2. *Let G and H be abelian groups and let*

$$\ldots \;\to\; F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \;\to\; 0$$

be a free resolution of H. Then for each $n \in \mathbb{N}_0$ there exists a canonical isomorphism

$$\operatorname{Ext}_n(H,G) \;\xrightarrow{\cong}\; H^n(F_*,G).$$

In particular if

$$0 \;\to\; F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \;\to\; 0$$

is a free resolution of H of length one, then there exist canonical isomorphisms

$$\operatorname{Ext}_1(H,G) \;\xrightarrow{\cong}\; \operatorname{coker}\big\{\operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)\big\}$$

and

$$\operatorname{Ext}_0(H,G) \;\xrightarrow{\cong}\; \ker\big\{\operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)\big\}$$

In Lemma 42.16 we saw that the higher torsion groups of abelian groups vanish. The same statement, with basically the same proof, also holds for Ext groups. More precisely, we have the following corollary.

Corollary 51.3. *For every choice of abelian groups G and H we have* $\mathrm{Ext}_n(H,G)=0$ *for* $n\geq 2$.

Proof. As we pointed out in the proof of Lemma 42.16 any abelian group H admits a free resolution of length 2:

$$0 \;\to\; F_1 \;\xrightarrow{f_1}\; F_0 \;\xrightarrow{f_0}\; H \;\to\; 0.$$

The corollary now follows immediately from Proposition 51.2. □

In Lemma 42.15 we saw that the 0-th Torsion functor is an old acquaintance, namely we saw that for all abelian groups G and H there exists a natural isomorphism $\mathrm{Tor}_0(H,G)\xrightarrow{\cong} H\otimes G$. The following lemma gives us the analogous result for the 0-th Ext functor.

Lemma 51.4. *Let G and H be abelian groups, then there exists a natural*[914] *isomorphism*

$$\mathrm{Ext}_0(H,G) \;\xrightarrow{\cong}\; \mathrm{Hom}(H,G).$$

Proof $(*)$. The proof of Lemma 51.4 is, not surprisingly, rather similar to the proof of Lemma 42.15. Let G and H be abelian groups. Let

$$\cdots\to\; F_2 \;\xrightarrow{f_2}\; F_1 \;\xrightarrow{f_1}\; F_0 \;\xrightarrow{f_0}\; H \;\to\; 0$$

be the canonical free resolution of H. We know from Lemma 51.1 that the sequence

$$(*)\qquad 0 \;\to\; \mathrm{Hom}(H,G) \;\xrightarrow{f_0^*}\; \mathrm{Hom}(F_0,G) \;\xrightarrow{f_1^*}\; \mathrm{Hom}(F_1,G)$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned}\mathrm{Ext}_0(H,G) \;&\overset{\text{by definition}}{\overset{\downarrow}{=}}\; H^0\big(\cdots\leftarrow \mathrm{Hom}(F_1,G)\xleftarrow{f_1^*}\mathrm{Hom}(F_0,G)\leftarrow 0\big)\\ &=\; \ker\big(\mathrm{Hom}(F_1,G)\xleftarrow{f_1^*}\mathrm{Hom}(F_0,G)\big)\;\underset{\substack{\uparrow\\ \text{isomorphism since }(*)\text{ is exact}}}{\xrightarrow{f_0^*}}\;\mathrm{Hom}(H,G).\end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. □

Definition. Given abelian groups G and H we refer to

$$\mathrm{Ext}(H,G) \;:=\; \mathrm{Ext}_1(H,G)$$

as *the* G-Ext-group of H.

[914]Here by "natural" we mean that these isomorphisms define a natural isomorphism from the contravariant functor $H\mapsto \mathrm{Ext}_0(H,G)$ to the contravariant functor $H\mapsto \mathrm{Hom}(H,G)$ and that they also define a natural transformation from the covariant functor $G\mapsto \mathrm{Ext}_0(H,G)$ to the covariant functor $G\mapsto \mathrm{Hom}(H,G)$.

Remark. Let G and H be two abelian groups. An *extension of H by G* is a short exact sequence of the form

$$0 \longrightarrow G \xrightarrow{\alpha} K \xrightarrow{\beta} H \longrightarrow 0.$$

We say that two such extensions given by K and K' are equivalent if there exists a homomorphism $\varphi\colon K \to K'$ that makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, "\alpha"] \arrow[d, "="] & K \arrow[r, "\beta"] \arrow[d, "\varphi"] & H \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & G \arrow[r, "\alpha'"] & K' \arrow[r, "\beta'"] & H \arrow[r] & 0. \end{tikzcd}$$

In [HS, Theorem II.3.4] it is shown that there is a natural bijection from the set of equivalence classes of extensions of H by G with the set $\mathrm{Ext}(H, G)$. This fact is the origin of the name "Ext-group".

In light of Corollary 51.3 and Lemma 51.4 the remaining task is to compute the Ext-group $\mathrm{Ext}(H, G) = \mathrm{Ext}_1(H, G)$. The following lemma allows us in particular to determine the Ext-group for all finitely generated abelian groups.

Lemma 51.5. *Let G, H and $G_1, \ldots, G_k, H_1, \ldots, H_k$ be abelian groups. The following statements hold:*

(1) *There exists a natural isomorphism*

$$\mathrm{Ext}\left(\bigoplus_{i=1}^{k} H_i, G\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(H_i, G).$$

(2) *There exists a natural isomorphism*

$$\mathrm{Ext}\left(H, \bigoplus_{i=1}^{k} G_i\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(H, G_i).$$

(3) *If H is a free abelian group, then* $\mathrm{Ext}(H, G) = 0$.
(4) *For all $n \in \mathbb{N}$ we have a natural isomorphism* $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$.
(5) *If H is a finitely generated abelian group, then we have a natural isomorphism*

$$\mathrm{Ext}(H, \mathbb{Z}) \cong \textit{torsion subgroup of } H := \{h \in H \mid \textit{there exists an } n \in \mathbb{N} \textit{ with } nh = 0\}.$$

In particular $\mathrm{Ext}(H, \mathbb{Z})$ *is a finite group.*
(6) *For $n, m \in \mathbb{N}$ we have* $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

Remark.

(a) The first two statements of the lemma also generalize to the case of infinitely many groups, but some care needs to be taken. Indeed, some of the finite direct sums are secretly finite direct products and hence they generalize to infinite direct products. More precisely, the following holds:

(1') If $\{H_i\}_{i\in I}$ is a family of abelian groups and if G is an abelian group, then

$$\operatorname{Ext}\Big(\bigoplus_{i\in I} H_i, G\Big) \;\cong\; \prod_{i\in I} \operatorname{Ext}(H_i, G).$$

(2') If H is an abelian group and if $\{G_i\}_{i\in I}$ is a family of abelian groups, then

$$\operatorname{Ext}\Big(H, \prod_{i\in I} G_i\Big) \;\cong\; \prod_{i\in I} \operatorname{Ext}(H, G_i).$$

We refer to [Rot2, Propositions 7.21 and 7.22] for a proof. We will not make use of these more general statements.

(b) The statements of Lemma 51.5 look very similar to the statements of Lemma 42.17. But there are some differences:
   (i) In Lemma 42.17 we saw that for every abelian group we have $\operatorname{Tor}(H, \mathbb{Z}) = 0$ whereas we have now shown in Lemma 51.5 (5) that for every finite abelian group we have $\operatorname{Ext}(H, \mathbb{Z}) \cong H$.
   (ii) In Lemma 42.17 we saw that $\operatorname{Tor}(H, G) = 0$ not only whenever H is a free abelian group but also for all torsion-free abelian groups H. Here the situation is different, the statement of Lemma 51.5 (3) does not generalize to torsion-free abelian groups. For example in [Rot2, p. 428] it is shown that there exists an abelian group G such that $\operatorname{Ext}(\mathbb{Q}, G) \neq 0$. But this issue is of no concern to us.

Proof. The first three statements are proved almost the same way as we proved Lemma 42.17. We leave the details as an exercise to the reader. We turn to the final three statements.

(4) Let G be an abelian group and let $n \in \mathbb{N}$. We have

$$\operatorname{Ext}(\mathbb{Z}_n, G) \;\overset{\substack{\text{Proposition 51.2 applied to the free resolution } 0\to\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}\to\mathbb{Z}_n\to 0}}{=}\; \operatorname{coker}\big\{\operatorname{Hom}(\mathbb{Z}, G) \xrightarrow{(\cdot n)^*} \operatorname{Hom}(\mathbb{Z}, G)\big\} \;\underset{\substack{\text{by the isomorphism } \operatorname{Hom}(\mathbb{Z},G)=G \text{ given by } f\mapsto f(1)}}{=}\; \operatorname{coker}\big\{G \xrightarrow{\cdot n} G\big\} \;=\; G/nG.$$

(5) This statement follows immediately from (1), (3) and (4) together with the classification of finitely generated abelian groups, see Theorem 11.4.

(6) This statement follows immediately from (4) and the calculation on page 1017. □

We have one more general result for calculating Ext-groups. Before we can state the result we need one more definition.

Definition. An abelian group G is called *divisible* if for any $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$.

Examples.

(1) The groups $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are divisible.

(2) Every quotient of a divisible group is divisible, for example the quotient group $\mathbb{Q}/\mathbb{Z}$ is also divisible.[915]
(3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group.

Proposition 51.6. *Let G and H be abelian groups. If G is divisible, then* $\mathrm{Ext}(H, G) = 0$.

In the proof of Proposition 51.6 it is useful to introduce the following notion.

Definition. An abelian group G is called *injective* if for all abelian groups A and B, every injective homomorphism $\varphi\colon B \to A$ and every homomorphism $\beta\colon B \to G$ there exists a homomorphism $\alpha\colon A \to G$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "\varphi"] \arrow[d, "\beta"'] & A \arrow[dl, dashed, "\alpha"] \\ & G. & \end{tikzcd}$$

The following lemma shows why we are interested in injective groups.

Lemma 51.7. *Let G and H be abelian groups. If G is injective, then* $\mathrm{Ext}(H, G) = 0$.

Proof. Let H be an abelian group. As we pointed out in the proof of Lemma 42.16 the abelian group H admits a free resolution of length 2:

$$0 \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} H \to 0.$$

By Proposition 51.2 the group $\mathrm{Ext}(H, G)$ is isomorphic to the cokernel of the map

$$\mathrm{Hom}\begin{pmatrix} F_1 \\ \downarrow \\ G \end{pmatrix} \xleftarrow{\varphi_1^*} \mathrm{Hom}\begin{pmatrix} F_0 \\ \downarrow \\ G \end{pmatrix}.$$

But by definition of G being injective this map φ_1^* is in fact an epimorphism. Thus we have shown that $\mathrm{Ext}(H, G) = 0$. □

To provide a proof for Proposition 51.6 it remains to prove the following proposition.

Proposition 51.8. *Every abelian group that is divisible is also injective.*

Remark. It is straightforward to prove the converse to Proposition 51.8, namely every injective abelian group is also divisible.

In the proof of Proposition 51.8 we will need Zorn's Lemma which we now recall for the reader's convenience.

Lemma 51.9. (Zorn's Lemma) *Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.*

We also recall the definitions in the statement of Zorn's Lemma:

[915] Odd as it may sound, this example will actually be important later on.

(1) A *partially ordered set* $(P, \leq)$ is a set together with a preorder $\leq$ (see page 481) that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.
(2) A *chain* in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
(3) An *upper bound* of a subset Q is an element $q \in Q$ such that $p \leq q$ for all $q \in Q$.
(4) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.[916]

We refer to [Ci, Theorem 4.3.4] for a proof of Zorn's Lemma. Perhaps it is more fun to look at Zorn's Lemma in "The Simpsons", see

`http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/`

Now we can give the proof of Proposition 51.8.

Proof of Proposition 51.8 $(*)$. Let G be an abelian group that is divisible. Suppose we are given a diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "\varphi"] \arrow[d, "\beta"'] & A \arrow[dl, "?\,\alpha\,?"] \\ & G. & \end{tikzcd}$$

Since φ is a monomorphism we can assume that B is in fact a subgroup of A and that φ is the inclusion map.

We denote by P the set of all pairs (C_i, γ_i) where C_i is a subgroup of A which contains B and where $\gamma_i \colon C_i \to G$ is a homomorphism such that $\gamma_i|_B = \beta$. This family of pairs is non-empty since (B, β) satisfies the required conditions. We define $(C_i, \gamma_i) < (C_j, \gamma_j)$ if $C_i \subset C_j$ and if $\gamma_j|_{C_i} = \gamma_i$. Given a chain $(C_j, \gamma_j)_{j \in J}$ in P we get an upper bound by setting

$$C := \bigcup_{j \in J} C_j \quad \text{and} \quad \begin{array}{rcl} \gamma \colon C & \to & G \\ c & \mapsto & \gamma_j(c) \text{ if } j \in J \text{ with } c \in C_j. \end{array}$$

Since $(C_j, \gamma_j)_{j \in J}$ is a chain one can easily verify that C is in fact a subgroup of A which contains B and that γ is a homomorphism with $\gamma|_B = \beta$.

Therefore we can apply Zorn's Lemma to this family and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim.

Claim. We have $C_m = A$.

Suppose that this is not the case. Then there exists an $a \in A \setminus C_m$. We denote by $\langle\langle C_m, a \rangle\rangle$ the subgroup of A generated by C_m and a, i.e. $\langle\langle C_m, a \rangle\rangle$ is the smallest subgroup of A that contains C_m and a. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta \colon \langle\langle C_m, a \rangle\rangle \to G$ that extends γ_m. We distinguish two cases:

(1) If there exists no $n \in \mathbb{N}$ such that $n \cdot a \in C_m$, then one can show easily that we have $\langle\langle C_m, a \rangle\rangle = C_m \oplus \langle a \rangle$. We extend γ_m to $\langle\langle C_m, a \rangle\rangle = C_m \oplus \langle a \rangle$ by setting $\delta(a) = 0$.

[916] Aren't "upper bound" and "maximal element" the same?

(2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $n \cdot a \in C_m$. We denote by $r \in \mathbb{N}_0$ the order of $n \cdot a \in C_m$. Here we say that the order is 0 if $n \cdot a \in C_m$ has infinite order. We consider the maps

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}_r & \xrightarrow{\Phi} & C_m \oplus \mathbb{Z}_{nr} & \xrightarrow{\Psi} & \langle\langle C_m, a\rangle\rangle \to 0 \\ & & k & \mapsto & (k \cdot n \cdot a, -k \cdot n) & & \\ & & & & (c, l) & \mapsto & c + l \cdot a \end{array}$$

It is clear that Φ is injective and that $\Psi \circ \Phi = 0$. Note that the image of Ψ lies a priori in A, but it is straightforward to see that the image of Ψ is precisely $\langle\langle C_m, a\rangle\rangle$. By definition of n we have that $\ker(\Psi) \subset \operatorname{im}(\Phi)$. Put differently, the sequence is exact. Since G is divisible there exists an $h \in G$ such that $n \cdot h = \gamma(n \cdot a)$. We denote by $\alpha\colon C_m \oplus \mathbb{Z}_{nr} \to G$ the homomorphism that is given by setting $\alpha(c) = \gamma(c)$ for $c \in C_m$ and $\alpha(k) = k \cdot h$ for $k \in \mathbb{Z}$. One can easily verify that $\operatorname{im}(\Phi)$ is contained in $\ker(\alpha)$. It is now straightforward to show that the homomorphism

$$\begin{array}{ccccc} \delta\colon \langle\langle C_m, a\rangle\rangle & \to & (C_m \oplus \mathbb{Z}_{nr})/\operatorname{im}(\Phi) & \to & G \\ x & \mapsto & [\Psi^{-1}(x)] & & \\ & & [y] & \mapsto & \alpha(y) \end{array}$$

has the desired properties.

This concludes the proof of the claim and thus of the lemma. □

The following proposition gives arguably slightly weird example.

Proposition 51.10. *The group* $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ *is isomorphic to the additive group* $(\mathbb{R}, +)$.

Remark. Note that this proposition is not totally pointless, for example it follows from the existence of Moore spaces, see Proposition 34.11, that there exist topological spaces X with $H_n(X) \cong (\mathbb{Q}, +)$. Thus even unexpected abelian groups can appear in topology.

Proof. This proposition is proved in [Wie]. Alternatively the proposition is almost proved in [Bre2, Lemma 14.8]. □

This example also shows that Ext-groups can be quite large. In fact the following proposition holds.

Proposition 51.11. *Let G be an abelian group. If G is not finitely generated, then* $\operatorname{Hom}(G, \mathbb{Z})$ *is uncountable or* $\operatorname{Ext}(G, \mathbb{Z})$ *is uncountable.*

Proof. We will barely make use of this proposition, thus we refer to [Hat, Proposition 3.F.12] instead for a proof. □

51.2. **The Universal Coefficient Theorem for Cohomology Groups.** Finally we want to explore the connection between the cohomology groups of a chain complex C and the usual homology groups of the chain complex. The key to relating homology groups to cohomology groups is given by the following evaluation homomorphism.

Definition. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. We refer to the map[917]

$$\begin{array}{rcl} \mathrm{ev}\colon H^n(C;G) & \to & \mathrm{Hom}(H_n(C),G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} H_n(C) & \to & G \\ [c] & \mapsto & \langle [\varphi],[c]\rangle = \varphi(c) \end{array} \right) \end{array}$$

as the *evaluation homomorphism.*

Remark. It follows immediately from Lemma 50.6 that the evaluation homomorphisms in natural in the chain complex C_* in the sense that if $f\colon C_* \to D_*$ is a chain map, then for any $n \in \mathbb{N}_0$ and any abelian group G the following diagram commutes

$$\begin{array}{ccc} H^n(C;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_n(C),G) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle (f_*)^*} \\ H^n(D;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_n(D),G). \end{array}$$

Furthermore, it follows easily from the definitions that the evaluation map is natural in G, in the sense that if $\varphi\colon G_1 \to G_2$ is homomorphism of abelian groups, then for any $n \in \mathbb{N}_0$ and any chain complex C_* the following diagram commutes

$$\begin{array}{ccc} H^n(C;G_1) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_n(C),G_1) \\ {\scriptstyle \varphi_*}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_*} \\ H^n(C;G_2) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_n(C),G_2). \end{array}$$

Here the map on the left is the one we had introduced on page 1163.

Recall that on page 845 we saw that $H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0$ and that on page 1197 we showed that $H^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z}_2$. This implies that the evaluation homomorphism $\mathrm{ev}\colon H^n(X;G) \to \mathrm{Hom}(H_n(X);G)$ is in general not a monomorphism. The following theorem relates homology groups and cohomology groups via the evaluation homomorphism and the Ext-groups.

Theorem 51.12. (Universal Coefficient Theorem for Cohomology Groups) *Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \;\to\; \mathrm{Ext}(H_{n-1}(C),G) \;\to\; H^n(C;G) \;\xrightarrow{\mathrm{ev}}\; \mathrm{Hom}(H_n(C),G) \to 0$$

which is natural in the chain complex C_* and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism[918]

$$H^n(C;G) \;\cong\; \mathrm{Ext}(H_{n-1}(C),G) \oplus \mathrm{Hom}(H_n(C),G).$$

[917]This map is well-defined by Lemma 50.5.

[918]Once again the short exact sequence is natural, whereas the isomorphism is not natural. More precisely, the discussion on page 1033 can easily be modified to show that there exists no natural isomorphism

$$H^n(C;G) \;\cong\; \mathrm{Ext}(H_{n-1}(C),G) \oplus \mathrm{Hom}(H_n(C),G).$$

We will discuss this issue in Exercise 51.1.

Before we provide the proof of the above Universal Coefficient Theorem for Cohomology Groups 51.12 let us formulate the most important application, namely let us apply it to the chain complex of a pair of topological spaces.

Theorem 51.13. (Universal Coefficient Theorem for Cohomology Groups) *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \;\to\; \mathrm{Ext}(H_{n-1}(X,A),G) \;\to\; H^n(X,A;G) \;\xrightarrow{\mathrm{ev}}\; \mathrm{Hom}(H_n(X,A),G) \to 0$$

which is natural in the chain complex C_* and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism

$$H^n(X,A;G) \;\cong\; \mathrm{Ext}(H_{n-1}(X,A),G) \oplus \mathrm{Hom}(H_n(X,A),G).$$

Proof of Theorem 51.13 assuming Theorem 51.12. As the reader will have noticed there is really not much to say, except that we recall that we pointed out on page 775 that the chain groups $C_n(X,A)$, $n \in \mathbb{N}_0$ are free abelian groups. $\square$

Remark. Here is one straightforward but interesting consequence to the Universal Coefficient Theorem 51.13 for Cohomology Groups. Given any pair (X, A) of topological spaces we see that

$$\begin{array}{ccccccc} H^1(X;\mathbb{Z}) & \cong & \mathrm{Ext}(H_0(X,A),\mathbb{Z}) \oplus \mathrm{Hom}(H_1(X,A),\mathbb{Z}) & = & \mathrm{Hom}(H_1(X,A),\mathbb{Z}) & = & \text{torsion-free.} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Universal Coefficient Theorem 51.13} & & \text{by Lemma 51.5 and Corollary 29.14} & & \begin{array}{c}\text{since } \mathrm{Hom}(G,\mathbb{Z}) \text{ is} \\ \text{always torsion free}\end{array} & \end{array}$$

Now we turn to the proof of the purely algebraic Universal Coefficient Theorem 51.12 for Cohomology Groups. The proof is, perhaps not surprisingly, rather similar to the proofs of the Universal Coefficient Theorem 42.19 and Proposition 42.22.

Proof $(*)$. Let (C_*, ∂_*) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. Note that it follows from Lemma 11.2 that B_n and Z_n are free abelian groups.

As in the proof of the Universal Coefficient Theorem 42.19 we consider the following short exact sequence of vertical chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Now let G be an abelian group. We apply the functor $\mathrm{Hom}(-,G)$ to the above short exact sequence of chain complexes and we obtain the following diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1},G) \arrow[r, "\partial_n^*"] \arrow[u] & \mathrm{Hom}(C_n,G) \arrow[r] \arrow[u] & \mathrm{Hom}(Z_n,G) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_{n-2},G) \arrow[r, "\partial_{n-1}^*"] \arrow[u, "0"] & \mathrm{Hom}(C_{n-1},G) \arrow[r] \arrow[u, "\partial_n^*"] & \mathrm{Hom}(Z_{n-1},G) \arrow[r] \arrow[u, "0"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

This is again a sequence of cochain complexes. Since the Z_n are free abelian we obtain from Lemma 48.4 that the above horizontal sequences of cochain complexes are still exact. Therefore we can apply Proposition 48.6 to obtain the following long exact sequence

$$\to \mathrm{Hom}(Z_{n-1},G) \xrightarrow{d_n} \mathrm{Hom}(B_{n-1},G) \to H^n(C;G) \xrightarrow{\partial_n} \mathrm{Hom}(Z_n,G) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_n,G) \to \dots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. One can show easily that the connecting homomorphism $d_n\colon \mathrm{Hom}(Z_n,G) \to \mathrm{Hom}(B_n,G)$ is the map $d_n = i_n^*$ where $i_n\colon B_n \to Z_n$ is the inclusion map. By Lemma 33.4 we obtain the following short exact sequence

$$0 \to \mathrm{coker}(\mathrm{Hom}(Z_{n-1},G) \to \mathrm{Hom}(B_{n-1},G)) \to H^n(C;G) \to \ker(\mathrm{Hom}(Z_n,G) \to \mathrm{Hom}(B_n,G)) \to 0.$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from Proposition 51.2 (together with Lemma 51.4 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $H_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to H_l(C)$. We have thus shown the existence of the desired short exact sequence

$$0 \;\to\; \mathrm{Ext}(H_{n-1}(C),G) \;\to\; H^n(C;G) \;\xrightarrow{\mathrm{ev}}\; \mathrm{Hom}(H_n(C),G) \to 0.$$

The statement that the short exact sequence is natural follows easily from the definitions and Lemma 42.13 (2). We leave the verification of the details as an exercise to the reader.

It remains to show that the above short exact sequence splits. By definitions this means that we need to construct a homomorphism $\eta\colon \mathrm{Hom}(H_n(C),G) \to H^n(C;G)$ such that $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}(H_n(C),G)$.

We consider again the short exact sequence

$$0 \;\to\; Z_n \;\hookrightarrow\; C_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0.$$

Since B_{n-1} is a free abelian group we obtain from Lemmas 33.1 and 33.2 that there exists a homomorphism $p_n\colon C_n \to Z_n$ such that p_n is the identity on $Z_n \subset C_n$.

In the proof of Proposition 42.22 we showed that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift left, "p_{n+1}"] \arrow[dr, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift left, "p_n"] \arrow[dr, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, shift left, "p_{n-1}"] & \dots \\
& Z_{n+1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n+1}"] & Z_n \arrow[u, hook, shift left] \arrow[d, two heads, "q_n"] & Z_{n-1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n-1}"] & \\
\dots \arrow[r] & \underbrace{H_{n+1}(C)}_{=Z_{n+1}/B_{n+1}} \arrow[r, "0"] & \underbrace{H_n(C)}_{=Z_n/B_n} \arrow[r, "0"] & \underbrace{H_{n-1}(C)}_{=Z_{n-1}/B_{n-1}} \arrow[r, "0"] & \dots
\end{tikzcd}$$

where the maps q_n are just the obvious projection maps.

The top and the bottom horizontal sequences of the above diagram are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

Now we apply the functor $\mathrm{Hom}(-, G)$ to the top and the bottom horizontal sequences and we obtain the following diagram of cochain complexes.

$$\begin{tikzcd}
\dots & \mathrm{Hom}(C_{n+1}, G) \arrow[l] & \mathrm{Hom}(C_n, G) \arrow[l, "\partial^*_{n+1}"'] & \mathrm{Hom}(C_{n-1}, G) \arrow[l, "\partial^*_n"'] & \dots \arrow[l] \\
\dots & \mathrm{Hom}(H_{n+1}(C), G) \arrow[l] \arrow[u, "(q_{n+1} \circ p_{n+1})^*"'] & \mathrm{Hom}(H_n(C), G) \arrow[l, "0"'] \arrow[u, "(q_n \circ p_n)^*"'] & \mathrm{Hom}(H_{n-1}(C), G) \arrow[l, "0"'] \arrow[u, "(q_{n-1} \circ p_{n-1})^*"'] & \dots \arrow[l]
\end{tikzcd}$$

The induced maps on the cohomology groups of these two cochain complexes give us a homomorphism

$$\eta\colon \mathrm{Hom}(H_n(C), G) = H^n(\text{lower sequence}) \ \to \ H^n(\text{upper sequence}) = H^n(C; G).$$

It remains to prove the following claim.

Claim. The composition $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}(H_n(C), G)$.

Let $\varphi\colon H_n(C) \to G$ be homomorphism. We need to show that $(\mathrm{ev} \circ \eta)(\varphi) = \varphi$ in $\mathrm{Hom}(H_n(C), G)$. So let $[\sigma] \in H_n(C)$ with $\sigma \in Z_n$. Then

$$\big((\mathrm{ev} \circ \eta)(\varphi)\big)([\sigma]) \underset{\substack{\uparrow \\ \text{definition of ev}}}{=} \langle \overbrace{\eta(\varphi)}^{\in H^n(C;G)}, [\sigma] \rangle \underset{\substack{\uparrow \\ \text{definition of } \eta \text{ and } \langle\,,\,\rangle}}{=} (\overbrace{\varphi \circ q_n \circ p_n}^{\in \mathrm{Hom}(C_n, G)})(\sigma) \underset{\substack{\uparrow \\ \text{since } p_n = \mathrm{id} \text{ on } Z_n}}{=} (\varphi \circ q_n)(\sigma) \underset{\substack{\uparrow \\ \text{since } q_n(\sigma) = [\sigma]}}{=} \varphi([\sigma]).$$

This concludes the proof of the claim. $\square$

Remark. The Universal Coefficient Theorem 51.12 for Cohomology Groups also gives a new proof for Corollary 49.3 (2) which says that if two topological spaces have isomorphic homology groups, then they also have isomorphic cohomology groups.

Example. We consider again the projective space $P := \mathbb{R}\mathrm{P}^2$. As a reminder, we showed on page 846 that $H_2(P) = 0$, $H_1(P) \cong \mathbb{Z}_2$ and $H_0(P) \cong \mathbb{Z}$. It follows from the Universal Coefficient Theorem 51.13 and from the properties of the Ext-group that we had listed in Lemma 51.5 that

$$\begin{array}{lllll} H^2(P;\mathbb{Z}) & \cong \operatorname{Hom}(H_2(P),\mathbb{Z}) \oplus \operatorname{Ext}(H_1(P),\mathbb{Z}) & \cong \operatorname{Hom}(0,\mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{Z}_2,\mathbb{Z}) & = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \\ H^1(P;\mathbb{Z}) & \cong \operatorname{Hom}(H_1(P),\mathbb{Z}) \oplus \operatorname{Ext}(H_0(P),\mathbb{Z}) & \cong \operatorname{Hom}(\mathbb{Z}_2,\mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{Z},\mathbb{Z}) & = 0 \oplus 0 = 0 \\ H^0(P;\mathbb{Z}) & \cong \operatorname{Hom}(H_0(P),\mathbb{Z}) & \cong \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) & = \mathbb{Z} \end{array}$$

and

$$\begin{array}{lllll} H^2(P;\mathbb{Z}_2) & \cong \operatorname{Hom}(H_2(P),\mathbb{Z}_2) \oplus \operatorname{Ext}(H_1(P),\mathbb{Z}_2) & \cong \operatorname{Hom}(0,\mathbb{Z}_2) \oplus \operatorname{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) & = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \\ H^1(P;\mathbb{Z}_2) & \cong \operatorname{Hom}(H_1(P),\mathbb{Z}_2) \oplus \operatorname{Ext}(H_0(P),\mathbb{Z}_2) & \cong \operatorname{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \operatorname{Ext}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2 \\ H^0(P;\mathbb{Z}_2) & \cong \operatorname{Hom}(H_0(P),\mathbb{Z}_2) & \cong \operatorname{Hom}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2. \end{array}$$

Fortunately this coincides with the calculations from page 1197.

Example. We return to the study of lens spaces. In Lemma 40.12 we saw that

$$H_n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0,3 \\ 0, & \text{if } n = 2 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

As in the previous example we can now use the Universal Coefficient Theorem 51.13 and Lemma 51.5 to compute the cohomology groups of $L(p,q)$. We leave it as an exercise to the reader to verify that[919]

$$H^n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0,3 \\ 0, & \text{if } n = 1 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 2. \end{cases}$$

This shows, as was already foreshadowed by Corollary 49.4, that cohomology groups also cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have not made any progress on Question 7.8 since the introduction of the fundamental group. All the extra invariants, namely higher homotopy groups, homology groups and cohomology groups (with any coefficients) have proved useless.

Proposition 51.14. *Let M be a connected n-dimensional topological manifold. Let G be an abelian group. Then the following statements hold:*

(1) *Let $x \in M \setminus \partial M$. For $i \neq n$ we have $H_i(M, M \setminus \{x\}; G) = 0$. Furthermore the map*
$$\operatorname{ev}\colon H^n(M, M \setminus \{x\}; G) \;\to\; \operatorname{Hom}(H_n(M, M \setminus \{x\}; \mathbb{Z}), G)$$
is an isomorphism.
(2) *If M is compact but not closed, then $H^n(M;G) = 0$.*
(3) *If M is closed, orientable and non-empty, then $H^n(M;G) \cong G$.*
(4) *If M is closed, non-orientable and non-empty, then $H^n(M;G) \cong G/2G$.*
(5) *For any $k > n$ we have $H^k(M;G) = 0$.*

[919]Note that the $\mathbb{Z}_p$ shifted from $n = 1$ in homology to $n = 2$ in cohomology.

Proof. Let M be a compact connected n-dimensional topological manifold and let G be an abelian group.

(1) This statement follows immediately from Lemma 45.16 and the Universal Coefficient Theorem 51.13.
(2) We suppose that M has non-empty boundary. Then

$$\begin{array}{ccccccc} H^n(M;G) & \cong & \mathrm{Hom}(H_n(M;\mathbb{Z}),G) \oplus \mathrm{Ext}(H_{n-1}(M;\mathbb{Z}),G) & = & \mathrm{Ext}(\mathbb{Z}^k,G) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Universal Coefficient Theorem 51.13} & & \begin{array}{c}\text{by Theorem 46.4 we have } H_n(M;\mathbb{Z})=0 \text{ and} \\ \text{by Proposition 46.21 (1) we have } H_{n-1}(M;\mathbb{Z})\cong\mathbb{Z}^k \\ \text{for some } k\in\mathbb{N}_0\end{array} & & \text{Lemma 51.5} & \end{array}$$

(3) Suppose that M is closed, orientable and non-empty. The argument is very similar to the proof of (2). We just need to use that we know from Theorem 46.2 that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and that from Proposition 46.21 (1) we know that $H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k$ for some k.
(4) Next suppose that M is closed, non-orientable and non-empty. This time we use that we know from Theorem 46.2 and Proposition 46.21 (2) that $H_n(M;\mathbb{Z}) \cong 0$ and that $H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathbb{Z}_2$ for some k. Here we also use Lemma 51.5 (4) which says that $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$.
(5) So let $k > n$. This time we use that by Theorem 46.4 we know that $H_k(M;\mathbb{Z}) = 0$ and that we know from Theorems 46.2 and 46.4 that $H_{k-1}(M;\mathbb{Z})$ is either zero or isomorphic to $\mathbb{Z}$. The desired statement follows again from the Universal Coefficient Theorem 51.13 and the fact that by Lemma 51.5 we have $\mathrm{Ext}(\mathbb{Z}, G) = 0$. □

Lemma 51.15. *Let M be a compact oriented connected n-dimensional non-empty topological manifold. Then we have $H^n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$. Furthermore there exists a unique generator $[M]^* \in H^n(M,\partial M;\mathbb{Z})$ with $\langle [M]^*, [M]\rangle_M = 1$.*[920]

Proof (∗). Let M be a compact oriented connected n-dimensional non-empty topological manifold. We start out with the following claim.

Claim. The homology group $H_{n-1}(M,\partial M;\mathbb{Z})$ is a free abelian group.

The proof of the claim is basically identical to the proof of Proposition 46.21. For completeness' sake we give the details. First we note that by Propositions 32.1 (3) and 45.2 (3) we know that $H_{n-1}(M,\partial M;\mathbb{Z})$ is finitely generated. Therefore by Lemma 46.22 it suffices to show that $\mathrm{Tor}(H_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p) = 0$ for all primes p.

[920]Here $[M] \in H_n(M,\partial M;\mathbb{Z})$ denotes the fundamental class provided by Corollary 46.5.

So let p be a prime. We have

$$\begin{array}{ccccc} & & \text{Universal Coefficient Theorem 42.23} & & \\ & & \downarrow & & \\ \mathbb{F}_p & \cong & H_n(M,\partial M;\mathbb{F}_p) & \cong & H_n(M,\partial M;\mathbb{Z})\otimes\mathbb{F}_p \oplus \operatorname{Tor}(H_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p) \\ & & \uparrow & & \cong \mathbb{F}_p \oplus \operatorname{Tor}(H_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p). \\ & & \text{by Proposition 46.20 (1),} & & \uparrow \\ & & \text{since by Lemma 45.25} & & \text{by Theorem 46.2 (3) we} \\ & & M \text{ is also } \mathbb{F}_p\text{-orientable} & & \text{have } H_n(M,\partial M;\mathbb{Z})\cong\mathbb{Z} \end{array}$$

It follows from Lemma 11.8 that $\operatorname{Tor}(H_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p)=0$. This concludes the proof of the claim.

It follows from the Universal Coefficient Theorem 51.13 for Cohomology Groups together with the above claim that the map

$$H^n(M,\partial M;\mathbb{Z}) \xrightarrow{\text{ev}} \operatorname{Hom}(\underbrace{H_n(M,\partial M;\mathbb{Z})}_{=\mathbb{Z}\cdot[M]},\mathbb{Z})$$

is an isomorphism. By Corollary 46.5 (3) we have the equality $H_n(M,\partial M;\mathbb{Z})=\mathbb{Z}\cdot[M]$. Hence there exists a unique element $[M]^*\in H^n(M,\partial M;\mathbb{Z})$ with $\text{ev}([M]^*)([M])=1$, i.e. with $\langle [M]^*,[M]\rangle_M=1$. □

Definition.

(1) Given a compact oriented connected non-empty n-dimensional topological manifold M we refer to the uniquely determined element $[M]^*\in H^n(M,\partial M;\mathbb{Z})$ from Lemma 51.15 as the *dual fundamental class* of M.

(2) Given a compact oriented n-dimensional non-empty topological manifold M with finitely many components $M_1,\dots,M_k$ we define its *dual fundamental class* to be

$$[M]^* := [M_1]^*+\cdots+[M_k]^* \in \underbrace{H^n(M_1,\partial M_1;\mathbb{Z})\oplus\cdots\oplus H^n(M_k,\partial M_k;\mathbb{Z})}_{=H^n(M,\partial M;\mathbb{Z})}.$$

Example. We consider the manifold S^1 with the usual orientation. We saw on page 1118 that the singular simplex $\mu\colon\Delta^1\to S^1$ that is given by $(1-t,t)\mapsto e^{2\pi it}$ represents the fundamental class of S^1. On page 1167 we introduced a cocycle $\theta_{\mathbb{Z}}\in C^1(S^1;\mathbb{Z})$ and we had explicitly calculated that $\langle[\theta_{\mathbb{Z}}],[S^1]\rangle=\theta_{\mathbb{Z}}(\mu)=1$. This shows immediately that $[\theta_{\mathbb{Z}}]\in H^1(S^1;\mathbb{Z})$ is the dual fundamental class of S^1.

Later on we will need the following lemma.

Lemma 51.16. *Let $f\colon M\to N$ be a map between closed oriented connected non-empty n-dimensional topological manifolds. Then*

$$f^*([N]^*) = \deg(f)\cdot[M]^*.$$

Proof. Let $f\colon M\to N$ be a map between closed, oriented connected n-dimensional non-empty topological manifolds. It follows from Lemma 51.15 that any class in $\varphi=H^n(M;\mathbb{Z})$

is of the form $\varphi = k \cdot [M]^*$ for some $k \in \mathbb{Z}$ and it follows from $\langle [M]^*, [M] \rangle = 1$ and the bilinearity of the Kronecker pairing that $k = \langle \varphi, [M] \rangle_M$. In our case we have

$$\langle f^*([N]^*), [M] \rangle \underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=} \langle [N]^*, f_*([M]) \rangle \underset{\substack{\uparrow \\ \text{by definition of } \deg(f)}}{=} \langle [N]^*, \deg(f) \cdot [N] \rangle \underset{\substack{\uparrow \\ \text{since } \langle\, , \rangle \text{ is bilinear}}}{=} \deg(f) \cdot \underbrace{\langle [N]^*, [N] \rangle}_{=1} = \deg(f).$$

It follows from the above discussion that $f^*([N]^*) = \deg(f) \cdot [M]^*$. □

Next we record the following useful consequence of the Universal Coefficient Theorem 51.13 for Cohomology Groups:

Proposition 51.17. *Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. Then the Hurewicz homomorphism $\pi_1(X, x_0) \to H_1(X)$ from page 924 and the evaluation homomorphism* $\mathrm{ev}\colon H^1(X; G) \to \mathrm{Hom}(H_1(X); G)$ *induce natural isomorphisms*

$$H^1(X; G) \xrightarrow{\cong} \mathrm{Hom}(H_1(X); G) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(X, x_0), G).$$

One often uses the natural isomorphisms from Proposition 51.17 to identify these groups.

Proof. As we pointed out on page 1220 and Proposition 38.2 (4), both maps are natural. So it remains to show that both maps are isomorphisms.

We insert one extra map into the above sequence of maps to see that the maps are indeed isomorphisms":

$$H^1(X; G) \underset{\substack{\uparrow \\ \text{isomorphism by Theorem 51.13} \\ \text{since } H_0(X) \cong \mathbb{Z} \text{ by Proposition 29.5}}}{\xrightarrow{\mathrm{ev}}} \mathrm{Hom}(H_1(X); G) \underset{\substack{\uparrow \\ \text{Hurewicz Theorem 38.5}}}{\xrightarrow{\cong}} \mathrm{Hom}(\pi_1(X, x_0)_{\mathrm{ab}}, G) \underset{\substack{\uparrow \\ \text{by Proposition 13.10 (2)} \\ \text{since } G \text{ is abelian}}}{\xleftarrow{\cong}} \mathrm{Hom}(\pi_1(X, x_0), G).$$

□

Before we continue we introduce a slight variation of the Kronecker pairing defined in Lemma 50.5. More precisely, let C_* be a chain complex and let R be a commutative ring. We consider the map

$$\begin{aligned} \langle\, , \,\rangle\colon H^n(C; R) \times H_n(C; R) &\to R \\ \Big([\varphi], \Big[\sum_{i=1}^k c_i \otimes a_i\Big]\Big) &\mapsto \sum_{i=1}^k \varphi(c_i) \cdot a_i. \end{aligned}$$

We leave it as an exercise to verify that this map is well-defined and bilinear. We refer to this pairing also as the *Kronecker pairing*.

The next proposition relates homology and cohomology with coefficients in a field $\mathbb{F}$.

Proposition 51.18. *Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a field. The map*

$$\begin{aligned} \mathrm{ev}\colon H^n(X, A; \mathbb{F}) &\to \mathrm{Hom}_{\mathbb{F}}(H_n(X, A; \mathbb{F}), \mathbb{F}) \\ \varphi &\mapsto \begin{pmatrix} H_n(X, A; \mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{pmatrix} \end{aligned}$$

is a natural isomorphism of $\mathbb{F}$*-vector spaces.*[921]

We postpone the proof of Proposition 51.18 to the next section. In most applications we are only interested in the following corollary to Proposition 51.18.

Corollary 51.19. *Let* X *be a topological space and let* $k \in \mathbb{N}_0$ *such that* $H_k(X;\mathbb{Z})$ *and* $H_{k-1}(X;\mathbb{Z})$ *are finitely generated. (By Proposition 45.2 we know that every compact topological manifold satisfies this condition.) Furthermore let* $\mathbb{F}$ *be a field. Then the homology and cohomology groups* $H_k(X;\mathbb{F})$ *and* $H^k(X;\mathbb{F})$ *are finite-dimensional* $\mathbb{F}$*-vector spaces with*

$$\dim_{\mathbb{F}}(H_k(X;\mathbb{F})) \;=\; \dim_{\mathbb{F}}(H^k(X;\mathbb{F})).$$

Remark. Note that in Corollary 51.19 we only claim that the homology and cohomology groups are abstractly isomorphic. We do not claim that there is a natural isomorphism. In fact it does not make sense to ask for a natural isomorphism since the homology groups of a topological space X with $\mathbb{F}$-coefficients are covariant in X whereas the cohomology groups of X with $\mathbb{F}$-coefficients are contravariant in X.

Proof. Let X be a topological space and let $k \in \mathbb{N}_0$ such that $H_k(X;\mathbb{Z})$ and $H_{k-1}(X;\mathbb{Z})$ are finitely generated. Let p be a prime number. By the classification of finitely generated abelian groups, see Theorem 11.4, we can make the identifications

$$H_{k-1}(X;\mathbb{Z}) \;=\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{\alpha}\mathbb{Z}_{a_i} \oplus \bigoplus_{i=1}^{\beta}\mathbb{Z}_{b_i} \quad \text{and} \quad H_k(X;\mathbb{Z}) \;=\; \mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma}\mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta}\mathbb{Z}_{d_i}$$

where all the a_i, b_i, c_i and d_i are non-zero, and where each a_i and each c_i is divisible by p and none of the b_i and d_i are divisible by p. We start out with the following claim:

Claim. We have

$$\dim_{\mathbb{F}}(H_k(X,\mathbb{F})) \;=\; \begin{cases} s, & \text{if } \mathbb{F} \text{ is a field of characteristic zero,} \\ s+\alpha+\gamma, & \text{if } \mathbb{F} \text{ is a field of characteristic } p. \end{cases}$$

First assume that $\mathbb{F}$ is a field of characteristic zero. In this case we have

$$H_k(X;\mathbb{F}) \;\underset{\substack{\uparrow \\ \text{by Theorem 42.19 and Lemma 42.17 (4) since } \mathbb{F} \text{ is torsion-free}}}{\cong}\; H_k(X;\mathbb{Z}) \otimes \mathbb{F} \;=\; \Big(\mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma}\mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta}\mathbb{Z}_{d_i}\Big) \otimes \mathbb{F} \;\underset{\substack{\uparrow \\ \text{see page 1004}}}{\cong}\; \mathbb{F}^s.$$

Now suppose that $\mathbb{F}$ is a field of characteristic p. We make the following observation. For any $n \in \mathbb{N}$ we have

$$\mathbb{Z}_n \otimes \mathbb{F} \underset{\substack{\uparrow \\ \text{Lemma 42.3 (3)}}}{=} \operatorname{coker}\big(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\big) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \operatorname{Tor}(\mathbb{Z}_n,\mathbb{F}) \underset{\substack{\uparrow \\ \text{Lemma 42.17 (6)}}}{=} \ker\big(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\big) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else.} \end{cases}$$

[921]Note that "natural" makes sense since $(X,A) \mapsto H^k(X,A;\mathbb{F})$ and $(X,A) \mapsto \operatorname{Hom}_{\mathbb{F}}(H_k(X,A;\mathbb{F}),\mathbb{F})$ both define *contravariant* functors from the category of pairs of topological spaces to the category of $\mathbb{F}$-vector spaces.

Then

$$\begin{array}{ccccc} H_k(X;\mathbb{F}) & \cong & H_k(X;\mathbb{Z})\otimes\mathbb{F}\oplus\mathrm{Tor}(H_{k-1}(X;\mathbb{Z}),\mathbb{F}) & \cong & \mathbb{F}^s\oplus\mathbb{F}^\gamma\oplus\mathbb{F}^\alpha. \\ & \uparrow & & \uparrow & \\ & \text{by Theorem 42.19} & & \begin{array}{c}\text{by Lemma 42.17, the discussion on page 1004}\\ \text{and the above observations}\end{array} & \end{array}$$

This concludes the proof of the claim.

Now we have

$$\begin{array}{ccccc} \dim_{\mathbb{F}}(H^k(X;\mathbb{F})) & = & \dim_{\mathbb{F}}(\mathrm{Hom}(H_k(X;\mathbb{F}),\mathbb{F})) & = & \dim_{\mathbb{F}}(H_k(X;\mathbb{F})) \\ & \uparrow & & \uparrow & \\ & \text{Proposition 51.18} & & \text{since by the claim } H_k(X;\mathbb{F}) \text{ is finite dimensional} & \end{array}$$

We could also prove the equality $\dim_{\mathbb{F}}(H^k(X;\mathbb{F})) = \dim_{\mathbb{F}}(H_k(X;\mathbb{F}))$ without appealing to Proposition 51.18. Indeed, using Theorem 51.13, the fact that for any abelian group G we have $\mathrm{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{\cdot n} G)$ and Proposition 51.6 one can easily compute $\dim_{\mathbb{F}}(H^k(X;\mathbb{F}))$ and one ends up with the same calculation as in the above claim. We leave it to the reader to fill in the details. □

51.3. **Proof of Proposition 51.18** $(*)$**.** In this section we give the proof of Proposition 51.18. Before we can provide the proof we need to prepare the ground.

Let $\mathbb{F}$ be a field. We define a *chain complex over* $\mathbb{F}$ to be a chain complex (V_*, v_*) where all chain groups are $\mathbb{F}$-vector spaces and all boundary maps are $\mathbb{F}$-homomorphisms. For example, if (C_*, ∂_*) is a chain complex of abelian groups, then $(C_*\otimes\mathbb{F}, \partial_*\otimes\mathrm{id})$ is easily seen to be a chain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over the field $\mathbb{F}$ and given $n\in\mathbb{N}_0$ its n-homology

$$H_n(V) \;:=\; \frac{\ker\{v_n\colon V_n\to V_{n-1}\}}{\mathrm{im}\{v_{n+1}\colon V_{n+1}\to V_n\}}.$$

is an $\mathbb{F}$-vector space in a natural way.

We also define cochain complexes and cochain maps over $\mathbb{F}$ in the obvious way. For example, if (C_*, c_*) is a chain complex, then the dual cochain complex $(\mathrm{Hom}(C_*(X,\mathbb{F})), c^*)$ is a cochain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over $\mathbb{F}$ we obtain a cochain complex by considering $(\mathrm{Hom}_{\mathbb{F}}(V_*;\mathbb{F}), v^*)$ and we define its n-th cohomology group to be the $\mathbb{F}$-vector space

$$H^n(V;\mathbb{F}) \;:=\; \frac{\ker\{v^*_{n+1}\colon \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F})\to\mathrm{Hom}_{\mathbb{F}}(V_{n+1},\mathbb{F})\}}{\mathrm{im}\{v^*_n\colon \mathrm{Hom}_{\mathbb{F}}(V_{n-1},\mathbb{F})\to\mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F})\}}.$$

The proof of the Universal Coefficient Theorem 51.12 for Cohomology Groups can be modified fairly easily to provide the following theorem.

Theorem 51.20. *Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. Then the map*

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon H^k(V;\mathbb{F}) & \mapsto & \mathrm{Hom}_{\mathbb{F}}(H_k(V),\mathbb{F}). \\ (\varphi\colon V_k\to\mathbb{F}) & \mapsto & \left(\begin{array}{rcl} H_k(V) & \to & \mathbb{F} \\ {[c]} & \mapsto & \varphi(c)\end{array}\right) \end{array}$$

is well-defined and it is an isomorphism.

A proof of that theorem is provided by any self-respecting book on homological algebra, see e.g. [DK, Corollary 2.31].

We will also need the following lemma.

Lemma 51.21. *Let C be an abelian group and let $\mathbb{F}$ be a field. Then the map*[922]

$$\begin{array}{rcl} \Phi_C\colon \operatorname{Hom}_{\mathbb{Z}}(C,\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) \\ (\varphi\colon C\to\mathbb{F}) & \mapsto & \left(\begin{array}{ccc} C\otimes\mathbb{F} & \to & \mathbb{F} \\ \sum\limits_{i=1}^{k} c_i\otimes a_i & \mapsto & \sum\limits_{i=1}^{k}\varphi(c_i)\cdot a_i \end{array}\right) \end{array}$$

is an isomorphism.

Proof. It is straightforward to verify that an inverse map to Φ_C is given by the following homomorphism of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Psi_C\colon \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{Z}}(C,\mathbb{F}) \\ (\varphi\colon C\otimes\mathbb{F}\to\mathbb{F}) & \mapsto & \big(C \xrightarrow{c\mapsto c\otimes 1} C\otimes\mathbb{F} \xrightarrow{\varphi} \mathbb{F}\big). \end{array}$$

□

Corollary 51.22. *Let C_* be a chain complex and let $\mathbb{F}$ be a field. The maps*

$$\Phi_{C_k}\colon \operatorname{Hom}_{\mathbb{Z}}(C_k,\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(C_k\otimes\mathbb{F},\mathbb{F}) \quad \textit{with } k\in\mathbb{N}_0$$

from the previous lemma define an isomorphism of cochain complexes of $\mathbb{F}$-vector spaces, in particular for each $k\in\mathbb{N}_0$ they induce an isomorphism

$$\Phi\colon H^k(C;\mathbb{F}) \to H^k(C\otimes\mathbb{F};\mathbb{F})$$

of $\mathbb{F}$-vector spaces.

Proof. By Lemma 51.21 the maps Φ_{C_k}, $k\in\mathbb{N}_0$, are isomorphisms of the cochain vector spaces. It is straightforward to see that these maps are also cochain maps. □

Now we can finally provide the proof of Proposition 51.18.

Proof of Proposition 51.18. Let (X,A) be a pair of topological spaces and let $\mathbb{F}$ be a field. We write $C_*=C_*(X,A)$. Given $k\in\mathbb{N}_0$ we consider the diagram

$$\begin{tikzcd} & H^k(C\otimes\mathbb{F};\mathbb{F}) \arrow[dr, "\mathrm{ev}_{\mathbb{F}}"] & \\ H^k(C;\mathbb{F}) \arrow[ur, "\Phi"] \arrow[rr, "\mathrm{ev}"'] & & \operatorname{Hom}_{\mathbb{F}}(H_n(C\otimes\mathbb{F}),\mathbb{F}). \end{tikzcd}$$

It follows easily from the definitions that for any representative in $H^k(C;\mathbb{F})$ the image under the composition of the two diagonal maps agrees with the image under the map ev. This implies that ev is well-defined and that the diagram commutes.

[922]Note that to the left we consider homomorphisms of $\mathbb{Z}$-modules, i.e. homomorphisms of abelian groups whereas to the right we consider homomorphisms of $\mathbb{F}$-vector spaces.

Also note that the left diagonal map is an isomorphism by Corollary 51.22 whereas the right diagonal map is an isomorphism by Theorem 51.20. In particular the bottom map is an isomorphism. But that is exactly what we had wanted to show. □

51.4. **Change of coefficients in cohomology** $(*)$**.** Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. By the Algebraic Universal Coefficient Theorem 42.18 we can relate the homology groups of $C_* \otimes G$ to the homology groups of C_*. Furthermore, by the Universal Coefficient Theorem for Cohomology Groups 51.12 we can relate the cohomology groups $H^n(C_*; G)$ to the homology groups $H_*(C)$. The following question arises.

Question 51.23. *Can we relate the cohomology groups $H^n(C; G)$ to the cohomology groups $H^n(C; \mathbb{Z})$?*

It turns out that the question is slightly subtle. To formulate the answer we need the following lemma.

Lemma 51.24. *Let G be an abelian group.*

(1) *For any abelian group H the map*

$$\begin{array}{rcl} \mu\colon \operatorname{Hom}(H,\mathbb{Z}) \otimes G & \to & \operatorname{Hom}(H,G) \\ \sum\limits_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{ccc} H & \to & G \\ h & \mapsto & \sum\limits_{i=1}^{m} \varphi_i(h) \cdot g_i \end{array} \right) \end{array}$$

is a well-defined natural *homomorphism.*

(2) *Given a chain complex (C_*, ∂_*) the map*

$$\begin{array}{rcl} H^n(C;\mathbb{Z}) \otimes G & \xrightarrow{\mu} & H^n(C;G) \\ \sum\limits_{i=1}^{m} [\varphi_i] \otimes g_i & \mapsto & \sum\limits_{i=1}^{m} [\mu(\varphi_i \otimes g_i)] \end{array}$$

is well-defined.

Proof. The proof is elementary and is left to the reader. □

The following theorem gives an affirmative answer to Question 51.23, albeit only only under an extra hypothesis.

Theorem 51.25. (Universal Coefficient Theorem) *Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. We suppose that all homology groups $H^n(C_*; \mathbb{Z})$ are finitely generated. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence of the form*

$$\begin{array}{ccccccccc} 0 & \to & H^n(C;\mathbb{Z}) \otimes G & \xrightarrow{\mu} & H^n(C;G) & \to & \operatorname{Tor}(H^{n+1}(C;\mathbb{Z}), G) & \to & 0. \\ & & [\varphi] \otimes g & \mapsto & [\mu(\varphi \otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.[923]

The key to proving the Universal Coefficient Theorem 51.25 is the following elementary but subtle lemma.

Lemma 51.26. *Let G be an abelian group. If H is a free abelian group of* finite rank*, then the map*

$$\mu\colon \operatorname{Hom}(H,\mathbb{Z})\otimes G \;\to\; \operatorname{Hom}(H,G)$$

is an isomorphism.

Proof. Let $v_1,\dots,v_n$ be a basis for the finitely generated free abelian group H. We denote by $\varphi_1,\dots,\varphi_n$ the dual basis of $\operatorname{Hom}(H,\mathbb{Z})$. In other words, these are the homomorphisms that are uniquely determined by $\varphi_i(v_j)=\delta_{ij}$. We leave it to the reader to verify that the map

$$\begin{array}{rcl} \nu\colon \operatorname{Hom}(H,G) & \to & \operatorname{Hom}(H,\mathbb{Z})\otimes G \\ \psi & \mapsto & \sum\limits_{i=1}^{n}\varphi_i\otimes\psi(v_i) \end{array}$$

is an inverse to the homomorphism μ. □

We will also need the following lemma.

Lemma 51.27. *Let (C_*,∂_*) be a chain complex of free abelian groups. If all cohomology groups $H^n(C_*;\mathbb{Z})$ are finitely generated, then all homology groups $H_n(C_*)$ are also finitely generated.*

Proof of Lemma 51.27. Let (C_*,∂_*) be a chain complex of free abelian groups such that all cohomology groups $H^n(C_*;\mathbb{Z})$ are finitely generated. By the Universal Coefficient Theorem 51.12 for Cohomology Groups there exists an isomorphism

$$H^n(C;\mathbb{Z}) \;\cong\; \operatorname{Ext}(H_{n-1}(C),\mathbb{Z})\oplus\operatorname{Hom}(H_n(C),\mathbb{Z}) \qquad \text{for every } n\in\mathbb{N}_0.$$

We see that all the groups $\operatorname{Ext}(H_{n-1}(C),\mathbb{Z})$ and $\operatorname{Hom}(H_n(C),\mathbb{Z})$ are finitely generated, in particular they are countable. But by Proposition 51.11 this implies that all the homology groups $H_n(C_*)$ are finitely generated. □

Proof of the Universal Coefficient Theorem 51.25. Let G be an abelian group. Furthermore let (C_*,∂_*) be a chain complex of free abelian groups such that all cohomology groups $H^n(C_*;\mathbb{Z})$ are finitely generated. It follows from Lemma 51.27 that all homology groups $H_n(C_*)$ are also finitely generated. Thus we obtain from Lemma 49.8 that there exists a chain homotopy equivalence $h\colon C_*\to D_*$ from (C_*,∂_*) to a chain complex (D_*,∂_*) such

[923]As always, the splitting is not natural. In fact in this theorem we do *not* even claim that the short exact sequence is natural. The reason is that the construction of the short exact sequence involves a choice of the chain complex D_* and we do not intend to discuss the question how that choice affects the map on the right hand side.

that each chain group D_n is a finitely generated free abelian group. Now we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overbrace{H_n(\mathrm{Hom}(D_*,\mathbb{Z}))}^{=H^n(D;\mathbb{Z}) \text{ by def.}} \otimes G \arrow[r, "{[\varphi]\otimes g \mapsto [\varphi\otimes g]}"] & H_n(\mathrm{Hom}(D_*,\mathbb{Z});G) \arrow[r] & \mathrm{Tor}(\overbrace{H_{n+1}(\mathrm{Hom}(D_*,\mathbb{Z}))}^{=H^{n+1}(D;\mathbb{Z}) \text{ by def.}}, G) \arrow[r] & 0. \\
& H^n(D;\mathbb{Z}) \otimes G \arrow[u, "="] \arrow[r, "\mu"] & H_n(\mathrm{Hom}(D_*,G)) \arrow[d, "\mu", "\cong"'] & \mathrm{Tor}(H^{n+1}(D;\mathbb{Z}),G) \arrow[u, "="] & \\
& H^n(D;\mathbb{Z}) \otimes G \arrow[u, "="] \arrow[r, "\mu"] \arrow[d, "h^*", "\cong"'] & H^n(D_*,G)) \arrow[u, "="] \arrow[d, "h^*", "\cong"'] & \mathrm{Tor}(H^{n+1}(D;\mathbb{Z}),G) \arrow[u, "="] \arrow[d, "h^*", "\cong"'] & \\
& H^n(C;\mathbb{Z}) \otimes G \arrow[r, "\mu"] & H^n(C,G) & \mathrm{Tor}(H^{n+1}(D;\mathbb{Z}),G). &
\end{tikzcd}$$

We make the following clarifications:

(1) The sequence on the top is the short exact sequence obtained from the Algebraic Universal Coefficient Theorem 42.18 applied to the chain complex $E_* = \mathrm{Hom}(D_{-*},\mathbb{Z})$. We need to invert the sign of the degrees twice, thus we end up with the degree $n+1 = -((-n)-1)$ on the right hand side. We know by Proposition 42.22 that the short exact sequence splits.
(2) The middle vertical map between the first two rows is an isomorphism by Lemma 51.26. Here we use that each chain group of D_* is a free abelian group of finite rank.
(3) One can easily verify that the diagram commutes.
(4) By Lemma 48.8 (4) the bottom vertical maps are isomorphisms.

It is now clear that we obtain the desired short exact sequence. □

The following theorem gives a practical "topological" version of the Universal Coefficient Theorem 51.25.

Theorem 51.28. (Universal Coefficient Theorem) *Let X be a topological space and let G be an abelian group. If all homology groups $H_n(X)$ of X are finitely generated (e.g. X could be a compact topological manifold or X could be a CW-complex that has only finitely many cells in each dimension*[924]*), then for each $n \in \mathbb{N}_0$ there exists a short exact sequence of the form*

$$\begin{array}{ccccccc} 0 & \to & H^n(X;\mathbb{Z}) \otimes G & \xrightarrow{\mu} & H^n(X;G) & \to & \mathrm{Tor}(H^{n+1}(X;\mathbb{Z}),G) \to 0. \\ & & [\varphi] \otimes g & \mapsto & [\mu(\varphi \otimes g)] & & \end{array}$$

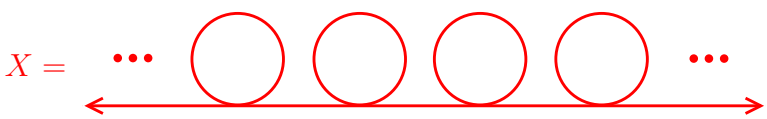
Furthermore the short exact sequence splits.

Proof. This theorem follows immediately from applying the Universal Coefficient Theorem 51.25 to the singular chain complex of X. □

Remark. In general the hypothesis on X that we put into the Universal Coefficient Theorem 51.28 cannot be dropped. For example consider the infinite string of circles X as

[924]By Proposition 45.2 (2) and Proposition 35.5 (1) such topological spaces qualify.

shown in Figure 692 and we take $G = \mathbb{Q}$. By Lemma 42.17 (4) we know that all Tor-groups $\mathrm{Tor}(H, \mathbb{Q})$ vanish. In Exercise 51.9 we will see that the groups $H^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $H^1(X;\mathbb{Q})$ are not isomorphic. This shows that in this setting the sequence given in Universal Coefficient Theorem 51.28 is not exact.



the groups
$H^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $H^1(X;\mathbb{Q})$
are not isomorphic

FIGURE 692.

The following lemma gives a convenient interpretation of the map μ appearing in the Universal Coefficient Theorem 51.28.

Lemma 51.29. *Let X be a topological space, let G be an abelian group and let $f\colon \mathbb{Z} \to G$ be a homomorphism. The following diagram commutes:*

$$\begin{tikzcd} H^n(X;\mathbb{Z}) \arrow[d, "a \mapsto a \otimes f(1)"'] \arrow[dr, "f_*"] & \\ H^n(X;\mathbb{Z}) \otimes G \arrow[r, "\mu"'] & H^n(X;G). \end{tikzcd}$$

Proof. The statement follows immediately from explicitly writing down the homomorphisms. □

Example. Let $n \in \mathbb{N} \cup \{\infty\}$. We denote by $f\colon \mathbb{Z} \to \mathbb{Z}_2$ the obvious group homomorphism. It follows from the Universal Coefficient Theorem 51.28, from Lemma 51.29, from the calculations of the cohomology groups $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ on page 1198, and from the properties of Tor-groups, see Lemma 42.17, that for any $k \in \mathbb{N}_0$ the map $f_*\colon H_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$ is an epimorphism.

We conclude this chapter with the following lemma which unwittingly gets used frequently.

Lemma 51.30. $(*)$ *We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ the obvious inclusion. Let $n \in \mathbb{N}_0$.*

(1) *Let X be a topological space such that all homology groups are finitely generated. We suppose that $H^n(X;\mathbb{Z})$ is a finitely generated free abelian group. If $v_1, \dots, v_m$ is a basis of $H^n(X;\mathbb{Z})$, then $\varphi_*(v_1), \dots, \varphi_*(v_m)$ is a basis of $H^n(X;\mathbb{Q})$.*

(2) *Let $f\colon X \to Y$ be a map between topological spaces. We suppose that all homology groups of X and Y are finitely generated and we suppose that $H^n(X;\mathbb{Z})$ and $H^n(Y;\mathbb{Z})$ are free abelian groups. We consider the following commutative diagram:*

$$\begin{tikzcd} H^n(Y;\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "\varphi_*"'] & H^n(X;\mathbb{Z}) \arrow[d, "\varphi_*"] \\ H^n(Y;\mathbb{Q}) \arrow[r, "f^*"] & H^n(X;\mathbb{Q}). \end{tikzcd}$$

Suppose $v_1, \dots, v_r$ is a basis for $H^n(X;\mathbb{Z})$ and that $w_1, \dots, w_s$ is a basis for $H^n(Y;\mathbb{Z})$. The matrix that represents $f^*\colon H^n(Y;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ with respect to the given bases also represents $f^*\colon H^n(Y;\mathbb{Q}) \to H^n(X;\mathbb{Q})$ with respect to the bases $\varphi_*(v_1), \dots, \varphi_*(v_r)$ and $\varphi_*(w_1), \dots, \varphi_*(w_s)$.

Proof $(*)$.

(1) This statement follows easily from the Universal Coefficient Theorem 51.28 together with Lemma 42.17 (4) and Lemma 51.29.
(2) The fact that the diagram commutes is an immediate consequence of Lemma 48.9. The remaining statement is just a psychologically helpful reinterpretation of the fact that the diagram commutes. □

Exercises for Chapter 51.

Exercise 51.1. Let C_* be a chain complex of free abelian groups and let $n \in \mathbb{N}$. Show that there does not exist a *natural* isomorphism

$$H^n(C;G) \;\cong\; \operatorname{Ext}(H_{n-1}(C), G) \oplus \operatorname{Hom}(H_n(C), G).$$

Exercise 51.2. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_0 and A_1 are free abelian groups. Show that there exists a path-connected CW-complex X such that for any $n \in \mathbb{N}$ we have $H^n(X;\mathbb{Z}) \cong A_n$.
Hint: Use Proposition 34.10.

Exercise 51.3. Let $(P, \leq)$ be a partially ordered set and let Q be a subset. Is an upper bound necessarily a maximal element? And conversely, is a maximal element necessarily an upper bound?

Exercise 51.4. Redo Exercise 50.2. More precisely, use the Universal Coefficient Theorem 51.13 for Cohomology Groups to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 51.5. Give an explicit example of a cocycle $\varphi\colon C_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ that represents a generator of $H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$.

Exercise 51.6. Let M be a compact oriented connected n-dimensional topological manifold with non-empty connected boundary. Show that $\delta([\partial M]^*) = [M]^* \in H^n(M, \partial M;\mathbb{Z})$ where δ denotes the connecting homomorphism in the long exact sequence of cohomology groups of the pair $(M, \partial M)$.
Hint: use Proposition 46.26.

Exercise 51.7.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.
(b) Show that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.

(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.
(d) Use (a) and (b), together with the existence of Moore spaces, see Proposition 34.11, and together with the Universal Coefficient Theorem 51.13 for Cohomology Groups and Proposition 51.10 to conclude that there exist topological spaces X and Y such that $H^n(X;\mathbb{Z}) \cong H^n(Y;\mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $H_1(X;\mathbb{Z})$ is not isomorphic to $H_1(Y;\mathbb{Z})$.

Remark: This exercise shows that in Corollary 49.4 the roles of homology and cohomology are not symmetric.

Exercise 51.8. Let H be a free abelian group and let G be an abelian group. As in Lemma 51.26 we consider the map

$$\begin{array}{rcl} \mathrm{Hom}(H,\mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H,G) \\ \sum\limits_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{rcl} H & \to & G \\ h & \mapsto & \sum\limits_{i=1}^{m} \varphi_i(h) \cdot g_i \end{array} \right). \end{array}$$

Show that this map is in general not an isomorphism.

Exercise 51.9. We consider the topological space X shown in Figure 692. Show that the groups $H^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $H^1(X;\mathbb{Q})$ are not isomorphic.

52. Cohomology groups and limits

52.1. **Direct limits of direct systems.** In the intermediate future direct limits of direct systems and inverse limits of inverse systems will play a large role. To fortify us for these discussions we recall the definitions and we prove some fairly elementary but useful lemmas regarding limits.

First we recall the following definition from page 481.

Definition. A *directed set* is a set I together with a relation $\leq$ on I which satisfies the following three conditions:

(1) the relation is reflexive, i.e. for any $i \in I$ we have $i \leq i$.
(2) the relation is transitive, i.e. for any $i, j, k \in I$ we have
$$i \leq j \text{ and } j \leq k \quad \Longrightarrow \quad i \leq k.$$
(3) for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

Examples.

(a) Evidently the natural numbers $\mathbb{N}$ together with the usual "$\leq$" relation form a directed set $(\mathbb{N}, \leq)$.
(b) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then the relation given by "$\subseteq$" turns $\mathcal{P}(M)$ into a directed set. To prove this claim it only remains to verify (3). But this also holds, since for any $X, Y \in \mathcal{P}(M)$, i.e. for any subsets X, Y of M we can take $Z = X \cup Y$ and we then have $X \subseteq Z$ and $Y \subseteq Z$.

Next we recall the following definition from page 482.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A *direct system over the directed set*[925] $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij} \colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ik} & = & f_{jk} \circ f_{ij} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Example. Let $\mathcal{C}$ be a category and let
$$X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3 \xrightarrow{g_3} X_4 \xrightarrow{g_4} \dots$$
be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}_0$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ij} = g_{j-1} \circ \dots \circ g_{i+1} \circ g_i \colon X_i \to X_j$. It is clear that $(\{X_i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ is a direct system over the directed set $(\mathbb{N}, \leq)$. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i \in \mathbb{N}}, \{g_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{N}})$ also as a direct system.

Finally we also recall the definition of a direct limit from page 483.

[925] On page 482 we had also considered direct systems that are defined over preordered sets, but for the remainder of this course we are only interested in direct systems over directed sets.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ in the category $\mathcal{C}$. A *direct limit* of the direct system is an object $\varinjlim X_i$ in $\mathcal{C}$ together with morphisms $h_i\colon X_i \to \varinjlim X_i, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $k \leq l$ we have $h_l \circ f_{kl} = h_k\colon X_k \to \varinjlim X_i$, i.e. for all $k \leq l$ the following diagram commutes

$$\begin{tikzcd} X_l \arrow[r, "h_l"] & \\ X_k \arrow[u, "f_{kl}"] \arrow[r, "h_k"'] & \varinjlim X_i. \end{tikzcd}$$

(2) If we are given another object Y and morphisms $h'_i\colon X_i \to Y, i \in I$ that satisfy (1), then there exists a unique morphism $F\colon \varinjlim X_i \to Y$ such that for all $k \leq l$ the following diagram commutes

$$\begin{tikzcd} X_l \arrow[r, "h_l"] \arrow[rr, bend left, "h'_l"] & & \\ X_i \arrow[u, "f_{kl}"] \arrow[r, "h_k"] \arrow[rr, bend right, "h'_k"'] & \varinjlim X_k \arrow[r, dashed, "F", "\exists!"'] & Y. \end{tikzcd}$$

As we pointed out on page 484, the direct limit of direct system, if it exists, is in an appropriate sense unique up to isomorphism.

The following proposition is a variation of Proposition 16.1.

Proposition 52.1. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system over one of the following categories:*

(1) *the category of abelian groups,*
(2) *the category of R-modules, where R is a commutative ring,*
(3) *the category of chain complexes,*
(4) *the category of cochain complexes.*

Then[926]

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \textit{where } x \sim f_{ij}(x) \textit{ for all } i \leq j,\ x \in X_i$$

with the obvious maps $X_i \to \varinjlim X_i$ is a direct limit. Here we view $\varinjlim X_i$ as a group as follows: if we are given $a \in X_i$ and $b \in X_j$, then by definition of a directed set there exists

[926]The construction of the direct limit that we gave in Proposition 16.1 was different, and arguably more complicated, since there we had considered direct systems over arbitrary preordered sets.

a $k \in I$ with $i \leq k$ and $j \leq k$, and we define[927]

$$[a] + [b] \;:=\; \underbrace{\big[\underbrace{f_{ik}(a)}_{\in X_k} + \underbrace{f_{jk}(b)}_{\in X_k}\big]}_{\textit{addition in } X_k} \;\in\; \varinjlim X_i.$$

Depending on the category we are dealing with we can also equip $\varinjlim X_i$ with an R-module structure or we can equip it with (co) boundary maps.

Proof. This explicit construction of the direct limit was already given on page 487. The proof of the proposition is a not overly exciting exercise in going through all the definitions. We leave this task to the reader. □

Now we give three lemmas which will be helpful in getting direct limits "under control".

Lemma 52.2. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in Proposition 52.1. Then the following four statements hold:*

(1) *Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.*
(2) *Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.*
(3) *Let $a_j \in X_j$. The image of a_j in $\varinjlim X_i$ is zero if and only if there exists a $k \geq j$ such that $f_{jk}(a_j) = 0 \in X_k$.*
(4) *If all maps $f_{ij}\colon X_i \to X_j$ are isomorphisms, then for any $j \in I$ the obvious map $X_j \to \varinjlim X_i$ is an isomorphism.*

Proof.

(1) The first statement follows immediately from the explicit description of the direct limit given in Proposition 52.1.
(2) The second statement will be proved in Exercise 52.1. For the exercise it is useful to note that the symbol "$\sim$" in the definition of $\varinjlim X_i$ in Proposition 52.1 is only a relation and we really have to work with the equivalence relation generated by this relation, see page 114 for more information.
(3) The third statement is a special case of the second statement.
(4) The last statement will be proved in Exercise 52.1. □

The second lemma is useful for simplifying calculations.

Lemma 52.3. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in Proposition 52.1. Let J be a subset of I such that $(J, \leq)$ is again a directed set.*

[927] It is straightforward to verify that this definition does not depend on the choice of k.

(1) *There exists a unique map*

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

such that for any $j \in J$ the following diagram commutes:

$$\begin{tikzcd} & X_j \arrow[dl] \arrow[dr] & \\ \varinjlim_{j\in J} X_j \arrow[rr] & & \varinjlim_{i\in I} X_i. \end{tikzcd}$$

(2) *If J has the property that given any $i \in I$ there exists a $j \in J$ with $i \leq j$, then the above map*

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

is an isomorphism.

The first part of the lemma is just an immediate consequence of the definition of the direct limit over the directed set J. As we will see, the second statement is a fairly straightforward consequence of the next lemma. But it is also a good exercise to prove the above second statement "by hand".

The following definition is a generalization of the definition on page 867.

Definition. Let $(I, \leq)$ and $(J, \leq)$ be directed sets.

(1) A *morphism* between the directed sets $(I, \leq)$ and $(J, \leq)$ is a map $\varphi\colon I \to J$ such that for any $i_1 \leq i_2$ in I we have $\varphi(i_1) \leq \varphi(i_2)$.
(2) A morphism $\varphi\colon I \to J$ is called *cofinal* if given any $j \in J$ there exists an $i \in I$ with $\varphi(i) \geq j$.
(3) Let $(\{K_i\}_{i\in I}, \{\chi_{i_1 i_2}\}_{i_1 \leq i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1 j_2}\}_{j_1 \leq j_2})$ be direct systems in one of the four categories that we had considered in Proposition 52.1. A *homomorphism between the direct systems* is a family $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ of homomorphisms that has the property that for any $i_1 \leq i_2$ the following diagram commutes:

$$\begin{tikzcd} K_{i_2} \arrow[r, "f_{i_2}"] & L_{\varphi(i_2)} \\ K_{i_1} \arrow[u, "\chi_{i_1 i_2}"] \arrow[r, "f_{i_1}"] & L_{\varphi(i_1)} \arrow[u, "\lambda_{\varphi(i_1)\varphi(i_2)}"'] \end{tikzcd}$$

(4) The maps $K_i \xrightarrow{f_i} L_{\varphi(i)} \to \varinjlim L_j$ induce, by the universal property of a direct limit, a map

$$\varinjlim_{i\in I} K_i \to \varinjlim_{j\in J} L_j$$

which is uniquely determined by the property that for any $i \in I$ the following diagram commutes

$$\begin{tikzcd} K_i \arrow[r, "f_i"] \arrow[d] & L_{\varphi(i)} \arrow[d] \\ \varinjlim_{i\in I} K_i \arrow[r] & \varinjlim_{j\in J} L_j. \end{tikzcd}$$

Lemma 52.4. *Let $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ be a homomorphism between two direct systems $(\{K_i\}_{i\in I}, \{\chi_{i_1 i_2}\}_{i_1\le i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1 j_2}\}_{j_1\le j_2})$ in one of the four categories that we had considered in Proposition 52.1. Suppose the following two conditions are satisfied:*

(1) *the morphism $\varphi\colon I \to J$ is cofinal, and*
(2) *for each $i \in I$ the map $f_i\colon K_i \to L_{\varphi(i)}$ is an isomorphism.*

Then the induced map

$$\varinjlim_{i\in I} K_i \ \to\ \varinjlim_{j\in J} L_j$$

is an isomorphism.

Sketch of proof. Let $j \in J$. Since $\varphi\colon I \to J$ is cofinal we can pick $i \in I$ with $f(i) \ge j$. We define

$$\begin{aligned} g_j\colon L_j &\to \varinjlim_{i\in I} K_i \\ x &\mapsto [f_i^{-1}(\lambda_{j,f(i)}(x))]. \end{aligned}$$

It is straightforward to verify that this map does *not* depend on the choice of i. Using this observation it is elementary to verify that for any $j_1 \le j_2$ the following diagram commutes:

$$\begin{tikzcd} L_{j_2} \arrow[dr, "g_{j_2}"] & \\ & \varinjlim_{i\in I} K_i. \\ L_{j_1} \arrow[uu, "\lambda_{j_1 j_2}"] \arrow[ur, "g_{j_1}"'] & \end{tikzcd}$$

But by the universal property of the direct limit this implies that we get an induced

$$\varinjlim_{j\in J} L_j \ \to\ \varinjlim_{i\in I} K_i.$$

We leave it as an exercise to verify that this map is an inverse to the map under consideration. □

52.2. **Inverse limits of inverse systems.** We recall the following definition from page 495.

Definition. Let $(I, \le)$ be a directed set and let $\mathcal{C}$ be a category. An *inverse system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms

$\{f_{ji}\colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[928] such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ki} & = & f_{ji} \circ f_{kj} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Example. Let $\mathcal{C}$ be a category and let

$$\ldots \xrightarrow{g_4} X_4 \xrightarrow{g_3} X_3 \xrightarrow{g_2} X_2 \xrightarrow{g_1} X_1$$

be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ji} = g_i \circ \cdots \circ g_{j-1}\colon X_j \to X_i$. It is clear that $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ is an inverse system over the directed set $(\mathbb{N}, \leq)$. It is also clear that any inverse system over $(\mathbb{N}, \leq)$ arises that way. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ also as an inverse system.

Using the convention from the previous example we can now reformulate in our context the original definition of an inverse limit from page 495.

Definition. Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system in a category $\mathcal{C}$. An *inverse limit* of the inverse system is an object $\varprojlim X_i$ in $\mathcal{C}$ together with morphisms $h_i\colon \varprojlim X_i \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \in \mathbb{N}$ we have $g_i \circ h_{i+1} = h_i\colon \varprojlim X_i \to X_i$, i.e. for all $i \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} & X_{i+1} \arrow[d, "g_i"] \\ \varprojlim X_i \arrow[ur, "h_{i+1}"] \arrow[r, "h_i"'] & X_i. \end{tikzcd}$$

(2) If we are given another object Y and morphisms $h'_i\colon Y \to X_i$, $i \in \mathbb{N}$, that satisfy (1), then there exists a unique morphism $F\colon Y \to \varprojlim X_i$ such that for all $i \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} & & X_{i+1} \arrow[dd, "g_i"] \\ Y \arrow[urr, bend left, "h'_{i+1}"] \arrow[r, dashed, "F", "\exists!"'] \arrow[drr, bend right, "h'_i"'] & \varprojlim X_i \arrow[ur, "h_{i+1}"] \arrow[dr, "h_i"] & \\ & & X_i. \end{tikzcd}$$

[928] So in an inverse system, given $i \leq j$ we have a morphism $X_j \to X_i$, whereas in a direct system we had a morphism $X_i \to X_j$.

Throughout the reminder of this course we will only be interested in inverse systems over the directed set $(\mathbb{N}, \leq)$. In the following proposition we give an explicit description of inverse limits for this special case. The proposition can be viewed as a slight variation of Proposition 16.7.

Proposition 52.5. *Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system over one of the following categories:*

1. *the category of abelian groups,*
2. *the category of R-modules, where R is a commutative ring,*
3. *the category of chain complexes,*
4. *the category of cochain complexes.*

Then

$$\varprojlim X_i \;:=\; \Big\{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \,\Big|\, g_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{N}\Big\}$$

with the obvious projection maps $\varprojlim X_i \to X_i$ is a direct limit.

Proof. This proposition is implicitly proved on page 501. □

Remark. Proposition 52.5 can be summarized as saying that, under the hypothesis of the proposition, we have

$$\varprojlim X_i \;=\; \text{all infinite sequences } (x_1, x_2, x_3, \dots) \text{ of the form } \begin{array}{ccc} x_3 & \in & X_3 \\ \downarrow & & \downarrow g_2 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_1 \\ x_1 & \in & X_1. \end{array}$$

Example. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and we denote by

$$\begin{array}{rcl} g_i\colon \mathbb{Z}^{i+1} & \to & \mathbb{Z}^i \\ (x_1, \dots, x_{i+1}) & \mapsto & (x_1, \dots, x_i) \end{array}$$

the projection map. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}})$ forms an inverse system in the category of abelian groups. As we had already pointed out on page 501, using the description of the inverse limit provided in Proposition 52.5 one can show fairly easily that

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}^i \;=\; \{(x_1, x_2, x_3, \dots) \mid x_j \in \mathbb{Z}\} \;\cong\; \mathbb{Z}^{\mathbb{N}}.$$

We conclude this section with the following analogue of Lemma 16.2.

Lemma 52.6. *Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Then $\varprojlim X_n$ exists and the natural map $\varprojlim X_n \to X_N$ is an isomorphism.*

Proof. We leave it to the reader to prove this elementary lemma. □

52.3. **Cohomology groups and inverse limits.** In Propositions 16.4 and 34.4 we saw that in favorable situations the fundamental group and the homology groups of a union of topological spaces are the direct limit of the corresponding fundamental groups and homology groups. In this section we will try to extend these results to cohomology groups. The results will not only be useful for computing cohomology groups, but they will also play an essential role in many proofs later on. For example the proof that de Rham cohomology is isomorphic to singular cohomology with real coefficients will build on the somewhat technical results of this section.

We recall that Proposition 34.4 was a consequence of the following two lemmas. The first lemma is of topological nature.

Lemma 34.5. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that every compact subset of X is already contained in one of the X_i. Then the chain complexes $C_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $C_*(X_i) \to C_*(X_j)$ for $i \leq j$ form a direct system of chain complexes. Furthermore the inclusion induced maps $C_*(X_i) \to C_*(X)$ induce an isomorphism*

$$\varinjlim C_*(X_i) \xrightarrow{\cong} C_*(X)$$

of chain complexes.

The second lemma is purely algebraic.

Lemma 34.6. *Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{H_n(C_i)\}_{i\in\mathbb{N}}, \{g_{i*}\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $H_n(C_i) \to H_n\big(\varinjlim C_i\big)$ induce an isomorphism*

$$\varinjlim H_n(C_i) \xrightarrow{\cong} H_n\big(\varinjlim C_i\big).$$

Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. The last lemma can be summarized as follows: for every $j \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ we have the following diagram

$$\begin{tikzcd}
H_n(C_{j+1}) \arrow[dr] \arrow[drr, bend left] & & \\
& \varinjlim H_n(C_i) \arrow[r, "\cong"] & H_n\big(\varinjlim C_i\big) \\
H_n(C_j) \arrow[uu, "g_{j*}"] \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Now we turn to cohomology groups. Let G be an abelian group. Whereas homology is a covariant functor, cohomology is a contravariant functor. This implies that from the above direct system of chain complexes we obtain for each $n \in \mathbb{N}_0$ an *inverse system*

$(\{H^n(C_i;G)\}_{i\in\mathbb{N}}, \{g_i^*\colon H^n(C_{i+1};G) \to H^n(C_i;G)\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore the maps $C_i \to \varinjlim C_i$ induce maps $H^n(\varinjlim C_i;G) \to H^n(C_i;G)$. Put differently, for any $j \in \mathbb{N}$ we are in the following situation

$$\begin{tikzcd}
 & & H^n(C_{j+1};G) \arrow[dd, "g_j^*"] \\
H^n\big(\varinjlim C_i;G\big) \arrow[urr, bend left] \arrow[rr, dashed] \arrow[drr, bend right] & & \varprojlim H^n(C_i;G) \arrow[ur] \arrow[dr] & \\
 & & H^n(C_j;G)
\end{tikzcd}$$

where the dotted arrow exists by the definition of the inverse limit of the groups $H^n(C_i;G)$. The question is whether the dotted arrow is in fact an isomorphism, put differently, the following question arises.

Question 52.7. *Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i\colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. Is the above map*

$$H^n\big(\varinjlim C_i;\mathbb{Z}\big) \;\to\; \varprojlim H^n(C_i;\mathbb{Z})$$

always an isomorphism?

It turns out that this is not necessarily the case. To formulate the relationship between the two sides of the above question it is useful to rewrite the inverse limit from Proposition 52.5. More precisely, given an inverse system $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups we consider the map

$$\begin{array}{rcl}
\mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\
(x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}.
\end{array}$$

Proposition 52.5 can be reformulated as saying that

$$\varprojlim D_i \;=\; \ker(\mu).$$

This leads us to the following definition.

Definition. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. Using the above notation we define[929]

$$\varprojlim{}^1 D_i \;:=\; \operatorname{coker}(\mu).$$

Example. We consider the inverse system

$$D_*\colon\; \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \;\to\; 0.$$

[929]If R is a commutative ring and if $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ is an inverse system of R-modules, then $\varprojlim^1 D_i$ is also an R-module in an obvious way.

In Exercise 52.2 we will see that $\varprojlim D_* = 0$ and $\varprojlim^1 D_* = 0$. Furthermore, we also consider the inverse system

$$E_*\colon\ \dots \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to 0.$$

In Exercise 52.2 we will see that $\varprojlim^1 E_* \neq 0$. Furthermore in Exercise 52.3 we will see that this inverse system arises "in nature", i.e. we will see that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $H^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $H^1(X_{i+1};\mathbb{Z}) \to H^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.

We cannot proceed without introducing the following somewhat technical definition.

Definition. We say an inverse system of abelian groups $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ satisfies the *Mittag-Leffler condition* if given any $i \in \mathbb{N}$ there exists an $n \geq i$ such that for all $s \geq n$ we have

$$\operatorname{im}\{g_i \circ \dots \circ g_{s-1}\colon D_s \to D_i\} \ = \ \operatorname{im}\{g_i \circ \dots \circ g_{n-1}\colon D_n \to D_i\}.$$

.

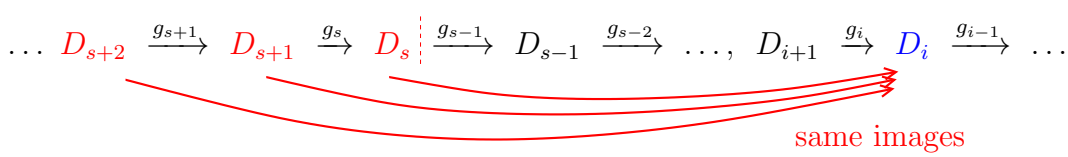


FIGURE 693.

Example. It is straightforward to verify that the above inverse system D_* satisfies the Mittag-Leffler condition whereas the above inverse system E_* does not satisfy the Mittag-Leffler condition.

In many situations one can ignore the precise definition of the Mittag-Leffler condition and instead one can appeal to the following lemma which in fact follows immediately from the definitions.

Lemma 52.8. *Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If there exists an $m \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ the map $g_i\colon D_{i+1} \to D_i$ is an epimorphism, then the inverse system satisfies the Mittag-Leffler condition.*

Our main interest in the Mittag-Leffler condition arises from the fact that if a direct system satisfies this condition, then the pesky $\varprojlim^1$-limit vanishes.

Lemma 52.9. *Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If this inverse system satisfies the Mittag-Leffler condition, then*

$$\varprojlim^1 D_i \ = \ 0.$$

Proof $(*)$. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups that satisfies the Mittag-Leffler condition. This means that given given any $i \in \mathbb{N}$ there exists an $n(i) \geq i$ such that for all $s \geq n(i)$ we have

$$\operatorname{im}\{g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i\} \;=\; \operatorname{im}\{g_i \circ \cdots \circ g_{n(i)-1}\colon D_{n(i)} \to D_i\}.$$

Without loss of generality we can assume that $n(i)$ is monotonously increasing in i. We have to show that the map

$$\begin{array}{rcl} \mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}} \end{array}$$

is an epimorphism.

Given $j \geq i$ we now write $f_{ji} := g_i \circ \cdots \circ g_{j-1}\colon D_j \to D_i$. First we consider $(a_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ with the property that for all $i \in \mathbb{N}$ we have $a_i \in \operatorname{im}\{f_{n(i),i}\colon D_{n(i)} \to D_i\}$. To show that $\{a_i\}_{i\in\mathbb{N}}$ lies in the image of μ it suffices to prove the following claim.

Claim. There exist $b_i \in \operatorname{im}\{f_{n(i),i}\colon D_{n(i)} \to D_i\}$ with $i \in \mathbb{N}$ such that for each i we have $a_i = b_i - g_i(b_{i+1})$.

We define the b_i inductively. We set $b_0 = a_0$. Suppose $b_0, \ldots, b_k$ are already defined with the desired properties. By our hypothesis we have $a_k \in \operatorname{im}(f_{n(k),k})$ and by our choice of b_k we have $b_k \in \operatorname{im}(f_{n(k),k})$. By the Mittag-Leffler condition and since we arranged that $n(k+1) \geq n(k)$ we have $\operatorname{im}(f_{n(k+1),k}) = \operatorname{im}(f_{n(k),k})$. Thus can we pick $c \in D_{n(k+1)}$ with $f_{n(k+1),k}(c) = a_k - b_k$. Now we put

$$b_{k+1} \;=\; -f_{n(k+1),k+1}(c).$$

Then

$$b_k - g_{k+1}(b_{k+1}) = b_k + g_{k+1}(f_{n(k+1),k+1}(c)) = b_k + (\underbrace{g_{k+1} \circ f_{n(k+1),k+1}}_{=f_{(n(k+1),k)}})(c) = b_k + (a_k - b_k) = a_k.$$

We have thus shown that $b_0, \ldots, b_{k+1}$ have the desired properties. This concludes the proof of the claim.

Now let $(b_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ be an arbitrary element. For each $i \in \mathbb{N}$ we set

$$c_i \;=\; b_i + \sum_{j=i+1}^{n(i)} f_{j,i}(b_j).$$

Claim. For each $i \in \mathbb{N}$ we have $b_i - (c_i - g_i(c_{i+1})) \in \operatorname{im}(f_{n(i),i})$.

Let $i \in \mathbb{N}$. We have

$$\begin{aligned} b_i - (c_i - g_i(c_{i+1})) \;&=\; b_i - b_i - \sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + g_i\Big(b_{i+1} + \sum_{j=i+2}^{n(i+1)} f_{j,i+1}(b_j)\Big) \\ &=\; -\sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + f_{i+1,i}(b_{i+1}) + \sum_{j=i+2}^{n(i+1)} f_{j,i}(b_j) \;=\; \sum_{j=n(i)+1}^{n(i+1)} f_{j,i}(b_j). \end{aligned}$$

But by definition of $n(i)$ we know that for each $j \in \{n(i)+1, \dots, n(i+1)\}$ we have $f_{j,i}(b_j) \in \operatorname{im}(f_{n(i),i})$. This concludes the proof of the claim.

It follows from the claim and the above discussion that $(b_i - (c_i - g_i(c_{i+1})))_{i\in\mathbb{N}}$ lies in the image of μ. Evidently $(c_i - g_i(c_{i+1}))_{i\in\mathbb{N}}$ lies in the image of μ. Since μ is a homomorphism we see that $(b_i)_{i\in\mathbb{N}}$ also lies in the image of μ. □

We continue with the following definition.

Definition. Let R be a commutative ring and let $(X_i, f_i)_{i\in\mathbb{N}}$ and $(Y_i, g_i)_{i\in\mathbb{N}}$ be inverse systems of R-modules. A *map of inverse systems* from (X_i, f_i) to (Y_i, g_i) consists of a sequence of module homomorphisms $u_i\colon X_i \to Y_i$, $i \in \mathbb{N}$ such that for each i we have $g_i \circ u_i = u_{i-1} \circ f_i$.

Remark. A map $u\colon (X_i, f_i) \to (Y_i, g_i)$ of inverse systems induces a map $u_*\colon \varprojlim X_i \to \varprojlim Y_i$ and also a map $u_*\colon \varprojlim^1 X_i \to \varprojlim^1 Y_i$ in the obvious way.[930]

Now we can formulate the following proposition.

Proposition 52.10. *Let $(\{C_i^*\}_{i\in\mathbb{N}}, \{f_i\colon C_{i+1}^* \to C_i^*\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes over $\mathbb{N}$. If for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{C_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, then there exists a natural[931] short exact sequence of the form*

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^1 H^{n-1}(C_i^*) \;\to\; H^n\big(\varprojlim_{i\in\mathbb{N}} C_i^*\big) \;\to\; \varprojlim_{i\in\mathbb{N}} H^n(C_i^*) \;\to\; 0$$

where the right-hand map is the obvious map.[932] *If R is a commutative ring and if the C_i^* are cochain complexes of R-modules, then the maps in the short exact sequence are natural R-module homomorphisms.*

Lemma 52.11. *Given any short exact sequence*

$$0 \;\to\; (X_i, f_i) \;\xrightarrow{u_i}\; (Y_i, g_i) \;\xrightarrow{v_i}\; (Z_i, h_i) \;\to\; 0$$

of inverse systems of abelian groups there exists a natural exact sequence

$$0 \to \varprojlim X_i \xrightarrow{u_*} \varprojlim Y_i \xrightarrow{v_*} \varprojlim Z_i \xrightarrow{\delta} \varprojlim{}^1 X_i \xrightarrow{u_*} \varprojlim{}^1 Y_i \xrightarrow{v_*} \varprojlim{}^1 Z_i \to 0.$$

If R is a commutative ring and if all cochain complexes and all maps are in the category of R-modules, then the maps in short exact sequence are natural R-module homomorphisms.

---

[930] Whatever obvious means.

[931] As so often it requires a few moments of pondering to figure out what "natural" is actually supposed to mean.

[932] Here it takes a little effort to figure out what the "obvious" map actually is supposed to be. By definition an inverse limit of cochain complexes comes with cochain maps $\varprojlim C_i^* \to C_i^*$. Since cohomology of cochain complexes is covariant in cochain maps we get induced maps $H^n(\varprojlim C_i) \to H^n(C_i^*)$ which then, by definition of the inverse limit, give rise to a map $H^n(\varprojlim C_i) \to \varprojlim H^n(C_i^*)$.

Proof. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \\
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0.
\end{array}$$

The horizontal sequences are evidently exact. We can view the rows as chain complexes. The desired natural exact sequence now follows from the snake lemma that we proved in Exercise 31.5. □

Now we can provide the proof of Proposition 52.10.

Proof of Proposition 52.10 (∗). Let $(\{C_i^*\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes such that for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{C_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. Given $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$ we denote by $B_i^m = \mathrm{im}\{C_i^{m-1} \to C_i^m\}$ the image of the $(m-1)$-st coboundary map and we denote by $Z_i^m = \ker\{C_i^m \to C_i^{m+1}\}$ the kernel of the m-th coboundary map. These give rise to the following two short exact sequences of inverse systems

$$\text{(i)} \qquad 0 \;\to\; B_i^m \;\to\; Z_i^m \;\to\; H^m(C_i) \;\to\; 0$$

and

$$\text{(ii)} \qquad 0 \;\to\; Z_i^m \;\to\; C_i^m \;\to\; B_i^{m+1} \;\to\; 0.$$

Let $m \in \mathbb{N}_0$. By our hypothesis the inverse systems $\{C_i^m\}_{i\in\mathbb{N}}$ satisfy the Mittag-Leffler condition. It is straightforward to see that then the inverse system $\{B_i^m\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition. It follows from Lemma 52.9 that

$$\varprojlim{}^1 B_i^{m+1} \;=\; 0 \quad \text{and} \quad \varprojlim{}^1 C_i^m \;=\; 0.$$

(Here and throughout the proof all limits are taken over $i \in \mathbb{N}$.) Using this data we obtain from Lemma 52.11, applied to the two short exact sequences (i) and (ii) of inverse systems above, the following two exact sequences

$$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim H^m(C_i) \xrightarrow{\delta} \underbrace{\varprojlim{}^1 B_i^m}_{=0} \to \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 H^m(C_i) \to 0$$

and

$$0 \to \varprojlim Z_i^m \to \varprojlim C_i^m \to \varprojlim B_i^{m+1} \xrightarrow{\delta} \varprojlim{}^1 Z_i^m \to \underbrace{\varprojlim{}^1 C_i^m}_{=0} \to \varprojlim{}^1 B_i^{m+1} \to 0$$

From these sequence we deduce that the maps

$$\text{(i)} \qquad \varprojlim{}^1 Z_i^m \;\to\; \varprojlim{}^1 H^m(C_i)$$

and

$$\text{(ii)} \qquad \varprojlim Z_i^m \;\to\; Z^m\big(\varprojlim C_i\big) := \ker\big(\varprojlim C_i^m \to \varprojlim B_i^{m+1}\big)$$

are isomorphisms and that the following sequences are exact

$$\text{(i')} \qquad 0 \;\to\; \varprojlim B_i^m \;\to\; \varprojlim Z_i^m \;\to\; \varprojlim H^m(C_i) \;\to\; 0.$$

and

$$\text{(ii')} \qquad 0 \;\to\; \underbrace{B^m\big(\varprojlim C_i\big)}_{=:\ker(\varprojlim C_i^m \to \varprojlim B_i^{m+1})} \;\to\; \varprojlim B_i^m \;\to\; \varprojlim{}^1 Z_i^{m-1} \;\to\; 0.$$

From (ii') and the isomorphism (ii) we obtain the inclusions

$$B^m\big(\varprojlim C_i\big) \;\subset\; \varprojlim B_i^m \;\subset\; \varprojlim Z_i^m \;=\; Z^m\big(\varprojlim C_i\big)$$

which gives rise to the short exact sequence[933]

$$0 \;\to\; \underbrace{\frac{\varinjlim B_i^m}{B^m(\varinjlim C_i)}}_{=\varinjlim{}^1 H^{m-1}(C_i)\ \text{by (ii') and (i)}} \;\to\; \underbrace{\frac{Z^m(\varinjlim C_i)}{B^m(\varinjlim C_i)}}_{=H^m(\varinjlim C_i)\ \text{by def.}} \;\to\; \underbrace{\frac{\varinjlim Z_i^m}{\varinjlim B_i^m}}_{=\varinjlim H^m(C_i)\ \text{by (i')}} \;\to\; 0.$$

□

52.4. **The cohomology group of an ascending sequence of topological spaces.** The following proposition can be viewed as the analogue of Proposition 34.4 for cohomology groups.

Proposition 52.12. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:*

1. *every compact subset of X is contained in one of the X_i, or*
2. *each X_i is open in X, or*
3. *X is a CW-complex and each X_i is a subcomplex.*

Then for any $n \in \mathbb{N}_0$ and any abelian group R there exists a natural short exact sequence

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^1 H^{n-1}(X_i;R) \;\xrightarrow{\tau}\; H^n(X;R) \;\to\; \varprojlim_{i\in\mathbb{N}} H^n(X_i;R) \;\to\; 0$$

where the right-hand map is induced by the restriction maps $H^n(X;R) \to H^n(X_i;R)$. If R is a commutative ring, then all maps are R-module homomorphisms.

[933] Here we use that a filtration $P \subset Q \subset R$ of three abelian groups induces a short exact sequence $0 \to Q/P \to R/P \to R/Q \to 0$.

Remark. In many situations the term $\varprojlim^1 H^{n-1}(X_i;R)$ will be zero. But it can be non-zero and one should view the proposition as a warning that in general it is not true that "limits commute with everything".

Proof. By Corollary 26.18 and Lemma 16.5 we only have to prove the proposition under the hypothesis (1). Therefore suppose that we are given a topological space X and subspaces $X_1 \subset X_2 \subset X_3 \subset \dots$ with $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that the following condition holds:

(1) every compact subset of X is contained in one of the X_i.

Furthermore let R be a commutative ring.

For $i \in \mathbb{N}$ we now consider the cochain complex $D_i^* := \mathrm{Hom}(C_*(X_i), R)$ and for $i \in \mathbb{N}$ we denote by $f_i\colon D_{i+1}^* \to D_i^*$ the map that is induced by the inclusion $X_i \to X_{i+1}$.

Claim. For each $k \in \mathbb{N}$ the inverse system of cochain groups $(\{D_i^k\}_{i\in\mathbb{N}}, \{f_i\colon D_{i+1}^k \to D_i^k\}_{i\in\mathbb{N}})$ satisfies the Mittag-Leffler condition.

Let $k \in \mathbb{N}_0$. By Lemma 52.8 it suffices to show that for every $i \in \mathbb{N}$ the map

$$D_{i+1}^k \;=\; \mathrm{Hom}(C_n(X_{i+1});R) \;\xrightarrow{f_i}\; \mathrm{Hom}(C_n(X_i);R) \;=\; D_i^k$$

is an epimorphism. Since $X_i \subset X_{i+1}$ we know that the set of singular chains in X_i is a subset of the set of singular chains in X_{i+1}. It follows from the discussion on page 1157 that the map $\mathrm{Hom}(C_n(X_{i+1});R) \to \mathrm{Hom}(C_n(X_i);R)$ is indeed an epimorphism. This concludes the proof of the claim.

Let $n \in \mathbb{N}_0$. From Proposition 52.10 we obtain the short exact sequence

$$0 \;\to\; \varprojlim{}^1 H^{n-1}(D_i^*) \;\to\; H^n\big(\varprojlim D_i^*\big) \;\to\; \varprojlim H^n(D_i^*) \;\to\; 0$$

of R-modules. Let $i \in \mathbb{N}$. We make the following two observations:

(1) By definition we have $H^k(D_i^*) = H^k(X_i;R)$ for every $k \in \mathbb{N}_0$.
(2) We have natural isomorphisms

$$\varprojlim D_i^* \;\underset{\substack{\uparrow\\ \text{by definition of the } D_i}}{=}\; \varprojlim \mathrm{Hom}(C_*(X_i);R) \;\underset{\substack{\uparrow\\ \text{Lemma 48.3}}}{=}\; \mathrm{Hom}\big(\varinjlim C_*(X_i), R\big) \;\underset{\substack{\uparrow\\ \text{by Lemma 34.5 and hypothesis (1)}}}{=}\; \mathrm{Hom}(C_*(X),R).$$

Combining all of the above we obtain our desired short exact sequence

$$0 \;\to\; \varprojlim{}^1 H^{n-1}(X_i;R) \;\to\; H^n(X;R) \;\to\; \varprojlim H^n(X_i;R) \;\to\; 0.$$

□

We provide a few examples for the above proposition.

Examples.

(1) As in Lemma 34.7 we consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X \;:=\; (\mathbb{R} \cup S^2 \times \mathbb{Z})/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. (See Figure 694 on the left.) Furthermore, given $k \in \mathbb{N}$ we consider the open subset

$$X_k \;:=\; \big(\big(-k-\tfrac{1}{2}, k+\tfrac{1}{2}\big) \cup S^2 \times \{-k,\dots,k\}\big)\big)/\sim .$$

We leave it as an exercise to verify that we can make the identifications[934]

$$H^n(X_k) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

such that for each $k \in \mathbb{N}$ the inclusion maps $X_k \to X_{k+1}$ induce the identity map on $\mathbb{Z} = H^0(X_{k+1}) \to H^0(X_k) = \mathbb{Z}$ and such that the inclusions induce the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+3} = H^2(X_{k+1}) & \to & H^2(X_k) = \mathbb{Z}^{2k+1} \\ (a_1,\dots,a_{2k+3}) & \mapsto & (a_2,\dots,a_{2k+2}) \end{array}$$

It follows from Lemma 52.8 and the above calculations that for each $n \in \mathbb{N}_0$ the direct system of cohomology groups $\{H^n(X_k)\}_{k\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$H^n(X) \;\underset{\substack{\uparrow\\ \text{Proposition 52.12 and Lemma 52.9}}}{=}\; \varprojlim H^n(X_k) \;=\; \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else}\end{cases} \;\underset{\substack{\uparrow\\ \text{see page 1243}}}{=}\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

Here recall that $\mathbb{Z}^{\mathbb{N}} = \{(x_1,x_2,\dots) \mid x_i \in \mathbb{Z}\}$. In Lemma 34.7 we saw that $H_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$. In this case we see that $H_2(X) \not\cong H^2(X)$ since the former group is countable whereas the latter group is uncountable.[935]

(2) We consider again the surface Σ_∞ of infinite genus that we had already encountered on pages 492 and 897. (See also Figure 694 on the right.) Using a variation of example (1) and the proof of Lemma 16.6 one can show that

$$H^n(\Sigma_\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=1,\\ 0, & \text{if } n\geq 2.\end{cases}$$

We leave the details as an exercise to the valiant reader.

[934] One possible approach to showing these statements is to use the Mayer–Vietoris Theorem 50.15 for Cohomology Groups.

[935] Of course we could have obtained the same observation from the Universal Coefficient Theorem 51.13 for Cohomology Groups.

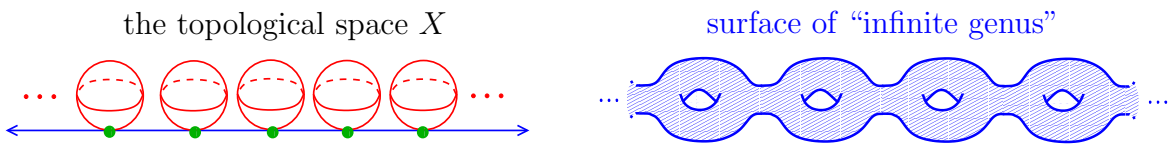


FIGURE 694.

We conclude this chapter with the following variation on Proposition 52.12. It can be viewed as the cohomological analogue of Lemma 35.1 (3).

Proposition 52.13. *Let X be a CW-complex and let $n \in \mathbb{N}$. For every $k > n$ and every abelian group G the inclusion $i\colon X^k \to X$ induces an isomorphism*

$$i^*\colon H^n(X;G) \;\to\; H^n(X^k;G).$$

Remark. We will give a very different proof of this proposition in Proposition **??**.

Proof. By Proposition 52.12 there exists a natural short exact sequence

$$0 \;\to\; \lim_{\substack{\leftarrow \\ j\in\mathbb{N}}}{}^1 H^{n-1}(X_j;G) \;\xrightarrow{\tau}\; H^n(X;G) \;\to\; \lim_{\substack{\leftarrow \\ j\in\mathbb{N}}} H^n(X_j;G) \;\to\; 0$$

where the right-hand map is induced by the restriction maps $H^n(X;G) \to H^n(X_i;G)$. It follows immediately from Lemmas 52.6 and Lemma 52.8 that is suffices to prove the following claim.

Claim. For any $j \geq k > n$ the inclusion induced map $H^n(X^j;G) \to H^n(X^k;G)$ is an isomorphism.

It follows from Lemma 35.1, the Universal Coefficient Theorem 51.13 for Cohomology Groups that for every $m \geq n$ we have $H^n(X^{m+1}, X^m; G) = 0$. Using the long exact sequence in cohomology of the pairs $(X^{m+1}, X^m), (X^{m+2}, X^m), \ldots$ one can now easily show that for any $j \geq k > n$ the inclusion induced map $H^n(X^j;G) \to H^n(X^k;G)$ is an isomorphism. This concludes the proof of the claim. □

Exercises for Chapter 52.

Exercise 52.1. Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system of abelian groups over I.

(a) Let $a_j \in X_j$ and let $a_k \in X_k$. Prove the following statement: if the images of a_j and a_k in $\varinjlim X_i$ agree, then there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.

(b) Show that if all maps $f_{ij}\colon X_i \to X_j$ are isomorphisms, then for any $j \in I$ the obvious map $X_j \to \varinjlim X_i$ is an isomorphism.

Exercise 52.2. We consider the inverse systems

$$D_*\colon\ \dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to\ 0$$

and

$$E_*\colon\ \dots \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to\ 0.$$

(a) Show that $\varprojlim D_* = 0$,

(b) show that $\varprojlim^1 D_* = 0$,

(c) show that $\varprojlim^1 E_* \neq 0$.

Exercise 52.3. Show that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $H^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $H^1(X_{i+1};\mathbb{Z}) \to H^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.
Put differently, we want topological spaces and maps such that the inverse system

$$H^1(X_4) \ \to\ H^1(X_3) \ \to\ H^1(X_2) \ \to\ H^1(X_1)$$

is isomorphic to the inverse system E_* from Exercise 52.2.

53. Cohomology with compact support and bounded cohomology

53.1. **The definition of cohomology with compact support and basic properties.** In this section we introduce "cohomology with compact support", which is an interesting variation on the usual cohomology groups. Later on it will be a key tool in proving the Poincaré Duality Theorem, which is the most important result regarding homology and cohomology groups of topological manifolds.

We start out with the following definition.

Definition. Let X be a topological space and let G be an abelian group.

(1) Given a subset $A \subset X$ we say that a cochain $\varphi \in C^i(X;G) = \mathrm{Hom}(C_i(X), G)$ *vanishes on* A if $\varphi(c) = 0$ for all singular i-simplices $c\colon \Delta^i \to X$ in A, i.e. if φ vanishes on the image of $C_i(A) \to C_i(X)$.

(2) Given a subset $B \subset X$ we say that a cochain *vanishes outside* B if the cochain vanishes on $X \setminus B$.

(3) We say that a cochain $\varphi \in C^i(X;G) = \mathrm{Hom}(C_i(X), G)$ has *compact support* if there exists a compact set $K \subset X$ such that φ vanishes outside of K. We denote the set of all i-cochains with compact support by $C^i_c(X;G)$.

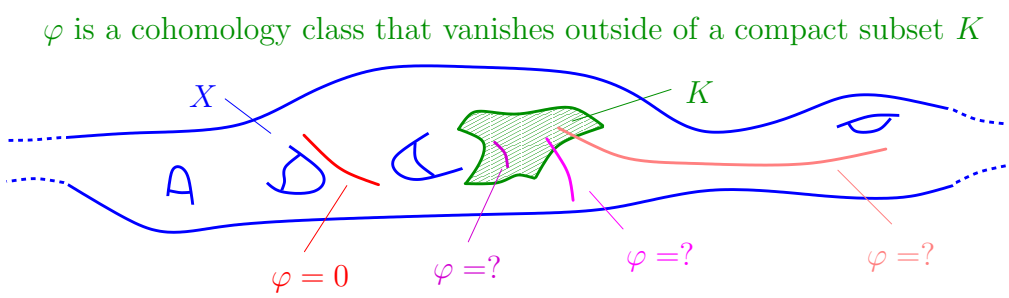


Figure 695.

Remark. Let X be a topological space and let G be an abelian group.

(1) Let A be a subset of X and let $n \in \mathbb{N}_0$. The inclusion $(X, \varnothing) \to (X, X \setminus A)$ induces an epimorphism $C_n(X) \to C_n(X, X \setminus A)$ and thus it also induces a monomorphism $C^n(X, X \setminus A; G) \to C^n(X;G)$. We have

$$\begin{aligned}\mathrm{im}(C^n(X, X\setminus A; G) \to C^n(X;G)) &= \text{all maps of the form } C_n(X) \to \overbrace{C_n(X, X\setminus A)}^{=C_n(X)/C_n(X\setminus A)} \to G \\ &= \{\varphi \in \mathrm{Hom}(C_n(X), G) \,|\, \varphi \text{ vanishes on } X \setminus A\ \} \\ &= \{\varphi \in \mathrm{Hom}(C_n(X), G) \,|\, \varphi \text{ vanishes outside } A\ \}.\end{aligned}$$

(2) On page 1165 we had made the identification

$$C^0(X;G) \;=\; \{\text{maps from } X \text{ to } G\}.$$

With this identification we can now write

$$C_c^0(X;G) = \{\text{maps } f\colon X \to G \mid \text{there exists a compact subset } K \text{ such that } f|_{X\setminus K} \equiv 0\}.$$

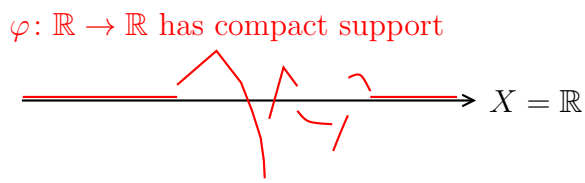


FIGURE 696.

The first question that arises is whether cochains with compact support give again rise to cohomology groups. The following lemma shows in particular that this is indeed the case.

Lemma 53.1. *Let X be a topological space and let G be an abelian group.*

(1) *A cochain $\varphi \in C^i(X;G)$ has compact support if and only if there exists a compact subset K such that φ lies in the image of the map $C^i(X, X\setminus K; G) \to C^i(X;G)$.*
(2) *The set of cochains that have compact support form a subcomplex $(C_c^*(X;G);\delta)$ of the cochain complex $(C^*(X;G);\delta)$.*
(3) *For any compact subset K of X the map $i_K\colon C^*(X, X\setminus K; G) \to C_c^*(X;G)$ from (1) is a cochain map.*

Proof. Let X be a topological space and let G be an abelian group. The first statement of the lemma follows immediately from the above remark.

We turn to the proof of the second statement.

Claim. Let $A, B \subset X$ be two subsets.

(a) If $\varphi \in C^i(X;G)$ vanishes outside A and if $\psi \in C^i(X;G)$ vanishes outside B, then for any $r, s \in \mathbb{Z}$ the linear combination $r\varphi + s\psi \in C^i(X;G)$ vanishes outside the union $A \cup B$.
(b) If $\varphi \in C^i(X;G)$ vanishes outside A, then $\delta\varphi \in C^{i+1}(X;G)$ also vanishes outside A.

The first statement is obvious. Now we turn to the proof of the second statement. Suppose we are given a cochain $\varphi \in C^i(X;G)$ that vanishes outside A. We need to show that $\delta\varphi \in C^{i+1}(X;G)$ also vanishes outside A. So let c be a singular $(i+1)$-simplex in $X \setminus A$. Then

$$(\delta\varphi)(c) \;=\; \varphi(\partial c) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \varphi(c \circ i_j^n) \;\underset{\substack{\uparrow \\ \text{the image of each } c\circ i_j^n \text{ also lies in } X\setminus A \text{ and } \varphi \text{ vanishes outside } A}}{=}\; 0.$$

This concludes the proof of the claim.

The second statement is now an immediate consequence of the above claim and the observation that given two compact subsets $K, L \subset X$ the union $K \cup L$ is again a compact subset of X.

Finally the last statement is an immediate consequence of the definitions. $\square$

Definition. Let X be a topological space and let G be an abelian group. We define the *n-th cohomology group with compact support* $H_c^n(X;G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e. we define

$$H_c^n(X;G) \;:=\; \frac{\ker\{\delta_n\colon C_c^n(X;G) \to C_c^{n+1}(X;G)\}}{\operatorname{im}\{\delta_{n-1}\colon C_c^{n-1}(X;G) \to C_c^n(X;G)\}}.$$

Remark.

(1) If X is a compact topological space, then every cochain has compact support, in particular the cohomology groups with compact support are just the usual cohomology groups.

(2) Let X be a path-connected topological space. Then

$$H_c^0(X;G) \underset{\uparrow}{=} \left\{ f\colon X \to G \,\middle|\, \begin{array}{c} f \text{ is constant and there exists a} \\ \text{compact subset } K \text{ with } \varphi|_{X\setminus K} \equiv 0 \end{array} \right\} = \begin{cases} 0, & \text{if } X \text{ non-compact,} \\ G, & \text{if } X \text{ compact.} \end{cases}$$

see proof of Proposition 48.11 and description of $C_c^0(X;G)$ on page 1256.

Example. We consider the topological space $X = \mathbb{R}$. As on page 1169 we consider the singular 0-cochain

$$\varphi_0\colon C_0(X) \;\to\; \mathbb{Z}$$

that is uniquely determined by the property that for any singular 0-simplex $x \in X$ we have

$$x \;\mapsto\; \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

It is clear that this 0-cochain does *not have* compact support. As on page 1169 we also consider the coboundary $\gamma_0 := \delta\varphi_0 \in C^1(\mathbb{R};\mathbb{Z})$. Since γ_0 is a coboundary, it is in particular a cocycle. By definition $\gamma_0 = \delta\varphi_0$ is uniquely determined by the property that for any singular 1-simplex $d\colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta\varphi_0)(d) \;=\; \varphi_0(\partial d) \;\underset{\uparrow}{=}\; \varphi_0(d(0,1)) - \varphi_0(d(1,0)).$$

see page 738

We claim that $\gamma_0 = \delta\varphi_0$ has compact support. In fact, we will now see that we can take K to be the compact set given by $K := \{0\}$. Indeed, if $d\colon \Delta^1 \to \mathbb{R} \setminus \{0\}$ is a singular 1-simplex, then it follows from the fact that Δ^1 is homeomorphic to the interval $[0,1]$ and the intermediate value theorem that either $d(0,1)$ and $d(1,0)$ both lie to the left of 0 or both lie to the right of 0. In both cases we obtain from the above formula that $\gamma_0(d) = (\delta\varphi_0)(d) = 0$. Thus we see that $\gamma_0 = \delta\varphi_0$ defines an element in $H_c^1(\mathbb{R})$.

Now we claim that $[\gamma_0] \in H^1_c(\mathbb{R};\mathbb{Z})$ is non-zero. This means that we have to show that there is no $\psi \in C^0_c(\mathbb{R};\mathbb{Z})$ with $\delta\psi = \gamma_0$. So let $\psi \in C^0_c(\mathbb{R};\mathbb{Z})$. Following the discussion on page 1256 we view ψ as a function $\mathbb{R} \to \mathbb{Z}$ which is zero outside of a compact subset of $\mathbb{R}$. This implies that there exists a $C \geq 0$ such that $\psi(x) = 0$ for all $x \notin [-C, C]$. Now we consider the singular 1-chain $d\colon \Delta^1 \to \mathbb{R}$ that is given by $d(1-t, t) = -2C + 4C \cdot t$ with $t \in [0,1]$. (See Figure 697 for an illustration.) Then we have

$$\gamma_0(d) \;=\; (\delta\varphi_0)(d) \;=\; \varphi_0(d(0,1)) - \varphi_0(d(1,0)) \;=\; \varphi_0(-2C) - \varphi_0(2C) \;=\; 0 - 1 \;=\; -1$$

but on the other hand we have

$$(\delta\psi)(d) \;=\; \psi(d(0,1)) - \psi(d(1,0)) \;=\; \psi(-2C) - \psi(2C) \;\underset{\substack{\uparrow \\ \text{since } \psi \text{ is zero outside } [-C,C]}}{=}\; 0 - 0 \;=\; 0.$$

This concludes the proof the $\gamma_0 = \delta\varphi_0$ is non-trivial in $H^1_c(\mathbb{R};\mathbb{Z})$.

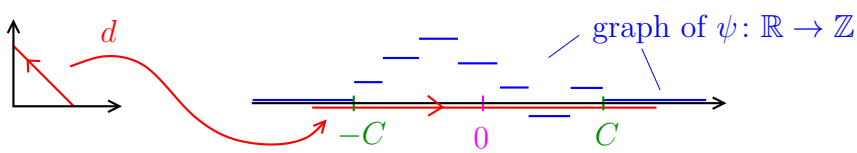


FIGURE 697.

The following lemma is an almost immediate consequence of Lemma 53.1 (1).

Lemma 53.2. *Let X be a topological space and let G be an abelian group. Given any $\varphi \in H^i_c(X;G)$ there exists a compact subset K such that φ lies in the image of the map $H^i(X, X \setminus K; G) \to H^i_c(X;G)$.*

Proof. Let $[\varphi] \in H^i_c(X;G)$. By Lemma 53.1 (1) there exists a compact subset K such that φ lies in the image of the map $C^i(X, X \setminus K; G) \to C^i_c(X;G)$. Since $C^*(X, X \setminus K; G)$ can be viewed as a subcomplex of $C^*_c(X;G)$ we see that φ is a also a cocycle in $C^*(X, X \setminus K; G)$, in particular φ lies in the image of the map $H^i(X, X \setminus K; G) \to H^i_c(X;G)$. $\square$

We conclude this section with the discussion of induced maps on cohomology groups with compact support.

If $f\colon X \to Y$ is a map between topological spaces and if G is any abelian group, then we saw that f induces a map $f^*\colon C^*(Y;G) \to C^*(X;G)$. But this map does in general *not* restrict to a map $f^*\colon C^*_c(Y;G) \to C^*_c(X;G)$. For example, if $f\colon \mathbb{R} \to \{P\}$ is the map that sends every point in $\mathbb{R}$ to the point P and if $\varphi\colon C_0(\{P\}) \to \mathbb{Z}$ is the map given by $P \mapsto 1$, then $f^*\varphi\colon C_0(\mathbb{R}) \to \mathbb{Z}$ corresponds to the map that takes the value 1 on all of $\mathbb{R}$, in particular it does not have compact support.

A moment of reflection shows, that if we want to get induced maps on cohomology groups with compact support, we have to restrict ourselves to "proper maps":

Definition. A map $f\colon X \to Y$ between topological spaces is called *proper* if the preimage of every compact subset of Y is a compact subset of X.

Examples.

(1) In Figure 698 we show graphs of functions $\mathbb{R} \to \mathbb{R}$, some of the functions are proper and some are not proper.

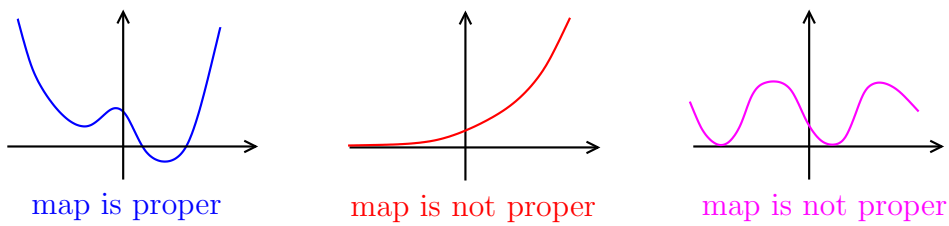


FIGURE 698.

(2) A map from a non-compact topological space to a compact topological space *cannot* be proper. In particular inclusion maps are not necessarily proper, for example the inclusion $i\colon (0,1) \to [-2,2]$ is not proper.
(3) Every homeomorphism is proper.
(4) Every map from a compact topological space to a Hausdorff space is proper. This statement follows immediately from Lemma 1.18 (1) and (2).

Remark. On page 330 we introduced the notion of a map from a compact topological manifold to some other topological manifold to be proper. As we pointed out in the last of the above examples, any such map is also proper in the above sense.

Lemma 53.3. *Let G be an abelian group.*

(1) *Let $f\colon X \to Y$ be a map between topological spaces. If f is a proper map, then the cochain map $f^*\colon C^*(Y;G) \to C^*(X;G)$ restricts to a cochain map*
$$f^*\colon C_c^*(Y;G) \quad \to \quad C_c^*(X;G).$$
In particular f^* induces a well-defined homomorphism
$$H_c^*(Y;G) \quad \to \quad H_c^*(X;G).$$
(2) *The maps $X \mapsto H_c^n(X;G)$ and $f \mapsto f^*$ define a contravariant functor*
$$\begin{array}{c}\textit{category where objects are topological spaces}\\ \textit{and morphisms are proper maps}\end{array} \quad \to \quad \textit{category of abelian groups.}$$
(3) *Homeomorphic topological spaces have isomorphic cohomology groups with compact support.*

Proof. Let $f\colon X \to Y$ be a proper map and let G be an abelian group. Let $\varphi\colon C_n(Y) \to G$ be a cocycle with compact support. We pick a compact subset K such that φ vanishes on $Y \setminus K$. Then $f^*\varphi$ vanishes on $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$, i.e. $f^*\varphi$ vanishes outside of the

subset $f^{-1}(K)$. But $f^{-1}(K)$ is compact since f is proper. This concludes the proof of the first statement. The second statement and the third statement are obvious. $\square$

53.2. **Cohomology with compact support and direct limits.** Given a topological space X we consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Furthermore, given any two compact subsets $K \subset L$ and any $i \in \mathbb{N}_0$ we get induced maps[936]

$$H^i(X, X \setminus K; G) \;\to\; H^i(X, X \setminus L; G).$$

Since cohomology groups are contravariantly functorial we see that these cohomology groups form a *direct* system over the directed set $(\mathcal{K}(X), \subset)$ and we can form the direct limit

$$\varinjlim_{K \in \mathcal{K}(X)} H^i(X, X \setminus K; G).$$

Recall that given any compact subset $K \subset X$ we have by Lemma 53.1 (3) a natural map $i_{K*}\colon H^i(X, X \setminus K; G) \to H^i_c(X; G)$. Given compact subsets $L \subset M$ these maps give rise to the following commutative diagram:

$$\begin{tikzcd}
H^i(X, X \setminus M; G) \arrow[dr, "\psi_M"'] \arrow[drr, bend left, "i_{M*}"] & & \\
& \varinjlim\limits_{K \in \mathcal{K}(X)} H^i(X, X \setminus K; G) \arrow[r, dashed, "\exists!", "\Phi"'] & H^i_c(X; G). \\
H^i(X, X \setminus L; G) \arrow[u] \arrow[ur, "\psi_L"] \arrow[urr, bend right, "i_{L*}"'] & &
\end{tikzcd}$$

Since this diagram commutes for all compact subsets $L \subset M$ we get the induced dotted arrow by the universal property of the direct limit.

The following proposition says that the above dotted arrow is in fact an isomorphism. This gives in particular an often convenient reformulation of the concept of cohomology with compact support.

Proposition 53.4. *Let X be a topological space and let G be an abelian group. Then the above map*

$$\Phi\colon \varinjlim_{K \in \mathcal{K}(X)} H^i(X, X \setminus K; G) \;\to\; H^i_c(X; G)$$

[936]The most confusing part about cohomology is to figure out which way the maps go. In this case $K \subset L$ implies $X \setminus L \subset X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \to (X, X \setminus K)$ given by inclusion, which by the discussion on page 1172 implies that we do indeed get an induced map $H^i(X, X \setminus K; G) \to H^i(X, X \setminus L; G)$.

defines a natural isomorphism.[937]

Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that Φ is surjective. So let $\varphi \in H^i_c(X;G)$. By Lemma 53.2 there exists a compact subset L such that φ lies in the image of $H^i(X, X \setminus L; G) \to H^i(X;G)$. It follows immediately from the above commutative diagram that φ also lies in the image of $\Phi\colon \varinjlim H^i(X, X \setminus K; G) \to H^i_c(X;G)$.

Now we show that the map Φ is injective. In the subsequent discussion, given a subset A of X, we identify the image of the monomorphism $C^j(X, A; G) \to C^j(X;G)$ with $C^j(X, A; G)$, i.e. we view each $C^j(X, A)$ as a subgroup of $C^j(X)$.

Now suppose we are given an element c in the kernel of Φ. By Lemma 52.2 (1) the element c lies in the image of some $H^i(X, X \setminus L; G)$. We pick a representative $\varphi \in C^i(X, X \setminus L; G)$. Since φ represents the zero class in $H^i_c(X;G)$ there exists a compactly supported $(i-1)$-cocycle η with $\delta\eta = \varphi$. We pick a compact subset M of X such that η vanishes on $X \setminus M$, i.e. according to the above notation and the discussion on page 1255 we have $\eta \in C^{i-1}(X, X \setminus M)$. We consider the following commutative diagram:

$$\begin{tikzcd}
C^{i-1}(X;G) \arrow[rrr, "\eta\to\varphi", "\delta"'] & & & C^i(X;G) \\
& C^{i-1}(X, X\setminus(M\cup L);G) \arrow[ul, hook] \arrow[r, "\delta"] & C^i(X, X\setminus(M\cup L);G) \arrow[ur, hook] & \\
C^{i-1}(X, X\setminus M;G) \arrow[uu, hook] \arrow[ur, hook] & & & C^i(X, X\setminus L;G). \arrow[uu, hook] \arrow[ul, hook]
\end{tikzcd}$$

It follows immediately from the diagram that in $C^i(X, X\setminus(M\cup L);G)$ we have $\delta\eta = \varphi$, which shows that $[\varphi]$ represents the zero class in $H^i(X, X\setminus(M\cup L);G)$. It follows from Lemma 52.2 (2) or alternatively from the commutative diagram from page 1260 that $\psi_L([\varphi]) = c$ is zero. $\square$

[937]Here the term "natural" refers to the fact that both sides define a contravariant functor from the category where objects are topological spaces and morphisms are proper maps to the category of abelian groups.

Example. Now we turn to a key example, namely for each $k \in \mathbb{N}_0$ we consider the cohomology with compact support $H^k_c(\mathbb{R}^n;\mathbb{Z})$ of $\mathbb{R}^n$. We have

$$\begin{array}{ccccccc}
 & & & & & & \vdots \\
 & \text{Proposition 53.4} & & & \text{Lemma 52.3} & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_2(0);\mathbb{Z}) \\
 & \downarrow & & & \downarrow & & \uparrow \\
H^k_c(\mathbb{R}^n;\mathbb{Z}) & = & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(\mathbb{R}^n)}} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K;\mathbb{Z}) & = & \lim\limits_{\substack{\longrightarrow \\ s\in\mathbb{N}}} & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_1(0);\mathbb{Z}) \\
 & & & & & & \uparrow \\
 & & & & & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0);\mathbb{Z}) \\
 & \cong & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0);\mathbb{Z}) & & \cong & & \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\
 & \uparrow & & & \uparrow & &
\end{array}$$

by Lemma 31.16 (4) all vertical maps in the direct system are isomorphisms, hence we can apply Lemma 52.2

same argument as in the proof of Lemma 45.16

We draw three interesting conclusions from this example:

(1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
(2) The example also shows that cohomology with compact support, in contrast to basically all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.
(3) Finally we consider the case $n = 1$ in more detail. We consider again the singular 1-cochain $\gamma_0 := \delta\varphi_0$ from page 1169. On pages 1176 and 1257 we saw that γ_0 defines an element in each $H^1(\mathbb{R}, \mathbb{R} \setminus [-s, s];\mathbb{Z})$ and an element in $H^1_c(\mathbb{R};\mathbb{Z})$. Furthermore on page 1176 we saw that for each $s \in N$ the class $[\gamma_0] \in H^1(\mathbb{R}, \mathbb{R} \setminus [-s, s];\mathbb{Z})$ is a generator. The above argument now implies that $[\gamma_0] \in H^1_c(\mathbb{R};\mathbb{Z})$ is also a generator.

Example. In Exercise 53.2 we will use cohomology with compact support to show that the topological spaces

$$X \ := \ \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y \ := \ \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in Figure 699 are not homeomorphic.

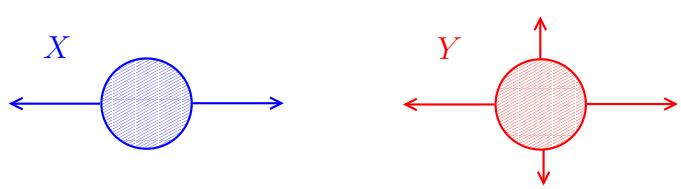


FIGURE 699.

In the previous examples cohomology with compact support turns out to be "bigger" than ordinary cohomology. On the other hand, Let us once again consider the topological space

$$X \;:=\; (\mathbb{R} \cup S^2 \times \mathbb{Z})/\sim$$

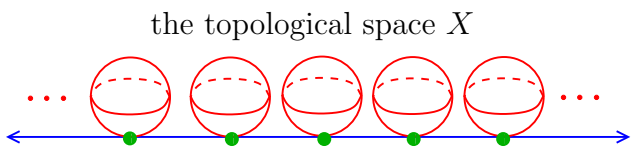
where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. (See Figure 53.2 for an illustration.) In Exercise 53.5 that

$$H^2_c(X) \;\cong\; \mathbb{Z}^{(\mathbb{N})},$$

in particular $H^2_c(X)$ is countable, whereas on page 1252 we saw that

$$H^2(X) \;\cong\; \mathbb{Z}^{\mathbb{N}}$$

is uncountable.



$H^2(X) \cong \mathbb{Z}^{\mathbb{N}}$ but $H^2_c(X) \cong \mathbb{Z}^{(\mathbb{N})}$

FIGURE 700.

In the following we will see the rather surprising fact that under fairly mild hypotheses an inclusion $U \to V$ of topological spaces induces for every $n \in \mathbb{N}_0$ a map $H^n_c(U) \to H^n_c(V)$ which "goes the same way".

We recall that given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Suppose U is an open subset of a topological space V and suppose that V is Hausdorff. We denote by $i_{UV*} = i_* \colon H^n_c(U) \to H^n_c(V)$ the composition of the following maps

$$H^n_c(U) \;\overset{\text{Proposition 53.4}}{=}\; \varinjlim_{K \in \mathcal{K}(U)} H^n(U, U \setminus K) \;\underset{\substack{\text{by the Excision Theorem 45.8 (2) the}\\ \text{map } H^n(V,V\setminus K) \to H^n(U,U\setminus K) \text{ is}\\ \text{an isomorphism for every compact subset } K \text{ of } U}}{\overset{\cong}{\longleftarrow}} \varinjlim_{K \in \mathcal{K}(U)} H^n(V, V \setminus K) \;\underset{\substack{\text{given by Lemma 52.3 (1) and the fact}\\ \text{that by Lemma 1.37 we have } \mathcal{K}(U) \subset \mathcal{K}(V)}}{\longrightarrow} \varinjlim_{K \in \mathcal{K}(V)} H^n(V, V \setminus K) \;\overset{\text{Proposition 53.4}}{=}\; H^n_c(V).$$

Note that, confusingly, the induced map on cohomology with compact support "goes the same direction" as the inclusion map $i \colon U \to V$.

Lemma 53.5. *Given a topological space X that is Hausdorff the above construction defines for each $n \in \mathbb{N}_0$ a* covariant *functor*

$$\begin{array}{c}\textit{category where the objects are open subsets}\\ \textit{of } X \textit{ and where the morphisms are inclusions}\end{array} \;\to\; \begin{array}{c}\textit{category of}\\ \textit{abelian groups.}\end{array}$$

Proof $(*)$. Let X be a topological space that is Hausdorff and let $U \subset V \subset W$ be open subsets. Let $n \in \mathbb{N}_0$. We have to show that $i_{UW*} = i_{VW*} \circ i_{UV*} \colon H^n_c(U) \to H^n_c(W)$. We consider the following diagram:

$$\begin{tikzcd}
\lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(U)}} H^n(U, U\setminus K) \arrow[rr, bend left, "i_{UV*}"] \arrow[drrr, bend right, "i_{UW*}"'] & & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(U)}} H^n(V, V\setminus K) \arrow[ll, "\cong"'] \arrow[r] & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(V)}} H^n(V, V\setminus K) \arrow[dd, bend left, "i_{VW*}"] \\
& & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(U)}} H^n(W, W\setminus K) \arrow[ull, "\cong"] \arrow[u, "\cong"] \arrow[r] \arrow[dr] & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(V)}} H^n(W, W\setminus K) \arrow[u, "\cong"'] \arrow[d] \\
& & & \lim\limits_{\substack{\longrightarrow \\ K\in\mathcal{K}(W)}} H^n(W, W\setminus K).
\end{tikzcd}$$

It follows from the explicit description of the direct limits in Proposition 52.1 and the uniqueness statement in the universal property of the direct limits that the two smaller triangles and the rectangle commute. Hence the diagram commutes and we get the desired statement. □

We conclude this section with the following proposition.

Proposition 53.6. *Let X be a topological space that is Hausdorff. Let $X_i, i \in \mathbb{N}$ be a sequence of subsets such that the following holds:*

1. *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$,*
2. *each X_i is open in X,*
3. *and $X = \bigcup\limits_{i\in\mathbb{N}} X_i$.*

Then for any $n \in \mathbb{N}_0$ the above inclusion induced maps $H^n_c(X_i) \to H^n_c(X)$ induce an isomorphism

$$\lim_{\longrightarrow} \underbrace{H^n_c(X_i)}_{\substack{\uparrow \\ \text{by Lemma 53.5 this is a direct system with the above inclusion induced maps}}} \xrightarrow{\cong} H^n_c(X).$$

Proof $(*)$. Using Proposition 53.4 we see that we have to show that the map

$$\Phi \colon \lim_{\substack{\longrightarrow \\ i\in\mathbb{N}}} \lim_{\substack{\longrightarrow \\ K\in\mathcal{K}(X_i)}} H^n(X_i, X_i \setminus K) \xrightarrow{\cong} \lim_{\substack{\longrightarrow \\ K\in\mathcal{K}(X)}} H^n(X, X \setminus K)$$

is an isomorphism. We need to show that Φ is injective and surjective. The key to the proof of these two statements are the following observations:

(a) it follows from (1) and (2), together with Lemma 1.38, that every compact subset L of X is already contained in some X_i,

(b) it follows from the Excision Theorem 45.8 (2) that for each compact subset $L \subset X_i$ we get an inclusion induced isomorphism $H^n(X, X \setminus L) \to H^n(X_i, X_i \setminus L)$.

Now we prove that Φ is surjective, the proof that Φ is injective is fairly similar and we leave that part of the proof to the reader.

So let c be an element of the limit on the right-hand side. By Lemma 52.2 there exists a compact subset L of X and some $\varphi \in H^n(X, X \setminus L)$ such that c is the image of φ under the map $H^n(X, X \setminus L) \to \varinjlim H^n(X, X \setminus K)$. By (a) there exists an $i \in \mathbb{N}$ with $L \subset X_i$. We consider the following commutative diagram

$$\begin{tikzcd}
\varinjlim\limits_{i\in\mathbb{N}} \varinjlim\limits_{K\in\mathcal{K}(X_i)} H^n(X_i, X_i \setminus K) \arrow[r, "\Phi"] & \varinjlim\limits_{K\in\mathcal{K}(X)} H^n(X, X \setminus K) \\
H^n(X_i, X_i \setminus L) \arrow[u, "\beta"] & H^n(X, X \setminus L). \arrow[l, "\alpha", "\cong"'] \arrow[u]
\end{tikzcd}$$

The bottom horizontal map is an isomorphism by (b). Since the diagram is commutative we now see that $\Phi(\beta(\alpha(\varphi))) = c$. $\square$

53.3. **Bounded cohomology** $(*)$**.** We had just seen that with a modest modification of the original definition of cohomology we obtain a new invariant which for non-compact topological spaces behaves quite differently from the usual cohomology groups.

In this section we will introduce another variation on cohomology, namely bounded cohomology. We will not make use of it later on, but it does play an important role in topology. A much more detail discussion of bounded cohomology is for example provided in [Lö2].

Definition. Let X be a topological space.

(1) Let $C \in \mathbb{R}$. We say that a cochain $\rho \in C_n(X;\mathbb{R}) = \mathrm{Hom}(C_n(X), \mathbb{R})$ is *C-bounded* if for all singular n-simplices $\sigma\colon \Delta^n \to X$ we have $|\rho(\sigma)| \leq C$.
(2) We say that a cochain $\rho \in C_n(X;\mathbb{R}) = \mathrm{Hom}(C_n(X), \mathbb{R})$ is *bounded* if it is C-bounded for some $C \in \mathbb{R}$.
(3) Given $i \in \mathbb{N}_0$ we write

$$C_b^n(X;\mathbb{R}) \ := \ \{\text{all bounded cochains in } C^n(X;\mathbb{R})\}.$$

Examples.

(1) Let X be a topological space. The 0-cochain that assigns to each point $x \in X$ the same value $c \in \mathbb{R}$ is bounded.
(2) The singular 1-cochain $\theta_{\mathbb{R}} \in C^1(S^1;\mathbb{R})$ defined on page 1166 is *not* bounded. Indeed, given $n \in \mathbb{N}$ we denote by $\mu_n\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t, t) \mapsto e^{2\pi i n t}$. It follows easily from the definition of $\theta_{\mathbb{R}}$ that $\theta_{\mathbb{R}}(\mu) = n$.

Lemma 53.7. *Let X be a topological space. The bounded cochains $C_b^*(X)$ form a subcomplex of the cochain complex $(C^*(X), \delta)$.*

Proof. Suppose that $\rho \in C^n(X;\mathbb{R})$ is a cochain that is C-bounded. For any $\varphi\colon \Delta^{n+1} \to X$ we obtain that

$$|(\delta\rho)(\varphi)| \;=\; |\rho(\partial\varphi)| \;\underset{\substack{\uparrow \\ \text{definition of } \partial \text{ and linearity of } \rho}}{=}\; \Big|\sum_{j=0}^{n+1}(-1)^j\cdot\rho(\varphi\circ i_j^{n+1})\Big| \;\underset{\substack{\uparrow \\ \text{triangle inequality}}}{\le}\; \sum_{j=0}^{n+1}|\rho(\varphi\circ i_j^{n+1})| \;\underset{\substack{\uparrow \\ \text{since } \rho \text{ is } C\text{-bounded}}}{\le}\; (n+2)\cdot C.$$

This shows that $\delta\rho$ is $C\cdot(n+2)$-bounded, in particular it is bounded. This shows that bounded cochains $C_b^*(X)$ form a subcomplex of the cochain complex $(C^*(X),\delta)$. □

This lemma leads us straight to the following definition.

Definition. Let X be a topological space X and let $n \in \mathbb{N}_0$. The *n-th bounded cohomology group of X* is defined as

$$H_b^n(X;\mathbb{R}) \;:=\; \frac{\ker\{\delta_n\colon C_b^n(X;\mathbb{R}) \to C_b^{n+1}(X;\mathbb{R})\}}{\operatorname{im}\{\delta_{n-1}\colon C_b^{n-1}(X;\mathbb{R}) \to C_b^n(X;\mathbb{R})\}}.$$

Examples.

(1) The proof of Proposition 48.11 can be used, without any changes, to show that 0-th bounded cohomology of a path-connected topological space agrees with 0-th cohomology with real coefficients.[938]
(2) Every cochain on a topological space with finitely many points is bounded. Thus it follows from Lemma 48.12 that for a topological space $\{x_0\}$ consisting of a single point x_0 we have

$$H_b^n(\{x_0\};\mathbb{R}) \;=\; H^n(\{x_0\};\mathbb{R}) \;=\; \begin{cases} 0, & \text{if } n \ge 1, \\ \mathbb{R}, & \text{if } n = 0. \end{cases}$$

Proposition 53.8.

(1) *Let $f\colon X \to Y$ be a map between topological spaces. The map $f^*\colon C^*(Y;\mathbb{R}) \to C^*(X;\mathbb{R})$ on singular cochains restricts to a cochain map $f^*\colon C_b^*(Y;\mathbb{R}) \to C_b^*(X;\mathbb{R})$. In particular the map f^* induces a map*

$$f^*\colon H_b^n(Y;\mathbb{R}) \;\to\; H_b^n(X;\mathbb{R}).$$

(2) *For each $n \in \mathbb{N}_0$ the maps $X \mapsto H_b^n(X;\mathbb{R})$ and $f \mapsto f^*$ define a contravariant functor from the category of topological spaces to the category of real vector spaces.*
(3) *Homotopic maps induce the same maps on bounded cohomology.*
(4) *If $f\colon X \to Y$ is a homotopy equivalence, then the induced map*

$$f^*\colon H_b^n(Y;\mathbb{R}) \;\xrightarrow{\cong}\; H_b^n(X;\mathbb{R})$$

is in fact an isomorphism.

Proof.

[938] Is this statement also true for topological spaces that are not path-connected?

(1) It is obvious that the pull-back of a bounded cohomology class is again bounded. The first statement is an immediate consequence of this observation.
(2) This statement is obvious.
(3) Let $f, g\colon X \to Y$ be two maps between topological spaces X and Y. Furthermore let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. As on page 758 we define

$$P_n\colon C_n(X) \;\to\; C_{n+1}(Y)$$

to be the map that is given by

$$(\sigma\colon \Delta^n \to X) \;\mapsto\; \sum_{j=0}^{n} (-1)^j \cdot F \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]}.$$

In the proof of Proposition 30.5 we saw that the maps P_n define a chain homotopy between f_* and g_*. It follows from this explicit description of P_n that for any cochain $\varphi \in C^{n+1}(Y;\mathbb{R})$ that is C-bounded the chain $P^*\varphi \in C^n(X;\mathbb{R})$ is $(n+1)\cdot C$-bounded, the map $P^*\colon C^{n+1}(Y;\mathbb{R}) \to C^n(X;\mathbb{R})$ restricts to a map $C_b^{n+1}(Y;\mathbb{R}) \to C_b^n(X;\mathbb{R})$. As in the proof of Lemma 48.8 (3) one sees easily that the maps P^* form a cochain homotopy between $f^*, g^*\colon C_b^*(Y;\mathbb{R}) \to C_b^*(X;\mathbb{R})$. The third statement now follows from Lemma 48.5.
(4) This statement follows immediately from (2) and (3). □

So far it seems like bounded cohomology might be quite similar to usual cohomology with real coefficients. It turns out that this guess is dramatically misguided.

For example the following theorem was proved by Mikhail Gromov [Grom2, p. 40], see also [Iv, Theorem 4.3].[939]

Theorem 53.9. *Let $f\colon X \to Y$ be a map between two connected CW-complexes. If the induced map $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an epimorphism of fundamental groups such that the kernel* $\ker\{f_*\colon \pi_1(X) \to \pi_1(Y)\}$ *is an amenable group, then for any $n \in \mathbb{N}_0$ the induced map*

$$f^*\colon H_b^n(Y;\mathbb{R}) \;\xrightarrow{\cong}\; H_b^n(X;\mathbb{R})$$

is an isomorphism.

Remark. In the formulation of Theorem 53.9 we used the notion of an "amenable group". We will not give the definition of an "amenable group", instead we refer to [Pi] or [CSC, Section 4] for the definition. What matters to us are the following two facts:

(1) Groups that admit a finite index subgroup that is solvable are amenable, see [Pi, Propositions 13.3 and 13.4]. In particular finite groups and solvable groups are amenable.
(2) Groups that contain a free group on two generators are not amenable, see [Pi, Proposition 14.1].

[939]Mikhail Gromov (*1943) is a Russian-French mathematician who is easily one of the most influential geometers alive. He won the Abel prize in 2009.

Now we can formulate the following rather surprising corollary.

Corollary 53.10. *Let X be a connected CW-complex such that $\pi_1(X)$ is amenable, e.g. such that $\pi_1(X)$ is solvable. Then $H_b^n(X;\mathbb{R}) = 0$ for any $n \geq 1$.*

Example. If follows from Corollary 53.10 that the higher bounded cohomology groups of the n-tori $(S^1)^n$, the spheres S^n and lens spaces all vanish. In particular this new invariant is also not useful in our attempt to answer Question 7.8.

Proof. We consider the map $f\colon X \to \{x_0\}$ that is given by sending every point in X to the point x_0. Then for any $n \geq 1$ we have

$$\begin{array}{ccccc} H_b^n(X;\mathbb{R}) & \xrightarrow[\cong]{f^*} & H_b^n(\{x_0\};\mathbb{R}) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{by Theorem 53.9 since} & & \text{see above example} & \\ & \ker(f_*) = \pi_1(X) \text{ is amenable} & & & \end{array}$$

We have thus proved the desired result. □

The following vanishing result is in fact fairly easy to show, see e.g. [Frig, Chapter 2].

Lemma 53.11. *For any CW-complex X we have $H_b^1(X;\mathbb{R}) = 0$.*

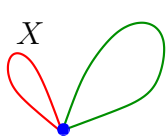
On the other hand, if the fundamental group is not amenable, then we also see some totally unexpected behavior as is shown in the following proposition.

Proposition 53.12. *Let X be the wedge of two circles. Then $H_b^2(X;\mathbb{R})$ and $H_b^3(X;\mathbb{R})$ are vector spaces of uncountable dimension.*

Proof. The statement regarding $H_b^2(X;\mathbb{R})$ is proved in [EF] and the statement regarding $H_b^3(X;\mathbb{R})$ is proved in [So, Theorem 3]. □

Somewhat shockingly the following question is still open:

Question 53.13. *Let X be the wedge of two circles. What are the bounded cohomology groups $H_b^k(X;\mathbb{R}) \neq 0$ for $k \geq 4$? Is any of the groups non-zero? Are all of them non-zero? Are all of them of uncountable dimension?*



$$\begin{aligned} \dim(H_b^1(X;\mathbb{R})) &= 0 \\ \dim(H_b^2(X;\mathbb{R})) &= \infty \\ \dim(H_b^3(X;\mathbb{R})) &= \infty \\ \dim(H_b^4(X;\mathbb{R})) &= ??? \end{aligned}$$

FIGURE 701.

Exercises for Chapter 53.

Exercise 53.1. We consider $X = \mathbb{R}$ and the singular 0-cochain

$$\varphi_0 \colon C_0(X) \ \to \ \mathbb{Z}$$

that is uniquely determined by the property that for any singular 0-simplex $x \in X = \mathbb{R}$ we have

$$x \ \mapsto \ \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$

On page 1257 we saw that $\gamma_0 = \delta\varphi_0 \colon C_1(\mathbb{R}) \to \mathbb{Z}$ defines a class in $H^1_c(\mathbb{R};\mathbb{Z})$. Prove, only using the definitions, that this class is non-zero.

Exercise 53.2. What is the cohomology with compact support of the topological spaces

$$X \ := \ \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y \ := \ \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in Figure 699.

Exercise 53.3.

(a) Give an example of a path-connected topological space X such that $H^1(X) = 0$ but such that $H^1_c(X) \cong \mathbb{Z}^{(\mathbb{N})}$. You do not have to justify why your example has the correct properties.

(b) Give an example of a path-connected topological space X such that $H^2(X) = 0$ but such that $H^2_c(X) \cong \mathbb{Z}^3$. You do not have to justify why your example has the correct properties.

Exercise 53.4. Let M be a *compact* topological manifold. We write $\overset{\circ}{M} = M \setminus \partial M$. The cohomology group $H^n_c(\overset{\circ}{M})$ is homeomorphic to some other cohomology group that might look more familiar. What is it? You do not have to justify your answer.

Exercise 53.5. Let us once again consider the topological space

$$X \ := \ (\mathbb{R} \cup S^2 \times \mathbb{Z})/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. On page 1252 we saw that $H^2(X) \cong \mathbb{Z}^{\mathbb{N}}$. Show that $H^2_c(X) \cong \mathbb{Z}^{(\mathbb{N})}$.

Exercise 53.6. Let $f, g \colon X \to Y$ be two maps between topological spaces. Show that if f and g are homotopic, then the induced maps on bounded cohomology are the same.

54. De Rham cohomology of manifolds

In this chapter we give a concise introduction to the de Rham cohomology groups of a manifold. In particular we will prove several statements for de Rham cohomology which are reminiscent of properties that we had already proved for singular cohomology. In the next chapter we will then explore the relationship between de Rham cohomology and singular cohomology.

54.1. **Alternating forms and the wedge-product.** An *alternating k-form* on a real vector space V is a map

$$\begin{array}{rcl} \omega\colon V^k & \to & \mathbb{R} \\ (v_1,\dots,v_k) & \mapsto & \omega(v_1,\dots,v_k) \end{array}$$

with the following properties:

(i) ω is linear in each argument, i.e. for all $v, v' \in V$ and $\lambda \in \mathbb{R}$ we have

$$\begin{array}{rcl} \omega(\dots, v+v', \dots) & = & \omega(\dots, v, \dots) + \omega(\dots, v', \dots) \\ \omega(\dots, \lambda v, \dots) & = & \lambda \cdot \omega(\dots, v, \dots), \end{array}$$

(ii) swapping two arguments flips the signs, i.e. for any $v, v' \in V$ we have

$$\omega(\dots, v, \dots, v', \dots) \;=\; -\omega(\dots, v', \dots, v, \dots).$$

We denote the vector space of alternating k-forms on V by $\wedge^k V^*$. A 0-form on V is by definition a real number, i.e. $\wedge^0 V^* = \mathbb{R}$.

Example.

(1) An alternating 1-form is by definition simply a linear map $V \to \mathbb{R}$. In particular $\wedge^1 V^* = V^* := \mathrm{Hom}(V, \mathbb{R})$ is the dual space of V.

(2) For $V = \mathbb{R}^n$ we denote by dx_i the 1-form which is given by

$$dx_i(v_1,\dots,v_n) \;=\; v_i.$$

Put differently, $dx_1, \dots, dx_n$ is the basis of $(\mathbb{R}^n)^*$ that is dual to the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$.

(3) If $V = \mathbb{R}^n$, then for each $\lambda \in \mathbb{R}$ the map

$$\begin{array}{rcl} (\mathbb{R}^n)^n & \to & \mathbb{R} \\ (v_1,\dots,v_n) & \mapsto & \lambda \cdot \det(v_1 \;\dots\; v_n) \end{array}$$

is an alternating n-form. It follows easily from the uniqueness of the determinant, see e.g. [Lan2, Proposition XIII.4.6] that that each alternating n-form on $\mathbb{R}^n$ is of this form, i.e. $\wedge^n(\mathbb{R}^n)^* \cong \mathbb{R}$.

Now we recall the wedge product of 1-forms. In the following let V be an n-dimensional real vector space and let $\varphi_1, \dots, \varphi_k \in V^*$ be 1-forms. We consider the map

$$\begin{aligned} \varphi_1 \wedge \cdots \wedge \varphi_k \colon V^k &\to \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \det \begin{pmatrix} \varphi_1(v_1) & \dots & \varphi_1(v_k) \\ \vdots & & \vdots \\ \varphi_k(v_1) & \dots & \varphi_k(v_k) \end{pmatrix}. \end{aligned}$$

One can easily verify that this is an alternating k-form. We refer to this alternating k-form as the *wedge product* of $\varphi_1, \dots, \varphi_k$.

Next let $\varphi_1, \dots, \varphi_n$ be a basis for V^* and let $k \in \{0, \dots, n\}$. It follows from [Le2, Proposition 14.8] or [Loe, Chapter 20.13] that the wedge products

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \quad \text{with} \quad i_1 < i_2 < \cdots < i_k$$

form a basis of $\wedge^k V^*$ and that $\wedge^k V^* = 0$ for $k > \dim(V)$.

Now we can introduce the wedge product of general alternating forms. Let $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$. Then by the above we can write

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} \cdot \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$$

and

$$\sigma = \sum_{j_1 < \cdots < j_l} b_{j_1, \dots, j_l} \cdot \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}.$$

Finally we define

$$\omega \wedge \sigma := \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_l} a_{i_1, \dots, i_k} b_{j_1, \dots, j_l} \cdot \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}.$$

One can show easily that $\omega \wedge \sigma$ is an alternating $(k+l)$-form. The following lemma summarizes some of the key properties of the wedge product of alternating forms. The lemma is proved in [Le2, Proposition 14.11], but it is also straightforward to deduce the statements of the lemma from the definitions.

Lemma 54.1. *Let V be a real vector space.*

(1) *For any $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$ we have*

$$\omega \wedge \sigma = (-1)^{kl} \cdot \sigma \wedge \omega.$$

(2) *The map*

$$\wedge^k V^* \times \wedge^l V^* \to \wedge^{k+l} V^*$$

is bilinear and distributive.

(3) *For any $\omega \in \wedge^k V^*$, $\sigma \in \wedge^l V^*$ and $\tau \in \wedge^m V^*$ we have*

$$(\omega \wedge \sigma) \wedge \tau = \omega \wedge (\sigma \wedge \tau).$$

54.2. **The de Rham cohomology of a manifold.** What are cohomology groups good for? The universal coefficient theorem 51.13 for cohomology group says that the cohomology groups do not contain more information than the homology groups. Furthermore cohomology groups have the "disadvantage" that the induced maps go the "wrong" direction which often leads to total confusion.

In this chapter we want to sketch one answer: the cohomology groups with $\mathbb{R}$-coefficients of a manifold are isomorphic to the de Rham cohomology groups which are defined using differential forms.

Before we can show that such an isomorphism exists we first need to give the definition of de Rham cohomology groups and we need to state and prove some properties of de Rham cohomology groups.

Let M be an n-dimensional manifold.

(1) A *differential form k-form on M*assigns to each point $P \in M$ an alternating k-form on the tangent space $T_P M$. Using charts one can introduce the notions of continuity and smoothness of differential k-forms. We refer to [Le2, Chapter 14], [Tu, Chapter 18.3] or [Frie2] for details.

(2) One of the key reasons for studying differential forms on manifolds is that these are the "right" objects on a manifold that can be integrated. Somewhat more precisely, if M is a compact oriented k-dimensional manifold and if ω is a continuous differential k-form on M, then one can introduce the integral

$$\int_M \omega \ \in \ \mathbb{R}.$$

We refer to [Le2, Chapter 16] or to [Tu, Chapter 22.3] for details. The integral has the following simple, but nonetheless essential property: if M is a compact 0-dimensional manifold, i.e. $M = \{P_1, \dots, P_s\}$, where all points are given the positive orientation, then the integral of a differential 0-form, i.e. of a function $f\colon M \to \mathbb{R}$, is just the sum $f(P_1) + \dots + f(P_s)$.

(3) Let $k \in \mathbb{N}_0$. We define the k-th de Rham group as the real vector space[940][941]

$$C^k_{\mathrm{dR}}(M) \ := \ \{\text{smooth differential } k\text{-forms on } M\}.$$

[940]Georges de Rham (1903-1990) was a Swiss mathematician. The subscript "dR" in our notation $C^k_{\mathrm{dR}}(M)$ stands for de Rham. We use the subscript do distinguish it from the singular cochains $C^k(M;\mathbb{R})$.

[941]In most references, see e.g. [Le2, BT, Frie2] one writes $\Omega^k(M)$ for the set of all smooth differential k-forms on M. Since in a second we will view $C^k_{\mathrm{dR}}(M) = \Omega^k(M)$ as the groups in a cochain complex we prefer the notation $C^k_{\mathrm{dR}}(M)$.

If $f\colon M \to N$ is a smooth map between manifolds, then it follows from [Le2, p. 360] that f induces a homomorphism[942]

$$\begin{array}{rcl} f^*\colon C^k_{\mathrm{dR}}(N) & \to & C^k_{\mathrm{dR}}(M) \\ \omega & \mapsto & \text{the differential } k\text{-form } f^*\omega \text{ which at a point } P \in M \text{ is given by} \\ & & \begin{array}{rcl} (T_PM)^k & \to & \mathbb{R} \\ (v_1,\dots,v_k) & \mapsto & \omega_{f(P)}(Df_P(v_1),\dots,Df_P(v_k)) \end{array} \end{array}$$

(4) We refer to [Le2, p. 363] for the definition of the differential[943]

$$d_k\colon C^k_{\mathrm{dR}}(M) \ \to\ C^{k+1}_{\mathrm{dR}}(M).$$

Often, when the subscript k is understood from the context, we just write d instead of d_k. A smooth form ω with $d\omega = 0$ is called *closed*. A smooth form ω for which there exists a smooth form μ with $d\mu = \omega$ is called *exact*.

The following proposition summarizes some of the key properties of the integral of forms and of the differential.

Proposition 54.2.

(1) *Let $f\colon M \to N$ be a diffeomorphism between compact oriented connected k-dimensional manifolds and let ω be a smooth differential k-form on N. Then*

$$\int_M f^*\omega \ = \ \epsilon \cdot \int_N \omega \qquad \textit{where } \epsilon = \begin{cases} +1, & \textit{if } f \textit{ is orientation-preserving,} \\ -1, & \textit{if } f \textit{ is orientation-reversing.} \end{cases}$$

(2) *Let M be a compact oriented k-dimensional manifold and let ω be a smooth differential k-form on M. If $M = A \cup B$ where A and B are compact k-dimensional submanifolds such that $A \cap B$ is a $(k-1)$-dimensional submanifold, then*

$$\int_M f^*\omega \ = \ \int_A f^*\omega + \int_B f^*\omega.$$

(3) *Let M be a manifold. Then for any $k \in \mathbb{N}_0$ the map $d \circ d\colon C^k_{\mathrm{dR}}(M) \to C^{k+2}_{\mathrm{dR}}(M)$ is the zero map.*

---

[942]It is clear that $f^*\omega$ is a differential k-form on M, what one needs to show is that $f^*\omega$ is in fact a smooth differential k-form.

[943]Here is a quick summary of the definition. Using a partition of unity and charts it suffices to define the differential for differential k-forms on an open subset $U \subset \mathbb{R}^n$. We consider a differential k-form

$$\omega \ = \ \sum_{i_1<\cdots<i_k} f_{i_1\dots i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $f_{i_1\dots i_k}$ are real-valued smooth functions on U. The differential of ω is then defined as the differential $(k+1)$-form

$$d\omega \ := \ \sum_{i_1<\cdots<i_k} df_{i_1\dots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(4) *Let $f\colon M \to N$ be a smooth map between two manifolds. Then for any smooth differential k-form ω on N we have*

$$d(f^*\omega) \;=\; f^*(d\omega).$$

Proof. The first statement is [Le2, Proposition 16.6]. The second statement follows fairly easily from the definitions and basic properties of the Lebesgue integral. We leave it to the reader to fill in the details. Finally the last two statements are proved in [Le2, Theorem 14.24 and Proposition 14.26]. □

The next theorem is one of the key results on differential forms. We refer to [Le2, Theorem 16.11] for a proof.

Theorem 54.3. (Stokes' Theorem) *If M is a compact oriented k-dimensional manifold with boundary ∂M and if $\omega \in C^{k-1}_{\mathrm{dR}}(M)$, then*[944]

$$\int_M d\omega \;=\; \int_{\partial M} \omega.$$

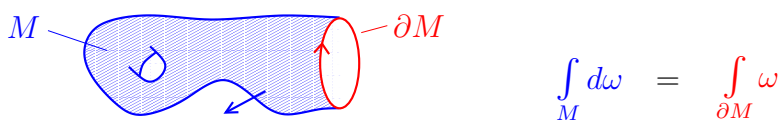


FIGURE 702.

Let M be a manifold. As we had just pointed out in Proposition 54.2 (3), the differentials $d_k\colon C^k_{\mathrm{dR}}(M) \to C^{k+1}_{\mathrm{dR}}(M)$ have the property that $d_{k+1} \circ d_k = 0$. Thus we obtain a cochain complex

$$0 \;\to\; C^0_{\mathrm{dR}}(M) \;\xrightarrow{d_0}\; C^1_{\mathrm{dR}}(M) \;\xrightarrow{d_1}\; C^2_{\mathrm{dR}}(M) \;\xrightarrow{d_2}\; \dots$$

in the sense of the discussion on page 1159. The *k–th de Rham cohomology group of M* is defined as the cohomology of this cochain complex, i.e. it is defined as

$$H^k_{\mathrm{dR}}(M) \;:=\; \frac{\ker\left\{d_k\colon C^k_{\mathrm{dR}}(M) \to C^{k+1}_{\mathrm{dR}}(M)\right\}}{\operatorname{im}\left\{d_{k-1}\colon C^{k-1}_{\mathrm{dR}}(M) \to C^k_{\mathrm{dR}}(M)\right\}}.$$

Note that $H^k_{\mathrm{dR}}(M)$ is the quotient of a real vector space by a subspace, in particular $H^k_{\mathrm{dR}}(M)$ is itself a real vector space.

The following proposition summarizes some basic properties of de Rham cohomology groups.

Proposition 54.4. *Let M be an n-dimensional manifold. Then the following hold:*

[944]Here the right-hand side is understood to be zero if $\partial M = \varnothing$.

(1) *If $f\colon M \to N$ is a smooth map to another manifold, then*

$$\begin{array}{rcl} f^*\colon H^k_{\mathrm{dR}}(N) & \to & H^k_{\mathrm{dR}}(M) \\ [\omega] & \mapsto & [f^*\omega] \end{array}$$

is a well-defined map.

(2) *If M is connected, then the map*

$$\begin{array}{rcl} \mathbb{R} & \to & H^0_{\mathrm{dR}}(M) \\ r & \mapsto & [\textit{constant function } x \mapsto r] \end{array}$$

is an isomorphism. We use this map to make the identification $H^0_{\mathrm{dR}}(M) = \mathbb{R}$.

(3) *If $f\colon M \to N$ is a smooth map between connected manifolds, then the induced map $f^*\colon H^0_{\mathrm{dR}}(N) \to H^0_{\mathrm{dR}}(M)$ is an isomorphism.*

(4) *$H^k_{\mathrm{dR}}(M) = 0$ for $k > n$.*

(5) *Suppose M consists of finitely many components $M_1, \dots, M_r$. Given $j \in \{1, \dots, r\}$ we denote by $i_j\colon M_i \to M$ the inclusion maps. Then for each $k \in \mathbb{N}_0$ the map*

$$\bigoplus_{j=1}^{r} i_j^*\colon H^k_{\mathrm{dR}}(M) \to \bigoplus_{j=1}^{r} H^k_{\mathrm{dR}}(M_j)$$

is an isomorphism.

(6) *If M has finitely many components, then $\dim_{\mathbb{R}}(H^0_{\mathrm{dR}}(M))$ equals the number of components of M.*

(7) *If M is closed and oriented, then the map*

$$\begin{array}{rcl} H^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_M \omega \end{array}$$

is well-defined. If M is non-empty, then the map is an epimorphism.

(8) *If M is closed orientable and non-empty, then $H^n_{\mathrm{dR}}(M) \neq 0$.*

Sketch of proof.

(1) The first statement is an immediate consequence of Proposition 54.2 (4) together with Lemma 48.5 (1).
(2) The second statement follows fairly easily from the definitions and elementary arguments.
(3) The third statement follows immediately from the second statement.
(4) The fourth statement is an immediate consequence of the fact that for any vector space V we have $\wedge^k V^* = 0$ for $k > \dim(V)$, see page 1271.
(5) The fifth statement follows easily from the definition.
(6) The sixth statement is an immediate consequence of (2) and (5).
(7) The seventh statement is a consequence of Stokes' Theorem and a straightforward calculation.
(8) Finally the last statement follows immediately from the seventh statement. □

Example. If $M = \{x\}$ is a manifold consisting of a single point x, then it follows immediately from Proposition 54.4 (2) and (4) that

$$H^k_{\mathrm{dR}}(\{x\}) \;=\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now we denote by $\mathcal{M}$ the category of manifolds which is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{M}) &:= \text{all manifolds,} \\ \mathrm{Mor}(M,N) &:= \text{all smooth maps from } M \text{ to } N. \end{aligned}$$

It is straightforward to verify that for any $k \in \mathbb{N}_0$ the maps

$$\begin{aligned} M &\mapsto H^k_{\mathrm{dR}}(M) \\ (f\colon M \to N) &\mapsto (f^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)) \end{aligned}$$

define a contravariant functor from the category of manifolds to the category $\mathcal{V}$ of real vector spaces.

54.3. **Partitions of unity.** In this short section we have to introduce the notion of a "partition of unity". This is a key technical tool when studying functions and forms on manifolds.

Definition. Let X be a topological space.

(1) The *support* of a function $f\colon X \to \mathbb{R}$ is defined as

$$\mathrm{supp}(f) \;:=\; \overline{\{x \in X \mid f(x) \neq 0\}}.$$

The same way we also define the support of a differential k-form if X is a manifold.

(2) A *partition of unity* on X is a family of maps[945] $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) for each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i. In particular for each i there exist only finitely many $i \in I$ such that $x \in \mathrm{supp}(f_i)$,
   (b) for every $x \in X$ we have[946]

$$\sum_{i\in I} f_i(x) \;=\; 1.$$

(3) Let $\{U_i\}_{i\in I}$ be an open covering. A *partition of unity subordinate to the open covering* is a partition of unity $\{f_j\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\mathrm{supp}(f_j) \subset U_i$.

Example. In Figure 703 we consider the topological space $X = \mathbb{R}$ with the open covering given by the intervals $U_i = (i-1, i+1)$, $i \in \mathbb{Z}$. Also we illustrate a smooth partition of unity subordinate to this open covering.

The following theorem guarantees the existence of (smooth) partitions of unity.

[945] As always, all maps are assumed to be continuous.

[946] Note that by (1) there are only finitely many non-zero summands, hence the expression on the left-hand side makes sense even if I is infinite.

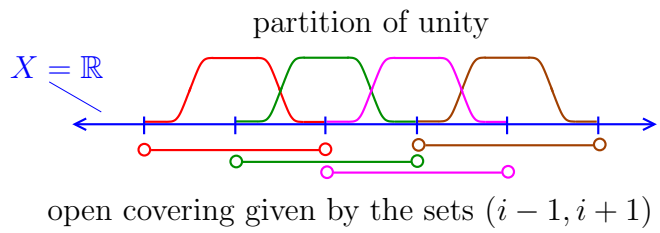


FIGURE 703.

Theorem 54.5. *Let M be a topological manifold. For any open covering of M there exists a partition of unity subordinate to the given open covering. If M is in fact a manifold, then the partition of unity can be chosen to be smooth.*

Proof. The theorem is proved for manifolds in [Tu, Theorem 13.7]. The proof for topological manifolds is basically the same. □

54.4. **The Poincaré Lemma.** In the following two sections we will see that de Rham cohomology has properties that are quite similar to singular cohomology with real coefficients.

For example, in Lemma 48.13 we saw that homotopic maps $f, g\colon X \to Y$ between topological spaces induce the same map on cohomology groups. Now we will obtain a similar result for de Rham cohomology groups, but we will have to restrict ourselves to *smooth homotopies*.

Proposition 54.6. *Let M be a manifold without boundary and let N be any manifold. Let f and g be two smooth maps between M and N. If there exists a smooth homotopy between f and g, then for any $k \in \mathbb{N}_0$ we have*

$$f^* = g^*\colon H^k_{\mathrm{dR}}(N) \;\to\; H^k_{\mathrm{dR}}(M).$$

The key to the proof of the proposition is the following lemma.

Lemma 54.7. *Let M be a manifold without boundary. We denote by $i_r\colon M \to M \times [0,1]$, $r = 0, 1$, the inclusion maps given by $i_r(z) = (z, r)$. Then for any $k \in \mathbb{N}_0$ we have*

$$i_0^* \;=\; i_1^*\colon H^k_{\mathrm{dR}}(M \times [0,1]) \;\to\; H^k_{\mathrm{dR}}(M).$$

We will first show that Lemma 54.7 implies Proposition 54.6.

Proof of Proposition 54.6 assuming Lemma 54.7. So suppose that $F\colon M \times [0,1] \to N$ is a smooth homotopy between two maps $f, g\colon M \to N$. Let $i_r\colon M \to M \times [0,1]$, $r = 0, 1$, be

the inclusion maps given by $i_r(z) = (z, r)$.[947] Then

$$\begin{array}{ccccccccc} f^* & = & (F \circ i_0)^* & = & i_0^* \circ F^* & = & i_1^* \circ F^* & = & (F \circ i_1)^* & = & g^*. \\ & & & \uparrow & & \uparrow & & \uparrow & & & \\ & & & \text{by functoriality} & & \text{by Lemma 54.7} & & \text{by functoriality} & & & \end{array}$$

We have thus shown that the two maps f^* and g^* agree. □

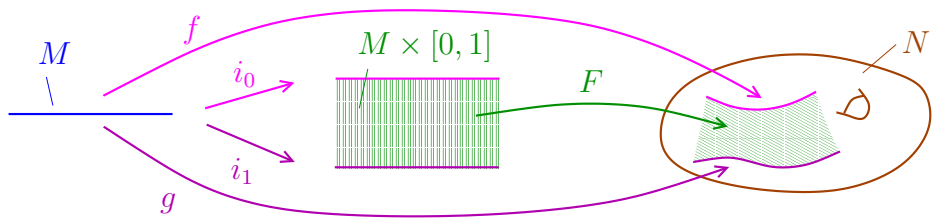


FIGURE 704. Illustration for the proof of Proposition 54.6.

Proof of Lemma 54.7. Let M be an m-dimensional manifold without boundary. By Lemma 48.5 it suffices to show that there exists a cochain homotopy

$$P_*\colon C^*_{\mathrm{dR}}(M \times [0,1]) \ \to\ C^{*-1}_{\mathrm{dR}}(M)$$

from i_0^* to i_1^*, i.e. we have to find maps $\{P_n\colon C^n_{\mathrm{dR}}(M \times [0,1]) \ \to\ C^{n-1}_{\mathrm{dR}}(M)\}_{n \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$d_{n-1} \circ P_n + P_{n+1} \circ d_n \ = \ i_0^* - i_1^*\colon C^n_{\mathrm{dR}}(M \times [0,1]) \ \to\ C^n_{\mathrm{dR}}(M).$$

For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:

$$\begin{tikzcd}
\arrow[r, "d_{n+1}"'] & C^{n+1}_{\mathrm{dR}}(M \times [0,1]) \arrow[d, "i_0^*, i_1^*"] \arrow[dr, "P_{n+1}"] & C^n_{\mathrm{dR}}(M \times [0,1]) \arrow[l, "d_n"'] \arrow[d, "i_0^*, i_1^*"] \arrow[dr, "P_n"] & C^{n-1}_{\mathrm{dR}}(M \times [0,1]) \arrow[l, "d_{n-1}"'] \arrow[d, "i_0^*, i_1^*"] & \arrow[l, "d_{n-2}"'] \\
\arrow[r, "d_{n+1}"'] & C^{n+1}_{\mathrm{dR}}(M) & C^n_{\mathrm{dR}}(M) \arrow[l, "d_n"] & C^{n-1}_{\mathrm{dR}}(M) \arrow[l, "d_{n-1}"] & \arrow[l, "d_{n-2}"]
\end{tikzcd}$$

Given any $(x,t) \in M \times [0,1]$ we denote by $h \in T_{(x,t)}(M \times [0,1])$ the tangent vector that is represented by the "horizontal" curve $s \mapsto (x, t+s)$. Given any $t \in [0,1]$ we have the obvious inclusion map $M \to M \times [0,1]$. Given any $x \in M$ this map induces an inclusion $T_xM \to T_{(x,t)}(M \times [0,1])$. We use this map to view T_xM as a subspace of $T_{(x,t)}(M \times [0,1])$. Now we can write $T_{(x,t)}(M \times [0,1]) = T_xM \oplus \mathbb{R} \cdot h$.[948] This decomposition is illustrated in Figure 705. Given $n \in \mathbb{N}_0$ we consider the map

[947] Note that the smooth structure of the manifold $M \times [0,1]$ from Proposition 56.9 has the property that i_0 and i_1 are smooth maps. This implies that the maps i_0^* and i_1^* are actually defined.

[948] Why do we have this decomposition?

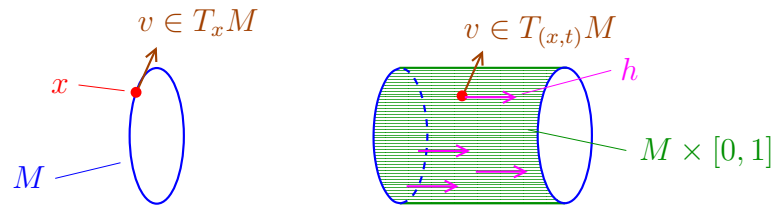


FIGURE 705.

$$\begin{aligned} P_n\colon C^n_{\mathrm{dR}}(M\times[0,1]) \;&\to\; C^{n-1}_{\mathrm{dR}}(M) \\ &\qquad \text{the } (n-1)\text{-form that at a point } x\in M \text{ is given by} \\ \omega \;&\mapsto\; \underbrace{(v_1,\ldots,v_{n-1})}_{(n\text{-}1)\text{ vectors in } T_xM} \mapsto (-1)^{n+1}\cdot\int_{t=0}^{t=1}\omega_{(x,t)}(\underbrace{v_1,\ldots,v_{n-1},h}_{n\text{ vectors in } T_{(x,t)}(M\times[0,1])})\,dt. \end{aligned}$$

It is straightforward to verify that this map is well-defined, i.e. the right-hand side is an $(n-1)$-form on M, and that this map is $\mathbb{R}$-linear. It remains to show that these maps P_n define a cochain homotopy, i.e. we have to verify that for all $n\in\mathbb{N}_0$ we have

$$(*)\qquad d_{n-1}\circ P_n + P_{n+1}\circ d_n \;=\; i_1^* - i_0^*\colon C^n_{\mathrm{dR}}(M\times[0,1]) \;\to\; C^n_{\mathrm{dR}}(M).$$

The idea of the proof is to first prove the equality $(*)$ for $M=\mathbb{R}^m$, and then to reduce the case of a general m-dimensional manifold to the case $M=\mathbb{R}^m$ via charts and a smooth partition of unity. So we start out with the following claim.

Claim. For any $n\in\mathbb{N}_0$ we have

$$d_{n-1}\circ P_n + P_{n+1}\circ d_n \;=\; i_1^* - i_0^*\colon C^n_{\mathrm{dR}}(\mathbb{R}^m\times[0,1]) \;\to\; C^n_{\mathrm{dR}}(\mathbb{R}^m).$$

We first prove the claim for $n\geq 1$. For any point in $\mathbb{R}^m\times[0,1]$ we can identify its tangent space with $\mathbb{R}^{m+1}$ and we equip its dual space with the basis given by $dx_1,\ldots,dx_m,dt$. It follows from the discussion of the basis of $\wedge^k(\mathbb{R}^{m+1})^*$ on page 1271 that for $n\geq 1$ every differential n-form on $\mathbb{R}^m\times[0,1]$ is a real linear combination of forms of the form

$$\eta \;=\; \varphi\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_n} \qquad\text{and}\qquad \mu \;=\; \psi\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_{n-1}}\wedge dt$$

where $\varphi,\psi\colon\mathbb{R}^n\times[0,1]\to\mathbb{R}$ are smooth functions. By the linearity of the terms in the claim it suffices to prove the claim for each such η and each such μ. It is straightforward to see that at each point $(x_1,\ldots,x_m)\in\mathbb{R}^m$ we have

$$P_n(\eta) \;=\; 0 \qquad\text{and}\qquad P_n(\mu) \;=\; (-1)^p\cdot\left(\int_{t=0}^{t=1}\psi(x_1,\ldots,x_m,t)\,dt\right)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_{n-1}}.$$

From this we obtain that $(d_{n-1} \circ P_n)(\eta) = 0$ and that

$$\begin{aligned}
(P_{n+1} \circ d_n)(\eta) &= P_{n+1}(d_0(\varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}) \\
&= P_{n+1}\Big(\Big(\sum_{i=1}^{p} \tfrac{\partial \varphi}{\partial x_i} \cdot dx_i + \tfrac{\partial \varphi}{\partial t} \cdot dt\Big) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) \\
&= (-1)^n \cdot P_{n+1}\Big(\tfrac{\partial \varphi}{\partial t} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m} \wedge dt\Big) \\
&= \Big(\int_0^1 \tfrac{\partial \varphi}{\partial t} dt\Big) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_m} \underset{\substack{\uparrow \\ \text{fundamental theorem of calculus}}}{=} (i_1^*)(\eta) - (i_0^*)(\eta).
\end{aligned}$$

We have thus proved the desired equality for η.

Now we turn to μ. We have $i_0^*(\mu) = i_1^*(\mu) = 0$ since $i_0^*(dt) = i_1^*(dt) = 0$. We obtain that

$$\begin{aligned}
(d_{n-1} \circ P_n)(\mu) &= (-1)^n \cdot \sum_{i=1}^{m} \Big(\int_{t=0}^{t=1} \frac{\partial \psi}{\partial x_i} dx_i\Big) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}, \\
(P_{n+1} \circ d_n)(\mu) &= (-1)^{n+1} \cdot \sum_{i=1}^{m} \Big(\int_{t=0}^{t=1} \frac{\partial \psi}{\partial x_i} dx_i\Big) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}.
\end{aligned}$$

We have thus proved the claim for μ and $n \geq 1$. It remains to prove the claim for $n = 0$. This follows immediately from the calculation that

$$(d_{-1} \circ P_0)(\varphi) + (P_1 \circ d_0)(\varphi) = 0 + P_1\Big(\sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} \cdot dx_i + \frac{\partial \varphi}{\partial t} \cdot dt\Big) = \int_{t=0}^{t=1} \frac{\partial \varphi}{\partial t} dt \underset{\substack{\uparrow \\ \text{fundamental theorem of calculus}}}{=} (i_1^*)(\varphi) - (i_0^*)(\varphi).$$

This concludes the proof of the claim.

Now we return to the case that M is any m-dimensional manifold. We pick a smooth atlas $\{\Phi_j \colon U_j \to \mathbb{R}^m\}_{j \in J}$ for the manifold M.[949] By Theorem 54.5 we can pick a smooth partition of unity $\{f_j \colon M \to \mathbb{R}\}_{j \in J}$ that is subordinate to the open covering $\{U_j\}_{j \in J}$. We consider the diagram

$$\begin{tikzcd}
C^n_{\mathrm{dR}}(M \times [0,1]) \arrow[r, "(d_{n-1} \circ P_n + P_{n+1} \circ d_n) - (i_1^* - i_0^*)"] \arrow[d, hook, "\omega \mapsto \prod f_j \omega|_{U_j}"] & C^n_{\mathrm{dR}}(M) \arrow[d, hook, "\omega \mapsto \prod f_j \omega|_{U_j}"] \\
\prod_{j \in J} C^n_{\mathrm{dR}}(U_j \times [0,1]) \arrow[r, "\prod (d_{n-1} \circ P_n + P_{n+1} \circ d_n) - (i_1^* - i_0^*)"] \arrow[d, "\cong", "\omega_j \mapsto (\Phi_j^{-1})^*(\omega_j)"'] & \prod_{j \in J} C^n_{\mathrm{dR}}(U_j) \arrow[d, "\cong", "\omega_j \mapsto (\Phi_j^{-1})^*(\omega_j)"'] \\
\prod_{j \in J} C^n_{\mathrm{dR}}(\mathbb{R}^n \times [0,1]) \arrow[r, "\prod (d_{n-1} \circ P_n + P_{n+1} \circ d_n) - (i_1^* - i_0^*)"] & \prod_{j \in J} C^n_{\mathrm{dR}}(\mathbb{R}^n).
\end{tikzcd}$$

[949]Here by a smooth atlas for M we mean an atlas $\{\Phi_j \colon U_j \to \mathbb{R}^n\}_{j \in J}$ such that all maps are smooth with respect to the given smooth structure on M. Using the fact that any open ball is diffeomorphic to $\mathbb{R}^n$ one can easily show that such an atlas exists.

One can show easily that the maps P_n and i_0^* and i_1^* are natural, i.e. they commute with induced maps given by smooth maps between manifolds. This observation, together with Proposition 54.2 (4) implies that the horizontal maps are natural, i.e. they commute with smooth maps between manifolds. Therefore it follows that the diagram commutes. The top vertical maps are monomorphisms since the f_j form a smooth partition of unity. The bottom vertical maps are isomorphisms since the Φ_j are diffeomorphisms. We had just shown in the claim that the bottom map is the zero map. Hence the top map is also the zero map. But this is precisely what we had wanted to show. □

The following result is known as the Poincaré Lemma. This humble name belies the fact that it is an extremely useful statement.

Lemma 54.8. (Poincaré Lemma) *If $U \subset \mathbb{R}^n$ is an open star-shaped set, then*

$$H^k_{\mathrm{dR}}(U) \;=\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

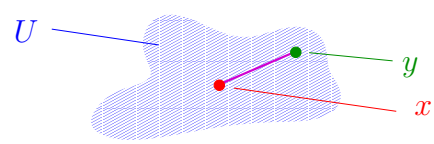


FIGURE 706.

Proof. By definition of a star-shaped subset of $\mathbb{R}^n$ there exists an $x \in U$ such that for every $y \in U$ the segment $\{xt + y(1-t) \,|\, t \in [0,1]\}$ lies in U. Let $i\colon \{x\} \to U$ be the inclusion map and let $p\colon U \to \{x\}$ the unique map from U to $\{x\}$. Evidently $p \circ i = \mathrm{id}_{\{x\}}$. Furthermore $i \circ p$ and id_U are smoothly homotopic, in fact a smooth homotopy is given by $F(y,t) = xt + y(1-t)$. It follows immediately from Proposition 54.6 and the functoriality of de Rham cohomology that $i^*\colon H^k_{\mathrm{dR}}(U) \to H^k_{\mathrm{dR}}(\{x\})$ and $p^*\colon H^k_{\mathrm{dR}}(\{x\}) \to H^k_{\mathrm{dR}}(U)$ are isomorphisms. The corollary now follows from the calculation of the de Rham cohomology groups for a point that we gave on page 1276. □

The following corollary is an immediate consequence of the Poincaré Lemma 54.8 and the functoriality of de Rham cohomology.

Corollary 54.9. *If M is an n-dimensional manifold that is diffeomorphic to an open star-shaped subset of $\mathbb{R}^n$, e.g. that is diffeomorphic to $\mathbb{R}^n$ or the open n-ball B^n, then*

$$H^k_{\mathrm{dR}}(M) \;=\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

54.5. **Applications of the Poincaré Lemma to classical vector calculus** $(*)$**.** In this section we apply the Poincaré Lemma to classical vector calculus. Let $U \subset \mathbb{R}^3$ be an open set. We denote by

$$C^\infty(U) \;:=\; \text{all smooth maps } U \to \mathbb{R} \quad \text{and} \quad \mathcal{V}(U) \;:=\; \text{all smooth maps } U \to \mathbb{R}^3$$

the set of all smooth real-valued functions respectively smooth vector fields on U. We recall that there are three classical maps relating smooth functions and smooth vector fields:

$$\begin{aligned} \operatorname{grad}\colon C^\infty(U) &\to \mathcal{V}(U) & \operatorname{curl}\colon \mathcal{V}(U) &\to \mathcal{V}(U) & \operatorname{div}\colon \mathcal{V}(U) &\to C^\infty(U) \\ f &\mapsto \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} & F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} &\mapsto \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ -\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} & F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} &\mapsto \tfrac{\partial F_x}{\partial x} + \tfrac{\partial F_y}{\partial y} + \tfrac{\partial F_z}{\partial z}. \end{aligned}$$

It is straightforward to verify that $\operatorname{curl} \circ \operatorname{grad} = 0$ and $\operatorname{div} \circ \operatorname{curl} = 0$. Put another way, the following sequence of maps is in fact a chain complex:

$$0 \;\to\; C^\infty(U) \;\xrightarrow{\text{grad}}\; \mathcal{V}(U) \;\xrightarrow{\text{curl}}\; \mathcal{V}(U) \;\xrightarrow{\text{div}}\; C^\infty(U) \;\to\; 0.$$

Now we will relate the above discussion to forms. A smooth function is by definition the same as a differential 0-form. A smooth function also gives rise to a 3-form, more precisely we have the following simple-minded isomorphism

$$\begin{aligned} \Theta\colon C^\infty(U) &\to C^3_{\mathrm{dR}}(U) \\ f &\mapsto f \cdot dx \wedge dy \wedge dz. \end{aligned}$$

More interestingly, a smooth vector field gives rise to a differential 1-form on U and also a 2-form on U. More precisely we have the following maps

$$\begin{aligned} \Phi\colon \mathcal{V}(U) &\to C^1_{\mathrm{dR}}(U) \\ &\quad \text{differential 1-form that} \\ &\quad \text{at } x \text{ is given by} \\ w &\mapsto T_xU = \mathbb{R}^3 \to \mathbb{R} \\ &\qquad\qquad v \mapsto \underbrace{v \cdot w}_{\text{scalar product}} \end{aligned} \qquad \begin{aligned} \Psi\colon \mathcal{V}(U) &\to C^2_{\mathrm{dR}}(U) \\ &\quad \text{differential 2-form that} \\ &\quad \text{at } x \text{ is given by} \\ w &\mapsto (T_xU)^2 = (\mathbb{R}^3)^2 \to \mathbb{R} \\ &\qquad\qquad (v_1, v_2) \mapsto \underbrace{(v_1 \times v_2)}_{\text{cross product}} \cdot w. \end{aligned}$$

It is straightforward to see that these maps are well-defined and that they are isomorphisms.

Summarizing we obtain the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & C^\infty(U) \arrow[r, "\text{grad}"] \arrow[d, "\cong", "\text{id}"'] & \mathcal{V}(U) \arrow[r, "\text{curl}"] \arrow[d, "\cong", "\Phi"'] & \mathcal{V}(U) \arrow[r, "\text{div}"] \arrow[d, "\cong", "\Psi"'] & C^\infty(U) \arrow[r] \arrow[d, "\cong", "\Theta"'] & 0 \\ 0 \arrow[r] & C^0_{\mathrm{dR}}(U) \arrow[r, "d"] & C^1_{\mathrm{dR}}(U) \arrow[r, "d"] & C^2_{\mathrm{dR}}(U) \arrow[r, "d"] & C^3_{\mathrm{dR}}(U) \arrow[r] & 0. \end{tikzcd}$$

A heroic calculation, see e.g. [Wein, Theorem 3.8], shows that the diagram commutes.

If U is star-shaped then we showed in the Poincaré Lemma 54.8 that the cohomology of the lower sequence vanishes for $k = 1, 2, 3$. Since the diagram commutes and since all vertical maps are isomorphisms we have thus proved the following classical theorem.

Theorem 54.10. *Let U be a star-shaped open subset of $\mathbb{R}^3$.*

(1) *Every smooth vector field on U with vanishing rotation is the gradient of a smooth function.*
(2) *Every smooth vector field on U with vanishing divergence is the rotation of a smooth vector field.*
(3) *Every smooth function on U is the divergence of a smooth vector field.*

54.6. **More properties of the de Rham cohomology groups.** In this section we will continue the theme from the Section 54.4 and we will prove two more statements regarding de Rham cohomology groups that we had already proved for singular cohomology groups.

One of the key tools in computing singular homology and cohomology groups are the Mayer–Vietoris Theorems 33.5 and 50.15. The following theorem is the de Rham cohomology version of these two theorems.

Theorem 54.11. (Mayer–Vietoris Theorem for de Rham cohomology groups *Let M be a manifold and let U, V be open subsets with $M = U \cup V$. Let $j_U \colon U \to M$, $j_V \colon V \to M$, $i_U \colon U \cap V \to U$ and $i_V \colon U \cap V \to V$ be the inclusion maps. Then the following two statements hold:*

(1) *The following sequence of cochain complexes is exact*

$$0 \;\to\; C^*_{\mathrm{dR}}(M) \;\xrightarrow{j_U^* \oplus -j_V^*}\; C^*_{\mathrm{dR}}(U) \oplus C^*_{\mathrm{dR}}(V) \;\xrightarrow{i_U^* + i_V^*}\; C^*_{\mathrm{dR}}(U \cap V) \;\to\; 0.$$

(2) *There exists a natural long exact sequence of the form*

$$\cdots \to H^k_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus -j_V^*} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i_U^* + i_V^*} H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{k+1}_{\mathrm{dR}}(M) \to \dots$$

In the proof of Theorem 54.11 we will require the following lemma.

Lemma 54.12. *Let W be a manifold, let $U \subset W$ be an open subset and let φ be a smooth form on U. We define*

$$\epsilon(\varphi, W) \;:=\; \textit{the form on } W \textit{ that is given by } \varphi \textit{ on } U \textit{ and that is zero on } W \setminus U.$$

If $\operatorname{supp}(\varphi) \subset U$, *then $\epsilon(\varphi, M)$ is a smooth form on W.*

FIGURE 707.

Proof. To show that a form μ on a manifold W is smooth it suffices to show that there exists an open covering $\{W_i\}_{i\in I}$ such that the restriction of μ to each W_i is smooth. In our case we consider the open covering of U given by U and by $W \setminus \operatorname{supp}(\varphi)$. Note that this is indeed a covering since $\operatorname{supp}(\varphi) \subset U$. The restriction of the form $\epsilon(\varphi, W)$ to U equals φ, hence it is smooth. The restriction of $\epsilon(\varphi, W)$ to $W \setminus \operatorname{supp}(\varphi)$ is also smooth since it is by definition identically zero on $W \setminus \operatorname{supp}(\varphi)$. □

Now we can provide the proof of Theorem 54.11.

Proof of Theorem 54.11. Not surprisingly we first prove statement (1). Since $M = U \cup V$ it is clear that the left-hand map $j_U^* \oplus -j_V^*$ is a monomorphism. It is obvious that the composition of the two maps is zero and that the image of the left-hand map $j_U^* \oplus -j_V^*$ is the kernel of $i_U^* + i_V^*$. It remains to prove the following claim.

Claim. The map $i_U^* + i_V^* \colon C^*_{\mathrm{dR}}(U) \oplus C^*_{\mathrm{dR}}(V) \to C^*_{\mathrm{dR}}(U \cap V)$ is in fact an epimorphism.

Let $\eta \in C^*_{\mathrm{dR}}(U \cap V)$. By Theorem 54.5 there exists a smooth partition of unity φ_U, φ_V that is subordinate to the open covering of M given by U and V. We refer to Figure 708 for an illustration.

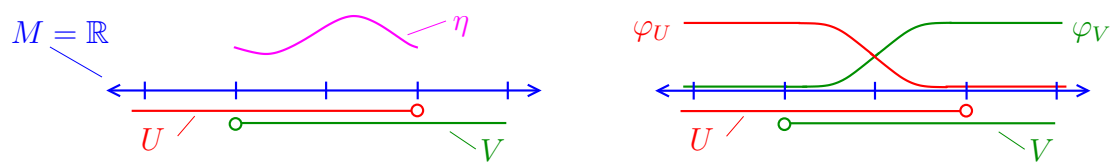


FIGURE 708.

We consider the forms $\epsilon(\eta\cdot\varphi_V, U)$ on U and $\epsilon(\eta\cdot\varphi_U, V)$. Both are smooth by Lemma 54.12. Now we have

$$(i_U^*+i_V^*)(\epsilon(\eta\cdot\varphi_V, U)+\epsilon(\eta\cdot\varphi_U, V)) \underset{\substack{\uparrow \\ \text{by definition of } i_U^* \text{ and } i_V^*}}{=} \epsilon(\eta\cdot\varphi_V, U)|_{U\cap V}+\epsilon(\eta\cdot\varphi_U, V)|_{U\cap V} \underset{\substack{\uparrow \\ \text{by definition of } \varphi_U \text{ and } \varphi_V}}{=} \eta\cdot\varphi_V + \eta\cdot\varphi_U \underset{\substack{\uparrow \\ \text{since } \varphi_U+\varphi_V \equiv 1}}{=} \eta.$$

This concludes the proof of the claim and thus also of statement (1).

The existence and naturality of the long exact sequence of statement (2) is an immediate consequence of statement (1) together with Proposition 48.6. □

Using the Mayer–Vietoris Theorem 54.11 for de Rham cohomology groups and some of our earlier results one can now compute many de Rham cohomology groups. The following lemma gives a sample calculation.

Lemma 54.13. *For any $n \geq 1$ we have*

$$H^k_{\mathrm{dR}}(S^n) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By induction we will prove the following statement: for any $n \geq 1$ we have

$$H^k_{\mathrm{dR}}(S^n) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{R}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 0$ the statement holds by Proposition 54.4 (5) and the calculation of de Rham cohomology for a point that we gave on page 1276.

Given S^n with $n \geq 1$ we write

$$U := \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \neq -1\} \quad \text{and } V = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \neq +1\}.$$

The stereographic projection that we defined in Lemma 1.41 gives diffeomorphisms from U and V to $\mathbb{R}^n$. We refer to Figure 709 for an illustration. In particular we know from Corollary 54.9 that

$$H^k_{\mathrm{dR}}(U) \;\cong\; H^k_{\mathrm{dR}}(V) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Using Proposition 54.6 it is also fairly straightforward to show that the inclusion map

$$\begin{aligned} \iota\colon S^{n-1} &\to U \cap V \\ P &\mapsto (P, 0) \end{aligned}$$

induces an isomorphism of de Rham cohomology groups.

Now we perform the induction step from $n = 0$ to $n = 1$. We consider the long exact Mayer–Vietoris sequence coming from Theorem 54.11 that corresponds to the decomposition $S^1 = U \cup V$:

$$0 \to \underbrace{H^0_{\mathrm{dR}}(S^1)}_{\cong\mathbb{R}} \to \underbrace{H^0_{\mathrm{dR}}(U)}_{\cong\mathbb{R}} \oplus \underbrace{H^0_{\mathrm{dR}}(V)}_{\cong\mathbb{R}} \to \underbrace{H^0_{\mathrm{dR}}(U \cap V)}_{\cong\mathbb{R}^2} \xrightarrow{\delta} H^1_{\mathrm{dR}}(S^1) \to \underbrace{H^1_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{H^1_{\mathrm{dR}}(V)}_{=0} \to \ldots$$

We recall that in Proposition 54.4 (3) we saw that inclusion induced maps between connected manifolds induce isomorphisms on 0-th de Rham cohomology. Using this observation and using the long exact sequence it is now straightforward to show that the de Rham cohomology of S^1 is precisely as predicted.[950]

The induction step for $n \geq 2$ from $n - 1$ to n is even easier, since one does not have to worry about 0-th de Rham cohomology. More precisely, for $k \geq 1$ the above Mayer–Vietoris sequence now looks as follows:

$$\begin{tikzcd} \ldots \arrow[r] & \underbrace{H^k_{\mathrm{dR}}(S^n)}_{\cong\mathbb{R}} \arrow[r] & \underbrace{H^k_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{H^k_{\mathrm{dR}}(V)}_{=0} \arrow[r] & H^k_{\mathrm{dR}}(U \cap V) \arrow[r, "\delta"] & H^{k+1}_{\mathrm{dR}}(S^n) \arrow[r] & \underbrace{H^{k+1}_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{H^{k+1}_{\mathrm{dR}}(V)}_{=0} \arrow[r] & \ldots \\ & & & H^k_{\mathrm{dR}}(S^{n-1}). \arrow[u, "\iota_*", "\cong"'] & & & \end{tikzcd}$$

It is now straightforward to prove the induction step. □

[950] Alternatively one can also use Lemma 40.4 to deduce that $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.

FIGURE 709. Illustration for the proof of Lemma 54.13.

Using the calculation from Lemma 54.13 and the Mayer–Vietoris Theorem 54.11 for de Rham cohomology groups one can now fairly easily modify the arguments in Lemmas 33.9, 33.12 and 33.14 to compute the de Rham cohomology of the real projective plane $\mathbb{R}\mathrm{P}^2$, the torus $T = S^1 \times S^1$ and the Klein bottle K:

$$H^n_{\mathrm{dR}}(\mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 1, \\ \mathbb{R}, & \text{if } n = 0 \end{cases} \quad H^n_{\mathrm{dR}}(T) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, 2, \\ \mathbb{R}^2, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases} \quad H^n_{\mathrm{dR}}(K) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, \\ \mathbb{R}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We conclude this discussion of de Rham cohomology with the following proposition which is very similar in nature to Proposition 52.12 for singular cohomology.

Proposition 54.14. *Let M be a manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:*

(1) *for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,*
(2) *for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and*
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.

Then for every $n \in \mathbb{N}_0$ there exists a natural short exact sequence

$$0 \;\to\; \varprojlim{}^1 H^{n-1}_{\mathrm{dR}}(U_i) \;\xrightarrow{\tau}\; H^n_{\mathrm{dR}}(M) \;\to\; \varprojlim H^n_{\mathrm{dR}}(U_i) \;\to\; 0$$

where the right-hand map is induced by the obvious maps $H^n_{\mathrm{dR}}(M) \to H^n_{\mathrm{dR}}(U_i)$.

Proof. For each $k \in \mathbb{N}_0$ we consider the inverse system

$$C^k_{\mathrm{dR}}(U_1) \;\leftarrow\; C^k_{\mathrm{dR}}(U_2) \;\leftarrow\; C^k_{\mathrm{dR}}(U_3) \;\leftarrow\; \dots$$

where all the maps are induced by the inclusion, put differently, they are all restriction maps.

Claim. The inverse system satisfies the Mittag-Leffler condition from page 1246.

Given any $j \geq i$ we denote by $\rho_{ji}\colon C^k_{\mathrm{dR}}(U_j) \to C^k_{\mathrm{dR}}(U_i)$ the restriction map. Now let $i \in \mathbb{N}$. We have to show that there exists an $n \geq i$ such that for all $r \geq n$ we have

$$\operatorname{im}\left\{\rho_{ri}\colon C^k_{\mathrm{dR}}(U_r) \to C^k_{\mathrm{dR}}(U_i)\right\} \;=\; \operatorname{im}\left\{\rho_{ni}\colon C^k_{\mathrm{dR}}(U_n) \to C^k_{\mathrm{dR}}(U_i)\right\}.$$

Since $\overline{U}_i$ is compact, since the U_j are nested and since $\overline{U}_i$ is contained in the union of all U_j there exists an n such that $\overline{U}_i \subset U_n$. We refer to Figure 710 for an illustration. We

claim that n has the desired property. So let $r \geq n$. We have to show that

$$\operatorname{im}\{\rho_{ri}\colon C^k_{\mathrm{dR}}(U_r) \to C^k_{\mathrm{dR}}(U_i)\} \;=\; \operatorname{im}\{\rho_{ni}\colon C^k_{\mathrm{dR}}(U_n) \to C^k_{\mathrm{dR}}(U_i)\}.$$

Clearly we have the inclusion "$\subset$". It remains to prove the reverse inclusion. So suppose $\eta \in C^k_{\mathrm{dR}}(U_i)$ is the restriction of some $\mu \in C^k_{\mathrm{dR}}(U_n)$. Note that U_n and $M \setminus \overline{U}_i$ form an open covering for M. It follows immediately from Theorem 54.5 that there exists a smooth function $\varphi\colon M \to \mathbb{R}$ with $\varphi|_{\overline{U}_i} \equiv 1$ and such that $\operatorname{supp}(\varphi) \subset U_n$. Let $r \geq n$. We see that

$$\eta \;=\; \mu|_{U_i} \;\underset{\substack{\uparrow \\ \text{since } \varphi \equiv 1 \text{ on } U_i}}{=}\; (\varphi \cdot \mu)|_{U_i} \;=\; \overbrace{\epsilon(\varphi \cdot \mu, U_r)}^{\text{smooth by Lemma 54.12}}\;|_{U_i}.$$

We have thus shown that η is also the restriction of some smooth form on U_r. This concludes the proof of the claim.

By the claim we can apply Proposition 52.10 to the inverse system $(\{C^*_{\mathrm{dR}}(U_i)\}_{i\in\mathbb{N}}, \{\rho_{ij}\}_{i\geq j})$ of cochain complexes over $\mathbb{N}$ and we obtain the short exact sequence

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^{1} \underbrace{H^{n-1}(C^*_{\mathrm{dR}}(U_i))}_{=H^{n-1}_{\mathrm{dR}}(U_i)} \;\to\; \underbrace{H^n\Big(\overbrace{\varprojlim_{i\in\mathbb{N}} C^*_{\mathrm{dR}}(U_i)}^{=\,C^*_{\mathrm{dR}}(M),\ \text{see page 502}}\Big)}_{=H^n_{\mathrm{dR}}(M)} \;\to\; \varprojlim_{i\in\mathbb{N}} \underbrace{H^n(C^*_{\mathrm{dR}}(U_i))}_{=H^n_{\mathrm{dR}}(U_i)} \;\to\; 0.$$

We have thus obtained the desired short exact sequence. □

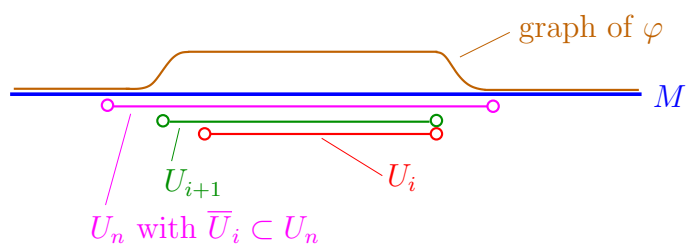


FIGURE 710. Illustration for the proof of Proposition 54.14.

Exercises for Chapter 54.

Exercise 54.1. Let Σ be the surface of genus 2 equipped with some orientation. Furthermore let $\gamma\colon S^1 \to \Sigma$ be an embedding. We write $C = \gamma(S^1)$ and we view C as an oriented curve on Σ. By the Tubular Neighborhood Theorem 10.19 we can pick a trivial tubular neighborhood $C \times [-2,2]$ for C. We denote by dt the usual differential 1-form on $[-2,2] \subset \mathbb{R}$. Let $f\colon [-2,2] \to \mathbb{R}_{\geq 0}$ be a smooth function with $\operatorname{supp}(f) \subset [-1,1]$ and

$\int_{t=0}^{t=1} f(t)\,dt = 1$. We denote by $p\colon C \times [-2,2] \to [-2,2]$ the projection. Finally denote by ω the smooth closed differential 1-form on Σ that is given by $p^*(f \cdot dt)$ on $C \times [-2,2]$ and that is extended to all of Σ by setting it zero outside of $C \times [-2,2]$. (Some of the objects are illustrated in Figure 711.)

Given a closed oriented curve $D \subset \Sigma$ we obtain the integral $\int_D \omega$.

(a) Show that the integral is an integer.
(b) What is the geometric interpretation of this integer?

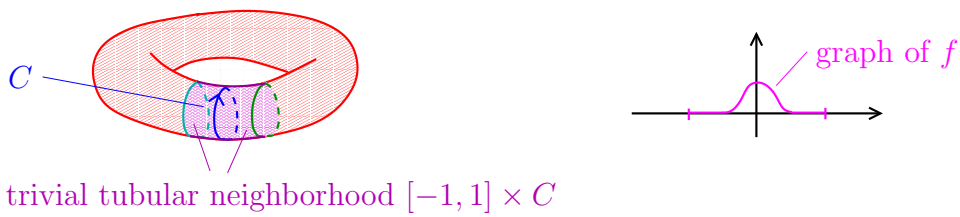


FIGURE 711.

Exercise 54.2. Let M be a manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:

(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.

In the proof of Proposition 54.14 we show that for each $k \in \mathbb{N}_0$ the inverse system

$$C^k_{\mathrm{dR}}(U_1) \;\leftarrow\; C^k_{\mathrm{dR}}(U_2) \;\leftarrow\; C^k_{\mathrm{dR}}(U_2) \;\leftarrow\; \dots$$

satisfies the Mittag-Leffler condition. Now let $i \in \mathbb{N}$. Does there exist an $n \geq i$ such that for all $r \geq n$ the map

$$C^k_{\mathrm{dR}}(U_r) \;\to\; C^k_{\mathrm{dR}}(U_i)$$

is an epimorphism?

55. The relationship between de Rham cohomology and singular cohomology

In this chapter we want to find the connection between the de Rham cohomology groups from the previous chapter and the singular cohomology groups that we defined earlier in Chapter 48.

To avoid a repetitive use of suffixes we will use the following convention.

Convention. In this chapter by a "manifold" we always mean a manifold *without boundary*, unless we say explicitly otherwise.

55.1. **Smooth cohomology theories.** To a manifold M we can associate the singular cohomology groups $H^i(M;\mathbb{R})$ and also the de Rham cohomology groups $H^i_{\mathrm{dR}}(M)$. In the last chapter we had in particular seen that these two cohomologies have many properties in common, for example they both satisfy a Mayer–Vietoris type theorem. Now we want to put both cohomologies in a common framework.

Definition. A *smooth cohomology theory* is a contravariant functor[951952]

$$\begin{array}{rcl} \mathcal{H}^*\colon \text{category of manifolds} & \to & \text{category of } \mathbb{N}_0\text{-graded real vector spaces} \\ M & \mapsto & \mathcal{H}^*(M) = \bigoplus\limits_{n\in\mathbb{N}_0} \mathcal{H}^n(M) \\ (f\colon M\to N) & \mapsto & \{f^* = \mathcal{H}^n(f)\colon \mathcal{H}^n(N)\to\mathcal{H}^n(M)\}_{n\in\mathbb{N}_0} \end{array}$$

that satisfies the following three axioms:

(1) (Smooth homotopy invariance) Let $f,g\colon M\to N$ be smooth maps between manifolds. If f and g are smoothly homotopic, then for all $n\in\mathbb{N}_0$ we have
$$\mathcal{H}^n(f) = \mathcal{H}^n(g)\colon \mathcal{H}^n(N) \;\to\; \mathcal{H}^n(M).$$

(2) (Mayer–Vietoris sequence) If M is a manifold and U and V are open subsets such that $M = U\cup V$, then there exists a natural long exact sequence of the form
$$\ldots \to \mathcal{H}^k(M) \xrightarrow{j_U^*\oplus -j_V^*} \mathcal{H}^k(U)\oplus\mathcal{H}^k(V) \xrightarrow{i_U^*+i_V^*} \mathcal{H}^k(U\cap V) \xrightarrow{\delta} \mathcal{H}^{k+1}(M) \to \ldots$$
where $j_U\colon U\to M$, $j_V\colon V\to M$, $i_U\colon U\cap V\to U$ and $i_V\colon U\cap V\to V$ denote the inclusion maps.

(3) (Limit property) Let M be a manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
   (a) for each $i\in\mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
   (b) for each $i\in\mathbb{N}$ we have $U_i\subset U_{i+1}$, and
   (c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

---

[951] Given a smooth map $f\colon M\to N$ the contravariant functor provides for each $n\in\mathbb{N}_0$ a morphism $\mathcal{H}^n(f)\colon \mathcal{H}^n(N)\to\mathcal{H}^n(M)$. As usual, when there is no danger of confusion we just write f^* instead of $\mathcal{H}^n(f)$.

[952] We refer to page 1037 for the definition of the category of $\mathbb{N}_0$-graded real vector spaces.

Then for every $n \in \mathbb{N}_0$ there exists a natural short exact sequence

$$0 \;\to\; \varprojlim{}^1 \mathcal{H}^{n-1}(U_i) \;\xrightarrow{\tau}\; \mathcal{H}^n(M) \;\to\; \varprojlim \mathcal{H}^n(U_i) \;\to\; 0$$

where the right-hand map is induced by the maps $\mathcal{H}^n(M) \to \mathcal{H}^n(U_i)$ which in turn are induced by the inclusions $U_i \to M$.

This time it is easy to give examples for our new definition.

Proposition 55.1.

(1) *Singular cohomology with real coefficients is a smooth cohomology theory.*
(2) *De Rham cohomology is a smooth cohomology theory.*

Proof.

(1) It follows from Lemma 48.13, Theorem 50.15, together with Proposition 50.12 and Proposition 52.12 (1), that singular cohomology with real coefficients is a smooth cohomology theory.
(2) It follows from Proposition 54.6, Theorem 54.11 and Proposition 54.14 that de Rham cohomology is a smooth cohomology theory. □

One might now be optimistic and hope that smooth cohomology theories are necessarily isomorphic. As it stands this thought though is too optimistic. For example other smooth cohomology theories are given by $\mathcal{H}^n(M) = 0$ for all $n \in \mathbb{N}_0$ or by $\mathcal{H}^n(M) = H^{n+5}(M;\mathbb{R})$ or by $\mathcal{H}^n(M) = H^n(M;\mathbb{R}) \oplus H^n(M;\mathbb{R})$. This shows that if we want to prove that smooth cohomology theories are isomorphic we have to add some extra hypotheses, for example we could demand that the 0-th cohomology groups of a point are isomorphic.

Notation. In the following $\star$ denotes the topological space consisting of a single point. Evidently $\star$ is a 0-dimensional manifold.

On page 251 we introduced the notion of a natural transformation between two *covariant* functors. Almost the same definition can also be applied to contravariant functors. For completeness' sake we write down the formal definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two contravariant functors. A *natural transformation between the contravariant functors F and G* assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F(Y) \arrow[r, "F^*(f)"] \arrow[d, "\Phi_Y"'] & F(X) \arrow[d, "\Phi_X"] \\ G(Y) \arrow[r, "G^*(f)"] & G(X). \end{tikzcd}$$

If all the maps Φ_X are isomorphisms we also say it is a *natural isomorphism.*

Example. Given $n \in \mathbb{N}_0$ we consider the two contravariant functors $X \mapsto H^n(X;\mathbb{Z})$ and $X \mapsto \mathrm{Hom}(H_n(X),\mathbb{Z})$ from the category of topological spaces to the category of abelian groups. It follows from the discussion on page 1220 that the evaluation homomorphisms $\mathrm{ev}\colon H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X),\mathbb{Z})$ define a natural transformation between the two contravariant functors, i.e. given any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{tikzcd} H^n(Y;\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "\mathrm{ev}"'] & H^n(X;\mathbb{Z}) \arrow[d, "\mathrm{ev}"] \\ \mathrm{Hom}(H_n(Y),\mathbb{Z}) \arrow[r, "f^*"] & \mathrm{Hom}(H_n(X),\mathbb{Z}). \end{tikzcd}$$

With this notation and this definition we can now formulate the following theorem.

Theorem 55.2. *Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories and let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation. Suppose t has the following three properties:*

(i) *the natural transformation t is compatible with the connecting homomorphism in the Mayer-Vietoris sequence, i.e. with the obvious notation we always have a commutative diagram*

$$\begin{tikzcd} \dots \arrow[r] & \mathcal{H}^n(M) \arrow[r, "j_U^*\oplus -j_V^*"] \arrow[d, "t"] & \mathcal{H}^n(U)\oplus\mathcal{H}^n(V) \arrow[r, "i_U^*+i_V^*"] \arrow[d, "t\oplus t"] & \mathcal{H}^n(U\cap V) \arrow[r, "\delta"] \arrow[d, "t"] & \mathcal{H}^{n+1}(M) \arrow[r] \arrow[d, "t"] & \dots \\ \dots \arrow[r] & \mathcal{K}^n(M) \arrow[r, "j_U^*\oplus -j_V^*"] & \mathcal{K}^n(U)\oplus\mathcal{K}^n(V) \arrow[r, "i_U^*+i_V^*"] & \mathcal{K}^n(U\cap V) \arrow[r, "\delta"] & \mathcal{K}^{n+1}(M) \arrow[r] & \dots \end{tikzcd}$$

(ii) *the natural transformation t is compatible with the map τ appearing in the limit property, i.e. with the obvious notation we always have a commutative diagram*[953]

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim^1 \mathcal{H}^{n-1}(U_i) \arrow[r, "\tau"] \arrow[d, "t"] & \mathcal{H}^n(M) \arrow[r] \arrow[d, "t"] & \varprojlim \mathcal{H}^n(U_i) \arrow[r] \arrow[d, "t"] & 0 \\ 0 \arrow[r] & \varprojlim^1 \mathcal{K}^{n-1}(U_i) \arrow[r, "\tau"] & \mathcal{K}^n(M) \arrow[r] & \varprojlim \mathcal{K}^n(U_i) \arrow[r] & 0, \end{tikzcd}$$

(iii) *for all $n \in \mathbb{N}_0$ the map $t^n(\star)\colon \mathcal{H}^n(\star) \to \mathcal{K}^n(\star)$ is an isomorphism.*

Then for every manifold M and every $n \in \mathbb{N}_0$ the map

$$t\colon \mathcal{H}^n(M) \;\to\; \mathcal{K}^n(M)$$

is an isomorphism.

Proof. Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories, which means they both satisfy

(α) smooth homotopy invariance,
(β) the Mayer-Vietoris sequence,
(γ) the limit property.

[953]Note that the left and right vertical maps are given by the remark on page 1248.

Furthermore let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation that has the above properties (i), (ii) and (iii). We need to prove the following statement:

Statement. Let $m \in \mathbb{N}_0$. For every m-dimensional manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n(M)\colon \mathcal{H}^n(M) \ \to \ \mathcal{K}^n(M)$$

is an isomorphism.

We start out with the following two observations:

(A) Suppose M is an m-dimensional manifold and U and V are open subsets with $M = U \cup V$. If all t^n are isomorphisms for U, V and $U \cap V$, then it follows from (β), (i) and the Five Lemma 31.11 that all the t^n are also isomorphisms for M.

(B) Let M be an m-dimensional manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
   (a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
   (b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
   (c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

   If all the t^n are isomorphisms for all U_i, then it follows from (γ) and the Five Lemma 31.11 that all the t^n are also isomorphisms for M.

After these preparations we will prove the statement in four surprisingly easy steps:

(1) Let M be an open convex subset of $\mathbb{R}^m$. We pick an inclusion $f\colon \star \to M$. As in the proof of the Poincaré Lemma 54.8 we see that f is a smooth homotopy equivalence. Given $n \in \mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} \mathcal{H}^n(M) & \xrightarrow{f^*} & \mathcal{H}^n(\star) \\ {\scriptstyle t}\downarrow & & \cong\downarrow{\scriptstyle t} \\ \mathcal{K}^n(M) & \xrightarrow{f^*} & \mathcal{K}^n(\star). \end{array}$$

The diagram commutes since t is a natural transformation. It follows easily from (α) that the horizontal maps are isomorphisms. By (iii) the right vertical map is also an isomorphism. It follows that the left vertical map is, as desired, an isomorphism

(2) We claim that t is an isomorphism for every subset M of $\mathbb{R}^m$ that is the union of k open convex subsets of $\mathbb{R}^m$. We prove this by induction on the number k of open convex subsets:
   (a) If $k = 1$, then M itself is an open convex subset of $\mathbb{R}^m$, hence the statement follows from (1).
   (b) Suppose the statement holds whenever M is the union of $k-1$ open convex subsets. Now let M be the union of k open convex subsets $U_1, \dots, U_k$. We write $V = U_1 \cup \dots \cup U_{k-1}$. By induction the statement holds for V and also for U_k. Furthermore since the intersection of two convex open subsets is again convex and open we see that $V \cap U_k = (U_1 \cap U_k) \cup \dots \cup (U_{k-1} \cap U_k)$ is the union

of $k-1$ open convex subsets, i.e. by induction the statement holds for $V \cap U_k$. But then it follows from (A) that the statement also holds for U.

(3) Let M be an open subset of $\mathbb{R}^m$. By Lemma 1.9 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we define $U_i := \bigcup_{j \le i} B_j$. By (2) the statement holds for each U_i, but then it follows from (B) that the statement holds for M.

(4) Finally let M be any m-dimensional manifold. We say a subset W of M is *small* if it is open, if $\overline{W}$ is compact and if W is contained in the domain of some chart for M. We make the following observations:
   (a) Since $\mathcal{H}$ and $\mathcal{K}$ are diffeomorphism invariants we obtain from (2) that the statement holds for every small subset of W (viewed as a manifold in its own right).
   (b) The same induction argument as in (2) shows that the statement holds for finite unions of small subsets (here the key observation is that the intersection of two small subsets is again small).[954]
   (c) Finally it follows easily from the fact that the manifold M is second-countable, the existence of charts around every point and Lemma 1.26 that there exist small sets $W_i, i \in \mathbb{N}$ which cover M. As in (3) we consider $U_i := \bigcup_{j \le i} W_j$. We had just seen in (b) that the statement holds for all the U_i. But then it follows from (B) that the statement holds for M. □

55.2. **Smooth singular cohomology.** One might think that it follows immediately from Proposition 55.1 and Theorem 55.2 (together with the calculations of the cohomology groups of a point in Lemma 48.12 and on page 1276) that the singular real valued cohomology and the de Rham cohomology groups of manifolds are always isomorphic.

But recall that to apply Theorem 55.2 we need to find a natural transformation

$$t\colon \text{de Rham cohomology} \quad \to \quad \text{real valued singular cohomology}$$

that has the property that

$$H^n_{\mathrm{dR}}(\star) \quad \xrightarrow{\cong} \quad H^n(\star;\mathbb{R}) \qquad \text{for all } n \in \mathbb{N}_0.$$

Here is an idea: on page 1272 we had mentioned that given a compact oriented n-dimensional manifold W and a smooth differential n-form ω on W there exists an

$$\int_W \omega \ \in \ \mathbb{R}$$

[954]Here we implicitly use the following statement: if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact. We will prove this statement in Exercise 55.2.

that has many good properties. So given some manifold M (with no restrictions on the dimension) we could try to define a map

$$\begin{array}{rcl} t\colon C^n_{\mathrm{dR}}(M) & \to & \mathrm{Hom}(C_n(M);\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{ccc} C_n(M) & \to & \mathbb{R} \\ \sum\limits_{i=1}^k a_i\cdot(f_i\colon \Delta^n\to M) & \mapsto & \sum\limits_{i=1}^k a_i\cdot\int\limits_{\Delta^n} f_i^*\omega \end{array}\right). \end{array}$$

Unfortunately, there are several problems with this "definition":

(1) The first question that arises is the following: what does it mean to integrate a form over Δ^n? After all, it is not clear in what sense Δ^n is a manifold. With some luck that should not be too much of an issue, since Δ^n is clearly pretty close to being a manifold.
(2) Let us suppose that we have resolved (1). Now let $f\colon \Delta^n \to M$ be a singular n-chain. If the map f is not smooth, then $f^*\omega$ just does not make any sense.[955]
(3) After fixing the previous issues, why should t be a cochain map?

Arguably the most serious issue is (2). The only way around it is to only work with smooth singular simplices. This immediately leads us to the following definition:

Definition. Let $n \in \mathbb{N}_0$.

(1) We write
$$\widetilde{\Delta}^n \;:=\; \Big\{(t_1,\dots,t_n)\in\mathbb{R}^n_{\geq 0}\;\Big|\;\sum_{i=1}^n t_i\in[0,1]\Big\}.$$
(2) We denote by Θ_n the homeomorphism[956]
$$\begin{array}{rcl} \Theta_n\colon \widetilde{\Delta}^n & \to & \Delta^n \\ (t_1,\dots,t_n) & \mapsto & \Big(t_1,\dots,t_n,1-\sum\limits_{i=1}^n t_i\Big). \end{array}$$

FIGURE 712.

Now let M be a manifold.

[955]Recall that if $f\colon N\to M$ is a smooth map between manifolds and ω is a smooth differential n-form on M, then $f^*\omega$ is defined to be the differential n-form that for $v_1,\dots,v_n\in T_PN$ has the value
$$(f^*\omega)_P(v_1,\dots,v_n) \;=\; \omega_{f(P)}(Df_P(v_1),\dots,Df_P(v_n)).$$
But if f is not differentiable, then there is no differential $Df_P\colon T_PN\to T_{f(P)}M$.

[956]Put differently, Θ_n is the inverse of the obvious projecton map $\Delta^n\to\widetilde{\Delta}^n$.

(3) Recall that by the definition on page 184 a map $f\colon \widetilde{\Delta}^n \to M$ is *smooth* if given any $a \in \widetilde{\Delta}^n$ there exists an open neighborhood U in $\mathbb{R}^n$ of a and a smooth map $\widetilde{f}\colon U \to M$ which coincides with f on A. We define the differential of f at $a \in \widetilde{\Delta}^n$ to be the differential of $\widetilde{f}$.[957]

(4) We say a singular n-simplex $\sigma\colon \Delta^n \to M$ is *smooth* if the map $\sigma \circ \Theta_n\colon \widetilde{\Delta}^n \to M$ is smooth. We illustrate the definition in Figure 713.

(5) We write

$$C_n^{\text{smooth}}(M) \;=\; \begin{array}{c}\text{the free abelian group generated by the}\\ \text{set of smooth singular } n\text{-simplices in } M.\end{array}$$

This is a subgroup of $C_n(M)$ and it is straightforward to verify that the boundary map $\partial_n\colon C_n(M) \to C_{n-1}(M)$ restricts to a map $\partial_n\colon C_n^{\text{smooth}}(M) \to C_{n-1}^{\text{smooth}}(M)$.

(6) Given an abelian group G we write

$$C^n_{\text{smooth}}(M;G) \;:=\; \operatorname{Hom}\big(C_n^{\text{smooth}}(M), G\big)$$

and we define $\delta_n\colon C^n_{\text{smooth}}(M;G) \to C^{n+1}_{\text{smooth}}(M;G)$ as the map $\delta_n := \partial^*_{n+1}$.

(7) We define the *n-th smooth singular cohomology group of a manifold M with G-coefficients* as

$$H^n_{\text{smooth}}(M;G) \;:=\; \frac{\ker\big\{\delta_n\colon C^n_{\text{smooth}}(M;G) \;\to\; C^{n+1}_{\text{smooth}}(M;G)\big\}}{\operatorname{im}\big\{\delta_{n-1}\colon C^{n-1}_{\text{smooth}}(M;G) \;\to\; C^n_{\text{smooth}}(M;G)\big\}}.$$

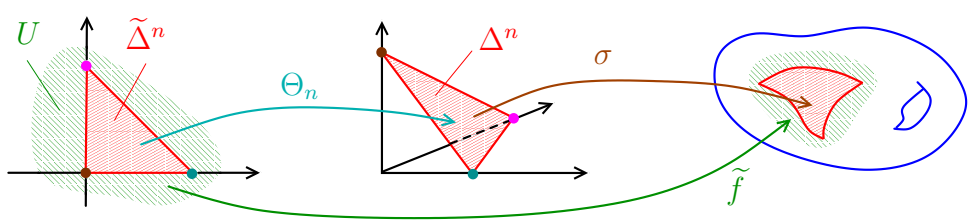


FIGURE 713.

Remark.

(1) Let $f\colon M \to N$ be a *smooth* map between two manifolds. Evidently the induced map $f_*\colon C^n(M) \to C^n(N)$ restricts to a map $f_*\colon C_n^{\text{smooth}}(M) \to C_n^{\text{smooth}}(N)$. In particular we get an induced map $f^*\colon C^n_{\text{smooth}}(N,G) \to C^n_{\text{smooth}}(M,G)$. Thus we obtain for each $n \in \mathbb{N}_0$ an induced map

$$f^*\colon H^n_{\text{smooth}}(N;G) \;\to\; H^n_{\text{smooth}}(M;G).$$

(2) Every map $\Delta^n \to \star$ is smooth, so there is no difference between ordinary singular cohomology and smooth singular cohomology for manifold $\star$.

[957] It is straightforward to verify that this definition does not depend on the choice of U and $\widetilde{f}$.

Proposition 55.3. *Smooth singular cohomology with real coefficients is a smooth cohomology theory.*

Sketch of proof. We need to verify that smooth singular cohomology with real coefficients satisfies the properties of a smooth cohomology theory.

(0) It is clear that smooth singular cohomology with real coefficients is a contravariant functor from the category of manifolds to the category of $\mathbb{N}_0$-graded real vector spaces.

(1) Suppose $f, g\colon M \to N$ are two smooth maps between two manifolds and suppose we are given a *smooth* homotopy F between f and g. Some thought shows that the chain homotopy

$$P_n\colon C_n(M) \;\to\; C_{n+1}(N), \quad n \in \mathbb{N}_0$$

from the proof of Proposition 30.5 restricts to a chain homotopy

$$C_n^{\text{smooth}}(M) \;\to\; C_{n+1}^{\text{smooth}}(N), \quad n \in \mathbb{N}_0.$$

Now we can once again appeal to Lemma 48.8 to conclude that the induced maps on smooth singular cohomology are the same.

(2) Let M be a manifold and let U and V be open subsets with $M = U \cup V$. The key to proving the existence of the Mayer–Vietoris sequence for ordinary singular cohomology was to show that the inclusion map

$$C_*^{\{U,V\}}(M) \;\to\; C_*(M)$$

is a chain homotopy equivalence. This in turn we showed using Proposition 31.21. In the proof of Proposition 31.21 we introduced the subdivision maps

$$u_n\colon C_n(M) \;\to\; C_n(M), \quad n \in \mathbb{N}_0.$$

It follows immediately from the definitions that these maps restrict to maps

$$u_n\colon C_n^{\text{smooth}}(M) \;\to\; C_n^{\text{smooth}}(M), \quad n \in \mathbb{N}_0.$$

Using this observation the proof of the existence of a Mayer–Vietoris sequence for smooth singular cohomology is otherwise identical to the proof for ordinary singular cohomology.

(3) The limit property is proved the same way as Proposition 52.12 (1). We only have to replace the fact, proved in Lemma 34.5 that $C_*(M) = \varinjlim C_*(U_i)$ by the observation that the same argument as in Lemma 34.5 shows that $C_*^{\text{smooth}}(M) = \varinjlim C_*^{\text{smooth}}(U_i)$.

□

Now we can easily show that smooth singular cohomology with real coefficients and ordinary singular cohomology with real coefficients are naturally isomorphic. More precisely, we have the following proposition.

Proposition 55.4. *There exists a natural isomorphism from smooth singular cohomology with real coefficients to ordinary singular cohomology with real coefficients. In particular*

for any manifold M and any $n \in \mathbb{N}_0$ we have a natural isomorphism

$$H^n(M;\mathbb{R}) \xrightarrow{\cong} H^n_{\text{smooth}}(M;\mathbb{R}).$$

Proof. For any manifold M we have the natural inclusion

$$C_*^{\text{smooth}}(M) \to C_*(M)$$

which evidently respects the boundary maps. These maps thus give rise to natural maps

$$C^*(M;\mathbb{R}) \to C^*_{\text{smooth}}(M;\mathbb{R})$$

which respect the coboundary maps. Thus we obtain a natural map

$$t(M)\colon H^*(M;\mathbb{R}) \to H^*_{\text{smooth}}(M;\mathbb{R}).$$

As we pointed out above, for $\star$ there is no difference between smooth and ordinary singular cohomology, i.e.

$$t(\star)\colon H^*(\star;\mathbb{R}) \to H^*_{\text{smooth}}(\star;\mathbb{R})$$

is an isomorphism. Since the natural transformation is induced by the inclusion map it follows easily that the natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. The proposition now follows immediately from Theorem 55.2. □

Now we also want to relate smooth singular cohomology to de Rham cohomology. The key for doing so is the following definition and the subsequent lemma.

Definition. Given a manifold M, given a smooth differential n-form ω on M and a smooth singular n-simplex $\sigma\colon \Delta^n \to M$ we define[958][959]

$$\int_\sigma \omega := (-1)^n \cdot \int_{\widetilde{\Delta}^n} (\sigma\circ\Theta_n)^*(\omega) := (-1)^n \cdot \int_{x\in\widetilde{\Delta}^n} \underbrace{\omega_{(\sigma\circ\Theta_n)(x)}(D(\sigma\circ\Theta_n)_x(e_1),\ldots,D(\sigma\circ\Theta_n)_x(e_n))}_{\text{continuous function in } x}.$$

We extend this definition linearly to define the integral of the smooth differential n-form ω over a smooth singular n-chain $\sum_{i=1}^k a_i \circ \sigma_i$.

Lemma 55.5. *Let M be a manifold, let ω be a smooth $(n-1)$-form and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then*[960]

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

[958]The Lebesgue integral is defined since $\widetilde{\Delta}^n$ is a compact subset of $\mathbb{R}^n$ and since the integrand is continuous.

[959]Recall that on page 935 we had equipped $\mathring{\Delta}^n$ with an orientation. The factor $(-1)^n$ is motivated by the fact that the restriction of the map $\Theta_n\colon \widetilde{\Delta}^n \to \Delta^n$ to the interiors is orientation-preserving if and only if n is even.

[960]Note that on the left we integrate a smooth differential n-form over a smooth singular n-simplex and that on the right we integrate a smooth $(n-1)$-form over a smooth singular $(n-1)$-chain.

In the following we provide a sketch of the proof of Lemma 55.5. A detailed proof is given in [War, p. 147].

Sketch of the proof. The idea of the proof is of course to use Stokes' Theorem 54.3. The fact that Δ^n, or equivalently $\widetilde{\Delta}^n$ are not manifolds in an obvious way is a slight nuisance. Here is one way how we can circumvent this problem. In [Le2, Chapter 16] or alternatively [Wall2, p. 30] the theory of manifolds with corner is developed. Prime examples are cubes $[0,1]^n$, half-balls $\{(x_1,\dots,x_n)\in \overline{B}^n \mid x_n \geq 0\}$ and most interestingly for us, the standard n-simplex Δ^n and equivalently also $\widetilde{\Delta}^n$. Furthermore there is a version of Stokes' Theorem 54.3 for manifolds with corner, see [Le2, Theorem 16.25].

In the following we provide the proof of Lemma 55.5 with an intuitive understanding of the concepts. All arguments can be made rigorous using the techniques developed in [Le2, Chapter 16].

Let M be a manifold, let ω be a smooth $(n-1)$-form and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then

$$\int_\sigma d\omega \underset{\substack{\uparrow\\ \text{by definition}}}{=} (-1)^n\cdot\int_{\widetilde{\Delta}^n}(\sigma\circ\Theta)^*(d\omega) \underset{\substack{\uparrow\\ \text{Proposition 54.2 (4)}}}{=} (-1)^n\cdot\int_{\widetilde{\Delta}^n} d(\sigma\circ\Theta)^*(\omega) \underset{\substack{\uparrow\\ \text{Stokes Theorem for manifolds}\\ \text{with corner applied to } \widetilde{\Delta}^n}}{=} (-1)^n\cdot\int_{\partial\widetilde{\Delta}^n}(\sigma\circ\Theta_n)^*(\omega)$$

$$\underset{\substack{\uparrow\\ \text{since } \Theta_n^{-1}\circ i_j^n\circ\Theta_{n-1}\colon \widetilde{\Delta}^{n-1}\to\partial\widetilde{\Delta}^n \text{ is}\\ \text{orientation-reversing if and only if } j \text{ is even}\\ \text{(here we use Proposition 54.2 (1))}}}{=} \sum_{j=0}^n(-1)^j\cdot\int_{\widetilde{\Delta}^{n-1}}(-1)^{n-1}\cdot(\sigma\circ i_j^n\circ\Theta_{n-1})^*(\omega) \underset{\substack{\uparrow\\ \text{by definition of the integral}\\ \text{of a smooth } (n-1)\text{-form over a}\\ \text{smooth singular } (n-1)\text{-simplex}}}{=} \sum_{j=0}^n(-1)^j\cdot\int_{\sigma\circ i_j^n}\omega \underset{\substack{\uparrow\\ \text{by definition of } \partial\sigma}}{=} \int_{\partial\sigma}\omega.$$

We refer to Figure 714 for an illustration of the proof. □

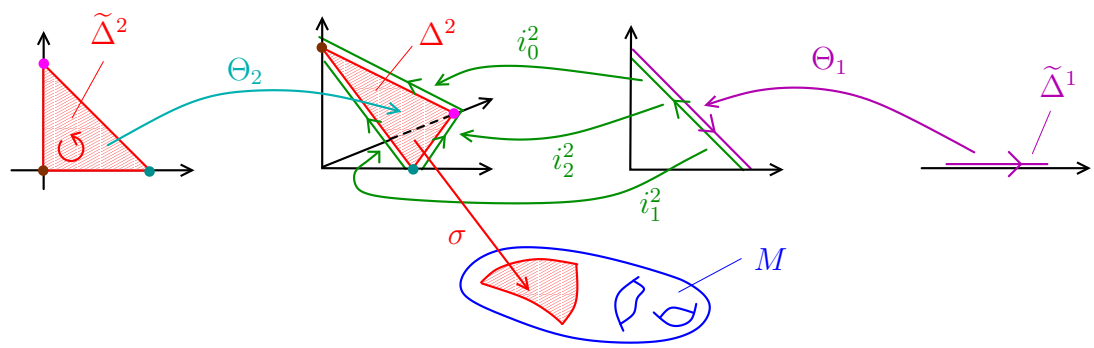


FIGURE 714. Illustration for the proof of Lemma 55.5.

55.3. **The isomorphism between singular and de Rham cohomology.** After the preparations in the previous sections we can now easily prove the theorem that we had been aiming for.

Theorem 55.6. (De Rham Theorem) *There exists a natural isomorphism u from de Rham cohomology to singular cohomology with real coefficients. In particular for any manifold M and any $k \in \mathbb{N}_0$ we have a natural isomorphism*

$$u\colon H^k_{\mathrm{dR}}(M;\mathbb{R}) \xrightarrow{\cong} H^k(M;\mathbb{R})$$

which has the property that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle \underbrace{u([\omega])}_{\in H^k(M;\mathbb{R})}, \underbrace{[\sigma]}_{\in H_k(M;\mathbb{Z})} \rangle_M = \int_\sigma \omega.$$

(*Here we stress, that according to the convention from page 1289 it is understood that the manifolds have empty boundary.*)

Remark.

(1) Let M be a closed oriented n-dimensional manifold and let $\omega \in \Omega_n(M)$ be an differential n-form. It follows quite easily from Proposition 39.3, the proof of Proposition 39.7, Lemma 46.7 and Theorem 55.6 that

$$\langle u([\omega]), [M]\rangle_M = \int_M \omega.$$

(2) A rather different proof for Theorem 55.6 is given in [War, p. 205] and [Ram, Chapte 4.6]. More precisely, in [War, Ram] the equivalence of de Rham cohomology and real valued singular cohomology is proved by relating both to another type of cohomology theory, namely sheaf cohomology. We also refer to [BT, Theorems 8.9 and 15.8] and [Dup, Theorem 1.15] for other proofs.

Proof. We denote by s the natural isomorphism from real-valued singular cohomology to real-valued smooth singular cohomology that we had constructed in the proof of Proposition 55.4.

Now we construct a natural isomorphism t from de Rham cohomology to real-valued smooth singular cohomology. For any manifold M and any $k \in \mathbb{N}_0$ we have, by the above discussion, a map

$$\begin{array}{rcl} C^k_{\mathrm{dR}}(M) & \to & C^k_{\mathrm{smooth}}(M;\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} C^{\mathrm{smooth}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int_\sigma \omega \end{array}\right) \end{array}$$

Note that it follows basically immediately from the definitions that this map is natural in M. Also note that it is an immediate consequence of Lemma 55.5 that for given M all these maps together form a cochain map $C^*_{\mathrm{dR}}(M) \to C^*_{\mathrm{smooth}}(M;\mathbb{R})$. Summarizing we see that we have constructed a natural map

$$t(M)\colon H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{smooth}}(M;\mathbb{R}).$$

Furthermore, in the special case of $M = \star$ the map

$$t(\star)\colon H^*_{\mathrm{dR}}(\star) \to H^*_{\mathrm{smooth}}(\star;\mathbb{R})$$

is easily seen to be an isomorphism. Once one has unraveled all the definitions it becomes apparent that this natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. Now we obtain from Theorem 55.2 that t is a natural isomorphism from de Rham cohomology to real-valued smooth singular cohomology.

Finally we set $u := s^{-1} \circ t$. By the above u is a natural isomorphism from de Rham cohomology to singular cohomology with real coefficients. It follows easily from the definitions of s and t that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle \underbrace{u([\omega])}_{\in H^k(M;\mathbb{R})} , \underbrace{[\sigma]}_{\in H_k(M;\mathbb{Z})} \rangle_M = \int_\sigma \omega.$$

□

Theorem 55.6 allows us to transport information from de Rham cohomology to real-valued singular cohomology and vice-versa. For example the many computational techniques that we had developed, e.g. cellular cohomology, for singular cohomology can now be used to calculate isomorphism types of de Rham cohomology groups. In particular we can easily prove the following corollary.

Corollary 55.7. *If M is a closed, oriented connected, non-empty n-dimensional manifold, then the map*

$$\begin{array}{rcl} H^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_M \omega \end{array}$$

is an isomorphism.

Proof. By Proposition 54.4 (7) the map is well-defined and it is an epimorphism. On the other hand, by Proposition 51.14 and Theorem 55.6 we know that $H^n_{\mathrm{dR}}(M) \cong \mathbb{R}$, which implies that the map is an isomorphism. □

55.4. **Realizing cohomology classes by maps to S^1.** We start out this section with the following definition.

Definition. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) = G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is *realized by a map* $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

In the following we will mostly discuss the case $G = \mathbb{Z}$ and $Y = S^1$. Before we can continue we introduce the following conventions.

Convention.

(1) We use the diffeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ given by $[t] \mapsto e^{2\pi i t}$ to make the identification $\mathbb{R}/\mathbb{Z} = S^1$.

(2) By Proposition 7.18 we have an isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ given by sending 1 to the equivalence class of the loop $t \mapsto e^{2\pi i t}$. We use this isomorphism to make the identification $\pi_1(\mathbb{R}/\mathbb{Z}, 0) = \pi_1(S^1, 1) = \mathbb{Z}$.
(3) Let X be a path-connected topological space and let $x_0 \in X$. As shown in Proposition 51.17 we can use the Hurewicz homomorphism and the evaluation homomorphism to make the identifications $H^1(X; \mathbb{Z}) = \mathrm{Hom}(H_1(X); \mathbb{Z}) = \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z})$.

Now we can formulate the following question.

Question 55.8. *Let X be a path-connected topological space that is "reasonable", e.g. that is a manifold or a CW-complex. Let $x_0 \in X$ be a base point. Can every homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z} = \pi_1(S^1, 1)$ be realized by a map $(X, x_0) \to (S^1, 1)$?*

The following proposition gives an affirmative answer if X is a manifold.

Proposition 55.9. *Let M be a connected manifold (here for once we allow M to have boundary) and let P be a base point. Every $\phi \in H^1(M; \mathbb{Z}) = \mathrm{Hom}(\pi_1(M, P), \mathbb{Z})$ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$.*

Remark. Sometimes the following reformulation of Proposition 55.9 is quite useful: Let M be a connected manifold and let P be a base point. Given any $\phi \in H^1(M; \mathbb{Z})$ there exists a smooth map $g\colon M \to S^1$ and a generator $\psi \in H^1(S^1; \mathbb{Z})$ with $\phi = g^*(\psi)$.

Examples.

(1) We consider the torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and we make the usual identification $H_1(T; \mathbb{Z}) = \mathbb{Z}^2$. We consider the cohomology classes in $H^1(T; \mathbb{Z}) = \mathrm{Hom}(H_1(T), \mathbb{Z})$ that are given by

$$\begin{array}{rcl} \phi\colon H_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & m \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon H_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & n. \end{array}$$

These two cohomology classes are evidently realized by the projection $S^1 \times S^1 \to S^1$ onto the first, respectively the second factor. In Exercise 55.3 the goal will be to give an example of an explicit map $g\colon T \to S^1$ that realizes the homomorphism $\varphi\colon H_1(T; \mathbb{Z}) = \mathbb{Z}^2 \to \mathbb{Z}$ given by $\varphi(m, n) = 6m + 4n$.
(2) In Figure 715 we consider the surface Σ of genus 2. As we saw on page 891, the oriented curves x_1, y_1, x_2, y_2 represent a basis for $H_1(\Sigma)$. We consider some random epimorphism $\phi\colon H_1(\Sigma) \to \mathbb{Z}$, e.g. we could consider the epimorphism $\phi\colon H_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By Proposition 55.9 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. But can one "visualize" such a map? We will return to this question at a later stage.

Proof. Let M be a connected manifold. We first assume that M has no boundary. We start out with the following definition. Let ω be a smooth differential 1-form on M and let

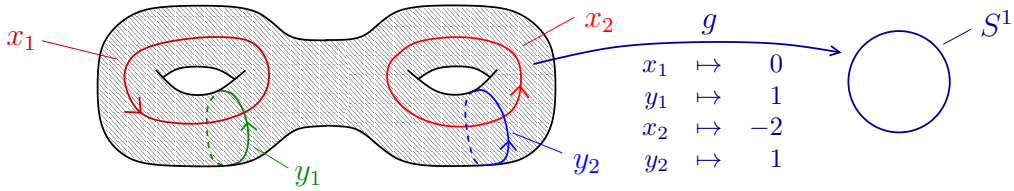


FIGURE 715.

$\gamma\colon [a,b] \to M$ be a piecewise smooth path. We pick $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each subinterval $[t_i, t_{i+1}]$ is smooth. We define

$$\int_{[a,b]} \gamma^*\omega \;=\; \sum_{i=0}^{n-1} \int_{[t_i,t_{i+1}]} \gamma^*\omega.$$

It follows immediately from Proposition 54.2 (2) that this definition does not depend on the choice of the subdivision.

After this preparation we start with the actual proof of the proposition. We fix a base point $P \in M$. Furthermore let $\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}(\pi_1(M,P),\mathbb{Z})$. We denote by u the natural isomorphism from de Rham cohomology to singular cohomology with real coefficients from Theorem 55.6. We pick a smooth closed differential 1-form ω on M with $u([\omega]) = \phi$.

Claim. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. Then

$$\int_{[0,1]} \delta^*\omega \;=\; \phi\big(\underbrace{[\delta]}_{\in \pi_1(M,P)}\big) \;\in\; \mathbb{R}.$$

Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. One can show easily that after possibly performing an orientation-preserving reparametrization we can write $\delta = \delta_1 * \cdots * \delta_k$ as a product of smooth paths $\delta_i\colon [0,1] \to M$, $i = 1,\dots,k$. As in the definition of the Hurewicz homomorphism we use the map $t \mapsto (1-t,t)$ to make the identification $[0,1] = \Delta^1$. Using this identification we view each δ_i also as a smooth 1-simplex. In particular we have $\delta_1 + \cdots + \delta_k \in C_1^{\text{smooth}}(M)$. Then

$$\int_{[0,1]} \delta^*\omega \;\underset{\substack{\text{by the definitions and}\\ \text{by Proposition 54.2}\\ \text{(1) and (2)}}}{=}\; \int_{\delta_1+\cdots+\delta_k} \omega \;\underset{\substack{\text{by Theorem 55.6}\\ \text{since } u([\omega]) = \phi}}{=}\; \langle \phi, [\delta_1 + \cdots + \delta_k]\rangle \;\underset{\substack{\text{via the identification}\\ H^1(M;\mathbb{Z}) = \operatorname{Hom}(\pi_1(M,P),\mathbb{Z})}}{=}\; \phi([\delta]).$$

This concludes the proof of the claim.

We define

$$\begin{aligned} g\colon M &\to S^1 = \mathbb{R}/\mathbb{Z} \\ Q &\mapsto \Big[\int_{[0,1]} \gamma^*\omega\Big] \quad \text{where } \gamma \text{ is a smooth path}^{961} \text{ from } P \text{ to } Q. \end{aligned}$$

We claim that the map g has the required properties. It remains to prove the following four statements:

(1) The map g is well-defined.
(2) We have $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.
(3) The map g is smooth.
(4) We have $g_* = \phi\colon \pi_1(M, P) \to \pi_1(S^1, 1) = \mathbb{Z}$.

Now we turn to the proofs of the statements.

(1) Let Q in M. Suppose we are given two piecewise smooth paths β and γ from the base point P to Q. (See the left of Figure 716 for an illustration.) Then

$$\int_{[0,1]} \beta^*\omega - \int_{[0,1]} \gamma^*\omega \underset{\substack{\uparrow \\ \text{follows from the definitions} \\ \text{and Proposition 54.2 (1) and (2)}}}{=} \int_{[0,1]} (\beta * \overline{\gamma})^*\omega \underset{\substack{\uparrow \\ \text{by the above claim, since } \beta * \overline{\gamma} \\ \text{is a piecewise smooth loop}}}{=} \phi([\beta * \overline{\gamma}]) \in \mathbb{Z}.$$

This shows that g is indeed well-defined.

(2) We determine $g(P)$ by taking the constant path from P to P. It follows immediately from the definition of g that $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.

(3) Let $Q \in M$. We need to show that g is smooth in an open neighborhood of Q. We pick a path γ from P to Q. We pick a chart $\Phi\colon U \to B_r(0)$ around Q with $\Phi(Q) = 0$. Given a point $R \in U$ we denote by δ_R the piecewise smooth path that is given by the concatenation of the following two paths:
   (a) the path γ from P to Q,
   (b) the "radial path" from Q to R that corresponds, under the diffeomorphism Φ, to the radial path from 0 to $\Phi(R) \in B_r(0)$.

   (See the right of Figure 716 for an illustration.) Using these paths δ_R it is now relatively straightforward to show that g is smooth in U. We leave the verification of the details to the reader.

(4) Suppose we are given an element in $\pi_1(M, P)$. By Lemma 10.24 (1) we can represent this element by a smooth loop $\delta\colon [0,1] \to M$. We have the following equality:

$$\begin{aligned} g_*([\delta]) &\overset{\substack{\text{by Theorem 7.17} \\ \downarrow}}{=} \text{endpoint of the lift of } g \circ \delta\colon [0,1] \to \mathbb{R} \text{ to the universal covering } \mathbb{R} \text{ of } \mathbb{R}/\mathbb{Z} \\ &= \text{endpoint of the lift of the path } t \mapsto \Big[\int_{[0,t]} \delta^*\omega\Big] \text{ to the universal covering } \mathbb{R} \\ &= \text{endpoint of the path } t \mapsto \int_{[0,t]} \delta^*\omega = \text{the point } \int_{[0,1]} \delta^*\omega \underset{\substack{\uparrow \\ \text{by the above claim}}}{=} \phi([\delta]). \end{aligned}$$

We have thus shown that $g_* = \phi\colon \pi_1(M, P) \to \pi_1(S^1, 1) = \mathbb{Z}$.

[961] By Lemma 10.24 (2) any two points on a connected manifold, that do not lie on the boundary, can be connected by a smooth path. Here we again use that M has no boundary.

This concludes the proof of the theorem in the case that M has empty boundary.

Finally let M be a manifold with non-empty boundary. We consider $W := M \cup \partial M \times [0,1)$ where we make the obvious identification of $P = P \times \{0\}$ for $P \in \partial M$. Using the Collar Neighborhood Theorem 10.10 one can easily verify that W is a manifold such that M is a submanifold and such that W has no boundary. Evidently M is a deformation retract of W. It follows from Lemma 9.10 and Proposition 9.12 (1) that the induced map $i_*\colon \pi_1(M,P) \to \pi_1(W,P)$ is an isomorphism. Now let $\phi \in \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$. By the previously considered case we can realize $\phi \circ (i_*)^{-1}\colon \pi_1(W,P) \to \mathbb{Z}$ by a smooth map $g\colon W \to S^1$. The restriction of g to M has all the desired properties. $\square$

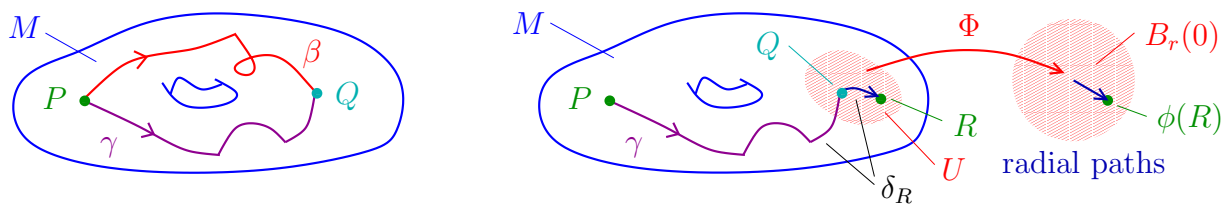


FIGURE 716.

Of course one can ask many variations on Question 55.8. Since arbitrary topological spaces can be quite weird we restrict ourselves in the following questions to CW-complexes.

Question 55.10. *Let X be a connected CW-complex.*

(1) *Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a map $X \to \Sigma_g$?*

(2) *Can every homomorphism $\pi_1(X) \to \pi_1\Big(\bigvee_{i=1}^{k} S^1\Big) = \langle x_1, \dots, x_k \rangle$ be realized by a map $X \to \bigvee_{i=1}^{k} S^1$?*

(3) *Can every homomorphism $\pi_1(X) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a map $X \to \mathbb{R}\mathrm{P}^2$?*

55.5. **The wedge product of differential forms** $(*)$**.** For a real vector space V we had already introduced on page 1271 the bilinear wedge product

$$\wedge^k V^* \times \wedge^l V^* \quad \to \quad \wedge^{k+l} V^*.$$

Given a manifold M we can define this wedge product for each tangent space $V = T_P M$. Using charts one can show that this wedge product sends smooth forms to smooth forms. More precisely, we have a bilinear map

$$\wedge\colon C^k_{\mathrm{dR}}(M) \times C^l_{\mathrm{dR}}(M) \quad \to \quad C^{k+l}_{\mathrm{dR}}(M).$$

We refer [Tu, p. 205] for details.

Lemma 55.11. *Let M be a manifold. For any $k, l \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \wedge : H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(M) & \to & H^{k+l}_{\mathrm{dR}}(M) \\ ([\omega],[\sigma]) & \mapsto & [\omega] \wedge [\sigma] := [\omega \wedge \sigma] \end{array}$$

is well-defined.

Proof. By [Tu, Proposition 4.7] we know that for any smooth differential k-form ω and any smooth differential l-form σ we have

$$(*) \qquad d(\omega \wedge \sigma) \;=\; d\omega \wedge \sigma + (-1)^k \cdot \omega \wedge d\sigma.$$

Now let $\omega \in C^k_{\mathrm{dR}}(M)$ and $\sigma \in C^l_{\mathrm{dR}}(M)$ be smooth forms.

(1) It follows immediately from $(*)$ that $\omega \wedge \sigma$ is again closed.
(2) If $\omega' = \omega + d\tau$ is another representative of $[\omega]$, then

$$\omega' \wedge \sigma - \omega \wedge \sigma \;\underset{\substack{\uparrow \\ \text{since “}\wedge\text{” is bilinear}}}{=}\; (\omega' - \omega) \wedge \sigma \;=\; d\tau \wedge \sigma \;\underset{\substack{\uparrow \\ \text{by } (*) \text{ and since } d\sigma = 0}}{=}\; d(\tau \wedge \sigma).$$

This shows that $\omega \wedge \sigma$ and $\omega' \wedge \sigma$ represent the same de Rham cohomology class, i.e. the wedge product does not depend on the choice of the representative of the de Rham cohomology class of the first entry.
(3) The proof that the wedge product does not depend on the choice of the representative of the second de Rham cohomology class is basically the same as the proof of (2). □

At this point it is helpful to introduce the following definition.

Definition.

(1) A *graded ring* is a ring[962] R together with a direct sum decomposition[963] $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ into subgroups such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map
$$R_m \times R_n \;\to\; R_{m+n}.$$
(2) A graded ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a) \cdot \deg(b)} \cdot b \cdot a$ holds is called a *superalgebra*.[964]
(3) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ and $S = \bigoplus_{n \in \mathbb{N}_0} S_n$ be two graded rings. A *morphism between the two graded rings* is a ring homomorphism $\varphi\colon R \to S$ such that for any $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subset S_n$.
(4) We define the *category of graded rings* to be the category where the objects are graded rings and the morphisms are morphisms of graded rings. Similarly we define the category of graded superalgebras.

[962]Here we mean by a ring an abelian group together with a multiplicative structure which is associative, distributive and that admits a multiplicatively neutral element. We do not demand that a ring is commutative.

[963]In particular, R viewed as an abelian group is $\mathbb{N}_0$-graded.

[964]The word “algebra” in our context is justified by the fact that any ring is a $\mathbb{Z}$-algebra. We could also have referred to such an object as a superring, but this terminology seems to be uncommon.

Example. We consider the polynomial ring $R = \mathbb{Q}[x]$. We can view this ring as a graded ring by defining $R_n := \mathbb{Q} \cdot x^n$ for $n \in \mathbb{N}_0$.

Recall that for a given manifold M we write

$$H^*_{\mathrm{dR}}(M) \;=\; \bigoplus_{i=0}^{\infty} H^i_{\mathrm{dR}}(M).$$

We consider the following map

$$\begin{array}{rcl} \wedge\colon H^*_{\mathrm{dR}}(M) \times H^*_{\mathrm{dR}}(M) & \to & H^*_{\mathrm{dR}}(M) \\ \left(\sum\limits_{i=0}^{\infty} x_i, \sum\limits_{i=0}^{\infty} y_i\right) & \mapsto & \sum\limits_{i,j=0}^{\infty} x_i \wedge y_j. \end{array}$$

where it is understood that each x_i and y_i is homogeneous of degree i.

The following proposition summarizes the key features of the wedge product on de Rham cohomology.

Proposition 55.12. *The maps*

$$\begin{array}{rcl} M & \mapsto & (H^*_{\mathrm{dR}}(M), \wedge) \\ (f\colon M \to N) & \mapsto & (f^*\colon H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category of manifolds to the category of graded superalgebras.

Proof.

(1) First note that it follows almost immediately from Lemma 54.1 that for a given manifold M the wedge product turns $H^*_{\mathrm{dR}}(M)$ into a graded superalgebra. The multiplicatively neutral element in de Rham cohomology is hereby the 0-dimensional cohomology class that is represented by the constant function $f\colon M \to \mathbb{R}$, $x \mapsto 1$.

(2) If $f\colon M \to N$ is a smooth map between manifolds, then it follows immediately from the definitions that for any choice of $\omega \in C^k_{\mathrm{dR}}(N)$ and $\sigma \in C^l_{\mathrm{dR}}(N)$ the following equality holds:

$$(f^*\omega \wedge f^*\sigma) \;=\; f^*(\omega \wedge \sigma).$$

Therefore the smooth map f induces a morphism $f^*\colon H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)$ of graded rings. □

Example. We consider the differential 1-forms dx and dy on $\mathbb{R}^2$ which are given by[965] $dx(v,w) = v$ and $dy(v,w) = w$. Furthermore we identify the 2-torus T with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ and we denote the projection $\mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ by p. It is straightforward to see that dx and dy induce differential 1-forms on T.[966] We denote these also by dx

[965] As always we identify $T_P\mathbb{R}^2$ with $\mathbb{R}^2$.

[966] Details are given in [Frie2, p. 248].

and dy.[967] The 1-forms dx and dy are closed, i.e. $d(dx) = d(dy) = 0$.[968] Thus they define cohomology classes $[dx], [dy] \in H^1_{\mathrm{dR}}(T)$. A straightforward calculation, see e.g. [Frie2, p. 248], shows that if we integrate the 2-form $dx \wedge dy$ on T with the standard orientation, then we obtain 1. It follows from Proposition 54.4 that $[dx \wedge dy] = [dx] \wedge [dy]$ is a non-trivial element in $H^2_{\mathrm{dR}}(T)$.

From the fact that $H^*_{\mathrm{dR}}(T)$ is a superalgebra we know that

$$[dx] \wedge [dy] \;=\; (-1)^{1\cdot 1} \cdot [dy] \wedge [dx] \;=\; -[dy] \wedge [dx].$$

Since this wedge product is non-zero we see that $(H^*_{\mathrm{dR}}(T), \wedge)$ is a non-commutative ring.

This discovery of a ring structure on de Rham cohomology raises the question whether one can also define such a ring structure using singular cohomology and whether such a ring structure exists on singular cohomology for more general spaces. We will answer these questions in the next chapter.

Exercises for Chapter 55.

Exercise 55.1. Let $\mathcal{H}^*$ be a smooth cohomology theory with $\mathcal{H}^n(\star) = 0$ for $n \geq 1$. We denote by $f\colon S^1 \to S^1$ the map that is given by $(x, y) \mapsto (x, -y)$. Show, only using the axioms of a smooth cohomology theory, that $f^*\colon \mathcal{H}^1(S^1) \to \mathcal{H}^1(S^1)$ is given by multiplication by -1.

Exercise 55.2. Let M be a manifold. Show that if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact.

Exercise 55.3. We make the usual identification $\pi_1(\mathbb{R}^2/\mathbb{Z}^2) = \mathbb{Z}^2$. We consider the homomorphism $\phi\colon \mathbb{Z}^2 = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) \to \mathbb{Z}$ given by $\phi((1,0)) = 6$ and $\phi((0,1)) = 4$. Provide an *explicit* map $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1$ such that $f_* = \phi$.

Exercise 55.4. Let M be a manifold and let $\phi \in H^1(M; \mathbb{Z})$. Show that $\phi \cup \phi = 0$.

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[967] More precisely, we showed that there exists a unique 1-form dx on T such that for each point $P \in \mathbb{R}^2$ and $v \in T_P\mathbb{R}^2$ we have $dx_P(v) = dx_{p(P)}(p_*(v))$. Of course the same statement also holds for dy.

[968] This is clear for dx and dy viewed as differential 1-forms on $\mathbb{R}^2$. But the projection map $p\colon \mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism. Since the differential commutes with smooth maps, see Proposition 54.2 (4), we also see that $d(dx) = d(dy) = 0$ on $\mathbb{R}^2/\mathbb{Z}^2$.

56. The Eilenberg-Zilber theorem and the cross product

In Section 55.5 we saw that de Rham cohomology of a manifold carries a product structure which turns de Rham cohomology into a ring. It is therefore natural to try to find such a product structure in singular homology and/or in singular cohomology.

The key to doing so will be the Eilenberg-Zilber Theorem 43.4 that we had stated in algebraic topology II without providing a proof. In this chapter we will give a slightly more precise statement and we provide the proof. In this and the next chapter we will see how it can be used to find product structures on homology and cohomology groups.

56.1. **The Eilenberg-Zilber theorem.** We start out by recalling and introducing some definitions.

Definition.

(1) Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C} = (C'_*, \partial'_*)$ be two chain complexes. On page 1041 we defined the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes where the n-th chain group is defined as

$$(\mathcal{C} \otimes \mathcal{C}')_n \quad := \quad \bigoplus_{p+q=n} C_p \otimes C'_q$$

and where the boundary homomorphisms are given by

$$\begin{array}{rcl} \partial^{\otimes} \colon (\mathcal{C} \otimes \mathcal{C}')_n & \to & (\mathcal{C} \otimes \mathcal{C}')_{n-1} \\ c_p \otimes c'_q & \mapsto & \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q. \end{array}$$

(2) Given a point x in a topological space X we denote by x also the singular 0-simplex given by sending the unique point of Δ^0 to x.

(3) Given two topological spaces X and Y we refer to the maps

$$\begin{array}{rcl} C_0(X) \otimes C_0(Y) & \to & C_0(X \times Y) \\ x \otimes y & \mapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{rcl} C_0(X \times Y) & \to & C_0(X) \otimes C_0(Y) \\ (x, y) & \mapsto & x \otimes y \end{array}$$

as the *canonical maps*. We refer to the identity maps on $C_0(X) \otimes C_0(Y)$ and $C_0(X \times Y)$ also as the *canonical maps*.

The following theorem is a slightly more precise formulation of the original Eilenberg-Zilber Theorem 43.4.

Theorem 56.1. (Eilenberg-Zilber)

(1) *Given any topological spaces X and Y there exist natural chain maps*

$$\Upsilon \colon C_*(X) \otimes C_*(Y) \quad \to \quad C_*(X \times Y)$$

and

$$\Theta \colon C_*(X \times Y) \quad \to \quad C_*(X) \otimes C_*(Y),$$

which on the 0-level are the above canonical maps.

(2) *The maps* Υ *and* Θ *are unique up to natural chain homotopies*[969]
(3) *The maps* Υ *and* Θ *are naturally*[970] *chain homotopy inverses of one another.*

We refer to the maps Υ and Θ as *Eilenberg-Zilber* maps. In the notation and our discussions we will usually ignore the fact that these maps are well-defined only up to a natural homotopy. We will only work with the properties of Υ and Θ stated in the theorem, so a different choice of maps Υ and Θ does not affect anything we say afterwards.

56.2. **Proof of the Eilenberg-Zilber Theorem 56.1.** The proof of the Eilenberg-Zilber Theorem 56.1 breaks up into three parts: we first construct Υ, then we construct Θ and finally we show that these maps are unique in the above sense and that they have the desired properties.

We start out with the following definition.

Definition. A topological space X is called *acyclic* if $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$ and if all higher homology groups are zero.

Example. In Corollary 30.8 we showed that every contractible topological space is acyclic. In particular every standard simplex Δ^n is contractible and thus acyclic. Furthermore, all products $\Delta^p \times \Delta^q$ of standard simplices are contractible and thus acyclic.[971]

We also fix some notation.

Notation. Let X be a topological space and let $x \in X$. If $\sigma\colon \Delta^n \to Y$ is a singular n-simplex in a topological space Y, then we denote by $x \times \sigma\colon \Delta^n \to X \times Y$ the singular n-simplex given by $v \mapsto (x, \sigma(v))$. We use the analogous notation for the roles of X and Y swapped.

Now we can formulate the following theorem.

Theorem 56.2. *Given any two topological spaces X and Y and any $p, q \in \mathbb{N}_0$ there exists a homomorphism*

$$\Upsilon_{p,q}\colon C_p(X) \otimes C_q(Y) \;\to\; C_{p+q}(X \times Y)$$

such that the following conditions are satisfied:

(1) *for $x \in X$ and $\sigma\colon \Delta^q \to Y$ we have $\Upsilon_{0,q}(x \otimes \sigma) = x \times \sigma$, the same statement also holds with the roles of X and Y swapped,*

[969]Here "natural chain homotopy" means that given two such maps Υ and Υ' there exists a choice of chain homotopies $(C_*(X) \otimes C_*(Y))_k \to C_{k+1}(X \times Y)$, which is natural in X and Y, between Υ and Υ'. An analogous statement holds for maps Θ and Θ'.

[970]Here "naturally" means that $\Upsilon \circ \Theta$ and $\Theta \circ \Upsilon$ are naturally homotopic to the identity maps, i.e. there exist natural chain homotopies from $\Upsilon \circ \Theta$ to the identity map and from $\Theta \circ \Upsilon$ to the identity map.

[971]This can be seen in many ways, for example pick points $x \in \Delta^p$ and $y \in \Delta^q$, it is straightforward to show that there exists a deformation retraction r from Δ^p to $\{x\}$ and that there exists a deformation retraction s from Δ^q to $\{y\}$. But these two deformation retractions also define a deformation retraction from $\Delta^p \times \Delta^q$ to (x, y).

(2) *the map* $\Upsilon_{p,q}$ *is natural, i.e. if* $f\colon X' \to X$ *and* $g\colon Y' \to Y$ *are maps between topological spaces, then for any* $a \in C_p(X')$ *and any* $b \in C_q(Y')$ *we have*

$$(f \times g)_*(\underbrace{\Upsilon_{p,q}(a \otimes b)}_{\in C_{p+q}(X' \times Y')}) \;=\; \underbrace{\Upsilon_{p,q}(f_*(a) \otimes g_*(b))}_{\in C_{p+q}(X \times Y)},$$

(3) *the maps* $\Upsilon_{p,q}$ *form a chain map, i.e. for any* $a \in C_p(X)$ *and any* $b \in C_q(Y)$ *we have*

$$\partial(\Upsilon_{p,q}(a \otimes b)) \;=\; \underbrace{\Upsilon_{p-1,q}(\partial a \otimes b) + (-1)^p \cdot \Upsilon_{p,q-1}(a \otimes \partial b)}_{=\Upsilon(\partial^{\otimes}(a \otimes b))}.$$

Proof. We prove the existence of the maps $\Upsilon_{p,q}$ by induction on $p \cdot q$. For clarity we henceforth suppress the subscript from $\Upsilon_{p,q}$.

Clearly for $p = 0$ or $q = 0$ the maps from (1) have the properties (2) and (3). So now suppose that we are given $p, q \in \mathbb{N}_0$ with $p \cdot q > 0$ such that the maps Υ have been defined for all p', q' with $p' \cdot q' < p \cdot q$.

Given any $n \in \mathbb{N}_0$ we denote by $\mathrm{id}_n\colon \Delta^n \to \Delta^n$ the identity map. We first want to come up with a reasonable definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)$. If we want (3) to be satisfied we have to define $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) = z$ where $z \in C_{p+q}(\Delta^p \times \Delta^q)$ satisfies

$$(*) \qquad \partial z \;=\; \Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \;\in\; C_{p+q-1}(\Delta^p \times \Delta^q).$$

As we pointed out on page 1309, the product $\Delta^p \times \Delta^q$ is acyclic, in particular we have $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$, which means that every singular $(p+q-1)$-chain *that is a cycle* is in fact the boundary of some $(p+q)$-chain z. So we have to show that the right-hand side of $(*)$ is indeed a cycle. We compute that

$$\begin{aligned}
&\overbrace{\partial(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q))}^{\text{boundary of our hypothetical } \Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)} \overset{\text{linearity of } \partial}{=} \partial(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot \partial(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&= \Upsilon(\underbrace{\partial\partial\,\mathrm{id}_p}_{=0} \otimes \mathrm{id}_q) + (-1)^{p-1} \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \\
&\qquad\qquad + (-1)^p \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) + (-1)^p \cdot (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \underbrace{\partial\partial\,\mathrm{id}_q}_{=0}) = 0.
\end{aligned}$$

by our induction hypothesis we can apply the boundary formula (3) to $\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)$ and $\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)$

This discussion gives us the definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) := z$.

Now let X and Y be topological spaces and let $\varphi\colon \Delta^p \to X$ and $\psi\colon \Delta^q \to Y$ be singular simplices. If we want (2) to be satisfied then we have to define

$$\Upsilon(\underbrace{\varphi \otimes \psi}_{\in C_p(X) \otimes C_q(Y)}) \;:=\; (\varphi \times \psi)_*(\underbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}_{\in C_{p+q}(\Delta^p \times \Delta^q)}) \;\in\; C_{p+q}(X \times Y).$$

We extend this map linearly to all of $C_p(X) \otimes C_q(Y)$. It follows almost immediately from the definition of Υ that the map Υ is natural in X and Y.

It remains to prove (3). This property holds basically by definition. But for completeness' sake we carry out the proof. So let $\varphi\colon \Delta^p \to X$ and $\psi\colon \Delta^q \to Y$ be singular simplices. We then have the following equality in $C_{p+q-1}(X \times Y)$:

$$\begin{aligned}
\partial(\Upsilon(\varphi \otimes \psi)) &= \partial\big((\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q))\big) \overset{\text{since } (\varphi\times\psi)_* \text{ is a chain map}}{=} (\varphi \times \psi)_*(\partial(\overbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}^{=z})) \\
&= (\varphi \times \psi)_*(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot (\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&\uparrow \\
&\text{by definition of } z \text{ and the linearity of } (\varphi \times \psi)_* \\
&= (\Upsilon(\varphi_*(\partial\,\mathrm{id}_p) \otimes \psi_*(\mathrm{id}_q))) + (-1)^p \cdot \Upsilon(\varphi_*(\mathrm{id}_p) \otimes \psi_*(\partial\,\mathrm{id}_q)) \\
&\uparrow \\
&\text{by induction we can apply the naturality of } \Upsilon \\
&= \Upsilon(\partial(\varphi) \otimes \psi) + (-1)^p \cdot \Upsilon(\varphi \otimes \partial\psi)) \\
&\uparrow \\
&\text{since } \varphi_* \text{ and } \psi_* \text{ are chain maps and since } \varphi_*(\mathrm{id}_p) = \varphi \text{ and } \psi_*(\mathrm{id}_q) = \psi
\end{aligned}$$

□

Now we turn to the construction of the second map in the Eilenberg-Zilber Theorem 56.1.

Theorem 56.3. *Given any two topological spaces X and Y and any $k \in \mathbb{N}_0$ there exists a homomorphism*

$$\Theta_k\colon C_k(X \times Y) \;\to\; (C_*(X) \otimes C_*(Y))_k \;=\; \bigoplus_{p+q=k} C_p(X) \otimes C_q(Y)$$

such that the following conditions are satisfied:

(1) *the map Θ_0 is the canonical map on the 0-level, i.e. for every $x \in X$ and $y \in Y$ we have $\Theta_0((x, y)) = x \otimes y$,*
(2) *the map Θ_k is natural in X and Y, i.e. for any two maps $f\colon X' \to X$ and $g\colon Y' \to Y$ of topological spaces and any $c \in C_k(X' \times Y')$ we have*

$$\Theta_k((f \times g)_*(c)) \;=\; (f_* \otimes g_*)(\Theta_k(c)),$$

(3) *the maps Θ_* form a chain map, i.e. for any $k \in \mathbb{N}_0$ we have*

$$\partial_k^{\otimes} \circ \Theta_k \;=\; \Theta_{k-1} \circ \partial_k.$$

In the proof of Theorem 56.3 we will need the following lemma.

Lemma 56.4. *Let X and Y be two acyclic topological spaces. For every $k \geq 1$ we have*

$$H_k(C_*(X) \otimes C_*(Y)) \;=\; 0.$$

Furthermore we have an isomorphism

$$\begin{aligned}
\epsilon\colon H_0(C_*(X) \otimes C_*(Y)) &\overset{\cong}{\to} \mathbb{Z} \\
\Big[\sum_{i=1}^{k} \underbrace{a_i}_{\in\mathbb{Z}} \cdot \underbrace{x_i}_{\in X} \otimes \underbrace{y_i}_{\in Y}\Big] &\mapsto \sum_{i=1}^{k} a_i.
\end{aligned}$$

Proof. The lemma is an immediate consequence of our hypotheses, Proposition 29.5 and the Künneth Theorem 43.7 for Chain Complexes. (Note that this Künneth Theorem is purely algebraic.) □

Proof of Theorem 56.3. As we will see the idea behind the proof of Theorem 56.3 is very similar to the idea of the proof of Theorem 56.2. We start out with the outline of proof:

(a) We use (1) to define Θ_0,
(b) we suppose the map $\Theta_0, \dots, \Theta_{k-1}$ are defined, we then use (3) to make an educated guess for $\Theta_k(d_k)$ where we consider
  (i) the topological spaces $X = Y = \Delta^k$ and
  (ii) the singular k-simplex $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ that is given by the diagonal map $d_k(x) = (x, x)$.
(c) we then use the naturality statement of (2) to define Θ_k for all topological spaces X and Y and all singular simplices in $C_*(X \times Y)$.

Now we carry out the above program. First of all, given a 0-simplex (x, y) in $X \times Y$ we define $\Theta_0((x, y)) = x \otimes y$. Then we extend this definition linearly to get the desired map $\Theta_0\colon C_0(X \times Y) \to C_0(X) \otimes C_0(Y)$.

Now suppose we have already constructed the maps $\Theta_0, \dots, \Theta_{k-1}$. We want to define Θ_k and we start out by defining $\Theta_k(d_k\colon \Delta^k \to \Delta^k \times \Delta^k)$. Since we need (3) to be satisfied we need to define $\Theta_k(d_k) := c$ where $c \in (C_*(\Delta^k) \otimes C_*(\Delta^k))_k$ satisfies $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial d_k)$. In the chain complex $C_*(\Delta^k) \otimes C_*(\Delta^k)$ we calculate that

$$\partial^{\otimes}(\overbrace{\Theta_{k-1}(\partial_k(d_k))}^{\text{defined by induction}}) \underset{\substack{\uparrow \\ \text{by induction we can apply (3)}}}{=} \Theta_{k-2}(\underbrace{\partial_{k-1}\partial_k}_{=0} d_k) = 0,$$

i.e. the chain $\Theta_{k-1}(\partial_k d_k) \in (C_*(\Delta^k) \otimes C_*(\Delta^k))_{k-1}$ is in fact a cycle. Since Δ^k is acyclic it follows immediately from Lemma 56.4 that this cycle is null-homologous. (For $k = 1$ one easily verifies, using the explicit description of Θ_0, that the cycle lies in the kernel of the isomorphism $\epsilon\colon H_0(C_*(\Delta^k) \otimes C_*(\Delta^k)) \to \mathbb{Z}$.) This means that there exists a singular chain $c \in (C_*(\Delta^k) \otimes C_*(\Delta^k))_k$ with $\partial^{\otimes}(c) = \Theta_{k-1}(\partial_k d_k)$. We define $\Theta_k(d_k) := c$.

Now we define Θ_k for arbitrary topological spaces X and Y and an arbitrary singular k-simplex $\sigma\colon \Delta^k \to X \times Y$. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Note that we have a commutative diagram

$$\begin{tikzcd} & \Delta^k \times \Delta^k \arrow[d, "(p\circ\sigma)\times(q\circ\sigma)"] \\ \Delta^k \arrow[ur, "d_k"] \arrow[r, "\sigma"'] & X \times Y, \end{tikzcd}$$

i.e. we have $\sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k = ((p \circ \sigma) \times (q \circ \sigma))_*(d_k)$. For the naturality condition (2) to be satisfied we need to define

$$\Theta_k(\sigma) := ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_k(d_k)) \in (C_*(X) \otimes C_*(Y))_k.$$

With this definition one readily verifies that the map Θ_k is natural in X and Y.

Finally we have to show that Θ_k satisfies the boundary formula (3). So as before let $\sigma\colon \Delta^k \to X \times Y$ be a singular k-simplex. Then

$$\begin{array}{rcl}
 & & \text{since } (p\circ\sigma)_* \otimes (q\circ\sigma)_*\colon C_*(\Delta^k)\otimes C_*(\Delta^k) \to C_*(X)\otimes C_*(Y) \\
\text{by definition of } \Theta_k(\sigma) & & \text{is a chain map by Lemma 43.3} \\
\downarrow & & \downarrow \\
\partial^{\otimes}\Theta_k(\sigma) & = & \partial^{\otimes}\big(((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\Theta_k(d_k))\big) \quad = \quad ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\partial^{\otimes}\Theta_k(d_k)) \\
 & = & ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\Theta_{k-1}(\partial d_k)) \quad = \quad \Theta_{k-1}\big(((p\circ\sigma)\times(q\circ\sigma))_*(\partial d_k)\big) \\
\uparrow & & \uparrow \\
\text{by definition of } \Theta_k(d_k) & & \text{by naturality of } \Theta_{k-1} \\
 & = & \Theta_{k-1}\partial\big(((p\circ\sigma)\times(q\circ\sigma))_*(d_k)\big) \quad = \quad \Theta_{k-1}\partial\sigma. \\
\uparrow & & \uparrow \\
\text{since } ((p\circ\sigma)\times(q\circ\sigma))_* \text{ is a chain map} & & \text{since } \sigma = ((p\circ\sigma)\times(q\circ\sigma))\circ d_k
\end{array}$$

We have now verified that Θ_k has all the desired properties. □

As we saw in the proofs of Theorem 56.2 and Theorem 56.3, the natural chain maps $\Upsilon\colon C_*(X)\otimes C_*(Y) \to C_*(X\times Y)$ and $\Theta\colon C_*(X\times Y) \to C_*(X)\otimes C_*(Y)$ depend on many choices. The next theorem says that different choices in the proofs lead to natural chain maps that are naturally homotopy equivalent to one another.

Theorem 56.5. *Let X and Y be topological spaces. Any two natural chain maps*

$$C_*(X\times Y) \;\to\; C_*(X)\otimes C_*(Y)$$

that are the canonical maps on the 0-level are in fact naturally chain homotopic. The same conclusion also holds for any two natural chain maps

$$C_*(X)\otimes C_*(Y)\to C_*(X)\otimes C_*(Y),\;\; C_*(X\times Y)\to C_*(X\times Y),\;\; C_*(X)\otimes C_*(Y)\to C_*(X\times Y)$$

that are the canonical maps on the 0-level.

Proof. The proof of this theorem is in many ways quite similar to the proofs of Theorem 56.2 and 56.3.

In the following we show the first statement. The proofs of the other three statements are very similar. So let

$$F, G\colon C_*(X\times Y) \;\to\; C_*(X)\otimes C_*(Y)$$

be two natural chain maps that are the canonical maps on the 0-chains, i.e. which satisfy $F((x,y)) = x\otimes y = G((x,y))$ for all $x\in X, y\in Y$. Given any $k\in\mathbb{Z}_{\geq -1}$ we have to define a map

$$P_k\colon C_k(X\times Y) \;\to\; (C_*(X)\otimes C_*(Y))_{k+1}$$

that satisfies the following three properties:

(1) the map P_{-1} is the zero map,
(2) the map P_k is natural in X and Y,
(3) we have $\partial^{\otimes}\circ P_k + P_{k-1}\circ\partial = F - G$.

So suppose we have already defined the maps $P_{-1}, \dots, P_{k-1}$ with $k \geq 0$. As before we denote by $d_k \colon \Delta^k \to \Delta^k \times \Delta^k$ the singular k-simplex that is given by the diagonal map $d_k(x) := (x, x)$. We first want to define $P_k(d_k) \in (C_*(\Delta^k) \otimes C_*(\Delta^k)_{k+1}$. In the chain complex $C_*(\Delta^k) \otimes C_*(\Delta^k)$ we calculate that

$$\begin{aligned}\partial^{\otimes}((F - G - P_{k-1}\partial)(d_k)) &= \partial^{\otimes} F(d_k) - \partial^{\otimes} G(d_k) - \partial^{\otimes} P_{k-1}(\partial d_k) \\ &= F(\partial d_k) - G(\partial d_k) - (F - G - P_{k-2}\partial)(\partial d_k) = P_{k-2}\partial\partial d_k = 0. \\ &\uparrow \\ &\text{since } F \text{ and } G \text{ are chain maps and since inductively we know that } \partial^{\otimes} \circ P_{k-1} + P_{k-1} \circ \partial = F - G\end{aligned}$$

It follows from Lemma 56.4 that there exists a $(k+1)$-chain $c \in (C_*(\Delta^k) \otimes C_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}(c) = (F - G - P_{k-1}\partial)(d_k)$. (For $k = 0$ we need to show that $\epsilon((F - G)(d_0)) = 0$, to conclude this we need to use that F and G agree on 0-simplices.) We define $P_k(d_k) := c$.

Now let X and Y be two topological spaces and let $\sigma \in C_k(X \times Y)$. As in the proof of Theorem 56.2 we denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the obvious projection maps and we set

$$P_k(\sigma) \ := \ ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(P_k(d_k)) \ \in \ (C_*(X) \otimes C_*(Y))_{k+1}.$$

Basically by definition this map is natural in X and Y. As in the proof of Theorem 56.2 one easily verifies that condition (3) is satisfied. □

Now we can easily provide the proof of the Eilenberg-Zilber Theorem 56.1.

Proof of the Eilenberg-Zilber Theorem 56.1. We pick natural chain maps

$$\Upsilon \colon C_*(X) \otimes C_*(Y) \ \to \ C_*(X \times Y)$$

and

$$\Theta \colon C_*(X \times Y) \ \to \ C_*(X) \otimes C_*(Y)$$

as provided by Theorems 56.2 and 56.3. These maps are the canonical maps on the 0-level. By Theorem 56.5 the maps Υ and Θ are thus unique up to a natural chain homotopy.

It remains to show that the maps Υ and Θ are naturally chain homotopy inverses of one another. The map $\Theta \circ \Upsilon \colon C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$ is a natural chain map that is the canonical map on the 0-level. But this is also the case for the identity map. It follows from Theorem 56.5 that $\Theta \circ \Upsilon$ is naturally chain homotopic to the identity. The same argument shows that $\Upsilon \circ \Theta$ is naturally chain homotopic to the identity. □

56.3. **The relative Eilenberg–Zilber map Υ.** In this short section we introduce a relative version of the Eilenberg–Zilber map Υ. We start out with the following elementary but essential remark.

Remark. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We denote by $i\colon A \to X$ and $j\colon B \to Y$ the inclusion maps. Then the diagram

$$\begin{tikzcd} C_*(A) \otimes C_*(B) \arrow[r, hook, "i_* \otimes j_*"] \arrow[d, "\Upsilon"] & C_*(X) \otimes C_*(Y) \arrow[d, "\Upsilon"] \\ C_*(A \times B) \arrow[r, hook, "(i \times j)_*"] & C_*(X \times Y) \end{tikzcd}$$

commutes by the naturality of Υ. In the following we do not explicitly mention the names of the various inclusion induced maps.

Proposition 56.6. *Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.*

(1) *The Eilenberg-Zilber map Υ induces a natural chain homotopy equivalence*[972]

$$\begin{array}{rcl} C_*(X,A) \otimes C_*(Y,B) & \to & C_*(X \times Y)/(C_*(A \times Y) + C_*(X \times B)) \\ \sum\limits_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum\limits_{i=1}^{n} [\Upsilon(c_i \otimes d_i)]. \end{array}$$

(2) *The Eilenberg-Zilber map Υ descends to a well-defined chain map*

$$\begin{array}{rcl} C_*(X,A) \otimes C_*(Y,B) & \to & C_*(X \times Y, A \times Y \cup X \times B) \\ \sum\limits_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum\limits_{i=1}^{n} \big[\Upsilon(c_i \otimes d_i)\big]. \end{array}$$

Proof. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & 0 \\ 0 \arrow[r] & C_*(A \times Y) \arrow[r] & C_*(X \times Y) \arrow[r] & C_*(X \times Y)/C_*(A \times Y) \arrow[r] & 0. \end{tikzcd}$$

The top sequence is given by tensoring a short exact sequence by the free abelian groups $C_*(Y)$, Lemma 42.9 implies that the resulting sequence is again exact. The bottom sequence is exact by definition. The two left vertical maps are homotopy equivalences by Theorem 56.1 and the diagram commutes by the naturality of Υ, as explained in the remark preceding the proposition. Finally note that it follows from Corollary 49.6 that the right vertical map is also a chain homotopy equivalence.

We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*(X,A) \otimes C_*(B) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y,B) \arrow[r] \arrow[d, "\Upsilon"] & 0 \\ 0 \arrow[r] & C_*(X \times B)/C_*(A \times B) \arrow[r] & C_*(X \times Y)/C_*(A \times Y) \arrow[r] & \dfrac{C_*(X \times Y)}{C_*(A \times Y) + C_*(X \times B)} \arrow[r] & 0. \end{tikzcd}$$

[972]Note that $C_*(A \times Y) + C_*(X \times B)$ is in fact a subcomplex of the chain complex $C_*(X \times Y)$, so the right-hand side is indeed a chain complex.

As above, it follows from the fact that each $C_k(X,A)$ is a free abelian group and Lemma 42.9 that the top sequence is exact. It follows easily from[973]

$$C_*(X\times B)\cap C_*(A\times Y) \;=\; C_*((X\times B)\cap(A\times Y)) \;=\; C_*(A\times B)$$

that the bottom sequence is also exact. The middle vertical map is a chain homotopy equivalence by the above discussion. Exactly the same argument also shows that the left vertical map is a chain homotopy equivalence. The diagram commutes by the naturality of Υ. It follows from Corollary 49.6 that the right vertical map is also a homotopy equivalence.

(2) The second statement follows immediately from (1) together with the observation that $C_*(A\times Y)+C_*(X\times B)$ is a subcomplex of $C_*(A\times Y\cup X\times B)$. □

56.4. **The cross product on homology.** Recall that one of our goals is to find some type of product structure in homology or cohomology. We start our search for a product structure in homology, since we are more comfortable with homology groups.

The following definition already gives us some type of product structure on homology.

Definition. Let (X,A) and (Y,B) be two pairs of topological spaces. We refer to the map[974]

$$\begin{array}{rcl} \times\colon H_p(X,A)\otimes H_q(Y,B) & \xrightarrow{\times} & H_{p+q}(X\times Y, A\times Y\cup X\times B). \\ [c]\otimes[d] & \mapsto & [c]\times[d] := [\Upsilon(c\otimes d)] \end{array}$$

as the *cross product*.

In the following lemma we summarize some elementary properties of the cross product.

Lemma 56.7.

(1) *Let X and Y be two topological spaces and let $x\in X$ be a point. We denote by $f\colon Y\to X\times Y$ the map that is given by $y\mapsto(x,y)$ and we denote by $[x]\in H_0(X)$ the homology class defined by x. Then for any $q\in\mathbb{N}_0$ the maps*

$$H_q(Y) \underset{f_*}{\overset{\sigma\mapsto[x]\times\sigma}{\rightrightarrows}} H_q(X\times Y)$$

agree. The statement holds with the roles of X and Y reversed.

(2) *The cross product is natural, i.e. if $f\colon(X,A)\to(X',A')$ and $g\colon(Y,B)\to(Y',B')$ are maps between pairs of topological spaces, then for any $p,q\in\mathbb{N}_0$ the following*

[973]Why does this equality of chain complexes hold?

[974]Note that the map is well-defined by Lemmas 43.6 and 56.6 (2) and it is independent of the choice of Υ by Theorem 56.1 (2).

diagram commutes:

$$\begin{array}{ccc} H_p(X,A)\otimes H_q(Y,B) & \xrightarrow{\times} & H_{p+q}(X\times Y, A\times Y\cup X\times B) \\ {\scriptstyle f_*}\downarrow \quad \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ H_p(X',A')\otimes H_q(Y',B') & \xrightarrow{\times} & H_{p+q}(X'\times Y', A'\times Y'\cup X'\times B'). \end{array}$$

Proof. The lemma follows immediately from the definitions and the properties of Υ given in Theorem 56.2 (1) and (2). □

We have the following variation of the Künneth Theorem 43.8 for topological spaces.

Theorem 56.8. (The Künneth Theorem) *Let (X,A) and (Y,B) be pairs of topological spaces. We assume that one of the following two statements holds:*

(1) *A and B are open subsets (note that A and B can possibly be empty), or*
(2) *(X,A) and (Y,B) are pairs of CW-complexes.*

Then for each $n\in\mathbb{N}$ there exists a natural short exact sequence

$$\begin{aligned} 0 \;\to\; \bigoplus_{p+q=n} H_p(X,A)\otimes H_q(Y,B) \;&\xrightarrow{\times}\; H_n(X\times Y, A\times Y\cup X\times B) \\ &\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X,A),H_q(Y,B)) \;\to\; 0. \end{aligned}$$

Furthermore the above short exact sequence splits.

Examples.

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Let $x\in M\setminus\partial M$ and let $y\in N\setminus\partial N$. Then it follows from the relative Künneth Theorem 56.8 (2), Lemma 45.16 and Lemma 42.17 that the cross product defines an isomorphism

$$H_m(M,M\setminus\{x\})\otimes H_n(N,N\setminus\{y\}) \xrightarrow{\times} H_{m+n}(M\times N, \underbrace{(M\setminus\{x\})\times N\cup M\times(N\setminus\{y\})}_{=(M\times N)\setminus\{(x,y)\}}).$$

(2) Let X and Y be path-connected topological spaces. Let $x\in X$ and let $y\in Y$. We denote by $i\colon X\to X\times Y$ the inclusion map that is given by $i(x)=(x,y)$ and similarly we denote by $j\colon Y\to X\times Y$ the inclusion map that is given by $j(y)=(x,y)$. It follows from Proposition 29.5, Lemma 56.7 (1) and the Künneth Theorem 56.8 that the map

$$i_*\oplus j_*\colon H_1(X;\mathbb{Z})\oplus H_1(Y;\mathbb{Z}) \;\to\; H_1(X\times Y;\mathbb{Z})$$

is an isomorphism. Let G be any abelian group. It follows easily from the above, the naturality of the Universal Coefficient Theorem 42.19 and Lemma 42.17 (3) that the map

$$i_*\oplus j_*\colon H_1(X;G)\oplus H_1(Y;G) \;\to\; H_1(X\times Y;G)$$

is also an isomorphism.

Proof. We consider the following maps

$$\begin{array}{c} 0 \to \bigoplus_{p+q=n} H_p(X,A)\otimes H_q(Y,B) \to H_n(C_*(X,A)\otimes C_*(Y,B)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X,A),H_q(Y,B)) \to 0. \\ \downarrow \Upsilon \\ H_n(C_*(X\times Y)/(C_*(A\times Y)+C_*(X\times B))) \\ \downarrow \\ H_n(C_*(X\times Y)/C_*(A\times Y\cup X\times B)) \\ \downarrow = \\ H_n(X\times Y, A\times Y\cup X\times B) \end{array}$$

We first note that the top sequence is exact by the Künneth Theorem for Chain Complexes 43.7. (Here we use again that relative chain groups are free abelian groups.) Here the top vertical map is an isomorphism by Proposition 56.6. Recall that we assume that one of the following two statements holds:

(1) A and B are open subsets, or
(2) (X,A) and (Y,B) are pairs of CW-complexes.

We then know from Proposition 50.12 (4) respectively (6) that the inclusion induced map

$$\underbrace{C_*^{\{A\times Y, X\times B\}}(A\times Y\cup X\times B)}_{=C_*(A\times Y)+C_*(X\times B)} \;\to\; C_*(A\times Y\cup X\times B)$$

is a chain homotopy equivalence. The same argument as in the proof of Corollary 31.16 shows that this piece of information implies that the second vertical map is indeed an isomorphism. It follows immediately from the definition of the cross product that the induced "diagonal map"

$$\bigoplus_{p+q=n} H_p(X,A)\otimes H_q(Y,B) \;\to\; H_n(X\times Y, A\times Y\cup X\times B)$$

is precisely the cross product. □

56.5. **Product orientations.** In this second to last section we want to consider products of (topological) manifolds. We start out with the following, arguably slightly surprising proposition. It can be viewed as a generalization of Propositions 3.4, 3.42 and Proposition 32.1 (6).

Proposition 56.9.

(1) *Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. The topological space $M\times N$ is an $(m+n)$-dimensional topological manifold where the boundary is given by*

$$\partial(M\times N) \;=\; \partial M\times N \cup M\times \partial N.$$

(2) *Given any two manifolds M and N we can equip $M\times N$ with a natural smooth structure such that the following conditions are satisfied:*

(a) *If $\partial M = \varnothing$ or $\partial N = \varnothing$, then the smooth structure on $M \times N$ is the obvious smooth structure given by taking products of charts.*

(b) *For any two manifolds M and N the following statements hold:*

  (i) *the inclusion map $\overset{\circ}{M} \times \overset{\circ}{N} \to M \times N$ is smooth,*[975]

  (ii) *for any $P \in M$ and $Q \in N$ the inclusion maps*

$$\begin{array}{rcl} M & \to & M \times N \\ x & \mapsto & (x, Q) \end{array} \quad \text{and} \quad \begin{array}{rcl} N & \to & M \times N \\ y & \mapsto & (P, y) \end{array}$$

are smooth.

Example. Let $M = \overline{B}^2$ and $N = [0,1]$, then $M \times N = \overline{B}^2 \times [0,1]$ is the solid cylinder which is a 3-dimensional topological manifold with boundary

$$\partial\big(\overline{B}^2 \times [0,1]\big) \;=\; \partial\overline{B}^2 \times [0,1] \cup \overline{B}^2 \times \partial([0,1]) \;=\; S^1 \times [0,1] \cup \overline{B}^2 \times \{0,1\}.$$

We refer to Figure 717 for an illustration.

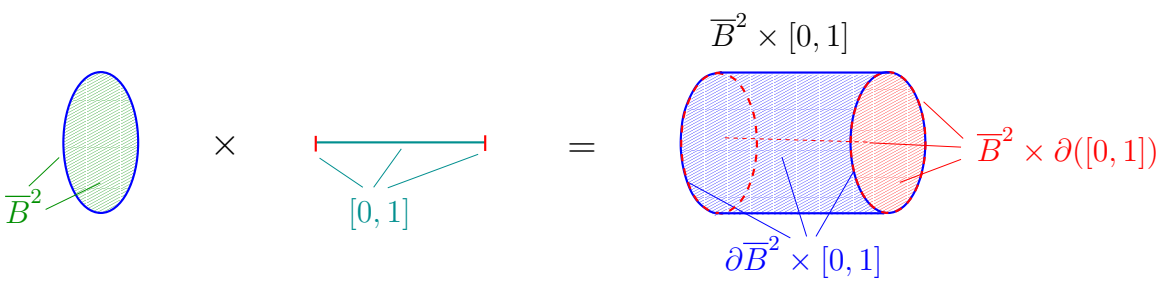


FIGURE 717.

Proof. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. We consider the topological space $M \times N$. By Lemma 1.55 we remarked that the product of two Hausdorff spaces is again Hausdorff. Thus we see that $M \times N$ is Hausdorff. It follows from Lemma 3.1 that $M \times N$ is second-countable.

It remains to find charts for $M \times N$. More precisely, we have to find a chart of type (i) for each point in $(M \times N) \setminus (\partial M \times N \cup M \times \partial N)$ and we have to find a chart of type (ii) for each point in $\partial M \times N \cup M \times \partial N$. So let $(x, y) \in M \times N$. We distinguish three cases:

(1) If $x \in M \setminus \partial M$ and $y \in N \setminus \partial N$ then taking "product charts" it is straightforward to find a chart of type (i) for $(x, y) \in M \times N$.

(2) If $x \in M \setminus \partial M$ and $y \in \partial N$ or if $x \in \partial M$ and $y \in N \setminus \partial N$ then it is equally straightforward to find a chart of type (ii) for $(x, y) \in M \times N$.

[975]Note that it follows from Proposition 32.1 that $\overset{\circ}{M} = M \setminus \partial M$ and $\overset{\circ}{N} = N \setminus \partial N$ do not have boundary. Hence the smooth structure on $\overset{\circ}{M} \times \overset{\circ}{N}$ is given by the products of charts.

(3) Finally suppose that $x \in \partial M$ and $y \in \partial N$. We have to find a chart of type (ii) for $(x,y) \in M \times N$. It is straightforward to see that there exists an open neighborhood U of $(x,y) \in M \times N$ and a homeomorphism $\Phi\colon U \to \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. We then compose this homeomorphism with the homeomorphism

$$\begin{array}{rcl} \Psi\colon \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^{m+n-2} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(\varphi), r\sin(\varphi)}_{=:z\in\mathbb{C}}) & \mapsto & (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(2\varphi), r\sin(2\varphi)}_{=z^2}). \end{array}$$

It is straightforward to see that $\Psi \circ \Phi$ is a chart of type (ii) around (x,y).

If we start out with a smooth atlas for M and N then one can easily verify that the atlas that we have just constructed is also smooth and that it has the properties stated in (a) and (b). The statement that the smooth structure on $M \times N$ is natural is shown in [Wall2, Proposition 2.6.2]. □

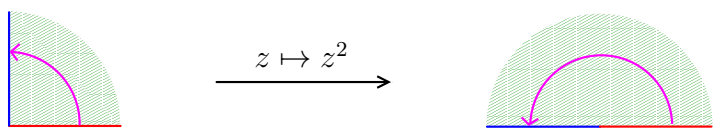


FIGURE 718. Illustration of the proof of Proposition 56.9.

The cross product now allows us to formulate the following proposition.

Proposition 56.10. *Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Suppose M is equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ and that N is equipped with an orientation $\{\nu_y\}_{y\in N\setminus\partial N}$.*

(1) *Given $(x,y) \in (M \times N) \setminus \partial(M \times N)$ we denote by $\rho_{(x,y)}$ the image of $\mu_x \otimes \nu_y$ under the isomorphism*

$$H_m(M, M \setminus \{x\}) \otimes H_n(N, N \setminus \{y\}) \xrightarrow{\times} H_{m+n}(M \times N, (M \times N) \setminus \{(x,y)\})$$

from page 1317. Then these classes $\rho_{(x,y)}$ define an orientation on $M \times N$.

(2) *Suppose both M and N are compact and connected, then the map*

$$H_m(M, \partial M) \otimes H_n(N, \partial N) \xrightarrow{\times} H_{m+n}(M \times N, \partial(M \times N))$$

given by the cross product is an isomorphism and it has the property that

$$[M] \times [N] \;=\; [M \times N] \;\in\; H_{m+n}(M \times M, \partial(M \times N))$$

where $[M \times N]$ denotes the fundamental class of the compact topological manifold $M \times N$ equipped with the orientation defined in (1).

Proof. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.

(1) The first statement of the proposition is a straightforward consequence of the definitions and the naturality of the cross product, see Lemma 56.7. For completeness' sake we sketch the proof. To simplify the notation we assume that $\partial M = \partial N = \varnothing$. Now let $(x, y) \in M \times N$. We pick an open neighborhood U of x and a class $\mu_U \in H_m(M, M\backslash U)$ and an open neighborhood V of y and a class $\nu_V \in H_n(N, N\backslash V)$ as given by the definition on page 1095. By the naturality of the cross product we obtain the following commutative diagram

$$\begin{tikzcd}
H_m(M, M\backslash U) \times H_n(N, N\backslash V) \arrow[r, "\times"] \arrow[d, "\mu_U \mapsto \mu_x"', "\nu_V \mapsto \nu_y"] & H_{m+n}(M \times N, M \times N\backslash(U \times V)) \arrow[d] \\
H_m(M, M\backslash\{x\}) \times H_n(N, N\backslash\{y\}) \arrow[r, "\times", "(\mu_x, \nu_y) \mapsto \rho_{(x,y)}"'] & H_{m+n}(M \times N, M \times N\backslash\{(x, y)\}).
\end{tikzcd}$$

It is now straightforward to verify that $\phi_{U\times V} := \mu_U \times \nu_V$ has the properties desired by the definition on page 1095.

(2) Now suppose that both M and N are compact. By Lemma 1.55, the product $M \times N$ is then also compact. Now let $x \in M \setminus \partial M$ and let $y \in N \setminus \partial N$. By Proposition 56.9 we have $(x, y) \in (M \times N) \setminus \partial(M \times N)$. Furthermore, by Lemma 56.7 we have the following commutative diagram

$$\begin{tikzcd}
H_m(M, \partial M) \otimes H_n(N, \partial N) \arrow[r, "\times"] \arrow[d] & H_{m+n}(M \times N, \partial(M \times N)) \arrow[d] \\
H_m(M, M \setminus \{x\}) \otimes H_n(N, N \setminus \{y\}) \arrow[r, "\times"] & H_{m+n}(M \times N, M \times N \setminus \{(x, y)\}).
\end{tikzcd}$$

The vertical maps are isomorphisms by Theorem 46.2. On page 1317 we saw that the bottom horizontal map is an isomorphism. It follows that, as claimed, the top horizontal map is also an isomorphism. The statement regarding the fundamental classes is an immediate consequence of the commutativity of the diagram and the definitions. □

Definition. Let M and N be oriented topological manifolds. We refer to the orientation on $M \times N$ defined in Proposition 56.10 (1) as the *product orientation.*

Remark. Recall that given any open subset $U \subset \mathbb{R}^k$, $k \in \mathbb{N}_0$, we introduced on page 1101 the standard orientation for U viewed as a topological manifold. (See also Figure 719 for a reminder.) The following lemma says that the above definition of the product orientation gives us the "expected" answer when we deal with open subsets of $\mathbb{R}^k$'s.

Proposition 56.11. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open non-empty subsets. We equip U and V with the standard orientation. Then the product orientation of $U \times V$, defined above, agrees with the standard orientation of $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.*

We postpone the proof of Proposition 56.11 to the next section.

Examples.

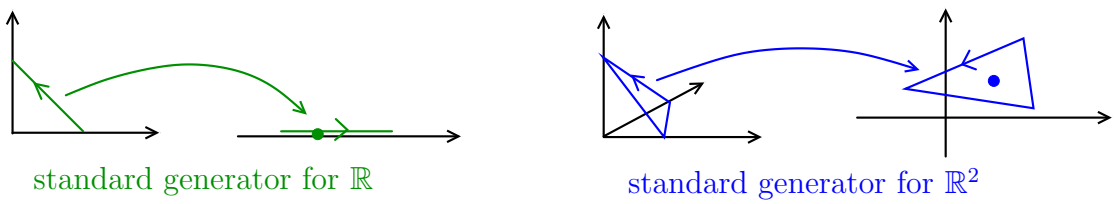


FIGURE 719.

(1) Let M and N be two oriented manifolds such that at most one of them has non-empty boundary. We have the following diagram

$$\begin{tikzcd}
\text{oriented manifolds } M, N \arrow[r, squiggly, "\text{Proposition 45.22}"] \arrow[d, squiggly, "\text{page 205}"] & \text{oriented topological manifolds } M, N \arrow[d, squiggly, "\text{see above}"] \\
\text{oriented manifold } M \times N \arrow[r, squiggly, "\text{Proposition 45.22}"] & \text{oriented topological manifold } M \times N.
\end{tikzcd}$$

It follows fairly easily from Proposition 56.11 that this diagram commutes.

(2) We write $I = [-1, 1]$. We view I as a 1-dimensional topological manifold and we equip I with the orientation coming from $\mathbb{R}$. Given $k \in \mathbb{N}_0$ we write

$$\partial I^k \;:=\; \{(x_1, \dots, x_k) \,|\, \text{there exists an } i \text{ with } x_i = -1 \text{ or } x_i = 1\}.$$

Given any $m, n \in \mathbb{N}$ one sees easily that

$$\partial(I^m \times I^n) \;=\; \partial I^m \times I^n \cup I^m \times \partial I^n \;=\; \partial(I^{m+n}).$$

(See also Figure 720 for an illustration.) It follows from Proposition 56.9 and the above discussion that we can view each $I^k = I \times \cdots \times I$ as a k-dimensional topological manifold with boundary given by the above subset ∂I^k. We equip I^k with the product orientation. It follows immediately from Proposition 56.11 that the orientation on the open subset $I^k \setminus \partial I^k \subset \mathbb{R}^k$ is the standard orientation.[976] As usual we denote by $[I^k] \in H_k(I^k, \partial I^k; \mathbb{Z})$ the fundamental class.

Now let $m, n \in \mathbb{N}_0$. We get a cross product

$$\overbrace{H_m(I^m, \partial I^m)}^{=\mathbb{Z}\cdot[I^m]} \otimes \overbrace{H_n(I^n, \partial I^n)}^{=\mathbb{Z}\cdot[I^n]} \;\to\; \overbrace{H_{m+n}(I^{m+n}, \partial I^{m+n})}^{=\mathbb{Z}\cdot[I^{n+n}]}$$

which by Proposition 56.10 (2) has the property that

$$[I^m] \times [I^n] \;=\; [I^{m+n}].$$

[976]Note that the homeomorphism

$$f\colon (I^k, \partial I^k) \;\to\; (\overline{B}^k, S^{k-1})$$

from Proposition 1.47 (2) has the property that the restriction to a dense open subset of $I^k \setminus \partial I^k \to B^k$ is an orientation-preserving diffeomorphism. In particular this homeomorphism is thus an orientation-preserving homeomorphism from the oriented topological manifold I^k to the oriented topological manifold $\overline{B}^k$.

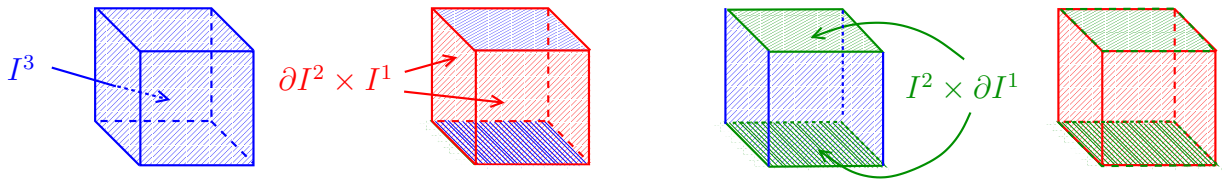


FIGURE 720.

56.6. **The shuffle product and the proof of Proposition 56.11** (∗)**.** In this section we will provide the proof of Proposition 56.11. The difficulty in the proof is that the definition of the cross product relies on the map Υ from Theorem 56.2. We gave a non-constructive proof of the existence of the map Υ. In this section we give an *explicit* description of a map ∇ which has all the properties that we had asked for in Theorem 56.2. We will then use this map ∇ to prove Proposition 56.11.

First we recall some notation from page 736.

Notation. For $i = 0, \dots, n$ we write

$$v_i \;:=\; (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \dots, a_k \in \Delta^n$ we consider the map

$$\begin{aligned} [a_0, \dots, a_k] \colon \Delta^k &\to \Delta^n \\ \sum_{j=0}^{k} \lambda_j v_j &\mapsto \sum_{j=0}^{k} \lambda_j a_j. \end{aligned}$$

The map $[a_0, \dots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex "spanned by $a_0, \dots, a_k$" in Δ^n.

Definition. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-*shuffle* is a pair (μ, ν) of disjoint sets of integers

$$1 \le \mu_1 < \mu_2 < \dots < \mu_m \le m+n \quad \text{and} \quad 1 \le \nu_1 < \nu_2 < \dots < \nu_n \le m+n.$$

We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n)$. We denote by $\eta^\mu \colon \Delta^{m+n} \to \Delta^m$ the unique affine linear map which satisfies $\eta^\mu(v_i) = v_j$ for $\mu_j \le i < \mu_{j+1}$.[977] The same way we define $\eta^\nu \colon \Delta^{m+n} \to \Delta^n$.

(2) Given two topological spaces X and Y we define

$$\begin{aligned} \nabla_{mn} \colon C_m(X) \otimes C_n(Y) &\to C_{m+n}(X \times Y) \\ (\sigma \otimes \tau) &\mapsto \sum_{(\mu,\nu)} \operatorname{sign}(\mu,\nu) \cdot \underbrace{(\overbrace{\sigma \circ \eta^\mu}^{\Delta^{m+n} \to X} \times \overbrace{\tau \circ \eta^\nu}^{\Delta^{m+n} \to Y})}_{\text{map } \Delta^{m+n} \to X \times Y} \end{aligned}$$

where we take the sum over all (m,n)-shuffles.

Examples.

[977] Here we implicitly use that $\mu_0 = 0$ and $\mu_{m+1} = m+n+1$.

(1) We consider the case $X = Y = \mathbb{R}$ and $m = n = 1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. Note that $\eta^1\colon \Delta^2 \to \Delta^1$ equals the map $[v_0, v_1, v_1]$ and that $\eta^2\colon \Delta^2 \to \Delta^1$ equals the map $[v_0, v_0, v_1]$. In Figure 721 we illustrate $\nabla_{11}(\sigma \otimes \tau)$ for two affine linear singular 1-simplices $\sigma\colon \Delta^1 \to X = \mathbb{R}$ and $\tau\colon \Delta^1 \to Y = \mathbb{R}$.

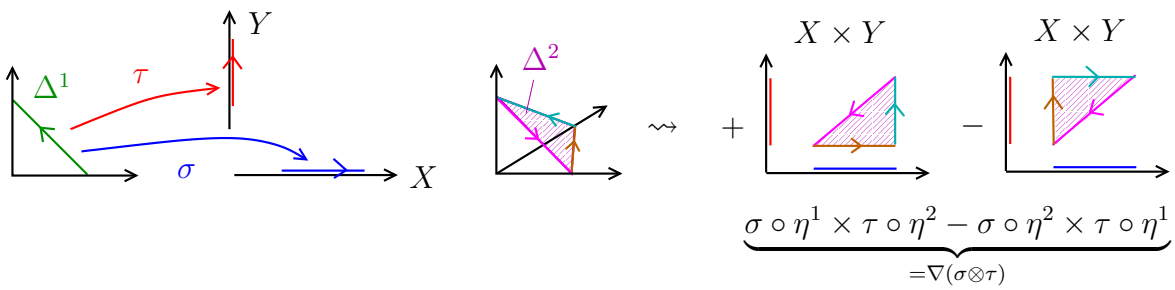


FIGURE 721. Illustration of the definition of the shuffle map.

(2) We consider the case $X = Y = S^1$ and $m = n = 1$. Furthermore we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t, t) = e^{2\pi i t}$. On page 1118 we saw that μ represents the fundamental class of S^1 with the standard orientation. Comparing Figures 654 and 721 we see that the shuffle $\nabla(\mu \otimes \mu)$ is precisely the usual representative for the fundamental class of the 2-dimensional torus $S^1 \times S^1 = ([0,1] \times [0,1])/\sim$.

Proposition 56.12. *The maps ∇ that we had just defined have all the properties that we had mentioned in Theorem 56.2, i.e. the maps $\nabla\colon C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ are natural chain maps that on the 0-level equal the canonical maps that we introduced on page 1308.*

Proof. It follows immediately from the definition that the maps ∇_{mn} are natural and that on the 0-level they agree with the canonical maps. It is elementary (albeit rather painful) to verify that these maps are chain maps. We refer to [Dol, p. 184] and [EM2] for details and proofs. □

Now we can finally provide the proof of Proposition 56.11.

Proof of Proposition 56.11. It suffices to prove the statement for $U = \mathbb{R}^m$ and $V = \mathbb{R}^n$. Given any $k \in \mathbb{N}_0$ we consider the singular chain $\sigma_k = (-1)^k \cdot [v_0, \dots, v_k, 0]$. (In plain English the singular simplex $[v_0, \dots, v_k, 0]\colon \Delta^k \to \mathbb{R}^k$ is just given by the obvious projection.) We make the following observation:

(∗) Let $z \in \mathbb{R}^k$ be any point that lies in the interior of the image of σ_k. It follows from the discussion on page 1089 that the chain σ_k represents the standard generator of $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{z\})$.

Next we consider the (m,n)-shuffle that is given by the sets $\mu_0 = \{1, \dots, m\}$ and by $\nu_0 = \{m+1, \dots, m+n\}$. It follows easily from the definitions that

$$\eta^{\mu_0} = [v_0, \dots, v_m, \dots, v_m]\colon \Delta^{m+n} \to \Delta^m \quad \text{and} \quad \eta^{\nu_0} = [v_m, \dots, v_m, \dots, v_{m+n}]\colon \Delta^{m+n} \to \Delta^n.$$

Also note that it is a straightforward consequence of the definitions that we have the following equality of singular $(m+n)$-chains in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$:

$$(\sigma_m \circ \eta^{\mu_0}) \times (\sigma_n \circ \eta^{\nu_0}) \;=\; (-1)^m \cdot (-1)^n \cdot [v_0, \ldots, v_{m-1}, v_m, \ldots, v_{m+n-1}, 0] \;=\; \sigma_{m+n}.$$

We pick a point $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ that lies in the interior of the image of σ_{m+n}. Note that x lies in the interior of the image of σ_m and that y lies in the interior of the image of σ_n.

It follows from the observation $(*)$, the definition of the product orientation on page 1321 and Proposition 45.17 (1) that it suffices to prove the following claim.

Claim. The following equality holds in $H_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{(x,y)\})$:

$$[\sigma_m] \times [\sigma_n] \;=\; [\sigma_{m+n}].$$

We have

$$\begin{aligned}
[\sigma_m] \times [\sigma_n] \;&\underset{\substack{\uparrow \\ \text{Proposition 56.12 and Theorem 56.5}}}{=}\; [\nabla_{mn}(\sigma_m \otimes \sigma_n)] \;\underset{\substack{\uparrow \\ \text{by definition}}}{=}\; \Big[\sum_{(\mu,\nu)} \operatorname{sign}(\mu,\nu) \cdot (\sigma_m \circ \eta^\mu \times \sigma_n \circ \eta^\nu)\Big] \\
&\underset{\substack{\uparrow \\ \text{for all other shuffles } (\mu,\nu) \text{ the point } (x,y) \text{ does not lie in the image of } \sigma_m \circ \eta^\mu \times \sigma_n \circ \eta^\nu}}{=}\; \Big[\underbrace{\operatorname{sign}(\mu_0,\nu_0)}_{=1} \cdot \underbrace{(\sigma_m \circ \eta^{\mu_0} \times \sigma_n \circ \eta^{\nu_0})}_{=\sigma_{m+n}}\Big] \;=\; [\sigma_{m+n}].
\end{aligned}$$

This concludes the proof of the claim and thus of the proposition. □

56.7. **The Acyclic Model Theorem.** In this section we will prove two "Acyclic Model Theorems" which on many occasions will allow us to construct maps and homotopies. We start out with the following fairly friendly Acyclic Model Theorem. We will need this particular Acyclic Model Theorem later on when we construct Steenrod operations.

Theorem 56.13. (Acyclic Model Theorem) *Let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized*[978] *chain complexes. Let $G\colon \mathcal{T} \to \mathcal{C}$ be a coinvariant functor. Furthermore S be a natural transformation from the functor $X \mapsto C_*(X)$ to the functor $X \mapsto G(X)$. If $H_i(G(\Delta^k)) = 0$ for any $k \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$, then there exists a natural chain homotopy P from S to the zero map. In other words, for any topological space X and any $k \in \mathbb{N}$ there exists a natural map $P_X\colon C_k(X) \to G_{k+1}(X)$, such that for any $k \in \mathbb{N}_0$ we have*

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon C_k(X) \;\to\; G_k(X).$$

Sketch of proof. The proof of this theorem is very similar to the proof of Statement (2) of Theorem 56.5. First recall that the fact that S is a natural transformation means that for

[978] Here "generalized" just means, see page 745, that we also allow chain groups in negative degrees.

any topological space X we get a commutative diagram as follows:

$$\begin{tikzcd}
\dots \arrow[r] & C_2(X) \arrow[r, "\partial_2"] \arrow[d, "S"] & C_1(X) \arrow[r, "\partial_1"] \arrow[d, "S"] & C_0(X) \arrow[r] \arrow[d, "S"] & 0 \arrow[r] \arrow[d, "S"] & 0 \arrow[r] \arrow[d, "S"] & \dots \\
\dots \arrow[r] & G_2(X) \arrow[r, "\partial_2"] & G_1(X) \arrow[r, "\partial_1"] & G_0(X) \arrow[r, "\partial_0"] & G_{-1}(X) \arrow[r, "\partial_{-1}"] & G_{-2}(X) \arrow[r, "\partial_{-2}"] & \dots
\end{tikzcd}$$

Now we consider $\mathrm{id}_{\Delta^0} \in C_0(\Delta^0)$. It is a cycle in the chain complex $C_*(\Delta^0)$. Since the above diagram commutes we see that $S(\mathrm{id}_{\Delta^0})$ is also cycle in $G_0(\Delta^0)$. Since we assume that $H_0(G(\Delta^0)) = 0$ we see that there exists some $\mu_1 \in G_1(\Delta^0)$ with $\partial_1(\mu_1) = S(\mathrm{id}_{\Delta^0})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_0 \colon C_0(X) & \to & G_1(X) \\
\sum_{i=1}^m a_i \cdot \underset{\substack{\uparrow \\ \text{map } \Delta^0 \to X}}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_1).
\end{array}$$

We iterate this procedure. Namely suppose that for $i = 0, \dots, k-1$ we have already constructed natural maps $P_i \colon C_i(X) \to G_{i+1}(X)$ such that $\partial \circ P_i + P_{i-1} \circ \partial = S_i$ for $i = 0, \dots, k-1$. For reference we consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & C_{k+1}(X) \arrow[r, "\partial_{k+1}"] \arrow[d, "S_{k+1}"] & C_k(X) \arrow[r, "\partial_k"] \arrow[d, "S_k"] \arrow[dl, dashed, "P_k"] & C_{k-1}(X) \arrow[r] \arrow[d, "S_{k-1}"] \arrow[dll, "P_k - 1"] & \dots \\
\dots \arrow[r] & G_{k+1}(X) \arrow[r, "\partial_{k+1}"'] & G_k(X) \arrow[r, "\partial_k"'] & G_{k-1}(X) \arrow[r, "\partial_{k-1}"] & \dots
\end{tikzcd}$$

Next we need to construct a suitable map P_k. We calculate that

$$\begin{aligned}
\partial_k((S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta_k})) &= \partial S_k(\mathrm{id}_{\Delta_k}) - \partial P_{k-1}(\partial\, \mathrm{id}_{\Delta_k}) \\
&\underset{\substack{\uparrow \\ \text{since } S \text{ is a chain map and since } \partial \circ P_{k-1} + P_{k-1} \circ \partial = S_i}}{=} S_k(\partial\, \mathrm{id}_{\Delta_k}) - (S_k - P_{k-2}\partial)(\partial\, \mathrm{id}_{\Delta_k}) = (P_{k-2} \circ \overbrace{\partial \circ \partial}^{=0})(\mathrm{id}_{\Delta_k}) = 0 \,.
\end{aligned}$$

Since we assume that $H_k(G(\Delta^k)) = 0$ we see that there exists some $\mu_{k+1} \in G_{k+1}(\Delta^k)$ with $\partial_{k+1}(\mu_{k+1}) = (S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta_k})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_k \colon C_k(X) & \to & G_{k+1}(X) \\
\sum_{i=1}^m a_i \cdot \underset{\substack{\uparrow \\ \text{map } \Delta^k \to X}}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_{k+1}).
\end{array}$$

Using the fact that S is natural one can easily verify that P_k has the desired properties. $\square$

The above version of the Acyclic Model Theorem will not work for the situations that we need to deal with in the near future. There are two problems with the above Acyclic Model Theorem:

(1) On several occasions we will be interested in natural transformations of the form $C_*(X) \to C_*(X) \otimes C_*(X)$, but for $X = \Delta^k$ the chain complexes to the right does not satisfy the condition that the homology groups for $X = \Delta^k$ vanish in dimensions ≥ 0.
(2) We will also need to deal with situations where to the left we have more complicated chain complexes to the left, e.g. we might want to study natural transformations of the type $C_*(X) \times C_*(X) \to C_*(X \times X)$.

To formulate a version of the Acyclic Model Theorem that deals with the above two issues we need a few more definitions.

Definition.

(1) An *augmented chain complex* is a chain complex
$$(C_*, \partial_*) \;=\; \ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$
together with an epimorphism $\epsilon\colon C_0 \to \mathbb{Z}$ such that $\epsilon \circ \partial_1 \colon C_1 \to \mathbb{Z}$ is the zero map.
(2) We say that an augmented chain complex $((C_*, \partial_*), \epsilon\colon C_0 \to \mathbb{Z})$ is *acyclic* if the generalized chain complex
$$\begin{array}{ccccccccc} \ldots \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \to 0 \\ & & & & & & & \uparrow & \\ & & & & & & & \text{degree } -1 & \end{array}$$
is acyclic, i.e. if all homology groups are zero.
(3) Let $((C_*, \partial_*), \epsilon)$ and $((C'_*, \partial'_*), \epsilon')$ be two augmented chain complexes. A *chain map of augmented chain complexes* is a chain map $f_*\colon C_* \to C'_*$ such that the following diagram commutes:
$$\begin{array}{ccc} C_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle =} \\ C'_0 & \xrightarrow{\epsilon'} & \mathbb{Z}. \end{array}$$
(4) We refer to the category $\mathcal{A}$ with
$$\begin{aligned} \mathrm{Ob}(\mathcal{A}) &= \text{augmented chain complexes} \\ \mathrm{Mor}((C,\epsilon),(C',\epsilon')) &= \begin{array}{c}\text{set of chain maps of augmented chain complexes} \\ \text{from } (C,\epsilon) \text{ to } (C',\epsilon')\end{array} \end{aligned}$$
as the *category of augmented chain complexes.*

Examples.

(1) Let X be a non-empty topological space. We denote by $\epsilon_X\colon C_0(X) \to \mathbb{Z}$ the augmentation that we introduced on page 742, i.e. ϵ_X is the map that sends each singular 0-simplex to 1. It follows from Lemma 29.4 (1) that $(C_*(X), \epsilon_X)$ is an augmented chain complex. If X is a contractible topological space, e.g. if $X = \Delta^k$, then it follows from Lemma 31.1 (7) that the corresponding augmented chain complex is acyclic.

(2) If (C_*, μ) and (D_*, ν) are two augmented chain complexes then the tensor product $C_* \otimes D_*$ of chain complexes, as defined in Lemma 43.2, together with the map

$$\begin{array}{rcl} \mu \otimes \nu \colon (C_* \otimes D_*)_0 = C_0 \otimes D_0 & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{k} c_i \otimes d_i & \mapsto & \sum\limits_{i=1}^{k} \mu(c_i) \cdot \nu(d_i) \end{array}$$

is easily seen to be an augmented chain complex.[979]

Definition. We say a chain complex C_* is *almost acyclic* if $H_i(C_*) = 0$ for $i \geq 1$ and if $H_0(C_*) \cong \mathbb{Z}$.

Example. If X is a contractible topological space, then it follows from Corollary 30.8 that $C_*(X)$ is almost acyclic.

Proposition 56.14. *Let C_* and D_* be two chain complexes. If C_* and D_* are almost acyclic and if C_* is free, i.e. if each chain group C_k is a free abelian group, then $C_* \otimes D_*$ is also almost acyclic.*

Proof. Let $n \in \mathbb{N}_0$. Since C_* is free we can apply the Künneth Theorem for Chain Complexes 43.7 which gives us the following short exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \xrightarrow{\Theta} H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C_*), H_q(D_*)) \to 0.$$

The proposition is an immediate consequence of this short exact sequence and of some very elementary statements regarding tensor products and torsion groups, see Lemma 42.3 and 42.17. □

The following lemma gives a convenient criterion for showing that augmented chain complexes are acyclic.

Lemma 56.15. *Let $((C_*, \partial_*), \epsilon)$ be an augmented chain complex. If the chain complex (C_*, ∂_*) is almost acyclic, then the augmented chain complex $((C_*, \partial_*), \epsilon)$ is acyclic.*

Proof $(*)$. We denote by $\widetilde{C}_*$ the generalized chain complex

$$\ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

We need to show that this chain complex is acyclic. It is clear that for $i \geq 1$ we have $H_i(\widetilde{C}_*) = H_i(C_*) = 0$. It remains to show that $H_0(\widetilde{C}_*) = 0$. Since $\mathbb{Z}$ is a free abelian group

[979] Indeed, by definition $(C_* \otimes D_*)_1$ is generated by elements $c_1 \otimes d_0$ with $C_1 \in C_1$ and $d_0 \in D_0$ and by elements $c_0 \otimes d_1$ with $C_0 \in C_0$ and $d_1 \in D_1$. We calculate that

$$((\mu \otimes \nu) \circ \partial_1)(c_1 \otimes d_0) = (\mu \otimes \nu)(\partial c_1 \otimes d_0 + (-1)^1 \cdot c_1 \otimes \underbrace{\partial d_0}_{=0}) = (\mu \otimes \nu)(\partial c_1 \otimes d_0) = \mu(\partial c_1) \cdot \nu(d_0) = 0 \cdot \nu(d_0) = 0.$$

The same argument shows that $((\mu \otimes \nu) \circ \partial_1)(c_0 \otimes d_1) = 0$.

and since $\epsilon C_0 \to \mathbb{Z}$ is an epimorphism we can pick a map $s\colon \mathbb{Z} \to C_0$ with $\epsilon \circ s = \mathrm{id}_{\mathbb{Z}}$. As in Lemma 31.1 (4) one can easily show that the map

$$\begin{array}{rcl} H_0(\widetilde{C}_*) \oplus \mathbb{Z} & \to & H_0(C_*) \\ [\sigma] \oplus n & \mapsto & [\sigma] + s(n) \end{array}$$

is an isomorphism. Since $H_0(C) \cong \mathbb{Z}$ it follows from Exercise 11.2 (b) (see also more generally Proposition 11.2) that $H_0(\widetilde{C}_*) = 0$. □

Definition. Let $\mathcal{K}$ be a category (in all applications $\mathcal{K}$ will be the category of non-empty topological spaces or the category of tuples of non-empty topological spaces) and let $F\colon \mathcal{K} \to \mathcal{A}$ be a covariant functor from $\mathcal{K}$ to the category $\mathcal{A}$ of augmented chain complexes. Let $\mathcal{M}$ be a collection of objects of $\mathcal{K}$.

(1) We say the functor F is *acyclic relative to* $\mathcal{M}$ if given any $X \in \mathcal{M}$ the augmented chain complex $F(X)$ is acyclic.
(2) We say the functor F is *free relative to* $\mathcal{M}$ if for each $k \in \mathbb{N}_0$ there exists $\mathcal{M}_k \subset \mathcal{M}$ and if for each $M \in \mathcal{M}_k$ there exists an element $\sigma_M \in F_k(M)$[980] such that given any $X \in \mathcal{K}$ the following two statements hold:
   (a) The set

$$\{\underbrace{F(f)}_{\substack{\uparrow \\ \text{the morphism } f\colon M \to X \text{ induces a homomorphism } F(f)\colon F_k(M) \to F_k(X)}}(\sigma_M) \in F_k(X) \mid \text{ with } M \in \mathcal{M}_k \text{ and } f \in \mathrm{Mor}(M, X)\}$$

   is a basis for the abelian group $F_k(X)$.[981]
   (b) The description of the set in (a) has no double elements, much more precisely, for $(M, f) \neq (M', f')$ we have $F(f)(\sigma_M) \neq F(f')(\sigma_{M'})$.

Remark. The objects in a collection $\mathcal{M}$ as in the previous definition are often called *models*.

The definition of "free relative to $\mathcal{M}$" is arguably somewhat intimidating. The following example should convince the reader that in practice it is not that bad.

Example. Let $\mathcal{K}$ be the category of non-empty topological spaces and let F be the covariant functor that assigns to each $X \in \mathcal{K}$ the augmented singular chain complex $(C_*(X), \epsilon_X)$. We consider

$$\mathcal{M} \ := \ \{\Delta^k \mid k \in \mathbb{N}_0\} \ = \ \text{the set of all standard simplices.}$$

We claim that F is acyclic and free relative to $\mathcal{M}$:

(1) By Lemma 31.1 (7) we know that for every $k \in \mathbb{N}_0$ we have $\widetilde{H}_*(\Delta^k) = 0$. But this just means that $\mathcal{M}$ is acyclic relative to F.

[980] Here $F_k(M)$ is the the k-chain group of the chain complex $F(M)$.
[981] In particular this implies that each $F_k(X)$ is a free abelian group.

(2) Let $k \in \mathbb{N}_0$. We set $\mathcal{M}_k = \{\Delta_k\}$ and we consider $\sigma_k = \mathrm{id}_{\Delta^k} \in C_k(\Delta_k)$. Now recall that given any topological space X and given any $k \in \mathbb{N}_0$ we have

$$\begin{array}{rcl} & \text{by definition of } C_k(X)\text{, see page 736} & \\ & \downarrow & \\ C_k(X) & = & \text{free abelian group generated by the set of singular } k\text{-simplices in } X \\ & = & \text{free abelian group generated by the set of all maps } \underbrace{f\colon \Delta^k \to X}_{=f_*(\mathrm{id}_{\Delta^k})}. \\ & \uparrow & \\ & \text{by definition of a singular } k\text{-simplex, see page 736} & \end{array}$$

But this means that the set $\{f_*(\mathrm{id}_{\Delta^k})\}_{f\colon \Delta^k \to X}$ is a basis for $C_k(X)$.

Now we can formulate the main theorem of this section.

Theorem 56.16. (Acyclic Model Theorem) *Let $\mathcal{K}$ be a category with a collection of objects $\mathcal{M}$. Furthermore let $F\colon \mathcal{K} \to \mathcal{A}$ be two covariant functors from $\mathcal{K}$ to the category of augmented chain complexes. If F is free relative to $\mathcal{M}$ and if G is acyclic relative to $\mathcal{M}$, then the following statements hold:*

(1) *There exists a natural transformation T from F to G. In other words, given any topological space X we have a chain map $T_X\colon F(X) \to G(X)$ of augmented chain complexes such that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:*[982]

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"'] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "F(f)"] & G(Y). \end{tikzcd}$$

(2) *Given two natural transformations S_X and T_X from F to G there is a natural chain homotopy P_X between them.*

Proof. The proof is, ultimately very much the same as the proof of Theorem 56.5. We decline to provide a proof and we follow the lead given by Munkres [Mun, p. 185], namely like Munkres we assign the proof as an exercise to the reader. Alternatively the reader can find the proof in [Sp, p. 169], [Vic, Theorem 5.3] or [Dol2, p. 175]. □

As an example for the Acyclic Model Theorem 56.16 we will prove a statement that we will anyway need shortly. To formulate the application we need the following definition.

Definition. A *diagonal approximation* is a natural chain map

$$\varphi\colon C_*(X) \;\to\; C_*(X) \otimes C_*(X)$$

[982]Note that the zero map is *not* a chain map of augmented chain complexes, since it does not respect the augmentation.

such that for each $x \in X$, viewed as a singular 0-simplex, we have

$$\varphi(x) \;=\; x \otimes x \;\in\; \underbrace{C_0(X) \otimes C_0(X)}_{=(C_*(X)\otimes C_*(X))_0}.$$

Proposition 56.17. *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

Proof. We let $\mathcal{K}$ be the category of non-empty topological spaces. As on page 1329 we consider the collection

$$\mathcal{M} \;:=\; \{\Delta^k \,|\, k \in \mathbb{N}_0\} \;=\; \text{the set of all standard simplices.}$$

We consider the covariant functors[983]

$$\begin{array}{rcl} F\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (C_*(X), \epsilon_X) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (C_*(X) \otimes C_*(X), \epsilon_X \otimes \epsilon_X). \end{array}$$

Now let $\Phi, \Psi\colon C_*(X) \to C_*(X) \otimes C_*(X)$ be two diagonal approximations. By definition these are natural transformations from F to G. The hypothesis that Φ and Ψ are diagonal approximations implies that Φ and Ψ define chain maps of augmented chain complexes.

By the Acyclic Model Theorem 56.16 (2) it remains to prove the following claim.

Claim.

(1) The functor F is free relative to $\mathcal{M}$.
(2) The functor G is acyclic relative to $\mathcal{M}$.

We prove these two statements:

(1) Conveniently enough we showed on page 1330 that F is free relative to $\mathcal{M}$.
(2) We need to show that given any $k \in \mathbb{N}_0$ the corresponding augmented chain complex $(C_*(\Delta^k) \otimes C_*(\Delta^k), \epsilon_X \otimes \epsilon_X)$ is acyclic. By Lemma 56.15 it suffices to show that the chain complex $C_*(\Delta^k) \otimes C_*(\Delta^k)$ is almost acyclic. But this is a consequence of the fact that $C_*(\Delta^k)$ is almost acyclic, see page 1328, and Proposition 56.14.

This concludes the proof of the claim. □

[983] It follows from the discussion on page 1327 that both F and G are indeed functors to the category of augmented chain complexes.

57. The cup product

57.1. **A first definition of the cup product.** Let X be a topological space. The cross product from page 1316 applied to the same topological spaces $X = Y$ gives us in particular the cross product[984]

$$\begin{array}{rcl} \times\colon \overbrace{H_p(C_*(X)) \otimes H_q(C_*(X))}^{=H_p(X)\otimes H_q(X)} & \to & H_{p+q}(C_*(X) \otimes C_*(X)) \xrightarrow{\Upsilon_*} \overbrace{H_{p+q}(C_*(X \times X))}^{=H_{p+q}(X\times X)}. \\ [c] \otimes [d] & \mapsto & [c \otimes d]. \end{array}$$

This is very nice, but this construction does not give us a product on the homology groups of X itself, since we end up in the homology of a different topological space.

Now we try our luck with cohomology groups. Before we can study maps on cohomology groups we need a little bit of algebraic preparation.

Definition. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes.

(1) Given $p, q \in \mathbb{N}_0$ we denote by

$$\Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} = \bigoplus_{i+j=p+q} \mathcal{C}_i \otimes \mathcal{D}_j \ \to\ \mathcal{C}_p \otimes \mathcal{D}_q$$

the obvious projection map.

(2) Let R be a commutative ring. We denote by Ξ the map

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}_p, R) \times \mathrm{Hom}(\mathcal{D}_q, R) &\to \mathrm{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ (\varphi, \psi) &\mapsto \left((\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\Pi_{p,q}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi\otimes\psi} \begin{array}{rcl} R\otimes R & \to & R \\ a\otimes b & \mapsto & a\cdot b \end{array} \right). \end{aligned}$$

Lemma 57.1. *Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. Furthermore let R be a commutative ring. The map*

$$\begin{array}{rcl} H^p(\mathcal{C}_*; R) \times H^q(\mathcal{D}; R) & \to & H^{p+q}(\mathcal{C}_* \otimes \mathcal{D}_*; R) \\ ([\varphi], [\psi]) & \mapsto & [\Xi(\varphi, \psi)] \end{array}$$

is well-defined.

Proof. We leave the verification of the lemma as an exercise to the reader. □

We return to topology. So let X be a topological space. We consider the map[985]

$$\begin{array}{rcl} \overbrace{H^p(C_*(X); \mathbb{Z}) \otimes H^q(C_*(X); \mathbb{Z})}^{=H^p(X;\mathbb{Z})\otimes H^q(X;\mathbb{Z})} & \xrightarrow{\Xi} & H^{p+q}(C_*(X) \otimes C_*(X); \mathbb{Z}) \xrightarrow{\Theta^*} \overbrace{H^{p+q}(C_*(X \times X); \mathbb{Z})}^{=H^{p+q}(X\times X;\mathbb{Z})}. \\ [\varphi] \otimes [\psi] & \mapsto & [\varphi \otimes \psi \circ \Pi_{p,q}]. \end{array}$$

Summarizing we get a map

$$H^p(X; \mathbb{Z}) \otimes H^q(X; \mathbb{Z}) \ \to\ H^{p+q}(X \times X; \mathbb{Z})$$

[984]Here $\Upsilon\colon C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ denotes the Eilenberg-Zilber map from Theorem 56.1.

[985]Here $\Theta\colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ denotes the Eilenberg-Zilber map from Theorem 56.1 (1). It follows from Theorem 56.1 (2) and Lemma 48.8 (3) that the definition below does not depend on the choice of Θ.

which is sometimes called the *cross product on cohomology*. Again, this is nice, but exactly as in the case of the cross product on homology, we end up in the cohomology of $X \times X$, but we really would like to end up in the cohomology of X.

Now we have to find a way how to go back from the topological space $X \times X$ to the original space X. In general there does not exist an "interesting"[986] map $X \times X \to X$. But, as we saw before, there does exist an interesting map in the opposite direction, namely the diagonal map

$$\begin{array}{rcl} d\colon X & \to & X \times X \\ x & \mapsto & (x, x). \end{array}$$

This diagonal map d induces a map

$$d_*\colon H_*(X;\mathbb{Z}) \;\to\; H_*(X \times X;\mathbb{Z})$$

on homology, and, much more interesting in our context, it also induces a map

$$d^*\colon H^*(X \times X;\mathbb{Z}) \;\to\; H^*(X;\mathbb{Z}).$$

on cohomology groups that goes in the "opposite direction". This discussion leads us to the following definition.

Definition. Let X be a topological space. We define the *cup product*[987] on the cohomology groups of X as the composition of the following maps:

$$H^p(X;\mathbb{Z}) \times H^q(X;\mathbb{Z}) \;\to\; H^p(X;\mathbb{Z}) \otimes H^q(X;\mathbb{Z}) \;\xrightarrow{\Theta^* \circ \Xi}\; H^{p+q}(X \times X;\mathbb{Z}) \;\xrightarrow{d^*}\; H^{p+q}(X;\mathbb{Z}).$$

For the resulting pairing we use the notation

$$\begin{array}{rcl} H^p(X;\mathbb{Z}) \times H^q(X;\mathbb{Z}) & \to & H^{p+q}(X;\mathbb{Z}) \\ (a, b) & \mapsto & a \smile b \end{array}$$

and we refer to $a \smile b$ as the *cup product of a and b*.

57.2. **An alternative definition of the cup product.** The definition of the cup product in the previous section was very natural and we could now develop the theory of the cup product using that definition. If one takes that route then one has to deal with the awkward fact that the Eilenberg-Zilber map Θ is not very explicit.

Now we give an alternative definition of the cup product which is more ad hoc, but which works surprisingly well in practice.

Definition. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$ be cochains. We define the *cup product* of φ and ψ to

[986]The projection from $X \times X$ onto one of the two factors does not count as "interesting", since it does not "mix" the two factors. In fact in Exercise 57.1 we will see that the map

$$H_*(X) \otimes H_*(X) \;\xrightarrow{\times}\; H_*(X \times X) \;\xrightarrow{p_*}\; H_*(X),$$

where $p\colon X \times X \to X$ denotes the projection on the first factor, is the zero map.

[987]The name "cup product" comes from the notation "$a \smile b$" for the composition since $\cup$ looks like a cup.

be the singular cochain[988]

$$\varphi \cup \psi \colon C_{p+q}(X) \quad \to \quad R$$

that is given by[989]

$$(\sigma \colon \Delta^{p+q} \to X) \quad \mapsto \quad \underbrace{\varphi(\sigma \circ [v_0, \dots, v_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [v_p, \dots, v_{p+q}])}_{\in R}.$$

The[990] cup product thus defines a map

$$\cup \colon C^p(X;R) \times C^q(X;R) \;\to\; C^{p+q}(X;R).$$

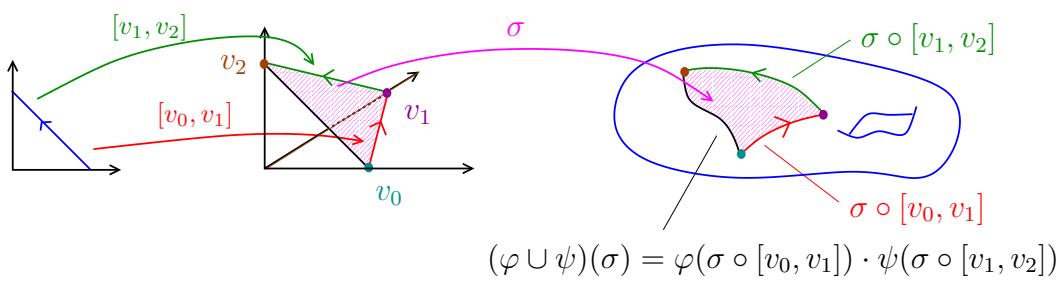


FIGURE 722. Illustration of the definition of the cup product.

Remark. The sign conventions for the cup product for cohomology classes $\varphi \in H^p(X;R)$ and $\psi \in H^q(X;R)$ differ in the literature. More precisely, given a topological space X and cochains $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$ the cup product $\varphi \cup \psi$, applied to a singular $(p+q)$-simplex $\sigma \colon \Delta^{p+q} \to X$ is defined as follows:

		$(\varphi \cup \sigma)(\sigma)$
(a)	Hatcher [Hat, p. 206]	$\varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$
(b)	Bredon [Bre, p. 328]	$(-1)^{pq} \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$
(c)	Dold [Dol, p. 222]	$(-1)^{pq} \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$
(d)	Spanier [Sp, p. 251]	$\varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$
(e)	Munkres [Mun, p. 288]	$\varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$

So we see that our sign convention agrees with the conventions of Hatcher, Spanier and Munkres, but they differ by the sign $(-1)^{pq}$ from the conventions used in Bredon and Dold.

Example. Since the second definition of the cup product is initially strange and not very intuitive we consider an example in slightly more detail. In the following we consider the torus $T = S^1 \times S^1 = [0,1] \times [0,1]/\sim$ with the two singular 2-simplices σ_1 and σ_2 illustrated in Figure 723 which we already saw in Figure 654. We recall the following notation:

[988]Here, when we write $[v_0, \dots, v_p] \colon \Delta^p \to \Delta^{p+q}$ and $[v_p, \dots, v_{p+q}] \colon \Delta^q \to \Delta^{p+q}$, we use the notation from page 1323.

[990]In the literature, given a singular n-simplex $\sigma \colon \Delta^n \to X$ one often writes ${}_p\lfloor\sigma = \sigma \circ [v_0, \dots, v_p]$ and $\sigma\rfloor_p = \sigma \circ [v_{n-p}, \dots, v_n]$.

(1) we denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor,
(2) we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex given by $\mu(1-t,t) = e^{2\pi i t}$,
(3) furthermore we denote by $\nu\colon \Delta^1 \to S^1$ the constant singular 1-simplex that is given by $\nu(1-t,t) = 1$.
(4) finally we recall that on page 1166 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in C^1(S^1;\mathbb{Z})$ and we showed that $\theta(\mu) = 1$, also note that $\theta(\nu) = 0$.

We want to determine the values of $p^*(\theta) \cup q^*(\theta) \in C^2(T;\mathbb{Z}) = \mathrm{Hom}(C_2(T),\mathbb{Z})$ on the singular 2-simplices σ_1 and σ_2. We have

$$\begin{aligned}(p^*(\theta) \cup q^*(\theta))(\sigma_1) &= p^*(\theta)(\sigma_1 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_1 \circ [v_1,v_2]) \\ &= \theta(\underbrace{p \circ \sigma_1 \circ [v_0,v_1]}_{=\nu}) \cdot \theta(\underbrace{q \circ \sigma_1 \circ [v_1,v_2]}_{=\nu}) \quad = \quad 0 \cdot 0 \quad = \quad 0.\end{aligned}$$

Much more interestingly we have

$$\begin{aligned}(p^*(\theta) \cup q^*(\theta))(\sigma_2) &= p^*(\theta)(\sigma_2 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_2 \circ [v_1,v_2]) \\ &= \theta(\underbrace{p \circ \sigma_2 \circ [v_0,v_1]}_{=\mu}) \cdot \theta(\underbrace{q \circ \sigma_2 \circ [v_1,v_2]}_{=\mu}) \quad = \quad 1 \cdot 1 \quad = \quad 1.\end{aligned}$$

To aid the reader we also recall in Figure 724 the notation that we had initially introduced on page 1117 and that we use in Figure 723.

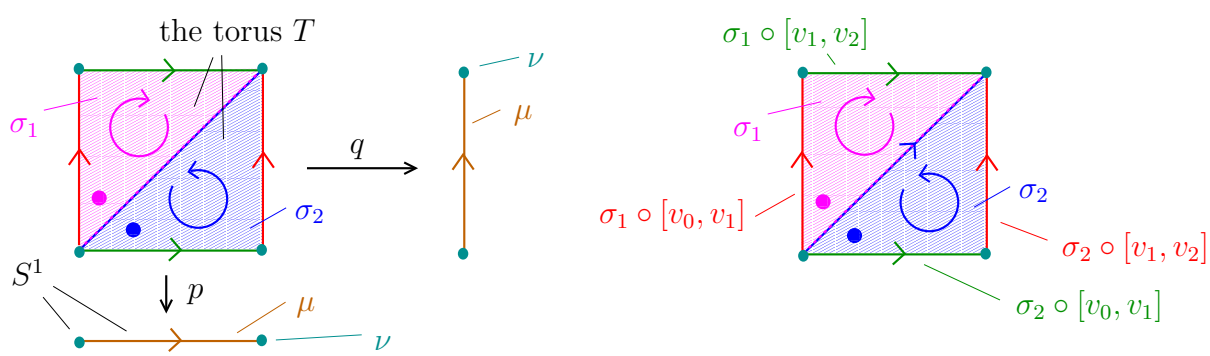


FIGURE 723.

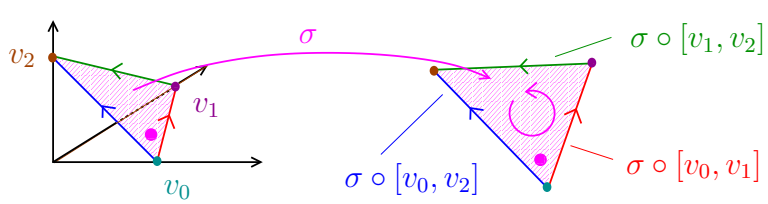


FIGURE 724. Notation of the "dotted triangle".

The definition of the cup product on cochains might look strange and arbitrary. But the following lemma shows that it behaves well with respect to coboundary maps.

Lemma 57.2. *Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$. Then*

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi \,+ (-1)^p \cdot \varphi \cup \delta\psi \;\in\; C^{p+q+1}(X;R).$$

Remark. A similar looking statement holds for differential forms. More precisely, given a manifold M and two differential forms $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ it follows from [Le2, Theorem 14.24 (ii)] that

$$d(\omega \wedge \eta) \;=\; d\omega \wedge \eta \,+\, (-1)^p \cdot \omega \wedge d\eta \;\in\; \Omega^{p+q+1}(M).$$

The (co-) boundary formula of Lemma 57.2 is of course also very similar to the definition of the boundary map in the tensor product of two chain complexes.

Proof $(*)$. The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$ be cochains. We have to prove the following equality:

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi \,+ (-1)^p \cdot \varphi \cup \delta\psi \;\in\; C^{p+q+1}(X;R) = \mathrm{Hom}(C_{p+q+1}(X), R).$$

So let $\sigma\colon \Delta^{p+q+1} \to X$ be a singular $(p+q+1)$-simplex. Then we have the following three equalities in R:[991]

$$\begin{aligned}
\text{(a)}\quad (\delta(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi)\Big(\sum_{j=0}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) \\
&= (\varphi \cup \psi)\Big(\sum_{j=0}^{p} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) + (\varphi \cup \psi)\Big(\sum_{j=p+1}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) \\
&= \sum_{j=0}^{p} (-1)^j \cdot (\varphi \cup \psi)(\sigma \circ i_j^{p+q+1}) + (-1)^p \cdot \sum_{k=1}^{q+1} (-1)^k \cdot (\varphi \cup \psi)(\sigma \circ i_{k+p}^{p+q+1}) \\
&= \sum_{j=0}^{p} (-1)^j \cdot \varphi(\underbrace{\sigma \circ [v_0, \dots, v_{p+1}] \circ i_j^{p+1}}_{\text{map } \Delta^p \to X}) \cdot \psi(\underbrace{\sigma \circ [v_{p+1}, \dots, v_{p+q+1}]}_{\text{map } \Delta^q \to X}) \\
&\quad + (-1)^p \cdot \sum_{k=1}^{q+1} (-1)^k \cdot \varphi(\underbrace{\sigma \circ [v_0, \dots, v_p]}_{\text{map } \Delta^p \to X}) \cdot \psi(\underbrace{\sigma \circ [v_p, \dots, v_{p+q+1}] \circ i_k^{q+1}}_{\text{map } \Delta^q \to X}).
\end{aligned}$$

[991]Here in the argument we use on several occasions that for a singular n-cochain ϕ and a singular $(n+1)$-simplex μ we have by definition

$$(\delta\phi)(\mu) \;=\; \phi(\partial\mu) \;=\; \phi\Big(\sum_{l=0}^{n+1} (-1)^l \cdot \mu \circ i_l^{n+1}\Big).$$

$$\begin{aligned}
\text{(b)}\quad (\delta\varphi \cup \psi)(\sigma) &= (\delta\varphi)(\sigma \circ [v_0, \dots, v_{p+1}]) \cdot \psi(\sigma \circ [v_{p+1}, \dots, v_{p+q+1}]) \\
&= \varphi\Big(\sum_{j=0}^{p+1} (-1)^j \cdot \underbrace{\sigma \circ [v_0, \dots, v_{p+1}] \circ i_j^{p+1}}_{\text{map } \Delta^p \to X}\Big) \cdot \psi(\underbrace{\sigma \circ [v_{p+1}, \dots, v_{p+q+1}]}_{\text{map } \Delta^q \to X}) \\
&= \sum_{j=0}^{p+1} (-1)^j \cdot \varphi(\sigma \circ [v_0, \dots, v_{p+1}] \circ i_j^{p+1}) \cdot \psi(\sigma \circ [v_{p+1}, \dots, v_{p+q+1}])
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\quad (-1)^p \cdot (\varphi \cup \delta\psi)(\sigma) &= (-1)^p \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot (\delta\psi)(\sigma \circ [v_p, \dots, v_{p+q+1}]) \\
&= (-1)^p \cdot \varphi(\underbrace{\sigma \circ [v_0, \dots, v_p]}_{\text{map } \Delta^p \to X}) \cdot \psi\Big(\sum_{k=0}^{q+1} (-1)^k \cdot \underbrace{\sigma \circ [v_p, \dots, v_{p+q+1}] \circ i_k^{q+1}}_{\text{map } \Delta^q \to X}\Big) \\
&= (-1)^p \cdot \sum_{k=0}^{q+1} (-1)^k \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q+1}] \circ i_k^{q+1}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j = p+1)$-summand in (b) cancels with the $(k = 0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. □

Lemma 57.3. *Let X be a topological space and let R be a commutative ring. Then the map*

$$\begin{array}{rcl}
\cup\colon H^p(X;R) \times H^q(X;R) & \to & H^{p+q}(X;R) \\
([\varphi],[\psi])] & \mapsto & [\varphi] \cup [\psi] := [\varphi \cup \psi]
\end{array}$$

is well-defined.

Definition. We refer to the pairing of Lemma 57.3 also as the *cup product*.

As we will see, the proof of Lemma 57.3 is basically identical to the proof of Lemma 55.11.

Proof. Let $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$ be cocycles.

(1) It follows immediately from Lemma 57.2 that $\varphi \cup \psi$ is again a cocycle.

(2) If $\varphi' = \varphi + \delta\tau$ is another representative of $[\varphi]$, then

$$\varphi' \cup \psi - \varphi \cup \psi \;\underset{\substack{\uparrow \\ \text{since “}\cup\text{” is clearly bilinear}}}{=}\; (\varphi' - \varphi) \cup \psi \;=\; \delta\tau \cup \psi \;\underset{\substack{\uparrow \\ \text{by Lemma 57.2 and since } \delta\psi = 0}}{=}\; \delta(\tau \cup \psi).$$

This shows that $\varphi \cup \psi$ and $\varphi' \cup \psi$ represent the same cohomology class, i.e. the cup product does not depend on the choice of the representative of the cohomology class of the first entry.

(3) The proof that the cup product does not depend on the choice of the representative of the second cohomology class is basically the same as the proof of (2). □

The following proposition now says that the original definition of the cup product and the new definition actually agree.

Proposition 57.4. *Let X be a topological space. Then for any $\varphi \in H^p(X;\mathbb{Z})$ and any $\psi \in H^q(X;\mathbb{Z})$ we have*

$$\varphi \,\tilde{\cup}\, \psi \;=\; \varphi \cup \psi \;\in\; H^{p+q}(X;\mathbb{Z}).$$

In the proof of Proposition 57.4 we need the following definition and the subsequent lemma.

Definition. A *diagonal approximation* consists of a natural chain map

$$\Phi\colon C_*(X) \;\to\; C_*(X)\otimes C_*(X)$$

for each topological space X, such that for each $x\in X$, viewed as a singular 0-simplex, the following equality holds in $(C_*(X)\otimes C_*(X))_0 = C_0(X)\otimes C_0(X)$:

$$\Phi(x) \;=\; x\otimes x.$$

Example. Given any topological space X we denote by $d\colon X\to X\times X$ the diagonal map given by $d(x)=(x,x)$. Then the induced chain maps

$$\Theta\circ d_*\colon C_*(X) \;\xrightarrow{d_*}\; C_*(X\times X) \;\underset{\substack{\uparrow\\ \text{Eilenberg-Zilber map from Theorem 56.1}}}{\xrightarrow{\Theta}}\; C_*(X)\otimes C_*(X)$$

form a diagonal approximation in the above sense.

The following lemma gives another diagonal approximation which will be very useful for doing actual calculations.

Lemma 57.5. *Given any topological space X we consider the map*

$$\Delta\colon C_*(X) \;\to\; C_*(X)\otimes C_*(X)$$

which for each singular n-simplex $\sigma\colon\Delta^n\to X$ is given by

$$\sigma \;\mapsto\; \sum_{p=0}^{n}\sigma\circ[v_0,\dots,v_p]\otimes\sigma\circ[v_p,\dots,v_n].$$

These maps form a diagonal approximation.

The diagonal approximation from Lemma 57.5 is called the *Alexander-Whitney diagonal approximation.* [992][993]

Proof. It follows almost immediately from the definition that the maps Δ are natural. Furthermore it is obvious that on the 0-chains the maps are of the desired form. It remains to show that for a given topological space X the maps actually form a chain map.

This follows from an elementary and uneventful calculation. For completeness' sake we provide the argument. Let $\mu\colon\Delta^n\to X$ be a singular n-simplex. Given $i_1,\dots,i_r$ we write $[i_1,\dots,i_r]:=\mu\circ[v_{i_1},\dots,v_{i_r}]$. We also use the convention that the empty singular simplex

[992]James Alexander (1888-1971) was an American mathematician whom we had already met on page 906.

[993]Hassler Whitney (1907–1989) was an American mathematician. We will encounter him again later on.

[ ] is 0. We calculate that

$$\begin{aligned}\Delta(\partial[0,\dots,n]) &= \Delta\Big(\sum_{i=0}^{n}(-1)^i\cdot[0,\dots,\widehat{i},\dots n]\Big)\\ &\underset{\uparrow}{=} \sum_{0\le p<i\le n}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n]+\sum_{0\le i<q\le n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,q]\otimes[q,\dots,n]\end{aligned}$$

in the second sum we secretly used the substitution $q=p+1$

and

$$\begin{aligned}\partial(\Delta[0,\dots,n]) &= \partial\Big(\sum_{p=0}^{n}[0,\dots,p]\otimes[p,\dots,n]\Big)\\ &\underset{\uparrow}{=} \sum_{0\le i\le p\le n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,p]\otimes[p,\dots,n]+\sum_{0\le p\le i\le n}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n].\end{aligned}$$

here we use the sign convention for the boundary in a tensor product

We see that the difference between the two terms equals

$$\begin{aligned}&\sum_{i=p}(-1)^i\cdot[0,\dots,\widehat{i},\dots,p]\otimes[p,\dots,n]+\sum_{i=p}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n]\\ &\underset{\uparrow}{=} \sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{i=0}^{n-1}(-1)^i\cdot[0,\dots,i]\otimes[i+1,\dots,n]\end{aligned}$$

here we use that $[\ ]=0$

$$= \sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{j=1}^{n}(-1)^{j-1}\cdot[0,\dots,j-1]\otimes[j,\dots,n] \;=\; 0.$$

We have thus shown that Δ is indeed a chain map. □

Proposition 57.6.

(1) *There exist diagonal approximations.*
(2) *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

Proof.

(1) We had just given two examples of diagonal approximations.
(2) The uniqueness statement was proved in Proposition 57.6. □

Now we are ready to provide the proof of Proposition 57.4.

Proof of Proposition 57.4. We start out with the following claim.

Claim. For any two cochains $\varphi\in C^p(X;\mathbb{Z})$ and $\psi\in C^q(X;\mathbb{Z})$ we have[994]

$$\varphi\cup\psi \;=\; (\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta \;\in\; C^{p+q}(X;\mathbb{Z})=\mathrm{Hom}(C_{p+q}(X),\mathbb{Z}).$$

[994]Here $\Delta\colon C_{p+q}(X)\to(C_*(X)\otimes C_*(X))_{p+q}$ denotes the above Alexander-Whitney diagonalization.

We have to show that the two sides are the same maps $C_{p+q}(X) \to \mathbb{Z}$. Therefore let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-simplex. We calculate that

$$\begin{aligned}
(\varphi \otimes \psi \circ \Pi_{p,q} \circ \Delta)(\sigma) &\overset{\text{definition of the Alexander-Whitney diagonalization}}{=} (\varphi \otimes \psi)\big(\Pi_{p,q}\big(\sum_{i+j=p+q} \overbrace{\sigma \circ [v_0, \dots, v_i] \otimes \sigma \circ [v_i, \dots, v_{p+q}]}^{\in C_i(X) \otimes C_j(X)}\big)\big) \\
&= (\varphi \otimes \psi)(\sigma \circ [v_0, \dots, v_p] \otimes \sigma \circ [v_q, \dots, v_{p+q}]) \\
&= \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_n]) \\
&= (\varphi \cup \psi)(\sigma).
\end{aligned}$$

This concludes the proof of the claim.

Now suppose that we are given cocycles $\varphi \in C^p(X;\mathbb{Z})$ and $\psi \in C^q(X;\mathbb{Z})$. We obtain the following equality in $H^{p+q}(X;\mathbb{Z})$:[995]

$$\begin{aligned}
[\varphi \widetilde{\cup} \psi] \overset{\text{by definition}}{=} [(\varphi \otimes \psi) \circ \Pi_{p,q} \circ (\Theta \circ d_*)] &= (\Theta \circ d_*)^*\big(\overbrace{[(\varphi \otimes \psi) \circ \Pi_{p,q}]}^{\in H^{p+q}(C_*(X) \otimes C_*(X))}\big) \\
&\underset{\substack{\uparrow \\ \text{by Proposition 57.6 the maps } \Theta \circ d_*, \Delta\colon C_*(X) \to C_*(X) \otimes C_*(X) \text{ are chain homotopic} \\ \text{so by Lemma 48.8 (3) they induce the same map on cohomology}}}{=} \Delta^*\big([(\varphi \otimes \psi) \circ \Pi_{p,q}]\big) = [(\varphi \otimes \psi) \circ \Pi_{p,q} \circ \Delta] \underset{\substack{\uparrow \\ \text{by the claim}}}{=} [\varphi \cup \psi].
\end{aligned}$$

We have thus obtained the desired equality of cohomology classes. □

Henceforth we will mostly work with the second definition of the cup product. In the following proposition we summarize some of the key properties of the cup product.

Proposition 57.7. *Let X be a topological space and let R be a commutative ring. Then the following statements hold:*

(1) *The cup product on $H^*(X;R)$ is R-bilinear and associative.*
(2) *The abelian group $H^*(X;R) = \bigoplus_{n \in \mathbb{N}_0} H^n(X;R)$ together with the map*

$$\cup\colon H^*(X;R) \times H^*(X;R) \to H^*(X;R)$$

is a ring[996] *where the multiplicatively neutral element is given by*

$$1_X \ := \ \big[\textit{constant map } X \to R \textit{ given by } x \mapsto 1\big] \ \in \ H^0(X;R).$$

Definition. Given a topological space X and a commutative ring R we refer to $(H^*(X;R), \cup)$ as the *R-cohomology ring of X*.

Proof. Let X be a topological space.

[995]Note that it follows from Lemma 57.1 that for two cocycles φ and ψ the composition $(\varphi \otimes \psi) \circ \Pi_{p,q}$ is indeed a cocycle, i.e. $(\varphi \otimes \psi) \circ \Pi_{p,q}$ defines an element in $H^{p+q}(C_*(X) \otimes C_*(X);\mathbb{Z})$.

[996]Recall that in Footnote 371 we had stated that in the lecture notes a ring is understood to be associative and it is understood to have a multiplicatively neutral element. We do not demand that a ring be commutative.

(1) It is clear from the second definition of the cup product that it is R-bilinear. It is also almost obvious, using the second definition, that the cup product is associative, but for completeness' sake we carry out the argument. Thus let $\alpha \in C^p(X;R)$, $\beta \in C^q(X;R)$ and $\gamma \in C^r(X;R)$ be cochains and let $\sigma\colon \Delta^{p+q+r} \to X$ be a singular simplex. Then

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)(\sigma) &= (\alpha \cup \beta)(\sigma \circ [v_0, \dots, v_{p+q}]) \cdot \gamma(\sigma \circ [v_{p+q}, \dots, v_{p+q+r}]) \\ &= \alpha(\sigma \circ [v_0, \dots, v_p]) \cdot \beta(\sigma \circ [v_p, \dots, v_{p+q}]) \cdot \gamma(\sigma \circ [v_{p+q}, \dots, v_{p+q+r}]) \\ &= (\alpha \cup (\beta \cup \gamma))(\sigma). \\ &\uparrow \\ &\text{same argument backwards}\end{aligned}$$

Thus we see that the cup product is already associative for cochains, so it is in particular associative for cohomology classes.

(2) Let $\eta\colon X \to R$ be the map that assigns to each point in X the value $1 \in R$. As usual we view η as a singular 0-cochain. It follows from Proposition 48.11 that η is a 0-cocycle and that we obtain an element $1_X := [\eta] \in H^0(X;R)$. It is clear from the definition that it is the neutral element with respect to the cup product. Together with (1) we have now shown that the cup product defines a ring structure on $H^*(X;R)$. □

Now that we have shown that $(H^*(X;R), \cup)$ is a ring the question arises whether it is commutative or whether it is not commutative. It turns out that the answer, as for the wedge product on de Rham cohomology, lies somewhere in between. More precisely, we have the following proposition.

Proposition 57.8. *Let X be a topological space and let R be a commutative ring. For any $a \in H^k(X;R)$ and $b \in H^l(X;R)$ we have*

$$a \cup b \;=\; (-1)^{kl} \cdot b \cup a.$$

Remark.

(1) If $R = \mathbb{F}_2$, then we are in the pretty situation that $-1 = 1$, i.e. the cup product is commutative. Put differently, the $\mathbb{F}_2$-cohomology ring $H^*(X;\mathbb{F}_2)$ of a topological space X is a commutative ring.

(2) The proof for Proposition 57.8 that we provide below is based on the second definition of the cup product. It is elementary but unfortunately unpleasant to read. In a later chapter we will provide a different, more conceptual proof based on the first definition of the cup product.

Proof $(*)$. We give a proof of the proposition following [Hat, p. 216-217]. In Hatcher's book more motivations for the steps in the proof are given.

So let $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ be two cocycles. Given a singular n-simplex $\sigma\colon \Delta^n \to X$ we denote by $\overline{\sigma}$ the singular n-simplex obtained from σ by precomposing σ with the affine linear map $[v_n, \dots, v_0]\colon \Delta^n \to \Delta^n$ that reverses the order of the vertices.

Furthermore we write $\epsilon_n = (-1)^{n(n+1)/2}$. We note that for any $m, n \in \mathbb{N}_0$ an elementary calculation shows that $\epsilon_{m+n} = (-1)^{mn} \cdot \epsilon_m \epsilon_n$ and that $\epsilon_1 = -1$.

Claim. The maps

$$\begin{array}{rcl} \rho\colon C_n(X) & \to & C_n(X) \\ \sigma & \mapsto & \epsilon_n \cdot \overline{\sigma} \end{array}$$

form a chain map that is chain homotopic to the identity.

We first show that the claim implies the statement of the proposition. Let $[\varphi] \in H^k(X;R)$ and $[\psi] \in H^l(X;R)$. Since by the claim ρ is chain homotopic to the identity it follows from Lemma 48.8 that the induced map $\rho^*\colon C^*(X;R) \to C^*(X;R)$ induces the identity on cohomology groups. So to deduce the statement of the proposition it suffices to show that

$$\rho^*(\psi \cup \varphi) \;=\; (-1)^{kl} \cdot \rho^*\varphi \cup \rho^*\psi \;\in\; C^{k+l}(X;R).$$

Now let σ be a $(k+l)$-chain. Then

$$\begin{array}{rcl} (\rho^*(\psi \cup \varphi))(\sigma) & = & (\psi \cup \varphi)(\rho_*(\sigma)) \;=\; (\psi \cup \varphi)(\epsilon_{k+l} \cdot \sigma \circ [v_{k+l}, \dots, v_0]) \\ & = & \epsilon_{k+l} \cdot \psi(\sigma \circ [v_{k+l}, \dots, v_k]) \cdot \varphi(\sigma \circ [v_k, \dots, v_0]) \\ & = & (-1)^{kl} \cdot \varphi(\epsilon_k \cdot \sigma \circ [v_k, \dots, v_0]) \cdot \psi(\epsilon_l \cdot \sigma \circ [v_{k+l}, \dots, v_k]) \\ & \uparrow & \\ & \text{since } \epsilon_{k+l} = (-1)^{kl} \cdot \epsilon_k \cdot \epsilon_l \text{ and since } R \text{ is a } \textit{commutative} \text{ ring} & \\ & = & (-1)^{kl} \cdot (\rho^*\varphi \cup \rho^*\psi)(\sigma). \end{array}$$

We have thus shown the statement of the proposition modulo the claim.

We turn to the proof of the claim. It is straightforward to verify that ρ is a chain map, indeed, for any singular n-simplex σ we have

$$\begin{array}{rcl} \partial\rho(\sigma) & = & \partial(\epsilon_n \cdot \sigma \circ [v_n, \dots, v_0]) \;=\; \epsilon_n \cdot \sum_{i=0}^{n} (-1)^i \cdot \sigma \circ [v_n, \dots, \widehat{v_{n-i}}, \dots, v_0], \\ \rho\partial(\sigma) & = & \rho\Big(\sum_{i=0}^{n} (-1)^i \cdot \sigma \circ [v_0, \dots, \widehat{v_i}, \dots, v_n]\Big) \underset{\uparrow}{=} \epsilon_{n-1} \cdot \sum_{j=0}^{n} (-1)^{n-j} \cdot \sigma \circ [v_n, \dots, \widehat{v_{n-j}}, \dots, v_0]. \\ & & \text{definition of } \rho \text{ and substitution } j = n-i \end{array}$$

From $\epsilon_n = (-1)^n \cdot \epsilon_{n-1}$ we now obtain that $\partial\rho = \rho\partial$, i.e. we have shown that ρ is a chain map.

We still have to show that ρ is chain homotopic to the identity. We proceed with ideas similar to the ideas used in the proof of Proposition 30.5. As in that proof, given $i \in \{0, \dots, n\}$ we write

$$v_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{0\} \;\in\; \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{1\} \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

We justify the reassignment of v_i to a different point by saying that we identify Δ^n with $\Delta^n \times \{0\} \in \mathbb{R}^{n+1} \times \mathbb{R}$. We denote by $\pi\colon \Delta^n \times [0,1] \to \Delta^n$ the projection map. We consider

the map

$$\begin{aligned} P\colon C_n(X) &\to C_{n+1}(X) \\ \sigma &\mapsto \sum_{i=0}^{n}(-1)^i\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,w_i]. \end{aligned}$$

Now we want to show that $\partial P + P\partial = \rho - \mathrm{id}$. We calculate

$$\begin{aligned} (P\partial)(\sigma) &= \sum_{i<j}(-1)^i(-1)^j\cdot\epsilon_{n-i-1}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i] \\ &\quad+\sum_{i>j}(-1)^{i-1}(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i], \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\partial P)(\sigma) &= \sum_{j\le i}(-1)^i(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i] \\ &\quad+\sum_{j\ge i}(-1)^i(-1)^{i+1+n-j}\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i]. \end{aligned}$$

Now using that $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$ we see that the sum over the $j \neq i$ terms agrees precisely with $-P\partial$ calculated above (with the roles of i and j reversed). It remains to show that the sum over the $i = j$ terms equals $(\rho - \mathrm{id})(\sigma)$. This is indeed the case since

$$\begin{aligned} &\text{sum of the } i=j \text{ summands in the above formula for } (\partial P)(\sigma) \\ &= \epsilon_n\cdot(\sigma\circ\pi)\circ[w_n,\dots,w_0] + \sum_{i=1}^{n}\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_{i-1},w_n,\dots,w_i] \\ &\qquad+\sum_{i=0}^{n-1}(-1)^{n+i+1}\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,w_{i+1}] - (\sigma\circ\pi)\circ[v_0,\dots,v_n] \\ &= \epsilon_n\cdot(\sigma\circ\pi)\circ[w_n,\dots,w_0] - (\sigma\circ\pi)\circ[v_0,\dots,v_n] \;=\; (\rho-\mathrm{id})(\sigma). \\ &\uparrow \end{aligned}$$

in the second sum we substitute $j = i+1$, we obtain a new sign $(-1)^{n+j}\epsilon_{n-j+1} = -\epsilon_{n-j}$, thus we see that the two sums over i cancel

We have now completed the proof that $\partial P + P\partial = \rho - \mathrm{id}$, in particular we have shown the claim and thus also the proposition. □

Let $k \in \mathbb{N}$ be odd. It follows from Proposition 57.8 that given any topological space X and any $\varphi \in H^k(X;\mathbb{Z})$ we have $\varphi\cup\varphi = -(1)^{k\cdot k}\cdot\varphi\cup\varphi$, put differently, we have $2\cdot(\varphi\cup\varphi) = 0$. Thus we see that $\varphi\cup\varphi$ is 2-torsion. The next lemma makes the amusing observation that for $k = 1$ this "square" is in fact always zero.

Lemma 57.9. *Let X be a topological space. For any $\varphi \in H^1(X;\mathbb{Z})$ we have $\varphi\cup\varphi = 0$.*

Remark. There is no higher-dimensional analogue of Lemma 57.9, for example on [Hat, p. 403] it is shown that given any odd k with $k \ge 3$ there exists a class $\gamma \in H^k(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ with $\gamma^m \neq 0$ for all $m \in \mathbb{N}$.

Proof. Let X be a topological space and let $\varphi \in C^1(X;\mathbb{Z})$ be a cocycle. We need to show that $\varphi \cup \varphi \in C^2(X;\mathbb{Z})$ is a coboundary. We consider the 1-dimensional cochain

$$\mu\colon C_1(X) \;\to\; \mathbb{Z}$$

that is determined by

$$(\sigma\colon \Delta^1 \to X) \;\mapsto\; \tfrac{1}{2}\underbrace{(\varphi(\sigma) - \varphi(\sigma)^2)}_{\in 2\mathbb{Z}}.$$

It remains to prove the following claim.

Claim. We have $\varphi \cup \varphi = \delta\mu\colon C_2(X) \to \mathbb{Z}$.

We only need to prove this equality for singular 2-simplices. So suppose we are given $\tau\colon \Delta^2 \to X$. Given $j \in \{0,1,2\}$ we write $i_j = i_j^2\colon \Delta^1 \to \Delta^2$. Note that

$$(*) \qquad \varphi(\tau\circ i_0) - \varphi(\tau\circ i_1) + \varphi(\tau\circ i_2) \;=\; \varphi(\partial\tau) \underset{\substack{\uparrow \\ \text{definition of } \delta\varphi}}{=} (\delta\varphi)(\tau) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is a cocycle}}}{=} 0.$$

Now we compute that

$$\begin{aligned}
(\delta\mu)(\tau) \;&=\; \mu(\partial\tau) \;=\; \mu\big((\tau\circ i_0) - (\tau\circ i_1) + (\tau\circ i_0)\big)\\
&=\; \tfrac{1}{2}\big(\varphi(\tau\circ i_0) - \varphi(\tau\circ i_1) + \varphi(\tau\circ i_2) + \varphi(\tau\circ i_0)^2 - \varphi(\tau\circ i_1)^2 + \varphi(\tau\circ i_2)^2\big)\\
&=\; \tfrac{1}{2}\big(\varphi(\tau\circ i_0)^2 - (\varphi(\tau\circ i_0) + \varphi(\tau\circ i_2))^2 + \varphi(\tau\circ i_2)^2\big)\\
&\quad\uparrow \\
&\quad \text{by } (*) \text{ we have } \varphi(\tau\circ i_1) = \varphi(\tau\circ i_0) + \varphi(\tau\circ i_2)\\
&=\; \varphi(\tau\circ i_2)\cdot\varphi(\tau\circ i_0) \;=\; (\varphi\cup\varphi)(\tau).\\
&\quad\uparrow \qquad\qquad\qquad\qquad \uparrow \\
&\quad \text{elementary algebra} \qquad \text{definition of the cup-product}
\end{aligned}$$

This concludes the proof of the claim and thus of the lemma. □

We continue with the following lemma that shows that the cup product behaves well under maps between topological spaces.

Lemma 57.10. *Let R be a commutative ring.*

(1) *Let $f\colon X \to Y$ be a map between topological spaces. For any $c \in H^p(Y;R)$ and $d \in H^q(Y;R)$ we have*

$$f^*(c)\cup f^*(d) \;=\; f^*(c\cup d).$$

In particular the map

$$f^*\colon H^*(Y;R) \;\to\; H^*(X;R)$$

is a homomorphism of graded rings[997] *which satisfies $f^*(1_Y) = 1_X$.*

[997]Recall that a *graded ring* is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m,n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$. A homomorphism $f\colon R \to S$ between graded rings is a ring homomorphism with $f(R_n) \subset S_n$ for all $n \in \mathbb{N}_0$

(2) *The maps*

$$\begin{array}{rcl} X & \mapsto & (H^*(X;R),\cup) \\ (f\colon X\to Y) & \mapsto & (f^*\colon H^*(Y;R)\to H^*(X;R)) \end{array}$$

define a contravariant functor from the category $\mathcal{T}$ of topological spaces to the category $\mathcal{GR}$ of graded rings.

Proof. The lemma follows basically immediately from the definitions. But for completeness' sake we write down the proof. So let $c=[\varphi]\in H^p(Y;R)$ and let $b=[\psi]\in H^q(Y;R)$. Furthermore let $\sigma\colon \Delta^{p+q}\to X$ be a singular $(p+q)$-chain. Then

$$\begin{array}{rcl} (f^*\varphi\cup f^*\psi)(\sigma) & = & (f^*\varphi)(\sigma\circ[v_0,\dots,v_p])\cdot(f^*\psi)(\sigma\circ[v_p,\dots,v_{p+q}]) \\ & = & \varphi(f\circ(\sigma\circ[v_0,\dots,v_p]))\cdot\psi(f\circ(\sigma\circ[v_p,\dots,v_{p+q}])) \\ & = & \varphi((f\circ\sigma)\circ[v_0,\dots,v_p))\cdot\psi((f\circ\sigma)\circ[v_p,\dots,v_{p+q}]) \\ & = & (\varphi\cup\psi)(f\circ\sigma) \;=\; (f^*(\varphi\cup\psi))(\sigma). \end{array}$$

Finally note that it follows immediately from the definitions of 1_X, 1_Y and f^* that we have the equality $f^*(1_Y)=1_X$. Statement (2) follows immediately from statement (1) and the fact that cohomology groups with R-coefficients form a contravariant functor of abelian groups. □

We can use Lemma 57.10 to calculate the cup product of the wedge of finitely many spheres.

Lemma 57.11. *For any wedge of finitely many spheres all cup products in degrees ≥ 1 are zero.*

Proof. Let $S^{r_1}\vee\cdots\vee S^{r_k}$ be the wedge of finitely many spheres and let R be a commutative ring. For each $i\in\{1,\dots,k\}$ we denote by $p_i\colon S^{r_1}\vee\cdots\vee S^{r_k}\to S^{r_i}$ the obvious projection map. By Proposition 50.17 we know that for every $n\in\mathbb{N}$ the map

$$p_1^*\oplus\cdots\oplus p_k^*\colon H^n(S^{r_1};R)\oplus\cdots\oplus H^n(S^{r_k};R)\xrightarrow{\cong} H^n(S^{r_1}\vee\cdots\vee S^{r_k};R)$$

is an isomorphism. It remains to show that for any choice of $i,j\in\{1\dots,k\}$ with $r_i>0$ and $r_j>0$ we have $p_i^*([S^{r_i}]^*)\cup p_j^*([S^{r_j}]^*)=0$. We prove the statement for $i\neq j$. The proof of the statement for $i=j$ is basically the same. So suppose that $i\neq j$. We denote by $g\colon S^{r_1}\vee\cdots\vee S^{r_k}\to S^{r_i}\vee S^{r_j}$ the obvious projection map and we denote by $q_i\colon S^{r_i}\vee S^{r_j}\to S^{r_i}$ and $q_j\colon S^{r_j}\vee S^{r_j}\to S^{r_j}$ the obvious projection maps. Then

$$p_i^*([S^{r_i}]^*)\cup p_j^*([S^{r_j}]^*) \underset{\substack{\uparrow \\ \text{since } q_i\circ g=p_i \text{ and } q_j\circ g=p_j}}{=} g^*(q_i^*([S^{r_i}]^*))\cup g^*(q_j^*([S^{r_j}]^*)) \underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=} g^*(\overbrace{q_i^*([S^{r_i}]^*)\cup q_j^*([S^{r_j}]^*)}^{\in H^{r_i+r_j}(S^{r_i}\vee S^{r_j})=0}) = 0.$$

We have thus proved that all cup products in degrees ≥ 1 are zero. □

Later on, after we will have found methods for computing the cup product, we will often use the following corollary to show that certain topological spaces are not homotopy equivalent.

Corollary 57.12. *Let X and Y be topological spaces. If X and Y are homotopy equivalent, then for any commutative ring R there exists an isomorphism*

$$(H^*(Y;R),\cup) \xrightarrow{\cong} (H^*(X;R),\cup)$$

of graded rings.

Proof. Let $f\colon X \to Y$ be a homotopy equivalence. In Lemma 48.13 we saw that the induced maps $f^*\colon H^n(Y;R) \to H^n(X;R)$ on cohomology *groups* are isomorphisms. But it follows from Lemma 57.10 that these isomorphisms preserve the ring structure, i.e. the map f^* is in fact an isomorphism of graded rings. □

57.3. **The cup product of the torus.** Evidently our goal now is to determine the cup product of some of our favorite different topological spaces. We start out with two elementary observations regarding the cup product.

Observation 57.13. *Let R be a commutative ring.*

(1) *Let X be a path-connected topological space. Then it follows from Proposition 48.11 that $H^0(X;R) = R\cdot 1_X$. For any $r\cdot 1_X \in H^0(X;R)$ and $\varphi \in H^k(X;R)$ we have*

$$\begin{array}{ccccccc} (r\cdot 1_X)\cup\varphi & = & r\cdot(1_X\cup\varphi) & = & r\cdot\varphi & = & \varphi\cup(r\cdot 1_X). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } \cup \text{ is } R\text{-bilinear} & & \text{Proposition 57.7} & & \text{the same argument backwards} & \end{array}$$

(2) *If M is an m-dimensional topological manifold, then for any p,q with $p+q>m$ the cup product $H^p(M;R)\times H^q(M;R)$ is zero, since it takes values in the group $H^{p+q}(M;R)$, which is zero by Theorem 46.4.*
(3) *By Proposition 57.7 the cup product is R-bilinear, so it suffices to determine the cup product on a generating system.*

In this section we want to determine the cup product on the torus $T = S^1\times S^1$. By Observation 57.13 it remains to determine the cup product

$$\cup\colon H^1(T;\mathbb{Z})\times H^1(T;\mathbb{Z}) \to H^2(T;\mathbb{Z}).$$

We first need to understand the cohomology groups $H^1(T;\mathbb{Z})$ and $H^2(T;\mathbb{Z})$ explicitly. We recall some notation and results from before:

(1) We denote by p respectively q the projection $S^1\times S^1 \to S^1$ onto the first respectively second factor. Furthermore, we recall that on page 1166 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in C^1(S^1;\mathbb{Z})$, we showed in Lemma 48.10 that θ is a cocycle and we showed on page 1200 that θ represents a generator of $H^1(S^1;\mathbb{Z})$.
(2) In Lemma 50.9 we showed that $H^1(T;\mathbb{Z}) \cong \mathbb{Z}^2$ and that the two cohomology classes $\alpha := p^*([\theta])$ and $\beta := q^*([\theta])$ form a basis for $H^1(S^1\times S^1;\mathbb{Z})$.

(3) On page 1118 we showed that the cycle $-\sigma_1+\sigma_2$, with the notation from Figure 654, represents the fundamental class of T.
(4) In Lemma 50.9 and 51.15 we showed that

$$\begin{array}{rcl} H^2(T;\mathbb{Z}) & \to & \mathbb{Z} \\ [\varphi] & \mapsto & \langle \varphi, [T]\rangle = \varphi(-\sigma_1+\sigma_2) \end{array}$$

is in fact an isomorphism. We will use this isomorphism to make the identification $H^2(T;\mathbb{Z}) = \mathbb{Z}$.

By Oberservation 57.13 (3) and the above discussion it suffices to determine all entries in the following matrix

$$\begin{pmatrix} \langle \alpha\cup\alpha, [T]\rangle & \langle \alpha\cup\beta, [T]\rangle \\ \langle \beta\cup\alpha, [T]\rangle & \langle \beta\cup\beta, [T]\rangle \end{pmatrix}.$$

Now we determine these four entries separately.

(1) We first consider the entry $\langle \alpha\cup\beta, [T]\rangle$. Fortunately we already did all the work on page 1335, more precisely, we have

$$\langle \alpha\cup\beta, [T]\rangle = (p^*\theta\cup q^*\theta)(-\sigma_1+\sigma_2) = (p^*\theta\cup q^*\theta)(-\sigma_1) + (p^*\theta\cup q^*\theta)(\sigma_2) \underset{\substack{\uparrow \\ \text{see calculations on page 1335}}}{=} 0+1 = 1.$$

(2) We could do the same argument as in (1) with α and β reversed. But it is easier to apply Proposition 57.8 which gives us that $\beta\cup\alpha = (-1)^{1\cdot 1}\cdot\alpha\cup\beta = -\alpha\cup\beta = -1$.
(3) There are many different ways for showing that $\alpha\cup\alpha = 0$:
  (a) We could do the calculation as in (1), but that is the least efficient approach.
  (b) We could use the symmetry from Proposition 57.8 to argue that $\alpha\cup\alpha = -\alpha\cup\alpha$, and hence $\alpha\cup\alpha = 0$ since $H^2(T;\mathbb{Z})$ is torsion-free.
  (c) We could just use Lemma 57.9 and remove all the fun.
  (d) We can also argue that

$$\langle \alpha\cup\alpha, [T]\rangle \;=\; \langle [p^*\theta]\cup[p^*\theta], [T]\rangle \underset{\substack{\uparrow \\ \text{by Lemma 57.10}}}{=} \langle p^*([\theta]\cup[\theta]), [T]\rangle \underset{\substack{\uparrow \\ \text{since } [\theta]\cup[\theta]\in H^2(S^1;\mathbb{Z})=0}}{=} 0.$$

(4) The same way as in (3) we see that $\beta\cup\beta = 0$.

Summarizing we have proved the following lemma.

Lemma 57.14. *Let T be the 2-dimensional torus. We use the above notation. The cup product $H^1(T;\mathbb{Z})\times H^1(T;\mathbb{Z}) \to H^2(T;\mathbb{Z})$ with respect to the basis α,β of $H^1(T;\mathbb{Z})$ and the identification $H^2(T;\mathbb{Z}) = \mathbb{Z}\cdot[T]^* = \mathbb{Z}$*[998] *is given by the matrix*

$$\begin{pmatrix} \alpha\cup\alpha & \alpha\cup\beta \\ \beta\cup\alpha & \beta\cup\beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[998]Here $[T]^* \in H^2(T;\mathbb{Z})$ denotes the dual fundamental class that we introduced on page 1226, it is uniquely determined by $\langle [T]^*, [T]\rangle = 1$.

Thus we have shown that the cup product on cohomology is in general non-zero and that it is in general not commutative.

Remark.

(1) We have just shown that the 2-cocycle $p^*(\theta) \cup q^*(\theta) \in C^2(T;\mathbb{Z})$ represents a generator of $H^2(T;\mathbb{Z})$. We have thus answered Question 50.10.

(2) We have now performed enough calculations to see that the cohomology rings contain more information than the homology or the cohomology groups. More precisely, it follows from Lemma 33.12 and Propositions 31.3 and 34.8 that for the 2-dimensional torus $S^1 \times S^1$ and for $S^1 \vee S^1 \vee S^2$ we have

$$H_i(S^1 \times S^1;\mathbb{Z}) \;\cong\; H_i(S^1 \vee S^1 \vee S^2;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0,2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i \geq 3. \end{cases}$$

We deduce from the Universal Coefficient Theorem 51.13 for Cohomology Groups that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ also have isomorphic cohomology groups. In Lemma 57.14 we saw that the cup product on $H^1(S^1 \times S^1;\mathbb{Z})$ is non-zero, whereas we showed in Lemma 57.11 that the cup product on $H^1(S^1 \vee S^1 \vee S^2;\mathbb{Z})$ is zero.[999]

In Exercise 57.2 we will use the above calculation of the cup product of the torus to prove the following slightly subtle lemma.

Lemma 57.15. *Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. If f is orientation-preserving, then*

$$\det\big(f_*\colon H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z})\big) \;=\; \det\big(f^*\colon H^1(T;\mathbb{Z}) \to H^1(T;\mathbb{Z})\big) \;=\; +1.$$

Conversely, if f is orientation-reversing, then both determinants equal -1.

57.4. **The cup product of the surface of genus 2.** In this section we calculate the cup product for the surface Σ of genus 2. By Observation 57.13 it remains to determine the cup product

$$H^1(\Sigma;\mathbb{Z}) \times H^1(\Sigma;\mathbb{Z}) \;\to\; H^2(\Sigma;\mathbb{Z})$$

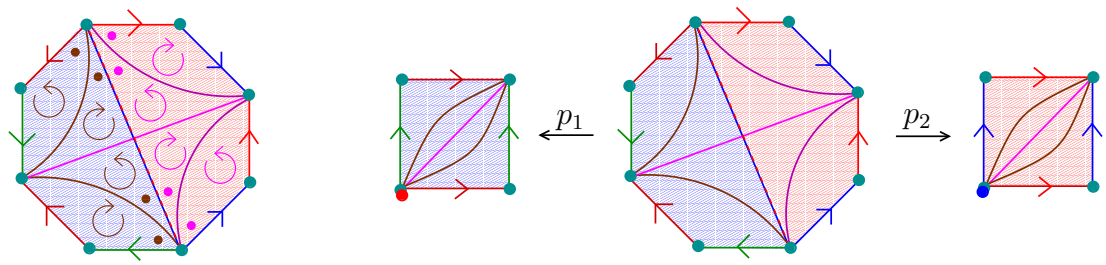
on a basis of $H^1(\Sigma;\mathbb{Z})$. We recall that on page 1202 we introduced maps $p_i\colon \Sigma \to T = S^1 \times S^1$ for $i = 1,2$. We refer to Figure 725 on the right for a reminder of the definition of the maps. We denote by $\alpha, \beta \in H^1(T;\mathbb{Z})$ the basis from Lemma 50.9 that we had also used in the previous section.

In Lemma 50.11 we saw that the map

$$p_1^* \oplus p_2^*\colon H^1(T;\mathbb{Z}) \oplus H^1(T;\mathbb{Z}) \;\to\; H^1(\Sigma;\mathbb{Z})$$

is an isomorphism. We deduce that a basis for $H^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1,2$. Now we can use the definition of the cup product and the explicit

[999]Of course we do not need to use cup products to distinguish the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$, this can already be done using fundamental groups. Nonetheless it is encouraging to see that we found a new way to prove this statement.



fundamental class of Σ projections from Σ to the torus

FIGURE 725.

description of a representative of the fundamental class $[\Sigma] \in H_2(\Sigma;\mathbb{Z})$ in Figure 725 on the left (see also Figure 657 for a more detailed discussion) to compute the cup product $H^1(\Sigma;\mathbb{Z}) \times H^1(\Sigma;\mathbb{Z}) \to H^2(\Sigma;\mathbb{Z}) = \mathbb{Z}$.

A not particularly exciting calculation leads us eventually to the following lemma.[1000]

Lemma 57.16. *Let Σ be the surface of genus 2. We use the above notation. The cup product $H^1(\Sigma;\mathbb{Z}) \times H^1(\Sigma;\mathbb{Z}) \to H^2(\Sigma;\mathbb{Z})$ with respect to the basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $H^1(\Sigma;\mathbb{Z})$ and the identification $H^2(\Sigma;\mathbb{Z}) = \mathbb{Z} \cdot [\Sigma]^* = \mathbb{Z}$ is given by the matrix*

$$\begin{pmatrix} \alpha_1 \cup \alpha_1 & \alpha_1 \cup \beta_1 & \alpha_1 \cup \alpha_2 & \alpha_1 \cup \beta_2 \\ \beta_1 \cup \alpha_1 & \beta_1 \cup \beta_1 & \beta_1 \cup \alpha_2 & \beta_1 \cup \beta_2 \\ \alpha_2 \cup \alpha_1 & \alpha_2 \cup \beta_1 & \alpha_2 \cup \alpha_2 & \alpha_2 \cup \beta_2 \\ \beta_2 \cup \alpha_1 & \beta_2 \cup \beta_1 & \beta_2 \cup \alpha_2 & \beta_2 \cup \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

An analogous statement holds for any surface of genus $g \geq 1$, i.e. the cup product on the first cohomology of the surface of genus $g \geq 1$ can be represented by a block diagonal matrix consisting of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark. In the next section we will obtain a more conceptual proof of Lemma 57.16.

Before we continue with more calculations of cup products we return to a question that we had asked in Algebraic Topology II.

Question 39.30.

(2) Let $g \geq 2$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?[1001]

[1000] We have to evaluate the cup products of the explicit 1-cochains on the eight singular 2-simplices illustrated in Figure 725. It is straightforward to see that all 16 cup products evaluate to zero on the four central singular 2-simplices. For the remaining cup products the calculation is almost the same as for the torus on page 1335.

[1001] Recall that if $f\colon M \to N$ is a map between two closed, oriented connected, non-empty n-dimensional topological manifolds, then the degree $\deg(f) \in \mathbb{Z}$ is defined as the unique integer $\deg(f)$ that satisfies $f_*([M]) = \deg(f) \cdot [N]$.

Now we have the tools to answer this question. In this context it is useful to introduce the following definition.

Definition. A closed, oriented connected, non-empty topological manifold M is called *flexible* if given any $k \in \mathbb{Z}$ there exists a map $f\colon M \to M$ of degree k.

Example. In Lemma 32.15 we saw that S^1 is flexible. The combination of this fact with Lemma 32.18 showed that in fact any sphere S^n, $n \geq 1$ is flexible.

Now we can determine which closed oriented connected 2-dimensional manifolds are flexible. More precisely, we have the following proposition which gives a negative answer to Question 39.30 (2).

Proposition 57.17. *Let $g \geq 1$. We denote by Σ_g the surface of genus g.*

(1) *The torus $\Sigma_1 = S^1 \times S^1$ is flexible.*
(2) *For $g \geq 2$ the surface Σ_g is not flexible. In fact if $k \neq -1, 0, 1$, then there is no map $\Sigma_g \to \Sigma_g$ of degree k.*

Proof.

(1) We first show that the torus $S^1 \times S^1$ is flexible. So let $k \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^k$. We claim that $f \times \mathrm{id}\colon S^1 \times S^1 \to S^1 \times S^1$ has degree k. In fact by the naturality of the short exact sequence of the Künneth-Theorem 43.8 we have a commutative diagram

$$\begin{tikzcd} H_1(S^1) \otimes H_1(S^1) \arrow[r, "\Theta", "\cong"'] \arrow[d, "f_* \otimes \mathrm{id}"] & H_2(S^1 \times S^1) \arrow[d, "(f \times \mathrm{id})_*"] \\ H_1(S^1) \otimes H_1(S^1) \arrow[r, "\Theta", "\cong"'] & H_2(S^1 \times S^1) \end{tikzcd}$$

where the horizontal maps are isomorphisms. By Lemma 32.15 the map on the left is given by multiplication by k. Hence the map on the right is also given by multiplication by k.

(2) Now let $g \geq 2$ and let $f\colon \Sigma_g \to \Sigma_g$ be a map of degree $k \neq 0$. We have to show that $k \in \{\pm 1\}$. We write $\pi := \pi_1(\Sigma_g)$. We start out with the following claim.

Claim. The map $f_*\colon \pi \to \pi$ is an epimorphism.

First note that the argument in the proof of Proposition 39.26 shows that $f_*(\pi)$ is a finite-index subgroup of π. (Here we used that $k \neq 0$.) We denote by $n \in \mathbb{N}$ the index of the subgroup $f_*(\pi) \subset \pi$. In the proof of Proposition 40.11 we used the multiplicativity of the Euler characteristic under finite covers to show that the abelianization of $f_*(\pi)$ is isomorphic to $\mathbb{Z}^{2n\cdot(g-1)+2}$. On the other hand the epimorphism $\pi \to f_*(\pi)$ induces evidently an epimorphism of the abelianization of π onto the abelianization of $f_*(\pi)$, i.e. we have an epimorphism from $\mathbb{Z}^{2g}$ onto $\mathbb{Z}^{2n\cdot(g-1)+2}$. Since $g \geq 2$ this is only possible if $n = 1$. This concludes the proof of the claim.

Claim. The map $f^*\colon H^1(\Sigma_g; \mathbb{Z}) \to H^1(\Sigma_g; \mathbb{Z})$ is an isomorphism.

By the first claim we now that the map $f_*\colon \pi \to \pi$ is an epimorphism. It follows from the naturality of the Hurewicz Isomorphism, see Propositions 38.3 and 38.2 (4), that the map $f_*\colon H_1(\Sigma_g) \to H_1(\Sigma_g)$ is also an epimorphism. But any epimorphism from a finitely generated abelian group to itself is in fact an isomorphism[1002], so we see that $f_*\colon H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ is in fact an isomorphism. By Proposition 51.17 the evaluation homomorphism $\mathrm{ev}\colon H^1(\Sigma_g;\mathbb{Z}) \cong \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{Z})$ is a natural isomorphism, i.e. we have a commutative diagram

$$\begin{array}{ccc} H^1(\Sigma_g;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}) \\ \downarrow{\scriptstyle f^*} & & \cong\downarrow{\scriptstyle (f_*)^*} \\ H^1(\Sigma_g;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}). \end{array}$$

The right-hand map is an isomorphism, it follows from the commutative diagram that the left-hand map $f^*\colon H^1(\Sigma_g;\mathbb{Z}) \to H^1(\Sigma_g;\mathbb{Z})$ is also an isomorphism. This concludes the proof of the second claim.

By Lemma 57.16 we know that there exist $a, b \in H^1(\Sigma_g;\mathbb{Z})$ with $a \cup b = [\Sigma_g]^*$. Since $f^*\colon H^1(\Sigma_g;\mathbb{Z}) \to H^1(\Sigma_g;\mathbb{Z})$ is an isomorphism we can in particular find cohomology classes $c, d \in H^1(\Sigma_g;\mathbb{Z})$ with $f^*(c) = a$ and $f^*(d) = b$. Now we have

$$\begin{array}{ccccccccc} & & & & & \text{since } c\cup d \in H^2(\Sigma_g;\mathbb{Z}) \text{ and since } [\Sigma_g]^* \in H^2(\Sigma_g;\mathbb{Z}) \text{ is a generator such an } m \text{ exists} & & & \\ & & & & & \downarrow & & & \\ [\Sigma_g]^* & = & a\cup b & = & f^*(c)\cup f^*(d) & = & f^*(c\cup d) & = & f^*(m\cdot[\Sigma_g]^*) & = & \deg(f)\cdot m\cdot[\Sigma_g]^*. \\ & & & & & \uparrow & & & & \uparrow & \\ & & & & & \text{Lemma 57.10} & & & & \text{by Lemma 51.16 we have } f^*([\Sigma_g]^*) = \deg(f)\cdot[\Sigma_g]^* & \end{array}$$

Evidently this is only possible if $\deg(f) \in \{\pm 1\}$. □

57.5. **The cup product of the real projective plane** $(*)$**.** Now we want to understand the cup product in the $\mathbb{F}_2$-cohomology of the real projective plane $\mathbb{R}\mathrm{P}^2$. Recall that on page 1197 we saw that $H^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$[1003] for $i = 0, 1, 2$ and that the $\mathbb{F}_2$-cohomology vanishes in all other dimensions. By Observation 57.13 it remains to determine the cup product on the first cohomology.

To compute the cup product we need to get a good understanding of the cohomology groups. Recall that on 1121 we had seen that $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in C_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a representative of the $\mathbb{F}_2$-fundamental class $[\mathbb{R}\mathrm{P}^2]$ of $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$.[1004] (See Figure 726 for the definition of $\sigma_1, \sigma_2, \tau_1$ and τ_2.) It follows from Proposition 51.18 and the above that the map

$$\begin{array}{rcl} \eta\colon H^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \to & \mathbb{F}_2 \\ [\varphi \in C^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)] & \mapsto & [\varphi(\sigma_1) + \varphi(\sigma_2) + \varphi(\tau_1) + \varphi(\tau_2)] \end{array}$$

[1002]Why is that?

[1003]Why is it safe to write $H^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$? Would it not be much more appropriate to write $H^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$?

[1004]Here we use the identification $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$ from page 113.

is an isomorphism.

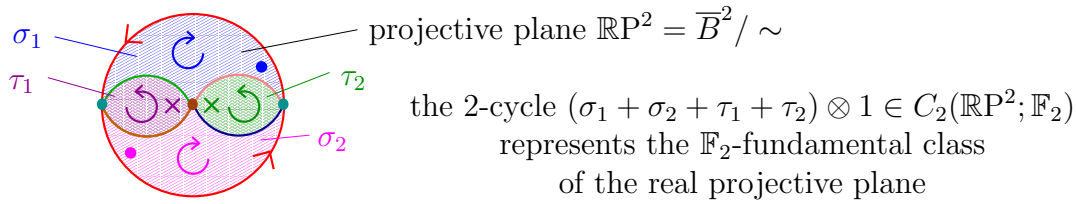


FIGURE 726.

Now we also need to find an explicit representative of $x \in H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. We will find such a representative in a similar fashion to the construction of the singular 1-cocycle $\theta_{\mathbb{Z}} \in H^1(S^1;\mathbb{Z})$ on page 1167. More precisely, we consider the map

$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{F}_2 \\ (x,y,z) & \mapsto & \begin{cases} 1, & \text{if } (z>0) \text{ or } (z=0 \text{ and } y>0) \text{ or } (z=y=0 \text{ and } x=1), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Note that for any point $P \in S^2$ we have $f(-P) = 1 - f(P)$.[1005] For a singular 1-simplex $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$ of σ and we define

$$\varphi(\sigma) \ := \ f\big(\widetilde{\sigma}(0,1)\big) - f\big(\widetilde{\sigma}(1,0)\big) \ \in \ \mathbb{F}_2.$$

We refer to Figure 727.

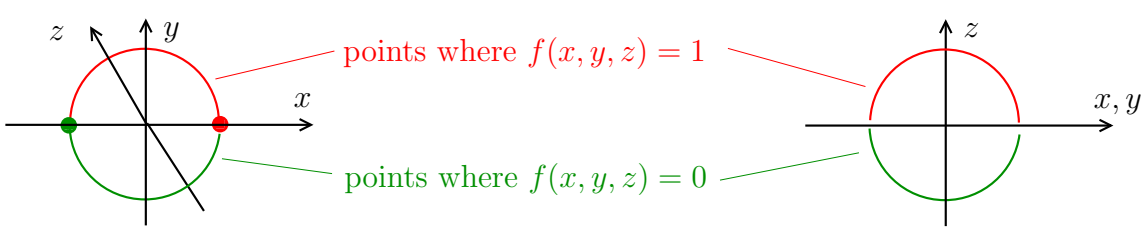


FIGURE 727.

Note that φ is well-defined since the only other lift of σ is given by multiplying $\widetilde{\sigma}$ by -1,[1006] but then it follows from $f(-P) = 1 - f(P)$ that the difference of the evaluations at the endpoints does not change. (Here we use that we work with $\mathbb{F}_2$-coefficients instead of $\mathbb{Z}$-coefficients.) The argument of Lemma 48.10 can easily be adapted to show that the singular 1-cochain $\varphi\colon C_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle.

[1005] In fact for the subsequent discussion any map $f\colon S^2 \to \mathbb{F}_2$ with this property would work. We just picked this example for concreteness.

[1006] Why is that true?

Example. We consider the map $\mu\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 = S^2/\sim$ given by $\mu(1-t,t) = [e^{\pi i t}]$. Then $\mu \in C_1(\mathbb{R}\mathrm{P}^2)$. Thus we can apply the Kronecker pairing $H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times H_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ from Lemma 50.5 to $[\varphi] \in H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ and $[\mu] \in H_1(\mathbb{R}\mathrm{P}^2)$ and we calculate that

$$\langle [\varphi],[\mu]\rangle \;=\; \varphi(\mu) \;\underset{\substack{\uparrow \\ \text{since a lift of } \mu(1-t,t)=[e^{\pi i t}] \text{ to } S^2 \text{ is given by } \widetilde{\mu}(1-t,t)=e^{\pi i t}}}{=}\; f(e^{\pi i}) - f(e^0) \;=\; 1 \;\in \mathbb{F}_2.$$

Since this is non-zero we have now actually shown two statements at once: the cohomology class $[\varphi] \in H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is non-zero and we have also certified that the homology class $[\mu] \in H_1(\mathbb{R}\mathrm{P}^2)$ is non-zero.[1007]

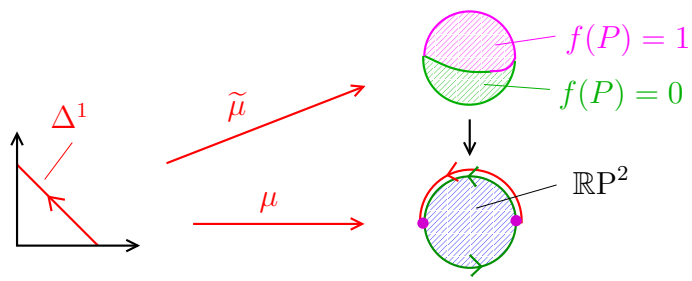


FIGURE 728.

Now we have the tools to prove the following lemma.

Lemma 57.18. *Under the identifications $H^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$ the cup product*

$$\cup\colon H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \;\to\; H^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$$

is given by

$$\begin{array}{rcl} \mathbb{F}_2 \times \mathbb{F}_2 & \to & \mathbb{F}_2 \\ (a,b) & \mapsto & a\cdot b. \end{array}$$

Remark. Note that the calculation in Lemma 57.18 that the statement of Lemma 57.9 does not hold for $\mathbb{F}_2$-coefficients.

Proof. By the discussion above it suffices to show that $\eta([\varphi]\cup[\varphi]) \neq 0$. Put differently, by definition of η we need to show that

$$(\varphi\cup\varphi)(\sigma_1) + (\varphi\cup\varphi)(\sigma_2) + (\varphi\cup\varphi)(\tau_1) + (\varphi\cup\varphi)(\tau_2) \;=\; 1 \;\in \mathbb{F}_2.$$

In Exercise 57.3 we will determine all four summands and we will see that the sum is indeed 1. □

The above calculation of the cup product of the real projective plane $\mathbb{R}\mathrm{P}^2$ is very explicit and hands on, but not very conceptual. Later on we will give a very different calculation of the cup product on $\mathbb{F}_2$-cohomology of all real projective spaces $\mathbb{R}\mathrm{P}^n$.

[1007] Do you know any other methods for showing that $[\mu] \neq 0 \in H_1(\mathbb{R}\mathrm{P}^2)$?

57.6. **The cup product and the wedge product** $(*)$**.** As a reminder, we denote by $\mathcal{M}$ the category of manifolds. As we already saw, the maps

$$\begin{array}{rcl} M & \mapsto & (H^k_{\mathrm{dR}}(M), \cup) \\ (f\colon M \to N) & \mapsto & (f^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category $\mathcal{M}$ of manifolds to the category of graded rings $\mathcal{GR}$.

If we assign to a manifold its k-th real cohomology groups, then we saw in Theorem 55.6 that it is naturally isomorphic to the k-th de Rham cohomology group. Now we can generalize Theorem 55.6.

Theorem 57.19. *The natural isomorphisms u from Theorem 55.6 are in fact ring homomorphisms, i.e. they define a natural isomorphism from the functor*

$$\begin{array}{rcl} \mathcal{M} & \to & \mathcal{GR} \\ M & \mapsto & (H^*_{\mathrm{dR}}(M), \wedge) \end{array} \quad \textit{to the functor} \quad \begin{array}{rcl} \mathcal{M} & \to & \mathcal{GR} \\ M & \mapsto & (H^*(M;\mathbb{R}), \cup). \end{array}$$

Example. As usual we denote by dt on $\mathbb{R}$ the 1-form that for every $v \in T_P\mathbb{R} = \mathbb{R}$ is given by $dt(v) = v$. We denote by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ the obvious projection map. The closed 1-form dt descends to a closed 1-form on S^1, i.e. there exists a unique 1-form $\overline{dt}$ on $\mathbb{R}/\mathbb{Z} = S^1$ whose pull-back equals dt. Let $\sigma\colon \Delta^1 \to \mathbb{R}/\mathbb{Z} = S^1$ be a smooth singular 1-simplex. Since Δ^1 is simply-connected we can lift this singular 1-simplex, i.e. there exists a map (which is in fact smooth) $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ with $\pi \circ \widetilde{\sigma} = \sigma$. We have

$$\int_\sigma \overline{dt} \underset{\substack{\uparrow \\ \text{see definition on page 1297}}}{=} -\int_{s=0}^{s=1} \overline{dt}(\sigma'(t, 1-t))ds \underset{\substack{\uparrow \\ \text{definition of } \overline{dt}}}{=} -\int_{s=0}^{s=1} \widetilde{\sigma}'(t, 1-t)ds \underset{\substack{\uparrow \\ \text{Fundamental Theorem} \\ \text{of Calculus}}}{=} -\widetilde{\sigma}(1,0) + \widetilde{\sigma}(0,1) \underset{\substack{\uparrow \\ \text{see page 1166}}}{=} \theta_{\mathbb{R}}(\sigma).$$

This little calculation shows that under the natural isomorphisms from Theorem 55.6 the class $[\overline{dt}] \in H^1_{\mathrm{dR}}(M)$ gets sent to our favorite generator $[\theta_{\mathbb{R}}] \in H^1(M;\mathbb{R})$, i.e. we have $u([\overline{dt}]) = [\theta_{\mathbb{R}}]$.

Next we consider the torus $M = S^1 \times S^1$. As before we denote $p\colon S^1 \times S^1 \to S^1$ respectively $q\colon S^1 \times S^1 \to S^1$ the projection onto the first, respectively the second factor. A direct calculation shows that, with the usual product orientation on $M = S^1 \times S^1$ we have

$$\int_{S^1 \times S^1} p^*(\overline{dt}) \wedge q^*(\overline{dt}) \;=\; +1.$$

On the other hand, by the naturality of u we have $u(p^*([\overline{dt}])) = p^*(u([\overline{dt}])) = p^*([\theta_{\mathbb{R}}])$ and similarly we also have $u(q^*([\overline{dt}]) = q^*([\theta_{\mathbb{R}}])$. The calculation of the above integral is now consistent with the calculation in Lemma 1347 where we had, implicitly, shown that $\langle p^*([\theta_{\mathbb{R}}]) \cup q^*([\theta_{\mathbb{R}}]), [S^1 \times S^1]\rangle = +1$.

Proof. It is shown in [Dup, Theorem 2.4], through a rather lengthy calculation, that the natural isomorphisms from Theorem 55.6 are ring homomorphisms. Unfortunately it is

rather difficult to make sure that all the different sign conventions match.[1008] The above example gives us at least some confidence that the signs are right. We also refer to [War, Theorem 5.45] for a proof of the theorem. □

Exercises for Chapter 57.

Exercise 57.1. Let X be a topological space. We consider the map

$$\Phi\colon H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X) \xrightarrow{p_*} H_*(X)$$

where $p\colon X \times X \to X$ denotes the projection on the first factor. Show that Φ is the trivial map.

Exercise 57.2. Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. Show that if f orientation-preserving, then the following two statements hold:

(a) $f_*\colon H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z})$ has determinant one,
(b) $f^*\colon H^1(T;\mathbb{Z}) \to H^1(T;\mathbb{Z})$ has determinant one.

Exercise 57.3. We consider $\mathbb{R}\mathrm{P}^2 = S^2/\sim$ which we identify with $\overline{B}^2/\sim$ via the homeomorphism that is induced by the projection map $S^2 \to \overline{B}^2$. Let $\varphi \in C^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle that we defined on page 1352. Determine *each of the four summands* in the sum

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2).$$

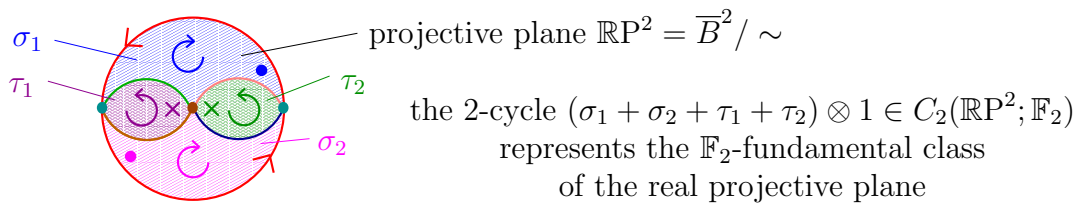


FIGURE 729.

Exercise 57.4. Let K be the Klein bottle.

(a) Is the cup product $H^1(K;\mathbb{Z}) \times H^1(K;\mathbb{Z}) \to H^2(K;\mathbb{Z})$ non-trivial?
(b) Is the cup product $H^1(K;\mathbb{F}_2) \times H^1(K;\mathbb{F}_2) \to H^2(K;\mathbb{F}_2)$ non-trivial?

[1008] For example, on [Dup, p. 20] it says "let $\cup$" be the usual cup-product", but as we had discussed on page 1334, it is not so clear what "usual cup-product" is supposed to mean.

58. The relative cup product

Our goal in this chapter is to introduce a relative version of the cup product. More precisely, given an excisive triad (X, A, B) and a commutative ring R we will introduce a cup product

$$H^p(X, A; R) \times H^q(X, B; R) \xrightarrow{\quad \cup \quad} H^{p+q}(X, A \cup B; R)$$

that is functorial in an appropriate sense and that equals the previous definition of the cup product for the special case that $A = B = \varnothing$. This relative cup product will help us in determining the cup products of wedges of topological spaces and of the suspension of a topological space. Also it will help us to determine the mysterious sounding Lusternik-Schnirelmann category.

58.1. **Definition of the relative cup product.** In this section we will work a lot with relative cochains. Before we do so we recall the following convention from page 1170.

Convention.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D, G) & \xrightarrow{\cong} & \{f\colon C \to G \,|\, f \text{ vanishes on } D\} \\ (\varphi\colon C/D \to G) & \mapsto & (C \to C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) In particular, given a pair of topological spaces (X, A), a commutative ring R and $n \in \mathbb{N}_0$ we make the identification

$$C^n(X, A; R) \xrightarrow{\cong} \{f\colon C_n(X) \to R \,|\, f(\sigma) = 0 \text{ for all } \sigma\colon \Delta^n \to A\}.$$

The definition of the relative cup product requires some preparation. We start out with the following definition.

Definition. Let (X, A, B) be a triad of topological spaces. As in the definition on page 1203 we write

$$C_n^{\{A,B\}}(A \cup B) \;:=\; \left\{ \sum_{j=1}^{k} n_j \sigma_j \in C_n(A \cup B) \;\middle|\; \begin{array}{c} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } A \text{ or it lies in } B \end{array} \right\}$$

The usual boundary map on the groups $C_*(A \cup B)$ restricts to a boundary map on the groups $C_*^{\{A,B\}}(A \cup B)$. Given a commutative ring R we consider

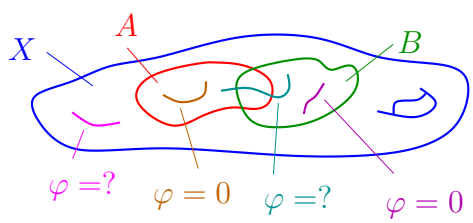
$$C^n(X, \{A, B\}; R) \;:=\; \ker\big\{C^n(X; R) \to \mathrm{Hom}\big(\underbrace{C_n^{\{A,B\}}(A \cup B)}_{\text{subgroup of } C_n(X)}, R\big)\big\}.$$

This is again a cochain complex and we denote the corresponding cohomology groups by $H^*(X, \{A, B\}; R)$.

Remark. With the above notation we could write that

$$C^n(X,\{A,B\};R) \;=\; \{\varphi\colon C_n(X)\to R \,|\, \varphi(\sigma)=0 \text{ for any } \sigma\colon \Delta^n\to A \text{ and any } \sigma\colon \Delta^n\to B\}.$$

The definition of $C^n(X,\{A,B\};R)$ is illustrated in Figure 730.



$\varphi \in H^1(X,\{A,B\};R)$

FIGURE 730.

Lemma 58.1. *Let (X,A,B) be an excisive triad of topological spaces. Given any commutative ring R the obvious*[1009] *map*

$$C^*(X,A\cup B;R) \quad\to\quad C^*(X,\{A,B\};R)$$

induces for every $n\in\mathbb{N}_0$ an isomorphism

$$H^n(X,A\cup B;R) \quad\xrightarrow{\cong}\quad H^n(X,\{A,B\};R).$$

Proof. We have the following commutative diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(A\cup B) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A\cup B) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{=\mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & C_*^{\{A,B\}}(A\cup B) & \longrightarrow & C_*(X) & \longrightarrow & \dfrac{C_*(X)}{C_*^{\{A,B\}}(A\cup B)} & \longrightarrow & 0.
\end{array}$$

By our hypothesis the triad (X,A,B) is excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It now follows from Corollary 49.6 that the right vertical map is also a chain homotopy equivalence.

Recall that by the convention on page 1356 we have an identification

$$\mathrm{Hom}\left(\frac{C_n(X)}{C_n^{\{A,B\}}(A\cup B)},R\right) \;=\; C^n(X,\{A,B\};R).$$

The lemma follows from the above vertical chain homotopy equivalence, the above observation and Lemma 48.8. □

The following lemma is the motivation for introducing the rather technical cohomology modules $H^*(X;\{A,B\};R)$.

[1009] What is the "obvious" map?

Lemma 58.2. *Let (X, A, B) be a triad of topological spaces and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} C^p(X,A;R) \times C^q(X,B;R) & \to & C^{p+q}(X,\{A,B\};R) \\ (\varphi,\psi) & \mapsto & \left[\begin{array}{rcl} C_{p+q}(X) & \to & R \\ \sigma & \mapsto & \varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}]) \end{array}\right]. \end{array}$$

is well-defined and it induces an R-bilinear map

$$\cup\colon H^p(X,A;R) \times H^q(X,B;R) \to H^{p+q}(X,\{A,B\};R).$$

Proof. The lemma follows easily from the following claim.

Claim. Given $\varphi \in C^p(X,A;R)$ and $\psi \in C^q(X,B;R)$ the homomorphism

$$\begin{array}{rcl} C_{p+q}(X) & \to & R \\ (\sigma\colon \Delta^{p+q} \to X) & \mapsto & \varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}]) \end{array}$$

lies actually in $C^{p+q}(X,\{A,B\})$.

By the remark on page 1357 we have to show that the above cochain vanishes on all singular $(p+q)$-simplices that lie entirely in A or that lie entirely in B. But this is clear:

(1) given $\sigma\colon \Delta^{p+q} \to A$ the first factor $\varphi(\sigma\circ[v_0,\dots,v_p])$ is necessarily zero since we have $\varphi \in C^p(X,A;R)$, and similarly,
(2) given $\sigma\colon \Delta^{p+q} \to B$ the second factor $\psi(\sigma\circ[v_p,\dots,v_{p+q}])$ is necessarily zero since we have $\psi \in C^q(X,B;R)$.

This concludes the proof of the claim and thus also of the lemma. □

Definition. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We refer to the composition

$$H^p(X,A;R) \times H^q(X,B;R) \xrightarrow{(c,d)\mapsto c\cup d} H^{p+q}(X,\{A,B\};R) \xleftarrow{\cong} H^{p+q}(X,A\cup B;R)$$

$$\uparrow$$

isomorphism by Lemma 58.1

again as a *cup product* (or sometimes as a *relative cup product*) and we denote it as usual by the "$\cup$"-symbol.

Remark. Let X be a topological space, let $A, B \subset X$ be subsets and let R be a commutative ring. Suppose that $C_*^{\{A,B\}}(A\cup B) = C_*(A\cup B)$. (This is for example the case if $A = B = \varnothing$ or more generally, if A is a subset of B or if B is a subset of A.) Then the above definition of the cup product is just the "naive" definition given by

$$\begin{array}{rcl} C^p(X,A;R)\times C^q(X,B;R) & \to & C^{p+q}(X,A\cup B;R) \\ (\varphi,\psi) & \mapsto & \left(\begin{array}{rcl} C_{p+q}(X,A\cup B) & \to & R \\ \sigma & \mapsto & \underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in R}\cdot\underbrace{\psi(\sigma\circ[v_p,\dots,v_{p+q}])}_{\in R} \end{array}\right). \end{array}$$

The first part of following proposition is a generalization of Lemma 57.10.

Proposition 58.3.

(1) *Let (X, A, B) and (Y, C, D) be two excisive triads and let R be a commutative ring. Let $f\colon (X, A, B) \to (Y, C, D)$ be a map. Then for all $p, q \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} H^p(Y, C; R) \times H^q(Y, D; R) \arrow[r, "\cup"] \arrow[d, "f^* \times f^*"] & H^{p+q}(Y, C \cup D; R) \arrow[d, "f^*"] \\ H^p(X, A; R) \times H^q(X, B; R) \arrow[r, "\cup"] & H^{p+q}(X, A \cup B; R). \end{tikzcd}$$

(2) *Let (X, A, B) be an excisive triad. If $\varphi\colon R \to S$ is a homomorphism between commutative rings, then for all $p, q \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} H^p(X, A; R) \times H^q(X, B; R) \arrow[r, "\cup"] \arrow[d, "\varphi_* \times \varphi_*"] & H^{p+q}(X, A \cup B; R) \arrow[d, "\varphi_*"] \\ H^p(X, A; S) \times H^q(X, B; S) \arrow[r, "\cup"] & H^{p+q}(X, A \cup B; S). \end{tikzcd}$$

Here the vertical maps to the left are the maps defined on page 1163.

Proof.

(1) It follows easily from the definitions that the following diagram commutes:

$$\begin{tikzcd} H^p(Y, C; R) \times H^q(Y, D; R) \arrow[r, "\cup"] \arrow[d, "f^* \times f^*"] & H^{p+q}(Y, \{C, D\}; R) \arrow[d, "f^*"] & H^{p+q}(Y, C \cup D; R) \arrow[l, "\cong"'] \arrow[d, "f^*"] \\ H^p(X, A; R) \times H^q(X, B; R) \arrow[r, "\cup"] & H^{p+q}(X, \{A, B\}; R) & H^{p+q}(X, A \cup B; R). \arrow[l, "\cong"'] \end{tikzcd}$$

The desired statement follows immediately from this observation.

(2) This statement follows, similar to (1), basically immediately from the definitions. □

We record a particularly useful special case of Proposition 58.3 (1) as a corollary.

Corollary 58.4. *Let (X, A, B) be an excisive triad and let R be a commutative ring. Then the following diagram commutes:*

$$\begin{tikzcd} H^p(X, A; R) \times H^q(X, B; R) \arrow[r, "\cup"] \arrow[d] & H^{p+q}(X, A \cup B; R) \arrow[d] \\ H^p(X; R) \times H^q(X; R) \arrow[r, "\cup"] & H^{p+q}(X; R). \end{tikzcd}$$

In particular, if $\alpha \in H^p(X; R)$ lies in the image of $H^p(X, A; R)$ and if $\beta \in H^q(X; R)$ lies in the image of $H^q(X, B; R)$ and if $A \cup B = X$, then $\alpha \cup \beta = 0$.

Proof. The first part of the corollary follows immediately from Proposition 58.3 (1) applied to the obvious map $f\colon (X, \varnothing, \varnothing) \to (X, A, B)$ of excisive triads.

We turn to the proof of the second part. We denote by $\sigma\colon H^p(X, A; R) \to H^p(X; R)$, $\tau\colon H^p(X, B; R) \to H^p(X; R)$ and by $\upsilon\colon H^{p+q}(X, A \cup B; R) \to H^{p+q}(X; R)$ the obvious

maps. Let $\widehat{\alpha} \in H^p(X, A; R)$ and $\widehat{\beta} \in H^q(X, B; R)$. We write $\alpha = \sigma(\widehat{\alpha})$ and $\beta = \tau(\widehat{\beta})$. We suppose that $A \cup B = X$. Then

$$\alpha \cup \beta \;=\; \sigma(\widehat{\alpha}) \cup \tau(\widehat{\beta}) \;\underset{\substack{\uparrow \\ \text{by the commutative diagram}}}{=}\; \upsilon\big(\widehat{\alpha} \cup \widehat{\beta}\big) \;\underset{\substack{\uparrow \\ \text{since } \widehat{\alpha} \cup \widehat{\beta} \text{ lies in the group} \\ H^{p+q}(X, A \cup B; R) = H^{p+q}(X, X; R) = 0}}{=}\; \upsilon(0) \;=\; 0.$$

□

Before we state the next lemma we introduce one more convention.

Convention. Let A, B, C, D, X and Y be sets, let $f\colon C \to A$, $g\colon B \to D$ and $\mu\colon X \to Y$ be maps and let $\varphi\colon A \times B \to X$ and $\psi\colon C \times D \to Y$ be maps. We say that the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[d, "g"] & X \arrow[d, "\mu"'] \\ C \times D \arrow[r, "\psi"] \arrow[u, "f"] & Y \end{tikzcd}$$

commutes, if for all $c \in C$ and $b \in B$ we have

$$\psi(c, g(b)) \;=\; \mu(\varphi(f(c), b)).$$

A similar convention holds with the directions of f and g reversed. Furthermore, if we are dealing with group homomorphisms, then there is an obvious notion of "commutes up to a given sign".

Later on, on one occasion we will need the following lemma.

Lemma 58.5. *Let (X, A) be a pair of topological spaces and let R be a commutative ring. For any $n, k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} H^p(A; R) \times H^q(A; R) \arrow[r, "\cup"] \arrow[d, "\delta"] & H^{p+q}(A; R) \arrow[d, "\delta"] \\ H^p(X; R) \times H^{q+1}(X, A; R) \arrow[r, "\cup"] \arrow[u, "i^*"] & H_{p+q+1}(X, A; R) \end{tikzcd}$$

up to the sign $(-1)^p$. Here $i\colon B \to X$ denotes the inclusion map and δ denotes the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A).

Proof $(*)$. We pick cocyles $\varphi \in C^p(X; R)$ and $\psi \in C^q(A; R)$. We denote by $\widetilde{\psi}\colon C^q(X) \to R$ the extension of ψ that is defined to be zero on all singular k-simplices that do not lie in

the subset A. We have the following equalities:

$$\begin{array}{rcl}
\delta([i^*\varphi]\cup[\psi]) & = & \delta([i^*\varphi\cup\psi]) \\
 & & \underset{\substack{\text{by the discussion of the connecting homomorphism in the remark on page 1174 and} \\ \text{since } \varphi\cup\widetilde{\psi} \text{ is the trivial extension of } i^*\varphi\cup\psi\colon C_{p+q}(A)\to R \text{ to a homomorphism } C_{p+q}(X)\to R}}{\downarrow} \\
 & & = [\delta(\varphi\cup\widetilde{\psi})] \\
 & & \underset{\substack{\text{Lemma 57.2}}}{\uparrow} \\
 & & = [\delta\varphi\cup\widetilde{\psi}+(-1)^p\cdot(\varphi\cup\delta\widetilde{\psi})] \\
 & \underset{\substack{\text{since } \varphi \text{ is a cocycle}}}{\uparrow} & \\
 & = (-1)^p\cdot[\varphi\cup\delta\widetilde{\psi}] & = (-1)^p\cdot([\varphi]\cup[\delta\widetilde{\psi}]) \underset{\substack{\text{again by the remark on page 1174}}}{\uparrow} = (-1)^p\cdot([\varphi]\cup\delta[\psi]).
\end{array}$$

This equality implies that the diagram commutes up to the sign $(-1)^p$. □

We conclude this section with the definition of the cup product on cohomology with compact support. We will need this definition at a later stage.

Definition. Let X be a topological space and let $\varphi\in H^k_c(X)$ and let $\psi\in H^l_c(X)$. By Lemma 53.2 there exists a compact subset $K\subset X$ and a cohomology class $\widetilde{\varphi}\in H^k(X,X\backslash K)$ such that the image of $\widetilde{\varphi}$ in $H^k_c(X)$ equals φ. Analogously we find a compact set L and $\widetilde{\psi}\in H^l(X,X\setminus L)$. We define

$$\varphi\cup\psi \;:=\; \text{the image of } \varphi\cup\psi\in H^{k+l}(X,X\setminus(K\cup L)) \text{ in } H^{k+l}_c(X).$$

It follows easily from Corollary 58.4 that this definition does not depend on any of the choices. We refer to $\varphi\cup\psi$ as the *cup product in cohomology with compact support.*

58.2. **Calculations using the relative cup product.** In this section we will use Proposition 58.3 (1) to calculate cup products of several topological spaces. We start out with the suspension $\Sigma(X)=X\times[-1,1]/\sim$ of a topological space X. We refer to page 452 for the precise definition and we refer to Figure 731 for a reminder.

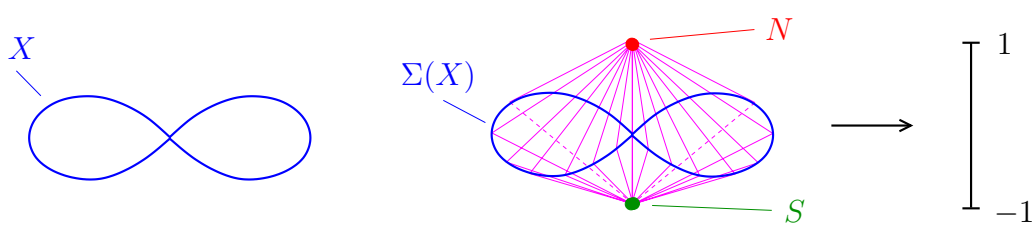


FIGURE 731.

The following lemma shows that the cup product of the suspension of a topological space is quite dull.

Lemma 58.6. *Let X be a path-connected topological space and let R be a commutative ring.*

(1) *We have*

$$H^k(\Sigma(X);R) \;\cong\; \begin{cases} R, & \text{if } k=0,\\ 0, & \text{if } k=1,\\ H^{k-1}(X;R), & \text{if } k\geq 2.\end{cases}$$

(2) *For any $k,l\geq 1$ the cup product*

$$H^k(\Sigma(X);R)\times H^l(\Sigma(X);R) \;\to\; H^{k+l}(\Sigma(X);R)$$

is the zero map.

Proof. We denote by $p\colon X\times[-1,1]\to\Sigma(X)$ the obvious projection map. We denote by $N:=[X\times\{1\}]$ the "North Pole" and we denote by $S:=[X\times\{-1\}]$ the "South Pole". We refer to Figure 731 for an illustration. We write $U=\Sigma(X)\setminus\{S\}$ and $V=\Sigma(X)\setminus\{N\}$. By the same argument as in Lemma 15.1 that U admits deformation retraction from U to the North Pole N. This shows that U is contractible. The same argument shows that V is contractible.

(1) Note that U and V are open subsets of $\Sigma(X)$.[1010] It follows from Proposition 50.12 (4) that $(\Sigma(X),U,V)$ is excisive. Therefore we can apply the Mayer–Vietoris Theorem 50.15 for Cohomology Groups and we obtain the following long exact sequence

$$\ldots\to H^i(\Sigma(X);R)\to H^i(U;R)\oplus H^i(V;R)\to H^i(U\cap V;R)\to H^{i+1}(\Sigma(X);R)\to\ldots$$

The desired isomorphism now follows easily from the above long exact sequence and the following:
   (a) above we had observed that U and V are contractible,
   (b) the intersection $U\cap V$ is homeomorphic to $X\times(-1,1)$, in particular it is homotopy equivalent to X,
   (c) Lemmas 48.14 and 48.12 and the discussion of induced maps on 0-th cohomology on page 1172.

(2) Now let $k,l\geq 1$. It follows immediately from the long exact sequence of cohomology groups corresponding to the pair $(\Sigma(X),U)$, the fact that U is homotopy equivalent to a point (together with Lemmas 48.12 and 48.13) that the inclusion induced map $H^k(\Sigma(X),U;R)\to H^k(\Sigma(X);R)$ is an isomorphism. Evidently the same also holds if we replace U by V.

By Corollary 58.4, and the fact that $(\Sigma(X),U,V)$ is excisive, we have the following commutative diagram

$$\begin{tikzcd}
H^k(\Sigma(X),U;R)\times H^l(\Sigma(X),V;R) \arrow[r, "\cup"] \arrow[d, "\cong"] & H^{k+l}(\Sigma(X),\overbrace{U\cup V}^{=\Sigma(X)};R) \arrow[d] \\
H^k(\Sigma(X);R)\times H^l(\Sigma(X);R) \arrow[r, "\cup"] & H^{k+l}(\Sigma(X);R).
\end{tikzcd}$$

[1010] Why is that?

Since $U \cup V = \Sigma(X)$ we see that the cohomology group on the top-right is zero. Furthermore, since the vertical map on the left is an isomorphism we see that the cup product on the bottom is zero. $\square$

Let X and Y be two topological spaces and let $x \in X$ and $y \in Y$ be good points. (See page 387 for the definition of a good point.) We use these points to define the wedge $X \vee Y$ of X and Y. Furthermore let R be a commutative ring. We denote by $p\colon X \vee Y \to X$ and $q\colon X \vee Y \to Y$ the obvious projection maps. By Proposition 50.17 we know that for every $n \in \mathbb{N}$ the map

$$p^* \oplus q^*\colon H^n(X;R) \oplus H^n(Y;R) \xrightarrow{\cong} H^n(X \vee Y;R)$$

is an isomorphism. The following lemma determines the cup product of $X \vee Y$ in terms of the cup products of X and Y. It can be viewed as a generalization of Lemma 57.11.

Lemma 58.7. *We continue with the above notation. Then for any $\alpha, \beta \in H^*(X;R)$ and $\varphi, \psi \in H^*(Y;R)$ in degrees ≥ 1 the following statements hold in $H^*(X \vee Y;R)$:*

$\cup$	$p^*(\beta)$	$q^*(\psi)$
$p^*(\alpha)$	$p^*(\alpha \cup \beta)$	0
$q^*(\varphi)$	0	$q^*(\varphi \cup \psi)$

Proof $(*)$. The statement regarding the "diagonal" terms of the table follows immediately from Lemma 57.10. It remains to prove that the off-diagonal terms are zero. We show that the top-right entry is zero. The proof that the bottom-left entry is zero is basically identical.

Thus let $\alpha \in H^k(X;R)$ and $\psi \in H^l(Y;R)$ with $k, l \geq 1$. We view X and Y as subsets of $X \vee Y$. We denote by $f\colon (X \vee Y, \varnothing) \to (X \vee Y, X)$ and $g\colon (X \vee Y, \varnothing) \to (X \vee Y, Y)$ the two inclusion maps of pairs of topological spaces.

Claim. There exist $\widetilde{\alpha} \in H^k(X \vee Y, Y;R)$ with $g^*(\widetilde{\alpha}) = p^*(\alpha)$ and $\widetilde{\psi} \in H^l(X \vee Y, X;R)$ with $f^*(\widetilde{\psi}) = q^*(\psi)$.

To prove the claim we consider the following two diagrams[1011]

$$\begin{tikzcd} H^k(X \vee Y;R) & H^k(X \vee Y, Y;R) \arrow[l, "g^*"'] \arrow[d] \\ H^k(X;R) \arrow[u, "p^*"] & H^k(X, \{x\};R) \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^l(X \vee Y;R) & H^l(X \vee Y, X;R) \arrow[l, "f^*"'] \arrow[d] \\ H^l(Y;R) \arrow[u, "q^*"] & H^l(Y, \{y\};R). \arrow[l] \end{tikzcd}$$

Here the maps that are unlabeled are induced by the obvious inclusions. It is easy to verify that the diagrams commute. It follows easily from the long exact sequence in cohomology of a pair of topological spaces that the two bottom horizontal maps are isomorphisms. Recall that in Lemma 50.16 we showed that the triad $(X \vee Y, X, Y)$ is excisive. Therefore

[1011] Recall that in $X \vee Y$ the points x and y get identified, so we can view x as an element in Y and we can view y as an element in X.

it follows from Lemma 50.13 that the vertical maps on the right are isomorphism. The claim is an immediate consequence of these observations.

Since, as we had just seen, the triad $(X \vee Y, X, Y)$ is excisive we can apply Corollary 58.4 and we obtain that

$$p^*(\alpha) \cup q^*(\psi) \;=\; g^*(\widetilde{\alpha}) \cup f^*(\widetilde{\psi}) \;\underset{\substack{\uparrow \\ \text{by Corollary 58.4 since } X \cup Y = X \vee Y}}{=}\; 0.$$

□

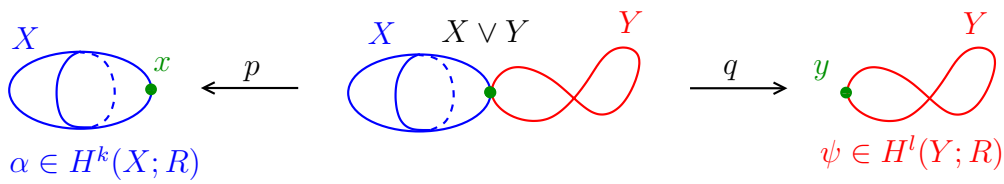


FIGURE 732. Illustration for the proof of Lemma 58.7.

Example. Now we can also give a new proof of Lemma 57.16, more precisely, we can give a new and somewhat more conceptual proof of the calculation of the cup product of the surface Σ of genus 2. First we recall that on page 1202 we introduced two projection maps $p_i \colon \Sigma \to T_i = S^1 \times S^1$ for $i = 1, 2$. It follows easily from Lemma 47.4 that these projection maps are degree-one maps, i.e. they send the fundamental class $[\Sigma]$ to the fundamental class $[T_i]$. In Lemma 50.11 we saw that the map

$$p_1^* \oplus p_2^* \colon H^1(T_1;\mathbb{Z}) \oplus H^1(T_2;\mathbb{Z}) \;\to\; H^1(\Sigma;\mathbb{Z})$$

is an isomorphism. We make the following observations:

(1) Let $\varphi, \psi \in H^1(T_i;\mathbb{Z})$, $i = 1, 2$. Then

$$\langle \varphi \cup \psi, [T_i]\rangle_{T_i} \;\underset{\substack{\uparrow \\ \text{since } p_i \text{ is of degree one}}}{=}\; \langle \varphi \cup \psi, p_{i*}([\Sigma])\rangle_{T_i} \;\underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=}\; \langle p_i^*(\varphi \cup \psi), [\Sigma]\rangle_{\Sigma} \;\underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=}\; \langle p_i^*(\varphi) \cup p_i^*(\psi), [\Sigma]\rangle_{\Sigma}.$$

(2) We denote by $q_i \colon T_1 \vee T_2 \to T_i$, $i = 1, 2$ the obvious projections. The two projection maps p_1, p_2 give rise to a map $p_1 \vee p_2 \colon \Sigma \to T_1 \vee T_2$ which has the property that $q_i \circ (p_1 \vee p_2) = p_i$, $i = 1, 2$. (See Figure 733 for an illustration.) Now let $\varphi_1 \in H^1(T_1;\mathbb{Z})$ and $\varphi_2 \in H^1(T_2;\mathbb{Z})$. Then

$$p_1^*(\varphi_1) \cup p_2^*(\varphi_2) \;\underset{\substack{\uparrow \\ \text{by Lemma 57.10, since } q_i \circ (p_1 \vee p_2) = p_i}}{=}\; (p_1 \vee p_2)^*(q_1^*(\varphi_1) \cup q_2^*(\varphi_2)) \;\underset{\substack{\uparrow \\ \text{Lemma 58.7}}}{=}\; (p_1 \vee p_2)^*(0) \;=\; 0.$$

Precisely the same proof shows also that $p_2^*(\varphi_2) \cup p_1^*(\varphi_1) = 0$.

These two claims, together with the previous calculation of the cup product on the torus $T = S^1 \times S^1$, see Lemma 57.14, give a new proof of Lemma 57.16.

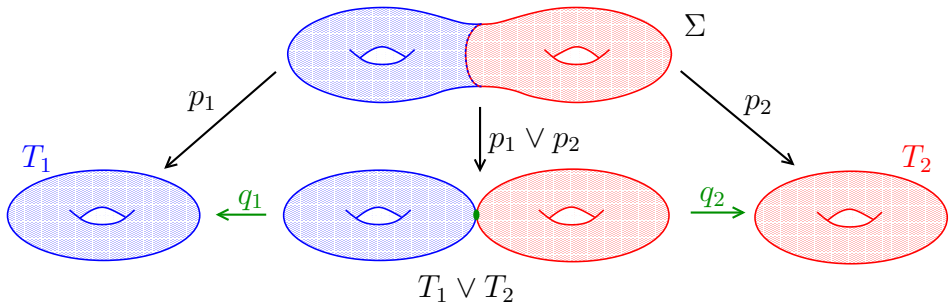


FIGURE 733.

58.3. **The Lusternik-Schnirelmann category.** We start out with the following definition.

Definition. Let M be an m-dimensional manifold.

(1) A *parametrization for* M is a smooth map $\Psi\colon V \to M$ with the following properties:
   (a) the domain V is either an open ball in $\mathbb{R}^m$, or an open rectangle, or all of $\mathbb{R}^m$,[1012]
   (b) the map Ψ is a diffeomorphism onto its image.
(2) We say a family $\{\Psi_i\colon V_i \to M\}_{i\in I}$ of parametrizations *covers* M if $\bigcup_{i\in I} \Psi_i(V_i) = M$.

The following slightly vague question arises:

Question 58.8. *For a given manifold M, how many parametrizations do we need to cover all of M?*

Examples.

(1) Let $N = (0, \dots, 0, 1)$ be the North Pole and let $S = (0, \dots, 0, -1)$ be the South Pole of the n-dimensional sphere S^n. We can cover S^n by the two open subsets $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$. Stereographic projection, see Figure 23, gives rise to two diffeomorphisms $\Phi_1\colon S^n \setminus \{N\} \to \mathbb{R}^n$ and $\Phi_2\colon S^n \setminus \{S\} \to \mathbb{R}^n$. The inverses of these two maps define parametrizations that cover S^n.
   Since S^n is not diffeomorphic to any of the admissible domains of a parametrization we see that the above collection of parametrizations is optimal in the sense that we cannot parametrize S^n with fewer parametrizations.
(2) We consider the n-torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection map. We pick real numbers $0 < s_1 < s_2 < \cdots < s_{n+1} < 1$. For $i = 1, \dots, n+1$

[1012]The condition on V sounds slightly arbitrary, and it is, but it is reasonable that for practical purposes one prefers domains that are "easy to describe".

we consider the map

$$\begin{array}{rcl} \Psi_i\colon \overbrace{(s_i, s_i+1)\times\ldots\times(s_i, s_i+1)}^{=:V_i} & \to & \mathbb{R}^n/\mathbb{Z}^n \\ (t_1,\ldots,t_n) & \mapsto & p(t_1,\ldots,t_n). \end{array}$$

For $i = 1,\ldots,n+1$ we write $U_i = \Psi_i(V_i)$. It is straightforward to see that the maps $\Psi_1,\ldots,\Psi_{n+1}$ are parametrizations that cover $\mathbb{R}^n/\mathbb{Z}^n$.[1013] We illustrate the images of the parametrizations for the 2-torus in Figure 734.

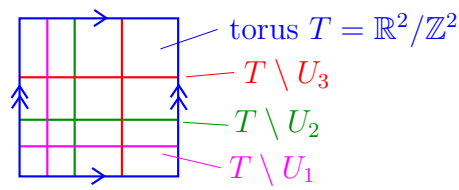


FIGURE 734.

(3) For $i = 1,\ldots,n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon \mathbb{R}^n & \to & \mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1}\setminus\{0\})/\sim \\ (x_1,\ldots,x_n) & \mapsto & \underbrace{[x_1:\ldots:x_{i-1}:1:x_i:\ldots:x_n]}_{\text{point in } \mathbb{R}\mathrm{P}^n \text{ represented by } (x_1,\ldots,x_{i-1},1,x_i,\ldots,x_n)}. \end{array}$$

It is straightforward to see that the maps $\Psi_1,\ldots,\Psi_{n+1}$ are parametrizations that cover the real projective space $\mathbb{R}\mathrm{P}^n$.

(4) Almost the same construction as in (3) shows that we can cover the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\sim$ by $n+1$ parametrizations such that the domain of each parametrization is diffeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

The following question is now a more precise version of Question 58.8.

Question 58.9.

(1) *Can we cover $(S^1)^n$ by fewer than $n+1$ parametrizations?*
(2) *Can we cover $\mathbb{R}\mathrm{P}^n$ by fewer than $n+1$ parametrizations?*
(3) *Can we cover $\mathbb{C}\mathrm{P}^n$ by fewer than $n+1$ parametrizations?*

Now we will generalize this discussion to arbitrary topological spaces by introducing the following definition.

Definition.

(1) We say that a map $f\colon Y \to X$ between topological spaces is *null-homotopic* if it is homotopic to a constant map.

[1013] Hmm, why is it true that $U_1 \cup \cdots \cup U_{n+1} = \mathbb{R}^n/\mathbb{Z}^n$?

(2) The *Lusternik-Schnirelmann category* $\mathrm{cat}(X)$[1014] of a topological space X is defined as the smallest $n \in \mathbb{N}_0$ such that there exists an open covering $U_1, \ldots, U_{n+1}$ of X such that each inclusion map $U_i \to X$ is null-homotopic.[1015] If no such n exists, then we define $\mathrm{cat}(X) = \infty$.

Remark.

(1) Suppose that a given manifold M can be covered by $n+1$ parametrizations. Since the domain of a parametrization is contractible and since a parametrization is a diffeomorphism onto its image we see that $\mathrm{cat}(M) \leq n$.[1016]
(2) The sets U_i in the definition of the Lusternik-Schnirelmann category do not have to be contractible, for example if X itself is contractible, then for any subset $U \subset X$ the inclusion map $U \to X$ is null-homotopic.
(3) The Lusternik-Schnirelmann category plays an important role in topological robotics, see e.g. [Far] for more details.

Examples.

(1) For a non-empty topological space X we have $\mathrm{cat}(X) = 0$ if and only if $\mathrm{id}\colon X \to X$ is null-homotopic, i.e. if X is contractible.
(2) Statement (1) together with Figure 735 shows that for a wedge X of $n \geq 1$ circles we have $\mathrm{cat}(X) = 1$.

FIGURE 735.

(3) The above discussion of parametrizations of S^n together with statement (1) shows that $\mathrm{cat}(S^n) = 1$.
(4) The discussion preceding Question 58.9 gives us the inequalities $\mathrm{cat}((S^1)^n) \leq n$, $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and also $\mathrm{cat}(\mathbb{C}\mathrm{P}^n) \leq n$.
(5) For any $g \geq 2$ the surface Σ of genus g satisfies $\mathrm{cat}(\Sigma) \leq 2$, see Figure 736 for an illustration in the case $g = 2$.
(6) In [Hau, p. 147] or alternatively in [CLOT, Chapter 1.2] it is shown that for any n-dimensional connected CW-complex X the inequality $\mathrm{cat}(X) \leq n$ holds. This general statement generalizes all of the above statements (2) to (5).

[1014]The word "category" with its modern meaning was first used by Eilenberg-Maclane [EM] in 1945. On the other hand the term "Lusternik-Schnirelmann category" was already established by 1941, see [Fo].

[1015]Note the fairly well-hidden "+1" in the subscript of U_{n+1}. Sometimes, especially in the older literature the Lusternik-Schnirelmann category is defined without the adjustment given by the "+1".

[1016]Hmm, why is the image of a parametrization an open subset of the manifold?

(7) It follows from statement (6), together with Lemma 39.2 and Proposition 39.3 that for any compact connected n-dimensional manifold M we have $\operatorname{cat}(M) \leq n$. By the discussion on page 1075 the same statement also holds for closed connected n-dimensional topological manifolds, as long as $n \neq 4$.[1017]

(8) If X and Y are two path-connected CW-complexes with finite Lusternik-Schnirelmann categories, then $\operatorname{cat}(X \times Y) \leq \operatorname{cat}(X) + \operatorname{cat}(Y)$, see e.g. [Fo, Theorem 9] or [CLOT, Theorem 1.37]. The Ganea Conjecture from 1971 predicted that for any CW-complex X one has $\operatorname{cat}(X \times S^n) = \operatorname{cat}(X) + \operatorname{cat}(S^n) = \operatorname{cat}(X) + 1$. But this conjecture was eventually disproved by Norio Iwase [Iw, Iw2] in 1998.

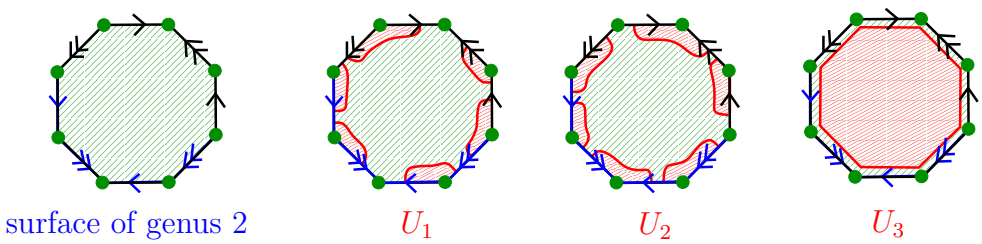


FIGURE 736.

As we saw in the above examples, one can obtain an upper bound on the Lusternik-Schnirelmann category of a given topological space by writing down explicit open subsets. To obtain lower bounds on the Lusternik-Schnirelmann category we introduce the following definition.

Definition. Given a topological space X we define its *cup length* $c(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows:

$$c(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } x_1, \dots, x_n \text{ in degrees} \geq 1 \text{ with } x_1 \cup \dots \cup x_n \neq 0 \end{array} \right\}.$$

Examples.

(1) If X is an n-dimensional CW-complex, then it follows immediately from Proposition 50.4 that $c(X) \leq n$.

(2) In Lemma 57.14 and Lemma 57.16 we saw that for any surface Σ of genus $g \geq 1$ there exist $x, y \in H^1(\Sigma; \mathbb{Z})$ with $x \cup y \neq 0$, which together with (1) shows that $c(\Sigma) = 2$.

(3) In Lemma 57.18 we saw that there exists an $x \in H_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ with $x \cup x \neq 0$, which shows that $c(\mathbb{R}\mathrm{P}^2) = 2$.

The following proposition shows that the cup length is a lower bound on the Lusternik-Schnirelmann category.

[1017] I do not know whether the case of 4-dimensional topological manifolds is known.

Proposition 58.10. *For every topological space X we have*

$$c(X) \;\leq\; \mathrm{cat}(X).$$

Example. It follows immediately from Proposition 58.10 and the above examples that

$$\mathrm{cat}(S^1 \times S^1) \;=\; \mathrm{cat}(\text{surface of genus } g \geq 2) \;=\; \mathrm{cat}(\mathbb{R}\mathrm{P}^2) \;=\; 2.$$

Proof. Let X be a topological space, let R be a commutative ring and let $x_j \in H^{s_j}(X;R)$, $j = 1, \ldots, n$ be cohomology classes in dimensions $s_j \geq 1$ such that $x_1 \cup x_2 \cup \cdots \cup x_n \neq 0$. We have to show that $\mathrm{cat}(X) \geq n$.

Thus let $k \leq n$ and let $U_1, \ldots, U_k$ be open subsets in X with $U_1 \cup \cdots \cup U_k = X$ such that each inclusion map $U_i \to X$ is null-homotopic. We have to show that $U_1 \cup \cdots \cup U_k$ is a *proper* subset of X.

Claim. For any $j \in \{1, \ldots, k\}$ the obvious map $\rho_j \colon H^{s_j}(X, U_j; R) \to H^{s_j}(X;R)$ is in fact an isomorphism.

By our hypothesis the inclusion map $i_j \colon U_j \to X$ is null-homotopic, it follows that the restriction map $i_j^* \colon H^k(X;R) \to H^k(U_j;R)$ is the zero map for $k \geq 1$ and an epimorphism for $k = 0$.[1018] We deduce from this observation, the long exact sequence of cohomology groups corresponding to the pair (X, U_j) and the fact that $s_j \geq 1$ that the obvious map $\rho_j \colon H^{s_j}(X, U_j; R) \to H^{s_j}(X;R)$ is in fact an isomorphism. This concludes the proof of the claim.

Given $j \in \{1 \ldots, k\}$ we write $y_j := \rho_j^{-1}(x_j) \in H^{s_j}(X, U_j; R)$ and we set $s = s_1 + \cdots + s_k$. By Corollary 58.4[1019] we have the following commutative diagram

$$\begin{tikzcd}
H^{s_1}(X, U_1; R) \times \cdots \times H^{s_k}(X, U_k; R) \arrow[r, "\cup"] \arrow[d, "{(y_1, \ldots, y_k) \mapsto (x_1, \ldots, x_k)}"'] & H^s(X, U_1 \cup \cdots \cup U_k; R) \arrow[d] \\
H^{s_1}(X;R) \times \cdots \times H^{s_k}(X;R) \arrow[r, "\cup"] & H^s(X;R).
\end{tikzcd}$$

By our hypotheses the cup product $x_1 \cup \cdots \cup x_k$ is non-zero, which then implies, by the commutativity of the diagram, that the cup product $y_1 \cup \cdots \cup y_k$ is also non-zero. But this implies that the group $H^s(X, U_1 \cup \cdots \cup U_k; R)$ is non-zero, which in turn implies that $U_1 \cup \cdots \cup U_k$ is a proper subset of X. $\square$

Proposition 58.10 gives us a tool to attack Question 58.9. In fact we saw right after Proposition 58.10 that we can now answer the question for the 2-dimensional torus $(S^1)^2$ and the real projective plane $\mathbb{R}\mathrm{P}^2$.

The following questions naturally arise:

[1018] This can be seen as follows: the map $i_j \colon U_j \to X$ is homotopic to a constant map that takes the value $P_j \in X$. We denote by $c_j \colon U_j \to \{P_j\}$ the constant map and we denote by $k_j \colon \{P_j\} \to X$ the inclusion map. We statement now follows from $i_j^* = (k_j \circ c_j)^* = c_j^* \circ k_j^*$ and Lemmas 48.13 and 48.12.

[1019] Strictly speaking Corollary 58.4 was formulated for the cup product of two terms, but one can easily reduce the general case of k terms to the special case of 2 terms.

Question 58.11.

(1) *What is the* $\mathbb{Z}$*-cohomology ring of* $(S^1)^n$*?*
(2) *What is the* $\mathbb{F}_2$*-cohomology ring of* $\mathbb{R}\mathrm{P}^n$*?*
(3) *What is the* $\mathbb{Z}$*-cohomology ring of* $\mathbb{C}\mathrm{P}^n$*?*

We will answer these questions in the coming chapters.

Exercises for Chapter 58.

Exercise 58.1. Let X and Y be non-empty topological spaces. Show that

$$c(X \times Y) \ \geq \ \max\{c(X), c(Y)\}.$$

Here c denotes the cup length.

59. The cap product

In this chapter we introduce the cap product that relates the homology and the cohomology groups. It can be viewed as a generalization of the Kronecker pairing. Initially it might not be clear that the cap product is good for, but later on we will see that it plays an essential role in many calculations and in the formulation of the Poincaré Duality Theorem.

59.1. **The definition of the cap product and calculations.** We start out by introducing a convention that simplifies our notation later on.

Convention. Let R be a commutative ring and let S be a non-empty set. We consider

$$R^{(S)} \ := \ \text{all maps from } S \text{ to } R \text{ which are non-zero for only finitely many } s \in S$$

with the obvious R-module structure. We refer to $R^{(S)}$ as the *free R-module generated by S*. For example, the free $\mathbb{Z}$-module generated by S is precisely the same as the free abelian group generated by S. We use the same convention as on page 366 to identify elements in $R^{(S)}$ with finite formal linear combinations $r_1 s_1 + \cdots + r_k s_k$ where $r_1, \ldots, r_k \in R$ and $s_1, \ldots, s_k \in S$. By Lemma 42.3 the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} \otimes R & \to & R^{(S)} \\ \sum\limits_{i=1}^{n} n_i \cdot s_i \otimes r_i & \mapsto & \sum\limits_{i=1}^{n} \underbrace{n_i r_i}_{\in R} \cdot s_i \end{array}$$

is an isomorphism. We use this isomorphism to identify the left-hand side with the right-hand side. In particular, given a topological space X and $k \in N_0$ we write

$$C_k(X;R) \ := \ \mathbb{Z}^{S_k(X)} \otimes R \ = \ R^{S_k(X)} \ = \ \left\{ \sum_{i=1}^{n} r_i \cdot \sigma_i \,\middle|\, r_1, \ldots, r_n \in R \text{ and } \sigma_i \colon \Delta^k \to X \right\},$$

where $S_k(X)$ denotes the set of singular k simplices in X.

The following definition is inspired by the second definition of the cup product that we gave on page 1334.

Definition. Let X be a topological space and let R be a commutative ring. Given a singular cochain $\varphi \in C^k(X;R)$ and a singular simplex $\sigma \colon \Delta^l \to X$ with $k \leq l$ we define the *cap product* as[1020]

$$\varphi \cap \sigma \ := \ \underbrace{\varphi(\sigma \circ [v_0, \ldots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \ldots, v_l]}_{\text{map } \Delta^{l-k} \to X}.$$

We extend this definition to singular R-chains and we obtain the cap product

$$\cap \colon C^k(X;R) \times C_l(X;R) \ \to \ C_{l-k}(X;R)$$

For $k > l$ we define the cap product to be the zero map.

[1020]Here we use, as usual, the notation from page 1323.

As for the cup product, the first thing we want to verify is that this map on the cochain and chain level defines a map on the level of cohomology groups and homology groups. The key to doing so is the following elementary lemma.

Lemma 59.1. *Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in C^k(X;R)$ and let $\sigma \in C_l(X;R)$. Then*

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta\varphi \cap \sigma \,+\, \varphi \cap \partial\sigma) \;\in\; C_{l-k-1}(X;R).$$

In particular this implies that $\varphi \cap \partial\sigma$ and $\delta\varphi \cap \sigma$ are homologous.

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. Clearly it suffices to prove the equality for the case that $\sigma\colon \Delta^l \to X$ is a singular l-simplex. This case follows from an elementary calculation. Indeed, we have

$$\begin{aligned}
&\text{(a)} & \varphi \cap \partial\sigma &= \sum_{i=0}^{k} (-1)^i \cdot \varphi(\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \dots, v_l] \\
& & &\quad + \sum_{i=k+1}^{l} (-1)^i \cdot \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, \widehat{v}_i, \dots, v_l] \\
&\text{(b)} & \delta\varphi \cap \sigma &= \sum_{i=0}^{k+1} (-1)^i \cdot \varphi(\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \dots, v_l] \\
&\text{(c)} & (-1)^k \cdot \partial(\varphi \cap \sigma) &= \sum_{i=k}^{l} (-1)^i \cdot \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, \widehat{v}_i, \dots, v_l].
\end{aligned}$$

Note that the summand of (b) corresponding to $i = k+1$ equals *minus* the summand of (c) corresponding to $i = k$. Adding (b) and (c) and comparing it to (a) we see that

$$\varphi \cap \partial\sigma \;=\; \delta\varphi \cap \sigma + (-1)^k \cdot \partial(\varphi \cap \sigma)$$

which gives the desired equality. □

The following lemma says that "$\cap$" descends to a map on homology and cohomology.

Lemma 59.2. *Let X be a topological space and let R be a commutative ring. Then for any $k, l \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} H^k(X;R) \times H_l(X;R) & \to & H_{l-k}(X;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi \cap \sigma] \end{array}$$

is well-defined and R-bilinear.

We refer to the pairing from Lemma 59.2 as the *cap product.*[1021]

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. The statement now follows easily from Lemma 59.1. Indeed, let $\varphi \in C^k(X;R)$ be a cocycle and let $\sigma \in C_l(X;R)$ be a cycle.

[1021]The name "cap product" comes from the notation "$a \cap b$" for the composition since $\cap$ looks like, just a little bit, like the "a" in cap. The other reason is that the cup product was first introduced, and mathematicians needed a different name for a somewhat related object. Also mathematicians are well-known for their great sense of humor.

(1) It follows from Lemma 59.1 that $\varphi \cap \sigma$ is a cycle, i.e. it represents a homology class.
(2) If $\sigma' = \sigma + \partial\tau$ is another representative of $[\sigma]$, then

$$\varphi \cap \sigma - \varphi \cap \sigma' \;=\; \varphi \cap (\sigma - \sigma') \;=\; \varphi \cap \partial\tau \;\underset{\substack{\uparrow \\ \text{by Lemma 59.1 and since } \delta\varphi = 0}}{=}\; \pm\partial(\varphi \cap \tau).$$

This shows that $\varphi \cap \sigma$ and $\varphi \cap \sigma'$ represent the same homology class, i.e. the cap product does not depend on the choice of the representative of the homology class.
(3) The proof that the cap product does not depend on the choice of representative of the cohomology class is basically the same as the proof of (2).

This shows that the map is well-defined. It is clear that the map is indeed R-bilinear. □

Remark. Rather depressingly there is an endless variety of definitions of the cap product in the literature. More precisely, given a cochain $\varphi \in C^k(X;R)$ and a singular simplex $\sigma\colon \Delta^l \to X$ the following definitions of $\varphi \cap \sigma$ can be found in the literature:

		$\varphi \cap \sigma$
(a)	Hatcher [Hat, p. 239]	$\varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l]$
(b)	Bredon [Bre, p. 335]	$(-1)^{k(l-k)} \cdot \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l]$
(c)	Dold [Dol, p. 335]	$(-1)^{k(l-k)} \cdot \sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$
(d)	Spanier [Sp, p. 254]	$\sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$
(e)	Munkres [Mun, p. 389]	$\sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$

To the best of my understanding, see [Sp, p. 250] the singular chains

$$\varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l] \quad \text{and } (-1)^{k(l-k)} \cdot \sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$$

are homologous. This shows that on the level of homology and cohomology groups the definition (a), which is also our definition, agrees with (c), but that the sign differs for (b), (d) and (e). For the most part this is not an issue, but one needs to be careful when one cites formulas from the literature. For example, as we had already mentioned on page 1162, Bredon [Bre] defined the coboundary map as

$$\delta_k^{\mathrm{Bre}} \;:=\; (-1)^{k+1} \cdot \partial_{k+1}^* \;=\; (-1)^{k+1} \cdot \delta_k$$

and as we saw above, the cap product is also defined differently. Thus our Lemma 59.1, which says that for $\varphi \in C^k(X;R)$ and $\sigma\colon \Delta^l \to X$ we have

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (\varphi \cap \partial\sigma \;-\; \delta\varphi \cap \sigma)$$

becomes the following equality in [Bre, Proposition VI.5.1]:[1022]

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot \varphi \cap \partial\sigma \;+\; \delta^{\mathrm{Bre}}(\varphi) \cap \sigma.$$

The following lemma follows immediately from the definitions.

[1022]The lesson you should draw is that if you ever have to write algebraic topology lectures notes, stick to a single book and do not try to mix books.

Lemma 59.3. *Let X be a topological space and let R be a commutative ring. Then for any $\sigma \in H_k(X;R)$ we have*

$$1_X \cap \sigma \;=\; \sigma.$$

For future applications we state the special case $k = l$ in the cap product in the following lemma.

Lemma 59.4. *Let X be a topological space and let R be a commutative ring.*

(1) *The composition*

$$H^k(X;R) \times H_k(X;R) \;\xrightarrow{\cap}\; H_0(X;R) \;\xrightarrow[\text{from page 1009}]{\text{augmentation } \epsilon}\; R$$

is precisely the Kronecker pairing from pages 1199 and 1227.

(2) *If X is path-connected, and if we make the usual identification $H_0(X;R) = R$ given by the augmentation map, see Proposition 29.5, then for any $\varphi \in H^k(X;R)$ and $\sigma \in H_k(X;R)$ we have*

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle \;\in\; R.$$

Proof. Both statements follow immediately from recalling all the relevant definitions. □

Example. We consider some examples of the cap product in detail to get some feeling for what it does.

(1) Let $T = S^1 \times S^1 = [0,1] \times [0,1]/\sim$ be the torus. We recall the following notations and results that we have by now used on many occasions. (We refer to Figure 737 for some of the notation.)
   (a) We denote by σ_1, σ_2 the singular 2-simplices from Figure 737. As we pointed out on page 1118, the singular 2-chain $-\sigma_1 + \sigma_2$ is a representative of the fundamental class $[T]$.
   (b) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
   (c) We denote by $\theta := \theta_{\mathbb{Z}} \in C^1(S^1;\mathbb{Z})$ the singular 1-cocycle that we introduced on page 1166.
   (d) As before we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = e^{2\pi i t}$. By the discussion on page 1118 we know that μ represents the fundamental class $[S^1]$ of S^1 equipped with the standard orientation. By a slight abuse of notation we also denote by $[1 \times S^1]$ the image of the fundamental class under the monomorphism $H_1(1 \times S^1;\mathbb{Z}) \to H_1(S^1 \times S^1;\mathbb{Z})$.

   We calculate that

$$p^*(\theta) \cap \sigma_1 \;=\; \underbrace{p^*(\theta)(\sigma_1 \circ [v_0,v_1])}_{=\theta(p\circ\sigma_1\circ[v_0,v_1])=0} \cdot \sigma_1 \circ [v_1,v_2] \;=\; 0$$

   and similarly we calculate that

$$p^*(\theta) \cap \sigma_2 \;=\; \underbrace{p^*(\theta)(\sigma_2 \circ [v_0,v_1])}_{=\theta(p\circ\sigma_2\circ[v_0,v_1])=1} \cdot \sigma_2 \circ [v_1,v_2] \;=\sigma_2\circ[v_1,v_2] = 1\times\mu.$$

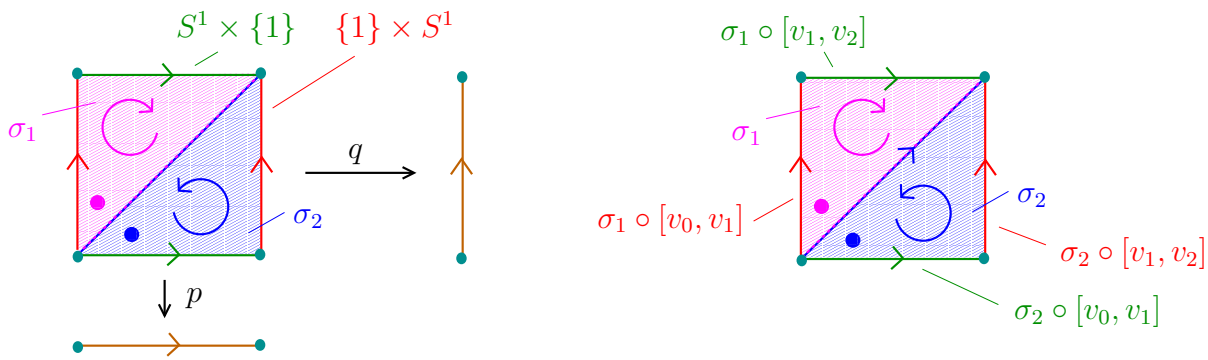


FIGURE 737.

Putting these calculations together we obtain that

$$[p^*(\theta)] \cap [T] \;=\; [p^*(\theta) \cap (-\sigma_1 + \sigma_2)] \;=\; [1 \times S^1].$$

Basically the same calculation shows that

$$[q^*(\theta)] \cap [T] \;=\; [q^*(\theta) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times 1].$$

So we see that the map

$$\cap[T] \colon H^1(T;\mathbb{Z}) \;\to\; H_1(T;\mathbb{Z})$$

is non-trivial, in fact we have just shown that this map satisfies

$$\begin{array}{rcr} [p^*(\theta)] & \mapsto & [1 \times S^1] \\ [q^*(\theta)] & \mapsto & -[S^1 \times 1] \end{array}$$

which shows that "capping with the fundamental class" is in this case an isomorphism.

(2) In Exercise 59.2 we will deal with the surface Σ of genus 2 and there we will also see that the map

$$\cap[\Sigma] \colon H^1(\Sigma;\mathbb{Z}) \;\to\; H_1(\Sigma;\mathbb{Z})$$

is an isomorphism.

(3) Finally we consider again the real projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$. We introduce the following notation:
   (a) We denote by $\varphi \in C^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ the cocycle that we introduced on page 1352.
   (b) We denote by $\gamma \colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$ the singular 1-simplex that is given by $\gamma(t, 1-t) = [e^{\pi i t}]$. Note that γ is a cycle in $C_1(\mathbb{R}\mathrm{P}^2)$ and note that $\varphi(\gamma) = 1$.
   (c) In Figure 659 we already saw a representative for the $\mathbb{F}_2$-fundamental class, but we will work with a different one that is "smaller" and that is shown in Figure 738. In Figure 738 we see the singular 2-simplex σ that sends the face $[v_0, v_2]$ to a point P and the singular 2-simplex τ that sends all points on Δ^2 to P. It is straightforward to see that $(\sigma + \tau) \otimes 1 \in C_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a cycle. Using Proposition 46.8 we see that it represents the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$.

It is straightforward to see that

$$[\varphi]\cap[\mathbb{R}\mathrm{P}^2] \;=\; [\varphi]\cap[(\sigma+\tau)\otimes 1] \;=\; [\underbrace{\varphi\cap\sigma}_{=\gamma}\otimes 1+\underbrace{\varphi\cap\tau}_{=0}\otimes 1] \;=\; [\gamma\otimes 1]$$

We saw on page 1353 that $[\gamma\otimes 1]$ is the unique non-trivial element of $H_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. Thus in this case we see that the map

$$\cap[\mathbb{R}\mathrm{P}^2]\colon H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \;\to\; H_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$$

is also an isomorphism.

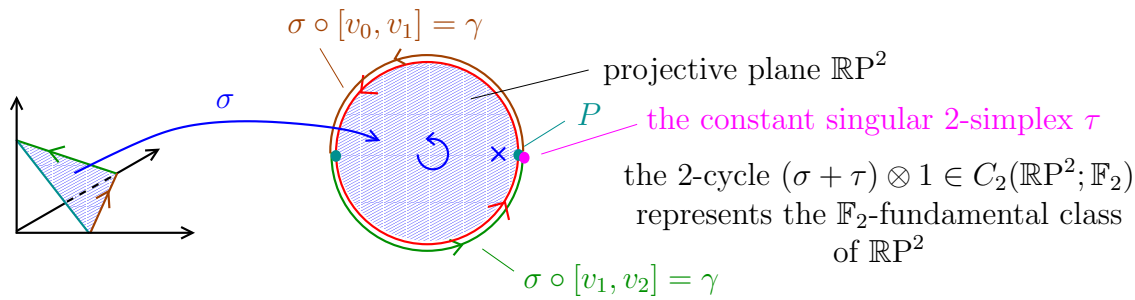


FIGURE 738.

Remark. In the last three examples we saw for the closed 2-dimensional manifolds $F = S^1\times S^1$, $F=\Sigma$, $F=\mathbb{R}\mathrm{P}^2$ and for $R=\mathbb{Z}$ respectively $R=\mathbb{F}_2$ that the map

$$-\cap[F]_R\colon H^1(F;R) \;\to\; H_1(F;R)$$

is an isomorphism. In the next chapter we will see that it is not a coincidence but that this observation can be generalized to all closed manifolds.

59.2. **The relative cap product and more properties of the cap product.** After considering the various examples of the cap product we continue with developing the theory.

Similar to the case of the cup product, we intend to extend the notion of cap product to relative homology and cohomology groups. As in the case of the cup product it requires some delicate footwork to get to the right notion of a relative cap product. We start out with the following rather technical definition.

Definition. Let (X,A,B) be a triad of topological spaces and let R be a commutative ring. We write

$$C_n(X,\{A,B\};R) \;:=\; \frac{C_n(X)\otimes R}{C_n^{\{A,B\}}(A\cup B)\otimes R}.$$

The boundary map on $C_*(X;R)$ descends to a boundary map on $C_*(X,\{A,B\};R)$ which we can thus view as a chain complex. We denote the corresponding homology groups by $H_*(X,\{A,B\};R)$.

This technical definition allows us to define a first version of a relative cap product.

Lemma 59.5. *Let (X, A, B) be a triad of topological spaces, let $k \leq l$ and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} \cap\colon C^k(X, A; R) \times C_l(X, \{A, B\}; R) & \to & C_{l-k}(X, B; R) \\ (\varphi, [\sigma\colon \Delta^l \to X]) & \mapsto & \big[\underbrace{\varphi(\sigma \circ [v_0, \ldots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \ldots, v_l]}_{\text{map } \Delta^{l-k} \to X}\big] \end{array}$$

is well-defined and it descends to a map

$$\cap\colon H^k(X, A; R) \times H_l(X, \{A, B\}; R) \;\to\; H_{l-k}(X, B; R)$$

that is R-bilinear.

Proof. We first show that the cap product on (co-) chains is well-defined. By definition of

$$C_n(X, \{A, B\}; R) \;:=\; \frac{C_n(X) \otimes R}{C_n^{\{A,B\}}(A \cup B) \otimes R} \;=\; \frac{C_n(X) \otimes R}{C_n(A) \otimes R + C_n(B) \otimes R}$$

we have to show that if we are given a singular l-simplex $\sigma\colon \Delta^l \to X$ such that

(1) the image of σ lies in A, or
(2) the image of σ lies in B,

then the product $\varphi(\sigma \circ [v_0, \ldots, v_k]) \cdot \sigma \circ [v_0, \ldots, v_k]$ is zero in $C_{l-k}(X, B; R)$. But this is indeed the case:

(1) first, if the image of σ lies in A, then it follows immediately from the hypothesis that φ lies in $C^k(X, A; R)$ that $\varphi(\sigma \circ [v_0, \ldots, v_k]) = 0$, and
(2) if the image of σ lies in B, then the image of $\sigma \circ [v_0, \ldots, v_k]$ also lies in B. Hence the term is zero in $C_{l-k}(X, B; R)$.

This concludes the proof of the first part of the lemma.

The argument that the map on (co-) chain complexes descends to a map on the level of homology and cohomology groups is the same as in the proof of Lemma 59.2. It is obvious that the map $\cap$ is R-bilinear. □

Evidently we are not very interested in the group $H_n(X, \{A, B\}; R)$. The following lemma allows us to replace this group by the much more agreeable group $H_n(X, A \cup B; R)$, at least for reasonable triads.

Lemma 59.6. *If (X, A, B) is an excisive triad of topological spaces, then given any commutative ring R the obvious map*

$$C_*(X, \{A, B\}; R) \;\to\; C_*(X, A \cup B; R)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X, \{A, B\}; R) \;\xrightarrow{\cong}\; H_n(X, A \cup B; R).$$

Proof. In the proof of Lemma 58.1 we already showed that the map

$$C_*(X, \{A, B\}; \mathbb{Z}) \;\to\; C_*(X, A \cup B; \mathbb{Z})$$

on chain complexes is a chain homotopy equivalence. By tensoring all maps and chain homotopies with R we see that the map

$$C_*(X, \{A, B\}; R) \;\to\; C_*(X, A \cup B; R)$$

is also a chain homotopy equivalence. The second statement is then an immediate consequence of Corollary 30.3. □

Lemma 59.6 leads to the following definition.

Definition. Let (X, A, B) be an excisive triad of topological spaces, let $k \le l$ and let R again be a commutative ring. We refer to the map

$$\begin{array}{ccccc} H^k(X,A;R) \times H_l(X, A\cup B; R) & \overset{\cong}{\leftarrow} & H^k(X,A;R) \times H_l(X,\{A,B\};R) & \xrightarrow{\cap} & H_{l-k}(X,B;R) \\ & & \uparrow & & \uparrow \\ & & \text{isomorphism given by Lemma 59.6} & & \text{map from Lemma 59.5} \end{array}$$

as the *cap product* (sometimes we refer to it as the *relative cap product*) that we also denote by "$\cap$". For $k > l$ we define the cap product to be the zero map.

Remark. In many examples, e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$, then the slightly mysterious first map in the definition of the cap product is just the identity. In particular, for any subset A of X we obtain cap products

$$H^k(X;R) \;\times\; H_l(X,A;R) \;\xrightarrow{\cap}\; H_{l-k}(X,A;R)$$

and

$$H^k(X,A;R) \;\times\; H_l(X,A;R) \;\xrightarrow{\cap}\; H_{l-k}(X;R)$$

and both of them are induced, on the (co-) chain level, by the formula

$$(\varphi, \sigma\colon \Delta^l \to X) \quad\mapsto\quad \varphi(\sigma\circ[v_0,\dots,v_k]) \cdot \sigma\circ[v_k,\dots,v_l].$$

This observation also shows that we can easily generalize Lemma 59.4 to the relative case in the obvious way. For example, if X is path-connected, and if we make the usual identification $H_0(X;\mathbb{Z}) = \mathbb{Z}$ given by the augmentation map, see Proposition 29.5, then for any $\varphi \in H^k(X,A;\mathbb{Z})$ and $\sigma \in H_k(X,A;\mathbb{Z})$ we have

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle \;\in\; \mathbb{Z}.$$

For the record we point out that it follows from this observation and Lemma 51.15 that if M is a compact oriented path-connected n-dimensional topological manifold, then for any point $P \in M$ we have

$$[M]^* \cap [M] \;=\; [P]$$

and $[M]^*$ is the only element in $H^n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ that has that property.

Example. Let $X = S^1 \times [0,1]$ be the annulus. In Figure 739 we show two singular 2-simplices σ_1, σ_2. It follows easily from Proposition 46.8 that $-\sigma_1 + \sigma_2$ represents the fundamental

class $[X] \in H_2(X, \partial X; \mathbb{Z})$ of the annulus X with the usual orientation. We denote by $p\colon X \to [0,1]$ the projection map. As on page 1171 we consider the function

$$\begin{array}{rcl} f\colon [0,1] & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain

$$\begin{array}{rcl} \rho\colon C_1([0,1]) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to [0,1]) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

It follows from the discussion on page 1171 that $p^*(\rho) \in C^1(X, \partial X; \mathbb{Z})$ is a cochain, thus it represents an element in $H^1(X, \partial X; \mathbb{Z})$. We calculate that

$$\begin{array}{rclcl} [p^*(\rho)] \cap \sigma_1 & = & \underbrace{p^*(\sigma_1 \circ [v_0, v_1])}_{=1} \cdot \underbrace{\sigma_1 \circ [v_1, v_2]}_{=[S^1 \times 1]} & = & [S^1 \times 1] \\ [p^*(\rho)] \cap \sigma_2 & = & \underbrace{p^*(\sigma_2 \circ [v_0, v_1])}_{=0} \cdot \sigma_2 \circ [v_1, v_2] & = & 0. \end{array}$$

Adding up we see that

$$[p^*(\rho)] \cap [X] \;=\; [p^*(\rho) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times 1].$$

Since the inclusion $S^1 \times 1 \to X$ is a homotopy equivalence we see that $H_1(X; \mathbb{Z}) \cong \mathbb{Z}$ and that $[S^1 \times 1]$ is a generator. A straightforward argument, e.g. using the long exact sequence of the pair $(X, \partial X)$, shows that $H^1(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$. The above calculation thus shows that the map

$$\begin{array}{rcl} H^1(X, \partial X; \mathbb{Z}) & \to & H_1(X; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is an epimorphism. Since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism and as a bonus we obtain that $[\rho] \in H^1(X, \partial X; \mathbb{Z})$ is a generator. In Exercise 59.3 we will show that the map

$$\begin{array}{rcl} H^1(X; \mathbb{Z}) & \to & H_1(X, \partial X; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is also an isomorphism.

The following lemma relates the cup product to the cap product.

Lemma 59.7. *Let X be a topological space and let R be a commutative ring.*

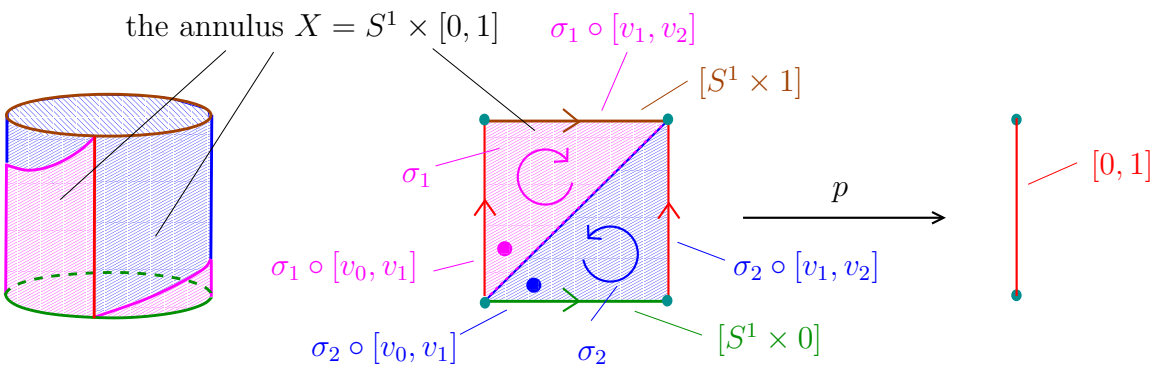


FIGURE 739.

(1) *Let* $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$ *and let* $\sigma \in C_n(X;R)$. *Then*[1023]
$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; C_{n-k-l}(X;R).$$
(2) *If* $A \subset X$ *is a subset and* $C, D \in \{\varnothing, A\}$ *we have for any* $\varphi \in H^k(X,C;R)$, $\psi \in H^l(X,D;R)$ *and* $\sigma \in H_n(X, C \cup D;R)$ *that*
$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; H_{n-k-l}(X;R).$$

Proof. The case $k + l > n$ is trivial. So suppose that $k + l \leq n$.

(1) So let $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$. It suffices to consider the case that $\sigma\colon \Delta^n \to X$ is a singular n-simplex. We have
$$\begin{aligned}
\varphi \cap (\psi \cap \sigma) &= \varphi \cap \big(\psi(\sigma \circ [v_0, \dots, v_l]) \cdot \sigma \circ [v_l, \dots, v_n]\big) \\
&= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi \cap (\sigma \circ [v_l, \dots, v_n]) \\
&= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi(\sigma \circ [v_l, \dots, v_{l+k}]) \cdot \sigma \circ [v_{l+k}, \dots, v_n] \\
&= (\psi \cup \varphi)(\sigma \circ [v_0, \dots, v_{l+k}]) \cdot \sigma \circ [v_{l+k}, \dots, v_n] \\
&= (\psi \cup \varphi) \cap \sigma.
\end{aligned}$$
(2) The statement regarding the (co-) homology classes follows almost immediately from the above result on (co-) chains obtained in (1). □

Now suppose we are given a map of excisive triads of topological spaces. We get induced maps on homology and cohomology and the following lemma tells us the relationship of these induced maps and the cap product.

Lemma 59.8. *Let R be a commutative ring.*

[1023]Note that on the right-hand side it says $(\psi \cup \varphi) \cap \sigma$ instead of $(\varphi \cup \psi) \cap \sigma$. In fact if one picks a different sign convention for the cap product this sign goes away. For example with the sign conventions used by Bredon, see [Bre, Chapter 5], the equation becomes
$$\varphi \cap (\psi \cap \sigma) \;=\; (\varphi \cup \psi) \cap \sigma$$
which looks much more appealing. Once we consider (co-) homology classes we obtain from Proposition 57.8 that the difference between these two expressions $(\psi \cup \varphi) \cap \sigma$ and $(\varphi \cup \psi) \cap \sigma$ is just a factor of $(-1)^{kl}$.

(1) *If* $f\colon (X,A,B) \to (\widetilde{X},\widetilde{A},\widetilde{B})$ *is a map between two excisive triads, then for any* $k,l \in \mathbb{N}_0$ *the following diagram commutes*[1024]

$$\begin{array}{ccccc} H^k(X,A;R) & \times & H_l(X,A\cup B;R) & \xrightarrow{\cap} & H_{l-k}(X,B;R) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H^k(\widetilde{X},\widetilde{A};R) & \times & H_l(\widetilde{X},\widetilde{A}\cup\widetilde{B};R) & \xrightarrow{\cap} & H_{l-k}(\widetilde{X},\widetilde{B};R), \end{array}$$

i.e. for any $\varphi \in H^k(\widetilde{X},\widetilde{A};R)$ *and any* $\sigma \in H_l(X,A\cup B;R)$ *we have*

$$\varphi \cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi)\cap\sigma}_{\in H_{l-k}(X,B;R)}) \;\in\; H_{l-k}(\widetilde{X},\widetilde{B};R).$$

(2) *Let* X *and* $\widetilde{X}$ *be path-connected topological spaces and let* $B \subset X$ *and* $\widetilde{B} \subset \widetilde{X}$ *be subsets. If* $f\colon (X,B) \to (\widetilde{X},\widetilde{B})$ *is a map, then for any* $\varphi \in H^k(\widetilde{X},\widetilde{B};R)$ *and any* $\sigma \in H_k(X,B;R)$ *we have*

$$\varphi \cap f_*(\sigma) \;=\; f^*(\varphi)\cap\sigma \;\in\; \underbrace{H_0(X;R) = R = H_0(\widetilde{X};R)}_{\substack{\text{identification from}\\ \text{Proposition 29.5}}}.$$

Remark. By Lemma 59.4 (1) we can view the above as a generalization of Lemma 50.6.

Proof. The first statement follows immediately from the definitions. We leave the details to the skeptical reader. The second statement is an immediate consequence of the first statement. □

The next lemma says that cup and cap products behave in the obvious way with respect to the path-components. The statements in the lemma follow easily from the definitions. Therefore we leave the proof of the lemma to the reader.

Lemma 59.9. $(*)$ *Let* X *be a topological space and let* R *be a commutative ring. We denote by* $\{X_j\}_{j\in J}$ *the path-components of* X*. By Lemmas 29.13 and 48.14 the inclusion maps* $\iota_j\colon X_j \to X$*,* $j \in J$*, induce for any* $n \in \mathbb{N}_0$ *isomorphisms*

$$\Phi := \bigoplus_{j\in J}(\iota_j)_*\colon \bigoplus_{j\in J} H_n(X_j;G) \;\xrightarrow{\cong}\; H_n(X;G)$$

and

$$\Psi := \prod_{j\in J}\iota_j^*\colon H^n(X;G) \;\xrightarrow{\cong}\; \prod_{j\in J} H^n(X_j;G).$$

Then the following two statements hold:

(1) *For any* $i \neq j$ *and any* $\alpha \in H^m(X_i;R)$ *and* $\beta \in H^n(X_j;R)$ *we have*

$$\Psi^{-1}(\alpha) \,\cup\, \Psi^{-1}(\beta) \;=\; 0.$$

[1024]Here "the diagram commutes" is understood in the sense of the convention from page 1360.

(2) *For any $i \neq j$ and any $\alpha \in H^m(X_i; R)$ and $\beta \in H_n(X_j; R)$ we have*

$$\Psi^{-1}(\alpha) \cap \Phi(\beta) = 0.$$

The same type of statements hold for cup and cap products on relative (co-) homology groups.

In the next lemma of the chapter we relate the relative cap product to (co-) boundary maps on (co-) homology. More precisely, we have the following lemma which is similar in spirit to Lemma 58.5.

Lemma 59.10. *Let R be a commutative ring.*

(1) *Let (X, B, A) be a triple of topological spaces. For any $n, k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
H^k(B, A; R) \times H_{n-1}(B, A; R) \arrow[r, "\cap"] \arrow[d, "\delta"'] & H_{n-k-1}(B; R) \arrow[d, "i_*"] \\
H^{k+1}(X, B; R) \times H_n(X, B; R) \arrow[r, "\cap"] & H_{n-k-1}(X; R). \arrow[u, "\partial"']
\end{tikzcd}$$

Here $i\colon B \to X$ denotes the inclusion map and δ and ∂ denote the connecting homomorphisms in the long exact sequences in (co-) homology corresponding to the triple (X, B, A).

(2) *Let (X, A, B) be an excisive triad and let $n \in \mathbb{N}_0$. Note that by Lemma 50.13 the map $j_*\colon H_{n-1}(B, A \cap B; R) \to H_{n-1}(A \cup B, A; R)$ is an isomorphism. For any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^{k+1}$:*

$$\begin{tikzcd}
H^k(X, A; R) \times H_n(X, A \cup B; R) \arrow[r, "\cap"] \arrow[d, "i^*"'] & H_{n-k}(X, B; R) \arrow[dd, "\partial"] \\
H^k(A \cup B, A; R) \times H_{n-1}(A \cup B, A; R) \arrow[d, "j^*"', "\cong"] & \\
H^k(B, A \cap B; R) \times H_{n-1}(B, A \cap B; R) \arrow[r, "\cap"] & H_{n-k-1}(B; R).
\end{tikzcd}$$

Here $i\colon (A \cup B, A) \to (X, A)$ and $j\colon (B, A \cap B) \to (A \cup B, A)$ denote the inclusion maps and ∂ denotes the connecting homomorphisms in the long exact sequences in homology corresponding to the triple $(X, A \cup B, A)$ respectively the pair (X, B).

Proof $(*)$.

(1) We pick $\psi \in H^k(B, A; R)$ and $\tau \in H_n(B, A; R)$. We make the following observations:
   (a) We can find $\sigma \in C_n(B; R)$ such that the image of σ in $C_n(B, A; R)$ is a cycle that represents τ.
   (b) We can pick a singular cochain $\varphi \in C^k(B; R) = \mathrm{Hom}(C_k(B), R)$ that vanishes on $C_k(A)$ and such that the corresponding element in $C^k(B, A; R)$ represents

ψ. We denote by $\widetilde{\varphi}\colon C_k(X) \to R$ the extension of φ that is defined to be zero on all singular k-simplices that do not lie in B.

We have the following equalities:

$$\delta\psi \cap \tau \underset{\substack{\uparrow \\ \text{see Lemma 48.15 and the} \\ \text{discussion on page 1174}}}{=} \big[\delta\widetilde{\varphi}\big] \cap [\sigma] \overset{\substack{\text{by the discussion of relative cap products on page 1378} \\ \downarrow}}{=} \big[\delta\widetilde{\varphi} \cap \sigma\big] \underset{\substack{\uparrow \\ \text{by Lemma 59.1 the classes } \delta\varphi \cap \sigma \\ \text{and } \varphi \cap \partial\sigma \text{ are homologous}}}{=} [\widetilde{\varphi} \cap \partial\sigma] = [\varphi \cap \partial\sigma] \overset{\downarrow}{=} [\varphi] \cap [\partial\sigma] \underset{\substack{\uparrow \\ \text{Proposition 31.13}}}{=} \psi \cap \partial\tau.$$

This equality implies that the diagram commutes.[1025]

(2) We pick $\psi \in H^k(X, A; R)$ and $\tau \in H_n(X, A \cup B; R)$. We make the following observations:

(a) It follows from our hypothesis that the triad (X, A, B) is excisive together with Lemma 59.6 that we can find $\sigma \in C_n(X; R)$ such that $\partial\sigma = \alpha + \beta$ with $\alpha \in C_{n-1}(A; R)$ and $\beta \in C_{n-1}(B; R)$ and such that the image of σ in $C_n(X, A \cup B; R)$ is a cycle that represents τ.

(b) As on page 1356 we make the identification

$$C^k(X, A; R) \xrightarrow{=} \text{all homomorphisms } C_k(X) \to R \text{ that vanish on } C_k(A).$$

We can pick a singular cochain $\varphi \in C^k(X; R) = \mathrm{Hom}(C_k(X), R)$ that vanishes on $C_k(A)$ and such that the corresponding element in $C^k(X, A; R)$ is a cocycle that represents ψ. Note that $\delta\varphi = 0$.

We have the following equalities:

$$\begin{aligned}
\partial(\psi \cap \tau) &\overset{\substack{\text{choice of } \sigma \text{ and definition} \\ \text{of relative cap product} \\ \downarrow}}{=} \partial([\varphi \cap \sigma]) \overset{\substack{\text{Proposition 31.13} \\ \downarrow}}{=} [\partial(\varphi \cap \sigma)] \overset{\substack{\text{Lemma 59.1} \\ \downarrow}}{=} (-1)^k \cdot [-\delta\varphi \cap \sigma + \varphi \cap \partial\sigma] \\
&\underset{\substack{\uparrow \\ \text{since } \delta\varphi = 0 \text{ and } \partial\sigma = \alpha + \beta}}{=} (-1)^k \cdot ([\varphi \cap \alpha + \varphi \cap \beta]) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ vanishes on } C_k(A) \text{ and} \\ \text{since } \alpha \circ [v_0, \ldots, v_k] \in C_k(A)}}{=} (-1)^k \cdot ([\varphi \cap \beta]) \\
&\underset{\substack{\uparrow \\ \text{by Proposition 31.13 we have } j_*([\beta]) = \partial[\sigma]}}{=} (-1)^k \cdot ([j^* i^* \varphi] \cap j_*^{-1}(\partial[\sigma])).
\end{aligned}$$

This equality implies that the diagram commutes. □

As we had mentioned before, a frequently used trick in the study of topological manifolds is to replace a topological manifold with non-empty boundary by its double. When doing this trick the following lemma can at times be useful.

[1025] With the various quotient groups, lifts, extensions, inclusions and so on it is admittedly quite tricky to figure out in between where any of the equalities hold.

Lemma 59.11. *Let M be a topological manifold. We denote by $r\colon DM \to M$ the folding map from page 1086 and given $l \in \mathbb{N}_0$ we denote by $D\colon H_l(M,\partial M;\mathbb{Z}) \to H_l(DM;\mathbb{Z})$ the doubling homomorphism that we introduced in Lemma 45.13. Then for any $\phi \in H^k(M;\mathbb{Z})$ and any $\sigma \in H_l(M,\partial M;\mathbb{Z})$ we have*

$$r^*\phi \cap D(\sigma) \;=\; D(\phi \cap \sigma) \;\in\; H_{l-k}(DM;\mathbb{Z}).$$

Proof. We will show that the desired equality actually holds already on the (co-) chain level. So let $\phi \in C^k(M;\mathbb{Z})$ and let $\sigma\colon \Delta^l \to M$ be a singular l-simplex. If $k > l$, then both sides of the lemma are trivially zero. Hence we can assume that $k \le l$. For $i = 1, 2$ we denote by $j_i\colon M \to M \times \{i\} \subset DM$ the obvious map. Then

$$\begin{aligned}
r^*\phi \cap D(\sigma) \;&\overset{\text{by definition of } D(\sigma)}{=}\; r^*\phi \cap (j_1 \circ \sigma - j_2 \circ \sigma) \;\overset{\text{since } \cap \text{ is bilinear}}{=}\; r^*\phi \cap j_1 \circ \sigma - r^*\phi \cap j_2 \circ \sigma \\
&= \phi(\underbrace{r \circ j_1 \circ \sigma}_{=\sigma} \circ [v_0, \dots, v_k]) \cdot j_1 \circ \sigma \circ [v_k, \dots, v_l] - \phi(\underbrace{r \circ j_2 \circ \sigma}_{=\sigma} \circ [v_0, \dots, v_k]) \cdot j_2 \circ \sigma \circ [v_k, \dots, v_l] \\
&\uparrow \\
&\text{definition of cap product and definition of } r^*\phi \\
&= \phi(\sigma \circ [v_0, \dots, v_k]) \cdot j_1 \circ (\sigma \circ [v_k, \dots, v_l]) - \phi(\sigma \circ [v_0, \dots, v_k]) \cdot j_2 \circ (\sigma \circ [v_k, \dots, v_l]) \\
&= j_{1*}(\phi \cap \sigma) - j_{2*}(\phi \cap \sigma) \;=\; D(\phi \cap \sigma). \\
&\uparrow \\
&\text{definition of } j_{1*},\, j_{2*} \text{ and the cap product}
\end{aligned}$$

□

Exercises for Chapter 59.

Exercise 59.1. Let $X = S^1 \vee S^2$. We denote by $p\colon X \to S^1$ and $q\colon X \to S^2$ the obvious projection maps and we denote by $i\colon S^1 \to X$ and $j\colon S^2 \to X$ the obvious inclusion maps. Determine $p^*([\theta_{\mathbb{Z}}]) \cap j_*([S^2])$.

Exercise 59.2. We denote by Σ the surface of genus two. Furthermore we denote by $p_1, p_2\colon \Sigma \to T = S^1 \times S^1$ the two projection maps and we denote by $p, q\colon S^1 \times S^1 \to S^1$ the projection onto the first respectively the second factor. As usual we write $\alpha = p^*(\theta)$ and $\beta = q^*(\theta)$. We saw that $[p_1^*(\alpha)], [p_1^*(\beta)], [p_2^*(\alpha)], [p_2^*(\beta)]$ form a basis for $H^1(\Sigma;\mathbb{Z})$. Furthermore we denote by $x_1, y_1, x_2, y_2 \in H_1(\Sigma;\mathbb{Z})$ the standard basis. For each of the four basis elements of $H^1(\Sigma;\mathbb{Z})$ determine the cap product with $[\Sigma]$ in terms of the basis for $H_1(\Sigma;\mathbb{Z})$.
Remark: This exercise is not as bad as it might sound initially.

Exercise 59.3. We consider the annulus $X = S^1 \times [0,1]$. Determine the following map:

$$\begin{array}{rcl} H^1(X;\mathbb{Z}) & \to & H_1(X,\partial X;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

More precisely, give a basis for $H^1(X;\mathbb{Z})$ and $H_1(X,\partial X;\mathbb{Z})$ and express the map $- \cap [X]$ in terms of that basis.

60. The Product Theorem and the Künneth Theorem for cohomology

In this chapter we will deal with the cup and cap product of product topological spaces. In particular at the end of the chapter we will have determined the cup product of the product $S^m \times S^n$ of two spheres and of the n-dimensional torus $(S^1)^n$. Furthermore we will have determined the Lusternik-Schnirelmann category of the n-dimensional torus.

60.1. **The Product Theorem.** The following theorem relates the cross product to the cap product and the cup product.

Theorem 60.1. (Product Theorem) *Let (X, A) and (Y, B) be two pairs of topological spaces such that the triad $(X\times Y, A\times Y, X\times B)$ is excisive. We denote by $p\colon X\times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Furthermore let*[1026] *$A_1, A_2 \in \{\varnothing, A\}$ and let $B_1, B_2 \in \{\varnothing, B\}$. Then for any $\alpha \in H^k(X, A_1; \mathbb{Z})$, $\mu \in H_m(X, A_1\cup A_2; \mathbb{Z})$, $\beta \in H^l(Y, B_1; \mathbb{Z})$ and $\nu \in H_n(Y, B_1 \cup B_2; \mathbb{Z})$ we have the equality*[1027]

$$\big(p^*(\alpha) \ \cup \ q^*(\beta)\big) \ \cap (\mu \times \nu) \ = \ (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$$

in $H_{m+n-k-l}(X \times Y, X \times B_2 \cup A_2 \times Y; \mathbb{Z})$.

Remark. Even though the equality in the Product Theorem 60.1 is easy to write down, it is much harder to figure out in which (co-) homology groups the various objects actually live. In fact the equality is understood as follows:

$$\underbrace{\Big(\underbrace{p^*(\alpha)}_{\in H^k(X\times Y, A_1\times Y)} \ \cup \ \underbrace{q^*(\beta)}_{\in H^l(X\times Y, X\times B_1)}\Big)}_{\in H^{k+l}(X\times Y, A_1\times Y\cup X\times B_2)} \cap \overbrace{(\mu \times \nu)}^{\substack{\text{lies in } H_{m+n}(X\times Y, (A_1\cup A_2)\times Y \cup X\times(B_1\cup B_2))\\ \downarrow}} = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot \underbrace{\underbrace{(\alpha \cap \mu)}_{\in H_{m-k}(X, A_2)} \times \underbrace{(\beta \cap \nu)}_{\in H_{n-l}(Y, B_2)}}_{\in H_{m+n-k-l}(X\times Y, X\times B_2\cup A_2\times Y)}.$$

In this formula we have to ensure that all products are in fact defined. This follows easily from Proposition 50.12, our hypothesis that $A_1, A_2 \in \{\varnothing, A\}$ and $B_1, B_2 \in \{\varnothing, B\}$ and our hypothesis that the triad $(X \times Y, A \times Y, X \times B)$ is excisive.

Remark.

(1) The condition in the Product Theorem 60.1 that the triad $(X \times Y, A \times Y, X \times B)$ needs to be excisive is arguably a little awkward. By Proposition 50.12 this condition is satisfied if we are in one of the following situations:
   (a) $A = \varnothing$ or $B = \varnothing$,
   (b) (X, A) and (Y, B) are pairs of CW-complexes,

[1026] By $A_1, A_2 \in \{\varnothing, A\}$ we mean that $A_1 = \varnothing$ or $A_1 = A$ and that $A_2 = \varnothing$ or $A_2 = A$.

[1027] Here, and throughout this section, we denote by p also the map of pairs $(X\times Y, A\times Y) \to (X, A)$ given by projection onto the first factor and similarly we denote by q the map of pairs $(X \times Y, X \times B) \to (Y, B)$ given by projection onto the second factor.

(c) $(X, A) = (X, \partial X)$ and $(Y, B) = (Y, \partial Y)$ where X and Y are topological manifolds,[1028]
(d) A and B are open subsets of X respectively Y.

(2) In fact a more general statement of the Product Theorem 60.1 holds if we deal with excisive triads (X, A_1, A_2) and (Y, B_1, B_2) and (co-) homology classes $\alpha \in H^k(X, A_1; \mathbb{Z})$, $\mu \in H_m(X, A_2; \mathbb{Z})$, $\beta \in H^l(Y, B_1; \mathbb{Z})$ and $\nu \in H_n(Y, B_2; \mathbb{Z})$. But some care needs to be taken to see which other triads need to be excisive to ensure that all the cap and cup products are defined. We will not attempt to formulate this more general statement and we will not make use of it. (See also [Dol, p. 240 and p. 244] for a few more details.)

(3) The cross product is defined the same way in all of the usual text books in algebraic topology. But we saw on pages 1334 and 1373 that in the literature different sign conventions are used for the cup product and the cap product. These also lead to different formulations of the Product Theorem 60.1. More precisely, given topological spaces X and Y and $\alpha \in H^k(X; \mathbb{Z})$, $\mu \in H_m(X; \mathbb{Z})$, $\beta \in H^l(Y; \mathbb{Z})$ and $\nu \in H_n(Y; \mathbb{Z})$ the following equations can be found in the literature:

$$\begin{array}{rrcl}
\text{[Bre, p. 337]} & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu) \\
\text{[Dol, p. 240]} & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu) \\
\text{[Sp, p. 255]} & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\alpha)\cdot(\deg(\nu)-\deg(\beta))} \cdot (\alpha \cap \mu) \times (\beta \cap \nu).
\end{array}$$

With the aforementioned sign conventions the formulas of Dold and Spanier are consistent with our formula, whereas the formula of Bredon differs from the formula of Dold and Spanier by the term $(-1)^{\deg(\alpha)\cdot\deg(\nu)+\deg(\beta)\cdot\deg(\mu)}$.[1029]

(4) The conclusion of the Product Theorem 60.1 also holds, with verbatim the same proof, if throughout we replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients.

The proof of the Product Theorem 60.1 is somewhat lengthy and technical. Therefore we first discuss some consequences of Theorem 60.1 which will then hopefully convince us that it is worth the effort to prove Theorem 60.1.

The following is one of the key calculations of the cup product and the cap product. As we will later see, many other calculations can be reduced to this calculation.

Proposition 60.2. *Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Then the following two statements hold:*

[1028] Let X be an m-dimensional topological manifold and let Y be an n-dimensional manifold. It follows from Propositions 56.9 and 32.1 that $W := \partial(X \times Y) = \partial X \times Y \cup X \times \partial Y$ is an $(m+n-1)$-dimensional topological manifold. It is straightforward to verify that $\partial X \times Y$ and $X \times \partial Y$ are submanifolds of W such that

$$(\partial X \times Y) \cap (X \times \partial Y) \;=\; \partial X \times \partial Y = \partial(\partial X \times Y) \;=\; \partial(X \times \partial Y).$$

Therefore it follows from Proposition 50.12 (5) that the triad $(X \times Y, \partial X \times Y, X \times \partial Y)$ is excisive.

[1029] Mathematicians with a long experience of teaching cohomology groups sidestep all issues with signs by working with $\mathbb{F}_2$-coefficients, see [Hau].

(1)
$$\underbrace{p^*([X]^*)}_{\in H^m(X\times Y,\partial X\times Y)} \quad \cup \quad \underbrace{q^*([Y]^*)}_{\in H^n(X\times Y,X\times\partial Y)} \quad = \quad \underbrace{[X\times Y]^*}_{\in H^{m+n}(X\times Y,\partial(X\times Y))}.$$

(2) *If $P \in X$ and $Q \in Y$ are points, then*[1030]
$$\overbrace{p^*([X]^*)}^{\in H^m(X\times Y,\partial X\times Y)} \quad \cap \quad \overbrace{[X\times Y]}^{\in H_{m+n}(X\times Y,\partial(X\times Y))} \quad = \quad \overbrace{[\{P\}\times Y]}^{\in H_n(X\times Y,X\times\partial Y)}$$
and similarly
$$\underbrace{q^*([Y]^*)}_{\in H^n(X\times Y,X\times\partial Y)} \quad \cap \quad \underbrace{[X\times Y]}_{\in H_{m+n}(X\times Y,\partial(X\times Y))} \quad = \quad (-1)^{mn}\cdot \underbrace{[X\times\{Q\}]}_{\in H_m(X\times Y,\partial X\times Y)}.$$

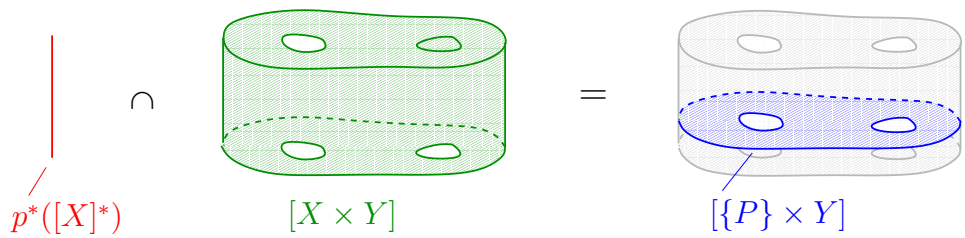


FIGURE 740.

Remark. As we pointed out above, the conclusion of the Product Theorem 60.1 also holds if we work with $\mathbb{F}_2$-coefficients. It follows easily that there is also an $\mathbb{F}_2$-coefficient analogue of Proposition 60.2.

Example. If we apply Proposition 60.2 (2) to $X = S^1$ and $Y = S^1$ we see that
$$p^*([S^1]^*)\cap[S^1\times S^1] = [1\times S^1] \quad \text{and} \quad q^*([S^1]^*)\cap[S^1\times S^1] = -[S^1\times 1].$$
This matches our down-to-earth calculation from page 1375.

Proof. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Furthermore let P be a point in X and let Q be a point in Y. We start out with the following two observations:

(a) For any compact oriented connected topological k-dimensional manifold M and any point $R \in M$ we obtain from the discussion on page 1378 that $[M]^*\cap[M] = [R]$ and that $[M]^*$ is the unique element in $H^k(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ with $[M]^*\cap[M] = [R]$ in $H_0(M;\mathbb{Z}) = \mathbb{Z}\cdot[R]$.

[1030]Here we use the following suggestive notation: We can view $\{P\}\times Y$ as a submanifold of $X\times Y$. It is canonically homeomorphic to Y, thus can view $\{P\}\times Y$ as a compact oriented topological manifold. We denote by $[\{P\}\times Y]$ its fundamental class, and by a slight abuse of notation we also denote by $[\{P\}\times Y]$ the image of the fundamental class under the inclusion $\{P\}\times Y \to X\times Y$.

(b) By Proposition 56.10 we have $[X \times Y] = [X] \times [Y]$.

Now we turn to the actual proof of the two statements.

(1) We perform the following calculation in $H_0(X \times Y; \mathbb{Z})$:

$$\begin{array}{rcl} & & \text{observation (b)} \\ & & \downarrow \\ (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X \times Y]) & = & (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ = (-1)^{mn+mn} \cdot ([X]^* \cap [X]) \times ([Y]^* \cap [Y]) & = & [P] \times [Q] \quad = [(P,Q)]. \\ \uparrow & & \uparrow \qquad\qquad \uparrow \\ \text{by Theorem 60.1 applied} & & \text{observation (a)} \quad \text{see Lemma 56.7 (2)} \\ \text{to } (X, A_1, A_2) = (X, \partial X, \partial X) & & \\ \text{and } (Y, B_1, B_2) = (Y, \partial Y, \partial Y) & & \end{array}$$

This implies, by observation (a), that $p^*([X]^*) \cup q^*([Y]^*) = [X \times Y]^*$.

(2) Let $P \in X$. Then

$$\begin{array}{rcl} p^*([X]^*) \cap ([X \times Y]) & = & (p^*([X]^*) \cup q^*(1_Y)) \cap ([X] \times [Y]) \\ & \uparrow & \\ \text{by Proposition 58.3 (1) we have } q^*(1_Y) = 1_{X \times Y} & \text{and by observation (b) we have } [X \times Y] = [X] \times [Y] & \\ & = & ([X]^* \cap [X]) \times (1_Y \cap [Y]) \quad = [P] \times [Y] \quad = [\{P\} \times Y]. \\ & & \uparrow \qquad\qquad\qquad\qquad \uparrow \qquad\qquad \uparrow \\ & & \text{by Theorem 60.1 applied to } (Y, B_1, B_2) = (Y, \varnothing, \partial Y) \quad \text{by Lemma 59.3} \quad \text{Lemma 56.7 (2)} \\ & & \text{note that here we use that } 1_Y \text{ lies in } H^0(Y; \mathbb{Z}) \quad \text{and observation (a)} \end{array}$$

This proves the first equality of (2). Now let $Q \in Y$. Then, very similar to the above, we have

$$\begin{aligned} q^*([Y]^*) \cap ([X \times Y]) &= (p^*(1_X) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ &= (-1)^{mn} \cdot (1_X \cap [X]) \times ([Y]^* \cap [Y]) \\ &= (-1)^{mn} \cdot [X] \times [Q] \;=\; (-1)^{mn} \cdot [X \times \{Q\}]. \end{aligned}$$

□

Using Proposition 60.2 (1) we can now easily compute the cup product of the product $S^m \times S^n$ of two spheres.

Lemma 60.3. *Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Let $1 \in H^0(S^m \times S^n; \mathbb{Z})$ be the neutral element from Proposition 57.7.*

(1) *The cohomology of $S^m \times S^n$ is given as follows:*

$$H^*(S^m \times S^n; \mathbb{Z}) \;=\; \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*([S^m]^*) \oplus \mathbb{Z} \cdot q^*([S^n]^*) \oplus \mathbb{Z} \cdot [S^m \times S^n]^*.$$

(2) *The multiplication table for the cup product is given by*

$\cup$	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
1	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
$p^*([S^m]^*)$	$p^*([S^m]^*)$	0	$[S^m \times S^n]^*$	0
$q^*([S^n]^*)$	$q^*([S^n]^*)$	$(-1)^{mn} \cdot [S^m \times S^n]^*$	0	0
$[S^m \times S^n]^*$	$[S^m \times S^n]^*$	0	0	$0.$

Here the table shows the cup product of the term in the first column with the term in the first row (and not the other way round).

Remark.

(1) For $m = n = 1$, i.e. for the torus $S^1 \times S^1$ we had calculated the cup product in Lemma 57.14.
(2) If we use Proposition 60.2 (1) with $\mathbb{F}_2$-coefficients then we can also determine the cup product on $H^*(S^m \times S^n; \mathbb{F}_2)$ and we obtain basically the same result.

Proof.

(1) Using Proposition 31.3, the Künneth Theorem 43.8 for topological spaces and the Universal Coefficient Theorem 51.13 for Cohomology Groups one can easily determine the isomorphism type of $H^*(S^m \times S^n; \mathbb{Z})$. Furthermore, using Lemma 50.7 and the homology classes given by the fundamental classes of S^m, S^n and $S^m \times S^n$ it is straightforward to verify that the given cohomology classes do indeed form a basis for $H^*(S^m \times S^n; \mathbb{Z})$.
(2) By Observation 57.13 it remains to show that the red-colored and the blue-colored entries are correct.
   (a) First we consider the red-colored entries. We have
$$p^*([S^m]^*) \cup p^*([S^m]^*) \underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=} p^*([S^m]^* \cup [S^m]^*) \underset{\substack{\uparrow \\ \text{since } [S^m]^* \cup [S^m]^* \in H^{2m}(S^m;\mathbb{Z}) = 0}}{=} p^*(0) = 0.$$
   The same argument also shows that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.
   (b) We turn to the blue-colored entries. We calculate that
$$(-1)^{mn} \cdot q^*([S^n]^*) \cup p^*([S^m]^*) \underset{\substack{\uparrow \\ \text{Proposition 57.8}}}{=} p^*([S^m]^*) \cup q^*([S^n]^*) \underset{\substack{\uparrow \\ \text{Proposition 60.2 (1)}}}{=} [S^m \times S^n]^*.$$
□

In the following we consider the wedge $X := S^2 \vee S^2 \vee S^4$ and the product $Y := S^2 \times S^2$. It follows from Lemma 33.12 and Propositions 31.3 and 34.8 and the Universal Coefficient Theorem 51.13 for Cohomology Groups that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ have isomorphic homology and cohomology groups. Furthermore one obtains easily from Propositions 5.14, 7.21 and 12.3 that both X and Y are simply connected.

It is quite easy to show that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ are not homeomorphic. Indeed, suppose there exists a homeomorphism $\phi\colon S^2 \vee S^2 \vee S^4 \to S^2 \times S^2$. We denote by P we wedge point of $X = S^2 \vee S^2 \vee S^4$ and we write $Q = f(P)$. Then $X \setminus \{P\}$ and $Y \setminus \{Q\}$ would also be homeomorphic. But the former topological space has three path-components whereas the latter has just one path-component.[1031] We refer to Figure 741 for a dimensionally reduced illustration.

The following lemma is much deeper than the above discussions.

[1031] How can you prove rigorously that given any point $Q \in S^2 \times S^2$ the complement $S^2 \times S^2 \setminus \{Q\}$ is still path-connected?

FIGURE 741.

Lemma 60.4. *The topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not even homotopy equivalent.*

Proof. We had just seen in Lemma 60.3 that there exist cohomology classes α and β in $H^2(S^2 \times S^2; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. On the other hand in Lemma 57.11 we saw that the cup product

$$\cup\colon H^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \times H^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \;\to\; H^4(S^2 \vee S^2 \vee S^4; \mathbb{Z})$$

is the zero-map. It follows from Corollary 57.12 that the topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not homotopy equivalent. □

In Proposition 39.21 we saw that there exists a degree-one map from $S^2 \times S^2$ onto the 4-dimensional sphere S^4. In Question 39.28 we had asked whether there exists a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$. Now we can answer this question in the negative.

Lemma 60.5. *There is no map $f\colon S^4 \to S^2 \times S^2$ of non-zero degree.*

Proof. Let $f\colon S^4 \to S^2 \times S^2$ be a map. We have to show that $\deg(f) = 0$. By Lemma 51.16 it suffices to show that $f^*([S^2 \times S^2]^*) = 0$. In Lemma 60.3 we showed that there exist cohomology classes $a, b \in H^2(S^2 \times S^2; \mathbb{Z})$ such that $a \cup b = [S^2 \times S^2]^* \in H^4(S^2 \times S^2; \mathbb{Z})$. It follows that

$$\begin{array}{ccccccc} f^*([S^2 \times S^2]^*) & = & f^*(a \cup b) & = & f^*(a) \cup f^*(b) & = & 0 \cup 0 \;= 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } [S^2 \times S^2]^* = a \cup b & & \text{Lemma 57.10} & & \text{since } f^*(a), f^*(b) \in H^2(S^4; \mathbb{Z}) = 0 & \end{array}$$

□

60.2. **Proof of the Product Theorem 60.1.** Now that we are fully convinced of the value of Theorem 60.1 it is time to actually provide a proof thereof.

Initially we introduced only the tensor product $C_* \otimes D_*$ of two chain complexes C_* and D_*. The definition of the boundary map in the tensor product $C_* \otimes D_*$ is *not* symmetric in the two terms. But it is straightforward to see that the tensor product of chain complexes is associative. More precisely, given chain complexes $C_*^1, \dots, C_*^k$ the boundary map on the tensor product $C_*^1 \otimes \cdots \otimes C_*^k$ is always determined by

$$\partial(c_1 \otimes c_2 \otimes \cdots \otimes c_k) \;=\; \sum_{i=1}^{k} (-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})} \cdot c_1 \otimes \cdots \otimes \partial c_i \otimes \cdots \otimes c_k,$$

regardless of the way how one breaks the tensor product into the tensor product of pairs of two chain complexes.

The following lemma is basically the same as Lemma 43.3, but we state it here for the convenience of the reader.

Lemma 60.6. *Let $f_i\colon C_*^i \to D_*^i$, $i = 1, \dots, k$ be chain maps between chain complexes. Then the map*

$$f_1 \otimes \cdots \otimes f_k\colon C_*^1 \otimes \cdots \otimes C_*^k \quad \to \quad D_*^1 \otimes \cdots \otimes D_*^k$$

that is determined by

$$c_1 \otimes \cdots \otimes c_k \quad \mapsto \quad f(c_1) \otimes \cdots \otimes f(c_k)$$

is also a chain map.

Remark. Even the definition of $f_1 \otimes \cdots \otimes f_k$ is, oddly enough, not the same everywhere. For example [Bre, p. 315] adds a sign to the map.

Proof. Since in the proof of the Product Theorem 60.1 there are lots of steps later on which can make us nervous, let us convince ourselves that the statement is indeed correct. We have

$$\begin{aligned}
\partial\big((f_1 \otimes \ldots \otimes f_k)(c_1 \otimes \ldots \otimes c_k)\big) &= \partial(f_1(c_1) \otimes \ldots \otimes f_k(c_k)) \\
&= \sum_{i=1}^{k} (-1)^{\deg(f_1(c_1)) + \cdots + \deg(f_{i-1}(c_{i-1}))} \cdot f_1(c_1) \otimes \ldots \otimes \overbrace{f_i(\partial c_i)}^{=\partial f_i(c_i)} \otimes \ldots \otimes f_k(c_k) \\
&\underset{\substack{\uparrow \\ \text{since } \deg(f_i(c_i)) = \deg(c_i)}}{=} (f_1 \otimes \ldots \otimes f_k)\Big(\sum_{i=1}^{k} (-1)^{\deg(c_1) + \cdots + \deg(c_{i-1})} \cdot c_1 \otimes \ldots \partial c_i \otimes \ldots \otimes c_k\Big) \\
&= (f_1 \otimes \ldots \otimes f_k)(\partial(c_1 \otimes \ldots \otimes c_k)).
\end{aligned}$$

□

Note though that the naive swapping of two factors in the tensor product of two chain complexes is not necessarily a chain map. But we have the following lemma.

Lemma 60.7. *Let A_* and B_* be chain complexes. The unique map*

$$\tau\colon A_* \otimes B_* \quad \to \quad B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \quad \mapsto \quad (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Remark. In Exercise 60.1 we will see that we can use Lemma 60.7 to give a new proof of Proposition 57.8, i.e. we can give a new proof that the cup product is (anti-) commutative.

Proof. The difficult bit about the lemma is coming up with the statement. The verification is now almost trivial. So let $a \in A_k$ and $b \in B_l$. Then we have

$$\begin{aligned}
\tau(\partial(a \otimes b)) &= \tau\big(\partial a \otimes b + (-1)^k \cdot a \otimes \partial b\big) &&= \tau(\partial a \otimes b) + (-1)^k \cdot \tau(a \otimes \partial b) \\
&= (-1)^{(k-1)l} \cdot b \otimes \partial a + (-1)^{k+k(l-1)} \cdot \partial b \otimes a &&= (-1)^{kl} \cdot \partial b \otimes a + (-1)^{l+kl} \cdot b \otimes \partial a \\
&= \partial((-1)^{kl} \cdot b \otimes a) \;=\; \partial(\tau(a \otimes b)).
\end{aligned}$$

□

Now we have to introduce a plethora of new definitions and notations.

Notation.

(1) We denote by $\mathbb{Z}$ also the chain complex given by the group $\mathbb{Z}$ in degree zero and where all other chain groups are zero.
(2) Given a chain complex C_* and $k \in \mathbb{N}_0$ we denote by $C_*[k]$ the chain complex which is the result of "shifting C_* by k to the left", i.e. it is the chain complex for which the m-th chain groups $(C_*[k])_m$ is given by C_{m-k}.[1032] For example, $\mathbb{Z}[3]$ is the following chain complex:

$$\begin{array}{ccccccccccc} \dots & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to & 0 & \to & 0. \\ & & & & \uparrow & & & & & & \\ & & & & \text{degree } 3 & & & & & & \end{array}$$

(3) If $\alpha\colon C_* \to D_*$ is a chain map, then for any $k \in \mathbb{N}_0$ we denote by $\alpha[k]\colon C_*[k] \to D_*[k]$ the obvious shift of the chain map. Evidently this is again a chain map.
(4) Let C_* be a chain complex and let $k \in \mathbb{N}_0$.
   (a) Given a cocycle $\varphi\colon C_k \to \mathbb{Z}$ we denote by $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ the map that is given by φ on C_k and zero on all other chain groups.
   (b) Conversely, given a chain map $\psi\colon C_* \to \mathbb{Z}[k]$ we denote by $\psi_k\colon C_k \to \mathbb{Z}$ the map that is the k-th map of the chain map.

Lemma 60.8. *Let C_* be a chain complex and let $k \in \mathbb{N}_0$. The following statements hold:*

(1) *Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. The map $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ is a chain map.*
(2) *Let $\psi\colon C_* \to \mathbb{Z}[k]$ be a chain map. The map $\psi_k\colon C_k \to \mathbb{Z}$ is a cocycle.*
(3) *Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps. If α and β are chain homotopic, then α_k and β_k are cohomologous.*

Proof.

(1) Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. We consider the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{k+1} & \xrightarrow{\partial} & C_k & \xrightarrow{\partial} & C_{k-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots. \end{array}$$

The hypothesis that φ is a cocycle means that $\delta\varphi = \varphi \circ \partial = 0$, which implies immediately that $\Pi_k(\varphi)$ is a chain map.
(2) The proof of statement (2) is almost the same as the proof of statement (1).
(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps and let $\{P_m\colon C_m \to \mathbb{Z}[k]_{m+1}\}_{m\in\mathbb{N}_0}$ be a chain homotopy between α and β. We summarize all relevant maps in the following

[1032] Once again, conventions in the literature differ. It is fairly common to define the boundary maps on $C_*[k]$ to be the boundary maps of C_* multiplied by $(-1)^k$, see e.g. [Weib, p. 10].

diagram:

$$\begin{tikzcd}
\dots \arrow[r] & C_{k+1} \arrow[r, "\partial"] \arrow[d] & C_k \arrow[r, "\partial"] \arrow[d, "\alpha_k-\beta_k"] \arrow[dl, "P_k"'] & C_{k-1} \arrow[r] \arrow[d] \arrow[dl, "P_{k-1}"'] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots.
\end{tikzcd}$$

By definition of a chain homotopy we obtain that $\alpha_k - \beta_k = P_{k-1} \circ \partial$. But this means that $\alpha_k - \beta_k = \delta P_{k-1}$, i.e. α_k and β_k are cohomologous. $\square$

Lemma 60.9. *Let C_* be a chain complex and let $k \in \mathbb{Z}$. The maps*

$$\begin{array}{rcl} \eta_m \colon (\mathbb{Z}[k] \otimes C_*)_m = \mathbb{Z} \otimes C_{m-k} & \to & (C_*[k])_m = C_{m-k} \\ a \otimes c_{m-k} & \mapsto & (-1)^{k(m-k)} \cdot a \cdot c_{m-k} \end{array}$$

form an isomorphism

$$\eta \colon \mathbb{Z}[k] \otimes C_* \xrightarrow{\cong} C_*[k]$$

of chain complexes.

Example. Let $k, l \in \mathbb{N}_0$. The chain map

$$\eta \colon \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l]$$

is determined by the fact that for $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and $b \in (\mathbb{Z}[l])_l = \mathbb{Z}$ we have

$$\eta(a \otimes b) = (-1)^{kl} \cdot a \cdot b.$$

Proof. It follows from Lemma 42.3 (3) that each η_m is an isomorphism. But we still need to verify that the maps η_m form a chain map $\eta \colon \mathbb{Z}[k] \otimes C_* \to C_*[k]$. Thus let $k \in \mathbb{N}_0$, let $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and let $c_{m-k} \in C_{m-k}$. Then

$$\begin{array}{rcl} & \text{since } a \text{ is of degree } k \text{ and since } \partial a = 0 & \text{since } \partial c_{m-k} \text{ is of degree } m-k-1 \\ & \downarrow & \downarrow \\ \eta_{m-1}(\partial(a \otimes c_{m-k})) & = & \eta_{m-1}((-1)^k \cdot a \otimes \partial c_{m-k}) \quad = \quad (-1)^k \cdot (-1)^{k(m-1-k)} \cdot a \cdot \partial c_{m-k} \\ & = & \partial((-1)^{k(m-k)} \cdot a \cdot c_{m-k}) \quad = \quad \partial(\eta_m(a \otimes c_{m-k})). \end{array}$$

It follows immediately that the maps η_m form a chain map. $\square$

Notation.

(1) Given topological spaces X and Y we denote by $\Upsilon \colon C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ and $\Theta \colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ the chain maps from the Eilenberg-Zilber Theorem 56.1.
(2) Given a topological space W we denote by $\Delta_W \colon C_*(W) \to C_*(W) \otimes C_*(W)$ the Alexander-Whitney diagonalization from page 1338. Recall that Δ_W is also a chain map.

In the following two lemmas we will reinterpret the cup product and the cap product as chain maps.

Lemma 60.10. *Let W be a topological space and let $\alpha \in C^k(W;\mathbb{Z})$ and $\beta \in C^l(W;\mathbb{Z})$ be cocycles. Then we have the following equality of chain maps:*

$$(-1)^{kl} \cdot \Pi_{k+l}\big(\underbrace{\alpha \cup \beta}_{\text{cocycle in } C^{k+l}(W;\mathbb{Z})}\big) \;=\; \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Delta_W \colon C_*(W) \to \mathbb{Z}[k+l].$$

Proof. We introduce the following notation:

(1) We denote by $\mu\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism of abelian groups given by $\mu(a \otimes b) = a \cdot b$.
(2) We denote by $\Pi_{k,l}\colon (C_*(W) \otimes C_*(W))_{k+l} \to C_k(W) \otimes C_l(W)$ the obvious projection map.

In the proof of Proposition 57.4 on page 1339 we showed that

$$\alpha \cup \beta \;=\; \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Delta_W \colon C_{k+l}(W) \;\to\; \mathbb{Z}.$$

The lemma follows immediately from this equality and the following claim.

Claim. [1033]

$$(-1)^{kl} \cdot \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \;=\; \underbrace{(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}}_{\substack{\text{the restriction to the } (k+l)\text{-th chain group of the} \\ \text{chain map } C_*(W) \otimes C_*(W) \to \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l] \\ a \otimes b \mapsto (-1)^{kl} \cdot ab}} \;\colon (C_*(W) \otimes C_*(W))_{k+l} \;\to\; \mathbb{Z}.$$

Let $\sigma_i \in C_i(W)$ and $\sigma_j \in C_j(W)$ with $i + j = k + l$. If $(i,j) \neq (k,l)$, then both maps are zero. So now suppose that $(i,j) = (k,l)$. Then it is clear that both maps send $\sigma_k \otimes \sigma_l$ to $(-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l)$. (Note that the sign $(-1)^{kl}$ for the map on the right-hand side stems from the definition of the chain map η.) This concludes the proof of the claim. □

Given a topological space W and given $\varphi \in C^k(W;\mathbb{Z})$ we obtain for each $m \in \mathbb{N}_0$ a map

$$\begin{aligned} \varphi \cap -\colon C_m(W) &\;\to\; C_{m-k}(W) = (C_*(W)[k])_m \\ \sigma &\;\mapsto\; \varphi \cap \sigma. \end{aligned}$$

Unfortunately in general these maps do *not* form a chain map from the chain complex $C_*(W)$ to the chain chain complex $C_*(W)[k]$. This motivates introducing the following slight variation of the cap product.

Definition. Let W be a topological space. Given $\varphi \in C^m(W;\mathbb{Z})$ and given $\sigma \in C_n(W)$ we define[1034]

$$\varphi \,\widetilde{\cap}\, \sigma \;:=\; (-1)^{m(n-m)} \cdot \varphi \cap \sigma.$$

In particular for $m = n$ we have $\varphi \,\widetilde{\cap}\, \sigma = \varphi \cap \sigma$.

Lemma 60.11. *Let W be a topological space.*

[1033] Here, given two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$ the map $\Pi_{k,l}\colon \mathcal{C}_* \otimes \mathcal{D}_* \to \mathcal{C}_k \otimes \mathcal{B}_l$ is the obvious projection map that we introduced on page 1332.

[1034] If we compare this definition with the discussion on page 1373 we see that the definition of $\widetilde{\cap}$ is precisely the definition of the cap product in Bredon [Bre] and Dold [Dol].

(1) *If $\varphi \in C^m(W;\mathbb{Z})$ is a cocycle, then the map*

$$\begin{array}{rcl} \varphi \mathbin{\widetilde{\cap}} -\colon C_*(W) & \to & C_*(W)[m] \\ \sigma & \mapsto & \varphi \mathbin{\widetilde{\cap}} \sigma \end{array}$$

is a chain map.

(2) *If $\varphi, \psi \in C^m(W;\mathbb{Z})$ are cohomologous cocycles, then the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$ are chain homotopic.*

Proof. Let W be a topological space. We start out with the following claim.

Claim. For any cochain $\tau \in C^l(W;\mathbb{Z})$ and any chain $\mu \in C_n(W;\mathbb{Z})$ we have

$$\partial(\tau \mathbin{\widetilde{\cap}} \mu) \;=\; (-1)^{l+n} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial\mu.$$

The claim is an immediate consequence of Lemma 59.1. Since we are worried about signs we write down the detailed calculation. We have

$$\begin{array}{rcl} \partial(\tau \mathbin{\widetilde{\cap}} \mu) & \underset{\substack{\uparrow \\ \text{definition of } \widetilde{\cap}}}{=} & (-1)^{l(n-l)} \cdot \partial(\tau \cap \mu) \underset{\substack{\uparrow \\ \text{Lemma 59.1}}}{=} (-1)^{l(n-l)} \cdot (-1)^l \cdot (-\delta\tau \cap \mu + \tau \cap \partial\mu) \\ & = & (-1)^{l(n-l)+l} \cdot \big(-(-1)^{(l+1)\cdot(n-l-1)} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + (-1)^{l\cdot(n-l-1)} \cdot \tau \mathbin{\widetilde{\cap}} \partial\mu \big) \\ & = & (-1)^{l+n} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial\mu. \end{array}$$

This concludes the proof of the claim.

Now we turn to the actual proof of the two statements.

(1) Let $\varphi \in C^m(W;\mathbb{Z})$ be a cocycle. Furthermore let $\sigma \in C_n(W)$. By the definition of the cap product and by definition of the shifting of chain complexes we have $\varphi \cap \sigma \in C_{n-m}(W) = (C_*(W)[m])_n$. This shows that $\varphi \mathbin{\widetilde{\cap}} -$ preserves degrees. It remains to show that the map $\varphi \mathbin{\widetilde{\cap}} -$ commutes with the boundary maps. But this is an immediate consequence of the above claim and the fact that φ is a cocycle.

(2) So let $\varphi, \psi \in C^m(W;\mathbb{Z})$ be cocycles. We assume they are cohomologous which means that there exists a $\tau \in C^{m-1}(W;\mathbb{Z})$ such that $\varphi = \psi + \delta\tau$. For any $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} P_k\colon C_k(W) & \to & (C_*(W)[m])_{k+1} = C_{k-m+1}(W) \\ \sigma & \mapsto & (-1)^{m+k} \cdot \tau \mathbin{\widetilde{\cap}} \sigma. \end{array}$$

We calculate that for any $\sigma \in C_k(W)$ we have

$$\begin{array}{rcl} \varphi \mathbin{\widetilde{\cap}} \sigma - \psi \mathbin{\widetilde{\cap}} \sigma \;=\; \delta\tau \mathbin{\widetilde{\cap}} \sigma & \underset{\substack{\text{by the claim} \\ \downarrow}}{=} & (-1)^{m+k} \cdot \partial(\tau \mathbin{\widetilde{\cap}} \sigma) + (-1)^{m+k-1} \cdot \tau \mathbin{\widetilde{\cap}} \partial\sigma \\ & = & (\partial \circ P_k)(\sigma) + (P_{k-1} \circ \partial)(\sigma). \end{array}$$

This shows that the maps $\{P_k\}_{k\in\mathbb{N}_0}$ form a chain homotopy between the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$. □

In the proof of Theorem 60.1 we will also need the following reformulation of the cap product.

Proposition 60.12. *Let W be a topological space and let $\varphi \in C^m(W;\mathbb{Z})$ be a cocycle. Then we have the following equality of chain maps*

$$\varphi \,\widetilde{\cap}\, - \;=\; \eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W \colon C_*(W) \;\to\; C_*(W)[m].$$

Proof. Let W be a topological space, let $\varphi \in C^m(W)$ be a cocycle and let $\sigma\colon \Delta^n \to W$ be a singular n-simplex. We have the following equalities in $C_*(W)[m]_n = C_{n-m}(W)$:

$$\begin{aligned}
\big(\eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\big)(\sigma) &\overset{\substack{\text{definition of the Alexander-Whitney diagonalization}\\ \downarrow}}{=} \eta\Big((\Pi_m(\varphi) \otimes \mathrm{id})\Big(\sum_{j=0}^{n} \sigma \circ [v_0,\dots,v_j] \otimes \sigma \circ [v_j,\dots,v_n]\Big)\Big) \\
&\underset{\substack{\uparrow \\ \text{by definition of the chain map } \Pi_m(\varphi)\otimes \mathrm{id}\colon C_*(W)\otimes C_*(W) \to \mathbb{Z}[m]\otimes C_*(W)}}{=} \eta\big(\underbrace{\varphi(\sigma \circ [v_0,\dots,v_m])}_{\in \mathbb{Z}[m]_m} \otimes \underbrace{\sigma \circ [v_m,\dots,v_n]}_{\in C_{n-m}}\big) \\
&\underset{\substack{\uparrow \\ \text{definition of } \eta \text{ in Lemma 60.9}}}{=} (-1)^{m(n-m)} \cdot \underbrace{\varphi(\sigma\circ[v_0,\dots,v_m])\cdot\sigma\circ[v_m,\dots,v_n]}_{=\varphi\cap\sigma} \underset{\substack{\uparrow \\ \text{definition of } \widetilde{\cap} \text{ on page 1394}}}{=} \varphi \,\widetilde{\cap}\, \sigma.
\end{aligned}$$

We have thus verified that the desired equality holds. □

The following statement is just the definition of the cross product.

Tautology 60.13. *Let X and Y two topological spaces. For any $[\mu] \in H_m(X;\mathbb{Z})$ and $[\nu] \in H_n(Y;\mathbb{Z})$ we have*

$$[\mu] \times [\nu] \;=\; [\Upsilon(\mu \otimes \nu)].$$

We need one more slightly delicate chain map before we can finally turn to the proof of the Product Theorem 60.1.

Lemma 60.14. *Let C_* and D_* be chain complexes and let $k, l \in \mathbb{N}_0$. The map*

$$\Psi_{k,l}\colon C_*[k] \otimes D_*[l] \;\to\; (C_* \otimes D_*)[k+l]$$

which is defined by the property that for $c_m \in C_m$ and $d_n \in D_n$ we have

$$c_m \otimes d_n \;\mapsto\; (-1)^{k(l+n)} \cdot c_m \otimes d_n.$$

is an isomorphism of chain complexes.

Proof. First note that it is clear that $\Psi_{k,l}$ preserves degrees and that on each degree it is an isomorphism. It remains to show that the map $\Psi_{k,l}$ is a chain map. Let $c_m \in C_m$ and

$d_n \in D_n$. We calculate that

$$\begin{aligned}
\Psi_{k,l}(\partial(c_m \otimes d_n)) &\overset{\substack{\text{since } c_m \text{ has degree } m+k \text{ in } C_*[k] \\ \downarrow}}{=} \Psi_{k,l}(\partial c_m \otimes d_n + (-1)^{m+k} \cdot c_m \otimes \partial d_n) \\
&= (-1)^{k(l+n)} \cdot \partial c_m \otimes d_n + (-1)^{m+k} \cdot (-1)^{k(l+n-1)} \cdot c_m \otimes \partial d_n \\
&= (-1)^{k(l+n)} \cdot (\partial c_m \otimes d_n + (-1)^m \cdot c_m \otimes \partial d_n) \underset{\substack{\uparrow \\ \text{since } c_m \text{ has degree } m \text{ in } C_*}}{=} (-1)^{k(l+n)} \cdot \partial(c_m \otimes d_n) \\
&= \partial((-1)^{k(l+n)} \cdot c_m \otimes d_n)) = \partial(\Psi_{k,l}(c_m \otimes d_n)).
\end{aligned}$$

This calculation implies that $\Psi_{k,l}$ is a chain map. □

The following theorem is the main technical result of this section. As we will see shortly, the Product Theorem 60.1 is a straightforward consequence of Theorem 60.15.

Theorem 60.15. *Let X and Y be topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projection maps. Let $\alpha \in C^k(X;\mathbb{Z})$ and $\beta \in C^l(Y;\mathbb{Z})$ be cocycles. Then the chain maps*

$$(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon \quad \text{and} \quad \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)$$

from $C_*(X) \otimes C_*(Y)$ to $C_*(X \times Y)[k+l]$ are chain homotopic.

Proof. Let X and Y be topological spaces and let $\alpha \in C^k(X;\mathbb{Z})$ and $\beta \in C^l(Y;\mathbb{Z})$ be cocycles. We consider the following diagram of chain maps

$$\begin{tikzcd}
C_*X \otimes C_*Y \arrow[r, "\Upsilon"] \arrow[d, "\Delta_X \otimes \Delta_Y"] & C_*(X \times Y) \arrow[r, "\Delta_{X\times Y}"] & C_*(X \times Y) \otimes C_*(X \times Y) \arrow[d, "\Theta \otimes \mathrm{id}"] \\
C_*X \otimes C_*X \otimes C_*Y \otimes C_*Y \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[d, "\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id}"'] & C_*X \otimes C_*Y \otimes C_*X \otimes C_*Y \arrow[r, "\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon"] \arrow[d, "\Pi_k(\alpha) \otimes \Pi_l(\beta) \otimes \mathrm{id} \otimes \mathrm{id}"] & C_*X \otimes C_*Y \otimes C_*(X \times Y) \arrow[d, "\Pi_k(\alpha) \otimes \Pi_l(\beta) \otimes \mathrm{id}"] \\
\mathbb{Z}[k] \otimes C_*X \otimes \mathbb{Z}[l] \otimes C_*Y \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[dd, "\eta \otimes \eta"'] & \mathbb{Z}[k] \otimes \mathbb{Z}[l] \otimes C_*X \otimes C_*Y \arrow[r, "\mathrm{id} \otimes \mathrm{id} \otimes \Upsilon"] \arrow[d, "\eta \otimes \mathrm{id} \otimes \mathrm{id}"] & \mathbb{Z}[k] \otimes \mathbb{Z}[l] \otimes C_*(X \times Y) \arrow[d, "\eta \otimes \mathrm{id}"] \\
 & \mathbb{Z}[k+l] \otimes (C_*X \otimes C_*Y) \arrow[r, "\mathrm{id} \otimes \Upsilon"] \arrow[d, "\eta"] & \mathbb{Z}[k+l] \otimes C_*(X \times Y) \arrow[d, "\eta"] \\
C_*X[k] \otimes C_*Y[l] \arrow[r, "\Psi_{k,l}"] & (C_*X \otimes C_*Y)[k+l] \arrow[r, "\Upsilon[k+l]"] & C_*(X \times Y)[k+l].
\end{tikzcd}$$

Claim. The above diagram commutes up to chain homotopies.

To prove the claim we consider the various smaller squares and rectangles.

(1) The two paths along the top rectangle from the top-left corner to the bottom-right corner are both natural chain maps. Both have the property that for $x \in X$ and $y \in Y$ the element $x \otimes y \in C_*(X) \otimes C_*(Y)$ gets sent to $x \otimes y \otimes (x, y)$ in $C_*(X) \otimes C_*(Y) \otimes C_*(X \times Y)$. In particular both paths agree on the 0-level.

It follows from Acyclic Model Theorem 56.16 (2) that the rectangle commutes up to a chain homotopy. More precisely, we apply the Acyclic Model Theorem 56.16 as follows:

(a) We take $\mathcal{C}$ to be the category of tuples (X, Y) of topological spaces.
(b) We consider the two functors $F\colon (X,Y) \mapsto (C_*(X) \otimes C_*(Y), \epsilon_X \otimes \epsilon_Y)$ and $G\colon (X,Y) \mapsto (C_*(X \times Y), \epsilon_{X\times Y})$.
(c) We take $\mathcal{M} = \{\Delta^k \times \Delta^l\}_{k,l\in\mathbb{N}_0}$.
(d) We consider the two natural transformations that are given by the two extremal paths in the diagram (shifted downward in degrees).

(2) It is clear that the two squares between the second and the third row commute.
(3) It is straightforward to verify that the two squares at the bottom right commute.
(4) It remains to consider the big square to the bottom left of the diagram. Starting with $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$, $b \in C_m(X)$, $c \in (\mathbb{Z}[l])_l = \mathbb{Z}$ and $d \in C_n(Y)$ and keeping in mind the sign conventions of Lemmas 60.7, 60.9 and 60.14 we get the following results for $a \otimes b \otimes c \otimes d$:
   (a) going down-right gives $(-1)^{km} \cdot (-1)^{ln} \cdot (-1)^{k(l+n)} \cdot ac \cdot (b \otimes d)$,
   (b) going right-down-down gives $(-1)^{ml} \cdot (-1)^{kl} \cdot (-1)^{(k+l)(m+n)} \cdot ac \cdot (b \otimes d)$.
   One easily verifies that the signs agree. So we see that the bottom left square also commutes.

This concludes the proof of the claim.

In light of the previous claim it suffices to prove the following claim.

Claim.

(1) the "left-bottom route" equals the chain map $\Upsilon[k+l] \circ \Psi_{k,l} \circ ((\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -))$,
(2) the "top-right route" is chain homotopic to $(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon$.

We start out with the proof of statement (1). First we note that

$$\begin{aligned}(\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -) &\underset{\substack{\uparrow \\ \text{by Proposition 60.12}}}{=} (\eta \circ (\Pi_k(\alpha) \otimes \mathrm{id}) \circ \Delta_X) \otimes (\eta \circ (\Pi_l(\beta) \otimes \mathrm{id}) \circ \Delta_Y) \\ &\underset{\substack{\uparrow \\ \text{follows from the observation that for chain maps } a,b,c \text{ and } d \\ \text{we have } (a\circ b)\otimes(c\circ d) = (a\otimes b)\circ(c\otimes d)}}{=} (\eta \otimes \eta) \circ ((\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id})) \circ (\Delta_X \otimes \Delta_Y).\end{aligned}$$

If we compose this equality with $\Upsilon[k+l] \circ \Psi_{k,l}$ then we obtain the "bottom left route" in the above diagram. This concludes the proof of statement (1).

We continue with the proof of statement (2). In that proof we will make use of the following subclaim.

Subclaim. The two cocycles[1035]

$$(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \quad \text{and} \quad (-1)^{kl} \cdot p^*(\alpha) \cup q^*(\beta)$$

in $C^{k+l}(X \times Y; \mathbb{Z})$ are cohomologous.

We consider the following diagram

$$\begin{tikzcd}
C_{k+l}(X \times Y) \arrow[r, hook] & C_*(X \times Y) \arrow[r, "\Theta"] \arrow[d, "\Delta_{X \times Y}"] & C_*(X) \otimes C_*(Y) \arrow[d, "\Pi_k(\alpha) \otimes \Pi_l(\beta)"] & \\
& C_*(X \times Y) \otimes C_*(X \times Y) \arrow[ur, "p_* \otimes q_*"] \arrow[r, "\Pi_k(p^*(\alpha)) \otimes \Pi_l(q^*(\beta))"'] & \mathbb{Z}[k] \otimes \mathbb{Z}[l] \arrow[r, "\eta"'] & \mathbb{Z}[k+l].
\end{tikzcd}$$

Note that by the above discussion all maps except for the top left map are chain maps. It follows immediately from the definitions that the bottom triangle of chain maps commutes "on the nose". The top triangle of chain maps commutes, up to a chain homotopy, by Theorem 56.5.[1036] It follows from Lemma 60.8 (3) that the restrictions of these two chain maps to maps $C_{k+l}(X \times Y) \to \mathbb{Z} = \mathbb{Z}[k+l]$ are cohomologous cochains. But the restriction of the "top route" equals evidently the left-hand side of the subclaim. On the other hand, by Lemma 60.10 the restriction of the "bottom route" equals the right-hand side of the subclaim. This concludes the proof of the subclaim.

[1035]Recall that $\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta$ is a chain map $C_*(X \times Y) \to \mathbb{Z}[k+l]$. Furthermore recall that the map $(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \colon C_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ is the $(k+l)$-th map of the chain map.

[1036]More precisely, given $x \in X$ and $y \in Y$ both routes along the triangle send (x, y) to $x \otimes y$. Furthermore both routes are natural chain maps. Thus we can apply Theorem 56.5.

Now we see that

$$
\begin{aligned}
(-1)^{kl}\cdot((p^*\alpha\cup q^*\beta)\,\widetilde{\cap}\,-)\circ\Upsilon \;&\overset{\substack{\text{the above subclaim and Lemma 60.11 (2)}\\ \downarrow}}{\simeq}\; \big((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}\,\widetilde{\cap}\,-\big)\circ\Upsilon\\
&= \eta\circ\big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l})\otimes\mathrm{id}\big)\circ\Delta_{X\times Y}\circ\Upsilon\\
&\quad\uparrow\\
&\quad\text{Proposition 60.12 applied to the } (k+l)\text{-cocycle } (\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}\\
&= \eta\circ\big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l})\circ\Theta\otimes\mathrm{id}\big)\circ\Delta_{X\times Y}\circ\Upsilon\\
&\quad\uparrow\\
&\quad\text{since for every chain map } \Omega\colon C_*(X)\times C_*(Y)\to\mathbb{Z}[k+l] \text{ we}\\
&\quad\text{have } \Pi_{k+l}((\Omega\circ\Theta)_{k+l}) = \Pi_{k+l}(\Omega_{k+l})\circ\Theta \text{ as maps } C_*(X\times Y)\to\mathbb{Z}[k+l],\\
&\quad\text{applies this observationc to } \Omega=\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\\
&= \eta\circ((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)\otimes\mathrm{id})\circ\Delta_{X\times Y}\circ\Upsilon\\
&\quad\uparrow\\
&\quad\text{here we use the equality of chain maps}\\
&\qquad\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l}) = \eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\colon C_*(X)\otimes C_*(Y)\to\mathbb{Z}[k+l]\\
&\quad\text{which is obvious once one unravels the definitions}\\
&= \text{the “top right route” in the above diagram.}\\
&\quad\uparrow\\
&\quad\text{follows from } (f\otimes\mathrm{id})\circ(g\otimes\mathrm{id}) = (g\circ f)\otimes\mathrm{id}
\end{aligned}
$$

This concludes the proof of the claim and thus also of the theorem. □

Now we can finally provide the proof of Theorem 60.1.

Proof of Theorem 60.1. To simplify the notation we only consider the case $A = B = \varnothing$. For the general case one needs to take more care, we refer to [Dol, p. 240 and p. 244] for details. So let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projection maps. Let $[\alpha]\in H^k(X;\mathbb{Z})$, $[\beta]\in H^l(Y;\mathbb{Z})$ and let $[\mu]\in H_m(X;\mathbb{Z})$ and $[\nu]\in H_n(Y;\mathbb{Z})$. We have to show that

$$
\big(\underbrace{[p^*(\alpha)]}_{\in H^k(X\times Y)}\cup\underbrace{[q^*(\beta)]}_{\in H^l(X\times Y)}\big)\cap\underbrace{\big([\mu]\times[\nu]\big)}_{\in H_{m+n}(X\times Y)} = (-1)^{kl+lm}\cdot\underbrace{([\alpha]\cap[\mu])}_{\in H_{m-k}(X)}\times\underbrace{([\beta]\cap[\nu])}_{\in H_{n-l}(Y)}
$$

We have the following equality in $H_{m+n-k-l}(X \times Y)$:

$$\begin{aligned}
(p^*([\alpha]) \cup q^*([\beta])) \cap ([\mu] \times [\nu]) &\overset{\substack{\text{Tautology 60.13 and the definition of } \widetilde{\cap}\text{, see page 1394}\\ \downarrow}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ([p^*\alpha \cup q^*\beta]) \,\widetilde{\cap}\, [\Upsilon(\mu \otimes \nu)] \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (((p^*\alpha \cup q^*\beta) \,\widetilde{\cap}\, -) \circ \Upsilon)_*([\mu \otimes \nu]) \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot ((-1)^{kl} \cdot \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \,\widetilde{\cap}\, - \otimes \beta \,\widetilde{\cap}\, -))_*([\mu \otimes \nu]) \\
&\underset{\substack{\uparrow\\ \text{by Theorem 60.15 and the fact that chain homotopic maps}\\ \text{induce the same map on homology groups}}}{=} \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot \Big[\Upsilon[k+l] \circ \Psi_{k,l} \circ \big(\underbrace{\underbrace{\alpha \,\widetilde{\cap}\, \mu}_{=(-1)^{k(m-k)} \cdot \alpha \cap \mu}}_{\in C_{m-k}(X)[k]} \otimes \underbrace{\underbrace{\beta \,\widetilde{\cap}\, \nu}_{=(-1)^{l(n-l)} \cdot \beta \cap \nu}}_{\in C_{n-l}(Y)[l]} \big)\Big] \\
&\underset{\substack{\uparrow\\ \text{definition of } \Psi_{k,l}\text{, see page 1396, and of the shifting from } \Upsilon \text{ to } \Upsilon[k+l]}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot (-1)^{k(l+n-l)} \cdot (-1)^{k(m-k)} \cdot (-1)^{l(n-l)} \cdot \big[\Upsilon(\alpha \cap \mu \otimes \beta \cap \nu)\big] \\
&\underset{\substack{\uparrow\\ \text{cleaning up signs and Tautology 60.13}}}{=} (-1)^{kl+lm} \cdot [\alpha \cap \mu \times \beta \cap \nu] = (-1)^{kl+lm} \cdot ([\alpha] \cap [\mu] \times [\beta] \cap [\nu]).
\end{aligned}$$

□

60.3. **The Künneth Theorem for cohomology groups.** Given two topological spaces X and Y the Künneth Theorem 56.8 for topological spaces determines the homology groups of $X \times Y$ in terms of the homology groups of X and Y. The following proposition gives us, under some mild hypothesis, a similar answer for cohomology groups.

Theorem 60.16. (The Künneth Theorem for cohomology groups) *Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. If all the homology groups of X are finitely generated*[1037]*, then given any $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \to \bigoplus_{k+l=n} H^k(X;\mathbb{Z}) \otimes H^l(Y;\mathbb{Z}) \to H^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \operatorname{Tor}(H^k(X;\mathbb{Z}), H^l(Y;\mathbb{Z})) \to 0$$
$$\varphi \otimes \psi \mapsto p^*\varphi \cup q^*\psi$$

which is natural with respect to the topological spaces X and Y. If all the homology groups of Y are also finitely generated, then the short exact sequence splits (albeit not naturally).

Remark. Let X and Y be two topological spaces. In the literature the map

$$\begin{aligned} H^k(X;\mathbb{Z}) \times H^l(Y;\mathbb{Z}) &\to H^{k+l}(X \times Y;\mathbb{Z}) \\ (\varphi, \psi) &\mapsto p^*\varphi \cup q^*\psi \end{aligned}$$

is often referred to as the *cohomology cross product* and it is written as $\varphi \times \psi := p^*\varphi \cup q^*\psi$. We will not make use of this notation.

[1037] By this we mean that for every $k \in \mathbb{N}_0$ the abelian group $H_k(X;\mathbb{Z})$ is finitely generated.

The proof of the theorem rests on a purely algebraic theorem that, given two chain complexes $\mathcal{C}$ and $\mathcal{C}'$, relates the cohomology groups of $\mathcal{C}$ and $\mathcal{C}'$ to the cohomology groups of $\mathcal{C}\otimes\mathcal{C}'$. Before we can state the theorem we need to introduce one more definition, which is closely related to the definitions introduced on page 1332.

Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes and let $p,q\in\mathbb{N}_0$.

(1) We denote by

$$\Pi_{p,q}\colon (\mathcal{C}_*\otimes\mathcal{C}'_*)_{p+q}=\bigoplus_{i+j=p+q} C_i\otimes C'_j \;\to\; C_p\otimes C'_q$$

the obvious projection map.

(2) We denote by $\mu\colon \mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$ the obvious isomorphism given by $\mu(a\otimes b)=ab$.

(3) We consider the map:

$$\begin{aligned}\Xi_{p,q}\colon \operatorname{Hom}(C_p,\mathbb{Z})\otimes\operatorname{Hom}(C'_q,\mathbb{Z}) \;&\to\; \operatorname{Hom}\big((\mathcal{C}\otimes\mathcal{C}')_{p+q},\mathbb{Z}\big)\\ \varphi_p\otimes\varphi'_q \;&\mapsto\; \Bigg(\bigoplus_{i+j=p+q} C_i\otimes C'_j \xrightarrow{\Pi_{p,q}} C_p\otimes C'_q \xrightarrow{\varphi_p\otimes\varphi'_q} \mathbb{Z}\otimes\mathbb{Z}\xrightarrow{\mu}\mathbb{Z}\Bigg).\end{aligned}$$

(4) It is straightforward to verify that the map $\Xi_{p,q}$ induces a well-defined map on cohomology groups, i.e. that the map

$$\begin{aligned}\Xi_{p,q}\colon H^p(\mathcal{C};\mathbb{Z})\otimes H^q(\mathcal{C}';\mathbb{Z}) \;&\to\; H^{p+q}(\mathcal{C}\otimes\mathcal{C}';\mathbb{Z})\\ ([\varphi_p]\otimes[\varphi'_q]) \;&\mapsto\; [\Xi_{p,q}(\varphi_p\otimes\varphi'_q)]\end{aligned}$$

is well-defined. Furthermore it follows immediately from the definition that $\Xi_{p,q}$ is natural[1038] with respect to $\mathcal{C}$ and $\mathcal{C}'$.

Now we can state the promised purely algebraic theorem.

Theorem 60.17. (The Künneth Theorem for cochain complexes) *Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n\in\mathbb{N}_0$ there exists a short exact sequence*

$$0\to\bigoplus_{p+q=n} H^p(\mathcal{C};\mathbb{Z})\otimes H^q(\mathcal{C}';\mathbb{Z})\xrightarrow{\Xi_{p,q}} H^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{Z})\to\bigoplus_{p+q=n+1}\operatorname{Tor}(H^p(\mathcal{C};\mathbb{Z}),H^q(\mathcal{C};\mathbb{Z}))\to 0$$

which is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$. If all the chain groups of $\mathcal{C}'$ are also free abelian and all the homology groups of $\mathcal{C}'$ are also finitely generated, then the short exact sequence splits (albeit not naturally).

[1038] In this case this means that for chain maps $f\colon\mathcal{C}\to\mathcal{D}$ and $f'\colon\mathcal{C}'\to\mathcal{D}'$ the following diagram commutes:

$$\begin{array}{ccc} H^p(\mathcal{D};\mathbb{Z})\otimes H^q(\mathcal{D}';\mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & H^{p+q}(\mathcal{D}\otimes\mathcal{D}';\mathbb{Z})\\ \big\downarrow{\scriptstyle f^*\otimes(f')^*} & & \big\downarrow{\scriptstyle (f\otimes f')^*}\\ H^p(\mathcal{C};\mathbb{Z})\otimes H^q(\mathcal{C}';\mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & H^{p+q}(\mathcal{C}\otimes\mathcal{C}';\mathbb{Z}).\end{array}$$

In the proof of the Künneth Theorem 60.17 for cochain complexes we need one technical lemma. To state the lemma we recall that on page 1041 we introduced the tensor product of two chain complexes. Basically the same definition also allows us to define the tensor product of two cochain complexes. With this preamble we can formulate the next lemma.

Lemma 60.18. *Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. We assume that all chain groups of $\mathcal{C}$ are finitely generated free abelian groups. Then the map*

$$\begin{array}{rcl} \Omega\colon \operatorname{Hom}(\mathcal{C}_*,\mathbb{Z})\otimes\operatorname{Hom}(\mathcal{C}'_*,\mathbb{Z}) & \to & \operatorname{Hom}(\mathcal{C}_*\otimes\mathcal{C}'_*,\mathbb{Z}) \\ (\varphi_p\colon C_p\to\mathbb{Z})\otimes(\varphi'_q\colon C'_q\to\mathbb{Z}) & \mapsto & (\mu\circ(\varphi_p\otimes\varphi'_q)\circ\Pi_{p,q}\colon (\mathcal{C}_*\otimes\mathcal{C}'_*)_{p+q}\to\mathbb{Z}) \end{array}$$

is an isomorphism of cochain complexes.

Proof $(*)$. We start out the proof of the lemma with the following elementary well-known observation:

$(*)$ For finitely many abelian groups $A_1,\dots,A_k$ we have a canonical isomorphism

$$\operatorname{Hom}(A_1\oplus\cdots\oplus A_k,\mathbb{Z}) = \operatorname{Hom}(A_1,\mathbb{Z})\oplus\cdots\oplus\operatorname{Hom}(A_k,\mathbb{Z}).$$

Now we turn to the actual proof of the lemma. It is elementary to verify that the map Ω is in fact a cochain map. So it remains to show that for every $n\in\mathbb{N}_0$ the map

$$\begin{array}{rcl} \bigoplus_{p+q=n}\operatorname{Hom}(C_p,\mathbb{Z})\otimes\operatorname{Hom}(C'_q,\mathbb{Z}) & \to & \operatorname{Hom}\Big(\bigoplus_{p+q=n}C_p\otimes C'_q,\mathbb{Z}\Big) \\ \varphi_p\otimes\varphi'_q & \mapsto & \mu\circ(\varphi_p\otimes\varphi'_q)\circ\Pi_{p,q} \end{array}$$

is an isomorphism. Applying $(*)$ to the right-hand side we see that it suffices to prove the following claim.

Claim. For every choice of $p,q\in\mathbb{N}_0$ the map

$$\begin{array}{rcl} \operatorname{Hom}(C_p,\mathbb{Z})\otimes\operatorname{Hom}(C'_q,\mathbb{Z}) & \to & \operatorname{Hom}(C_p\otimes C'_q,\mathbb{Z}) \\ \varphi_p\otimes\varphi'_q & \mapsto & \mu\circ(\varphi_p\otimes\varphi'_q) \end{array}$$

is an isomorphism.

Let $p,q\in\mathbb{N}_0$. By our hypothesis on $\mathcal{C}_*$ the chain group C_p is a finitely generated free abelian group. Therefore we can find a basis $a_1,\dots,a_k$ for C_p. We denote by $\widehat{a}_1,\dots,\widehat{a}_k$ the dual basis of $\operatorname{Hom}(C_p,\mathbb{Z})$. We can thus write

$$\operatorname{Hom}(C_p,\mathbb{Z}) = \bigoplus_{i=1}^{k}\mathbb{Z}\cdot\widehat{a}_i \quad\text{and}\quad C_p\otimes C'_q = \bigoplus_{i=1}^{k}\mathbb{Z}\cdot a_i\otimes C'_q.$$

Applying $(*)$ to the above decompositions and both sides of the claim one sees easily that the given homomorphism is an isomorphism. This concludes the proof. □

Now we can finally prove the Künneth Theorem 60.17 for cochain complexes.

Proof of the Künneth Theorem 60.17 for cochain complexes. We only provide an outline of the proof. Let $\mathcal{C}=(C_*,c_*)$ and $\mathcal{C}'=(C'_*,c'_*)$ be chain complexes such that all the chain groups C_n are free abelian groups and such that all the homology groups $H_k(\mathcal{C};\mathbb{Z})$ are finitely generated.

The idea is that one can turn any cochain complex into a chain complex by reversing the signs in the degrees. Thus one can try to reduce the theorem to the Künneth Theorem 43.7. Towards the end of the proof we need to insert Lemma 60.18 to make everything work.

The proof proceeds in three steps:

(1) Since $\mathcal{C}$ is a free chain complex and since all homology groups of $\mathcal{C}$ are finitely generated we can use Lemma 49.8 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups.

(2) Given $n \in \mathbb{Z}_{\leq 0}$ we write $E_n := \operatorname{Hom}(C_{-n}, \mathbb{Z})$, $e_n := \partial^*_{-n+1} \colon E_n \to E_{n-1}$ and similarly we write $E'_n := \operatorname{Hom}(\mathcal{C}'_{-n}, \mathbb{Z})$ and $e'_n := \partial^*_{-n+1} \colon E'_n \to E'_{n-1}$. It follows immediately from the definitions that $\mathcal{E}_* := (E_*, e_*)$ and $\mathcal{E}'_* := (E'_*, e'_*)^*)$ form chain complexes in negative degrees. (We defined chain complexes to be given by chain groups indexed by $n \in \mathbb{N}_0$ but evidently the same theory also works for chain groups indexed by $\mathbb{Z}$ and $\mathbb{Z}_{\leq 0}$.) Since the groups E_n are free abelian we can apply the Künneth Theorem 43.7 (which also applies to chain complexes in non-positive degrees) to obtain for each $n \in \mathbb{Z}_{\leq 0}$ a natural short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus\limits_{p+q=n} H_p(\mathcal{E}) \otimes H_q(\mathcal{E}') & \to & H_n(\mathcal{E} \otimes \mathcal{E}') & \to & \bigoplus\limits_{p+q=n-1} \operatorname{Tor}(H_p(\mathcal{E}), H_q(\mathcal{E}')) & \to & 0. \\ & & [\varphi] \otimes [\psi] & : & & & & & \end{array}$$

Put differently, for each $n \in \mathbb{N}_0$ we have a natural short exact sequence

$$0 \to \bigoplus_{-p-q=n} \underbrace{H_{-p}(\mathcal{E})}_{=H^p(\mathcal{C};\mathbb{Z})} \otimes \underbrace{H_{-q}(\mathcal{E}')}_{=H^q(\mathcal{E}';\mathbb{Z})} \to H_{-n}(\mathcal{E} \otimes \mathcal{E}') \to \bigoplus_{-p-q=n+1} \operatorname{Tor}(\underbrace{H_{-p}(\mathcal{E})}_{=H^p(\mathcal{C};\mathbb{Z})}, \underbrace{H_{-q}(\mathcal{E}')}_{=H^p(\mathcal{C};\mathbb{Z})}) \to 0.$$

(3) For each $n \in \mathbb{N}_0$ we have isomorphisms

$$\begin{array}{ccccc} H_{-n}(\mathcal{E}_* \otimes \mathcal{E}'_*) & = & H^n(\mathcal{E}_{-*} \otimes \mathcal{E}'_{-*}) & = & H^n\big(\overbrace{\operatorname{Hom}(\mathcal{C}_*, \mathbb{Z})}^{=\mathcal{E}_{-*}} \otimes \overbrace{\operatorname{Hom}(\mathcal{C}'_*, \mathbb{Z})}^{=\mathcal{E}'_{-*}}\big) \xrightarrow{\Omega} H^n(\mathcal{C}_* \otimes \mathcal{C}'_*; \mathbb{Z}). \\ & \uparrow & & \uparrow & \\ & \text{by definition} & & \text{by definition} & \text{by Lemma 60.18 and our hypothesis on } \mathcal{C}_* \end{array}$$

We use this isomorphism to replace the middle term of the second short exact sequence in (2) and we obtain the desired short exact sequence.

Finally suppose that all the chain groups of $\mathcal{C}'$ are also free abelian and that all the homology groups of $\mathcal{C}'$ are also finitely generated. As in (1) we can use Lemma 49.8 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups. It follows that $\mathcal{E}_*$ is a free chain complex which implies, by the last statement of the Künneth Theorem 43.7, that the short exact sequence splits. □

Now we are in a position to prove the Künneth Theorem 60.16 for cohomology groups.

Proof of the Künneth Theorem 60.16 for cohomology groups $(*)$. Let X and Y be two topological spaces such that all the homology groups of X are finitely generated. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the obvious projection maps. Throughout the proof we use all the definitions that we introduced in Section 60.2.

It follows from the Eilenberg-Zilber Theorem 56.1, together with comparing the conclusion of the Künneth Theorem 60.17 for cochain complexes with the desired statement of the Künneth Theorem 60.16 for cohomology groups, that it remains to prove the following claim.

Claim. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} H^k(X;\mathbb{Z}) \otimes H^l(Y;\mathbb{Z}) & \to & H^{k+l}(X \times Y;\mathbb{Z}) \\ [\alpha] \otimes [\beta] & \mapsto & [p^*\alpha \cup q^*\beta] \end{array}$$

agrees with the map that is given by

$$[\alpha] \otimes [\beta] \quad \mapsto \quad [(\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Theta].$$

Let $\alpha \in C^k(X;\mathbb{Z})$ and $\beta \in C^l(Y;\mathbb{Z})$ be cocycles. We have to show that the two cocycles

$$\underbrace{p^*\alpha \cup q^*\beta}_{=:\aleph} \quad \text{and} \quad \underbrace{(\alpha \otimes \beta) \circ \Pi_{k,l}}_{=:\beth} \circ \Theta$$

are cohomologous.

As usual, given a topological space W we denote by $\epsilon_W \colon C_0(W) \to \mathbb{Z}$ the augmentation map. Clearly ϵ_W is a cocycle so by Lemma 60.8 (2) we can extend ϵ_W to a chain map $C_*(W) \to \mathbb{Z}$ which, by a slight abuse of notation, we also denote by ϵ_W. It follows from the Eilenberg-Zilber Theorem 56.1, Theorem 60.15 and Lemma 60.8 (2) that the maps

$$\underbrace{(-1)^{kl} \cdot \epsilon_{X\times Y}[k+l] \circ ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -)}_{=:\aleph'} \colon C_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

and

$$\underbrace{\epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)}_{=:\beth'} \circ \Theta \colon C_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

are cocycles that are cohomologous. It remains to prove the following two subclaims.

Subclaim. We have

$$\aleph' = (-1)^{kl} \cdot \aleph \colon C_{k+l}(X \times Y) \ \to \ \mathbb{Z}.$$

Let $\sigma \in C_{k+l}(X \times Y)$. Then

$$\aleph'(\sigma) \ \underset{\substack{\uparrow \\ \text{under the identification } \mathbb{Z}[k+l] = \mathbb{Z} \text{ we} \\ \text{have } \epsilon_{X\times Y}[k+l] = \epsilon_{X\times Y}}}{=} \ (-1)^{kl} \cdot \epsilon_{X\times Y}((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} \sigma) \ \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\cap}\text{, see page 1394,} \\ \text{together with Lemma 59.4 (1)}}}{=} \ (-1)^{kl} \cdot (p^*\alpha \cup q^*\beta)(\sigma) \ = \ (-1)^{kl} \cdot \aleph(\sigma).$$

This concludes the proof of the subclaim.

Subclaim. We have

$$\beth' = (-1)^{kl} \cdot \beth \colon (C_*(X) \otimes C_*(Y))_{k+l} \ \to \ \mathbb{Z}.$$

We have to show that both sides take the same value on any tensor product of the form $\sigma_i \otimes \sigma_j$ with $i+j=k+l$ and $\sigma_i \in C_i(X)$ and let $\sigma_j \in C_j(Y)$. It follows easily from the definition that both sides are zero for $\sigma_i \otimes \sigma_j$ with $(i,j) \neq (k,l)$. So let $\sigma_k \in C_k(X)$ and let $\sigma_l \in C_l(Y)$. Then

$$\begin{aligned}
\beth'(\sigma_k \otimes \sigma_l) &= (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} \sigma_k \otimes \beta \mathbin{\widetilde{\cap}} \sigma_l) \\
&\quad \uparrow \\
&\quad \text{since } \epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] = (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \\
&= (-1)^{kl} \cdot (\epsilon_X \otimes \epsilon_Y) \circ (\alpha \cap \sigma_k \otimes \beta \cap \sigma_l) && = (-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l) \\
&\quad \uparrow && \uparrow \\
&\quad \text{we use } \epsilon_{X\times Y} \circ \Upsilon = \epsilon_X \otimes \epsilon_Y \colon (C_*(X) \otimes C_*(Y))_0 \to \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}, && \text{by Lemma 59.4} \\
&\quad \text{the definition of } \Psi_{k,l} \text{ in Lemma 60.14 and the definition of } \widetilde{\cap} \text{ on page 1394} \\
&= (-1)^{kl} \cdot ((\alpha \otimes \beta) \circ \Pi_{k,l})(\sigma_k \otimes \sigma_l) && = (-1)^{kl} \cdot \beth(\sigma_k \otimes \sigma_l).
\end{aligned}$$

This concludes the proof of the second subclaim, thus also of the claim and therefore also of the theorem. □

60.4. **The cohomology ring of product topological spaces.** In this section we will determine the cohomology ring of the m-dimensional torus $(S^1)^m$ and more generally, under some mild technical hypothesis, we will determine the cohomology ring of a product $X \times Y$ of topological spaces in terms of the cohomology rings of X and Y.

We recall the following definition.

Definition.

(1) A *graded ring* is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m,n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$. We say that an element r of R that lies in some R_n is *homogeneous* and we refer to $\deg(r) := n$ as the degree of r.

(2) A homomorphism $\varphi\colon A \to B$ between graded rings is a ring homomorphism that preserves the gradings, i.e. that satisfies $\varphi(A_n) \subset B_n$ for all $n \in \mathbb{N}_0$.

(3) A graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} \cdot b \cdot a$ holds is called a *superalgebra*.

Examples.

(1) Proposition 57.8 says in particular that for any topological space X the cohomology ring $(H^*(X;\mathbb{Z}), \cup)$ is a superalgebra. Furthermore Lemma 57.10 implies that for any map $f\colon X \to Y$ the induced map $f^*\colon H^*(Y;\mathbb{Z}) \to H^*(X;\mathbb{Z})$ is a homomorphism of superalgebras.

(2) The polynomial ring $\mathbb{F}_2[a] = \bigoplus_{i\in\mathbb{N}_0} \mathbb{F}_2 \cdot a^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(a^i) := i$.

(3) The polynomial ring $\mathbb{Z}[x] = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z} \cdot x^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(x^i) := 2i$.

(4) More generally, the polynomial ring $\mathbb{Z}[x_1, \dots, x_k]$ in k variables $x_1, \dots, x_k$ can be viewed as a superalgebra by setting $\deg(x_1^{r_1} \cdot \dots \cdot x_k^{r_k}) = 2(r_1 + \dots + r_k)$.

(5) The ring $\mathbb{Z}[x]/(x^2) = \mathbb{Z} \oplus \mathbb{Z} \cdot x$ is a superalgebra by viewing elements in $\mathbb{Z}$ as elements of degree 0 and by viewing elements in $\mathbb{Z} \cdot x$ as elements of degree 1. Note that the map

$$\begin{array}{rcl} \mathbb{Z}[x]/(x^2) & \to & H^*(S^1; \mathbb{Z}) \\ a \cdot 1 + b \cdot x & \mapsto & a \cdot 1_{S^1} + b \cdot [S^1]^* \end{array}$$

is an isomorphism of superalgebras.

(6) We consider the ring

$$\Lambda[x_1, \dots, x_k] := \mathbb{Z}[x_1, \dots, x_k]/\text{ideal generated by } x_1^2, \dots, x_k^2, \{x_i x_j + x_j x_i\}_{i,j=1,\dots,k}.$$

In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". For example for any $i, j \in \{1, \dots, k\}$ we have $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. This ring can be viewed in a unique way as a superalgebra such that $\deg(x_j) = 1$ for $j = 1, \dots, k$. The ring $\Lambda[x_1, \dots, x_k]$ is often referred to as the *exterior algebra on generators* $x_1, \dots, x_k$, see e.g. [Hat, p. 217] and [Mac, p. 183].[1039] We leave it as an amusing exercise to verify that $\Lambda[x_1, \dots, x_k]$ is a free abelian group of rank 2^k where a basis is given by

$$\{x_{n_1} \wedge \dots \wedge x_{n_l} \,|\, n_1 < \dots < n_l\}.$$

In Exercise 51.2 we saw that any sequence $\{A_n\}_{n \geq 1}$ of finitely generated abelian groups, with A_1 torsion-free, can appear as the sequence of cohomology groups $H^*(X; \mathbb{Z})$ of some CW-complex. It is natural to ask whether any superalgebra structure can be realized as the cohomology ring of a topological space. More precisely, we have the following question, to which we will return later.

Question 60.19. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be superalgebra such that ${}_0 \cong \mathbb{Z}$, R_1 is torsion-free and such that all R_n are finitely generated. Does there exist a topological space X such that $(H^*(X; \mathbb{Z}), \cup)$ is isomorphic to R?*

Since there is not much to say right now about this question we continue with the next definition.

[1039] If we replace the $\mathbb{Z}$-coefficients by $\mathbb{R}$-coefficients we obtain a ring that we denote by $\Lambda_{\mathbb{R}}[x_1, \dots, x_k]$. This ring is isomorphic to the ring $\wedge^*(\mathbb{R}^k)^* = \bigoplus_{j=0}^{k} \wedge^j (\mathbb{R}^k)^*$ of alternating forms on $\mathbb{R}^k$, where the ring structure on $\wedge^*(\mathbb{R}^k)^*$ is given by the wedge product of alternating forms. More precisely a ring isomorphism is given by

$$\begin{array}{rcl} \Lambda_{\mathbb{R}}[x_1, \dots, x_k] & \to & \wedge^*(\mathbb{R}^k)^* \\ x_{i_1} \wedge \dots \wedge x_{i_r} & \to & dy_{i_1} \wedge \dots \wedge dy_{i_r} \end{array}$$

where $dy_i \colon \mathbb{R}^k \to \mathbb{R}$ is the projection onto the i-th coordinate.

Definition. The tensor product of two superalgebras $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ and $B = \bigoplus_{n\in\mathbb{N}_0} B_n$ is defined as the graded abelian group

$$A \otimes B \;:=\; \bigoplus_{n\in\mathbb{N}_0} \underbrace{\bigoplus_{p+q=n} A_p \otimes B_q}_{=:(A\otimes B)_n}$$

where for $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$ the multiplication is given by

$$(a_k \otimes b_l) \cdot (a_m \otimes b_n) \;=\; (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n.$$

Lemma 60.20.

(1) *The tensor product of two superalgebras is again a superalgebra.*
(2) *The tensor product of superalgebras is associative, i.e. for any three superalgebras A, B and C the map*

$$(A \otimes B) \otimes C \;\to\; A \otimes (B \otimes C)$$

that is induced by

$$(a \otimes b) \otimes c \;\mapsto\; a \otimes (b \otimes c)$$

is an isomorphism of superalgebras.

It follows immediately from Lemma 60.20 (2) that we can ignore parentheses when we consider the tensor product of several superalgebras.

Proof $(*)$. Let A, B and C be superalgebras and let $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$.

(1) We have the following equalities:

$$\begin{aligned}(a_k \otimes b_l)\cdot(a_m \otimes b_n) \;&=\; (-1)^{lm}\cdot a_k\cdot a_m \otimes b_l\cdot b_n \;=\; (-1)^{lm+km+ln}\cdot a_m\cdot a_k \otimes b_n\cdot b_l\\ &=\; (-1)^{lm+km+ln+kn}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l)\\ &=\; (-1)^{(m+n)(k+l)}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l)\\ &=\; (-1)^{\deg(a_m\otimes b_n)\cdot\deg(a_k\otimes b_l)}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l).\end{aligned}$$

It follows immediately that $A \otimes B$ is again a superalgebra.

(2) It suffices to show that the indicated map is a homomorphism of superalgebras. Thus let $c_r \in C_r$ and $c_s \in C_s$. Then

$$\begin{aligned}((a_k \otimes b_l) \otimes c_r)\cdot((a_m \otimes b_n) \otimes c_s) \;&=\; (-1)^{r(m+n)}\cdot((a_k \otimes b_l)\cdot(a_m \otimes b_n)) \otimes (c_r\cdot c_s)\\ &=\; (-1)^{r(m+n)+lm}\cdot(a_k\cdot a_m \otimes b_l\cdot b_n \otimes c_r\cdot c_s)\\ &=\; (-1)^{m(l+r)}\cdot(a_k\cdot a_m \otimes (b_l \otimes c_r)\cdot(b_n \otimes c_s))\\ &=\; (a_k \otimes (b_l \otimes c_r))\cdot(a_m \otimes (b_n \otimes c_s)).\end{aligned}$$

□

Examples.

(1) As above we view the polynomial rings $\mathbb{F}_2[a]$ and $\mathbb{F}_2[b]$ as superalgebras by setting $\deg(a^i) = \deg(b^i) = i$. We consider the map

$$\begin{aligned}\mathbb{F}_2[a] \otimes \mathbb{F}_2[b] \;&\to\; \mathbb{F}_2[a,b]\\ a^r \otimes b^s \;&\mapsto\; a^r \cdot b^s.\end{aligned}$$

Using Lemma 42.3 (1) it is straightforward to show that this map is an isomorphism of abelian groups. One can easily verify that the map is a homomorphism of superalgebras. Summarizing we have convinced ourselves that this map is in fact an isomorphism of superalgebras.

(2) As before we view the polynomial rings $\mathbb{Z}[x_j]$, $j = 1, \dots, k$ as superalgebras with $\deg(x_j^i) = 2i$. We consider the map

$$\begin{array}{rcl} \mathbb{Z}[x_1] \otimes \cdots \otimes \mathbb{Z}[x_k] & \to & \mathbb{Z}[x_1, \dots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \cdot \dots \cdot x_k^{r_k}. \end{array}$$

The same argument as in (1) shows that this map is an isomorphism of superalgebras.

(3) As above we view the rings $\mathbb{Z}[x_j]/(x_j^2)$ as superalgebras with $\deg(x_j) = 1$. We leave it as an exercise to verify that the map

$$\begin{array}{rcl} \mathbb{Z}[x_1]/(x_1^2) \otimes \cdots \otimes \mathbb{Z}[x_k]/(x_k^2) & \to & \Lambda[x_1, \dots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \wedge \cdots \wedge x_k^{r_k} \end{array}$$

is an isomorphism of superalgebras.

The following proposition explains our sudden interest in tensor products of superalgebras.

Proposition 60.21. *Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps.*

(1) *The map*

$$\begin{array}{rcl} \Phi\colon H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) & \to & H^*(X \times Y;\mathbb{Z}) \\ \varphi \otimes \psi & \mapsto & p^*\varphi \cup q^*\psi \end{array}$$

is a homomorphism of superalgebras.

(2) *If all the homology groups of X are finitely generated and torsion-free, then the homomorphism from* (1) *is in fact an isomorphism.*

Proof. Let X and Y be two topological spaces.

(1) Let $\varphi_k \in H^k(X;\mathbb{Z})$, $\psi_l \in H^l(Y;\mathbb{Z})$, $\varphi_m \in H^m(X;\mathbb{Z})$ and $\psi_n \in H^n(Y;\mathbb{Z})$. Then

$$\begin{array}{rcl} \overbrace{\Phi\big((-1)^{lm}\cdot(\varphi_k \cup \varphi_m) \otimes (\psi_l \cup \psi_n)\big)}^{=\Phi((\varphi_k\otimes\psi_l)\cdot(\varphi_m\otimes\psi_n))} & & = (-1)^{lm}\cdot p^*(\varphi_k \cup \varphi_m) \cup q^*(\psi_l \cup \psi_n) \\ = (-1)^{lm}\cdot(p^*(\varphi_k) \cup p^*(\varphi_m) \cup q^*(\psi_l) \cup q^*(\psi_n)) & & = \underbrace{p^*(\varphi_k) \cup q^*(\psi_l) \cup p^*(\varphi_m) \cup q^*(\psi_n)}_{=\Phi(\varphi_k\otimes\psi_l)\cup\Phi(\varphi_m\otimes\psi_n))}. \\ \uparrow & & \uparrow \\ \text{Proposition 57.8} & & \text{Lemma 57.10} \end{array}$$

It follows immediately that Φ is a homomorphism of superalgebras.

(2) Now suppose that all the homology groups of X are finitely generated and torsion-free. This hypothesis allows us to apply the Künneth Theorem 60.16 for cohomology

groups and we obtain the following short exact sequence:

$$0 \to \bigoplus_{k+l=n} H^k(X;\mathbb{Z}) \otimes H^l(Y;\mathbb{Z}) \to H^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \operatorname{Tor}(H^k(X;\mathbb{Z}), H^l(Y;\mathbb{Z})) \to 0.$$
$$\varphi \otimes \psi \mapsto p^*\varphi \cup q^*\psi$$

Since the homology groups of X are torsion-free we obtain from the Universal Coefficient Theorem 51.13 for Cohomology Groups that the cohomology groups $H^k(X;\mathbb{Z})$ are also torsion-free. But by Lemma 42.17 this implies that all the Tor-terms on the right-hand side of the short exact sequence vanish. □

The following proposition in particular provides an answer to Question 58.11 (1).

Proposition 60.22. *Let $n \in \mathbb{N}_0$. We denote by $T = (S^1)^n$ the n-dimensional torus. For $i = 1, \dots, n$ we denote by $p_i \colon T \to S^1$ the projection onto the i-th factor. Then there exists a unique isomorphism*

$$\psi \colon \Lambda[x_1, \dots, x_n] \quad \to \quad H^*(T;\mathbb{Z})$$

of superalgebras which has the property that for each $i \in \{1, \dots, n\}$ we have

$$\psi(x_i) \quad = \quad p_i^*([S^1]^*).$$

Proof. We consider the following isomorphism of superalgebras

$$\begin{array}{ccccccc} \Lambda[x_1, \dots, x_n] & \xleftarrow{\cong} & \mathbb{Z}[x_1]/(x_1^2) \otimes \dots \otimes \mathbb{Z}[x_k]/(x_k^2) & \xrightarrow{\cong} & H^*(S^1;\mathbb{Z}) \otimes \dots \otimes H^*(S^1;\mathbb{Z}) & \xrightarrow{\cong} & H^*((S^1)^n;\mathbb{Z}). \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{isomorphism from page 1409} & & \text{isomorphism from page 1407} & & \text{Proposition 60.21} & & \end{array}$$

It is clear from the definitions of the various maps that the map from left to right sends each x_i to $p_i^*([S^1]^*)$. The composition of the isomorphisms thus gives us the desired isomorphism. The uniqueness follows immediately from the fact that the x_i generate $\Lambda[x_1, \dots, x_n]$ as a ring. □

The following corollary gives in particular a negative answer to Question 58.9.

Corollary 60.23. *For the n-dimensional torus $T = (S^1)^n$ the Lusternik-Schnirelmann category* $\operatorname{cat}(T)$ *equals n.*

Proof. On page 1367 we showed that $\operatorname{cat}(T) \leq n$. In the following we will show that the reverse inequality holds. Let $\psi \colon \Lambda[x_1, \dots, x_n] \to H^*(T;\mathbb{Z})$ be the isomorphism of graded rings provided by Proposition 60.22. By the discussion on page 1407 we know that $x_1 \wedge \dots \wedge x_n \neq 0$. Since ψ is an isomorphism we deduce that $\psi(x_1) \cup \dots \cup \psi(x_n) \neq 0$, i.e. the cup length $c(T)$ of T is at least n. We know from Proposition 58.10 that $\operatorname{cat}(T) \geq c(T)$. Thus we obtain that $\operatorname{cat}(T) \geq n$. □

We conclude this chapter with the following variation of Proposition 60.21.

Proposition 60.24. *Let X and Y be two topological spaces. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the obvious projection maps. If all the homology groups of X are finitely*

generated, then the map

$$\begin{array}{rcl} \Phi\colon H^*(X;\mathbb{F}_2)\otimes H^*(Y;\mathbb{F}_2) & \to & H^*(X\times Y;\mathbb{F}_2) \\ \varphi\otimes\psi & \mapsto & p^*\varphi\cup q^*\psi \end{array}$$

is an isomorphism of superalgebras.

In the proof of Proposition 60.24 we will need the following theorem which is a variation of the Künneth Theorem 60.17 for cochain complexes.

Theorem 60.25. *Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n\in\mathbb{N}_0$ there exists a natural isomorphism*

$$\bigoplus_{k+l=n} H^k(\mathcal{C};\mathbb{F}_2)\otimes H^l(\mathcal{C}';\mathbb{F}_2) \xrightarrow{\Xi} H^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{F}_2).$$

The proof of Theorem 60.25 is evidently very similar to the proof of the Künneth Theorem 60.17 for cochain complexes. We refer to [Mun, Theorem 60.6] for a proof.

Proof of Proposition 60.24. The proof is almost the same as the proof of Proposition 60.21. More precisely, we first note that if in the proof of the Künneth Theorem 60.16 for cohomology groups we replace the Künneth Theorem 60.17 for cochain complexes by Theorem 60.25 we obtain that the map

$$\begin{array}{rcl} \bigoplus_{k+l=n} H^k(X;\mathbb{F}_2)\otimes H^l(Y;\mathbb{F}_2) & \to & H^n(X\times Y;\mathbb{F}_2) \\ \varphi\otimes\psi & \mapsto & p^*\varphi\cup q^*\psi \end{array}$$

is an isomorphism. But the same argument as in the proof of Proposition 60.21 shows that this map is in fact a homomorphism of superalgebras. □

Exercises for Chapter 60.

Exercise 60.1. Let X be a topological space. Recall that Proposition 57.8 says that for any $a\in H^k(X;\mathbb{Z})$ and $b\in H^l(X;\mathbb{Z})$ we have

$$a\cup b \;=\; (-1)^{kl}\cdot b\cup a.$$

Provide a proof of this statement using the original definition of the cup product from page 1333 and using Lemma 60.7.

Exercise 60.2. Let $g\in\mathbb{N}$ and denote by Σ_g the surface of genus g. Show that the Lusternik-Schnirelmann category of $S^1\times\Sigma_g$ is greater or equal than three.

Exercise 60.3. Let Σ be the surface of genus 2, let $q\colon\Sigma\times S^1\to\Sigma$ be the projection map and let $p\colon\Sigma\to S^1$ be the map shown in Figure 742. Determine the cap product

$$q(p^*([S^1]^*))\cap[\Sigma\times S^1] \;\in\; H_2(\Sigma\times S^1;\mathbb{Z})$$

up to sign.

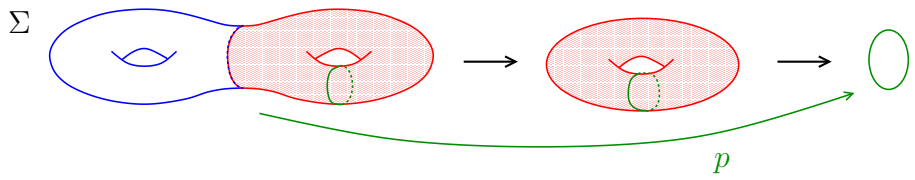


FIGURE 742.

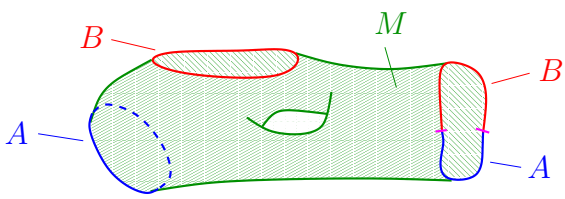
61. Poincaré-duality

61.1. **The Poincaré Duality Theorem.** The following theorem is by far the most important theorem regarding homology and cohomology groups of topological manifolds.[1040]

Theorem 61.1. (Poincaré Duality Theorem) *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in H_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map*[1041]

$$\begin{array}{rcl} H^k(M, A; R) & \xrightarrow{\cong} & H_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.



for all commutative rings R we have
$H^k(M, A; R) \xrightarrow{\cong} H_{2-k}(M, B; R)$

Figure 743. Illustration of the Poincaré Duality Theorem 61.1.

We proceed as follows: in this section and the next section we discuss several special cases and applications of the Poincaré Duality Theorem 61.1. In the subsequent section we will first provide a "pictorial" proof for closed manifolds and $\mathbb{F}_2$-coefficients. In the last two sections of the chapter we will then provide a rigorous, but arguably less intuitive proof, of the theorem. In the coming chapters we will then use the Poincaré Duality Theorem 61.1 to prove many more statements.

Notation. The isomorphism of the Poincaré Duality Theorem 61.1 is called the *Poincaré Duality isomorphism*. We denote it and its inverse both by PD_M, i.e. with the notation of

[1040]The theorem is named after the French mathematician Henri Poincaré whom we had already encountered twice before. He had formulated a simplified version of the theorem in 1895. But it took much longer to come up with a rigorous proof of the theorem as it is stated here.

[1041]Recall that Proposition 50.12 (5) implies that (M, A, B) is an excisive triad. In particular, following the definition of the relative cap product on page 1378, we obtain a relative cap product

$$H^k(M, A; R) \times \underbrace{H_n(M, \overbrace{A \cup B}^{=\partial M}; R)}_{\text{contains } [M]} \xrightarrow{\cap} H_{n-k}(X, B; R).$$

This shows that the map in the Poincaré Duality Theorem is in fact defined.

the Poincaré Duality Theorem 61.1 we have the isomorphism[1042]

$$\begin{array}{rcl} \mathrm{PD}_M\colon H^k(M,A;R) & \xrightarrow{\cong} & H_{n-k}(M,B;R) \\ \sigma & \mapsto & \sigma\cap[M] \end{array}$$

together with the inverse

$$\mathrm{PD}_M\colon H_{n-k}(M,B;R) \xrightarrow{\cong} H^k(M,A;R).$$

The isomorphisms PD_M thus allow us to go back and forth between homology and cohomology of an R-oriented manifold.

Remark. Let M be a compact n-dimensional topological manifold.

(1) Proposition 45.24 says that M is $\mathbb{F}_2$-orientable, i.e. the duality statement holds with $\mathbb{F}_2$-coefficients without any further assumptions on M.
(2) If M is in fact orientable, then it follows from Lemma 45.25 that M is R-orientable for any commutative ring R. In particular the duality statement holds for any commutative ring.
(3) The two most important instances of A and B are that we have either $A=\varnothing$ or $A=\partial M$. Under the hypothesis of the Poincaré Duality Theorem 61.1 we then get isomorphisms

$$\begin{array}{rcl} H^k(M;R) & \xrightarrow{\cong} & H_{n-k}(M,\partial M;R) \\ \sigma & \mapsto & \sigma\cap[M] \end{array} \quad \text{and} \quad \begin{array}{rcl} H^k(M,\partial M;R) & \xrightarrow{\cong} & H_{n-k}(M;R) \\ \sigma & \mapsto & \sigma\cap[M]. \end{array}$$

(4) For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the *Poincaré-Lefschetz Duality Theorem.*
(5) Let P be a point in M and suppose that M is connected. As we pointed out on page 1378, we have $[M]^*\cap[M]=[P]\in H_0(M;\mathbb{Z})$. Therefore we obtain that $\mathrm{PD}_M([P])=[M]^*$.

Examples. Earlier we had already performed several explicit calculations:

(1) On page 1375 we saw that for any surface Σ the map

$$-\cap[\Sigma]\colon H^1(\Sigma;\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z})$$

is an isomorphism. On page 1376 we had obtained similar results for $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients.
(2) On page 1379 we saw that for the annulus $X=S^1\times[0,1]$ the map

$$-\cap[X]\colon H^1(X;\mathbb{Z}) \to H_1(X,\partial X;\mathbb{Z})$$

is an isomorphism.

These calculations can be viewed as special case of the Poincaré Duality Theorem 61.1.

[1042]When M is understood, then we drop it from the notation.

It is also instructive to consider the cases $k = 0$ and $k = 1$ of the Poincaré Duality Theorem 61.1. For example, for $k = 0$ and the notation of the theorem we obtain for an compact, connected, non-empty n-dimensional topological manifold that

$$H_n(M, \partial M; \mathbb{Z}) \cong H^0(M; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 48.11}}}{\cong} \mathbb{Z} \text{ and } H_n(M; \mathbb{Z}) \cong H^0(M, \partial M; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 48.11 and page 1171}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } \partial M = \varnothing, \\ 0, & \text{if } \partial M \neq \varnothing. \end{cases}$$

This calculation is consistent with the results from Theorem 46.2 and Theorem 46.4 (1).

Next we consider the case $k = 1$ in the Poincaré Duality Theorem 61.1. As we will see, we obtain the following corollary which gives a new proof of Proposition 39.17 and Proposition 46.21 (1) and of the claim on page 1225.

Corollary 61.2. *Let M be a compact orientable n-dimensional topological manifold. Then the homology groups $H_{n-1}(M; \mathbb{Z})$ and $H_{n-1}(M, \partial M; \mathbb{Z})$ are free abelian groups.*

Proof. We have

$$\begin{aligned} H_{n-1}(M; \mathbb{Z}) &\underset{\substack{\uparrow \\ \text{Poincaré Duality Theorem 61.1}}}{\cong} H^1(M, \partial M; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{Universal Coefficient Theorem 51.13}}}{\cong} \operatorname{Hom}(H_1(M, \partial M), \mathbb{Z}) \oplus \operatorname{Ext}(\overbrace{H_0(M, \partial M)}^{\cong \mathbb{Z}^r \text{ for some } r \in \mathbb{N}_0}, \mathbb{Z}) \\ &\underset{\substack{\uparrow \\ \text{Lemma 51.5 (3)}}}{\cong} \operatorname{Hom}(H_1(M, \partial M), \mathbb{Z}) \underset{\substack{\uparrow \\ \text{by Proposition 45.2 the group } H_1(M, \partial M) \text{ is finitely generated, hence} \\ \text{it is isomorphic to } \mathbb{Z}^k \oplus T \text{ for some } k \in \mathbb{N} \text{ and some finite group } T}}{\cong} \operatorname{Hom}(H_1(\mathbb{Z}^k \oplus T), \mathbb{Z}) \cong \mathbb{Z}^k. \end{aligned}$$

Basically the same argument also shows that $H_{n-1}(M, \partial M; \mathbb{Z})$ is a free abelian group. $\square$

The following theorem says that for a closed topological manifold the homology and cohomology groups with coefficients in a *field* are symmetric around the middle dimension.

Theorem 61.3. (Poincaré Duality Theorem with field coefficients) *Let M be a compact n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *We have*[1043]

$$\dim_{\mathbb{F}_2}(H_k(M; \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(H_{n-k}(M, \partial M; \mathbb{F}_2))$$

and

$$\dim_{\mathbb{F}_2}(H^k(M; \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(H^{n-k}(M, \partial M; \mathbb{F}_2)).$$

(2) *If M is orientable, then for any field $\mathbb{F}$ we have*

$$\dim_{\mathbb{F}}(H_k(M; \mathbb{F})) = \dim_{\mathbb{F}}(H_{n-k}(M, \partial M; \mathbb{F}))$$

and

$$\dim_{\mathbb{F}}(H^k(M; \mathbb{F})) = \dim_{\mathbb{F}}(H^{n-k}(M, \partial M; \mathbb{F})).$$

[1043]The statement implies of course that $H^k(M; \mathbb{F})$ and $H^{n-k}(M, \partial M; \mathbb{F})$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

Examples.

(1) We consider the real projective space $\mathbb{R}\mathrm{P}^n$. On page 1028 we had determined that

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k=0,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

So the homology groups with $\mathbb{F}_2$-coefficients are in particular symmetric, which is exactly what the above Poincaré Duality Theorem 61.3 with $\mathbb{F}_2$-coefficients predicts.

(2) Towards the end of the 19th century, when Henri Poincaré first developed what became known as Poincaré Duality he thought that for any closed orientable connected n-dimensional manifold the groups $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ are isomorphic. A Danish mathematician, Poul Heegaard, pointed out to him that this is not true in general, for example for the lens space $L(p,q)$ we have by Lemma 40.12 that

$$H_1(L(p,q);\mathbb{Z}) \;\cong\; \mathbb{Z}_p \quad \text{but} \quad H_2(L(p,q);\mathbb{Z}) \;=\; 0.$$

This shows that the homology groups with $\mathbb{Z}$-coefficients are *not* symmetric.

Proof. Let M be a compact n-dimensional topological manifold. We have

$$\begin{aligned}\dim_{\mathbb{F}_2}(H_k(M;\mathbb{F}_2)) \;&\underset{\substack{\uparrow\\ \text{Theorem 61.1}}}{=}\; \dim_{\mathbb{F}_2}(H^{n-k}(M,\partial M;\mathbb{F}_2)) \;\underset{\substack{\uparrow\\ \text{Proposition 51.18}}}{=}\; \dim_{\mathbb{F}_2}(\mathrm{Hom}(H_{n-k}(M,\partial M;\mathbb{F}_2),\mathbb{F}_2))\\ &\underset{\substack{\uparrow\\ \text{since Proposition 45.2 (3) says that } H_{n-k}(M,\partial M;\mathbb{F}_2) \text{ is finite-dimensional}}}{=}\; \dim_{\mathbb{F}_2}(H_{n-k}(M,\partial M;\mathbb{F}_2)).\end{aligned}$$

The proofs of all other statements are basically the same. □

Remark. Let M be a compact topological n-dimensional manifold and let $\mathbb{F}$ be a field. In Proposition 45.2 (2) we had mentioned that all homology groups $H_k(M;\mathbb{F})$ are finite-dimensional. But this proof relied on the slightly mysterious Theorem 45.1. Now we can give an alternative argument if M is closed and orientable. Indeed, we have

$$H_k(M;\mathbb{F}) \;\underset{\substack{\uparrow\\ \text{Theorem 61.1}}}{\cong}\; H^{n-k}(M;\mathbb{F}) \;\underset{\substack{\uparrow\\ \text{Proposition 51.18}}}{\cong}\; H_{n-k}(M;\mathbb{F})^* \;\underset{\substack{\uparrow\\ \text{Theorem 61.1}}}{\cong}\; H^k(M;\mathbb{F})^* \;\underset{\substack{\uparrow\\ \text{Proposition 51.18}}}{\cong}\; (H_k(M;\mathbb{F})^*)^*.$$

(Here, as always given an $\mathbb{F}$-vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ its dual vector space.) But by [Jacb, p. 244-248] (see also [DF, Chapter 11.4]) a vector space V is isomorphic to its double dual $(V^*)^*$ if and only if the vector space is finite dimensional.

61.2. **Consequences of the Poincaré Duality Theorem 61.1.** In this section we present a few fairly immediate consequences of the Poincaré Duality Theorem 61.1.

Proposition 61.4. *For every compact odd-dimensional topological manifold M we have*

$$\chi(M) \;=\; \tfrac{1}{2}\chi(\partial M).$$

In particular if M is closed, then $\chi(M)=0$.

Proof. Let M be a compact topological manifold of dimension $2n+1$. Then

$$\begin{aligned}
\chi(M) &\underset{\substack{\uparrow \\ \text{Proposition 45.7 (4)}}}{=} \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(H_i(M,\mathbb{F}_2)) \underset{\substack{\uparrow \\ \text{by Theorem 61.3}}}{=} \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(H_{2n+1-i}(M,\partial M,\mathbb{F}_2)) \\
&\underset{\substack{\uparrow \\ \text{follows from the substitution } j=2n+1-i \\ \text{together with the fact that } (-1)^{2n+1}=-1}}{=} -\sum_{j=0}^{2n+1}(-1)^j\cdot\dim_{\mathbb{F}_2}(H_j(M,\partial M,\mathbb{F}_2)) \underset{\substack{\uparrow \\ \text{Proposition 45.7 (4)}}}{=} -\chi(M,\partial M) \underset{\substack{\uparrow \\ \text{Proposition 45.7 (4)}}}{=} -\chi(M)+\chi(\partial M).
\end{aligned}$$

Summarizing we have shown that $\chi(M) = -\chi(M) + \chi(\partial M)$. The desired equality follows from solving for $\chi(M)$. □

Proposition 61.4 raises the question, what can we say about Euler characteristics of closed even-dimensional manifolds. By the classification of closed oriented connected 2-dimensional manifolds, see Theorem 14.10, we know that the Euler characteristic of such a manifold is even. In higher dimensions there is evidently much more freedom for constructing manifolds.

Question 61.5. *Let $n \geq 4$ be even. Is every integer the Euler characteristic of a closed oriented connected n-dimensional manifold?*

Using Proposition 61.4 we easily obtain the following surprising (?) corollary.

Corollary 61.6. *The 2-dimensional manifold $\mathbb{R}\mathrm{P}^2$ is not the boundary of a compact 3-dimensional manifold.*

Proof. Suppose there was a compact 3-dimensional manifold M with $\partial M = \mathbb{R}\mathrm{P}^2$. We would obtain that

$$\chi(M) \underset{\substack{\uparrow \\ \text{Proposition 61.4}}}{=} \tfrac{1}{2}\chi(\partial M) = \tfrac{1}{2}\chi(\mathbb{R}\mathrm{P}^2) \underset{\substack{\uparrow \\ \text{see page 966}}}{=} \tfrac{1}{2}.$$

But the Euler characteristic of a compact manifold is by definition an integer. □

Remark. In Exercise 61.4 we will study the question, whether the Klein bottle is the boundary of a compact 3-dimensional manifold M. It follows from the CW-structure on page 647 that the Euler characteristic of the Klein bottle K is zero, therefore Proposition 61.4 does not obstruct the existence of such an M.

Clearly non-orientable manifolds exhibit some unusual behavior. So the following question arises.

Question 61.7. *Is every closed* orientable *manifold the boundary of a compact manifold?*

Note that in Corollary 8.4 we showed that if N is a non-orientable manifold, then $\pi_1(N)$ admits an epimorphism onto $\mathbb{Z}_2$.[1044] It follows from the Hurewicz Theorem 38.5 that

[1044] If in the proof of Corollary 8.4 we replace Proposition 8.3 by Proposition 45.26 we see that the statement also holds for non-orientable topological manifolds.

there exists also an epimorphism $H_1(N;\mathbb{Z}) \to \mathbb{Z}_2$. For closed 3-dimensional manifolds the following Lemma 61.8 is thus a considerable strengthening of Corollary 8.4.

Lemma 61.8. *Let M be a connected closed 3-dimensional manifold. If M is non-orientable, then $H_1(M;\mathbb{Z})$ is infinite.*

Proof. Let M be a connected closed 3-dimensional manifold. We have

$$\begin{array}{ccccccccc} 0 & = & \chi(M) & = & \sum\limits_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(M) & = & b_0(M) - b_1(M) + b_2(M) & = & 1 - b_1(M) + b_2(M). \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 61.4} & & \text{see page 1079} & & \text{Theorems 46.2 and 46.4} & & \text{by Proposition 29.5} & \end{array}$$

This equality can only hold if $b_1(M) > 0$, i.e. if $H_1(M;\mathbb{Z})$ is infinite. □

Definition. Let $f\colon M \to N$ be a map between closed, oriented connected non-empty n-dimensional topological manifolds. We consider the following diagram of maps:

$$\begin{tikzcd} H_k(M;\mathbb{Z}) \arrow[r, shift left, "\mathrm{PD}_M"] \arrow[d, "f_*"'] & H^{n-k}(M;\mathbb{Z}) \arrow[l, shift left, "\cap[M]"] \\ H_k(N;\mathbb{Z}) \arrow[r, shift left, "\mathrm{PD}_N"] & H^{n-k}(N;\mathbb{Z}). \arrow[l, shift left, "\cap[N]"] \arrow[u, "f^*"'] \end{tikzcd}$$

We refer to

$$\begin{array}{rcl} f^!\colon H_k(N;\mathbb{Z}) & \to & H_k(M;\mathbb{Z}) \\ \sigma & \mapsto & f^*(\mathrm{PD}_N(\sigma)) \cap [M] \end{array}$$

as the *Umkehr map* of f.[10451046]

The following lemma shows that in some circumstances the Umkehr map is just the transfer map that we introduced on page 1064.

Lemma 61.9. *Let $p\colon \widetilde{N} \to N$ be a finite cover of a closed, oriented, non-empty n-dimensional topological manifold. If we equip $\widetilde{N}$ with the orientation given by Lemma 45.28 (3), then for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon H_k(N;\mathbb{Z}) \to H_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.*

Proof. We start our proof with the following claim.

Claim. Let $r \le s$. Given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in C^r(N;\mathbb{Z})$ we have

$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$

[1045]This map is indeed called "Umkehr map" in English. The name stems from the fact that it is a map on homology that "goes the wrong way".

[1046]Wikipedia writes: "In category theory [...] certain unusual functors are denoted $f^!$ and $f_!$, with the exclamation mark used to indicate that they are exceptional in some way. They are thus accordingly sometimes called shriek maps, with "shriek" being slang for an exclamation mark [...]."

We write $m = [\widetilde{N} : N]$. We denote by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_m$ the lifts of σ. Then

$$\begin{aligned} p^*(\phi) \cap p^*(\sigma) &= p^*(\phi) \cap \sum_{i=1}^m \widetilde{\sigma}_i = \sum_{i=1}^m p^*(\phi)(\widetilde{\sigma}_i \circ [v_0, \dots, v_r]) \cdot \widetilde{\sigma}_i \circ [v_r, \dots, v_s] \\ &= \sum_{i=1}^m \phi((p_* \circ \widetilde{\sigma}_i) \circ [v_0, \dots, v_r]) \cdot \widetilde{\sigma}_i \circ [v_r, \dots, v_s] \\ &= p^*\big(\phi(\sigma \circ [v_0, \dots, v_r]) \cdot \sigma \circ [v_r, \dots, v_s]\big) = p^*(\phi \cap \sigma) \\ &\uparrow \end{aligned}$$

since $p_* \circ \widetilde{\sigma}_i = \sigma$ and since $\widetilde{\sigma}_i \circ [v_0, \dots, v_r]$, $i = 1, \dots, m$ are the lifts of $\sigma \circ [v_0, \dots, v_r]$

This concludes the proof of the claim.

Now we turn to the actual proof of the lemma. We pick an n-cycle τ that is a representative for the fundamental class $[N] \in H_n(N; \mathbb{Z})$ of N. By Proposition 46.24 we know that $p^*(\tau)$ represents the fundamental class of $\widetilde{N}$. By definition of the Umkehr map it suffices to prove the following claim.

Claim. For any cocycle $\phi \in C^l(N; \mathbb{Z})$ we have

$$f^*(\phi) \cap p^*(\tau) = p^*(\phi \cap \tau).$$

The claim is an immediate consequence of the first claim. □

Finally we want to study degree-one maps between manifolds. We recall that on page 959 we defined the degree of a map $f\colon M \to N$ between closed, oriented connected non-empty n-dimensional topological manifolds to be the unique integer $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$. We had obtained the following results:

(1) In Proposition 39.21 we showed that for any closed, oriented, non-empty n-dimensional manifold M there exists a degree-one map $M \to S^n$.[1047]
(2) In Proposition 39.26 we saw that if $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.
(3) In Lemma 60.5 we had used cup products to show that there does not exist a map $f\colon S^4 \to S^2 \times S^2$ of non-zero degree.

Now we add the following proposition to our list on results on the non-existence of degree-one maps.

Proposition 61.10. *Let $f\colon M \to N$ be a degree-one map between closed, oriented, non-empty n-dimensional topological manifolds. Then for every $k \in \mathbb{N}_0$ the following holds:*

(1) *the map $f_* \circ f^!$ is the identity, in particular the map $f^!\colon H_k(N; \mathbb{Z}) \to H_k(M; \mathbb{Z})$ splits,*
(2) *the image $f^!(H_k(N; \mathbb{Z}))$ is a subsummand of $H_k(M; \mathbb{Z})$, i.e. there exists a subgroup Q of $H_k(M; \mathbb{Z})$ such that $H_k(M; \mathbb{Z}) = f^!(H_k(N; \mathbb{Z})) \oplus Q$.*

[1047] Basically the same proof, together with say Theorem 46.2 (4) and an excision argument, shows that the same statement also holds for topological manifolds. We leave it to the reader to fill in the details.

Proof. Let $f\colon M \to N$ be a degree-one map between closed, oriented, non-empty n-dimensional topological manifolds. Let $\sigma \in H_k(N;\mathbb{Z})$. We have

$$(f_* \circ f^!)(\sigma) \underset{\substack{\uparrow \\ \text{definition of } f^!}}{=} f_*\big(f^*(\mathrm{PD}_N(\sigma)) \cap [M]\big) \underset{\substack{\uparrow \\ \text{Lemma 59.8}}}{=} \mathrm{PD}_N(\sigma) \cap f_*([M]) \underset{\substack{\uparrow \\ \text{since } \deg(f)=1}}{=} \mathrm{PD}_N(\sigma) \cap [N] \underset{\substack{\uparrow \\ \text{definition of } \mathrm{PD}_N}}{=} \sigma.$$

The second statement follows from combining the first statement with Lemma 33.2. □

It follows from remark (2) above that there is no degree-one map from $\mathbb{RP}^3$ to $S^1 \times S^2$. In Question 39.29 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$. Now we can give a negative answer.

Corollary 61.11. *There is no degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$.*

Proof. The corollary follows immediately from Proposition 61.10 and the observation that the group $H_1(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}_2$ is not a subsummand of $H_1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. □

61.3. **A geometric approach to the Poincaré Duality Theorem 61.1** (∗)**.** In this section we will provide a sketch of a geometric proof for the following weaker version of Poincaré Duality Theorem 61.1.

Theorem 61.12. (Poincaré Duality Theorem) *If M is a closed n-dimensional*[1048] *manifold, then for every $k \in \mathbb{N}_0$ there exists an isomorphism*

$$\mathrm{PD}_M\colon H_k(M;\mathbb{F}_2) \xrightarrow{\cong} H^{n-k}(M;\mathbb{F}_2).$$

The proof we provide for Theorem 61.12 has the great advantage that it is very pictorial and one "can see" what is happening. Also, much later, when we introduce a new invariant called Reidemeister torsion, the method of the proof of Theorem 61.12 will be useful. In the next two sections we will give a rigorous, more formal proof of Poincaré Duality Theorem 61.12, therefore we will take the liberty of skipping many steps in the following proof.

Sketch of the proof of Theorem 61.12. Before we start with the actual proof of Theorem 61.12 we want to recall and introduce several definitions for the standard simplex Δ^m:

(1) We denote by $B_m := (\frac{1}{m+1}, \dots, \frac{1}{m+1}) \in \Delta^m$ the *barycenter* of the standard m-simplex Δ^m.
(2) Given points $P_0, \dots, P_k$ in the simplex Δ^m we denote by

$$\begin{aligned} [P_0,\dots,P_k]\colon \Delta^k &\to \Delta^m \\ (t_0,\dots,t_k) &\mapsto \sum_{i=0}^k t_i \cdot P_i \end{aligned}$$

the singular k-simplex that sends each vertex v_i of Δ^k to P_i. We also write

$$\langle P_0,\dots,P_k\rangle := \text{image of } [P_0,\dots,P_k].$$

[1048]Note that here we do not allow topological manifolds.

(3) An r-*dimensional face* of Δ^m is defined as a subset of the form $\langle v_{n_1}, \dots, v_{n_r}\rangle$ where $0 \le n_1 < \cdots < n_r \le m$. We refer to the image of the barycenter B_r under the map $[v_{n_1}, \dots, v_{n_r}] \to \Delta^m$ as the *barycenter of the face*.
(4) The barycenter of a face τ is denoted by $\underline{\tau}$.

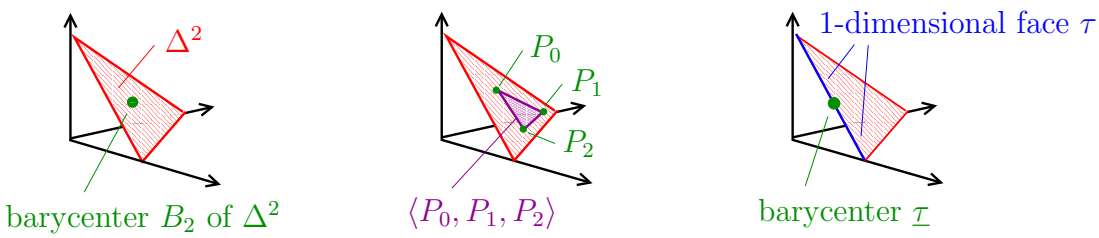


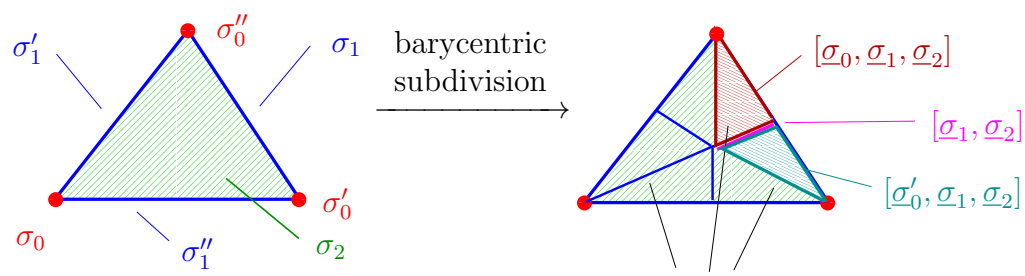
FIGURE 744.

We continue with some new definitions.

(1) Let σ and τ be faces of Δ^m. If $\sigma \subset \tau$, then we write $\sigma \le \tau$. If $\sigma \ne \tau$, then we write $\sigma < \tau$.
(2) We refer to the set of singular simplices

$$[\underline{\sigma}_0, \dots, \underline{\sigma}_k] \quad \text{where } \sigma_0, \dots, \sigma_k \text{ are faces with } \sigma_0 < \sigma_1 < \cdots < \sigma_k \le \sigma$$

as the *barycentric subdivision of* Δ^m. The definition is illustrated in Figure 745.



2-dimensional simplex σ_2 with three 0-dimensional faces $\sigma_0, \sigma_0', \sigma_0''$ and three 1-dimensional faces $\sigma_1, \sigma_1', \sigma_1''$

barycentric subdivision of a 2-simplex σ_2.

FIGURE 745.

Now let M be a closed, oriented n-dimensional manifold. According to Proposition 39.3 the manifold M admits a simplicial structure $\{\varphi_i \colon \Delta^{n_i} \to M\}_{i \in I}$. We refer to pages 936 and 938 for the precise definition of a simplicial structure. In the following we summarize some of the key features and definitions of a simplicial structure.

(1) For each $i \in I$ the map $\varphi_i \colon \Delta^{n_i} \to M$ is injective.

(2) The image of $\varphi_i\colon \Delta^{n_i} \to M$ is called an *n_i-dimensional simplex*. We refer to it also by φ_i. We use the notation that we introduced above for the standard simplices in the obvious way for the simplices of M:
   (a) The image of the interior of Δ^{n_i} is called the interior of φ_i.
   (b) The image of a face of Δ^{n_i} is called a face of φ_i. Similarly we define the barycenter $\underline{\tau}$ of a face τ of φ_i.
   (c) Given points $P_0, \dots, P_k \in \varphi_i(\Delta^{n_i})$ we define

$$[P_0, \dots, P_k] := \sigma_i \circ [\varphi_i^{-1}(P_0), \dots, \varphi_i^{-1}(P_k)]\colon \Delta^k \to M.$$

   We denote the image of $[P_0, \dots, P_k]$ again by $\langle P_0, \dots, P_k\rangle$.

(3) For each $x \in M$ there exists a unique simplex such that x lies in the interior of the simplex.
(4) Every face of every simplex of M is again a simplex of M.
(5) If the images of φ_i and φ_j intersect, then there exist faces F_i of Δ^{n_i} and F_j of Δ^{n_j} such that the following two conditions are satisfied:
   (a) we have $\varphi_i(\Delta^{n_i}) \cap \varphi_j(\Delta^{n_j}) = \varphi_i(F_i) = \varphi_j(F_j)$,
   (b) the map $\varphi_j^{-1} \circ \varphi_i\colon F_i \to F_j$ is an affine homeomorphism.

Now we apply the barycentric subdivision to each simplex of M and we obtain a new set of singular simplices. We call this set *the barycentric subdivision of M*.[1049] For a k-simplex σ of M we define

$$\sigma^d := \bigcup_{\sigma=\sigma_0<\sigma_1<\dots<\sigma_s} \langle \underline{\sigma}_0, \underline{\sigma}_1, \dots, \underline{\sigma}_s\rangle.$$

We refer to σ^d as the *cell dual to σ* and we refer to $n-k$ as the dimension of the dual cell. Furthermore we define

$$\partial\sigma^d := \bigcup_{\sigma<\sigma_1<\dots<\sigma_s} \langle \underline{\sigma}_1, \dots, \underline{\sigma}_s\rangle \subset \sigma^d.$$

We illustrate σ^d and $\partial\sigma^d$ in Figure 746. We see in Figure 746 that in both examples the dual cell is a disk and that $\partial\sigma^d$ is the boundary of σ^d. The following claim says that this not a coincidence.

Claim 1. For each k-dimensional simplex σ the pair $(\sigma^d, \partial\sigma^d)$ is homeomorphic to the pair $(\overline{B}^{n-k}, S^{n-k})$.

In the proof of the claim one needs to use that M is a closed n-dimensional manifold. A proof of the claim can for example be found in [RS, Chapter 2].

In Figure 746 we see two simplices σ and τ, where $\sigma \subset \partial\tau$. In the dual cells this inclusion is reversed and we have $\tau^d \subset \partial\sigma^d$. Again this is not a coincidence, in fact the following claim holds in general.

[1049] It is straightforward to verify that the barycentric subdivision of M is again a simplicial structure for M. We will not make use of this fact.

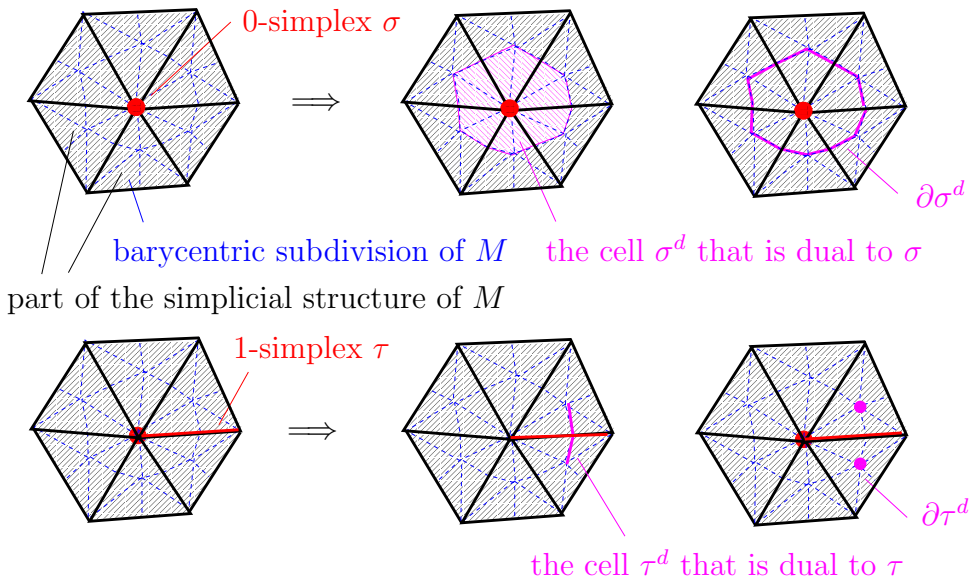


FIGURE 746. The dual cells of a 0-simplex and a 1-simplex.

Claim 2. Let σ and τ be two simplices of M. Then the following holds

$$\tau \text{ is a face of } \sigma \quad \Longleftrightarrow \quad \sigma^d \subset \tau^d.$$

Let τ be a face of σ. Then each sequence $\sigma = \sigma_0 < \sigma_1 \cdots < \sigma_k$ of faces can be extended to a sequence $\tau < \sigma = \sigma_0 < \sigma_1 \cdots < \sigma_k$. In particular we have $\langle \underline{\sigma}_0, \ldots, \underline{\sigma}_k \rangle \subset \langle \underline{\tau}, \underline{\sigma}_0, \ldots, \underline{\sigma}_k \rangle$, thus $\sigma^d \subset \tau^d$. The converse statement is proved exactly the same way. Thus we have proved the claim.

In the following we will equip M with two different CW-structures:

(1) Given $k \in \mathbb{N}_0$ we denote by $S_k(M)$ the set of k-simplices of M.
(2) In Lemma 39.2 we saw that the simplicial structure of M gives rise to a CW-structure X for M where the k-cells of X correspond precisely to the k-dimensional simplices of M. Put differently, we have a canonical identification

$$\text{set of the } k\text{-cells of } X \;=\; \{\sigma \,|\, \sigma \in S_k(M)\}.$$

(3) It follows fairly easily from the previous two claims that the dual cells form in fact a CW-structure for M. We denote the corresponding CW-complex by Y and refer to is as *the dual CW-complex structure*. Note that we have a canonical identification

$$\text{set of the } k\text{-cells of } Y \;=\; \{\sigma^d \,|\, \sigma \in S_{n-k}(M)\}.$$

(4) Given a k-cell y of Y we define its dual y^d to be the unique $(n-k)$-simplex τ with $y = \tau^d$.

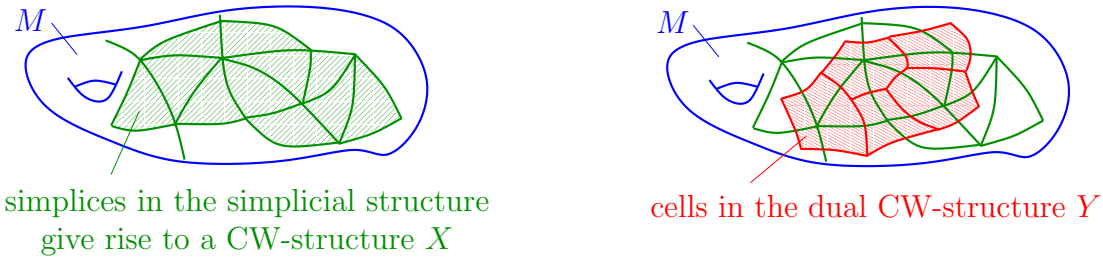


FIGURE 747.

Given $\sigma \in S_k(M)$ we define

$$\mathrm{PD}(\sigma)\colon C^{\mathrm{CW}}_{n-k}(Y) \ \to\ \mathbb{F}_2$$

to be the uniquely defined homomorphism that sends σ^d to one and that sends all other $(n-k)$-dimensional cells of Y to zero. The definition is illustrated in Figure 748.

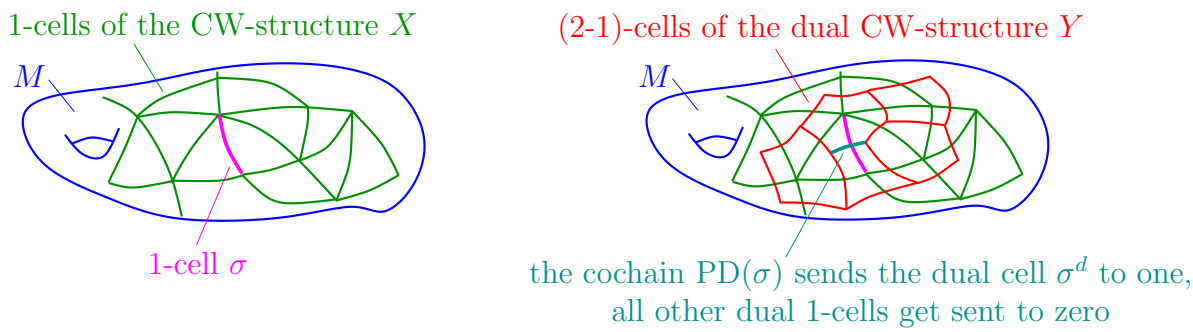


FIGURE 748.

Claim 3. The maps

$$\begin{array}{rcl} \mathrm{PD}\colon C^{\mathrm{CW}}_k(X)\otimes\mathbb{F}_2 & \to & \mathrm{Hom}\left(C^{\mathrm{CW}}_{n-k}(Y),\mathbb{F}_2\right) \\ \sum\limits_{i=1}^m a_i\cdot\sigma_i & \mapsto & \sum\limits_{i=1}^m a_i\cdot\mathrm{PD}(\sigma_i) \end{array}$$

induce isomorphisms

$$H^{\mathrm{CW}}_k(X;\mathbb{F}_2) \ \xrightarrow{\cong}\ H^{n-k}_{\mathrm{CW}}(Y;\mathbb{F}_2).$$

Before we provide the proof of the claim we note that we now obtain that

$$\begin{array}{ccccccc} H_k(M;\mathbb{F}_2) & \cong & H^{\mathrm{CW}}_k(X;\mathbb{F}_2) & \cong & H^{n-k}_{\mathrm{CW}}(Y;\mathbb{F}_2) & \cong & H^{n-k}(M;\mathbb{F}_2). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 35.4} & & \text{by the above claim} & & \text{Proposition 50.4} & \end{array}$$

We have thus proved the theorem, modulo the proof of the claim.

We turn to the proof of the claim. First note that it is straightforward to verify that the maps

$$\begin{array}{rcl} \mathrm{Hom}\left(C^{\mathrm{CW}}_{n-k}(Y), \mathbb{F}_2\right) & \to & C^{\mathrm{CW}}_k(X) \otimes \mathbb{F}_2 \\ f & \mapsto & \sum\limits_{\sigma \in S_k(M)} f(\sigma^d) \cdot \sigma \end{array}$$

are an inverse to the maps PD under consideration. It remains to show that the following diagram commutes:

$$\begin{array}{ccc} C^{\mathrm{CW}}_k(X; \mathbb{F}_2) & \xrightarrow{\ \mathrm{PD}\ } & \mathrm{Hom}\left(C^{\mathrm{CW}}_{n-k}(Y), \mathbb{F}_2\right) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial^*} \\ C^{\mathrm{CW}}_{k-1}(X; \mathbb{F}_2) & \xrightarrow{\ \mathrm{PD}\ } & \mathrm{Hom}\left(C^{\mathrm{CW}}_{n-(k-1)}(Y), \mathbb{F}_2\right). \end{array}$$

To show that the diagram commutes it suffices to show that for each k-simplex $\sigma \in S_k(M)$ and each $(k-1)$-simplex $\tau \in S_{k-1}(M)$ we have

$$(\mathrm{PD}(\partial\sigma))(\tau^d) \ = \ (\mathrm{PD}(\sigma))(\partial\tau^d) \ \in \ \mathbb{F}_2.$$

We denote by $y_1, \ldots, y_{n-k+2}$ the $(n-k)$-cells of Y that appear in the boundary of the $(n-k+1)$-cell τ^d. Then we have[1050]

$$\begin{array}{rcll} & & \text{by definition of the cellular boundary map and of PD} & \text{Claim 1} \\ & & \downarrow & \downarrow \\ (\mathrm{PD}(\partial\sigma))(\tau^d) = 1 & \iff & \tau \text{ is a face of } \sigma & \iff \sigma^d \subset \tau^d \\ & \iff & \text{there exists an } i \text{ with } y_i = \sigma^d & \\ & & \uparrow & \\ & & \text{since } y_1, \ldots, y_{n-k+2} \text{ are the only } (n-k)\text{-cells contained in } \tau^d & \\ & \iff & \text{there exists an } i \text{ with } (y_i)^d = \sigma \iff (\mathrm{PD}(\sigma))(\partial\tau^d) = 1. & \end{array}$$

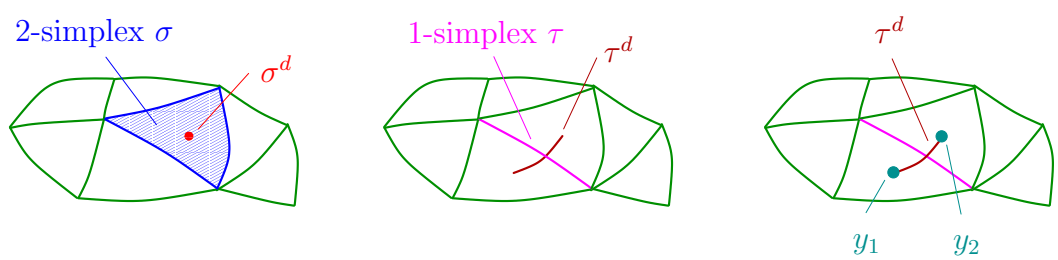


FIGURE 749.

Since $\mathbb{F}_2 = \{0, 1\}$ consists of precisely two elements the above equivalences prove that $(\mathrm{PD}(\partial\sigma))(\tau^d) = (\mathrm{PD}(\sigma))(\partial\tau^d)$. This concludes the proof of the claim. $\square$

[1050]Even though we justify the various equivalences one could argue that more details are required to turn the argument into a rigorous proof.

We only sketched the proof of Theorem 61.12, but it is not too difficult to turn the argument into a rigorous proof, see e.g. [SZ]. Nonetheless there are several issues with the statement and the proof of Theorem 61.12.

(1) Right now the proof only deals with closed manifolds. But this is not a big problem, with more care one can also prove a relative version. We refer to [SZ, Chapter 14.8] for details, at least in the special case that $A = \varnothing$ or $A = \partial M$.
(2) For closed orientable manifolds one would like to have isomorphisms of homology and cohomology groups with $\mathbb{Z}$-coefficients. In fact this can be arranged by a more careful discussion of orientations and signs. We refer to [Tur, p. 73], [Mun, Chapter 65] or [SZ, Chapter 14.2] for details on how to promote the above argument to a proof that deals with $\mathbb{Z}$-coefficients.
(3) We had stated Theorem 61.12 for manifolds. The only point where we used that we are working with a manifold was when we cited Proposition 39.3 to deduce that the manifold admits a simplicial structure. Put differently, the proof applies to any closed topological manifold that admits a simplicial structure. But it was shown by Ciprian Manolescu [Man, Man2] in 2013 that for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure.
(4) Even if one cares only about closed manifolds the proof of Theorem 61.12 is somewhat unsatisfactory. The proof relies on introducing a simplicial structure and the isomorphism between homology and cohomology groups relies on this choice. The statement of the Poincaré Duality Theorem 61.1 is much nicer since it states explicitly the isomorphism.

FIGURE 750. Ciprian Manolescu (on the right).

61.4. **The proof of Poincaré Duality for topological manifolds without boundary.** In this section we will prove the Poincaré Duality Theorem 61.1 for closed topological manifolds. More precisely, we will prove the following theorem.

Theorem 61.13. *Let M be a closed, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in H_n(M; R)$*

the R-fundamental class. Then for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{PD}_M \colon H^k(M;R) & \xrightarrow{\cong} & H_{n-k}(M;R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

In order to simplify the notation a little bit we will work throughout the section with $R = \mathbb{Z}$. The proof for other commutative rings is verbatim the same.

The key idea is somewhat similar to the strategy of the proof of Theorem 46.10. More precisely, using a Mayer–Vietoris argument and a limit argument we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$.

To carry out the above strategy we need a version of Poincaré Duality for every topological manifold (not necessarily compact) with empty boundary. Formulating this generalized Poincaré Duality Theorem requires some preparations. First we recall some definitions and results from earlier on:

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Let $k \in \mathbb{N}_0$. By Proposition 53.4 we have an identification

$$\varinjlim_{K \in \mathcal{K}(X)} H^k(X, X \setminus K) \;=\; H^k_c(X).$$

(2) Let M be an n-dimensional topological manifold with empty boundary. If M is equipped with an orientation $\{\mu_x\}_{x \in M}$, then we saw in Theorem 46.10 (i) that given any compact subset K of M there exists a unique class $\mu_K^M \in H_n(M, M \setminus K)$ such that for any $x \in K$ the image of μ_K^M under the map

$$H_n(M, M \setminus K) \;\to\; H_n(M, M \setminus \{x\})$$

equals the given orientation μ_x at x. If M is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ_K^M.

Lemma 61.14. *Let M be an oriented n-dimensional topological manifold with empty boundary.*

(1) *Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL} \colon (M, M \setminus L) \to (M, M \setminus K)$ the inclusion map of pairs.*
   (a) *The map $f_{KL*} \colon H_n(M, M \setminus L) \to H_n(M, M \setminus K)$ sends μ_L to μ_K.*
   (b) *For any $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} H^k(M, M \setminus K) \arrow[d, "f_{KL}^*"] \arrow[dr, "\cap \mu_K"] & \\ H^k(M, M \setminus L) \arrow[r, "\cap \mu_L"'] & H_{n-k}(M). \end{tikzcd}$$

(2) *For any $k \in \mathbb{N}_0$ there exists a unique map*

$$\mathrm{PD}_M \colon H^k_c(M) \to H_{n-k}(M)$$

such that for each compact subset $K \subset M$ the following diagram commutes[1051]

$$\begin{tikzcd} H^k(M, M \setminus K) \arrow[rr, "\cap \mu_K"] \arrow[dr] & & H_{n-k}(M). \\ & H^k_c(M) \arrow[ur, "\mathrm{PD}_M"'] & \end{tikzcd}$$

Proof. Let M be an oriented n-dimensional topological manifold with empty boundary.

(1) (a) This is precisely Lemma 46.11 (1).
(b) Let $\varphi \in H^k(M, M \setminus K)$. We see that

$$\varphi \cap \mu_K \underset{\substack{\uparrow \\ \text{part (1a)}}}{=} \varphi \cap f_{KL*}(\mu_L) \underset{\substack{\uparrow \\ \text{Lemma 59.8}}}{=} f^*_{KL}(\varphi) \cap \mu_L.$$

But this means precisely that the diagram under consideration commutes.

(2) This statement follows immediately from (1b) and the universal property of the direct limit. □

Throughout the remainder of this section we use the following notation.

Notation. Let $k \in \mathbb{N}_0$ and let X be a topological space.

(1) Given any subset A of X we denote by $i_*\colon H_k(A) \to H_k(X)$ and $i^*\colon H_k(X) \to H_k(A)$ the inclusion induced map.
(2) Suppose that X is Hausdorff and that U is an open subset of X. Then we denote by $i\colon H^k_c(U) \to H^k_c(X)$ the map defined by

$$H^n_c(U) = \varinjlim_{K \in \mathcal{K}(U)} H^n(U, U \setminus K) \xleftarrow{\cong} \varinjlim_{K \in \mathcal{K}(U)} H^n(X, X \setminus K) \to \varinjlim_{K \in \mathcal{K}(X)} H^n(X, X \setminus K) = H^n_c(X).$$

We refer to Lemma 53.5 for the proof that these inclusion maps are functorial with respect to compositions of inclusions.

We will prove the following generalization of Theorem 61.13.[1052] This more general version of Poincaré Duality will not only be useful for the divide-et-impera approach to the proof of Theorem 61.13, but it will also be of interest in its own right later on.

Theorem 61.15. *Let M be an oriented n-dimensional topological manifold with empty boundary. Then for each $k \in \mathbb{N}_0$ the map*

$$\mathrm{PD}_M\colon H^k_c(M) \xrightarrow{\cong} H_{n-k}(M)$$

from Lemma 61.14 is an isomorphism.

[1051] Note that for a class $\varphi \in H^k(M, M \setminus K)$ and $\mu_K \in H_n(M, M \setminus K)$ the cap product $\varphi \cap \mu_K$ does indeed lie in $H_{n-k}(M)$.

[1052] Why is it a generalization of Theorem 61.13?

Example. Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Then

$$\begin{array}{ccccc} H_n(M;\mathbb{Z}) & \cong & H_c^0(M;\mathbb{Z}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases} \\ & \uparrow & & \uparrow & \\ & \text{Theorem 61.15} & & \text{see page 1257} & \end{array}$$

This calculation is consistent with the results obtained in Theorems 46.2 and 46.4.

Remark.

(1) Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Following the convention introduced on page 1413 we denote the inverse of the isomorphism $\mathrm{PD}_M\colon H_c^k(M) \xrightarrow{\cong} H_{n-k}(M)$ also by PD_M.

(2) Given a possibly non-compact oriented manifold without boundary Theorem 61.15 allows us to relate singular homology to cohomology with compact support. There is also a different version of Poincaré Duality that relates singular cohomology to a different flavor of homology, namely Borel-Moore homology. We refer to [Gre], [Bre2, Chapter V.9] and [PS, p. 332] for details.

As a warm up for the proof of Theorem 61.15 we will deal with the case $M = B^n$ by hand.[1053] More precisely, we will prove the following lemma.

Lemma 61.16. *For each $k \in \mathbb{N}_0$ the map*

$$\mathrm{PD}_{B^n}\colon H_c^k(B^n) \xrightarrow{\cong} H_{n-k}(B^n)$$

is an isomorphism.

Proof. Let $k \in \mathbb{N}_0$. Given any $s \in (0,1)$ we write $\overline{B}_s^n := \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and we also write $S_s^{n-1} := \partial\overline{B}_s^n$. For the remainder of the proof we fix $0 < r < s < 1$. We denote by $i\colon (\overline{B}_s^n, S_s^n) \to (B^n, B^n \setminus \overline{B}_r^n)$ the inclusion map of pairs.

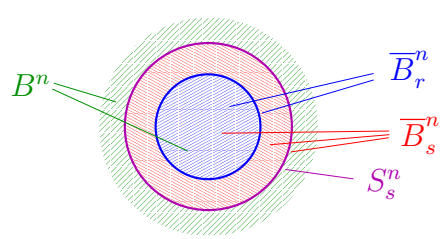


FIGURE 751. Illustration for the proof of Lemma 61.16.

[1053]We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation induced from the standard orientation of $\mathbb{R}^n$, see Lemma 45.28.

We consider the following diagram:

$$\begin{tikzcd}
H^k\big(\overline{B}^n_s, S^n_s\big) \arrow[d, "\cap[\overline{B}^n_s]"] & H^k\big(B^n, B^n \setminus \overline{B}^n_r\big) \arrow[l, "i^*"'] \arrow[r] \arrow[d, "\cap\mu_{\overline{B}^n_r}"] & H^k_c(B^n) \arrow[d, "\mathrm{PD}_{B^n}"] \\
H_{n-k}\big(\overline{B}^n_s\big) \arrow[r, "i_*"] & H_{n-k}(B^n) \arrow[r, "="] & H_{n-k}(B^n).
\end{tikzcd}$$

The lemma is an immediate consequence of the following claim.

Claim.

(1) The diagram commutes,
(2) the left vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

We turn to the proof of the three statements.

(1) The square on the right commutes by definition of PD_{B^n}. So it remains to show that the square on the left commutes. Thus let $\varphi \in H^k(B^n, B^n \setminus \overline{B}^n_r)$. We have

$$i_*\big(i^*\varphi \cap [\overline{B}^n_s]\big) \underset{\substack{\uparrow \\ \text{Lemma 59.8}}}{=} \varphi \cap i_*([\overline{B}^n_s]) \underset{\substack{\uparrow \\ \text{since the images of } [\overline{B}^n_s] \text{ and } \mu_{\overline{B}^n_r} \\ \text{in } H_n(B^n, B^n \setminus \{0\}) \text{ agree}}}{=} \varphi \cap \mu_{\overline{B}^n_r} \in H_{n-k}(B^n).$$

(2) We consider the left vertical map of the diagram. Clearly both groups are zero for $k \neq n$. Evidently a generator for $H^n(\overline{B}^n_s, S^n_s)$ is given by the dual fundamental class $[\overline{B}^n_s]^*$. Since $\overline{B}^n_s$ is path-connected we see, following the discussion on page 1378, that $[\overline{B}^n_s]^* \cap [\overline{B}^n_s] = \langle [\overline{B}^n_s]^*, [\overline{B}^n_s] \rangle = 1 \in H_0(\overline{B}^n_s) = \mathbb{Z}$. In particular the left vertical map is an isomorphism.

(3) It follows easily from Corollary 31.16 and Lemma 48.13 that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. We consider the following commutative diagram

$$\begin{tikzcd}
H^k(B^n, B^n \setminus \overline{B}^n_r) \arrow[rr] \arrow[dr] & & \lim\limits_{\substack{\longrightarrow \\ K \in \mathcal{K}(B^n)}} H^k(B^n, B^n \setminus K) = H^k_c(B^n). \\
& \lim\limits_{\substack{\longrightarrow \\ t \in (0,1)}} H^k(B^n, B^n \setminus \overline{B}^n_t) \arrow[ur] &
\end{tikzcd}$$

We make the following two observations:

(a) Every compact subset of B^n is contained in a closed ball $\overline{B}^n_t$ with $t \in (0,1)$. Therefore it follows from Lemma 52.3 that the right diagonal map is an isomorphism.
(b) It follows easily from Lemmas 48.13 and 52.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim. □

In the proof of Theorem 61.15 we will need the following two lemmas.

Lemma 61.17. *Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Then given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.*

FIGURE 752. Illustration of Lemma 61.17.

Proof (∗). Let M be an n-dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Furthermore let C be a compact subset of M. Using charts one can easily show that given any point $x \in C$ there exists a compact set A_x with the following properties:

(1) the point x lies in the interior of A_x,
(2) if $x \in U$ then the set A_x is contained in U,
(3) if $x \in V$ then the set A_x is contained in V.

It follows from (1), the fact that the sets A_x cover all of C together with the hypothesis that C is compact that there exist finitely many $y_1, \ldots, y_r \in C$ with $C \subset \mathring{A}_{y_1} \cup \cdots \cup \mathring{A}_{y_r}$. One can now easily verify that $K := \bigcup_{y_i \in U} (A_{y_i} \cap C)$ and $L := \bigcup_{y_i \in V} (A_{y_i} \cap C)$ have the desired properties. □

Lemma 61.18. *Let M be a compact oriented n-dimensional topological manifold and let U, V be open subsets such that $M = U \cup V$ and let $K \subset U$ and $L \subset V$ be compact subsets. Then there exist singular n-chains $\alpha_{U \setminus L}$, $\alpha_{U \cap V}$ and $\alpha_{V \setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:*

$$\begin{array}{lrcll} \text{(a)} & \mu^M_{K \cup L} & = & [\alpha_{U \setminus L} + \alpha_{U \cap V} + \alpha_{V \setminus K}] & \textit{in } H_n(M, M \setminus (K \cup L)) \\ \text{(b)} & \mu^{U \cap V}_{K \cap L} & = & [\alpha_{U \cap V}] & \textit{in } H_n(U \cap V, U \cap V \setminus (K \cap L)) \\ \text{(c)} & \mu^U_K & = & [\alpha_{U \setminus L} + \alpha_{U \cap V}] & \textit{in } H_n(U, U \setminus K). \end{array}$$

Proof. We pick a representative $\alpha \in C_n(M)$ for $\mu^M_{K \cup L}$. Note that the open sets $U \setminus L, U \cap V$ and $V \setminus K$ cover M. It follows from Lemmas 31.25 and 31.28 that there exist singular chains $\alpha_{U \setminus L}$, $\alpha_{U \cap V}$ and $\alpha_{V \setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that α is homologous in $C_k(M)$

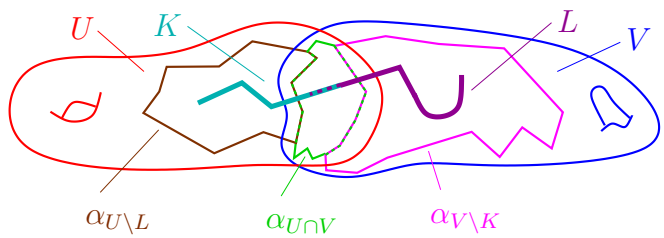


FIGURE 753. Illustration of Lemma 61.18.

to $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$. The chains are in particular homologous in $C_k(M, M \setminus (K \cup L))$ which implies that (a) holds.[1054]

Next note that we have

$$\mu^M_{K\cap L} \underset{\substack{\uparrow \\ \text{Lemma 61.14}}}{=} \text{image of } \mu^M_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] \underset{\substack{\uparrow \\ \text{since } \alpha_{U\setminus L}, \alpha_{V\setminus K} \text{ lie in } C_n(M\setminus(K\cap L))}}{=} [\alpha_{U\cap V}] \in H_n(M, M \setminus (K \cap L)).$$

But $\alpha_{U\cap V}$ lies in $U \cap V$, therefore it follows from Lemma 46.11 (2) that $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $H_n(U \cap V, U \cap V \setminus (K \cap L))$. We have thus verified (b).

Finally the proof of equality (c) is basically the same as the proof of equality (b). □

We recall the following notation and convention:

Convention. Let X be a topological space, let A be a subset and let $k \in \mathbb{N}_0$. As on page 1356 we make the identification

$$C^k(X, A) \xrightarrow{=} \text{all homomorphisms } C_k(X) \to \mathbb{Z} \text{ that vanish on } C_k(A).$$

Furthermore, given another subset B of X we write

$$C^k(X, \{A, B\}) := \text{all homomorphisms } C_k(X) \to \mathbb{Z} \text{ that vanish on } C_k(A) \text{ and on } C_k(B).$$

If A and B are open subsets of X then we use the isomorphism

$$H^k(X, A \cup B) \xrightarrow{\cong} H^k(X, \{A, B\})$$

from Lemma 58.1 to identify the two groups.

In the proof of Theorem 61.15 we will also need the following "inverted" Mayer–Vietoris Theorem, which is very similar to Theorem 33.7.

Theorem 61.19. (The inverted Mayer–Vietoris Theorem for Cohomology groups) *Let X be a topological space and let A and B be open subsets.*

(1) *For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in C^n(X, A \cup B)$ there exist $\varphi_A \in C^n(X, A)$ and $\varphi_B \in C^n(X, B)$ such that $\varphi_A + \varphi_B = \varphi$.*

[1054] Why is $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$ a cycle in $C_*(M, M \setminus (K \cup L))$?

(2) *For all $n \in \mathbb{N}_0$ there exists a unique homomorphism*

$$\delta\colon H^n(X, A \cap B) \;\to\; H^{n+1}(X, A \cup B)$$

which has the following property: for every $\varphi_A \in C^n(X, A)$ and $\varphi_B \in C^n(X, B)$ such that $\varphi_A + \varphi_B$ is a cocycle in $C^n(X, A \cap B)$ we have

$$\delta([\varphi_A + \varphi_B]) \;=\; [\delta\varphi_A] \;=\; -[\delta\varphi_B] \;\in\; H^n(X, \{A, B\}) = H^n(X, A \cup B).$$

(3) *The sequence*

$$\ldots \to H^n(X, A \cup B) \xrightarrow{i^* \oplus -i^*} H^n(X, A) \oplus H_n(X, B) \xrightarrow{i^* \oplus i^*} H^n(X, A \cap B) \xrightarrow{\delta} H^{n+1}(X, A \cup B) \to \ldots$$

is exact.

Proof. We start out with the following two observations:

(1) It is straightforward to verify that the following sequence of cochain maps between cochain complexes is exact:

$$0 \;\to\; C^*(X, \{A, B\}) \;\xrightarrow{i^* \oplus -i^*}\; C^*(X, A) \oplus C^*(X, B) \;\xrightarrow{i^* \oplus i^*}\; C^*(X, A \cap B) \;\to\; 0.$$

(2) Since the subsets A and B are open we know from Proposition 50.12 (4) that the triad (X, A, B) is excisive. Therefore it follows from Lemma 58.1 that the obvious map $C^*(X, A \cup B) \to C^*(X, \{A, B\})$ induces for every $n \in \mathbb{N}_0$ an isomorphism $H^n(X, A \cup B) \xrightarrow{\cong} H^n(X, \{A, B\})$.

With these two observations it is straightforward to modify the proof of Theorem 33.5 to obtain the theorem. We leave it to the reader to fill in the details. □

The key to the proof of Theorem 61.15 is the following lemma.

Lemma 61.20. *Let M be an oriented n-dimensional topological manifold with empty boundary that is the union of two open sets U and V. Then there exists a diagram*

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H_c^k(U \cap V) & \xrightarrow{i \oplus -i} & H_c^k(U) \oplus H_c^k(V) & \xrightarrow{i + i} & H_c^k(M) & \longrightarrow & H_c^{k+1}(U \cap V) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & & \downarrow{\scriptstyle \mathrm{PD}_U \oplus \mathrm{PD}_V} & & \downarrow{\scriptstyle \mathrm{PD}_M} & & \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & \\
\ldots \longrightarrow & H_{n-k}(U \cap V) & \xrightarrow{i \oplus -i} & H_{n-k}(U) \oplus H_{n-k}(V) & \xrightarrow{i + i} & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U \cap V) & \longrightarrow \ldots
\end{array}$$

with the following properties:

(1) *the first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$,*
(2) *the lower exact sequence is the long exact Mayer–Vietoris sequence for $M = U \cup V$,*
(3) *the upper horizontal sequence is also exact.*

Proof. Let M be an oriented n-dimensional topological manifold with empty boundary. Throughout the proof, given any two subsets $X \subset Y$ and $i \in \mathbb{N}_0$ we write as before $H^i(Y|X) := H^i(Y, Y \setminus X)$. We note that if U is open and $K \subset U$ is compact we obtain a map

$$- \cap \mu_K^U\colon H^k(U|K) \;\to\; H_{n-k}(U).$$

Now let let U and V be open subsets with $M = U \cup V$. Furthermore let $K \subset U$ and $L \subset V$ be compact subsets. We consider the following diagram

$$\begin{array}{ccccccccc}
\dots \to & H^k(M|K\cap L) & \to & H^k(M|K) \oplus H^k(M|L) & \longrightarrow & H^k(M|K\cup L) & \xrightarrow{\delta} & H^{k+1}(M|K\cap L) & \to \dots \\
& \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong & \\
& H^k(U\cap V|K\cap L) & & H^k(U|K) \oplus H^k(V|L) & & \downarrow \cap\mu^M_{K\cup L} & & H^{k+1}(U\cap V|K\cap L) & \\
& \downarrow \cap\mu^{U\cap V}_{K\cap L} & & \downarrow \cap\mu^U_K \oplus \cap\mu^V_L & & & & \downarrow \cap\mu^{U\cap V}_{K\cap L} & \\
\dots \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \xrightarrow{\partial} & H_{n-k-1}(U\cap V) & \to \dots
\end{array}$$

which we now explain in more detail:

(1) We write $A = M \setminus K$ and $B = M \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from Theorem 61.19.
(2) The lower horizontal sequence is the Mayer-Vietoris sequence coming from Theorem 33.5, corresponding to the decomposition $M = U \cup V$.
(3) The three vertical maps on the top are induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision Theorem 45.8 (2).

The following is the key claim of the proof.

Claim. The first two rectangles commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.

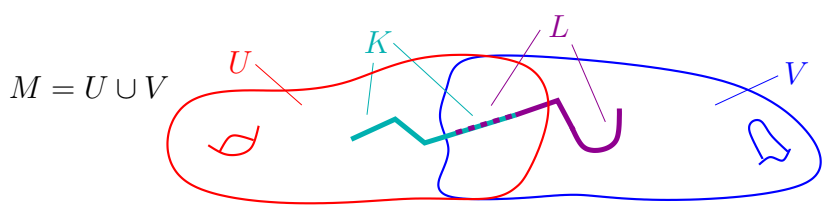


FIGURE 754.

Before we prove the claim we show that it implies Theorem 61.15. For each group appearing in the above diagram we take the direct limit over the directed set[1055]

$$I \;=\; \{(K, L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct limits that were introduced on page 867. We make the following remarks regarding this new diagram:

[1055]We view I as a directed set given by the relation $(K, L) \le (K', L')$ if $K \subset K'$ and $L \subset L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

(1) Using Lemma 52.4 one can show that the groups that we obtain taking the direct limit over I are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\varinjlim_{(K,L)\in I} H^k(U\cap V|K\cap L) \;=\; \varinjlim_{C\in\mathcal{K}(U\cap V)} H^k(U\cap V|C) \;=\; H^k_c(U\cap V).$$

we apply Lemma 52.4 to the cofinal morphism of directed sets given by $(K,L)\mapsto K\cap L$

and

$$\varinjlim_{(K,L)\in I} H^k(M|K\cup L) \;=\; \varinjlim_{C\in\mathcal{K}(M)} H^k(M|C) \;=\; H^k_c(M).$$

we apply Lemma 52.4 to the morphism of directed sets given by $(K,L)\mapsto K\cup L$, this morphism is cofinal by Lemma 61.17

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct limits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in Lemma 61.14 one can easily verify that the direct limit of the diagram does indeed give rise to the diagram specified in the statement of the lemma.

(4) Proposition 34.2 says that the direct limit of exact sequences is again exact.

All these remarks put together show that the direct limit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the lemma.

We turn to the proof the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of Lemma 59.8. The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

Subclaim. The first rectangle of the diagram on page 1434 commutes.

So we need to show that the following diagram commutes:

$$\begin{tikzcd}
H^k(M, M\setminus K\cap L) \arrow[r, "i\oplus -i"] \arrow[d, "\cong"] \arrow[dr, "i\oplus -i"] & H^k(M, M\setminus K)\oplus H^k(M, M\setminus L) \arrow[d, "\cong"] \\
H^k(U\cap V, U\cap V\setminus K\cap L) \arrow[d, "\cap\mu^{U\cap V}_{K\cap L}"] & H^k(U, U\setminus K)\oplus H^k(V, V\setminus L) \arrow[d, "\cap\mu^U_K\oplus\cap\mu^V_L"] \\
H_{n-k}(U\cap V) \arrow[r, "i\oplus -i"] & H_{n-k}(U)\oplus H_{n-k}(V).
\end{tikzcd}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand

summands. So let $\phi \in H^k(M, M \setminus K \cap L)$. We consider the following diagram of maps between pairs of topological spaces

$$\begin{tikzcd}
(U \cap V, U \cap V \setminus K \cap L) \arrow[r, "\alpha"] \arrow[d, "f \circ \alpha"] & (U, U \setminus K \cap L) \arrow[dl, "f"'] \\
(M, M \setminus K \cap L) & (U, U \setminus K). \arrow[l, "f \circ \beta"] \arrow[u, "\beta"]
\end{tikzcd}$$

With this notation we need to show that

$$\alpha_*\big((f \circ \alpha)^*(\phi) \cap \mu^{U \cap V}_{K \cap L}\big) \;=\; (f \circ \beta)^*(\phi) \cap \mu^U_K.$$

But this is indeed the case, since

$$\begin{array}{ccccc}
\alpha_*\big(\underbrace{(f \circ \alpha)^*}_{=\alpha^* \circ f^*}(\phi) \cap \mu^{U \cap V}_{K \cap L}\big) & = & f^*(\phi) \cap \alpha_*(\mu^{U \cap V}_{K \cap L}) & = & f^*(\phi) \cap \mu^U_{K \cap L} \\
 & & \uparrow & & \uparrow \\
 & & \text{Lemma 59.8} & & \text{by Lemma 46.11 we have } \alpha_*(\mu^{U \cap V}_{K \cap L}) = \mu^U_{K \cap L} \\
 & = & \beta_*\big((f \circ \beta)^*(\phi) \cap \mu^U_K\big) & = & (f \circ \beta)^*(\phi) \cap \mu^U_K \\
 & & \uparrow & & \uparrow \\
 & & \begin{array}{c}\text{same argument backwards} \\ \text{with the roles of } \alpha \text{ and } \beta \text{ exchanged}\end{array} & & \text{since } \beta \text{ on the first coordinate is the identity}
\end{array}$$

This concludes the proof of the subclaim.

Subclaim. The second rectangle of the diagram on page 1434 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. We leave it as an exercise to the reader to fill in the details.

Subclaim. The third rectangle of the diagram on page 1434 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{tikzcd}
H^k(M, M \setminus K \cup L) \arrow[r, "\delta"] \arrow[d, "\cap \mu_{K \cup L}"] & H^{k+1}(M, M \setminus K \cap L) \arrow[r, "\cong"] & H^{k+1}(U \cap V, U \cap V \setminus K \cap L) \arrow[d, "\cap \mu_{K \cap L}"] \\
H_{n-k}(M) \arrow[rr, "\partial"] & & H_{n-k-1}(U \cap V).
\end{tikzcd}$$

So let $[\varphi] \in H^k(M, M \setminus K \cup L) = H^k(M, A \cap B)$. We start out with the following preparations:

(1) By Theorem 61.19 (1) we can find $\varphi_A \in C^k(M, A)$ and $\varphi_B \in C^k(M, B)$ such that $\varphi = \varphi_A + \varphi_B$.

(2) By Lemma 61.18 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

$$\begin{array}{llcll}
\text{(2a)} & \mu_{K\cup L} & = & [\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}] & \text{in } H_n(M, M\setminus(K\cup L)) \\
\text{(2b)} & \mu_{K\cap L}^{U\cap V} & = & [\alpha_{U\cap V}] & \text{in } H_n(U\cap V, U\cap V\setminus(K\cap L)) \\
\text{(2c)} & \mu_K^U & = & [\alpha_{U\setminus L}+\alpha_{U\cap V}] & \text{in } H_n(U, U\setminus K).
\end{array}$$

After these preparations we obtain the following equalities in $H_{n-k-1}(U\cap V)$:[1056]

$$\begin{array}{l}
\text{maps in the rectangle applied clockwise to } [\varphi] \;=\; \delta[\varphi]\cap\mu_{K\cap L}^{U\cap V} \overset{\substack{\text{Theorem 61.19 and (2b)}\\ \downarrow}}{=} [\delta\varphi_A\cap\alpha_{U\cap V}] \\
\underset{\substack{\uparrow\\ \text{Lemma 59.1}}}{=} [\varphi_A\cap\partial\alpha_{U\cap V}] \underset{\substack{\uparrow\\ \text{note that (2c) implies that } \partial(\alpha_{U\cap V}+\alpha_{U\setminus L}) \text{ is a singular chain in } U\setminus K,\\ \text{but } \varphi_A \text{ vanishes on singular chains in } A=M\setminus K}}{=} [\varphi_A\cap\partial\alpha_{U\cap V} - \underbrace{\varphi_A\cap\partial(\alpha_{U\cap V}+\alpha_{U\setminus L})}_{=0}] = [-\varphi_A\cap\partial\alpha_{U\setminus L}] \\
\underset{\substack{\uparrow\\ \text{since } \varphi_B \text{ vanishes on singular chains in } B=M\setminus L}}{=} [-\varphi_A\cap\partial\alpha_{U\setminus L} - \underbrace{\varphi_B\cap\partial\alpha_{U\setminus L}}_{=0}] = [-(\varphi_A+\varphi_B)\cap\partial\alpha_{U\setminus L}] \underset{\substack{\uparrow\\ \text{since } \varphi=\varphi_A+\varphi_B}}{=} [-\varphi\cap\partial\alpha_{U\setminus L}] \\
\underset{\substack{\uparrow\\ \text{by Lemma 59.1 and}\\ \text{since } \delta\varphi=0}}{=} [(-1)^{k+1}\cdot\partial(\varphi\cap\alpha_{U\setminus L})] \underset{\substack{\uparrow\\ \text{by the definition of the connecting homomorphism } \partial \text{ in the}\\ \text{Mayer–Vietoris sequence, see Theorem 33.5 (1)}}}{=} (-1)^{k+1}\cdot\partial[\underbrace{(\varphi\cap\alpha_{U\setminus L})}_{\in C_{n-k}(U)} + \underbrace{(\varphi\cap\alpha_{U\cap V}+\varphi\cap\alpha_{V\setminus K})}_{\in C_{n-k}(V)}] \\
\underset{\substack{\uparrow\\ \text{by equality (2a)}}}{=} (-1)^{k+1}\cdot\partial\big([\varphi]\cap\mu_{K\cup L}^M\big) = (-1)^{k+1}\cdot\text{maps in the rectangle applied counterclockwise to } [\varphi].
\end{array}$$

This concludes the proof of the claim and thus also of the lemma. □

Lemma 61.20 actually contains an interesting and useful statement which is worth jotting down separately.

Lemma 61.21. *Let M be an oriented n-dimensional topological manifold with empty boundary and let U be an open subset of M. We denote by $i\colon U\to M$ the inclusion. For any $k\in\mathbb{N}_0$ the following two diagrams commute:*

$$\begin{tikzcd}
H_c^k(U) \arrow[r, "i_*"] \arrow[d, "\mathrm{PD}_U"] & H_c^k(M) \arrow[d, "\mathrm{PD}_M"] \\
H_{n-k}(U) \arrow[r, "i_*"] & H_{n-k}(M)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
H_k(U) \arrow[r, "i_*"] \arrow[d, "\mathrm{PD}_U"] & H_k(M) \arrow[d, "\mathrm{PD}_M"] \\
H_c^{n-k}(U) \arrow[r, "i_*"] & H_c^{n-k}(M).
\end{tikzcd}$$

Proof. The statement that the diagram on the left commutes for any k is contained in Lemma 61.20 (1). Since the maps PD_U and PD_M are isomorphisms the statement that the

[1056]The maps δ and ∂ written in blue are the connecting homomorphisms in the Mayer–Vietoris sequences.

diagram on the right commutes is just a reformulation of the fact that the diagrams on the left commutes. □

Now we turn to the actual proof of Theorem 61.15.

Proof of Theorem 61.15. We fix a dimension n. Throughout this proof by a "manifold" we mean an oriented n-dimensional topological manifold without boundary. We say *Poincaré Duality holds for a manifold M* if for each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_M \colon H_c^k(M) \ \to \ H_{n-k}(M)$$

from Lemma 61.14 is an isomorphism. Our goal is to show that Poincaré Duality holds for all manifolds.

We start out with the following claim.

Claim. Let W be a manifold.

(A) If Poincaré Duality holds for W equipped with an orientation, then it also holds for any other orientation of W.
(B) If W is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for W.
(C) Suppose that W is the union of two open subsets U and V. If Poincaré Duality holds for U, V and $U \cap V$, then it also holds for W.
(D) If W is the union of a nested sequence $U_1 \subset U_2 \subset \dots$ of open subsets such that Poincaré Duality holds for each U_i, then it also holds for W.

So let W be a manifold.

(A) Clearly it suffices to consider the case that W is path-connected. By Lemma 45.21 the path-oriented connected manifold W admits only one other orientation, namely the reverse orientation. Evidently Poincaré Duality also holds for that orientation.
(B) This statement is obvious.
(C) If all maps $\mathrm{PD}_U, \mathrm{PD}_V$ and $\mathrm{PD}_{U\cap V}$ are isomorphisms, then it follows immediately from Lemma 61.20 and the Five Lemma 31.11 that also each PD_W is an isomorphism, i.e. Poincaré Duality holds for $W = U \cup V$.
(D) Let $k \in \mathbb{N}_0$. We suppose that for every $i \in \mathbb{N}$ the map $\mathrm{PD}_{U_i} \colon H_c^k(U_i) \to H_{n-k}(U_i)$ is an isomorphism. Using Lemmas 59.8 and 61.14 it is straightforward to verify that for any $i \le j$ the following diagram commutes:

$$\begin{array}{ccc} H_c^k(U_i) & \xrightarrow{\mathrm{PD}_{U_i}} & H_{n-k}(U_i) \\ \downarrow & & \downarrow \\ H_c^k(U_j) & \xrightarrow{\mathrm{PD}_{U_j}} & H_{n-k}(U_j). \end{array}$$

We obtain isomorphisms

$$\begin{array}{ccccccc} H_c^k(W) & = & \varinjlim H_c^k(U_i) & \xrightarrow{\varinjlim \mathrm{PD}_{U_i}} & \varinjlim H_{n-k}(U_i) & = & H_{n-k}(W). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Proposition 53.6} & & \text{isomorphism by our hypothesis and since the above diagram commutes} & & & & \text{Proposition 34.4} \end{array}$$

We leave it as an exercise to verify that the map from left to right is indeed given by PD_W.

This concludes the proof of the claim.

The following claim completes the proof of the theorem.

Claim.

(1) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.
(2) For every $m \in \mathbb{N}$ Poincaré Duality holds for the union of m open bounded convex subsets of $\mathbb{R}^n$.
(3) Poincaré Duality holds for every open subset of $\mathbb{R}^n$.
(4) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(5) Poincaré Duality holds for all manifolds.

Now we provide the proof of the claim.

(1) Let M be an open bounded convex subset of $\mathbb{R}^n$. By Proposition 1.47 (1) there exists a homeomorphism $M \to B^n$. It follows from Lemma 61.16 and (B) that Poincaré Duality also holds for M.
(2) We prove the claim by induction on m. By (1) we know that the statement holds for $m = 1$. Now suppose it holds for all unions of $m - 1$ open bounded convex subsets of $\mathbb{R}^n$. Let M be the union of m open bounded convex subsets $U_1, \dots, U_m$. We set $V = U_1 \cup \dots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for $V \cap U_m = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$, which is the union of $(m-1)$ open bounded convex subsets.[1057] Hence by (C) the statement also holds for
$$M = V \cup U_m = (U_1 \cup \dots \cup U_{m-1}) \cup U_m.$$
(3) Let M be an arbitrary open subset of $\mathbb{R}^n$. By Lemma 1.9 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \le i} B_j$. By (2) Poincaré Duality holds for each $U_i \in \mathbb{N}$. But then it follows from (D) that Poincaré Duality also holds for M.
(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (3) and (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m - 1$ charts. Let M be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\dots,m}$ with m charts. We write $U = U_1 \cup \dots \cup U_{m-1}$ and $V = U_m$. Both are open

[1057] Here we once again use the convenient fact that the intersection of two convex sets is convex.

subsets of a manifold, hence they are also manifolds. Furthermore U and V are manifolds that admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$ is a manifold that admits an atlas with $m-1$ charts. By induction Poincaré Duality holds for U, V and $U \cap V$. Thus by (C) Poincaré Duality also holds for M itself.

(5) Finally let M be any manifold. We saw in Exercise 3.7 (1), using the fact that M is second-countable and using Lemma 1.26, that M admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j\in\mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j\le i} U_j$. By (4) Poincaré Duality holds for each X_i. It follows from (D) that Poincaré Duality also holds for M. □

61.5. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Poincaré Duality Theorem 61.1. For the reader's convenience we recall the statement.

Theorem 61.1. (Poincaré Duality Theorem) *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in H_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} H^k(M, A; R) & \xrightarrow{\cong} & H_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

Proof of Theorem 61.1 for $A = \partial M$ and $B = \varnothing$. In this proof we consider the case that $A = \partial M$ and $B = \varnothing$. To simplify the notation we will once again only consider the case that $R = \mathbb{Z}$. The proof of the general case is basically the same. We introduce the following objects:

(1) From the Topological Collar Neighborhood Theorem 45.3 we obtain a collar neighborhood $\partial M \times [-1, 0]$ of $\partial M = M \times \{0\}$. We refer to it as the "internal collar".

(2) We define

$$W := M \cup (\partial M \times [0, 1))/\sim \text{ with } x \sim (x, 0) \text{ for every } x \in \partial M.$$

Sometimes $\partial M \times [0, 1)$ is called an *external collar of M*. Furthermore, given any $t \in [-1, 1)$ we write

$$K_t := (M \setminus \partial M \times [-1, 0]) \cup (\partial M \times [-1, t])/\sim .$$

By elementary arguments and using Proposition 32.1 (1) one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary,

(B) W admits an orientation such that M is a submanifold and such that $M \to W$ is orientation-preserving,

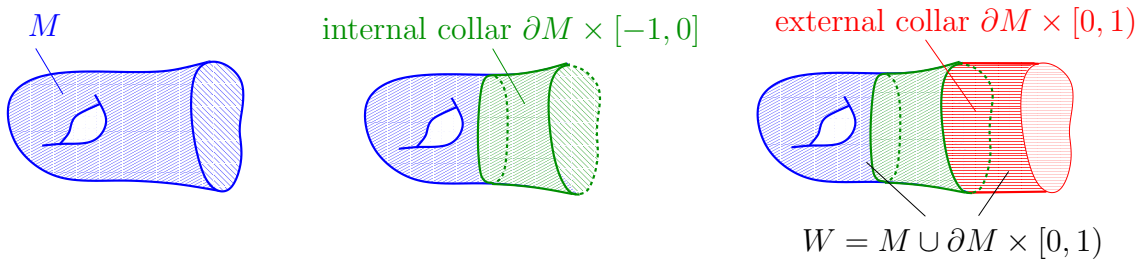


FIGURE 755.

(C) every K_t is a compact subset of W and given any compact subset K of W there exists a $t \in [-1,1)$ with $K \subset K_t$.

We denote by $i\colon (M, \partial M) \to (W, W \setminus K_{-1})$ the obvious inclusion map. We obtain the following diagram:

$$\begin{tikzcd}
H^k(M,\partial M) \arrow[d, "\cap[M]"] & H^k(W, W \setminus K_{-1}) \arrow[l, "i^*"'] \arrow[r] \arrow[d, "\cap \mu_{K_t}"] & H^k_c(W) \arrow[d, "\mathrm{PD}_W"] \\
H_{n-k}(M) \arrow[r, "i_*"] & H_{n-k}(W) \arrow[r, "="] & H_{n-k}(W).
\end{tikzcd}$$

It remains to prove the following claim which resembles closely the claim in the proof of Lemma 61.16.

Claim. The following three statements hold:

(1) the diagram commutes,
(2) the right vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

We turn to the proof of the claim.

(1) The right-hand square commutes by definition of PD_W. Using (B) we can apply the same argument as in the proof of the claim in Lemma 61.16 to show that the left-hand square commutes.
(2) Theorem 61.15 implies that the right vertical map is an isomorphism.
(3) It follows easily from Corollary 31.16 and Lemma 48.13 that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. Similar to the proof of Lemma 61.16 we consider the following

commutative diagram

$$\begin{tikzcd}
H^k(W, W \setminus K_{-1}) \arrow[rr] \arrow[dr] & & \lim\limits_{\substack{\longrightarrow \\ K \in \mathcal{K}(W)}} H^k(W, W \setminus K) = H^k_c(W). \\
& \lim\limits_{\substack{\longrightarrow \\ t \in [-1,1)}} H^k(W, W \setminus K_t) \arrow[ur] &
\end{tikzcd}$$

We make the following two observations:

(a) By (C) every compact subset of W is contained in some K_t. Therefore it follows from Lemma 52.3 that the right diagonal map is an isomorphism.

(b) It follows easily from Lemmas 48.13 and 52.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of Theorem 61.1 in the special case that $A = \partial M$ and $B = \varnothing$. □

Proof of Theorem 61.1 for the general case. Now we finally consider the general case of Theorem 61.1. So we suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We recall that by Proposition 50.12 (5) we know that the triad (M, A, B) is excisive. In this proof, once again we simplify the notation and we only work with $\mathbb{Z}$-coefficients.

We equip ∂M and then also B with the orientations coming from Proposition 45.30 and Lemma 45.28. We consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^k(M, \partial M) \arrow[r] \arrow[d, "\cong"', "-\cap[M]"] & H^k(M, A) \arrow[r] \arrow[d, "-\cap[M]"] & H^k(\partial M, A) \arrow[r, "\delta"] \arrow[d, "\cong"', "i^*"] & H^{k+1}(M, \partial M) \arrow[r] \arrow[d, "\cong"', "-\cap[M]"] & \dots \\
\dots \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k}(M, B) \arrow[r, "\partial"] & H^k(B, \partial B) \arrow[d, "\cong"', "-\cap[B]"] & & \\
& & & H_{n-k-1}(B) \arrow[r] & H_{n-k-1}(M) \arrow[r] & \dots
\end{tikzcd}$$

We make the following observations:

(1) The top horizontal sequence is the long exact sequence in cohomology of the triple $(M, \partial M, A)$ provided by Lemma 48.15.

(2) The bottom horizontal sequence is the long exact sequence in homology of the pair (M, B) provided by Corollary 31.14.

(3) The above vertical map $i^* \colon H^k(\partial M, A) \to H^k(B, \partial B)$ is induced by the inclusion $i \colon (B, \partial B) \to (\partial M, A)$. It is an isomorphism by Proposition 50.12 (5) together with Lemma 50.13 (3).

(4) The other maps decorated with "≅" are isomorphisms by the special case of the Poincaré Duality Theorem 61.1 that we have already dealt with.

By the Five Lemma 31.11 it remains to prove the following claim.

Claim. The above diagram commutes up to signs, more precisely:

(1) the first rectangle commutes,
(2) the second rectangle commutes up to the sign $(-1)^k$,
(3) the third rectangle commutes.

We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{tikzcd}
H^k(M,\partial M) \times H_n(M,\partial M) \arrow[r] & H_{n-k}(M) \arrow[d] \\
H^k(M,\partial M) \times H_n(M,\partial M) \arrow[u, "="] \arrow[d] \arrow[r] & H_{n-k}(M,B) \\
H^k(M,A) \times H_n(M,\partial M) \arrow[u, "="] \arrow[r] & H_{n-k}(M,B) \arrow[u, "="]
\end{tikzcd}$$

We can convince ourselves that the diagram commutes by applying Lemma 59.8 to the maps $(M,\partial M,\varnothing) \to (M,\partial M,B)$ and $(M,A,B) \to (M,\partial M,B)$ of excisive triads. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $j\colon (B,\partial B) \to (\partial M, A)$ the inclusion map and we denote by $\partial\colon H_n(M,\partial M) \to H_{n-1}(\partial M, A)$ the connecting homomorphism in the long exact sequence of the triple $(M,\partial M,A)$. It follows from Proposition 46.26 (1), Lemma 46.23 and the remark on page 780 that the following equality holds

$$(*) \qquad i_*([B]) \;=\; \partial[M] \in H_{n-1}(\partial M, A).$$

Now we continue with the proof of the claim.

(2) We apply Lemma 59.10 (2) to the excisive triad (M,A,B). (Recall that $A \cup B = \partial M$ and $A \cap B = \partial B$.) We obtain that for any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{tikzcd}
H^k(M,A) \times H_n(M,\partial M) \arrow[r, "\cap"] \arrow[d, "i^*"'] & H_{n-k}(M,B) \arrow[dd, "\partial"] \\
H^k(\partial M,A) \times H_{n-1}(\partial M,A) \arrow[d, "j^*"', "\cong"] & \\
H^k(B,\partial B) \times H_{n-1}(B,\partial B) \arrow[r, "\cap"] & H_{n-k-1}(B).
\end{tikzcd}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{array}{ccc}
H^k(B,\partial B)\times H_{n-1}(B,\partial B) & \xrightarrow{\cap} & H_{n-k-1}(B) \\
{\scriptstyle i^*}\uparrow \quad \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
H^k(\partial M,A)\times H_{n-1}(\partial M,A) & \xrightarrow{\cap} & H_{n-k-1}(\partial M) \\
{\scriptstyle \delta}\downarrow \quad \uparrow{\scriptstyle \partial} & & \downarrow{\scriptstyle i_*} \\
H^{k+1}(M,\partial M)\times H_n(M,\partial M) & \xrightarrow{\cap} & H_{n-k-1}(M).
\end{array}$$

The upper part of this diagram commutes by Lemma 59.8 applied to the map $(B,\partial B,\varnothing)\to(\partial M,A,\varnothing)$ between excisive triads. The lower part commutes by Lemma 59.10 (1) applied to the triple $(M,\partial M,A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 61.1. □

Remark. It follows from the Poincaré Duality Theorem 61.1 and Corollary 57.12 that if X is a topological space that is homotopy equivalent to a closed orientable connected n-dimensional manifold, then there exists a $\phi\in H^n(X;\mathbb{Z})$ such that for any $k\in\mathbb{N}_0$ the map

$$-\cap\phi\colon H^k(X;\mathbb{Z})\ \to\ H_{n-k}(X;\mathbb{Z})$$

is an isomorphism. (The same condition also has to hold for all finite covers of X.) One can ask to what degree the converse holds. But in [DP, p. 21] (see also [MM, p. 32]) it is shown that there exists a simply connected 5-dimensional CW-complex X such that the cohomology ring is isomorphic to the cohomology ring of $S^2\times S^3$ but such that X is not homotopy equivalent to a compact manifold.

If M is a compact topological n-dimensional manifold with non-empty boundary then the following proposition relates capping with the fundamental class of M to capping with the fundamental class of ∂M.

Proposition 61.22. *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. As usual we denote by $[M]\in H_n(M,\partial M;R)$ the R-fundamental class. We consider the following diagram:*

$$\begin{array}{ccccccc}
\ldots\to H^k(M,\partial M;R) & \longrightarrow & H^k(M;R) & \longrightarrow & H^k(\partial M;R) & \xrightarrow{\delta} & H^{k+1}(M,\partial M;R)\to\ldots \\
\downarrow{\scriptstyle -\cap[M]} & & \downarrow{\scriptstyle -\cap[M]} & & \downarrow{\scriptstyle -\cap[\partial M]} & & \downarrow{\scriptstyle -\cap[M]} \\
\ldots\longrightarrow H_{n-k}(M;R) & \longrightarrow & H_{n-k}(M,\partial M;R) & \to & H_{n-k-1}(\partial M;R) & \longrightarrow & H_{n-k-1}(M;R)\xrightarrow{\partial}\ldots
\end{array}$$

where the horizontal sequences are the long exact sequences in (co-) homology of the pair $(M,\partial M)$. The first and the third square commute, furthermore the second square commutes up to the sign $(-1)^k$.

Proof. We had implicitly proved the proposition in the above proof of the Poincaré Duality Theorem 61.1 applied to the special case that $A=\varnothing$ and $B=\partial M$. □

Exercises for Chapter 61.

Exercise 61.1.

(a) Let M be a connected closed 3-dimensional manifold. Show that if M is non-orientable, then $H_1(M;\mathbb{Z})$ is infinite.
(b) Is the hypothesis in (a) that M is closed necessary? Justify your answer.

Exercise 61.2.

(a) Let M be an n-dimensional closed orientable connected topological manifold with $\pi_1(M) \cong S_5$. What can you say about $H^{n-1}(M;\mathbb{Z})$? Does the answer depend on whether or not M has boundary?
(b) Let M be a non-compact orientable topological manifold. Show that the group $H_{n-1}(M, \partial M; \mathbb{F}_2)$ is non-zero.

Exercise 61.3. Let $p\colon \widetilde{N} \to N$ be a finite covering of a closed, oriented n-dimensional topological manifold. We equip $\widetilde{N}$ with the orientation given by Lemma 45.28

(a) Let $r \leq s$. Show that given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in C^r(N;\mathbb{Z})$ we have
$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$
Here p^* for homology classes denotes the transfer map.
(b) Show that for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon H_k(N;\mathbb{Z}) \to H_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.
Hint: Proposition 46.24 might be useful.

Exercise 61.4. Is the Klein bottle the boundary of a compact 3-dimensional manifold? Justify your answer.

Exercise 61.5.

(a) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Show that given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.
(b) Does statement (a) hold for any topological space?

Exercise 61.6. Let M be a closed connected 4-dimensional topological manifold that is simply connected and with $\chi(M) = 2$. What can you say about the homeomorphism type of M?

Exercise 61.7. Let W be a closed orientable topological 4-manifold with $\chi(W) = -3$. Show that W admits a finite covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.

Exercise 61.8. Let M be a closed 4-manifold.

(a) Show that if M is simply connected, then there exist at most finitely many isomorphism classes of finite groups which can act discretely on M.

(b) Is the hypothesis in (a) that M is simply connected necessary?

Exercise 61.9. Let M be a compact orientable 4-manifold with $\partial M = A \sqcup B$ where A and B are two connected 3-manifolds with $H_1(A;\mathbb{Z})$ finite. We suppose that the inclusion induced map $H_1(A;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is an epimorphism. Show that the inclusion induced map $H_1(B;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is a monomorphism.

62. The cup product of projective spaces

In this chapter we will use the Poincaré Duality Theorem 61.1 to determine the cup product on $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. These calculations will allow us to prove several interesting consequences which at first glance have very little to do with cup products.

62.1. **Non-singular pairings and the cup product.** Before we start drawing more conclusions from the Poincaré Duality Theorem 61.1 we need to introduce some purely algebraic definitions.

Definition. Let R be a commutative ring and let V, W and S be R-modules.

(1) A *pairing* on $V \times W$ is a map

$$V \times W \ \to \ S$$

that is R-bilinear, i.e. that is R-linear in V and that is R-linear in W.[1058]

(2) We say that a pairing $\langle\ ,\ \rangle\colon V \times W \to S$ is *non-singular* if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}$$

are isomorphisms.

We start out with the following elementary lemma.

Lemma 62.1. *Let R be a commutative ring. Let V and W be free R-modules of the same rank and let*

$$\langle\ ,\ \rangle\colon V \times W \to R$$

be a pairing.

(1) *We denote by n the rank of V. Let $v_1,\dots,v_n$ be a basis for V and let $w_1,\dots,w_n$ be a basis for W. Then the following statement holds:*

$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad\Longleftrightarrow\quad \det(\langle v_i,w_j\rangle)_{i,j=1,\dots,n}) \text{ is a unit of } R.$$

(2) *Suppose V and W are of rank one. If v and w are generators of V respectively W, then the following statement holds:*

$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad\Longleftrightarrow\quad \langle v,w\rangle \text{ is a unit of the ring } R.$$

Proof. The second statement is evidently just a special case of the first statement. So it suffices to prove the first statement. Let $v_1,\dots,v_n$ be a basis for V and let $w_1,\dots,w_n$ be a basis for W. We denote by $v_1^*,\dots,v_n^* \in \mathrm{Hom}(V,R)$ and $w_1^*,\dots,w_n^* \in \mathrm{Hom}(W,R)$ the dual bases. We consider the maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,R) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & R \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & R \\ v & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}.$$

[1058] In most examples we will be interested in $S = R$, but later on we will, oddly enough, also develop a fondness for $R = \mathbb{Z}$ and $S = \mathbb{Q}/\mathbb{Z}$.

Since $\langle\, ,\, \rangle$ is R-bilinear both maps are R-homomorphisms. Furthermore, it is straightforward to show that with respect to these bases both the R-homomorphisms are represented by the transpose of the matrix $(\langle v_i, w_j\rangle_{i,j=1,\dots,n})$. The lemma is an immediate consequence of this observation. $\square$

Remark. The proof of Lemma 62.1 shows that if V and W are free R-modules of finite rank, then the pairing is non-singular if and only if either of the two maps $V \to \operatorname{Hom}(W, R)$ or $W \to \operatorname{Hom}(V, R)$ is an isomorphism.[1059]

Now we return to topology.

Proposition 62.2. *Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *Suppose M is oriented with fundamental class $[M]$. If $H_{k-1}(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$ are torsion-free, then the cup product pairing*
$$\begin{array}{rcl} H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

(2) *Let $\mathbb{F}$ be a field. If M is $\mathbb{F}$-oriented with fundamental class $[M]$, then for any $k \in \mathbb{N}_0$ the cup product pairing*
$$\begin{array}{rcl} H^k(M;\mathbb{F}) \times H^{n-k}(M;\mathbb{F}) & \to & \mathbb{F} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

Proof. Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We suppose that M is oriented with fundamental class $[M]$ and we suppose that $H_{k-1}(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$ are torsion-free. It follows from Proposition 45.2, our hypothesis on $H_{k-1}(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$, the classification of finitely generated abelian groups, Lemma 51.5 (3) and the Universal Coefficient Theorem 51.13 that $H_{k-1}(M;\mathbb{Z})$, $H_{n-k-1}(M;\mathbb{Z})$, $H^k(M;\mathbb{Z})$ and $H^{n-k}(M;\mathbb{Z})$ are finitely generated free abelian groups. Therefore by the remark on page 1448 it suffices to show that the map
$$\begin{array}{rcl} \Omega\colon H^{n-k}(M;\mathbb{Z}) & \to & \operatorname{Hom}(H^k(M;\mathbb{Z}), \mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M]\rangle) \end{array}$$
is an isomorphism. We start out with the following claim.

Claim. The following diagram commutes:[1060]
$$\begin{tikzcd} H^{n-k}(M;\mathbb{Z}) \arrow[rr, "\Omega"] \arrow[dr, "\mathrm{ev}"'] & & \operatorname{Hom}(H^k(M;\mathbb{Z}), \mathbb{Z}). \\ & \operatorname{Hom}_{\mathbb{Z}}(H_{n-k}(M;\mathbb{Z}), \mathbb{Z}) \arrow[ur, "(\cap[M])^*"'] & \end{tikzcd}$$

[1059] Is the hypothesis on V and W necessary to draw this conclusion?

[1060] The right diagonal is the dual map corresponding to the map $H^k(M;\mathbb{Z}) \xrightarrow{\cap[M]} H_{n-k}(M;\mathbb{Z})$.

We start out with the observation that for any $\psi \in H^{n-k}(M;\mathbb{Z})$ and $\varphi \in H^k(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:

$$\Omega(\psi)(\varphi) \underset{\substack{\uparrow \\ \text{by definition of } \Omega \\ \text{and Lemma 59.4}}}{=} (\varphi \cup \psi) \cap [M] \underset{\substack{\uparrow \\ \text{Lemma 59.7}}}{=} \psi \cap (\varphi \cap [M]) \underset{\substack{\uparrow \\ \text{Lemma 59.4}}}{=} \langle \psi, \varphi \cap [M] \rangle = \mathrm{ev}(\psi)(\varphi \cap [M]).$$

But this observation immediately shows that the diagram commutes. This concludes the proof of the claim.

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms. The left diagonal map is an isomorphism by the Universal Coefficient Theorem 51.13 for Cohomology Groups together with Lemma 51.5 (3) and the fact, pointed out above, that $H_{n-k-1}(M;\mathbb{Z})$ is free abelian. Furthermore the second map is an isomorphism by the Poincaré Duality Theorem 61.1.

(2) Now let $\mathbb{F}$ be a field. We suppose that M is $\mathbb{F}$-oriented with corresponding fundamental class $[M]$. By Proposition 51.18 the map

$$\begin{array}{rcl} \mathrm{ev}\colon H^k(M;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(H_k(M;\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} H_k(M;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{array} \right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces. The proof of (2) is now basically the same as the proof of (1). □

The following corollary will be surprisingly effective in determining cup products on topological manifolds.

Corollary 62.3. *Let M be a closed, connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *Suppose that M is oriented and that $H_{k-1}(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$ are torsion-free. Furthermore suppose that $H^k(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H^{n-k}(M;\mathbb{Z}) \cong \mathbb{Z}$. If $\varphi \in H^k(M;\mathbb{Z})$ and $\psi \in H^{n-k}(M;\mathbb{Z})$ are generators, then*

$$\varphi \cup \psi \;=\; \epsilon \cdot [M]^* \;\in\; H^n(M;\mathbb{Z})$$

for some $\epsilon \in \{-1, 1\}$.

(2) *Let $\mathbb{F}$ be a field such that M is $\mathbb{F}$-orientable. Suppose that $H^k(M;\mathbb{F}) \cong \mathbb{F}$ and $H^{n-k}(M;\mathbb{F}) \cong \mathbb{F}$. If $\varphi \in H^k(M;\mathbb{F})$ and $\psi \in H^{n-k}(M;\mathbb{F})$ are non-zero, then $\varphi \cup \psi$ is non-zero in $H^n(M;\mathbb{F}) \cong \mathbb{F}$.*

Proof. Let M be a closed, connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) By our hypothesis and by Proposition 62.2 (1) the cup product pairing

$$\begin{array}{rcl} H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. It follows from this fact and Lemma 62.1 (2) that $\langle \varphi \cup \psi, [M] \rangle = \pm 1$. But by Lemma 51.15 this just means that $\varphi \cup \psi \in H^n(M;\mathbb{Z})$ is a generator.

(2) This proof is almost identical to the proof of (1), we just need to replace Proposition 62.2 (1) by Proposition 62.2 (2). □

Example. Let $m \neq n \in \mathbb{N}$. As usual we denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Let $1 \in H^0(S^m \times S^n;\mathbb{Z})$ be the neutral element from Proposition 57.7. In Lemma 60.3 we showed that the cohomology of $S^m \times S^n$ is given as follows:

$$H^*(S^m \times S^n;\mathbb{Z}) \;=\; \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot p^*([S^m]^*) \oplus \mathbb{Z}\cdot q^*([S^n]^*) \oplus \mathbb{Z}\cdot [S^m \times S^n]^*.$$

It follows from Corollary 62.3 (1)[1061] that $p^*([S^m]^*) \cup q^*([S^n]^*) = \pm[S^m \times S^n]^*$. This result recovers (up to the sign indeterminacy) the calculation of the non-trivial cup product on $S^m \times S^n$ that we had initially performed in Lemma 60.3 using the somewhat mysterious Product Theorem 60.1. In Exercise 62.3 we will see, that with slightly more effort, one can also prove the same type of result for $m = n$.

Next we consider the topological spaces $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$. It follows from Propositions 5.14 and 12.3 and the discussion on page 697 that $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are both simply connected. Furthermore it follows from Propositions 31.3 and 34.8, the calculation on page 884 and the Universal Coefficient Theorem 51.13 for Cohomology Groups that for any $k \in \mathbb{N}_0$ we have

$$H_k(S^2 \vee S^4;\mathbb{Z}) \;\cong\; H^k(S^2 \vee S^4;\mathbb{Z}) \;\cong\; H^k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\cong\; H_k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4, \\ 0, & \text{otherwise.} \end{cases}$$

We can thus view $S^2 \vee S^4$ as a (co-) homological doppelgänger to $\mathbb{C}\mathrm{P}^2$.

Using Corollary 62.3 (1) we can easily prove the following lemma which implies in particular that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}\mathrm{P}^2$.

Lemma 62.4. *The topological space $S^2 \vee S^4$ is not homotopy equivalent to any closed topological manifold.*

Proof. Let X be a closed topological manifold whose homology and cohomology groups are isomorphic to the corresponding groups of $S^2 \vee S^4$. It follows from the above calculation of homology groups, our hypothesis that X is compact, from Propositions 29.5 and 46.21 and Theorems 46.2 and 46.4 that X is a closed oriented closed connected 4-dimensional topological manifold. But then Corollary 62.3 (1) implies that the cup product

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \;\to\; H^4(X;\mathbb{Z})$$

is non-zero.

On the other hand in Lemma 57.11 we saw that the cup product

$$\cup\colon H^2(S^2 \vee S^4;\mathbb{Z}) \times H^2(S^2 \vee S^4;\mathbb{Z}) \;\to\; H^4(S^2 \vee S^4;\mathbb{Z})$$

[1061] Of course here we also use the fact, with which we are very familiar by now, that the homology groups of $S^m \times S^n$ are torsion-free.

is the zero-map. By Corollary 57.12 and the above calculation of the cup product of X this implies that $S^2 \vee S^4$ and X are not homotopy equivalent. $\square$

Remark. One can ask whether the adjective "closed" in Lemma 62.4 is necessary. It turns out that this is the case. In fact every wedge of finitely many spheres can be viewed as a deformation retract of an open subset of a sufficiently high-dimensional $\mathbb{R}^n$. For the wedge $S^1 \vee S^1$ we sketch the argument in Figure 756. With slightly more care one can also show that this wedge of finitely many spheres is also a deformation retract of some compact n-dimensional manifold of $\mathbb{R}^n$. We leave it as an exercise to provide a formal proof for these statements.

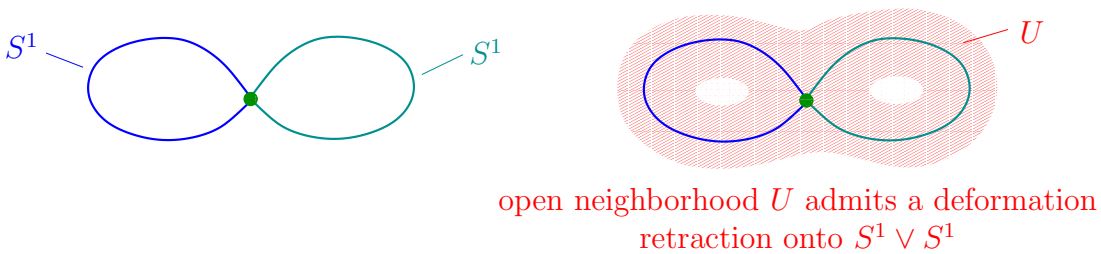


FIGURE 756.

We conclude this section with the following application to de Rham cohomology. We will not make use of it later on.

Corollary 62.5. *If M is a closed oriented n-dimensional manifold, then for any $k \in \mathbb{N}_0$ the pairing*

$$\begin{array}{rcl} H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\varphi],[\psi]) & \mapsto & \int\limits_M \varphi \wedge \psi \end{array}$$

is non-singular.

Since there is no "de Rham homology" the above corollary is usually called the Poincaré Duality Theorem for de Rham cohomology.

Remark. Evidently there are also more direct proofs of Corollary 62.5 that take place within the setting of de Rham cohomology. We refer to [BT, p. 44] or alternatively [War, Theorem 6.13] for such a proof.

Proof. It is straightforward to see that it suffices to deal with the case that M is connected. Theorem 57.19 and the remark on page 1299 give us the following commutative diagram

with vertical isomorphisms:

$$\begin{array}{ccc} H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) & \xrightarrow{(\varphi,\psi)\mapsto \int_M \varphi\wedge\psi} & \mathbb{R} \\ \cong\downarrow \qquad \downarrow\cong & & \downarrow= \\ H^k(M;\mathbb{R}) \times H^{n-k}(M;\mathbb{R}) & \xrightarrow{(\varphi,\psi)\mapsto\langle\varphi\cup\psi,[M]\rangle} & \mathbb{R}. \end{array}$$

By Proposition 62.2 (2) the pairing on the bottom is non-singular. Thus the pairing on the top is also non-singular. □

62.2. **The cup product of $\mathbb{C}\mathrm{P}^n$ and applications.** In this section we apply our acquired wisdom to the complex projective spaces $\mathbb{C}\mathrm{P}^n$. Recall that in Lemma 21.6 we showed that $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional manifold that comes with a canonical orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as an *oriented* manifold. In particular $\mathbb{C}\mathrm{P}^n$ comes with a canonical fundamental class $[\mathbb{C}\mathrm{P}^n] \in H_{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and a canonical dual fundamental class $[\mathbb{C}\mathrm{P}^n]^* \in H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

An astute observer might already have noticed that Corollary 62.3 (1) can be used to compute the cup product on $H^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. In fact in this section we will use Corollary 62.3 (1) and the naturality of the cup product to compute the cohomology ring of any complex projective space $\mathbb{C}\mathrm{P}^n$. Throughout this section, given $m \le n$ we will view $\mathbb{C}\mathrm{P}^m$ as a submanifold of $\mathbb{C}\mathrm{P}^n$ via the embedding

$$\begin{array}{rcl} i\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_m] & \mapsto & [z_0 : \cdots : z_m : 0 : \cdots : 0]. \end{array}$$

We prepare the ground with the following elementary lemma.

Lemma 62.6.

(1) *For every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have*

$$H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; H^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4,\dots,2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let $m \le n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. For any $k \le 2m$ the induced maps $i_*\colon H_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $i^*\colon H^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ are isomorphisms.*

Proof $(*)$. First we recall that in Lemma 26.2 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0,2,\dots,2n$ and which admits no cells in any other dimension. Furthermore we saw that the CW-structure can be chosen in such a way that for $k \in \{0,\dots,n\}$ the $(2k+1)$-skeleton on $\mathbb{C}\mathrm{P}^n$ equals $\mathbb{C}\mathrm{P}^k$.

Furthermore recall that by Propositions 35.4 we know that singular and cellular homology are *naturally* isomorphic. Together with the above discussion of the CW-structures it follows that for every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have

$$H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\xrightarrow{\cong}\; H^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4,\dots,2n, \\ 0, & \text{otherwise} \end{cases}$$

and that for every $m \leq n$ the inclusion induced maps $i_*\colon H_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms for $k \leq 2m$.

In Proposition 50.4 we had also seen that singular and cellular cohomology are naturally isomorphic. So the same logic applies to cohomology groups. □

We also introduce the following notation.

Notation.

(1) Let X be a topological space, let R be a commutative ring and let $\varphi \in H^k(X;R)$. We define $\varphi^0 = 1_X$ and given any $n \in \mathbb{N}$ we define

$$\varphi^n \;=\; \underbrace{\varphi \cup \cdots \cup \varphi}_{n \text{ times}} \;\in\; H^{kn}(X;R).$$

(2) Recall that we denote by $[\mathbb{C}\mathrm{P}^1]^* \in H^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ the dual fundamental class of $\mathbb{C}\mathrm{P}^1$. For any $n \geq 1$ we denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above inclusion map. By Lemma 62.6 we can define $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We refer to $x \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as the *standard generator*.

Now we have the tools to determine the cup product of all complex projective spaces.

Proposition 62.7. *Let $n \in \mathbb{N}$.*

(1) *For every $k \in \{0,\dots,n\}$ the k-th power $x^k \in H^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.*
(2) *If we view $\mathbb{Z}[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*

$$\mathbb{Z}[x]/(x^{n+1}) \;\to\; (H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}), \cup)$$

is an isomorphism of superalgebras.

Proof. It follows immediately from the definitions and Proposition 57.8 that the map in (2) is a homomorphism of superalgebras. This observation, together with Lemma 62.6, shows that (2) is a consequence of (1). Thus it remains to prove (1)

We prove (1) by induction on n. Evidently the statement holds for $n = 1$. Now suppose it holds for some $n-1$ with $n \geq 2$. We denote by $i\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Note that $i^*(x) = x \in H^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$. It follows easily from Lemma 62.6 together with Lemma 57.10 and our induction hypothesis that for each $k \in \{1,\dots,n-1\}$ the cohomology class $x^k \in H^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator. It remains to show that $x^n \in H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is also a generator. This is an immediate consequence of Corollary 62.3 (1), applied to the generator $\varphi := x \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ and the generator $\psi := x^{n-1} \in H^{2(n-1)}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$. □

The previous proof leaves it open whether nor not $x^n \in H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equals the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$. We formulate this as a question which we will answer later on, when we will have obtained more tools for calculating cup products.

Question 62.8. *Let $n \in \mathbb{N}$. Does the n-th power $x^n \in H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equal the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$?*

In Lemma 62.4 we had in particular shown that $\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to its (co-) homological doppelgänger $S^2 \vee S^4$. On page 1050 we pointed out that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ is another example of a pair of topological spaces that cannot be distinguished using the fundamental group and using homology groups. This motivated us to ask in Question 43.9 whether the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are homeomorphic. Now we can answer the question in the negative, in fact we can prove an even stronger result.

Corollary 62.9. *The topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent.*

Proof. In Lemma 60.3 we saw that the cup product

$$H^2(S^2 \times S^4; \mathbb{Z}) \times H^2(S^2 \times S^4; \mathbb{Z}) \quad \to \quad H^4(S^2 \times S^4; \mathbb{Z})$$

is zero, whereas we saw in Proposition 62.7 that the cup product

$$H^2(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \times H^2(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \quad \to \quad H^4(\mathbb{C}\mathrm{P}^3; \mathbb{Z})$$

is non-zero. It follows from Corollary 57.12 that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. □

We recall the following definition from page 1149.

Definition. Let M be a compact oriented topological manifold. We say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*.

On page 1152 we saw that many manifolds are amphichiral, for example we showed that all compact oriented 2-dimensional topological manifolds are amphichiral.

In Question 47.6 we had asked whether there exist compact oriented topological manifolds that are chiral. Now we will see that the complex projective space $\mathbb{C}\mathrm{P}^2$ is chiral which gives an affirmative answer to Question 47.6. Note that the following corollary also gives a negative answer to Question 39.30 (3).

Corollary 62.10. *The manifold $\mathbb{C}\mathrm{P}^2$ is chiral, i.e. there does not exist a self-homeomorphism $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ with $f_*([\mathbb{C}\mathrm{P}^2]) = -[\mathbb{C}\mathrm{P}^2]$.*

Remark. In Exercise 62.4 we will discuss for which n the complex projective space $\mathbb{C}\mathrm{P}^n$ is chiral.

Proof. As usual we denote by $x \in H^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$ the standard generator. By Proposition 62.7 we know that $x \cup x \in H^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$ is a generator. This implies that $x \cup x = \epsilon \cdot [\mathbb{C}\mathrm{P}^2]^*$ for some $\epsilon \in \{-1, 1\}$.

Now let $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ be a self-homeomorphism. Since we want to show that $\mathbb{C}\mathrm{P}^2$ is chiral we need to show that $f_*([\mathbb{C}\mathrm{P}^2]) = [\mathbb{C}\mathrm{P}^2] \in H_4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$, or equivalently, that $f^*([\mathbb{C}\mathrm{P}^2]^*) = [\mathbb{C}\mathrm{P}^2]^* \in H^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$.

Now we see that

$$\begin{aligned} f^*([\mathbb{C}\mathrm{P}^2]^*) &= \epsilon \cdot f^*(x \cup x) \underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=} \epsilon \cdot (f^*(x) \cup f^*(x)) \underset{\substack{\uparrow \\ f^*(x) \text{ is also a generator of } H^2(\mathbb{C}\mathrm{P}^4;\mathbb{Z}), \\ \text{hence } f^*(x) = \eta \cdot x \text{ for some } \eta \in \{-1,1\}}}{=} \epsilon \cdot (\eta \cdot x \cup \eta \cdot x) \\ &\underset{\substack{\uparrow \\ \cup \text{ is bilinear}}}{=} \epsilon \cdot \eta^2 \cdot (x \cup x) \underset{\substack{\uparrow \\ \text{since } \eta \in \{-1,1\}}}{=} \epsilon \cdot (x \cup x) = [\mathbb{C}\mathrm{P}^2]^*. \end{aligned}$$

□

Next we turn to the calculation of the cohomology ring of the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty = \varinjlim \mathbb{C}\mathrm{P}^n$.

Lemma 62.11.

(1) *For every $k \in \mathbb{N}$ we have*

$$H^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong H_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(2) *Let $n \in \mathbb{N}$. We denote by $i\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. For any $k \leq 2n$ the induced maps $i_*\colon H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ and $i^*\colon H^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to H^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms.*

Proof. The statements in (1) and (2) regarding homology groups follow from the discussion on page 885. The statements in (1) and (2) regarding cohomology groups then follow from the Universal Coefficient Theorem 51.13 for Cohomology Groups. □

Remark. One could also prove the lemma using Lemma 62.6 and a suitable generalization of Proposition 52.12.

Notation. We denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. By Lemma 62.11 we can set $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. We refer to x as the *standard generator* of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

Proposition 62.12.

(1) *For every $k \in \mathbb{N}$ the k-th power $x^k \in H^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.*
(2) *If we view $\mathbb{Z}[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*

$$\mathbb{Z}[x] \to (H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}), \cup)$$

is an isomorphism of superalgebras.

Proof. Exactly as in the proof of Proposition 62.7 we see that (2) is a consequence of (1). Thus it remains to prove (1) So let $k \in \mathbb{N}$. We consider the inclusion map $j\colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^\infty$. By Proposition 62.7 and Lemma 57.10 we know that $j^*(x^k) = j^*(x)^k = x^k \in H^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ is a generator. It follows from Lemma 62.11 that $x^k \in H^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is also a generator. □

Remark. It follows from Proposition 60.21 and the discussion on page 1409 that for each $k \in \mathbb{N}$ we also have an isomorphism

$$(H^*(\underbrace{\mathbb{C}\mathrm{P}^\infty \times \ldots \times \mathbb{C}\mathrm{P}^\infty}_{k \text{ copies}}; \mathbb{Z}), \cup) \;\cong\; \mathbb{Z}[x_1, \ldots, x_k]$$

of superalgebras. Here $x_1, \ldots, x_k$ are given by preimages of the standard generator of the group $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ under the k obvious projection maps.

The fact that so far we worked with $\mathbb{Z}$-coefficients was mostly irrelevant. In fact most of the results carry over, with obvious modifications, to the cohomology groups $H^*(\mathbb{C}\mathrm{P}^n; R)$ where R is an arbitrary commutative ring. For future reference we state the following proposition.

Proposition 62.13. *Let $n \in \mathbb{N}$.*

(1) *For every every $k \in \mathbb{N}_0$ we have*

$$H_k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \;\cong\; H^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *We denote by $x \in H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2)$ the unique non-trivial element. If we view the ring $\mathbb{Z}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*

$$\mathbb{Z}_2[x]/(x^{n+1}) \;\to\; (H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2), \cup)$$

is an isomorphism of superalgebras.

The obvious analogues for $\mathbb{C}\mathrm{P}^\infty$ also hold.

Proof. This proposition can be proved in an almost identical way as Proposition 62.12. Alternatively we can deduce the proposition from Proposition 62.12, the Universal Coefficient Theorem 51.25, Lemma 51.29 and Proposition 58.3 (2). We leave it to the impeccably conscientious reader to fill in the details. □

Towards the end of this section we recall the definition of a retract. Thus let A be a subset of a topological space X. We denote by $i\colon A \to X$ the inclusion map. Recall that we say that A is a *retract* of X if there exists a map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. It follows immediately from the functoriality of homology groups and Lemma 33.2 that for each k the group $H_k(A; \mathbb{Z})$ is a subsummand of $H_k(X; \mathbb{Z})$.

Let $m < n$. In Question 38.12 we had asked whether $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is a retract of $\mathbb{C}\mathrm{P}^n$. Note that Lemma 62.6 shows that a study of homology groups does not preclude that for $m < n$ the complex projective space $\mathbb{C}\mathrm{P}^m$ is a retract of $\mathbb{C}\mathrm{P}^n$. Nonetheless, the final result of the section gives a negative answer to Question 38.12, i.e. we show that $\mathbb{C}\mathrm{P}^m$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proposition 62.14. *If $1 \le m < n$, then $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.*

Proof. Let $1 \leq m < n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Suppose there exists a retraction $r\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$. We denote by $x \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ the standard generator. We have

$$\begin{array}{ccccccccc} x^n & = & (r^*(i^*(x)))^n & = & r^*(i^*(x)^n) & = & r^*(0) & = & 0 \in H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}). \\ & \uparrow & & \uparrow & & \uparrow & & & \\ & \begin{array}{c}\text{by Lemma 62.6 the map} \\ i^*\colon H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \\ \text{is an isomorphism, together} \\ \text{with } i^* \circ r^* = (r \circ i)^* = \mathrm{id} \\ \text{we obtain that } r^* \circ i^* = \mathrm{id}\end{array} & & \begin{array}{c}\text{Lemma 57.10}\end{array} & & \begin{array}{c}\text{note that } n > m \text{ implies that } i^*(x)^n \text{ lies} \\ \text{in the trivial group } H^n(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = 0\end{array} & & & & \end{array}$$

But this contradicts the fact shown in Proposition 62.7 that $x^n \neq 0$. □

62.3. **The cup product of $\mathbb{R}\mathbf{P}^n$ and applications.** In this section we intend to determine the cup product of the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. Evidently the approach will be very similar to the one taken in the previous section. First we recall that on page 192 we saw that the real projective space $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional manifold. Given $m \leq n$ we will view $\mathbb{R}\mathrm{P}^m$ as a submanifold of $\mathbb{R}\mathrm{P}^n$ via the embedding

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_m : 0 : \cdots : 0]. \end{array}$$

The following lemma plays the role of Lemma 62.6.

Lemma 62.15.

(1) *For every $n \in \mathbb{N}$ we have*

$$H^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let $m \leq n$. For any $k \leq m$ the induced maps $i_*\colon H_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) \to H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ and $i^*\colon H^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to H^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$ are isomorphisms.*

Proof. The calculation of the isomorphism types of the (co-) homology groups and of the induced maps on homology is basically the same as in the proof of Lemma 62.15. The argument that determines the induced map on $\mathbb{F}_2$-cohomology is slightly different. This time we use Proposition 51.18 which gives us the following commutative diagram:

$$\begin{array}{ccc} H^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\mathbb{F}_2) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ H^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2),\mathbb{F}_2). \end{array}$$

As we had just pointed out, the maps on the right are isomorphisms for $k \leq m$. It follows that the maps on the left are also isomorphisms for $k \leq m$. □

Notation. Given $n \in \mathbb{N}$ we denote by $x \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ the unique non-trivial element.

The following proposition is the analogue of Propositions 62.7 and 62.12.

Proposition 62.16. *Let $n \in \mathbb{N}$.*

(1) *For every $k \in \{0, \dots, n\}$ the k-th power $x^k \in H^k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \mathbb{F}_2$ is non-trivial.*
(2) *If we view $\mathbb{F}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map*
$$\mathbb{F}_2[x]/(x^{n+1}) \ \to \ (H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2), \cup)$$
is an isomorphism of superalgebras.
(3) *If we view $\mathbb{F}_2[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map*
$$\mathbb{F}_2[x] \ \to \ (H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2), \cup)$$
is an isomorphism of superalgebras.

Proof. First recall that Proposition 45.24 implies that any topological manifold, in particular $\mathbb{R}\mathrm{P}^n$, is $\mathbb{F}_2$-orientable. With this observation the proof of the proposition is basically identical to the proofs of Proposition 62.7 and 62.12. We just need to replace Lemma 62.6 by Lemma 62.15 and we need to replace Corollary 62.3 (1) by Corollary 62.3 (2). □

Remark.

(1) In principle one could have tried to compute the cup product on $\mathbb{R}\mathrm{P}^n$ using the same approach that we had taken in Section 57.5. More precisely, suppose one has found an explicit $\mathbb{F}_2$-fundamental class σ for $\mathbb{R}\mathrm{P}^n$. We denote by $\varphi \in C^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ the 1-cocycle from page 1352. Now we know from Proposition 62.16 that $\langle \varphi^n, \sigma \rangle = 1$. But more interestingly, if one shows with an explicit calculation that $\langle \varphi^n, \sigma \rangle = 1$, then this gives a new and elementary proof of Proposition 62.16.
(2) We have just computed the cup product on $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$. We leave it as an exercise to use the same approach to determine the cup product on $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$. This cup product is also determined, using a somewhat different approach, in [Hat, p. 214].
(3) In Corollary 61.11 we showed that there is no map $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$ of degree one, thus giving a negative answer to Question 39.29. On page 1389 and in Proposition 62.16 we had determined the cup product of $S^1 \times S^2$ and $\mathbb{R}\mathrm{P}^3$ on cohomology with $\mathbb{F}_2$-coefficients. Using an argument similar to the proof of Lemma 60.5 one can easily give an alternative proof of Corollary 61.11.

For completeness' sake we mention the following analogue of Proposition 62.14.

Proposition 62.17. *If $1 \le m < n$, then $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.*

Proof. The proof is totally analogous to the proof of Proposition 62.14. In the proof we just need to replace Lemma 62.6 by Lemma 62.15. □

Remark. We can also give an alternative proof of Proposition 62.17 using higher homotopy groups. More precisely, let $1 \le m < n$. By Proposition 28.9 together with Proposition 28.13 we know that $\pi_m(S^n) = 0$. Furthermore by Lemma 38.10 together with Proposition 28.13

we know that $\pi_m(S^m) \neq 0$. Since higher homotopy groups are functorial, see Proposition 28.6, we can use the standard argument, see e.g. the proof of Lemma 6.2, to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

One of the many consequences of our calculations from this chapter is that we can now calculate the Lusternik-Schnirelmann category for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. More precisely, we have the following proposition that gives a complete answer to Question 58.9.

Proposition 62.18. *For any $n \in \mathbb{N}_0$ we have* $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) = \mathrm{cat}(\mathbb{C}\mathrm{P}^n) = n$.

Proof. By the discussion on page 1366 we know that $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and $\mathrm{cat}(\mathbb{C}\mathrm{P}^n) \leq n$. On the other hand it follows from Propositions 62.7, 62.16 and 58.10 that $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) \geq n$ and $\mathrm{cat}(\mathbb{C}\mathrm{P}^n) \geq n$. □

We conclude this section with a new proof for the Borsuk-Ulam Theorem. First we recall the statement of the Borsuk-Ulam Theorem.

Theorem 44.16 (Borsuk-Ulam). *For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.*

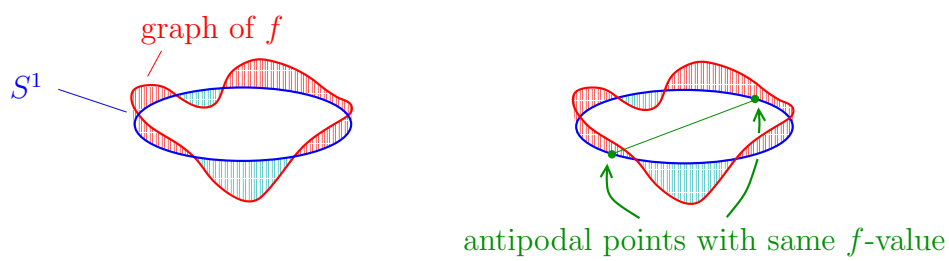


FIGURE 757. Illustration of the Borsuk-Ulam Theorem.

In the new proof of the Borsuk-Ulam Theorem below we will need the following Lemma.

Lemma 62.19.

(1) *Let $k \geq 2$. As usual we denote by $p\colon S^k \to \mathbb{R}\mathrm{P}^k$ the obvious projection map. If $\gamma\colon [0,1] \to S^k$ is a path with $\gamma(1) = -\gamma(0)$, then $p \circ \gamma$ represents the unique non-trivial element in $\pi_1(\mathbb{R}\mathrm{P}^k)$.*

(2) *Let $k, l \geq 2$. Furthermore let $h\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ be a map. If $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then the induced maps $h_*\colon H_1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2) \to H_1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2)$ and $h^*\colon H^1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2) \to H^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ are also isomorphisms.*

Proof of Lemma 62.19. First we recall that for $k \geq 2$ we know from Corollary 7.19 that $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2$.

(1) The first statement is an immediate consequence of Lemma 7.16 (1) and the fact that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

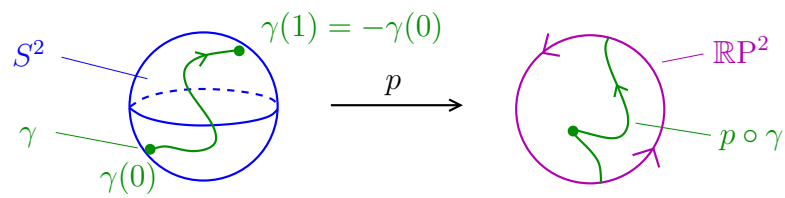


FIGURE 758. Illustration of Lemma 62.19.

(2) Since $k, l \geq 2$ both fundamental groups are isomorphic to $\mathbb{Z}_2$. In particular if $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then this map is already an isomorphism. It follows from the naturality of the Hurewicz homomorphism, see Proposition 38.2 (4), together with the Hurewicz Theorem 38.5 that the induced map $h_*\colon H_1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2) \to H_1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2)$ is also an isomorphism. Finally it follows from the naturality of the isomorphisms in Proposition 51.17 that the induced map $h^*\colon H^1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2) \to H^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ is also an isomorphism. □

Proof of the Borsuk-Ulam 44.16 for $n \geq 2$. Suppose there exists a map $f\colon S^n \to \mathbb{R}^n$ such that for any $x \in S^n$ we have that $f(x) \neq f(-x)$. We consider the map

$$\begin{array}{rcl} g\colon S^n & \to & S^{n-1} \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

By our hypothesis this map is defined. It is clear that $g(-x) = -g(x)$ for all $x \in S^n$. Therefore this map g descends to a map

$$\begin{array}{rcl} h\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [x] & \mapsto & [g(x)] \end{array}$$

i.e. we get the following commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\quad g \quad} & S^{n-1} \\ \downarrow{\scriptstyle p} & & {\scriptstyle q}\downarrow \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{[x]\mapsto h(x):=[g(x)]} & \mathbb{R}\mathrm{P}^{n-1} \end{array}$$

where p and q denote the obvious projection maps.[1062]

Claim. The map $h^*\colon H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is an isomorphism.

By Lemma 62.19 (2) we only have to show that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$ has non-trivial image. Let $\gamma\colon [0,1] \to S^n$ be a path with $\gamma(1) = -\gamma(0)$. Note that $p \circ \gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^n$ is a

[1062]The map $q \circ g\colon S^n \to \mathbb{R}\mathrm{P}^{n-1}$ is continuous. It follows from Lemma 1.66 that the map h is also continuous.

loop. We calculate that

$$h_*([p\circ\gamma]) \;=\; [h\circ p\circ\gamma] \;\underset{\text{by the above commutative diagram}}{\overset{\downarrow}{=}}\; [q\circ(g\circ\gamma)] \;\underset{\text{by Lemma 62.19 (1) since } (g\circ\gamma)(1)=g(\gamma(1))=g(-\gamma(0))=-g(\gamma(0))=-(g\circ\gamma)(0)}{\overset{\uparrow}{\neq}}\; e \;\in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

This concludes the proof of the claim.

We denote by $y \in H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $x \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ the unique non-trivial elements. We have the following equality in $H^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:[1063]

$$0 \;\underset{\text{Proposition 62.16}}{\overset{\uparrow}{\neq}}\; x^n \;\underset{\text{by the claim}}{\overset{\uparrow}{=}}\; (h^*(y))^n \;\underset{\text{Lemma 57.10}}{\overset{\uparrow}{=}}\; h^*(y^n) \;\underset{\text{Proposition 62.16}}{\overset{\uparrow}{=}}\; h^*(0) \;=\; 0.$$

We have thus obtained the desired contradiction. □

62.4. **The cup product of quaternionic and octonion projective spaces** (∗)**.** In this very short section we discuss the cup product of the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ that we introduced on page 1062. We will also calculate the cup product of the octonion projective plane $\mathbb{O}\mathrm{P}^2$ that we defined on page 1063. These calculations might not sound all that interesting, but later on we will use them to show that certain homotopy groups of spheres are non-trivial.

The following proposition is the only, and thus also the most interesting, result of this section. The statement is of course, in light of Proposition 62.7, entirely predictable.

Proposition 62.20.

(1) *Given any $n \in \mathbb{N}$ there exists an isomorphism*

$$\mathbb{Z}[z]/(z^{n+1}) \;\to\; (H^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}),\cup)$$

of superalgebras, where z is an element of degree 4.

(2) *There exists an isomorphism*

$$\mathbb{Z}[z]/(z^3) \;\to\; (H^*(\mathbb{O}\mathrm{P}^2;\mathbb{Z}),\cup)$$

of superalgebras, where z is an element of degree 8.

In both cases the obvious analogues with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$ also hold.

Proof.

(1) Let $n \in \mathbb{N}_0$. By Lemma 44.10 we know that $\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable manifold which admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$. The proof of Proposition 62.7 can be used, with very minor modifications, to prove the desired result.

[1063] The argument is very similar to the argument in the proof of Proposition 62.14.

(2) In Proposition 44.11 we showed that the octonion projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional manifold which admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Once again the proof of Proposition 62.7 clinches the deal. □

Propositions 62.7 and 62.20 do raise a few questions, which can be viewed as a more specific versions of Question 60.19.

The first question asks whether we can realize polynomial rings in other degrees.

Question 62.21. *Let m be an even number and $n \in \mathbb{N}$. Does there exist a topological space X such that $(H^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^{n+1})$ where z is now an element of degree m?*

The following question is even more specific, it asks whether there exists a "cohomological $\mathbb{O}\mathrm{P}^3$.

Question 62.22. *Does there exist a topological space X such that $(H^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^4)$ where z is now an element of degree 8?*

It will note take the reader very long to see that a negative answer to Question 62.22 would also tragically give a negative answer to Question 44.12.

62.5. **Division algebras.** We recall that on page 1054 we defined a *real division algebra* to be an algebra over $\mathbb{R}$ such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$. In Section 44.2 we gave several examples of real division algebras, namely $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ (which are not commutative) and the Cayley Octonions $\mathbb{O}$ (which are neither commutative nor associative). These real division algebras are defined over $\mathbb{R}^n$ with $n = 1, 2, 4$ and 8.

These dimensions are powers of 2 and the following theorem, which was first proved by Heinz Hopf [Hopf2] in 1940, shows that this is not a coincidence.

Theorem 62.23. (Hopf 1940) *If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.*

In the proof of Theorem 62.23 we will need the following lemma which is interesting in its own right.

Lemma 62.24. *Let $k \in \mathbb{N}$. We denote by $\varphi \in H^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ the unique non-trivial element. We denote by p and q the two obvious projection maps $\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$ and we write $\alpha = p^*(\varphi)$ and $\beta = q^*(\varphi)$. Then the obvious homomorphism*

$$\mathbb{F}_2[\alpha, \beta]/(\alpha^{k+1}, \beta^{k+1}) \;\to\; H^*(\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$$

of superalgebras is an isomorphism.

Proof. The lemma is an immediate consequence of Propositions 62.16 and Proposition 60.24 and the discussion on page 1409. □

Now we can provide the proof of Theorem 62.23.

Proof of Theorem 62.23. Let "$*$" be a real division algebra structure on $\mathbb{R}^n$. We want to show that n is a power of 2. Since 1 and 2 are powers of 2 we can assume that $n \geq 3$. As in the proof of Theorem 44.5 we consider the map

$$\begin{array}{rcl} \widetilde{f}\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & \widetilde{f}(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

We make the following observations:

(1) In the proof of Theorem 44.5 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined, i.e. that we never divide by zero.
(2) In the proof of Theorem 44.5 we had also shown that the map $\widetilde{f}$ is continuous.
(3) Since "$*$" defines in particular an algebra we have

$$\widetilde{f}(-x,y) = -\widetilde{f}(x,y) \quad \text{and} \quad \widetilde{f}(x,-y) = -\widetilde{f}(x,y) \quad \text{for any } x,y \in S^{n-1}.$$

From (3) it follows that the above map $\widetilde{f}$ descends to a map[1064]

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [\widetilde{x}], [\widetilde{y}] & \mapsto & [\widetilde{f}(x,y)]. \end{array}$$

We pick a point $z \in S^{n-1}$. We consider the inclusion maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \\ x & \mapsto & (x,[z]) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \\ x & \mapsto & ([z],x). \end{array}$$

Our first goal is to prove the following claim.

Claim 1. The map

$$(f \circ i)_*\colon H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \to H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$$

is an isomorphism. The same conclusion holds with i_* replaced by j_*.

We provide the proof for the first statement. The proof with i_* replaced by j_* is basically the same. By Lemma 62.19 (2) it suffices to show that the map

$$(f \circ i)_*\colon \pi_1(\mathbb{R}\mathrm{P}^{n-1}) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$$

has non-trivial image. We introduce the following notation:

(1) We denote by $\pi\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ the obvious projection map.
(2) We pick a path $\gamma\colon [0,1] \to S^{n-1}$ with $\gamma(1) = -\gamma(0)$. Note that $\pi \circ \gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^{n-1}$ is a loop.
(3) We denote by $\widetilde{i}$ the inclusion map $S^{n-1} \to S^{n-1} \times S^{n-1}$ that is given by $\widetilde{i}(x) = (x,z)$.

[1064]The argument of Footnote 1062 shows that f is continuous.

Note that we get the following commutative diagram

$$\begin{array}{ccccccc}
[0,1] & \xrightarrow{\gamma} & S^{n-1} & \xrightarrow{\widetilde{i}} & S^{n-1}\times S^{n-1} & \xrightarrow{\widetilde{f}} & S^{n-1} \\
 & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi\times\pi} & & \downarrow{\scriptstyle\pi} \\
 & & \mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{i} & \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{f} & \mathbb{R}\mathrm{P}^{n-1}.
\end{array}$$

Finally we note that

$$(f\circ i)_*([\pi\circ\gamma]) \;=\; [f\circ i\circ\pi\circ\gamma] \;\overset{\substack{\text{by the above commutative diagram}\\ \downarrow}}{=}\; \big[\pi\circ(\widetilde{f}\circ\widetilde{i}\circ\gamma)\big] \;\underset{\substack{\uparrow\\ \text{this follows from Lemma 62.19 (1) since the following equality holds:}\\ (\widetilde{f}\circ\widetilde{i}\circ\gamma)(1)=\widetilde{f}(\gamma(1),z)=\widetilde{f}(-\gamma(0),z)=-\widetilde{f}(\gamma(0),z)=-(\widetilde{f}\circ\widetilde{i}\circ\gamma)(0)}}{\neq}\; e \;\in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

This concludes the proof of the claim.

We denote by $\varphi\in H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ the unique non-trivial element. Furthermore, as so often we denote by p and q the two obvious projection maps $\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1}\to\mathbb{R}\mathrm{P}^{n-1}$. Finally we write $\alpha=p^*(\varphi)$ and $\beta=q^*(\varphi)\in H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$.

Claim 2. The following equality holds

$$f^*(\varphi) \;=\; \alpha+\beta \;\in\; H^1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2).$$

We start out the proof of the claim with the following three observations:

(1) Proposition 51.18 says that for any topological space W the map

$$\begin{array}{rcl}
\mathrm{ev}\colon H^1(W;\mathbb{F}_2) & \to & \mathrm{Hom}(H_1(W;\mathbb{F}_2),\mathbb{F}_2) \\
\psi & \mapsto & (\sigma\mapsto\langle\psi,\sigma\rangle)
\end{array}$$

is an isomorphism.

(2) Let $x\in H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ be the unique non-trivial element. By the discussion on page 1317 the map

$$\begin{array}{rcl}
H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)\oplus H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) & \to & H_1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \\
a\oplus b & \mapsto & i_*(a)\oplus j_*(b)
\end{array}$$

is an isomorphism. In particular we see that $H_1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ is generated by $i_*(x)$ and $j_*(x)$.

(3) Since $H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ both contain a unique non-trivial element, namely φ and x, we obtain from Observation (1) that $\mathrm{ev}(\varphi)(x)=\langle\varphi,x\rangle=1$.

It follows from (1) and (2) that it suffices to show that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(x)$ and $j_*(y)$. In fact we have

$$\mathrm{ev}(f^*(\varphi))(i_*(x)) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \langle f^*(\varphi), i_*(x)\rangle \underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=} \langle \varphi, f_*(i_*(x))\rangle \underset{\substack{\uparrow \\ \text{by Claim 1 since } x \text{ is the unique non-zero element of } H_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)}}{=} \langle \varphi, x\rangle \underset{\substack{\uparrow \\ \text{observation (3)}}}{=} 1.$$

We also calculate that

$$\mathrm{ev}(\alpha+\beta)(i_*(x)) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \langle p^*\varphi + q^*\varphi, i_*(x)\rangle \underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=} \langle \varphi, p_*(i_*(x))\rangle + \langle \varphi, q_*(i_*(x))\rangle \underset{\substack{\uparrow \\ \text{since } p\circ i = \mathrm{id} \text{ and } q\circ i \text{ is a constant map}}}{=} \langle \varphi, x\rangle = 1.$$

We have thus shown that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(x)$. The same argument also works with $i_*(x)$ replaced by $j_*(y)$. This concludes the proof of the claim.

Using Lemma 62.24 we can and will make the identification

$$H^*(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) = \mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n).$$

Using this identification we see that

$$0 = f^*(0) \underset{\substack{\uparrow \\ \text{by Proposition 62.16} \\ \text{since } \varphi\in H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)}}{=} f^*(\varphi^n) \underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=} (f^*(\varphi))^n \underset{\substack{\uparrow \\ \text{by the previous claim}}}{=} (\alpha+\beta)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot\alpha^k\beta^{n-k} \underset{\substack{\uparrow \\ \text{since } \alpha^n=\beta^n=0}}{=} \sum_{k=1}^{n-1}\binom{n}{k}\cdot\alpha^k\beta^{n-k}.$$

This equality holds in the polynomial ring $\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n)$. In this ring the $n-1$ monomials $\alpha\beta^{n-1}, \alpha^2\beta^{n-2}, \dots, \alpha^{n-1}\beta$ are linearly independent over the field $\mathbb{F}_2$. This implies that for $k = 1,\dots,n-1$ we have $\binom{n}{k} \equiv 0 \bmod 2$. Thus it remains to prove the following purely algebraic claim.

Claim. If $n\in\mathbb{N}$ satisfies that $\binom{n}{k}\equiv 0 \bmod 2$ for $k=1,\dots,n-1$, then n is a power of 2.

We can and will write n as a sum $n = n_1+\dots+n_r$ where each n_i is a power of 2 and with $n_1<\dots<n_r$. In the polynomial ring $\mathbb{F}_2[x]$ we obtain that

$$1+x^n \overset{\substack{\text{by our hypothesis} \\ \downarrow}}{=} \sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n \overset{\substack{\text{since } n=n_1+\dots+n_r \\ \downarrow}}{=} (1+x)^{n_1}\cdot\dots\cdot(1+x)^{n_r}$$
$$\underset{\substack{\uparrow \\ \text{since each } n_i \text{ is a power of 2 and} \\ \text{since in } \mathbb{F}_2[x] \text{ we have } (a+b)^2=a^2+b^2}}{=} (1+x^{n_1})\cdot\dots\cdot(1+x^{n_r}) \underset{\substack{\uparrow \\ \text{since } n_1<\dots<n_r}}{=} 1+x^{n_1}+\text{higher powers}.$$

But this is only possible if $n=n_1$, i.e. if n itself is a power of 2. This concludes the proof of the claim. $\square$

In fact, with more fancy topological methods one can actually strengthen the previous theorem to obtain the following theorem which gives a complete answer to Question 44.8. Unfortunately the proof is well beyond the scope of these modest lecture notes.

Theorem 62.25. (Bott-Kervaire-Milnor 1958) *Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.*

Proof. The theorem was first proved using topological methods by Michel Kervaire [Kerv] and John Milnor [Mi3, Corollary 1], building on the "Bott Periodicity Theorem" proved by Raoul Bott [Bot, Bot2].[1065] Full details for the proof can also be found in [Hat2, Theorem 2.16]. As of right now there is no purely algebraic proof for the statement. □

Remark. One might ask whether any finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The statement holds under the stronger hypothesis that the algebra is "normed" or "alternative". We refer to [BD, p. 73], [EHHKMNPR, p. 262] and [Bae] for results in this direction. As is shown in [Hat, p. 59], [Bru, Os], without any extra hypotheses one cannot expect a uniqueness statement.

Exercises for Chapter 62.

Exercise 62.1. Let V and W be free $\mathbb{Z}$-modules of finite rank, i.e. V and W are free abelian groups. Furthermore let $\varphi\colon V \times W \to \mathbb{Z}$ be a pairing.

(a) Let $v_1, \dots, v_n$ be a basis for V. Show that the following two statements are equivalent:
  (a) There exists a basis $w_1, \dots, w_n$ for W such that $\lambda(v_i, w_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$.[1066]
  (b) The pairing λ is non-singular.
(b) Suppose that λ is non-degenerate. Show that V and W have the same rank.
(c) Suppose that V and W are of rank n and suppose we are given $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$ such that $\det(\lambda(v_i, w_j)) = \pm 1$. Show that $v_1, \dots, v_n$ is a basis for V and that $w_1, \dots, w_n$ is a basis for W.

Exercise 62.2. Let V be a free $\mathbb{Z}$-module of finite rank, i.e. V is a free abelian group. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form. Given a subgroup $U \subset V$ we write

$$U^\perp \;=\; \{v \in V \mid \varphi(u, v) = 0 \text{ for all } u \in U\}.$$

We suppose that the form φ is non-singular. Show that for any subgroup $U \subset V$ we have $\operatorname{rank}(U^\perp) = \operatorname{rank}(V) - \operatorname{rank}(U)$.

[1065] The Bott Periodicity Theorem says that for any $n \in \mathbb{N}$ we have $\pi_n(\mathrm{O}(\infty)) \cong \pi_{n+8}(\mathrm{O}(\infty))$. Here $\mathrm{O}(\infty)$ is defined as the direct limit of all orthogonal spaces $\mathrm{O}(n)$, $n \in \mathbb{N}$, where $\mathrm{O}(n) \to \mathrm{O}(n+1)$ is the obvious inclusion map. We refer to [Bot, Bot2] and also [Stro, Chapter 32.6] for a proof of this result.

[1066] The basis $w_1, \dots, w_n$ is in fact unique.

Exercise 62.3. Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Use Poincaré Duality to show that

$$p^*([S^m]^*) \cup q^*([S^n]^*) \;=\; \pm[S^m \times S^n]^*.$$

Put differently, you should show the equality without using the Product Theorem 60.1.

Exercise 62.4. For which $n \in \mathbb{N}$ is the complex projective space $\mathbb{C}\mathrm{P}^n$ chiral?

Exercise 62.5. Let $1 \leq m < n$. Show that $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Exercise 62.6. Does there exist a degree one map from $\mathbb{C}\mathrm{P}^2$ to $S^2 \times S^2$?

Exercise 62.7. Is $S^3 \vee \mathbb{R}\mathrm{P}^2$ homotopy equivalent to a manifold?

Exercise 62.8. Show that there exists a connected CW-complex X with base point x_0 and a homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2, *)$ which cannot be realized by a map $g\colon X \to \mathbb{R}\mathrm{P}^2$, i.e. such that there is no map $f\colon X \to \mathbb{R}\mathrm{P}^2$ with $f(x_0) = *$ such that $f_* = \varphi$. Remark: This gives a negative answer to Question 55.10 (3).

63. The Hopf invariant

In the previous chapters we have seen that the introduction and the calculation of the cup product allowed us to answer many questions on topological spaces that we could not deal with before. In this section we want to use the cup product to address the following question that we formulated a long time ago.

Question 28.17. *Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?*

In this chapter we will use the calculation of the cup product of $\mathbb{C}\mathrm{P}^2$ to give an affirmative answer to this question. Furthermore we will use the quaternions and octonions to define higher dimensional analogues of the Hopf map which also represent non-trivial elements in homotopy groups of spheres.

63.1. **The Hopf map is not null-homotopic.** First we recall the definition of the Hopf map. On page 113 we gave an explicit identification $\mathbb{C}\mathrm{P}^1 = S^2$. Furthermore on page 726 we made the following definition.

Definition. We refer to the map[1067]

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the *Hopf map*.

Since the Hopf map is so much fun we will introduce two higher-dimensional analogues of the Hopf map. More precisely, we will repeat the trick with $\mathbb{C}$ replaced by the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. First we note that we can use the homeomorphisms from Lemma 44.10 (2) and Proposition 44.11 (1) to make the identifications $\mathbb{H}\mathrm{P}^1 = S^4$ and $\mathbb{O}\mathrm{P}^1 = S^8$. Now we can introduce the quaternion and octonionic Hopf maps.

Definition. We refer to

$$\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 = \{(z_1, z_2) \in \mathbb{H}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^4 = \mathbb{H}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the *quaternionic Hopf map* and we refer to

$$\begin{array}{rcl} H_{\mathbb{O}}\colon S^{15} = \{(z_1, z_2) \in \mathbb{O}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^8 = \mathbb{O}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the *octonionic Hopf map*.

The idea for answering Question 28.17 is to translate this question about maps to a question about topological spaces for which we can then bring our new invariants to bear. To carry out this approach we recall the following definitions from earlier on:

[1067]See also Figure 461 for an illustration of the map.

Definition.

(1) On page 450 we defined the *cone* of a topological space A to be the topological space $\mathrm{Cone}(A) := (A \times [0,1])/\sim$ where $(a,0) \sim (a',0)$ for all $a, a' \in A$.

(2) On page 458 we defined the *mapping cone* of a map $f\colon A \to X$ between topological spaces as
$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \cup X)/\sim$$
where $(a,1) \sim f(a)$ for all $a \in A$.

Both definitions are illustrated in Figure 759.

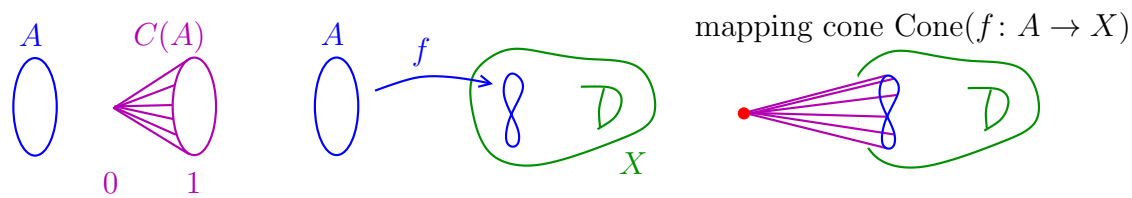


FIGURE 759.

Examples.

(1) It follows from Proposition 1.40 and Lemma 1.66 that the map
$$\begin{array}{rcl} \mathrm{Cone}(S^{n-1}) & \to & \overline{B}^n \\ [(P,t)] & \mapsto & t \cdot P \end{array}$$
is a homeomorphism. We use this homeomorphism to identify $\mathrm{Cone}(S^{n-1})$ with $\overline{B}^n$. In particular, if X is a topological space and if $f\colon S^{n-1} \to X$ is a map, then we get an identification
$$\mathrm{Cone}(f\colon S^{n-1} \to X) \;=\; \overline{B}^n \cup_f X.$$
It follows immediately from this observation that any finite CW-complex can be viewed as an iterated mapping cone (where we start out with a finite discrete set).

(2) In Exercise 63.1 we will see that the mapping cone of the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^2$ is a topological space that we are very familiar with.

The next lemma gives us another elementary but oddly useful example.

Lemma 63.1. *If $f\colon S^k \to X$ is a constant map, then there exists a homeomorphism*
$$\mathrm{Cone}(f\colon S^k \to X) \;\xrightarrow{\cong}\; S^{k+1} \vee X,$$
where we perform the wedge for suitable points in S^{k+1} and X.

Proof. Suppose $f\colon S^k \to X$ is a constant map. It follows easily from the example on page 1469 and the homeomorphism $\overline{B}^{k+1}/S^k \cong S^{k+1}$ from page 99 that $\mathrm{Cone}(f\colon S^k \to X)$ is homeomorphic to a wedge $S^{k+1} \vee X$. We leave it to the reader to fill in the details. $\square$

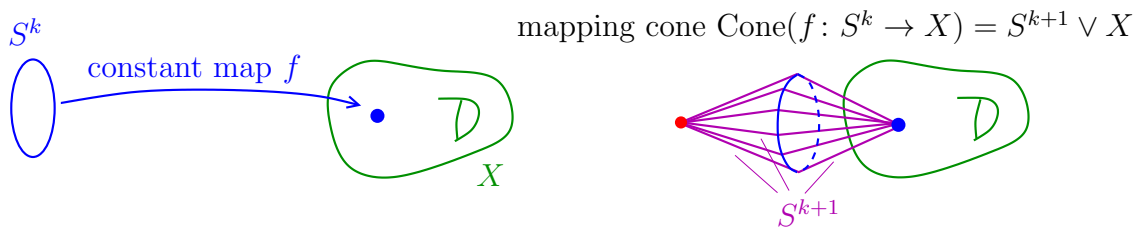


FIGURE 760. Illustration of Lemma 63.1.

The following lemma gives a particularly interesting example of a mapping cone.

Lemma 63.2. *There exist homeomorphisms*

$$\begin{array}{lrcl} (1) & \mathrm{Cone}(H\colon S^3 \to S^2) & \cong & \mathbb{C}\mathrm{P}^2 \\ (2) & \mathrm{Cone}(H_{\mathbb{H}}\colon S^7 \to S^4) & \cong & \mathbb{H}\mathrm{P}^2 \\ (3) & \mathrm{Cone}(H_{\mathbb{O}}\colon S^{15} \to S^8) & \cong & \mathbb{O}\mathrm{P}^2. \end{array}$$

Proof.

(1) In Lemma 26.2 we saw that $\mathbb{C}\mathrm{P}^2$ is a CW-complex that is obtained from the CW-complex $\mathbb{C}\mathrm{P}^1 = S^2$ by attaching a 4-cell. The attaching map $S^3 \to \mathbb{C}\mathrm{P}^1$ of the 4-cell is precisely the Hopf map. This observation, together with the discussion on page 1469, already implies the statement. For completeness' sake we give the argument more explicitly and in more detail. We consider the maps[1068]

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^3) & \to & \mathbb{C}\mathrm{P}^2 \\ [(z_0, z_1), r] & \mapsto & [(r \cdot z_0, r \cdot z_1, 1-r)] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0 : z_1] & \mapsto & [z_0 : z_1 : 0]. \end{array}$$

It follows immediately from the definitions that these two maps define a map

$$\underbrace{\mathrm{Cone}(H\colon S^3 \to S^2)}_{=(\mathrm{Cone}(S^3)\cup\mathbb{C}\mathrm{P}^1)/\sim} \to \mathbb{C}\mathrm{P}^2$$

which is easily seen to be a bijection. It follows from Lemma 1.66 that this map is continuous. Furthermore, using Proposition 1.40 and Lemma 1.81 one sees that the map is in fact a homeomorphism.

(2) The proof of (2) is basically identical to the proof of (1).

(3) The proof of (3) is similar to the proof of (1), we refer to [Lackm, p. 4] for details. □

Now we can finally answer Question 28.17 in the affirmative. More precisely we have the following theorem. The statement regarding the "classical" Hopf map was first proved by Heinz Hopf [Hopf] in 1931.

Theorem 63.3.

(1) *The Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.*

[1068] It follows from Lemma 1.66 and Lemma 1.65 (3) that these two maps are continuous.

(2) *We have* $[H_{\mathbb{H}}] \neq 0 \in \pi_7(S^4)$.
(3) *We have* $[H_{\mathbb{O}}] \neq 0 \in \pi_{15}(S^8)$.

Proof. First we prove (1). Recall that for two topological spaces X and Y we write $X \simeq Y$ if they are homotopy equivalent. We have

$$\begin{array}{ccccccc} \text{Cone(Hopf map } H) & \cong & \mathbb{C}\mathrm{P}^2 & \not\simeq & S^2 \vee S^4 & \cong & \text{Cone(constant map)}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 63.2 (1)} & & \text{Lemma 62.4} & & \text{Lemma 63.1} & \end{array}$$

But by Lemma 15.11 this means that the Hopf map is not homotopic to the constant map. In particular the Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.

The proof of the other two statements is quite similar. We again use Lemma 63.2 and we use the calculations of the cup products of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, see Proposition 62.20, to conclude that the mapping cones of the quaternion and octonionic Hopf maps are not homotopy equivalent to a wedge of spheres. □

63.2. **The Hopf invariant.** It is natural to ask how far we can push the methods of Theorem 63.3 to gain more insight into homotopy groups of spheres. In this section, given $n \geq 2$ we will introduce the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \ \to \ \mathbb{Z}$$

which satisfies $\mathrm{Hopf}(H) = \pm 1$, $\mathrm{Hopf}(H_{\mathbb{H}}) = \pm 1$ and $\mathrm{Hopf}(H_{\mathbb{O}}) = \pm 1$.

To introduce the Hopf invariant we need a careful study of the cohomology groups of a mapping cone. Recall that in Lemma 33.18 we saw that the homology groups of a mapping cone fit naturally into a long exact sequence of homology groups. The following lemma shows that the analogous statement also holds for the cohomology groups of a mapping cone.

Lemma 63.4. *Let R be a commutative ring.*

(1) *Let $f\colon A \to X$ be a map between topological spaces.*
   (a) *We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$ the obvious inclusion map.*
   (b) *We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also Figure 761.) We consider the following diagram*

$$\begin{tikzcd} \dots \arrow[r] & H^{k-1}(U \cap V; R) \arrow[r, "\delta"] \arrow[d, "="] & H^k(\mathrm{Cone}(f); R) \arrow[r] & H^k(U; R) \oplus H^k(V; R) \arrow[r] & \dots \\ & H^{k-1}(A; R). \arrow[urr, "=:\delta^{k-1}(f)"'] & & & \end{tikzcd}$$

Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 50.15 for Cohomology Groups[1069] *and the vertical map comes from the obvious identifications* $H^{k-1}(U \cap V; R) = H^{k-1}(A \times \frac{1}{2}; R) = H^{k-1}(A; R)$. *From the above diagram we obtain the map*

$$\delta^{k-1}(f)\colon H^{k-1}(A; R) \ \to \ H^k(\mathrm{Cone}(f); R).$$

The following sequence is exact

$$\ldots \to \widetilde{H}^k(\mathrm{Cone}(f); R) \xrightarrow{j(f)^*} H^k(X; R) \xrightarrow{f^*} H^k(A; R) \xrightarrow{\delta^k(f)} \widetilde{H}^{k+1}(\mathrm{Cone}(f); R) \to \ldots$$

(2) *The connecting homomorphism in* (1) *is natural. In particular given any commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{X} \end{array}$$

of maps between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots \to & \widetilde{H}^k(\mathrm{Cone}(f); R) & \xrightarrow{j(f)^*} & H^k(X; R) & \xrightarrow{f^*} & H^k(A; R) & \xrightarrow{\delta^k(f)} & \widetilde{H}^{k+1}(\mathrm{Cone}(f); R) & \to \ldots \\ & \uparrow & & {\scriptstyle\psi^*}\uparrow & & {\scriptstyle\varphi^*}\uparrow & & \uparrow & \\ \ldots \to & \widetilde{H}^k(\mathrm{Cone}(\widetilde{f}); R) & \xrightarrow{j(\widetilde{f})^*} & H^k(\widetilde{X}; R) & \xrightarrow{\widetilde{f}^*} & H^k(\widetilde{A}; R) & \xrightarrow{\delta^k(\widetilde{f})} & \widetilde{H}^{k+1}(\mathrm{Cone}(\widetilde{f}); R) & \to \ldots \end{array}$$

Here the vertical maps between the cohomology groups of the mapping cones are induced by the maps from Lemma 15.9 (3).

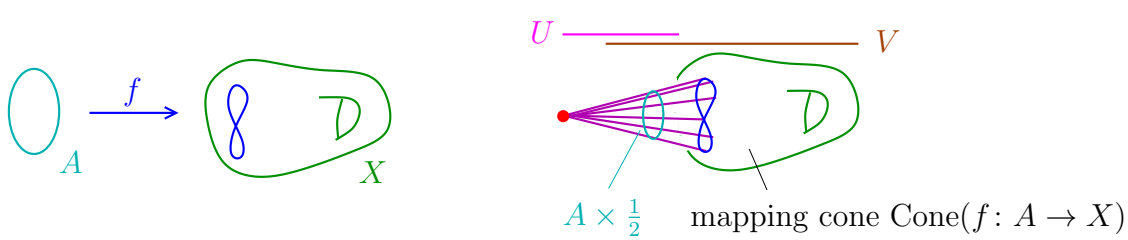


FIGURE 761. Illustration of Lemma 63.4.

Proof. The proof is basically identical to the proof of Lemma 33.18. The meticulous reader who filled in the details of the proof of Lemma 33.18 will have no troubles working out the details of the present proof. □

The following lemma is an immediate consequence of Lemma 63.4.

[1069]Note that U and V are open subsets of $\mathrm{Cone}(f)$. Thus it follows from Proposition 50.12 that we can in fact apply the theorem.

Lemma 63.5. *Let $f\colon S^m \to S^n$ be a map. If $m \geq n+1$ and $n \geq 1$, then the maps*

$$H^n(\mathrm{Cone}(f);\mathbb{Z}) \xrightarrow{j(f)^*} H^n(S^n;\mathbb{Z})$$

and

$$H^m(S^m;\mathbb{Z}) \xrightarrow{\delta^{2n-1}(f)} H^{m+1}(\mathrm{Cone}(f);\mathbb{Z})$$

are isomorphisms and all other reduced cohomology groups of $\mathrm{Cone}(f)$ *are zero. An analogous statement holds for homology groups. In particular we have isomorphisms*

$$H_k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \cong H^k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, n, m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Throughout this section we use the notation that we had already introduced in the formulation of Lemma 63.4. More precisely, given a map $f\colon A \to X$ between two topological spaces we denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$ the obvious inclusion map and furthermore we denote by $\delta^{k-1}(f)\colon H^{k-1}(A;\mathbb{Z}) \to H^k(\mathrm{Cone}(f);\mathbb{Z})$ the coboundary map in the long exact sequence in cohomology of Lemma 63.4.

Definition. Let $f\colon S^{2n-1} \to S^n$ be a map with $n \geq 2$. The *Hopf invariant* $\mathrm{Hopf}(f)$ is defined to be the unique integer that satisfies[1070]

$$\underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\in H^n(\mathrm{Cone}(f);\mathbb{Z})} \cup \underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\in H^n(\mathrm{Cone}(f);\mathbb{Z})} = \mathrm{Hopf}(f) \cdot \underbrace{\delta^{2n-1}(f)([S^{2n-1}]^*)}_{\in H^{2n}(\mathrm{Cone}(f);\mathbb{Z})}$$

in $H^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$.

Remark. It follows immediately from the (anti-) commutativity of the cup product, see Proposition 57.8, that the Hopf invariant is zero for all odd n.

Example.

(1) As usual we denote by $H\colon S^3 \to S^2$ the Hopf map. In Lemma 63.2 we showed that the mapping cone $\mathrm{Cone}(H)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$. By Lemma 63.5 we know that $(j(H)^*)^{-1}([S^2]^*) \in H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and $\delta^3(f)([S^3]^*) \in H^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ are generators. It follows from the calculation of the cup product on $\mathbb{C}\mathrm{P}^2$, see Proposition 62.7, that $\mathrm{Hopf}(H) = -1$ or $\mathrm{Hopf}(H) = 1$.[1071]

(2) Using Proposition 62.20 one can show, by the same argument as in (1), that the Hopf invariants of the quaternionic Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$ and the octonionic Hopf map $H_{\mathbb{O}}\colon S^{15} \to S^8$ are equal to ± 1.

Lemma 63.6. *Let $n \geq 2$ and let $f_0, f_1\colon S^{2n-1} \to S^n$ be two maps. If f_0 and f_1 are homotopic, then* $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(f_1)$.

[1070] Here we implicitly use that in Lemma 63.5 we showed that $j(f)^*$ and $\delta^{2n-1}(f)$ are isomorphisms.

[1071] With more care one can show that in fact $\mathrm{Hopf}(H) = 1$, but this is irrelevant for our purpose.

Proof $(*)$. Let $H\colon S^{2n-1}\times[0,1]\to S^n$ be a homotopy between f_0 and f_1. We denote by $i_0,i_1\colon S^{2n-1}\to S^{2n-1}\times[0,1]$ the two obvious inclusion maps. We obtain the following two commutative diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_0"] \arrow[d, "i_0"'] & S^n \arrow[d, "="] \\ S^{2n-1}\times[0,1] \arrow[r, "H"] & S^n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S^{2n-1} \arrow[r, "f_1"] \arrow[d, "i_1"'] & S^n \arrow[d, "="] \\ S^{2n-1}\times[0,1] \arrow[r, "H"] & S^n. \end{tikzcd}$$

By Lemma 15.9 (3) the diagrams give rise to inclusion maps $\Xi_0\colon \operatorname{Cone}(f_0)\to\operatorname{Cone}(H)$ and $\Xi_1\colon \operatorname{Cone}(f_1)\to\operatorname{Cone}(H)$. From Lemma 63.4 we obtain the following commutative diagram of long exact sequences:

$$\begin{tikzcd} \dots \arrow[r] & \widetilde{H}^k(\operatorname{Cone}(f_0)) \arrow[r, "j(f_0)^*"] & H^k(S^n) \arrow[r, "f_0^*"] & H^k(S^{2n-1}) \arrow[r, "\delta"] & \widetilde{H}^{k+1}(\operatorname{Cone}(f_0)) \arrow[r] & \dots \\ \dots \arrow[r] & \widetilde{H}^k(\operatorname{Cone}(H)) \arrow[r, "j(H)^*"] \arrow[u, "\Xi_0^*"] \arrow[d, "\Xi_1^*"'] & H^k(S^n) \arrow[r, "H^*"] \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & H^k(S^{2n-1}\times[0,1]) \arrow[r, "\delta"] \arrow[u, "\cong"', "i_0^*"] \arrow[d, "\cong"', "i_1^*"] & \widetilde{H}^{k+1}(\operatorname{Cone}(H)) \arrow[r] \arrow[u, "\Xi_0^*"] \arrow[d, "\Xi_1^*"'] & \dots \\ \dots \arrow[r] & \widetilde{H}^k(\operatorname{Cone}(f_1)) \arrow[r, "j(f_1)^*"] & H^k(S^n) \arrow[r, "f_1^*"] & H^k(S^{2n-1}) \arrow[r, "\delta"] & \widetilde{H}^{k+1}(\operatorname{Cone}(f_1)) \arrow[r] & \dots \end{tikzcd}$$

Note that the vertical maps between the cohomology groups of the mapping cones are isomorphisms. This can be shown using the Five Lemma 31.11, or alternatively, in a more fancy way using Lemma 15.11.

Next we consider the generator $\Theta := (i_0^*)^{-1}([S^{2n-1}]^*) = (i_1^*)^{-1}([S^{2n-1}]^*)$ of the group $H^{2n}(\operatorname{Cone}(H);\mathbb{Z})$. We define a number $\operatorname{Hopf}(H)$ via the equality

$$(j(H)^*)^{-1}([S^n]^*)\cup(j(H)^*)^{-1}([S^n]^*) \;=\; \operatorname{Hopf}(H)\cdot\delta^{2n-1}(H)(\Theta)$$

in $H^{2n}(\operatorname{Cone}(H);\mathbb{Z})\cong\mathbb{Z}$. We apply the above commutative diagram and we apply Lemma 57.10 to the maps $\Xi_0\colon \operatorname{Cone}(f_0)\to\operatorname{Cone}(H)$ and $\Xi_1\colon \operatorname{Cone}(f_1)\to\operatorname{Cone}(H)$ and we see that $\operatorname{Hopf}(f_0)=\operatorname{Hopf}(H)=\operatorname{Hopf}(f_1)$. □

It follows from Lemma 63.6 that the Hopf invariant descends to a well-defined map on homotopy groups:

Definition. It follows from Lemma 63.6 that for each $n\geq 2$ the map

$$\begin{array}{rcl} \pi_{2n-1}(S^n,*) & \to & \mathbb{Z} \\ [f] & \mapsto & \operatorname{Hopf}(f) \end{array}$$

is well-defined. This map is called the *Hopf invariant*.

Proposition 63.7. *For any $n\geq 2$ the Hopf invariant* $\operatorname{Hopf}\colon \pi_{2n-1}(S^n,*)\to\mathbb{Z}$ *is a homomorphism.*

Together with the calculation of the Hopf invariant for the Hopf maps H, $H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ on page 1473 we obtain the following immediate corollary to Proposition 63.7.

Corollary 63.8. *The Hopf invariants*

$$\begin{array}{lrcl} (1) & \mathrm{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ (2) & \mathrm{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ (3) & \mathrm{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms.

The technically slightly dicey proof of Proposition 63.7 will require the remainder of this section. The reader is cordially invited to skip the argument and to jump to the much more entertaining content of the next section.

Our proof[1072] of Proposition 63.7 will require a few preparations. First of all, on several occasions we will need the following construction.

Construction 63.9. *Suppose we are given the following commutative diagram of topological spaces:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y}. \end{tikzcd}$$

Then, as discussed in Lemma 15.9 (3), we obtain an induced map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(f\colon \widetilde{X} \to \widetilde{Y}) \\ [P] & \mapsto & \begin{cases} [(\varphi(x), t)], & \text{if } P = (x,t) \text{ with } x \in X,\ t \in [0,1], \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}$$

By Lemma 15.9 (4) the map fits into the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "\psi"'] & \mathrm{Cone}(f\colon X \to Y) \arrow[r, "p"] \arrow[d] & \Sigma(X) \arrow[d, "\Sigma(\varphi)"] \\ \widetilde{Y} \arrow[r, hook, "j"] & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) \arrow[r, "p"] & \Sigma(\widetilde{X}) \end{tikzcd}$$

where the horizontal maps to the left are the inclusions maps i introduced in Lemma 15.8 (1) and where the horizontal maps p on the right are given by the projection and the homeomorphism from Lemma 15.8 (2).

Finally we will need the following slight reinterpretation of the Hopf invariant.

Lemma 63.10. *Let $f\colon S^{2n-1} \to S^n$ be a map. We denote by $j\colon S^n \to \mathrm{Cone}(f)$ the inclusion map and we denote by $p\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f)/S^n = \Sigma(S^{2n-1})$ the map given by the projection and homeomorphism from Lemma 15.8 (2). Then*

$$(j^*)^{-1}([S^n]^*) \cup (j^*)^{-1}([S^n]^*) \ = \ \mathrm{Hopf}(f) \cdot p^*(\Sigma([S^{2n-1}]^*)) \ \in \ H^{2n}(\mathrm{Cone}(f); \mathbb{Z}).$$

Proof of Lemma 63.10. The statement follows basically immediately from the discussion on page 1209[1073] and the cohomological analogue of Lemma 33.19. □

[1072] Our proof is based on the proof in [MTa, p. 35]. An alternative proof is sketched in [Stro, p. 557].

[1073] Note that here we use the convenient fact that $(-1)^{(2n-1)+1} = +1$. In reality the sign is largely irrelevant, as long as we can be sure that the sign depends only on n.

Proof of Proposition 63.7. We start out with a few preparations.

(1) Given a pointed topological space (X, x_0) we define $X \vee X$ by identifying two copies of X along the base point. We refer to the map $\varphi\colon X \vee X \to X$ which is given by the identity on either of the two copies as the *folding map*.

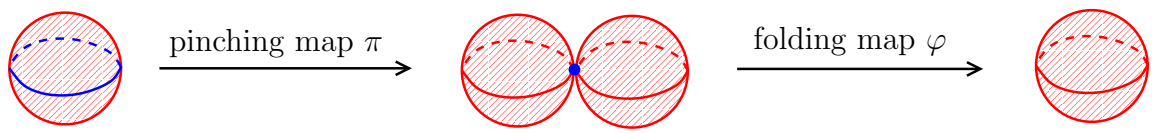


FIGURE 762.

(2) Let $k \in \mathbb{N}_{\geq 2}$. On page 715 we introduced a pinching map $\pi\colon S^k \to S^k \vee S^k$. The precise definition is rather irrelevant to us right now. What matters is that we saw on page 716 that given any pointed topological space (X, x_0) and given any two maps $f, g\colon (S^k, *) \to (X, x_0)$ we have

$$[f_1 * f_2] \;=\; \left[S^k \xrightarrow{\pi} S^k \vee S^k \xrightarrow{f_1 \vee f_2} X \vee X \xrightarrow{\varphi} X\right] \;\in\; \pi_k(X, x_0).$$

Now we turn to our actual situation. Let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given two maps $f, g\colon (S^{2n-1}, *) \to (S^n, *)$. For $i = 1, 2$ we consider the following diagram

$$\begin{tikzcd}
S^n \arrow[r, "j"] & \mathrm{Cone}(\varphi \circ (f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n) \arrow[r, "p"] & \Sigma S^{2n-1} \\
S^n \vee S^n \arrow[u, "\varphi"] \arrow[d, "="'] \arrow[r, "j"] & \mathrm{Cone}((f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n \vee S^n) \arrow[u, "\varphi"'] \arrow[d, "\psi"] \arrow[r, "p"] & \Sigma S^{2n-1} \arrow[u, "="'] \arrow[d, "\Sigma(\pi)"] \\
S^n \vee S^n \arrow[d, "="'] \arrow[r, "j"] & \mathrm{Cone}(f_1 \vee f_2\colon S^{2n-1} \vee S^{2n-1} \to S^n \vee S^n) \arrow[r, "p"] & \Sigma(S^{2n-1} \vee S^{2n-1}) \\
S^n \vee S^n \arrow[u, "="] \arrow[d, "q_i"'] \arrow[r, "j \vee j"] & \mathrm{Cone}(f_1\colon S^{2n-1} \to S^n) \vee \mathrm{Cone}(f_2\colon S^{2n-1} \to S^n) \arrow[u, "\Xi"'] \arrow[d, "q_i"] \arrow[r, "p \vee p"] & \Sigma S^{2n-1} \vee \Sigma S^{2n-1} \arrow[u, "\Omega"'] \arrow[d, "q_i"] \\
S^n \arrow[r, "j"] & \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n) \arrow[r, "p"] & \Sigma S^{2n-1}.
\end{tikzcd}$$

The top three vertical maps are illustrated in Figure 763. We make the following clarifications:

(1) The horizontal maps to the left are the inclusions maps j introduced in Lemma 15.8 (1) and the horizontal maps p on the right are given by the projection and the homeomorphism from Lemma 15.8 (2).

(2) The vertical maps between the first and the second and between the second and the third row are obtained from Construction 63.9 applied to the following two commutative diagrams

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "(f_1 \vee f_2) \circ \pi"] \arrow[d, "="'] & S^n \vee S^n \arrow[d, "\varphi"] \\
S^{2n-1} \arrow[r, "\varphi \circ (f_1 \vee f_2) \circ \pi"] & S^n
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
S^{2n-1} \arrow[r, "(f_1 \vee f_2) \circ \pi"] \arrow[d, "\pi"'] & S^n \vee S^n \arrow[d] \\
S^{2n-1} \vee S^{2n-1} \arrow[r, "f_1 \vee f_2"] & S^n \vee S^n.
\end{tikzcd}$$

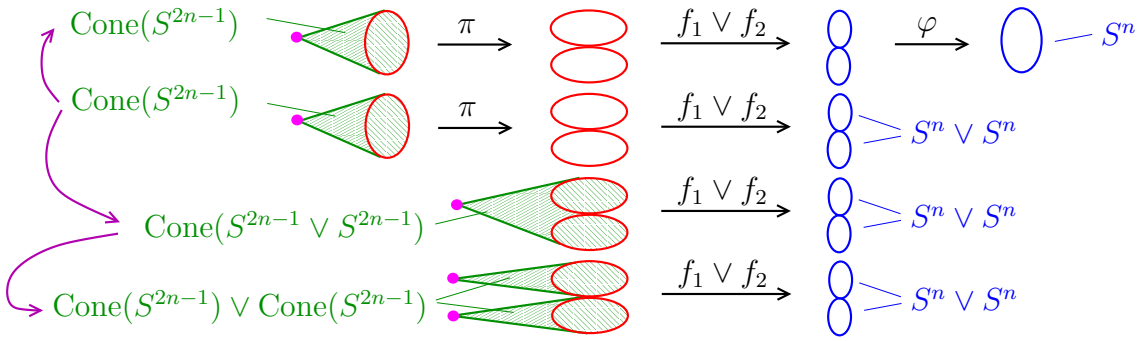


FIGURE 763. Illustration for the proof of Proposition 63.7.

In particular we see that the four upper squares commute.

(3) Next we consider the two diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f_i} & S^n \\ \downarrow & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1 \vee f_2} & S^n \vee S^n \end{array}$$

which are given by the injections into the first respectively second factor. By Construction 63.9 these diagrams give rise to two sequences of maps. We "wedge" the two sequences and we obtain the vertical maps from the fourth to the third row.

(4) At the bottom, in each column the map q_i is the projections onto the first factor.

Next we apply the cohomology functor $X \mapsto H^k(X) = H^k(X;\mathbb{Z})$ with $k = n$ and $k = 2n$ to the above diagram (except that that at the bottom we consider both maps q_1 and q_2). We obtain the following commutative diagram:[1074]

$$\begin{array}{ccccc}
H^k(S^n) & \xleftarrow[\cong]{j^*} & H^k(\mathrm{Cone}(\varphi\circ(f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n)) & \xleftarrow[\cong]{p^*} & H^k(\Sigma S^{2n-1}) \\
{\scriptstyle\varphi^*}\downarrow & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle =} \\
H^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & H^k(\mathrm{Cone}((f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & H^k(\Sigma S^{2n-1}) \\
{\scriptstyle =}\uparrow & & \uparrow{\scriptstyle\psi^*} & & \uparrow{\scriptstyle\Sigma(\pi)^*} \\
H^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & H^k(\mathrm{Cone}(f_1\vee f_2 : S^{2n-1}\vee S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & H^k(\Sigma(S^{2n-1}\vee S^{2n-1})) \\
{\scriptstyle =}\downarrow & & \cong\downarrow{\scriptstyle\Xi^*} & & \cong\downarrow{\scriptstyle\Omega^*} \\
H^k(S^n\vee S^n) & \xleftarrow[\cong]{(j\vee j)^*} & H^k(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{(p\vee p)^*} & H^k(\Sigma S^{2n-1}\vee\Sigma S^{2n-1}) \\
{\scriptstyle q_1^*\oplus q_2^*}\uparrow\cong & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} \\
H^k(S^n)\oplus H^k(S^n) & \xleftarrow[\cong]{j_1^*\oplus j_2^*} & H^k(\mathrm{Cone}(f_1))\oplus H^k(\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{p_1^*\oplus p_2^*} & H^k(\Sigma S^{2n-1})\oplus H^k(\Sigma S^{2n-1}).
\end{array}$$

We make the following observations:

[1074] Any reader who knows of a program which can automatically turn the latex-code for a diagram of topological spaces into the latex-code for the corresponding diagram of cohomology groups should contact me urgently.

(1) It follows almost immediately from Lemma 63.4 that the horizontal maps to the left are isomorphisms in degree n. Since these isomorphisms are only in degree n they are colored in red.
(2) It follows easily from Lemma 63.4 and the cohomological analogue of Lemma 33.19 that the horizontal maps to the right are isomorphisms for $k = 2n$. Since these isomorphisms are only in degree $2n$ they are colored in green.
(3) Similar to the calculation in Exercise 34.3 we see that the map Ω^* is an isomorphism.
(4) It follows from (1), (2) and (3) that the map Ξ^* is an isomorphism.
(5) The vertical maps at the very bottom are induced by the obvious projection maps. The maps are isomorphisms by Proposition 50.17.

Next we denote by $\alpha = [S^n]^* \in H^n(S^n)$ the dual fundamental class. Furthermore we write $\beta = ([\Sigma([S^{2n-1}]^*)]) \in H^{2n}(\Sigma S^{2n-1})$.

Claim 1. We have $\varphi^*(\alpha) = q_1^*(\alpha) + q_2^*(\alpha)$ and $((\Sigma(\pi)^* \circ (\Omega^*)^{-1})(q_1^*(\beta) + q_2^*(\beta)) = \beta$.

We leave it to the reader to verify this claim.

The key calculation takes place in the fourth row. More precisely, we have the following claim.

Claim 2. The following equality holds in $H^{2n}(\mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2))$:

$$\big((j \vee j)^*)^{-1}(q_1^*(\alpha) + q_2^*(\alpha))\big)^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).$$

We calculate that

$$\begin{aligned}
&\big((j \vee j)^{*-1}(q_1^*\alpha + q_2^*\alpha)\big)^2 \overset{\substack{\text{since } j_i \circ q_i = q_i \circ (j \vee j)\colon S^n \vee S^n \to \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n)\\ \downarrow}}{=} \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big) \cup \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big)\\
&\underset{\substack{\uparrow\\ \text{by Lemma 58.6 the “mixed” terms are zero}}}{=} q_1^*(j_1^{*-1}\alpha) \cup q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha) \cup q_2^*(j_2^{*-1}\alpha) \underset{\substack{\uparrow\\ \text{since the cup product is natural, see Lemma 57.10}}}{=} q_1^*(j_1^{*-1}\alpha \cup j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha \cup j_2^{*-1}\alpha)\\
&\underset{\substack{\uparrow\\ \text{by the reinterpretation of } \mathrm{Hopf}(f_1)\\ \text{and } \mathrm{Hopf}(f_2) \text{ given in Lemma 63.10}}}{=} q_1^*(\mathrm{Hopf}(f_1) \cdot p_1^*(\beta)) + q_2^*(\mathrm{Hopf}(f_2) \cdot p_2^*(\beta)) \underset{\substack{\uparrow\\ \text{since } p_i \circ q_i = q_i \circ (p \vee p)\colon\ \mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2) \to \Sigma S^{2n-1}}}{=} (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).
\end{aligned}$$

This concludes the proof of Claim 2.

It follows from $[f_1 * f_2] = [\varphi \circ (f_1 \vee f_2) \circ \pi]$, the definition of α and β and the reinterpretation of the Hopf invariant given in Lemma 63.10, that it remains to prove the following claim.

Claim 3. The following equality holds in the first row:

$$((j^*)^{-1}(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

The idea is of course to push the equality from Claim 2 through the above commutative diagram towards the top. The key tool hereby is the naturality of the cup product, see Lemma 57.10. First we note that it follows from Lemma 57.10 and the fact that φ^* is an

isomorphism in degree $2n$ that it remains to prove that the following equality holds in the second row:

$$(*) \qquad (j^*)^{-1}(\varphi^*(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

To prove this equality we note that it follows from Claims 1 and 2 together with Lemma 57.10 that the following equality holds in the third row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta))).$$

Again, using Lemma 57.10 we see that the following equality holds in the second row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\underbrace{\Sigma(\pi)^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta)))}_{=\,\beta \text{ by Claim 1}}).$$

But using Claim 1 we see that this is precisely the desired equality that we formulated in $(*)$. This concludes the proof of Claim 3. □

63.3. **The Hopf invariant and multiplication maps.** By a *multiplication map* we mean any map of the form $f\colon S^k \times S^k \to S^k$. In this section we will see that such multiplication maps give rise to potentially interesting maps $\Theta_f\colon S^{2k+1} \to S^k$. In the key calculation of this section we show how we can determine the Hopf invariant $\mathrm{Hopf}(\Theta_f)$ directly in terms of homological data of the original map f.

We start out with the following not terribly inspired definition.

Definition. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as[1075]

$$\mathrm{bideg}(f) := (p, q) \;\text{ where } p := \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & f(x, y_0) \end{pmatrix} \text{ and } q := \deg\begin{pmatrix} S^n & \to & S^n \\ y & \mapsto & f(x_0, y) \end{pmatrix}.$$

Examples.

(1) The map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication has bidegree $(1, 1)$. This can be seen as follows: taking $x_0 = y_0 = 1$, we see that the two maps $S^1 \to S^1$ given by $z \mapsto f(z, 1)$ and $z \mapsto f(1, z)$ are in fact the identity.
(2) The same argument as in (1) shows that the quaternion multiplication $S^3 \times S^3 \to S^3$ is a map of bidegree $(1, 1)$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ is also a map of bidegree $(1, 1)$.

The previous example shows that the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ give rise to maps of bidegree $(1, 1)$. This result can be generalized to all division algebras.

Lemma 63.11. *Let $n \in \mathbb{N}_{\geq 2}$. If $\mathbb{R}^n$ admits the structure of a division algebra*[1076]*, then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p, q) where both p and q are odd.*

[1075] Why does the definition not depend on the choice of x_0 and y_0?

[1076] Recall that by the definition on page 1054 a division algebra is an algebra that has left and right inverses. We do not assume that division algebras are associative and we do not assume that they preserve the scalar product.

Proof. Let $*$ be the multiplication map of a division algebra on $\mathbb{R}^n$. As in the proof of Theorem 44.5 we consider the map

$$\begin{array}{rcl} f\colon S^{n-1}\times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & f(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

In the proof of Theorem 44.5 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined and that it is continuous. Now we pick a point $* \in S^{n-1}$ and we consider the maps

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x,*) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon S^{n-1} & \to & S^{n-1} \\ y & \mapsto & f(*,y). \end{array}$$

It follows easily from the fact that $*$ is the multiplication of a real algebra that for all $z \in S^{n-1}$ we have $\varphi(-z) = -\varphi(z)$ and $\psi(-z) = -\psi(z)$. By Proposition 44.17 this implies that $\deg(\varphi)$ and $\deg(\psi)$ are odd. Conveniently enough that is exactly what he had set out to show. $\square$

Before we continue we recall a few conventions that we had introduced along the way. Most of them are also illustrated in Figure 764.

(1) Given a topological space we consider the cone $\mathrm{Cone}(X)$, the suspension $\Sigma(X)$ and we consider the two copies $\mathrm{Cone}_\pm(X)$ of the cone in the suspension.[1077]
(2) Let $m \in \mathbb{N}_0$. By the discussion on page 450 we can make the identification $\mathrm{Cone}(S^m) = \overline{B}^{m+1}$ and by the discussion on page 453 we can make the identification $\Sigma(S^m) = S^{m+1}$.
(3) Let $m, n \in \mathbb{N}_0$. By the discussion on page 116 we can make the identification

$$(S^m \times \overline{B}^{n+1}) \sqcup_{\mathrm{id}_{S^m\times S^n}} (\overline{B}^{m+1} \times S^n) \xrightarrow{=} S^{m+n+1}.$$

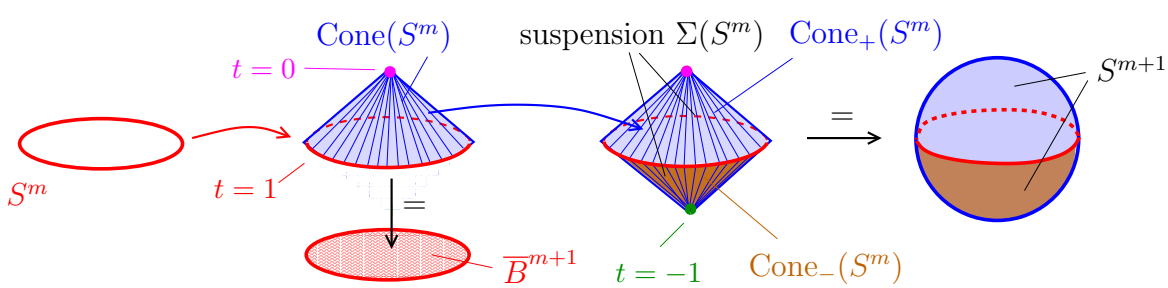


FIGURE 764.

Now we can move on to the promised construction of potentially interesting maps between spheres.

[1077] Here we implicitly use Lemma 15.2.

Definition. Let $f\colon S^p \times S^q \to S^r$ be a map. We consider the maps

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}_+(S^r) \\ (P,[(Q,t)]) & \mapsto & [(f(P,Q),1-t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(S^p) \times S^q & \to & \mathrm{Cone}_-(S^r) \\ ([(P,t)],Q) & \mapsto & [(f(P,Q),t-1)]. \end{array}$$

We will see in Exercise 63.2 that these two maps are continuous. Evidently the two maps agree on the overlap $S^p \times S^q$. Thus by Lemma 1.83 (3) these two maps define a continuous map

$$\Theta_f\colon \underbrace{(S^p \times \mathrm{Cone}(S^q)) \sqcup_{S^p\times S^q} (\mathrm{Cone}(S^p) \times S^q) = S^{p+q+1}}_{=(S^p\times \overline{B}^{q+1})\sqcup_{S^p\times S^q}(\overline{B}^{p+1}\times S^q)} \to \underbrace{\Sigma(S^r)}_{=S^{r+1}}.$$

We refer to Figure 765 for a moderately successful attempt at illustrating the map Θ_f.

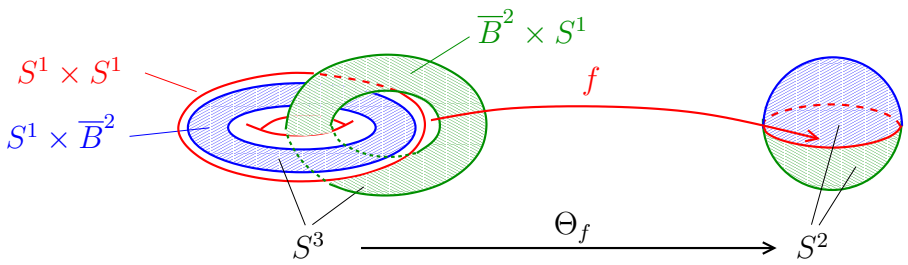


FIGURE 765.

Example.

(1) Again we consider the map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication. We obtain by the above construction a map $\Theta_f\colon S^3 \to S^2$. The heroically inclined reader will surely rise to the challenge of showing that this map is homotopic to the Hopf map.[1078]

(2) Similarly to (1) one can show that the quaternion multiplication $S^3 \times S^3 \to S^3$ gives rise to the quaternionic Hopf map $S^7 \to S^4$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ gives rise to the quaternionic Hopf map $S^{15} \to S^8$.

Given a map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ the next proposition computes the Hopf invariant of Θ_f in terms of the bidegree of f.

Proposition 63.12. *Let $n \in \mathbb{N}_{\geq 2}$. If $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ is a map of bidegree (r,s), then the induced map $\Theta_f\colon S^{2n-1} \to S^n$ has Hopf invariant $\pm rs$.*[1079]

Proof. We make the following preparations:

(1) Throughout the proof all cohomology groups are understood to be with $\mathbb{Z}$-coefficients.

(2) To simplify the notation we write $\mathrm{C}(\Theta_f)$ instead of $\mathrm{Cone}(\Theta_f)$.

[1078] To be on the safe side, let us more cautiously say that the map Θ_f is either homotopic to the Hopf map or that it is homotopic to the composition of the Hopf map with a reflection in a hyperplane.

[1079] We leave it to the sign aficionados to determine the correct sign.

(3) By Lemma 15.5 we know that the inclusion $S^n \to \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n)$ is actually a homeomorphism onto its image, thus we can identify S^n with its image in $\mathrm{C}(\Theta_f)$.
(4) We denote by $\overline{B}^n_+$ the upper hemisphere of S^n and we denote by $\overline{B}^n_+$ the lower hemisphere of S^n.
(5) By the discussion on page 450 we can make the identification

$$\overline{B}^n \times \overline{B}^n \;=\; \overline{B}^{2n} \;=\; \mathrm{Cone}\big(\underbrace{(S^{n-1} \times \overline{B}^n) \sqcup_{S^{n-1}\times S^{n-1}} (\overline{B}^n \times S^{n-1})}_{=S^{2n-1}}\big).$$

We denote by

$$F\colon \overline{B}^n \times \overline{B}^n \;\to\; \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n) \;=\; \underbrace{\mathrm{Cone}(S^{2n-1})}_{=\overline{B}^n\times\overline{B}^n=\overline{B}^{2n}} \sqcup_{\Theta_f} S^n$$

the obvious map which is given by the identification $\overline{B}^n \times \overline{B}^n = \mathrm{Cone}(S^{2n-1})$ followed by the projection.

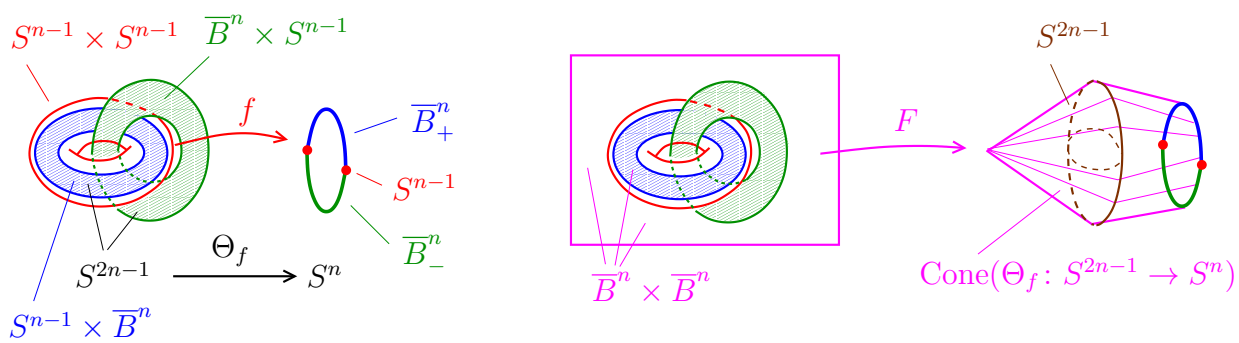


FIGURE 766. Illustration for the proof of Proposition 63.12.

Next we pick generators for various cohomology groups:
(1) We denote by $x \in H^n(\mathrm{C}(\Theta_f))$ and $z \in H^{2n}(\mathrm{C}(\Theta_f))$ the generators that are given in the definition of the Hopf invariant, see page 1473.
(2) The long exact sequence in cohomology of the pair $(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ shows that the maps $H^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm) \to H^n(\mathrm{C}(\Theta_f)))$ are isomorphisms. Let $x_\pm \in H^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ be the generators that correspond to x under the given isomorphisms.
(3) We denote by $[S^{n-1}]^* \in H^n(S^n)$ and $[\overline{B}^n]^* \in H^n(\overline{B}^n, S^{n-1})$ the standard dual generators. For $i = 1, 2$ we denote by $p_i\colon \overline{B}^n \times \overline{B}^n \to \overline{B}^n$ the projection onto the first respectively second coordinate. We write

$$\begin{aligned} y_1 &= p_1^*([\overline{B}^n]^*) \in H^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \quad \text{and} \\ y_2 &= p_2^*([\overline{B}^n]^*) \in H^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}). \end{aligned}$$

The following claim contains the key calculation of the proof.

Claim. The map F induces the following maps on relative cohomology:

$$\begin{array}{llcl} \text{(i)} & H^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & \xrightarrow{F^*} & H^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ & x_+ & \mapsto & \pm r \cdot y_1 \\ \text{(ii)} & H^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) & \xrightarrow{F^*} & H^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}) \\ & x_- & \mapsto & \pm s \cdot y_2. \end{array}$$

Before we turn to the actual calculation we make the following two preparations:

(a) We point out that the restrictions of F to various subsets gives us the following commutative diagram

$$\begin{tikzcd} \overline{B}^n \times \overline{B}^n \arrow[r, "F"] & \mathrm{C}(\Theta_f) \\ \overline{B}^n \times S^{n-1} \arrow[u, hook] \arrow[r, "\Theta_f"] & \overline{B}^n_- \arrow[u, hook] \\ S^{n-1} \times S^{n-1} \arrow[u, hook] \arrow[r, "f"] & S^{n-1}. \arrow[u, hook] \end{tikzcd}$$

(b) We pick a point $* \in S^{n-1}$.

To provide the actual proof of the claim we consider the following diagram:

$$\begin{tikzcd} H^n(\mathrm{C}(\Theta_f)) \arrow[d, "\cong"] & & & H^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) \arrow[lll, "\cong"'] \arrow[r, "F^*"] \arrow[dl] \arrow[d] & H^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \arrow[d, "\cong"] \\ H^n(S^n) & H^n(S^n, \overline{B}^n_+) \arrow[l, "\cong"'] \arrow[r, "\cong", "\text{exc.}"'] & & H^n(\overline{B}^n_-, S^{n-1}) \arrow[r, "\Theta_f^*"] & H^n(\overline{B}^n \times *, S^{n-1} \times *) \\ & & & H^{n-1}(S^{n-1}) \arrow[u, "\delta", "\cong"'] \arrow[r, "f^*", "{[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*}"'] & H^{n-1}(S^{n-1} \times *). \arrow[u, "\delta", "\cong"'] \end{tikzcd}$$

We make the following clarifications and observations:

(1) The maps that are not specified are induced by the obvious inclusions of (pairs) of topological spaces.
(2) The inclusion induced maps that are decorated by "$\cong$" are isomorphisms for the following reasons:
  (a) by the long exact sequence in cohomology for the corresponding pair,
  (b) the maps are induced by a homotopy equivalence,
  (c) by the long exact sequence provided by Lemma 63.4,
  (d) by an appropriate version of the Excision Theorem, see the remark on page 1192.
(3) The quadrilateral to the left commutes since all maps are induced by inclusions.
(4) It follows from (2) and (3) that the diagonal map is an isomorphism.
(5) The triangle towards the center commutes since all maps are induced by inclusions.
(6) It follows from (2), (4) and (5) that the middle vertical map is an isomorphism.
(7) The squares to the right commute by the commutative diagram shown in (a) and by the naturality of the connecting homomorphisms.

(8) The connecting homomorphisms δ are easily seen to be isomorphisms since the other groups in the corresponding long exact sequence are zero.
(9) Basically by the definition of the bidegree of the map f the bottom horizontal map is given by $[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*$.

The above arguments show that the vertical maps of the blue part of the diagram are isomorphisms. Since the bottom map is given by multiplication by r we see that the top map, with whatever choice of generators is also given by multiplication by $\pm r$. This concludes the proof of part (i) of the claim. The proof of part (ii) is evidently basically the same. This concludes the proof of the claim.

Finally we consider the following diagram

$$\begin{tikzcd}
H^n(\mathrm{C}(\Theta_f)) \times H^n(\mathrm{C}(\Theta_f)) \arrow[d, "\cup"', "(x,x)\mapsto x^2=\mathrm{Hopf}(\Theta_f))\cdot z"] & H^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) \times H^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) \arrow[l, "\substack{x\leftarrow\!\shortmid x_- \\ x\leftarrow\!\shortmid x_+}", "\cong"'] \arrow[r, "\substack{x_-\mapsto \pm r\cdot y_1 \\ x_+\mapsto \pm s\cdot y_2}", "F^*"'] \arrow[d, "\cup"] & H^n(\overline{B}^n \times \overline{B}^n, S^{n-1}\times \overline{B}^n) \times H^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}) \arrow[d, "\cup"', "(y_1,y_2)\mapsto y_1\cup y_2=[\overline{B}^n\times\overline{B}^n]^*"] \\
H^{2n}(\mathrm{C}(\Theta_f)) & H^{2n}(\mathrm{C}(\Theta_f), S^n) \arrow[l, "\cong"] \arrow[r, "F^*", "\cong"'] & H^{2n}(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1} \cup S^{n-1} \times \overline{B}^n). \arrow[ll, bend left, "\pm z\leftarrow\!\shortmid [\overline{B}^n\times\overline{B}^n]^*"]
\end{tikzcd}$$

We make the following comments:

(1) It follows almost immediately from Proposition 50.12 (5) that $(\mathrm{C}(\Theta_f), \overline{B}^n_-, \overline{B}^n_+)$ and $(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n, \overline{B}^n \times S^{n-1})$ are excisive triads. Thus we see that the relative cup product in the center and to the right are actually defined.
(2) It follows from the naturality of the cup product, see Proposition 58.3, that the two squares commute.
(3) By Proposition 60.2 (1) we have $y_1 \cup y_2 = p_1^*([\overline{B}^n]) \cup p_2^*([\overline{B}^n]) = [\overline{B}^n \times \overline{B}^n]^*$.
(4) By the above claim we know that the top right horizontal map is in fact given by $x_- \mapsto \pm r \cdot y_1$ and $x_+ \mapsto \pm s \mapsto s \cdot y_2$.
(5) Since the bottom horizontal maps are isomorphisms we see that going right to left the generator $[\overline{B}^n \times \overline{B}^n]^*$ gets sent to $\pm z$.

It follows immediately from the above comments that $\mathrm{Hopf}(\Theta_f) = \pm r \cdot s$. □

Since the Hopf invariant gives us a homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$ we are now interested in finding interesting maps $S^{n-1} \times S^{n-1} \to S^{n-1}$. Perhaps somewhat surprisingly it is not that hard to find interesting maps.

Proposition 63.13. *If n is even, then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p, q) such that $p \cdot q$ is even.*

Proof. We consider the map

$$\begin{aligned} f\colon S^{n-1} \times S^{n-1} &\to S^{n-1} \\ (x, y) &\mapsto y - \langle x, y\rangle \cdot x = \text{reflection of } y \text{ in the hyperplane } x^\perp. \end{aligned}$$

We refer to Figure 767 for an illustration of the map f. $\square$

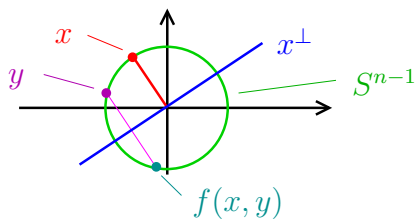


FIGURE 767. Illustration for the proof of Proposition 63.13.

The combination of Propositions 63.7 and 63.13 gives us the following interesting corollary.

Corollary 63.14. *Given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.*

As always in mathematics, every result opens up lots of new questions. Here are a few:

Question 63.15.

(1) *We have now seen that there exists an epimorphism $\pi_3(S^2) \to \mathbb{Z}$. Is this map an isomorphism?*
(2) *We have now seen that there exists an epimorphism $\pi_7(S^4) \to \mathbb{Z}$. Is this map an isomorphism?*

The previous question points to a problem: we have now developed some tools for showing that certain homotopy groups of spheres are non-trivial, but we have no tools yet for computing any interesting homotopy groups.

Here is another question.

Question 63.16. *Given any $n \in \mathbb{N}$ we know by Lemma 38.10 and Corollary 63.14 that $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$ admit an epimorphism onto $\mathbb{Z}$. Are there are any other homotopy groups of spheres that admit an epimorphism onto $\mathbb{Z}$?*

In light of Lemma 63.11 the following question becomes interesting:

Question 63.17. *For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f) = 1$?*[1080]

[1080]Note that Propositions 63.7 and 63.13 tell us that this question is equivalent to the question, for which $n \in \mathbb{N}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f)$ is odd?

Initially one might not appreciate Question 63.17. But rather mysteriously it is not that hard to come up with maps of Hopf degree 2, but it is strikingly difficult to come up with maps of Hopf degree 1. To underline our point we conclude this chapter with another construction of a map of Hopf degree 2.

Example. Let $n \in \mathbb{N}$. We start out with the following two reminders:

(1) We can view S^n as a CW-complex with one 0-cell and one n-cell. In particular, if we denote by S^n_a and S^n_b two copies of S^n then we view S^n_a as a CW-complex with one 0-cell P and one n-cell that we denote by S^n_a. Similarly we view S^n_b as a CW-complex with one 0-cell Q and one n-cell S^n_b.

(2) Given two finite CW-complexes X and Y we introduced on page 669 the product CW-structure on $X \times Y$. Recall that the n-cells of $X \times Y$ correspond precisely to products $e_x \times f_y$ where e_x is a p-cell of X and f_y is an $(n-p)$-cell of Y.

It follows from (1) and (2) that we can view $S^n_a \times S^n_b$ as a CW-complex with one 0-cell (P, Q), two n-cells $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$ and one $2n$-cell $S^n_a \times S^n_b$. We consider the maps

$$S^{2n-1} \xrightarrow{\varphi} \overbrace{S^n_a \times \{Q\} \cup \{P\} \times S^n_b}^{=S^n_a \vee S^n_b} \xrightarrow{p} S^n.$$

φ: attaching map of the $2n$-cell to the $(2n-1)$-skeleton of $S^n_a \times S^n_b$ which is precisely $S^n_a \times \{Q\} \cup \{P\} \times S^n_b$

p: the obvious projection on $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$

The composition of the maps φ and p defines a map $f\colon S^{2n-1} \to S^n$. It is clear from

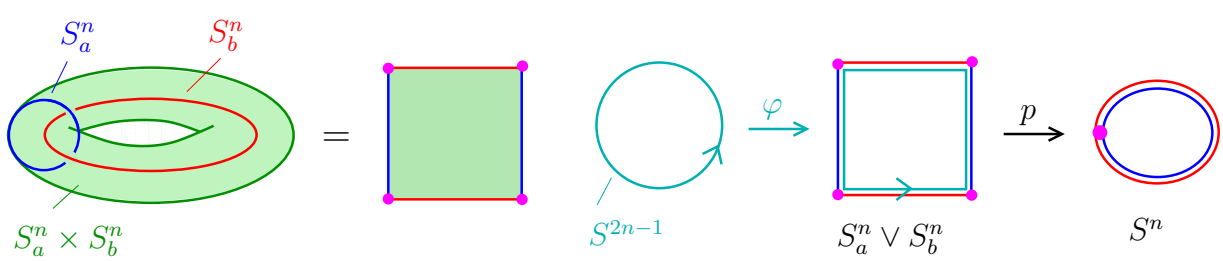


FIGURE 768.

Figure 768 that this map is not very interesting for $n = 1$. But using fairly elementary arguments one can show that for all even n we have $\mathrm{Hopf}(f) = \pm 2$. We refer to [Hat, p. 428] or [Vic, p. 138] for details. Alternatively we leave it as a brain teaser for the reader to perform this calculation.

Exercises for Chapter 63.

Exercise 63.1. What is the homeomorphism type of the mapping cone of the map $S^1 \to S^1$ given by $z \mapsto z^2$?
Hint: It is a topological space that we are quite familiar with.

Exercise 63.2. Let $f\colon S^p \times S^q \to S^r$ be a map. (As always, f is understood to be continuous.) Show that the map

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q),t)] \end{array}$$

is continuous.
Hint: Use Theorem 2.15 and Lemma 2.16 (2).

Exercise 63.3. Let $f\colon S^k \times S^k \to S^k$ be the projection onto the first factor. Show that the corresponding map $\Theta_f\colon S^{2k+1} \to S^{k+1}$, that is defined on page 1481, is homotopic to a constant map.
Hint: The exercise is very easy.

64. Differential topology II: Regular and Tubular Neighborhoods

Before we continue our long discussion of the impact of the Poincaré Duality Theorem 61.1 for manifolds it is useful to arm ourselves with several results regarding manifolds which will come in handy on many occasions. This arming process will happen in this and the subsequent chapter.

In this chapter we will introduce the notion of a regular neighborhood of a subset of a manifold and we will show that submanifolds admit tubular neighborhoods and we will show that tubular neighborhoods are regular. The results of this chapter can be viewed as a generalization of the Tubular Neighborhood Theorem 10.19. For the most part it might be OK to just skip this chapter and work with a naive understanding of regular and tubular neighborhoods. Nonetheless, the reader should be aware that these notions can be defined properly, that the definitions are not entirely self-evident, and that existence of such neighborhoods is non-trivial.

64.1. **Regular neighborhoods.** We start out with the following definition.

Definition. Let M be an n-dimensional topological manifold. We say a subset $W \subset M$ is a *k-dimensional submanifold with corner*[1081] if given any $P \in W$ there exists a chart of the type (α), (β) and (γ) from page 179 or if

(δ) there exists a chart $\Phi\colon U \to V$ of type (ii)[1082] for M such that

$$\Phi(U \cap W) \ \subset \ \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_1,\dots,x_k \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$

and with $\Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0) \,|\, x_1,\dots,x_{k-2} \in \mathbb{R}\}$.

If M is an n-dimensional manifold, then we demand that the charts come from the maximal smooth atlas. We refer to the set of points in W that admit a chart of type (δ) as the *corner set of* W and we denote it by $\partial_c W$.

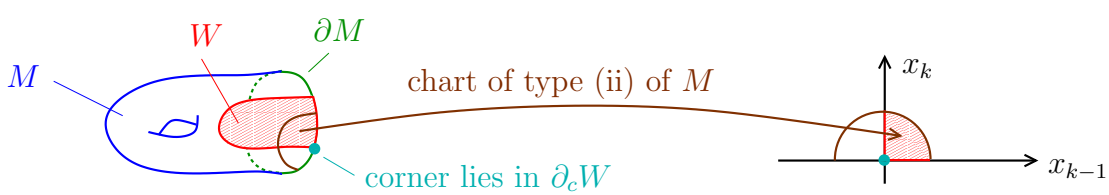


Figure 769.

The following lemma also introduces a notation that we will use throughout these notes.

[1081] Often in the literature such objects are called "submanifold with corners". Following [Wall, p. 30] we write "submanifold with corner" instead.

[1082] See page 171 for the definition of a chart of type (ii).

Lemma 64.1. *Let M be an n-dimensional (topological) manifold and let W be a compact n-dimensional submanifold with corner. We write*

$$\partial_0 W \ := \ W \cap \overline{M \setminus W} \quad \text{and } \partial_1 W \ := \ W \cap \partial M.$$

Then the following two statements hold:

(1) *The subset W itself is a (topological) manifold.*
(2) *The boundary of the (topological) manifold W equals $\partial_0 W \cup \partial_1 W$. Furthermore $\partial_0 W$ and $\partial_1 W$ are $(n-1)$-dimensional submanifolds of ∂W with $\partial_0 W \cap \partial_1 W = \partial_c W$.*
(3) *The set $\mathring{W} := W \setminus \partial_0 W$ is the interior of W in M, in particular $\mathring{W}$ is an open subset of M.*
(4) *The subset $M \setminus \mathring{W}$ is an n-dimensional submanifold of M with corner given by $\partial_c(M \setminus \mathring{W}) = \partial_c W$. Furthermore $\partial_0(M \setminus \mathring{W}) = \partial_0 W$.*

Remark.

(1) Note that if W is a compact submanifold "without corners", then the definition of $\partial_0 W$ and $\partial_1 W$ given in Lemma 64.1 coincides with the definition given on page 805. In particular Lemma 64.1 can be seen as a variation on Proposition 32.2.
(2) Given a manifold X the Excision Theorem for Manifolds 32.7 was initially formulated for a codimension zero submanifold B of X. We leave it as an exercise to verify that the conclusion of the theorem also holds if B is a codimension zero submanifold *with corner*.

Sketch of proof. In the proof that W itself is a (topological) submanifold we only have to worry about the corners. But for these we can use the approach taken in Proposition 56.9 to find a (smooth) atlas. The remaining statements follow fairly easily from Proposition 32.1 and the definitions. We leave it as an exercise to fill in the details. □

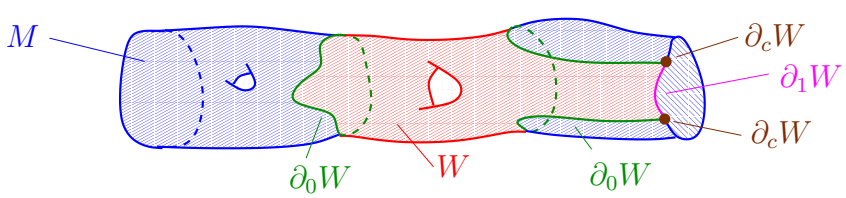


FIGURE 770. Illustration of Lemma 64.1.

We continue with a somewhat technical definition.

Definition. Let M be an m-dimensional topological manifold and let K be a compact subset. A *regular neighborhood* of K is an m-dimensional compact submanifold with corner N of M with the following properties:

(1) K is contained in the interior $\mathring{N}$ of N,
(2) K is a deformation retract of N,

(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

We refer to $\mathring{N}$ as an *open regular neighborhood of* K.

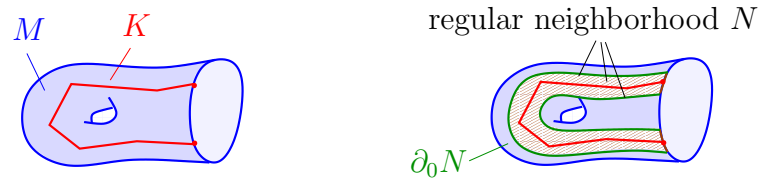


FIGURE 771.

Remark. The notion of a "regular neighborhood" is often used in the literature. For example Rourke–Sanderson [RS, p. 33] define a notion of "regular neighborhood" of a "compact polyhedron in a PL-manifold M" and in [RS, Theorem 3.24] they prove a uniqueness statement. Unfortunately from their definition it is not clear that what they call a regular neighborhood is indeed a regular neighborhood in the above sense. We also refer to [Hem, p. 7f] for a discussion of "regular neighborhoods".

Note that in the definition of a regular neighborhood we did not demand that K is a submanifold, in particular K could be any subset. In fact we have the following proposition.

Proposition 64.2. *Every finite spatial graph G in $\mathbb{R}^n$, in the sense of the definition on page 134, admits a regular neighborhood.*

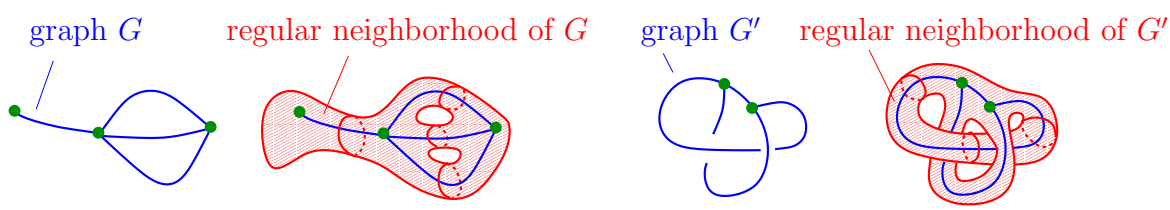


FIGURE 772. Regular neighborhoods for spatial graphs in $\mathbb{R}^3$.

Proof. This proposition is proved in [FH, FH2]. □

We continue this section with the following lemma which can be viewed as a generalization of Lemma 40.7 (3).

Lemma 64.3. *Let K be a compact subset of a topological manifold M. If N is a regular neighborhood of K, then the following statements hold:*

(1) *$M \setminus \mathring{N}$ is a deformation retract of $M \setminus K$,*
(2) *the inclusion $M \setminus \mathring{N} \to M \setminus K$ is a homotopy equivalence, in particular it induces an isomorphism of homotopy groups (for suitable base points) and homology groups,*

(3) *$M \setminus \mathring{N}$ is path-connected if and only if $M \setminus K$ is path-connected,*
(4) *the inclusion map $M \setminus \mathring{N} \to M \setminus K$ induces isomorphisms of fundamental groups and (co-) homology groups.*

Proof.

(1) By definition of a regular neighborhood there exists a deformation retraction r from $N \setminus K$ to $\partial_0 N$. We extend this deformation retraction to all of $M \setminus K$ in the obvious way. More precisely, we consider the map

$$\begin{array}{rcl} f\colon (M \setminus K) \times [0,1] & \to & M \setminus K \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin N, \\ r(P,t), & \text{if } P \in N \setminus K. \end{cases} \end{array}$$

Using Lemma 1.32 (2) and Proposition 10.2 one can easily show that this map is indeed continuous. Once continuity is verified it is straightforward to show that f is indeed a deformation retraction from $M \setminus K$ to $M \setminus \mathring{N}$.

(2) It follows from (1) and Lemma 9.10 that the inclusion is a homotopy equivalence. The statement regarding homotopy groups and homology groups follow from Proposition 28.7 and Corollary 30.8.

(3) This statement follows from (2) and Lemma 9.9.

(4) This last statement follows from (2) together with Proposition 9.12 (1), Corollary 30.8 and Lemma 48.13 (3). □

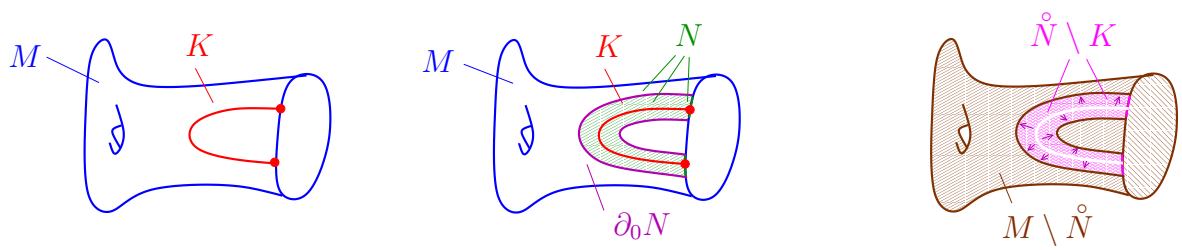


FIGURE 773. Illustration of Lemma 64.3.

The following corollary, which in the literature is often used implicitly or explicitly, gives one application of the theory of regular neighborhoods. In the next chapter we will see more applications.

Lemma 64.4. *Let M be a compact topological manifold and let $K \subset M$ be a compact subset. If K admits a regular neighborhood, then the following two statements hold:*

(1) *the fundamental group $\pi_1(M \setminus K)$ is finitely presented, and*
(2) *for each $k \in \mathbb{N}_0$ the homology group $H_k(M \setminus K; \mathbb{Z})$ is finitely generated.*

Proof. By Lemma 64.3 the topological space $M \setminus K$ is homotopy equivalent to $M \setminus \mathring{N}$. Since M is compact it follows from Lemma 64.1 (3) that $M \setminus \mathring{N}$ is also compact. Furthermore,

by Lemma 64.1 (4) we know that $M \setminus \mathring{N}$ is an n-dimensional manifold. Summarizing we have seen that $M \setminus \mathring{N}$ is a compact n-dimensional manifold. The lemma now follows from Proposition 45.2 and Lemma 64.3 (4). $\square$

In the Tubular Neighborhood Theorem 10.19 we showed that if M is an orientable manifold, then every closed 1-dimensional submanifold admits a trivial tubular neighborhood, which is easily seen to be a regular neighborhood. It is thus natural to guess that something similar should also hold for higher-dimensional submanifolds. In fact we have the following theorem.

Theorem 64.5. (Regular Neighborhood Theorem) *Let M be a manifold. Every compact proper submanifold of M admits a regular neighborhood.*

An outline of the proof will be sketched in the next section.

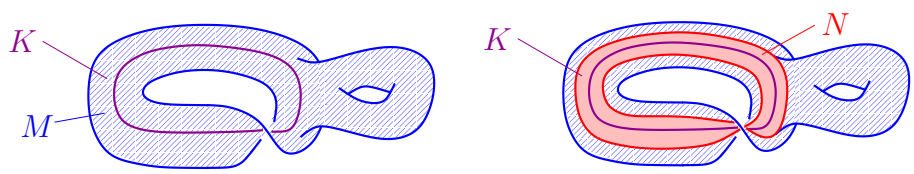


FIGURE 774. Illustration of the Regular Neighborhood Theorem 64.5.

64.2. **The Tubular Neighborhood Theorem II.** In Section 10.6 we introduced the notion of a "trivial tubular neighborhood" of a submanifold and we saw in Theorem 10.19 that some types of submanifolds admit trivial tubular neighborhoods.

In this section we will first introduce the general notion of a tubular neighborhood of a submanifold. We will then discuss existence and uniqueness, before we then show that tubular neighborhoods give rise to regular neighborhoods.

Definition. Let $m \in \mathbb{N}_0$.

(1) Let K be a manifold. A *smooth $\overline{B}^m$-bundle over K* is a manifold N together with a smooth map $p\colon N \to K$ with the following property: given any $x \in K$ there exists a *trivialization of p around x*, i.e. there exists an open neighborhood U of x and a diffeomorphism $\Phi\colon p^{-1}(U) \to \overline{B}^m \times U$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & \overline{B}^m \times U \arrow[dl, "(x,v)\mapsto x"] \\ & U. & \end{tikzcd}$$

A smooth $\overline{B}^1$-bundle is often called a *smooth interval bundle* and a smooth $\overline{B}^2$-bundle is often called a *smooth disk bundle*.

(2) An *isomorphism* between two smooth $\overline{B}^m$-bundles $p\colon M \to K$ and $q\colon N \to K$ is defined to be a diffeomorphism $\Phi\colon M \to N$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & N \arrow[dl, "q"] \\ & K. & \end{tikzcd}$$

If such an isomorphism exists we say that the two smooth $\overline{B}^m$-bundles are *isomorphic*.

The following gives two trivial but nonetheless extremely important examples of constructions of smooth $\overline{B}^m$-bundles.

Example.

(1) Let K be a manifold and let $m \in \mathbb{N}_0$. We denote by $p\colon \overline{B}^m \times K \to K$ the obvious projection map. This map is evidently a smooth $\overline{B}^m$-bundle over K and we refer to it as the *trivial smooth $\overline{B}^m$-bundle over K*.

(2) Let $p\colon N \to K$ be a smooth $\overline{B}^m$-bundle. If $Y \subset K$ is a submanifold, then the restriction $p\colon p^{-1}(Y) \to Y$ is also a smooth $\overline{B}^m$-bundle.

Definition.

(1) Let W be a manifold and let $p\colon \overline{B}^m \times W \to W$ and $q\colon \overline{B}^m \times W \to W$ be two smooth $\overline{B}^m$-bundles. We say an isomorphism $\Phi\colon \overline{B}^m \times W \to \overline{B}^m \times W$ of smooth $\overline{B}$-bundles is *linear* if there exists a map $g\colon W \to \mathrm{O}(m)$ such that for any $(v, w) \in \overline{B}^m \times W$ we have

$$\Phi(v, w) \;=\; (g(w) \cdot v, w).$$

(2) A *linear $\overline{B}^m$-bundle* is a smooth $\overline{B}^m$-bundle $p\colon N \to K$ together with a family of trivializations $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to \overline{B}^m \times U_i)\}_{i\in I}$ such that for any $i, j \in I$ the isomorphism $\Phi_j \circ \Phi_i^{-1}\colon \overline{B}^m \times (U_i \cap U_j) \to \overline{B}^m \times (U_i \cap U_j)$ of smooth $\overline{B}^m$-bundles over $U_i \cap U_j$ is linear. Henceforth, by a trivialization of a linear $\overline{B}^m$-bundle we always mean a trivialization from the above family.

(3) An *isomorphism of two linear $\overline{B}^m$-bundles* $p\colon M \to K$ and $q\colon N \to K$ is defined as an isomorphism $\Theta\colon M \to N$ of smooth $\overline{B}^m$-bundles such that for any trivializations $(U, \Phi\colon p^{-1}(U) \to \overline{B}^m \times U)$ of p and $(V, \Psi\colon q^{-1}(V) \to \overline{B}^m \times V)$ of q the isomorphism $\Phi \circ \Theta \circ \Psi^{-1}\colon \overline{B}^m \times (U \cap V) \to \overline{B}^m \times (U \cap V)$ of smooth $\overline{B}^m$-bundles over $U \cap V$ is linear.

The trivializations of a linear $\overline{B}^m$-bundle play a role similar to the smooth atlas of a manifold. In the case of a manifold we usually suppress the specific atlas from the notation. This leads us to the following convention.

Convention. In the following we will usually suppress the family of trivializations in the notation of a linear $\overline{B}^m$-bundle.

Examples.

(1) Any trivial $\overline{B}^m$-bundle is a linear $\overline{B}^m$-bundle in the obvious way.

(2) We consider the Möbius band $M = ([0,1] \times [-1,1])/(0,x) \sim (1,-x)$ together with the projection map

$$\begin{array}{rcl} p\colon M = ([0,1] \times [-1,1])/\sim & \to & [0,1]/0 \sim 1 \\ [(x,y)] & \mapsto & [x]. \end{array}$$

In Exercise 65.7 we will verify that this map defines a linear interval bundle. We refer to Figure 775 for an illustration.

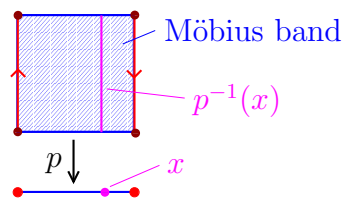


FIGURE 775.

For future reference we formulate the following lemma that classifies isomorphisms of some linear $\overline{B}^m$-bundles.

Lemma 64.6.

(1) *Let N be an oriented n-dimensional manifold and let $k \in \mathbb{N}$. The map*

$$\begin{array}{rcl} \{\textit{maps } N \to \mathrm{SO}(k)\}/\textit{homotopy} & \to & \left\{ \begin{array}{c} \textit{orientation-preserving} \\ \textit{isomorphisms of } \overline{B}^k \times N \end{array} \right\} \Big/ \textit{isotopy} \\ [f\colon N \to \mathrm{SO}(k)] & \to & \left(\begin{array}{rcl} \overline{B}^k \times N & \to & \overline{B}^k \times N \\ (v,P) & \mapsto & (\underbrace{f(P)}_{\in \mathrm{SO}(k)} \cdot v, P) \end{array} \right) \end{array}$$

is a bijection.

(2) *Let N be an oriented n-dimensional manifold. The identity is the only orientation-preserving linear $\overline{B}^1$-isomorphism of the trivial bundle $\overline{B}^1 \times N$.*

(3) *Up to isotopy every isomorphism of the linear bundle $\overline{B}^2 \times S^1$ is of the form*

$$\begin{array}{rcl} \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (w,z) & \mapsto & (w \cdot z^k, w) \end{array}$$

for some $k \in \mathbb{Z}$.

Sketch of proof.

(1) This statement follows fairly easily from the definitions. We leave the details to the reader.

(2) This follows from (1) and the trivial observation that $\mathrm{SO}(1) = \{1\}$, thus there exists precisely one map from N to $\mathrm{SO}(1)$.
(3) This statement follows from (1) and the observation that any map $S^1 \to \mathrm{SO}(2) = S^1$ is homotopic to a map of the form $z \mapsto z^k$ for some $k \in \mathbb{Z}$. This in turn is a fairly straightforward consequence of the fact that $\pi_1(S^1, 1) \cong \mathbb{Z}$, see Proposition 7.18.[1083]

□

Now we can formulate the definition that we had been aiming for.

Definition. Let M be an n-dimensional manifold and let K be a compact proper k-dimensional submanifold. A *tubular neighborhood* for K is a pair $(N, p\colon N \to K)$ with the following properties:

(1) N is a compact codimension zero submanifold with corner.
(2) N is a neighborhood of K.
(3) The map $p\colon N \to K$ is a linear $\overline{B}^{n-k}$-bundle such that $p(x) = x$ for all $x \in K$.
(4) We have $\partial_1 N = p^{-1}(\partial K)$.[1084]

Remark. There are many books that introduce the notion of a tubular neighborhood, see e.g. [Kos, BJ, Bre, Lan4] and [Wall2]. The definitions and results tend to be subtly different and it requires some effort to go back and forth between these books. Our treatment is, by accident, very similar to the one given in [Wall2, Chapter 2.5].

Example.

(1) If K is a closed 1-dimensional manifold of an orientable n-dimensional manifold, then one can verify easily that the above concept of a tubular neighborhood is equivalent to the notion that we introduced in the Tubular Neighborhood Theorem 10.19.
(2) As above we consider the Möbius band $M = [0,1] \times [-1,1]/\sim$. Furthermore let $K = [0,1] \times \{0\} \subset M$ be the submanifold that is given by the central circle.[1085] In Figure 776 we illustrate a tubular neighborhood for K.

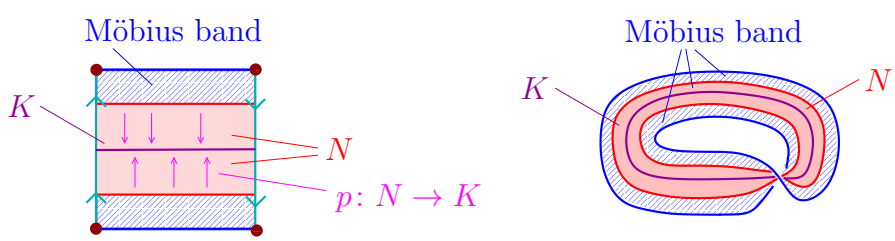


FIGURE 776.

[1083]So why don't we have to worry about base points?
[1084]Note that in Proposition 32.1 we showed that for the proper submanifold K we have $\partial K = \partial M \cap K$.
[1085]Could we not just take all of M to be a tubular neighborhood?

The following theorem is a generalization of the earlier Tubular Neighborhood Theorem 10.19.

Theorem 64.7. (General Tubular Neighborhood Theorem) *Let M be a manifold and let K be a proper submanifold. If K is compact, then K admits a tubular neighborhood.*

Proof. The sketch for the existence of a tubular neighborhood that we provided for the previous Tubular Neighborhood Theorem 10.19 is also valid in this more general context. As before a full proof is provided in [Wall2, Theorem 2.3.3]. Closely related and similar results are also proved in [Kos, Section III.2], [Bre, Theorem II.11.14], [Le2, Theorem 6.24], [BJ, Theorem 12.11] and [Lan4, Chapter IV]. □

In Theorem 10.19 we already saw that certain types of submanifolds admit a trivial tubular neighborhood. To this list we add one more example.

Proposition 64.8. $(*)$ *Let M be an orientable manifold and let K be a proper 2-dimensional submanifold. If each component of K has non-empty boundary, then K admits a trivial tubular neighborhood.*

Proof. By the General Tubular Neighborhood Theorem 64.7 it remains to show that a tubular neighborhood is already trivial. Note that with our hypothesis the topological space K admits a retraction onto a 1-dimensional CW-complex.[1086] Now it follows fairly easily from Propositions 9.1.1 and Proposition 9.2.3 of [Hau] that the tubular neighborhood is indeed trivial. □

The following proposition gives an appropriate uniqueness of tubular neighborhoods. The proposition can be viewed as a generalization of Proposition 10.20.

Proposition 64.9. (Uniqueness of tubular neighborhoods) *Let M be an n-dimensional manifold and let K be a compact proper k-dimensional submanifold. Furthermore let $p\colon U \to K$ and $q\colon V \to K$ be two tubular neighborhoods of K. Then there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:*

1. *$F(P,0) = P$ for all $P \in M$,*
2. *$F(P,t) = P$ for all $P \in K$ and $t \in [0,1]$,*
3. *the diffeomorphism $\Phi = F_1\colon M \to M$ restricts to diffeomorphisms $\Phi\colon M \setminus \mathring{U} \to M \setminus \mathring{V}$ and $\Phi\colon U \to V$, and the latter map is an isomorphism of linear $\overline{B}^{n-k}$-bundles,*
4. *the diffeotopy F has compact support.*

Proof. As in the case of Proposition 10.20 a proof is provided in [Wall2, Chapter 2.5]. □

Convention. By the uniqueness of tubular neighborhoods that we had just proved in Proposition 64.9 we usually do not have to worry about the precise choice of a tubular neighborhood. Given a compact proper submanifold of a manifold M we usually denote by $N(K)$ some choice of a tubular neighborhood.

[1086] To be super-precise, this follows from the classification of 2-dimensional manifolds with boundary, see Theorem 14.15 and Lemma 14.20.

Remark. The General Tubular Neighborhood Theorem 64.7 is stated for manifolds instead of the more general context of topological manifolds. With slight modifications the definition of a tubular neighborhood also makes sense for a k-dimensional proper submanifold K of an n-dimensional topological manifold M.

(1) By [FHNOP, Theorem 4.6] tubular neighborhoods exist for any compact proper submanifold of a 4-dimensional topological manifold. If the submanifold is one-dimensional and the topological manifold is orientable, then there exists a trivial tubular neighborhood. Furthermore, by [FHNOP, Theorem 4.7] such a tubular neighborhood is unique in an appropriate sense.
(2) If $k = n - 2$, i.e. if K is a codimension 2 submanifold, then some type of tubular neighborhood seems to exist, see e.g. [DV, Theorem 6.8.1] and [FQ, p. 137] for details. We also refer to [Ped] and [Joha] for some positive results. Note though that when it comes to topological manifolds one should be very cautious with references. It is advisable to very carefully check the definitions used in papers on topological manifolds.
(3) If $k = n - 3$, i.e. if K is a codimension 3 submanifold, then, according to [DV, p. 347] and [Edw3, p. 9], a tubular neighborhood does not exist in general.

Fortunately for the most part we will not deal with topological manifolds and the General Tubular Neighborhood Theorem 64.7 will be good enough for our purposes.

Definition.

(1) A linear $\overline{B}^m$-bundle $p\colon N \to K$ is called *trivial* if it is isomorphic, as a linear $\overline{B}^m$-bundle, to the trivial $\overline{B}^m$-bundle $\overline{B}^m \times K \to K$.
(2) Let M be an m-dimensional manifold and let K be a k-dimensional submanifold. A tubular neighborhood $(N, p\colon N \to K)$ of K is called *trivial* if the linear $\overline{B}^m$-bundle $p\colon N \to K$ is trivial.

Example. As on page 1495 we consider again the Möbius band $M = [0,1] \times [-1,1]/\sim$ and the submanifold $K = [0,1] \times \{0\} \subset M$. At least from the picture it is pretty clear that K does not admit a trivial tubular neighborhood.[1087]

The following lemma gives some other instances where the tubular neighborhood is non-trivial.[1088]

Lemma 64.10. *Let M be an orientable manifold and let K be a proper submanifold. If K is non-orientable, then the tubular neighborhood is non-trivial.*

Sketch of proof. We prove the converse, if the K admits a trivial tubular neighborhood, then K is orientable. So suppose that K is a proper k-dimensional submanifold of an orientable m-dimensional manifold M and suppose that K admits a trivial neighborhood

[1087] What would be a rigorous proof for the statement that K does not admit a trivial tubular neighborhood?

[1088] To be more precise, we still need examples of say closed non-orientable manifolds that are embedded in orientable manifolds. We will shortly see that we can embed any real projective space $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{2n}$.

$\overline{B}^{m-k} \times K$. Let $P \in K \subset M$. We identify $T_P K$ and $T_0 \overline{B}^{m-k}$ with its images in $T_{(0,P)}M$ under the obvious inclusion maps $K \to \{0\} \times K \to M$ and $\overline{B}^{m-k} \to \overline{B}^{m-k} \times \{P\} \to P$. We define

$$\mathcal{B} \subset T_P K \text{ is a positive basis} \quad :\Longleftrightarrow \quad \begin{array}{l} \text{a positive basis of } \overbrace{T_0 \overline{B}^{m-k}}^{=\mathbb{R}^{m-k}} \text{ followed by } \mathcal{B} \\ \text{is a positive basis of } T_{(0,P)}M. \end{array}$$

One easily verifies that this defines an orientation for K. □

If we get away from the pesky non-orientable (sub-) manifolds we get some more positive results.

Proposition 64.11. *Let M be an oriented*[1089] *m-dimensional manifold and let K be a compact oriented proper submanifold.*

(1) *If K is of dimension one and $m \geq 2$, then there exists a trivial tubular neighborhood $N(K) = \overline{B}^{m-1} \times K$ such that the product orientation of $\overline{B}^{m-1} \times K$ matches the orientation of $N(K) \subset M$.*
(2) *If K is of codimension one, then there exists a trivial tubular neighborhood of the form $N(K) = [-1,1] \times K$ such that the product orientation of $[-1,1] \times K$ matches the orientation of M.*[1090]

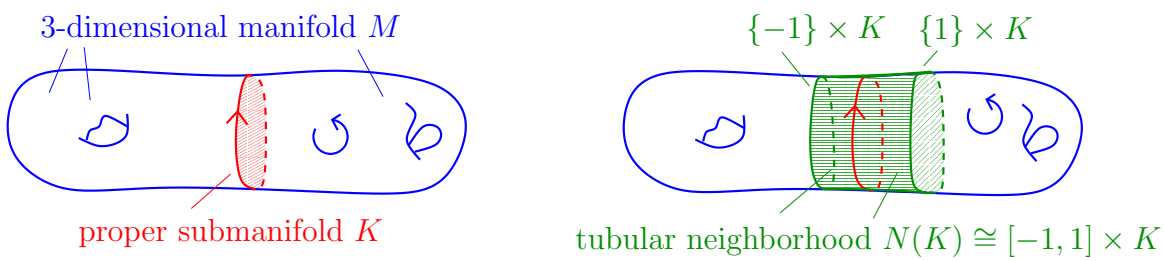


FIGURE 777. Illustration of Proposition 64.11.

Proof. In both cases it follows from the General Tubular Neighborhood Theorem 64.7 together with [Hau, Proposition 9.2.3] and [MiS, Axiom 3] that K admits a trivial neighborhood.[1091] The statement regarding the orientations can be achieved by possibly using an orientation reversing diffeomorphism of $\overline{B}^{m-1}$ or $[-1,1]$. □

[1089] Is the hypothesis that K is orientable, necessary? Put differently, is there an example of an orientable manifold and a compact proper codimension one submanifold such that the tubular neighborhood is non-trivial?

[1090] The reason why we prefer $[-1,1] \times K$ over $K \times [-1,1]$ will become apparent later.

[1091] We had mentioned the case $\dim(K) = 1$ explicitly in the original Tubular Neighborhood Theorem 10.19.

Proposition 64.11 raises the following question.

Question 64.12. *Let M be an orientable manifold and let K be a compact orientable proper submanifold. Is the tubular neighborhood $N(K)$ necessarily trivial?*

After this long discussion of tubular neighborhoods one can be excused for having lost track of what they are actually supposed to be good for. The following proposition finally implies Theorem 64.5.

Proposition 64.13. *Let M be a manifold and let K be a compact proper submanifold of M. Every tubular neighborhood of K is also a regular neighborhood of K.*

Proof. Let M be an n-dimensional manifold and let K be a compact proper k-dimensional submanifold of M. Let $(N, p\colon N \to K)$ be a tubular neighborhood of K. Recall that this means that the following three statements are satisfied:

(1) N is a compact codimension zero submanifold with corner of M that is a neighborhood of K.
(2) The map $p\colon N \to K$ is a linear $\overline{B}^{n-k}$-bundle such that for all $x \in K$ we have $p(x) = x$. We denote by $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to \overline{B}^{n-k} \times U_i)\}_{i\in I}$ the corresponding family of trivializations.
(3) We have $\partial_1 N = p^{-1}(\partial K)$, put differently, we have $\partial M \cap N = p^{-1}(\partial M \cap K)$.

We start out the actual proof of the proposition with the following definition and construction. Let $v \in N$. We write $x := p(v)$. We pick an $i \in I$ with $p(v) \in U_i$ and we write $\Phi_i(v) = (w, x)$ where $w \in \overline{B}^m$.

(1) Given $s \in [0,1]$ we write $\varphi_s(v) := \Phi_i^{-1}(s \cdot w, x)$.
(2) Suppose $v \notin K$. This is equivalent to saying that $w \neq 0$. Then given $t \in [0,1]$ we write $\psi_t(v) := \Phi_i^{-1}\big((1-t)\cdot w + t \cdot \frac{w}{\|w\|}, x\big)$.

We leave it as an exercise to verify that these definitions are independent of the choice of the trivialization Φ_i.[1092] From this independence it also follows easily that the maps

$$\begin{array}{rcl} \Phi\colon N \times [0,1] & \to & N \\ (v,s) & \mapsto & \varphi_s(v) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon (N \setminus K) \times [0,1] & \to & N \setminus K \\ (v,t) & \mapsto & \psi_t(v) \end{array}$$

are continuous.[1093] We refer to Figure 778 for an illustration of the definition of Φ and Ψ. Finally recall that by definition of a regular neighborhood we have to prove the following claim.

Claim.

(1) K is contained in the interior of N,
(2) K is a deformation retract of N,

[1092] At this point it is essential that we work with a linear $\overline{B}^{n-k}$-bundle.

[1093] Evidently we only have to show continuity o-locally. So let $v \in N$. We can pick a trivialization that contains x and then it follows immediately from the definitions that the maps Φ and Ψ are continuous in the neighborhood of x corresponding to the trivialization.

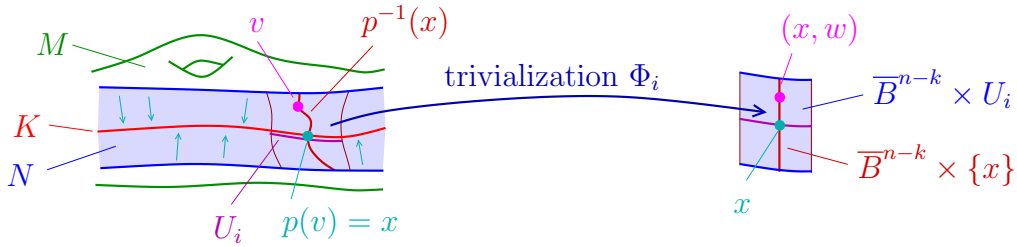


FIGURE 778. Illustration for the proof of Proposition 64.13.

(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

We turn to the proof of the claim.

(1) By definition of a neighborhood we know that K is contained in the interior of N.
(2) The above map $\Phi\colon N \times [0,1] \to N$ is easily seen to be a deformation retraction from N to K.
(3) The above map $\Psi\colon (N \setminus K) \times [0,1] \to N \setminus K$ is easily seen to be a deformation retraction from $N \setminus K$ to $\partial_0 N$.

This concludes the proof of the claim and thus also of the proposition. □

Remark. One of the frustrating aspects about tubular neighborhoods is that every book seems to work with a different definition. IMHO a tubular neighborhood should have the following two features:

(1) it should be a regular neighborhood, and
(2) it should satisfy a uniqueness statement.

By Propositions 64.9 and 64.13 our definition has these two properties. It is not always apparent that the definitions in the literature have these two features.

We conclude this chapter with a lemma which says that for transverse submanifolds one can find "transverse" tubular neighborhoods. We will not really make use of the lemma in these notes, but it seems to me that this lemma often gets used implicitly in the literature.

Lemma 64.14. $(*)$ *Let M be a manifold and let X and Y be proper submanifolds of M. If the submanifolds X and Y are compact and if they intersect transversally, then there exist tubular neighborhoods $p\colon U \to X$ and $q\colon V \to Y$ such that*[1094]

$$U \cap V \;=\; p^{-1}(V \cap X) = q^{-1}(U \cap Y).$$

Proof. I could not find a reference for the third statement in any of the standard books on differential topology [Le2, Wall2, Kos, GP]. But the standard proof for the existence of tubular neighborhoods can easily be adapted to prove the statement. The restriction to compact X and Y is almost certainly not necessary. □

[1094]From Figure 779 it should be pretty clear what we want to say. But does the statement of the lemma capture this idea? It is at times difficult to turn a picture into a precise mathematical statement.

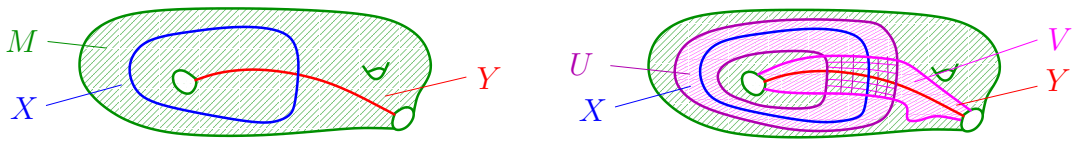


FIGURE 779. Illustration of Lemma 64.14.

Exercises for Chapter 64.

Exercise 64.1. Let G and G' be two spatial graphs in $\mathbb{R}^3$ with regular neighborhoods $N(G)$ and $N(G')$. Which of the following two implications is correct?

$$G \text{ and } G' \text{ are isotopic} \quad \Longleftrightarrow \quad N(G) \text{ and } N(G') \text{ are isotopic?}$$

Exercise 64.2. Let M be a 2-dimensional manifold. Furthermore let $f\colon [0,1] \to M$ be an embedding such that $f(0) \in \partial M$ and such that $f((0,1]) \subset M \setminus \partial M$. Show that the inclusion $M \setminus f([0,1]) \to M$ is a homotopy equivalence.
Hint: Use the Extension Theorem 10.26 and the General Tubular Neighborhood Theorem 64.7.

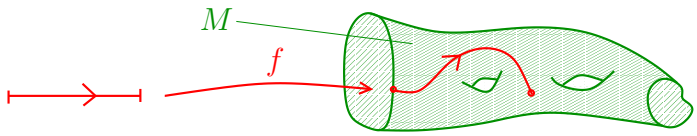


FIGURE 780.

65. Differential topology III: Whitney Approximation and Whitney Embedding

We continue with our attempt to expand our toolkit for dealing with manifolds.

65.1. **The Regular Value Theorem II** (∗)**.** In this section we formulate several refinements of the Regular Value Theorem 3.44. The refinements will play an important role later on, but perhaps it is best to skip the section at first reading.

Theorem 65.1. (Regular Value Theorem) *Let M be an m-dimensional manifold, let N be an n-dimensional manifold without boundary, let $f\colon M \to N$ be a smooth map and let $s \in N$ be a regular value of f. Then the following statements hold:*

1. *The preimage $X := f^{-1}(s)$ is a proper $(m-n)$-dimensional submanifold of M.*
2. *If M is orientable, then X is also orientable.*
3. *If M is compact, then X is also compact.*

For the last statement we assume that M is compact and orientable.

4. *There exists an open neighborhood U of s such that for any embedding $\varphi\colon \overline{B}^n \to U$ with $\varphi(0) = s$ there exists a tubular neighborhood $\Phi\colon \overline{B}^n \times X \to M$ such that the following two statements hold:*
   (a) *we have $f^{-1}(\varphi(\overline{B}^n)) = \Phi(\overline{B}^n \times X)$,*
   (b) *the following diagram commutes:*

$$\begin{tikzcd} \overline{B}^n \times X \arrow[r, "\Phi"] \arrow[d, "(p,x)\mapsto p"] & M \arrow[d, "f"] \\ \overline{B}^n \arrow[r, "\varphi"] & N. \end{tikzcd}$$

If M is oriented, then the orientation on X can be chosen such that Φ is orientation-preserving.

The first three statements of the above theorem are precisely the content of the original Regular Value Theorem 3.44.

Example. In Figure 113 we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x,y,z) = z$. This function has precisely four critical points which are indicated in Figure 113 on the left. On the right we show a regular value s and its preimage and we show a critical value t and its preimage $f^{-1}(t)$ which in this case is *not* a submanifold.

Sketch of proof. As mentioned above, the first three statements of the theorem form precisely the content of the original Regular Value Theorem 3.44. So it remains to deal with the final statement. This can be deduced quite easily from the Submersion Theorem 3.46. We leave it to the reader to fill in the details. □

For the reader's convenience we discuss the cases that $N = \mathbb{R}$ and $N = S^1$ separately. These two special cases also allow us to elucidate the somewhat confusing fourth statement of the Regular Value Theorem 65.1.

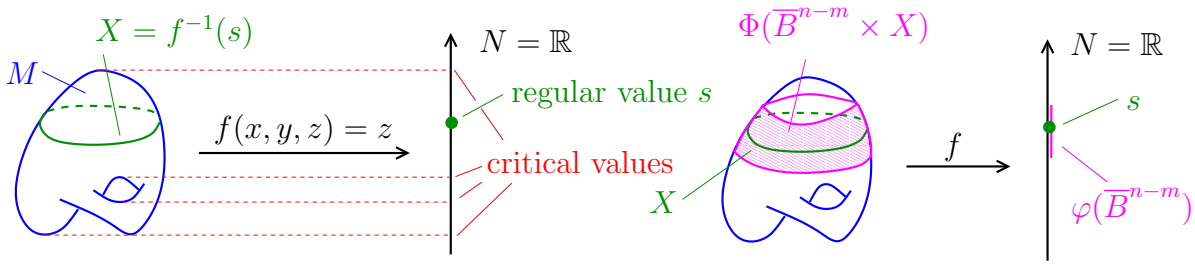


FIGURE 781. Illustration of the Regular Value Theorem 65.2.

Theorem 65.2. (Regular Value Theorem for $N = \mathbb{R}$) *Let M be an n-dimensional manifold, let $f\colon M \to \mathbb{R}$ be a smooth map and let $s \in \mathbb{R}$ be regular value of f. Then beyond statements (1), (2), (3) of the Regular Value Theorem 65.1 the following two statements hold:*

(4) *If M is compact and orientable, then there exists an $\epsilon > 0$ and a tubular neighborhood $[-\epsilon, \epsilon] \times X$ of X such that $f^{-1}([s-\epsilon, s+\epsilon]) = [-\epsilon, \epsilon] \times X$ and such that the following diagram commutes*

$$\begin{tikzcd} {[-\epsilon,\epsilon] \times X} \arrow[r, hook] \arrow[d, "(t,x)\mapsto t"] & M \arrow[d, "f"] \\ {[-\epsilon,\epsilon]} \arrow[r, "t\mapsto t+s"] & \mathbb{R}. \end{tikzcd}$$

If M is oriented, then the orientation on X can be chosen such that the embedding $[-\epsilon,\epsilon]\times \to M$ is orientation-preserving.

(5) *Suppose that $f^{-1}(s) \cap \partial M = \varnothing$. Then the preimage $W := f^{-1}((-\infty, s])$ is an n-dimensional submanifold of M with $\partial_0 W = f^{-1}(s)$.*

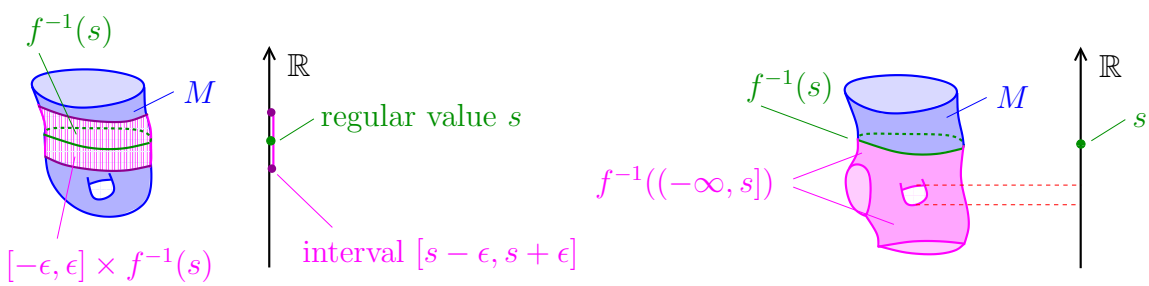


FIGURE 782. Illustration of the Regular Value Theorem 65.2.

Sketch of proof.

(4) Note that given any neighborhood U of $s \in \mathbb{R}$ there exists an $\epsilon > 0$ such that $[s-\epsilon, s+\epsilon] \subset U$. The statement now follows immediately from the Regular Value Theorem 65.1 (4) applied to the map $\varphi\colon [-\epsilon,\epsilon] \to \mathbb{R}$ given by $t \mapsto t+s$.

(5) If one takes a look at the proof of the Regular Value Theorem 65.1 (1) one sees that it actually provides submanifold charts of type (β) for $W := f^{-1}((-\infty, s])$ and any point on $f^{-1}(s)$. Also note that $f^{-1}((-\infty, s))$ is an open subset of M. We leave it to the reader to assemble from this observations the full proof of the statement. □

We also mention the following variation on the Regular Value Theorem 65.1.

Theorem 65.3. (Regular Value Theorem for $N = S^1$) *Let M be an n-dimensional manifold, let $f\colon M \to S^1$ be a smooth map and let $z = e^{2\pi i t} \in S^1$ be a regular value. Then beyond statements* (1), (2), (3) *of the Regular Value Theorem 65.1 the following statement holds:*

(4) *If M is compact and orientable, then there exists an $\epsilon > 0$ and a tubular neighborhood $[-\epsilon, \epsilon] \times X$ of X such that $f^{-1}(\{e^{2\pi i(t+s)} \mid t \in [-\epsilon, \epsilon]\}) = [-\epsilon, \epsilon] \times X$ and such that the following diagram commutes*

$$\begin{tikzcd} {[-\epsilon,\epsilon]\times X} \arrow[r, hook] \arrow[d, "(t,x)\mapsto t"] & M \arrow[d, "f"] \\ {[-\epsilon,\epsilon]} \arrow[r, "s\mapsto e^{2\pi i(t+s)}"] & S^1. \end{tikzcd}$$

If M is oriented, then the orientation on X can be chosen such that the embedding $[-\epsilon, \epsilon]\times \to M$ is orientation-preserving.

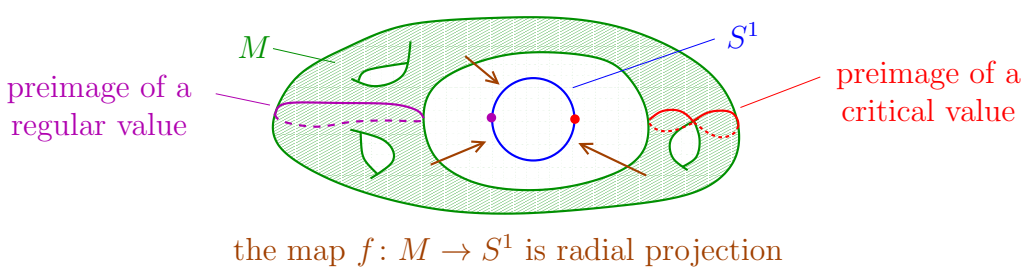


FIGURE 783. Illustration of the Regular Value Theorem 65.3.

Example. Let N be a manifold and let $\varphi\colon N \to N$ be a diffeomorphism. We consider the mapping torus $M := M(N, \varphi) = (N \times [0, 1])/(x, 0) \sim (\varphi(x), 1)$ together with the canonical projection $f\colon M \to S^1$ given by $f([p, x)]) = e^{2\pi i x}$. Then every $z = e^{2\pi i x} \in S^1$ is a regular value and the preimage $f^{-1}(z) = N \times \{x\}$ is just a "parallel" copy of N.

Proof. As in the case of Theorem 65.2 (4) this statement can be deduced easily from the Regular Value Theorem 65.1 (4). □

The following proposition often allows us to reduce a statement about non-compact manifolds to a statement about compact manifolds.

Proposition 65.4. *Let W be an n-dimensional manifold with compact boundary.*[1095] *There exists a sequence $X_1, X_2, \dots$ of n-dimensional submanifolds of W with the following three properties:*

(1) *the sequence is nested, i.e. for each $i \in N$ we have $X_i \subset X_{i+1}$,*
(2) *each X_i is compact and each X_i contains the boundary ∂W,*
(3) *we have $\bigcup_{i\in\mathbb{N}} \overset{\circ}{X}_i = W$.*

In the proof of Proposition 65.4 we will need the following useful lemma which is proved in [Le2, Proposition 2.28].

Lemma 65.5. *Let W be an n-dimensional manifold with compact boundary. There exists a smooth function $f\colon W \to \mathbb{R}_{\geq 0}$ with the property that for any $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of W.*

Examples.

(1) Let $W = [0,1)$ the half-open interval viewed as a 1-dimensional manifold. In this case the function defined by $f(x) = \frac{1}{1-x}$ has the desired properties
(2) We consider the surface of infinite genus illustrated in Figure 784. In this case the function $f(x,y,z) = x$ has the desired properties of Lemma 65.5. [1096]

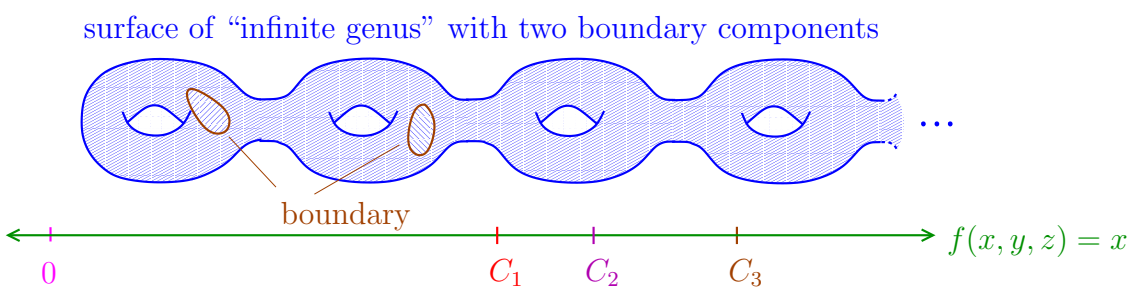


FIGURE 784. Illustration of the proof of Proposition 65.4

Proof of Proposition 65.4. Let W be an n-dimensional manifold with compact boundary. By Lemma 65.5 there exists a smooth function $f\colon W \to \mathbb{R}_{\geq 0}$ with the property that for any $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of W.

Since ∂W is compact it follows from Lemma 1.37 that there exists a $C \in \mathbb{R}_{\geq 0}$ such that $\partial W \subset f^{-1}([0,C])$. It follows from Sard's Theorem 3.48 and Proposition 3.47 (2) that there exists an increasing sequence $C \leq C_1 \leq C_2 \leq C_3 \leq \dots$ of regular values of f such that $\lim_{i\to\infty} C_i = \infty$. For each $i \in \mathbb{N}$ we set $X_i := f^{-1}((-\infty, C_i])$. By the Regular Value

[1095] Note that if the boundary is empty, it is also compact.

[1096] We consider the surface of infinite genus illustrated in Figure 784. Let us ignore the two boundary components so that we get a manifold without boundary. Is the result homeomorphic to the surface of infinite genus illustrated in Figure 297?

Theorem 65.2 (5) and by our choice of f each X_i is a compact n-dimensional submanifold of W with $\partial_0 X_i = f^{-1}(C_i)$. In particular we have by Lemma 64.1 that $\mathring{X}_i = f^{-1}((-\infty, C_i))$. By construction this sequence satisfies the desired conditions (1), (2) and (3). □

65.2. **The Whitney Approximation Theorem.** As we will see later on, there are many ways to construct interesting continuous maps between two manifolds. The following theorem allows us to "replace" such a continuous map by a smooth map.

Theorem 65.6. (Whitney Approximation Theorem) *Let M be manifold (with or without boundary) and let N be a manifold without boundary.*

(1) *Let $f\colon M \to N$ be a map that is smooth on a (possibly empty) closed subset A.*[1097] *There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:*
   (a) *The homotopy is a homotopy rel A.*
   (b) *We have $F_0 = f$.*
   (c) *The map $F_1\colon M \to N$ is smooth.*

   Suppose that we are given a continuous function $\rho\colon M \to \mathbb{R}_{>0}$ and suppose that we are given a metric d on N. Then we can find a homotopy F as above that also satisfies the following extra condition:
   (d) *For every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$.*

(2) *Let $f, g\colon M \to N$ be two smooth maps that agree on a (possibly empty) closed subset A. If f and g are homotopic rel A, then there exists also a smooth homotopy between f and g that is constant on A.*

Remark. Suppose we are in the setting of the Whitney Approximation Theorem 65.6. Let $f\colon M \to N$ be a map that is injective (respectively surjective, bijective). We now know that we can find a homotopy from f to some smooth map $F_1 = g\colon M \to N$, but the theorem does not give us any control over whether the smooth map $F_1 = g$ is also injective (respectively surjective, bijective).

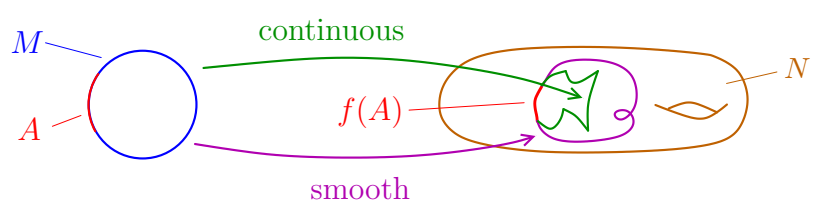


FIGURE 785. Illustration of Theorem 65.6 (1).

Proof.

[1097] Let M and N be manifolds and let $A \subset M$ be an arbitrary subset. On page 184 we defined what it means for a map $f\colon A \to N$ to be smooth.

(1) The first statement, without Condition (d), is the content of [Le2, Theorem 6.26]. The full statement including Condition (d) follows by almost the same argument as in the proof of [Le2, Theorem 6.26], one just needs o make full use of [Le2, Theorem 6.21]. We refer to [MTo, Lemma A.9] for a closely related argument. A weaker version of (1) is also proved in [Hirs, Lemma 5.1.5] and in [DFN, Theorem 10.1.2].
(2) The proof of Statement (2) can reduced to Statement (1) without too many problems. We refer to [Le2, Theorem 6.29] or alternatively to [DFN, Theorem 12.1.4] for details. □

We continue with the following pleasant corollary to the Whitney Approximation Theorem 65.6.

Corollary 65.7. *Let $n \in \mathbb{N}$. We pick a base point $*$ on S^n. Furthermore let M be a manifold without boundary. We pick a base point x_0. The obvious map*

$$\{\textit{all smooth maps } f\colon (S^n, *) \to (M, x_0) \textit{ up to smooth homotopies}\} \ \to \ \pi_n(M, x_0)$$

is an isomorphism.

Remark. In Exercise 65.1 we will use this isomorphism, together with Proposition 3.47 to give a quick proof of Proposition 28.9, i.e. we will reprove that for $k < n$ we have $\pi_k(S^n) = 0$.

Proof. Note that it follows immediately from the definition on page 184 that the restriction of any map $S^n \to M$ to any one-point subset $\{*\}$ is smooth. Thus the corollary follows immediately from the Whitney Approximation Theorem 65.6, applied to $A = \{*\}$. □

We conclude this section with two and a half more technical consequences of the Whitney Approximation Theorem 65.6. These results are interesting in their own right, but the reader might want to skip them initially.

Proposition 65.8. $(*)$ *Let M be manifold (with or without boundary) and let N be a manifold without boundary. Furthermore let $f\colon M \to N$ be a map. Finally let A and X be disjoint closed subsets such that the restriction of f to A is smooth and let $U \subset N$ be an open subset that contains $f(M \setminus X)$. If $X \neq \varnothing$, then there exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:*

(1) *The homotopy is a homotopy rel $X \cup A$.*
(2) *We have $F_0 = f$.*
(3) *The restriction of F_1 to the open subset $M \setminus X$ is smooth.*[1098]
(4) *For each $t \in [0,1]$ we have $F_t(M \setminus X) \subset U$.*

Proof. First we consider the case that $U = N$. By Theorem 3.10 we can pick a metric c on M such that the topology from the metric c agrees with the given topology on M. The

[1098]Since X is closed we see that $M \setminus X$ is an open subset of M, thus we can view $M \setminus X$ as a manifold in its own right.

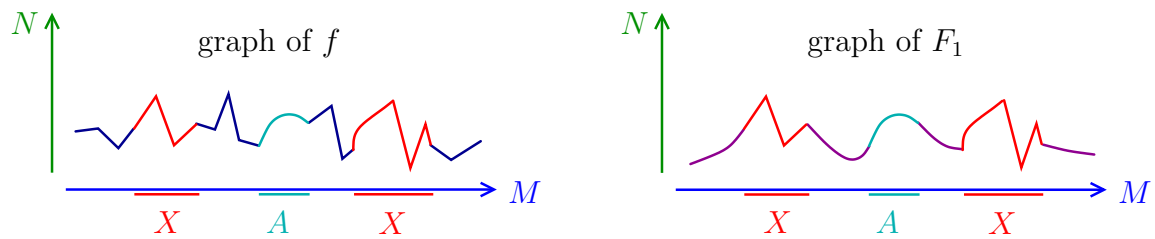


FIGURE 786. Illustration of Proposition 65.8.

same way we can pick a metric d on N. We consider the open subset $W := M \setminus X$ and we view it as a manifold. Since $X \neq \varnothing$ we can consider the function

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) := c(w, X) = \inf\{c(w,x) \,|\, x \in X\}. \end{array}$$

Since X is closed and since the topology defined by c agrees with the topology on M we see that $\rho(w) > 0$ for every $w \in W$. By Lemma 1.57 we know that ρ is continuous. Since N is a manifold without boundary we can apply the Whitney Approximation Theorem 65.6 to the map $f\colon W \to N$ and we see that there exists a homotopy $G\colon W \times [0,1] \to N$ with the following properties:

(a) The homotopy is a homotopy rel A.
(b) We have $G_0 = f|_W$.
(c) The map $G_1\colon W \to N$ is smooth.
(d) For every $(w,t) \in W \times [0,1]$ we have $d(G(w,t), f(w)) \leq \rho(w)$.

It follows easily from (d) and the definition of ρ that the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in W, \\ f(x), & \text{if } x \in X \end{cases} \end{array}$$

is continuous. Now it is straightforward to verify that this map has all the desired properties.

Finally we consider the case that $U \neq N$. In this case we consider the map

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) := \min\{c(w,X), d(f(w), N \setminus U)\}. \end{array}$$

Since U is open and since $f(W) \subset U$ we see that ρ is again non-zero every. The rest of the argument is basically the same and it follows from (d) that for each $t \in [0,1]$ we have $F_t(M \setminus X) \subset U$. $\square$

Loosely speaking the following proposition says that a map $M \to X$ from a manifold to a topological space X can be made smooth with respect to open balls B^n contained in X.

Proposition 65.9. *Let M be a manifold and let $f\colon M \to X$ be a map to a topological space X that is Hausdorff. Let $\varphi\colon \overline{B}^n \to X$ be an injective map such that $\varphi(B^n)$ is an open subset of X. Given any $r \in (0,1)$ there exists a map $g\colon M \to X$ with the following properties:*

1. *g is homotopic to f rel the complement of $f^{-1}(\varphi(B^n_r))$ in M,*
2. *the map*
$$\begin{array}{rcl} f^{-1}(\varphi(B^n_r)) & \to & B^n_r \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$
is smooth.

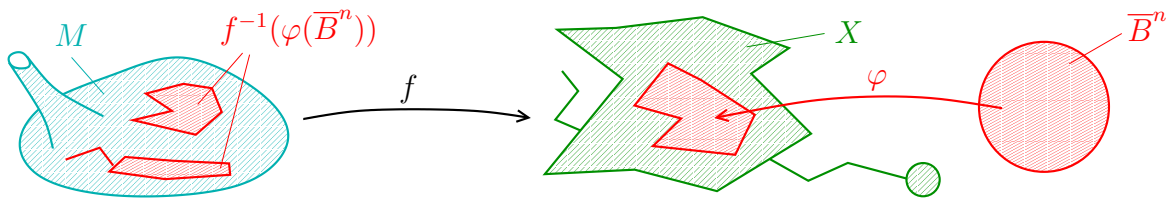


FIGURE 787.

Proof. Since $\overline{B}^n$ is compact, since X is Hausdorff and since $\varphi\colon \overline{B}^n \to X$ is injective we know by Proposition 1.40 that φ is a homeomorphism onto its image. Thus we might as well identify $\overline{B}^n$ with its image under the map $\varphi\colon \overline{B}^n \to X$. Now let $r \in \mathbb{R}_{>0}$. By Lemma 1.35 we know that B^n_r is also an open subset of X. Furthermore, note that it follows from the fact that X is Hausdorff, together with Lemma 1.18 (1), that $\overline{B}^n_r$ and $\overline{B}^n$ are a closed subsets of X. We consider the map

$$\begin{array}{rcl} \alpha\colon M & \to & \overline{B}^n/S^{n-1} \\ x & \mapsto & \begin{cases} f(x), & \text{if } x \in f^{-1}(B^n), \\ [S^{n-1}], & \text{if } x \in M \setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that α is continuous.[1099] As discussed on page 99 we can view $\overline{B}^n/S^{n-1}$ as a manifold such that B^n is a submanifold. Therefore we can apply Proposition 65.8 to the map $\alpha\colon M \to \overline{B}^n/S^{n-1}$ with $A = \varnothing$, with the closed subset $X = M \setminus f^{-1}(B^n_r)$[1100] and with the open subset $U = B^n_r \subset N$. We obtain a homotopy $F\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ with the following properties:

1. The homotopy is a homotopy rel $X = M \setminus f^{-1}(B^n_r)$.
2. We have $F_0 = \alpha$.
3. The restriction of F_1 to $M \setminus X = f^{-1}(B^n_r)$ is smooth.

[1099] Indeed, the restrictions of α to the closed subset $M \setminus f^{-1}(B^n)$ is a constant map and the restriction of α to the closed subset $f^{-1}(\overline{B}^n)$ is given by the composition of the map f with the projection map $\overline{B}^n \to \overline{B}^n/S^{n-1}$. Therefore we obtain from Lemma 1.32 (2) that α is indeed continuous.

[1100] Note the discrete subscript "r".

(4) For each $t \in [0,1]$ we have $F_t(f^{-1}(B^n_r)) \subset B^n_r$.

Finally we consider the map

$$\begin{array}{rcl} G\colon M \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,t), & \text{if } x \in f^{-1}(B^n_r) \\ f(x), & \text{if } x \in M \setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that the restrictions of G to the closed subsets $(M \setminus f^{-1}(B^n_r)) \times [0,1]$ and $f^{-1}(\overline{B}^n_r) \times [0,1]$ are continuous.[1101] Thus we obtain from Lemma 1.32 (2) that G is continuous. It is now straightforward to verify that $g := G_1$ has all the desired properties. □

The following proposition is a slight variation on Proposition 65.9.

Proposition 65.10. *Let M be a manifold and let $f\colon M \to X$ be a map to a topological space X that is Hausdorff. Let $n \in \mathbb{N}$ and let $* \in S^n$ be a point. Finally let $\varphi\colon S^n \to X$ be an injective map such that $\varphi(S^n \setminus \{*\})$ is an open subset of X. There exists a map $g\colon M \to X$ with the following properties:*

(1) *g is homotopic to f rel the complement of $f^{-1}(\varphi(S^n \setminus \{*\}))$ in M,*
(2) *the map*

$$\begin{array}{rcl} f^{-1}(\varphi(S^n \setminus \{*\})) & \to & S^n \setminus \{*\} \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$

is smooth.

Proof. We leave it to the reader to modify the proof of Proposition 65.9 to obtain the desired result. As the reader will notice, in this case there is no need to work with some $r \in (0,1)$. In particular the conclusion we get is slightly more satisfactory. □

Example. Let $\bigvee_{i=1}^{k} S^{n_i}$ be a wedge of spheres. We denote by $*$ the wedge point. Let M be a manifold and let $f\colon M \to \bigvee_{i=1}^{k} S^{n_i}$ be a map. By iteratively applying Proposition 65.10 altogether k times we see that f is homotopic rel $f^{-1}(\{*\})$ to a map that is smooth on the open sets $f^{-1}(S^{n_i} \setminus \{*\})$, $i = 1, \dots, k$.

65.3. **Embeddings of (topological) manifolds into $\mathbb{R}^n$.** In this and the coming section we study the existence of embeddings of (topological) manifolds into some sufficiently high-dimensional $\mathbb{R}^n$. For the reader's convenience we recall the following definitions from page 330.

[1101] For nervous people, like the author, let us verify in slightly greater detail that the restriction of G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ is indeed continuous. First note that by (4) the restriction of the map F to $f^{-1}(B^n_r) \times [0,1]$ takes values in B^n_r. It follows from Exercise 1.5 (3) and Exercise 1.25 that the restriction of the map F to $f^{-1}(\overline{B}^n_r) \times [0,1]$ takes values in the closure of B^n_r in $\overline{B}^n / S^{n-1}$. Since $r \in (0,1)$ we see that the closure equals $\overline{B}^n_r \subset B^n$. We now see that the restriction of the map G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ equals the composition of the map $F\colon f^{-1}(\overline{B}^n_r) \times [0,1] \to \overline{B}^n_r$ with the inclusion $\overline{B}^n_r \to X$.

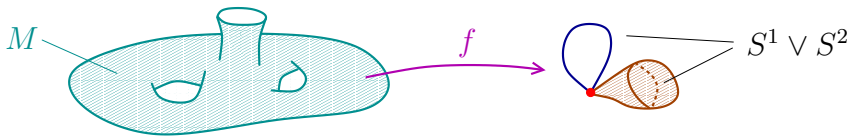


FIGURE 788.

Definition.

(1) A map $\varphi\colon N \to M$ between two topological manifolds is called a *topological embedding* if $\varphi\colon N \to \varphi(N)$ is a homeomorphism and if $\varphi(N)$ is a submanifold of M. (Here we view $\varphi(N)$ as a topological space equipped with the subspace topology coming from M.)

(2) A map $\varphi\colon N \to M$ between two manifolds is called an *immersion* if the following conditions are satisfied:
   (a) φ is smooth,
   (b) for each $P \in N$ the induced map $\varphi_*\colon T_PN \to T_{\varphi(P)}M$ is a monomorphism,
   (c) for each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$,
   (d) for each $P \in \partial N$ with $\varphi(P) \subset \partial M$ the image $\varphi_*(T_PN)$ is *not* contained in $T_{\varphi(P)}(\partial M)$.[1102]

   An immersion that also satisfies the following extra condition
   (e) $\varphi\colon N \to \varphi(N)$ is a topological embedding,
   is called an *embedding*.

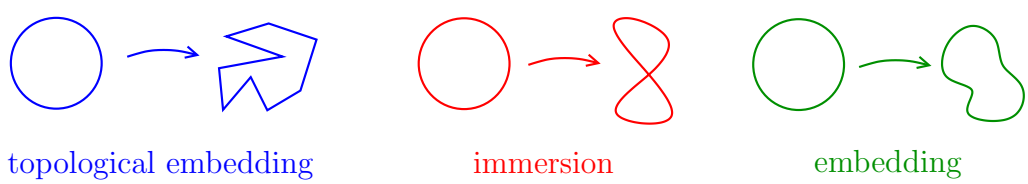


FIGURE 789.

Remark.

(1) Let $\varphi\colon N \to M$ be an injective map between topological manifolds. As is shown in Figure 790 such a map is not necessarily an embedding. But it follows from Lemma 1.37 that such a map is a topological embedding under the added hypothesis that N is compact. On several occasions we will use this observation without saying it explicitly.

(2) The super-meticulous reader will have noticed that the above definition of an embedding between manifolds differs slightly from the definition we gave on page 330.

[1102] Another way of expressing this condition is that the map $T_PN \to T_{\varphi(P)}M/T_{\varphi(P)}\partial M$ is an epimorphism.

In the original definition we had not demanded that the image of an embedding is (a topological) submanifold, but instead we saw in Proposition 10.1 that this property follows already from the a priori slightly weaker notion of an embedding that we had introduced on page 330. So any fears of incoherence in these notes just mercifully dissipated.

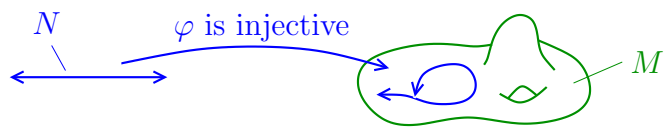


FIGURE 790.

Examples.

(1) It is straightforward to verify that the map

$$\begin{array}{rcl} S^1 \times S^1 & \to & \mathbb{R}^3 \\ (e^{i\theta}, e^{i\varphi}) & \mapsto & ((2+\sin(\theta))\cdot\cos(\varphi), (2+\sin(\theta))\cdot\sin(\varphi), \cos(\theta)) \end{array}$$

is an embedding of the 2-dimensional torus into $\mathbb{R}^3$.

(2) Let M be a closed non-empty n-dimensional topological manifold. In Corollary 36.8 we showed that there is no topological embedding of M into $\mathbb{R}^n$.

The question arises whether every (topological) manifold can be embedded into some $\mathbb{R}^n$. In the following proposition we give an affirmative answer for compact (topological) manifolds.

Proposition 65.11. *Given any compact (topological) manifold M there exists an $n \in \mathbb{N}$ and a (topological) embedding $M \to \mathbb{R}^n$. Furthermore, if M is an m-dimensional manifold*[1103]*, then we can arrange that $n = 2m+1$.*

The proof of the "furthermore" statement of Proposition 65.11 requires the following lemma, which is of interest in its own right.

Lemma 65.12. *Let M be an m-dimensional submanifold of $\mathbb{R}^n$. If $n > 2m+1$, then there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space*[1104] *T_PM and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.*

[1103]Just to avoid any confusion, note that the second statement only refers to manifolds. The proof that we give does not apply to topological manifolds.

[1104]Here we use Proposition 3.32 to interpret the tangent space T_PM as a subvector space of $\mathbb{R}^n$.

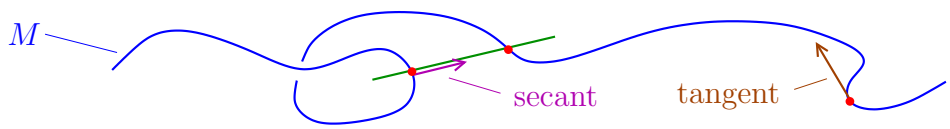


FIGURE 791. Illustration for Lemma 65.12.

Proof of Lemma 65.12. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. We suppose that $n > 2m+1$. We consider the maps

$$\begin{array}{rcl} f\colon \{(P,w) \mid P \in M, w \in T_P M \setminus \{0\}\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (P,w) & \mapsto & [w] \end{array}$$

and

$$\begin{array}{rcl} g\colon \{(x,y) \in M \times M \mid x \neq y\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x,y) & \mapsto & [x-y]. \end{array}$$

By the discussion on page 192 we can view $\mathbb{R}\mathrm{P}^{n-1}$ as an $(n-1)$-dimensional manifold. One can easily verify that the left-hand sides of both f and g are manifolds of dimension $2m$ and that the two maps are smooth. Since $2m < n-1$ we obtain from Proposition 3.47 that the complement of the images of f and g is dense, in particular non-empty. This means that there exists a $v \in \mathbb{R}\mathrm{P}^{n-1}$ such that the preimages of v under f and g are both the empty set. But that means that v has the desired properties. □

Now we turn to the proof of Proposition 65.11.

Proof of Proposition 65.11.

(a) First let M be a compact m-dimensional manifold. Since M is compact we can pick a finite smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,s}$. By Theorem 54.5 we can find a smooth partition of unity subordinate to the open covering $\{U_i\}_{i=1,\dots,s}$ of M. Put differently, we can find smooth functions $\nu_i\colon M \to [0,1]$, $i = 1,\dots,s$ such that for each i we have $\operatorname{supp}(\nu_i) \subset U_i$ and such that for all $x \in M$ we have $\nu_1(x) + \cdots + \nu_s(x) = 1$. For $i = 1,\dots,s$ we define

$$\begin{array}{rcl} \widetilde{\Phi}_i\colon M & \to & \mathbb{R}^m \\ x & \mapsto & \begin{cases} \Phi_i(x) \cdot \nu_i(x), & \text{if } x \in U_i, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It follows easily[1105] from Lemma 3.15 (1) that each of these maps $\widetilde{\Phi}_i$ is smooth. Finally note that the map

$$\begin{array}{rcl} \varphi\colon M & \to & \mathbb{R}^{s\cdot m+s} \\ x & \mapsto & (\widetilde{\Phi}_1(x),\dots,\widetilde{\Phi}_s(x),\nu_1(x),\dots,\nu_s(x)) \end{array}$$

[1105]Hmm, how does the smoothness of $\widetilde{\Phi}_i$ follow from Lemma 3.15 (1)?

is an embedding.[1106]

(b) The proof for topological manifolds is basically the same as in (a). The only difference is that from Theorem 54.5 we only obtain a continuous partition of unity subordinate to the open covering $\{U_i\}_{i=1,\dots,s}$ of M. In particular the map $\varphi\colon M \to \mathbb{R}^{s\cdot m+s}$ that we constructed in (a) is only a topological embedding.

(c) Finally assume that M is a compact m-dimensional manifold. We claim that we can can embed M into $\mathbb{R}^{2m+1}$. By (a) we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m+1$ there is nothing to prove, so we may assume that $n > 2m+1$. By Lemma 65.12 there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space T_PM and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

We consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

It follows easily from the claim that the restriction of the projection to M is an embedding. Thus we have embedded M into a real vector space of dimension $n-1$. Iterating this process we get an embedding of M into $\mathbb{R}^{2m+1}$. □

In the remainder of this section and in the subsequent chapter we will discuss refinements of Proposition 65.13 for manifolds. Many of the results also have more or less obvious analogues for topological manifolds, but to simplify the discussion we will not spell out the statements for topological manifolds.

Our first goal is to consider manifolds with boundary in greater detail. We recall the following definition and notation.

Definition.

[1106] For completeness' sake we provide the proof that $\varphi\colon M \to \mathbb{R}^{s\cdot m+s}$ is an embedding, even though the interest in the full details might be quite muted.

(a) First we show that φ is injective. Let $x, x' \in M$. We pick i with $\nu_i(x) \neq 0$. If $\nu_i(x') = 0$, then evidently $\varphi(x) \neq \varphi(x')$. If $\nu_i(x) \neq 0$, then $x, x' \in U_i$. Note that $\Phi_i(x) \neq \Phi_i(x')$, which is equivalent to $\nu_i(x)^{-1} \cdot \widetilde{\Phi}_i(x) \neq \nu_i(x')^{-1} \cdot \widetilde{\Phi}_i(x')$. But this means that $(\widetilde{\Phi}_i(x), \nu_i(x)) \neq (\widetilde{\Phi}_i(x'), \nu_i(x'))$, which implies that $\varphi(x) \neq \varphi(x')$.

(b) We need to show that for each $x \in M$ the differential $D\varphi_x\colon T_xM \to \mathbb{R}^{s\cdot m+s}$ is a monomorphism. We pick i with $\nu_i(x) \neq 0$. We consider the map

$$\begin{array}{rcccl} U_i & \to & \{(v,t) \in \mathbb{R}^m \times \mathbb{R} \mid t \neq 0\} & \to & \mathbb{R}^m \\ y & \mapsto & (\widetilde{\Phi}_i(y), \nu_i(y)) & & \\ & & (v,t) & \mapsto & \frac{1}{t} \cdot v. \end{array}$$

Note that the composition is just the diffeomorphism Φ_i, thus the differential at x is a monomorphism. But this implies that the differential of the first map at x is a monomorphism, which implies that the differential of φ at x is a monomorphism.

(1) Let N be a compact manifold and let M be any manifold. A map $\varphi\colon N \to M$ is called *proper* if $\varphi^{-1}(\partial M) = \partial N$.
(2) Given $n \in \mathbb{N}$ we write

$$H_n \;:=\; \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}.$$

For manifolds with boundary the following proposition refines the statement of Proposition 65.11.

Proposition 65.13. *Let M be a compact m-dimensional manifold.*

(1) *Given any $n \geq 2m+1$ there exists a proper embedding $M \to H_n$.*
(2) *Suppose we are given an embedding $f\colon \partial M \to \mathbb{R}^k$ of the boundary of M. Given any $n \geq 2m+k+1$ there exists a proper embedding $F\colon M \to H_n$ such that the following diagram commutes:*

$$\begin{tikzcd} \partial M \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d, "F"] \\ \mathbb{R}^k \arrow[r, "x \mapsto (x,0)"] & H_n. \end{tikzcd}$$

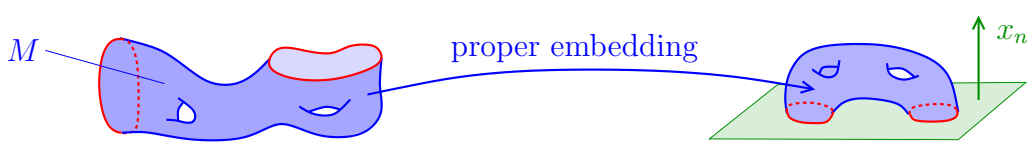


FIGURE 792. Illustration of Proposition 65.13 (1).

Proof $(*)$. Both statements of the proposition follow from the following claim.[1107]

Claim. Let M be a compact m-dimensional manifold and let Y be a (possibly empty) union of components of ∂M. Suppose we are given an embedding $f\colon Y \to \mathbb{R}^k$ of the boundary of M. Let $n \geq 2m+k+1$. Then there exists a proper embedding $F\colon M \to H_n$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d, "F"] \\ \mathbb{R}^k \arrow[r, "x \mapsto (x,0)"] & H_n. \end{tikzcd}$$

Evidently the proof of the claim is quite similar to the proof of Proposition 65.11. Therefore we will only indicate in what ways the proof needs to be modified.

[1107]Indeed, the first statement of the proposition follows from $Y = \varnothing$ and $k = 0$. The second statement follows from setting $Y = \partial M$.

(a) By the Collar Neighborhood Theorem 10.10 we can pick a collar $\partial M \times [0,1]$. Note that using Lemma 3.12 one can easily construct a smooth non-decreasing function $\eta\colon [0,1] \to [0,1]$ with $\eta(t) = t$ for all $t \in [0, \frac{1}{4}]$ and with $\eta(t) = 1$ for all $t \in [\frac{1}{2}, 1]$. We define[1108]

$$\begin{array}{rcl} \mu\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P = (Q,t) \text{ with } Q \in \partial M \text{ and } t \in [0,1], \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and we define

$$\begin{array}{rcl} \widetilde{f}\colon M & \to & \mathbb{R}^k \\ P & \mapsto & \begin{cases} f(Q) \cdot (1 - \eta(t)), & \text{if } P = (Q,t) \text{ with } Q \in \partial M \text{ and } t \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

In the following we use the same notation and definitions as in the proof of Proposition 65.11. In particular we pick a finite atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,s}$, we pick a corresponding smooth partition of unity $\nu_i\colon M \to [0,1]$, $i = 1, \dots, s$ and we define $\widetilde{\Phi}_i\colon M \to \mathbb{R}^m$, $i = 1, \dots, s$ the same way as before. Similar to the proof of Proposition 65.11 one can verify that the map

$$\begin{array}{rcl} F\colon M & \to & \mathbb{R}^k \times \mathbb{R}^{s\cdot m+s} \times \mathbb{R}_{\geq 0} = H_{k+s\cdot m+s+1} \\ x & \mapsto & \big(\underbrace{\widetilde{f}(x)}_{\in\mathbb{R}^k}, \underbrace{\mu(x) \cdot \big(\widetilde{\Phi}_1(x), \dots, \widetilde{\Phi}_s(x), \nu_1(x), \dots, \nu_s(x)\big)}_{\in\mathbb{R}^{s\cdot m+s}}, \underbrace{\mu(x)}_{\in\mathbb{R}_{\geq 0}}\big) \end{array}$$

is a proper embedding. It is clear that the map F makes the diagram in the claim commute.

(b) To prove the claim it now remains to show that given such a proper embedding $F\colon M \to H_n$ with $n > 2m+k+1$ we can find a lower-dimensional proper embedding with the same properties. Without loss of generality we can assume that M is already a submanifold of H_n.

Similar to the proof of Proposition 65.11 we can find a non-zero vector $v \in \mathbb{R}^n$ with the following three properties:

(a) the vector does not lie in any tangent space T_xM,
(b) for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v,
(c) the vector is orthogonal to $\mathbb{R}^k \times \{0\}$.[1109]

Note that projection onto the hyperplane that is orthogonal to v is the identity on $\mathbb{R}^k \times \{0\}$. Thus as in the proof of Proposition 65.11 we can lower the dimension by one without losing any of the desired properties. □

As happens so often in mathematics, the proof of a result immediately invites new questions. For example, in light of Propositions 65.11 and 65.13 the following question arises.

[1108] It follows from Lemma 3.15 (2) that these two maps are smooth.

[1109] Here we need that we have "extra k dimensions", i.e. we need that $n > 2m + 1 + k$.

Question 65.14. *For a given (topological) manifold M, e.g. $M = \mathbb{R}\mathrm{P}^n$, what is the minimal $k \in \mathbb{N}$ such that M admits a (topological) embedding into $\mathbb{R}^k$?*

We will address this question in the following chapter.

We conclude this section with a discussion of the uniqueness of embeddings of a given manifold M into some $\mathbb{R}^n$. Let us consider the three embeddings $f_0, f_1, f_2 \colon S^1 \to \mathbb{R}^3$ that are shown in Figure 793. The embeddings f_0, f_1 are "essentially the same", in the sense that they are isotopic. On the other hand it follows from Lemma 18.6, Exercise 12.6 and the Isotopy Extension Theorem 10.21 that the two embeddings f_0 and f_2 are not isotopic.

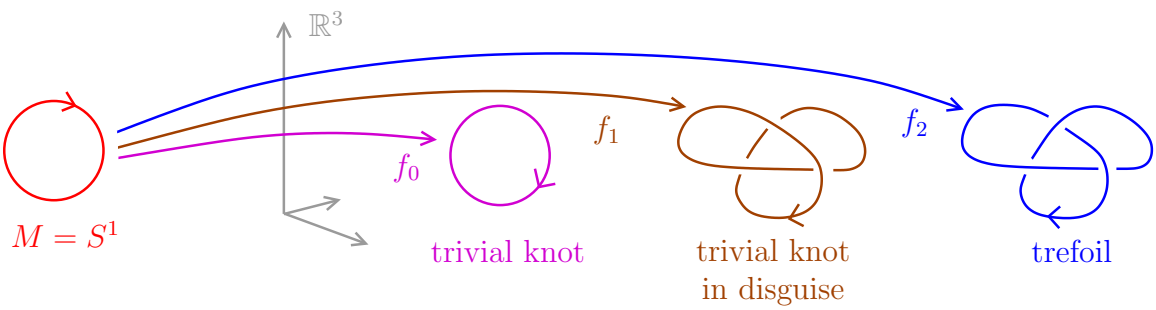


FIGURE 793.

Now we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Therefore we can view the above embeddings f_0, f_1, f_2 as embeddings $f_0, f_2 \colon S^1 \to \mathbb{R}^4$. In Figure 794 we do our best to illustrate an isotopy $F \colon S^1 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^4$ from f_2 to f_1. The key to the understanding of the figure is that we use "pinkness" as the fourth coordinate, i.e. two points in $\mathbb{R}^3$ with different levels of "pinkness" differ in the fourth coordinate. With this convention the map $F \times S^1 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ shown in Figure 794 has the property that each F_t is an embedding. We have thus shown that f_2, f_1, viewed as embeddings into $\mathbb{R}^4$, are isotopic. In Exercise 65.5 we will in fact see that all embeddings of S^1 into $\mathbb{R}^3$ become isotopic once we view them as embeddings into $\mathbb{R}^4$.

In the above longish example we have seen that non-isotopic embeddings can become isotopic once we add extra dimensions. The following proposition shows that this phenomenon holds in general. More precisely, the proposition says that any two embeddings of a given closed manifold into some $\mathbb{R}^k$ become isotopic if we are allowed to increase the dimensions.

Proposition 65.15. *Let M be a closed manifold. Suppose we are given two embeddings $f_0, f_1 \colon M \to \mathbb{R}^k$. There exists a smooth isotopy $F \colon M \times [0,1] \to \mathbb{R}^{2k}$ such that for $i = 0, 1$*

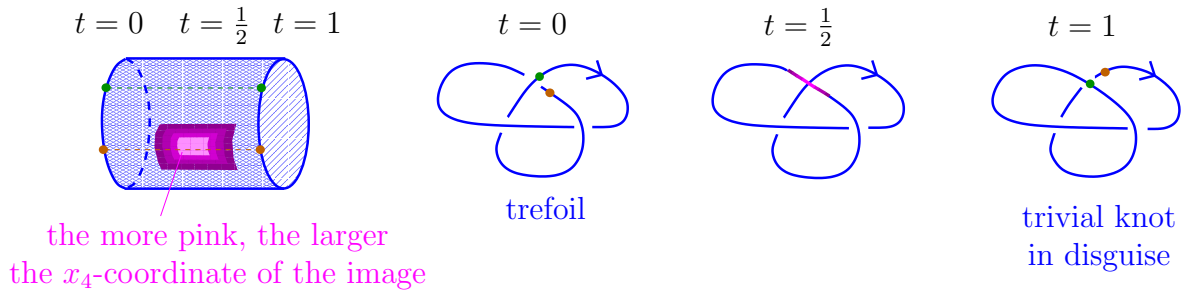


FIGURE 794.

the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, "f_i"'] \arrow[dr, "F_i"] & \\ \mathbb{R}^k \arrow[rr, "x\mapsto(x,0)"'] & & \mathbb{R}^{2k}. \end{tikzcd}$$

Proof. By Lemma 3.12 there exists a smooth function $\nu\colon [0,1]\to[0,1]$ with $\nu(t)=1$ for $t\in[0,\frac{1}{4}]$ and $\nu(t)=0$ for $t\in[\frac{3}{4},1]$. We consider the map

$$\begin{aligned} F\colon M\times[0,1] &\to \mathbb{R}^{2k} \\ (x,t) &\mapsto (\underbrace{f_0(x)\cdot\nu(t)+f_1(x)\cdot(1-\nu(t))}_{\in\mathbb{R}^k}, \underbrace{f_0(x)\cdot(1-\nu(t))\cdot\nu(t)}_{\in\mathbb{R}^k}). \end{aligned}$$

It is straightforward to verify that F is a smooth isotopy and that F makes the given diagram commute for $i=0,1$. □

The final proposition of this section is the "boundary version" of Proposition 65.15. This proposition will play an important role later on. But till then it is best to ignore it.

Proposition 65.16. $(*)$ *Let M be a compact manifold and let Y be a (possibly empty) union of boundary components of M. Suppose we are given two proper embeddings $f_0, f_1\colon M\to H_k$.*

(1) *If $f_0|_Y=f_1|_Y$, then there exists a proper smooth isotopy $F\colon M\times[0,1]\to H_{2k}$ rel Y such that for $i=0,1$ the following diagram commutes*

$$\begin{tikzcd} & M \arrow[dl, "f_i"'] \arrow[dr, "F_i"] & \\ H_k \arrow[rr, "x\mapsto(0,x)"'] & & H_{2k}. \end{tikzcd}$$

(2) *If $f_0(Y)=f_1(Y)$, then there exists a proper smooth isotopy $F\colon M\times[0,1]\to H_{2k}$ rel Y such that for $i=0,1$ the image of $f_i(M)$ under the obvious inclusion $H_k\to H_{2k}$ agrees with $F_i(M)$.*

The analogous statements also hold if we replace H_k by $[0,1]\times\mathbb{R}^{k-1}$ and if we replace H_{2k} by $[0,1]\times\mathbb{R}^{2k-1}$.

Proof $(*)$.

(1) We suppose that $f_0|_Y = f_1|_Y$. As in the proof of Proposition 65.15 we pick a smooth function $\nu\colon [0,1]\to[0,1]$ with $\nu(t)=1$ for $t\in[0,\frac{1}{4}]$ and $\nu(t)=0$ for $t\in[\frac{3}{4},1]$. Furthermore, as in the proof of Proposition 65.13 we pick a collar $\partial M\times[0,1]$, we pick a non-decreasing smooth function $\eta\colon[0,1]\to[0,1]$ with $\eta(t)=t$ for all $t\in[0,\frac{1}{4}]$ and with $\eta(t)=1$ for all $t\in[\frac{1}{2},1]$ and we consider the smooth functions

$$\begin{array}{rcl} \mu\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P=(Q,t) \text{ with } Q\in Y \text{ and } t\in[0,1], \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and

$$\begin{array}{rcl} \epsilon\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P=(Q,t) \text{ with } Q\in \partial M \text{ and } t\in[0,1], \\ 1, & \text{otherwise.} \end{cases} \end{array}$$

Finally we consider the map

$$\begin{aligned} F\colon M\times[0,1] &\to \mathbb{R}^k\times H_k = H_{2k} \\ (x,t) &\mapsto (\underbrace{\mu(x)\cdot(f_0(x)\cdot(1-\eta(t))\cdot\eta(t))}_{\in\mathbb{R}^k}, \underbrace{f_0(x)\cdot\eta(t)+f_1(x)\cdot(1-\eta(t))+\epsilon(x)\cdot e_k}_{\in H_k}). \end{aligned}$$

It is straightforward to verify that F is a proper smooth isotopy rel Y and that F makes the given diagram commute for $i=0,1$.

(2) Now we only suppose that $f_0(Y)=f_1(Y)$. We consider the map $\varphi := f_1^{-1}\circ f_0\colon M\to M$. By Proposition 10.1 (2) we know that φ is a diffeomorphism of M. Note that the embeddings $f_0\colon M\to H_k$ and $f_1\circ\varphi\colon M\to H_k$ agree on Y. Thus we can apply (1). The proper smooth isotopy that we obtain this way evidently has the desired property.

The above proofs also apply, with minuscule modifications, if we replace H_k by $[0,1]\times\mathbb{R}^{k-1}$ and if we replace H_{2k} by $[0,1]\times\mathbb{R}^{2k-1}$. □

Example. We consider the two embeddings $f_0, f_1\colon M=[0,1]\to H_3$ shown in Figure 795. They agree on $Y=\{0\}$. By Proposition 65.16 these two embeddings become isotopic rel Y if we view f_0 and f_1 as embeddings into H_6.

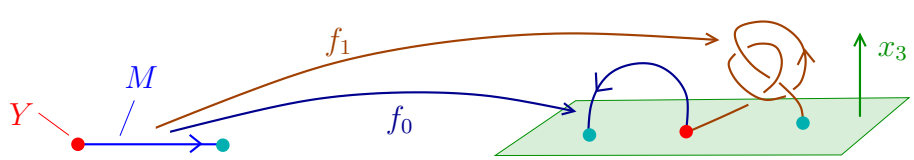


FIGURE 795.

65.4. **The Whitney Embedding Theorem** (∗)**.** In this section we state several propositions and theorems that generalize the results from the previous section to larger classes of manifolds and that give slightly improved bounds on the dimensions required to obtain (unique) embeddings. We will not make use of the results, therefore we will not provide any proofs.

The following proposition can be viewed as a moderately difficult generalization of Proposition 65.11.

Proposition 65.17.

(1) *Every n-dimensional manifold M without boundary admits a proper embedding $f\colon M \to \mathbb{R}^{2n+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2n+1}$.*
(2) *Every n-dimensional manifold M with non-empty boundary admits a proper embedding $f\colon M \to H_{2n+1}$ such that $f(M)$ is a closed subset of H_{2n+1}.*

Proof. As indicated above, it is an entertaining and slightly challenging exercise to modify the proofs of Proposition 65.11 and 65.13 to obtain the results. The argument is carried out in [Why, Theorem 1], [Bre, Theorem II.10.8], [Hirs, Theorems 1.3.5 and 1.4.3] and [Le2, Theorem 6.15]. □

Remark. Note that in Proposition 65.17 it is essential that manifolds are, by definition, second-countable. If we did not make this assumption, then any set with the discrete topology would be a 0-dimensional manifold, but there are sets whose cardinality is bigger than the cardinality of any $\mathbb{R}^n$. More precisely, by Cantor's Theorem, see page 255, the power set of $\mathbb{R}$ does not admit an injection into any $\mathbb{R}^n$.

In Propositions 65.11, 65.13 and 65.17 we had in particular seen that every n-dimensional manifold admits an embedding into $\mathbb{R}^{2n+1}$. The following theorem decreases the dimension by one. Even though this looks like a minor achievement, the proof of the Whitney Embedding Theorem is quite intricate.

Theorem 65.18. (Whitney Embedding Theorem) *Every n-dimensional manifold admits an embedding into $\mathbb{R}^{2n}$.*

Proof. This result was first proved by Hassler Whitney [Why2, p. 236] in 1944. For closed manifolds more recent accounts of the proof can be found in [Adac, Theorem II.2.11] or [Pra, Chapter 6.3]. □

Remark.

(1) By [Why3, p. 247] every n-dimensional manifold, with $n \geq 2$, can be immersed into $\mathbb{R}^{2n-1}$.
(2) Under some extra conditions one can obtain better results. For example we saw on page 210 that every closed orientable 2-dimensional manifold embeds into $\mathbb{R}^3$. Furthermore Hirsch [Hirs2] and Wall [Wall5] showed that every 3-dimensional manifold embeds into $\mathbb{R}^5$. Finally for $n > 4$ Haefliger-Hirsch [HH, Theorem 1.2] showed that every closed orientable n-dimensional manifold embeds into $\mathbb{R}^{2n-1}$.

(3) In [Mun3, p. 315] it is shown that every n-dimensional topological manifold admits a topological embedding into $\mathbb{R}^{2n+1}$. At least for compact topological manifolds one can actually find a topological embedding into $\mathbb{R}^{2n}$. We refer to [MeS, Corollary 4.14] and [BrM] for details.
(4) If M and N are two manifolds that can be embedded (respectively immersed) into $\mathbb{R}^n$, then the construction in [Hae, Chapter 1] [Hae2, Chapter 3] shows that the connected sum $M\#N$ can also be embedded in $\mathbb{R}^n$.

The following theorem is a refinement of Propositions 65.15 and 65.16.

Theorem 65.19.

(1) *Let M be a closed n-dimensional manifold. If $m \geq 2n+1$ and $m \geq n+3$, then any two embeddings $f_0, f_1 \colon M \to \mathbb{R}^m$ are smoothly isotopic.*
(2) *Let M be a compact n-dimensional manifold. If $m \geq 2n+1$ and $m \geq n+3$, then any two proper embeddings $f_0, f_1 \colon M \to H_m$ are related by a proper smooth isotopy.*

Proof. Both statements follow from [Hud, Theorem 4]. For connected manifolds the first statement also follows from [Hae, p. 47]. □

Remark.

(1) We consider again the two embeddings $f_0, f_2 \colon S^1 \to \mathbb{R}^3$ from Figure 793. As we pointed out on page 1517, these two embeddings are not isotopic. This shows that in Theorem 65.19 we not only need the condition that $m \geq 2n+1$ (which is satisfied in this example) but also the condition that $m \geq n+3$.
(2) The statements of Theorem 65.19 can be generalized considerably, in particular the target manifolds do not necessarily need to be $\mathbb{R}^m$ and H_m. We refer to [Hae, p. 47], [Hud] and [Sko] for many generalizations.

65.5. **Neighborhood retracts.** We recall two definitions from pages 307 and 130 and we introduce two related definitions.

Definition. Let X be a topological space.

(1) We say X is *contractible* if X is homotopy equivalent to a point.
(2) We say X is *locally contractible* if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible.
(3) We say that a subset U of X is *contractible in X* if the inclusion map $U \to X$ is homotopic to a constant map.
(4) We say X is *weakly o-locally contractible* if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible in U.

Examples.

(1) It follows almost immediately from the definitions that a topological space that is o-locally contractible is also weakly o-locally contractible.

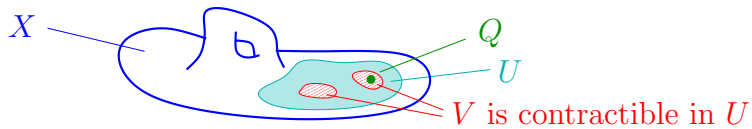


FIGURE 796.

(2) It is straightforward to verify that topological manifolds are o-locally contractible.
(3) In Proposition 26.11 (6) we showed that CW-complexes are o-locally contractible. In particular topological graphs and spatial graphs are o-locally contractible.
(4) It is easy to show that $\mathbb{R}^2 \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not weakly o-locally contractible.
(5) It is an amusing exercise to show that the topologist's sine curve

$$\big\{(0,y) \,\big|\, y \in [-1,1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x})) \,\big|\, x \in (0,\pi]\big\} \;\subset\; \mathbb{R}^2,$$

which we had initially introduced on page 123, is not weakly o-locally contractible.
(6) It is an amusing (but not entirely trivial) exercise to come up with a subset of some $\mathbb{R}^n$ that is weakly o-locally contractible but not o-locally contractible.

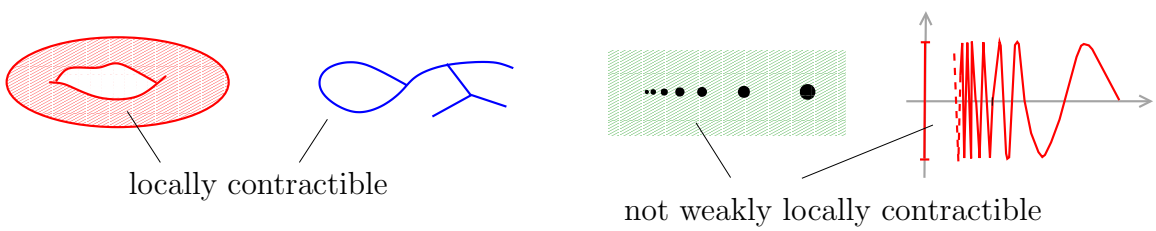


FIGURE 797.

Definition. Let X be a topological space and let A be a subset. We say A is a *neighborhood retract* if there exists a neighborhood N of A such that A is a retract of N.

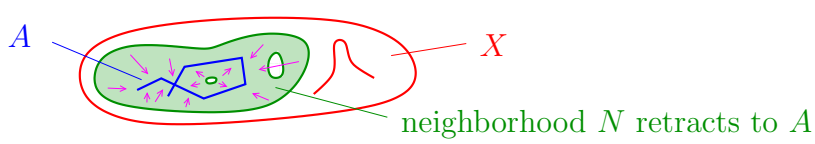


FIGURE 798.

Examples.

(1) It follows from Proposition 64.2 that every finite spatial graph in $\mathbb{R}^n$ is a neighborhood retract.

(2) Let M be a manifold. By the Regular Neighborhood Theorem 64.5 every compact proper submanifold of M is a neighborhood retract.
(3) We will see in Exercise 65.4 that $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, viewed as a subset of $X = \mathbb{R}^2$, is not a neighborhood retract.

We continue with the following fairly elementary lemma.

Lemma 65.20. *Let X be a subset of $\mathbb{R}^n$. If X is a neighborhood retract, then it is also weakly o-locally contractible.*

Proof $(*)$. Let W be a neighborhood of X that admits a retraction $r\colon W \to X$. Let $Q \in X$ and let U be a neighborhood of Q in X. Since W is open there exists an $s > 0$ such that $B_s(Q) \subset r^{-1}(U)$. We denote by $F\colon B_r(Q) \times [0,1] \to B_r(Q)$ the obvious homotopy from the identity to the constant map. Then $r \circ F$ restricts to a homotopy $(B_s(Q) \cap X) \times [0,1] \to U$ from the identity to a constant map. In particular $V := B_s(Q) \cap X$ is contractible in U. $\square$

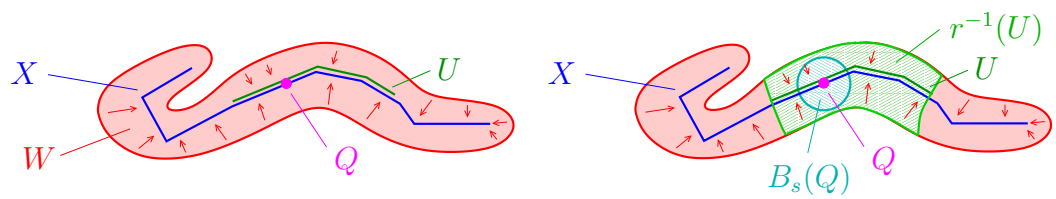


FIGURE 799. Illustration of the proof of Lemma 65.20.

It is rather stunning that for compact subsets the converse to Lemma 65.20 holds. More precisely, the following theorem was proved by Karol Borsuk [Bor] in 1932.[1110]

Theorem 65.21. (Borsuk Theorem *Let X be a compact subset of $\mathbb{R}^n$. If X is weakly o-locally contractible, then it is a neighborhood retract.*[1111]

Remark.

(1) It is fairly easy to modify the proof of Theorem 65.21 to show that the statement also holds for subsets of $\mathbb{R}^n$ that are weakly o-locally contractible and that are *locally* compact, see [Bre, Theorem E.3] for details. We will not make use of this more general statement.

[1110]Karol Borsuk (1905-1982) was a Polish mathematician. He lost his job at the university of Warsaw after the German invasion in 1939. To survive financially he developed a board game and produced and sold it during the war. Almost all copies of the game were lost during the Warsaw uprising in 1944, but in the 1990s one surviving copy was found and the game was republished. We refer to

`https://en.wikipedia.org/wiki/Animal_Husbandry_(game)`

for more details.

[1111]Is is worth stressing that the proposition does *not* claim that X is a *deformation retract* of a neighborhood of X.

(2) In the literature, see e.g. [Dol, Bre], a topological space X is called a *Euclidean neighborhood retract* if there exists a subset Y of some $\mathbb{R}^n$ which is homeomorphic to X and that is a neighborhood retract. In this language Theorem 65.21, with the slight strengthening of (1) says that every o-locally compact, weakly o-locally contractible topological space that admits a topological embedding into some $\mathbb{R}^n$ is a Euclidean neighborhood retract.

Before we provide the proof of Theorem 65.21 it is convenient to introduce one definition and to prove one lemma.

Definition. Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. A $\frac{1}{2^k}$*-cube* is a subset C of $\mathbb{R}^n$ of the form

$$C \;=\; \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \dots, a_n \in \mathbb{Z}.$$

Put differently, a $\frac{1}{2^n}$-cube is a cube with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.

Lemma 65.22. *Let $x \in \mathbb{R}^n$ and let $r > 0$. If $k \in \mathbb{N}_0$ is chosen such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$, then there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \dots, W_m$ such that*

$$x \;\in\; \textit{interior of } W_1 \cup \cdots \cup W_m \;\subset\; W_1 \cup \cdots \cup W_m \;\subset\; B_r(x).$$

Proof. We write $x = (x_1, \dots, x_n)$ and we write $\epsilon = \frac{1}{2^k}$. For $i = 1, \dots, n$ we denote by y_i the result of rounding x_i down to the closest number in $\mathbb{Z} \cdot \epsilon$. Clearly we have

$$(x_1, \dots, x_k) \;\in\; \big[y_1, y_1{+}\epsilon\big] \times \cdots \times \big[y_n, y_n{+}\epsilon\big] \;\subset\; \text{interior of } \underbrace{\big[y_1{-}\epsilon, y_1{+}2\epsilon\big] \times \ldots \times \big[y_n{-}\epsilon, y_n{+}2\epsilon\big]}_{=:V}.$$

Since V is the union of finitely many ϵ-cubes it suffices to show that V is contained in $B_r(x)$. Thus let $y \in V$. We have

$$d(y,x) \;\underset{\substack{\uparrow \\ \text{since } x \in V}}{\leq}\; \text{diameter of } V \;\underset{\substack{\uparrow \\ \text{see page 137}}}{=}\; 3\sqrt{n} \cdot \epsilon \;=\; 3\sqrt{n} \cdot \tfrac{1}{2^k} \;\underset{\substack{\uparrow \\ \text{choice of } k}}{<}\; r$$

which means of course that $y \in B_r(x)$. Thus we have shown that $V \subset B_r(x)$. □

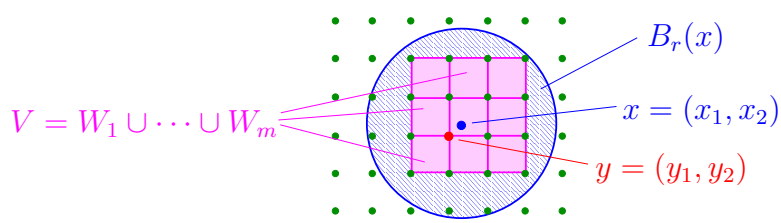


FIGURE 800. Illustration of the proof of Lemma 65.22.

Proof of Theorem 65.21. Let X be a compact, weakly o-locally contractible subset of $\mathbb{R}^n$. Evidently we only have to consider the case that X is non-empty. We introduce the following notation.

(1) We define C_1 to be the union of all 1-cubes that do not intersect X.
(2) We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that do not intersect X.
(3) We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that do not intersect X.

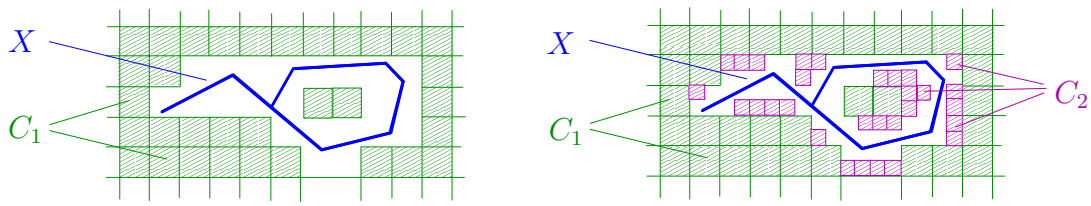


FIGURE 801.

Claim.

(1) We have $\bigcup_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$.
(2) Given any $x_0 \in \mathbb{R} \setminus X$ there exists an open neighborhood U that intersects only finitely many cubes of $C_1, C_2, \ldots$.

Let $x_0 \in \mathbb{R}^n \setminus X$. Since X is compact we can apply Corollary 1.56 to find an $r > 0$ such that $B_r(x_0) \subset \mathbb{R}^n \setminus X$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{3\sqrt{n}\cdot 2^k} < r$. By Lemma 65.22 there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_m$ such that

$$x_0 \in \underbrace{\text{interior of } W_1 \cup \cdots \cup W_m}_{:=U} \subset B_r(x).$$

This implies in particular that x_0 is contained in C_k or it is contained already in one of $C_1, \ldots, C_{k-1}$. Furthermore it is fairly easy to see that if U intersects a $\frac{1}{2^m}$-cube with $m \geq k$ it already intersects a $\frac{1}{2^k}$-cube. It follows from this observation and the iterative definition of the C_i that U intersects only finitely many cubes of $C_1, C_2, \ldots$. This concludes the proof of the claim.

We view each C_i as a CW-complex in the obvious way: the n-cells correspond precisely to the cubes and the lower dimensional cells correspond precisely to the faces of the cubes. For each k the union $C_1 \cup \cdots \cup C_k$ is also a CW-complex in an obvious way and finally we can also view the union $C := \bigcup_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$ as a CW-complex. We leave it as an exercise to verify, using statement (2) of the claim, that the topology on this CW-complex, given by the weak topology defined on page 654, agrees with the usual subspace topology on $\mathbb{R}^n \setminus X$.

We continue with the following definition: given a cell σ of C and a map $f\colon \sigma \to X$ we define[1112]

$$\rho(f) \;:=\; \max\{d(z, f(z)) \,|\, z \in \sigma\}.$$

We start with the iterative definition of a subset of $\mathbb{R}^n$ together with a map to X.

(a) We denote by A_0 the 0-skeleton of C. Next we define a map $r_0\colon A_0 \to X$. Let $a \in A_0$. Since X is compact it follows from Corollary 1.56 that we can find a point $r_0(a) \in X$ with $d(a, r_0(a)) = d(a, X)$.

(b) Now suppose inductively that we have defined a map $r_i\colon A_i \to X$ on a union A_i of some of the i-cells of C. We define the next set A_{i+1} and the corresponding map $r_{i+1}\colon A_{i+1} \to X$ as follows:

(b$_1$) Let σ be an $(i+1)$-cell of C for which r_i is defined on $\partial\sigma$. Suppose that σ can be extended to a map $r_{i+1}\colon \sigma \to X$ that agrees with r_i on $\partial\sigma$. Then we pick a map $r_{i+1}\colon \sigma \to X$ with $r_{i+1}|_{\partial\sigma} = r_i$ such that

$$\rho(r_{i+1}) \;\le\; 2 \cdot \underbrace{\inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ is a map with } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}}_{\ge\, d(\sigma, X) \text{ which in turn is } > 0 \text{ by Corollary 1.56}}.$$

(b$_2$) We denote by A_{i+1} the union of all $(i+1)$-cells as in (b$_1$) and we define r_{i+1} on A_{i+1} as in (b$_1$).

(c) We set $A := \bigcup_{i\in\mathbb{N}} A_i \cup X$ and we define $r\colon A \to X$ to be the unique map that equals r_i on each A_i and that is the identity on X.

We refer to Figure 802 for an illustration. It remains to show that the set A is a neighbor-

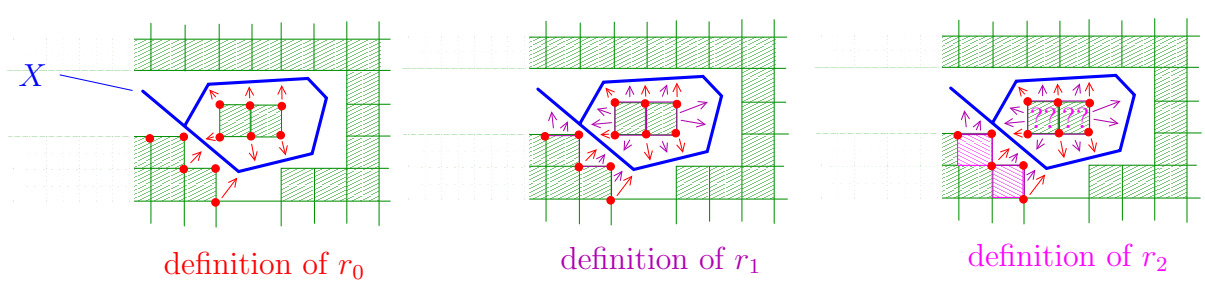


FIGURE 802.

hood of X and that the map $r\colon A \to X$ is continuous. Both statements follow from the following claim.

Claim. Given any $x \in X$ and any $\epsilon_0 > 0$ there exists an open neighborhood U that is contained in A and such that $|\underbrace{r(x)}_{=x} - r(y)| < \epsilon_0$ for all $y \in U$, i.e. such that $r(U) \subset B_{\epsilon_0}(x)$.

[1112]Note hat σ is compact, hence the maximum exists by Lemma 1.37 (2).

So let $x \in X$ and let $\epsilon_0 > 0$. We adopt the following notation: given $\mu > 0$ we write $W(\mu) = X \cap B_\mu(x)$.

Subclaim. We can find $\epsilon_1, \dots, \epsilon_{2n}$ with the following two properties:

(1) for each $i \in \{0, \dots, 2n-1\}$ we have $\epsilon_{i+1} < \frac{1}{3} \cdot \epsilon_i$,
(2) for every $i = 1, \dots, 2n$ the subset $W(\epsilon_i)$ is contractible in $W(\epsilon_{i-1})$.

So suppose that we have already found $\epsilon_0, \dots, \epsilon_i$ for some $i \in \{0, \dots, 2n-1\}$ with the desired properties. Since X is weakly o-locally contractible we can find an open neighborhood U of x in X that is contained in $W(\epsilon_i)$ and that is contractible in $W(\epsilon_i)$. Since U is open there exists an $\epsilon_{i+1} < \frac{1}{3}\epsilon_i$ such that $W(\epsilon_{i+1}) \subset U$. Since U is contractible in $W(\epsilon_i)$ the subset $W(\epsilon_{i+1})$ is also contractible in $W(\epsilon_i)$. We refer to Figure 803 for an illustration. This concludes the proof of the subclaim.

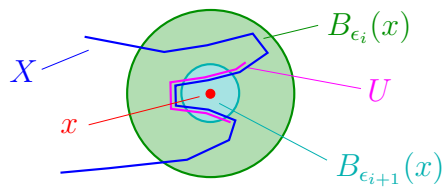


FIGURE 803.

We set $U := B_{\frac{1}{4}\epsilon_{2n}}(x)$. It suffices to prove the following two subclaims.

Subclaim 1. For each $i = 0, \dots, n$ the map r is defined for each i-cell σ of the CW-complex C that lies in U and for each such σ we have $r(\sigma) \subset W(\epsilon_{2n-2i})$.

Subclaim 2. The union of X with all cells of C that lie in U is a neighborhood of x.

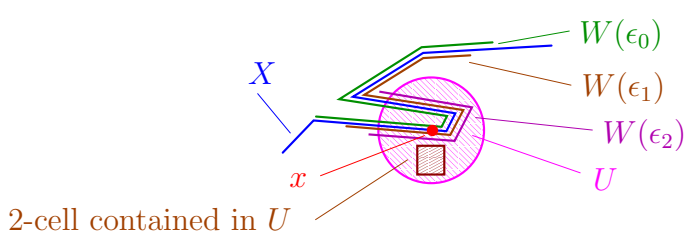


FIGURE 804.

First we prove Subclaim 1. We prove Subclaim 1 iteratively for $i = 0, 1, \dots, n$. We start out with $i = 0$. Thus we have to show that $r(U \cap A_0) \subset W(\epsilon_{2n})$. Therefore let $y \in U \cap A_0$. Then we have

$$d(x, r(y)) \underset{\substack{\uparrow \\ \text{triangle inequality and } r(y) = r_0(y)}}{\leq} d(x,y) + d(y, r_0(y)) \underset{\substack{\uparrow \\ \text{by (a) since } x \in X}}{\leq} d(x,y) + d(y,x) = 2d(x,y) \underset{\substack{\uparrow \\ \text{since } y \in U = B_{\frac{1}{4}\epsilon_{2n}}(x)}}{<} 2 \cdot \frac{1}{4} \cdot \epsilon_{2n} = \frac{1}{2} \cdot \epsilon_{2n}.$$

But since $r(y) \in X$ this means precisely that $r(y) \in W(\epsilon_{2n}) = X \cap B_{\epsilon_{2n}}(x)$. This shows that Subclaim 1 holds for $i = 0$.

Now suppose that Subclaim 1 holds for some $i \in \{0, \dots, n-1\}$. Let σ be an $(i+1)$-cell of the CW-complex C inside U. Since $r(\partial\sigma) \subset W(\epsilon_{2n-2i})$ and since $W(\epsilon_{2n-2i})$ is contractible in $W_{2n-2i-1}$ it follows from Proposition 1.47 and Proposition 28.7 (2) that we can extend $r|_{\partial\sigma} = r_i|_{\partial\sigma}$ to a map $s\colon \sigma \to W(\epsilon_{2n-2i-1})$. By definition of A_{i+1} this implies that $\sigma \subset A_{i+1}$. In particular r is defined on σ. It remains to show that $r(\sigma) \subset W(\epsilon_{2n-2i-2})$. Thus let $y \in \sigma$. We have

$$\begin{aligned}
d(x, r(y)) &\underset{\substack{\uparrow \\ \text{since } r(y) = r_{i+1}(y)}}{\leq} d(x, r_{i+1}(y)) \underset{\substack{\uparrow \\ \text{by definition}}}{\leq} \rho(r_{i+1}) \underset{\substack{\uparrow \\ \text{by } (\mathrm{b}_1)}}{\leq} 2 \cdot \inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ and } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}. \\
&\underset{\substack{\uparrow \\ \text{since } s \text{ is such a map}}}{\leq} 2\cdot\rho(s) = 2\cdot\max\{d(z, s(z)) \,|\, z \in \sigma\} \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} 2\cdot\max\{d(z, x) + d(x, s(z)) \,|\, z \in \sigma\} \\
&\underset{\substack{\uparrow \\ \text{since } \sigma \subset U = B_{\frac{1}{4}\epsilon_{2n}}(x) \text{ and since for each} \\ z \in \sigma \text{ we have } s(z) \in W(\epsilon_{2n-2i-1}) \subset B_{\epsilon_{2n-2i-1}}(x)}}{\leq} 2\cdot\left(\frac{1}{4}\epsilon_{2n} + \epsilon_{2n-2i-1}\right) = \frac{1}{2}\cdot\epsilon_{2n} + 2\cdot\epsilon_{2n-2i-1} \underset{\substack{\uparrow \\ \text{since for each } j \text{ we have } \epsilon_{j+1} \leq \frac{1}{3}\epsilon_j}}{\leq} \frac{2}{9}\cdot\epsilon_{2n-2i-2} + \frac{2}{3}\cdot\epsilon_{2n-2i-2} \leq \epsilon_{2n-2i-2}.
\end{aligned}$$

This concludes the proof that $r(\sigma) \subset W(\epsilon_{2n-2i-2})$. In particular this concludes the proof of Subclaim 1.

Finally it remains to prove Subclaim 2. We set $s = \frac{1}{4}\epsilon_{2n}$. As a reminder, we have to show that the union of X with all cells of C that lie in $B_s(x)$ is a neighborhood of x. Let W be a $\frac{1}{2^k}$-cube with $W \cap B_{\frac{s}{2}}(x) \neq \varnothing$ and $W \cap (\mathbb{R}^n \setminus B_s(x)) \neq \varnothing$. It follows that the diameter of W is at least $\frac{s}{2}$ which in turn implies that $\frac{\sqrt{n}}{2^k} \geq \frac{s}{2}$. We set $k_0 = \lfloor \ln_2(\frac{2\sqrt{n}}{s}) \rfloor$. We see that

$$\begin{aligned}
&\text{union of } X \text{ with all cells of } C \text{ that lie in } B_s(x) \\
&\supset \text{union of } X \text{ with all cubes of } C \text{ that lie in } B_s(x) \\
&\supset B_s(x) \setminus \text{union of all cubes of } C \text{ that are not contained in } B_s(x) \\
&\underset{\substack{\uparrow \\ \text{since } \mathbb{R}^n \setminus X = C}}{=} B_s(x) \setminus \text{union of all cubes of } C \text{ that are not contained in } B_s(x) \text{ but intersect } B_s(x) \\
&\underset{\substack{\uparrow \\ \text{definition of } k_0}}{\supset} B_s(x) \setminus \underbrace{\text{union of all } \tfrac{1}{2^k}\text{-cubes of } C \text{ with } k \leq k_0}_{\text{closed subset of } \mathbb{R}^n \text{ that does not contain } x}.
\end{aligned}$$

The last set is clearly an (open) neighborhood of x. But then the first set is also a neighborhood of x. This concludes the proof of Subclaim 2. □

The following proposition is a fairly straightforward consequence of Theorem 65.21. The proposition is a weak version of Theorem 45.1 that we had not proved.

Proposition 65.23. *Every compact topological manifold is a retract of a finite CW-complex.*

Using Proposition 65.23 we can now provide a proof for Proposition 45.2.

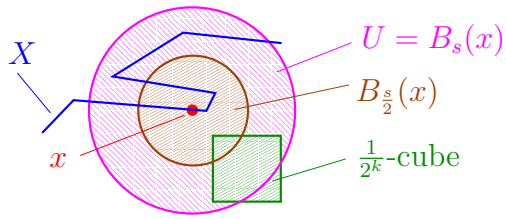


FIGURE 805. Illustration for the proof of Theorem 65.21.

Proof of Proposition 45.2. Let M be a compact, connected topological manifold. It follows from Proposition 65.23 that there exists a finite CW-complex X that contains M and which admits a retraction $r\colon X \to M$. If we denote by $i\colon M \to X$ the inclusion map, then $r \circ i = \mathrm{id}_X$. This implies that the map $r_*\colon \pi_1(X) \to \pi_1(M)$ is an epimorphism and it implies that for every abelian group G the maps $r_*\colon H_k(X;G) \to H_k(M;G)$ are epimorphisms. It follows from Proposition 27.12, Lemma 13.8, Proposition 35.5 and and the discussion on page 1010 that the fundamental group $\pi_1(M)$ is finitely presented and that the homology groups $H_k(M;G)$ are finitely generated. □

Before we prove Proposition 65.23 we make the following elementary observation.

Observation 65.24. *Let Z be a topological space and let K be a subset.*

(1) *Let $K \subset U \subset V \subset Z$ be subsets. If K is a retract of V, then K is also a retract of U.*
(2) *If K is a retract of a neighborhood it is also a retract of an open neighborhood.*

Proof. The first statement follows from the observation that the restriction of any retraction $r\colon V \to K$ to U is again a retraction. The second statement follows immediately from the first statement. □

Proof of Proposition 65.23. Let M be a compact topological manifold. It follows from Proposition 65.11 that there exists an $n \in \mathbb{N}$ and a topological embedding $M \to \mathbb{R}^n$. Since topological manifolds are weakly o-locally contractible we can apply Theorem 65.21. Together with Observation 65.24 (2) we obtain an open neighborhood U of $f(M)$ such that $f(M)$ is a retract of U. By the above, by Observation 65.24 (1) and by Lemma 1.37 it remains to prove the following claim.

Claim. Given any compact set K of an open subset U of $\mathbb{R}^n$ there exists a finite CW-complex X with $K \subset X \subset U$.

The idea is to define X to be the union of finitely many cubes. We set $r := d(K, \mathbb{R}^n \setminus U)$. Since K is compact and since U is open it follows easily[1113] from Corollary 1.56 that $r > 0$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} \cdot r$. We denote by X the union of all $\frac{1}{2^k}$-cubes that intersect K. (See page 1524 for the definition of a $\frac{1}{2^k}$-cube.) Since K is compact we see

[1113] What is the "easy" argument?

that X is the union of finitely[1114] many $\frac{1}{2^k}$ cubes. The same argument as in the proof of Lemma 65.22 shows that each $\frac{1}{2^k}$-cube is contained in U. As in the proof of Theorem 65.21 we can view X as a finite CW-complex. This concludes the proof of the claim. □

FIGURE 806. Illustration of the proof of Proposition 65.23.

We conclude this chapter with the following proposition.

Proposition 65.25. *Let $f\colon M \to \mathbb{R}^n$ be a topological embedding of a compact topological manifold. There exists a compact n-dimensional submanifold X of $\mathbb{R}^n$ that contains $f(M)$ and such that $f(M)$ is a retract of X.*

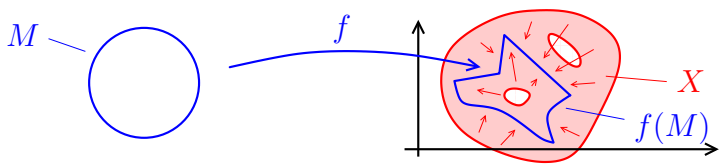


FIGURE 807. Illustration of Proposition 65.25.

Proof. Let $f\colon M \to \mathbb{R}^n$ be a topological embedding of a compact topological manifold. By Theorem 65.21 there exists an open neighborhood W of $f(M)$ such that $f(M)$ is a retract of W. Since W is an open subset of $\mathbb{R}^n$ it is in particular an n-dimensional manifold W. Therefore we can apply Proposition 65.4 and we obtain a sequence $X_1, X_2, \dots$ of subsets of W with the following three properties:

(a) the sequence is nested, i.e. for each $i \in N$ we have $X_i \subset X_{i+1}$,
(b) each X_i is a compact n-dimensional submanifold of W,
(c) we have $\bigcup_{i\in\mathbb{N}} \mathring{X}_i = W$ where $\mathring{X}_i := X_i \setminus \partial_0 X_i$.

We make the following observations:

(1) By Lemma 64.1 (3) each $\mathring{X}_i$ is an open subset of W and thus also of $\mathbb{R}^n$.
(2) It follows from (a), (c) and (1), the hypothesis that M is compact and Lemma 1.38 that there exists an $i \in \mathbb{N}$ such that $f(M) \subset X_i$.
(3) Since W is an open subset of $\mathbb{R}^n$ every submanifold of W is in fact also a submanifold of $\mathbb{R}^n$.

[1114]Why is that? Why do we need only finitely many such cubes?

It follows from this discussion and property (2) and Observation 65.24 (1) that $X := X_i$ has all the desired properties. $\square$

65.6. **Representing one-dimensional homology classes by submanifolds.** We start out with the following definition.

Definition. Let M be a manifold and let X be compact manifold. Let f and g be proper maps from X to M. A *proper homotopy* is a homotopy $F\colon [0,1] \times X \to M$ between f and g such that for each $t \in [0,1]$ the map $x \mapsto F(t,x)$ is a proper map. If such a proper homotopy exists, then we say that f and g are *properly homotopic*.

Now we can state the following theorem.

Theorem 65.26. *Let M be a manifold and let X be compact manifold. Furthermore let $f\colon X \to M$ be a proper map.*

(1) *The map f is properly homotopic to a proper smooth map $\widetilde{f}\colon X \to M$.*[1115]
(2) *If $\dim(M) = 2\cdot\dim(X)$, then the map f is properly homotopic to a proper immersion $\widetilde{f}\colon X \to M$ which has furthermore the property that there exist points $P_1, \dots, P_k$ in M with the following properties:*
   (a) *for each i the preimage $\widetilde{f}^{-1}(P_i)$ consists of precisely two points $Q_i^{\pm}$ which furthermore satisfy that $f_*(T_{Q_i^+}X) + f_*(T_{Q_i^-}X) = T_{P_i}(M)$,*
   (b) *the restriction of $\widetilde{f}$ to $X \setminus \{Q_1^{\pm}, \dots, Q_k^{\pm}\}$ is injective.*
(3) *If $\dim(M) \geq 2 \cdot \dim(X) + 1$, then the map f is properly homotopic to a proper embedding $\widetilde{f}\colon X \to M$.*[1116]

Furthermore, if f is already smooth on ∂X or if f is already an embedding on ∂X, then in (1) and (3) we can find a homotopy that is the identity on ∂X.

Example. In Figure 808 we consider $M = [-1,1] \times \mathbb{R}^2$ and a proper map $f\colon [-1,1] \to M$. We first see that f is homotopic to a proper smooth map and then we see that it is homotopic to a proper embedding.

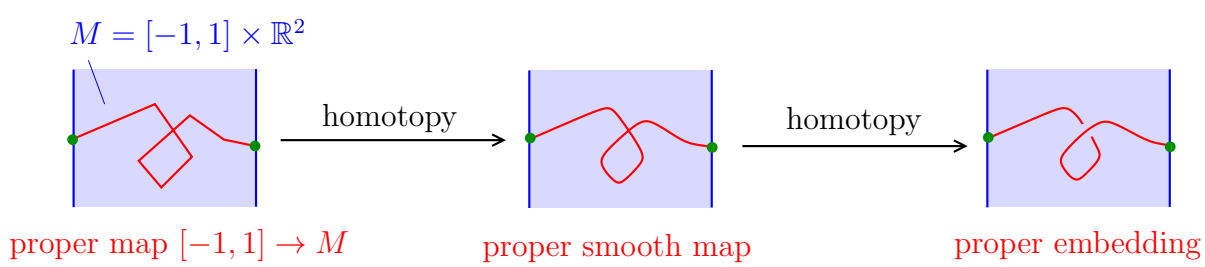


FIGURE 808. Illustration of Theorem 65.26.

[1115]This statement is very close to the statement of the Whitney Approximation Theorem 65.6.
[1116]Note that in general this statement is not true for $\dim(M) = 2 \cdot \dim(X)$. Why not?

Proof. The two statements are modest generalizations of [Hirs, Chapter 2, Theorem 2.6] and [Hirs, Chapter 2, Theorem 2.13]. □

We can now easily prove the following proposition.

Proposition 65.27. *Let M be an m-dimensional manifold with $m \geq 2$ and let A be a compact codimension zero submanifold of ∂M. Given any homology class in $\sigma \in H_1(M, A; \mathbb{Z})$ there exists an oriented proper 1-dimensional submanifold P with $\partial P \subset A$ such that $[P] = \sigma$.*

Sketch of proof. We only consider the case that $A = \varnothing$. We leave it to the reader to modify the proof to deal with the general case. Without loss of generality we can assume that M is connected. So let $x \in H_1(M; \mathbb{Z})$. By the Hurewicz Theorem 38.5, there exists a map $f\colon S^1 \to M$ such that $f_*([S^1]) = x$. Now we distinguish two cases.

(1) First we consider the case $m \geq 3$. Finally it follows from Theorem 65.26 (3) that f is homotopic to an embedding $g\colon S^1 \to M$. By Proposition 10.1 we know that the image $g(S^1)$ is a submanifold of M. It follows from Proposition 30.5 that $[g(S^1)] = g_*([S^1]) = f_*([S^1]) = x$.

(2) Now we consider the case $m = 2$. We apply Theorem 65.26 (2) to f to obtain a new map $\widetilde{f}\colon S^1 \to M$. This map is not an embedding but it has finitely many "transverse double points". We remove double points with the method shown in Figure 809. This way we obtain a submanifold of M. We leave it as an exercise to verify that it represents the homology class x, i.e. we leave it as an exercise to verify that this method from Figure 809 does not change the homology class. □

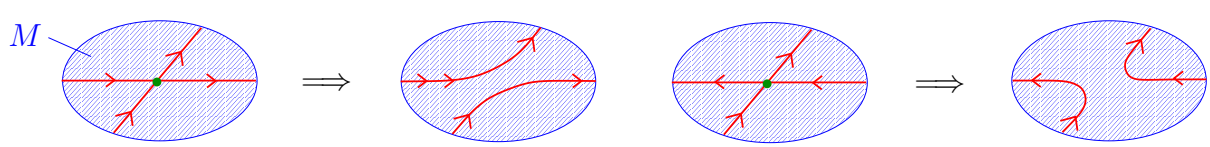


FIGURE 809.

We leave the proof of the following corollary to Proposition 65.27 as an exercise.

Corollary 65.28. *Let M be a connected manifold and let P, Q be two distinct points in M. Then there exists an embedding $\gamma\colon [0,1] \to M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.*

Exercises for Chapter 65.

Exercise 65.1. Let $k < n$. Use the Whitney Approximation Theorem 65.6 and Proposition 3.47 to give a quick proof that $\pi_k(S^n) = 0$.

Exercise 65.2. Let M be a manifold and let V and W be two disjoint closed subsets. Show that for any $\epsilon > 0$ there exists a smooth function $f\colon M \to \mathbb{R}$ such that $f(V) \subset (-\epsilon, \epsilon)$ and $f(W) \subset (1-\epsilon, 1+\epsilon)$.
Hint: Use Urysohn's Lemma 1.45.

Exercise 65.3. Provide a proof for Proposition 65.10.

Exercise 65.4. We consider $\mathbb{Q}$ with the usual topology and we view it as a subset of $\mathbb{R}$. Show that there is no open neighborhood of $\mathbb{Q}$ that admits a retraction onto $\mathbb{Q}$.

Exercise 65.5. Let $f_0, f_1\colon S^1 \to \mathbb{R}^3$ be two embeddings. As usual we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. By Theorem 65.19 we know that $f_0, f_1\colon S^1 \to \mathbb{R}^4$ are isotopic. Show that f_0 and f_1 are also isotopic as maps $S^1 \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$.
Hint: Consider Figure 794. Furthermore it might be useful to think of knots in terms of diagrams and to throw overboard any hopes of being 100% rigorous.

Exercise 65.6. Let M be a compact m-dimensional manifold. Show by hand that there exists an immersion $f\colon M \to \mathbb{R}^{2m}$.
Hint: Modify the proof of Proposition 65.11.

Exercise 65.7. We consider the Möbius band $M = ([0,1] \times [-1,1])/(0,x) \sim (1,-x)$ together with the projection map

$$\begin{array}{rcl} p\colon M = ([0,1] \times [-1,1])/\sim & \to & [0,1]/0 \sim 1 \\ [(x,y)] & \mapsto & [x]. \end{array}$$

Show that this map defines a linear interval bundle.

66. Alexander Duality

In this chapter we will study the Alexander Duality Theorem which, given a "reasonable" subset X of S^n relates the homology groups of $S^n \setminus X$ to the cohomology groups of X.

66.1. **The basic Alexander Duality Theorem.** We start out with the basic Alexander Duality Theorem which will be surprisingly easy to prove and which is already very helpful.

Theorem 66.1. (Alexander Duality Theorem) *Let K be a compact non-empty subset of S^n that admits a regular neighborhood. For any $i \in \mathbb{N}_0$ there exist isomorphisms*

$$\text{(a)} \qquad \widetilde{H}_i(S^n \setminus K; \mathbb{Z}) \;\cong\; \widetilde{H}^{n-i-1}(K; \mathbb{Z})$$

and

$$\text{(b)} \qquad \widetilde{H}^i(S^n \setminus K; \mathbb{Z}) \;\cong\; \widetilde{H}_{n-i-1}(K; \mathbb{Z}).$$

By the Regular Neighborhood Theorem 64.5 this conclusion holds for any closed non-empty submanifold of S^n. In this particular case the isomorphisms are natural.

Proof. Let K be a compact non-empty subset of S^n that admits a regular neighborhood N. We first consider statement (a) and the case that $i > 0$. In this case we have the following isomorphisms:

$$\begin{array}{ccccccc}
 & & & \text{Poincaré Duality Theorem 61.1 applied to } S^n \setminus \mathring{N}\text{, which is a compact manifold by Lemma 64.1 (4)} & & & \\
 & & & \downarrow & & & \\
\widetilde{H}_i(S^n \setminus K; \mathbb{Z}) & \xleftarrow{\cong} & \widetilde{H}_i(S^n \setminus \mathring{N}; \mathbb{Z}) & \xleftarrow{\cong} & H^{n-i}(S^n \setminus \mathring{N}, \partial N; \mathbb{Z}) & \xleftarrow{\cong} & H^{n-i}(S^n, N; \mathbb{Z}) \\
 & & \uparrow & & & & \uparrow \\
 & & \text{Lemmas 64.3 and 48.16 (6)} & & & & \text{Excision Theorem 45.8} \\
 & & \xleftarrow{\cong} \widetilde{H}^{n-i-1}(N; \mathbb{Z}) & & \xrightarrow{\cong} \widetilde{H}^{n-i-1}(K; \mathbb{Z}). & & \\
 & & \uparrow & & \uparrow & & \\
 & & \begin{array}{c}\text{by the long exact sequence in} \\ \text{reduced cohomology of the} \\ \text{pair } (S^n, N)\text{, see Lemma 48.17} \\ \text{here we use that } i > 0\end{array} & & \begin{array}{c}\text{by Lemma 48.16 (6) since } K \text{ is a} \\ \text{deformation retract of } N\end{array} & &
\end{array}$$

The case $i = 0$ is proved almost the same way. We leave it as an exercise to verify this case. This concludes the proof of (a). The proof of the existence of the isomorphism (b) is basically identical.

Finally suppose that K is a closed non-empty submanifold of S^n. It follows from the General Tubular Neighborhood Theorem 64.7 and Proposition 64.13 that we can take a tubular neighborhood as a regular neighborhood. By Proposition 64.9 tubular neighborhoods are unique in an appropriate sense which then allows us to get a natural isomorphism. □

Remark.

(1) The argument in the proof of the Alexander Duality Theorem 66.1 implies in particular that given any compact, non-empty n-dimensional submanifold W of S^n there

FIGURE 810. Illustration for the proof of the Alexander Duality Theorem 66.1.

exists an isomorphism

$$\widetilde{H}_i(S^n \setminus \mathring{W}; \mathbb{Z}) \;\cong\; \widetilde{H}^{n-i-1}(W; \mathbb{Z}).$$

(2) In Alexander Duality Theorem 66.1 we explicitly do not write "natural" isomorphism. The isomorphism that we construct in the proof depends on the choice of a regular neighborhood. It is not clear to me whether two regular neighborhoods are necessarily in any way "equivalent".

In practice we often like to consider objects that lie in $\mathbb{R}^n$ instead of S^n. The following variation of the Alexander Duality Theorem 66.1 deals with this situation.

Corollary 66.2. *Let K be a compact non-empty subset of $\mathbb{R}^n$. If K admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism*

$$\widetilde{H}_i(\mathbb{R}^n \setminus K) \;\cong\; H^{n-i-1}(K).$$

By the Regular Neighborhood Theorem 64.5 this conclusion holds for any compact submanifold of $\mathbb{R}^n$.

Proof. Let K be a compact non-empty subset of $\mathbb{R}^n$ that admits a regular neighborhood N in $\mathbb{R}^n$. Recall that N is by definition compact. It follows easily that N is also a regular neighborhood for K in S^n. Now let $i \in \mathbb{N}_0$. Then

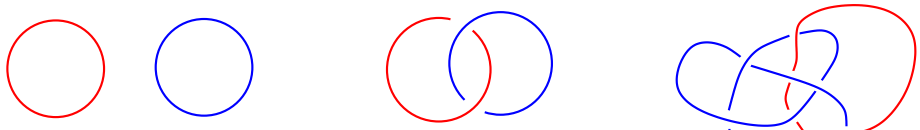
$$\begin{array}{ccccc} \widetilde{H}^{n-i-1}(K) & \cong & \widetilde{H}_i(S^n \setminus K) & \cong & \begin{cases} \widetilde{H}_i(\mathbb{R}^n \setminus K), & \text{if } i \neq n-1, \\ \widetilde{H}_{n-1}(\mathbb{R}^n \setminus K) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases} \\ & & \uparrow & & \uparrow \\ & & \begin{array}{c}\text{Alexander Duality Theorem 66.1} \\ \text{which we can apply since } K \\ \text{admits a regular neighborhood in } S^n\end{array} & & \text{see Exercise 33.5} \end{array}$$

The corollary now follows from Lemmas 48.16 and 11.8 (1). □

In the remainder of this section we discuss a few applications of the Alexander Duality Theorem 66.1. The following definition generalizes the notion of a knot.

Definition. An *m-component link* is a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m circles.

The following lemma computes the homology groups of a link complement. It can be viewed as a generalization of Lemma 40.7 (5).



2-component trivial link Hopf link

FIGURE 811. Examples of 2-component links.

Lemma 66.3. *For any m-component link L in S^3 we have*

$$H_i(S^3 \setminus L; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let L be an m-component link. We calculate that

$$\widetilde{H}_i(S^3 \setminus L; \mathbb{Z}) \;\underset{\substack{\uparrow \\ \text{Alexander Duality Theorem 66.1}}}{\cong}\; \widetilde{H}^{2-i}(L; \mathbb{Z}) \;\cong\; \widetilde{H}^{2-i}(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m\text{-copies}}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The lemma follows from this calculation together with Lemma 48.16 (3). □

For any 2-component link $L = K_1 \cup K_2$, for example for any of the links shown in Figure 811, we see that

$$H_1(S^3 \setminus L; \mathbb{Z}) \;\cong\; \mathbb{Z}^2 \quad \text{and} \quad H_2(S^3 \setminus L; \mathbb{Z}) \;\cong\; \mathbb{Z}.$$

The problem with this calculation is that we do not obtain any understanding of what the homology classes "look like". It seems reasonable to guess, along the lines of Lemma 40.7, that the first homology group is generated by the two meridians. But (a), we do not know how to prove it, and (b), we have no idea what a generator of the second homology group looks like. So the following question arises:

Question 66.4. *Given a link L, what are explicit bases for the homology groups of $S^3 \setminus L$?*

We will return to this question in a later chapter. We continue with another example.

Example. We consider the connected graph G with two vertices and three edges that is illustrated in Figure 812 on the left. On the right of Figure 812 we show a spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is homeomorphic to G. We have the following isomorphisms:

$$\widetilde{H}_i(S^3 \setminus G'; \mathbb{Z}) \;\underset{\substack{\uparrow \\ \text{by Proposition 64.2 we can apply} \\ \text{the Alexander Duality Theorem 66.1}}}{\cong}\; \widetilde{H}^{2-i}(G'; \mathbb{Z}) \;\cong\; \widetilde{H}^{2-i}(G; \mathbb{Z}) \;\underset{\substack{\uparrow \\ \text{similar to Lemma 40.17}}}{\cong}\; \begin{cases} \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In this case it is also rather unclear what a basis of $H_1(S^3 \setminus G; \mathbb{Z})$ should look like. Later on we will find an explicit basis for $H_1(S^3 \setminus G; \mathbb{Z})$.

graph G — the spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$

FIGURE 812.

We conclude this section with the following variation on the above Alexander Duality Theorem 66.1. It will come in handy at times. But arguably it is best to study it only when it is actually needed.

Theorem 66.5. (Alexander Duality Theorem) $(*)$ *Let $n \in \mathbb{N}_0$ and let $j \in \{0, \dots, n\}$. Given any compact oriented n-dimensional manifold X with $H_j(X; \mathbb{Z}) = H_{j+1}(X; \mathbb{Z}) = 0$ and given any compact proper submanifold G of X there exist natural isomorphisms*

$$\alpha_G \colon H_j(X \setminus G; \mathbb{Z}) \xrightarrow{\cong} H^{n-j-1}(G, \partial G; \mathbb{Z}) \quad \text{and} \quad \alpha_G \colon H^j(X \setminus G; \mathbb{Z}) \xrightarrow{\cong} H_{n-j-1}(G, \partial G; \mathbb{Z})$$

such that the following conditions are satisfied:

(1) *If $X = S^n$, then the isomorphisms are precisely the isomorphisms from the Alexander Duality Theorem 66.1.*
(2) *The isomorphisms decompose in the obvious way if X is disconnected.*
(3) *If W is a compact oriented n-dimensional manifold with*

$$H_j(W; \mathbb{Z}) = H_{j+1}(W; \mathbb{Z}) = H_j(\partial W; \mathbb{Z}) = H_{j+1}(\partial W; \mathbb{Z}) = 0,$$

and if F is a compact proper submanifold of W, then we have commutative diagrams

$$\begin{array}{ccc} H_j(W \setminus F; \mathbb{Z}) & \xrightarrow[\alpha_F]{\cong} & H^{n-j-1}(F, \partial F; \mathbb{Z}) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle \delta} \\ H_j(\partial W \setminus \partial F; \mathbb{Z}) & \xrightarrow[\alpha_{\partial F}]{\cong} & H^{n-j-2}(\partial F; \mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^j(W \setminus F; \mathbb{Z}) & \xrightarrow[\alpha_F]{\cong} & H_{n-j-1}(F, \partial F; \mathbb{Z}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \partial} \\ H^j(\partial W \setminus \partial F; \mathbb{Z}) & \xrightarrow[\alpha_{\partial F}]{\cong} & H_{n-j-2}(\partial F; \mathbb{Z}) \end{array}$$

where we denote by $i \colon \partial W \setminus \partial F \to W \setminus F$ the inclusion map and where the right-hand vertical maps are the connecting homomorphisms of the long exact sequences in (co-) homology of the pair $(F, \partial F)$.

Example. The above Alexander Duality Theorem 66.5 will be applied later on to the manifolds $W = [0,1] \times S^n$ and to $W = \overline{B}^n$. For example, suppose that F is a connected properly embedded surface in $\overline{B}^4$ such that $\partial F = K$ is a knot. We can apply the Alexander Duality Theorem 66.5 for $j = 1$. Together with Proposition 61.22 we obtain the following

commutative diagram:

$$\begin{tikzcd}
H_1(\overline{B}^4 \setminus F;\mathbb{Z}) \arrow[r, "\alpha_F", "\cong"'] & H^2(F,K;\mathbb{Z}) \arrow[r, "-\cap[F]", "\cong"'] & H_0(F;\mathbb{Z}) \arrow[d, "i_*"] \\
H_1(S^3 \setminus K;\mathbb{Z}) \arrow[u, "i_*"] \arrow[r, "\alpha_K", "\cong"'] & H^1(K;\mathbb{Z}) \arrow[u, "\delta"'] \arrow[r, "-\cap[K]", "\cong"'] & H_0(K;\mathbb{Z}).
\end{tikzcd}$$

The right hand vertical map is of course an isomorphism and both groups are isomorphic to $\mathbb{Z}$. We now see that both groups on the left are isomorphic to $\mathbb{Z}$ and that the inclusion induced map is an isomorphism.

Sketch of proof. Let $n \in \mathbb{N}_0$ and let $j \in \{0,\dots,n\}$. Let X be a compact orientable n-dimensional manifold with $H_j(X;\mathbb{Z}) = H_{j+1}(X;\mathbb{Z}) = 0$ and let G be a compact proper submanifold of X. We pick a tubular neighborhood N for G as provided by the General Tubular Neighborhood Theorem 64.7. Recall that N is a submanifold with corner. We define $\mathring{N}$ as in Lemma 64.1. We write $M := N \cap \partial X$. Note that M is a tubular neighborhood for $\partial F \subset \partial X$. We refer to Figure 813 for an illustration. We consider the following diagram

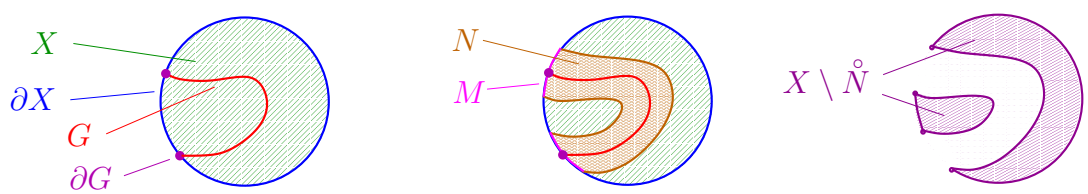


FIGURE 813.

$$\begin{tikzcd}
H_j(X\setminus\mathring{N}) \arrow[d, "\cong"'] & H^{n-j}(X\setminus\mathring{N},\partial(X\setminus\mathring{N})) \arrow[l, "\cong", "\mathrm{PD}"'] & H^{n-j}(X,N\cup\partial X) \arrow[l, "\cong", "\mathrm{exc}"'] & H^{n-j-1}(N\cup\partial X,\partial X) \arrow[l, "\cong", "\delta"'] \\
 & & & H^{n-j-1}(N,M) \arrow[u, "\mathrm{exc}", "\cong"'] \arrow[d, "\cong"] \\
H_j(X\setminus G) \arrow[rrr, dashed, "=:\alpha_G"] & & & H^{n-j-1}(G,\partial G).
\end{tikzcd}$$

We make the following comments:

(a) All maps where no map is specified are induced by inclusions. It follows from Proposition 64.13, together with Corollary 30.8 and Lemma 48.13 (4) that the vertical inclusion maps to the left and bottom right are isomorphisms.

(b) The two maps that are decorated with "exc" are inclusion induced maps that are isomorphisms by "excision". We leave it to the reader to verify that these inclusion maps are indeed isomorphisms. As an example, if $\partial X = \varnothing$, then it follows from the Excision Theorem 45.8 that the top "exc"-map is an isomorphism.

(c) The map on the top right indicated by "δ" is the connecting homomorphism of the long exact sequence in cohomology of the triple $(X, N\cup\partial X, \partial X)$. By our hypothesis we have $H_j(X;\mathbb{Z}) = H_{j+1}(X;\mathbb{Z}) = 0$. By the Poincaré Duality Theorem 61.1 this implies that $H^{n-j}(X,\partial X;\mathbb{Z}) = H^{n-j-1}(X,\partial X;\mathbb{Z}) = 0$, which in turn implies that the connecting homomorphism $\delta\colon H^{n-j-1}(N\cup\partial X,\partial X;\mathbb{Z}) \to H^{n-j}(X, N\cup\partial X;\mathbb{Z})$ is an isomorphism.

We have now seen that all maps in the diagram are isomorphisms. Thus we can define α_G to be the map on the bottom. A priori the definition of α_G depends on the choice of the tubular neighborhood N. But by Proposition 64.9 tubular neighborhoods are unique in an appropriate sense. This implies that isomorphisms do not depend on the choice of the tubular neighborhood. We have to verify that these homeomorphisms have the desired properties.

(1) This statement is basically clear once one compares the definitions.
(2) This statement follows from all the usual results for the behavior of invariants under the disjoint union.
(3) Let W be a compact oriented n-dimensional manifold with

$$(*) \qquad H_j(W;\mathbb{Z}) = H_{j+1}(W;\mathbb{Z}) = H_j(\partial W;\mathbb{Z}) = H_{j+1}(\partial W;\mathbb{Z}) = 0.$$

Furthermore let F be a compact proper submanifold of W. We pick a tubular neighborhood N for F. We define $\mathring{N}$ as above. We write $M := N\cap\partial W$. Note that M is a tubular neighborhood for ∂F. We consider the following diagram

$$\begin{tikzcd}
H_j(W\setminus\mathring{N}) \arrow[d, "\cong"'] & H^{n-j}(W\setminus\mathring{N},\partial(W\setminus\mathring{N})) \arrow[l, "\cong"', "\mathrm{PD}"] & H^{n-j}(W, N\cup\partial W) \arrow[l, "\cong"', "\mathrm{exc}"] & H^{n-j-1}(N\cup\partial W,\partial W) \arrow[l, "\cong"', "\delta"] \\
 & & & H^{n-j-1}(N,M) \arrow[u, "\mathrm{exc}"', "\cong"] \arrow[d, "\cong"] \\
H_j(W\setminus F) \arrow[rrr, blue, "\alpha_F"] & & & H^{n-j-1}(F,\partial F) \\
H_j(\partial W\setminus\partial F) \arrow[u, "i_*"] \arrow[rrr, blue, "\alpha_{\partial F}"] & & & H^{n-j-2}(\partial F) \arrow[u, "\delta"] \\
H_j(\partial W\setminus\mathring{M}) \arrow[u, "\cong"] & H^{n-j}(\partial W\setminus\mathring{M},\partial M) \arrow[l, "\cong"', "\mathrm{PD}"] & H^{n-j-1}(\partial W, M) \arrow[l, "\cong"', "\mathrm{exc}"] & H^{n-j-2}(M). \arrow[l, "\cong"', "\delta"] \arrow[u, "\cong"']
\end{tikzcd}$$

We make the following observations:
(a) By our hypothesis $(*)$ and the discussion above the horizontal maps α_G and $\alpha_{\partial G}$ are defined.
(b) One can easily see that the bottom "$\cup$" of the diagram is indeed the map $\alpha_{\partial F}$.
(c) If one ignores the blue arrows, then the brave reader can verify that the diagram does indeed commute.

Basically the same argument also provides the desired statements with the roles of homology and cohomology switched. □

66.2. **Immersions and embeddings of manifolds into $\mathbb{R}^n$.** In Question 1.85 we had asked whether the projective space $\mathbb{R}\mathrm{P}^2$ or the Klein bottle can be viewed as subsets of $\mathbb{R}^3$. We can reinterpret this question in three different ways:

Question 66.6.

(1) *Does* $\mathbb{R}\mathrm{P}^2$ (*respectively the Klein bottle*) *admit an immersion into* $\mathbb{R}^3$*?*
(2) *Does* $\mathbb{R}\mathrm{P}^2$ (*respectively the Klein bottle*) *admit an embedding into* $\mathbb{R}^3$*?*
(3) *Does* $\mathbb{R}\mathrm{P}^2$ (*respectively the Klein bottle*) *admit a topological embedding into* $\mathbb{R}^3$*?*

We start out with Question 66.6 (1).

(a) We saw in Figure 47 that there exists an immersion of the Klein bottle K into $\mathbb{R}^3$. In Figure 814 we show the images of two other immersions of the Klein bottle into $\mathbb{R}^3$. This gives a positive answer to Question 66.6 (1) for the Klein bottle.

Figure 814.

(b) The case of the real projective plane is much more interesting. In 1901 David Hilbert, arguably the greatest German mathematician of his generation, gave his PhD student Werner Boy [Boy][1117] the task to prove that there is no immersion of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$. To everyone's surprise Boy did the opposite, he showed that it was possible to immerse $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$. In Figure 815 we give several illustrations of the image of the immersion. A detailed description of the immersion is given in [Kir5, Ap]. But perhaps it is more helpful to just watch the following video:

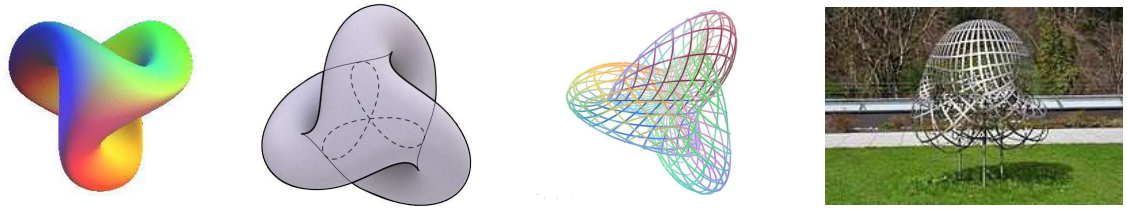
`https://www.youtube.com/watch?v=uiq-EcQz_uU`

A fun description of Boy's surface, and of many other concepts from topology, can also be found here:

`http://www.savoir-sans-frontieres.com/JPP/telechargeables/English/Topo_the_world_eng.pdf`

Lemma 3.39 says that the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are non-orientable. Therefore the following proposition gives in particular a negative answer to Question 66.6 (2).

[1117]Werner Boy (1879-1914) was a German mathematician who died in the first weeks of World War I.



Boy's surface at Oberwolfach

FIGURE 815.

Proposition 66.7. *Let N be a closed n-dimensional connected manifold. If N is non-orientable, then there is no embedding of N into $\mathbb{R}^{n+1}$.*

Proof. Let N be a closed n-dimensional connected non-orientable manifold. Suppose that N can be embedded into $\mathbb{R}^{n+1}$. We can thus view N as a submanifold of $\mathbb{R}^{n+1}$. We obtain that

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \cong & H^n(N;\mathbb{Z}) & \cong & \widetilde{H}_0(\mathbb{R}^{n+1}\setminus N;\mathbb{Z}) & \cong & \mathbb{Z}^{\text{number of path-components of } \mathbb{R}^{n+1}\setminus N \text{ minus } 1}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 51.14} & & \begin{array}{c}\text{Alexander Duality}\\ \text{Theorem 66.2}\end{array} & & \text{Proposition 29.5} & \end{array}$$

But this is absurd. We have thus obtained the desired contradiction. □

In the previous chapter we had also raised the following related question.

Question 65.14. *Given $n \in \mathbb{N}$, what is the minimal $k \in \mathbb{N}$ such that $\mathbb{R}\mathrm{P}^n$ admits a (topological) embedding into $\mathbb{R}^k$?*

Recall that it follows from Theorem 46.2 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^n$ in Proposition 35.11 that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. In particular it follows from Proposition 66.7 that for n even it is not possible to embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. But this raises the question, whether for n odd one can embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. It turns out that this is also not the case:

Proposition 66.8. *If $n \in \mathbb{N}_{\geq 2}$, then there is no embedding from $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.*

In the proof of Proposition 66.8 we will need the following lemma which is interesting in its own right.

Lemma 66.9. *If K is a closed orientable connected $(n-1)$-dimensional submanifold of the manifold S^n, then there exist two compact n-dimensional submanifolds A and B of S^n such that $A \cap B = \partial A = \partial B = K$.*

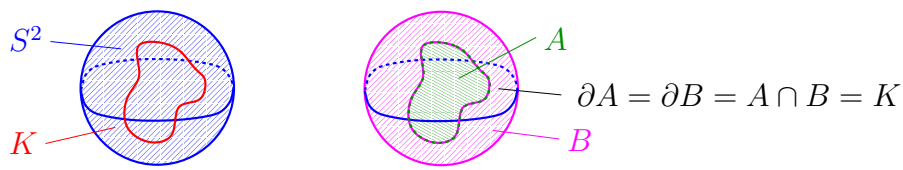


FIGURE 816.

Proof $(*)$. First we note that by Proposition 64.11 the submanifold K admits a tubular neighborhood $[-1,1] \times K$. Our hypothesis that K is connected implies that $[-1,0] \times K$ and $[0,1] \times K$ are also connected.

We first calculate that

$$\begin{array}{ccccccc} H_0(S^n \setminus K;\mathbb{Z}) & \cong & \widetilde{H}_0(S^n \setminus K;\mathbb{Z}) \oplus \mathbb{Z} & \cong & H^{n-1}(K;\mathbb{Z}) \oplus \mathbb{Z} & \cong & \mathbb{Z} \oplus \mathbb{Z}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 31.1 (4a)} & & \text{Theorem 66.1} & & \text{Theorem 46.2} & \end{array}$$

It follows from Proposition 29.5 that $S^n \setminus K$ has precisely two path-components A' and B'. Since $[-1,0] \times K$ is connected and since it is contained in $S^n \setminus K = A' \cup B'$ it follows that $[-1,0] \times K$ is either contained in A' or it is contained in B'. Without loss of generality we can assume that $[-1,0] \times K$ is contained in A'.

Claim. The subset $[0,1] \times K$ is contained in B'.

We consider the set $A'' := A' \cup [-1,1] \times K$. Note that $S^n = A'' \cup B'$. Since both A'' and B' are open and since S^n is connected it follows that $A'' \cap B' \neq \varnothing$. But this implies that $[0,1] \times K \cap B' \neq \varnothing$. But since $[0,1] \times K$ is connected we see that $[0,1] \times K$ is contained in B'. This concludes the proof of the claim.

We set $A := A' \cup K = A' \cup [-1,0] \times K$ and $B := B' \cup K = B' \cup [0,1] \times K$. It is easy to show that A and B are compact submanifolds of S^n such that $A \cap B = \partial A = \partial B = K$. $\square$

Now we can provide the proof of Proposition 66.8.

Proof of Proposition 66.8. As we had discussed just before the proposition, it remains to deal with the case that n is odd. So suppose that $n \geq 3$ is odd and that there exists an embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. We identify $\mathbb{R}\mathrm{P}^n$ with its image under the embedding, i.e. we view $\mathbb{R}\mathrm{P}^n$ as a submanifold of S^{n+1}. Note it follows from $n \geq 3$ and the discussion on page 1193 that $H^1(S^n;\mathbb{F}_2) = H^2(S^n;\mathbb{F}_2) = 0$.

Since n is odd we know that $\mathbb{R}\mathrm{P}^n$ is orientable. Thus we can apply Lemma 66.9 which tells us that there exist two compact $(n+1)$-dimensional submanifolds A and B of S^{n+1} with $A \cap B = \partial A = \partial B = K$. We denote by $i\colon \mathbb{R}\mathrm{P}^n \to A$ and by $j\colon \mathbb{R}\mathrm{P}^n \to B$ the inclusion maps. As usual we denote by $x \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ the unique generator.

By Proposition 50.12 (5) we can apply the Mayer–Vietoris Theorem 50.15 for Cohomology Groups to $S^{n+1} = A \cup B$ and we obtain the long exact sequence

$$\cdots \to \underbrace{H^1(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n\geq 3} \to H^1(A;\mathbb{F}_2) \oplus H^1(B;\mathbb{F}_2) \xrightarrow{i^*+j^*} \underbrace{H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)}_{=\mathbb{F}_2\cdot x} \to \underbrace{H^2(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n\geq 3} \to \dots$$

Thus we see that the map $i^* + j^* \colon H^1(A;\mathbb{F}_2) \oplus H^1(B;\mathbb{F}_2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2 \cdot x$ is an isomorphism. Without loss of generality we can assume that i^* is non-trivial which means that there exists an $a \in H^1(A;\mathbb{F}_2)$ with $i^*(a) = x$. We have

$$i^*(a^n) \underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=} (i^*(a))^n = x^n \underset{\substack{\uparrow \\ \text{Proposition 62.16}}}{\neq} 0 \in H^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).$$

In particular the map $i^* \colon H^n(A;\mathbb{F}_2) \to H^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is non-zero. Now we consider the other end of the above Mayer-Vietoris sequence and we obtain the exact sequence

$$\begin{array}{ccccccc} H^n(A;\mathbb{F}_2) \oplus H^n(B;\mathbb{F}_2) & \xrightarrow{i^*+j^*} & \overbrace{H^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)}^{=\mathbb{F}_2\cdot x^n} & \to & \overbrace{H^{n+1}(S^{n+1};\mathbb{F}_2)}^{\cong\mathbb{F}_2} & \to & \underbrace{H^{n+1}(A;\mathbb{F}_2) \oplus H^{n+1}(B;\mathbb{F}_2)}_{\substack{\text{both are zero by Proposition 51.14,} \\ \text{since } A \text{ and } B \text{ are compact, connected} \\ (n+1)\text{-dimensional manifolds} \\ \text{with non-empty boundary}}} \\ a^n \oplus 0 & \mapsto & x^n & & & & \end{array}$$

Thus we see that the map on the left is an epimorphism and that the group on the right is trivial. But then the fact that $H^{n+1}(S^{n+1};\mathbb{F}_2) \neq 0$ shows that the sequence cannot possibly be exact. We have thus obtained the desired contradiction. □

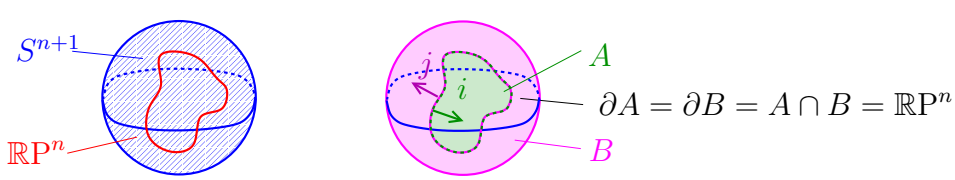


FIGURE 817. Illustration for the Proof of Proposition 66.8.

Remark. The above proposition does not give a complete answer to Question 65.14, we have only shown that $\mathbb{R}\mathrm{P}^n$ cannot embed into S^{n+1}. More lower bounds on the minimal dimension of some $\mathbb{R}^k$ into which we can embed $\mathbb{R}\mathrm{P}^n$ have been obtained by Michael Atiyah [At].

66.3. **The generalized Alexander Duality Theorem.** In a bout of optimism one might hope that the statement of Alexander Duality Theorem 66.1 holds for any (compact) subset of S^n. But this is not the case. More precisely, once again we consider the topologist's sine curve

$$X \;=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\}\cup\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\pi]\big\}\;\subset\;\mathbb{R}^2$$

that we had initially introduced on page 123. It is fairly straightforward to see that X is a closed subset of $\mathbb{R}^2$. Evidently X is bounded. Thus the Heine-Borel Theorem 1.21 implies that X is in fact a compact subset of $\mathbb{R}^2$ and thus of $S^2=\mathbb{R}^2\cup\{\infty\}$. Back on page 123 we saw that X is not path-connected, which by Proposition 48.11 and Lemma 48.14 implies that $\widetilde{H}^0(X;\mathbb{Z})\neq 0$. On the other hand, with some effort one can show that there exists a deformation retraction from $S^2\setminus X=(\mathbb{R}^2\setminus X)\cup\{\infty\}$ to ∞, which implies that $H_1(S^2\setminus X;\mathbb{Z})=0$. But this shows that $\widetilde{H}_1(S^2\setminus X;\mathbb{Z})\not\cong\widetilde{H}^0(X;\mathbb{Z})$.

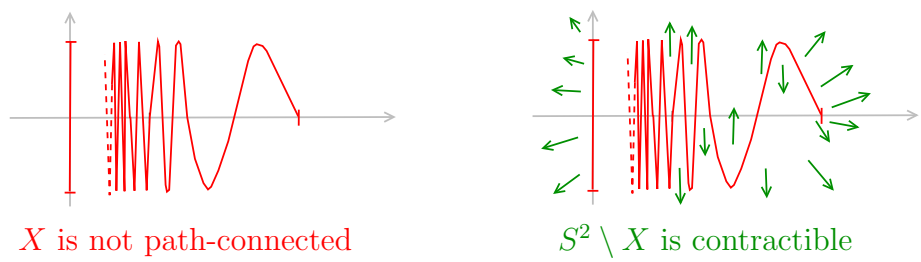


FIGURE 818.

Perhaps rather surprisingly one only needs a fairly modest hypothesis to ensure that Alexander Duality holds after all. More precisely, we have the following theorem which, by Lemma 65.20, is a significant generalization of the earlier Alexander Duality Theorem 66.1 (a).

Theorem 66.10. (Generalized Alexander Duality Theorem) *Let $n\geq 2$. If $X\neq S^n$ is a weakly o-locally contractible, non-empty compact subset of S^n, then for any $j\in\mathbb{N}_0$ we have a natural*[1118] *isomorphism*

$$\widetilde{H}_j(S^n\setminus X;\mathbb{Z})\;\xrightarrow{\cong}\;\widetilde{H}^{n-j-1}(X;\mathbb{Z}).$$

Example. In Figure 819 we show a subset X of $\mathbb{R}^3$ that is homeomorphic to the compact interval $[0,1]$. (We had already encountered this subset in Figure 550.) We leave it as an exercise to convince oneself that X does not admit a regular neighborhood. In particular this is a situation where the Generalized Alexander Duality Theorem 66.10 applies, even though our initial Alexander Duality Theorem 66.1 fails.

Remark. We leave it to the reader to determine whether the obvious generalization of Alexander Duality Theorem 66.1 (b) holds.

[1118] What does "natural" mean in this context? What are the categories? What are the functors?

weakly locally contracible compact subset of $\mathbb{R}^3$

FIGURE 819.

For most applications of Alexander Duality we will actually need the following straightforward consequence of the above theorem.

Theorem 66.11. (Alexander Duality Theorem for Manifolds) *Let $n \geq 2$ and let $k < n$. Furthermore let X be a compact, non-empty k-dimensional topological manifold. If $f\colon X \to S^n$ is an injective map, then for any $j \in \mathbb{N}_0$ we have a natural isomorphism*

$$\widetilde{H}_j(S^n \setminus f(X); \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-j-1}(X; \mathbb{Z}).$$

Proof of the Alexander Duality Theorem 66.11 for Manifolds. Let $f\colon X \to S^n$ be an injective map from a compact, non-empty k-dimensional topological manifold to S^n where $k < n$. It is a consequence of Proposition 1.40 that $f\colon X \to f(X)$ is a homeomorphism. From $k < n$ and Proposition 32.5 it follows that $f(X) \neq S^n$. We have the following isomorphisms:

$$\widetilde{H}_j(S^n \setminus f(X); \mathbb{Z}) \underset{\substack{\uparrow \\ \text{we can apply the Generalized Alexander Duality Theorem 66.10} \\ \text{since } f(X) \text{ is weakly o-locally contractible and since } f(X) \neq S^n}}{\xrightarrow{\cong}} \widetilde{H}^{n-j-1}(f(X); \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-j-1}(X; \mathbb{Z}).$$

□

Example. Let $n \geq 2$ and let $k < n$. Furthermore let $f\colon S^k \to S^n$ be an injective map. We obtain that

$$\widetilde{H}_j\big(S^n \setminus f(S^k); \mathbb{Z}\big) \underset{\substack{\uparrow \\ \text{Theorem 66.11}}}{\cong} \widetilde{H}^{n-j-1}(S^k; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{see page 1196 and Lemma 48.16}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

This calculation gives a new proof for Proposition 36.1 (2). Basically the same approach also gives a new proof of Proposition 36.1 (1).

Remark. The Generalized Alexander Duality Theorem 66.10 can be generalized even further. More precisely, an analogous statement also holds for any compact subset K of S^n, e.g. the topologist's sine curve. But hereby one needs to replace the usual singular cohomology on the right-hand side by the *Čech cohomology* of K. We refer to [Hat, p. 256] or [Pra, p. 271] for more details.

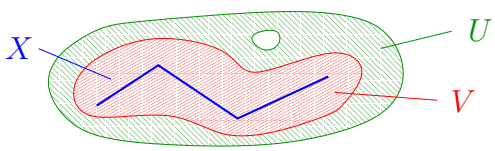
Proof of the Generalized Alexander Duality Theorem 66.10. Let $n \geq 2$ and let $X \neq S^n$ be a weakly o-locally contractible, non-empty compact subset of the sphere $S^n = \mathbb{R}^n \cup \{\infty\}$.

Without loss of generality we can assume that $X \subset \mathbb{R}^n \subset S^n$.[1119] We recall the following definition:

(1) Given a topological space Y we denote by $\mathcal{K}(Y)$ the set of all compact subsets of Y. Given $K, L \in \mathcal{K}(Y)$ we write $K \leq L$ if $K \subset L$. The set $\mathcal{K}(X)$ together with the relation "$\leq$" is a directed set since the union of two compact subsets is again compact.

We also introduce a new definition and notation:

(2) We denote by $\mathcal{N}(X)$ the set of all open neighborhoods of X. Given $U, V \in \mathcal{N}(X)$ we write $U \leq V$ if $V \subset U$. (Note the reversal of V and U.) The set $\mathcal{N}(X)$ together with the relation "$\leq$" is a directed set since the intersection of two open neighborhoods of X is again an open neighborhood of X.[1120]
(3) Given any subsets $A \subset B$ of S^n we denote by $i_{A,B}\colon A \to B$ the inclusion map.
(4) It follows from our definition of "$\leq$" on $\mathcal{N}(X)$ that for any $m \in \mathbb{N}_0$ the cohomology groups $\{\widetilde{H}^m(U)\}_{U \in \mathcal{N}(X)}$ together with the inclusion induced maps form a *direct* system of abelian groups.



we have $U \leq V$ and $i^*_{V,U}\colon \widetilde{H}^m(U) \to \widetilde{H}^m(V)$

FIGURE 820.

We continue with the key technical part of the theorem.

Claim. For any $m \in \mathbb{N}_0$ the map

$$\Phi\colon \varinjlim_{U \in \mathcal{N}(X)} \widetilde{H}^m(U) \;\to\; \widetilde{H}^m(X),$$

that is induced by the restriction maps $i^*_{X,U}\colon \widetilde{H}^m(U) \to \widetilde{H}^m(X)$, is an isomorphism.

By our hypothesis that X is compact and weakly o-locally contractible, Theorem 65.21 and Observation 65.24 (2) there exists an open neighborhood $W \subset \mathbb{R}^n$ of $X \subset \mathbb{R}^n$ for which there exists a retraction $r\colon W \to X$. (We refer to Figure 821 for an illustration.)

[1119]Indeed, by our hypothesis that $X \neq S^n$ there exists a point $P \in S^n$ that does not lie in X. By Lemma 1.73 (3) we know that the group $\mathrm{O}(n+1)$ acts transitively on S^n. In particular there exists a homeomorphism h of S^n with $h(P) = \infty$. Thus we can replace X by $h(X)$ which evidently lies in $\mathbb{R}^n$.

[1120]Here the idea is that the neighborhoods "approximate" X, in particular $U \leq V$ means that V is a "better approximation of X", which just means that $V \subset U$.

We consider the following commutative diagram

$$\begin{tikzcd}
 & \widetilde{H}^m(W) \arrow[dl] \arrow[dr, "i^*_{X,W}"] & \\
\lim\limits_{\substack{\longrightarrow \\ U\in\mathcal{N}(X)}} \widetilde{H}^m(U) \arrow[rr, "\Phi"'] & & \widetilde{H}^m(X). \arrow[ul, bend right, "r^*"']
\end{tikzcd}$$

From $(i^*_{X,W} \circ r^*) = (r \circ i_{X,W})^* = \mathrm{id}^*_X$ we deduce that the map $i^*_{X,W}$ is an epimorphism. It follows that the horizontal map Φ is also an epimorphism. It remains to show that Φ is a monomorphism.

By Lemma 52.2 (3) it remains to show that for any $U \in \mathcal{N}(X)$ there exists a $V \in \mathcal{N}(X)$ with $U \leq V$, i.e. with $V \subset U$, such that

$$\ker\left\{i^*_{X,U}\colon \widetilde{H}^m(U) \to \widetilde{H}^m(X)\right\} \ \subset\ \ker\left\{i^*_{V,U}\colon \widetilde{H}^m(U) \to \widetilde{H}^m(V)\right\}.$$

So let $U \in \mathcal{N}(X)$. After possibly replacing U by $U \cap W$ we can without loss of generality assume that $U \subset W$. We continue with the following subclaim.

Subclaim. There exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion $i_{V,U}\colon V \to U$ and the retraction $r\colon V \to X \subset U$ are homotopic maps from V to U.

We consider the homotopy

$$\begin{array}{rcl} F\colon U \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot x + t \cdot r(x) \end{array}$$

from the identity id_U to the retraction $r\colon U \to X$. Note that $F(X \times [0,1]) = X$. Since X is compact and since F is continuous there exists an open neighborhood V of X, contained in U, such that $F(V \times [0,1]) \subset U$.[1121] But then the restriction of F to $F\colon V \times [0,1] \to U$ defines a homotopy from the inclusion $V \to U$ to the retraction $r\colon V \to X \subset U$. (We refer to Figure 821 for an illustration.) This concludes the proof of the subclaim.

We consider the following diagram

$$\begin{tikzcd}
\widetilde{H}^m(U) \arrow[rr, "i^*_{X,U}"] \arrow[dr, "i^*_{V,U}"'] \arrow[dr, bend left, "r^*"] & & \widetilde{H}^m(X) \arrow[dl, "r^*"'] \\
 & \widetilde{H}^m(V). &
\end{tikzcd}$$

The above subclaim says that the two maps $i_{V,U}\colon V \to U$ and $r\colon V \to X \subset U$ are homotopic. It follows from Lemma 48.13 that the two maps $i^*_{V,U}$ and r^* from $\widetilde{H}^m(U)$ to

[1121] This can be seen as follows. The set $W := F^{-1}(U) \subset U \times [0,1]$ is open and it evidently contains $X \times [0,1]$. Since X and $[0,1]$ are compact and since W is open one sees easily that there exists an $r > 0$ such that for any $z \in X$ we have $B_r(z) \times [0,1] \subset W$. But then $V := \bigcup\limits_{x\in X} B_r(x)$ has the property that $V \times [0,1] \subset W$.

$\widetilde{H}^m(V)$ agree. Since r is the identity on X we see that $r\colon U \to X \subset V$ equals the map $i_{X,U} \circ r\colon U \to X$. This shows that the diagram commutes. It follows immediately from these observations that V has the desired property. This concludes the proof of the claim.

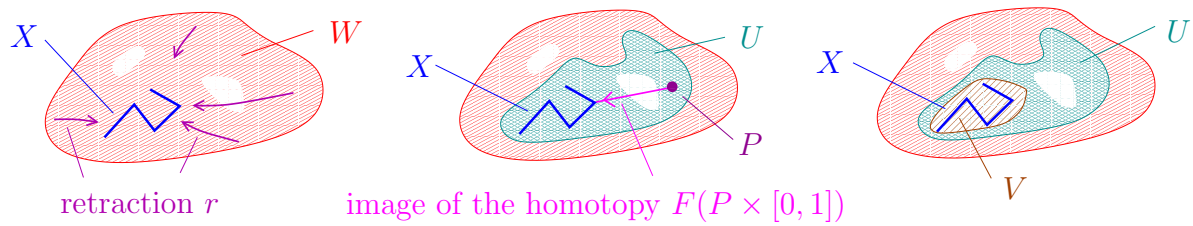


FIGURE 821.

We turn to the actual proof of the theorem. First let $j \neq 0$. The desired isomorphism will be given by a sequence of natural isomorphisms as follows:

$$H_j(S^n\setminus X) \underset{(1)}{\overset{\cong}{\Longleftarrow}} H_c^{n-j}(S^n\setminus X) \underset{(2)}{\overset{\cong}{\Longleftarrow}} \lim_{\substack{\longrightarrow \\ K\in\mathcal{K}(S^n\setminus X)}} H^{n-j}(S^n\setminus X, \overbrace{(S^n\setminus X)\setminus K}^{=(S^n\setminus K)\setminus X}) \underset{(3)}{\overset{\cong}{\Longleftarrow}} \lim_{\substack{\longrightarrow \\ U\in\mathcal{N}(X)}} H^{n-j}(S^n\setminus X, U\setminus X)$$
$$\underset{(4)}{\overset{\cong}{\Longleftarrow}} \lim_{\substack{\longrightarrow \\ U\in\mathcal{N}(X)}} H^{n-j}(S^n, U) \underset{(5)}{\overset{\cong}{\Longleftarrow}} \lim_{\substack{\longrightarrow \\ U\in\mathcal{N}(X)}} \widetilde{H}^{n-j-1}(U) \underset{(6)}{\overset{\cong}{\Longrightarrow}} \widetilde{H}^{n-j-1}(X).$$

We explain the natural isomorphisms in more detail:

(1) Since X is compact we know from Lemma 1.18 (2) that $S^n \setminus X$ is an open subset of $S^n \setminus X$. In particular it is also an n-dimensional manifold and via Lemma 45.28 it inherits an orientation from S^n. Also note that it follows from Proposition 32.1 that $S^n \setminus X$ has no boundary. After this preamble the first isomorphism is the Poincaré Duality isomorphism from Theorem 61.15 corresponding to the oriented manifold $S^n \setminus X$. By Lemma 61.21 this isomorphism is natural.

(2) The second isomorphism is provided by Proposition 53.4.

(3) The third isomorphisms comes from the straightforward observation that the map $\mathcal{N}(X) \to \mathcal{K}(S^n \setminus X)$ given by $U \mapsto S^n \setminus U$ is an isomorphism of directed sets which gives us an isomorphism of direct systems.

(4) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccc} H^{n-j}(S^n\setminus X, V\setminus X) & \longleftarrow & H^{n-j}(S^n, V) \\ \uparrow & & \uparrow \\ H^{n-j}(S^n\setminus X, U\setminus X) & \longleftarrow & H^{n-j}(S^n, U) \end{array}$$

where all the maps are induced by the obvious inclusions of pairs of topological spaces. It follows from the Excision Theorem 50.1 for Cohomology Groups that the

horizontal maps are isomorphisms. The desired isomorphism of direct limits is now an immediate consequence of Lemma 52.4.

(5) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccc} H^{n-j}(S^n, V) & \xleftarrow{\ \delta\ } & \widetilde{H}^{n-j-1}(V) \\ \uparrow & & \uparrow \\ H^{n-j}(S^n, U) & \xleftarrow{\ \delta\ } & H^{n-j-1}(U) \end{array}$$

where the vertical maps are induced by the obvious inclusions and where the horizontal maps are the connecting homomorphisms of the long exact sequences in reduced cohomology provided by Lemma 48.17. Since $j \geq 1$ we obtain from these long exact sequences and the calculation of the cohomology groups of S^n that the horizontal maps are in fact isomorphisms. As in (4) we appeal to Lemma 52.4 to obtain the desired isomorphism of direct limits.

(6) The last map is the isomorphism of the above claim.

It still remains to deal with the case $j = 0$. In this case all of the above maps are still isomorphisms, except for map (5). To address this issue we consider the following diagram:

$$\begin{array}{ccccccc} H_0(S^n \backslash X) & \xrightarrow[\cong]{(1)} & H^n_c(S^n \backslash X) & \xrightarrow[\cong]{(2),(3)} & \varinjlim\limits_{U \in \mathcal{N}(X)} H^n(S^n \backslash X, U \backslash X) & \xrightarrow[\cong]{(4)} & \varinjlim\limits_{U \in \mathcal{N}(X)} H^n(S^n, U) \\ \downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\ H_0(S^n) & \xrightarrow[\cong]{(1)} & H^n_c(S^n) & \xrightarrow[\cong]{(2),(3)} & \varinjlim\limits_{U \in \mathcal{N}(X)} H^n(S^n, U) & \xrightarrow[\cong]{(4)} & \varinjlim\limits_{U \in \mathcal{N}(X)} H^n(S^n). \end{array}$$

Here the second vertical map is the inclusion induced map defined by Lemma 53.5. The first and the fourth vertical maps are the obvious inclusion induced maps. Furthermore the third vertical map is the inverse of the isomorphism which is induced by the isomorphisms $H^n(S^n, U) \to H^n(S^n \setminus X, U \setminus X)$ that are given by the Excision Theorem 50.1 for Cohomology Groups. The top horizontal maps are precisely the above maps (1), (2), (3) and (4). The bottom horizontal maps are defined completely analogously with $X = \varnothing$. Since the maps defined in (1), (2), (3) and (4) are natural we see that the diagram commutes. It follows that

$$\begin{array}{ccccc} & \text{follows easily from } H_0(S^n) \cong \mathbb{Z} & & \text{by the above commutative diagram} & \\ & \downarrow & & \downarrow & \\ \widetilde{H}_0(S^n \backslash X) & \cong & \ker\{H_0(S^n \backslash X) \to H_0(S^n)\} & \cong & \ker\left\{\varinjlim H^n(S^n, U) \to \varinjlim H^n(S^n)\right\} \\ & \cong & \varinjlim \ker\{H^n(S^n, U) \to H^n(S^n)\} & \cong & \varinjlim \widetilde{H}^{n-1}(U) \ \cong\ \widetilde{H}^{n-1}(X). \\ & \uparrow & & \uparrow & \uparrow \\ & \text{Corollary 34.3} & & \text{by the long exact sequence of Lemma 48.17 and since } n \geq 2 & \text{by the claim} \end{array}$$

□

The following proposition finally gives a negative answer to Question 66.6 (3).

Proposition 66.12. *Let M be a closed connected n-dimensional topological manifold. If M is not orientable, then there is no* topological *embedding of M into $\mathbb{R}^{n+1}$.*

Proof. The proof of Proposition 66.12 is verbatim the same as the proof of Proposition 66.7, we only need to replace the original Alexander Duality Theorem 66.1 by the Alexander Duality Theorem 66.11 for Manifolds.

We also give an alternative argument that does not require the slightly obscure Generalized Alexander Duality Theorem 66.10. Thus let M be a closed connected n-dimensional topological manifold which admits a topological embedding $f\colon M \to \mathbb{R}^{n+1}$. We have to show that M is orientable. As usual we view $\mathbb{R}^{n+1}$ as a subset of S^{n+1}. We make the following observations:

(1) By Proposition 65.25 there exists a compact $(n+1)$-dimensional submanifold X of $\mathbb{R}^{n+1}$ that contains $f(M)$ and such that there exists a retraction $r\colon X \to f(M)$. We denote by $i\colon f(M) \to X$ the inclusion map.
(2) Since $r \circ i = \mathrm{id}_{f(M)}$ we see that the map $r^*\colon H^n(f(M);\mathbb{Z}) \to H^n(X;\mathbb{Z})$, induced by the retraction, is a monomorphism.
(3) By the remark on page 1535 there is an isomorphism $H^n(X;\mathbb{Z}) \cong H_0(S^{n+1} \setminus X;\mathbb{Z})$. It follows in particular from Proposition 29.5 that $H^n(X;\mathbb{Z})$ is a free abelian group.
(4) It follows from (2) and (3) that $H^n(f(M);\mathbb{Z})$ is a torsion-free abelian group.
(5) Since $f\colon M \to f(M)$ is a homeomorphism we obtain from (4) that $H^n(M;\mathbb{Z})$ is a torsion-free abelian group.
(6) We deduce from (5) and Proposition 51.14 that the closed connected n-dimensional topological M is orientable. □

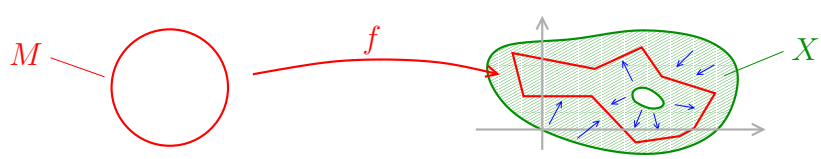


FIGURE 822. Illustration of the proof of Proposition 66.12.

The following proposition is a strengthening of Proposition 66.8.

Proposition 66.13. *If $n \in \mathbb{N}_{\geq 2}$, then there is no topological embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.*

Proof. First let n be even. In this case $\mathbb{R}\mathrm{P}^n$ is not orientable and the result is an immediate consequence of Proposition 66.12.

Now let n be odd. This case is significantly harder and it was first proved by René Thom [Tho3, p. 180] in 1952. We cannot provide the argument. But it seems like the argument of Thom is somewhat reminiscent of the argument in the proof of Proposition 66.8. So suppose there exists a topological embedding $f\colon \mathbb{R}\mathrm{P}^n \to S^{n+1}$. It follows from Proposition 29.5, the Generalized Alexander Duality Theorem 66.10 and the fact that $H^n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ (here we use that n is odd), that $S^{n+1} \setminus f(\mathbb{R}\mathrm{P}^n)$ consists of two path

components. We denote the closures of the path components by A and B. The problem is that the triad (S^{n+1}, A, B) is in general not excisive so we cannot apply the Mayer–Vietoris Theorem 50.15 for Cohomology Groups. The idea in the proof is to replace usual singular cohomology by Alexander-Spanier cohomology for which the above triad is excisive. We refer to [Mas3, Chapter 8] or [Mas5] for an introduction to Alexander-Spanier cohomology and we refer to [Mas3, Theorem II.8.15] for the proof that in this setting the above triad is excisive. □

66.4. **The square-peg problem.** We start out with the following definition.

Definition.

(1) The image of an injective map $S^1 \to \mathbb{R}^2$ is called a *Jordan curve*.[1122]
(2) Given a Jordan curve C we say that a rectangle R is *inscribed in* C if all vertices of R belong to C.

The following question was asked by Otto Toeplitz [Toep] in 1911.[1123]

Question 66.14. *Does every Jordan curve admit an inscribed square?*

This question is known as the "inscribed square problem" or the "square-peg problem". Lev Schnirelmann [Schni][1124] gave a positive answer if the curve is smooth, in fact he gave a proof if the curve is C^2, i.e. if the map $S^1 \to \mathbb{R}^2$ is twice continuously differentiable. But the general case is still open.[1125] A detailed discussion of the question and its history is given in [Matsc].

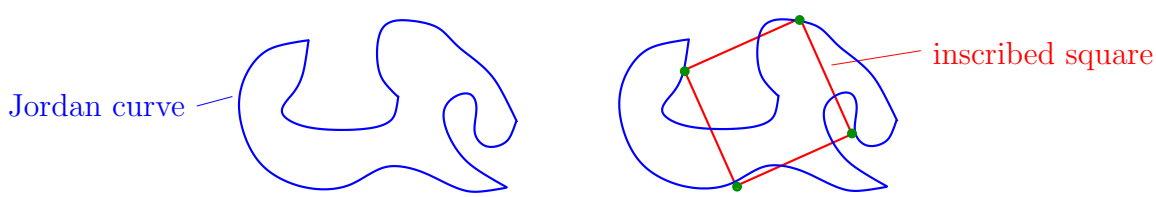


FIGURE 823.

We conclude this chapter with the following proposition.

Proposition 66.15. *Every Jordan curve admits an inscribed rectangle.*

[1122]Note that this definition is slightly different from the definition we gave on page 584 where we defined a Jordan curve to be an injective map $S^1 \to \overline{\mathbb{C}} = S^2$.

[1123]Otto Toeplitz (1881-1940) was a Jewish German mathematician who was forced to emigrate in 1939. He is mainly known for his contributions to analysis.

[1124]We had already encountered Lev Schnirelmann in Theorem 44.18.

[1125]Recently the Fields medalist Terence Tao [Tao] wrote a long paper in which he outlined an approach to the square-peg problem. See also the following video:

`https://www.youtube.com/watch?v=RGxnWgy5i80`

FIGURE 824. The more common square peg problem.

In [Mey] the subsequent ingenious proof of Proposition 66.15 is attributed to Herbert Vaughan. In preparation for the proof we need the following elementary lemma.

Lemma 66.16. *Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z, w) \sim (w, z)$. The quotient space $T/\sim$ is homeomorphic to the Möbius band and*

$$\partial(T/\sim) \;=\; \{[(z,z)] \,|\, z \in S^1\}.$$

Proof. The homeomorphism is illustrated in Figure 825. □

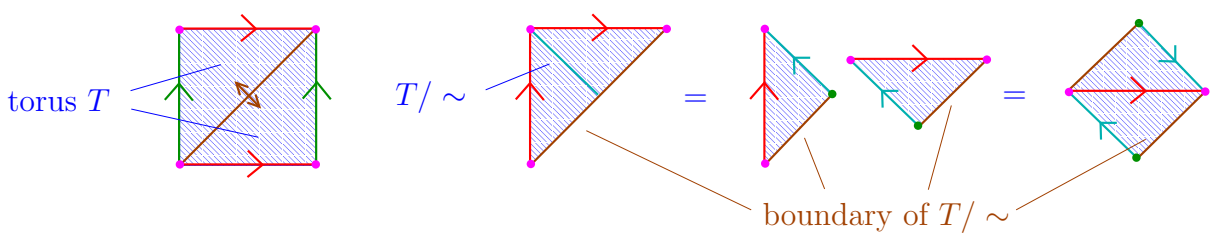


FIGURE 825.

Now we are already ready to prove Proposition 66.15.

Proof of Proposition 66.15. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an injective map. We have to show that $\gamma(S^1)$ admits an inscribed rectangle. We consider the map

$$\begin{aligned} \Phi\colon T := S^1 \times S^1 &\to \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \\ (z,w) &\mapsto \left(\tfrac{\gamma(z)+\gamma(w)}{2}, |\gamma(z)-\gamma(w)|\right). \end{aligned}$$

Since for all $z, w \in S^1$ we have $\Phi(z,w) = \Phi(w,z)$ we see that Φ descends to a map

$$\Psi\colon T/\sim \;\to\; \mathbb{R}^3$$

where $\sim$ denotes of course the equivalence relation generated by $(z,w) \sim (w,z)$.

Claim. The map $\Psi\colon T/\sim \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is not injective.

We first show that the claim implies the proposition. So now that we know that Ψ is not injective we know that there exist distinct pairs of points $\{z,w\}$ and $\{u,v\}$ with $\Phi(z,w) = \Phi(u,v)$. We consider the quadrilateral given by the points z, u, v, w. It follows

from $\Phi(z,w) = \Phi(u,v)$ that the diagonals bisect each other and that the diagonals have the same length. But such a quadrilateral is necessarily a rectangle.[1126] We refer to Figure 826 for an illustration.

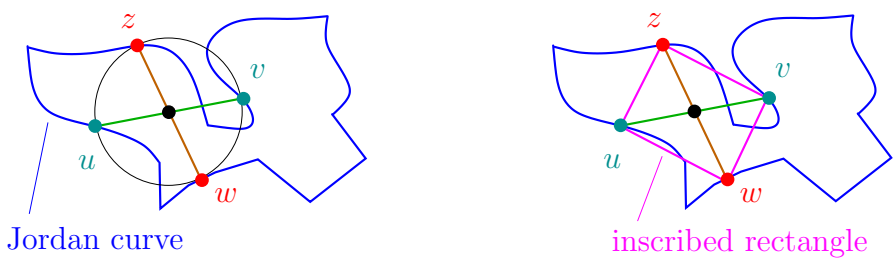


FIGURE 826.

It remains to prove the claim. Let us assume that $\Psi\colon T/\sim \to H := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is injective. By Lemma 66.16 the quotient $M := T/\sim$ is homeomorphic to the Möbius band and the boundary ∂M is given by $\{[(z,z)] \,|\, z \in S^1\}$. We see immediately from the definition of Ψ that for a point $[(z,w)] \in M = T/\sim$ we have $\Psi([(z,w)]) \in \partial H = \mathbb{R} \times \mathbb{R} \times \{0\}$ if and only if $[(z,w)] \in \partial M$. We denote by M_1 and M_2 two copies of M and we denote by $DM = M_1 \cup_{\partial M_1 = \partial M_2} M_2$ the double of M. It follows from the discussion on page 435 that DM is in fact homeomorphic to the Klein bottle, in particular DM is a closed, non-orientable 2-dimensional manifold.

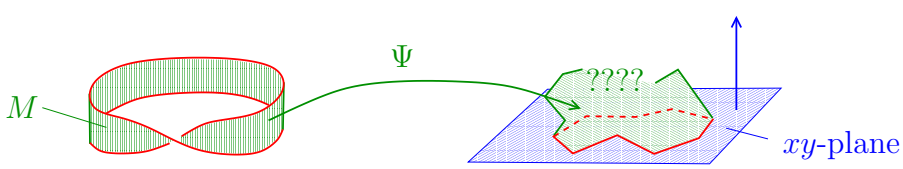


FIGURE 827.

We denote by $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the xy-plane, i.e. ρ is the map that is given by $\rho(x,y,z) = (x,y,-z)$. We consider the map

$$DM = M_1 \cup_{\partial M_1 = \partial M_2} M_2 \quad \to \quad \mathbb{R}^3$$

that is given by

$$\begin{array}{rcll} P & \mapsto & \Psi(P), & \text{if } P \in M_1, \\ P & \mapsto & \rho(\Psi(P)), & \text{if } P \in M_2. \end{array}$$

It follows immediately from the above discussion that this map is well-defined and injective. Furthermore it follows from Lemma 1.66 that the map is in fact continuous. But now

[1126] Hmm, why is that?

we have constructed an injective map from the Klein bottle to $\mathbb{R}^3$. On page 1511 we pointed out that such a map is in fact a topological embedding. But we had just proved in Proposition 66.12 that such a topological embedding cannot exist. This concludes the proof of the claim. $\square$

Remark.

(1) The above proof of Proposition 66.15 is also explained in the following video:

`https://www.youtube.com/watch?v=AmgkSdhK4K8`

(2) There are many variations on Question 66.14. For example, given $r \neq 1$ one can ask whether every Jordan curve admits an inscribed rectangle with aspect ratio r. For $r \neq 1$ this is an open problem even for smooth curves. Arguably the best result is the recent observation of Cole Hugelmeyer [Hug] that a deep result in knot theory, due to Josh Batson [Bats], implies that every smooth Jordan curve admits an inscribed rectangle of aspect ratio $\sqrt{3}$. The key idea of [Hug] is rather similar to the proof of Proposition 66.15.

Exercises for Chapter 66.

Exercise 66.1. Let $X \subset S^3$ be a closed orientable connected 2-dimensional submanifold. We view S^3 as a subset of S^4 in the usual way.

(a) Determine the homology groups of $S^4 \setminus X$.
(b) Now let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in Figure 828. Give an example of a closed curve in $S^4 \setminus X$ that is not null-homotopic.

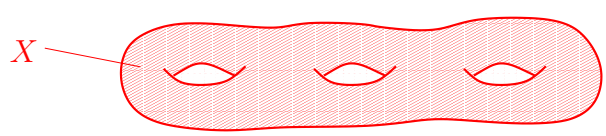


FIGURE 828.

Exercise 66.2. This exercise is a continuation of Exercise 66.1.

(a) Let $X \subset S^n$ be a closed submanifold. We view S^n as a subset of S^{n+1} in the usual way. Show that $S^{n+1} \setminus X$ is homotopy equivalent to $\Sigma(S^n \setminus X)$. (Here $\Sigma(S^n \setminus X)$ denotes the suspension of $S^n \setminus X$.)
Remark: You will need to use at some point that X is a "reasonable subset", e.g. that X is a closed submanifold. The statement does not hold for all subsets of S^n.
(b) Again let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in Figure 828. As above we view it as a submanifold of S^4. Determine $\pi_1(S^4 \setminus X)$.

Exercise 66.3.

(a) Let F be an orientable connected 2-dimensional manifold with non-empty boundary and let $i\colon F \to \mathbb{R}^3$ be an injective map. Is the complement $\mathbb{R}^3 \setminus F$ necessarily path-connected?

(b) Let F be an orientable connected surface non-compact 2-dimensional manifold with empty boundary and let $i\colon F \to S^3$ be an injective map. Is the complement $S^3 \setminus F$ necessarily path-connected?

67. The cup product and algebraic intersection numbers

In this chapter we will introduce the algebraic intersection number of two oriented submanifolds of a manifold M and we then relate this algebraic intersection number to the cup product of suitable cohomology classes of M.

67.1. **The Transversality Theorems.** The following definition is a harmless generalization of the definition from page 207.

Definition. Let M be a manifold and let Y be a proper submanifold of M. Let X be some other manifold and let $f\colon X \to M$ be a smooth map. We say f *intersects* Y *transversally* if for every $P \in X$ with $f(X) \in M$ the following two conditions are satisfied:

(1) We have $P \notin \partial X$ and $f(P) \notin \partial Y$.
(2) For every $P \in X$ with $f(P) \in Y$ we have the following equality[1127]

$$f_*(T_P X) + T_{f(P)} Y \;=\; T_P M.$$

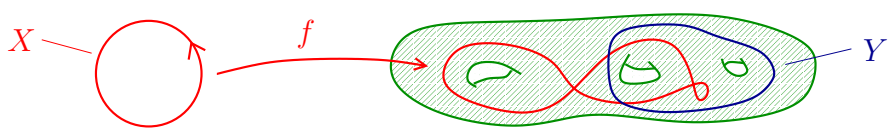


Figure 829.

Examples.

(1) Let M be a manifold and let X and Y be two proper submanifolds of M. We denote by $i\colon X \to M$ the inclusion map. It follows immediately from the definitions that i intersects Y transversally if and only if X intersects Y transversally in the sense of the definition on page 207.
(2) Let X be a manifold and let $f\colon X \to \mathbb{R}^n$ be a smooth map. Furthermore let $y \in \mathbb{R}^n$. By definition the map f is transverse to the 0-dimensional submanifold $\{y\}$ if and only if y is a regular value of f.

The following lemma might ring a bell. Considering the above examples we see that it is quite close to the statement of Lemma 67.1 and to the statement of the Regular Value Theorem 3.44.

Lemma 67.1. *Let M be a manifold and let Y be a proper submanifold of M of codimension l. Let X be some other manifold and let $f\colon X \to M$ be a smooth map. If f intersects Y transversely, then the following statements hold:*

(1) *The preimage $f^{-1}(Y)$ is a proper submanifold of X of codimension l.*
(2) *If M, X and Y are orientable, then $f^{-1}(Y)$ is orientable.*
(3) *If X is compact, then so is $f^{-1}(Y)$.*

[1127] Since Y is a submanifold of M we can view the tangent space $T_P Y$ as a subspace of $T_P M$.

Proof. The proof of this lemma is not overly hard. For space reasons we skip the proof and instead we refer to [Le2, Theorem 6.30] for a proof of (1). We leave the task of verifying the remaining statements to the reader. □

The following definition is a variation of the definition that we gave on page 334.

Definition. Let M be a manifold.

(1) Let W, W' be two proper submanifolds and let Z be (possibly empty) subset of W. We say that W and W' are (*smoothly*) *isotopic rel* Z, if there exists a (*smooth*) *isotopy from* W *to* W' *rel* Z, i.e. a map

$$\begin{array}{rcl} H\colon W \times [0,1] & \to & M \\ (z,t) & \mapsto & H(z,t) \end{array}$$

such that the following hold:
   (a) for each $P \in W$ we have $H(P,0) = P$,
   (b) for each $P \in Z$ and any $t \in [0,1]$ we have $H(P,t) = P$,
   (c) the map

$$\begin{array}{rcl} W & \to & M \\ z & \mapsto & H(z,1) \end{array}$$

   defines a homeomorphism (diffeomorphism) from W to W',
   (d) for each $t \in [0,1]$ the map $W \to M$ given by $P \mapsto H(P,t)$ is a proper homeomorphism onto its image (is a proper embedding).

   If W and W' are two oriented proper submanifolds, then we also want the following condition to be satisfied:
   (e) the diffeomorphism $W \to W'$ in (c) is orientation-preserving.

(2) Let X and Y be two submanifolds of M. We say X *and* Y *can be made transverse* (*respectively disjoint*) if there exist transverse (respectively disjoint) submanifolds X' and Y' of M such that X' is smoothly isotopic to X and Y' is smoothly isotopic to Y.

The next theorem says in particular that we can make any two proper submanifolds of a given manifold transverse.

Theorem 67.2. (Transversality Theorem) *Let M be a manifold and let Y be a proper submanifold of M.*[1128]

(1) *If X is a manifold and if $f\colon X \to M$ is a smooth map, then f is homotopic to a map that is transverse to Y.*
(2) *Let X be a proper submanifold of M. Given any neighborhood U of X there exists a smooth isotopy $F\colon X \times [0,1] \to U$ rel ∂X from X to a proper submanifold X' that is transverse to Y.*

Proof.

[1128]Note that the two statements are similar, but at least a priori they are independent in the sense that neither implies the other statement.

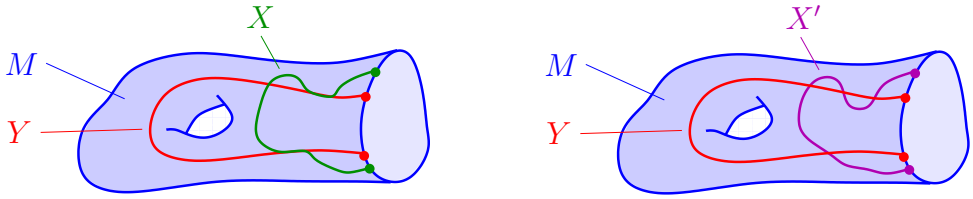


FIGURE 830. Illustration of Theorem 67.2 (2).

(1) This statement is proved in [Le2, Theorem 6.36]. A slightly less general statement is proved in [GP, p. 70].
(2) The theorem follows from Proposition 4.4.4, Theorem 4.5.6 and Proposition 4.5.7 of [Wall2].[1129] □

Example. Let M be a manifold. Furthermore let X and Y be two proper submanifolds. If $\dim(X) + \dim(Y) < \dim(M)$, then one deduces easily from the definitions that X and Y intersect transversally if and only if X and Y are disjoint. With this observation the Transversality Theorem 67.2 (2) says in particular that any two proper 1-dimensional submanifolds X and Y of the 3-dimensional ball $\overline{B}^3$ can be made disjoint after an isotopy rel boundary. This statement is illustrated in Figure 831.

FIGURE 831.

We conclude this section with a short discussion of self-transversal maps.

Definition. Let M and X be manifolds and let $\varphi\colon (X, \partial X) \to (M, \partial M)$ be a proper smooth map. We say φ *is self-transverse* if for any two points $P, Q \in X$ with $\varphi(P) = \varphi(Q)$ we have

$$\varphi_*(T_P X) + \varphi_*(T_P Y) \;=\; T_{\varphi(P)} M$$

and if for any two points $P, Q \in \partial X$ with $\varphi(P) = \varphi(Q)$ we have

$$\varphi_*(T_P X) + \varphi_*(T_P Y) \;=\; T_{\varphi(P)}(\partial M).$$

The proof of the following lemma is similar to the proof of Lemma 3.43. It is left as an exercise to the reader.

[1129] A first look at [Wall2, Theorem 4.5.6] might be slightly confusing since it demands that Y is a "submanifold of $J^r(M, N)$" for some $r \in \mathbb{N}_0$. But as is explained on page 101 of [Wall2], $J^0(M, X) = M \times X$, and $Y \times \{*\}$ is clearly a submanifold.

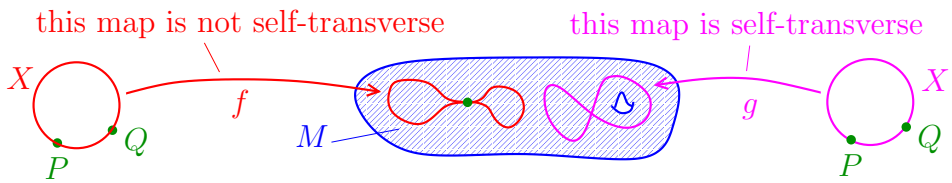


FIGURE 832.

Lemma 67.3. *Let M be an n-dimensional manifold and let a X be an m-dimensional manifold. Furthermore let $\varphi\colon (X, \partial X) \to (M, \partial M)$ be a proper smooth map. If φ is a self-transverse immersion, then the set*

$$\{P \in X \mid \textit{there exists a } Q \neq P \textit{ with } \varphi(P) = \varphi(Q)\}$$

is a proper submanifold of M of dimension $n - 2m$.

The following is a variation on the Transversality Theorem 67.2 (2).

Theorem 67.4. (Self-Transversality Theorem) *Let M be a manifold, let X be a manifold and let $\varphi\colon (X, \partial X) \to (M, \partial M)$ be an immersion. Given any neighborhood U of $\varphi(X)$ there exists a smooth isotopy $F\colon X \times [0,1] \to U$ from φ to a self-transverse immersion.*

Proof. The theorem is a straightforward consequence of [Wall2, Proposition 4.6.6]. □

67.2. **Algebraic intersection numbers of submanifolds.** In the following sections we want to study the question whether we can always make submanifolds disjoint. We start out with an example.

Example. We consider the three submanifolds X, Y and Z of the torus that are illustrated in Figure 833 on the left. As is shown in the same figure on the right, one can make X and Y disjoint.

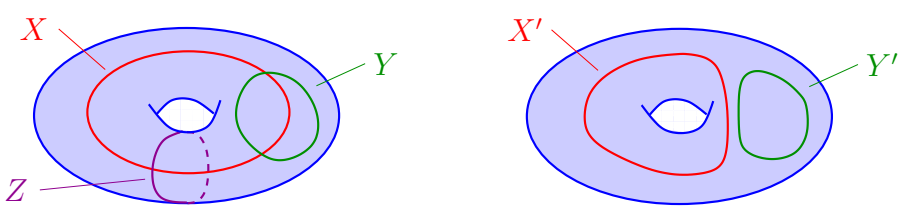


FIGURE 833.

It does not look like we can make the submanifolds X and Z disjoint. But how can we prove it? We will record this question for later.

Question 67.5. *Can we make the submanifolds X and Z, that are illustrated in Figure 833, disjoint?*

Lemma 67.6. *Let M be an m-dimensional manifold, let $X \subset M$ be a proper submanifold of dimension k and let $Y \subset M$ be a proper submanifold of the complementary dimension $m-k$. If X and Y intersect transversally and if X and Y are compact, then the intersection $X \cap Y$ consists of finitely many points.*

Proof. By Lemma 3.43 we know that $X \cap Y$ is a 0-dimensional submanifold of M. From our hypothesis we know that $X \cap Y$ is also compact. But any compact 0-dimensional submanifold of a manifold M is a finite set.[1130] □

Definition. Let M be an m-dimensional oriented manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k and let $Y \subset M$ be a compact oriented proper submanifold of the complementary dimension $m-k$. Suppose that X and Y intersect transversally.

(1) Let P be an intersection point of X and Y. We assign to P the sign $\sigma_P = 1$, if

$$\text{positive basis of } T_PX \text{ followed by positive basis of } T_PY \;=\; \text{positive basis of } T_PM.$$

Otherwise we put $\sigma_P = -1$.[1131]

(2) We define[1132]

$$\textit{geometric intersection number} \text{ of } X \text{ and } Y \;:=\; \#(X \cap Y)$$

and

$$\textit{algebraic intersection number} \text{ of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \sigma_P \in \mathbb{Z}.$$

If X and Y do not intersect, then the algebraic intersection number of X and Y is of course zero.

Examples.

(1) In Figure 834 we show pieces of oriented submanifolds of the oriented manifold $\mathbb{R}^2$. On the left the sign of the intersection point is positive, on the right it is negative.

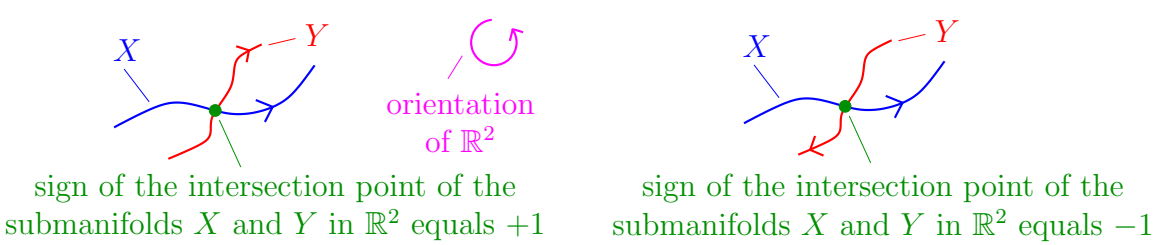


FIGURE 834.

[1130] Why is that?

[1131] It is a straightforward exercise in linear algebra to show that the definition of the sign σ_P does not depend on the choice of the positive bases of T_PU and T_PV.

[1132] It follows from Lemma 67.6 that the geometric intersection number is finite and that the algebraic intersection number is given by a finite sum.

(2) We consider the surface of genus 2 with the orientation where the normal vector "sticks outward" furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in Figure 835. For the submanifolds A and B we have

$$\text{geometric intersection number of } A \text{ and } B \;=\; 1,$$

the sign of the intersection point is negative, hence

$$\text{algebraic intersection number of } A \text{ and } B \;=\; -1.$$

For the submanifolds A and C we have

$$\text{geometric intersection number of } A \text{ and } C \;=\; 2,$$

the intersection points have opposite signs, therefore

$$\text{algebraic intersection number of } A \text{ and } C \;=\; 0.$$

Finally for the submanifolds D and E we have

$$\text{geometric intersection number of } D \text{ and } E \;=\; 2,$$

here both intersection points have negative sign, thus

$$\text{algebraic intersection number of } D \text{ and } E \;=\; -2.$$

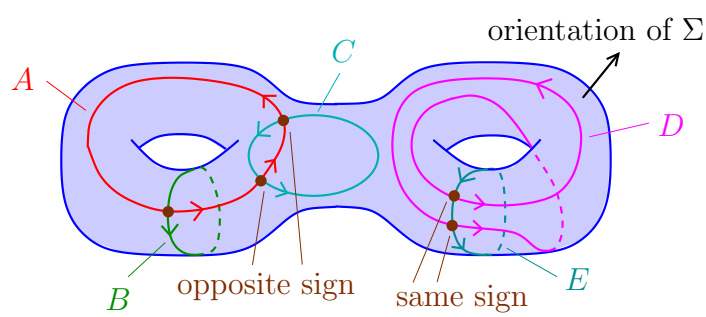


FIGURE 835.

(3) Now suppose that M and N are oriented manifolds of dimensions m respectively n. We assume that at most one of the two manifolds has non-empty boundary. We saw in Proposition 3.42 (1) that $M \times N$ is naturally[1133] a $(m+n)$-dimensional manifold. It is straightforward to show that for each $P \in M \setminus \partial M$ and $Q \in N \setminus \partial N$ the obvious map[1134]

$$T_P M \oplus T_Q N \;\to\; T_{(P,Q)}(M \times N)$$

[1133]What does he mean by "naturally"?
[1134]What is the obvious map?

is an isomorphism. If $\{v_1, \dots, v_m\}$ is a positive basis for $T_P M$ and $\{w_1, \dots, w_n\}$ is a positive basis for $T_Q N$, then we endow $T_{(P,Q)}(M \times N)$ with the orientation that makes the above isomorphism orientation-preserving.[1135]

Given $P \in M$ and $Q \in N$ it follows easily from the definitions that

$$\text{algebraic intersection number of } M \times \{Q\} \text{ and } \{P\} \times N \;=\; 1.$$

In Figure 836 we illustrate the statement that in the torus $S^1 \times S^1$ we have

$$\text{algebraic intersection number of } S^1 \times \{1\} \text{ and } \{1\} \times S^1 \;=\; 1.$$

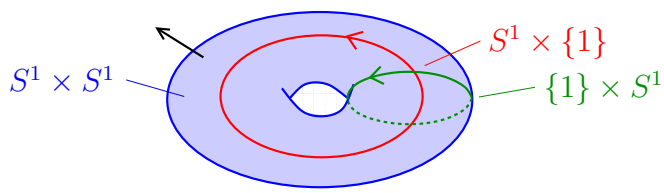


FIGURE 836.

(4) The above example (2) suggests that if two submanifolds have non-zero algebraic intersection number, then they cannot be made disjoint. This hunch is definitely not true if we allow *both* of the proper submanifolds to have boundary. More precisely in Figure 837 on the left we show the torus with one open disk removed and we show two proper submanifolds X and Y with boundary. The algebraic intersection number is $+1$, but as we see in Figure 837 on the right, these two submanifolds can be made disjoint. Therefore in the following discussion we will always assume that at least one of the two submanifolds is closed.[1136]

Before we continue, and in light of the last example, we introduce the following definition.

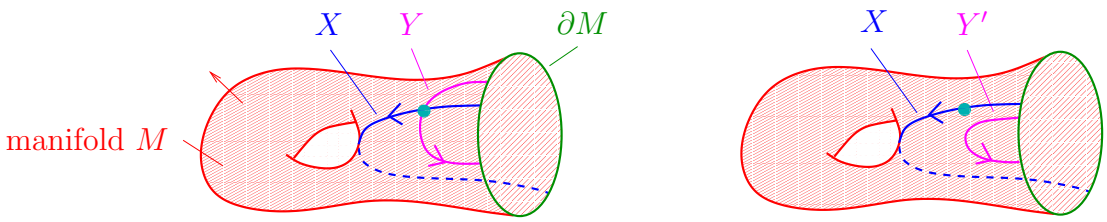
Definition. Let M be a compact oriented manifold and let X and Y be two submanifolds. We say X *and* Y *are complementary* if the following four conditions are satisfied:

(1) X is closed and Y is a proper submanifold,
(2) both X and Y are oriented,
(3) we have $\dim(X) + \dim(Y) = \dim(M)$,
(4) the submanifolds X and Y intersect transversally.

We would be able to answer Question 67.5 if we could prove the following statement.

[1135] One can easily verify that these orientations on the tangent spaces of $M \times N$ define an orientation on $M \times N$. In fact it follows fairly easily from Proposition 56.11 that this definition of the product orientation agrees with the definition we gave on page 1321.

[1136] Of course one could reformulate the question and consider only smooth isotopies rel the boundary of M. If we restrict ourselves to such isotopies then X and Y cannot be made disjoint.



the algebraic intersection number of X and Y equals $+1$

FIGURE 837.

Statement 67.7. *Let M be a compact oriented manifold and let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. Then the following holds:*

$$X, X' \text{ are smoothly isotopic and } Y, Y' \text{ are smoothly isotopic} \quad \Longrightarrow \quad X \cdot Y \;=\; X' \cdot Y'.$$

But how can we prove such a statement? In Figure 838 we sketch an idea: We break the smooth isotopies from X to X' and from Y to Y' into "smaller isotopies" such that each smaller isotopy either does not change the type of intersection points, or two intersection points of opposite sign get created or erased. Even though the idea is fairly convincing, it is very hard to turn this idea into a proper proof.[1137][1138] So for the time being we will not attempt to provide a proof for Statement 67.7.

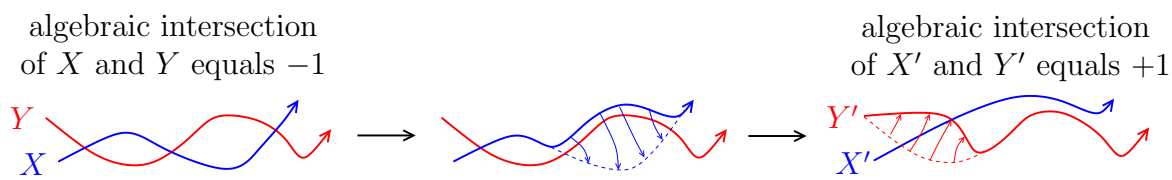


FIGURE 838.

We continue with developing the theory of algebraic intersection numbers a little further. We have the following elementary lemma which follows quite easily from the definitions and some basic linear algebra.

Lemma 67.8. *Let M be a compact oriented manifold. If X and Y are complementary submanifolds of M, then*

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \cdot X \cdot Y.$$

[1137] You will see it is already hard to prove the statement in dimension 2, let alone in higher dimensions. In fact I am not aware of a direct geometric proof of Statement 67.7.

[1138] Even though we will not follow this approach, it was certainly this type of picture mathematicians had in mind when they developed the concept of algebraic intersection numbers in the early 20th century.

Sketch of proof. The statement follows easily from the definitions and the observation that the matrix that represents the linear homeomorphism that swaps the first k vectors with the last $(m-k)$ vectors of a basis of an m-dimensional vector space is represented by the matrix $\begin{pmatrix} 0 & \mathrm{id}_k \\ \mathrm{id}_{m-k} & 0 \end{pmatrix}$. One easily verifies that the determinant of this matrix equals $(-1)^{k(m-k)}$. □

The first part of the following definition dates back to page 579.

Definition.

(1) An *n-dimensional complex manifold* is a $2n$-dimensional manifold M without boundary together with a smooth atlas $\{\phi_i \colon U_i \to V_i\}_{i\in I}$ such that all transition maps are biholomorphisms.
(2) We define the *maximal complex atlas* of a complex manifold the same way as we defined the maximal smooth atlas of a manifold, see page 184.
(3) We say $W \subset M$ is a *k-dimensional complex submanifold* if for every point $P \in W$ there exists a chart $\phi_i \colon U_i \to V_i$ from the maximal complex atlas with $P \in U_i$ and which satisfies that $\phi_i(U_i \cap W) \subset \mathbb{C}^k \times \{0\}$.

Remark.

(1) Let M be a complex manifold. Note that every complex submanifold of M is a complex manifold in its own right.
(2) Proposition 21.4 shows that every complex manifold comes with a natural orientation. We will use this orientation throughout.

Now we can formulate this elementary but rather useful lemma.

Lemma 67.9. *Let M be a closed complex manifold and let X and Y be two closed complex submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. Then the following hold:*

(1) *The sign at every transverse intersection point of X and Y is equal to $+1$.*
(2) *If X and Y intersect transversally, then the algebraic intersection number equals the geometric intersection number.*

Proof.

(1) We write $k = \dim(X)$ and $l = \dim(Y)$. Suppose that P is a transverse intersection point of X and Y. We pick a totally real basis $v_1, \dots, v_k$ for T_PX and we pick a totally real basis $w_1, \dots, w_l$ for T_PY. Then $v_1, \dots, v_k, w_1, \dots, w_l$ is evidently a totally real basis for $T_PX \oplus T_PY = T_PM$. We have
$$\underbrace{v_1, iv_1, \dots, v_k, iv_k}_{\text{positive basis for } T_PX}, \underbrace{w_1, iw_1, \dots, w_k, iw_k}_{\text{positive basis for } T_PY} = \underbrace{v_1, iv_1, \dots, v_k, iv_k, w_1, iw_1, \dots, w_l, iw_l}_{\text{positive basis for } T_PM}.$$
This shows that $\sigma_P = +1$. We have concluded the proof of the first statement.
(2) The second statement is an immediate consequence of the first statement. □

Example. We consider the complex projective plane $\mathbb{C}\mathrm{P}^n$. In Lemma 21.6 we saw that $\mathbb{C}\mathrm{P}^n$ is a complex manifold of complex dimension n. It follows easily from the definitions that for any $k \leq n$ the subset

$$X_k \;:=\; \{[z_0 : \ldots : z_k : 0 : \ldots : 0] \mid (z_0, \ldots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is a k-dimensional complex submanifold. Similarly

$$Y_k \;:=\; \{[0 : \ldots : 0 : z_{n-k} : \ldots : z_n] \mid (z_{n-k}, \ldots, z_n) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is also a k-dimensional complex submanifold. Evidently both X_k and Y_k are biholomorphic to $\mathbb{C}\mathrm{P}^k$. Furthermore

$$X_k \cap Y_{n-k} \;=\; [0 : \ldots : 0 : \underset{\substack{\uparrow \\ \text{the } (k+1)\text{-st entry}}}{1} : 0 : \ldots : 0]$$

consists of a single point P. We leave it as an exercise to verify that X_k and Y_{n-k} intersect transversally at P. It follows from Lemma 67.9 that $X_k \cdot Y_{n-k} = 1$. The same way we also see that $Y_k \cdot X_{n-k} = 1$.

67.3. **The cup product and algebraic intersection numbers.** We recall the following notation.

Notation.

(1) Given a compact oriented n-dimensional topological manifold W we denote by $[W] \in H_n(W, \partial W; \mathbb{Z})$ its fundamental class. If W is a proper submanifold of some topological manifold M, then sometimes we denote by $[W] \in H_n(M, \partial M; \mathbb{Z})$ the image of $[W]$ under the inclusion induced map $H_n(W, \partial W; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{Z})$.

(2) For the reader's convenience we recall the statement of the Poincaré Duality Theorem 61.1. Let M be a compact, non-empty n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map

$$H^k(M, A; \mathbb{Z}) \;\xrightarrow{\cap [M]}\; H_{n-k}(M, B; \mathbb{Z})$$

is an isomorphism. We denote by

$$\mathrm{PD}_M \colon H_{n-k}(M, B; \mathbb{Z}) \;\longrightarrow\; H^k(M, A; \mathbb{Z})$$

its inverse. If M is understood from the context then often we just write PD instead of PD_M.

The following theorem relates the cup product on a manifold to algebraic intersection numbers. Or conversely, it expresses the algebraic intersection number of submanifolds as a cup product.

Theorem 67.10. *Let M be a compact oriented connected m-dimensional manifold. Let X and Y be a complementary pair of submanifolds of M. We write $k = \dim(X)$ and we*

denote by $i\colon X \to M$ and $j\colon Y \to M$ the obvious inclusion maps. Furthermore we denote by $[X] \in H_k(X)$ and $[Y] \in H_{m-k}(Y, \partial Y)$[1139] *the fundamental classes of X and Y. Then*

$$X \cdot Y \;=\; \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in H_k(M)})}_{\in H^{m-k}(M,\partial M)} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in H_{m-k}(M,\partial M)})}_{\in H^k(M)}}^{\in H^m(M,\partial M)}, \overbrace{[M]}^{\in H_m(M,\partial M)} \Big\rangle \;\in\; \mathbb{Z}.$$

In particular with the abbreviation $[X] = i_*([X])$ and $[Y] := j_*([Y])$ we have the slightly more readable equality

$$X \cdot Y \;=\; \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle.$$

Remark.

(1) Note that if in the setting of Theorem 67.10 the manifold M had non-empty boundary and if both X and Y were proper submanifolds with non-empty boundary, then the cohomological term in the statement of Theorem 67.10 would be

$$\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in H_k(M,\partial M)})}_{\in H^{m-k}(M)} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in H_{m-k}(M,\partial M)})}_{\in H^k(M)}$$

but this is an element in $H^m(M)$ which is zero by Proposition 46.4 since M has non-empty boundary. After the informal discussion on page 1562 this is now a more formal reason why we cannot get any meaningful information if both X and Y have boundary.

(2) One easily verifies that the (anti-) commutativity of the algebraic intersection number of Lemma 67.8 corresponds precisely to the (anti-) commutativity of the cup product from Proposition 57.8.

(3) The following citation from [Kir4, p. 21] encapsulates the roles of algebraic intersection numbers and cup products:

> "Think with [algebraic] intersection [numbers], prove with cup products."

More precisely, algebraic intersection numbers are very intuitive and often they can be used to guess a statement. But when it comes to proving statements, e.g. why does the intersection number only depend on the isotopy type of a submanifold?, it is usually much easier to work with cup products.

Before we move on to the somewhat lengthy proof of Theorem 67.10 we want to convince ourselves that proving the theorem is worth the effort.

One application of Theorem 67.10 is that now we can provide a proof of a strengthened version of Statement 67.7. More precisely, we have the following proposition.

Proposition 67.11. *Let M be a compact oriented manifold.*

[1139] Recall that by definition of a complementary pair X is closed whereas Y is allowed to have boundary.

(1) *Let $\{X, Y\}$ and $\{X', Y'\}$ be pairs of complementary submanifolds of M. If X, X' are isotopic and if Y, Y' are isotopic, then $X \cdot Y = X' \cdot Y'$.*
(2) *If X and Y are two complementary submanifolds of M that can be made disjoint, then $X \cdot Y = 0$.*

Example. Now we can answer Question 67.5. The submanifolds X and Z, that are illustrated in Figure 833, cannot be made disjoint since their algebraic intersection number is non-zero.

Proof. Let M be a compact oriented manifold. We start out with the following claim.

Claim. If Z and Z' are isotopic compact oriented proper s-dimensional submanifolds of M, then $[Z] = [Z'] \in H_s(M, \partial M; \mathbb{Z})$.

We denote by $i\colon Z \to M$ and $i'\colon Z' \to M$ the inclusion maps. Since Z and Z' are isotopic there exists in particular an orientation-preserving homeomorphism $f\colon Z \to Z'$ such that the maps $i, i' \circ f\colon (Z, \partial Z) \to (M, \partial M)$ are homotopic. We obtain that

$$[Z] \;=\; i_*([Z]) \;\underset{\substack{\uparrow \\ \text{Proposition 31.15}}}{=}\; (i' \circ f)_*([Z]) = i'_*(f_*([Z])) \;\underset{\substack{\uparrow \\ \text{by Proposition 47.3 since } f \text{ is an} \\ \text{orientation-preserving homeomorphism}}}{=}\; i'_*([Z']) \;=\; [Z']$$

This concludes the proof of the claim.

Now let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. We assume that the submanifolds X, X' are isotopic and that the manifolds Y, Y' are isotopic. We obtain that

$$X \cdot Y \;\underset{\substack{\uparrow \\ \text{Theorem 67.10}}}{=}\; \langle \mathrm{PD}_M([X]), \mathrm{PD}_M([Y]), [M] \rangle \;\underset{\substack{\uparrow \\ \text{by the claim}}}{=}\; \langle \mathrm{PD}_M([X']), \mathrm{PD}_M([Y']), [M] \rangle \;\underset{\substack{\uparrow \\ \text{Theorem 67.10}}}{=}\; X' \cdot Y'.$$

This concludes the proof of the first statement. The second statement is an immediate consequence of the first statement. □

Now we will use Theorem 67.10 "the other way", i.e. we will apply the theorem to use the algebraic intersection number to draw conclusions on (co-) homology classes and to determine cup products using our geometric understanding of submanifolds.

Example.

(1) Let M be the complement in S^3 of the two open 3-dimensional tori shown in Figure 839 on the left. On the right of that figure we show an annulus Y that is properly embedded in M. We will now see that $[Y] \in H_2(M, \partial M; \mathbb{Z})$ is non-trivial. Indeed, in the figure on the right we also see a closed 1-dimensional submanifold X of M. The two submanifolds X and Y are clearly transverse and they intersect in one point. So with whatever orientation, we have $X \cdot Y = \pm 1$. By Theorem 67.10 this implies in particular that $\mathrm{PD}_M([Y]) \neq 0$. But this in turn implies of course that $[Y]$ itself is non-zero.

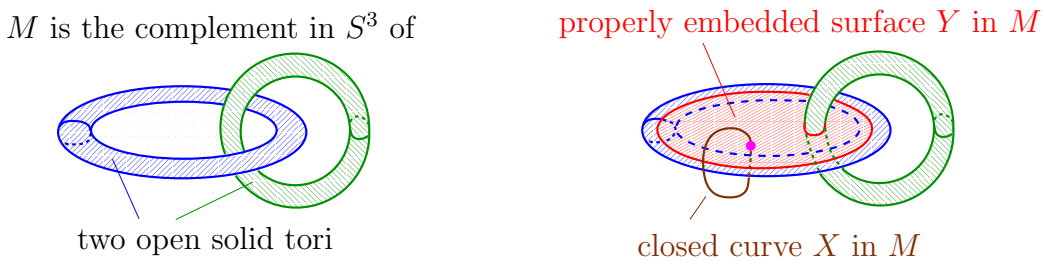


FIGURE 839.

(2) This time let M be the complement in S^3 of the two open 3-dimensional tori shown in Figure 840 on the left. Almost the same argument as in (1) shows that the surface X defines a non-trivial element in $H_2(M;\mathbb{Z})$.

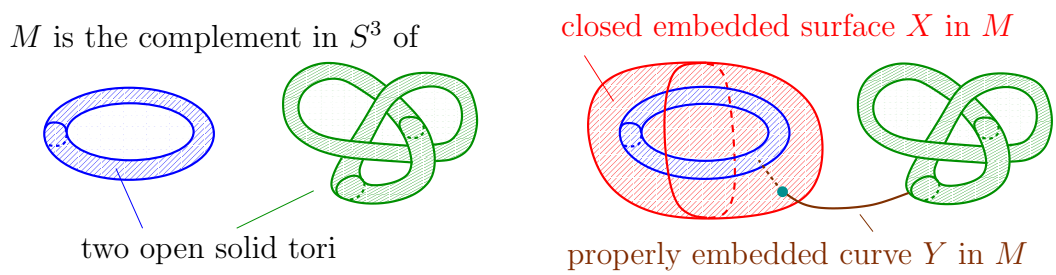


FIGURE 840.

Example. We consider the torus $T = S^1 \times S^1$ together with the submanifolds $X = S^1 \times \{1\}$ and $Y = \{1\} \times S^1$ which we endow with the usual orientations. By now we have many ways[1140] for showing that PD([X] and PD([Y]) form a basis for $H^1(T;\mathbb{Z})$. Thus if we want to understand the cup product on the torus, it suffices to understand the cup products involving these two cohomology classes. We calculate that

$$-\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([X]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{Proposition 57.8}}}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([Y]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{Theorem 67.10}}}{=} X \cdot Y \underset{\substack{\uparrow \\ \text{Figure 841}}}{=} +1.$$

We denote by $X' = S^1 \times \{z\}$ with $z \neq 1$ a "parallel copy" of X and similarly we define Y'. We calculate that

$$\langle \mathrm{PD}([X]) \cup \mathrm{PD}([X]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{by the claim in the proof of Proposition 67.11}}}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([X']), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{by Theorem 67.10, since } X \text{ and } X' \text{ are disjoint, in particular transverse}}}{=} X \cdot X' \underset{\substack{\uparrow}}{=} 0.$$

[1140]So, how many ways do you know?

and with the same argument we show that $\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle = 0$. We will now convince ourselves that these calculations match our previous calculations. To do so we denote by p_1 respectively p_2 the projection $S^1 \times S^1 \to S^1$ onto the first respectively the second factor and we denote by $\theta := \theta_{\mathbb{Z}} \in C^1(S^1; \mathbb{Z})$ the singular 1-cocycle that we introduced on page 1166. Now we see that

$$\begin{pmatrix} \mathrm{PD}([X]) \cup \mathrm{PD}([X]) & \mathrm{PD}([X]) \cup \mathrm{PD}([Y]) \\ \mathrm{PD}([Y]) \cup \mathrm{PD}([X]) & \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]) \end{pmatrix} \underset{\substack{\text{on page 1374 we already showed that} \\ [p_1^*\theta] \cap [T] = [1 \times S^1] = [Y] \text{ and } [p_2^*\theta] \cap [T] = -[S^1 \times 1] = -[X]}}{=} \begin{pmatrix} [p_2^*\theta] \cup [p_2^*\theta] & -[p_2^*\theta] \cup [p_1^*\theta] \\ -[p_1^*\theta] \cup [p_2^*\theta] & [p_1^*\theta] \cup [p_1^*\theta] \end{pmatrix} \underset{\text{see page 1374}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

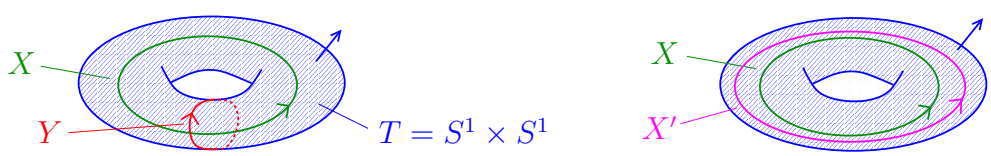


FIGURE 841.

We will do one more calculation of the cup product right after the proof of Theorem 67.10.

67.4. **Proof of Theorem 67.10.** In this section we will prove Theorem 67.10. Somewhat different proofs are given in [Bre, Theorem VI.11.9] or [DK, Theorem 10.32]. A quick outline of a proof is given in [GS, Proposition 1.2.5].

The proof of Theorem 67.10 will rely heavily on two lemmas that relate Poincaré duality in an oriented topological manifold M to Poincaré duality in a compact codimension 0 submanifold W.

The first lemma says that given a homology class in W we can apply Poincaré duality in M or W and essentially we end up with the same result:

Lemma 67.12. *Let M be a compact oriented m-dimensional topological manifold and let W be a compact m-dimensional submanifold with $M \cap \partial W = \varnothing$. We equip W with the induced orientation from Lemma 45.28. We write $\mathring{W} = W \setminus \partial W$.*[1141] *We fix the following notation:*

1. *We denote by w the inclusion map $W \to M$ and also the inclusion map of the pairs $(W, \partial W) \to (M, M \setminus \mathring{W})$,*
2. *we denote by $p\colon (M, \partial M) \to (M, M \setminus \mathring{W})$ the inclusion map of pairs.*

[1141] Recall that by Proposition 32.2 the set $\mathring{W} = W \setminus \partial W$ is indeed the interior of W in M.

Then for any $k \in \mathbb{N}_0$ the following diagram commutes[1142]

$$\begin{tikzcd}
H_k(W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] \arrow[d, "w_*"] & H^{m-k}(W,\partial W;\mathbb{Z}) \arrow[r, "(w^*)^{-1}", "\cong"'] & H^{m-k}(M, M\setminus \mathring{W};\mathbb{Z}) \arrow[d, "p^*"] \\
H_k(M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] & & H^{m-k}(M,\partial M;\mathbb{Z}).
\end{tikzcd}$$

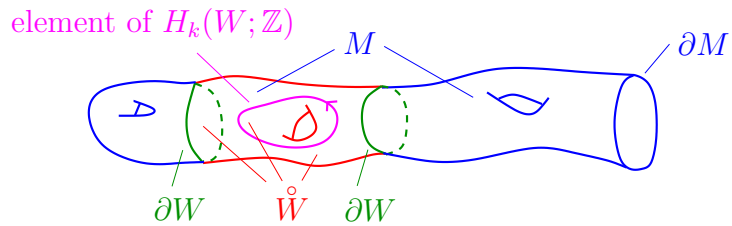


FIGURE 842. Illustration of Lemma 67.12.

Proof. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{tikzcd}
H_k(W;\mathbb{Z}) \arrow[d, "w_*"] & H^{m-k}(W,\partial W;\mathbb{Z}) \arrow[l, "\cap[W]"', "\cong"] & H^{m-k}(M, M\setminus \mathring{W};\mathbb{Z}) \arrow[l, "\cong", "w^*"'] \arrow[d, "p^*"] \\
H_k(M;\mathbb{Z}) & & H^{m-k}(M,\partial M;\mathbb{Z}). \arrow[ll, "\cap[M]", "\cong"']
\end{tikzcd}$$

So let $\beta \in H^{m-k}(M, M\setminus \mathring{W};\mathbb{Z})$. We have the following equality in $H_k(M;\mathbb{Z})$:

$$\begin{array}{ccccccc}
 & & \text{by Lemma 46.23 we have } p_*([M]) = w_*([W]) \in H_m(M, M\setminus\mathring{W}) & & & & \\
 & & \downarrow & & & & \\
w_*(w^*(\beta)\cap[W]) & = & \beta\cap w_*([W]) & = & \beta\cap p_*([M]) & = & p_*(\overbrace{p^*(\beta)\cap[M]}^{\in H_k(M)}) \quad = \quad p^*(\beta)\cap[M]. \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \begin{array}{c}\text{Lemma 59.8 applied to}\\ w\colon (W,\partial W)\to(M, M\setminus\mathring{W})\end{array} & & \begin{array}{c}\text{Lemma 59.8 applied to}\\ p\colon (M,\partial M)\to(M, M\setminus\mathring{W})\end{array} & & \begin{array}{c}\text{since } p\colon M\to M \text{ is id}\\ \text{we have } p_* = \mathrm{id}_* = \mathrm{id}\end{array}
\end{array}$$

We have thus shown that the above diagram does indeed commute. □

We take a short rest from more technicalities and enjoy the fact that we can already prove Theorem 67.10 in the special case that M is closed and that the submanifolds *do not intersect*.

Proof of Theorem 67.10 for disjoint submanifolds. Let M be a closed oriented connected m-dimensional manifold and let X and Y be a complementary pair of *disjoint* submanifolds

[1142]Here we use that $w\colon (W,\partial W)\to(M, M\setminus\mathring{W})$ induces an isomorphism on cohomology by excision, see page 1192 for details.

of M. Since the submanifolds are disjoint it suffices to show that the cup product of the Poincaré duals of $[X]$ and $[Y]$ is also zero.

We write $k = \dim(X)$. We apply the General Tubular Neighborhood Theorem 64.7 to the submanifold $X \cup Y$ of M. We obtain disjoint tubular neighborhoods A for X and B for Y. We denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{A})$ and $p\colon (M, \varnothing) \to (M, M \setminus \mathring{B})$ the obvious inclusion map of pairs of topological spaces.

We apply Lemma 67.12 to the codimension zero submanifold A and the homology class $[X] \in H_k(A; \mathbb{Z})$. We obtain an $\alpha \in H^{m-k}(M, M \setminus \mathring{A}; \mathbb{Z})$ with $p^*(\alpha) = \mathrm{PD}_M([X])$. Similarly we see that there exists a $\beta \in H^k(M, M \setminus \mathring{A}; \mathbb{Z})$ with $p^*(\beta) = \mathrm{PD}_M([Y])$.

We consider the following diagram

$$\begin{tikzcd}
H^{m-k}(M;\mathbb{Z}) \times H^k(M;\mathbb{Z}) \arrow[rr, "\cup"] & & H^m(M;\mathbb{Z}) \\
H^{m-k}(M, M\setminus\mathring{A};\mathbb{Z}) \times H^k(M, M\setminus\mathring{B};\mathbb{Z}) \arrow[rr, "\cup"] \arrow[u, "p^*\times p^*"'] \arrow[u, shift left=6, "(\mathrm{PD}_M([X]),\mathrm{PD}_M([Y]))"] \arrow[u, shift left=3, "\uparrow"] \arrow[u, shift left=1, "(\alpha,\beta)"] & & H^m(M, \underbrace{M\setminus\mathring{A} \cup M\setminus\mathring{B}}_{=M};\mathbb{Z}). \arrow[u]
\end{tikzcd}$$

The diagram commutes by Proposition 58.3 (1). It follows from the fact that the lower right group is zero that the cup product of $\mathrm{PD}_M([X])$ and $\mathrm{PD}_M([Y])$ is zero. $\square$

After this refreshing proof of the special case we continue with trying to prove the general case of Theorem 67.10. The following lemma also relates Poincaré Duality in a manifold M to Poincaré Duality in a compact submanifold W of codimension zero.

Lemma 67.13. *Let M be a compact oriented m-dimensional topological manifold and let W be a compact m-dimensional submanifold with corner. We equip W with the induced orientation from*[1143] *Lemma 45.28. We fix the following notation:*

1. *We write $\partial_1 W = W \cap \partial M$ and $\partial_0 W = \overline{\partial W \setminus \partial_1 W}$,*
2. *we write $\mathring{W} = W \setminus \partial_0 W$,*
3. *the inclusion maps $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $(W, \partial_1 W) \to (M, \partial M)$ are denoted by w, finally*
4. *we denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{W})$ the obvious map.*

[1143]Lemma 45.28 was not formulated for submanifolds with corner, but the same construction as in Lemma 45.28 endows $W \setminus \partial W$ with an orientation.

Then for any $k \in \mathbb{N}_0$ the following diagram commutes[1144] [1145]

$$\begin{tikzcd}
H_k(M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] \arrow[d, "p_*"] & & H^{m-k}(M,\partial M;\mathbb{Z}) \arrow[d, "w^*"] \\
H_k(M, M\setminus \mathring{W};\mathbb{Z}) \arrow[r, "(w_*)^{-1}", "\cong"'] & H_k(W,\partial_0 W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] & H^{m-k}(W,\partial_1 W;\mathbb{Z}).
\end{tikzcd}$$

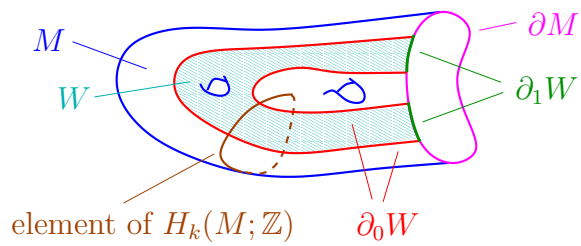


FIGURE 843. Illustration of Lemma 67.13.

Proof. In this proof we will twice use the full force of Lemma 59.8 applied to excisive triads. Since this can be seriously confusing we recall the general statement of the lemma. Let $f\colon (X,A,B) \to (\widetilde{X},\widetilde{A},\widetilde{B})$ be a map between two excisive triads. Then for any choice of $\varphi \in H^k(\widetilde{X},\widetilde{A};\mathbb{Z})$ and $\sigma \in H_m(X, A\cup B;\mathbb{Z})$ we have

$$\begin{array}{ccccc}
f_*\colon H_k(X,A\cup B) \to H_k(\widetilde{X},\widetilde{A}\cup\widetilde{B}) & & f^*\colon H^{m-k}(\widetilde{X},\widetilde{A}) \to H^{m-k}(X,A) & & \\
\downarrow & & \downarrow & & \\
\varphi \cap f_*(\sigma) & = & f_*(f^*(\varphi)\cap\sigma) & \in & H_k(\widetilde{X},\widetilde{B};\mathbb{Z}). \\
 & & \uparrow & & \\
 & & f_*\colon H_k(X,B) \to H_k(\widetilde{X},\widetilde{B}) & &
\end{array}$$

After this reminder we start out with the proof of the lemma in earnest. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{tikzcd}
H_k(M;\mathbb{Z}) \arrow[d, "p_*"] & & H^{m-k}(M,\partial M;\mathbb{Z}) \arrow[ll, "\cap[M]"', "\cong"] \arrow[d, "w^*"] \\
H_k(M, M\setminus \mathring{W};\mathbb{Z}) & H_k(W,\partial_0 W;\mathbb{Z}) \arrow[l, "w_*"', "\cong"] & H^{m-k}(W,\partial_1 W;\mathbb{Z}). \arrow[l, "\cap[W]"']
\end{tikzcd}$$

[1144]Here we use that by Lemma 64.1 we can apply the Poincaré Duality Theorem 61.1 to obtain that $\cap[W]\colon H^{m-k}(W,\partial_1 W;\mathbb{Z}) \to H_k(W,\partial_0 W;\mathbb{Z})$ is an isomorphism. We denote its inverse by PD_W.

[1145]Furthermore we use that $w\colon (W,\partial_0 W) \to (M, M\setminus\mathring{W})$ induces an isomorphism on cohomology by excision, see page 1192 for details.

So let $\beta \in H^{m-k}(M, \partial M; \mathbb{Z})$. We have the following equalities in $H_k(M, M \setminus \mathring{W}; \mathbb{Z})$:

$$\begin{array}{ccccccc} p_*(\beta \cap [M]) & = & \beta \cap p_*([M]) & = & \beta \cap w_*([W]) & = & w_*(w^*(\beta) \cap [W]). \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

Lemma 59.8 applied to the map $p\colon (M, \partial M, \varnothing) \to (M, \partial M, M \setminus \mathring{W})$ of excisive triads

by Lemma 46.23 we have $p_*([M]) = w_*([W])$ in $H_m(M, M \setminus \mathring{W})$

Lemma 59.8 applied to the map $w\colon (W, \partial_1 W, \partial_0 W) \to (M, \partial M, M \setminus \mathring{W})$ of excisive triads □

In the proof of Theorem 67.10 we will need the following proposition.

Proposition 67.14. *Let M be a compact oriented manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \dots, P_s$ the intersection points of X and Y. Then there exist*

(1) *compact submanifolds with corner A and B such that A is a neighborhood for X and B is a neighborhood for Y, and*
(2) *injective maps $\Psi_i\colon \overline{B}^k \times \overline{B}^l \to M$, $i = 1, \dots, s$*

such that the following statements hold:

(a) *For each $i \in \{1, \dots, s\}$ we have $\Psi_i((0,0)) = P_i$.*
(b) *The images $Q_i := \Psi_i\big(\overline{B}^k \times \overline{B}^l\big)$, $i = 1, \dots, s$ are disjoint.*
(c) *We have $Q_1 \cup \dots \cup Q_s = A \cap B$.*
(d) *For each $i \in \{1, \dots, s\}$ the following statements hold:*
  (d1) *the restriction of Ψ_i to $B^k \times B^l$ is a diffeomorphism onto its image,*
  (d2) *we have $Q_i \cap X = \Psi_i(\overline{B}^k \times \{0\})$ and the map $\Psi_i\colon \overline{B}^k \times \{0\} \to X$ is an orientation-preserving embedding,*
  (d3) *we have $Q_i \cap Y = \Psi_i(\{0\} \times \overline{B}^l)$ and the map $\Psi_i\colon \{0\} \times \overline{B}^l \to Y$ is an orientation-preserving embedding.*

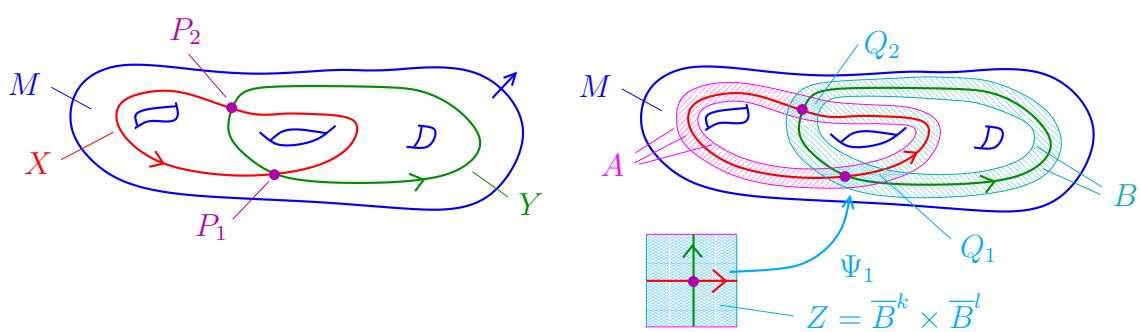


FIGURE 844. Illustration of Proposition 67.14.

Sketch of a proof. Let M be a compact oriented n-dimensional manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \dots, P_s$

the intersection points of X and Y. For $i = 1, \dots, s$ we can and will pick embeddings $\Psi_i \colon \overline{B}^n \to M \setminus \partial M$ such that the following conditions are satisfies:

(1) the images are disjoint,
(2) for each $i \in \{1, \dots, s\}$ the map Ψ_i restricts to orientation-preserving diffeomorphisms $B^k \times \{0\} \to \Psi_i(B^n) \cap X$ and $\{0\} \times B^l \to \Psi_i(B^n) \cap Y$.

We consider $W := M \setminus \bigcup_{i=1}^{s} \Psi_i(B^n)$. Note that $X' := W \cap X$ and $Y' := W \cap Y$ are proper submanifolds of the compact manifold W. Thus we can apply the General Tubular Neighborhood Theorem 64.7 to obtain tubular neighborhoods for X' and Y'. Using Proposition 10.27 we can arrange the intersections of the tubular neighborhoods with $\Psi_i(S^{n-1})$ to be "standard $(n-1)$-dimensional balls". At this stage it is fairly straightforward to extend the tubular neighborhoods to the interiors of the $\Psi_i(B^n)$ and to obtain all the desired objects. We have faith in the unwavering enthusiasm of our readers for filling in the details. □

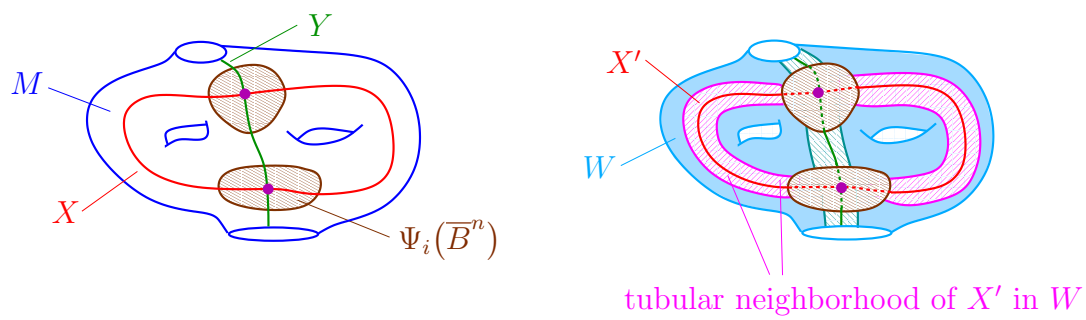


FIGURE 845. Illustration of the proof of Proposition 67.14.

Remark. An alternative approach to Proposition 67.14 would be to use tubular neighborhoods provided by Lemma 64.14.

Given the situation of Proposition 67.14 we can make one more statement.

Lemma 67.15. *We use the notation of Proposition 67.14. Let $i \in \{1, \dots, s\}$. We equip the topological manifold Q_i, which is a codimension zero submanifold of M, with the induced orientation coming from Lemma 45.28. Furthermore we equip the topological manifold $\overline{B}^k \times \overline{B}^l$ with the orientation coming from the standard orientation of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$. With these choices the following statement holds:*

(g) *If we denote by σ_i the sign of the intersection point P_i, then*

$$\Psi_{i*}\big([\overline{B}^k \times \overline{B}^l]\big) \;=\; \sigma_i \cdot [Q_i].$$

Proof. Let $i \in \{1, \dots, s\}$. If follows from (e) and (f) that at the point $(0,0)$ the diffeomorphism $\Psi_i \colon B^k \times B^l \to \mathring{Q}_i$ is orientation-preserving if and only if $\sigma_i = 1$. Now the statement follows easily from the definition of σ_i, Proposition 47.3 and Lemma 45.23. □

The last preparation we need before we can provide the actual proof of Theorem 67.10 is the following variation of Proposition 60.2.

Proposition 67.16. *Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. If $P \in X$ and $Q \in Y$ are points, then*

$$\mathrm{PD}_{X\times Y}([X \times \{Q\}]) \cup \mathrm{PD}_{X\times Y}([\{P\} \times Y]) \;=\; [X \times Y]^*.$$

Proof. We calculate that

$$\begin{aligned}
\mathrm{PD}_{X\times Y}([X\times\{Q\}])\cup\mathrm{PD}_{X\times Y}([\{P\}\times Y]) &\overset{\text{Proposition 57.8}}{\underset{\text{by Proposition 60.2 (1) and the definition of } \mathrm{PD}_{X\times Y}}{=}} (-1)^{mn}\cdot\mathrm{PD}_{X\times Y}([\{P\}\times Y])\cup\mathrm{PD}_{X\times Y}([X\times\{Q\}]) \\
&= p^*([X]^*) \cup q^*([Y]^*) \;\underset{\text{by Proposition 60.2 (1)}}{=}\; [X\times Y]^*.
\end{aligned}$$

□

We turn to the actual proof of Theorem 67.10.

Proof of Theorem 67.10. Let M be a compact oriented connected m-dimensional manifold. To simplify the notation we assume that M is closed. The case that M has boundary is proved in a rather similar fashion.[1146]

Let X and Y be a complementary pair of submanifolds of M. We introduce the following notation:

(1) We write $k = \dim(X)$ and $l = m - k$.
(2) As usual we denote by $[X] \in H_k(X)$ and $[Y] \in H_l(Y)$ the fundamental classes of X and Y. By a slight abuse of notation we denote by $[X]$ respectively $[Y]$ also the images of these homology classes in any subset of M that contain X respectively Y.
(3) We denote the intersection points of X and Y by $P_1, \dots, P_s$ and we denote the corresponding signs by $\sigma_1, \dots, \sigma_s$.
(4) As usual, given a topological space X we denote by $\epsilon_X = \epsilon\colon H_0(X) \to \mathbb{Z}$ the augmentation map.

With the notation that we just introduced, and keeping in mind Lemma 59.4 (1), the statement of Theorem 67.10 can be reformulated as the following statement.

Statement.

$$\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \;=\; \sum_{i=1}^{s} \sigma_i.$$

We introduce many more objects, notations and conventions:

[1146] Admittedly, the case that M has boundary also requires the generalization of Lemma 67.12 to the case that W is a codimension zero submanifold with corner. But that is not really a problem.

(5) We apply Proposition 67.14 to the submanifolds X and Y. In the remainder of the proof we use the notation that we introduced in that proposition. As a reminder, most of the relevant objects are illustrated in Figure 844.
(6) We denote by Z the "cylinder" $\overline{B}^k \times \overline{B}^l$.
(7) We write $\mathring{A} = A \setminus \partial A$ and $\mathring{B} = B \setminus \partial B$.
(8) We denote all of the inclusion maps $(M, \varnothing) \to (M, M \setminus \mathring{A})$, $(M, \varnothing) \to (M, M \setminus \mathring{B})$ and $(M, \varnothing) \to (M, M \setminus (\mathring{A} \cap \mathring{B}))$ by p.
(9) We write $C = A \cap B = Q_1 \cup \cdots \cup Q_s$. Furthermore we write $\partial_A C = \partial A \cap B$ and $\partial_B C = A \cap \partial B$. Note that $\partial C = \partial_A C \cup \partial_B C$. Also note that $(M \setminus \mathring{A}) \cup (M \setminus \mathring{B}) = M \setminus \mathring{C}$.
(10) We denote by $c\colon C \to M$ the inclusion map and we denote by c also all the corresponding inclusion maps of pairs where the first terms are C and M.
(11) Given $i \in \{1, \dots, s\}$ we write $\partial_A Q_i = \partial_A C \cap Q_i$ and $\partial_B Q_i = \partial_B C \cap Q_i$. Note that $\partial Q_i = \partial_A Q_i \cup \partial_B Q_i$.
(12) Given $i \in \{1, \dots, s\}$ we denote by $q_i\colon Q_i \to C$ the inclusion map and we denote by q_i also all the corresponding inclusion maps of pairs where the first terms are Q_i and C.
(13) Note that C and each Q_i is a codimension zero submanifold of M. We equip these submanifolds with the corresponding orientation introduced in Lemma 45.28.

We illustrate some of the definitions in Figure 846.

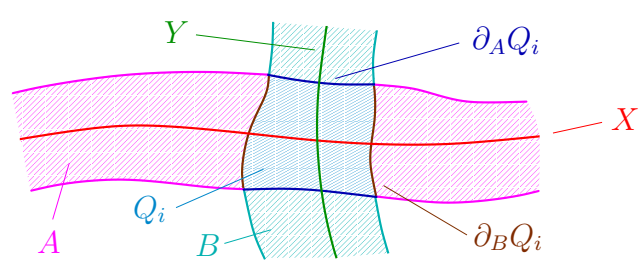


Figure 846.

Now we consider the following diagram:[1147]

$$\begin{tikzcd}
H^l(M) \times H^k(M) \arrow[r, "\cup"] & H^m(M) \arrow[r, "\cap[M]"] & H_0(M) \arrow[r, "\epsilon_M"] \arrow[d, "="] & \mathbb{Z} \arrow[d, "="'] \\
H^l(M, M\setminus \mathring{A}) \times H^k(M, M\setminus \mathring{B}) \arrow[r, "\cup"] \arrow[u, "p^*\times p^*"] \arrow[d, "c^*\times c^*"] & H^m(M, M\setminus \mathring{C}) \arrow[r, "\cap p_*([M])"] \arrow[u, "p^*"] \arrow[d, "c^*"] & H_0(M) \arrow[r, "\epsilon_M"] & \mathbb{Z} \\
H^l(C, \partial_A C) \times H^k(C, \partial_B C) \arrow[r, "\cup"] \arrow[d, "\bigoplus_{i=1}^s q_i^*\times q_i^*"] & H^m(C, \partial C) \arrow[r, "\cap[C]"] \arrow[d, "\bigoplus_{i=1}^s q_i^*"] & H_0(C) \arrow[r, "\epsilon_C"] \arrow[u, "c_*"] & \mathbb{Z} \arrow[u, "="] \\
\bigoplus_{i=1}^s H^l(Q_i, \partial_A Q_i) \times H^k(Q_i, \partial_B Q_i) \arrow[r, "\oplus\cup"] & \bigoplus_{i=1}^s H^m(Q_i, \partial Q_i) \arrow[r, "\bigoplus_{i=1}^s \cap[Q_i]"] & \bigoplus_{i=1}^s H_0(Q_i) \arrow[r, "\sum \epsilon_{Q_i}"] \arrow[u, "\sum_{i=1}^s q_{i*}"] & \mathbb{Z}. \arrow[u, "="]
\end{tikzcd}$$

Claim 1. The above diagram commutes.

In the following we deal with all of the nine squares of the diagram.

(a) The left squares in the first and second row commute by Proposition 58.3 (1).
(b) The left square in the third row commutes by Lemma 59.9 (1) together with the aforementioned Proposition 58.3 (1).
(c) The middle square in the first row commutes by Lemma 59.8 (1).
(d) It follows from our orientation conventions and Lemma 46.23 that $c_*([C]) = p_*([M])$. It follows from Lemma 59.8 (1) that the middle square in the second row commutes.
(e) It follows from the remark on page 1114 that $\sum_{i=1}^s q_{i*}([Q_i]) = [C]$. Now we obtain from Lemma 59.8 (1) and Lemma 59.9 (2) that the middle square in the third row commutes.
(f) The squares that involve the augmentation maps commute by the discussion on page 747.

This concludes the proof of the claim.

We introduce a few last pieces of notation:

(14) We denote by $a\colon (A, \partial A) \to (M, M \setminus \mathring{A})$ and $b\colon (B, \partial B) \to (M, M \setminus \mathring{B})$ the inclusion maps, note that a and b induce isomorphisms on cohomology by excision, see page 1192 for details.
(15) We write
$$\alpha = (a^*)^{-1}(\underbrace{\mathrm{PD}_A([X])}_{\in H^l(A,\partial A)}) \in H^l(M, M\setminus \mathring{A}) \text{ and } \beta = (b^*)^{-1}(\underbrace{\mathrm{PD}_B([Y])}_{\in H^k(B,\partial B)}) \in H^k(M, M\setminus \mathring{B}).$$
(16) Given $i \in \{1, \dots, s\}$ we write $X_i = X \cap Q_i$ and $Y_i = Y \cap Q_i$. It follows from Proposition 67.14 (e) and (f) that X_i and Y_i are codimension 0 submanifolds of X and Y. Since X and Y are oriented we can view X_i and Y_i also as oriented topological

[1147] Many of the vertical maps in the diagram are isomorphisms by excision, see page 1192 for details. But this is in fact irrelevant for our purpose.

manifolds. As usual we denote by $[X_i] \in H_k(X_i, \partial X_i)$ and $[Y_i] \in H_k(Y_i, \partial Y_i)$ the corresponding fundamental classes. With the same convention as before we denote by $[X_i]$ also the image under the inclusion induced map $H_k(X_i, \partial X_i) \to H_k(Q_i, \partial_B Q_i)$ and similarly we denote by $[Y_i]$ also the image under the inclusion induced map $H_k(Y_i, \partial Y_i) \to H_k(Q_i, \partial_A Q_i)$.

Claim 2. We have

(a) We have $p^*(\alpha) = \mathrm{PD}_M([X]) \in H^l(M)$ and $p^*(\beta) = \mathrm{PD}_M([Y]) \in H^k(M)$.

(b) For any $i \in \{1, \dots, s\}$ we have

$$(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i]) \in H^l(Q_i, \partial_A Q_i) \text{ and } (q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i]) \in H^k(Q_i, \partial_B Q_i).$$

(c) For any $i \in \{1, \dots, s\}$ we have

$$\epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) \;=\; \sigma_i.$$

We provide the proofs for the two statements of the claim.

(a) By Lemma 67.12 the following diagram commutes:

$$\begin{tikzcd}
H_k(A) \arrow[r, "\mathrm{PD}_A"'] \arrow[rr, bend left, "{[X]\mapsto\alpha}"] \arrow[d, "{[X]\mapsto[X]}"'] & H^l(A, \partial A) \arrow[r, "\cong", "(a^*)^{-1}"'] & H^l(M, M\setminus \mathring{A}) \arrow[d, "p^*"] \\
H_k(M) \arrow[rr, "\mathrm{PD}_M"] & & H^l(M).
\end{tikzcd}$$

We apply this commutative diagram to $[X] \in H_k(A)$ and we obtain immediately that $p^*(\alpha) = \mathrm{PD}_M([X])$. The same way one shows that $p^*(\beta) = \mathrm{PD}_M([Y])$.

(b) Let $i \in \{1, \dots, s\}$. We denote by $r_i \colon Q_i \to A$ the inclusion map. We consider the following diagram:

$$\begin{tikzcd}
H_k(A) \arrow[rr, "\mathrm{PD}_A"] \arrow[d] & & H^l(A, \partial A) \arrow[d, "r_i^*"] & H^l(M, M\setminus \mathring{A}) \arrow[l, "a^*"', "\cong"] \arrow[dl, "(q_i\circ c)^*"] \\
H_k(A, A\setminus \mathring{Q}_i) & H_k(Q_i, \partial_B Q_i) \arrow[l, "r_{i*}"', "\cong"] \arrow[r, "\mathrm{PD}_{Q_i}"] & H^l(Q_i, \partial_A Q_i). &
\end{tikzcd}$$

We make the following observations:

(α) The triangle on the right commutes since $a \circ r_i = c \circ q_i \colon Q_i \to M$ are both the inclusion map.

(β) It follows from Lemma 67.13, applied to the codimension zero submanifold Q_i of A, that the rectangle commutes.[1148]

(γ) By Lemma 46.23 we have $r_{i*}([X_i]) = [X] \in H_k(A, A \setminus \mathring{Q}_i)$.

[1148] One can easily convince oneself that $\partial_A Q_i = \partial_1 Q_i$ and $\partial_B Q_i = \partial_0 Q_i$ for Q_i viewed as a submanifold with corner of A.

(δ) It follows from a suitable variation of the excision theorem, see the remark on page 1489, that the map $r_{i*}\colon H_k(Q_i, \partial BQ_i) \to H_k(A, A \setminus \mathring{Q}_i)$ is an isomorphism.

The desired equality $(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i])$ follows from the above commutative diagram applied to $[X] \in H_k(A)$. Exactly the same way we can also show that $(q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i])$.

(c) Let $i \in \{1, \dots, s\}$. We recall that $Z = \overline{B}^k \times \overline{B}^l$. We consider the following diagram:

$$\begin{tikzcd}
H^l(Q_i, \partial_A Q_i) \times H^k(Q_i, \partial_B Q_i) \arrow[r, "\cup"] \arrow[d, "\Psi_i^* \times \Psi_i^*"] & H^m(Q_i, \partial Q_i) \arrow[r, "\cap [Q_i]"] \arrow[d, "\Psi_i^*"] & H_0(Q_i) \arrow[r, "\epsilon_{Q_i}"] & \mathbb{Z} \\
H^l(Z, \overline{B}^k \times \partial\overline{B}^l) \times H^k(Z, \partial\overline{B}^k \times \overline{B}^l) \arrow[r, "\cup"] & H^m(Z, \partial Z) \arrow[r, "\cap \sigma_i \cdot [Z]"] & H_0(Z) \arrow[r, "\epsilon_Z"] \arrow[u, "\Psi_i^*"] & \mathbb{Z}. \arrow[u, "="']
\end{tikzcd}$$

As in the previous diagram, the squares to the left and right commute. The middle square is slightly more interesting. Lemma 67.15 says that $\Psi_{i*}([Q_i]) = \sigma_i \cdot [Z]$. Using this observation we obtain once again from Lemma 59.8 (1) that the middle square commutes.

We calculate that

$$\begin{aligned}
\epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) &\overset{\text{by the above commutative diagram}}{=} \sigma_i \cdot \epsilon_Z\big((\Psi_i^*(\mathrm{PD}_{Q_i}([X_i])) \cup \Psi_i^*(\mathrm{PD}_{Q_i}([Y_i]))) \cap [Z]\big) \\
&= \sigma_i \cdot \epsilon_Z\big((\mathrm{PD}_Z([\overline{B}^k \times \{0\}]) \cup \mathrm{PD}_Z([\{0\} \times \overline{B}^l])) \cap [Z]\big) \\
&\underset{\text{Proposition 67.14 (d)}}{=} \sigma_i \cdot \epsilon_{\overline{B}^k \times \overline{B}^l}\big([\overline{B}^k \times \overline{B}^l]^* \cap [\overline{B}^k \times \overline{B}^l]\big) \underset{\text{definition of the dual fundamental class}}{=} \sigma_i.
\end{aligned}$$

Proposition 67.16

This concludes the proof of the claim.

After all these preparations we can prove the desired equality:

$$\begin{aligned}
\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) &\overset{\text{by the first row of the commutative diagram from Claim 1 and by Claim 2 (a)}}{=} \epsilon_M\big((\alpha \cup \beta) \cap p_*([M])\big) \\
&\underset{\text{by the second and third row of commutative diagram from Claim 1 and by Claim 2 (b)}}{=} \sum_{i=1}^s \epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) \underset{\text{by Claim (2c)}}{=} \sum_{i=1}^s \sigma_i.
\end{aligned}$$

This concludes the proof of the theorem. □

Remark. Towards the end of the proof of the theorem we marked in red color that we used Proposition 67.16. Most of the other statements that we used in the proof were fairly straightforward lemmas regarding cup and cap products. But arguably the key ingredient

in the proof is Proposition 67.16 which was a variation of Proposition 60.2 and which in turn was a fairly straightforward consequence of the Product Theorem 60.1.

67.5. **Further applications of Theorem 67.10.** In this section we will give two more applications of Theorem 67.10.

Notation. Let $n \in \mathbb{N}$ and let $m \in \{0, \dots, n\}$.

(1) We consider the inclusion map

$$\begin{array}{rcl} i_{m,n}\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \dots : z_m] & \mapsto & [z_0 : \dots : z_m : 0 : \dots : 0]. \end{array}$$

When m and n are understood we write i instead of $i_{m,n}$. In Lemma 62.6 we showed that for any $k \le 2m$ the induced maps $i_*\colon H_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $i^*\colon H^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ are isomorphisms.

(2) By a slight abuse of notation we abbreviate $(i_{m,n})_*([\mathbb{C}\mathrm{P}^m])$ to $[\mathbb{C}\mathrm{P}^m] \in H_{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. Furthermore, again by a slight abuse of notation, we write $[\mathbb{C}\mathrm{P}^m]^* \in H^{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ for the unique element with $\langle [\mathbb{C}\mathrm{P}^m]^*, [\mathbb{C}\mathrm{P}^m]\rangle = 1$. Finally we write $x = [\mathbb{C}\mathrm{P}^1]^*$.

In Proposition 62.7 we had used the Poincaré Duality Theorem 61.1 to show that for given $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$ the k-th power $x^k \in H^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. In the following proposition we give a somewhat different proof for this statement, and more interestingly, we can precisely nail down the cohomology class x^k.

Proposition 67.17. *Let $n \in \mathbb{N}$. For every $k \in \{0, \dots, n\}$ we have*

$$(\underbrace{[\mathbb{C}\mathrm{P}^1]^*}_{=x})^k \;=\; [\mathbb{C}\mathrm{P}^k]^* \;\in\; H^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Remark.

(1) In contrast to Proposition 67.17 our earlier Proposition 62.7 only proves the equality $x^k = \pm[\mathbb{C}\mathrm{P}^k]^*$.

(2) Proposition 67.17 gives in particular an affirmative answer to Question 62.8.

The key to the proof of Proposition 67.17 is the following lemma.

Lemma 67.18. *Let $n \in \mathbb{N}$. For any $k \in \{0, \dots, n\}$ we have*

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \;=\; [\mathbb{C}\mathrm{P}^n]^* \;\in\; H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$. For $k \in \{0, \dots, n\}$ we consider the inclusion map

$$\begin{array}{rcl} t_{k,n}\colon \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \dots : z_k] & \mapsto & [0 : \dots : 0 : z_0 : \dots : z_k]. \end{array}$$

We also consider the map[1149]

$$\begin{array}{rcl} \Phi\colon \mathbb{C}\mathrm{P}^k \times [0,1] & \to & \mathbb{C}\mathrm{P}^n \\ ([z], t) & \mapsto & \big[\big(z \cdot \cos^2\big(t \cdot \frac{\pi}{2}\big), 0\big) + \big(0, z \cdot \sin^2\big(t \cdot \frac{\pi}{2}\big)\big)\big]. \end{array}$$

[1149]It is straightforward to verify that this map is well-defined, i.e. it does not depend on the choice of the representative $z \in \mathbb{C}^{k+1} \setminus \{0\}$ and $\big(z \cdot \cos^2\big(t \cdot \frac{\pi}{2}\big), 0\big) + \big(0, z \cdot \sin^2\big(t \cdot \frac{\pi}{2}\big)\big) \neq 0$ for all $t \in [0,1]$.

Then

$$\begin{aligned}
\langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n]\rangle &= \langle \mathrm{PD}(i_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n]\rangle \\
&\underset{\substack{\uparrow \\ \text{by Proposition 30.5, since } \Phi \text{ is a homotopy from } i_{k,n} \text{ to } t_{k,n}}}{=} \langle \mathrm{PD}(t_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n]\rangle \\
&= \langle \mathrm{PD}([t_{k,n}(\mathbb{C}\mathrm{P}^k)]) \cup \mathrm{PD}([i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})]), [\mathbb{C}\mathrm{P}^n]\rangle \\
&\underset{\substack{\uparrow \\ \text{by Theorem 67.10 since the submanifolds} \\ t_{k,n}(\mathbb{C}\mathrm{P}^k) \text{ and } i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \text{ are transverse}}}{=} t_{k,n}(\mathbb{C}\mathrm{P}^k) \cdot i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \underset{\substack{\uparrow \\ \text{see page 1565 for details}}}{=} 1.
\end{aligned}$$

By definition of the dual fundamental class this implies the desired result. □

We record the following fairly immediate corollary.

Corollary 67.19. *Let $n \in \mathbb{N}$. For any $k \in \{0, \dots, n\}$ we have*

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) = [\mathbb{C}\mathrm{P}^{n-k}]^* \in H^{2n-2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Then

$$\begin{aligned}
\langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]), [\mathbb{C}\mathrm{P}^{n-k}]\rangle &\underset{\substack{\downarrow \\ \text{by definition of } \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}])}}{=} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]), \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]) \cap [\mathbb{C}\mathrm{P}^n]\rangle \\
&\underset{\substack{\uparrow \\ \text{Lemma 59.7 and Proposition 57.8}}}{=} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n]\rangle \underset{\substack{\uparrow \\ \text{Lemma 67.18}}}{=} 1.
\end{aligned}$$

By definition of the dual fundamental class this implies the desired result. □

Now we can provide the proof of Proposition 67.17.

Proof of Proposition 67.17. We want show that for every $n \in \mathbb{N}$ and every $k \in \{0, \dots, n\}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in H^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. We start out with the following observation that follows immediately from the discussion on page 1580 and Lemma 57.10.

Observation. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m\}$. If $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in H^{2k}(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$, then for any $n \geq m$ we also have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in H^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$

By the observation it suffices to prove the following claim.

Claim. For any $n \in \mathbb{N}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^n = [\mathbb{C}\mathrm{P}^n]^* \in H^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$.

We prove the claim by induction on n. For $n = 1$ the statement holds by definition. So suppose that the statement holds for some $n - 1$ with $n \geq 2$. Then the following equality holds in $H^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$:

$$([\mathbb{C}\mathrm{P}^1]^*)^n = ([\mathbb{C}\mathrm{P}^1]^*)^{n-1} \cup [\mathbb{C}\mathrm{P}^1]^* \underset{\substack{\uparrow \\ \text{by induction and the observation}}}{=} [\mathbb{C}\mathrm{P}^{n-1}]^* \cup [\mathbb{C}\mathrm{P}^1]^* \underset{\substack{\uparrow \\ \text{Corollary 67.19}}}{=} \mathrm{PD}([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-1}]) \underset{\substack{\uparrow \\ \text{Lemma 67.18}}}{=} [\mathbb{C}\mathrm{P}^n]^*.$$

This concludes the proof of the claim and thus of the proposition. □

Now we can answer Question 64.12 from an earlier chapter.

Proposition 67.20. *Every tubular neighborhood of* $\mathbb{C}\mathrm{P}^1 = i_{1,2}(\mathbb{C}\mathrm{P}^1)$ *in* $\mathbb{C}\mathrm{P}^2$ *is non-trivial.*[1150]

In Figure 102 we had implicitly seen that the tubular neighborhood of $\mathbb{R}\mathrm{P}^1$ in $\mathbb{R}\mathrm{P}^2$ is a Möbius band, and thus non-trivial. The above proposition is thus an analogue of this simpler, and more visual, observation.

Proof. Suppose that $\mathbb{C}\mathrm{P}^1$ admits a trivial tubular neighborhood in $\mathbb{C}\mathrm{P}^2$. By definition this means that there exists an embedding $\Phi\colon \mathbb{C}\mathrm{P}^1 \times \overline{B}^2 \to \mathbb{C}\mathrm{P}^2$ with $\Phi(P,0) = P$ for all $P \in \mathbb{C}\mathrm{P}^1$. We denote by $i = i_{1,2}\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the usual inclusion map and we denote by $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the embedding that is given by $P \mapsto \Phi(P,1)$. Then

$$\begin{aligned}
\pm 1 &\underset{\substack{\uparrow \\ \text{Proposition 67.17 or alternatively} \\ \text{Proposition 62.7}}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\substack{\uparrow \\ \text{by Proposition 30.5 since } i,f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2 \text{ are homotopic} \\ \text{via the homotopy } (P,t)\mapsto \Phi(P,t)}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(f_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \\
&\underset{\substack{\uparrow \\ \text{by Theorem 67.10}}}{=} i(\mathbb{C}\mathrm{P}^1)\cdot f(\mathbb{C}\mathrm{P}^1) \underset{\substack{\uparrow \\ \text{since } i(\mathbb{C}\mathrm{P}^1) \text{ and } f(\mathbb{C}\mathrm{P}^1) \text{ are disjoint}}}{=} 0.
\end{aligned}$$

We have thus obtained a contradiction. □

67.6. **Is the algebraic intersection number a complete invariant?** It is natural to ask whether the converse to Proposition 67.11 (2) holds. Playing around with curves on a surface shows that this looks dubious. More precisely, consider the oriented curves X and Y on the surface of genus 2 that are illustrated in Figure 847. Their algebraic intersection number is zero, but it is not clear how one can make them disjoint.[1151]

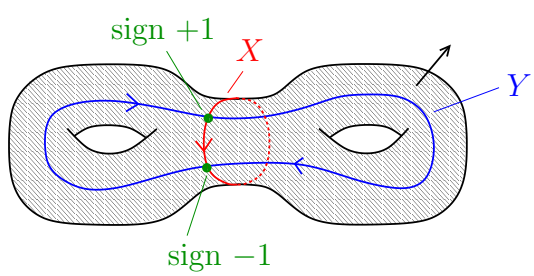


FIGURE 847.

[1150] We refer to page 1497 for the definition of a trivial tubular neighborhood.

[1151] Fun exercise: show that the curves cannot be made disjoint.

If one thinks for a while about the previous example, then one realizes that the issue in the example is that the manifold M is not simply connected. So it is more reasonable to demand some condition on the fundamental groups

Question 67.21. *Let M be an oriented m-dimensional manifold and let X and Y be complementary connected submanifolds of M. Suppose that $X \cdot Y = 0$ and that M is simply connected, does it follow that X and Y can be made disjoint?*

It turns out that the answer depends on the dimension of M.

Theorem 67.22. (Whitney) *Let M be an oriented simply connected manifold and let X and Y be complementary connected submanifolds of M.*

(1) *The answer to Question 67.21 is yes if* $\dim(X) \geq 3$ *and* $\dim(Y) \geq 3$.
(2) *The answer to Question 67.21 is yes if* $\dim(X) = 2$, $\dim(Y) \geq 3$ *and* $\pi_1(M \backslash Y) = 0$.

Remark. The fact that we have a positive answer to Question 67.21 in dimensions ≥ 6 turns out to be extremely useful, since, together with Theorem 67.10, it makes it possible to translate algebraic topological data, namely cup products, directly into geometric statements. This fact was a key ingredient in Hassler Whitney's proof that every n-dimensional manifold without boundary can be embedded into $\mathbb{R}^{2n}$, see Theorem 65.18. Furthermore, as explained in [Mi5], this fact played a crucial role in Steven Smale's proof of the Poincaré Conjecture in dimensions ≥ 5, see Theorem 28.22.

For dimensions ≥ 5 the theorem was proved by Hassler Whitney [Why2, Theorem 4] in 1944. An exposition of the proof is also given in [Kir3], [Sco, Chapter 1.5], [Mi5, Theorem 6.6], [GS, Theorem 9.2.7] or [RS, Theorem 5.12]. We only give a quick outline of the proof. Our main goal is to show where the restrictions on the dimensions enter.

Sketch of a proof of Theorem 67.22 (1). Let M be an oriented simply connected m-dimensional manifold and let X and Y be complementary connected submanifolds of M with $X \cdot Y = 0$ and with $\dim(X) \geq 3$ and $\dim(Y) \geq 3$. We do induction on the geometric intersection number $\#(X \cap Y)$. If $X \cap Y = \varnothing$, then there is nothing to prove. Now suppose that $X \cap Y \neq \varnothing$. Then there exist two intersection points P and Q of opposite signs. Since X is connected we obtain from Corollary 65.28 an embedded curve α on X from P to Q and since Y is connected we can similarly find an embedded curve β on Y from Q to P. Note that $\varphi := \alpha * \beta$ is in fact a loop. Since M is simply connected the curve $\varphi = \alpha * \beta \colon S^1 \to M$ is null-homotopic.

By Lemma 5.1 there exists a map $\Phi \colon \overline{B}^2 \to M$ with $\Phi|_{S^1} = \varphi$. Theorem 65.26 suggests[1152] that we can find a homotopy from Φ to a map $\Psi \colon \overline{B}^2 \to M$ such that $\Psi(S^1) = \Phi(S^1)$ and such that $\Psi \colon \overline{B}^2 \to M$ is an embedding.[1153] It follows from $\dim(X) \geq 3$ and $\dim(Y) \geq 3$ that $\dim(\Phi(\overline{B}^2)) + \dim(X) < \dim(M)$ and $\dim(\Phi(\overline{B}^2)) + \dim(Y) < \dim(M)$. Thus the

[1152]Here the exposition starts becoming rather vague.

[1153]Here there are some issues since the map $\alpha * \beta \colon S^1 \to M$ is not smooth at two points. In particular it is not quite clear what "embedding" is supposed to mean.

Transversality Theorem 67.2 suggests that we can find a further isotopy so that we end up a with an embedded disk D with $\partial D = |\varphi| = |\alpha| \cup |\beta|$ and such that the interior of the disk intersects neither X nor Y. One can now use this disk to find an isotopy of X that removes the two intersection points P and Q without creating new intersection points. The precise construction of the isotopy is rather delicate. In fact, as the perspicacious reader might have noticed, so far we have not yet used that P and Q have opposite signs. We need it in finding this last isotopy. We refer to the aforementioned references for the details. □

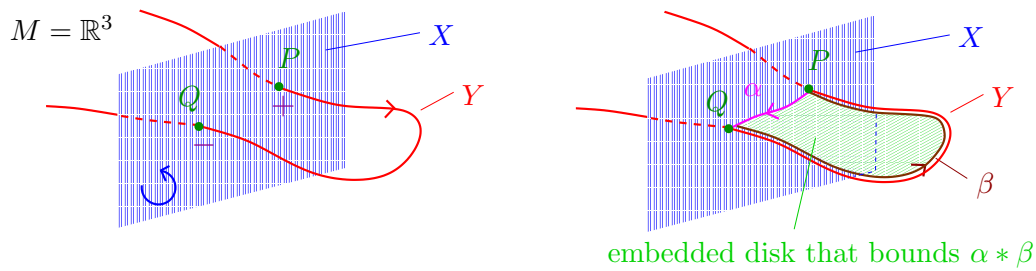


FIGURE 848. Illustration for the sketch of the proof of Theorem 67.22 (1).

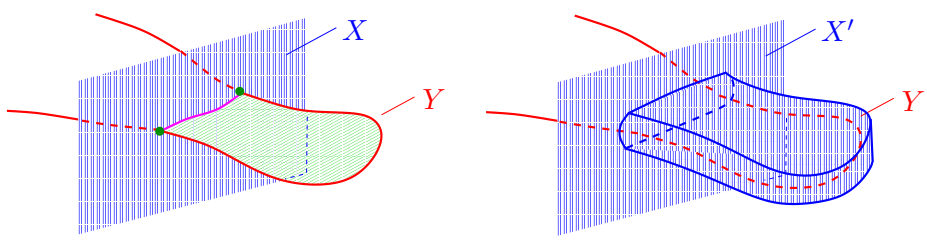


FIGURE 849. Second illustration for the sketch of the proof of Theorem 67.22 (1).

Remark. The approach taken in the proof of making submanifolds disjoint using embedded disks is often referred to as the "Whitney trick".

It turns out that we also have an affirmative answer to Question 67.21 in very low dimensions.

Theorem 67.23. *The answer to Question 67.21 is also yes if the dimension of the manifold is* ≤ 3.

Proof. The theorem is trivial for 1-dimensional manifolds. For 2-dimensional manifolds the proof is an amusing exercise using the fact, proved in Theorem 14.14, that every simply connected 2-dimensional manifold is diffeomorphic to S^2 or to $\overline{B}^2$.

For 3-dimensional manifolds the proof is similar to the proof in dimensions ≥ 5 except that one also needs to make use of the Loop Theorem (see [Pap] and [Hem]) to find the

embedded disk needed for the "Whitney trick". We leave it as a slightly challenging exercise to fill in the details in the 3-dimensional setting. □

So we are stilling missing 4-dimensional manifolds. Here the situation becomes tricky. It was known by 1961, see [KeM2], that the approach of the proof of Theorem 67.22, i.e. the Whitney trick, does not work for 4-dimensional manifolds. Later, see [Lacke], it was also shown that the Whitney trick does not work for topological 4-dimensional manifolds either.

In the 4-dimensional setting it is not only the case that the Whitney trick does not work. But more fundamentally, Question 67.21 has in fact a negative answer. For example it is known that there exist closed simply connected 4-manifolds and embedded spheres X, Y with zero intersection number that cannot be made disjoint by a smooth isotopy, in fact they cannot even be made disjoint by an isotopy that is not necessarily smooth, see e.g. [As, Theorem 1.2] for details. (More examples for the failure of the Whitney trick in dimension 4 are given in [Strl, Section 15.1] and [Gil].) As is pointed out in [As, Proposition 3.9], it is a consequence of the Freedman disk embedding theorem [FQ, Theorem 5.1.A] that under even stronger hypotheses the algebraic intersection number is in fact a complete invariant in the setting of topological manifolds.

The failure of the Whitney trick is at the heart of all the peculiar behavior of 4-dimensional manifolds which we had already encountered in Theorem 28.27.

67.7. **Higher dimensional analogue of Theorem 67.10.** We start out with the following definition which at first glance might look rather artificial.

Definition.

(1) An orientation for a 0-dimensional manifold M is a map $\epsilon\colon M \mapsto \{\pm 1\}$.
(2) Given an oriented 1-dimensional manifold W we define ∂W as the oriented 0-dimensional manifold given by the set ∂W and the sign $+1, -1$ assigned to the points depending on whether or not the orientation of W sticks "out" or "in" of W.[1154]
(3) The *fundamental class* of a compact oriented 0-dimensional manifold M consisting of points $\{P_1, \dots, P_k\}$ is defined as the homology class
$$\sum_{i=1}^{k} \epsilon(P_i) \cdot [P_i] \in H_0(M; \mathbb{Z}).$$
(4) Let N be an oriented k-dimensional topological manifold and let $M = \{P\}$ be a 0-dimensional manifold M consisting of a single point P that is equipped with an orientation $\epsilon \in \{-1, 1\}$. We equip the product topological manifolds $\{P\} \times N$ and $N \times \{P\}$ with the orientation that agrees with the orientation of N if $\epsilon(P) = +1$ and that is the opposite orientation if $\epsilon(P) = -1$. Similarly we define the product orientation of the product of any oriented topological manifold with any oriented 0-dimensional manifold.

[1154] We leave it to the reader to make this super-rigorous.

Example.

(1) The definition of the orientation of the boundary of an oriented 1-dimensional manifold is illustrated in Figure 850

FIGURE 850.

(2) The boundary of the 1-dimensional manifold $[0,1]$ with the usual orientation is given by $-\{0\} \cup \{1\}$.

(3) One can easily verify that the statement regarding product orientations from Proposition 3.42 (5) also holds in this context, i.e. given a 1-dimensional manifold M with boundary and a manifold N without boundary we have $\partial(M \times N) = (\partial M) \times N$ as *oriented* manifolds. In particular we have

$$\partial([0,1] \times N) \;=\; \underbrace{-\{0\} \times N}_{=\{0\}\times(-N)} \cup \{1\} \times N.$$

Let M be a compact oriented m-dimensional manifold, let $X \subset M$ be a closed oriented submanifold of codimension k and let Y be a compact oriented proper submanifold of the complementary dimension $m-k$. We suppose that X and Y are transverse. By assigning to each intersection point in $X \cap Y$ the sign introduced on page 1560 we can view $X \cap Y$ as a 0-dimensional oriented submanifold of M. With this convention we can reformulate the conclusion of Theorem 67.10 as follows:

Theorem 67.24. *With the above notation we have*

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \;=\; \mathrm{PD}_M([X \cap Y]) \;\in\; H^{k+l}(M, \partial M; \mathbb{Z}).$$

Proof. Using Lemma 59.9 one can easily verify that it suffices to prove the theorem for each connected component of M separately. Put differently, without loss of generality we can assume that M is connected. We denote by $P_1, \dots, P_n$ the points in $X \cap Y$ and we denote by $\sigma_1, \dots, \sigma_n \in \{\pm 1\}$ the corresponding signs. Now we calculate that

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \underset{\substack{\uparrow \\ \text{by Theorem 67.10 and since } M \text{ is connected}}}{=} (X \cdot Y) \cdot [M]^* = \sum_{i=1}^{n} \sigma_i \cdot [M]^* \underset{\substack{\uparrow \\ \text{see page 1414}}}{=} \sum_{i=1}^{n} \mathrm{PD}_M(\sigma_i \cdot [\{P_i\}]) = \mathrm{PD}_M([X \cap Y]).$$

□

The formulation of Theorem 67.24 has the nice feature that with some effort one can also generalize it to submanifolds that are not of complementary dimensions. Before we can formulate this generalization we need to formulate the following lemma.

Lemma 67.25. *Let M be an oriented m-dimensional manifold, let $X \subset M$ be an oriented proper submanifold and let Y be an oriented proper submanifold. Suppose that X and Y intersect transversally. By Lemma 3.43 we know that $X \cap Y$ is a submanifold. Given $P \in X \cap Y$ we have a natural short exact sequence induced by the various inclusion maps:*

$$0 \;\to\; T_P(X \cap Y) \;\to\; T_P X \oplus T_P Y \;\xrightarrow{\rho}\; T_P M \;\to\; 0.$$

We pick a set of vectors $\mathcal{C}$ in $T_P X \oplus T_P Y$ such that $\rho(\mathcal{C})$ is a positive basis for $T_P M$. We say that a basis $\mathcal{B}$ for $T_P(X \cap Y)$ is positive *if the basis $\{\mathcal{B}, \mathcal{C}\}$ is a positive basis for $T_P X \oplus T_P Y$. This definition is independent of the choices and it turns $X \cap Y$ into an oriented submanifold of M.*

Proof. We leave the elementary proof to the reader. □

Example. Let $n \in \mathbb{N}$ and let $0 \le k \le l \le n$. We consider $\mathbb{C}\mathrm{P}^n$ with the complex submanifolds

$$\begin{aligned} A &= \{[z_0 : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_0, \ldots, z_l) \in \mathbb{C}^{l+1} \setminus \{0\}\} \\ B &= \{[0 : \cdots : 0 : z_k : \cdots : z_n] \,|\, (z_k, \ldots, z_n) \in \mathbb{C}^{n+1-k} \setminus \{0\}\} \end{aligned}$$

and

$$C = \{[0 : \cdots : 0 : z_k : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_k, \ldots, z_l) \in \mathbb{C}^{l+1-k} \setminus \{0\}\}.$$

As usual we equip $\mathbb{C}\mathrm{P}^n$ and A, B, C with the natural orientations coming from Proposition 21.4. Then $A \cap B = C$ and one can easily show that the orientation of $A \cap B$ equals the orientation of C.

The following theorem is a generalization of Theorem 67.24 and thus also of Theorem 67.10.

Theorem 67.26. *Let M be a compact oriented m-dimensional manifold. Furthermore let $X \subset M$ be a closed oriented submanifold of codimension k and let Y be a compact oriented proper submanifold of codimension l. Then*[1155]

$$\underbrace{\mathrm{PD}_M([X])}_{\in H^k(M,\partial M)} \cup \underbrace{\mathrm{PD}_M([Y])}_{\in H^l(M)} \;=\; \mathrm{PD}_M([X \cap Y]) \;\in\; H^{k+l}(M, \partial M; \mathbb{Z}).$$

Proof. A proof is given in [Bre, Theorem VI.11.9]. □

Remark. Theorem 67.26 can be summarized as saying that cup products correspond to taking intersections of submanifolds. With this point of view, the symbol "$\cup$" for the cup product does not seem to be a great choice.

Examples.

[1155]The sign should be taken with a slight grain of salt. It is conceivable that the sign depends on the various dimensions.

(1) Let M be the manifold that is the complement in S^3 of the three open solid tori and the open ball that are shown in Figure 851. Furthermore let X and Y be the surfaces shown in that figure. Basically the same argument as on page 1568 shows that $[X]$ and $[Y]$ are non-trivial in $H_2(M;\mathbb{Z})$. On the other hand the class $[X \cap Y] \in H_1(M;\mathbb{Z})$ is clearly trivial. Therefore we deduce from Theorem 67.26 that $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) = 0$.

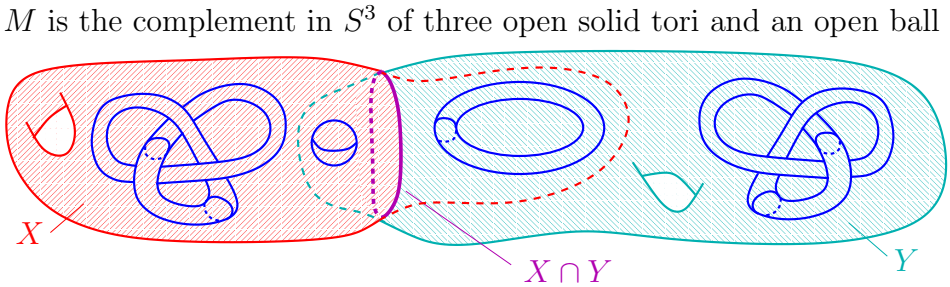


FIGURE 851.

(2) Using Theorem 67.26 and using the example preceding that theorem one can fairly easily give a completely geometric calculation of the cup product of $\mathbb{C}\mathrm{P}^n$.

Remark. Using Theorem 67.26 one can easily modify the proof of Proposition 67.20 to show that given any $n \geq 2$ every tubular neighborhood of $\mathbb{C}\mathrm{P}^{n-1}$ in $\mathbb{C}\mathrm{P}^n$ is non-trivial.

Exercises for Chapter 67.

Exercise 67.1. Let A, B be two 1-dimensional oriented submanifolds of S^2 with $A \cdot B = 0$. Sketch a proof for the claim that A and B can be made disjoint via isotopies of A and B.

Exercise 67.2. Let M be a closed oriented connected m-dimensional manifold. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering.

(a) Let X, Y be a complementary pair of submanifolds of M. We write $\widetilde{X} := p^{-1}(X)$ and $\widetilde{Y} := p^{-1}(Y)$. What is the relationship between $X \cdot Y$ and $\widetilde{X} \cdot \widetilde{Y}$? It suffices to give a short justification.
(b) Let X be a closed oriented submanifold of M. What is the relationship between $\mathrm{PD}_{\widetilde{M}}([\widetilde{X}])$ and $p^*(\mathrm{PD}_M([X])$? Justify your answer.
(c) Reprove (a) using (b) and Theorem 67.10.

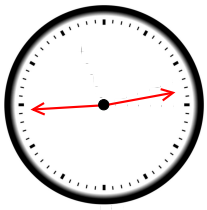
Exercise 67.3. Let M be an oriented m-dimensional manifold, let X be a k-dimensional oriented proper submanifold and let Y be an $(m-k)$-dimensional oriented proper submanifold of M. We suppose that $X \cap Y \cap \partial M = \varnothing$. What is the relationship between the algebraic intersection number $X \cdot Y$ in M and $DX \cdot DY$ in DM?
Here, as always, given a manifold N we denote by $DN = N \cup_{\partial N = \partial N'} N'$ the double of

N. We view DN as an oriented manifold where the orientation is chosen such that the orientation of DN agrees with the orientation on N.

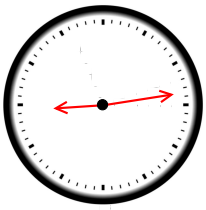
Exercise 67.4. Let $C \subset S^2 \times [1,2]$ be a closed surface. Suppose that there exists a $P \in S^2$ such that C intersects $\{P\} \times [1,2]$ transversally and that the intersection consists of a single point. Show that C intersects every path $\gamma\colon [0,1] \to S^2 \times [a,b]$ from a point in $S^2 \times \{a\}$ to a point in $S^2 \times \{b\}$.

Exercise 67.5. By accident you ended up buying a clock where both hands have the same length, so you can not distinguish the hand which shows the hour from the hand that indicates the minute. Fortunately, very often you can still tell the precise time. For example, in the configuration in Figure 852 it does not make sense for the hand that is indicating the hour to be just *before* 9 whereas the hand showing the minute is at 15.

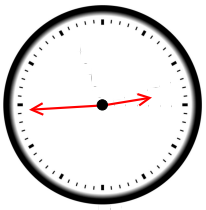
Of course there will be some configurations when it is indeed impossible to tell which hand is which. How many such confusions moments will there be in a 12 hour period? Hint: Consider a suitable map $\gamma\colon S^1 \to S^1 \times S^1$.



clock where both hands have the same length



this makes no sense



the time is 2:44

FIGURE 852.

Exercise 67.6. Let C and D be closed curves in $\mathbb{R}^2$ that intersect transversally. Give a rigorous proof that the number of intersection points between C and D is an even number.

68. Intersections of cycles and the cup product

In this chapter we generalize some of the results from the previous chapter. In particular we introduce the notion of "transverse cycles" in a topological manifold and we define the "algebraic intersection number" of transverse cycles. Afterwards we will show in Theorem 68.5 how this algebraic intersection number relates to the cup product. Later on we will see that this theorem greatly generalizes Theorem 67.10.

In the literature the pictorial "algebraic intersection number of transverse cycles" is often used as a substitute for the more forbidding cup product. This means that many authors implicitly use Theorem 68.5. To the best of my knowledge there is no published rigorous proof for Theorem 68.5, hence we decided to fill this gap in the literature.[1156]

Since the first part of the chapter is at times quite technical we start out with the following motivating question.

Question 68.1. *As shown in Figure 853 we consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm = (0, \pm 1)$. Do there exist disjoint paths α and β such that α connects P_- to P_+ and β connects Q_- to Q_+?*

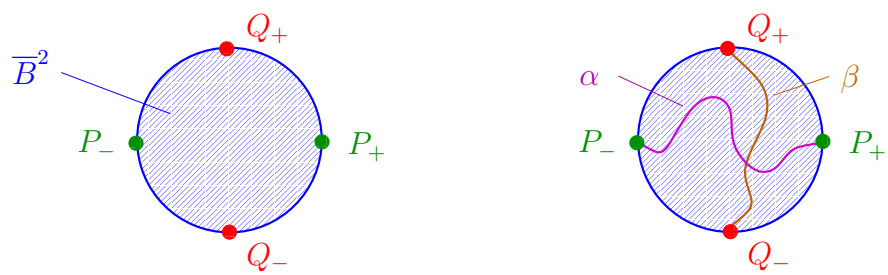


Figure 853.

It is pretty clear that the answer should be no. In fact once we have developed all the tools we will find it quite easy to answer Question 68.1 in the negative.

68.1. Intersections of cycles.

First we fix some notation.

Notation and Convention.

(1) Given a map $f\colon Y \to X$ between topological spaces we write $|f| = f(Y)$.
(2) We write $I = [-1, 1]$.
(3) Let M be an oriented topological manifold. We equip any codimension zero submanifold with the orientation coming from Lemma 45.28.

The following definition generalizes the concept of transversality from the last chapter.

[1156] In mathematics a statement that "everybody in a field knows", but for which no proof was ever written down, is called "folklore". It is a rather unfortunate aspect of mathematics that all too often statements which are very useful and which sound about right get used without bothering to write down a proof and without bothering to check whether a proof exists in the literature.

Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. We say f and g *are transverse* at a point $P \in |f| \cap |g|$ if the following happens: there exists a compact submanifold Z of M and a homeomorphism $\Phi\colon Z \to I^k \times I^{m-k}$ such that:

(1) We have $P \in Z$ and we have $\Phi(P) = (0,0) \in I^k \times I^{m-k}$.
(2) The map Φ restricts to a homeomorphism $|f| \cap Z \to I^k \times \{0\} = I^k$,
(3) The map Φ restricts to a homeomorphism $|g| \cap Z \to \{0\} \times I^{m-k} = I^{m-k}$,
(4) the restriction of f to a map $f^{-1}(|f| \cap Z) \to |f| \cap Z$ is a homeomorphism, we write $X_P := f^{-1}(|f| \cap Z)$ and we denote the homeomorphism $X_P \to |f| \cap Z$ by $\widetilde{f}$,
(5) the restriction of g to a map $g^{-1}(|g| \cap Z) \to |g| \cap Z$ is a homeomorphism, we write $Y_P := g^{-1}(|g| \cap Z)$ and we denote the homeomorphism $Y_P \to |g| \cap Z$ by $\widetilde{g}$.

If these conditions are satisfied and if M, X, Y are oriented, then we refer to the degree of the map [1157] [1158] [1159] [1160] [1161]

$$\begin{array}{c}
\overbrace{H_k(X, X\setminus\{f^{-1}(P)\})}^{=\mathbb{Z}} \otimes \overbrace{H_{m-k}(Y, Y\setminus\{g^{-1}(P)\})}^{=\mathbb{Z}} \\
\cong \uparrow \\
H_k(X_P, X_P\setminus\{f^{-1}(P)\}) \otimes H_{m-k}(Y_P, Y_P\setminus\{g^{-1}(P)\}) \\
\cong \downarrow \widetilde{f}_* \otimes \widetilde{g}_* \\
H_k(|f|\cap Z, (|f|\cap Z)\setminus\{P\}) \otimes H_{m-k}(|g|\cap Z, (|g|\cap Z)\setminus\{P\}) \\
\cong \downarrow \Phi_* \otimes \Phi_* \\
H_k(I^k, I^k\setminus\{0\}) \otimes H_{m-k}(I^{m-k}, I^{m-k}\setminus\{0\}) \\
\cong \downarrow \times \\
H_m(I^m, I^m\setminus\{0\}) \\
\cong \downarrow (\Phi_*)^{-1} \\
H_m(Z, Z\setminus\{P\}) \\
\cong \downarrow \\
\underbrace{H_m(M, M\setminus\{P\})}_{=\mathbb{Z}}
\end{array}$$

as the *sign at* P. We refer to Figure 854 for an illustration. We say the maps f and g are *transverse* if they are transverse at all points in $|f| \cap |g|$.

On several occasions we will employ the following, almost self-explaining convention:

Convention. We continue with the notation from the previous definition. If X is a submanifold of M, then often we do not distinguish between X and the inclusion map. For example we will say that X is transverse to $g\colon Y \to M$ at a point P. The same convention applies to Y being a submanifold of M.

Lemma 68.2. *Let M be an oriented m-dimensional topological manifold, let X be an oriented k-dimensional topological manifold and let Y be an oriented $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be maps and let $P \in |f| \cap |g|$ be a transverse intersection point of f and g.*

[1157] It follows from (1) together with Proposition 32.1 that $P \in M \setminus \partial M$. Similarly it follows from (2) and (4) together with Proposition 32.1 that $f^{-1}(P)$ consists of a single point of X that lies in $X \setminus \partial X$. The analogous statement holds for $g^{-1}(P)$.

[1158] We use the orientations on X, Y and M and Footnote 1157 to identify the groups $H_k(X, X\setminus\{f^{-1}(P)\})$, $H_{m-k}(Y, Y\setminus\{g^{-1}(P)\})$ and $H_m(M, M\setminus\{P\})$ with $\mathbb{Z}$. Furthermore we identify $\mathbb{Z}\otimes\mathbb{Z}$ with $\mathbb{Z}$ via the natural isomorphism $m \otimes n \mapsto m \cdot n$.

[1159] Note that by the Excision Theorem 45.8 the first and the last map are inclusion induced isomorphisms.

[1160] The map $\times\colon H_k(I^k, I^k\setminus\{0\}) \times H_{m-k}(I^{m-k}, I^{m-k}\setminus\{0\}) \to H_m(I^m, I^m\setminus\{0\})$ denotes the cross product that we introduced on page 1316. This map is an isomorphism by the discussion on page 1317.

[1161] A priori this definition of the sign depends on the choice of the chart Φ. We leave it as a slightly lengthy exercise to show that the definition of the sign is independent of this choice.

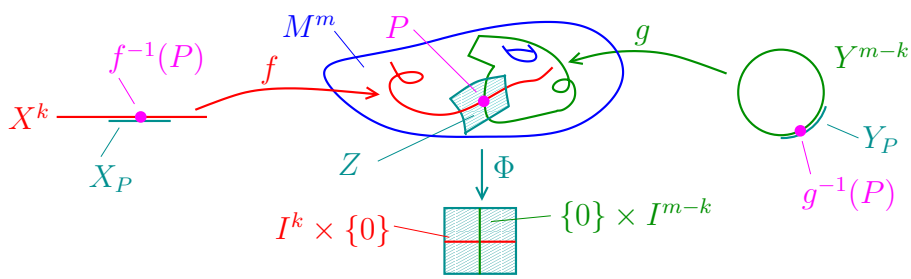


FIGURE 854.

(a) *We can find Z and Φ as in the above definition with the following two extra properties:*
  (6) *the homeomorphism $X_P \to I^k \times \{0\}$ is orientation-preserving, and*
  (7) *the homeomorphism $Y_P \to \{0\} \times I^{m-k}$ is orientation-preserving.*

(b) *If* (6) *and* (7) *are satisfied, then*

$$\Phi_*([Z]) \;=\; \textit{sign at } P \cdot [I^m] \;\in\; H_m(I^m, \partial I^m).$$

Proof.

(a) It is straightforward to see that it suffices to postcompose the homeomorphism $\Phi\colon Z \to I^k \times I^{m-k}$ from the definition of a transverse intersection point with the homeomorphism

$$(x_1, \ldots, x_k, y_1, \ldots, y_{m-k}) \;\mapsto\; (\epsilon \cdot x_1, x_2, \ldots, x_k, \mu \cdot y_1, y_2, \ldots, y_{m-k}),$$

for suitable signs $\epsilon, \mu \in \{-1, 1\}$, to obtain the desired homeomorphism. We leave it to the reader to fill in the details.

(b) This statement follows easily from the definitions together with Propositions 56.11 and 47.3. □

Lemma 68.3. *We continue with the above notation. If $f\colon X \to M$ and $g\colon Y \to M$ are transverse and if X and Y are compact, then $|f| \cap |g|$ is finite.*

Proof. A slight variation of the proof of Lemma 67.6 (2) shows that $|f| \cap |g|$ is indeed finite. We leave the details to the reader. □

Using Lemma 68.3 we can now introduce the following definition.

Definition. Let M be an oriented m-dimensional topological manifold. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g are transverse, then we define the *algebraic intersection number of f and g* to be

$$f \cdot g \;:=\; \sum_{P \in |f| \cap |g|} \text{sign at } P.$$

If X is actually a submanifold of M and if $f\colon X \to M$ is the inclusion map, then we write $X \cdot g$ instead of $f \cdot g$. The same way we proceed if Y is actually a submanifold of M.

The following lemma shows that the above definition is a generalization of the algebraic intersection number of submanifolds

Lemma 68.4. *Let M be an oriented m-dimensional manifold, let X be an oriented k-dimensional manifold and let Y be an oriented $(m-k)$-dimensional manifold. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. If the submanifolds are transverse in the sense of the definition on page 207, then the maps $i\colon X \to M$ and $j\colon Y \to M$ are transverse*[1162] *are and*

$$X \cdot Y \;=\; i \cdot j.$$

Sketch of proof. Suppose that the submanifolds X and Y are transverse. It follows almost immediately from Lemma 3.43 that the inclusion maps $i\colon X \to M$ and $j\colon Y \to M$ are transverse. Furthermore, after unraveling the definitions, it follows easily from Proposition 56.11 and the orientation conventions of Proposition 45.22 that $X \cdot Y = i \cdot j$. □

Now we turn to the study of singular chains in a topological manifold. To be on the safe side in our subsequent definitions we introduce the following convention.

Convention.

(1) Let X be a topological space. When we write "let $\sigma = c_1 \cdot \sigma_1 + \cdots + c_r \cdot \sigma_r$ be a singular m-chain in X" it is is understood that the σ_i are pairwise different and that all the c_i are non-zero. We write $|\sigma| = |\sigma_1| \cup \cdots \cup |\sigma_r|$.
(2) Given $n \in \mathbb{N}_0$ we equip the topological manifold Δ^n with the orientation given by the discussion on page 935 together with Proposition 45.22. In particular, given an oriented m-dimensional topological manifold M and singular simplices $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ it makes sense to say that they are transverse, and if they are transverse, we have an algebraic intersection number $\sigma \cdot \tau$.

Definition. Let M be an oriented m-dimensional topological manifold. Furthermore let $\sigma = c_1 \cdot \sigma_1 + \cdots + c_r \cdot \sigma_r \in C_k(M)$ be a singular k-chain and $\tau = d_1 \cdot \tau_1 + \cdots + d_s \cdot \tau_s \in C_{m-k}(M)$ be a singular $(m-k)$-chain. We say σ and τ *are transverse* if the following two conditions are satisfied:

(1) for all $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$ the maps $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k} \to M$ are transverse, and
(2) for all $(i,j) \neq (i',j') \in \{1, \dots, r\} \times \{1, \dots, s\}$ we have
$$(|\sigma_i| \cap |\tau_j|) \cap (|\sigma_{i'}| \cap |\tau_{j'}|) \;=\; \varnothing.$$
Put differently, any intersection point is the intersection point of a unique pair of singular simplices.

[1162]Does the converse to this statement hold? More precisely, if the maps $i\colon X \to M$ and $j\colon Y \to M$ are transverse in the sense of the definition on page 1591, does it follows that the submanifolds transverse in the sense of the definition on page 207?

If this is the case, then we define the *algebraic intersection number of* σ *and* τ to be

$$\sigma \cdot \tau \ := \ \sum_{i,j} (c_i \cdot d_j) \cdot (\sigma_i \cdot \tau_j).$$

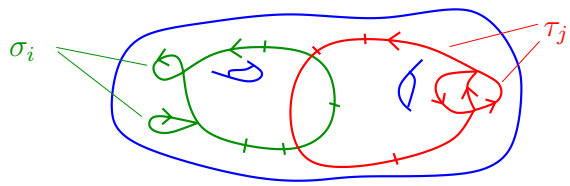


FIGURE 855.

Before we can state the main theorem of this chapter we recall the formulation of the Poincaré Duality theorem for (not necessarily compact) topological manifolds without boundary. Thus let W be an oriented m-dimensional topological manifold without boundary and let $K \subset W$ be a compact subset. By Theorem 46.10 there exists a unique lift of the orientation classes of W to a class in $H_m(W, W \setminus K)$. Given any $k \in \mathbb{N}_0$ we denote by

$$- \cap W \colon H^k(W, W \setminus K) \ \to \ H_{m-k}(W)$$

the map that is given by capping with this unique lift. Lemma 61.14 says that these maps define a well-defined map

$$- \cap W \colon H_c^k(W) \ \to \ H_{m-k}(W).$$

Furthermore the Poincaré Duality Theorem 61.15 says that this map is an isomorphism. We denote by

$$\mathrm{PD}_W \colon H_{m-k}(W) \ \to \ H_c^k(W)$$

the inverse map. Finally we remind the reader that given a topological space X we denote by $\epsilon_X = \epsilon \colon H_0(X) \to \mathbb{Z}$ the augmentation map.

Now we can formulate the main theorem of this chapter.

Theorem 68.5. *Let M be an oriented m-dimensional topological manifold without boundary. Let $\sigma \in C_k(M)$ and $\tau \in C_{m-k}(M)$ be two cycles. If σ and τ are transverse, then*[1163]

$$\epsilon_M\Big(\big(\overbrace{\mathrm{PD}_M([\sigma])}^{\in H_c^{m-k}(M)} \underbrace{\cup}_{\in H_c^m(M)} \overbrace{\mathrm{PD}_M([\tau])}^{\in H_c^k(M)}\big) \cap M\Big) \ = \ \sigma \cdot \tau.$$

[1163]We refer to page 1361 for the definition of the cup product on cohomology with compact support.

68.2. **Proof of Theorem 68.5.** Before we can provide the proof of Theorem 68.5 we need to prove several lemmas.

Definition. Let X be a topological space.

(1) Let A and B be disjoint subsets of X. We say U and V are *separating open neighborhoods for A and B* if U and V are disjoint, if U is an open neighborhood of A and if V is an open neighborhood of B.
(2) A topological space X is called *normal* if any two disjoint closed subsets A and B admit separating open neighborhoods.

Proposition 68.6.

(1) *Every topological manifold is normal.*
(2) *Let M be a topological manifold and let A and B be two closed disjoint subsets of M. If $U' \subset \partial M$ and $V' \subset \partial M$ are separating neighborhoods for $A \cap \partial M$ and $B \cap \partial M$, then there exist separating neighborhoods U and V of A and B such that $U \cap \partial M = U'$ and $V \cap \partial M = V'$.*

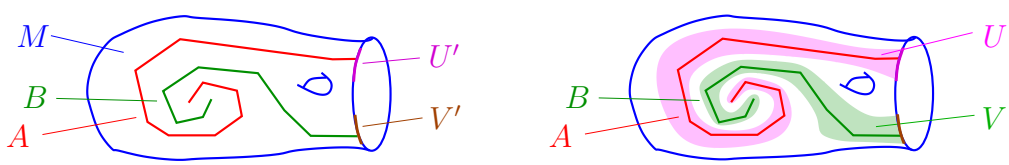


FIGURE 856. Illustration of Proposition 68.6.

Remark. For many situations it suffices to find separating open neighborhoods for *compact* subsets A and B of a Hausdorff space X. In this situation the existence of such open neighborhoods is an entertaining little exercise.

Sketch of proof.

(1) First note that by Theorem 3.10 the topological space M is metrizable, i.e. M admits a metric such that the topology from the metric agrees with the given topology. With some effort one can obtain the desired statement from this information. Alternatively we refer to [Wil, Theorems 20.9 and 20.10] for a proof that every metrizable topological space is normal.
(2) Let M be a topological manifold and let A and B be two closed disjoint subsets of M. We pick a collar neighborhood $\partial M \times [0,1]$ as given by the Topological Collar Neighborhood Theorem 45.3.

    Next suppose that we are given separating neighborhoods $U' \subset \partial M$ and $V' \subset \partial M$ for $A \cap \partial M$ and $B \cap \partial M$. By Proposition 32.1 the complement $M \setminus \partial M$ is again a manifold. By (1) it is normal. Thus there exist separating open neighborhoods $\widetilde{U}$ and $\widetilde{V}$ for the closed subsets $A \cap (M \setminus \partial M)$ and $B \cap (M \setminus \partial M)$. It follows from Lemma 3.6 (1) that $\widetilde{U}$ and $\widetilde{V}$ are open subsets of M. It is now straightforward

to verify that $U := \widetilde{U} \cup U' \times [0,1)$ and $V := \widetilde{V} \cup V' \times [0,1)$ have all the desired properties. $\square$

Lemma 68.7. *Let M be an oriented m-dimensional topological manifold without boundary and let $U \subset M$ be an open subset. Let $i\colon U \to M$ be the inclusion map. For any compact subset $K \subset U$ and any $k \in \mathbb{N}_0$ the map*

$$i^*\colon H^{m-k}(M, M \setminus K) \;\to\; H^{m-k}(U, U \setminus K)$$

is an isomorphism and the following diagram commutes:[1164]

$$\begin{tikzcd}
H_c^{m-k}(U) \arrow[rr, "\cap U = \mathrm{PD}_U^{-1}", "\cong"'] \arrow[d, "i_*"] & & H_k(U) \arrow[d, "i_*"] \\
 & H^{m-k}(U, U\setminus K) \arrow[ul] \arrow[ur, "\cap U"'] \arrow[d, "(i^*)^{-1}"] & \\
H_c^{m-k}(M) \arrow[rr, "\cap M = \mathrm{PD}_M^{-1}", "\cong"'] & & H_k(M). \\
 & H^{m-k}(M, M\setminus K) \arrow[ul] \arrow[ur, "\cap M"'] &
\end{tikzcd}$$

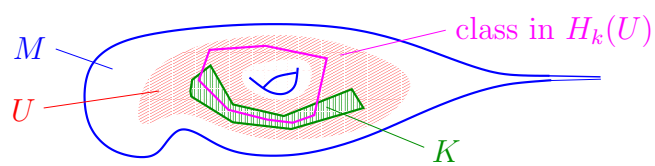


FIGURE 857. Illustration for Lemma 68.7.

Proof. We make the following comments and observations:

(i) By excision, see page 1192, the map $i^*\colon H^{m-k}(M, M \setminus K) \to H^{m-k}(U, U \setminus K)$ is an isomorphism.

(iii) The triangles on the top and the bottom commute by definition of PD_U and PD_M, see Lemma 61.14.

(iv) The square to the left commutes by definition of the map $i_*\colon H_c^{m-k}(U) \to H_c^{m-k}(M)$, see page 1263.

(v) It follows from Lemma 59.8 and Lemma 46.11 (2) that the square to the right commutes.

(vi) The rectangle in the back commutes by Lemma 61.21. $\square$

As in the case of Theorem 67.10 it is perhaps instructive to first prove Theorem 68.5 in the special case that the cycles do not intersect.

[1164] On page 135 we had mentioned that every finite graph can be realized by a spatial graph in $\mathbb{R}^3$. This has the helpful corollary that every finite diagram can in fact be drawn on paper together with indications for 3D-behavior.

Proof of Theorem 68.5 for disjoint cycles. Let M be an oriented m-dimensional topological manifold without boundary. Let $\sigma \in C_k(M)$ and $\tau \in C_{m-k}(M)$ be two cycles such that $|\sigma| \cap |\tau| = \varnothing$. We have to show that $\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]) = 0$.

By Proposition 68.6 we can find separating open neighborhoods U and V for $|\sigma|$ and $|\tau|$. Given a compact subset J of M and given $i \in \mathbb{N}_0$ we denote by $\varphi_J \colon H^i(M, M \setminus J) \to H^i_c(M)$ the natural map.

Claim.

(1) There exists a compact subset $K \subset U$ and some $\alpha \in H^{m-k}(M, M \setminus K)$ such that $\varphi_K(\alpha) = \mathrm{PD}_M([\sigma])$.
(2) There exists a compact subset $L \subset V$ and some $\beta \in H^k(M, M \setminus L)$ such that $\varphi_L(\beta) = \mathrm{PD}_M([\tau])$.

We consider $\mathrm{PD}_U([\sigma]) \in H^{m-k}_c(U)$. By definition of cohomology with compact support there exists a compact subset K of U such that $\mathrm{PD}_U([\sigma])$ lies in the image of the natural map $H^{m-k}(U, U \setminus K) \to H^{m-k}_c(U)$. The first statement of the claim now follows from Lemmas 68.7. The proof of the second statement is evidently basically the same. This concludes the proof of the claim.

We consider the following diagram

$$\begin{tikzcd}
H^{m-k}_c(M) \times H^k_c(M) \arrow[rr, "\cup"] & & H^m_c(M) \\
H^{m-k}(M, M\setminus K) \times H^k(M, M\setminus L) \arrow[u, "\varphi_K \times \varphi_L"] \arrow[rr, "\cup"] & & H^m(M, \underbrace{M\setminus K \cup M\setminus L}_{=M}). \arrow[u, "\varphi_M"] \\
& (\mathrm{PD}_M([\sigma]), \mathrm{PD}_M([\tau])) & \\
& (\alpha, \beta) \arrow[u, mapsto] &
\end{tikzcd}$$

The diagram commutes by the definition of the cup product on cohomology with compact support, see page 1361. It follows from the observation that the lower right group is zero that the cup product of $\mathrm{PD}_M([\sigma])$ and $\mathrm{PD}_M([\tau])$ is zero. $\square$

Convention. Let $n \in \mathbb{N}$. We use the homeomorphism from Proposition 1.47 (2) to make the identification $I^n = \overline{B}^n$ where $(\frac{1}{2}, \dots, \frac{1}{2})$ gets identified with the origin. Furthermore we use the homeomorphism from Lemma 29.1 to identify $\overline{B}^n$ with Δ^n. Summarizing we have canonical identifications $I^n = \overline{B}^n = \Delta^n$.

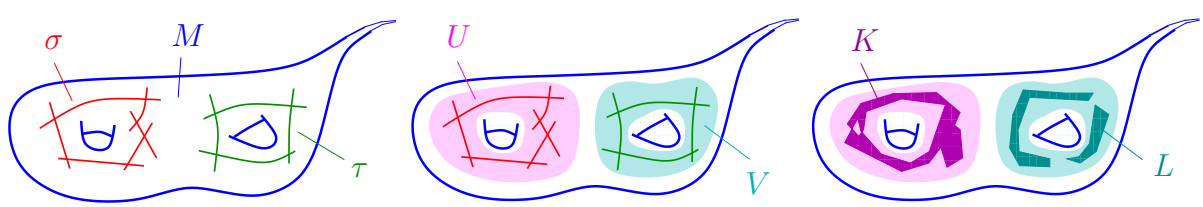


FIGURE 858. Illustration for the proof of Theorem 68.5 for disjoint cycles.

Lemma 68.8. *Let X be a compact oriented, n-dimensional topological manifold. Furthermore let $\varphi_i\colon I^n = \Delta^n \to X \setminus \partial X$, $i = 1, \dots, r$, be orientation-preserving topological embeddings such that the images are disjoint. We write $W = X \setminus (\varphi_1(\overset{\circ}{I}{}^n) \cup \dots \cup \varphi_r(\overset{\circ}{I}{}^n))$. Given any representative $\mu \in C_{n-1}(\partial X)$ of the fundamental class of ∂X there exists a singular n-chain $\sigma \in C_n(W)$ such that*

$$\partial(\sigma + \varphi_1 + \dots + \varphi_r) = \mu \quad \text{and with} \quad [\sigma + \varphi_1 + \dots + \varphi_r] \ = \ [X].$$

Example. We will employ Lemma 68.8 to the case that $X = \Delta^n$ itself is the standard n-simplex. This situation is illustrated in Figure 859.

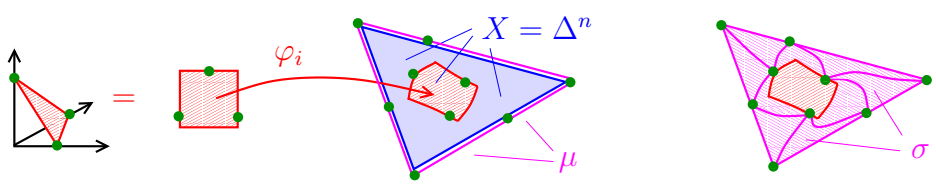


FIGURE 859.

Proof. Note that $\partial W = \varphi_1(\partial I^n) \cup \dots \cup \varphi_r(\partial I^n) \cup \partial X$. Since the φ_i are orientation-preserving we obtain from Proposition 46.26 that $-\partial\varphi_1 - \dots - \partial\varphi_r + \mu \in C_{n-1}(\partial W)$ represents the fundamental class of ∂W. Again by Proposition 46.26 we know that there exists a singular n-chain $\sigma \in C_n(W)$ with $\partial\sigma = -\partial\varphi_1 - \dots - \partial\varphi_r + \mu$. It follows from Proposition 46.8 that $\sigma + \varphi_1 + \dots + \varphi_r$ represents the fundamental class of X. □

The formulation of the following, rather technical lemma, might initially be somewhat intimidating. It can be viewed as a cousin to Lemma 67.12.

Lemma 68.9. *Let M be an oriented m-dimensional topological manifold, let Z be a compact codimension zero submanifold with a decomposition $\partial Z = A \cup B$ of the boundary into compact submanifolds with $\partial A = \partial B = A \cap B$. Let $U \subset M$ be an open subset with $U \cap \partial Z \subset A$ and let $K \subset U$ be a compact subset. We consider the three inclusion maps $p\colon (U, \varnothing) \to (U, U \setminus \overset{\circ}{Z})$, $g\colon (Z \cap U, A \cap U) \to (Z, A)$ and $z\colon (Z \cap U, A \cap U) \to (U, U \setminus \overset{\circ}{Z})$. We note that the induced map*

$$z_*\colon H_k(Z \cap U, A \cap U) \ \to \ H_k(U, U \setminus \overset{\circ}{Z})$$

is an isomorphism by the Excision Theorem 45.10 for topological manifolds. With this preparation the following diagram commutes

$$\begin{tikzcd}
 & H_k(M) & H^{m-k}(M, M\setminus K) \arrow[l, "\cap M"'] \arrow[dl] \arrow[dd] \\
H_k(U) \arrow[ur] \arrow[d, "p_*"] & H^{m-k}(U, U\setminus K) \arrow[l, "\cap U"'] & \\
H_k(U, U\setminus \mathring{Z}) \arrow[r, "(z_*)^{-1}"] & H_k(Z\cap U, A\cap U) \arrow[r, "g_*"] & H_k(Z, A) & H^{m-k}(Z, B). \arrow[l, "\cap [Z]"']
\end{tikzcd}$$

Here all the unmarked maps are induced by the obvious inclusion maps of (pairs) of topological spaces.

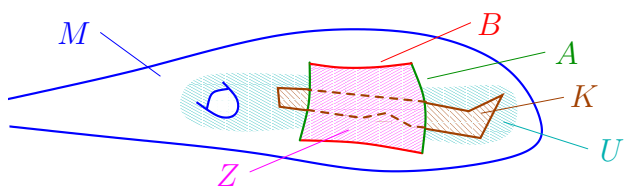


FIGURE 860. Illustration of Lemma 68.9.

Proof. We make the following preparations:

(a) We pick a representative $\nu_M \in H_m(M, M \setminus K)$ for the lift of the orientation of M and similarly we pick $\nu_U \in H_m(U, U \setminus K)$.

(b) We consider the two obvious inclusion maps $f\colon (Z, \partial Z) \to (Z, \overbrace{Z \setminus (\mathring{Z} \cap K)}^{=(\mathbb{Z}\setminus K)\cup A})$ and $g\colon (Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to (Z, Z \setminus (\mathring{Z} \cap K))$. It follows from the Excision Theorem 31.17 that the map $g_*\colon H_m(Z\cap U, (Z\cap U)\setminus(\mathring{Z}\cap K)) \to H_m(Z, Z\setminus(\mathring{Z}\cap K))$ is an isomorphism. We write $\nu_Z := g_*^{-1}(f_*([Z]))$.

Next we consider the following expanded version of the above diagram:

$$\begin{tikzcd}
 & H_k(M) & & H^{m-k}(M, M\setminus K) \arrow[ll, "\cap\nu_M"'] \arrow[dl] \arrow[ddd] \\
H_k(U) \arrow[ur] \arrow[d] & & H^{m-k}(U, U\setminus K) \arrow[ll, "\cap\nu_U"'] \arrow[d] & \\
H_k(U, U\setminus \mathring{Z}) \arrow[r, "(z_*)^{-1}"] & H_k(Z\cap U, A\cap U) \arrow[dr] & H^{m-k}(Z\cap U, (Z\cap U)\setminus K). \arrow[l, "\cap\nu_Z"'] & \\
 & & H_k(Z, A) \arrow[d, "="] & H^{m-k}(Z, Z\setminus K) \arrow[l, "\cap f_*([Z])"] \arrow[ul] \arrow[d] \\
 & & H_k(Z, A) & H^{m-k}(Z, B). \arrow[l, "\cap[Z]"]
\end{tikzcd}$$

It suffices to show that all the five regions of the diagram commute:

(1) The parallelogram on the top of the diagram commutes by Lemma 59.8 applied to the map $h\colon (U, U\setminus K, \varnothing) \to (M, M\setminus K, \varnothing)$, together with the observation that Lemma 46.23 implies that $h_*(\nu_U) = \nu_M$.
(2) The 4-gon on the right commutes since all maps are induced by inclusions.
(3) For the lower parallelogram we obtain commutativity from applying Lemma 59.8 to the map $g\colon (Z\cap U, Z\cap U\setminus K, A\cap U) \to (Z, Z\setminus K, A)$.
(4) To show commutativity of the lower rectangle we apply Lemma 59.8 to the map $f\colon (Z, B, A) \to (Z, Z\setminus K, A)$.

Finally we turn to the upper rectangle. We need to show that the following rectangle commutes:

$$\begin{tikzcd}
H_k(U) \arrow[d, "p_*"] & & H^{m-k}(U, U\setminus K) \arrow[ll, "\cap\nu_U"'] \arrow[d, "z^*"] \\
H_k(U, U\setminus \mathring{Z}) & H_k(Z\cap U, A\cap U) \arrow[l, "z_*"'] & H^{m-k}(Z\cap U, (Z\cap U)\setminus K). \arrow[l, "\cap\nu_Z"']
\end{tikzcd}$$

Now let $\beta \in H^{m-k}(U, U\setminus K)$. We have the following equalities in $H_k(U, U\setminus \mathring{Z})$:

$$\begin{array}{ccccccc}
 & & \text{it follows from Lemma 46.30 that} & & & & \\
 & & \text{we have } p_*(\nu_U) = z_*(\nu_Z) \text{ in } H_m(U, U\setminus K\cup U\setminus \mathring{Z}) & & & & \\
 & & \downarrow & & & & \\
p_*(\beta\cap\mu) & = & \beta\cap p_*(\mu) & = & \beta\cap z_*(\nu) & = & z_*(z^*(\beta)\cap\nu). \\
 & & \uparrow & & \uparrow & & \\
 & & \text{Lemma 59.8 applied to the map} & & \text{Lemma 59.8 applied to the map} & & \\
 & & p\colon (U, U\setminus K, \varnothing) \to (U, U\setminus K, U\setminus \mathring{Z}) & & z\colon (Z\cap U, Z\cap U\setminus K, A\cap U) \to (U, U\setminus K, U\setminus \mathring{Z}) & &
\end{array}$$

We have thus shown that the diagram commutes. □

After all these preparations we can finally attack Theorem 68.5.

Proof of Theorem 68.5. Let M be an oriented m-dimensional topological manifold without boundary. Also let $\sigma = c_1 \cdot \sigma_1 + \cdots + c_r \cdot \sigma_r \in C_k(M)$ and $\tau = d_1 \cdot \tau_1 + \cdots + d_s \cdot \tau_s \in C_{m-k}(M)$ be two singular cycles that are transverse. We introduce some notation.

(1) We denote by $P_1, \dots, P_n$ the points in $\bigcup_{i,j} |\sigma_i| \cap |\tau_j|$. Furthermore we denote by $\nu_1, \dots, \nu_n \in \{-1, 1\}$ the corresponding signs.
(2) We denote by $X_1, X_2, \dots, X_n$ distinct copies of I^k and we denote by $Y_1, Y_2, \dots, Y_n$ distinct copies of I^{m-k}.
(3) Given $t \in (0,1)$ we write $X_i(t) = \{x \in X_i \mid \|x\| \le t\}$ and $\mathring{X}_i(t) = \{x \in X_i \mid \|x\| < t\}$. The same way we also define $Y_i(t)$ and $\mathring{Y}_i(t)$.
(4) For each $i \in \{1, \dots, n\}$ we pick, as in the definition of transverse intersection, a compact submanifold Z_i of M that is a neighborhood of P_i and a homeomorphism $\Phi_i \colon Z_i \to X_i \times Y_i$. It follows from Lemma 68.2 we can assume that the maps $\sigma_i \colon \Delta^k \cap \sigma_i^{-1}(Z_i) \to X_i \times \{0\}$ and $\tau_i \colon \Delta^{m-k} \cap \tau_i^{-1}(Z_i) \to \{0\} \times Y_i$ are orientation-preserving. After possibly shrinking the Z_i we can and will assume that the Z_i are disjoint.
(5) We use the homeomorphism $\Phi_i \colon Z_i \to X_i \times Y_i$ to identify Z_i with $X_i \times Y_i$ as *unoriented* topological manifolds. Under this identification we have an *orientation-preserving* identification $X \cap Z_i = X_i \times \{0\}$ and $Y \cap Z_i = \{0\} \times Y_i$ of oriented topological manifolds.
(6) We write $Z = Z_1 \cup \cdots \cup Z_n$.

For $i = 1, \dots, r$ we apply Lemma 68.8 to $X = \Delta^k$ with the cubes given by the preimages of the Z_i under the maps σ_i. Similarly we proceed with the τ_i. After some renaming we can assume that we are in the following situation:[1165]

(7) For $i = 1, \dots, n$ the singular simplex σ_i is an orientation-preserving homeomorphism $\sigma_i \colon \Delta^k \to X_i \times \{0\}$. Furthermore for $i = n+1, \dots, r$ the image of each σ_i lies in $(M \setminus Z) \cup (X_i \times \{0\})$.
(8) For $j = 1, \dots, n$ the singular simplex τ_j is an orientation-preserving homeomorphism $\tau_j \colon \Delta^{m-k} \to \{0\} \times Y_j$. Furthermore for $j = n+1, \dots, s$ the image of each τ_j lies in $(M \setminus Z) \cup (\{0\} \times Y_i)$.

We introduce the following notation:

(9) We write $\widetilde{\sigma} = c_{n+1} \cdot \sigma_{n+1} + \cdots + c_r \cdot \sigma_r$ and we denote by $|\widetilde{\sigma}|$ the union of the images of $\sigma_{n+1}, \dots, \sigma_r$. Similarly we define $\widetilde{\tau}$ and $|\widetilde{\tau}|$.

We refer to Figure 861 on the left for an illustration.

Claim 1. There exist open subsets U and V of M, compact subsets K and L of M and a $t \in (0,1)$ with the following properties:

[1165]Evidently a lot of details are packed into this sentence. In particular we use that this replacing process does not affect the algebraic intersection number of cycles. We leave this as a fairly elementary, albeit somewhat lengthy exercise to the reader.

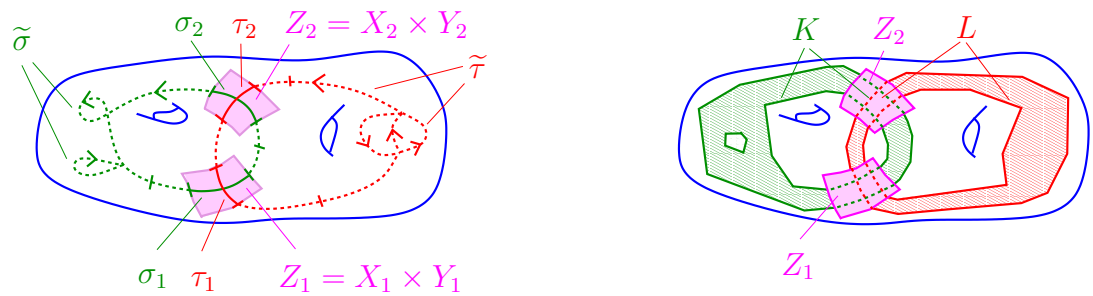


FIGURE 861.

(a) there exists a cohomology class $\varphi \in H^{m-k}(M, M \setminus K)$ such that the image of φ in $H_c^{m-k}(M)$ equals $\mathrm{PD}_M([\sigma])$,
(b) there exists a cohomology class $\psi \in H^k(M, M \setminus L)$ such that the image of ψ in $H_c^k(M)$ equals $\mathrm{PD}_M([\tau])$,
(c) we have $\widetilde{\sigma} \subset U$ and $\widetilde{\tau} \subset V$,
(d) we have $K \subset U$ and $L \subset V$,
(e) we have $K \cap L \cap (M \setminus Z) = \varnothing$,
(f) for each $i \in \{1, \ldots, n\}$ we have $U \cap Z_i = X_i \times \mathring{Y}_i(\frac{1}{2})$ and $V \cap Z_i = \mathring{X}_i(\frac{1}{2}) \times Y_i$,
(g) for each $i \in \{1, \ldots, n\}$ we have $K \cap Z_i = X_i \times Y_i(t)$ and $L \cap Z_i = X_i(t) \times Y_i$.

We refer to Figure 861 on the right for an illustration.

We consider the topological manifold $W := M \setminus \mathring{Z}$. Note that $|\widetilde{\sigma}| \cap \partial W = \partial X_i \times \{0\}$ and $|\widetilde{\tau}| \cap \partial W = \{0\} \times \partial Y_i$. It follows from Proposition 68.6 (2) that there exist disjoint open neighborhoods $\widetilde{U} \subset W$ of $|\widetilde{\sigma}|$ and $\widetilde{V} \subset W$ of $|\widetilde{\tau}|$ such that for each $i \in \{1, \ldots, n\}$ we have $\widetilde{U} \cap \partial Z_i = \partial X_i \times \mathring{Y}_i(\frac{1}{2})$ and $\widetilde{V} \cap \partial Z_i = \mathring{X}_i(\frac{1}{2}) \times \partial Y_i$. Finally we write $U = \widetilde{U} \cup \bigcup_{i=1}^{n} X_i \times \mathring{Y}_i(\frac{1}{2})$ and $V = \widetilde{V} \cup \bigcup_{i=1}^{n} \mathring{X}_i(\frac{1}{2}) \times \partial Y_i$. We refer to Figure 862 for an illustration.

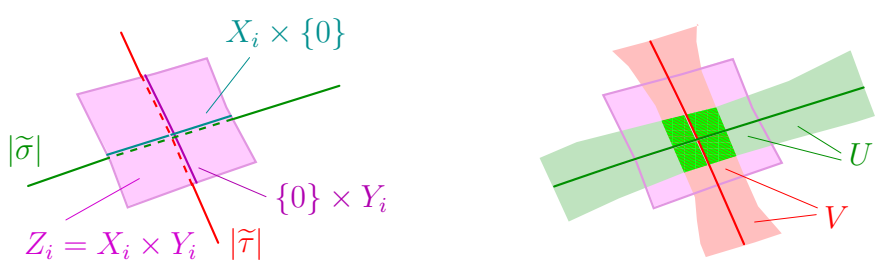


FIGURE 862.

We denote by $i \colon U \to M$ and $j \colon V \to M$ the inclusion maps. Since U is an open subset of M we can consider $\mathrm{PD}_U([\sigma]) \in H_c^{m-k}(U)$. By Lemma 53.2 there exists a compact subset

K' of U such that $\mathrm{PD}_U([\sigma])$ lies in the image of the map $H^{m-k}(U, U \setminus K') \to H_c^{m-k}(U)$. Similarly we can find a compact subset L' of V such that $\mathrm{PD}_V([\tau])$ lies in the image of the map $H^k(V, V \setminus L') \to H_c^k(V)$. Since each $K \cap Z_i$ and $L \cap Z_i$ is compact and since each $K \cap Z_i$ is contained in $U \cap Z_i = X_i \times \mathring{Y}_i(\frac{1}{2})$ and since each $L \cap Z_i$ is contained in $V \cap Z_i = \mathring{X}_i(\frac{1}{2}) \times Y_i$ we can and will pick a $t \in (0, \frac{1}{2})$ such that for all i we have $K' \cap Z_i \subset X_i \times Y_i(t)$ and $L' \cap Z_i \subset X_i(t) \times Y_i$. We set $K := K' \cup \bigcup_{i=1}^{n} X_i \times Y_i(t)$ and $L := L' \cup \bigcup_{i=1}^{n} X_i(t) \times Y_i$. Note that by construction the desired conditions (c), (d), (e), (f) and (g) are satisfied. It remains to prove (a) and (b). Since $K' \subset K$ we have the following commutative diagram

$$\begin{tikzcd}
 & H^{m-k}(U, U \setminus K') \arrow[dl] \arrow[dr] & \\
H^{m-k}(U, U \setminus K) \arrow[rr] & & H_c^{m-k}(U).
\end{tikzcd}$$

This shows that $\mathrm{PD}_U([\sigma])$ also lies in the image of $H^{m-k}(U, U \setminus K)$. The same way we see that $\mathrm{PD}_V([\tau])$ lies in the image of $H^{m-k}(V, V \setminus L)$. Statements (a) and (b) now follow directly from Lemma 68.7. This concludes the proof of the claim.

We continue with some more observations and notation.

(10) For $i = 1, \dots, n$ we write $K_i = K \cap Z_i$ and $L_i = L \cap Z_i$. By the above discussion we know that $K_i = X_i \times Y_i(t)$ and $L_i = X_i(t) \times Y_i$. Also note that by construction we have $K \cap L \subset \mathring{Z}$.

(11) Given any compact subset T of M we denote by $o\colon (M, \varnothing) \to (M, M \setminus T)$ the obvious map. Also we denote by $o^*\colon H^*(M, M \setminus T) \to H_c^*(M)$ the obvious map.

(12) We denote by $z\colon Z \to M$ the inclusion map and we denote by z also all the inclusion maps of pairs of topological spaces where the total spaces are Z and M.

(13) Given $i \in \{1, \dots, n\}$ we denote by $z_i\colon Z_i \to Z$ the inclusion map. We adopt the same convention as in (12) for pairs of topological spaces.

(14) We view Z as an m-dimensional submanifold of M and we equip Z with the orientation coming from Lemma 45.28. Given any subset $W \subset \mathring{Z}$ we denote by $[Z]$ also the image of the fundamental class of Z under the map $H_m(Z, \partial Z) \to H_m(Z, W)$. We adopt the same convention for $[Z_i]$.

(15) We refer to the map of pairs $(Z_i, X_i \times \partial Y_i) \to (Z_i, Z_i \setminus K_i)$ and also to the map $(Z_i, \partial X_i \times Y_i) \to (Z_i, Z_i \setminus L_i)$ as the canonical map c.

(16) By Theorem 46.10 there exists a unique lift $\mu \in H_m(M, M \setminus (K \cap L))$ of the orientation classes of M.

Next we consider the following diagram:

$$\begin{tikzcd}
H_c^{m-k}(M) \times H_c^k(M) \arrow[r, "\cup"] & H_c^m(M) \arrow[r, "\cap M"] & H_0(M) \arrow[r, "\epsilon_M"] \arrow[d, "="'] & \mathbb{Z} \arrow[d, "="'] \\
H^{m-k}(M, M\setminus K) \times H^k(M, M\setminus L) \arrow[r, "\cup"] \arrow[u, "o^*\times o^*"'] \arrow[d, "z^*\times z^*"] & H^m(M, M\setminus(K\cap L)) \arrow[r, "\cap\mu"] \arrow[u, "o^*"'] \arrow[d, "z^*"] & H_0(M) \arrow[r, "\epsilon_M"] & \mathbb{Z} \\
H^{m-k}(Z, Z\setminus K) \times H^k(Z, Z\setminus L) \arrow[r, "\cup"] \arrow[d, "\bigoplus_{i=1}^n z_i^*\times z_i^*"] & H^m(Z, Z\setminus K\cap L) \arrow[r, "\cap[Z]"] \arrow[d, "\bigoplus_{i=1}^n z_i^*"] & H_0(Z) \arrow[r, "\epsilon_Z"] \arrow[u, "z_*"'] & \mathbb{Z} \arrow[u, "="'] \\
\bigoplus_{i=1}^n H^{m-k}(Z_i, Z_i\setminus K_i) \times H^k(Z_i, Z_i\setminus L_i) \arrow[r, "\oplus\cup"] \arrow[d, "\bigoplus_{i=1}^n c^*\times c^*"] & \bigoplus_{i=1}^n H^m(Z_i, Z_i\setminus K_i\cap L_i) \arrow[r, "\oplus\cap[Z_i]"] \arrow[d, "\bigoplus_{i=1}^n c^*"] & \bigoplus_{i=1}^n H_0(Z_i) \arrow[r, "\sum\epsilon_{Z_i}"'] \arrow[u, "\sum_{i=1}^n z_{i*}"'] & \mathbb{Z} \arrow[u, "="'] \\
\bigoplus_{i=1}^n H^{m-k}(Z_i, X_i\times\partial Y_i) \times H^k(Z_i, \partial X_i\times Y_i) \arrow[r, "\oplus\cup"] & \bigoplus_{i=1}^n H^m(Z_i, \partial Z_i) \arrow[r, "\oplus\cap[Z_i]"] & \bigoplus_{i=1}^n H_0(Z_i) \arrow[r, "\sum\epsilon_{Z_i}"'] \arrow[u, "="'] & \mathbb{Z}. \arrow[u, "="']
\end{tikzcd}$$

Claim 2. The above diagram commutes.

In the following we deal with all of the twelve squares of the diagram.

(a) The square on the top left commutes by definition of the cup product on cohomology with compact support, see page 1361.
(b) The left squares in the second row and the fourth commute by Proposition 58.3 (1).
(c) The left square in the third row commutes by Lemma 59.9 (1) together with the aforementioned Proposition 58.3 (1).
(d) The middle square in the first row commutes by definition of the map $\cap M$.
(e) It follows from Lemma 46.11 that $z_*([Z]) = \mu \in H_m(M, M \setminus (K \cap L))$. Therefore we obtain from Lemma 59.8 (1) that the middle square in the second row commutes.
(f) It follows from the remark on page 1114 that $\sum_{i=1}^n z_{i*}([Z_i]) = [Z]$. Now we obtain from Lemma 59.8 (1) and Lemma 59.9 (2) that the middle square in the third row commutes.
(g) The squares that involve the augmentation maps commute by the discussion on page 747.
(h) It follows from Lemma 59.8 (1) and our conventions regarding the meaning of $[Z_i]$ that the middle square in the fourth row commutes.

This concludes the proof of the claim.

We introduce one last piece of notation:

(17) We denote by PD_{Z_i} and $\mathrm{PD}_{X_i\times Y_i}$ the isomorphisms from the Poincaré Duality 61.1 corresponding to the decomposition $\partial Z_i = \partial(X_i \times Y_i) = (\partial X_i \times Y_i) \cup (X_i \times \partial Y_i)$.

Claim 3. Let $i \in \{1, \ldots, n\}$.

(i) For the above $\varphi \in H^{m-k}(M, M \setminus K)$ we have $(c^* \circ z_i^* \circ z^*)(\varphi) = c_i \cdot \mathrm{PD}_{Z_i}([X_i \times \{0\}])$.
(ii) For the above $\psi \in H^k(M, M \setminus L)$ we have $(c^* \circ z_i^* \circ z^*)(\varphi) = d_i \cdot \mathrm{PD}_{Z_i}([\{0\} \times Y_i])$.

(iii) We have

$$\epsilon_{Z_i}\Big(\big((\mathrm{PD}_{Z_i}([X_i\times\{0\}])\cup \mathrm{PD}_{Z_i}([\{0\}\times Y_i])\big)\cap [Z_i]\Big) \;=\; \nu_i.$$

We provide the proof of the three statements:

(i) We introduce the following notation:

(α) We write $A=\partial X_i\times Y_i$ and $B=X_i\times\partial Y_i$.

(β) We denote by $p\colon (U,\varnothing)\to(U,U\setminus \mathring{Z}_i)$, $\widetilde{z}\colon (Z_i\cap U, A\cap U)\to(U,U\setminus\mathring{Z}_i)$ and $g\colon (Z_i\cap U, A\cap U)\to(Z_i,A)$ the obvious inclusion maps.

We make the following observation: We calculate that

$$(c^*\circ z_i^*\circ z^*)(\varphi)\cap[Z_i] \;\overset{\substack{\text{by Lemma 68.9 and since}\\ \varphi\cap M=[\sigma]\text{ with }|\sigma|\subset U}}{\overset{\downarrow}{=}}\; g_*(\widetilde{z}_*^{-1}(p_*([\sigma]))) \;\overset{\text{by (7) on page 1602}}{\overset{\downarrow}{=}}\; c_i\cdot g_*([X\cap Z_i]) \;\underset{\substack{\uparrow\\ \text{by the orientation-preserving identification } X\cap Z_i=X_i\times\{0\}}}{=}\; c_i\cdot[X_i\times\{0\}].$$

But by definition of the Poincaré duality isomorphism this is precisely what we had needed to show.

(ii) The proof of this statement is evidently the same as the proof of (2).

(iii) We recall that we made the identification $Z_i=X_i\times Y_i=I^k\times I^{m-k}=I^m$ as *unoriented* topological manifolds. The desired equality follows from the following observations:

(α) It is a consequence of our conventions and Lemma 68.2 that under the above identification we have $[Z_i]=\nu_i\cdot[X_i\times Y_i]$.

(β) By (i) and by the definition of the Poincaré Duality isomorphisms we have

$$\mathrm{PD}_{Z_i}([X_i\times\{0\}]) \;=\; \nu_i\cdot\mathrm{PD}_{X_i\times Y_i}([X_i\times\{0\}])$$

and

$$\mathrm{PD}_{Z_i}([\{0\}\times Y_i]) \;=\; \nu_i\cdot\mathrm{PD}_{X_i\times Y_i}([\{0\}\times Y_i]).$$

(γ) By Proposition 67.16 we have

$$\mathrm{PD}_{X_i\times Y_i}([X_i\times\{0\}])\cup\mathrm{PD}_{X_i\times Y_i}([\{0\}\times Y_i]) \;=\; [X_i\times Y_i]^*.$$

This concludes the proof of the claim.

After all these preparations we can prove the desired equality:

$$\epsilon_M\big((\mathrm{PD}_M([\sigma])\cup\mathrm{PD}_M([\tau]))\cap M\big) \;\overset{\substack{\text{by construction of }\varphi\text{ and }\psi\text{ in Claim 1 and the first row of the commutative diagram from Claim 2}\\ \downarrow}}{=}\; \epsilon_M((\varphi\cup\psi)\cap M)$$

$$\underset{\substack{\uparrow\\ \text{by the second, third and fourth row of the}\\ \text{commutative diagram from Claim 2}}}{=} \sum_{i=1}^n \epsilon_{Z_i}\Big(\big((c^*\circ z_i^*\circ z^*)(\varphi)\cup(c^*\circ z_i^*\circ z^*)(\psi)\big)\cap[Z_i]\Big) \;\underset{\substack{\uparrow\\ \text{by Claim 3}}}{=}\; \sum_{i=1}^n c_i\cdot d_i\cdot\nu_i \;=\; \sigma\cdot\tau.$$

This concludes the proof of the theorem. □

We conclude this section with the following variation of Theorem 68.5.

Theorem 68.10. *Let M be a compact orientable m-dimensional topological manifold. Let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let $\sigma \in C_k(M, A)$ and $\tau \in C_{m-k}(M, B)$ be two cycles. If σ and τ are transverse, then*

$$\epsilon_M\big(\big(\underbrace{\overbrace{\mathrm{PD}_M([\sigma])}^{\in H^{m-k}(M,B)} \cup \overbrace{\mathrm{PD}_M([\tau])}^{\in H^{k}(M,A)}}_{\in H^m(M,\partial M)}\big) \cap [M]\big) \;=\; \sigma \cdot \tau.$$

Sketch of proof. The proof of the theorem follows along the lines of the proof of Theorem 68.5. The key difference is that we need to use Poincaré Duality for non-compact topological manifolds with boundary. More precisely, let U be an oriented topological m-dimensional manifold with boundary. By Theorem 46.29, given any compact set K of U the orientation defines a natural homology class μ_K in $H_m(U, (U \setminus K) \cup \partial U)$. The same construction as in Lemma 61.14, this time using Lemma 46.30 instead of Lemma 46.11, shows that capping with these homology classes defines a well-defined map

$$\mathrm{PD}_U \colon H^k_c(U) \;\to\; H_{m-k}(U, \partial U).$$

A moderately heroic effort shows that the proof of the Poincaré Duality Theorem 61.1, as provided in Section 61.5 can be modified to show that the above map PD_U is an isomorphism. Once this is out of the way it is fairly straightforward to modify the proof of Theorem 68.5 to obtain a proof of Theorem 68.10. □

68.3. **Intersections of transverse maps of manifolds.** The following theorem is frequently used in the literature. I am not aware though of a published proof.

Theorem 68.11. *Let M be a compact oriented connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Finally let $f \colon X \to M$ and $g \colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g are transverse, then*

$$f \cdot g \;=\; \langle \mathrm{PD}_M(f_*[X]) \cup \mathrm{PD}_M(g_*[Y]), [M] \rangle.$$

Remark. It follows immediately from Lemma 68.4 that Theorem 68.11 contains our earlier Theorem 67.10 as a special case.

Proof. It follows easily from Lemma 68.8 that there exists a k-cycle $\sigma \in C_k(M, A)$ and an $(m-k)$-cycle $\tau \in C_{m-k}(M, B)$ with the following properties:

(1) we have $[\sigma] = f_*([X])$ and $[\tau] = g_*([Y])$,
(2) the cycles σ and τ are transverse,

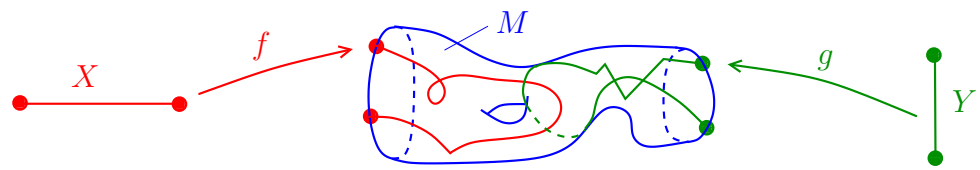


FIGURE 863. Illustration of Theorem 68.11.

(3) the equality $f \cdot g = \sigma \cdot \tau$ holds.

It follows that

$$f \cdot g \;=\; \sigma \cdot \tau \;\underset{\substack{\uparrow \\ \text{Theorem 68.10}}}{=}\; \langle \mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]), [M] \rangle \;\underset{\substack{\uparrow \\ \text{since } [\sigma] = f_*([X]) \text{ and } [\tau] = g_*([Y])}}{=}\; \langle \mathrm{PD}_M(f_*[X]) \cup \mathrm{PD}_M(g_*[Y]), [M] \rangle.$$

□

As promised, we now find it quite easy to answer Question 68.1 in the negative.

Lemma 68.12. *We consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm(0, \pm 1)$. If $\alpha\colon [-1,1] \to \overline{B}^2$ is a proper map with $\alpha(\pm 1) = P_\pm$ and if $\beta\colon [-1,1] \to \overline{B}^2$ is a proper map with $\beta(\pm 1) = Q_\pm$, then α and β intersect in at least one point.*

Proof. We write $I = [-1,1]$. Suppose that there exist proper maps $\alpha\colon I \to \overline{B}^2$ and $\beta\colon I \to \overline{B}^2$ with $\alpha(\pm 1) = P_\pm$ and with $\beta(\pm 1) = Q_\pm$ such that $|\alpha| \cap |\beta| = \varnothing$. We introduce the following notation:

(1) We consider the two "straight paths" $\alpha'\colon I \to \overline{B}^2$ given by $\alpha'(t) = (t, 0)$ and $\beta'\colon I \to \overline{B}^2$ given by $\beta'(t) = (0, t)$.
(2) We write $A = \left\{ e^{it} \,\middle|\, t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \right\}$ and $B = \left\{ e^{it} \,\middle|\, t \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \right\}$.
(3) We denote by PD_M the isomorphisms from the Poincaré Duality Theorem 61.1 corresponding to the decomposition $\partial \overline{B}^2 = A \cup B$.

We refer to Figure 864 for an illustration. We have

$$\begin{aligned} 0 \;\underset{\substack{\downarrow \\ \text{since } |\alpha| \cap |\beta| = \varnothing}}{=}\; \alpha \cdot \beta \;&\underset{\substack{\downarrow \\ \text{Theorem 68.11}}}{=}\; \left\langle \mathrm{PD}_M(\alpha_*([I])) \cup \mathrm{PD}_M(\beta_*([I]), [M] \right\rangle \\ &\underset{\substack{\uparrow \\ \text{by considering the long exact sequences in homology} \\ \text{of the pairs } (\overline{B}^2, A) \text{ and } (\overline{B}^2, B) \text{ we see that} \\ \alpha_*([I]) = \alpha'_*([I]) \text{ and } \beta_*([I]) = \beta'_*([I])}}{=}\; \left\langle \mathrm{PD}_M(\alpha'_*([I])) \cup \mathrm{PD}_M(\beta'_*([I]), [M] \right\rangle \underset{\substack{\uparrow \\ \text{Theorem 68.11}}}{=} \alpha' \cdot \beta' \underset{\substack{\uparrow \\ \alpha', \beta' \text{ intersect} \\ \text{transversely in} \\ \text{one point}}}{=} \pm 1. \end{aligned}$$

□

68.4. **The algebraic $\mathbb{F}_2$-intersection number** $(*)$**.** In this section we quickly discuss the "$\mathbb{F}_2$-analogue" of the results of the previous sections.

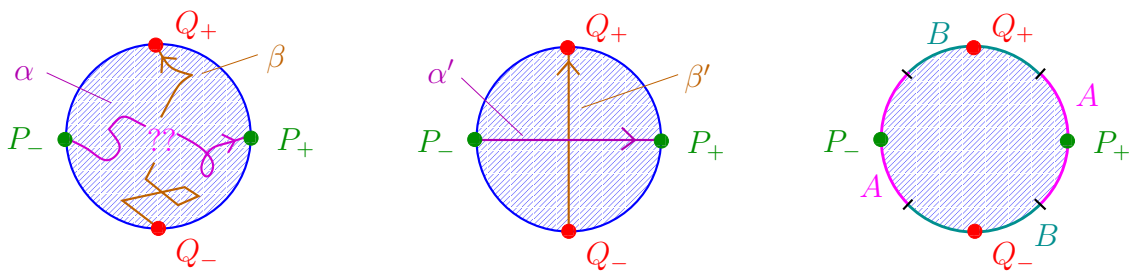


FIGURE 864.

Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g are transverse, then we define

$$\text{the algebraic } \mathbb{F}_2\text{-intersection number of } f \text{ and } g \;:=\; f \cdot_{\mathbb{F}_2} g \;:=\; \#(|f| \cap |g|) \bmod 2.$$

If X and Y are in fact submanifolds and f and g are the inclusion map, then we write $X \cdot_{\mathbb{F}_2} Y := f \cdot_{\mathbb{F}_2} g$.

We recall the following notation.

Notation.

(1) In Corollary 46.6 we saw that any compact n-dimensional manifold W has a unique $\mathbb{F}_2$-fundamental class $[W]_{\mathbb{F}_2} \in H_n(W, \partial W; \mathbb{F}_2)$. If W is in fact a proper submanifold of some other topological manifold M, then often times we denote by $[W]_{\mathbb{F}_2} \in H_n(M, \partial M; \mathbb{F}_2)$ the image of $[W]_{\mathbb{F}_2}$ under the inclusion induced map $H_n(W, \partial W; \mathbb{F}_2) \to H_n(M, \partial M; \mathbb{F}_2)$.

(2) In this section, given a compact topological manifold we denote by PD_M the Poincaré Duality isomorphisms with $\mathbb{F}_2$-coefficients provided by the Poincaré Duality Theorem 61.1.

Now we can formulate the following theorem.

Theorem 68.13. *Let M be a compact connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M such that $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g are transverse, then*

$$f \cdot_{\mathbb{F}_2} g \;=\; \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{f_*([X]_{\mathbb{F}_2})}_{\in H_k(M,A;\mathbb{F}_2)})}_{\in H^{m-k}(M,B;\mathbb{F}_2)} \cup \underbrace{\mathrm{PD}_M(\underbrace{g_*([Y]_{\mathbb{F}_2})}_{\in H_{m-k}(M,B;\mathbb{F}_2)})}_{\in H^{k}(M,A;\mathbb{F}_2)}}^{\in H^m(M,\partial M;\mathbb{F}_2)}, \overbrace{[M]_{\mathbb{F}_2}}^{\in H_m(M,\partial M;\mathbb{F}_2)} \Big\rangle \;\in\; \mathbb{F}_2.$$

In particular, if X and Y are already submanifolds of M, then

$$X \cdot_{\mathbb{F}_2} Y \;=\; \langle \mathrm{PD}_M([X]_{\mathbb{F}_2}) \cup \mathrm{PD}_M([Y]_{\mathbb{F}_2}), [M]_{\mathbb{F}_2} \rangle.$$

Proof. By Proposition 45.24 every topological manifold is $\mathbb{F}_2$-orientable. With this observation the proof of the theorem is a variation, and basically a simplification, of the proof of Theorem 67.10. We leave it to the skeptical reader to fill in the details. □

Example. Similar to the discussion on page 1567 we can use Theorem 68.13 to show that some homology classes are in fact non-zero. For example in Figure 865 we show the Klein bottle K together with two closed 1-dimensional submanifolds. These intersect in precisely one point, so their algebraic $\mathbb{F}_2$-intersection number is necessarily non-zero. Theorem 68.13 implies that $[X]_{\mathbb{F}_2}$ and $[Y]_{\mathbb{F}_2}$ are non-zero in $H_1(K;\mathbb{F}_2)$.

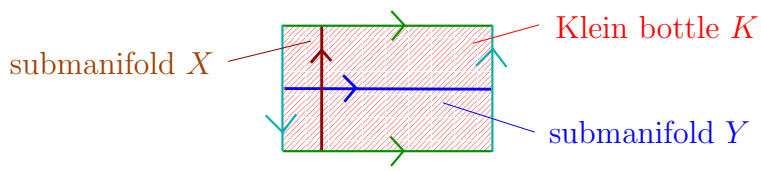


FIGURE 865.

Exercises for Chapter 68.

Exercise 68.1. Is every topological space that is Hausdorff also normal?

Exercise 68.2. We consider the definition of the sign of a transverse intersection point of two maps $f\colon X \to M$ and $g\colon Y \to M$ between oriented topological manifolds that we gave on page 1592. Did we implicitly assume that $\dim(X) > 0$ and $\dim(Y) > 0$, or does the definition also work for 0-dimensional topological manifolds?

Exercise 68.3. Can one use the Generalized Alexander Duality Theorem 66.10 to answer Question 68.1? Or some other of the earlier theorems? If yes, how would you do it? You could also try to restrict yourselves to the case that the map α is injective.

Exercise 68.4. Let M be a compact oriented connected n-dimensional topological manifold and let $N \subset M$ be a compact oriented proper submanifold of codimension one. Finally let $\gamma\colon S^1 \to M$ be a map that is transverse to N. Show that

$$\langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \;=\; \gamma \cdot N.$$

69. Homology, cohomology and submanifolds

In this chapter, given an oriented, compact manifold M we will relate the following objects:

(1) Codimension one submanifolds of M,
(2) cohomology classes in $H^1(M;\mathbb{Z})$ of dimension one,
(3) homology classes in $H_{n-1}(M,\partial M;\mathbb{Z})$ of codimension one,
(4) maps $M \to S^1$.

69.1. **Codimension one submanifolds.** Let M be a compact oriented connected n-dimensional topological manifold. Every compact oriented proper submanifold $Y \subset M$ of *codimension* k gives rise to a homomorphism

$$\begin{array}{rcl} H_k(M;\mathbb{Z}) & \to & \mathbb{Z} \\ \sigma & \mapsto & \langle \underbrace{\mathrm{PD}_M([Y])}_{\in H^k(M;\mathbb{Z})}, \sigma\rangle. \end{array}$$

In many circumstances the next proposition gives an explicit description of this homomorphism $H_k(M;\mathbb{Z}) \to \mathbb{Z}$:

Proposition 69.1. *Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of* codimension k. *Let $f\colon X \to M$ be a map from a closed k-dimensional topological manifold X to M. If f is transverse to Y, then*

$$\langle \underbrace{\mathrm{PD}_M([Y])}_{\in H^k(M;\mathbb{Z})}, \underbrace{f_*([X])}_{\in H_k(M;\mathbb{Z})} \rangle \;=\; f \cdot Y.$$

Remark. Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of codimension one. In Proposition 64.11 we saw that Y admits a tubular neighborhood of the form $[-1,1] \times Y$ such that the product orientation of $[-1,1] \times Y$ matches the orientation of M. If $f\colon S^1 \to M$ is a map, then loosely speaking, the algebraic intersection number $f \cdot Y$ counts (with signs) how often f goes from from $Y \times \{\mp 1\}$ to $Y \times \{\pm 1\}$.

Proof. We have

$$\begin{array}{ll} f \cdot Y = \epsilon_M\big((\mathrm{PD}_M(f_*([X])) \cup \mathrm{PD}_M([Y]))\cap[M]\big) & = \epsilon_M\big(\mathrm{PD}_M([Y]) \cap (\mathrm{PD}_M(f_*([X]) \cap [M]))\big) \\ \uparrow & \uparrow \\ \text{by Theorem 68.11 since } f \text{ and } Y \text{ are transverse} & \text{by Lemma 59.7 and the discussion on page 1378} \\ = \epsilon_M(\mathrm{PD}_M([Y]) \cap f_*([X])) & = \langle \mathrm{PD}_M([Y]), f_*([X])\rangle. \\ \uparrow & \uparrow \\ \text{definition of } \mathrm{PD}_M(f_*([X])) & \text{Lemma 59.4} \end{array}$$

□

Now we can prove the following proposition which gives an easy criterion for showing that a codimension one submanifold represents a non-trivial class in homology.

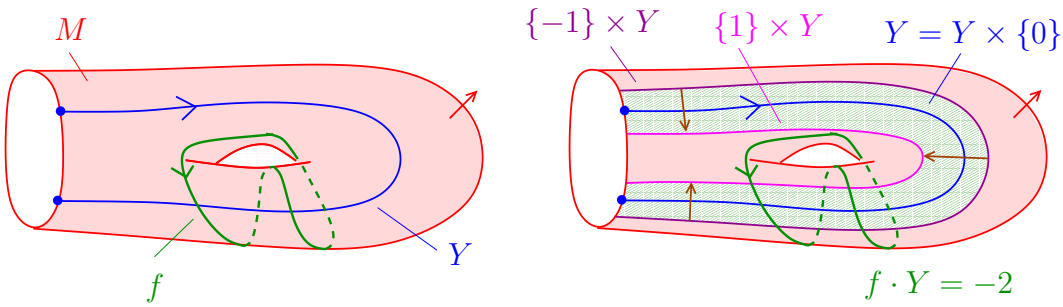


FIGURE 866.

Proposition 69.2. *Let M be a compact oriented n-dimensional manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If the complement $M \setminus N$ is connected, then $[N] \in H_{n-1}(M, \partial M; \mathbb{Z})$ is a primitive*[1166] *element of infinite order.*

Example. Proposition 69.2 can in particular be seen as a generalization of the argument given on page 1567.

Remark. In any statement of a lemma, proposition or theorem it is always a good exercise to see whether the hypothesis stated are actually necessary. In the following we will discuss whether the conditions in Proposition 69.2 that M needs to be compact and oriented are necessary.

(1) We consider the real projective plane $\mathbb{R}\mathrm{P}^2$ and the 1-dimensional submanifold N that is illustrated in Figure 867 to the left. We see that $\mathbb{R}\mathrm{P}^2 \setminus N$ is connected, but since $H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{F}_2$ we see that N cannot possibly represent an element of infinite order. On the other hand, an "$\mathbb{F}_2$-version" of Proposition 69.2, which by Proposition 45.24 also applies to non-orientable manifolds, shows that N represents a non-trivial element in $H_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.
(2) Let M be a non-compact oriented n-dimensional manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If ∂M is compact, then we will see in Exercise 69.1 using Proposition 65.4 and 69.2, that the homology class $[N] \in H_{n-1}(M; \mathbb{Z})$ is still an element of infinite order.

Proof. Let M be a compact oriented n-dimensional manifold and let N be a compact orientable proper submanifold of codimension one such that $M \setminus N$ is connected. By Proposition 64.11 we can find a trivial tubular neighborhood $[-1, 1] \times N$ in M. By hypothesis $M \setminus N$ is connected. It follows from Proposition 64.13, Lemma 64.3 and Lemma 3.8 that $M \setminus (-1, 1) \times N$ is path-connected.

It follows from our hypothesis that N is non-empty, together with Corollary 32.4, that we can pick a point $P \in N \setminus \partial N$. We write $P_\pm = P \times \{\pm 1\} \in [-1, 1] \times N$. We denote

[1166]Recall that an element g of an additive abelian group G is called primitive if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

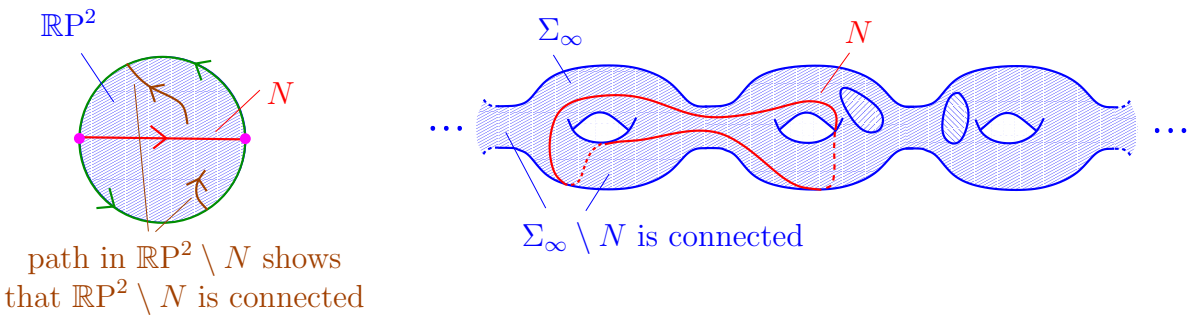


FIGURE 867.

by $\alpha\colon [0, \frac{1}{2}] \to [-1,1] \times N$ the "horizontal path" in $[-1,1] \times N$ from P_- to P_+. Since $M \setminus (-1,1) \times N$ is path-connected we can connect the two points P_+ to P_- via a path $\beta\colon [\frac{1}{2}, 1] \to M \setminus (-1,1) \times N$. We refer to Figure 868 for an illustration.

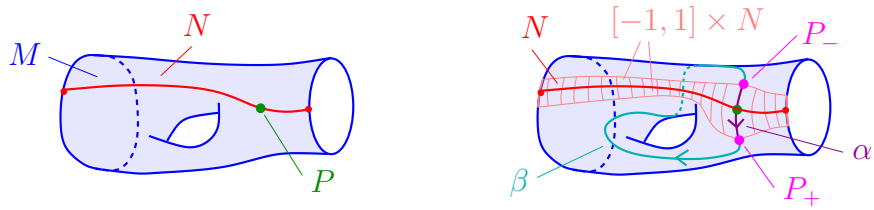


FIGURE 868. Illustration for the proof of Proposition 69.2.

We consider the map $\gamma\colon S^1 = [0,1] \to M$ that is defined by α on $[0, \frac{1}{2}]$ and that is given by β on $[\frac{1}{2}, 1]$. We consider the following homomorphism

$$\begin{aligned} \Psi\colon H_{n-1}(M, \partial M; \mathbb{Z}) &\to \mathbb{Z} \\ \sigma &\mapsto \langle \underbrace{\mathrm{PD}_M(\sigma)}_{\in H^1(M;\mathbb{Z})}, \underbrace{\gamma_*([S^1])}_{\in H_1(M;\mathbb{Z})} \rangle. \end{aligned}$$

We want to show that $[N] \in H_{n-1}(M, \partial M; \mathbb{Z})$ is a primitive element of infinite order. It suffices to show that $\Psi([N]) = \pm 1 \in \mathbb{Z}$. In fact we have

$$\begin{array}{ccccccc} \Psi([N]) & = & \langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle & = & \gamma \cdot N & = & \pm 1, \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by definition} & & \begin{array}{c}\text{by Proposition 69.1 since} \\ \gamma \text{ and } N \text{ are clearly transverse}\end{array} & & \text{by construction of } \gamma & \end{array}$$

as desired. □

We conclude this section with the following lemma.

Lemma 69.3. $(*)$ *Let M be a compact orientable n-dimensional manifold with at most one boundary component and which furthermore satisfies $H_1(M;\mathbb{Z}) = 0$. (For example we could take $M = \overline{B}^n$ or $M = S^n$). Then for every connected submanifold N of codimension one with $N \cap \partial M = \varnothing$ there exists an n-dimensional compact submanifold W of M with $\partial W = N$.*

Proof $(*)$. By Proposition 64.11 we can find a trivial tubular neighborhood $[-1,1] \times N$ in M. It follows from the argument of the proof of Proposition 69.2 and our assumption that M is orientable and the fact that M is compact that $X := M \setminus (-1,1) \times N$ is disconnected. Furthermore it follows from Proposition 3.24 that X is a submanifold of M with $\partial X = \partial M \cup (\{-1\} \times N) \cup (\{1\} \times N)$. Since ∂M consists of at most one component, since M is connected and since X is disconnected we see that there exists a component Y of X with $\partial X = \{\epsilon\} \times N$ for some $\epsilon \in \{-1,1\}$. By adding "half" of the tubular neighborhood to Y we obtain a submanifold with $\partial Y = M$. □

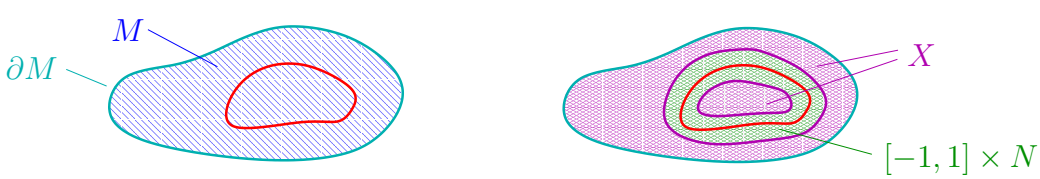


FIGURE 869. Illustration for the proof of Lemma 69.3.

69.2. **Representing homology classes by submanifolds.** The following definition is a slight generalization of the definition that we introduced on page 957.

Definition. Let M be a manifold and let A be a union of components of ∂M.

(1) If N is a compact oriented k-dimensional submanifold of M with $\partial N \subset A$, then, by a slight abuse of notation, we denote by $[N]$ the image of the fundamental class $[N]$ in $H_k(M,A;\mathbb{Z})$ under the inclusion induced map $H_k(N,\partial N;\mathbb{Z}) \to H_k(M,A;\mathbb{Z})$.
(2) We say that a homology class $z \in H_k(M,A;\mathbb{Z})$ *is represented by a submanifold* if there exists a compact oriented k-dimensional compact oriented proper submanifold N with $\partial N \subset A$ such that $[N] = z$.

Let M be an orientable, compact n-dimensional manifold. We saw in Theorem 67.10 that we can calculate the cup product of $\alpha \in H^k(M,\partial M;\mathbb{Z})$ and $\beta \in H^{n-k}(M;\mathbb{Z})$ if the Poincaré duals $\mathrm{PD}_M(\alpha) \in H_{n-k}(M;\mathbb{Z})$ and $\mathrm{PD}_M(\beta) \in H_k(M,\partial M;\mathbb{Z})$ are represented by transverse submanifolds. Thus the following question arises.

Question 69.4. *Let M be a manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $H_k(M,A;\mathbb{Z})$ be represented by submanifolds?*

In Proposition 65.27 we already saw that we can always represent one-dimensional homology classes by submanifolds. We work our way up by one dimension.

Proposition 69.5. *Let M be an n-dimensional manifold with $n \geq 5$ and let $A \subset \partial M$ be a union of components of ∂M. Every class in $H_2(M, A; \mathbb{Z})$ is represented by a submanifold.*

Remark. Below we will deal with 2-dimensional homology classes for orientable manifolds of dimension ≤ 4.

Proof. Let M be an n-dimensional manifold with $n \geq 5$ and let $\sigma \in H_2(M, A; \mathbb{Z})$. The fairly elementary argument of [Cal, p. 3] shows that there exists a compact oriented 2-dimensional manifold Σ and a proper map $f\colon \Sigma \to M$ with $f(\partial\Sigma) \subset A$ such that $f_*([\Sigma]) = \sigma$. Since $n \geq 5$ we obtain from Theorem 65.26 that f is properly homotopic to a proper embedding $g\colon \Sigma \to M$. It follows from Proposition 30.5 that the submanifold Σ represents the same homology class as $f_*([\Sigma])$, namely σ. $\square$

Instead of moving on to the next highest dimension, we now actually consider homology class of *codimension one*. More precisely, we want to prove the following proposition.

Proposition 69.6. *If M is a compact orientable n-dimensional manifold, then every class in $H_{n-1}(M, \partial M; \mathbb{Z})$ can be represented by a submanifold.*

Proof. Let M be a compact orientable n-dimensional manifold and let $\sigma \in H_{n-1}(M, \partial M; \mathbb{Z})$. We write $\phi = \mathrm{PD}_M(\sigma) \in H^1(M; \mathbb{Z})$. By Proposition 55.9 there exists a smooth map $f\colon M \to S^1$ such that $\phi = f^*([S^1]^*)$ where $[S^1]^* \in H^1(S^1; \mathbb{Z})$ is the dual fundamental class.

By Sard's Theorem 3.48 and Proposition 3.47 (2) there exists a regular value $z \in S^1$. To simplify the notation we assume that in fact $z = 1$ is a regular value. We write $N := f^{-1}(1)$. By the Regular Value Theorem 65.3 (1), (2), (3) we know that N is a compact orientable $(n-1)$-dimensional manifold. Furthermore we know from Theorem 65.3 (4) that there exists an $\epsilon \in (0, \pi)$ and a tubular neighborhood $[-\epsilon, \epsilon] \times N$ of N such that the following diagram commutes

$$\begin{tikzcd}
{[-\epsilon, \epsilon] \times N} \arrow[r, hook] \arrow[d, "(s,x) \mapsto s"] & M \arrow[d, "f"] \\
{[-\epsilon, \epsilon]} \arrow[r, "s \mapsto e^{2\pi i s}"] & S^1.
\end{tikzcd}$$

We pick the orientation for N such that the inclusion map $[-\epsilon, \epsilon] \times N \to M$ is orientation-preserving. It remains to prove that $[N] = \sigma \in H_{n-1}(M, \partial M; \mathbb{Z})$. Recall that $\phi = \mathrm{PD}_M(\sigma)$, i.e. $\sigma = \phi \cap [M]$. Therefore it suffices to show that $[N] = \phi \cap [M] \in H_{n-1}(M, \partial M; \mathbb{Z})$.

We introduce the following notation:

(1) We write $I = [-\epsilon, \epsilon]$ and $\mathring{I} = (-\epsilon, \epsilon)$,
(2) we denote by $i\colon I \times N \to M$ the inclusion map and we denote by i also all maps of pairs where the first map is given by the inclusion map,
(3) we denote by q all maps of pairs of topological spaces where the first map is just the identity on M,
(4) finally we denote by $p\colon I \times N \to I$ the projection map.

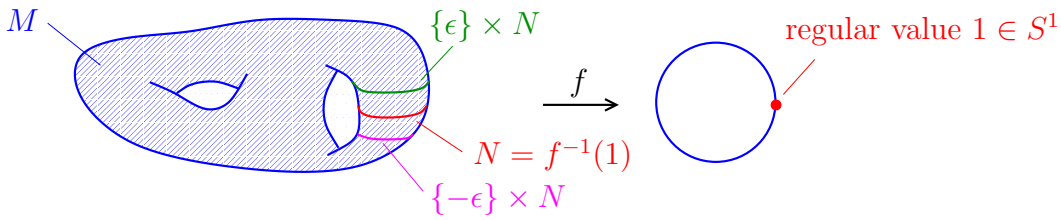


FIGURE 870. Illustration for the proof of Proposition 69.6.

We consider the following diagram[1167]

$$\begin{tikzcd}
H^1(M) & \times & H_n(M,\partial M) \arrow[r, "\cap"] \arrow[d, "q_*"] & H_{n-1}(M,\partial M) \arrow[d, "q_*"', "="] \\
H^1(M, M\setminus(\mathring{I}\times N)) \arrow[u, "q^*"] \arrow[d, "i^*"', "\cong"] & \times & H_n(M, M\setminus(\mathring{I}\times N)\cup\partial M) \arrow[r, "\cap"] & H_{n-1}(M,\partial M) \\
H^1(I\times N, \partial I\times N) \arrow[u, "p^*"] & \times & H_n(I\times N, \partial(I\times N)) \arrow[u, "i_*"'] \arrow[r, "\cap"] & H_{n-1}(I\times N, I\times\partial N) \arrow[u, "i_*"] \\
H^1(I,\partial I) \arrow[u, "p^*"] & \times & H_1(I,\partial I) \arrow[u, "-\times[N]"'] \arrow[r, "\cap"] & H_0(I). \arrow[u, "-\times[N]"]
\end{tikzcd}$$

The proposition follows from the following claim.

Claim.

(1) The diagram commutes.[1168]
(2) The map $i^*\colon H^1(M, M\setminus(\mathring{I}\times N)) \to H^1(I\times N, \partial I\times N)$ is an isomorphism.
(3) We have $(q^*\circ i_*^{-1}\circ p^*)([I]^*) = \phi$.
(4) We have $i_*([I]\times[N]) = q_*([M])$.

We turn to the proof of these four statements:

(1) The top two squares commute by Lemma 59.8 applied to the two maps

$$q\colon (M,\varnothing,\partial M) \;\to\; (M, M\setminus(\mathring{I}\times N), \partial M)$$

and

$$i\colon (I\times N, \partial I\times N, I\times\partial N) \;\to\; (M, M\setminus(\mathring{I}\times N), \partial M).$$

[1167]For completeness' sake we recall that the relative cap product is only defined for excisive triads, see page 1358. So one should think for a second about why the second cap product is in fact defined.

[1168]Note that here for the first three rows we use the convention of page 1360 for what "commutativity" actually means.

Finally regarding the bottom square, note that

$$p^*([I]^*) \cap ([I] \times [N]) \underset{\substack{\uparrow \\ \text{Proposition 56.10}}}{=} p^*([I]^*) \cap ([I \times N]) \underset{\substack{\uparrow \\ \text{Proposition 60.2 (2)}}}{=} [\{0\} \times N] = (\overbrace{[I]^* \cap [I]}^{=[\{0\}]}) \times [N].$$

Since $[I]^* \in H^1(I, \partial I)$ and $[I] \in H_1(I, \partial I)$ are generators it follows from this calculation that the bottom square commutes.

(2) It follows from excision for cohomology groups, see page 1192, that i^* is indeed an isomorphism. We leave it to the reader to verify the details.

(3) We identify the interval I with its image in S^1 under the orientation-preserving map defined by $s \mapsto e^{2\pi i s}$. We denote by $j\colon I \to S^1$ and $j\colon (I, \partial I) \to (S^1, S^1 \setminus \mathring{I})$ the inclusion maps. Furthermore we denote by $r\colon (S^1, \varnothing) \to (S^1, S^1 \setminus \mathring{I})$ the obvious map of pairs of topological spaces. We consider the following diagram

$$\begin{tikzcd}
H^1(S^1) \arrow[rr, "f^*"] & & H^1(M) \\
H^1(S^1, S^1 \setminus \mathring{I}) \arrow[u, "r^*"] \arrow[rr, "f^*"] \arrow[d, "j^*", "\cong"'] & & H^1(M, M \setminus (\mathring{I} \times N)) \arrow[u, "q^*"'] \arrow[d, "i^*"', "\cong"] \\
H^1(I, \partial I) \arrow[rr, "f^* = p^*"] & & H^1(I \times N, \partial I \times N).
\end{tikzcd}$$

Here the map j^* is an isomorphism by excision for cohomology groups. Also note that at the bottom horizontal map we have $f^* = p^*$ by the discussion on page 1615. Furthermore we note that by the functoriality of cohomology groups both squares commute. Finally note that it follows immediately from Lemma 46.23 that $r^*((j^*)^{-1}([I]^*)) = [S^1]^*$. The desired statement is a consequence of the observation that $\phi = f^*([S^1]^*)$.

(4) This statement follows immediately from Lemma 46.23. □

It is natural to ask what we can say about representing homology classes by submanifolds in the other dimensions. In the following two propositions we summarize what is known.

Proposition 69.7. *If M is a compact orientable n-dimensional manifold, then every class in $H_{n-2}(M, \partial M; \mathbb{Z})$ can be represented by a submanifold.*

Proof. The proof of this proposition is similar in spirit to the proof of Proposition 69.6. More precisely, given a class in $H_{n-2}(M, \partial M; \mathbb{Z}) = H^2(M; \mathbb{Z})$ one constructs a smooth map $M \to \mathbb{C}\mathrm{P}^1 = S^2$ and again one takes the preimage of a regular value. We will provide the full proof after we have developed much more theory, see Proposition 69.7. □

In contrast, for low-dimensional classes the following proposition was proved by René Thom. To the best of my knowledge the proof is much less constructive than the proofs of Propositions 69.6 and Proposition 69.7.

Proposition 69.8. *Let M be a compact orientable n-dimensional manifold and let $k \leq 6$ with $k < n$.*[1169] *Then every class in $H_k(M, \partial M; \mathbb{Z})$ can be represented by a submanifold.*

Remark. For $k \leq 5$ the statement can be found in [Tho4, Théorème 6][Tho, Théorème II.27]. The statement for $k = 6$ can be found in the footnote on page 173 of [Tho2].

As so often, the above references deal with "manifolds", without making it entirely clear whether or not this includes the case of manifolds with boundary. But using Lemma 45.11, Lemma 45.12 and the Transversality Theorem 67.2 one can reduce the case of a manifold with boundary to the case of a closed manifold. We leave it to the reader to fill in the details.

Finally note that by [BHK, Theorem 1], extending [Tho, p. 62f], there exists a 10-dimensional closed manifold and a class in $H_7(M; \mathbb{Z})$ that is not represented by a submanifold.

Remark. Propositions 69.6 and Proposition 69.7 also have analogues for topological manifolds. We refer to [FHNOP] for details.

69.3. **The basic Thom-Pontryagin Theory.** In the proof of Proposition 69.6 we saw that a map from a manifold N to S^1 gives rise to a codimension one submanifold. In this section we will go in the opposite direction.

Definition. Let M be a compact oriented n-dimensional manifold and let N be a proper oriented $(n-1)$-dimensional submanifold. By Proposition 64.11 there exists a trivial tubular neighborhood of the form $[-1, 1] \times N$ such that the product orientation of $[-1, 1] \times N$ matches the orientation of $[-1, 1] \times N$, viewed as a codimension zero submanifold of M. We refer to the map[1170]

$$\begin{array}{rcl} \rho_N \colon M & \to & S^1 \\ P & \mapsto & \begin{cases} e^{\pi i t}, & \text{if } P = (t, Q) \in [-1, 1] \times N, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

as the *Thom-Pontryagin map.*[1171] The definition of ρ_N is illustrated in Figure 871.

Remark. The Thom-Pontryagin map ρ_N that we had just constructed depends on the choice of a tubular neighborhood. But it follows from the uniqueness of tubular neighborhoods, see Proposition 64.9, together with Lemma 64.6 (2) that any other choice of a tubular neighborhood $[-1, 1] \times N$ leads to a map $M \to S^1$ that is homotopic to the original map.

Lemma 69.9. *Let M be a compact oriented n-dimensional manifold and let N be a proper oriented proper $(n-1)$-dimensional submanifold. We have*

$$\rho_{N*} \;=\; \mathrm{PD}_M([N]) \;\in\; \mathrm{Hom}(H_1(N; \mathbb{Z}), \mathbb{Z}) \;\underset{\substack{\uparrow \\ \textit{identification from Proposition 51.17}}}{=}\; H^1(N; \mathbb{Z}).$$

[1169] Why do we exclude the case $k = n$?

[1170] It follows easily from Proposition 10.2 and Lemma 1.32 (2') that this map is indeed continuous.

[1171] Lev Pontryagin (1908-1988) was a Soviet topologist, he made many major contributions to topology, despite having lost his eyesight in an accident when he was 14.

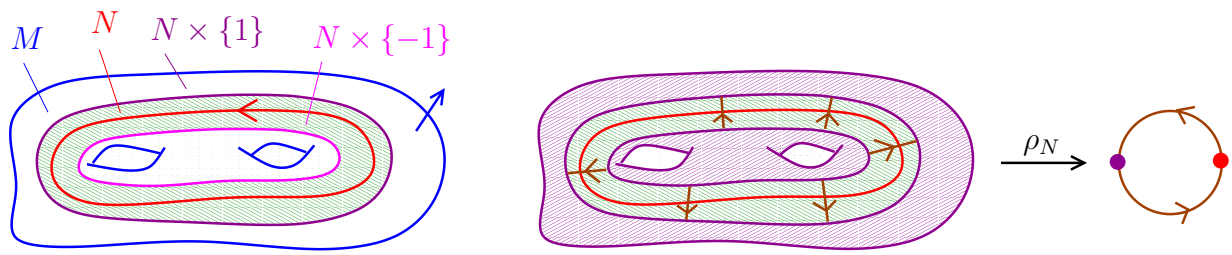


FIGURE 871.

Proof. We had implicitly verified this lemma in the above proof of Proposition 69.6. □

Example. We return to the example that we had discussed on page 1301. More precisely, let Σ be the surface of genus 2. The closed curves x_1, y_1, x_2, y_2 shown in Figure 872 represent a basis for $H_1(\Sigma;\mathbb{Z})$. We consider the epimorphism $\phi\colon H_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By Proposition 55.9 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. Now we will construct such a map explicitly.

We consider the closed curve C that is illustrated in Figure 872. We claim that the corresponding map $\rho_C\colon \Sigma \to S^1$ has the desired property. Put differently, we claim that $\rho_{C*} = \phi \in \operatorname{Hom}(H_1(N;\mathbb{Z}),\mathbb{Z})$. It suffices to show that the two homomorphisms agree on the basis $[x_1], [y_1], [x_2], [y_2]$ of $H_1(\Sigma;\mathbb{Z})$. For example we have

$$\begin{array}{ccccccccc} \rho_{C*}([x_2]) & = & \langle \mathrm{PD}_M([C]), [x_2]\rangle & = & x_2 \cdot C & = & -2 & = & \phi(x_2). \\ & \uparrow & & \uparrow & & \uparrow & & & \\ & \text{Lemma 69.9} & & \text{Proposition 69.1} & & \text{see Figure 872} & & & \end{array}$$

It is straightforward to verify that the same equalities hold for all the other basis elements.

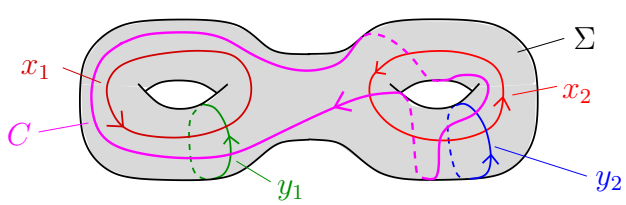


FIGURE 872.

In the remainder of this chapter we study an amusing generalization of the above Thom-Pontryagin map.

Definition. Let M be a compact oriented n-dimensional manifold and let $N_1, \dots, N_k$ be disjoint proper oriented $(n-1)$-dimensional submanifolds. As on page 1618 we pick a trivial tubular neighborhood of the form $[-1,1] \times (N_1 \cup \dots \cup N_k)$. We consider the wedge $\bigvee_{j=1}^{k} S^1_j$ given by identifying the points $-1 \in S^1_j$ to a single point $*$. We again refer to the

map

$$\begin{array}{rcl} \rho_{N_1,\dots,N_k}\colon M & \to & \bigvee_{j=1}^{k} S^1_j \\ P & \mapsto & \begin{cases} e^{\pi i t} \in S^1_j, & \text{if } P=(t,Q)\in[-1,1]\times N_j, \\ *, & \text{otherwise} \end{cases} \end{array}$$

as the *Thom-Pontryagin map*. The definition of ρ_N is illustrated in Figure 873.

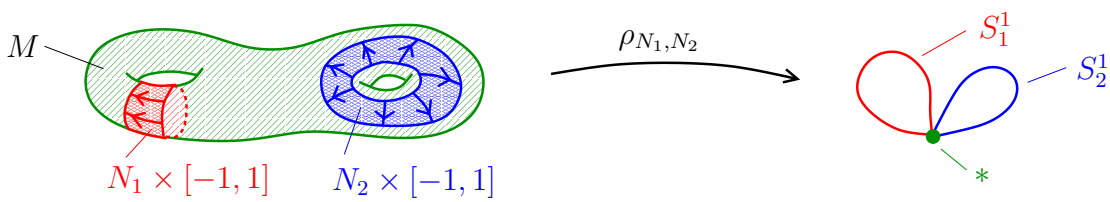


FIGURE 873.

Before we continue we recall that in Lemma 12.4 we gave an identification

$$\langle t_1,\dots,t_k\rangle \;=\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, *\Big)$$

where each t_j is represented by a loop that goes once around the j-th circle S^1_j in a positive direction.

Proposition 69.10. *Let M be a compact oriented connected n-dimensional manifold and let $N_1,\dots,N_k$ be disjoint proper oriented non-empty $(n-1)$-dimensional submanifolds. If $M\setminus(F_1\cup\dots\cup F_k)$ is connected, then for any $x_0\in M$ the map*

$$\rho_{N_1,\dots,N_k}\colon \pi_1(M,x_0) \;\to\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, *\Big) \;=\; \langle t_1,\dots,t_k\rangle$$

is an epimorphism.

Proof. We write $N=N_1\cup\dots\cup N_k$ and $\rho=\rho_{N_1,\dots,N_k}$. We pick a trivial tubular neighborhood of the form $[-1,1]\times N$. It follows from our hypothesis that $M\setminus N$ is connected, together with Proposition 64.13 and Lemma 64.3 that $M\setminus(-1,1)\times N$ is also connected. Since M is connected it follows from Exercise 6.4 that it suffices to consider a base point x_0 that lies in $M\setminus(-1,1)\times N$.

For $j=1,\dots,k$ we do the following:

(1) we pick a point $P_i\in N_i$,
(2) since $M\setminus(-1,1)\times N$ is connected we can pick a path α_j from x_0 to $(-1,P_j)$ and we can pick a path γ_j from $(1,P_j)$ to x_0,
(3) we denote by β_j the "horizontal" path in $[-1,1]\times\mathbb{N}_j$ from $(-1,P_j)$ to $(1,P_j)$,
(4) we write $\delta_j := \alpha_j * \beta_j * \gamma_j$.

It follows easily from the definitions that for $j = 1, \ldots, k$ we have $\rho_*([\delta_j]) = t_j$. We have thus shown that the generating set $\{t_1, \ldots, t_k\}$ lies in the image of ρ_*. In other words, ρ_* is indeed an epimorphism. $\square$

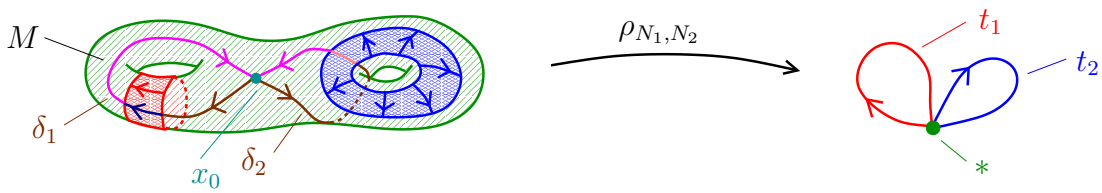


FIGURE 874. Illustration for the proof of Proposition 69.10.

Example. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. As should be pretty clear from considering Figure 875, the surface Σ_g contains g disjoint curves $N_1, \ldots, N_g$ such that the complement is connected. Thus it follows from Proposition 69.10 that there exists an epimorphism

$$\pi_1(\Sigma_g) \ \to \ \langle x_1, \ldots, x_g \rangle.$$

In fact, as we will see in Exercise 69.2, it is not difficult to show the existence of such an epimorphism on a more algebraic level.

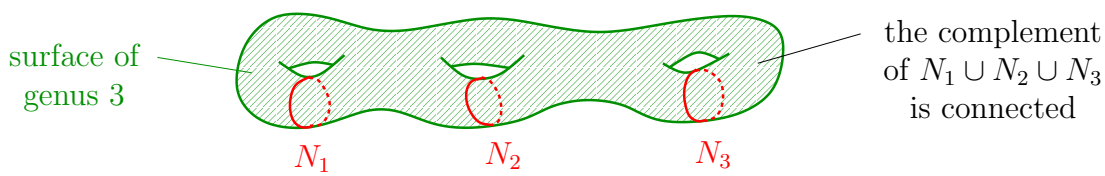


FIGURE 875.

The previous example raises the following thought-provoking question.

Question 69.11. *Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.*

(1) *What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \ldots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \cdots \cup F_k)$ is connected?*
(2) *What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of k generators?*

In summary, as of right now we know the following:

$$\begin{array}{ccccccc} g & \underset{\substack{\uparrow \\ \text{by the above} \\ \text{example}}}{\leq} & k(g) & \underset{\substack{\uparrow \\ \text{Proposition 69.10}}}{\leq} & l(g) & \underset{\substack{\uparrow \\ \text{Proposition 13.10}}}{\leq} & 2g. \end{array}$$

Exercises for Chapter 69.

Exercise 69.1. Let M be a *non-compact* oriented n-dimensional manifold with compact boundary. Furthermore let N be a compact orientable non-empty proper submanifold of codimension one such that $M \setminus N$ is connected. Show that $[N] \in H_{n-1}(M;\mathbb{Z})$ is an element of infinite order.
Hint: Use Propositions 65.4 and 69.2.

Exercise 69.2. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. By Proposition 14.3 we know that there exists an isomorphism

$$\pi_1(\Sigma_g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle.$$

Use this isomorphism to give a purely algebraic proof that $\pi_1(\Sigma_g)$ admits an epimorphism onto a free group on g generators.

Exercise 69.3. Here is one approach to answering Question 69.11 (1). Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. Let $F_1, \dots, F_k$ be disjoint curves on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected. By the Tubular Neighborhood Theorem 10.19 we can pick a tubular neighborhood $\bigcup_{i=1}^{k} [-1,1] \times F_i$ for the submanifold $F_1 \cup \dots \cup F_k \subset \Sigma_g$. We consider the compact orientable 2-dimensional manifold $X := \Sigma_g \setminus \bigcup_{i=1}^{k} (-1,1) \times F_i$. What can you say about the Euler characteristic of X? What does the classification of compact orientable 2-dimensional manifolds tell you?

70. Applications to knot theory

After so much theory we want to practice what we have learned by applying our recent results to knot theory.

70.1. **Curves on 2-dimensional manifolds** $(*)$**.** We recall the following definition from page 306: A *closed oriented curve* C in a manifold M is a closed oriented connected 1-dimensional submanifold of M or of ∂M. In this section we want to study curves on 2-dimensional manifolds. Our main focus will be curves on the torus.

Before we state the results we recall that on page 853 we saw that a closed oriented curve C in a manifold M defines a homology class $[C] \in H_1(M;\mathbb{Z})$. We start out with the following lemma.

Lemma 70.1. *Let $T = S^1 \times S^1$ be the torus and let C be a closed oriented curve on T.*

(1) *If $T \setminus C$ is connected, then $[C]$ is a primitive element of $H_1(T;\mathbb{Z})$.*
(2) *If $T \setminus C$ is disconnected, then C is the boundary of an embedded disk.*

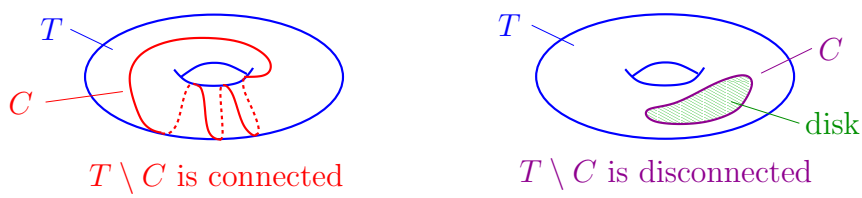


Figure 876. Illustration of Lemma 70.1.

Proof. If $T \setminus C$ is connected, then it follows from Proposition 69.2 that $[C]$ is a primitive element of $H_1(T;\mathbb{Z})$. Now suppose that $T \setminus C$ is disconnected. By the Tubular Neighborhood Theorem 10.19 there exists a tubular neighborhood $[-1,1] \times C$. From Lemma 64.3 and Proposition 64.13 we obtain that $T \setminus (-1,1) \times C$ is also disconnected. Since C is connected this implies that $T \setminus (-1,1) \times C$ has two components A and B, one of which contains $\{-1\} \times C$ and one which contains $\{1\} \times C$.[1172] By Proposition 3.24 we know that A and B are compact 2-dimensional manifolds, in particular they are finite CW-complexes. We have

$$\chi(A) + \chi(B) \underset{\substack{\uparrow \\ \text{Lemma 40.5}}}{=} \underbrace{\chi(T)}_{=0} + \underbrace{\chi(\{-1\} \times C)}_{=0} + \underbrace{\chi(\{1\} \times C)}_{=0} = 0.$$

Since A and B have boundary it follows from the classification of compact 2-dimensional manifolds and the calculation of Euler characteristics, see Lemma 40.6, that one of A or B is diffeomorphic to the disk $\overline{B}^2$. $\square$

[1172] Why is that?

Lemma 70.2. *Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. If C and D are disjoint and if $[C]$ and $[D]$ are both non-zero*[1173] *in $H_1(T;\mathbb{Z})$, then $[C] = \pm[D]$.*

Proof. From Theorem 67.10 together with our hypothesis that $C \cap D = \varnothing$ we obtain that $\mathrm{PD}_T([C]) \cup \mathrm{PD}_T([D]) = 0$. It follows easily from the calculation of the cup product of the torus, see Lemma 57.14, that $\mathrm{PD}_T([C]) = k \cdot \mathrm{PD}_T([D])$ for some $k \in \mathbb{Z}$. We obtain from Lemma 70.1 that $k \in \{-1, 1\}$. □

Lemma 70.3. *Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. Suppose that $[C] = [D] \neq 0$, then the following hold:*

1. *the two closed oriented curves are smoothly isotopic,*
2. *if C and D are furthermore disjoint, then there exists an embedded annulus $X \subset T$ with $\partial X = C \cup -D$.*

Furthermore, the same statements also hold if we replace the torus $T = S^1 \times S^1$ by the annulus $S^1 \times [-1, 1]$.

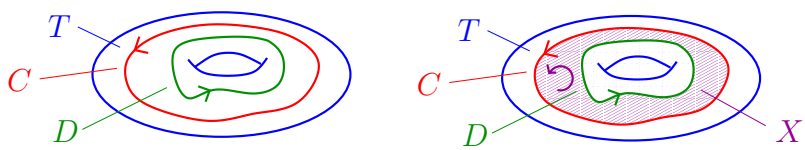


FIGURE 877. Illustration of Lemma 70.3 (2)

Sketch of proof. We provide a sketch of a proof for the various statements:

1. By the Transversality Theorem 67.2 (2) we can assume that C and D are already transverse. It follows from the calculation of the cup product of the torus and Theorem 67.10 that the algebraic intersection number is 0. The argument is now similar to the argument of the proof of Theorem 67.22, but this time we use Lemma 70.1 to find the desired embedded disks. We refer to Figure 878 for an illustration.

   It is a good, and not entirely trivial, mental exercise to fill in the details. The argument is sketched in more detail in [Rol, Lemma 5 of Chapter 2.C].
2. The proof of this statement is similar to the proof of Lemma 70.1 (2). We leave it to the reader to fill in the details.

The proof in the case that $T = S^1 \times [-1, 1]$ is easier. We will discuss it in Exercise 70.2. □

For completeness sake we state the following proposition, even though we will not make use of it.

Proposition 70.4. $(*)$ *Let F be a 2-dimensional manifold and let C and D be two closed oriented curves on F.*

[1173] Is this condition that $[C]$ and $[D]$ are both non-zero necessary?

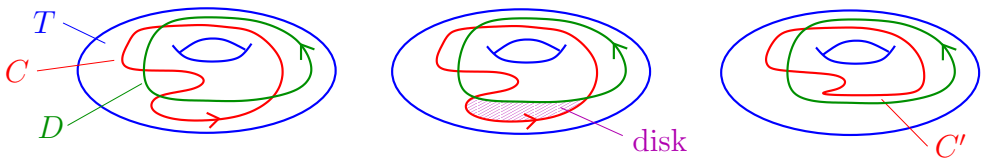


FIGURE 878.

(1) *If C and D are homotopic, then C and D are smoothly isotopic.*
(2) *if C and D are furthermore disjoint and if they are not null-homotopic, then there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$.*

Proof.

(1) This statement is [FaM, Proposition 1.10]. (Closely related statements are also proved in [Baer], [Eps2, Theorem 2.1].) Unfortunately all the references are somewhat cavalier when it comes to distinguishing continuous and smooth maps, but with some effort one can see that all the references provide smooth isotopies.
(2) If F is orientable, then we will provide the proof in Exercise 70.5. The proof of the non-orientable case is not much harder. □

In Exercise 70.1 we will use Proposition 10.17 to prove the following corollary to Lemma 70.3.

Corollary 70.5. *Let $T = S^1 \times S^1$ be the torus and let C and D be two closed oriented 1-dimensional submanifolds of T. If $[C] = [D] \in H_1(T;\mathbb{Z})$, then there exists a compact oriented proper 2-dimensional submanifold Ω of $T \times [0,1]$ such that*

$$\partial\Omega \;=\; C \times \{0\} \cup -D \times \{1\}.$$

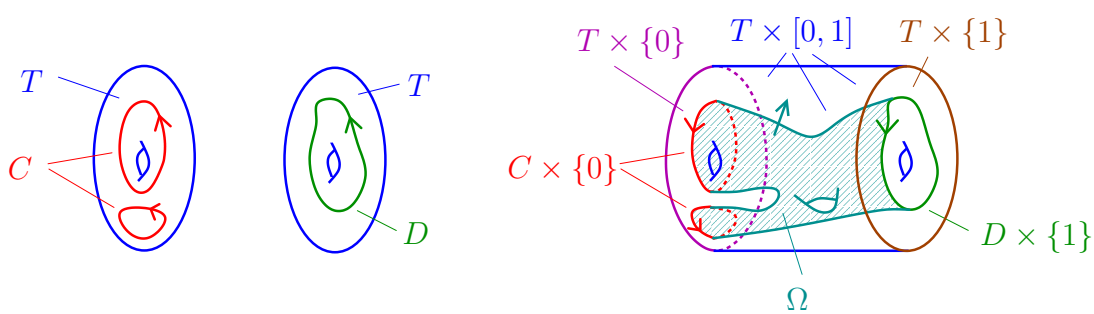


FIGURE 879. Illustration of Corollary 70.5.

70.2. **The longitude of a knot.** We start out our discussion of knots with the following definition.

Definition. Let $K \subset S^3$ be an oriented knot.

(1) A *fattening of* K is an orientation-preserving embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $F(0,P) = P$ for all $P \in K$.
(2) Given a fattening F of K we refer to $X_K := S^3 \setminus F(B^2 \times K)$ as the *knot exterior*.
(3) Given a fattening F of K and given $* \in K$ we refer to the oriented submanifold $\mu_K := F(S^1 \times \{*\})$ of $S^3 \setminus K$ as the *meridian* of K.
(4) By Lemma 40.7 (5) the inclusion induced map $H_1(X_K;\mathbb{Z}) \xrightarrow{\cong} H_1(S^3 \setminus K;\mathbb{Z})$ is an isomorphism and we have $H_1(X_K;\mathbb{Z}) = \mathbb{Z}\cdot[\mu_K]$. We denote by ϕ_K the isomorphism

$$\begin{array}{rcl} \phi_K\colon H_1(S^3 \setminus K;\mathbb{Z}) = H_1(X_K;\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ [\mu_K] & \to & 1. \end{array}$$

(5) Suppose we are given a fattening F for K. We define

$$\text{self-linking number of } F \ := \ \phi_K(\underbrace{F_*([\{1\} \times K])}_{\in H_1(X_K;\mathbb{Z})}).$$

Remark. It follows from the uniqueness of tubular neighborhoods, see Proposition 10.20, that the exterior X_K and the meridian μ_K are, in an appropriate sense, independent of the choice of the fattening. Therefore often we just say "the exterior" and "the meridian".

Lemma 70.6. *Let K be an oriented knot in S^3.*

(1) *Let $F\colon \overline{B}^2 \times S^1 \to S^3$ be a fattening of K and let $s \in \mathbb{Z}$. As in Lemma 64.6 we consider the homeomorphism*

$$\begin{array}{rcl} \Psi\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (w,z) & \mapsto & (w \cdot z^s, z). \end{array}$$

The map $F \circ \Psi$ is a fattening with

$$\textit{self-linking number of } F \circ \Psi \ = \ \textit{self-linking number of } F + s.$$

(2) *Given any $n \in \mathbb{Z}$ there exists a fattening of K with self-linking number n.*
(3) *If $F, F'\colon \overline{B}^2 \times K \to S^3$ are two fattenings with self-linking number n, then there exists a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1 \circ F = F'$.*

Given $K \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ the analogous statements hold if we consider fattenings that are contained in $\mathbb{R}^3$. Furthermore, in this setting the diffeotopy in (3) *of $\mathbb{R}^3$ can be assumed to have compact support.*

Remark. Given an oriented knot K and $n \in \mathbb{Z}$ sometimes we use the uniqueness statement of Lemma 70.6 to talk about *the fattening of K with self-linking number n*. Sometimes we shorten his cumbersome expression to *the n-fattening of K*.

Proof.

(1) We denote by m the self-linking number of F. We write $\lambda := F(\{1\} \times K)$ and $\mu := F(S^1 \times \{1\})$ and we view λ and μ as oriented 1-dimensional submanifolds with

the orientations given by K and S^1. We calculate that

$$\text{self-linking number of } F\circ\Psi = \phi_K(F_*(\Psi_*([\{1\}\times K])) \underset{\substack{\uparrow \\ \text{follows from Lemma 38.7 (2)}}}{=} \phi_K(s\cdot[\mu]+[\lambda]) \underset{\substack{\uparrow \\ \text{since } \phi_K(\mu)=1 \text{ and} \\ \text{by definition of } m}}{=} s+m.$$

(2) From the Tubular Neighborhood Theorem 10.19 we know that there exists a fattening $F\colon \overline{B}^2\times K\to S^3$. We denote by m the self-linking number of F. The desired statement now follows from (1) applied to $n-m$.

(3) The uniqueness statement is an immediate consequence of (1) together with Proposition 64.9 and Lemma 64.6 (3).

The same arguments as above also apply if we replace S^3 by $\mathbb{R}^3$. □

Definition. Let $K\subset S^3$ be an oriented knot. We pick a fattening $F\colon \overline{B}^2\times K\to S^3$ of self-linking number zero. We refer to $\lambda_K = F(\{1\}\times K)$ as *longitude* of K.[1174]

Examples.

(1) In Figure 880 we show the longitude and the meridian for the trivial knot and the Figure-8 knot.

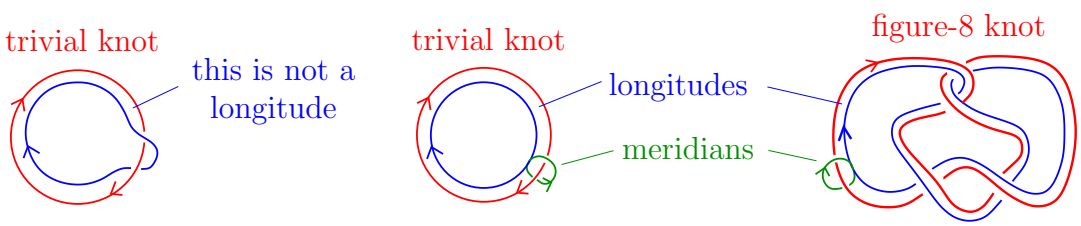


FIGURE 880.

(2) In Figure 881 we show the trefoil knot with three wannabe longitudes. It is not entirely clear which one is the right longitude. We will soon develop a tool with which it will be straightforward to settle this issue.

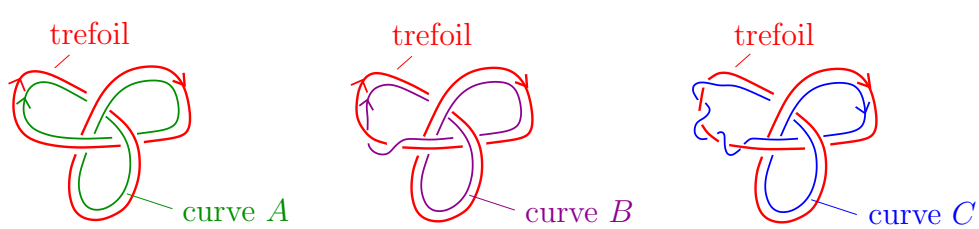


FIGURE 881.

[1174]It follows from Lemma 70.6 (3) that the longitude is in an appropriate way uniquely determined.

On several occasions the following lemma comes in handy.

Lemma 70.7. *Let K be an oriented knot and let $F\colon \overline{B}^2 \times K \to S^3$ be a fattening with self-linking number zero. We denote by $X_K = S^3 \setminus F(B^2 \times K)$ the corresponding knot exterior. Furthermore we denote as usual by $\mu_K = F(S^1 \times \{*\})$ and $\lambda_K = F(\{1\} \times K)$ the corresponding meridian and longitude.*

(1) *The classes $[\mu_K]$ and $[\lambda_K]$ form a basis for $H_1(\partial X_K; \mathbb{Z})$.*
(2) *There exists a short exact sequence of the following form*

$$\begin{array}{ccccccc} 0 & \to & H_2(X_K, \partial X_K; \mathbb{Z}) & \to & \overbrace{H_1(\partial X_K; \mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\lambda_K]} & \to & \overbrace{H_1(X_K; \mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]} \to 0 \\ & & & & [\mu_K] & \mapsto & [\mu_K] \\ & & & & [\lambda_K] & \mapsto & 0. \end{array}$$

(3) *The longitude λ_K is the unique, up to smooth isotopy, closed oriented curve on the torus $F(S^1 \times K)$ that is homologous to $K = \{0\} \times K$ in $F(\overline{B}^2 \times K)$ and that is null-homologous in X_K.*
(4) *If we view μ_K and λ_K as closed oriented curves on the oriented torus $F(S^1 \times K)$, then the algebraic intersection number equals $\mu_K \cdot \lambda_K = +1$.*
(5) *If denote by $i\colon \mu_K \to \partial X_K$ the inclusion map, then*

$$i^*([\mu_K]) \cap [\partial X_K] \;=\; -[\lambda_K] \;\in\; H_1(\partial X_K; \mathbb{Z}).$$

Proof.

(1) This statement follows from Lemma 33.13.
(2) We consider the long exact sequence of homology groups of the pair $(X_K, \partial X_K)$:

$$\underbrace{H_2(X_K; \mathbb{Z})}_{=0 \text{ by Lemma 40.7 (5)}} \to H_2(X_K, \partial X_K; \mathbb{Z}) \to H_1(\partial X_K; \mathbb{Z}) \xrightarrow{\phi} H_1(X_K; \mathbb{Z}).$$

By definition we have $\phi([\lambda_K]) = 0$. Furthermore we know from Lemma 40.7 that the right-hand map is an epimorphism. The statement follows immediately from these observations.
(3) It follows from (1) that the kernel of $H_1(\partial X_K; \mathbb{Z}) \to H_1(X_K; \mathbb{Z})$ equals $\mathbb{Z} \cdot [\lambda_K]$. Since $H_1(F(\overline{B}^2 \times K); \mathbb{Z}) = \mathbb{Z} \cdot [K] = \mathbb{Z} \cdot [\lambda_K]$ we see that there exists a unique homology class that has the properties we are interested in. The desired statement now follows from Lemma 70.3 (1).
(4) This statement follows from the definitions and the easy observation that the oriented submanifolds $S^1 \times \{1\}$ and $\{1\} \times S^1$ of the oriented manifold $S^1 \times S^1$ have algebraic intersection number one.

(5) We calculate that

$$\begin{array}{rcl} i^*([\mu_K]) \cap [\partial X_K] & = & i^*([\mu_K]) \cap F_*(-[S^1 \times K]) \\ & \uparrow & \\ & \text{since } F_*([S^1 \times S^1]) = [\partial F(\overline{B}^2 \times K)] = -[\partial X_K] & \\ & = & -F_*([\{1\} \times K]) \\ & \uparrow & \\ & \text{see page 1374} & \end{array} \quad \begin{array}{rcl} & = & F_*\big([S^1 \times \{*\}]^* \cap (-[S^1 \times K])\big) \\ & \uparrow & \\ & \text{Lemma 59.8} & \\ & = & -[\lambda_K]. \\ & \uparrow & \\ & \text{by definition of } \lambda_K & \end{array}$$

□

70.3. **Seifert surfaces.**

Definition. Let $K \subset S^3$ be a knot. A *Seifert surface*[1175] *for* K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented manifold.

Examples.

(1) In Figure 882 we show Seifert surfaces for the trivial knot, the trefoil and the Figure-8 knot.

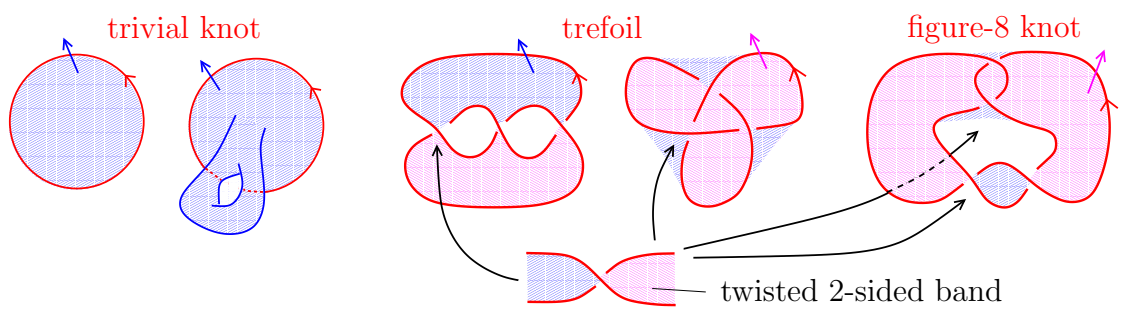


FIGURE 882.

(2) In Figure 883 we see two surfaces of minimal surface area that bound the trefoil, both were manufactured by Jürgen Neukirch. The one on the left is orientable, thus it is a Seifert surface. The one on the right is diffeomorphic to the Möbius band, thus it is not a Seifert surface.

The following proposition is arguably one of the most useful results in knot theory.

Proposition 70.8. *Every knot $K \subset S^3$ admits a Seifert surface.*

Remark.

[1175]Seifert surfaces are named after Herbert Seifert (1907-1996) whom we had already encountered at the Seifert-van Kampen Theorem 12.2.

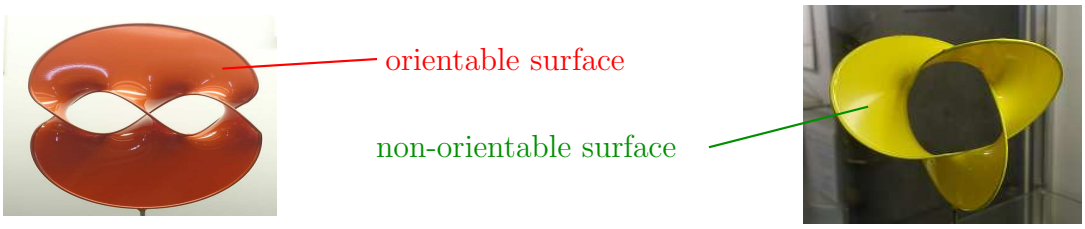


FIGURE 883.

(1) The proposition was first proved in 1930 by F. Frankl and Lev Pontryagin [FPo] in 1930. Shortly afterwards Herbert Seifert [Sei2] gave a different proof. More precisely Seifert found an algorithm that given a diagram for a knot produces an explicit Seifert surface for the knot. Seifert's algorithm is also described in most books on knot theory, see e.g. [Rol, p. 120] or [Lic3, Theorem 2.2].
(2) A slight generalization of the argument we give proves the following stronger statement: given any closed orientable n-dimensional manifold $V \subset S^{n+2}$ there exists a compact orientable $(n+1)$-dimensional submanifold $W \subset S^{n+2}$ such that $\partial W = V$. We refer to [Bre, Theorem VII.14.7], [Er, Lemma 2.2] and [BDR] for details. The proof in these two references is close to the approach taken in the proof of Proposition 70.8, but it circumvents the use of Proposition 69.6 by a more formal argument.

Proof. Let K be a knot. We need to show that there exists a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$.

By Lemma 70.6 (2) we can pick a fattening $2\overline{B}^2 \times K$ of the knot K with self-linking number zero. We denote by $X := S^3 \setminus (2B^2 \times K)$ the knot exterior. By Lemma 70.7 (2) we know that there exists a unique homology class $\sigma \in H_2(X, \partial X; \mathbb{Z})$ with $\partial\sigma = [\{2\} \times K] \in H_1(\partial X; \mathbb{Z})$. It follows from Proposition 69.6 that there exists a compact oriented proper 2-dimensional submanifold Σ of X such that $[\Sigma] = \sigma$. We then also have $[\partial\Sigma] = [\{2\} \times K] \in H_1(\partial X; \mathbb{Z})$.[1176]

We consider the torus $T = 2S^1 \times K$. Since $[\{2\} \times S^1] = [\partial\Sigma]$ we obtain from Corollary 70.5 a 2-dimensional submanifold Ω of

$$[1,2] \times S^1 \times K \;=\; (2\overline{B}^2 \setminus B^2) \times K \;=\; \{rz \,|\, r \in [1,2], z \in S^1\} \times K$$

[1176]Here is the argument for this claim. We denote by $i\colon \Sigma \to X$ the inclusion map. Recall that by definition of a proper submanifold we have $\partial\Sigma \subset \partial X$. We consider the following diagram

$$\begin{array}{ccc} H_2(\Sigma, \partial\Sigma; \mathbb{Z}) & \xrightarrow[{[\Sigma]\mapsto[\partial\Sigma]}]{\partial} & H_1(\partial\Sigma; \mathbb{Z}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ H_2(X, \partial X; \mathbb{Z}) & \xrightarrow[{\sigma\mapsto[\{2\}\times K]}]{\partial} & H_1(\partial X; \mathbb{Z}) \end{array}$$

given by the connecting homomorphisms of the pairs $(X, \partial X)$ and $(\Sigma, \partial\Sigma)$. In the diagram we already used that by Proposition 46.26 we have $\partial([\Sigma]) = [\partial\Sigma] \in H_1(\partial X; \mathbb{Z})$. The statement now follows from the observation that this diagram commutes by Proposition 31.13 (3).

such that $\partial\Omega = (\{1\} \times K) \cup -\partial\Sigma$. We apply Proposition 10.17 twice to

$$([0,1] \times K) \cup \Omega \cup \Sigma \;\subset\; (\overline{B}^2 \times K) \cup (\{rz \,|\, r \in [1,2], z \in S^1\} \times K) \cup X = S^3$$

and we obtain a compact orientable 2-dimensional submanifold Z whose boundary equals K. This submanifold might not be connected, but the component of Z that contains K is the desired Seifert surface. $\square$

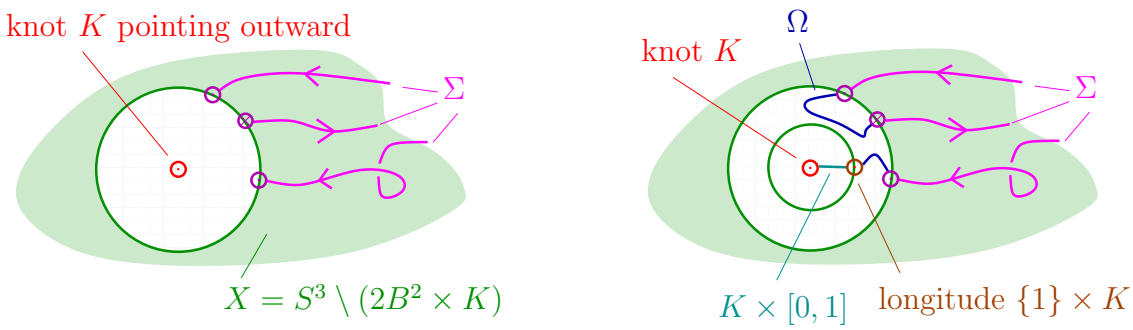


FIGURE 884. Illustration for the proof of Proposition 70.8.

Definition. Let F be a compact orientable connected 2-dimensional manifold with m boundary components. By Theorem 14.14 there exists a unique $g \in \mathbb{N}_0$ such that F is homeomorphic to the surface of genus g with m open disks removed. We refer to g as the *genus of* F.

Given a Seifert surface it is sometimes not entirely obvious how to determine the genus. The following lemma gives an easy criterion to determine the genus of a Seifert surface.

Lemma 70.9. *Let F be a compact orientable connected 2-dimensional submanifold with one boundary component. If $\Gamma \subset F$ is a graph that is a deformation retract of F, then*

$$\textit{genus of } F \;=\; \tfrac{1}{2}(1 - \chi(F)) \;=\; \tfrac{1}{2}(1 - \#\textit{vertices of } \Gamma + \#\textit{edges of } \Gamma).$$

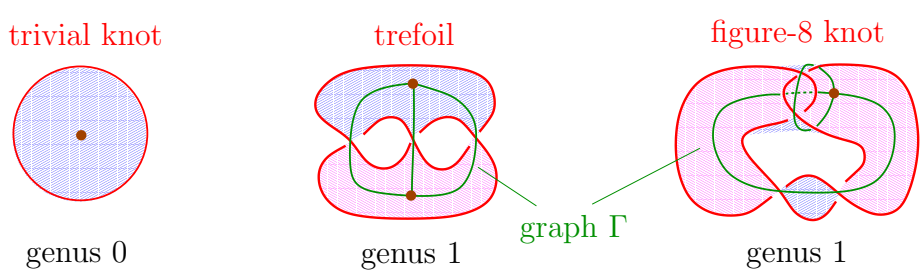


FIGURE 885.

Proof. We have

$$\text{genus of } F \underset{\substack{\uparrow \\ \text{by Proposition 35.10 we} \\ \text{have } \chi(F)=1-2g}}{=} \tfrac{1}{2}(1-\chi(F)) \underset{\substack{\uparrow \\ \text{by Corollary 40.2 since } \Gamma \text{ is a} \\ \text{deformation retract of } F}}{=} \tfrac{1}{2}(1-\chi(\Gamma)).$$

□

If a knot K bounds a Seifert surface of genus g, then the construction sketched in Figure 886 shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$.

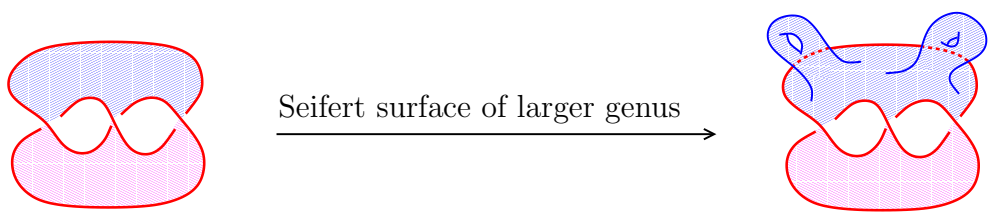


FIGURE 886.

So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition.

Definition. Given a knot $K \subset S^3$ we define

$$\text{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

As we have seen in Figure 885, the trivial knot has genus 0. The following lemma gives the converse to this observation.

Lemma 70.10. *Let K be a knot. If the genus of K is zero, then K is equivalent to the trivial knot.*

Sketch of proof. Let K be a knot with genus 0. This means that K bounds a disk, i.e. there exists an embedding $\varphi\colon \overline{B}^2 \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ with $\varphi(S^1) = K$. We write $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and we make the identification $\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{R}^3$. It suffices to show that there exists a smooth isotopy from φ to an embedding $\psi\colon \overline{B}^2 \to \mathbb{R}^3$ such that $\psi(S^1) = S^1 \subset \mathbb{C}$. We perform the following steps:

(0) Using that SO(4) is connected, see Lemma 1.96 (1), we can assume after a smooth isotopy that $\infty \notin \varphi(\overline{B}^2)$.
(1) Using that $\mathrm{SL}(3,\mathbb{R})$ is connected, see again Lemma 1.96 (2), we can find a smooth isotopy such that afterwards we have $\varphi(0) = 0$ and $(D_0\varphi)(T_0\overline{B}^2) = \mathbb{C} \times \{0\}$.
(2) We can shrink the disk such that the projection $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \{0\}$ is injective on the disk.[1177]

[1177] At this point we could just apply Theorem 10.28 to conclude that K is equivalent to the trivial knot. Just for fun we indicate how one can prove the lemma "by hand".

(3) We can find a smooth isotopy to end up with an embedding $\varphi\colon \overline{B}^2 \to \mathbb{C}$ with $\varphi(0) = 0$ and $D_0\varphi = \mathrm{id}$.
(4) After possibly further shrinking the disk, the image of the embedding $\varphi\colon \overline{B}^2 \to \mathbb{C}$ is convex.
(5) The map

$$\begin{array}{rcl} F\colon \overline{B}^2 \times [0,1] & \to & \mathbb{C} \\ z & \mapsto & \varphi(z)\cdot(1-t) + z\cdot t \end{array}$$

is the final smooth isotopy that we were required to find.

We illustrate the steps (2) to (5) in Figure 887. We leave it to the reader to fill in all the technical details and to make all steps rigorous.[1178] □

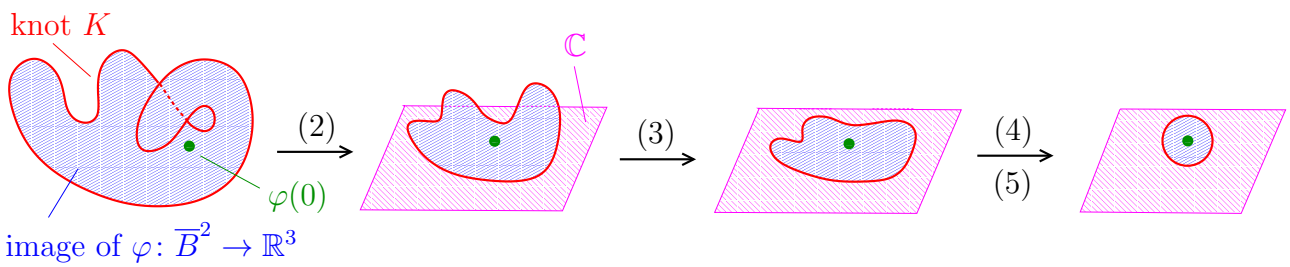


FIGURE 887.

Example. In Lemma 18.6 (2) we saw that the trefoil is not equivalent to the trivial knot. We had also just seen that the trefoil admits a Seifert surface of genus 1. It follows from Lemma 70.10 that

$$\mathrm{genus}(\text{trefoil}) \;=\; 1.$$

We conclude this section with the following question.

Question 70.11. *We consider the knot K in Figure 888. It admits a Seifert surface of genus two. Does it also admit a Seifert surface of genus one?*

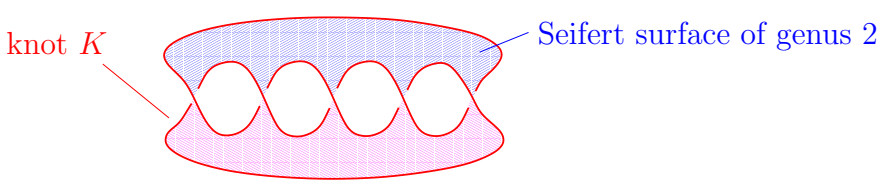


FIGURE 888.

[1178]It might be slightly painful, but it is not overly difficult to make all steps rigorous. Step (4) is a very nice exercise in real analysis.

70.4. **Linking numbers.** We recall the following definition from page 1535.

Definition. An *m-component link* is a closed 1-dimensional submanifold of S^3.

Remark. It follows from Proposition 3.53 that a link is the same as a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of finitely many copies of S^1.

Convention. When we write, "let $L = L_1 \sqcup \cdots \sqcup L_m$ be an *m-component link*", then it is understood that each L_i is non-empty. In particular each L_i is a knot in its own right.

Definition. We say that *two m-component links $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ are equivalent* if there exists a smooth isotopy from L to J which preserves the ordering of the components. If L and J are oriented links, then we also demand that the isotopy is orientation-preserving.

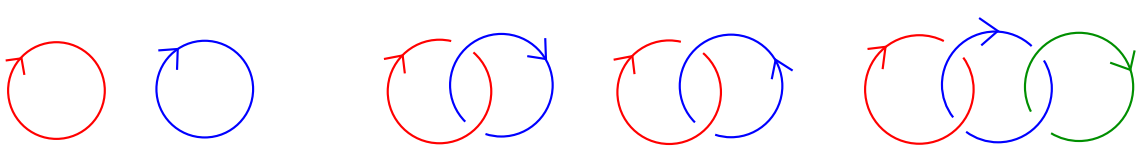
Examples.

(1) We say $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ is a *trivial link* if it lies in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and[1179] if it bounds m disjoint disks in $\mathbb{R}^2 \times \{0\}$. Using Theorem 10.28 one can easily verify that all trivial links are in fact equivalent.

(2) In Figure 889 we show four examples of oriented links: the 2-component trivial link, two versions $H^\pm$ of the Hopf link and one further link. The Hopf link is in fact related to the Hopf map

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

from page 726. More precisely, if P, Q are two distinct points in S^2, then the preimage $H^{-1}(\{P, Q\})$ is a 2-component link that is equivalent to the Hopf links without orientation. This can be seen as follows: for $P = [1 : 0]$ and $Q = [0 : 1]$ one can verify "by hand" that the preimage is equivalent to a Hopf link. Given any two other distinct points P' and Q' in S^2 we can find disjoint smooth paths from P to P' and Q to Q', the preimages of these paths give rise to a smooth isotopy of links.



2-component trivial link $\quad H^-$ Hopf links H^+ $\quad$ chain link

FIGURE 889.

Our goal now is to distinguish inequivalent links. It follows immediately from the definition that if two links $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ are equivalent, then

[1179] Is the second condition necessary?

the components are equivalent knots. Therefore we are mostly interested in distinguishing links with equivalent components.

The following proposition generalizes Proposition 10.33. The proof is verbatim the same.

Proposition 70.12. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component links. If they are equivalent, then the following two statements hold:*

(1) *There exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(L_i) = J_i$ for $i = 1, \dots, m$. If L and J are oriented, then there exists such a diffeomorphism Φ which for each $i \in \{1, \dots, m\}$ restricts to an orientation-preserving diffeomorphism $L_i \to J_i$.*

(2) *The link complements $S^3 \setminus L$ and $S^3 \setminus J$ are diffeomorphic, in particular homeomorphic.*

Remark. On page 361 we remarked that for knots the converse to Proposition 10.33 (2) holds. More precisely, if K and J are two knots such that there exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are equivalent. The analogous statement is no longer true for links. In Figure 890 we show two links $L = K \sqcup J$ and $L' = K' \sqcup J'$. In Exercise 70.7 we will see that $S^3 \setminus L$ and $S^3 \setminus L'$ are diffeomorphic. Even though we cannot prove it at the moment, it should be fairly clear that L and L' are not equivalent. This shows in particular that the problem of classifying links cannot be reduced to the problem of classifying the complements of links.

FIGURE 890.

To distinguish links with equivalent components we will now introduce an invariant of oriented links.

Definition. Let $K \subset S^3$ be an oriented knot. As before we denote by μ_K the meridian as defined on page 362 and as above we denote by ϕ_K the epimorphism $\phi_K\colon H_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$ that is given by $\phi_K([\mu_K]) = 1$. If $J \subset S^3$ is an oriented knot that is disjoint from K, then we define the *linking number of K and J* to be

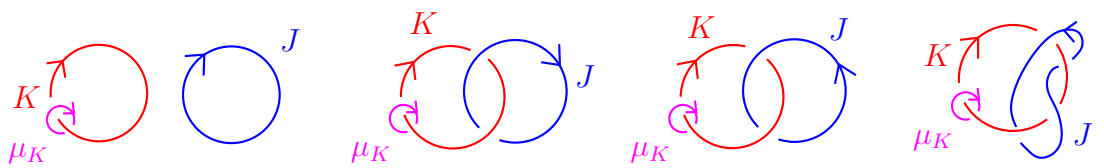
$$\operatorname{lk}(K, J) \;:=\; \phi_K([J])$$

where $[J] \in H_1(S^3 \setminus K; \mathbb{Z})$ denotes the image of the fundamental class of the oriented manifold J under the inclusion $J \to S^3 \setminus K$.

Remark. We continue with the above notation. If we denote by $[\mu_K^*] \in H^1(S^3 \setminus K; \mathbb{Z})$ the unique generator with $\langle [\mu_K^*], [\mu_K] \rangle = 1$, then it follows immediately from the definitions that for any oriented knot J disjoint from K we have

$$\mathrm{lk}(K, J) \;=\; \langle [\mu_K^*], [J] \rangle.$$

Example. If K is the trivial knot, then we saw in Proposition 18.4 that $S^3 \setminus K$ is diffeomorphic to the open solid torus $B^2 \times S^1$ and it is fairly straightforward to determine $\phi_K \colon H_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$ applied to a given curve. This allows us to determine the linking numbers of the oriented links shown in Figure 891.



$\mathrm{lk}(K, J) = 0$ $\mathrm{lk}(K, J) = -1$ $\mathrm{lk}(K, J) = 1$ $\mathrm{lk}(K, J) = 2$

FIGURE 891.

Proposition 70.13.

(1) *Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component oriented links. If L and J are equivalent, then for all $r, s \in \{1, \dots, m\}$ we have*

$$\mathrm{lk}(L_r, L_s) \;=\; \mathrm{lk}(J_r, J_s).$$

(2) *Let $K \sqcup J$ be an oriented 2-component link. If we change the orientation of one of the components of the link, then the sign of the linking number flips.*

Proof. The first statement is an immediate consequence of Proposition 70.12 (1). The second statement follows easily from the definitions. □

Example. It follows from Proposition 70.13 together with the calculation of the linking numbers in Figure 891 that the two Hopf links H^- and H^+ are not equivalent.

The following proposition says that the linking number $\mathrm{lk}(K, J)$ is symmetric in the oriented knots K and J.

Proposition 70.14. *For any oriented 2-component link $L = J \sqcup K$ we have*

$$\mathrm{lk}(J, K) \;=\; \mathrm{lk}(K, J).$$

Proof $(*)$. We pick a fattening for the link $L = J \sqcup K$ and use it to define the exteriors X_L, X_K and X_J. Throughout the proof we consider the following inclusion maps

$$\begin{tikzcd} \partial X_K \arrow[dr, "f"] & & X_J \\ & X_L \arrow[ur, "j"] \arrow[dr, "k"] & \\ \partial X_J \arrow[ur, "g"] & & X_K. \end{tikzcd}$$

Furthermore, throughout the proof we use the isomorphisms from Proposition 29.5 and page 747 to make the identifications $H_0(\partial X_J;\mathbb{Z}) = H_0(X_L;\mathbb{Z}) = H_0(X_J;\mathbb{Z}) = \mathbb{Z}$. We start out with the following claim.

Claim. We have

$$\mathrm{lk}(K,J) \;=\; \big(\underbrace{k^*([\mu_K^*])}_{\in H^1(X_L;\mathbb{Z})} \cup \underbrace{j^*([\mu_J^*])}_{\in H^1(X_L;\mathbb{Z})}\big) \cap \underbrace{g_*([\partial X_J])}_{\in H_2(X_L;\mathbb{Z})}.$$

We denote by $[J] \in H_1(J;\mathbb{Z})$ the fundamental class. We have the following equalities:

$$\begin{array}{rcl} & & \text{Proposition 57.8} \\ & & \downarrow \\ (k^*([\mu_K^*]) \cup j^*([\mu_J^*])) \cap g_*([\partial X_J]) & = & -(j^*([\mu_J^*]) \cup k^*([\mu_K^*])) \cap g_*([\partial X_J]) \\ = \; -k^*([\mu_K^*]) \cap (j^*([\mu_J^*]) \cap g_*([\partial X_J])) & = & -k^*([\mu_K^*]) \cap g_*(g^*j^*[\mu_J^*] \cap [\partial X_J]) \\ \uparrow & & \uparrow \\ \text{Lemma 59.7} & & \text{Lemma 57.10 applied to } g \end{array}$$

$$\begin{array}{rcrcrcrcl} = & k^*([\mu_K^*]) \cap g_*([\lambda_J]) & = & \langle [\mu_K^*], k_*g_*([\lambda_J])\rangle & = & \langle [\mu_K^*], k_*g_*([J])\rangle & = & \mathrm{lk}(K,J). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \text{Lemma 70.7 (3)} & & \text{Lemmas 50.6 and 59.4} & & \text{since } [\lambda_J]=[J]\in H_1(X_K) & & \text{see page 1636} & \end{array}$$

This concludes the proof of the claim.

Now we conclude that

$$\begin{array}{rcl} \mathrm{lk}(K,J) & = & (j^*([\mu_J^*]) \cup k^*([\mu_K^*])) \cap g_*([\partial X_J]) \\ & \uparrow & \\ & \text{by the claim} & \end{array} \quad \begin{array}{rcl} & = & (-k^*([\mu_K^*]) \cup j^*([\mu_J^*])) \cap (-f_*([\partial X_K])) \\ & \uparrow & \\ & \text{by Proposition 57.8 and since by Corollary 46.27} & \\ & \text{we have } g_*([\partial X_J]) + f_*([\partial X_K]) = 0 \in H_2(X_L;\mathbb{Z}) & \end{array}$$

$$\begin{array}{rcl} & = & \mathrm{lk}(J,K). \\ & \uparrow & \\ & \text{by the claim} & \end{array}$$

□

To determine the linking number in more complicated cases it is helpful to have different ways to calculate the linking number.

Proposition 70.15. *Let $K \sqcup J$ be an oriented 2-component link. If F is a Seifert surface for K that is transverse to J, then*

$$\mathrm{lk}(K,J) \;=\; F \cdot J$$

where $F \cdot J$ denotes the algebraic intersection number of F and J.

Example. In Figure 892 we show an oriented link $K \sqcup J$ and a Seifert surface F for K. Clearly we have $F \cdot J = 0$, thus it follows from Proposition 70.15 that $\mathrm{lk}(K, J) = 0$.

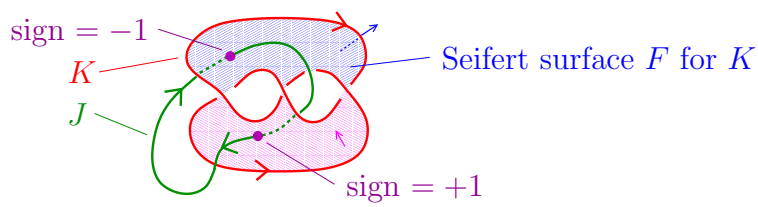


FIGURE 892.

Proof. Let $K \sqcup J$ be an oriented 2-component link. We pick a tubular neighborhood $\overline{B}^2 \times K$ for K. After possibly shrinking the tubular neighborhood we can and will assume that $S^1 \times K$ intersects F transversally in a single curve[1180] and that $\overline{B}^2 \times K$ does not intersect J. We write $X := S^3 \setminus (B^2 \times K)$ and we write $F' := F \cap X$. It follows from our choice of tubular neighborhood and from Lemma 3.43 that F' is a compact proper submanifold of F.

We start out with the following claim.

Claim. We have $\mathrm{PD}_X([F']) = [\mu_K^*] \in H^1(X; \mathbb{Z})$.

By definition of the Poincaré duality isomorphism PD_X the claim is equivalent to showing that $[\mu_K^*] \cap [X] = [F']$. From Lemma 70.7 (2) we know that the connecting homomorphism $\partial\colon H_2(X, \partial X; \mathbb{Z}) \to H_1(\partial X; \mathbb{Z})$ is a monomorphism. Therefore it suffices to show that $\partial([\mu_K^*] \cap [X]) = \partial[F']$. We denote by $i\colon \partial X \to X$ the inclusion map. We calculate that

$$\begin{array}{ccccccccc} \partial([\mu_K^*] \cap [X]) & \underset{\uparrow}{=} & -i^*([\mu_K^*]) \cap [\partial X] & \underset{\uparrow}{=} & [\lambda_K] & \underset{\uparrow}{=} & [\partial F'] & \underset{\uparrow}{=} & \partial([F']). \\ & & \text{Proposition 61.22} & & \text{Lemma 70.7 (5)} & & \text{Lemma 70.7 (4)} & & \text{Proposition 46.26} \end{array}$$

This concludes the proof of the claim.

Now we calculate that

$$\begin{array}{ccccccccc} & \text{Lemma 67.8} & & \text{since } J \subset X & & \text{Theorem 67.10} & & \text{Lemma 59.7} & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ F \cdot J & = & J \cdot F & = & J \cdot F' & = & (\mathrm{PD}_X([J]) \cup \mathrm{PD}_X([F'])) \cap [X] & = & \mathrm{PD}_X([F']) \cap (\mathrm{PD}_X([J]) \cap [X]) \\ & & & & & = & [\mu_K^*] \cap [J] \quad = \quad \langle [\mu_K^*], [J] \rangle & = & \mathrm{lk}(K, J). \\ & & & & & \uparrow & & \uparrow & \\ & & & & & \text{by the claim we have } \mathrm{PD}_X([F']) = [\mu_K^*] \text{ and} & & \text{see page 1636} & \\ & & & & & \text{by definition we have } \mathrm{PD}_X([J]) \cap [X] = [J] & & & \end{array}$$

We have thus proved the desired equality. □

[1180] This last statement requires a little bit of thought. We leave the technical details to the reader.

In practical situations the following proposition is often the most efficient way to compute linking numbers. We will formulate the proposition in a slightly informal way.

Proposition 70.16. *Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane.*[1181] *Then*

$$\mathrm{lk}(K,J) \;=\; \begin{array}{l} \tfrac{1}{2}\textit{number of positive crossings between } K \textit{ and } J \\ -\;\tfrac{1}{2}\textit{number of negative crossings between } K \textit{ and } J \end{array}$$

where a crossing P is called a positive (respectively negative) crossing if the direction of the upper strand together with the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$. We refer to Figure 893 for an illustration.

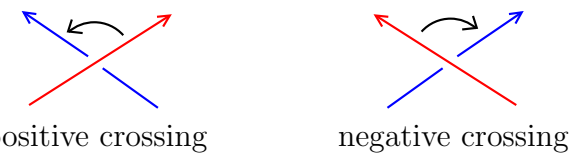


FIGURE 893.

Examples.

(1) In Figure 894 we show two oriented links and we compute the linking number using Proposition 70.16 by considering the signs of the crossings between K and J. As a reality check, on the left we consider the Hopf link that we had already considered in Figure 891. In both cases we obtain that the linking number equals -1.

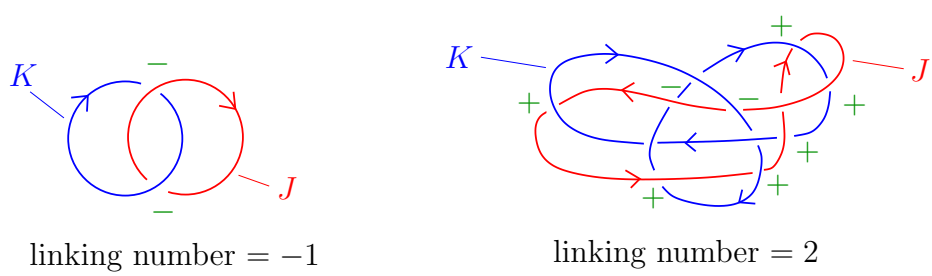


FIGURE 894.

(2) Now we can settle the question from page 1627, which of the three closed curves shown in Figure 895 is actually the longitude of the trefoil. More precisely, by definition the longitude λ_K satisfies $\phi_K([\lambda_K]) = 0$, but again by definition this means that $\mathrm{lk}(K, \lambda_K) = 0$. Conversely, loosely speaking any curve that runs "parallel to K"

[1181] We will not attempt to make the notion of a diagram precise and we work with a "naive" interpretation of the concept. We refer to the standard textbooks in knot theory, e.g. [**?**, Rol, BZH] for more details.

and that has linking number zero with K, is a longitude. Using Proposition 70.16 we calculate the linking numbers of the oriented knot K with the oriented curves A, B and C. We obtain that C is the only curve with linking number zero. Thus C is the longitude.

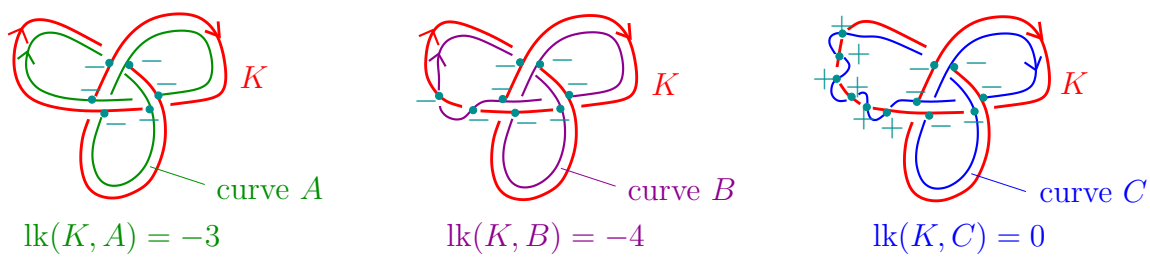


FIGURE 895.

Remark. Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane. An amusing exercise shows that

$$\begin{aligned} \operatorname{lk}(K,J) \;=\; & \text{number of positive crossings where } K \text{ crosses over } J \\ - \; & \text{number of negative crossings where } K \text{ crosses over } J. \end{aligned}$$

Sketch of proof. [1182] We introduce the following notation:

(1) Given $t \in \mathbb{R}$ we consider the map

$$\begin{aligned} \lambda_t \colon \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (x,y,z+t). \end{aligned}$$

We extend this definition to a homeomorphism of S^3 by setting $\lambda_t(\infty) = \infty$.[1183]

(2) By Proposition 70.8 the knot J admits a Seifert surface F.

(3) We pick $r \in \mathbb{R}$ such that the minimal z-coordinate of any point on $\lambda_r(F)$ is greater than the maximal z-coordinate of any point in K.

(4) We write $F' = ((J \times [0,1]) \cup F)/\sim$ where we identify any point in $J \subset F$ with the corresponding point in $J \times \{1\}$. Note that F' is a compact 2-dimensional manifold that inherits an orientation from F.

(5) We consider the map[1184]

$$\begin{aligned} \Psi \colon F' = ((J \times [0,1]) \cup F)/\sim \;&\to\; \mathbb{R}^3 \\ Q \;&\mapsto\; \begin{cases} \lambda_{t\cdot r}(P) & \text{if } Q = (P,t) \in J \times [0,1], \\ \lambda_r(Q), & \text{if } Q \in F. \end{cases} \end{aligned}$$

[1182] A proof can evidently only be as precise as the input. Since we did not bother to completely rigorously formulate the statement we can also not expect a completely rigorous proof. But with some effort one can formulate both the statement and the proof rigorously.

[1183] Why is this map continuous?

[1184] Note that the map $\Psi \colon F' \to S^3$ is not necessarily injective. In particular the image is not necessarily a Seifert surface for J.

Then

$$\begin{array}{ccccc} \mathrm{lk}(K,J) & = & K\cdot\Psi & = & K\cdot\Psi|_{J\times[0,1]}. \\ & \uparrow & & \uparrow & \\ & \text{by the same argument} & & \text{by choice of } r \text{ we have} & \\ & \text{as in Proposition 70.15} & & K\cap\Psi(F)=\varnothing & \end{array}$$

The right-hand side can be computed easily, in fact the points in $K\cap\Psi(J\times[0,1])$ correspond precisely to the crossings of the diagrams where K crosses over J and one can easily convince oneself that the signs agree. The proposition now follows from the reformulation of the linking number in the remark preceding the proof. □

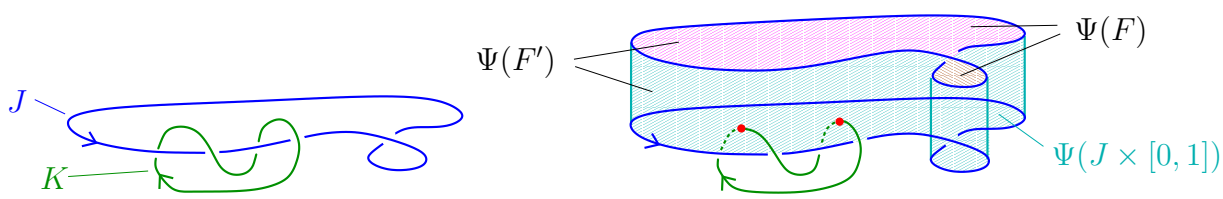


FIGURE 896. Illustration of the proof of Proposition 70.16.

In the following lemma we give a 4-dimensional interpretation of the linking number.

Lemma 70.17. *Let K, K' be two disjoint oriented knots.*

(1) *There exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse.*
(2) *Let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. If S and S' are transverse, then*

$$\mathrm{lk}(K,K') \;=\; S\cdot_{\overline{B}^4} S'.$$

Sketch of proof. Let K, K' be two disjoint oriented knots.

(1) By Proposition 70.8 and the Transversality Theorem 67.2 (2) there exists a Seifert surface F for K that is transverse to K'. Furthermore let F' be a Seifert surface for K'. We pick a collar neighborhood $S^3\times[0,2]$ for $S^3=\partial\overline{B}^4$. We use the collar neighborhood $S^3\times[0,1]$ to define the push-in[1185] $\widetilde{F}$ of F and we use the collar neighborhood $S^3\times[0,2]$ to define the push-in $\widetilde{F}'$ of F'. It follows easily from Lemma 10.18 that $\widetilde{F}$ and $\widetilde{F}'$ have the desired properties.
(2) First we note that

$$\begin{array}{ccccccc} \mathrm{lk}(K,K') & = & F\cdot_{S^3}K' & = & \widetilde{F}\cdot_{\overline{B}^4}\widetilde{F}' & = & -(-\widetilde{F})\cdot_{-\overline{B}^4}(-\widetilde{F}'). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 70.15} & & \text{by inspection of the} & & \text{follows easily from the definitions} & \\ & & & \text{intersection points} & & & \end{array}$$

[1185] We refer to page 346 for the definition of push-ins.

Now let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. We think of $S^4 = \overline{B}^4 \cup (-\overline{B}^4)$ as the union of two copies of $\overline{B}^4$ with opposite orientations, the first copy contains S, S' and the second contains $-\widetilde{F}, -\widetilde{F}'$. (See Figure 897 for an illustration.) It follows fairly easily from the Collar Neighborhood Theorem 10.15 that $S \cup -\widetilde{F}$ and $S' \cup -\widetilde{F}'$ are oriented submanifolds of S^4.[1186] We calculate that

$$S \cdot_{\overline{B}^4} S' - \operatorname{lk}(K, K') \underset{\substack{\uparrow \\ \text{by the above}}}{=} S \cdot_{\overline{B}^4} S' + (-\widetilde{F}) \cdot_{-\overline{B}^4} (-\widetilde{F}') \underset{\substack{\uparrow \\ \text{since } S \cap \widetilde{F}' = S' \cap \widetilde{F} = \varnothing}}{=} (S \cup -\widetilde{F}) \cdot_{S^4} (S' \cup -\widetilde{F}') \underset{\substack{\uparrow \\ \text{by Theorem 68.11 and} \\ \text{since } H^2(S^4;\mathbb{Z}) = 0}}{=} 0.$$

We have thus shown the desired equality $S \cdot_{\overline{B}^4} S' = \operatorname{lk}(K, K')$. □

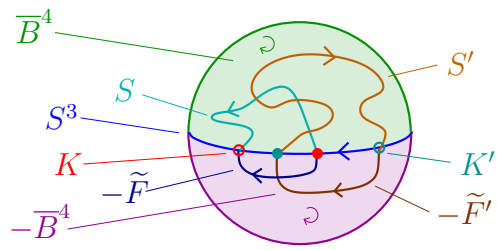


FIGURE 897. Illustration for the proof of Lemma 70.17.

The following proposition gives two more alternative ways to define or calculate or interpret the linking number.

Proposition 70.18. *Let $K \sqcup J \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be an oriented 2-component link.*

(1) *If $\varphi\colon S^1 \to K$ and $\psi\colon S^1 \to J$ are two orientation-preserving diffeomorphisms, then*[1187]

$$\operatorname{lk}(K, J) = \deg \begin{pmatrix} S^1 \times S^1 & \to & S^2 \\ (z, w) & \mapsto & \frac{\varphi(z) - \psi(w)}{\|\varphi(z) - \psi(w)\|} \end{pmatrix}.$$

(2) *Let $f\colon [0, 2\pi] \to K$ and $g\colon [0, 2\pi] \to J$ be two maps such that f and g descend to orientation-preserving diffeomorphisms from $[0, 2\pi]/0 \sim 2\pi$ to K and J. Then*

$$\operatorname{lk}(K, J) = \frac{1}{4\pi} \int_{s=0}^{s=2\pi} \int_{t=0}^{t=2\pi} \frac{\det \overbrace{\big((f(s) - g(t)) \;\; f'(s) \;\; g'(t)\big)}^{3 \times 3\text{-matrix}}}{\|f(s) - g(t)\|^3}\, dt\, ds.$$

Remark.

[1186] The word "submanifold of S^4" is used in a slightly ambiguous way since it depends on whether we view S^4 as a topological manifold or a manifold. In this case we view S^4 only as a topological manifold.

[1187] Here we use the notion of "degree" that we introduced on page 959.

(1) The linking number of an oriented 2-component link was initially introduced by Gauss in 1833 by the integral in Proposition 70.18. We refer to Figure 898 for a scan of the relevant paragraphs in [Gau, p. 605]. The term *geometria situs* is hereby the earlier name for topology. We refer to [Epp, p. 73] for more information on the work of Gauss regarding the linking number.

Von der *Geometria Situs*, die Leibnitz ahnte und in die nur einem Paar Geometern (Euler und Vandermonde) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z'-\mathrm{d}z\,\mathrm{d}y')+(y'-y)(\mathrm{d}z\,\mathrm{d}x'-\mathrm{d}x\,\mathrm{d}z')+(z-z')(\mathrm{d}x\,\mathrm{d}y'-\mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4m\pi$$

und m die Anzahl der Umschlingungen.

Figure 898.

(2) It is not immediately clear why the integral in Proposition 70.14 is an invariant of the equivalence class of the link $K \sqcup J$. How would you prove this statement, without using Proposition 70.18 (2)?
(3) It is clear from the second reformulation of the linking number given in Proposition 70.18 that the linking number is symmetric. How does the symmetry follow from the first reformulation?

Proof. The proposition is proved in [RN, Section 5] or alternatively, with fewer details, in [Rol, p. 133]. See also [tD, p. 284]. □

Example. Given any topological invariant the question that arises is, how good is it at distinguishing objects. It is quite easy to find examples of links which show that the linking number is far from being a complete invariant. In Figure 899 we show the Whitehead link and the Borromean rings. Both are links with zero linking numbers and unknotted components but which are appear to be non-trivial. The Whitehead link is named after

John Henry Constantine Whitehead who also introduced the concept of CW-complexes. The Borromean rings are named after the Borromeo family that carries the Borromean rings in its coat of arms since medieval days.

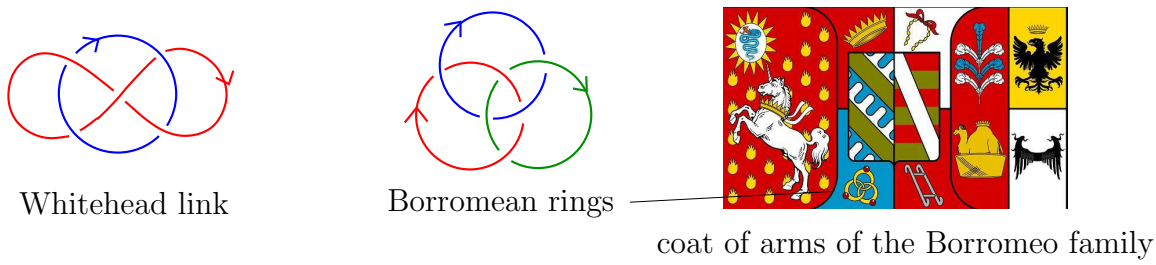


FIGURE 899.

We record the following question for later on.

Question 70.19. *How can we show that neither the Whitehead link nor the Borromean rings are equivalent to the trivial link?*

Remark. We can of course try to determine the fundamental groups of the complements of the trivial links and the Whitehead link and the Borromean rings. But then the question arises, how can we distinguish these groups?

70.5. **Alexander duality and linking numbers** $(*)$**.** Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood. (For example, by Propositions 64.2, 64.11 and 64.13 the subset Y could be a link or a finite spatial graph.) We can compute the isomorphism types of the first homology group of the complement $S^3 \setminus Y$ as follows:

$$\begin{array}{ccccccc} H_1(S^3 \setminus Y;\mathbb{Z}) & \xrightarrow{\cong} & H^1(Y;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(H_1(Y;\mathbb{Z}),\mathbb{Z}) & \cong & H_1(Y;\mathbb{Z})/\text{torsion}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Alexander Duality} & & \text{Universal Coefficient} & & & \\ & \text{Theorem 66.1} & & \text{Theorem 51.13} & & \text{non-canonical isomorphism} & \end{array}$$

The catch with the Alexander Duality Theorem 66.1 is that it is quite hard to "see" the isomorphism. For example, suppose we are given a basis for $H_1(Y;\mathbb{Z})/\text{torsion}$, how do we get a basis for $H_1(S^3 \setminus Y;\mathbb{Z})$? In many situations the following lemma resolves that issue.

Lemma 70.20. *Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N(Y)$. We suppose that $H_1(Y;\mathbb{Z})$ is a free abelian group*[1188] *of rank k. Let $C_1,\dots,C_k$ be oriented curves in $N(Y)$ and let $D_1,\dots,D_k$ be oriented curves in $S^3 \setminus N(Y)$. If*

$$\det\big(\underbrace{(\mathrm{lk}(C_i,D_j)_{i,j=1,\dots,k})}_{k\times k\text{-matrix}}\big) \;=\; \pm 1,$$

then $[D_1],\dots,[D_k]$ form a basis of $H_1(S^3 \setminus Y;\mathbb{Z})$.

[1188] Is it possible that $H_1(Y;\mathbb{Z})$ is not a free abelian group?

Proof. Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N = N(Y)$. We denote by $\mathring{N}$ the interior of N. We consider the homomorphisms

$$\begin{array}{rcl} \Omega\colon \mathbb{Z}^k & \to & H_1(S^3\setminus\mathring{N};\mathbb{Z}) \\ (a_1,\dots,a_k) & \mapsto & \sum\limits_{i=1}^{k} a_i\cdot[D_i] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Theta\colon H_1(S^3\setminus\mathring{N};\mathbb{Z}) & \to & \mathbb{Z}^k \\ \sigma & \mapsto & (\phi_{C_1}(\sigma),\dots,\phi_{C_k}(\sigma)). \end{array}$$

It follows from our hypothesis and the definition of linking numbers that $\Theta\circ\Omega$ is an epimorphism. Furthermore we know from the Alexander Duality Theorem 66.1 and the Universal Coefficient Theorem 51.13 that $H_1(S^3\setminus\mathring{N};\mathbb{Z})$ is also isomorphic to $\mathbb{Z}^k$. But by Lemma 11.8 (5) an epimorphism between two free abelian groups of the same rank is an isomorphism. Thus we see that Θ and Ω are isomorphisms.

Since $\{[D_1],\dots,[D_k]\}$ is the image under Ω of the standard basis of $\mathbb{Z}^k$ we see that $[D_1],\dots,[D_k]$ form a basis for $H_1(S^3\setminus\mathring{N};\mathbb{Z})$. But by Lemma 64.3 the inclusion induced map $H_1(S^3\setminus Y;\mathbb{Z})\to H_1(S^3\setminus\mathring{N};\mathbb{Z})$ is an isomorphism. Thus we see that $[D_1],\dots,[D_k]$ form a basis of $H_1(S^3\setminus Y;\mathbb{Z})$.[1189] □

Example. In Figure 900 we consider a spatial graph G together with a regular neighborhood N. By Lemma 40.17 we know that $H_1(G;\mathbb{Z})\cong\mathbb{Z}^2$. In Figure 900 we also see two oriented curves C_1 and C_2 in N and two oriented curves D_1 and D_2 in $S^3\setminus\mathring{N}$. Using Proposition 70.16 one can easily calculate that

$$\begin{pmatrix} \mathrm{lk}(C_1,D_1) & \mathrm{lk}(C_1,D_2) \\ \mathrm{lk}(C_2,D_1) & \mathrm{lk}(C_2,D_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows from Lemma 70.20 that $[D_1],[D_2]$ form a basis for $H_1(S^3\setminus G;\mathbb{Z})$.

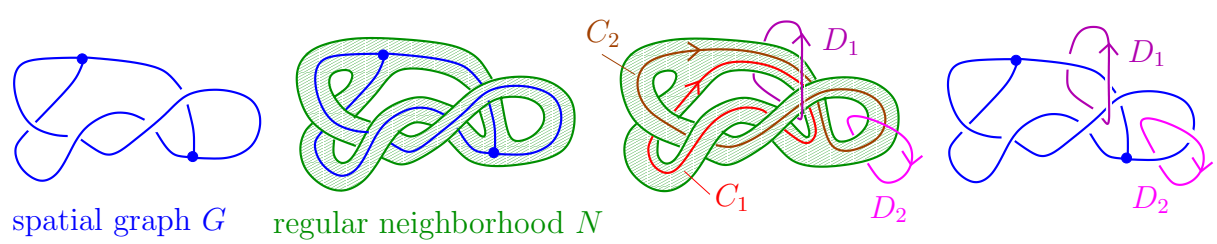


FIGURE 900.

In Lemma 66.3 we computed the homology groups of the complement of a link, but we could not determine explicit generators. The following lemma now solves that problem. In particular it gives an answer to Question 66.4.

Lemma 70.21. *Let $L = L_1\sqcup\cdots\sqcup L_m$ be an m-component link. We pick a fattening $\overline{B}^2\times L$ for L.*

[1189] The alert reader will have noticed that the proof resembles the argument given in Lemma 50.7.

(1) *A basis for* $H_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ *is given by the homology classes that are represented by the meridians* $\mu_1, \dots, \mu_m$.
(2) *A basis for* $H_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ *is given by considering the fundamental classes of all but one of the tori* $S^1 \times L$, $i \in \{1, \dots, m\}$. *More precisely, for any* $j \in \{1, \dots, m\}$ *a basis for* $H_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ *is given by*

$$[S^1 \times L_1], \dots, [S^1 \times L_{i-1}], [S^1 \times L_{i+1}], \dots, [S^1 \times L_m].$$

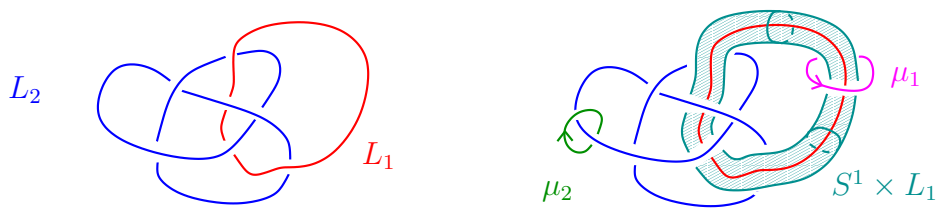


FIGURE 901.

Sketch of proof. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link. We know from Lemma 66.3 that $H_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ and that $H_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$. We pick a fattening $\overline{B}^2 \times L$ for L. By the Collar Neighborhood Theorem 10.10 we can extend this fattening to a fattening $2\overline{B}^2 \times L$.

(1) The first statement follows from Lemma 70.20 applied to $C_i = \{0\} \times L_i$, $i = 1, \dots, m$ and $D_i := \mu_i$, $i = 1, \dots, m$.
(2) To simplify the notation we consider the case $j = m$. We write $X' := S^3 \setminus \frac{1}{2}B^2 \times L$. For $i = 1, \dots, m$ we write $T_i := S^1 \times L_i$. For $i = 1, \dots, m$ we pick a point P_i on $\frac{1}{2}S^1 \times L_i \subset \partial X'$. Since $S^3 \setminus (2B^2 \times L)$ is path-connected we can pick for each $i \in \{1, \dots, m-1\}$ a path γ_i from P_i to P_m that intersects T_i transversally in a single point and that does not intersect any of the T_j with $j \neq i, m$. Now we consider the map

$$\begin{array}{rcl} \Psi\colon H_1(X', \partial X'; \mathbb{Z}) \times H_2(X'; \mathbb{Z}) & \to & \mathbb{Z} \\ (\sigma, \tau) & \mapsto & \langle \mathrm{PD}_{X'}(\sigma) \cup \mathrm{PD}_{X'}(\tau), [X'] \rangle. \end{array}$$

It follows easily from Theorem 68.11 that for $i, j \in \{1, \dots, m-1\}$ we have the equality $\Psi([\gamma_i], [T_j]) = \delta_{ij}$. By Lemma 50.8 this implies that $[T_1], \dots, [T_{m-1}]$ do indeed form a basis of $H_2(X'; \mathbb{Z})$. □

Exercises for Chapter 70.

Exercise 70.1. Provide a proof for Corollary 70.5.

Exercise 70.2. Let $A = S^1 \times [-1, 1]$ be the annulus. Let C and D be two closed oriented curves on A. Suppose that $[C] = [D] \neq 0$. Show that the following two statements hold:

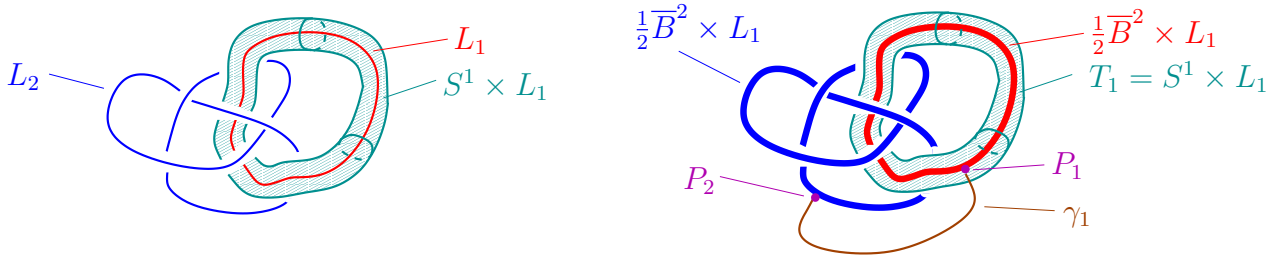


FIGURE 902.

(1) the two closed oriented curves are smoothly isotopic,
(2) if C and D are furthermore disjoint, then there exists an embedded annulus $X \subset A$ with $\partial A = C \cup -D$.

Hint: Using the fact that A is an annulus you can find a smooth isotopy that makes the curves disjoint. At that point it is perhaps better to first prove (2).

Exercise 70.3. Let F be an oriented connected 2-dimensional manifold and let C and D be distinct boundary curves. Suppose that C and D are homotopic. Show that F is diffeomorphic to the annulus $S^1 \times [-1, 1]$.

Exercise 70.4. Find explicit Seifert surfaces for the knots K_1 and K_2 shown in Figure 903. Remark: The answer to a question can only be as rigorous as the input. Since the input is a picture it suffices here of course to draw a convincing picture of a Seifert surface.

FIGURE 903.

Exercise 70.5. Let F be an oriented 2-dimensional manifold and let C and D be two closed oriented *disjoint* curves on F that are not null-homotopic. Assume that C and D are homotopic. Show that there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$. Hint: Use Exercise 70.3. You might also want to use some of the ideas of the proof of Lemma 70.1.

Exercise 70.6. Let $\varphi\colon \overline{B}^3 \to S^3$ and $\psi\colon \overline{B}^3 \to S^3$ be two embeddings with disjoint images. Let $K \subset \varphi(\overline{B}^3)$ and $J \subset \psi(\overline{B}^3)$ be oriented knots. Show that $\mathrm{lk}(K, J) = 0$.

Exercise 70.7. We consider the two links shown in Figure 890. Show that the link complements are diffeomorphic.

FIGURE 904.

Exercise 70.8. The Borromean rings, shown in Figure 899, have the property that if you remove one component, then the remaining two knots form a trivial link. Does there exist a four component link with the property that if you remove any of the four components, then you get a three component trivial link?

Exercise 70.9. Let $L = K \sqcup J$ and $L' = K' \sqcup J'$ be two links such that the complements are diffeomorphic. Does it follow that $\operatorname{lk}(K, J) = \pm \operatorname{lk}(K', J')$?

Exercise 70.10. An m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$ is called *split link* if there exist disjoint closed balls $B_1, \dots, B_m$ such that each L_i is contained in B_i. Furthermore L is called a *boundary link* if there exist m disjoint Seifert surfaces $F_1, \dots, F_m$.

(a) A split link is easily seen to be a boundary link. Do there exist boundary links that are not split?

(b) Show that if $L = L_1 \sqcup \cdots \sqcup L_m$ is a boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group of rank m.
Hint: You could try to find an interesting map from $S^3 \setminus L$ onto the wedge of m circles.

71. The intersection form of topological manifolds I: definition and calculations

In this chapter we will introduce and study the intersection form of compact oriented even-dimensional topological manifolds. Before we do so it is convenient to remind ourselves of a few basic notions and results in algebra.

71.1. **(Anti-) symmetric forms.** On page 1447 we introduced the notion of a pairing over a commutative ring R. We recall the definition and we introduce some new notions.

Definition. Let R be a commutative ring.

(1) A *pairing over* R is a pair (V, φ) where V is an R-module and φ is a map
$$\varphi\colon V \times V \ \to \ S$$
from $V \times V$ to some R-module S which has the property that φ is R-bilinear, i.e. that is R-linear in both arguments.

(2) We say that a pairing $\varphi\colon V \times V \to S$ is *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$. Furthermore we call the pairing *anti-symmetric* if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$.

(3) We say that two pairings (V, φ) and (W, ψ) are *isometric*, written $(V, \varphi) \cong (W, \psi)$, if there exists an isometry, i.e. an isomorphism $\varphi\colon V \to W$ of R-modules such that $\varphi(v_1, v_2) = \psi(\varphi(v_1), \varphi(v_2))$ for all $v_1, v_2 \in V$.

(4) Given a pairing $\varphi\colon V \times V \to S$ we denote by $-\varphi$ the pairing that is given by $(-\varphi)(v, w) := -\varphi(v, w)$ for all $v, w \in V$.

(5) Given two pairings $\varphi\colon V \times V \to S$ and $\psi\colon W \times W \to S$ we define the *direct sum* of φ and ψ to be the pairing
$$\begin{array}{rcl} \varphi \oplus \psi\colon (V \oplus W) \times (V \oplus W) & \to & S \\ (v \oplus w), (v' \oplus w') & \mapsto & \varphi(v, v') + \psi(w, w'). \end{array}$$

(6) Given a pairing (V, g) and $n \in \mathbb{Z}$ we define
$$n \cdot (V, g) \ := \ \begin{cases} \overbrace{(V, g) \oplus \cdots \oplus (V, g)}^{n\text{-summands}}, & \text{if } n \geq 0, \\ \underbrace{(V, -g) \oplus \cdots \oplus (V, -g)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{cases}$$

(7) Given a pairing (V, φ) and a submodule W we denote by $\varphi|_W$ the pairing that is given by restricting the map $\varphi\colon V \times V \to S$ to $W \times W \to S$.

In this chapter we will mostly be interested in the following special type of a pairing.

Definition. Let R be a commutative ring. A *form* is a finitely generated free R-module V together with a pairing $\varphi\colon V \times V \to R$ which is either symmetric or anti-symmetric. We refer to the rank of V also as the rank of the form.[1190]

[1190] The rank of a finitely generated free module over a commutative ring is well-defined, see e.g. [Lam, p. 4]. Note though that the rank of a finitely generated free module over a non-commutative ring is not

For the reader's convenience we recall the follow

Definition. Let R be a commutative ring. A form $\varphi\colon V \times V \to R$ is called *non-singular* if the two maps

$$\text{(a)} \quad \begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ v & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ w & \mapsto & \varphi(v,w) \end{array} \right) \end{array} \quad \text{and} \quad \text{(b)} \quad \begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ v & \mapsto & \varphi(v,w) \end{array} \right) \end{array}$$

are isomorphisms.

The following elementary lemma follows easily from the definitions and the fact that forms are by definition symmetric or anti-symmetric.

Lemma 71.1. *Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a pairing over R. if one of the two maps in the above definition is an isomorphism, then so is the other, hence the form is non-singular.*

In many situations it is convenient to represent a form by a matrix. We recall the relevant definition from the linear algebra courses.

Definition. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. Given any basis $\{v_1, \dots, v_k\}$ of the free R-module V we refer to the $(k \times k)$-matrix

$$((\varphi(v_i, v_j))_{i,j=1,\dots,k}$$

as a *matrix representing the form φ.*

The following elementary lemma is usually proved in linear algebra. It is also straightforward to verify it using only the relevant definitions.[1191]

Lemma 71.2. *Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. Let $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ be bases of the free R-module V. We write*

$$A \ := \ ((\varphi(v_i, v_j))_{i,j=1,\dots,k}, \quad B \ := \ ((\varphi(w_i, w_j))_{i,j=1,\dots,k}.$$

If $P = (p_{ij})_{i,j=1,\dots,k} \in \mathrm{GL}(k, R)$ denotes the base change matrix that is uniquely defined by the property that $v_i = \sum_{j=1}^{k} p_{ij} w_j$ for $i = 1, \dots, k$, then

$$A \ = \ P^T B P.$$

The following lemma is even more elementary, it follows basically immediately from the definitions.

Lemma 71.3.

(1) *Let $\varphi\colon V \times V \to R$ be a form. If A is a matrix that represents φ, then the matrix $-A$ represents the form $-\varphi$.*

---

necessarily well-defined. More precisely, there exists a non-commutative ring R and $m \neq n \in \mathbb{N}$ such that R^m is isomorphic to R^n. We refer to [Lam, p. 4] for such examples.

[1191] It is easy though to get confused.

(2) *Let* $\varphi\colon V\times V \to R$ *and* $\psi\colon W\times W \to R$ *be two forms. If* A *is a matrix that represents* φ *and if* B *is a matrix that represents* ψ*, then the matrix* $A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *represents the form* $\varphi \oplus \psi$*.*

(3) *Let* $\varphi\colon V \times V \to R$ *be a form and let* $n \in \mathbb{N}_0$*. If* A *is a matrix that represents* φ*, then the matrix* $n \cdot A := \underbrace{A \oplus \cdots \oplus A}_{n\text{-times}}$ *represents the form* $n \cdot \varphi$*.*

Before we can turn to topology we need to introduce one last elementary mathematical object.

Definition.

(1) Given a finitely generated abelian group H we denote by $\mathrm{F}H := H/\text{torsion subgroup}$ the *maximal torsion-free quotient* of H.

(2) Given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we denote the obvious induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

Convention. On many occasions we do not distinguish in the notation between an element in an abelian group H and the element it represents in the maximal torsion-free quotient FH.

The following lemma summarizes everything we need to know about maximal torsion-free quotients.

Lemma 71.4.

(1) *The maps* $H \to \mathrm{F}H$ *and* $\alpha \mapsto \alpha$ *define a functor from the category of finitely generated abelian groups to the category of finitely generated free abelian groups.*

(2) *For every homomorphism* $\alpha\colon A \to B$ *between two finitely generated abelian groups the following diagram commutes:*[1192]

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ FA \arrow[r, "\alpha"] & FB. \end{tikzcd}$$

(3) *Any (anti-) symmetric pairing*

$$\varphi\colon H \times H \to \mathbb{Z}$$

over a finitely generated abelian group descends to a well-defined form

$$\begin{aligned} \mathrm{F}H \times \mathrm{F}H &\to \mathbb{Z}. \\ ([g],[h]) &\mapsto \varphi(g,h). \end{aligned}$$

[1192]In fancy speak this means that the projection maps $H \to \mathrm{F}H$ define a natural transformation from the identity functor to the functor $H \mapsto \mathrm{F}H$.

Proof. All the statements of the lemma are basically elementary. The only ingredient we need to use is that by Theorem 11.4 every finitely generated free abelian group is in fact a free abelian group.[1193] □

71.2. **The definition of the intersection form of topological manifolds.** Let M be a compact oriented non-empty m-dimensional topological manifold. Recall that by the Poincaré Duality Theorem 61.1 we know that for any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} H^{m-k}(M, \partial M; \mathbb{Z}) & \to & H_k(M; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As before we denote by

$$\mathrm{PD}_M \colon H_k(M; \mathbb{Z}) \to H^{m-k}(M, \partial M; \mathbb{Z})$$

the inverse of this isomorphism. This leads us to the following definition.

Definition. Let M be a compact oriented non-empty topological manifold of the even dimension $2n$. We refer to the map[119411951196]

$$\begin{array}{rcl} Q_M \colon \mathrm{F}H_n(M; \mathbb{Z}) \times \mathrm{F}H_n(M; \mathbb{Z}) & \to & \mathbb{Z} \\ (a, b) & \mapsto & \langle \mathrm{PD}_M(a) \cup \mathrm{PD}_M(b), [M] \rangle \end{array}$$

as the *intersection form of M*.

Remark. Since $\mathrm{PD}_M \colon H_n(M; \mathbb{Z}) \to H^n(M, \partial M; \mathbb{Z})$ is an isomorphism it is clear that the intersection form is basically just a reformulation of the cup product pairing

$$\begin{array}{rcl} H^n(M, \partial M; \mathbb{Z}) \times H^n(M, \partial M; \mathbb{Z}) & \to & H^{2n}(M, \partial M; \mathbb{Z}) \\ (\alpha, \beta) & \mapsto & \alpha \cup \beta \end{array}$$

on cohomology groups in the middle dimension. The big psychological advantage of homology is that very often one can "see homology classes". For example, in many cases homology classes are represented by submanifolds and we can then use Theorem 67.10 to compute the intersection form of a manifold. Sometimes we can also represent homology classes by explicit cycles and we can use Theorem 68.5 to compute the intersection form.

In the following proposition we summarize some of the key properties of the intersection form. The proposition shows in particular that the intersection form is a form in the sense of the definition on page 1649.

[1193]It might sound silly to mention Theorem 11.4 explicitly, but it is worth noting that for more complicated commutative rings R there is usually no meaningful functor from the category of finitely generated R-modules to the category of free R-modules.

[1194]Note that $H_n(M; \mathbb{Z})$ is not necessarily torsion-free. For example $M = S^1 \times \mathbb{R}\mathrm{P}^3$ is orientable, but it follows easily from the Künneth Formula 43.8 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^3$ in Proposition 35.11 that $H_2(S^1 \times \mathbb{R}\mathrm{P}^3; \mathbb{Z}) \cong \mathbb{Z}_2$.

[1195]As mentioned above, in the notation usually we do not distinguish between an element in $H_n(M; \mathbb{Z})$ and the element it represents in $\mathrm{F}H_n(M; \mathbb{Z})$.

[1196]Note that $\mathrm{PD}(a)$ and $\mathrm{PD}(b)$ lie in $H^n(M, \partial M)$, hence the cup product lies in $H^{2n}(M, \partial M; \mathbb{Z})$. The fundamental class $[M]$ is an element in $H_{2n}(M, \partial M; \mathbb{Z})$, hence $\langle \mathrm{PD}(a) \cup \mathrm{PD}(b), [M] \rangle$ is indeed defined.

Proposition 71.5. *Let M be a compact oriented non-empty $2n$-dimensional topological manifold. Then the following hold:*

(1) *If n is even, then the intersection form Q_M is symmetric, otherwise it is antisymmetric.*
(2) *If M is closed, then the intersection form is non-singular.*

The proof of Proposition 71.5 (2) relies on the following reformulation of the intersection form, which will also be useful later on on a separate occasion.

Lemma 71.6. $(*)$ *Let M be a compact oriented connected $2n$-dimensional topological manifold. We denote by $i\colon (M,\varnothing) \to (M,\partial M)$ the inclusion map. We write*

$$\Phi\colon H_n(M;\mathbb{Z}) \xrightarrow{i_*} H_n(M,\partial M;\mathbb{Z}) \xrightarrow{\mathrm{PD}_M} H^n(M;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z}).$$

For any $a,b \in \mathrm{F}H_{2n}(M;\mathbb{Z})$ we have

$$Q_M(a,b) \;=\; \Phi(b)(a).$$

Proof of Lemma 71.6 $(*)$. First we consider the following diagram:

$$\begin{tikzcd}
H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \arrow[r, "\mathrm{id}\times i_*"] \arrow[d, "\mathrm{PD}_M\times\mathrm{PD}_M"'] & H_n(M;\mathbb{Z}) \times H_n(M,\partial M;\mathbb{Z}) \arrow[d, "\mathrm{PD}_M\times\mathrm{PD}_M"] \\
H^n(M,\partial M;\mathbb{Z}) \times H^n(M,\partial M;\mathbb{Z}) \arrow[r, "\mathrm{id}\times i^*"] \arrow[d, "\cup"] & H^n(M,\partial M;\mathbb{Z}) \times H^n(M;\mathbb{Z}) \arrow[d, "\cup"] \\
H^{2n}(M,\partial M;\mathbb{Z}) \arrow[r, "="] & H^{2n}(M,\partial M;\mathbb{Z}).
\end{tikzcd}$$

By Proposition 61.22 we know that the upper square commutes. We obtain from Proposition 58.3, applied to the inclusion map $f\colon (M,\varnothing,\partial M) \to (M,\partial M,\partial M)$, that the lower square also commutes.

Now let $a,b \in H_{2n}(M;\mathbb{Z})$. We calculate that

$$\begin{array}{rcl}
Q_M(b,a) & = & \langle \mathrm{PD}_M(b)\cup\mathrm{PD}_M(a),[M]\rangle \\
 & \overset{\text{Lemma 59.4 (1)}}{=} & (\mathrm{PD}_M(b)\cup\mathrm{PD}_M(a))\cap[M] \\
 & = & (\mathrm{PD}_M(b)\cup\mathrm{PD}_M(i_*(a)))\cap[M] \\
 & \underset{\text{Lemma 59.7}}{=} & (\mathrm{PD}_M(i_*(a))\cap(\mathrm{PD}_M(b)\cap[M]) \\
 & \underset{\text{by the above commutative diagram}}{=} & \mathrm{PD}_M(i_*(a))\cap b \\
 & \underset{\text{Lemma 59.4 (1)}}{=} & \langle \mathrm{PD}_M(i_*(a)),b\rangle \\
 & \underset{\text{definition of }\mathrm{PD}_M(b)}{=} & \Phi(a)(b).
\end{array}$$

$\square$

Proof of Proposition 71.5.

(1) In Proposition 57.8 we showed that for a given topological space X the cup product

$$H^n(X;\mathbb{Z}) \times H^n(X;\mathbb{Z}) \;\to\; H^{2n}(X;\mathbb{Z})$$

is commutative if n is even and anti-commutative if n is odd. The same proof also applies verbatim to any relative cup product

$$H^n(X,A;\mathbb{Z}) \times H^n(X,A;\mathbb{Z}) \;\to\; H^{2n}(X,A;\mathbb{Z})$$

The first statement is an immediate consequence of this observation.

(2) Now suppose that M is closed. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{F}H_k(M;\mathbb{Z}) \arrow[rr, "b\mapsto(a\mapsto Q_M(b,a))"] \arrow[dr, "\mathrm{PD}_M", "\cong"'] & & \mathrm{Hom}(\mathrm{F}H_k(M;\mathbb{Z}),\mathbb{Z}). \\
& \mathrm{F}H^k(M;\mathbb{Z}) \arrow[ur, "\mathrm{ev}"] &
\end{tikzcd}$$

We make the following observations:

(a) Since M is closed the map PD_M does indeed take values in the cohomology of M. Furthermore, by the Poincaré Duality Theorem 61.1 the map PD_M is an isomorphism.

(b) It follows from the Universal Coefficient Theorem 51.13 for Cohomology Groups together with Lemma 51.5 (5) and Proposition 45.2 (2) that the evaluation map

$$\mathrm{ev}\colon\; \mathrm{F}H^n(M;\mathbb{Z}) \;\to\; \mathrm{Hom}(\mathrm{F}H_n(M;\mathbb{Z}),\mathbb{Z})$$

is an isomorphism.

(c) The diagram commutes by Lemma 71.6.

(d) We have seen that the two diagonal maps are isomorphisms and that the diagram commutes. Thus we obtain that the top map is an isomorphism. Since we have seen in (1) that Q_M is symmetric that isomorphism takes care of all of our problems. □

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism type of the manifold.

Lemma 71.7. *Let $f\colon M \to N$ be a homeomorphism between two compact oriented non-empty even-dimensional topological manifolds. Then*

$$Q_M \;\cong\; \begin{cases} \;\;Q_N, & \text{if } f \text{ is orientation-preserving,} \\ -Q_N, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof. [1197] Let $f\colon M \to N$ be a homeomorphism between two compact oriented $2n$-dimensional topological manifolds. We write $\varepsilon = +1$ if f is orientation-preserving and $\varepsilon = -1$ if f is orientation-reversing. We start out with the following claim.

Claim. For any $c \in H_n(M;\mathbb{Z})$ we have $\mathrm{PD}_N(f_*(c)) = \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(c)) \in H^n(N,\partial N;\mathbb{Z})$.

[1197] The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is basically trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $H_k(M;\mathbb{Z}) \to H^{2n-k}(M,\partial M;\mathbb{Z})$.

So let $c \in H_n(M;\mathbb{Z})$. We have

$$(f^{-1})^*(\mathrm{PD}_M(c)) \cap [N] \underset{\substack{\uparrow \\ \text{by Proposition 47.3 and the definition of } \varepsilon}}{=} (f^{-1})^*(\mathrm{PD}_M(c)) \cap (\varepsilon f_*([M])) \underset{\substack{\uparrow \\ \text{Lemma 59.8}}}{=} \varepsilon(f_*(\mathrm{PD}_M(c) \cap [M]) \underset{\substack{\uparrow \\ \text{definition of } \mathrm{PD}_M}}{=} \varepsilon f_*(c).$$

It follows from the definition of PD_N that we get the desired equality. This concludes the proof of the claim.

Now let $a, b \in \mathrm{F}H_n(M;\mathbb{Z})$. Then

$$\begin{aligned} Q_N(f_*(a), f_*(b)) &\underset{\substack{\downarrow \\ \text{by definition of } Q_N}}{=} \langle \mathrm{PD}_N(f_*(a)) \cup \mathrm{PD}_N(f_*(b)), [N]\rangle \\ &\underset{\substack{\uparrow \\ \text{by the claim and by Proposition 47.3}}}{=} \langle (\varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(a)) \cup \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(b)), \varepsilon \cdot f_*([M])\rangle \\ &\underset{\substack{\uparrow \\ \text{by Propositition 58.3 (1) and since } \varepsilon^3 = \varepsilon}}{=} \varepsilon \cdot \langle (f^{-1})^*(\mathrm{PD}_M(a) \cup \mathrm{PD}_M(b)), f_*([M])\rangle \underset{\substack{\uparrow \\ \text{by Lemma 50.6 and the definition of } Q_M}}{=} \varepsilon \cdot Q_M(a,b). \end{aligned}$$

The lemma now follows from the fact that $f_*\colon \mathrm{F}H_n(M;\mathbb{Z}) \to \mathrm{F}H_n(N;\mathbb{Z})$ is an isomorphism. □

Examples.

(1) We first consider the $2n$-dimensional torus $M = S^n \times S^n$. By the Künneth Formula 43.8 we know that a basis for $H_n(S^n \times S^n;\mathbb{Z})$ is given by $a = [S^n \times \{0\}]$ and $b = [\{0\} \times S^n]$. We have two approaches for determining the intersection with respect to this basis:
   (a) We can use the Product Theorem 60.1 and its corollaries. More precisely, we first determine the Poincaré duals of $[S^n \times \{0\}]$ and $[\{0\} \times S^n]$ as in Proposition 60.2 (2) and then we can apply Lemma 60.3.
   (b) As on page 1568 we can use algebraic intersection numbers and Theorem 67.10 to determine the intersection form.

   In both cases we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix

$$\begin{pmatrix} Q_M(a,a) & Q_M(a,b) \\ Q_M(b,a) & Q_M(b,b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

(2) We consider the surface Σ_3 of genus 3. In principle we could obtain the intersection form from the calculation of the cup product that we had performed in Lemma 57.16. But it is much more fun and instructive to use Theorem 67.10 instead. More precisely, we consider the surface Σ_3 that is illustrated in Figure 905. We give it the usual orientation. We already saw in the discussion on page 891 that the closed oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in Figure 905 represent a basis for $H_1(\Sigma;\mathbb{Z})$. It

follows easily from Theorem 67.10 that with respect to this basis the intersection form is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

An analogous statement holds of course for any surface of genus $g \geq 1$.[1198]

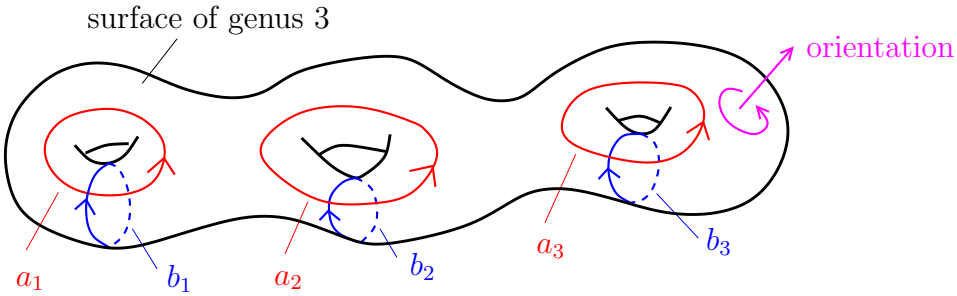


FIGURE 905.

(3) Next we consider the surface of genus two with three disks removed that is shown in Figure 906. By the discussion on page 896 we know that a basis for $H_1(\Sigma;\mathbb{Z})$ is represented by the curves a_1, b_1, a_2, b_2 together with two out of the three curve c_1, c_2, c_3. It follows from Theorem 67.10 that with respect to the basis $[a_1], [b_1], [a_2], [b_2], [c_1], [c_2]$ the intersection form is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from Proposition 21.4. It follows from Proposition 67.17 that the intersection form

$$Q_{\mathbb{C}\mathrm{P}^2}\colon H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \times H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \ \to\ \mathbb{Z}$$

is represented by the 1×1-matrix (1).

[1198]What do the curves a_i and b_i look like in the more precise description of the surface of genus 3 as a dodecagon with identified sides?

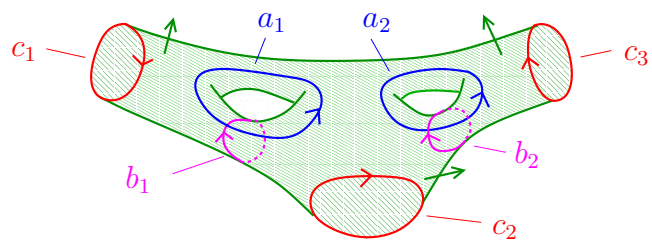


FIGURE 906.

(5) Let M be a closed oriented $2n$-dimensional topological manifold. As usual we denote by $-M$ the same manifold with the opposite orientation. It follows immediately from the definitions that $Q_{-M} = -Q_M$.

(6) For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. With this notation it follows from (4) and (5) that the intersection form

$$Q_{\overline{\mathbb{C}\mathrm{P}}^2}\colon H_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \times H_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \ \to\ \mathbb{Z}$$

is represented by the 1×1-matrix (-1).

71.3. **The intersection form and degree-one maps.** As we remarked on page 1652 the intersection form of a compact oriented $2n$-dimensional manifold contains basically the same information as the cup product pairing

$$\begin{array}{rcl} H^n(M,\partial M;\mathbb{Z}) \times H^n(M,\partial M;\mathbb{Z}) & \to & H^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) & \mapsto & \alpha \cup \beta. \end{array}$$

Given a map $f\colon X \to Y$ between topological spaces it follows from Lemma 57.10 that the cup product behaves well under the induced maps f^* on cohomology groups. It stands to reason that the intersection form should also behave well with respect to induced maps on homology.

So suppose we are given a degree-one map $f\colon X \to Y$ between closed oriented connected $2n$-dimensional topological manifolds. By the aforementioned Lemma 57.10 we know that for any $\alpha \in \mathrm{F}H^i(Y;\mathbb{Z})$ and $\beta \in \mathrm{F}H^j(Y;\mathbb{Z})$ we have $f^*\alpha \cup f^*\beta = f^*(\alpha \cup \beta)$. It seems reasonable to expect that given $\alpha \in H_n(X;\mathbb{Z})$ and $\beta \in H_n(X;\mathbb{Z})$ the following equality holds:

$$(*) \qquad Q_X(\alpha,\beta) \ = \ Q_Y(f_*(\alpha), f_*(\beta)).$$

In fact here is a diagram that looks awfully relevant:

$$\begin{tikzcd}
H_n(X;\mathbb{Z}) \times H_n(X;\mathbb{Z}) \arrow[r, "\mathrm{PD}\times\mathrm{PD}", "\cong"'] \arrow[d, "f_*\times f_*"] & H^n(X;\mathbb{Z}) \times H^n(X;\mathbb{Z}) \arrow[r, "\cup"] & H^{2n}(X;\mathbb{Z}) \arrow[r, "\langle -,[X]\rangle"] & \mathbb{Z} \arrow[d, "\mathrm{id}"'] \\
H_n(Y;\mathbb{Z}) \times H_n(Y;\mathbb{Z}) \arrow[r, "\mathrm{PD}\times\mathrm{PD}", "\cong"'] & H^n(Y;\mathbb{Z}) \times H^n(Y;\mathbb{Z}) \arrow[r, "\cup"] \arrow[u, "f^*\times f^*"] & H^{2n}(Y;\mathbb{Z}) \arrow[r, "\langle -,[Y]\rangle"] \arrow[u, "f^*", "\cong"'] & \mathbb{Z}.
\end{tikzcd}$$

If this diagram did commute, then $(*)$ would hold. One could now waste a lot of time trying to convince oneself that the diagram commutes. Or alternatively one might notice that $(*)$ just does not hold in general.

Example. We consider the map $f\colon S^1 \times S^1 \to S^2$ that is illustrated in Figure 907. It follows easily from Lemma 47.4 that this is a degree-one map. We consider the two homology classes α and β on the torus that are sketched in Figure 907. It follows easily from Theorem 67.10 that $Q_{S^1\times S^1}(\alpha,\beta) = +1$. On the other hand we have $f_*(\alpha) = f_*(\beta) = 0 \in H^1(S^2;\mathbb{Z})$, which evidently implies that $Q_{S^2}(f_*(\alpha), f_*(\beta)) = 0$. Thus in this case the equality $(*)$ does not hold.[1199]

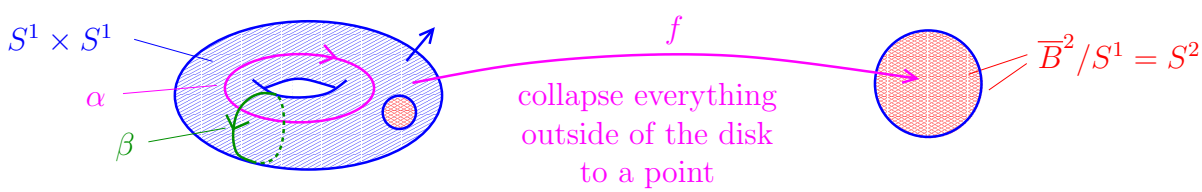


FIGURE 907.

Remark. With this discussion we also learned an interesting lesson:

Not every diagram commutes, not even up to sign.

71.4. **The connected sum of topological manifolds.** Recall that on page 355 we introduced the connected sum of two manifolds. In the following we generalize this definition to the connected sum of topological manifolds.

Definition. Let M and X be oriented n-dimensional topological manifolds. We say a map $\varphi\colon X \to M$ is *suitable* (respectively *anti-suitable*) if the following conditions are satisfied:

(1) we have $\varphi(X) \subset M \setminus \partial M$,
(2) the map φ is a topological embedding,
(3) the map φ is orientation-preserving (respectively orientation-reversing).[1200]

Now we can define the connected sum of two oriented topological manifolds.

Definition. Let $n \in \mathbb{N}$ and let M and N be two oriented non-empty n-dimensional topological manifolds. Given suitable a map $\varphi\colon \overline{B}^n \to M$ and given an anti-suitable map $\psi\colon \overline{B}^n \to N$ we define the *connected sum M and N* as

$$M\#N \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \varphi(P) = \psi(P) \text{ for all } P \in S^{n-1}.$$

We refer to Figure 908 for an illustration.

We have the following proposition which in the following we will use without explicitly referring to it.

[1199] This still begs the question, in the above diagram, which of the squares does not commute?

[1200] We refer to page 1099 for the definition of "orientation-preserving" and "orientation-reversing".

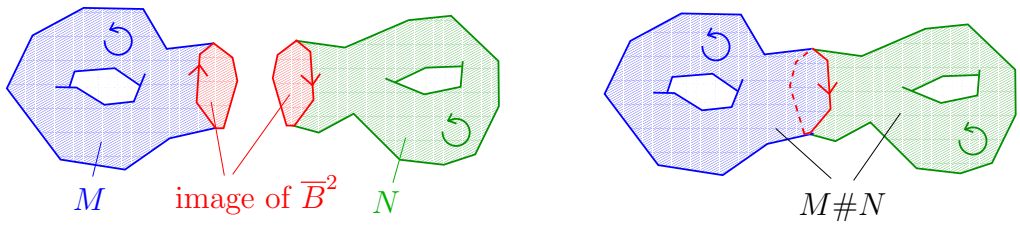


FIGURE 908.

Proposition 71.8. *Let $n \in \mathbb{N}$ and let M and N be two oriented connected n-dimensional topological manifolds.*

(1) *The homeomorphism type of $M\#N$ does not depend on the choice of (anti-) suitable maps.*
(2) *The connected sum $M\#N$ is an n-dimensional topological manifold.*
(3) *Given suitable maps $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to N$ we can equip the connected sum $M\#N$ with an orientation such that the two inclusion maps $M \setminus \varphi(B^n) \to M\#N$ and $M \setminus \psi(B^n) \to M\#N$ are orientation-preserving.*

Convention. We always equip the connected sum of two oriented connected n-dimensional topological manifolds with the orientation from Proposition 71.8 (3).

The proof of Proposition 71.8 (1) relies heavily on the following, surprisingly hard theorem which can be viewed as the topological analogue of Theorem 10.28

Theorem 71.9. *Let M be an oriented connected n-dimensional topological manifold. Let φ and ψ be two maps $\overline{B}^n_1 \sqcup \cdots \sqcup \overline{B}^n_s \to X$ from the disjoint union of s closed n-balls to X. Suppose the following statements hold:*

(1) *both φ and ψ are suitable or both are anti-suitable, and*
(2) *either $s = 1$ or $n \geq 2$,*

then there exists a homeomorphism F of M, isotopic to the identity, such that $F \circ \varphi = \psi$.

Proof. The theorem follows from combining the argument of [Rol, p. 42] together with the "annulus theorem" that was proved in 1969 by Rob Kirby [Kir] for $n \neq 4$ and that was proved in 1982 by Frank Quinn [Qu, Edw2] for $n = 4$. More details can be found in [FHNOP]. □

Now we can provide the proof of Proposition 71.8.

Proof of Proposition 71.8.

(1) The proof that the homeomorphism type of the connected sum of two oriented connected n-dimensional topological manifolds does not depend on the choice of (anti-) suitable maps is a fairly straightforward consequence of Theorem 71.9 applied twice with $s = 1$.

(2) It is fairly straightforward to show that the connected sum of two oriented connected topological manifolds is again a orientable connected topological manifold, see e.g. [Le3, Proposition 6.6] for details.
(3) The proof of the last statement is similar to the proof of Lemma 45.29. We leave the details as an exercise to the reader. □

Next we state how some of the key invariants behave under the connected sum operation.

Proposition 71.10. *Let $n \in \mathbb{N}$ and let M and N be two closed*[1201] *oriented connected n-dimensional topological manifolds and furthermore let $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to N$ be suitable maps. Then $H_0(M\#N;R) \cong H_n(M\#N;R) \cong R$. Furthermore, for any $i \neq 0, n$ and any commutative ring R the following inclusion induced maps are isomorphisms*

$$H_i(M\#N;R) \xleftarrow{\cong} H_i(M\backslash\varphi(B^n);R) \oplus H_i(N\backslash\psi(B^n);R) \xrightarrow{\cong} H_i(M;R) \oplus H_i(N;R).$$

Furthermore if $n \geq 3$, then the following inclusion induced maps are isomorphisms

$$\pi_1(M\#N) \xleftarrow{\cong} \pi_1(M \setminus \varphi(B^n)) * \pi_1(N \setminus \psi(B^n)) \xrightarrow{\cong} \pi_1(M) * \pi_1(N).$$

Proof. We proved the statements in Proposition 12.11 and Proposition 39.15 for manifolds. The proof for topological manifolds is very similar, we just need to make the following two modifications:

(1) The statement of the Seifert-van Kampen Theorem 12.9 also holds for topological manifolds, we just need to replace the Collar Neighborhood Theorem 10.10 by the Topological Collar Neighborhood Theorem 45.3.
(2) The Mayer–Vietoris argument in the proof of Lemma 39.16 works for topological manifolds by Proposition 50.12 (5) and the Mayer–Vietoris Theorem 50.14. □

71.5. **The intersection form and the connected sum operation.** The next proposition says that the isomorphism of homology groups from Proposition 71.10 induces an isometry of intersection forms.

Proposition 71.11. *Let $n \geq 1$, let M and N be closed oriented $2n$-dimensional topological manifolds and let $\varphi\colon \overline{B}^{2n} \to M$ and $\psi\colon \overline{B}^{2n} \to N$ be two suitable maps. The isomorphism $H_n(M\#N;\mathbb{Z}) \xrightarrow{\cong} H_n(M;\mathbb{Z}) \oplus H_n(N;\mathbb{Z})$ from Proposition 71.10 induces an isometry*

$$Q_{M\#N} \cong Q_M \oplus Q_N$$

of forms.

Example. We consider the surface Σ of genus two. In Figure 909 we recall that we can view Σ as the connected sum of two tori. It thus follows from Proposition 71.11, the discussion on page 1655 and Lemma 71.3 (2) that the intersection form on $H_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is represented

[1201] How does the proposition change if we allow topological manifolds with boundary?

by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Not surprisingly this recovers the result from page 1656.

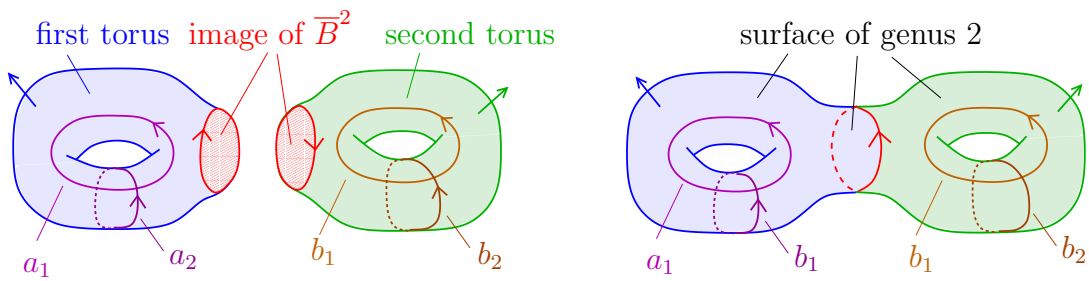


FIGURE 909.

Remark. Let $n \geq 1$, let M and N be closed oriented $2n$-dimensional topological manifolds and let $\varphi\colon \overline{B}^{2n} \to M$ and $\psi\colon \overline{B}^{2n} \to N$ be two suitable maps. Figure 909 suggests the following approach to proving Proposition 71.11:

(1) First we pick transverse n-dimensional cycles $a_1, \dots, a_k$ in $M \setminus \varphi(\overline{B}^{2n})$ such that $[a_1], \dots, [a_k]$ form a basis of $\mathrm{F}H_n(M; \mathbb{Z})$.
(2) Similarly we pick transverse n-dimensional cycles $b_1, \dots, b_l$ in $N \setminus \psi(\overline{B}^{2n})$ such that $[b_1], \dots, [b_l]$ form a basis of $\mathrm{F}H_n(N; \mathbb{Z})$.
(3) It follows immediately from Proposition 71.10 that $[a_1], \dots, [a_k], [b_1], \dots, [b_l]$ form a basis for $\mathrm{F}H_n(M\#N; \mathbb{Z})$.
(4) We use the basis from (3) to determine the intersection form of $M\#N$. Here we make the following observations:
   (a) By the orientation conventions from Proposition 71.8 (3) the algebraic intersection numbers of the a_i in M are the same as in $M\#N$,
   (b) similarly the algebraic intersection numbers of the b_i in N are the same as in $M\#N$,
   (c) the a_i do not intersect the b_j, hence $a_i \cdot b_j = 0$ for any i, j.

   Summarizing we obtain immediately from Theorem 68.5 that $Q_M \oplus Q_N \cong Q_{M\#N}$.

The above argument sounds very convincing. But there is a major catch: how can we find such $a_1, \dots, a_k$ and $b_1, \dots, b_l$? The problem is not that we need to avoid $\varphi(\overline{B}^{2n})$ and $\psi(\overline{B}^{2n})$.[1202] The problem is to find such classes that are transverse. Even though we have some tools, e.g. we have Theorem 65.26 and the Transversality Theorem 67.2, it is very hard to turn the above approach into a rigorous proof.

[1202] Why is this not a problem?

After the discussion in the previous remark we recall the advice from [Kir4, p. 21], see page 1566:

"Think with [algebraic] intersection [numbers], prove with cup products."

Thus we will use cup products to prove Proposition 71.11.

Proof of Proposition 71.11. Throughout the proof let $n \geq 1$. We start out with a lengthy preparation. Thus let X and Y be two oriented connected $2n$-dimensional topological manifolds with $\partial X \cong S^{2n-1}$ and $\partial Y \cong S^{2n-1}$. We equip ∂X and ∂Y with the orientation coming from Proposition 45.30. We identify ∂X and ∂Y via an orientation-reversing homeomorphism φ. We write $Z = X \cup_\varphi Y$. In the following we view X and Y as submanifolds of Z. We write $\mathring{X} = X \setminus \partial X$ and $\mathring{Y} = Y \setminus \partial Y$. Recall that by Proposition 71.8 we can equip Z with an orientation such that the induced orientations on X and Y are the given orientations.

We start out with the following claim.

Claim. The inclusions $i\colon X \to Z$ and $j\colon Y \to Z$ induce an isometry

$$Q_X \oplus Q_Y \xrightarrow{\cong} Q_Z.$$

As in Proposition 71.11 we see that the two inclusion maps i and j induce an isomorphism $H_n(X;\mathbb{Z}) \oplus H_n(Y;\mathbb{Z}) \to H_n(Z;\mathbb{Z})$. Evidently this isomorphism descends to an isomorphism $\mathrm{F}H_n(X;\mathbb{Z}) \oplus \mathrm{F}H_n(Y;\mathbb{Z}) \to \mathrm{F}H_n(Z;\mathbb{Z})$. Therefore it suffices to prove the following three statements.

(1) for $a, b \in \mathrm{F}H_n(X;\mathbb{Z})$ we have $Q_X(a,b) = Q_Z(i_*(a), i_*(b))$,
(2) for $a, b \in \mathrm{F}H_n(Y;\mathbb{Z})$ we have $Q_Y(a,b) = Q_Z(j_*(a), j_*(b))$,
(3) for $a \in \mathrm{F}H_n(X;\mathbb{Z})$ and $b \in \mathrm{F}H_n(Y;\mathbb{Z})$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

We start out with the proof of the first statement. So let $a, b \in \mathrm{F}H_n(X;\mathbb{Z})$. We denote by $p\colon (Z, \partial Z) \to (Z, Z \setminus \mathring{X})$ and $i\colon (X, \partial X) \to (Z, Z \setminus \mathring{X})$ the two obvious inclusion maps of pairs of topological spaces. Note that it is a consequence of the Excision Theorem 45.8 for Topological Manifolds that $i_*\colon \mathrm{F}H_k(X, \partial X;\mathbb{Z}) \to \mathrm{F}H_k(Z, Z \setminus \mathring{X};\mathbb{Z})$ is an isomorphism for

every $k \in \mathbb{N}_0$. Then

$$\begin{aligned}
Q_X(a,b) &\underset{\substack{\uparrow \\ \text{by definition of } Q_X}}{=} \langle \mathrm{PD}_X(a) \cup \mathrm{PD}_X(b), [X] \rangle \underset{\substack{\uparrow \\ \text{Lemma 50.6 applied to } i\colon (X,\partial X) \to (Z, Z \setminus \mathring{X}), \\ \text{together with Proposition 58.3 (1)}}}{=} \big\langle (i^*)^{-1}(\mathrm{PD}_X(a)) \cup (i^*)^{-1}(\mathrm{PD}_X(b)), i_*([X]) \big\rangle \\
&\underset{\substack{\uparrow \\ \text{first note that by Lemma 46.23 we have, } i_*([X]) = p_*([Z]), \text{ next apply} \\ \text{Lemma 50.6 to } p, \text{ finally apply Proposition 58.3 (1) to } p}}{=} \big\langle p^*((i^*)^{-1}(\mathrm{PD}_X(a))) \cup p^*((i^*)^{-1}(\mathrm{PD}_X(b))), [Z] \big\rangle \\
&\underset{\substack{\uparrow \\ \text{by Lemma 67.12 we know that} \\ p^*(i^*)^{-1}(\mathrm{PD}_X(c)) = \mathrm{PD}_Z(i_*(c))}}{=} \big\langle \mathrm{PD}_Z(i_*(a)) \cup \mathrm{PD}_Z(i_*(b)), [Z] \big\rangle \underset{\substack{\uparrow \\ \text{definition of } Q_Z}}{=} Q_Z(i_*(a), i_*(b)).
\end{aligned}$$

The second statement is of course proved the same way. We turn to the proof of the third statement. So let $a \in \mathrm{F}H_n(X;\mathbb{Z})$ and $b \in \mathrm{F}H_n(Y;\mathbb{Z})$. The above argument shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup product in Z of an element in the image $H^n(Z, Z \setminus \mathring{X}) \to H^n(Z)$ and an element in the image of $H^n(Z, Z \setminus \mathring{Y}; \mathbb{Z}) \to H^n(Z;\mathbb{Z})$. But $(Z \setminus \mathring{X}) \cup (Z \setminus \mathring{Y}) = Z$. So it follows from Proposition 58.3 (1) that this cup product is zero. This concludes the proof of the claim.

After these preparations it is very easy to prove the actual statement of the proposition. Thus let M and N be closed oriented $2n$-dimensional topological manifolds and let $\varphi\colon \overline{B}^{2n} \to M$ and $\psi\colon \overline{B}^{2n} \to N$ be two suitable maps. We write $X = M \setminus \varphi(B^{2n})$ and $Y = N \setminus \psi(B^{2n})$. It follows from the definition of a topological embedding that X and Y are compact topological manifolds. With this notation we can write $M = X \cup \overline{B}^{2n}$ and $N = Y \cup \overline{B}^{2n}$. It follows from the claim and the fact that $H_n(\overline{B}^{2n};\mathbb{Z}) = 0$ that the inclusions induce isometries

$$Q_{M\#N} \xleftarrow{\cong} Q_X \oplus Q_Y \xrightarrow{\cong} Q_M \oplus Q_N.$$

We have thus proved the desired statement. □

We recall the following question.

Question 43.10. *Are the topological spaces* $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ *and* $S^2 \times S^2$ *homeomorphic?*

These topological spaces are hard to distinguish since we saw that they are simply connected and that their (co-) homology groups agree. Now we want to look at the intersection forms to see whether they can help us in distinguishing these three manifolds.

Example. On page 1656 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the 1×1-matrix (1) and we saw that the intersection form of $\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the 1×1-matrix (-1). From Proposition 71.11 we thus obtain:

(1) the intersection form of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, whereas

(2) the intersection form of $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Finally we recall that on page 1655 we showed that

(3) the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Motivated by Question 43.10 and Lemma 71.7 the following question arises.

Question 71.12. *Are the forms over $\mathbb{Z}^2$ represented by the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

isometric (*possibly up to a sign reversal*)*?*

In the next chapter we will develop the theory of non-singular forms to answer Question 71.12.

Exercises for Chapter 71.

Exercise 71.1. Let M be a compact oriented topological manifold. Recall that we say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*. For which $k, l \in \mathbb{N}_0$ is the connected sum $k \cdot \mathbb{C}\mathrm{P}^2\#l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ chiral? Justify your answer. More precisely:

(a) for the values that you exclude, show that $k \cdot \mathbb{C}\mathrm{P}^2\#l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is not chiral,

(b) for all the other values show explicitly that $k \cdot \mathbb{C}\mathrm{P}^2\#l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is chiral.

Exercise 71.2. Let R be a commutative ring. A form $\varphi\colon V \times V \to R$ is called *non-degenerate* if the two maps

$$\text{(a)} \quad \begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ v & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ w & \mapsto & \varphi(v,w) \end{array} \right) \end{array} \quad \text{and} \quad \text{(b)} \quad \begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ v & \mapsto & \varphi(v,w) \end{array} \right) \end{array}$$

are *monomorphisms*. Now let M be a compact oriented 4-dimensional topological manifold such that ∂M is a rational homology sphere. Show that the intersection form Q_M is non-degenerate.

72. The intersection form of topological manifolds II: Algebra and 4-dimensional manifolds

In the previous chapter we introduced the intersection form of compact oriented even-dimensional topological manifolds. Towards the end of the last chapter it became clear that we have to invest some time to study (anti-) symmetric forms as purely algebraic objects.

We start out with the following lemma.

Lemma 72.1. *Let R be a commutative ring.*

(1) *Let A be a (skew-) symmetric*[1203] *$n \times n$-matrix over R. We consider the map*

$$\begin{array}{rcl} \varphi(A)\colon R^n \times R^n & \to & R \\ (v,w) & \mapsto & v^T A w. \end{array}$$

Then the following hold:

(a) *$\varphi(A)$ is an (anti-) symmetric form over R,*

(b) *the form $\varphi(A)$ is non-singular if and only if A is invertible, i.e. if $\det(A)$ is a unit in R,*

(c) *if $P \in \mathrm{GL}(n,R)$ is an invertible matrix, then the forms $\varphi(P^T AP)$ and $\varphi(A)$ are isometric.*

(2) *Let (V,ψ) be an (anti-) symmetric form over R of rank n. If A is a matrix that represents ψ, then $(V,\psi) \cong (R^n, \varphi(A))$. In particular every (anti-) symmetric form over R is isometric to $\varphi(A)$ for some (skew-) symmetric $n \times n$-matrix A over R.*

Proof.

(1) (a) This statement follows almost immediately from the definitions and the observation that for any $v, w \in R^n$ we have

$$\varphi(A)(w,v) \;=\; w^T A v \;=\; (v^T A^T w)^T \;\underset{\substack{\uparrow \\ \text{for a } (1\times 1)\text{-matrix } (x) \text{ we have } (x)^T = (x)}}{=}\; v^T A^T w \;=\; \varphi(A^T)(v,w).$$

(b) The proof of this statement is an instructive exercise in linear algebra. We will not rob the reader of the joy of doing this exercise.

(c) We claim that that the isomorphism $v \mapsto Pv$ defines an isometry from the form $(R^n, \varphi(P^T AP))$ to the form $(R^n, \varphi(A))$. Indeed, for any $v, w \in R^n$ we have

$$\varphi(A)(Pv, Pw) \;=\; (Pv)^T APw \;=\; v^T (P^T AP) w \;=\; \varphi(P^T AP)(v,w).$$

(2) Let $\psi\colon V \times V \to R$ be an (anti-) symmetric form over R. Let $v_1, \dots, v_n$ be a basis of V. We denote by $A = (\psi(v_i, v_j))_{i,j=1,\dots,n}$ the corresponding matrix representing the form ψ. We consider the isomorphism

$$\begin{array}{rcl} \Theta\colon R^k & \to & V \\ (\lambda_1, \dots, \lambda_n) & \mapsto & \lambda_1 v_1 + \dots + \lambda_n v_n. \end{array}$$

[1203] Recall that a matrix A is called *skew-symmetric* if $A^T = -A$.

We claim that Θ is an isometry from the form $(R^n, \varphi(A))$ to the form (V, ψ). Indeed, if we denote by $e_1, \dots, e_n$ the standard basis for R^n, then for any $i, j \in \{1, \dots, n\}$ we have

$$\psi(\Theta(e_i), \Theta(e_j)) \;=\; \psi(v_i, v_j) \;=\; e_i^T A e_j \;=\; \varphi(A)(e_i, e_j).$$

By bilinearity of forms we obtain the corresponding statement for all pairs of vectors in R^n. □

Convention. Lemma 72.1 allows us to go back and forth between (anti-) symmetric forms and (skew-) symmetric matrices. Thus in the following discussion we will blur the distinction between these two concepts. For example, given a (skew-) symmetric matrix A over a commutative ring we will often just refer to the (anti-) symmetric form $\varphi(A)$ as the *form* A.

72.1. **Non-singular anti-symmetric forms.** In this section we will completely classify non-singular anti-symmetric forms over the real numbers and over the integers.

First we consider the matrix

$$S \;:=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is skew-symmetric and it has determinant 1. From Lemma 72.1 we obtain an anti-symmetric form $\varphi(S)$. The following proposition shows that $\varphi(S)$ is in some sense the only example of a non-singular anti-symmetric form over the real numbers or over the integers.

Proposition 72.2. *If (V, φ) is a non-singular anti-symmetric form over $\mathbb{R}$ or over $\mathbb{Z}$, then there exists a $k \in \mathbb{N}_0$ such that*[1204] *$(V, \varphi) \cong k \cdot \varphi(S)$, i.e. φ is represented by the $2k \times 2k$-matrix*

$$\underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}}_{k \text{ diagonal blocks of } 2\times 2\text{-matrices}}.$$

In particular the rank of V is even.

Remark. Propositions 71.5 and 72.2 imply in particular that the isometry type of the intersection of a surface of genus g is determined by the formal properties of an intersection form. In particular we did not need to go through the discussion on page 1656 if we just need to know the *isomorphism type* of the intersection form.

Sketch of proof. First we remark that throughout the proof on many occasions we use Lemma 72.1 without saying so explicitly.

After this preamble, let (V, φ) be a non-singular anti-symmetric form over $\mathbb{R}$ of rank n. We pick an $(n \times n)$-matrix A that represents φ. Since φ is anti-symmetric we see that all

[1204] Here, when we write $k \cdot \varphi(S)$ we use the definition from page 1649.

terms of A on the diagonal are zero. We consider the following three types of elementary matrices:

$$D_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}, \lambda \neq 0, \quad P := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad Q_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

We make the following elementary observations:

(i) Replacing a matrix B by $D_\lambda^T B D_\lambda$ means that first we multiply a column by λ, and then we multiply the corresponding row by λ.
(ii) Replacing a matrix B by $P^T B P$ means that first we exchange two columns, and then we exchange the corresponding two rows.
(iii) Replacing a matrix B by $Q_\lambda^T B Q_\lambda$ means that first we add λ-times one column to another column, and then we do the same operation for the corresponding rows.

Now we can perform the following sequence of operations to A:

$$A \overset{(1)}{=} \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(2)}{\to} \begin{pmatrix} 0 & 1 & * & \dots & * \\ -1 & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(3)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & * & \dots & * \\ 0 & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & * & * & * & 0 \end{pmatrix} \overset{(4)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & 0 & * & * & 0 \end{pmatrix}$$

where we perform the following steps:

(1) We write out A, here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are skew-symmetric.
(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (ii) and (i) we can arrange that the $(1,2)$-entry equals 1.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.
(4) Similarly, we can use operations of type (iii) to "clean up" the remaining terms on the second row and column.

At this stage we can continue with the same process, applied to the $(n-2)\times(n-2)$-matrix at the right bottom. Eventually we will have turned A into a matrix of the promised type.

Now suppose that (V, φ) is a non-singular anti-symmetric form over $\mathbb{Z}$. In Exercise 72.4 we will modify the above proof to deal with this new situation. □

Even though Proposition 72.2 is a little disappointing, it is just good enough to answer Question 61.5 where we had asked which integers appear as Euler characteristics of closed orientable connected even-dimensional manifolds. Now we can give a complete answer to this question.

Proposition 72.3. *Let $k \in \mathbb{N}_0$.*

(1) *Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $4k$-dimensional manifold with Euler characteristic n.*
(2) *The Euler characteristic of any closed orientable connected topological manifold of dimension $4k+2$ is even. Conversely, any even number is the Euler characteristic of such a manifold.*

Proof. First we show that the Euler characteristic of a closed orientable topological manifold M of dimension $4k+2$ is even. Indeed, we have

$$\chi(M) \underset{\substack{\uparrow \\ \text{Proposition 45.7 (4)}}}{=} \sum_{i=0}^{4k+2}(-1)^i \cdot \dim_{\mathbb{R}}(H_i(M;\mathbb{R})) \underset{\substack{\uparrow \\ \text{by Theorem 61.3 we have } \dim_{\mathbb{R}}(H_i(M;\mathbb{R})) = \dim_{\mathbb{R}}(H_{4k+2-i}(M;\mathbb{R}))}}{=} \dim_{\mathbb{R}}(H_{2k+1}(M;\mathbb{R})) + \sum_{i=0}^{2k}(-1)^i \cdot 2\dim_{\mathbb{R}}(H_i(M;\mathbb{R}))$$

$$\underset{\substack{\uparrow \\ \text{by Proposition 71.5 and the facts that } M \text{ is closed and that } 2k+1 \text{ is odd we know} \\ \text{that the intersection on } H_{2k+1}(M;\mathbb{R}) \text{ is anti-symmetric and non-singular} \\ \text{by Proposition 72.2 we thus know that } \dim_{\mathbb{R}}(H_{2k+1}(M,\mathbb{R})) \text{ is even}}}{\equiv} 0 \mod 2.$$

Besides this restriction on the possible values of Euler characteristics, all other values can be realized. This follows easily from the following three facts:

(1) On page 966 we saw that

$$\chi(\mathbb{C}\mathrm{P}^n) = n+1 \equiv \begin{cases} 0 \bmod 2, & \text{if } n \text{ is odd.} \\ 1 \bmod 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Lemma 40.8 says that if M and N are two compact oriented connected *even-dimensional* manifolds, then $\chi(M\#N) = \chi(M) + \chi(N) - 2$.
(3) There is any number of reasons, e.g. we could use Lemma 40.9, why the Euler characteristic of the n-dimensional torus $(S^1)^n$ is zero.

Using these three facts one can show easily that any permissible value for an Euler characteristic of a $2k$-dimensional manifold is realized by an appropriate connected sum of copies of $\mathbb{C}\mathrm{P}^k$ and copies of the $2k$-dimensional torus. □

72.2. **Interlude: Sphere-packing.** In this short section we want to consider sphere packings. The relationship with our actual goal of dealing with symmetric forms is arguably tenuous but nonetheless fascinating.

Definition. Let $v_1, \dots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. We refer to the group

$$\mathbb{Z} \cdot v_1 + \dots + \mathbb{Z} \cdot v_n = \left\{ \sum_{i=1}^n a_i \cdot v_i \,\middle|\, a_1, \dots, a_n \in \mathbb{Z} \right\}$$

as the *lattice spanned by* $v_1, \dots, v_n$.[1205]

We start out with the following theorem. The fact that strictly speaking we did not define all the terms will not bother us.

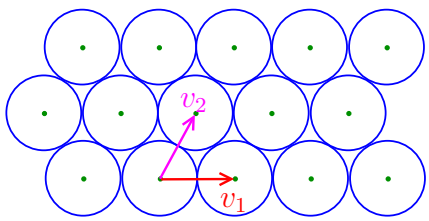
[1205] This is perhaps not the standard definition of a lattice, but who cares?

Theorem 72.4.

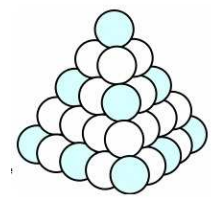
(1) *The densest packing of disks of radius* $\frac{1}{2}$ *in* $\mathbb{R}^2$ *is given by setting the centers on the "hexagonal" lattice spanned by* $v_1 = (1,0)$ *and* $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ *shown in Figure 910 on the left. The density equals* $\frac{\pi}{\sqrt{12}}$.

(2) *No packing of balls of the same radius in* $\mathbb{R}^3$ *has density greater than that of the "face-centered cubic packing" shown in Figure 910 in the middle and on the right. The density equals* $\frac{\pi}{\sqrt{18}}$.

Proof. The first statement was proved by L. Fejes Tóth [FT, FT2] in 1943. It is much harder to prove the second statement. This statement had been conjectured by Kepler in 1613 and it was finally proved by Thomas Hales [Hale, Hale2, Hale3] in 2005. □



densest circle packing



ball packings of highest density

FIGURE 910.

For a mathematician it is natural to wonder what are the highest densities of packings in higher dimensions. Not surprisingly this is a very problem. Astonishingly the next case that was dealt with is the 8-dimensional setting. More precisely, the following theorem was proved by Maryna Viazovska in 2016.

Theorem 72.5. *We consider the following matrix*

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

No packing of unit balls in $\mathbb{R}^8$ *has density greater than that given by the lattice spanned by the eight column vectors of the* E_8*-matrix. Furthermore, this packing has density* $\frac{\pi^4}{384}$.

Remark. We also refer to [Bae2] for other interesting appearances of the E_8-matrix, for example its relationship to the regular icosahedron

Shortly afterwards the 24-dimensional case was settled in [CKMRV], in this case a densest packing is given by the "Leech lattice". OK, that was fun, but now let us return to our actual problem, and let us see whether this section was of any use.

72.3. **Non-singular symmetric forms over the real numbers.** In an earlier section we saw that non-singular *anti-symmetric* forms are almost depressingly boring mathematical objects. In this and the next section we will consider non-singular *symmetric* forms. These will, fortunately, turn out to be much more interesting.

Examples. We list some symmetric integral matrices with determinant equal to ± 1. By Lemma 72.1 these define in particular non-singular forms over $\mathbb{R}$ but also over $\mathbb{Z}$.

(1) The 1×1-matrices $(+1)$ and (-1).
(2) The slightly mysterious 8×8-matrix E_8 from the previous section.
(3) The matrix
$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
is symmetric and it has determinant -1.

It is natural to ask whether there is some type of analogue of Proposition 72.2 for symmetric forms. In fact using the methods of the proof of Proposition 72.2 one can prove the following statement.

Proposition 72.6. *If (V, φ) is a non-singular symmetric form over $\mathbb{R}$ of rank n, then there exists a $k \in \mathbb{N}_0$ such that $(V, \varphi) \cong k \cdot (+1) \oplus (n-k) \cdot (-1)$.*

Remark. Recall that Proposition 72.2 dealt with anti-symmetric forms over $\mathbb{R}$ and also with anti-symmetric forms over $\mathbb{Z}$. In contrast Proposition 72.6 only deals with symmetric forms over $\mathbb{R}$. It is a very instructive exercise to run through the proof of Proposition 72.6 and to determine where it breaks down if we work over $\mathbb{Z}$.

Sketch of proof. Let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ of rank n. In the subsequent arguments below we use the conventions and notions we introduced in Proposition 72.2. In particular we will also use the operations of type (i), (ii) and (iii) that we introduced in that proof.

We pick an $(n \times n)$-matrix A that represents φ. Now we can perform the following sequence of operations to A:

$$A \overset{(1)}{=} \begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} \epsilon_1 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

where we perform the following steps:

(1) We write out A, here and throughout the steps we use the convention that $*$ can mean anything, the only condition being that all matrices are symmetric.

(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (iii) and (i) we can arrange that the $(1,1)$-entry equals some $\epsilon_1 \in \{-1, 1\}$.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

At this stage we can continue with the same process, applied to the $(n-1)\times(n-1)$-matrix at the right bottom. Eventually we will have turned A into a diagonal matrix where all entries lie in $\{-1, 1\}$. After operations of type (ii) we can arrange that the first diagonal entries are equal to $+1$ and the final ones are equal to -1. □

Let $n \in \mathbb{N}_0$. Proposition 72.6 raises the question whether or not the symmetric forms defined by $k \cdot (+1) \oplus (n-k) \cdot (-1)$ are isometric for different values of $k \in \{0, \ldots, n\}$. The following definition is the key to answering this question.

Definition. Let (V, φ) be a symmetric form over the real numbers.

(1) We say φ is *positive-definite* if $\varphi(v, v) > 0$ for all $v \neq 0$.
(2) We say φ is *negative-definite* if $\varphi(v, v) < 0$ for all $v \neq 0$.
(3) We say φ is *indefinite* if φ is neither positive-definite nor negative-definite.

Examples.

(1) The form $(+1)$ is positive-definite and the form (-1) is negative-definite.
(2) The form E_8 is positive-definite. Evidently one can show this by a brute force calculation. A more conceptual proof is given in [HGK, Theorem 1.16.14].
(3) The form H is indefinite, indeed, we have

$$\varphi(H)\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) \;=\; \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \;=\; 2$$

and

$$\varphi(H)\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) \;=\; \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \;=\; -2.$$

To distinguish non-isometric forms we need to introduce some invariants of forms.

Definition. Let (V, φ) be a symmetric form over the real numbers. We define[1206]

$$\begin{aligned} b^+(\varphi) &:= \text{ maximal dimension of a subspace } W \text{ of } V \text{ such that } \varphi|_W \text{ is positive-definite,} \\ b^-(\varphi) &:= \text{ maximal dimension of a subspace } W \text{ of } V \text{ such that } \varphi|_W \text{ is negative-definite.} \end{aligned}$$

Furthermore we define the *signature* $\operatorname{sign}(\varphi)$ as

$$\operatorname{sign}(\varphi) \;:=\; b^+(\varphi) - b^-(\varphi).$$

The following lemma is an immediate consequence of the definitions.

[1206]Just to make sure there is no misunderstanding, by "subspace" we mean a "vector subspace". The set of all vectors $v \in V$ with $\varphi(v, v) > 0$ is *not* a subspace, but some open subset of V.

Lemma 72.7. *Let (V,φ) and (W,ψ) be two non-singular symmetric forms over $\mathbb{R}$. If (V,φ) and (W,ψ) are isometric, then*

$$\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W), \quad b^{\pm}(\varphi) = b^{\pm}(\psi) \quad \text{and} \quad \operatorname{sign}(\varphi) = \operatorname{sign}(\psi).$$

In the following proposition we determine $b^{\pm}$ and the signature for a non-singular form that is given by a diagonal matrix.

Proposition 72.8. *If D is a diagonal $(n\times n)$-matrix where the diagonal entries $d_1,\dots,d_n$ are non-zero real numbers, then*

$$b^+(\varphi(D)) \;=\; \text{number of positive } d_i\text{'s} \quad \text{and} \quad b^-(\varphi(D)) \;=\; \text{number of negative } d_i\text{'s}.$$

In particular

$$\operatorname{sign}(\varphi(D)) \;=\; \text{number of positive } d_i\text{'s} \;-\; \text{number of negative } d_i\text{'s}.$$

Proof. After reordering the diagonal entries we can assume that the first k entries of D are positive and that the remaining $(n-k)$ entries are negative. We write

$$\begin{aligned} V^+ &:= \{(x_1,\dots,x_k,0,\dots,0)\in\mathbb{R}^n \mid x_1,\dots,x_k\in\mathbb{R}\} \quad \text{and} \\ V^- &:= \{(0,\dots,0,y_1,\dots,y_{n-k})\in\mathbb{R}^n \mid y_1,\dots,y_{n-k}\in\mathbb{R}\} \end{aligned}$$

It is clear that $\varphi(D)|_{V^+}$ is positive-definite and that $\varphi(D)|_{V^-}$ is negative-definite. Thus $b^+(\varphi(D))\ge k$ and $b^-(\varphi(D))\ge n-k$. If remains to prove the following claim.

Claim. For any symmetric form (W,ψ) over $\mathbb{R}$ we have $b^+(\psi)+b^-(\psi)\le\dim_{\mathbb{R}}(W)$.

Thus let W^+ be a positive-definite subspace of W and let W^- be a negative-definite subspace of W. The intersection $W^+\cap W^-$ is a subspace that is positive-definite and negative-definite. So for any non-zero vector $v\in W^+\cap W^-$ we have $\psi(v,v)>0$ and $\psi(v,v)<0$. This is only possible if no such v exists, i.e. we have shown that the intersection $W^+\cap W^-=\{0\}$. Now we see that

$$\dim(W^+)+\dim(W^-) \underset{\substack{\uparrow\\ \text{linear algebra}}}{=} \dim(W^++W^-)+\dim(W^+\cap W^-) \underset{\substack{\uparrow\\ W^+\cap W^-=\{0\}}}{=} \dim(W^++W^-) \underset{\substack{\uparrow\\ \text{since } W^++W^-\subset W}}{\le} \dim(W).$$

This concludes the proof of the claim. □

Corollary 72.9.

(1) *Let (V,φ) be a non-singular symmetric form over $\mathbb{R}$. We write $n=\dim_{\mathbb{R}}(V)$. The following statements hold:*
   (a) *we have $b^{\pm}(-\varphi)=b^{\mp}(\varphi)$, in particular we have $\operatorname{sign}(-\varphi)=-\operatorname{sign}(\varphi)$,*
   (b) *we have $b^+(\varphi)+b^-(\varphi)=\dim_{\mathbb{R}}(V)$,*
   (c) *we have $b^+(\varphi)=\frac{1}{2}(n+\operatorname{sign}(\varphi))$,*
   (d) *if we write $k=b^+(\varphi)$, then the form (V,φ) is isometric to $k\cdot(+1)\oplus(n-k)\cdot(-1)$.*

(2) *If (V,φ) and (W,ψ) are two non-singular symmetric forms over $\mathbb{R}$, then*

$$\operatorname{sign}(\varphi\oplus\psi) \;=\; \operatorname{sign}(\varphi)+\operatorname{sign}(\psi).$$

Proof.

(1) (a) This statement follows immediately from the definitions.
(b) By Proposition 72.6 and Lemma 72.7 it suffices to prove the statement for non-singular symmetric forms that are represented by diagonal matrices. But for those forms the statement follows immediately from Proposition 72.8.
(c) This statement is an immediate consequence of (b) and the definition of the signature.
(d) This statement follows from Proposition 72.6, Lemma 72.7 and Proposition 72.8.

(2) This statement follows immediately from statement (b), Proposition 72.6 applied to φ and ψ and Proposition 72.8. □

Example. Given a non-singular symmetric form φ on $\mathbb{R}^3$ one can visualize it by considering the "unit-sphere"

$$S(\varphi) \;:=\; \{v \in \mathbb{R}^3 \,|\, \varphi(v,v) = 1\}.$$

Corollary 72.9 (3) implies in particular that every non-singular symmetric form on $\mathbb{R}^3$ is isometric to one of the following four forms

(a) $3 \cdot (1)$, (b) $2 \cdot (1) \oplus (-1)$
(c) $2 \cdot (-1) \oplus (1)$ (d) $3 \cdot (-1)$.

In the last case the unit-sphere is the empty set. In the other cases the unit-spheres are sketched in Figure 911.

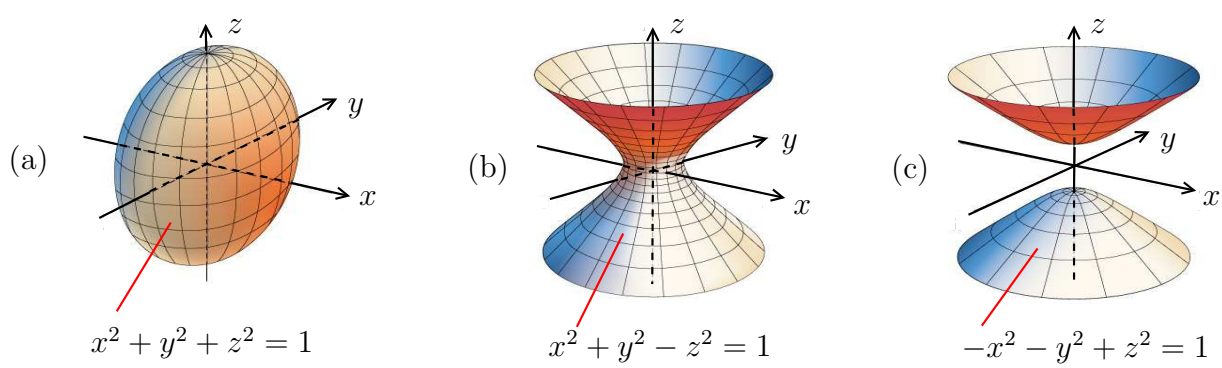


FIGURE 911.

The following theorem says that the dimension and the signature are a complete invariant for isometry types of non-singular symmetric forms over $\mathbb{R}$.

Theorem 72.10. (Sylvester's Theorem a. k. a. Sylvester's law of inertia) *Let* (V,φ) *and* (W,ψ) *be two non-singular symmetric forms over* $\mathbb{R}$. *Then*

$$(V,\varphi) \text{ and } (W,\psi) \text{ are isometric} \quad\Longleftrightarrow\quad \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W) \text{ and } \operatorname{sign}(\varphi) = \operatorname{sign}(\psi).$$

Proof. The "⇒"-direction is precisely Lemma 72.7. It remains to prove the "⇐"-direction. So suppose we are given two non-singular symmetric forms (V,φ) and (W,ψ) over $\mathbb{R}$ with same dimension and same signature. It follows from our hypothesis together with Corollary 72.9 (c) and (d) that there exist $n,k \in \mathbb{N}_0$ such that (V,φ) and (W,ψ) are both isometric to $k\cdot(+1)\oplus(n-k)\cdot(-1)$. But this implies that (V,φ) and (W,ψ) are themselves isometric. □

Remark. It is also very easy to classify symmetric forms over $\mathbb{R}$ that are not necessarily non-singular. More precisely, given a symmetric form (V,φ) over $\mathbb{R}$ we define the *nullity* $\sigma(\varphi)$ as

$$\nu(\varphi) \;:=\; \text{maximal dimension of a subspace } W \text{ of } V \text{ such that } g|_W \text{ is the zero form.}$$

It is an amusing little exercise to modify the proof of Sylvester's Theorem 72.10 to show that two symmetric forms over $\mathbb{R}$ are isometric if and only if their dimensions, signatures and nullities agree.

Now it is time to apply our recently acquired knowledge to topology.

Definition.

(1) If (V,φ) is a symmetric form over the integers, then we define

$$\begin{array}{rcl} g\otimes\mathbb{R}\colon V\otimes\mathbb{R}\times V\otimes\mathbb{R} & \to & \mathbb{R} \\ (v\otimes r, w\otimes s) & \mapsto & r\cdot\varphi(v,w)\cdot s \end{array}.$$

Furthermore we define

$$\operatorname{sign}(\varphi) \;:=\; \operatorname{sign}(\varphi\otimes\mathbb{R}).$$

(2) Let $k\geq 1$. Let M be a closed oriented non-empty $4k$-dimensional topological manifold. We define the *signature of M* to be[1207]

$$\operatorname{sign}(M) \;:=\; \operatorname{sign}(Q_M).$$

For convenience we define the signature of the empty manifold to be zero.

Lemma 72.11. *Let $k\in\mathbb{N}$.*

(1) *Let M be a closed oriented non-empty $4k$-dimensional manifold. The form $Q_M\otimes\mathbb{R}$ is isometric to the form*

$$\begin{array}{rcl} H^{2k}(M;\mathbb{R})\times H^{2k}(M;\mathbb{R}) & \to & \mathbb{R} \\ (\varphi,\psi) & \mapsto & \langle\varphi\cup\psi,[M]\rangle. \end{array}$$

(2) *The signature is additive under the connected sum operation and under disjoint union.*

(3) *For any closed oriented non-empty $4k$-dimensional topological manifold we have*

$$\operatorname{sign}(-M) \;=\; -\operatorname{sign}(M).$$

[1207]Recall that by Proposition 71.5 the intersection form Q_M is indeed symmetric, so it makes sense to consider its signature.

(4) *Let M and N be two closed oriented non-empty $4k$-dimensional topological manifolds. If M and N are homeomorphic, then* $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.

Proof.

(1) This statement follows easily from the various definitions and Corollary 42.20.
(2) It follows easily from lemma 59.9 that the signature is additive under disjoint union. Furthermore it follows easily from Proposition 71.11 and Corollary 72.9 that the signature is additive under the connected sum operation.
(3) The second statement is an immediate consequence of Lemma 71.7 and Corollary 72.9.
(4) The last statement is also an immediate consequence of Lemma 71.7 and Corollary 72.9. □

Examples. We have

$$\begin{array}{ccccccccc}
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2) + \operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1) + \operatorname{sign}(1) & = & 1+1 & = & 2 \\
 & \uparrow & & \uparrow & & & & & \\
 & \text{Lemma 72.11 (2)} & & \text{see page1656} & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2) - \operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1) - \operatorname{sign}(1) & = & 1-1 & = & 0.
\end{array}$$

Furthermore we have

$$\operatorname{sign}(S^2 \times S^2) \underset{\substack{\uparrow \\ \text{see page 1655}}}{=} \operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\substack{\uparrow \\ \text{Lemma 72.1}}}{=} \operatorname{sign}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \operatorname{sign}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \underset{\substack{\uparrow \\ \text{Proposition 72.8}}}{=} 0.$$

The above examples, together with Lemmas 71.7 and 72.11 (4) now allow us to give a partial answer to Question 43.10.

Corollary 72.12. *The topological space $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homeomorphic to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ nor to $S^2 \times S^2$.*

Remark. Corollary 72.12 shows in particular that in the definition of the connected sum of two orientable (topological) manifolds we needed to work with *oriented* (*topological*) *manifolds.*

Unfortunately we are not yet able to distinguish the two manifolds $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$. In fact Sylvester's Theorem 72.10 might suggest that we are again at a dead end. But this is not the case since Sylvester's Theorem is a statement about *real* symmetric forms, but we are really dealing with symmetric forms defined over the integers. We will exploit this extra piece of information in the next section.

72.4. **Non-singular symmetric forms over the integers.** In this section we want to study symmetric forms over the integers. We start out with the following slightly silly looking definition.

Definition. A symmetric form (V, φ) over the integers is called *even* if

$$\varphi(v, v) \equiv 0 \mod 2 \quad \text{for all } v \in V,$$

otherwise the intersection form is called *odd*. We refer to the property of φ being even or odd as the *parity* of the form.

The following lemma summarizes some basic facts about the parity of symmetric forms.

Lemma 72.13.

(1) *Isometric forms have the same parity.*
(2) *For a symmetric form φ the parities of φ and $-\varphi$ agree.*
(3) *The direct sum of an odd form with any other form is also odd.*
(4) *If A is a symmetric square matrix over the integers, then*

$$\varphi(A) \text{ is even} \quad \Longleftrightarrow \quad \text{all diagonal entries of } A \text{ are even.}$$

Examples. It follows from Lemma 72.13 that the forms $(+1)$, (-1) and $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ are odd and that the forms E_8 and $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are even.

Proof. The only statement which requires a little thought is the fact that if $A = (a_{ij})$ is a symmetric $(n \times n)$-matrix with even diagonal entries, then $\varphi(A)$ is even. Thus let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. We calculate that

$$\varphi(A)(v, v) = v^T A v = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \cdot a_{ij} \underset{\substack{\uparrow \\ \text{since } A \text{ is symmetric}}}{=} 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} v_i v_j \cdot a_{ij} + \sum_{i=1}^{n} v_i^2 a_{ii} \underset{\substack{\uparrow \\ \text{since the } a_{ii} \text{ are even}}}{\equiv} 0 \bmod 2. \qquad \square$$

It follows immediately from these examples and the earlier calculations of the intersection forms that the intersection form of $S^2 \times S^2$ is even whereas the intersection form of $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ is odd. This allows us to settle the remaining case of Question 43.10. More precisely, by Lemma 71.7 and Lemma 72.13 (1) and (2) we have the following proposition.

Proposition 72.14. *The topological space $S^2 \times S^2$ is not homeomorphic to $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$.*

We consider the matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows from Sylvester's Theorem 72.10 that the matrices give rise to *isometric forms over the real numbers*. On the other had we had just seen, using the parity, that the matrices give rise to *non-isometric forms over the integers*. The question arises, how much more complicated is the classification of non-singular symmetric forms over the integers than the classification of non-singular forms over the real numbers. In particular the following question arises.

Question 72.15. *Can we classify non-singular symmetric forms over the integers?*

The following is the key theorem in the classification of non-singular *indefinite* symmetric forms over $\mathbb{Z}$.

Theorem 72.16. *If φ_1 and φ_2 are two non-singular* indefinite *symmetric forms over the integers, then the following holds:*

$$\varphi_1 \text{ and } \varphi_2 \text{ are isometric} \quad \Longleftrightarrow \quad \varphi_1 \text{ and } \varphi_2 \text{ have the same rank, signature and parity.}$$

Example. Using the parity we had just seen that the intersection forms of the closed oriented 4-dimensional manifolds $M = S^2 \times S^2$ and $N = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, even up to sign. Now we consider the closed oriented 4-dimensional manifolds that we obtain from M and N by performing the connected sum with $\mathbb{C}\mathrm{P}^2$. In other words, we are considering $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$. It follows from Proposition 71.11 that the intersection forms of these two manifolds are represented by the matrices

$$A \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both forms are non-singular indefinite and odd. Thus it follows from Theorem 72.16 that the intersection forms are isometric. Thus we cannot distinguish $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ using their intersection forms. In fact, it turns out that there is a good reason why we have a hard time distinguishing them: as is shown in [GS, p. 151] the two manifolds are in fact diffeomorphic, in fact with good eyes one can write down an explicit diffeomorphism.

Proof. The proof of the theorem is provided in [MH, Theorem II.5.3] (see also [GS, Theorem 1.2.14] for an alternative summary). The proof relies crucially on a significant input from algebraic theory, namely the Hasse-Minkowski theorem. This theorem says that given $a_1, \ldots, a_n \in \mathbb{Q} \setminus \{0\}$ the equation

$$x_1^2 \cdot a_1 + \cdots + x_n^2 \cdot a_n \;=\; 0$$

has a non-trivial integral solution if and only if it has a non-trivial real solution and if for each prime p the equation has also has a non-trivial solution over the p-adic numbers $\mathbb{Q}_p$. A proof of the Hasse-Minkowski theorem is given in [Ser2, Chapter IV Theorem 8], [BSh, Section 7.1] or [Neuk, p. 385]. □

As an immediate corollary to Theorem 72.16 we obtain the classification of non-singular *indefinite odd* forms. More precisely, we have the following corollary.

Corollary 72.17. *If φ is a non-singular indefinite odd symmetric form over the integers, then*

$$\varphi \;\cong\; b^+(\varphi) \cdot (1) \oplus b^-(\varphi) \cdot (-1).$$

Proof. Let φ be a non-singular indefinite odd symmetric form. It follows from Proposition 72.8 that the rank, signature and parity of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by Theorem 72.16. $\square$

Now we turn to the study of even forms. We have the following, perhaps somewhat surprising proposition.

Proposition 72.18. *Let φ be a non-singular symmetric form over the integers. If φ is even, then* $\operatorname{sign}(\varphi) \equiv 0 \bmod 8$.

Example. As a reality check, the two even non-singular symmetric forms that we are acquainted with are $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8. These forms a non-singular even forms and the signatures are 0 and 8, so that matches Proposition 72.18.

Proof. Given a non-singular symmetric form (W, ψ) over $\mathbb{Z}$ we say that a vector $w \in W$ is *characteristic* if for all $u \in W$ we have

$$\varphi(u, w) \;\equiv\; \varphi(w, w) \mod 2.$$

We make the following observations:

(1) It follows easily from the definitions that (W, ψ) is even if and only if 0 is a characteristic vector.
(2) If $W = k \cdot (1) \oplus l \cdot (-1)$, then it is straightforward to show that a vector $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.

It follows from Observation (1) that the following claim implies in particular the desired statement of Proposition 72.18.

Claim. Let (V, φ) be a non-singular symmetric form over $\mathbb{Z}$. If $v \in V$ is a characteristic vector, then

$$\varphi(v, v) \;\equiv\; \operatorname{sign}(\varphi) \mod 8.$$

Let $v \in V$ be a characteristic vector of (V, φ). It is straightforward to verify that $v + e_- + e_+$ is a characteristic vector in $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, \varphi \oplus (1) \oplus (-1))$, where e_- and e_+ are generators of the first respectively the second $\mathbb{Z}$-summand. By Corollary 72.17 there exists an isometry[1208]

$$\Phi \colon \varphi \oplus (1) \oplus (-1) \;\xrightarrow{\cong}\; (b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1).$$

We write $u := \Phi(v + e_- + e_+)$. It is straightforward to verify that u is a characteristic vector for the form on the right-hand side. The above Observation (2) tells us that each

[1208]Note that both sides are non-singular odd indefinite forms, regardless of what non-singular symmetric form φ we actually consider.

entry of u is an odd number. Now we calculate that

$$\begin{aligned}
\varphi(v,v) &\overset{\substack{\text{straightforward calculation}\\ \downarrow}}{=} (\varphi \oplus (1) \oplus (-1))(v + e_- + e_+, v + e_- + e_+) \overset{\substack{\text{since } \Phi \text{ is an isometry}\\ \downarrow}}{=} \\
&= ((b^+(\varphi)+1)\cdot(1) \oplus (b^-(\varphi)+1)\cdot(-1))(u,u) \\
&= \sum_{i=1}^{b^+(\varphi)+1} (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1)\cdot(\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 \\
&\underset{\substack{\uparrow\\ \text{since for each odd number } k \text{ we have } k^2 \equiv 1 \bmod 8}}{\equiv} \sum_{i=1}^{b^+(\varphi)+1} 1 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) = (b^+(\varphi)+1) - (b_-(\varphi)+1) = \operatorname{sign}(\varphi) \bmod 8.
\end{aligned}$$

This concludes the proof of the claim. □

Now we can provide the classification of non-singular indefinite *even* symmetric forms.

Corollary 72.19. *If φ is a non-singular indefinite even symmetric form, then*

$$\varphi \cong \tfrac{\operatorname{sign}(\varphi)}{8} \cdot E_8 \oplus \tfrac{\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|}{2} \cdot H.$$

Proof. First note that it follows from Proposition 72.18 that $\operatorname{sign}(\varphi)$ is indeed divisible by 8. Furthermore it follows from $\operatorname{rank}(\varphi) = b_+(\varphi) + b_-(\varphi)$ and $\operatorname{sign}(\varphi) = b_+(\varphi) - b_-(\varphi)$ that $\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|$ is divisible by 2. So it makes sense to write down the right-hand side.

Next we recall that the two non-singular symmetric forms E_8 and H are even, that $\operatorname{rank}(E_8) = \operatorname{sign}(E_8) = 8$ and that $\operatorname{rank}(H) = 2$ and $\operatorname{sign}(H) = 0$. Now let φ be a non-singular indefinite even symmetric form. It follows from the above observation regarding E_8 and H that the rank, signature and parity of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by Theorem 72.16. □

Now we turn to the study of non-singular definite symmetric forms over the integers. We start out with some more examples of such forms. The following lemma is proved in [MH, Lemmas 6.1 and 6.2] and [Ser2, p. 51].

Lemma 72.20. *We denote by $e_1, \dots, e_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{Z}^{4m}$ spanned by the vectors $\{e_i + e_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(e_1 + \cdots + e_{4m})$. Then the following statements hold:*

1. *The restriction of the usual form $\langle\, ,\, \rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a non-singular positive-definite symmetric form.*
2. *The form Γ_{4m} is odd if and only if m is odd.*
3. *The form Γ_8 is isometric to E_8.*
4. *The form Γ_{16} is not isometric to $E_8 \oplus E_8$.*

[1209] Why is it true that for each odd number k we have $k^2 \equiv 1 \bmod 8$?

It turns out that there are many more definite symmetric forms than indefinite symmetric forms and that there is no known classification of such forms. For example, by [MH, p. 28] and [Ser2, p. 55] we have the following table

dimension	8	16	24	32	40
number of isometry types of non-singular positive-definite even symmetric forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Here in dimension 8, up to isometry, the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16}.

Similar results are also known for the number of isometry classes of non-singular positive-definite odd symmetric forms of a given rank. We refer to [CS, p. 416] and [Kin, Table 2] for details.

72.5. **The intersection forms of 4-dimensional (topological) manifolds.** We have now introduced lots of invariants of topological spaces, namely the fundamental group, higher homotopy groups, homology groups and the cohomology groups together with the cup product. The question arises, whether we are done at some point, or whether this list of ever new invariants continues forever.

In an attempt to respond to this question we now want to consider simply connected 4-dimensional (topological) manifolds and their intersection forms. We are interested in the following two questions:

Question 72.21.

(1) *Is the intersection form a complete invariant of a simply connected 4-dimensional (topological) manifold?*[1210]
(2) *Which isometry types of non-singular symmetric forms are realized by simply connected 4-dimensional (topological) manifolds?*

In the remainder of this section we give a mostly chronological account of the developments regarding the two (or possibly four?) questions posed above.

In the previous section we introduced the initially rather unexpected form E_8. One might wonder whether a form as strange as E_8 can ever appear "in nature". As we will now see, somewhat surprisingly this is indeed the case.

Definition. We refer to

$$K3 \;:=\; \left\{[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathrm{P}^3 \,\middle|\, z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\right\}$$

as the *K3-surface*.

[1210]Note that for a simply connected 4-dimensional topological manifold M we have $H_1(M;\mathbb{Z}) = 0$ by the Hurewicz Theorem 38.5 and consequently we have $H_3(M;\mathbb{Z}) = 0$ by the Poincaré Duality Theorem 61.1. Thus we see that for simply connected 4-dimensional topological manifolds the intersection form contains all the (co-) homological data we could ask for.

Remark. The odd name "K3-surface" comes about as follows. First of all, as we will see in Proposition 72.22, K3 is a *2-dimensional* complex manifold, i.e. it is a complex surface. According to [OTL] the definition of the K3-surface is already implicit in the work of Srinivasa Ramanujan in the 1910s. The K3-surface was (re-) discovered by André Weil [Weil] who named it in honor of three algebraic geometers, Kummer, Kähler and Kodaira, and also the mountain K2 in Pakistan which is the second highest mountain on earth.[1211]

The following proposition summarizes some of the key aspects of the K3-surface.

Proposition 72.22. *The K3-surface is a closed simply connected complex manifold of complex dimension 2 or equivalently of real dimension 4. Furthermore we have*

$$\textit{intersection form of the K3-surface} \cong 2 \cdot E_8 \oplus 3 \cdot H.$$

Proof. All the statements are proved in [GS, Theorem 1.3.8] or alternatively [McS, p. 176]. The intersection form is also calculated in [Huy, Proposition 3.5]. The calculation of the intersection form, perhaps somewhat disappointingly, makes use of Corollary 72.19. □

The K3-surface plays a big role in (algebraic) geometry, as is indicated by the fact that a whole book [Huy] is dedicated to the study of the K3-surfaces. In the following we want to discuss one particular aspect. Namely, the calculation of the intersection form of the K3-surface, of the intersection form of $S^2 \times S^2$ and Proposition 71.11 raises the question whether one can split off copies of $S^2 \times S^2$ form the K3-surface. More precisely the following question arose almost immediately after the introduction of the K3-surface.

Question 72.23.

(1) *Does there exist a closed oriented 4-dimensional topological manifold M such that the K3-surface is homeomorphic to $M \# S^2 \times S^2$?*
(2) *Does there exist a closed oriented 4-dimensional manifold M such that the K3-surface is diffeomorphic to $M \# S^2 \times S^2$?*

We will return to the question shortly. But before we do so, we continue with our chronological discussion. The following theorem of Vladimir Rokhlin from 1952 was the first result to address Question 72.21 (2).

Theorem 72.24. (Rokhlin 1952) *Let X be a closed oriented 4-dimensional manifold. If the intersection form Q_X is even and if $H_1(X;\mathbb{Z})$ has no 2-torsion, then* $\operatorname{sign}(X)$ *is divisible by 16.*[1212]

[1211] More precisely, André Weil wrote: "Dans la seconde partie de mon rapport, il s'agit des varietés kähleriennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire".

[1212] Recall that from Proposition 72.18 we know, for purely algebraic reasons, that the signature of a non-singular even symmetric form is divisible by 8.

Remark. The slightly odd condition on $H_1(X;\mathbb{Z})$ is actually necessary. Indeed, by [Hab, FS] there exists a closed oriented 4-dimensional manifold X with 2-torsion in $H_1(X;\mathbb{Z})$ and with an even intersection form Q_X that satisfies $\operatorname{sign}(X) = 8$.

Outline of proof. Let X be a closed oriented 4-dimensional manifold such that the intersection form Q_X is even and such that $H_1(X;\mathbb{Z})$ has no 2-torsion. It follows from [GS, Corollary 5.7.6] that the manifold X admits a "spin structure". The definition of a "spin structure" is for example given in [GS, Chapter 1.4.2], but it is of no concern to us. What matters is that Rokhlin [Rok][Rok2, p. 21] showed that for any closed oriented 4-dimensional manifold that admits a spin structure, the signature is divisible by 16. More modern proofs of Rokhlin's Theorem are given in [FK] and [Sav2, Theorem 2.1]. □

The following theorem was the first noteworthy result towards an affirmative answer to Question 72.21 (1).

Theorem 72.25. (Milnor 1958) *Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds. Then*

$$X \text{ and } Y \text{ are homotopy equivalent} \iff Q_X \text{ is isometric to } \pm Q_Y.$$

Proof. This theorem was proved for manifolds by John Milnor [Mi6, Theorem 3] in 1958, building on work of John Whitehead [Whd3]. Furthermore [GS, Theorem 1.2.25] sketches a proof that also works for topological manifolds. □

After these somewhat preliminary results by Vladimir Rokhlin and John Milnor the early 1980s saw two major breakthroughs by Michael Freedman and Simon Donaldson.

First, in 1981 Michael Freedman [Fre] gave a complete answer to Question 72.21 in the category of topological manifolds.[1213] More precisely, Michael Freedman [Fre] proved the following theorem.

Theorem 72.26. (Freedman 1981) *Let Q be a nonsingular symmetric form over the integers.*

(1) *If Q is even, then up to an orientation-preserving homeomorphism, there exists a unique closed oriented simply connected topological manifold whose intersection form is isometric to Q.*
(2) *If Q is odd, then up to orientation-preserving homeomorphisms, there exist precisely two closed oriented simply connected topological manifolds whose intersection forms are isometric to Q.*

Remark.

(1) Later we will see that for a simply connected topological space the Hurewicz homomorphism $\pi_2(X) \to H_2(X;\mathbb{Z})$ is an isomorphism. In particular, if $\pi_2(X)$ vanishes, then $H_2(X;\mathbb{Z}) = 0$. This fact shows that Theorem 72.26, applied to the form on the trivial group $\{0\}$, implies in particular Theorem 28.23.

[1213] Michael Freedman (*1951) is an American mathematician, in 1986 he was awarded the Fields medal for the proof of Theorem 72.26.

(2) On page 1656 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the (1×1)-matrix (1). Theorem 72.26 (2) says that there exists a closed oriented simply connected topological 4-manifold $\mathbb{C}\mathrm{P}^{2*}$ with the same intersection as $\mathbb{C}\mathrm{P}^2$ but that is not homeomorphic to $\mathbb{C}\mathrm{P}^2$. Furthermore $\mathbb{C}\mathrm{P}^{2*}$ is unique up to an orientation-preserving homeomorphism.
(3) The two homeomorphism types of Theorem 72.26 are distinguished by the Kirby-Siebenmann invariant, see [KSi2] or [Rudy2, Chapter 3.4]. This invariant is defined for any closed topological manifold X of any dimension and it takes values in $H^4(X;\mathbb{Z}_2)$.[1214] By [Rudy2, Theorem 3.4.3] this invariant vanishes if X admits a "piecewise linear structure". For manifolds of dimension ≤ 7 any manifold, i.e. any topological manifold with a smooth structure, also admits a "piecewise linear structure". We refer to [Cer, p. IX] and [Sco, p. 220] for more details.
(4) Freedman's Theorem also answers Question 72.23 (1) in the affirmative. More precisely, by the existence statement of Theorem 72.26 (1) there exists a closed oriented 4-dimensional topological manifold X with $Q_X \cong E_8$. Furthermore, by the uniqueness statement of Theorem 72.26 (1) there exists a homeomorphism of the K3-surface to $X\#X\#3\cdot(S^2\times S^2)$.

Among many other consequences we obtain the following rather surprising result.

Theorem 72.27. *There exist closed orientable 4-dimensional topological manifolds that do not admit a smooth structure.*

Proof. At this point we can combine the above results to give two types of examples of such manifolds:

(1) It follows from Theorem 72.26 (2) and the above Remark (3) that given any non-singular odd symmetric form Q there exists a closed oriented 4-dimensional topological manifold X with $Q_X \cong Q$ and non-trivial Kirby-Siebenmann invariant. As we pointed out above in Remark (3), this topological manifold does not admit a smooth structure.
(2) If we apply Freedman's Theorem 72.26 (1) to the non-singular positive-definite even form E_8 we obtain a certain 4-dimensional topological manifold which, by Rokhlin's Theorem 72.24, cannot admit a smooth structure. □

In a remarkable twist, shortly after Freedman proved his results on the intersection forms of topological 4-dimensional manifolds, Simon Donaldson[1215], see [Don, Theorem A] and

[1214]Here it is worth recalling that Proposition 51.14 (3) says that for a closed orientable 4-dimensional topological manifold we have $H^4(X;\mathbb{Z}_2)\cong\mathbb{Z}_2$.

[1215]Simon Donaldson (*1957) is an English mathematician. He proved Theorem 72.28 when he was a second year PhD student. The Fields medalist Michael Atiyah described in [At2] the work of Simon Donaldson as follows: "In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world." Simon Donaldson was awarded the Fields medal in 1986.

[Don2, Theorem 1], proved the following result regarding intersection forms of 4-dimensional manifolds.[1216]

Theorem 72.28. (Donaldson 1982) *Let X be a closed oriented 4-dimensional manifold. If Q_X is positive-definite, then Q_X is represented by the identity matrix.*

To understand the significance of Donaldson's Theorem, often referred to as *Donaldson's Theorem A*, recall that on page 1680 we saw that there are more than 10^{51} isometry classes of non-singular positive-definite symmetric forms of rank 40. By Freedman's Theorem 72.26 all of them arise as intersection forms of closed oriented connected topological 4-manifolds. By Donaldson's Theorem 72.28 the only one which appears as the intersection form of a closed oriented connected 4-dimensional manifold is the one represented by the identity matrix.[1217]

FIGURE 912. The Fields medalists Michael Freedman, Gerhard Faltings and Simon Donaldson at the ICM 1986 in Berkeley.

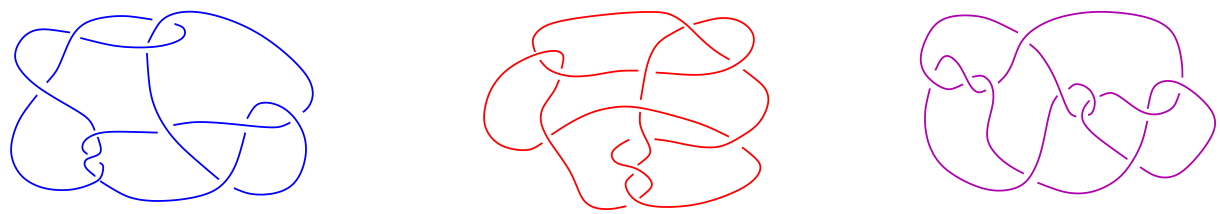
Remark. It is difficult to exaggerate the importance of Donaldson's Theorem in mathematics.[1218] For example it is at the heart of the proof of Theorem 28.27. But Donaldson's Theorem can also appear in totally unexpected situations. For example in 2017 Duncan McCoy [McC] used Donaldson's Theorem, and many other ingredients, to show that if K is an alternating knot that can be unknotted using a single crossing change, then such a crossing change can already be performed in an alternating diagram. For example, according to [CL] the three alternating knots shown in Figure 913 can be unknotted using a single crossing change in *some diagram*. By the aforementioned result we now know that we can turn these three knots into the trivial knot by changing a crossing in the alternating diagrams shown. Which crossings do you need to change?

Now that we have digested Donaldson's Theorem 72.28, the question arises, what can we say about *indefinite* intersection forms of closed oriented 4-dimensional manifolds? First recall that in Corollary 72.17 we saw that every non-singular indefinite odd symmetric form

[1216]An exposition of the proof is also given in [Law] and a different proof, using Seiberg-Witten invariants is given in [Ni2, Theorem 2.4.18] and a proof using Heegaard Floer homology is given in [OS, Theorem 9.1].

[1217]The identity matrix represents, by Proposition 71.11 and the discussion on page 1656, the intersection form of the connected sum of 40 copies of $\mathbb{C}P^2$.

[1218]Admittedly it is also hard to underestimate the role of Donaldson's Theorem outside of mathematics.



the knot $11a_{12}$ the knot $11a_{41}$ the knot $11a_{56}$

FIGURE 913.

is isometric to $m \cdot (1) \oplus n \cdot (-1)$ for some $m, n \in \mathbb{N}$. According to Proposition 71.11 and the discussion on page 1656 every such form is realized by the manifold $m \cdot \mathbb{CP}^2 \# n \cdot \overline{\mathbb{CP}}^2$.

Therefore, in the following we only need to discuss the realizability of non-singular indefinite *even* symmetric forms. Recall that in Corollary 72.19 we showed that every non-singular even indefinite symmetric form is isometric to $m \cdot H \oplus n \cdot E_8$ for some $m \in \mathbb{N}_0$ and $n \in \mathbb{Z} \setminus \{0\}$. The following theorem, proved by Mikio Furuta [Fu2] in 2001 gives some restrictions on the possible values of m and n.

Theorem 72.29. (Furuta's 10/8-Theorem) *If M is a closed oriented connected 4-dimensional manifold with* indefinite *even intersection form, then*

$$b_2(M) \ \geq \ \frac{10}{8} \cdot \operatorname{sign}(M) + 2.$$

In particular[1219] $Q_M \cong m \cdot H \oplus n \cdot E_8$ *with* $m \geq 2|n| + 1$.

Remark. Furuta's 10/8-Theorem allows us to answer Question 72.23 (2) in the negative. More precisely, suppose that we have a decomposition of the K3-surface as a connected sum $M \# N$. We make the following four observations:

(1) Without loss of generality we have $\operatorname{sign}(M) \geq \operatorname{sign}(N)$.
(2) It follows from Proposition 72.22, Lemma 72.13 (3) and Proposition 71.11 that Q_M and Q_N are even.
(3) From Lemma 72.11 (2) and Proposition 72.22 we obtain that $\operatorname{sign}(M) + \operatorname{sign}(N) = 16$.
(4) From Proposition 72.22 and Proposition 71.10 we know the equality $b_2(M) + b_2(N) = 22$.

If we combine these facts with Rokhlin's Theorem 72.24 we obtain easily that $\operatorname{sign}(M) = 16$. Furthermore, it follows from Donaldson's Theorem 72.28 that Q_M is indefinite. Now Furuta's 10/8-Theorem implies that $b_2(M) \geq \frac{10}{8} \cdot 16 + 2 = 22$, but by Proposition 72.22 this equals $b_2(K3)$. Thus we obtain from Proposition 71.10 that $b_2(N) = 0$.[1220] We can

[1219] This second statement follows from the first statement together with Corollary 72.19 and the calculation of the rank and signature of H and E_8.

[1220] By Proposition 71.10 and Proposition 72.22 we also know that $\pi_1(N) = 0$. As we will see later on, the Hurewicz Theorem shows that $\pi_1(N) = 0$ and $H_2(N; \mathbb{Z}) = 0$ implies that $\pi_2(N) = 0$. Thus it follows

conclude that we cannot split off a copy of $S^2 \times S^2$ from $K3$. This answers Question 72.23 (2) in the negative.

Furuta's 10/8 Theorem is not yet optimal since it does not quite close the gap between the forms which we can realize by manifolds and the forms which we can exclude. More precisely it follows from Proposition 71.11, the calculation of the intersection forms of the K3-surface, see Proposition 72.22, and of the intersection form of $S^2 \times S^2$ that for any $m \in \mathbb{Z}\setminus\{0\}$ and any $n \geq 3|m|$ there exists a closed oriented simply connected 4-dimensional manifold with intersection form isometric to $m \cdot E_8 \oplus n \cdot H$. More precisely, we have

$$\text{intersection form of } m \cdot \mathrm{K3} \,\#\, (n - 3|m|) \cdot (S^2 \times S^2) \;\cong\; m \cdot E_8 \oplus n \cdot H.$$

The following conjecture predicts that this result is optimal.

Conjecture 72.30. (11/8-Conjecture) *If M is a closed oriented 4-dimensional manifold with indefinite even intersection form, then*

$$b_2(M) \;\geq\; \tfrac{11}{8} \cdot \operatorname{sign}(M).$$

Equivalently, if $Q_M \cong m \cdot H \oplus n \cdot E_8$ with $n \neq 0$, then $n \geq 3|m|$.[1221]

Remark.

(1) A proof of the 11/8-Conjecture would imply, by Freedman's Theorem 72.26, that any closed oriented simply connected 4-dimensional manifold is *homeomorphic* to either a connected sum of the form $m \cdot \mathbb{C}\mathrm{P}^2 \# n \cdot \overline{\mathbb{C}\mathrm{P}}^2$ or alternatively to a connected sum of the form $m \cdot \mathrm{K3} \# n \cdot (S^2 \times S^2)$.

(2) For simply connected manifolds currently the best result is [HLSX, Corollary 1.13] which says that if M is a closed oriented simply-connected 4-manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite and even, then $b_2(M) \geq \frac{10}{8} \cdot |\operatorname{sign}(M)| + 4$.

The other question which arises in this context is whether closed orientable 4-dimensional manifolds that are homeomorphic are also necessarily diffeomorphic. It turns out that in many cases the answer is very much no. For example the following theorem was proved in 1998 by Ron Fintushel and Ron Stern [FS2].

Theorem 72.31. *There exist closed orientable 4-dimensional manifolds X_i, $i \in \mathbb{N}$ such that each X_i is homeomorphic to the K3-surface, but such that the X_i are pairwise non-diffeomorphic.*

Similar results have been obtained for many other 4-dimensional manifolds, e.g. for $\mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}}^2$ with $k = 9$ by Simon Donaldson [Don2] in 1987, $k = 8$ by Dieter Kotschick [Kot]

from Theorem 28.23 that N is homeomorphic to S^4. Since Question 28.26 is still open, we do not know whether N is diffeomorphic to S^4.

[1221]As before, the fact that these two statements are equivalent follows from Corollary 72.19 and the calculation of the rank and signature of H and E_8.

in 1989 and $k = 7$ by Jongil Park [Par] in 2005, and since then also for $k = 6, 5, 4, 3, 2$, see e.g. [AP, AP2] for more details on the history of this problem.

The following question is perhaps the most important problem in the theory of 4-dimensional manifolds.

Question 72.32. *Does every closed oriented connected 4-dimensional manifold M admit infinitely many pairwise non-diffeomorphic smooth structures? Put differently, given any such M, do there always exist infinitely many manifolds M_i, $i \in \mathbb{N}$ which are homeomorphic to M but which are pairwise non-diffeomorphic?*

Note that an affirmative answer to Question 72.32 would also give a positive answer to Question 28.26.

The following theorem, proved by C. T. C. Wall [Wall7] in 1964 (see also [Sco, p. 149]) shows that in some sense homeomorphic simply connected manifolds are not that far from being diffeomorphic.

Theorem 72.33. *Let M and N be two closed oriented simply-connected 4-manifolds. If M and N are homeomorphic, then there exists a $k \in \mathbb{N}$ such that $M\#k \cdot (S^2 \times S^2)$ and $N\#k \cdot (S^2 \times S^2)$ are diffeomorphic.*

This raises the question, whether in the above theorem $k = 1$ suffices. More precisely the following question is still open.

Question 72.34. *Let M and N be two closed oriented simply-connected 4-manifolds that homeomorphic. Is $M\#(S^2 \times S^2)$ diffeomorphic to $N\#(S^2 \times S^2)$?*

There are some indications, see [JZ], that the answer to the above question should be no.

Remark. We conclude with a quick foray into the high-dimensional world. For $n \geq 3$ Wall [Wall3] (see also [Ran, p. 269]) classified diffeomorphism classes of pairs $(M, \partial M)$ where M is an orientable $(n-1)$-connected $2n$-dimensional manifold and where the boundary ∂M is a homotopy sphere. The classification goes via the intersection form and one extra invariant which lies in the homotopy group $\pi_{n-1}(\mathrm{SO}(n))$.

72.6. **Symmetric forms over $\mathbb{F}_2$.** Considering the power of the intersection form of a closed oriented even-dimensional topological manifold it is natural to ask, whether one can obtain similar results for non-orientable topological manifolds.

In fact, suppose we are given a closed $2n$-dimensional topological manifold M (not necessarily orientable). We denote by $[M]_{\mathbb{F}_2} \in H_{2n}(M; \mathbb{F}_2)$ the $\mathbb{F}_2$-fundamental class given by Corollary 46.6. Given $k \in \mathbb{N}_0$ we denote by

$$\mathrm{PD}_M^{\mathbb{F}_2} \colon H_k(M; \mathbb{F}_2) \;\to\; H^{2n-k}(M; \mathbb{F}_2)$$

the inverse of the isomorphism given by the Poincaré Duality Theorem 61.1. With these we define the $\mathbb{F}_2$-*valued intersection form*

$$\begin{array}{rcl} H_n(M; \mathbb{F}_2) \times H_n(M; \mathbb{F}_2) & \to & \mathbb{F}_2 \\ (a, b) & \mapsto & \big\langle \mathrm{PD}_M^{\mathbb{F}_2}(a) \cup \mathrm{PD}_M^{\mathbb{F}_2}(b), [M]_{\mathbb{F}_2} \big\rangle. \end{array}$$

The above definition is completely analogous to the definition on page 1652. Basically the same argument as in Proposition 71.5 shows that the $\mathbb{F}_2$-valued intersection form is symmetric and non-singular. Furthermore, as in Lemma 71.7 one can show easily that the $\mathbb{F}_2$-valued intersection forms of homeomorphic closed topological manifolds are isometric.

Examples.

(1) Similar to the discussion on page 1655 one can show that the $\mathbb{F}_2$-valued intersection form of the torus $S^1 \times S^1$ is represented by the matrx $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2) Proposition 62.16, together with the obvious analogue of Proposition 71.11, implies that the $\mathbb{F}_2$-valued intersection form of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is represented by the matrx $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The question arises, whether we can classify non-singular symmetric forms over $\mathbb{F}_2$. For better or worse it turns out that there is a straightforward classification. To state the classification we need the following definition.

Definition. Let V be a finite-dimensional $\mathbb{F}_2$-vector space. We say a form $\lambda\colon V \times V \to \mathbb{F}_2$ is *even* if $\lambda(v, v) = 0 \in \mathbb{F}_2$ for all $v \in V$. Otherwise we say that the symmetric form is *odd*. We call the property of a form to be even or odd to be the *parity* of the form.

Example. As we had just seen, the $\mathbb{F}_2$-valued intersection form of the torus $S^1 \times S^1$ is even whereas the $\mathbb{F}_2$-valued intersection form of the torus $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is odd. Thus we have found a new, and particularly complicated, proof that these two 2-dimensional manifolds are not homeomorphism.

The following proposition gives a complete classification of non-singular symmetric forms over $\mathbb{F}_2$ up to isometry.

Proposition 72.35. *Two non-singular symmetric forms (V, κ) and (W, λ) are isometric if and only if $\dim(V) = \dim(W)$ and if κ and λ have the same parity.*

Proof. The proposition is proved in [Hopk, Theorem 2.7]. It can also be deduced from [Szy, Corollary 6.3.1]. □

The proposition shows that, unfortunately, we extract only very little information from the $\mathbb{F}_2$-valued intersection form.

Exercises for Chapter 72.

Exercise 72.1. Let φ be a symmetric form on a finite-dimensional real vector space V. Prove the following statement: If φ is positive-definite, then φ is non-singular.

Exercise 72.2. Let M be a closed oriented $4k$-dimensional manifold and let N be a closed oriented $4l$-dimensional manifold. Show that $\operatorname{sign}(M \times N) = \operatorname{sign}(M) \cdot \operatorname{sign}(N)$.

Exercise 72.3. Let M be a closed non-orientable $4k$-dimensional manifold. Let $p\colon \widetilde{M} \to M$ be the orientation covering that we introduced on page 294. Show that $\operatorname{sign}(\widetilde{M}) = 0$.

Exercise 72.4.

(a) Explain how to modify the proof of Proposition 72.2 so that it also holds for the integers.
(b) Explain why the proof of Proposition 72.6 cannot be generalized to the integers.

Exercise 72.5. Let A and C be invertible real $k \times k$-matrices. We assume that C is symmetric.

(a) Show that the signature of the matrix $\begin{pmatrix} 0 & A \\ A^T & C \end{pmatrix}$ is zero.
(b) Does the conclusion of (a) still hold if A and C are not invertible?

Exercise 72.6. For which $k \in \mathbb{Z}$ does there exist a closed oriented 3-manifold N with $\operatorname{sign}(S^1 \times N) = k$?

Exercise 72.7. Show that the map

$$\begin{array}{rcl} \{A \in \mathrm{GL}(n, \mathbb{R}) \,|\, A = A^T\} & \to & \mathbb{Z} \\ A & \mapsto & \operatorname{sign}(A) \end{array}$$

is continuous.
Hint: You can work either with the definition of the signature or you could use Proposition 1.78.

73. Handlebodies and Heegaard splittings of 3-dimensional manifolds

Given coprime $p, q \in \mathbb{N}$ we introduced the lens spaces $L(p,q)$ on page 266. Using the fundamental groups it was very easy to see that homeomorphic lens spaces have the same p-parameter. But so far the more advanced invariants, i.e. higher homotopy groups, homology groups and cohomology groups failed miserably at providing us with more information.

In the next chapter we will cleverly apply cup products to obtain information on lens spaces $L(p,q)$ which goes beyond detecting the first parameter p. But before we do so we find it convenient to introduce several more general definitions regarding manifolds.

73.1. **The boundary connected sum operation.** The following definition gives us a new way to construct a manifold out of two given manifolds.

Definition. Let $n \in \mathbb{N}_{\geq 2}$. Let M and M' be two n-dimensional oriented manifolds with non-empty boundary. Let $\varphi\colon \overline{B}^{n-1} \to \partial M$ and $\varphi'\colon \overline{B}^{n-1} \to \partial M'$ be two embeddings where φ is orientation-preserving and φ' is orientation-reversing. We define the *boundary connected sum of M and M'* as[1222]

$$M \#_b M' \;:=\; (M \sqcup M')/\sim \qquad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in \overline{B}^{n-1}.$$

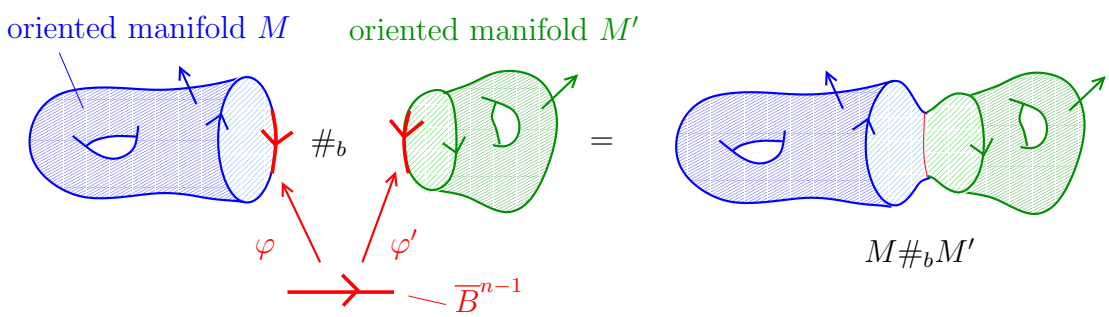


Figure 914.

In the following proposition we collect some basic facts on the boundary connected sum operation.

Proposition 73.1. *Let $n \in \mathbb{N}_{\geq 2}$. Let M and M' be two n-dimensional oriented manifolds with non-empty boundary and let A and A' be components of ∂M respectively $\partial M'$. Let $\varphi\colon \overline{B}^{n-1} \to A$ and $\varphi'\colon \overline{B}^{n-1} \to A'$ be two embeddings where φ is orientation-preserving and φ' is orientation-reversing.*

(1) *The boundary connected sum $M \#_b M'$ is naturally an n-dimensional manifold that contains M and M' as submanifolds with corner.*

[1222] Sometimes in the literature one also writes $M \sharp M'$ instead of $M \#_b M'$.

(2) *Given the boundary components A and A' the diffeomorphism type of the boundary connected sum $M\#_b M'$ does not depend on the choice of the two embeddings $\varphi\colon \overline{B}^{n-1} \to A$ and $\varphi'\colon \overline{B}^{n-1} \to A'$. In particular, if the boundary of M and M' is connected, then the diffeomorphism type of the boundary connected sum $M\#_b M'$ is well-defined.*

(3) *The boundary connected sum $M\#_b M'$ admits a unique orientation such that M and M' are oriented submanifolds with corner.*[1223]

(4) *We write $\partial M = A \sqcup B$ and $\partial M' = A' \sqcup B'$. Then*

$$\partial(M\#_b M') \;=\; A\#A' \;\sqcup\; B \sqcup B'.$$

In particular, if the boundary of M and M' is connected, then the boundary of $M\#_b M'$ is the connected sum of the boundaries of M and M'.

(5) *For any $i \geq 1$ and any commutative ring R the inclusion maps induce an isomorphism*

$$H_i(M;R) \oplus H_i(M';R) \;\xrightarrow{\cong}\; H_i(M\#_b M';R).$$

Furthermore, if M and M' are connected, then the inclusion maps induce an isomorphism

$$\pi_1(M) * \pi_1(M') \;\xrightarrow{\cong}\; \pi_1(M\#_b M').$$

Convention. We use Proposition 73.1 (3) to equip the boundary connected sum with an orientation.

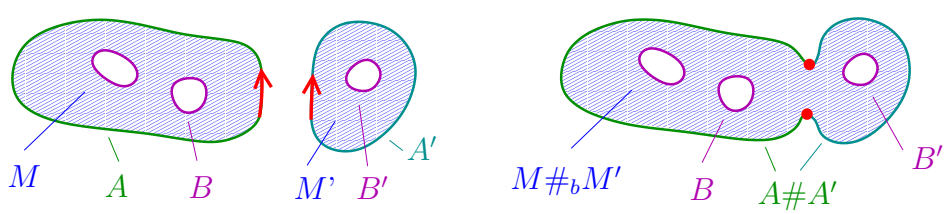


FIGURE 915. Illustration of Proposition 73.1.

Sketch of proof. Let M and M' be two oriented n-dimensional manifolds with non-empty connected boundary.

(1) This statement is a generalization of Proposition 10.27 (1). The tricky bit is to define the smooth atlas around the points given by $\varphi(S^{n-2}) = \varphi'(S^{n-2})$ in $M\#_b M'$. This is done by "smoothing the corners" as described for example in [Ni, p. 33] or in [Wall2, Proposition 2.6.2].

[1223]Basically the same construction as in Lemma 45.28 shows that given an oriented manifold, any codimension zero submanifold with corner inherits an orientation.

(2) This statement can be proved fairly easily using Theorem 10.28 and the Collar Neighborhood Theorem 10.10.[1224]
(3) This statement is proved in a very similar fashion as Proposition 10.27 (3).
(4) The final statement follows almost immediately from the definitions.
(5) It is pretty clear that both statements can be deduced from the Seifert-van Kampen 14.1 and the Mayer–Vietoris Theorem 33.5. The only technical issue that needs to be resolved is that we have to replace the subsets M and M' of $M\#_b M'$ by open subsets U and U' that admit M and M' as deformation retracts and such that $U \cap U'$ is a deformation retract of $M \cap M'$. This can be achieved using the Isotopy Extension Theorem 10.21 and the Collar Neighborhood Theorem 10.10.

We leave it as an exercise to provide full details for all of the above statements. See also [Kos, Theorem 3.1] for the same line of argument, given with more authority. □

Convention. In the following pages we will give many examples of boundary connected sums and related constructions. In many cases the objects we write down appear to be "manifolds with corner". There are two approaches to dealing with this problem:

(1) By "smoothing corners" we can turn any "manifold with corner" into a manifold, see e.g. [Wall2, Proposition 2.6.2] for details.
(2) Most of the examples that we discuss below are 3-dimensional manifolds. Theorem 28.24 says that every topological 3-dimensional manifold admits a unique smooth structure. Subsequently we apply this result as follows. We are given a topological space X that is homeomorphic to a 3-manifold M. By definition X is also a topological 3-dimensional manifold. Theorem 28.24 tells us that X admits a unique smooth structure, and that X, viewed as a manifold, is in fact diffeomorphic to M.[1225]

Examples.

(1) For any oriented n-dimensional manifold M with connected boundary we have a diffeomorphism $M\#_b\overline{B}^n \xrightarrow{\cong} M$ which is furthermore the identity outside of a collar neighborhood of ∂M. We refer to Figure 916 for an illustration and we refer to [Kos, p. 97] for a formal proof of this statement.
(2) We view the solid torus $S^1 \times \overline{B}^2$ as an oriented manifold where the orientation is given by the usual product orientation. Given $g \in \mathbb{N}$ we refer to

$$H_g \ := \ \text{boundary connected sum of } g \text{ copies of the solid torus } S^1 \times \overline{B}^2$$

as the *genus g handlebody*. As pointed out in the convention on page 1691, we view the genus g handlebody as an oriented manifold.

[1224]For this argument we need that ∂M and $\partial M'$ are connected.

[1225]We could also use a specific homeomorphism $f\colon X \to M$ to "transfer" a smooth atlas of M to X, but in most cases we are too lazy to write down a specific homeomorphism. Furthermore it is convenient to know that the smooth structure on X is independent of the choice of the homeomorphism f.

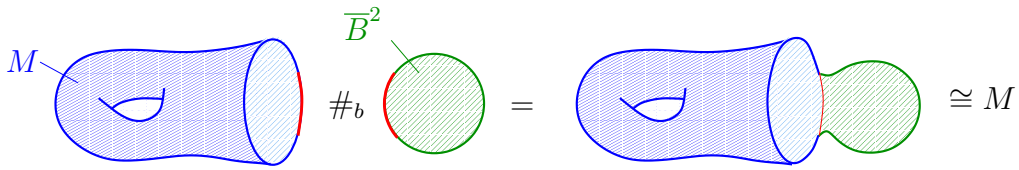


FIGURE 916.

By Proposition 73.1 (3) the boundary of H_g is the connected sum of g copies of the torus $S^1 \times S^1$. If we combine this fact with the discussion in Figure 224 we see that the boundary of the genus g handlebody is diffeomorphic to the surface of genus g. In the future we will at times identify ∂H_g with the surface of genus g. This discussion is illustrated in Figure 917.

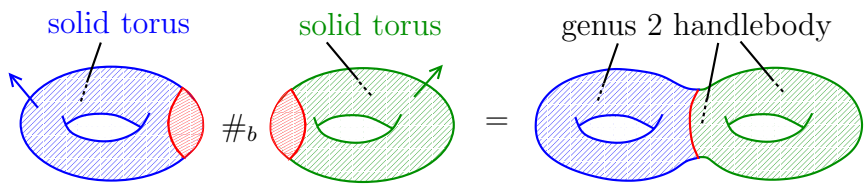


FIGURE 917.

(3) For "continuity reasons" we sometimes refer to the 3-ball $\overline{B}^3$ as a genus 0 handlebody.
(4) Handlebodies occur naturally in many contexts:
  (a) If we "thicken up" a spatial graph in S^3, then we obtain a topological space that is homeomorphic to a handlebody. We refer to Figure 918 for an illustration.

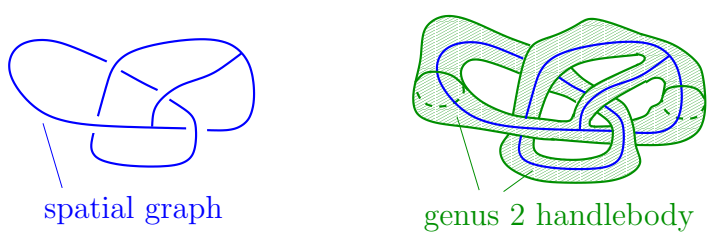


FIGURE 918.

  (b) One can obtain the genus g handlebody from the 3-ball $\overline{B}^3$ by either "attaching g handles" or by "drilling out g straight open tunnels". We refer to Figure 919 for an illustration.

We will not try to make either of the three constructions precise.

We conclude the discussion of handlebodies with the following elementary lemma.

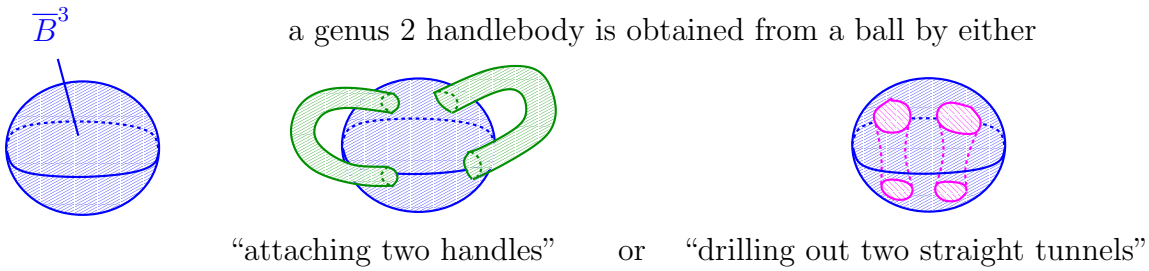


FIGURE 919.

Lemma 73.2. *The genus g handlebody H_g admits a deformation retraction to a connected graph Γ with $\chi(\Gamma) = 1-g$. In particular H_g is homotopy equivalent to the wedge of g circles and $\pi_1(H_g)$ is a free group on g generators.*

Remark. It follows from Lemma 73.2 that the Euler characteristic of the genus g handlebody H_g equals $1-g$. This is precisely half of the Euler characteristic of the surface Σ_g of genus g, which is consistent with Proposition 61.4.

Sketch of proof. Each solid torus in the construction of the genus g handlebody admits a deformation retraction to a circle such that the disks used in the definition of the boundary connected sum deformation retract to points on the circle. Thus we see that the genus g handlebody deformation retracts onto a graph G built out of g circles $C_1, \dots, C_g$ where for $i = 1, \dots, g-1$ the circle C_i is glued to C_{i+1} along a point. This is a graph of Euler characteristic $1-g$.[1226] We refer to Figure 920 for an illustration.

By Proposition 9.21 (3) the graph Γ is homotopy equivalent to the wedge of g circles. The statement regarding the fundamental groups follows from Lemma 12.5 and Proposition 9.12 (1). □

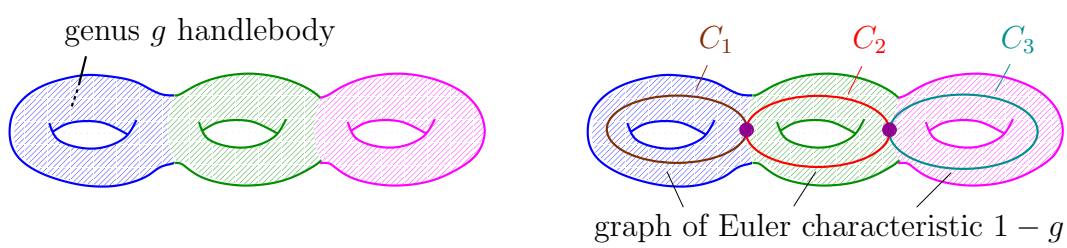


FIGURE 920.

Remark. We consider again the standard embedding of the surface Σ of genus three into $\mathbb{R}^3$ that is shown in Figure 921. Furthermore we consider the "obvious" genus 3 handlebody H that cobounds Σ and that is also shown in Figure 921. We already saw on page 891

[1226] Why is the Euler characteristic equal to $1-g$?

that the closed oriented curves $a_1, a_2, a_3, b_1, b_2, b_3$ form a basis for $H_1(\Sigma_3;\mathbb{Z})$. It follows from the proof of Lemma 73.2 that the closed oriented curves b_1, b_2, b_3 lie in the kernel of the inclusion induced map $H_1(\Sigma;\mathbb{Z}) \to H_1(H;\mathbb{Z})$ and that the oriented curves a_1, a_2, a_3 represent a basis for $H_1(H;\mathbb{Z})$.

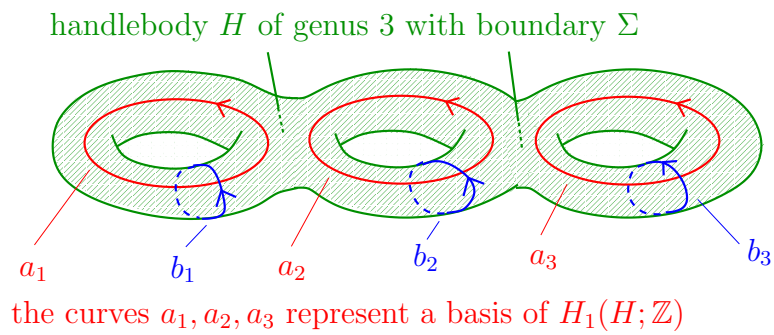


FIGURE 921.

73.2. **Heegaard splittings of 3-dimensional manifolds.**

Definition. Let M be a closed oriented connected 3-dimensional manifold.

(1) A *splitting of* M consists of a triple (X, Y, F) where X and Y are compact 3-dimensional submanifolds of M such that $M = X \cup Y$ and such that $F = X \cap Y = \partial X = \partial Y$. Given such a splitting we endow X and Y with the orientation coming from M and we endow F with the orientation coming from $F = \partial X$.[1227]

(2) A splitting (X, Y, F) of M is called a *Heegaard splitting* if X and Y are both diffeomorphic to handlebodies. We refer to the genus of the surface F as the *genus of the Heegaard splitting*.

Often, when there is no danger of confusion, we say "$M = X \cup Y$ is a splitting".

Remark. Heegaard splittings are named after the Danish mathematician Poul Heegaard (1871–1948) who proved the first important theorem regarding Heegaard splittings, see below. We had encountered Poul Heegaard before on page 1416.

Examples.

(1) Let $n \in \mathbb{N}$. On page 73 we saw that we can write the n-sphere as the union of the upper hemisphere $S^n_{\geq 0}$ and the lower hemisphere $S^n_{\leq 0}$, both of which are diffeomorphic[1228] to $\overline{B}^n$ and which intersect in an $(n-1)$-sphere. This observation gives us two examples of Heegaard splittings at once:

[1227] Note that the orientation coming from $F = \partial Y$ would be the opposite orientation.

[1228] Actually one has to think for a minute about why the hemispheres are *diffeomorphic* to $\overline{B}^n$. On page 73 we saw that the projection map from the hemispheres to $\overline{B}^n$ is a homeomorphism. But this is not a diffeomorphism (why not?). Fortunately stereographic projections, as defined in Lemma 1.41, can be used to give an explicit diffeomorphism.

(a) We obtain a genus 0 Heegaard splitting $S^3 = S^3_{\geq 0} \cup S^3_{\leq 0}$.
(b) We also obtain a genus 1 Heegaard splitting

$$S^1 \times S^2 \;=\; \underbrace{(S^1 \times S^2_{\geq 0})}_{\text{solid torus}} \cup \underbrace{(S^1 \times S^2_{\leq 0})}_{\text{solid torus}}.$$

(2) A 3-manifold can have many different Heegaard splittings. For example we consider $S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1\}$ and the following two subspaces of S^3:

$$X \;:=\; \{(z,w) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \tfrac{1}{2}\}$$

and

$$Y \;:=\; \{(z,w) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \tfrac{1}{2}\}.$$

In Lemma 18.1 we showed that both X and Y are diffeomorphic to the solid torus $S^1 \times \overline{B}^2$. In particular Lemma 18.1 shows that $(X, Y, \partial X = \partial Y)$ is a genus 1 Heegaard splitting of S^3.

(3) The sphere S^3 admits in fact a Heegaard splitting of arbitrarily high genus. This can be seen as follows: we start out with the genus 0 Heegaard decomposition given by decomposing S^3 into two closed 3-balls X and Y. Now we we "drill g unknotted open tunnels" out of X and we add (the closure of) what we just drilled out to Y, which has the effect of "adding g handles to Y". In this construction we turned X and Y into genus g handlebodies. This example is illustrated in Figure 922. In fact

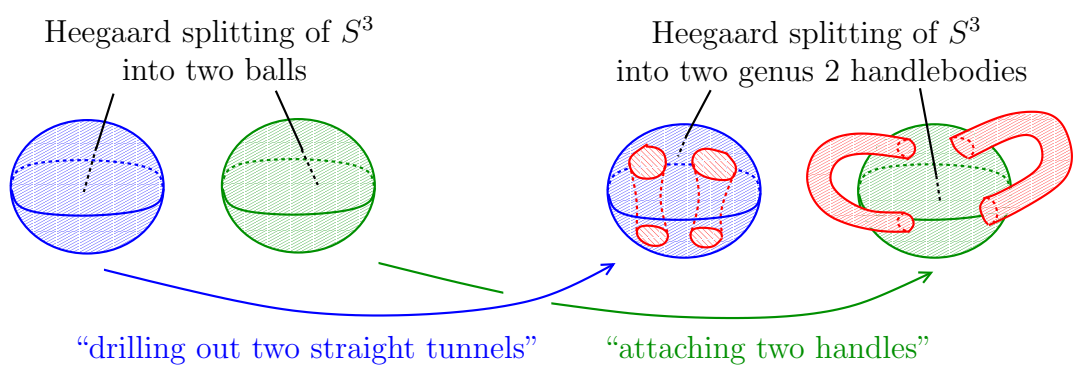


FIGURE 922.

we have now found all Heegaard splittings of S^3. More precisely, given a genus g Heegaard splitting (X, Y, F) of S^3 there exists a diffeomorphism f of S^3 which turns the Heegaard splitting into the Heegaard splittings of the same genus listed above. For $g = 0$ this statement is equivalent to the Generalized Smooth Schönflies Theorem 36.4 in dimension 3. For $g \geq 1$ the uniqueness of Heegaard splittings was shown in 1968 by Friedhelm Waldhausen [Wald, Schle].

(4) We consider the 3-dimensional cube $[0,1]^3$. We consider the equivalence relation $\sim$ that identifies the opposite sides of the cube. More precisely, on $[0,1]^3$ we denote by $\sim$ the equivalence relation that is generated by

$$(0,a,b) \sim (1,a,b), \quad (a,0,b) \sim (a,1,b) \quad \text{and} \quad (a,b,0) \sim (a,b,1) \quad \text{for } a,b \in [0,1].$$

It follows easily from Proposition 1.40 that the map $[0,1]^3/\sim \to S^1 \times S^1 \times S^1$ given by $(x,y,z) \mapsto (e^{2\pi i x}, e^{2\pi i y}, e^{2\pi i z})$ is a homeomorphism. We use this homeomorphism to identify $[0,1]^3/\sim$ with the 3-dimensional torus $S^1 \times S^1 \times S^1$. In Figure 923 we show the cube $[0,1]$ and we show the subset

$$X \;:=\; \big\{(x_1,x_2,x_3) \in [0,1]^3 \,\big|\, \text{at least two coordinates satisfy } \big|x_i - \tfrac{1}{2}\big| \le \tfrac{1}{4}\big\}.$$

We leave it as an exercise to show that $X/\sim$ is homeomorphic to the genus 3 handlebody and that $Y := \overline{([0,1]^3 \setminus X)/\sim}$ is also homeomorphic to the genus 3 handlebody. This shows that the 3-dimensional torus admits a genus 3 Heegaard splitting. It is natural to ask whether the 3-dimensional torus admits a Heegaard splitting of smaller genus. We study this question in Exercise 73.2

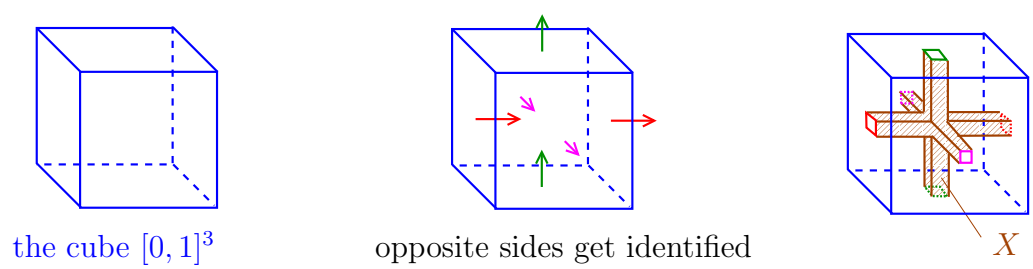


FIGURE 923.

(5) Let F be the surface of genus g and let $\varphi\colon F \to F$ be an orientation-preserving self-diffeomorphism. By Lemma 15.19 the mapping torus

$$M(F,\varphi) \;:=\; F \times [0,1] \,/\, (x,0) \sim (\varphi(x),1)$$

is a closed orientable 3-dimensional manifold. In Exercise 73.3 we will see that $M(F,\varphi)$ admits a Heegaard splitting of genus $2g+1$.

(6) Let X be a handlebody and let Y be a disjoint copy of the handlebody X. Furthermore let $\varphi\colon \partial X \to \partial Y$ be an orientation-reversing diffeomorphism. It follows from Proposition 10.12 that

$$M(X,Y,\varphi) \;:=\; (X \sqcup Y)/\sim \qquad \text{with } x \sim \varphi(x) \text{ for } x \in X$$

is a 3-dimensional manifold and that $M(X,Y,\varphi)$ admits a unique orientation such that the orientation of X, viewed as a codimension 0 submanifold of $M(X,Y,\varphi)$, agrees with the orientation of $M(X,Y,\varphi)$. Note that the images of X and Y in $M(X,Y,\varphi)$ define a Heegaard splitting.

In the following we want to consider Heegaard splittings of lens spaces in detail. Recall that given coprime $p, q \in \mathbb{N}$ the map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z, w)) & \mapsto & (e^{2\pi i k/p} \cdot z, e^{2\pi i k q/p} \cdot w) \end{array}$$

defines a free orientation-preserving action of the group $\mathbb{Z}_p$ on S^3. By definition we have $L(p, q) = S^3/\mathbb{Z}_p$ and, following the convention established on page 266, we equip $L(p, q)$ with the obvious orientation coming from S^3. It follows immediately from the definitions that this action by $\mathbb{Z}_p$ on $S^3 = X \cup Y$ preserves the solid tori X and Y from page 1696 setwise. In fact the quotients $X/\mathbb{Z}_p$ and $Y/\mathbb{Z}_p$ are again diffeomorphic to the solid torus. More precisely, we pick $r \in \mathbb{N}$ with $r \cdot q \equiv -1 \bmod p$. Similar to the proof of Lemma 18.1 it is fairly straightforward to verify that the maps

$$\begin{array}{rcl} \Phi\colon X/\mathbb{Z}_p = \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \frac{1}{2}\right\}/\mathbb{Z}_p & \to & S^1 \times \overline{B}^2 \\ [(z, w)] & \mapsto & \left(\frac{z^p}{|z^p|}, \frac{z^{-q} w}{|z^{1-q}|}\right) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon Y/\mathbb{Z}_p = \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \frac{1}{2}\right\}/\mathbb{Z}_p & \to & S^1 \times \overline{B}^2 \\ [(z, w)] & \mapsto & \left(\frac{w^p}{|w^p|}, \frac{z w^r}{|w^{1+r}|}\right) \end{array}$$

are well-defined diffeomorphisms and that Φ is orientation-preserving.[1229] This shows that

$$L(p, q) = X/\mathbb{Z}_p \cup Y/\mathbb{Z}_p$$

is a genus 1 Heegaard decomposition of the lens space $L(p, q)$. In the following lemma we describe this Heegaard splitting in greater detail.

Lemma 73.3. *Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write*

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \text{and we consider the orientation-reversing diffeomorphism} \quad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

If we make the usual identification $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, then there exists an orientation-preserving diffeomorphism

$$L(p, q) \xrightarrow{\cong} M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \varphi).$$

Proof. In this proof we continue to use the notation that we introduced before stating the lemma. Now we write $U = X/\mathbb{Z}_p$ and $V = Y/\mathbb{Z}_p$ and as above we consider the two diffeomorphisms $\Phi\colon U \to S^1 \times \overline{B}^2$ and $\Psi\colon V \to S^1 \times \overline{B}^2$. We denote by $\gamma\colon S^1 \times S^1 \to S^1 \times S^1$ the diffeomorphism that is given by

$$S^1 \times S^1 = \partial(S^1 \times \overline{B}^2) \xleftarrow{\Phi} \partial U = \partial V \xrightarrow{\Psi} \partial(S^1 \times \overline{B}^2) = S^1 \times S^1.$$

[1229] To show that Φ is orientation-preserving it suffices by Proposition 47.3 to show that Φ is orientation-preserving at a single point. Using that $p > 0$ one can calculate explicitly that Φ is orientation-preserving at the point $[(1, 0)] \in X/\mathbb{Z}_p$.

Since Φ and Ψ are diffeomorphisms the two maps together give rise to a well-defined diffeomorphism

$$\begin{array}{rcl} L(p,q) = U \cup V & \to & M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \gamma) = (S^1 \times \overline{B}^2) \cup_\gamma (S^1 \times \overline{B}^2) \\ P & \mapsto & \begin{cases} \Phi(P) \text{ in the first solid torus,} & \text{if } P \in U, \\ \Psi(P) \text{ in the second solid torus,} & \text{if } P \in V. \end{cases} \end{array}$$

It follows from the orientation convention on page 1697, together with the fact that Φ is orientation-preserving, that this diffeomorphism is in fact orientation-preserving.

We consider the following diagram of diffeomorphisms:

$$\begin{tikzcd} S^1 \times S^1 \arrow[d, "="] & \partial(S^1 \times \overline{B}^2) \arrow[l, "="'] & \partial U \arrow[l, "\Phi"'] \arrow[r, "="] & \partial V \arrow[r, "\Psi"] & \partial(S^1 \times \overline{B}^2) \arrow[r, "="] & S^1 \times S^1 \arrow[d, "="] \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[rrrrr, "\varphi"] & & & & & \mathbb{R}^2/\mathbb{Z}^2. \end{tikzcd}$$

Starting from ∂U, it is straightforward to verify that the diagram commutes. But this just shows that, under the identification $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, the two self-diffeomorphisms γ and φ of $S^1 \times S^1$ agree. $\square$

We have now given explicit Heegaard splittings of some basic examples of closed 3-manifolds. Perhaps somewhat surprisingly it is relatively straightforward to generalize these examples to all closed orientable 3-dimensional manifolds. More precisely, in 1898 Poul Heegaard proved the following theorem in his PhD thesis [Hee]. This result is one of the earliest results in the topology of 3-dimensional manifolds.

Theorem 73.4. *Every closed orientable connected 3-dimensional manifold admits a Heegaard splitting.*

Remark. As we had mentioned on page 1696, for each $g \geq 1$ the 3-sphere S^3 admits (up to diffeomorphism) a unique Heegaard splitting. In general though closed orientable connected 3-manifolds can admit inequivalent Heegaard splittings of a fixed genus, see e.g. [Engm, BGAM].

Sketch of proof. The idea of the proof of the theorem is quite simple. Let M be a closed orientable connected 3-dimensional manifold. By Proposition 39.3 the manifold M admits a triangulation, i.e. a simplicial structure. As usual we denote by M^1 the 1-skeleton of M. We pick a closed neighborhood C around the 1-skeleton M^1 such that the intersection of C with each 3-simplex is of the form shown in the middle of Figure 924. Furthermore we write $D = \overline{M \setminus C}$. The claim is that both C and D are handlebodies. The idea is that C and D are built out of pieces as sketched in Figure 924 and thus C and D are "obviously" handlebodies. But if one looks more carefully into the argument one sees that one needs the hypothesis that M is orientable to conclude that C and D are indeed handlebodies. We will not fill in the details of the argument. Unfortunately the proofs in the literature tend to be equally short on details, in particular the role of the orientability condition gets all

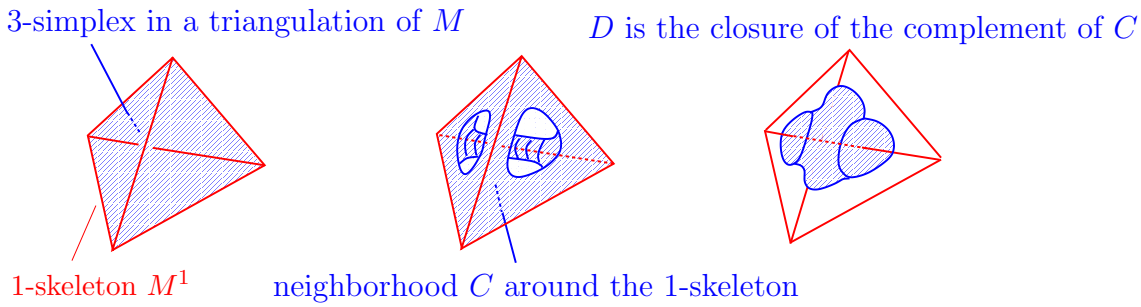


FIGURE 924.

too often swiped under the carpet. We refer to [Sav, Theorem 1.1], [Schar, p. 924] [Hem, Theorem 2.5] for proofs of the theorem which follow the above line of argumentations.

As so often in topology, there is another proof of the theorem, which might be less intuitive and less visual, but which is much more rigorous. This proof uses "Morse theory", which is a theory that explains how one can decompose a closed manifold of any dimension into "handles". We refer to [Mata, Mi4, Ni] for introductions to Morse theory, and we refer to [Johb, Theorem 4.29] or [Mata, Chapter 5.2] for details on how to obtain Heegaard splittings from Morse theory. □

Let $n \geq 4$. In Proposition 19.3 we saw that given any finitely presented group π there exists a closed orientable connected n-dimensional manifold with fundamental group isomorphic to π. Using Theorem 73.4 we will shortly see that the class of finitely presented groups that appear as fundamental groups of closed orientable connected 3-dimensional manifolds is much more restricted. Before we can state the result in detail we need to introduce the following group theoretic notion.

Definition. The *deficiency* of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$.

Examples.

(1) The free group $\langle x_1, \dots, x_k \rangle$ on k generators admits, by definition, a presentation of deficiency k.
(2) Let $m \in \mathbb{N}$. On page 409 we saw that there exists an isomorphism
$$\langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } i < j \in \{1, \dots, m\} \rangle \xrightarrow{\cong} \mathbb{Z}^m.$$
This shows that $\mathbb{Z}^m$ admits a presentation of deficiency $m - \frac{m(m-1)}{2}$.
(3) Let $r \in \mathbb{N}$ and $d_1, \dots, d_s \in \mathbb{N}$. It follows basically immediately from the above isomorphism in (2) that that there exists an isomorphism
$$\left\langle x_1, \dots, x_{r+s} \,\middle|\, \begin{array}{c} [x_i, x_j] = e \text{ for all } i < j \in \{1, \dots, r+s\} \\ \text{and } x_1^{d_1} = \dots = x_s^{d_s} = e \end{array} \right\rangle \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{d_i}.$$

This shows that $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{d_i}$ admits a presentation of deficiency $r - \frac{(r+s)(r+s-1)}{2}$.

Given a finite presentation for a group π it is straightforward to find a finite presentation of smaller deficiency: we just add trivial relations to a given presentation. This observation naturally leads us to the following definition:

Definition. Let π be a finitely presented group. We define the *deficiency* $\mathrm{def}(\pi)$ *of* π as the maximal deficiency of a finite presentation for π.

The following lemma shows, in particular, that the deficiency is in fact a finite natural number.

Lemma 73.5.

(1) *If π is a finitely presented group, then*[1230]
$$\mathrm{def}(\pi) \ \leq \ \mathrm{rank}(\pi_{\mathrm{ab}}).$$
(2) *The deficiency of the free group $\langle x_1, \ldots, x_k \rangle$ equals k.*
(3) *The deficiency of $\mathbb{Z}^m$ equals $m - \frac{m(m-1)}{2}$.*
(4) *The deficiency of an abelian group is at most one. Furthermore, up to isomorphism, we have that*
   (a) *the only abelian groups of deficiency 1 are $\mathbb{Z}$ and $\mathbb{Z}^2$, and*
   (b) *the only abelian groups of deficiency 0 are $\mathbb{Z}_n$ and $\mathbb{Z} \oplus \mathbb{Z}_n$ with $n \neq 0$ and $\mathbb{Z}^3$.*

Proof.

(1) Let π be a finitely presented group and let $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finite presentation. It follows from Proposition 13.10 (5) that there exists a short exact sequence of the form
$$0 \ \to \ A \ \to \ \mathbb{Z}^k \ \to \ \pi_{\mathrm{ab}} \ \to \ 0$$
where A is an abelian group that admits an epimorphism $\mathbb{Z}^l \to A$. We obtain that
$$k - l \ \underset{\substack{\uparrow \\ \text{Lemma 11.8 (6)}}}{\leq} \ k - \mathrm{rank}(A) \ \underset{\substack{\uparrow \\ \text{Lemma 11.6 (2)}}}{=} \ \mathrm{rank}(\pi_{\mathrm{ab}}).$$
It follows that $\mathrm{def}(\pi) \leq \mathrm{rank}(\pi_{\mathrm{ab}})$.
(2) Let $F = \langle x_1, \ldots, x_k \rangle$ be the free group on k generators. Evidently its deficiency is at least k. By Proposition 13.10 (4) we know that $F_{\mathrm{ab}} \cong \mathbb{Z}^k$. Thus it follows from (1) that the deficiency of F is also at most k.
(3) We had just seen that $\mathrm{def}(\mathbb{Z}^m) \geq m - \frac{m(m-1)}{2}$. It follows from [Eps, Lemma 1.3] that this inequality is in fact an equality.
(4) This statement also follows from from the above examples of presentations and [Eps, Lemma 1.3]. □

[1230]Here we denote by π_{ab} the abelianization of π that we introduced on page 415. Furthermore, given a finitely generated abelian group H we denote by $\mathrm{rank}(H)$ the rank that we introduced on page 371.

Now we have all the tools to state, and partially prove, the following theorem.

Theorem 73.6. *Let M be a closed orientable connected 3-dimensional manifold.*

(1) *We have* $\operatorname{def}(\pi_1(M)) \geq 0$,
(2) *If* $\pi_2(M) = 0$, *then* $\operatorname{def}(\pi_1(M)) = 0$.
(3) *If* $\pi_1(M)$ *is abelian, then* $\pi_1(M)$ *is either isomorphic to* $\mathbb{Z}^3$ *or to* $\mathbb{Z}_n$ *for some* $n \in \mathbb{N}_0$.

Remark. The list in (3) is complete, in the sense that all these abelian groups are realized by fundamental groups of closed orientable connected 3-dimensional manifolds. Indeed, given $n \in \mathbb{N}$, we obtain from Corollary 7.19 and Proposition 7.21 that

$$\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3 \quad \pi_1(L(n,1)) \cong \mathbb{Z}_n, \qquad \text{and } \pi_1(S^1 \times S^2) \cong \mathbb{Z}.$$

Proof. Let M be a closed orientable connected 3-dimensional manifold.

(1) By Theorem 73.4 there exists a genus g Heegaard splitting $M = X \cup_F Y$ for M. We pick a base point $x_0 \in F = X \cap Y$. By Lemma 73.2 and Proposition 14.3 we can make identifications $\pi_1(X, x_0) = \langle x_1, \dots, x_g \rangle$, $\pi_1(Y, x_0) = \langle y_1, \dots, y_g \rangle$ and

$$\pi_1(F, x_0) \;=\; \langle a_1, \dots, a_{2g} \,|\, [a_1, a_2] \cdot \dots \cdot [a_{2g-1}, a_{2g}] \rangle.$$

We denote by $\varphi\colon \pi_1(F, x_0) \to \pi_1(X, x_0)$ and $\psi\colon \pi_1(F, x_0) \to \pi_1(Y, x_0)$ the inclusion induced maps. Now we have

$$\begin{array}{rcl} & \text{Seifert-van Kampen Theorem 14.2 for manifolds} & \\ & \downarrow & \\ \pi_1(M, x_0) & \cong & \pi_1(X, x_0) *_{\pi_1(F, x_0)} \pi_1(Y, x_0) \\ & = & \langle x_1, \dots, x_g, y_1, \dots, y_g \,|\, \varphi(a_1) = \psi(a_1), \dots, \varphi(a_{2g}) = \psi(a_{2g}) \rangle. \\ & \uparrow & \\ & \text{Lemma 13.13} & \end{array}$$

We have thus shown that $\pi_1(M)$ admits a presentation of deficiency zero.

(2) This statement is proved in [Eps, Lemma 3.1].
(3) It follows from (1) and Lemma 73.5 (4), that the only isomorphism types of abelian groups that can occur are $\mathbb{Z}_n$ for some $n \in \mathbb{N}_0$, $\mathbb{Z}^2$, $\mathbb{Z} \oplus \mathbb{Z}_n$ for some $n \geq 2$ and $\mathbb{Z}^3$. With somewhat more effort, see e.g. [Eps, Theorem 3.3] or [Rei, p. 28] one can also exclude the groups $\mathbb{Z}^2$ and $\mathbb{Z} \oplus \mathbb{Z}_n$ with $n \geq 2$ from the list. □

We conclude this discussion of Heegaard splittings with a particularly fascinating aspect of Heegaard splittings: they can be used to give a purely group theoretic formulation of the Poincaré Conjecture in dimension three. More precisely, the following theorem was proved by John Stallings [Sta3, Chapter 3] and William Jaco [Jaco, Theorem 6.3]. The origin of the theorem lies in the description of the fundamental group of a 3-manifold in terms of a Heegaard splitting that we gave in the proof of Theorem 73.6.

Theorem 73.7. *We introduce the following notation and language:*

(a) *Given* $g \in \mathbb{N}_0$ *we denote by* π_g *the fundamental group of the surface of genus* g *and we denote by* F_g *the free group of rank* g.

(b) *We say a homomorphism* $\beta\colon G \to A * B$ *from a group* G *to the free product of two groups* A *and* B *is* essential*, if there is no* $h \in A * B$ *such that* $h\beta(G)h^{-1}$ *is contained in* A *or* B*.*

The following two statements are equivalent:

(1) *Every 3-dimensional homotopy sphere is diffeomorphic to* S^3*.*
(2) *For any* $g \in \mathbb{N}$ *any epimorphism* $\alpha\colon \pi_g \to F_g \times F_g$ *factors through an essential monomorphism* $\beta\colon \pi_g \to A * B$ *from* π_g *to the free product of two groups* A *and* B*.*

Put differently, by Perelman's proof of Theorem 28.25 we now know that the first statement of Theorem 73.7 holds, thus the second statement also holds. This is the only known proof of that purely group theoretic statement. Conversely, any proof of the second statement of Theorem 73.7 would immediately reprove Theorem 28.25. Surely that is a quick and easy way to mathematical glory.

Remark.

(1) Perhaps by now it is no longer surprising that there is a huge literature on Heegaard splittings, see e.g. [Schle, Chapter 6] for an introduction.
(2) Recently the notion of a Heegaard splitting has been modified by David Gay and Rob Kirby [GK] to give the notion of a *trisection* of a closed orientable connected 4-dimensional manifold. One remarkable feature of trisections is that by recent work of Aarons Abrams, David Gay and Rob Kirby [AGK] they allow for a purely group theoretic reformulation of the smooth Poincaré Conjecture in dimension 4 that we had formulated as Question 28.26. This can be viewed as an analogue of Theorem 73.7.

73.3. **The mapping class group** $(*)$**.** In this section we will introduce the mapping class group of manifolds and we will study the mapping class groups of surfaces in more detail. We will use this discussion to show that every 3-manifold with a genus 1 Heegaard decomposition is in fact a lens space.

We start out with the following definition.

Definition. Let M be an oriented manifold.

(1) We write $\mathrm{Diff}(M)$ for the group of all self-diffeomorphisms of M.
(2) We denote by $\mathrm{Diff}_0(M)$ the set of all self-diffeomorphisms of M that are diffeotopic to the identity.
(3) We write $\mathrm{Diff}^+(M)$ for the group of all orientation-preserving self-diffeomorphisms of M.

We have the following useful lemma.

Lemma 73.8. *Let* M *be an oriented manifold. The set* $\mathrm{Diff}_0(M)$ *is contained in* $\mathrm{Diff}^+(M)$ *and it is a normal subgroup of* $\mathrm{Diff}(M)$*.*

Proof. It follows easily from Lemma 10.6 that $\mathrm{Diff}_0(M)$ is a normal subgroup of $\mathrm{Diff}(M)$. Furthermore it follows from Lemma 45.19 and Proposition 45.22 that $\mathrm{Diff}_0(M)$ is actually contained in $\mathrm{Diff}^+(M)$.[1231] □

We recall the following definition.

Definition. Let M and N be two manifolds. Two diffeomorphisms $f, g\colon M \to N$ are called *diffeotopic*, if there exists a *diffeotopy* from f to g, i.e. a smooth map[1232]

$$\begin{array}{rcl} H\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) for all $x \in M$ we have $H(x,0) = f(x)$ and $H(x,1) = g(x)$, and
(2) for each $t \in [0,1]$ the map $M \to N$ given by $x \mapsto H(x,t)$ is a diffeomorphism.

In Lemmas 10.13 we already saw that for some constructions it does not make a difference if we modify a map by a diffeotopy.

In light of Lemma 73.8 it now makes sense to introduce the following definition.

Definition. Let M be an oriented manifold. The *mapping class group* $\mathrm{MCG}(M)$ of M is defined as

$$\mathrm{MCG}(M) \;:=\; \mathrm{Diff}(M)/\,\mathrm{Diff}_0(M)$$

and the *orientation-preserving mapping class group* $\mathrm{MCG}^+(M)$ of M is defined as

$$\mathrm{MCG}^+(M) \;:=\; \mathrm{Diff}^+(M)/\,\mathrm{Diff}_0(M).$$

Example. In this example we want to consider the orientation-preserving mapping class group of 1-dimensional manifolds. It is an amusing exercise to show that $\mathrm{MCG}^+(\mathbb{R})$, $\mathrm{MCG}^+([0,1]$ and $\mathrm{MCG}^+([0,1)$ are the trivial group. Furthermore, in Proposition 20.5 we showed that $\mathrm{MCG}^+(S^1)$ is also the trivial group.

For surfaces of genus 0 and 1, i.e. for the sphere S^2 and the torus $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ the following theorem gives a complete description of the mapping class groups.

Theorem 73.9.

(1) $\mathrm{MCG}^+(S^2)$ *is the trivial group.*
(2) *The group* $\mathrm{MCG}(S^2)$ *contains two elements, one is represented by the identity and the other one is represented by any reflection in a hyperplane.*
(3) *Let* $T = \mathbb{R}^2/\mathbb{Z}^2$ *be the 2-dimensional torus. We use the generators from Lemma 33.13 to make the usual identification* $H_1(T;\mathbb{Z}) = \mathbb{Z}^2$ *and we use this identification to*

[1231] Sharp eyes will have noticed that this proof relies on going through the notion of $\mathbb{Z}$-orientability of a topological manifold. How can one proof this lemma only using the language of manifolds and their orientations?

[1232] See page 333 for the definition of what it means that a map on $M \times [0,1]$ is smooth.

make the corresponding identifications $\mathrm{Aut}(H_1(T;\mathbb{Z})) = \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2,\mathbb{Z})$ *and* $\mathrm{SAut}(H_1(T;\mathbb{Z})) = \mathrm{SAut}(\mathbb{Z}^2) = \mathrm{SL}(2,\mathbb{Z})$. *The maps*

$$\begin{aligned} \Phi\colon \mathrm{MCG}(T) &\mapsto \mathrm{GL}(2,\mathbb{Z}) \\ [f\colon T\to T] &\mapsto \underbrace{(f_*\colon H_1(T)\to H_1(T))}_{\in \mathrm{Aut}(H_1(T;\mathbb{Z}))=\mathrm{GL}(2,\mathbb{Z})} \end{aligned} \quad \textit{and} \quad \begin{aligned} \Psi\colon \mathrm{GL}(2,\mathbb{Z}) &\mapsto \mathrm{MCG}(T) \\ A &\mapsto \begin{bmatrix} \mathbb{R}^2/\mathbb{Z}^2 \mapsto \mathbb{R}^2/\mathbb{Z}^2 \\ [v] \mapsto [Av] \end{bmatrix} \end{aligned}$$

are inverses of one another.

(4) *The same statement as in* (3) *holds if we replace* $\mathrm{MCG}^+(\mathbb{R}^2/\mathbb{Z}^2)$ *by* $\mathrm{MCG}(\mathbb{R}^2/\mathbb{Z}^2)$ *and if we replace* $\mathrm{SL}(2,\mathbb{Z})$ *by* $\mathrm{GL}(2,\mathbb{Z})$.[1233]

Remark. In 1969 by Jean Cerf [Cer] (see also [Eli, Theorem 2.4.1] and [Hat7]) showed that the obvious analogues of the first two statements of Theorem 73.9 also hold for S^3. As is pointed out in [Hat5], it follows from the existence of exotic spheres, see page 730 and the h-cobordism theorem [Sma3, Mi5], that in most higher dimensions the analogues do not hold.

Proof. The first two statements were proved in 1959 by Stephen Smale [Sma2].[1234] It is an amusing exercise to show that $\Phi\circ\Psi = \mathrm{id}$ on $\mathrm{GL}(2,\mathbb{Z})$. The fact that $\Psi\circ\Phi = \mathrm{id}$ on $\mathrm{MCG}(\mathbb{R}^2/\mathbb{Z}^2)$ is somewhat harder. We refer to [FaM, Theorem 2.5] or alternatively to [Rol, Chapter 2.D] for proofs. The claim regarding the orientation-preserving mapping class group follows from the above result together with Lemma 57.15. □

Now we can give the following characterization of lens spaces.

Corollary 73.10. *If M is a closed orientable connected 3-dimensional manifold, then the following two statements are equivalent:*

$$M \text{ admits a genus 1 Heegaard splitting} \iff M \text{ is a lens space or } M \cong S^1\times S^2.$$

Proof. By Lemma 73.3 and the discussion on page 1696 we know that lens spaces and $S^1\times S^2$ admit genus 1 Heegaard splittings.

Now suppose that M is a closed orientable connected 3-dimensional manifold that admits a genus 1 Heegaard splitting $M = S^1\times\overline{B}^2 \cup_\varphi S^1\times\overline{B}^2$ where φ is an orientation-reversing self-diffeomorphism of $S^1\times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. It is an immediate consequence of Theorem 73.9 (3), Lemma 57.15 and Lemma 10.13 that, after a diffeotopy, we can assume that φ is given by multiplication by a matrix $\begin{pmatrix} q & p \\ s & r\end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ with $qr - ps = -1$.

First assume that $p = 0$. We leave it as an amusing exercise to show that in this case M is diffeomorphic to $S^1\times S^2$. If $p > 0$, then it follows from Lemma 73.3 that M is diffeomorphic to $L(p,q)$. Finally we consider the case that $p < 0$. We denote by ψ the self-diffeomorphism

[1233] We showed in Lemma 57.15 that the map $\Phi\colon \mathrm{MCG}^+(T)\to\mathrm{GL}(2,\mathbb{Z})$ takes values in $\mathrm{SL}(2,\mathbb{Z})$.

[1234] We had already encountered Stephen Smale in the proof of the Poincaré Conjecture in dimensions ≥ 5, see Theorem 28.22.

of $\mathbb{R}^2/\mathbb{Z}^2$ that is given by multiplication by the matrix $\begin{pmatrix} q & -p \\ -s & r \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. The map

$$S^1 \times \overline{B}^2 \cup_\varphi S^1 \times \overline{B}^2 \;\to\; S^1 \times \overline{B}^2 \cup_\psi S^1 \times \overline{B}^2,$$

given by reflection on both of the S^1 factors is easily seen to be a well-defined diffeomorphism. But as we had just shown, the right-hand side is diffeomorphic to the lens space $L(-p,q)$. □

We had just seen that the mapping class group of the 2-dimensional sphere and the torus are groups we are very familiar with. It is natural to ask, whether we can also understand the mapping class groups of higher genus surfaces.

Throughout the remainder of this section, given $g \geq 1$ we denote by Σ_g the surface of genus g. To facilitate the discussion, we equip $H_1(\Sigma_g;\mathbb{Z})$ with the standard basis $x_1,\dots,x_g,y_1,\dots,y_g$ shown in Figure 925 and we use this basis to make the identification $H_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$.

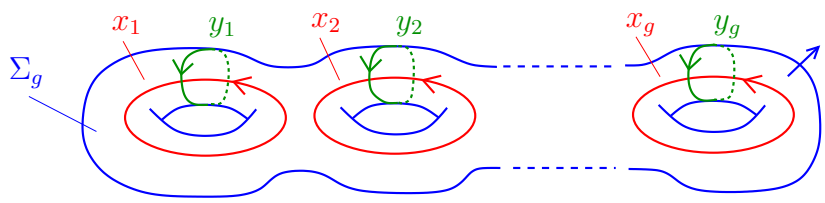


FIGURE 925.

Here is a somewhat optimistic interpretation of the above question.

Question 73.11. *Let $g \geq 2$. Is the map*

$$\begin{array}{rcl} \Phi_g\colon \mathrm{MCG}(\Sigma_g) & \mapsto & \mathrm{GL}(2g,\mathbb{Z}) \\ [f\colon \Sigma_g \to \Sigma_g] & \mapsto & \underbrace{\big(f_*\colon H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})\big)}_{\in \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}))=\mathrm{GL}(2g,\mathbb{Z})} \end{array}$$

an isomorphism?

It turns out that the answer to Question 73.11 is no. In fact we have already developed all the tools to show that Φ_g is not an epimorphism. At this point it is useful to introduce the following definition.

Definition.

(1) Given $g \in \mathbb{N}$ we write $J_g := \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}$.

(2) We say a matrix $A \in \mathrm{GL}(2g,\mathbb{Z})$ is *symplectic*[1235] if $A^T J A = J$.

[1235]The origin of the word "symplectic" is explained in the following webpage

`http://jeff560.tripod.com/mathword.html`

which explores the earliest known uses of some of the words of mathematics.

(3) We define the *symplectic group* $\mathrm{Sp}(2g,\mathbb{Z})$ to be the group[1236] of all symplectic matrices.

Now we can formulate the following proposition.

Proposition 73.12. *Let $g \in \mathbb{N}$. The homomorphism $\Phi_g\colon \mathrm{MCG}^+(\Sigma_g) \to \mathrm{GL}(2g,\mathbb{Z})$ takes values in $\mathrm{Sp}(2g,\mathbb{Z})$.*

Proof. Recall that we use the homology classes represented by the curves $x_1,\dots,x_g,y_1,\dots,y_g$ shown in Figure 925 to make the identification $\mathbb{Z}^{2g} = H_1(\Sigma_g;\mathbb{Z})$. It follows from the calculation of the intersection form of Σ_g on page 1656 that we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \arrow[r, "(v,w)\mapsto v^T J w"] \arrow[d, "="] & \mathbb{Z} \arrow[d, "="] \\
H_1(\Sigma_g;\mathbb{Z}) \times H_1(\Sigma_g;\mathbb{Z}) \arrow[r, "Q_{\Sigma_g}"] & \mathbb{Z}.
\end{tikzcd}$$

Now let f be an orientation-preserving self-homeomorphism of Σ_g. We denote by A the matrix that represents the induced map $f_*\colon H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ with respect to the above basis.

It is a consequence of Lemma 71.7 that the induced map f_* preserves the intersection form of Σ_g. Therefore we obtain from the commutative diagram that the induced map on $H_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$ also preserves the form on the top. But this means that for any two vectors $v,w \in \mathbb{Z}^g$ we have $v^T A^T J A w = (Av)^T J A w = v^T J w$. This in turn implies that $A^T J A = J$. □

The question now arises, how big is the symplectic group? The following proposition summarizes some known results:

Proposition 73.13. *Let $g \in \mathbb{N}$.*

(1) *We have $\mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{SL}(2g,\mathbb{Z})$.*
(2) *We have $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$.*
(3) *For $g \geq 2$ the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ is a proper subgroup of $\mathrm{SL}(2g,\mathbb{Z})$.*

Proof.

(1) The fact that every symplectic matrix has determinant equal to $+1$ is a somewhat non-trivial exercise in linear algebra. A proof is for example given in [Serd, Corollary 10.2] or [McS, Lemma 1.1.15].
(2) A short and amusing calculation (worth doing!) shows that every matrix in $\mathrm{SL}(2,\mathbb{Z})$ is in fact symplectic.

[1236] It is straightforward to verify that the set of all symplectic $(2g \times 2g)$-matrices is in fact a subgroup of $\mathrm{GL}(2g,\mathbb{Z})$.

(3) Let $g \geq 2$. Unless one is very unlucky, almost any randomly picked matrix in $\mathrm{SL}(2g, \mathbb{Z})$ will not lie in $\mathrm{Sp}(2g, \mathbb{Z})$. For example an elementary calculation shows that the matrix

$$A := \begin{pmatrix} 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

does not satisfy $A^T J A = J$, i.e. A does not lie in $\mathrm{Sp}(4, \mathbb{Z})$. A straightforward modification of this example shows that given any $g \geq 2$ there exist matrices in $\mathrm{SL}(2g, \mathbb{Z})$ that do not lie in $\mathrm{Sp}(2g, \mathbb{Z})$. □

In a last ditch effort to salvage Question 73.11 one could ask whether perhaps the map $\Phi_g \colon \mathrm{MCG}(\Sigma_g)^+ \to \mathrm{Sp}(2g, \mathbb{Z})$ is an isomorphism.

Now is perhaps a good moment to recall that we had already constructed diffeomorphisms of Σ_g. More precisely, recall that given an oriented curve C on Σ_g we introduced on page 472 the corresponding smooth Dehn twist $t_C \colon \Sigma_g \to \Sigma_g$. The following proposition shows that, in some sense, Dehn twists are all orientation-preserving diffeomorphisms we ever need.

Proposition 73.14. *The Dehn twists along the* $3g-1$ *curves shown in Figure 926 form a generating set for* $\mathrm{MCG}^+(\Sigma_g)$.

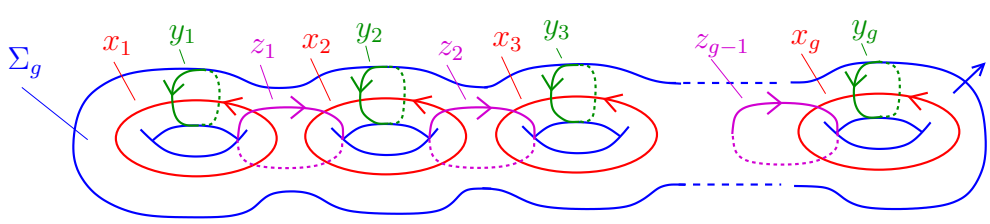


FIGURE 926.

Remark. It is significantly harder to show that $\mathrm{MCG}^+(\Sigma_g)$ admits a finite presentation, see [HT] and [Waj] for details.

Proof. This proposition proved in 1963 by Raymond Lickorish [Lic4], see also [FaM, Theorem 4.13] or [Bir, Theorem 4.3]. □

In fact this source of orientation-preserving diffeomorphisms is rich enough to prove the following proposition.

Proposition 73.15. *For any* $g \geq 1$ *the above map* $\Phi_g \colon \mathrm{MCG}^+(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ *is an epimorphism.*

Proof. The result is proved in [FaM, Theorem 6.4]. □

This leads us to the following definition.

Definition. Given $g \geq 1$ we refer to

$$T_g \ := \ \ker\{\Phi_g \colon \ \mathrm{MCG}(\Sigma_g)^+ \to \mathrm{Sp}(2g, \mathbb{Z})\}$$

as the *Torelli group* of Σ_g.[1237]

The question now becomes, whether the Torelli group T_g is non-trivial. In fact the next proposition shows precisely that.

Proposition 73.16. *Let $g \in \mathbb{N}$.*

(1) *If C is an oriented curve on Σ_g that is not null-homotopic, then the Dehn twist t_C defines a non-trivial element in $\mathrm{MCG}(\Sigma_g)$.*
(2) *If C is an oriented curve on Σ_g that is null-homologous, then the Dehn twist t_C defines an element of the Torelli group T_g.*
(3) *For $g \geq 2$ the Dehn twist along the curve C shown in Figure 927 defines a non-trivial element in the Torelli group T_g.*

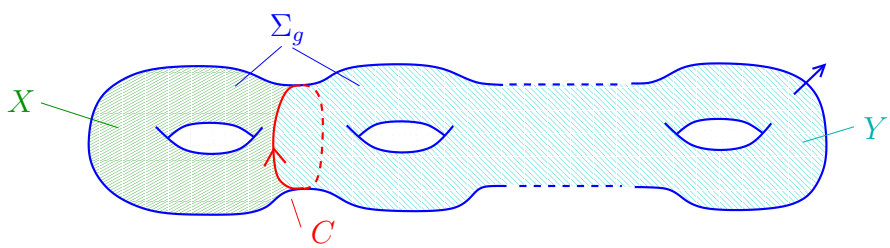


FIGURE 927.

Proof.

(1) The first statement is proved in [FaM, Proposition 3.1].
(2) This statement follows from [FaM, Proposition 3.2]. With some effort one could also prove this statement "by hand".
(3) By (1) and (2) it suffices to show that C is null-homologous but not null-homotopic. Note that C is the boundary of a submanifold X of Σ_g. We obtain from Corollary 46.27 (1) that $[C] = 0 \in H_1(X; \mathbb{Z})$, which implies that C is also null-homologous in Σ_g. On the other hand we saw on page 433 that C is not null-homotopic. $\square$

There are still lots of open questions regarding mapping class groups of surfaces. For example the following question is an interesting open problem.

Question 73.17. *Let Σ_g be the surface of genus $g \in \mathbb{N}_0$. Is the mapping class group $\mathrm{MCG}(\Sigma_g)$ linear, i.e. is it isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some large enough n?*

[1237] For slightly mysterious reasons this group is named after the Italian mathematician Ruggiero Torelli (1884–1915) who died in World War I.

We had just seen in Theorem 73.9 that the answer is yes for $g = 0$ and $g = 1$. Furthermore Stephen Bigelow and Ryan Budney [BB] showed in 2001 that the answer is yes for $g = 2$. The case $g \geq 3$ is still wide open.

As in almost any topic we have touched so far, there is again a tremendous amount of literature and work on mapping class groups. We refer to [Bir, FaM, Hat5] for more information regarding mapping class groups and diffeomorphism groups of manifolds in dimension 2 and in higher dimensions.

Exercises for Chapter 73.

Exercise 73.1. Let H_g be the genus g handlebody. Let $W = H_g \cup H_g$ be the double of H_g. This is a closed 3-dimensional manifold. Do we know what it is?

Exercise 73.2. In Chapter 65 we have seen that the 3-dimensional torus $S^1 \times S^1 \times S^1$ admits a Heegaard splitting of genus three. Does it also admit a Heegaard splitting of genus two?

Exercise 73.3. Let F be the surface of genus g and let $\varphi\colon F \to F$ be an orientation-preserving self-diffeomorphism. By Proposition 10.12 we know that the mapping torus

$$M(F, \varphi) \ := \ F \times [0,1] \,/\, (x, 0) \sim (\varphi(x), 1)$$

is a closed orientable connected 3-dimensional manifold. Show that $M(F, \varphi)$ admits a Heegaard splitting of genus $2g + 1$.

74. The linking pairing on rational homology spheres

74.1. **Rational homology spheres.** We recall the following lemma.

Lemma 7.7. *Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,*[1238] *then $L(p,q)$ and $L(p,r)$ are diffeomorphic.*

In Question 7.8 we had asked whether the converse to Lemma 7.7 holds. So far we have made little progress. On page 281 we saw, using fundamental groups, that if two lens spaces $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, then $p = \widetilde{p}$. Later on, on page 723 we saw that higher homotopy groups do not carry any additional information. In Lemma 40.12 we had determined the homology groups of lens spaces, more precisely we showed that

$$H_n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0,3, \\ \mathbb{Z}_p, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

From the Universal Coefficient Theorem for Cohomology Groups 51.13 it follows that

$$H^n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0,3, \\ \mathbb{Z}_p, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that homology groups and cohomology groups cannot be used to distinguish the lens spaces $L(p,q)$ and $L(p,q')$.

In Chapter 71 we had successfully applied the cup product to distinguish several manifolds that could not be distinguished just using cohomology groups. It is therefore tempting to try to use cup products to distinguish lens spaces. But here we immediately encounter the problem, that all non-trivial cup products are trivially trivial. More precisely, all cup products of cohomology classes in degree ≥ 1 are zero, since too many cohomology groups are zero. For example, the cup product on the non-trivial group $H^2(L(p,q);\mathbb{Z})$ with itself is zero since it takes values in $H^4(L(p,q);\mathbb{Z}) = 0$.

In this chapter we will develop an analogue of the intersection form, namely the *linking pairing*. This is an invariant defined for lens spaces, or more generally, for odd-dimensional rational homology spheres, which are defined as follows:

Definition. Let R be a commutative ring and let M be an n-dimensional topological manifold. We say M is an *R-homology sphere*, if for each $k \in \mathbb{N}_0$ we have $H_k(M;R) \cong H_k(S^n;R)$, i.e. we have

$$H_k(M;R) \;\cong\; \begin{cases} R, & \text{if } k = 0,n, \\ 0, & \text{otherwise.} \end{cases}$$

Instead of $\mathbb{Z}$-homology sphere we usually say *integral homology sphere* and instead of $\mathbb{Q}$-homology sphere we usually say *rational homology sphere*.

[1238] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s \equiv 1 \bmod p$. We write $r^{-1} := s$.

Examples.

(1) It follows from the aforementioned calculation of the homology groups of lens spaces and Corollary 42.20 (which is a straightforward consequence of the Universal Coefficient Theorem 42.18), that lens spaces are rational homology spheres.
(2) It follows from Proposition 71.10 that the connected sum of two n-dimensional R-homology spheres is again an n-dimensional R-homology sphere.

The following lemma gives an alternative description of rational homology spheres.

Lemma 74.1. *Let M be an n-dimensional topological manifold M. Then the following holds:*[1239]

$$M \text{ is a rational homology sphere} \iff \begin{array}{c} M \text{ is closed connected and orientable and} \\ H_i(M;\mathbb{Z}) \text{ is finite for } i = 1, \dots, \lfloor \frac{n-1}{2} \rfloor. \end{array}$$

Proof. The proof of the lemma is basically an exercise in combining all the standard facts on homology groups of topological manifolds that we have gathered so far. More precisely, the "$\Rightarrow$" direction follows from Corollary 42.20, Proposition 45.2 (2), Proposition 29.5 and Theorems 46.2 and 46.4. The proof of the "$\Leftarrow$" direction also requires the above ingredients, furthermore it also requires the Poincaré Duality Theorem 61.1. We leave it to the reader to connect the dots. □

74.2. **The Bockstein homomorphisms.** Before we can introduce the linking pairing of rational homology spheres we need to give some harmless extensions of our definitions and results from (co-) homology theory. We will do so in the coming two sections.

Let X be a topological space and let $f\colon A \to B$ be a homomorphism of abelian groups. We recall that in Lemma 42.7 we saw that the maps

$$\begin{array}{rcl} f_*\colon C_*(X;A) & \to & C_*(X;B) \\ \left[\sum_{i=1}^k \sigma_i \otimes a_i\right] & \mapsto & \left[\sum_{i=1}^k \sigma_i \otimes f(a_i)\right] \end{array}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$f_*\colon H_n(X;A) \to H_n(X;B).$$

Similarly, it ifollows from the discussion on page 1163 that the maps

$$\begin{array}{rcl} f_*\colon C^n(X;A) = \mathrm{Hom}(C_n(X),A) & \to & C^n(X;B) = \mathrm{Hom}(C_n(X),B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map, in particular we get for each $n \in \mathbb{N}_0$ an induced map

$$f_*\colon H^n(X;A) \to H^n(X;B)$$

In the following lemma we discuss one subtle example.

[1239]Here recall that given a real number r we denote by $\lfloor r \rfloor = \max\{n \in \mathbb{Z} \mid n \leq r\}$ the result of rounding r down.

Lemma 74.2. $(*)$ *Let $k \in \mathbb{Z}$. We denote by $f\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by k. Given any pair (X, A) of topological spaces and given any $n \in \mathbb{N}_0$ the maps $f_*\colon H_n(X, A; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$ and $f_*\colon H^n(X, A; \mathbb{Z}) \to H^n(X, A; \mathbb{Z})$ are given by multiplication by m.*

Proof $(*)$. It is arguable whether this lemma actually needs a proof. Nonetheless, since at one point we will make use of the statement for cohomology, we provide the proof for the induced maps on cohomology. Thus let $\gamma\colon C_k(X, A) \to \mathbb{Z}$ be a cochain. We see that

$$f_*([\gamma]) = [f \circ \gamma] = \begin{bmatrix} C_k(X, A) \to \mathbb{Z} \\ \sigma \mapsto \underbrace{n \cdot f(\sigma)}_{=(f\circ\gamma)(\sigma)} \end{bmatrix} \underset{\substack{\uparrow \\ \text{this is the point where something happens}}}{=} \begin{bmatrix} C_k(X, A) \to \mathbb{Z} \\ \sigma \mapsto (n \cdot f)(\sigma) \end{bmatrix} = [n \cdot f] = n \cdot [\gamma].$$

□

These observations lead us to the definitions of the Bockstein homomorphisms.

Definition. Let X be a topological space and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Let $n \in \mathbb{N}_0$. It follows from the fact that $C_n(X)$ is a free abelian group and from the natural isomorphisms of Lemma 42.3 that

$$0 \;\to\; C_n(X; A) \;\xrightarrow{i_*}\; C_n(X; B) \;\xrightarrow{p_*}\; C_n(X; C) \;\to\; 0$$

is also exact. Thus we obtain a short exact sequence

$$0 \;\to\; C_*(X; A) \;\xrightarrow{i_*}\; C_*(X; B) \;\xrightarrow{p_*}\; C_*(X; C) \;\to\; 0$$

of chain complexes. Furthermore, from Proposition 31.10 we obtain a long exact sequence

$$\ldots \;\xrightarrow[=:\beta_{n-1}]{\partial_{n-1}}\; H_n(X; A) \;\xrightarrow{i_*}\; H_n(X; B) \;\xrightarrow{p_*}\; H_n(X; C) \;\xrightarrow[=:\beta_n]{\partial_n}\; H_{n-1}(X; A) \;\to\; \ldots$$

which is called the *Bockstein sequence in homology.*[1240] Furthermore for each $n \in \mathbb{N}_0$ the connecting homomorphism $\beta = \partial_n\colon H_n(X; C) \to H_{n-1}(X; A)$ is called the *n-th Bockstein homomorphism in homology.*

The same setup also works in cohomology.

Definition. We continue with the notation from the previous definition. Since each $C_n(X)$ is a free abelian group one sees easily that

$$0 \;\to\; C^*(X; A) \;\xrightarrow{i}\; C^*(X; B) \;\xrightarrow{p_*}\; C^*(X; C) \;\to\; 0$$

is a short exact sequence of cochain complexes. From Proposition 48.6 we obtain a natural long exact sequence of cohomology groups

$$\ldots \;\xrightarrow[=:\beta_{n-1}]{\delta_{n-1}}\; H^n(X; A) \;\xrightarrow{i_*}\; H^n(X; B) \;\xrightarrow{p_*}\; H^n(X; C) \;\xrightarrow[=:\beta_n]{\delta_n}\; H^{n+1}(X; A) \;\to\; \ldots$$

[1240]The Bockstein homomorphism is named after the Russian mathematician Meyer Bockstein (1913-1990).

which is called the *Bockstein sequence in cohomology.* Furthermore for each $n \in \mathbb{N}_0$ the connecting homomorphism $\beta = \delta_n \colon H^n(X;C) \to H^{n+1}(X;A)$ is called the *n-th Bockstein homomorphism in cohomology.*

Examples. Let X be a topological space. The following two instances of the Bockstein homomorphisms are arguably the ones which are most frequently used:

(1) Let $k \in \mathbb{N}$. We consider the short exact sequence $0 \to \mathbb{Z}_k \xrightarrow{\cdot k} \mathbb{Z}_{k^2} \xrightarrow{p} \mathbb{Z}_k \to 0$ where p stands for the obvious projection map. Given a topological space X we obtain Bockstein homomorphisms $\beta_n \colon H_n(X;\mathbb{Z}_k) \to H_{n-1}(X;\mathbb{Z}_k)$, $n \in \mathbb{N}_0$. In Exercise 74.1 we will see that $\beta_{n-1} \circ \beta_n \colon H_n(X;\mathbb{Z}_k) \to H_{n-2}(X;\mathbb{Z}_k)$ is the zero map.
(2) Later on we will consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. It gives rise to Bockstein homomorphisms $\beta_n \colon H_n(X;\mathbb{Q}/\mathbb{Z}) \to H_{n-1}(X;\mathbb{Z})$ and $\beta_n \colon H^n(X;\mathbb{Q}/\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$.

For completeness' sake we mention the following lemma.

Lemma 74.3. *Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups.*

(1) *Let $n \in \mathbb{N}_0$. The n-th Bockstein homomorphisms β_n corresponding to the short exact sequence of abelian groups form a natural transformation from the contravariant functor $X \mapsto H^n(X;C)$ to the contravariant functor $X \mapsto H^{n+1}(X;A)$.*
(2) *Given a map $f \colon X \to Y$ between topological spaces the following diagram commutes:*

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\beta_{n-1}} & H^n(Y;A) & \xrightarrow{i_*} & H^n(Y;B) & \xrightarrow{p_*} & H^n(Y;C) & \xrightarrow{\beta_n} & H^{n+1}(Y;A) & \longrightarrow \dots \\
 & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\
\dots \xrightarrow{\beta_{n-1}} & H^n(X;A) & \xrightarrow{i_*} & H^n(X;B) & \xrightarrow{p_*} & H^n(X;C) & \xrightarrow{\beta_n} & H^{n+1}(X;A) & \longrightarrow \dots
\end{array}$$

Proof. The first statement is essentially a consequence of Lemma 31.9 (2). The second statement follows from the first statement together with Lemma 48.9. □

74.3. **The cup product and the cap product with coefficients.** On page 1333, given a topological space X, we introduced the cup product for the R-cohomology groups $H^*(X;R)$ provided the coefficients are given by a commutative ring R. In this section we will generalize the notion of the cup product where we no longer demand that the coefficients lie in a commutative ring R. Furthermore we also define a slight variation on the cap product. It is advisable to quickly look at the definitions and then move on to the next section. All the statements proved in this section are basically the obvious generalizations of earlier results.

Before we introduce the more general cup product, we recall the following notation from page 1323.

Notation. Given any $i \in \{0,\dots,n\}$ we write $v_i := (0,\dots,1,0,\dots,0) \in \mathbb{R}^{n+1}$ and secondly, given $a_0,\dots,a_s \in \Delta^n$ we denote by $[a_0,\dots,a_s]$ the unique affine linear map $\Delta^s \to \Delta^n$ that sends v_i to a_i for $i = 0,\dots,s$.

Now we can give the quite simple-minded definition of the more general cup product.

Definition. Let X be a topological space and let A and B be abelian groups. We define

$$\begin{aligned} C^p(X;A)\times C^q(X;B) \;&\to\; C^{p+q}(X;A\otimes B) \\ (\varphi,\psi) \mapsto &\left(\begin{array}{l} \varphi\cup_\otimes\psi\colon C_{p+q}(X)\to A\otimes B \\ \quad(\sigma\colon\Delta^{p+q}\to X)\mapsto \underbrace{\varphi(\sigma\circ[v_0,\ldots,v_p])}_{\in A}\otimes\underbrace{\psi(\sigma\circ[v_p,\ldots,v_{p+q}])}_{\in B} \end{array}\right). \end{aligned}$$

We have the following elementary lemma. The proof is basically verbatim the same as the proof of Lemma 57.2.

Lemma 74.4. *Let X be a topological space and let A and B be two abelian groups. If $\varphi\in C^p(X;A)$ and $\psi\in C^q(X;B)$, then*

$$\delta(\varphi\cup_\otimes\psi) \;=\; \delta\varphi\cup_\otimes\psi+(-1)^p\cdot\varphi\cup_\otimes\delta\psi \;\in\; C^{p+q+1}(X;A\otimes B).$$

Now we can easily prove the following lemma. The proof again is basically the same as the proof of Lemma 57.3, we just need to replace Lemma 57.2 by Lemma 74.4.

Lemma 74.5. *Let X be a topological space and let A and B be abelian groups. Then the map*

$$\begin{array}{rcl} \cup_\otimes\colon H^p(X;A)\times H^q(X;B) & \to & H^{p+q}(X;A\otimes B) \\ ([\varphi],[\psi]) & \mapsto & [\varphi\cup_\otimes\psi] \end{array}$$

is well-defined.

We refer to the pairing from Lemma 74.5 again as a *cup product*. The following lemma is an immediate consequence of the definitions.

Lemma 74.6. *Let X be a topological space.*

(1) *If $\alpha\colon A\to A'$ and $\beta\colon B\to B'$ are homomorphisms between two abelian groups, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes*

$$\begin{array}{ccc} H^k(X;A)\times H^l(X;B) & \xrightarrow{\cup_\otimes} & H^{k+l}(X;A\otimes B) \\ \downarrow{\scriptstyle\alpha_*\times\beta_*} & & \downarrow{\scriptstyle(\alpha\otimes\beta)_*} \\ H^k(X;A')\times H^l(X;B') & \xrightarrow{\cup_\otimes} & H^{k+l}(X;A'\otimes B'). \end{array}$$

(2) *If $f\colon X\to X'$ is a map to another topological space X' and if A and B are two abelian groups, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes*

$$\begin{array}{ccc} H^k(X';A)\times H^l(X';B) & \xrightarrow{\cup_\otimes} & H^{k+l}(X';A\otimes B) \\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*} \\ H^k(X;A)\times H^l(X;B) & \xrightarrow{\cup_\otimes} & H^{k+l}(X;A\otimes B). \end{array}$$

To formulate the relationship to the earlier definition of the cup product we need to introduce the following definition.

Definition. Let R be a commutative ring. We refer to the homomorphism[12411242]

$$\begin{array}{rcl} \mu\colon R\otimes R & \to & R \\ \sum\limits_{i=1}^{n} \underbrace{a_i}_{\in\mathbb{Z}} \cdot \underbrace{r_i}_{\in R} \otimes \underbrace{s_i}_{\in R} & \mapsto & \sum\limits_{i=1}^{n} a_i \cdot \underbrace{(r_i \cdot s_i)}_{\in R} \end{array}$$

as the *multiplication induced map.*

Now we can formulate the relationship between the two cup products.

Lemma 74.7. *Let X be a topological space. If R is a commutative ring, then we have the following commutative diagram*

$$\begin{tikzcd} & H^p(X;R)\times H^q(X;R) \arrow[dl, "\cup_\otimes"'] \arrow[dr, "\cup"] & \\ H^{p+q}(X;R\otimes R) \arrow[rr, "\mu_*"] & & H^{p+q}(X;R) \end{tikzcd}$$

where $\mu\colon R\otimes R\to R$ denotes the multiplication induced map.

The proof of the following proposition is almost identical to the proof of the earlier Proposition 57.8.

Proposition 74.8. *Let X be a topological space and let A and B be abelian groups. Furthermore let $\varphi\in H^p(X;A)$ and $\psi\in H^q(X;B)$. If we denote by $\Theta\colon A\otimes B\to B\otimes A$ the natural isomorphism induced by $a\otimes b\mapsto b\otimes a$, see Lemma 42.3* (1)*, then*

$$\Theta_*\big(\underbrace{a\cup_\otimes b}_{\in H^{p+q}(X;A\otimes B)}\big) \;=\; (-1)^{p\cdot q}\;\cdot\;\underbrace{b\cup_\otimes a.}_{\in H^{p+q}(X;B\otimes A)}$$

We also need a barely visible variation on the cap product.

Definition. Let X be a topological space, let U be a subset of X and let G be an abelian group. Furthermore let $k\le l$. We define the *cap product*

$$\begin{array}{rcl} C^k(X;G)\times C_l(X,U) & \to & C_{l-k}(X,U;G) = C_{l-k}(X,U)\otimes G \\ (\varphi,[\sigma\colon\Delta^l\to X]) & \mapsto & \varphi\cap\sigma := \underbrace{\sigma\circ[v_k,\dots,v_l]}_{\in C_{l-k}(X,U)}\otimes\underbrace{\varphi(\sigma\circ[v_0,\dots,v_k])}_{\in G}. \end{array}$$

For $k>l$ we define the cap product to be the zero map.

The following lemma summarizes the key properties of the more general cap product.

Lemma 74.9. *Let X be a topological space, let U be a subset of X, let G be an abelian group and let $k,l\in\mathbb{N}_0$.*

[1241]Note that this is indeed a homomorphism of abelian groups.

[1242]Here on the right-hand side we implicitly use that any abelian group, in particular R, is a $\mathbb{Z}$-module.

(1) *The map*

$$\begin{array}{rcl} H^k(X;G) \times H_l(X,U;\mathbb{Z}) & \to & H_{l-k}(X,U;G) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi] \cap [\sigma] \ := \ [\varphi \cap \sigma] \end{array}$$

is well-defined.

(2) *For any $\varphi \in H^k(X;G)$ and any $\sigma \in H_k(X;\mathbb{Z})$ we have*

$$\epsilon_X(\varphi \cap \sigma) \ = \ \langle \varphi, \sigma \rangle \ \in \ A,$$

where $\epsilon_X \colon H_0(X;G) \to G$ is the augmentation map and $\langle\ ,\ \rangle$ denotes the Kronecker pairing that we introduced in Lemma 50.5.

(3) *If $\rho \colon G \to B$ is a homomorphism of abelian groups, then for any $\varphi \in H^k(X;G)$ and any $\sigma \in H_l(X,U;\mathbb{Z})$ we have*

$$\rho_*(\varphi) \cap \sigma \ = \ \rho_*(\varphi \cap \sigma) \ \in \ H_{l-k}(X,U;B).$$

(4) *If $f \colon (X,U) \to (Y,V)$ is a map between pairs of topological spaces, then for any $\varphi \in H^k(Y;G)$ and any $\sigma \in H_l(X,U;\mathbb{Z})$ we have*

$$f_*(f^*(\varphi) \cap \sigma) \ = \ \varphi \cap f_*(\sigma) \ \in \ H_{l-k}(Y,V;G).$$

Note that here f^* denotes the induced map $f^* \colon H^k(Y;G) \to H^k(X;G)$ on absolute cohomology, whereas f_* denotes the induced maps $H_l(X,U;\mathbb{Z}) \to H_l(Y,V;\mathbb{Z})$ and $H_{l-k}(X,U;A) \to H_{l-k}(Y,V;A)$ on relative homology groups.

(5) *Let $\varphi \in H^k(X;\mathbb{Z})$, $\psi \in H^l(X;G)$ and furthermore let $\sigma \in H_n(X;\mathbb{Z})$. We denote by $c \colon G \otimes \mathbb{Z} \to G$ and $c \colon \mathbb{Z} \otimes G \to G$ the obvious isomorphisms given by Lemma 42.3. Then*

$$\varphi \cap (\psi \cap \sigma) \ = \ \underbrace{c_*(\overbrace{\psi \cup_\otimes \varphi}^{\in H^{k+l}(X;G\otimes\mathbb{Z})})}_{\in H^{k+l}(X;G)} \cap \sigma \ = \ (-1)^{kl} \cdot \underbrace{c_*(\overbrace{\varphi \cup_\otimes \psi}^{\in H^{k+l}(X;\mathbb{Z}\otimes G)})}_{\in H^{k+l}(X;G)} \cap \sigma \ \in \ H_{n-k-l}(X;G).$$

Proof. The proof of the first statement is entirely analogous to the proof of Lemma 59.2 or Lemma 43.6. The second statement and the third statement are an immediate consequence of the definitions. The fourth statement is a mild generalization of Lemma 59.8 and it is also a straightforward consequence of the definitions. Furthermore the first equality of the last statement is an equally mild generalization of Lemma 59.7. Finally the last equality of the last statement is a consequence of Proposition 74.8. □

Later on we will also need the following slight variation of the Poincaré Duality Theorem 61.1. The proof is verbatim the same as the proof of the Poincaré Duality Theorem 61.1.

Proposition 74.10. *Let M be a compact oriented non-empty n-dimensional topological manifold and let G be an abelian group. For each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} H^k(M;G) & \xrightarrow{\cong} & H_{n-k}(M,\partial M;G) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

The above proposition leads us to the following definition.

Definition. Let M be a closed oriented non-empty n-dimensional topological manifold and let G be an abelian group. We denote by

$$\mathrm{PD}_M^G\colon H_k(M;G) \xrightarrow{\cong} H^{n-k}(M;G)$$

the inverse of the isomorphism of the Poincaré Duality isomorphism given by Proposition 74.10.

The following lemma is an immediate consequence of the relevant definitions together with Lemma 74.9 (3).

Lemma 74.11. *Let M be a closed oriented non-empty n-dimensional topological manifold. If $\rho\colon G \to H$ is a homomorphism of abelian groups, then for every $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} H_k(M;G) & \xrightarrow{\mathrm{PD}_M^G} & H^{n-k}(M;G) \\ {\scriptstyle\rho_*}\downarrow & & \downarrow{\scriptstyle\rho_*} \\ H_k(M;H) & \xrightarrow{\mathrm{PD}_M^H} & H^{n-k}(M;H). \end{array}$$

74.4. **The linking pairing on odd-dimensional rational homology spheres.** Now we return to the study of rational homology spheres. We start out with the following elementary lemma.

Lemma 74.12. *Let $n \geq 1$. For every $(2n+1)$-dimensional rational homology sphere M the Bockstein homomorphism $\beta\colon H^n(M;\mathbb{Q}/\mathbb{Z}) \to H^{n+1}(M;\mathbb{Z})$ corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an isomorphism.*

Proof. It follows from our hypothesis together with Proposition 51.18 (or alternatively together with the Poincaré Duality Theorem 61.3), that for the rational homology sphere M we have $H^n(M;\mathbb{Q}) = H^{n+1}(M;\mathbb{Q}) = 0$. Thus it follows from the Bockstein sequence in cohomology that the Bockstein homomorphism $H^n(M;\mathbb{Q}/\mathbb{Z}) \to H^{n+1}(M;\mathbb{Z})$ is an isomorphism. □

We continue with the following purely algebraic definition.

Definition. A *linking pairing* on an abelian group H is a pairing that takes values in $\mathbb{Q}/\mathbb{Z}$, i.e. it is a bilinear map $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$.

Example. Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. The map

$$\begin{array}{rcl} \lambda\colon \mathbb{Z}_n \times \mathbb{Z}_n & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \frac{k}{n} \cdot a \cdot b \end{array}$$

is a symmetric linking pairing. It is an amusing exercise to show that this linking pairing is non-singular, in the sense of the definition on page 1447, if and only if k is coprime to n.

The following lemma is a variation of Lemma 62.1. It gives another rich source of linking pairings.

Lemma 74.13. *If $A \in M(n \times n, \mathbb{Z})$ is a symmetric matrix with* $\det(A) \neq 0$*, then the map*

$$\begin{array}{rcl} \lambda(A)\colon \mathbb{Z}^n/A\mathbb{Z}^n \times \mathbb{Z}^n/A\mathbb{Z}^n & \to & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \mapsto & v^T A^{-1} w \end{array}$$

is well-defined and it defines a non-singular symmetric linking pairing.

Since we will not make use of Lemma 74.13 we will only provide a sketch of a proof.

Sketch of proof. It follows easily from the symmetry of A that the linking pairing is well-defined, i.e. that it is independent of the choice of the representatives v and w. Furthermore it is straightforward to see that the linking pairing is symmetric. It remains to show that the linking pairing is non-singular. By the symmetry of the linking pairing it suffices to show that the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^n/A\mathbb{Z}^n & \to & \mathrm{Hom}(\mathbb{Z}^n/A\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \\ [v] & \mapsto & ([w] \mapsto v^T A^{-1} w) \end{array}$$

is an isomorphism. We make the following three observations:

(1) From Lemma 11.7 and $\det(A) \neq 0$ we know that $\mathbb{Z}^n/A\mathbb{Z}^n$ is a torsion-group.
(2) It follows fairly easily from the classification of finitely generated abelian groups, see Theorem 11.4, that for any finite abelian group H the groups $\mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ and H have the same number of elements.
(3) By (1) and (2) the map Φ is a homomorphism between finite abelian groups with the same number of elements. Thus it suffices to show that Φ is a monomorphism.

Now let $v \in \mathbb{Z}^n$ such that $\Phi([v]) = 0$. This implies that for the i-th standard basis vector e_i the product $v^T A^{-1} e_i =: \mu_i$ lies in $\mathbb{Z}$. But this means that $v^T A^{-1} = (\mu_1 \ \dots \ \mu_n) \in \mathbb{Z}^n$, i.e. using the symmetry of A, we obtain that

$$v \ = \ A^T(\mu_1 \ \dots, \mu_n)^T \ = \ A(\mu_1 \ \dots \ \mu_n)^T \ \in \ A \cdot \mathbb{Z}^n. \qquad \square$$

Before we continue we fix some notation that we will use throughout the remainder of this chapter.

Notation. Let S be a commutative ring and let R be a subring of S. We adopt the following notations:

(1) We denote by $i\colon R \to S$ the inclusion map and we denote by $p\colon S \to R/S$ the projection map.
(2) Given a topological space X we denote by β the Bockstein homomorphism corresponding to the short exact sequence $0 \to R \xrightarrow{i} S \xrightarrow{p} S/R \to 0$.
(3) Similar to the discussion on page 1716 we can consider the multiplication induced homomorphisms

$$\begin{array}{rcl} R \otimes S/R & \to & S/R \\ \sum\limits_{i=1}^{k} r_i \otimes [s_i] & \mapsto & \sum\limits_{i=1}^{k} [r_i \cdot s_i] \end{array} \quad \text{and} \quad \begin{array}{rcl} S/R \otimes R & \to & S/R \\ \sum\limits_{i=1}^{k} [s_i] \otimes r_i & \mapsto & \sum\limits_{i=1}^{k} [s_i \cdot r_i]. \end{array}$$

We denote both homomorphisms by ν.

The following definition can be viewed as an odd-dimensional analogue of the definition of the intersection form on a compact oriented even-dimensional manifold that we gave on page 1652.

Definition. Let M be an oriented rational homology sphere of dimension $2n+1$ with $n \geq 1$. We refer to the map

$$\begin{array}{rcl} \lambda_M \colon H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \nu_*\big(\underbrace{\overbrace{\beta^{-1}(\mathrm{PD}(a))}^{\in H^{n+1}(M;\mathbb{Q}/\mathbb{Z})} \cup_\otimes \overbrace{\mathrm{PD}(b)}^{\in H^{n+1}(M;\mathbb{Z})}}_{\in H^{2n+1}(M;\mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z})}\big), [M]\Big\rangle \end{array}$$

as the *linking pairing* λ_M *of* M.

Remark. With very minor efforts one can generalize the previous definition. More precisely, let M be a closed oriented $(2n+1)$-dimensional manifold. We denote by $TH_n(M;\mathbb{Z})$ the torsion subgroup of $H_n(M;\mathbb{Z})$. In an almost identical fashion to the above one can define a pairing

$$TH_n(M;\mathbb{Z}) \times TH_n(M;\mathbb{Z}) \ \to\ \mathbb{Q}/\mathbb{Z}.$$

Furthermore, with similar proofs one can generalize the subsequent Propositions 74.16 and Proposition 74.17 to this more general setup. But since we will not make use of this generalization we leave it to the reader to fill in the details.

Lemma 74.14. *Let $n \geq 1$ and let $f \colon M \to N$ be a homotopy equivalence between oriented $(2n+1)$-dimensional rational homology spheres. By Proposition 30.5 there exists an $\epsilon \in \{-1,1\}$ with $f_*([M]) = \epsilon \cdot [N] \in H_{2n+1}(N;\mathbb{Z})$. Then for any $x, y \in H_n(M;\mathbb{Z})$ the following equality holds:*

$$\lambda_N(f_*(x), f_*(y)) \ = \ \epsilon \cdot \lambda_M(x,y).$$

Proof. In the proof of Lemma 71.7 we had implicitly proved the following statement.[1243]

Statement 74.15. *For any $c \in H_k(M;\mathbb{Z})$ we have*

$$f^*(\mathrm{PD}_N^{\mathbb{Z}}(f_*(c))) \ = \ \varepsilon \cdot \mathrm{PD}_M^{\mathbb{Z}}(c) \in H^{2n+1-k}(M;\mathbb{Z}).$$

[1243]Strictly speaking, in the proof of Lemma 71.7 the map f was a homeomorphism, but that makes no difference.

Now we can prove the desired equality. More precisely, we have the following equalities in $\mathbb{Q}/\mathbb{Z}$:

$$\begin{aligned}
&\text{by definition of the linking pairing } \lambda_N \\
&\quad\downarrow \\
\lambda_M(x,y) &= \big\langle \beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(x)) \cup_{\otimes} \mathrm{PD}_M^{\mathbb{Z}}(y), [M]\big\rangle \\
&= \big\langle f^*(\beta^{-1}(\mathrm{PD}_N^{\mathbb{Z}}(f_*(x)))) \cup_{\otimes} f^*(\mathrm{PD}_N^{\mathbb{Z}}(f_*(y))), \epsilon\cdot(f_*)^{-1}([N])\big\rangle \\
&\quad\uparrow \\
&\text{by the above statement, the convenient truth that } \epsilon^2 = 1 \text{ and the definition of } \epsilon \\
&\text{we also used the naturality of } \beta\text{, which is a consequence of Proposition 48.6 (2)} \\
&= \big\langle \beta^{-1}(\mathrm{PD}_N^{\mathbb{Z}}(f_*(x))) \cup_{\otimes} \mathrm{PD}_M^{\mathbb{Z}}(f_*(y)), \epsilon\cdot[N]\big\rangle \;=\; \epsilon\cdot\lambda_N(f_*(x), f_*(y)). \\
&\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
&\text{by Lemma 50.6 and Lemma 74.6 (2)} \qquad\qquad\qquad\qquad \text{definition of } \lambda_N
\end{aligned}$$

□

In Proposition 71.5 we saw that the intersection form of a closed oriented even-dimensional topological manifold is (anti-) symmetric and non-singular. The following two propositions give us similar statements for linking pairings.

Proposition 74.16. *Let $n \geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. If n is odd, then the linking pairing λ_M on $H_n(M;\mathbb{Z})$ is symmetric otherwise it is anti-symmetric.*

Remark. Linking forms were initially introduced in 1933 by Herbert Seifert [Sei, p. 814] for oriented 3-dimensional rational homology spheres. He gave a somewhat informal argument why the linking pairing is symmetric. The fact that linking pairings are (anti-) symmetric is well-known and implicitly or implicitly used many times. But published proofs of this statement are few and far between. The subsequent proof of Proposition 74.16 first appeared in [CFH].

Proposition 74.17. *Let $n \geq 1$. The linking pairing λ_M of any oriented $(2n+1)$-dimensional rational homology sphere is non-singular.*

In these modest lecture notes we will not make use of Propositions 74.16 and 74.17. Since the proofs are somewhat technical we hide them in the next section.

74.5. **Proofs of Propositions 74.16 and 74.17.** We will see that Proposition 74.16 is a fairly straightforward consequence of the next lemma.

Lemma 74.18. *Let S be a commutative ring and let R be a subring of S. Let X be a topological space. For any $x \in H^k(X;S/R)$ and any $y \in H^l(X;S/R)$ we have*

$$\nu_*\big(\underbrace{\beta(x) \cup_{\otimes} y}_{\in H^{k+l+1}(X;R\otimes S/R)}\big) \;=\; (-1)^{k+1}\cdot\nu_*\big(\underbrace{x \cup_{\otimes} \beta(y)}_{\in H^{k+l+1}(X;S/R\otimes R)}\big) \;\in\; H^{k+l+1}(X;S/R).$$

Proof of Proposition 74.16 assuming Lemma 74.18. Let $n \geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. Furthermore let $a, b \in H_n(M;\mathbb{Z})$. We write

$x = \beta^{-1}(\mathrm{PD}(a)) \in H^n(M;\mathbb{Q}/\mathbb{Z})$ and $y = \beta^{-1}(\mathrm{PD}(b)) \in H^n(M;\mathbb{Q}/\mathbb{Z})$. Then

$$\begin{aligned}
\lambda_M(a,b) &\overset{\text{by definition of } \lambda_M}{=} \langle \nu_*(x \cup_\otimes \beta(y)), [M]\rangle \overset{\text{by Lemma 74.18}}{=} (-1)^{n+1}\cdot\langle \nu_*(\beta(x) \cup_\otimes y), [M]\rangle \\
&\underset{\text{by Proposition 74.8 since } n(n+1) \text{ is even}}{=} (-1)^{n+1}\cdot\langle \nu_*(y \cup_\otimes \beta(x)), [M]\rangle \underset{\text{by definition of } \lambda_M}{=} (-1)^{n+1}\cdot\lambda_M(b,a).
\end{aligned}$$

□

Proof of Lemma 74.18. First recall that we have a short exact sequence of cochain complexes:

$$0 \to C^*(X;R) \xrightarrow{i_*} C^*(X;S) \xrightarrow{p_*} C^*(X;S/R) \to 0.$$

In particular we can and will pick $f \in C^k(X;S)$ and $g \in C^l(X;S)$ such that $[p_*(f)] = x$ and such that $[p_*(g)] = y$. Next recall that the Bockstein homomorphisms in cohomology are the connecting homomorphisms corresponding to the above short exact sequence of cochain complexes. It follows immediately from the definition of the connecting homomorphism, see page 1160 and Proposition 48.6, that there exist unique cocycles $\beta(f) \in C^{k+1}(X;R)$ and $\beta(g) \in C^{l+1}(X;R)$ such that $i_*(\beta(f)) = \delta^k(f) \in C^{k+1}(X;S)$ and also such that $i_*(\beta(g)) = \delta^l(g) \in C^{l+1}(X;S)$. We have the following equality in $H^{k+l+1}(X;S\otimes S)$:

$$(*) \qquad \begin{aligned}
0 &\underset{\substack{\text{by definition}\\ \text{of cohomology}}}{=} [\delta(f \cup_\otimes g)] \overset{\text{see Lemma 74.4}}{=} [\delta(f) \cup_\otimes g + (-1)^k \cdot f \cup_\otimes \delta(g)] \\
&= \big[\underbrace{i_*(\beta(f)) \cup_\otimes g}_{\in C^{k+l+1}(X;R\otimes S)} + (-1)^k \cdot \underbrace{f \cup_\otimes i_*(\beta(g))}_{\in C^{k+l+1}(X;S\otimes R)}\big].
\end{aligned}$$

We consider the following diagram of group homomorphisms

$$\begin{tikzcd}
R\otimes S \arrow[r, "i\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes p"'] & S\otimes S \arrow[d, "\mu"] & S\otimes R \arrow[l, "\mathrm{id}\otimes i"'] \arrow[d, "p\otimes\mathrm{id}"] \\
R\otimes S/R \arrow[dr, "\nu"] & S \arrow[d, "p"] & S/R\otimes R \arrow[dl, "\nu"'] \\
& S/R &
\end{tikzcd}$$

where, as usual, ν and μ stand for the obvious maps that are induced by multiplication in the ring S. It is straightforward to see that the diagram commutes. We have the following

equalities in $H^{k+l+1}(X;S/R)$:

$$\begin{aligned}
&\text{by the above equality } (*)\\
&\quad\downarrow\\
0 &= (p\circ\mu)_*\big(\big[i_*(\beta(f))\cup_\otimes g \;+\; (-1)^k\cdot f\cup_\otimes i_*(\beta(g))\big]\big)\\
&= \big[\nu_*\big(\beta(f)\cup_\otimes p_*(g)\big) \;+\; (-1)^k\cdot\nu_*\big(p_*(f)\cup_\otimes\beta(g)\big)\big]\\
&\quad\uparrow\\
&\text{by the above commutative diagram of group homomorphisms and some basic observations}\\
&\quad\text{regarding the functoriality of the cup product in the coefficients}\\
&= \nu_*\big([\beta(f)]\cup_\otimes[p_*(g)]\big) \;+\; (-1)^k\cdot\nu_*\big([p_*(f)]\cup_\otimes[\beta(g)]\big)\\
&\quad\uparrow\\
&\text{by definition of } \cup_\otimes \text{ and since } \beta(f), p_*(g), p_*(f) \text{ and } \beta(g) \text{ are cocycles}\\
&= \nu_*(\beta(x)\cup_\otimes y) \;+\; (-1)^k\cdot\nu_*(x\cup_\otimes\beta(y)).\\
&\quad\uparrow\\
&\text{since } [\beta(f)]=\beta([p_*(f)])=:\beta(x) \text{ and } [p_*(g)]=y \text{ and similar for the second summand}
\end{aligned}$$

This of course implies the desired statement. □

Before we give the proof of Proposition 74.17 we prove the following lemma which gives a slight reformulation of the linking pairing.

Lemma 74.19. *Let $n\geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. For any choice of $a,b\in H_n(M;\mathbb{Z})$ we have*

$$\lambda_M(a,b) \;=\; \epsilon_M\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))\cap b\big) \;=\; \big\langle\beta^{-1}(\overbrace{\mathrm{PD}_M^{\mathbb{Z}}(a)}^{\in H^{n+1}(M;\mathbb{Z})}),b\big\rangle.$$

Proof. Let $a,b\in H_n(M;\mathbb{Z})$. We have the following equalities in $\mathbb{Q}/\mathbb{Z}$:

$$\begin{aligned}
&\text{by definition and Lemma 74.9 (2)}\\
&\quad\downarrow\\
\lambda_M(a,b) &= \epsilon_M\big(\overbrace{\nu_*\big(\underbrace{\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))}_{\in H^{n+1}(M;\mathbb{Q}/\mathbb{Z})}\cup_\otimes\underbrace{\mathrm{PD}_M^{\mathbb{Z}}(b)}_{\in H^n(M;\mathbb{Z})}\big)}^{\in H^{2n+1}(M;\mathbb{Q}/\mathbb{Z})}\cap[M]\big)\\
&= \epsilon_M\big(\underbrace{\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))}_{\in H^n(M;\mathbb{Q}/\mathbb{Z})}\cap\underbrace{(\mathrm{PD}_M^{\mathbb{Z}}(b)\cap[M])}_{\in H_n(M;\mathbb{Z})}\big) \qquad = \epsilon_M\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))\cap b\big).\\
&\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow\\
&\text{by Lemma 74.9 (5) and the convenient fact} \qquad \text{definition of } \mathrm{PD}_M^{\mathbb{Z}}(b)\\
&\quad\text{that } n(n+1) \text{ is always even}\\
&= \big\langle\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a)),b\big\rangle.\\
&\quad\uparrow\\
&\text{Lemma 74.9 (2)}
\end{aligned}$$

We have thus verified both desired equalities. □

We conclude this technical section with the proof of Proposition 74.17.

Proof of Proposition 74.17. Let M be an oriented $(2n+1)$-dimensional rational homology sphere. We need to show that the linking pairing $\lambda_M\colon H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is non-singular. To do so we consider the map

$$\Phi_M\colon H_n(M;\mathbb{Z}) \xrightarrow{\mathrm{PD}_M^{\mathbb{Z}}} H^{n+1}(M;\mathbb{Z}) \xrightarrow[\cong]{\beta^{-1}} H^n(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(H_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}).$$

Here the first map is given by Poincaré Duality, the second map is the inverse of the Bockstein isomorphism given by Lemma 74.12 and the third map is given by the evaluation map.

In Proposition 74.16 we showed that the linking pairing is (anti-) symmetric. Therefore it suffices to prove that the map

$$\begin{array}{rcl} H_n(M;\mathbb{Z}) & \to & \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\ a & \mapsto & (b \mapsto \lambda_M(a,b)) \end{array}$$

is an isomorphism. But this in turn follows immediately from the following claim.

Claim.

(1) The map Φ_M is an isomorphism.
(2) For any $a, b \in H_n(M;\mathbb{Z})$ we have $\lambda_M(a,b) = \Phi_M(a)(b)$.

First note that the second statement is an immediate consequence of Lemma 74.19. For the first statement it remains to show that the third map in the definition of Φ_M is an isomorphism. Recall that by the Universal Coefficient Theorem 51.13 for Cohomology Groups we have a short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \to H^n(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \to 0.$$

On page 1217 we remarked that the abelian group $\mathbb{Q}/\mathbb{Z}$ is divisible. By Proposition 51.6 this implies that the Ext-term to the left is zero. But this shows that the evaluation homomorphism to the right is an isomorphism. This concludes the proof of the claim. □

74.6. **Linking forms of lens spaces.** In the next section we will prove the following proposition.

Proposition 74.20. *Let $p, q \in \mathbb{N}$ be coprime. The linking pairing of the lens space $L(p,q)$ is isometric to*[1244][1245]

$$\begin{array}{rcl} \mathbb{Z}_p \times \mathbb{Z}_p & \to & \mathbb{Q}/\mathbb{Z} \\ (x,y) & \mapsto & -\frac{q}{p}\cdot x \cdot y \mod \mathbb{Z}. \end{array}$$

[1244]Recall that we have $L(p,q) = S^3/\mathbb{Z}_p$ and, that following the convention established on page 266, we equip $L(p,q)$ with the obvious orientation coming from S^3.

[1245]By now I am fairly confident that the sign in the calculation of the linking pairing of the lens space $L(p,q)$ is correct. But the fact that, in my attempts to calculate the linking pairing, the sign flipped on numerous occasions means that one should take the result with a grain of salt.

Remark. There are many different approaches to calculating the linking pairing of lens spaces. The first calculation was given by Seifert [Sei, p. 824], see also [ST, p. 290] and [ST, p. 279]. Alternative calculations are given in [Hat, p. 306], [Mun, Chapter 69] and [Bre, p. 364]. The proof of Proposition 74.20 that we give is rather different from the previous proofs. The approach taken below has the advantage that it can also be used to determine the linking pairing of any oriented 3-dimensional rational homology sphere in terms of a Heegaard splitting. We refer to [CFH] for details.

Before we turn to the somewhat lengthy, and not overly enjoyable proof, we want to use this section to state and prove several corollaries of Proposition 74.20.

Corollary 74.21. *Let $L(p,q)$ and $L(p,\widetilde{q})$ be lens spaces. Suppose there exists a homotopy equivalence $f\colon L(p,q) \to (p,\widetilde{q})$. By Proposition 30.5 there exists an $\epsilon \in \{-1,1\}$ such that $f_*([L(p,q)]) = \epsilon \cdot [L(p,\widetilde{q})]$. Then there exists $r \in \mathbb{Z}$, coprime to p, such that*

$$q \;=\; \epsilon \cdot r^2 \cdot \widetilde{q} \bmod p.$$

Proof. We pick isomorphisms $\varphi\colon \mathbb{Z}_p \to H_1(L(p,q);\mathbb{Z})$ and $\widetilde{\varphi}\colon \mathbb{Z}_p \to H_1(L(p,\widetilde{q});\mathbb{Z})$ as provided by Proposition 74.20. The map $\widetilde{\varphi}^{-1} \circ f_* \circ \varphi\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is an isomorphism, hence it is given by multiplication by some $r \in \mathbb{Z}$ that is coprime to p. Together with Lemma 74.14 we see that the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}_p \times \mathbb{Z}_p \arrow[r, "\varphi\times\varphi"] \arrow[rr, bend left, "(x,y)\mapsto -\frac{q}{p}\cdot x\cdot y"] \arrow[d, "(x,y)\mapsto(rx,ry)"] & H_1(L(p,q);\mathbb{Z}) \times H_1(L(p,q);\mathbb{Z}) \arrow[r, "\lambda_{L(p,q)}"] \arrow[d, "f_*\times f_*"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot(-1)"] \\
\mathbb{Z}_p \times \mathbb{Z}_p \arrow[r, "\widetilde{\varphi}\times\widetilde{\varphi}"'] \arrow[rr, bend right, "(x,y)\mapsto -\frac{\widetilde{q}}{p}\cdot x\cdot y"'] & H_1(L(p,\widetilde{q});\mathbb{Z}) \times H_1(L(p,\widetilde{q});\mathbb{Z}) \arrow[r, "\lambda_{L(p,\widetilde{q})}"] & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

If we start out with $(1,1) \in \mathbb{Z}_p \times \mathbb{Z}_p$ on the upper left corner and we compare the different routes to the lower right corner we obtain that $-\epsilon \cdot \frac{q}{p} = r^2 \cdot \frac{\widetilde{q}}{p}$. Multiplying by p we obtain the desired equality. □

Now we can give the classification of lens spaces up to homotopy equivalences:

Theorem 74.22. *Let $p \in \mathbb{N}$. If $q, \widetilde{q} \in \mathbb{N}$ are coprime to p, then*

the lens spaces $L(p,q)$ and $L(p,\widetilde{q})$ are homotopy equivalent	$\Longleftrightarrow$	*there exists an $r \in \mathbb{Z}$, coprime to p, such that $q = \pm r^2 \cdot \widetilde{q}^{\pm 1} \bmod p$.*

Proof. The "⇒"-direction is precisely the statement of Corollary 74.21. The "⇐"-direction was shown by Whitehead [Whd2, Theorem 10] (see also [Cohe, Chapter 29] or [DK, Theorem 11.1]), by writing down explicit homotopy equivalences. □

So, after a very long wait we have now finally made some serious progress towards answering Question 7.8. For example, one can easily verify, using Theorem 74.22 that the lens spaces $L(5,1)$ and $L(5,2)$ are not homotopy equivalent. Nonetheless, we have not yet completely answered Question 7.8. For example the lens spaces $L(7,1)$ and $L(7,4)$ are homotopy equivalent by Theorem 74.22. But it is not clear whether these two lens spaces are homeomorphic. We will look into this problem in Algebraic Topology IV.5.

Remark. For those who might get tired of chasing down lens spaces, I would like to quote from John F. Kennedy's speech at Rice University on September 12 1962, where he announced that the United States would attempt to land a man on the moon by the end of the decade:

> *But why, some say, the Moon? Why choose this as our goal? And they may well ask, why climb the highest mountain? Why, 35 years ago, fly the Atlantic? Why does Rice play Texas?*
>
> *We choose to go to the Moon! We choose to go to the Moon in this decade and do the other things, not because they are easy, but because they are hard; because that goal will serve to organize and measure the best of our energies and skills, because that challenge is one that we are willing to accept, one we are unwilling to postpone, and one we intend to win.*

John F. Kennedy speaking at Rice University in September 1962

Figure 928.

In Question 47.8 we had asked whether the lens space $L(3,1)$ is amphichiral, i.e. we had asked $L(3,1)$ admits an orientation-reversing self-homeomorphism.

Now we have the tools to address this question.

Proposition 74.23. *If there exists an* orientation-reversing *homeomorphism between two lens spaces $L(p,q)$ and $L(p,\widetilde{q})$, then there exists an $r \in \mathbb{Z}$ such that $q = -r^2 \cdot \widetilde{q}^{\pm 1} \bmod p$.*

Example. Note that $1 \neq -r^2 \bmod 3$ for any $r \in \mathbb{Z}$. Thus it follows from Proposition 74.23 that there is no orientation-reversing homeomorphism from the lens space $L(3,1)$ to itself, i.e. the lens space $L(3,1)$ is chiral. This now answers Question 47.8.

Proof. If $f\colon L(p,q) \to L(p,\widetilde{q})$ is an orientation-reversing homeomorphism, then we have $f_*([L(p,q)]) = -[L(p,\widetilde{q})]$, i.e. we have $\epsilon = -1$ in the statement of Corollary 74.21. The proposition now follows immediately from Corollary 74.21. $\square$

74.7. **The proof of Proposition 74.20.** This section is almost entirely dedicated to the proof of Proposition 74.20. Unless the reader suffers from an insatiable appetite for cup and cap products, it is recommended to skip this section.

We start out with the following lemma that relates the Bockstein homomorphisms in homology and cohomology via the Poincaré duality

Lemma 74.24. *Let M be a closed oriented m-dimensional topological manifold. If we are given a short exact sequence $0 \to P \xrightarrow{i} Q \xrightarrow{p} R \to 0$ of abelian groups, then for every $k \in \{0, \dots, m\}$ the two diagrams*

$$\begin{array}{ccc} H^k(M;R) & \xrightarrow{\cap[M]} & H_{m-k}(M;R) \\ \downarrow\beta & & \downarrow\beta \\ H^{k+1}(M;P) & \xrightarrow{\cap[M]} & H_{m-k-1}(M;P) \end{array} \quad \textit{and} \quad \begin{array}{ccc} H_{m-k}(M;R) & \xrightarrow{\mathrm{PD}_M^R} & H^k(M;R) \\ \downarrow\beta & & \downarrow\beta \\ H_{m-k-1}(M;P) & \xrightarrow{\mathrm{PD}_M^P} & H^{k+1}(M;P) \end{array}$$

commute up to the sign $(-1)^{k+1}$.

Proof. We pick a cycle $\sigma \in C_m(M;\mathbb{Z})$ that represents the fundamental class. Furthermore let $\psi \in C^k(M;R)$ be a cocycle. We pick $\varphi \in C^k(M;Q)$ with $p_*(\varphi) = \psi$. Then we have the following equalities in $H_{m-k-1}(M;P)$:[1246]

$$\beta([\psi]) \cap [M] \;\overset{\substack{\text{by definition of the connecting}\\ \text{homomorphism on page 1160}}}{=}\; [\delta(\varphi) \cap \sigma] \;\underset{\substack{\uparrow\\ \text{by Lemma 59.1 since } \sigma \text{ is a cycle}}}{=}\; [\partial(\varphi \cap \sigma)] \;\overset{\substack{\text{by definition of the connecting}\\ \text{homomorphism on page 770}}}{=}\; \beta([p_*(\varphi \cap \sigma)]) \;=\; \beta([\psi] \cap [M]).$$

But this implies that the left-hand diagram does indeed commute. The right-hand diagram is, by definition of the Poincaré duality isomorphisms, basically the same diagram. □

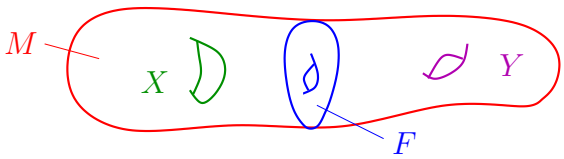
We continue with the following definition.

Definition. Let M be a 3-dimensional rational-homology sphere. Given a splitting (X, Y, F) of M, as defined on page 1695, we denote by $\Theta(X, Y, F)$ the homomorphism

$$H_1(M;\mathbb{Z}) \xrightarrow{\beta^{-1}} H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_2(M,X;\mathbb{Q}/\mathbb{Z}) \underset{\substack{\uparrow\\ \text{isomorphism by the Excision Theorem 45.8 for manifolds}}}{\leftarrow} H_2(Y,F;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H_1(F;\mathbb{Q}/\mathbb{Z}),$$

where "∂" denotes the connecting homomorphism of the long exact sequence of $\mathbb{Q}/\mathbb{Z}$-homology groups corresponding to the pair (Y, F). If the splitting (X, Y, F) is understood, then we just write Θ instead of $\Theta(X, Y, F)$.

[1246]The perspicacious reader will have noticed that the commutativity of the left-hand diagram has little to do with the hypothesis that we are capping with the fundamental class of a closed oriented topological manifold. Since we do not need the more general statements we leave the lemma as it is.



awfully schematic picture of a splitting of M

FIGURE 929.

The following lemma will allow us to compute the linking pairing of an oriented 3-dimensional rational homology sphere in terms of Poincaré duality on the surface F.

Lemma 74.25. *Let M be an oriented 3-dimensional rational homology sphere and let (X, Y, F) be a splitting of M, as defined on page 1695. We denote by $i\colon F \to M$ the inclusion map. Then the following diagram commutes:*

$$\begin{tikzcd}
H_1(M;\mathbb{Z}) \arrow[r, "\mathrm{PD}_M^{\mathbb{Z}}"] \arrow[d, "\Theta"] & H^2(M;\mathbb{Z}) \arrow[r, "\beta^{-1}"] & H^1(M;\mathbb{Q}/\mathbb{Z}) \arrow[d, "i^*"] \\
H_1(F;\mathbb{Q}/\mathbb{Z}) \arrow[rr, "\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}", "\cong"'] & & H^1(F;\mathbb{Q}/\mathbb{Z}).
\end{tikzcd}$$

Proof. First of all, we denote the inclusion map $F \to X$ by j. Furthermore we denote by $f\colon (M, \varnothing) \to (M, X)$ and $g\colon (Y, F) \to (M, X)$ the obvious maps between pairs of topological spaces. We write also $g\colon Y \to M$ for the inclusion of Y into M. We consider the following diagram:

$$\begin{tikzcd}
H_1(M;\mathbb{Z}) \arrow[dd, bend right=60, "\Theta"'] & & H^2(M;\mathbb{Z}) \arrow[ll, "\cap[M]", "\cong"'] \\
H_2(M;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\beta", "\cong"'] \arrow[d, "f_*"] & & H^1(M;\mathbb{Q}/\mathbb{Z}) \arrow[ll, "\cap[M]", "\cong"'] \arrow[u, "\beta"'] \arrow[dd, "g^*"'] \arrow[ddd, bend left=60, "i^*"] \\
H_2(M,X;\mathbb{Q}/\mathbb{Z}) & & \\
H_2(Y,F;\mathbb{Q}/\mathbb{Z}) \arrow[u, "g_*"'] \arrow[d, "\partial"] & & H^1(Y;\mathbb{Q}/\mathbb{Z}) \arrow[ll, "\cap[Y]", "\cong"'] \arrow[d, "j^*"'] \\
H_1(F;\mathbb{Q}/\mathbb{Z}) \arrow[d, "="'] & & H^1(F;\mathbb{Q}/\mathbb{Z}) \arrow[ll, "\cap[Y]", "\cong"'] \arrow[d, "="] \\
H_1(F;\mathbb{Q}/\mathbb{Z}) & & H^1(F;\mathbb{Q}/\mathbb{Z}). \arrow[ll, "\cap[F]", "\cong"']
\end{tikzcd}$$

We consider the commutativity of the four rectangles in the diagram separately.

(1) The top rectangle commutes by Lemma 74.24. (Strictly speaking Lemma 74.24 says that the diagram commutes up to the sign $(-1)^{1+1} = 1$.)

(2) We turn to the second rectangle (or equivalently the first and only pentagon). Let $\varphi \in H^1(M;\mathbb{Q}/\mathbb{Z})$. We have the following equalities in $H_2(M,X;\mathbb{Q}/\mathbb{Z})$:

$$\begin{array}{ccccccc} & & & \text{by Lemma 46.23 we have } f_*([M]) = g_*([Y]) & & & \\ & & & \downarrow & & & \\ f_*(\varphi \cap [M]) & = & \varphi \cap f_*([M]) & = & \varphi \cap g_*([Y]) & = & g_*(g^*(\varphi) \cap [Y]). \\ & & \uparrow & & & \uparrow & \\ & & \begin{array}{c}\text{Lemma 74.9 (4) applied to} \\ f\colon (M,\varnothing) \to (M,X) \text{ where we use} \\ \text{that } f^* = \mathrm{id}\colon H^1(M;\mathbb{Q}/\mathbb{Z}) \to H^1(M;\mathbb{Q}/\mathbb{Z})\end{array} & & & \begin{array}{c}\text{Lemma 74.9 (4) applied} \\ \text{to } g\colon (Y,F) \to (M,X)\end{array} & \end{array}$$

(3) The third rectangle anti-commutes[1247] by the obvious extension of Proposition 61.22 from $\mathbb{Z}$-coefficients to $\mathbb{Q}/\mathbb{Z}$-coefficients.
(4) It follows from our orientation convention from page 1695 that $F = -\partial Y$ as oriented manifolds. By the discussion on page 1139 this implies $[F] = -\partial[Y]$. We obtain from this observation that the bottom rectangle anti-commutes.

Summarizing we see that the two approaches of going from $H^2(M;\mathbb{Z})$ to $H_1(F;\mathbb{Q}/\mathbb{Z})$ agree. The lemma follows from the observation, that by definition PD_M is the inverse of the top horizontal map and that similarly $\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}$ is the inverse of the bottom horizontal map. □

Now we turn to lens spaces. In the following we will use the description of lens spaces that we had obtained in Lemma 73.3. At this point it is convenient to recall and introduce some notation:

Notation.

(1) We write $F = S^1 \times S^1$. We write $a = [S^1 \times 1]$ and $b = [1 \times S^1]$ and we use this ordered basis of $H_1(F;\mathbb{Z})$ to make the usual identification $H_1(F;\mathbb{Z}) = \mathbb{Z}^2$. By a slight abuse of notation we also denote by a and b the corresponding elements in $H_1(F;\mathbb{Q}) = H_1(F;\mathbb{Z}) \otimes \mathbb{Q}$.
(2) Let $p, q \in \mathbb{N}$ be coprime. We pick $r, s \in \mathbb{N}$ such that $ps - qr = 1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \text{and we consider the diffeomorphism} \quad \begin{array}{rcl} \varphi\colon F = \mathbb{R}^2/\mathbb{Z}^2 & \to & F = \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

By Lemma 38.7 (2) the induced map φ_* on the homology group $H_1(F;\mathbb{Z}) = \mathbb{Z}^2$ is given by multiplication by the matrix A.
(3) We write $X = S^1 \times \overline{B}^2$ and we denote by Y another disjoint copy of the solid torus $S^1 \times \overline{B}^2$. We equip X and Y with the usual product orientation. In Lemma 73.3 we gave an explicit orientation-preserving diffeomorphism

$$L(p,q) \xrightarrow{\cong} (X \sqcup Y)/x \sim \varphi(x) \quad \text{for } x \in F = \partial X$$

[1247]Here by "anti-commutes" we mean that the left-down path agrees with the down-left path except for a minus sign.

that we will use to identify $L(p,q)$ with the topological space to the right. By an abuse of notation we denote the images of the X and Y in $L(p,q)$ again by X and Y. Note that (X,Y,F) is a Heegaard splitting for $L(p,q)$.

(4) As usual, for want of a better notation, we denote by i the various inclusion maps $F \to X$, $X \to L(p,q)$ and $F \to L(p,q)$.
(5) We denote by $\rho\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the obvious projection map.
(6) Finally we denote by $\Theta\colon H_1(L(p,q);\mathbb{Z}) \to H_1(F;\mathbb{Q}/\mathbb{Z})$ the map that we introduced on page 1727.

Lemma 74.26. *The following statements hold:*

(1) $i_*(a) \in H_1(L(p,q);\mathbb{Z})$ *is a generator,*
(2) *the equality* $\Theta(i_*(a)) = \rho_*(a - \frac{q}{p}\cdot b)$ *holds in* $H_1(F;\mathbb{Q}/\mathbb{Z})$.

Proof. By Proposition 50.12 we can apply the Mayer–Vietoris Theorem 50.14 to the decomposition $L(p,q) = X \cup Y$ and we obtain the following long exact sequence

$$H_1(F;\mathbb{Z}) \xrightarrow[\text{-}i_*]{i_*} \begin{matrix} H_1(X;\mathbb{Z}) \\ H_1(Y;\mathbb{Z}) \end{matrix} \to H_1(L(p,q);\mathbb{Z}) \to 0.$$

Recall that we denote by $a, b \in H_1(F;\mathbb{Z}) = H_1(S^1 \times S^1;\mathbb{Z})$ the standard basis. Since $S^1 \times S^1$ is the boundary of $S^1 \times \overline{B}^2$ we denote by a also the corresponding generator of $H_1(X;\mathbb{Z}) = H_1(S^1 \times \overline{B}^2;\mathbb{Z})$. Furthermore we denote by $a' \in H_1(Y;\mathbb{Z}) = H_1(S^1 \times \overline{B}^2;\mathbb{Z})$ the obvious generator. It follows from the above discussion and conventions, in particular the above description of φ_*, that with respect to our bases the above long exact sequence Mayer–Vietoris sequence becomes

$$\begin{matrix} \mathbb{Z}\cdot a \\ \mathbb{Z}\cdot b \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ q & p \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot a \\ \mathbb{Z}\cdot a' \end{matrix} \to H_1(L(p,q);\mathbb{Z}) \to 0.$$

At this stage it is an elementary linear algebra exercise to show that the image of a in $H_1(L(p,q);\mathbb{Z})$ is a generator.[1248]

We turn to the proof of the second statement. We denote by $h\colon (L(p,q),\varnothing) \to (L(p,q),X)$ and $j\colon (Y,F) \to (L(p,q),X)$ the obvious maps of pairs of topological spaces. We recall

[1248]This can be seen as follows. We consider the map

$$\operatorname{coker}\left\{\begin{pmatrix} 1 & 0 \\ q & p \end{pmatrix}\colon \begin{matrix} \mathbb{Z}\cdot a \\ \mathbb{Z}\cdot b \end{matrix} \to \begin{matrix} \mathbb{Z}\cdot a \\ \mathbb{Z}\cdot a' \end{matrix}\right\} \to \mathbb{Z}_p$$
$$ra + sa' \mapsto qr - s.$$

It is straightforward to see that this map is well-defined. It follows from Lemma 11.7 that the group on the left has order p. Since the map is an epimorphism and since both groups have the same order, we see that the map is in fact an isomorphism. Since the image of a on the right-hand side is a generator (here we use that q is coprime to p) we see that a also represents a generator on the left.

that Θ is defined as the composition of the following maps:

$$\begin{array}{ccccccc} H_1(L(p,q);\mathbb{Z}) & \xrightarrow{\beta^{-1}} & H_2(L(p,q);\mathbb{Q}/\mathbb{Z}) & \xrightarrow{h_*} & H_2(L(p,q),X;\mathbb{Q}/\mathbb{Z}) & \xleftarrow[\cong]{j_*} & H_2(Y,F;\mathbb{Q}/\mathbb{Z}) \\ & & & & & & \downarrow \partial \\ & & & & & & H_1(F;\mathbb{Q}/\mathbb{Z}). \end{array}$$

As before we denote by $i\colon F \to L(p,q)$ the inclusion map. For clarity we recall in the following diagram all the maps that play a role in the definition of the Bockstein homomorphism $\beta\colon H_2(L(p,q);\mathbb{Q}/\mathbb{Z}) \to H_1(L(p,q);\mathbb{Z})$:

$$\begin{array}{ccccccc} & & C_2(L(p,q);\mathbb{Q}) & \xrightarrow{\rho_*} & C_2(L(p,q);\mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \partial & & & & \\ 0 \longrightarrow & C_1(L(p,q);\mathbb{Z}) & \longrightarrow & C_1(L(p,q);\mathbb{Q}). & & & \end{array}$$

Finally, we denote by $\widetilde{a}$ and $\widetilde{b}$ the obvious singular chains in F representing the homology classes a and b.

Now we turn to the actual proof of the equality $\Theta(i_*(a)) = \rho_*(a - \frac{q}{p}\cdot b)$ in $H_1(F;\mathbb{Q}/\mathbb{Z})$. We consider $a - \frac{q}{p}\cdot b \in H_1(F;\mathbb{Q})$. Note that

$$\varphi_*\big(a - \tfrac{q}{p}\cdot b\big) \underset{\substack{\uparrow \\ \text{by Lemma 38.7 and the naturality of the Universal Coefficient Theorem 42.19}}}{=} \begin{pmatrix} q & p \\ s & r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\frac{q}{p} \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix} \in H_1(F;\mathbb{Q}) = \begin{matrix} \mathbb{Q}\cdot a \\ \mathbb{Q}\cdot b \end{matrix},$$

i.e. the image of $\varphi_*\big(a - \frac{q}{p}\cdot b\big)$ in $H_1(Y;\mathbb{Q})$ is zero.

This implies that there exists a singular 2-chain $\sigma \in C_2(Y;\mathbb{Q})$ with $\partial(\sigma) = \widetilde{a} - \frac{q}{p}\cdot\widetilde{b}$. Furthermore b represents the zero element in $H_1(X;\mathbb{Z})$, hence $\widetilde{b}$ is null-homologous in X, which implies that there exists a singular 2-chain $\mu \in C_2(X;\mathbb{Z})$ with $\partial(\mu) = \widetilde{b}$. Now we calculate that

$$\beta\big(\big[\underbrace{\rho_*(\overbrace{\sigma + \tfrac{q}{p}\cdot\mu}^{\in C_2(L(p,q);\mathbb{Q})})}_{\text{cycle in } C_2(L(p,q);\mathbb{Q}/\mathbb{Z})}\big]\big) \underset{\substack{\uparrow \\ \text{definition of } \beta}}{=} \big[\partial\big(\sigma + \tfrac{q}{p}\cdot\mu\big)\big] \underset{\substack{\uparrow \\ \text{since } \partial(\sigma) = a - \frac{q}{p}\cdot b}}{=} \big[\widetilde{a} - \tfrac{q}{p}\cdot\widetilde{b} + \tfrac{q}{p}\cdot\widetilde{b}\big] = i_*(a) \in H_1(L(p,q);\mathbb{Q}/\mathbb{Z}).$$

Since $\mu \in C_2(X;\mathbb{Z})$ and since $\sigma \in C_2(Y;\mathbb{Q})$ we have the equality

$$h_*\big(\rho_*\big(\sigma + \tfrac{q}{p}\mu\big)\big) = j_*(\rho_*(\sigma)) \in C_2(L(p,q),X;\mathbb{Q}/\mathbb{Z}).$$

We conclude that we have the following equalities in $H_1(F;\mathbb{Q}/\mathbb{Z})$:

$$\begin{array}{ccccccc}
 & & & \text{since } h_*(\rho_*(\sigma+\frac{q}{p}\mu)) = j_*(\rho_*(\sigma)) & & \text{by the above} \\
 & & & \downarrow & & \downarrow \\
\Theta(i_*(a)) & = & \partial\big(j_*^{-1}\big(h_*\big(\big[\rho_*\big(\sigma+\frac{q}{p}\mu\big)\big]\big)\big)\big) & = & \partial([\rho_*(\sigma)]) & = & [\rho_*(\partial(\sigma))] \;=\; \big[\rho_*\big(\widetilde{a}-\frac{q}{p}\cdot\widetilde{b}\big)\big] \\
 & \uparrow & & & \uparrow & & \\
 & \text{since } \beta([\rho_*(\sigma+\frac{q}{p}\cdot\mu)]) = i_*(a) & & & \text{since } \rho_*(\sigma) \text{ lives in } C_2(Y;\mathbb{Q}/\mathbb{Z}) \text{ we can calculate the} & & \\
 & & & & \text{connecting homomorphism of } [\rho_*(\sigma)] \text{ and we obtain } [\rho_*(\partial(\sigma))] & & \\
 & = & \rho_*\big(a-\frac{q}{p}\cdot b\big). & & & &
\end{array}$$

□

Now it is straightforward to provide the proof of Proposition 74.20.

Proof of Proposition 74.20. Let $p,q\in\mathbb{N}$ be coprime. We use the notation F,X,Y and a,b as above. We write $L=L(p,q)$. We denote by $\rho\colon\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$ the obvious projection map. To simplify the notation we use the convention that given any abelian group A we make the usual identification $H_0(L;A)=A$ and $H_0(F;A)=A$ from page 1009.

We start out this proof with the following claim.

Claim. If $x,y\in H_1(F;\mathbb{Z})$, then

$$\mathrm{PD}_F^{\mathbb{Z}}(x)\cap y \;=\; Q_F(y,x) \;\in\; H_0(F;\mathbb{Z})=\mathbb{Z}.$$

This claim follows from a calculation which by now is standard fare for us. For completeness' sake we carry out the details. We have

$$\begin{array}{ccccc}
 & & \text{by definition of } \mathrm{PD}_F^{\mathbb{Z}} & & \text{Lemma 59.7} \\
 & & \downarrow & & \downarrow \\
\mathrm{PD}_F^{\mathbb{Z}}(x)\cap y & = & \mathrm{PD}_F^{\mathbb{Z}}(x)\cap\big(\mathrm{PD}_F^{\mathbb{Z}}(y)\cap[F]\big) & = & \big(\mathrm{PD}_F^{\mathbb{Z}}(y)\cup\mathrm{PD}_F^{\mathbb{Z}}(x)\big)\cap[F] \\
 & = & \big\langle\mathrm{PD}_F^{\mathbb{Z}}(y)\cup\mathrm{PD}_F^{\mathbb{Z}}(x),[F]\big\rangle & = & Q_F(y,x). \\
 & & \uparrow & & \uparrow \\
 & & \text{Lemma 59.4} & & \text{by definition of } Q_F
\end{array}$$

This completes the proof of the claim.

From Lemma 74.26 we know that the map

$$\begin{array}{ccc}
\mathbb{Z}_p & \to & H_1(L(p,q);\mathbb{Z}) \\
x & \mapsto & x\cdot i_*(a)
\end{array}$$

is an isomorphism. Thus it remains to prove the following claim.

Claim.

$$\lambda_L(i_*(a),i_*(a)) \;=\; \frac{q}{p}\;\in\;\mathbb{Q}/\mathbb{Z}.$$

We calculate that

$$\begin{array}{rcl}
\lambda_L(i_*(a), i_*(a)) & = & \overbrace{\beta^{-1}(\mathrm{PD}_L^{\mathbb{Z}}(i_*(a)))}^{\in H^1(L;\mathbb{Q}/\mathbb{Z})} \cap \overbrace{i_*(a)}^{\in H_1(L;\mathbb{Z})} \\
& \uparrow & \\
& \text{Lemma 74.19} & \\
& = & \mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}\big(\rho_*\big(a - \frac{q}{p}\cdot b\big)\big) \cap a \\
& \uparrow & \\
& \text{Lemmas 74.9 (4) and 74.26 (2)} & \\
& = & \rho\big(\mathrm{PD}_F^{\mathbb{Q}}\big(a - \frac{q}{p}\cdot b\big) \cap a\big) \\
& \uparrow & \\
& \text{Lemma 74.9 (3)} & \\
& = & \rho\big(Q_F(a,a) - \frac{q}{p} Q_F(a,b)\big) \\
& \uparrow & \\
& \text{by the claim} &
\end{array}
\qquad
\begin{array}{rcl}
& = & \overbrace{i^*\big(\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}(\Theta(i_*(a)))\big)}^{\in H^1(L;\mathbb{Q}/\mathbb{Z})} \cap \overbrace{i_*(a)}^{\in H_1(L;\mathbb{Z})} \\
& \uparrow & \\
& \text{Lemma 74.25 applied to } i_*(a) \in H_1(L;\mathbb{Z}) & \\
& = & \rho_*\big(\mathrm{PD}_F^{\mathbb{Q}}\big(a - \frac{q}{p}\cdot b\big)\big) \cap a \\
& \uparrow & \\
& \text{Lemma 74.11} & \\
& = & \rho\big(\mathrm{PD}_F^{\mathbb{Z}}(a) \cap a - \frac{q}{p}\cdot\big(\mathrm{PD}_F^{\mathbb{Z}}(b) \cap a\big)\big) \\
& \uparrow & \\
& \text{Lemma 74.11} & \\
& = & -\rho\big(\frac{q}{p}\big). \\
& \uparrow & \\
& \text{by page 1655 we have } Q_F(a,a) = 0 \text{ and } Q_F(a,b) = 1 &
\end{array}$$

□

74.8. **Computing the linking pairing by intersecting singular simplices.** Many working topologists would rather forgo the austere beauty of cup and cap products and they would rather work and calculate by "intersecting simplices". The following proposition shows that in many (most? all?) situations one can calculate linking pairings geometrically by determining algebraic intersection numbers of singular chains.

Proposition 74.27. *Let $n \geq 1$, let M be an oriented $(2n+1)$-dimensional rational homology sphere and let $x, y \in H_n(M;\mathbb{Z})$. Let σ and τ be cycles in $C_n(M;\mathbb{Z})$ that represent x and y. From Lemma 74.1 we know that $H_n(M;\mathbb{Z})$ is torsion, hence there exists a $d \in \mathbb{N}$ such that $d \cdot x = 0 \in H_n(M;\mathbb{Z})$ which means that $d \cdot \sigma$ is null-homologous. Suppose $\Sigma \in C_{n+1}(M;\mathbb{Z})$ is a singular chain with $\partial\Sigma = d \cdot \sigma$. If Σ and τ intersect transversally in the sense of the definition on page 1594, then*

$$\lambda_M(x,y) \;=\; (-1)^n \cdot \frac{1}{d} \cdot \underbrace{\Sigma \cdot \tau}_{\substack{\text{algebraic intersection} \\ \text{number as defined on page 1595}}} \;\in\; \mathbb{Q}/\mathbb{Z}.$$

Remark. Alas, all too often in the literature the right-hand side of Proposition 74.27 gets confused with the actual definition of the linking pairing, see e.g. [Frie, p. 899] for a typical example.

The proof of the proposition rests on a variation of Theorem 68.5. Before we can formulate that variation we make the following observation.

Observation. Let M be an oriented m-dimensional topological manifold and let A and B be abelian groups. Basically the same definitions as on page 1594 allow us to say that chains $\sigma \in C_k(M;A)$ and $\tau \in C_{m-k}(M;B)$ are *transverse* and they allow us to define the *algebraic intersection invariant* $\sigma \cdot_{\otimes} \tau \in A \otimes B$.

Now we can formulate the following slight variation of Theorem 68.5.

Proposition 74.28. *Let M be a closed oriented m-dimensional topological manifold, let A and B be abelian groups and let $\sigma \in C_k(M;A)$ and $\tau \in C_{m-k}(M;B)$ be two cycles. If σ and τ are transverse, then*

$$\epsilon_M\Big(\big(\underbrace{\overbrace{\mathrm{PD}_M^A([\sigma])}^{\in H^{m-k}(M;A)} \cup_\otimes \overbrace{\mathrm{PD}_M^B([\tau])}^{\in H^k(M;B)}}_{\in H^m(M;A\otimes B)}\big) \cap [M]\Big) \;=\; \sigma \cdot_\otimes \tau \;\in\; A\otimes B.$$

Proof. The proof of Theorem 68.5 generalizes without a hitch to give us this statement with more fancy coefficients. □

Now we can provide the proof of Proposition 74.27.

Proof of Proposition 74.27. As before we denote by $\nu\colon \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ the multiplication induced isomorphism. We perform the following calculation:

$$\begin{aligned}
\lambda_M(x,y) &\overset{\substack{\text{by the definition on page 1720, Lemma 74.9 (2) and Lemma 74.9 (3) applied to } \nu}}{=} \nu_*\big(\epsilon_M\big(\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(x)) \cup_\otimes \mathrm{PD}_M^{\mathbb{Z}}(y)\big) \cap [M]\big)\big)\\
&= (-1)^n \cdot \nu_*\big(\epsilon_M\big(\big(\mathrm{PD}_M^{\mathbb{Q}/\mathbb{Z}}(\beta^{-1}(x)) \cup_\otimes \mathrm{PD}_M^{\mathbb{Z}}(y)\big) \cap [M]\big)\big)\\
&\overset{\substack{\text{Lemma 74.24}}}{=} (-1)^n \cdot \nu_*\big(\big(\tfrac{1}{d}\otimes\Sigma\big) \cdot_\otimes \tau\big) \underset{\substack{\text{follows immediately from the definitions}}}{=} (-1)^n \cdot \tfrac{1}{d}\cdot\Sigma\cdot\tau \;\in\; \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

by Proposition 74.28 and the observation that $\frac{1}{d}\otimes\Sigma$ represents $\beta^{-1}(x)$

□

74.9. **Linking forms and intersection forms** $(*)$**.** Before we turn to the relationship between linking forms and intersection forms we intend to introduce a variation on the intersection form. Recall that given a compact oriented connected $2n$-dimensional topological manifold M we introduced on page 1652 the intersection form

$$\begin{aligned}
Q_M\colon \mathrm{F}H_n(M;\mathbb{Z}) \times \mathrm{F}H_n(M;\mathbb{Z}) &\to \mathbb{Z}\\
(a,b) &\mapsto \big\langle \underbrace{\underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{F}H^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{F}H^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{F}H^{2n}(M,\partial M;\mathbb{Z})}, [M] \big\rangle.
\end{aligned}$$

Here, given an abelian group H we write $\mathrm{F}H := H/\text{torsion subgroup}$ the maximal free abelian quotient of H. Now we introduce a non-symmetric version of the intersection form.

Definition. Let M be a compact oriented connected $2n$-dimensional manifold. We refer to the pairing

$$\begin{array}{rcl} \widetilde{Q}_M\colon \mathrm{F}H_n(M;\mathbb{Z}) \times \mathrm{F}H_n(M,\partial M;\mathbb{Z}) & \to & \mathbb{Z} \\ (v,w) & \mapsto & \underbrace{\big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{F}H^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{F}H^n(M;\mathbb{Z})}, [M] \big\rangle}_{\in \mathrm{F}H^{2n}(M,\partial M;\mathbb{Z})} \end{array}$$

as the *reduced intersection pairing of M.*

Example. Let M be the torus minus two open balls. We consider the four curves a, b, c, d on M that are shown in Figure 930. The curves a, b, c represent a basis for $H_1(M;\mathbb{Z})$, whereas the curves a, b, d represent a basis for $H_1(M,\partial M;\mathbb{Z})$.[1249] Using these bases and using Theorem 67.10 we see that

$$Q_M \text{ is represented by } \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and } \widetilde{Q}_M \text{ represented by } \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Note that the former matrix has determinant zero whereas the latter matrix has determinant. This calculation, together with Lemma 62.1 shows that the former pairing is degenerate whereas the latter pairing is non-singular.[1250] As we will see in the next proposition, it is not a coincidence that the reduced intersection pairing is non-singular.

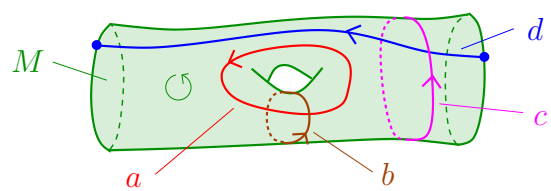


FIGURE 930.

Proposition 74.29. *Let M be a compact oriented connected $2n$-dimensional topological manifold. We denote by $p_*\colon H_n(M;\mathbb{Z}) \to H_n(M,\partial M;\mathbb{Z})$ the obvious map.*

(1) *The pairing $\widetilde{Q}_M\colon \mathrm{F}H_n(M;\mathbb{Z}) \times \mathrm{F}H_n(M,\partial M;\mathbb{Z}) \to \mathbb{Z}$ is non-singular.*
(2) *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{F}H_n(M;\mathbb{Z}) \times \mathrm{F}H_n(M;\mathbb{Z}) \arrow[r, "Q_M"] \arrow[d, "\mathrm{id} \times p_*"] & \mathbb{Z} \arrow[d] \\ \mathrm{F}H_n(M;\mathbb{Z}) \times \mathrm{F}H_n(M,\partial M;\mathbb{Z}) \arrow[r, "\widetilde{Q}_M"] & \mathbb{Z}. \end{tikzcd}$$

[1249] Why is that?

[1250] In fact, using this observation and Exercise 62.1 (c) one can easily answer the previous footnote.

(3) *Let $v_1, \dots, v_m$ be a basis of the free abelian group $\mathrm{F}H_n(M;\mathbb{Z})$. By* (1) *together with Exercise 62.1* (a) *there exists a unique basis $w_1, \dots, w_m$ of the free abelian group $\mathrm{F}H_k(M, \partial M;\mathbb{Z})$ with $\widetilde{Q}_M(v_i, w_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, m\}$. We denote by A the $(m \times m)$-matrix that represents the intersection form Q_M with respect to the basis $v_1, \dots, v_n$, i.e. we have $A = (a_{ij})$ with $a_{ij} = Q_M(v_i, v_j)$. This matrix represents the map $p_*\colon \mathrm{F}H_n(M;\mathbb{Z}) \to \mathrm{F}H_n(M, \partial M;\mathbb{Z})$ with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_m$.*

Proof.

(1) We consider the map

$$\Psi\colon \mathrm{F}H_n(M, \partial M;\mathbb{Z}) \xrightarrow{\mathrm{PD}_M} \mathrm{F}H^n(M;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{F}H_n(M;\mathbb{Z}), \mathbb{Z}).$$

Basically the same argument as in the proof of Lemma 71.6 shows that for any $a \in \mathrm{F}H_n(M;\mathbb{Z})$ and any $b \in \mathrm{F}H_{2n}(M, \partial M;\mathbb{Z})$ we have $\widetilde{Q}_M(a, b) = \Psi(b)(a)$. Since PD_M and $\mathrm{ev}\colon \mathrm{F}H^n(M;\mathbb{Z}) \to \mathrm{Hom}(\mathrm{F}H_n(M;\mathbb{Z}), \mathbb{Z})$ are isomorphisms we obtain by the same argument as in Proposition 71.5 (together with the remark on page 1448) that $\widetilde{Q}_M$ is non-singular.

(2) This statement follows easily from the definitions and Proposition 61.22. Alternatively one can use the descriptions of Q_M and $\widetilde{Q}_M$ that are given in Lemma 71.6 and in (1).

(3) Let $i \in \{1, \dots, m\}$. We write $p_*(v_i) = \sum_{j=1}^{m} b_{ji} \cdot w_j$. Let $j \in \{1, \dots, m\}$. We see that

$$b_{ji} \underset{\substack{\uparrow \\ \text{since } \widetilde{Q}_M(v_j, w_k) = \delta_{jk}}}{=} \widetilde{Q}_M\Big(v_j, \sum_{k=1}^{m} b_{ki} \cdot w_k\Big) = \widetilde{Q}_M(v_j, p_*(v_i)) \underset{\substack{\uparrow \\ \text{follows from (2)}}}{=} Q_M(v_j, v_i) = a_{ji}.$$

Since by definition the matrix (b_{ij}) represents the map p_* with respect to the given bases we obtain the desired result. □

In the following let M be a compact oriented connected 4-dimensional topological manifold. We suppose that $Y = \partial M$ is a rational homology sphere. We want to study, in favorable situations, the relationship between the intersection form Q_M on M and the linking pairing λ_Y on Y. More precisely, we want to prove the following proposition.

Proposition 74.30. *Let M be a compact oriented connected 4-dimensional topological manifold such that $Y = \partial M$ is a rational homology sphere. We suppose that $H_1(M;\mathbb{Z}) = 0$. The following statements hold:*

(1) *The groups $H_2(M, Y;\mathbb{Z})$ and $H_2(M;\mathbb{Z})$ are free abelian groups of finite rank.*

(2) *By* (1) *we can pick a basis $v_1, \dots, v_m$ for the free abelian group $H_2(M;\mathbb{Z})$. By Proposition 74.29 together with Exercise 62.1* (a) *there exists a unique basis $w_1, \dots, w_m$ for the free abelian group $H_2(M, Y;\mathbb{Z})$ with $\widetilde{Q}_M(v_i, w_j) = \delta_{ij}$. We denote by $A = (Q_M(v_i, v_j))$ the $(m \times m)$-matrix that represents the intersection form with respect to the basis $v_1, \dots, v_m$. The following two statements hold:*

(a) *If we denote by* $\Psi\colon \mathbb{Z}^m \to H_2(M,Y;\mathbb{Z})$ *the isomorphism that is given by the basis* $w_1,\dots,w_m$*, then the map*

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}^m/A\mathbb{Z}^m & \to & H_1(Y;\mathbb{Z}) \\ [u] & \mapsto & \partial(\Psi(u)) \\ & & \uparrow \\ & & \text{connecting homomorphism } \partial\colon H_2(M,Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z}) \text{ of the pair } (M,Y) \end{array}$$

is well-defined and it is an isomorphism.

(b) *The matrix* A *is symmetric and if satisfies* $\det(A) \neq 0$.

(c) *The isomorphism* Θ *from* (a) *is in fact an isometry from*[1251]

$$\begin{array}{rcl} \mathbb{Z}^m/A\mathbb{Z}^m \times \mathbb{Z}^m/A\mathbb{Z}^m & \xrightarrow{\lambda(A)} & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \longmapsto & v^T A^{-1} w \end{array} \quad \text{to} \quad \underbrace{H_1(Y;\mathbb{Z}) \times H_1(Y;\mathbb{Z}) \xrightarrow{\lambda_Y} \mathbb{Q}/\mathbb{Z}}_{\text{linking pairing}}.$$

FIGURE 931. Illustration of Proposition 74.30.

Remark.

(1) We leave it to the reader to figure out what is a correct higher-dimensional analogue of Proposition 74.30. The reader will found out, that determining the correct sign is a never ending source of hilarious entertainment.

(2) Often it is (psychologically) easier to compute intersection forms of manifolds, thus on many occasions in the literature linking pairings are computed by calculating the intersection form of a suitable bounding manifold.

The following lemma is the key ingredient in the proof of Proposition 74.30.

Lemma 74.31. *We continue with the notation from Proposition 74.30. We denote by* $i\colon Y \to M$ *the inclusion and we denote by* $p\colon C_*(M) \to C_*(M,Y)$ *the projection map. The following diagram commutes up to a minus sign:*

$$\begin{tikzcd} H^2(M;\mathbb{Z}) \arrow[r, two heads, "i^*"] \arrow[d] & H^2(Y;\mathbb{Z}) \arrow[r, "\beta^{-1}", "\cong"'] & H^1(Y;\mathbb{Q}/\mathbb{Z}) \arrow[d, "\delta"] \\ H^2(M;\mathbb{Q}) \arrow[r, "(p^*)^{-1}", "\cong"'] & H^2(M,Y;\mathbb{Q}) \arrow[r] & H^2(M,Y;\mathbb{Q}/\mathbb{Z}). \end{tikzcd}$$

[1251]It follows from (b) and Lemma 74.13 that the form $\lambda(A)$ is in fact defined.

Remark. The encyclopedic reader will not have failed to notice that Lemma 74.31 is a special case of Exercise 31.6.

Proof of Lemma 74.31. Throughout the proof we adopt the following conventions:

(1) Given a cochain ψ that takes values in $\mathbb{Q}$ we denote by $\overline{\psi}$ the corresponding cochain that takes values in $\mathbb{Q}/\mathbb{Z}$.
(2) If ψ is a cochain that takes values in $\mathbb{Z}$, then we view it also as a cochain with values in $\mathbb{Q}$.

Now let $\phi \in C^2(M;\mathbb{Z}) = \mathrm{Hom}(C_2(M),\mathbb{Z})$ be a cochain. Note that the Poincaré Duality Theorem 61.1 implies that $H^i(Y;\mathbb{Q}) \cong H_i(Y;\mathbb{Q}) = 0$ for $i = 1, 2$. Thus we obtain from the long exact sequence of the pair (M, Y) that the map $p^*\colon H^2(M, Y;\mathbb{Q}) \to H^2(M;\mathbb{Q})$ is an isomorphism there exists a cochain $\psi \in C^2(M, Y;\mathbb{Q})$ such that $[\phi] = [p^* \circ \psi] \in H^2(M;\mathbb{Q})$. This means that there exists some $\alpha \in C^1(M;\mathbb{Q})$ such that $p^* \circ \psi - \phi = \delta\alpha$. We write $\varphi := \alpha|_{C_1(Y)}$. We calculate that

$$\begin{array}{ccccccccc}
\delta\overline{\varphi} & = & \overline{\delta\alpha}|_{C_2(Y)} & = & \overline{p^* \circ \psi - \phi}|_{C_2(Y)} & = & \overline{\phi}|_{C_2(Y)} & = & 0 \in C_2(Y;\mathbb{Q}/\mathbb{Z}). \\
 & \uparrow & & & & \uparrow & & \uparrow & \\
 & \text{restriction is a cochain map} & & & \text{since } p^* \circ \psi \text{ vanishes on } C_2(Y) & & & \text{since } \phi \text{ takes integral values.} &
\end{array}$$

In other words, $\overline{\varphi}$ represents an element in $H^2(Y;\mathbb{Q}/\mathbb{Z})$. The following claim computes the two possible connecting homomorphisms.

Claim. We have $\delta([\overline{\varphi}]) = [p^*(\psi)]$ and $\beta([\overline{\varphi}]) = -[i^*(\phi)]$.

The claim follows from considering the following two sequences of maps and elements:

$$\begin{array}{ccc}
\overset{(4)\ \overline{\psi}}{C^2(M,Y;\mathbb{Q}/\mathbb{Z})} & \xrightarrow{p^*} & \overset{(3)\ \overline{p^*\psi-\phi}=\overline{p^*\psi}}{C^2(M;\mathbb{Q}/\mathbb{Z})} \\
 & & \uparrow{\scriptstyle\delta} \\
 & & \underset{(2)\ \overline{\alpha}}{C^1(M;\mathbb{Q}/\mathbb{Z})} \xrightarrow{i^*} \underset{(1)\ \overline{\varphi}}{C^1(Y;\mathbb{Q}/\mathbb{Z})}
\end{array}
\quad \& \quad
\begin{array}{ccc}
\overset{(4)\ -i^*\phi}{C^2(Y;\mathbb{Z})} & \to & \overset{(3)\ \overbrace{i^*p^*}^{=0}\psi-i^*\phi}{C^2(Y;\mathbb{Q})} \\
 & & \uparrow{\scriptstyle\delta} \\
 & & \underset{(2)\ \varphi=i^*\alpha}{C^1(Y;\mathbb{Q})} \to \underset{(1)\ \overline{\varphi}}{C^1(Y;\mathbb{Q}/\mathbb{Z})}.
\end{array}$$

This concludes the proof of the claim.

Now we are basically done: Namely, if we go down and right, then the claim shows that $[\phi] \in H^2(M;\mathbb{Z})$ gets sent to $[\overline{\varphi}] \in H^2(M, Y;\mathbb{Q}/\mathbb{Z})$. On the other hand, if we go right and down, then the claim shows that $[\phi]$ is sent to $-[\overline{\varphi}]$. $\square$

Proof of Proposition 74.30. We consider the following maps:

$$\begin{tikzcd}
H_2(M;\mathbb{Z}) \arrow[r, "p_*"] \arrow[d, "\cong"', "\mathrm{PD}_M"] & H_2(M,Y;\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\cong"', "\mathrm{PD}_M"] & H_1(Y;\mathbb{Z}) \arrow[r] & \overbrace{H_1(M;\mathbb{Z})}^{=0 \text{ by hypothesis}} \arrow[r] & H_1(M,Y;\mathbb{Z}) \arrow[r] & 0. \\
H^2(M,Y;\mathbb{Z}) \arrow[d, "\cong"', "\mathrm{ev}"] & H^2(M;\mathbb{Z}) \arrow[d, "\cong"', "\mathrm{ev}"] & & & & \\
\mathrm{Hom}(H_2(M,Y;\mathbb{Z}),\mathbb{Z}) & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}). & & & &
\end{tikzcd}$$

We make the following observations:

(a) The horizontal sequence is the long exact sequence in reduced homology of the pair (M,Y).
(b) By hypothesis we have $H_1(M;\mathbb{Z}) = 0$.
(c) It follows from (a) and (b) that $H_1(M,Y;\mathbb{Z}) = 0$.
(d) The vertical maps PD_M are the Poincaré duality isomorphisms given by the Poincaré Duality Theorem 61.1.
(e) It follows from (b), (c) and the Universal Coefficient Theorem 51.13 that both evaluation homomorphisms ev are isomorphisms.
(f) We obtain from the above discussion that $H_2(M;\mathbb{Z}) \cong \mathrm{Hom}(H_2(M,Y;\mathbb{Z}),\mathbb{Z})$ and that $H_2(M,Y;\mathbb{Z}) \cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z})$. It follows from Proposition 45.2 (2) together with the classification of finitely generated abelian groups, see Theorem 11.4, that $H_2(M;\mathbb{Z})$ and $H_2(M,Y;\mathbb{Z})$ are free abelian groups. In particular we have now shown statement (1) of the proposition.

We pick a basis $v_1,\dots,v_m$ for the free abelian group $H_k(M;\mathbb{Z})$. We denote by $w_1,\dots,w_m$ the unique basis for the free abelian group $H_2(M,Y;\mathbb{Z})$ with $\widetilde{Q}_M(v_i,w_j) = \delta_{ij}$. We denote by $A = (a_{ij})$ the $(n\times n)$-matrix that is given by $a_{ij} = Q_M(v_i,v_j)$. Note that it follows from Proposition 71.5 (1) that A is symmetric.

We denote by $\Phi\colon \mathbb{Z}^m \xrightarrow{\cong} H_2(M;\mathbb{Z})$ and $\Psi\colon \mathbb{Z}^m \xrightarrow{\cong} H_2(M,Y;\mathbb{Z})$ the isomorphisms that are given by the above bases. From the above long exact sequence together with Proposition 74.29 we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}^m \arrow[r, "u\mapsto A\cdot u"] \arrow[d, "\Phi"', "\cong"] & \mathbb{Z}^m \arrow[r] \arrow[d, "\cong"', "\Psi"] & \mathbb{Z}^m/A\mathbb{Z}^m \arrow[r] \arrow[d, dotted, "\Theta"] & 0 \\
H_2(M;\mathbb{Z}) \arrow[r, "p_*"] & H_2(M,Y;\mathbb{Z}) \arrow[r, "\partial"] & H_1(Y;\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Note that the lower sequence is exact by the discussion above. Furthermore note that the diagram commutes by Proposition 74.29 (2). It follows immediately that the map $\partial\circ\Psi\colon \mathbb{Z}^m \to H_1(Y;\mathbb{Z})$ descends to an isomorphism $\mathbb{Z}^m/A\mathbb{Z}^m \to H_1(Y;\mathbb{Z})$. This concludes the proof of statement (2a).

Next note that it follows from Proposition 71.5 (1) that A is symmetric. Furthermore note that it follows from (2a), the fact that $H_1(Y;\mathbb{Z})$ and Lemma 11.7 that $\det(A) \neq 0$. This concludes the proof of statement (2b).

The key to proving statement (2c) is the following claim:

Claim. There exists a pairing $\chi\colon H_2(M,Y;\mathbb{Z}) \times H_2(M,Y;\mathbb{Z}) \to \mathbb{Q}$ such that the following diagram commutes:

$$\begin{tikzcd}
H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \arrow[r, "Q_M"] \arrow[d, "p_* \times p_*"] & \mathbb{Z} \arrow[d, hook] \\
H_2(M,Y;\mathbb{Z}) \times H_2(M,Y;\mathbb{Z}) \arrow[r, "\chi"] \arrow[d, "\partial \times \partial"] & \mathbb{Q} \arrow[d] \\
H_1(Y;\mathbb{Z}) \times H_1(Y;\mathbb{Z}) \arrow[r, "\lambda_M"'] & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

We first explain why the claim implies statement (2c). If we equip $H_2(M;\mathbb{Z})$ and $H_2(M,Y;\mathbb{Z})$ with the above bases, then we know by Proposition 74.29 (3) that the map p_* is represented by the matrix A. Furthermore, by definition A represents the intersection form Q_M. Now, for purely algebraic reasons we see that χ is represented by the rational matrx A^{-1}.[1252] Statement (2c) follows immediately from this observation, once we have proved the claim.

Thus it remains to prove the claim. In the following we denote by $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ the inclusion. Next we consider the following diagram:

$$\begin{tikzcd}
H_2(M;\mathbb{Z}) \arrow[r, "p_*"] \arrow[d, "\cong", "\mathrm{PD}_M"'] & H_2(M,Y;\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\cong", "\mathrm{PD}_M"'] & H_1(Y;\mathbb{Z}) \arrow[d, "\cong", "\mathrm{PD}_Y"'] \\
H^2(M,Y;\mathbb{Z}) \arrow[r, "p^*"] & H^2(M;\mathbb{Z}) \arrow[r, "i^*"] & H^2(Y;\mathbb{Z}).
\end{tikzcd}$$

By Proposition 61.22 the diagram commutes.[1253] The diagram with the vertical isomorphisms allows us to translate our above problem into a different taks. Namely it follows from the diagram and the definition of the intersection form Q_M that it suffices to show that there exists a pairing $\chi\colon H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Q}$ such that the following diagram

[1252]The purely algebraic reason is the following. Suppose we are given a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{Z}^m \times \mathbb{Z}^m \arrow[r, "(x,y)\mapsto x^T P y"] \arrow[d, "(x,y)\mapsto(Cx,Dx)"] & \mathbb{Z} \arrow[d] \\
\mathbb{Z}^m \times \mathbb{Z}^m \arrow[r, "(x,y)\mapsto x^T Q y"] & \mathbb{Q}.
\end{tikzcd}$$

Then $P = C^T Q D$. If C and D are invertible over $\mathbb{Q}$, then $Q = (C^T)^{-1} P D^{-1}$. In our case we have $C = D = P = A$. Furthermore A is symmetric and we know that $\det(A) \neq 0$. Thus we see that $Q = A^{-1} A A^{-1} = A^{-1}$.

[1253]To be more precise, the square on the right commutes up to the $(-1)^2$, which happens to be $+1$.

commutes:

$$\begin{tikzcd}
H^2(M,Y\ \mathbb{Z}) \times H^2(M,Y;\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "p^*\times p^*"] & H^4(M,Y;\mathbb{Z}) \arrow[r, "\langle -,[M]\rangle"] & \mathbb{Z} \arrow[d, "\varphi"] \\
H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \arrow[rr, "\chi"] \arrow[d, "\mathrm{PD}_Y^{-1}\circ i^* \times \mathrm{PD}_Y^{-1}\circ i^*"] & & \mathbb{Q} \arrow[d] \\
H_1(Y;\mathbb{Z}) \times H_1(Y;\mathbb{Z}) \arrow[rr, "\lambda_M"'] & & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

The key to proving the existence of such a pairing χ is the following daunting diagram:

$$\begin{tikzcd}
H^2(M,Y;\mathbb{Z})\times H^2(M,Y;\mathbb{Z}) \arrow[rr, "\cup"] \arrow[dr, "\varphi_*\times\mathrm{id}"] \arrow[ddd, "p^*\times p^*"] & & H^4(M,Y;\mathbb{Z}) \arrow[r, "\langle -,[M]\rangle"] \arrow[d, "\varphi_*"] \arrow[dl, phantom, "\text{(I)}"] & \mathbb{Z} \arrow[d, "\varphi"] \arrow[dl, phantom, "\text{(II)}"] \\
& H^2(M,Y;\mathbb{Q})\times H^2(M;\mathbb{Z}) \arrow[r, "\cup"'] \arrow[d, "\mathrm{id}"'] \arrow[d, shift left=6, "p^*"] \arrow[dr, phantom, "\text{(III)}"] & H^4(M,Y;\mathbb{Q}) \arrow[r, "\langle -,[M]\rangle"'] \arrow[d, "="] \arrow[dr, phantom, "\text{(IV)}"] & \mathbb{Q} \arrow[d, "="] \arrow[dr, phantom, "\text{(V)}"] \\
& H^2(M,Y;\mathbb{Q})\times H^2(M;\mathbb{Z}) \arrow[r, "\cup"'] \arrow[d, "p^*"'] \arrow[d, shift left=6, "\mathrm{id}"] & H^4(M,Y;\mathbb{Q}) \arrow[r, "\langle -,[M]\rangle"'] & \mathbb{Q} \arrow[ddd, "="] \\
\boxed{H^2(M;\mathbb{Z})\times H^2(M;\mathbb{Z})} \arrow[r, "\varphi_*\times\mathrm{id}"] \arrow[ddd, "i^*\times i^*"] & \boxed{H^2(M;\mathbb{Q})\times H^2(M;\mathbb{Z})} \arrow[d, "(p^*)^{-1}"'] \arrow[d, shift left=6, "\mathrm{id}"] \arrow[dr, phantom, "\text{(VI)}"] & & \\
& \boxed{H^2(M,Y;\mathbb{Q})\times H^2(M;\mathbb{Z})} \arrow[r, "\cup"] \arrow[d] \arrow[d, shift left=6, "\mathrm{id}"] \arrow[dr, phantom, "\text{(VII)}"] & \boxed{H^4(M,Y;\mathbb{Q})} \arrow[r, "\langle -,[M]\rangle"] \arrow[d] \arrow[dr, phantom, "\text{(VIII)}"] & \boxed{\mathbb{Q}} \arrow[d] \arrow[dr, phantom, "\text{(IX)}"] \\
& H^2(M,Y;\mathbb{Q}/\mathbb{Z})\times H^2(M;\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "i^*"] & H^4(M,Y;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\langle -,[M]\rangle"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "="] \\
H^2(Y;\mathbb{Z})\times H^2(Y;\mathbb{Z}) \arrow[r, "\beta^{-1}\times\mathrm{id}"] & H^1(Y;\mathbb{Q}/\mathbb{Z})\times H^2(Y;\mathbb{Z}) \arrow[r, "\cup"] \arrow[u, "\delta"] \arrow[dr, phantom, "\text{(X)}"] & H^3(Y;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\langle -,[Y]\rangle"] \arrow[u, "\delta"] \arrow[dr, phantom, "\text{(XI)}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "="] \\
H_1(Y;\mathbb{Z}) \times H_1(Y;\mathbb{Z}) \arrow[u, "\mathrm{PD}_Y\times\mathrm{PD}_Y"] \arrow[rrr, "\lambda_M"'] & & \text{(XII)} & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

The promised pairing χ is given by the traveling along the route that is traced out by the boxes. Thus it remains to prove that the diagram commutes. We make the following clarifications and observations:

(I) The lower cup product is the composition of the cup product as defined in Lemma 74.5 with the obvious map $\mathbb{Z}\otimes\mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$. It follows basically immediately from the definitions that the square commutes, see also Proposition 58.3 (2).

(II) This square commutes by Lemma 50.5 (2).

(III) This square commutes since changing coefficients is a natural transformation, see Lemma 48.9.

(IV) This square commutes by a mild generalization of Proposition 58.3 (1).

(V) What a relief, there is nothing to show.
(VI) Note that the two vertical maps to the left of the square are inverses of one another. So we see that nothing happens in this square.
(VII) This recactangle commutes by Lemma 74.31.
(VIII) Similar to (I) the cup products are the composition of the cup product as defined in Lemma 74.5 with the obvious maps $\mathbb{Z}\otimes\mathbb{Q}\xrightarrow{\cong}\mathbb{Q}$ and $\mathbb{Z}\otimes\mathbb{Q}/\mathbb{Z}\xrightarrow{\cong}\mathbb{Q}/\mathbb{Z}$. As in (I) we see that the square commutes.
(IX) By the same argument as in (II) we see that the square commutes.
(X) A slight generalization of Lemma 58.5 to more general coefficients shows that the square icommutes up to the sign $(-1)^2 = 1$.
(XI) By Proposition 46.26 we have $\partial([M]) = [Y]$. It follows from a minute generalization of Lemma 59.10 (1) that this square commutes.
(XII) This rectangle commutes by definition of the linking pairing.

This concludes the proof of the claim. □

Exercises for Chapter 74.

Exercise 74.1. Let $k \in \mathbb{N}$. We consider the short exact sequence $0 \to \mathbb{Z}_k \xrightarrow{\cdot k} \mathbb{Z}_{k^2} \xrightarrow{p} \mathbb{Z}_k \to 0$ where p stands for the obvious projection map. Given a topological space X we obtain Bockstein homomorphisms $\beta_n\colon H_n(X;\mathbb{Z}_k) \to H_{n-1}(X;\mathbb{Z}_k)$, $n \in \mathbb{N}_0$. Show that

$$\beta_{n-1}\circ\beta_n\colon H_n(X;\mathbb{Z}_k) \;\to\; H_{n-2}(X;\mathbb{Z}_k)$$

is the zero map.

75. Branched coverings and chiral knots

After this long and technical, and basically picture free chapter, it is time to return to some more concrete topological questions. We recall the following question from some time ago.

Question 47.2. *Is the trefoil chiral, i.e. can the mirror of the trefoil be deformed into the trefoil itself? Put differently, are the two knots shown in Figure 75 smoothly isotopic in the sense of the definition on page 337?*

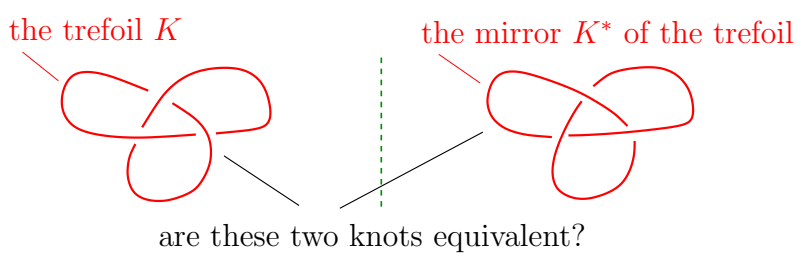


Figure 932.

In the last chapters we had developed several tools for showing that certain closed manifolds, e.g. the complex projective space $\mathbb{C}\mathrm{P}^2$ and the lens space $L(3,1)$, are chiral. In this section we will attack Question 47.2 by reducing it to a question that we had already dealt with. More precisely, the idea is to associate to a knot in S^3 an oriented rational homology sphere that retains at least some information on the knot. Then we will use the linking pairing of this rational homology sphere to draw conclusions on the chirality of the knot.

75.1. **Branched coverings.** In this section we will introduce branched coverings. We start out by recalling the following definition from page 1497.

Definition. Let M be a manifold and let $W \subset M$ be a proper codimension two submanifold. A *trivial tubular neighborhood* for W is an embedding

$$F\colon \overline{B}^2 \times W \;\to\; M$$

such that $F(0,P) = P$ for all $P \in W$ and such that $F(\overline{B}^2 \times \partial W) = \partial M \cap F(\overline{B}^2 \times W)$.

Examples.

(1) It is clear that every 0-dimensional submanifold of a 2-dimensional manifold admits a trivial tubular neighborhood.
(2) It follows from the Tubular Neighborhood Theorem 10.19 that every closed 1-dimensional submanifold of a closed orientable 3-dimensional manifold admits a trivial tubular neighborhood.
(3) In Proposition 67.20 we saw that the 2-dimensional submanifold $W = \mathbb{C}\mathrm{P}^1$ of the 4-dimensional manifold $M = \mathbb{C}\mathrm{P}^2$ does not admit a trivial tubular neighborhood.

Convention. Let $\overline{B}^2 \times W \to M$ be an embedding as above. On many occasions, by an abuse of notation, we will identify the image of the embedding $\overline{B}^2 \times W \to M$ with $\overline{B}^2 \times W$. Put differently, we often use the phrase let "$\overline{B}^2 \times W$ be a trivial tubular neighborhood".

Definition. Let M be a manifold and let $W \subset M$ be a compact proper submanifold of codimension two. We say a map $p\colon \widetilde{M} \to M$ is a *k-fold branched covering with branch locus W* if the following two conditions are satisfied:

(1) The restriction $p\colon \widetilde{M} \setminus p^{-1}(W) \to M \setminus W$ is a k-fold covering map in the usual sense.
(2) There exists a trivial tubular neighborhood $\overline{B}^2 \times W$ of W with the following property: for each component X of $p^{-1}(\overline{B}^2 \times W)$ there exists an $m \in \mathbb{N}_{\geq 2}$ and a diffeomorphism $\Psi\colon X \to \overline{B}^2 \times W$ such that the following diagram commutes[1254]

$$\begin{tikzcd} X \arrow[rr, "\Psi"] \arrow[dr, "p"'] & & \overline{B}^2 \times W \arrow[dl, "(z,w)\mapsto(z^m,w)"] \\ & \overline{B}^2 \times W. & \end{tikzcd}$$

We refer to m as the *branching degree of X*.

We say the branched covering is *simple* if $m = k$ throughout. Furthermore we say that the branched covering is *regular* if the covering map $p\colon \widetilde{M} \setminus p^{-1}(W) \to M \setminus W$ is regular in the sense of the definition on page 624. Otherwise we say that the branched covering is *irregular*.

Examples.

(1) The map $p\colon \widetilde{M} = \mathbb{C} \to M = \mathbb{C}$ given by $z \mapsto z^k$ is a simple k-fold branched covering with branch locus $W = \{0\}$. This is basically clear:
   (a) On page 260 we saw that the restriction of p to a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a k-fold covering map.
   (b) We take the obvious trivial tubular neighborhood $\overline{B}^2$ for 0 and we denote by Ψ the identity map. It is obvious that the above diagram commutes.
(2) The same argument as in (1) shows that the map

$$\begin{array}{rcl} \widetilde{M} = \mathbb{C} \setminus \{\{e^{2\pi i k/3} \,|\, k = 0,1,2\} & \to & \mathbb{C} \setminus \{1\} \\ z & \mapsto & z^3 \end{array}$$

that is illustrated in Figure 933 is a simple 3-fold branched covering with branch locus $W = \{0\}$.
(3) The projection map $\widetilde{M} = \mathbb{C} \to M := \mathbb{C}/z \sim -z$ is a simple 2-fold branched covering with branch locus $W = \{0\}$. To verify this statement we consider the

[1254]Here we view $\overline{B}^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ as a subset of $\mathbb{C}$.

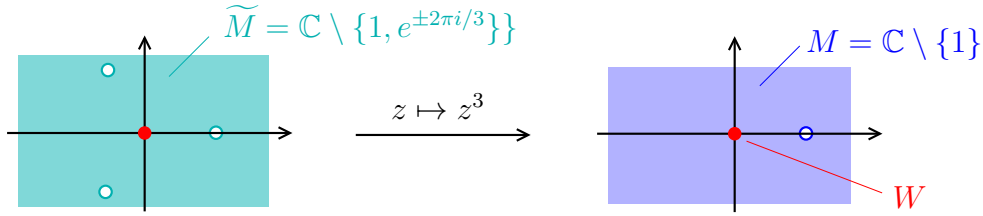


FIGURE 933.

following diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "z\mapsto[z]"] \arrow[dr, "z\mapsto z^2"'] & & \mathbb{C}/z \sim -z \arrow[dl, "{[z]\mapsto z^2}"] \\ & \mathbb{C}. & \end{tikzcd}$$

The diagram is clearly commutative and the right-hand diagonal map is easily seen to be a homeomorphism. We use this homeomorphism to equip $\mathbb{C}/z \sim -z$ with a smooth structure. In (1) we saw that the left diagonal map is a branched covering, thus the top horizontal map is also a branched covering.

(4) Let Σ be a closed complex 1-dimensional manifold (i.e. Σ is a closed 2-dimensional manifold with a complex structure) and let $f\colon \Sigma \to \mathbb{C}$ be a meromorphic non-constant function. We consider the map

$$\begin{array}{rcl} \Sigma & \to & \mathbb{C}\mathrm{P}^1 = S^2 \\ z & \mapsto & \begin{cases} [f(z):1], & \text{if } f(z) \neq \infty \\ [1:0], & \text{otherwise.} \end{cases} \end{array}$$

It is straightforward to see that this map is holomorphic. Using [Jä2, Satz 10] one can easily show that $f\colon \Sigma \to \mathbb{C}\mathrm{P}^1 = S^2$ is a branched covering.

Before we give more interesting examples of branched coverings we state and prove the following lemma that, given a simple branched covering $p\colon \widetilde{M} \to M$, relates the Euler characteristic of $\widetilde{M}$ to the Euler characteristic of M, the Euler characteristic of the branch locus and the degree of the branched covering.

The lemma carries a fancy name, even though it is quite elementary.

Lemma 75.1. (Riemann-Hurwitz formula) *Let M be a compact manifold and let W be a compact proper codimension 2 submanifold of M. If $p\colon \widetilde{M} \to M$ is a simple k-fold branched covering with branch locus W, then*

$$\chi\big(\widetilde{M}\big) \;=\; k \cdot \chi(M) + (1-k) \cdot \chi(W).$$

A similar statement also holds for non-simple branched coverings. The formulation becomes more messy, but the proof in this more general setting is equally trivial. Since we will not make use of this more general formula we decline to formulate it.

Proof. As usual, statements about Euler characteristics can be proved quite easily. First of all we pick a tubular neighborhood $\overline{B}^2 \times W$ and a diffeomorphism $\Psi\colon p^{-1}(\overline{B}^2 \times W) \to \overline{B}^2 \times W$ as in the definition of a branched covering. Now we calculate that

$$\begin{array}{rcl}
 & & \text{Proposition 45.7 (3) applied to } \widetilde{M} = \big(\widetilde{M} \setminus \Psi(B^2 \times W)\big) \cup \Psi\big(\overline{B}^2 \times W\big) \\
 & & \downarrow \\
\chi\big(\widetilde{M}\big) & = & \chi\big(\widetilde{M} \setminus \Psi(B^2 \times W)\big) + \chi\big(\Psi(\overline{B}^2 \times W)\big) - \chi\big(\Psi(S^1 \times W)\big) \\
 & = & k \cdot \chi(M \setminus (B^2 \times W)) + \chi(W) \\
 & & \uparrow \\
 & & \text{by Proposition 45.7 (5) since } p\colon \widetilde{M} \setminus \Psi(B^2 \times W) \to M \setminus (B^2 \times W) \text{ is a } k\text{-fold covering} \\
 & & \text{and using that Proposition 45.7 (2) implies that } \chi(\Psi(\overline{B}^2 \times W)) = \chi(W) \text{ and } \chi(\Psi(S^1 \times W)) = 0 \\
 & = & k \cdot \chi(M) - k \cdot \chi(\overline{B}^2 \times W) + k \cdot \chi(W \times S^1) + \chi(W) \;=\; k \cdot \chi(M) + (1-k) \cdot \chi(W). \\
 & & \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
 & & \text{by Proposition 45.7 (3) applied to } M = M \setminus (B^2 \times W) \cup \overline{B}^2 \times W \qquad \text{again by Proposition 45.7 (2)}
\end{array}$$

□

Remark. The concept of a map that is a covering outside of a sufficiently small subset also appears in other branches of mathematics. It is often also referred to as a "branched covering" or "ramified covering" or "ramification". In Figure 934 we show an extract of Neukirch's book on algebraic number theory [Neuk].

6. Extensions. Let $\mathcal{O}$ be a Dedekind domain with field of fractions K. Let $L|K$ be a finite separable extension, and $\mathcal{O}$ the integral closure of $\mathcal{O}$ in L. Let $Y = \operatorname{Spec}(\mathcal{O})$, $X = \operatorname{Spec}(\mathcal{O})$, and

$$f : X \longrightarrow Y$$

the morphism induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}$. If $\mathfrak{p}$ is a maximal ideal of $\mathcal{O}$ and

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$$

the prime decomposition of $\mathfrak{p}$ in $\mathcal{O}$, then $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ are the different points of X which are mapped to $\mathfrak{p}$ by f. The morphism f is a "ramified covering." It is graphically represented by the following picture:

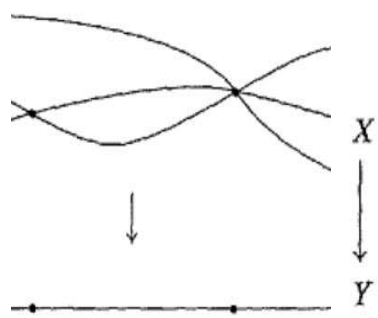


FIGURE 934.

75.2. **The hyperelliptic involution and branched coverings of surfaces.** Before we can provide the reader with new examples of branched coverings we need to introduce the following fun object.

Definition. As on page 967 we consider the surface Σ_g of genus g as the quotient of the regular $(4g+2)$-gon E_{4g+2} where we identify opposite sides by a translation. The map

$$\begin{array}{rcl}
\Theta\colon E_{4g+2}/\sim & \to & E_{4g+2}/\sim \\
[z] & \mapsto & [-z]
\end{array}$$

is called the *hyperelliptic involution*.[1255][1256] For $g = 2$ the surface $\Sigma_2 = E_{10}/\sim$ and the hyperelliptic involution are illustrated in Figure 935.

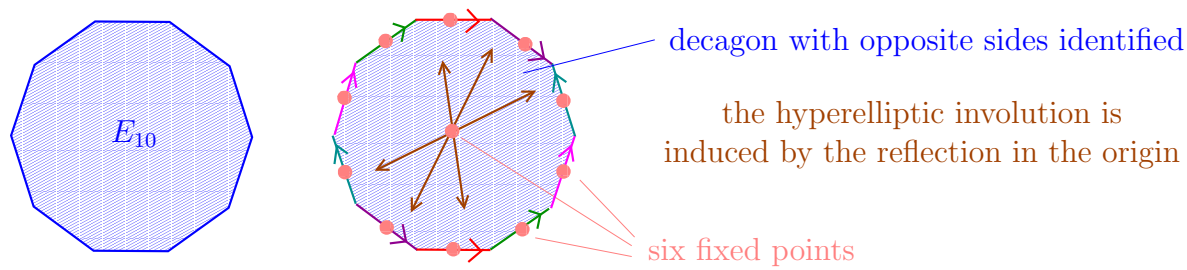


FIGURE 935.

The following lemma summarizes some properties of the hyperelliptic involution.

Lemma 75.2. *Let $g \in \mathbb{N}_0$.*

(1) *The hyperelliptic involution $\Theta\colon \Sigma_g \to \Sigma_g$ is orientation-preserving.*
(2) *The hyperelliptic involution $\Theta\colon \Sigma_g \to \Sigma_g$ has precisely $2g+2$ fixed points.*
(3) *The map $\Theta_*\colon H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ equals $-\operatorname{id}_{H_1(\Sigma_g;\mathbb{Z})}$.*
(4) *The quotient space $\Sigma_g/x \sim \Theta(x)$ is homeomorphic to the 2-sphere and it can be given a smooth structure such that the quotient map $\Sigma_g \to \Sigma_g/x \sim \Theta(x)$ is a 2-fold branched covering where the branch locus consists of $2g+2$ points.*

Sketch of proof.

(1) It follows easily from Proposition 47.3, applied to the origin, that the hyperelliptic involution Θ is orientation-preserving.
(2) It is straightforward to show that the fixed points of the hyperelliptic involution are given by the origin and the points that are represented by the $4g+2$ midpoints of the sides of the $(4g+2)$-gon. Since opposite midpoints get identified we see that we have precisely $1 + \frac{1}{2}(4g+2) = 2g+2$ fixed points.
(3) We equip E_{4g+2} with the obvious CW-structure with $2g+2$ 0-cells, $2g+2$ 1-cells and one 2-cell. Since the involution identifies cells with cells we can equip $\Sigma_g = E_{4g+2}/\sim$ with a corresponding quotient CW-structure. The hyperelliptic involution is a cellular map and one can easily compute that the induced map on $H_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z})$ is $-\operatorname{id}$. By Proposition 35.4 we know that singular homology is *naturally* isomorphic to cellular homology. In our setting this means that we have a commutative diagram

$$\begin{tikzcd} H_1(\Sigma_g;\mathbb{Z}) \arrow[r] \arrow[d, "f_*"'] & H_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z}) \arrow[d, "f_*"] \\ H_1(\Sigma_g;\mathbb{Z}) \arrow[r] & H_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z}), \end{tikzcd}$$

[1255]Recall that a map $f\colon X \to X$ is called an *involution* if $f \circ f = \operatorname{id}_X$.
[1256]I do not know where the fancy name "hyperelliptic" comes from.

where the horizontal maps are the same. Since the right vertical map is $-\mathrm{id}$ we now see that the left vertical map is also $-\mathrm{id}$.

(4) We leave it as an exercise to show that the quotient space $\Sigma_g/x \sim \Theta(x)$ is a closed orientable connected 2-dimensional manifold such that the quotient map $\Sigma_g \to \Sigma_g/x \sim \Theta(x)$ is a 2-fold branched covering with branch locus given by the images of the fixed points.[1257] Once we believe this fact we can calculate that

$$2-2g \;=\; \chi(\Sigma_g) \;\underset{\substack{\uparrow \\ \text{Lemma 75.1}}}{=}\; 2\cdot\chi(\Sigma_g/\sim) - \chi(\text{branch locus}) \;\underset{\substack{\uparrow \\ \text{by (1) there exist } 2g+2 \text{ fixed points}}}{=}\; 2\cdot\chi(\Sigma_g/\sim) - (2g+2).$$

Thus we see that $\chi(\Sigma_g/\sim) = 2$. It follows from Theorem 14.9 that $\Sigma_g/\sim$ is indeed diffeomorphic to the 2-dimensional sphere.[1258] □

More interestingly, the hyperelliptic involution is in some sense fully determined by being an orientation-preserving involution and satisfying Statement (2) or Statement (3) of Lemma 75.2. More precisely, the following proposition holds:

Proposition 75.3. *Let $g \in \mathbb{N}_0$ and let Ξ be an orientation-preserving involution of Σ_g. Suppose one of the following two conditions is satisfied:*

(1) *The map Ξ has precisely $2g+2$ fixed points.*
(2) *The map $\Xi_*\colon H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ equals $-\mathrm{id}$.*

Then Ξ and the hyperelliptic involution Θ represent conjugate elements in the orientation-preserving mapping class group $\mathrm{MCG}^+(\Sigma_g)$.

Proof. This statement is proved in Proposition 7.15 and the discussion on page 216 of [FaM]. □

Examples.

(1) Using Proposition 75.3 one can show easily that the maps

$$\begin{array}{ccc} \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [(x,y)] & \mapsto & [(-x,-y)] \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1\times S^1 & \to & S^1\times S^1 \\ (z,w) & \mapsto & (z^{-1},w^{-1}), \end{array}$$

under a suitable diffeomorphism between $\mathbb{R}^2/\mathbb{Z}^2 = S^1\times S^1$ and $E_6/\sim$, correspond to the hyperelliptic involution.[1259] In Lemma 1.58 we gave an explicit diffeomorphism between $S^1\times S^1$ and a specific submanifold of $\mathbb{R}^3$. Under this diffeomorphism the above hyperelliptic involution corresponds to rotation by π around a line in the xy-plane through the origin.

[1257] In the verification of this statement it can be useful to keep the second example on page 1744 in mind.

[1258] It always feels like cheating to use the calculation of Euler characteristics together with Theorem 14.9 to determine the diffeomorphism type of a given closed orientable connected 2-dimensional manifold. Can you find a more direct proof that $\Sigma_g/\sim$ is diffeomorphic to S^2?

[1259] Clearly both maps are orientation-preserving involutions. Thus it remains to show that these involutions of $\mathbb{R}^2/\mathbb{Z}^2$ and $S^1\times S^1$ have precisely four fixed points. What are they?

In Figure 936 we show the torus as a submanifold of $\mathbb{R}^3$ together with the hyperelliptic involution. We do our best to visualize that the quotient of the torus is indeed given by the sphere.

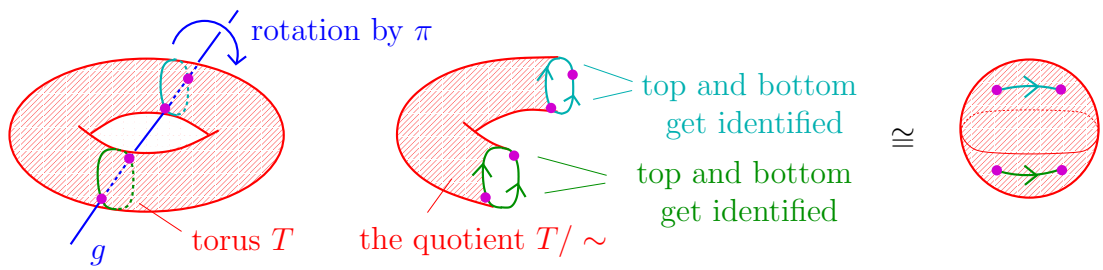


FIGURE 936.

(2) Using Proposition 75.3 we can now also visualize the hyperelliptic involution in our favorite "physical" model of the surface of genus g. More precisely, in Figure 937 on the right we show the surface of genus 2 as a submanifold of $\mathbb{R}^3$. Rotation by π around the x-axis is an orientation-preserving involution which has 6 fixed points. It follows from Proposition 75.3 that, under a suitable diffeomorphism to Σ_g, this involution corresponds to the hyperelliptic involution. For convenience we refer to this involution also as the hyperelliptic involution. In Figure 938 we indicate why the quotient is indeed the sphere.

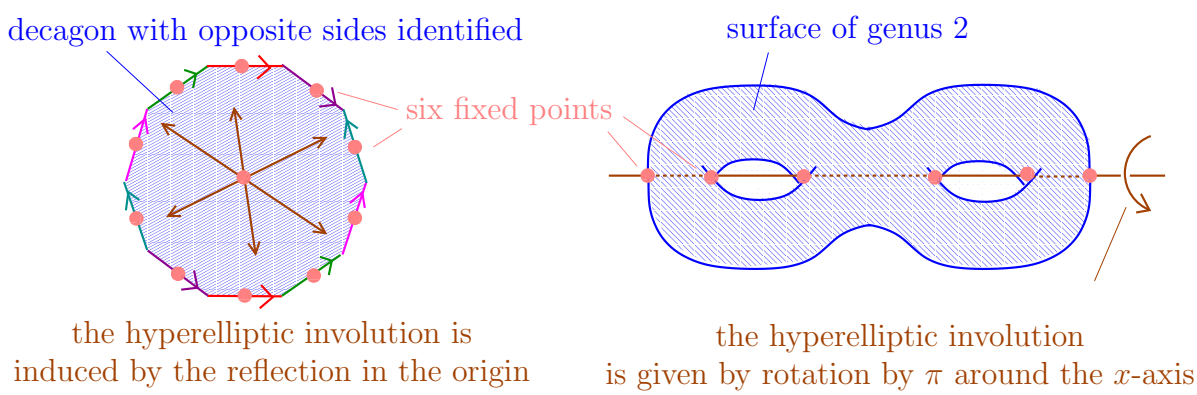


FIGURE 937.

75.3. **Branched coverings of 3-manifolds.** In this section we consider branched coverings of 3-manifolds. Before we start out with definitions and examples we provide a word of caution. Whenever one wants to deal with "real-life topological spaces" it is very hard to write down everything absolutely rigorously. In this chapter we will therefore take some liberties and we will not write down every argument in full detail.

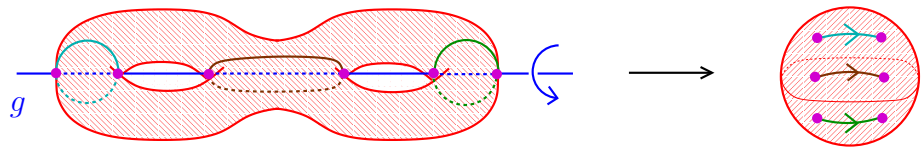


FIGURE 938.

As a balm to our mathematical conscience we recall that we pointed out on page 1692, topological 3-dimensional manifolds admit a unique smooth structure. Hence we do not need to distinguish between the concepts of being "homeomorphic" and "diffeomorphic".[1260]

Before we give a 3-dimensional example of a branched covering we need to introduce one more definition.

Definition.

(1) An *arc* in a 3-manifold M is a proper submanifold of M that is diffeomorphic to the interval $[0,1]$.

(2) We say arcs $A_1,\dots,A_m$ in $\overline{B}^3$ are *unknotted* if there exists a homeomorphism $f\colon \overline{B}^3 \to \overline{B}^2 \times [0,1]$ and distinct points $P_1,\dots,P_m$ such that for $i = 1,\dots,m$ we have $f(A_i) = P_i \times [0,1]$.

Example. In Figure 939 we show to the left an arc in the solid torus. To the right we show two arcs in $\overline{B}^3$ that are unknotted.

FIGURE 939.

Now we can give the promised examples of branched coverings of 3-dimensional manifolds.

Examples.

(1) We consider the map

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (re^{i\varphi}, z) & \mapsto & (re^{-i\varphi}, z^{-1}). \end{array}$$

[1260]This fact that by smoothing we can always replace a homeomorphism by a diffeomorphism should not be overgeneralized. For example, Bing [Bin2] gave an example of an involution f on S^3 that is not isotopic to a smooth involution. More precisely, we can isotope f to a smooth map, but we can no longer arrange that that map is an involution.

The map Θ is an involution that restricts to the hyperelliptic involution on the boundary. In Lemma 1.58 we had identified the solid torus $\overline{B}^2 \times S^1$ with a submanifold of $\mathbb{R}^3$. Under this identification the above involution Θ corresponds to the rotation by π around a line in the xy-plane. This map restricts to an involution on the solid torus $\overline{B}^2 \times S^1 \subset \mathbb{R}^3$. As is illustrated in Figure 940, the quotient $\overline{B}^2 \times S^1/x \sim \Theta(x)$ is the 3-ball and the quotient map $p\colon \overline{B}^2 \times S^1 \to \overline{B}^3$ is a 2-fold branched covering where the branch locus is given by two unknotted arcs.[1261]

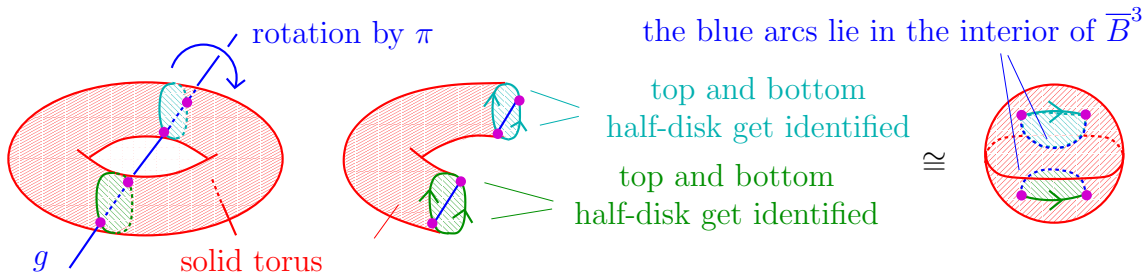


FIGURE 940.

(2) Above we saw that the surface of genus g is the branched covering of S^2 with branch locus given by $2g+2$ points. Similarly to (1) we can now replace the surface of genus g by the handlebody H_g of genus g and we obtain a branched covering of H_g over the 3-ball $\overline{B}^3$ where the branch locus is given by $g+1$ unknotted arcs.

It follows from the classification of closed connected 2-dimensional manifolds, see Theorem 14.10, together with discussion on page 1749 that every closed orientable connected 2-dimensional manifold Σ admits a simple branched covering over the 2-dimensional sphere. The following theorem says that a similar statement also holds in the 3-dimensional setting.

Theorem 75.4. (Hilden-Montesinos) *Every closed orientable connected 3-dimensional manifold M admits a map $p\colon M \to S^3$ which is a simple 3-fold branched covering (not necessarily regular) with branch locus a knot.*

Remark. The theorem has its origins in the work of James Alexander [Al3] from 1920. He had claimed that every closed orientable 3-dimensional manifold admits a map $p\colon M \to S^3$ which is a branched covering with branch locus a link. But the definition of a branched covering map in [Al3] is weaker than our notion and few details were provided in [Al3].

[1261] It is painful, but not impossible to prove this statement rigorously. Indeed, first of all we can replace $\overline{B}^2 \times S^1$ by $([-1,1]/-1 \sim 1) \times [0,2]^2$ where the involution is given by $(s,x,y) \sim (-s,x,2-y)$. The quotient space is then homeomorphic to $[0,1] \times [0,2]^2$ where $(0,x,y) \sim (0,x,2-y)$ and $(1,x,y) \sim (1,x,2-y)$. Using this description it is no longer that difficult to show that the quotient space is homeomorphic to $\overline{B}^2 \times [0,1]$ where the fixed point set is given by two arcs of the form $P_1 \times [0,1]$ and $P_2 \times [0,1]$. We leave it to the reader to fill in the details.

Proof. The theorem was independently proved by Hugh Hilden [Hild] and José Montesinos [Mon] in the early 1970s. An exposition of the proof is also given in [PrS, Theorem 23.1]. The definition of a branched covering used in the above references is slightly less restrictive, but it is clear from the proofs that they provide branched coverings in our sense. □

75.4. **Branched coverings defined by knots.** In the previous sections we saw that several maps $p\colon \widetilde{M} \to M$ between manifolds are branched coverings and we had determined the branch locus of those branched coverings.

In this section we go the other way round. We set $M = S^3$, we prescribe the branch locus as a given knot K and we describe a degree m. Given this data we will construct an essentially unique m-fold branched covering $p\colon \Sigma_m(K) \to S^3$ with branch locus K.

Definition. Let K be a knot in S^3 and let $m \in \mathbb{N}$.

(1) By Lemma 70.6 (2) we can pick a fattening $F\colon \overline{B}^2 \times K \to S^3$ with self-linking number 0.
   (a) As usual we write $X_K := S^3 \setminus F(B^2 \times K)$.
   (b) We equip X_K with the natural orientation coming from S^3.
   (c) We denote by $f\colon S^1 \times K \to \partial X_K = \partial F(\overline{B}^2 \times K)$ the diffeomorphism that is given by restricting F to the boundary torus $S^1 \times K$.
   (d) We denote by $\phi_K\colon \pi_1(X_K) \to H_1(X_K;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ one of the two epimorphisms. (It will not matter which one we take.)

(2) By Proposition 20.8 there exists a connected covering $p\colon \widetilde{X_K} \to X_K$ that corresponds to $\ker\{\pi_1(X_K) \xrightarrow{\phi_K} \mathbb{Z} \to \mathbb{Z}_m\}$, i.e. that satisfies

$$p_*(\pi_1(\widetilde{X_K})) \;=\; \ker\left\{\pi_1(X_K) \xrightarrow{\phi_K} \mathbb{Z} \to \mathbb{Z}_m\right\}.$$

Furthermore, by Proposition 20.9 this covering is in an appropriate sense unique. We equip $\widetilde{X_K}$ with the orientation coming from Proposition 8.1 and coming from the above orientation on X_K.

(3) We consider the branched covering

$$\begin{array}{rcl} q\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (z,p) & \mapsto & (z^m,p). \end{array}$$

Since the self-linking number of F was chosen to be zero, and by the uniqueness statement of Proposition 20.9, we see that the restriction of q to the boundary $S^1 \times K$ can be identified with the m-fold covering corresponding to

$$\ker\left\{\pi_1(S^1 \times K) \xrightarrow{f_*} \pi_1(X_K) \to \mathbb{Z} \to \mathbb{Z}_m\right\}.$$

(4) It follows from (3) that f lifts to a diffeomorphism $\widetilde{f}\colon S^1 \times K \to \partial\widetilde{X_K}$ such that the following diagram commutes

$$\begin{array}{ccc} S^1 \times K & \xrightarrow{\widetilde{f}} & \partial\widetilde{X_K} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times K & \xrightarrow{f} & \partial X_K. \end{array}$$

(5) We define the *m-fold branched covering of K* to be the topological space

$$\Sigma_m(K) \ := \ \left(\overline{B}^2 \times K\right) \cup_{\widetilde{f}} \widetilde{X_K}.$$

The following proposition summarizes some basic properties of branched coverings of knots.

Proposition 75.5. *Let K be a knot and let $m \in \mathbb{N}$.*

(1) *The m-fold branched covering $\Sigma_m(K)$ is a closed orientable 3-dimensional topological manifold. It admits a unique smooth structure and a unique orientation such that $\widetilde{X_K}$ is a submanifold and such that the orientation on the submanifold $\widetilde{X_K}$ agrees with the orientation coming from Proposition 8.1 and the natural orientation on $X_K \subset S^3$.*

(2) *The oriented diffeomorphism type of $\Sigma_m(K)$ is independent of all of our choices.*

(3) *The map*

$$\begin{array}{ccccc} \Sigma_m(K) & = & \overline{B}^2 \times K & \cup_{\widetilde{f}} & \widetilde{X_K}. \\ \downarrow & & \downarrow q & & \downarrow p \\ S^3 & = & \overline{B}^2 \times K & \cup_f & X_K \end{array}$$

is well-defined and it is a regular simple m-fold branched covering with branch locus given by the knot K.

(4) *If two knots K and J are equivalent, then there exists an orientation-preserving diffeomorphism $\Sigma_m(K) \to \Sigma_m(J)$.*

(5) *Let K be a knot in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Let $K^* \subset \mathbb{R}^3$ be a mirror image, i.e. $K^* = \rho(K)$ where ρ is the reflection in a hyperplane of $\mathbb{R}^3$. There exists an* orientation-reversing *diffeomorphism from $\Sigma_m(K)$ to $\Sigma_m(K^*)$.*

(6) *If K is amphichiral, then $\Sigma_m(K)$ is amphichiral.*

(7) *We have isomorphisms*

$$H_1(\Sigma_m(K);\mathbb{Z}) \oplus \mathbb{Z} \cong H_1(\widetilde{X}_K;\mathbb{Z}) \cong H_1\left(\begin{matrix}\text{covering of } S^3 \setminus K \text{ corresponding to} \\ \text{the epimorphism } \pi_1(S^3 \setminus K) \to \mathbb{Z}_m\end{matrix};\mathbb{Z}\right).$$

(8) *If p is a prime number and $r \in \mathbb{N}$, then the homology group $H_1(\Sigma_{p^r}(K);\mathbb{Z})$ is finite.*

Sketch of proof.

(1) The first statement follows from Proposition 10.12.

(2) The second statement follows fairly easily from Propositions 20.9 and 64.9.

(3) The third statement follows easily from the definitions.

(4) Suppose that K and J are equivalent. This means that there exists a smooth isotopy from the knot K to the knot J. As we pointed out in the proof of Proposition 10.33, the smooth isotopy can be extended to a smooth isotopy of S^3 carrying K to J. It now follows easily from the definitions that there exists an orientation-preserving diffeomorphism $\Sigma_m(K) \to \Sigma_m(J)$.
(5) Let ρ be a reflection in a hyperplane of $\mathbb{R}^3$. From Exercise 1.19 we obtain that, by setting $\rho(\infty) = \infty$, the map ρ extends to a homeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We denote this homeomorphism again by ρ. It follows from Proposition 47.3, say applied to the origin, that $\rho\colon S^3 \to S^3$ is orientation-reversing. It follows again easily from the definitions that ρ induces an orientation-reversing homeomorphism $\Sigma_m(K) \to \Sigma_m(K^*)$.
(6) This statement is an immediate consequence of (4) and (5).
(7) An amusing Meyer–Vietoris argument shows that there exists an isomorphism as stated on the left. The inclusion of the knot exterior into the knot complement is a homotopy equivalence, therefore it induces an isomorphism as stated on the right.
(8) The last statement is the only statement that is non-trivial. The case $p = 2$ and $r = 1$ is proved for example on [Rol, p. 213]. The general case is proved in [Gor, p. 362]. □

Now we can consider some examples of branched coverings of knots.

Examples.

(1) The 1-fold branched covering of a knot K is, basically by definition, given by S^3.
(2) For K the trivial knot it is an amusing exercise to verify that all branched coverings are again diffeomorphic to S^3.
(3) The following converse to (2) holds: If K is a knot and $m \in \mathbb{N}_{\geq 2}$ such that $\Sigma_m(K)$ is diffeomorphic to S^3, then K is equivalent to the trivial knot. This is a consequence of the *Smith Conjecture* which was proved in 1984, see [MB].[1262]
(4) More generally one can ask to what degree branched coverings remember the original knot K. The bad news is that by [Nak, Sak] there are many examples of triples (K, K', m) where K and K' are inequivalent "prime" knots and where $m \in \mathbb{N}_{\geq 2}$ such that $\Sigma_m(K)$ is diffeomorphic to $\Sigma_m(K')$. The good news is that it is shown in [Koj] that if K and K' are two "prime" knots such that $\Sigma_m(K)$ and $\Sigma_m(K')$ are diffeomorphic for infinitely many $m \in \mathbb{N}$, then K and K' are equivalent.

The idea now is of course to apply Proposition 75.5 to settle Question 47.2, namely to show that the trefoil is not amphichiral. Therefore we need to determine at least one interesting branched covering of the trefoil.

Definition. Let $J \subset S^3$ be a knot.

[1262]The Smith Conjecture, which was originally formulated in 1949 [Eil, Problem 36], says that if f is a diffeomorphism of S^3 of finite order, i.e. such that there exists an $n \in \mathbb{N}$ with $f^n = \mathrm{id}_{S^3}$, then the fixed point set of f cannot be a nontrivial knot. The conjecture was proved in 1984 [MB]. By [Gif] the analogous statement is wrong for 4-dimensional manifolds.

(1) We write

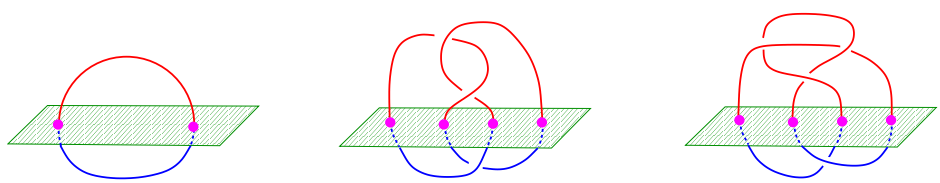
$$\begin{array}{rclcrcl} S^3_{\geq 0} & := & \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 \geq 0\} & \text{and} & J_{\geq 0} & := & J \cap S^3_{\geq 0} \\ S^3_{\leq 0} & := & \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 \leq 0\} & \text{and} & J_{\leq 0} & := & J \cap S^3_{\leq 0} \end{array}$$

and

$$S^3_{=0} := \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 = 0\} \quad \text{and} \quad J_{=0} := J \cap S^3_{=0}.$$

(2) We say J is in *m-bridge position* if the following three conditions are satisfied:
   (a) J intersects the sphere $S^3_{=0} = \text{xy-plane} \cup \{\infty\}$ transversally,
   (b) the intersection $J_{\geq 0} = J \cap S^3_{\geq 0}$ consists of m trivial arcs in the 3-ball $S^3_{\geq 0}$,
   (c) the intersection $J_{\leq 0} = J \cap S^3_{\leq 0}$ consists of m trivial arcs in the 3-ball $S^3_{\leq 0}$.

Example. In Figure 941 we show three knots which are isotopic to the trivial knot, the trefoil and the figure-8 knot.[1263] The first knot is in 1-bridge position whereas the latter two knots are in 2-bridge position.



trivial knot the trefoil figure-8 knot

FIGURE 941.

Lemma 75.6. *Every knot K in S^3 is equivalent to a knot J that is in m-bridge position for some m.*

Sketch of proof. If a knot K is represented by a diagram with m crossings, then it is already in an m-bridge position. We refer to [BZH, Chapter 2.D] for more details. □

Definition. Given a knot $K \subset S^3$ we define its *bridge number* as

$$b(K) := \min\{m \in \mathbb{N} \,|\, K \text{ is equivalent to a knot in } m\text{-bridge position}\}.$$

If $b(K) = m$, then we also just say that K is an *m-bridge knot*.

Examples.

(1) If K is a knot with bridge number 1, then K is equivalent to the trivial knot. This sounds pretty convincing, but as so often, it is rather painful to write down a completely rigorous argument.
(2) It follows from (1) and Figure 941 that the trefoil and the figure-8 knot are 2-bridge knots.

[1263] We leave it as an exercise to verify that the knots pictured in Figure 941 are in fact isotopic to the knots shown in Figure 225.

Proposition 75.7. *Let K be a knot. The 2-fold branched covering $\Sigma_2(K)$ admits a Heegaard splitting of genus $b(K)-1$.*

Sketch of proof. By Proposition 75.5(4) we can assume that K itself is already in m-bridge position. We denote by $p\colon \Sigma_2(K) \to S^3$ the branched covering with branch locus K. The restriction of p to $p^{-1}(S^3_{\geq 0}) \to S^3_{\geq 0}$ is a 2-fold branched covering with branch locus $K_{\geq 0}$. Since $K_{\geq 0}$ are unknotted arcs this branched covering is essentially the same as the branched covering considered on page 1751.[1264] But this implies that $p^{-1}(S^3_{\geq 0})$ is a genus $m-1$ handlebody. Similarly we see that $p^{-1}(S^3_{\leq 0})$ is a genus $m-1$ handlebody and that the intersection of these two handlebodies, which is given by $p^{-1}(S^3_{=0})$ is a surface of genus $m-1$. But that means precisely that $\Sigma_2(K)$ admits a genus $m-1$ Heegaard splitting. □

Corollary 75.8. *If K is a knot with bridge number $b(K)=2$, then $\Sigma_2(K)$ is diffeomorphic to a lens space.*

Proof. Let K be a knot with bridge number $b(K)=2$. It follows from Proposition 75.7 that the 2-fold branched covering $\Sigma_2(K)$ admits a Heegaard splitting of genus one. It follows from Proposition 75.5 (1) and Corollary 73.10 that $\Sigma_2(K)$ is either diffeomorphic to a lens space or to $S^1 \times S^2$. In Proposition 75.5 (8) we saw that $H_1(\Sigma_2(K);\mathbb{Z})$ is finite, thus we can exclude the possibility that $\Sigma_2(K)$ is diffeomorphic to $S^1 \times S^2$. □

We had just seen that the trefoil and the figure-8 knot are 2-bridge knots. So their 2-fold branched coverings are lens spaces. The following proposition tells us what these 2-fold branched coverings are.

Proposition 75.9. [1265]

(1) *The 2-fold branched covering of the trefoil is diffeomorphic to the lens space $L(3,2)$.*
(2) *The 2-fold branched covering of the figure-8 knot is diffeomorphic to the lens space $L(5,2)$.*

Remark. Horst Schubert [Schub] showed in 1956 that every lens space is the 2-fold branched covering of a 2-bridge knot.

Proof. Let K be a 2-bridge knot. There is a general algorithm that determines the lens space which equals the 2-fold branched covering $\Sigma_2(K)$, see e.g. [BZH, Proposition 12.3]. The statement about the trefoil is then implied by [BZH, Example 12.16]. The statement about the figure-8 knot is implicit in [BZH, Section 12].

There are also several other proofs for the first statement:

(1) Let K be the trefoil. It follows from the calculation on page 542, the Hurewicz Theorem 38.5, Proposition 75.5 and Lemma 11.8 (1) that $H_1(\Sigma_2(K);\mathbb{Z}) \cong \mathbb{Z}_3$.

[1264] An argument similar to the proof of Proposition 75.5 (2) shows that all 2-fold branched coverings of the 3-ball with branch locus given by two unknotted arcs are homeomorphic.

[1265] Since it is exceedingly painful to keep track of orientation conventions we will not attempt to determine whether there exist orientation-preserving or orientation-reversing diffeomorphisms between the branched coverings and the lens spaces.

Furthermore, by Corollary 75.8 we know that $\Sigma_2(K)$ is a lens space. Thus we see that $\Sigma_2(K)$ is diffeomorphic to $L(3,1)$ or to $L(3,2)$. But these two lens spaces are diffeomorphic by Lemma 7.7.

(2) We also refer to [Sti, Section 7.2.2] and [Rol, p. 304] for two alternative proofs of the first statement. □

Finally we can answer Question 47.2 in the negative.

Proposition 75.10. *The trefoil is chiral.*

Proof. Let K be the trefoil. By Proposition 75.9 (9) we know that $\Sigma_2(K)$ is diffeomorphic to the lens space $L(3,2)$. We know from Proposition 74.23 that $L(3,2)$ is chiral. But by Proposition 75.5 (6) this implies that K is also chiral. □

We have now shown that the trefoil is not equivalent to its mirror image. Therefore one often distinguishes in the literature between the left-handed trefoil (which equals our trefoil from before) and the right-handed trefoil. The names stem from the observation that if we equip the right-handed trefoil with either of the two orientations the following holds: if the thumb of the right-hand points in the direction of the overcrossing-strand, then the fingers point in the direction of the undercrossing-strand. (It is a fun exercise to verify this statement.)

left-handed trefoil right-handed trefoil

FIGURE 942.

Remark. On page 1154 we saw that $L(5,2)$ is amphichiral. This is consistent with the fact, illustrated in Figure 673, that the figure-8 knot is amphichiral and Propositions 75.9 and 75.5 (6).

Remark. In this chapter we studied questions about embeddings of a given closed k-dimensional manifold W into a given closed $(k+2)$-dimensional manifold M, namely of embeddings of $W = S^1$ into $M = S^3$, by associating to an embedding a branched covering. The same approach can also be used in many other contexts, for example it can and is being used to study embeddings of surfaces into closed 4-manifolds, see e.g. [HsS].

Exercises for Chapter 75.

Exercise 75.1. The definition of the m-fold branched cover of a knot easily in S^3 generalizes to the notion of an m-fold branched cover of a link in S^3.

(a) We consider the trivial 2-component link L_2. What is the diffeomorphism type of the corresponding 2-fold branched cover?
(b) We consider the trivial 2-component link L_2. What is the diffeomorphism type of the corresponding 5-fold branched cover?
(c) We consider the trivial 2-component link L_3. What is the diffeomorphism type of the corresponding 2-fold branched cover?

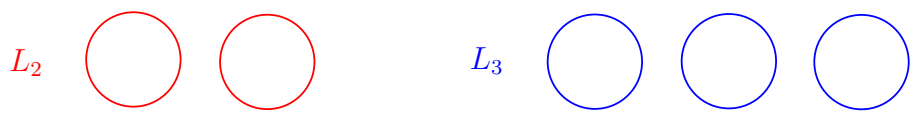


FIGURE 943.

76. Cobordisms

In Question 61.7 we had asked whether every closed orientable manifold is the boundary of a compact manifold. In this chapter we will see that the answer is very much no. In fact, we will "quantify" to what degree the answer is no by introducing the "cobordism group", which is an interesting object in its own right.

76.1. **The cobordism rings.** Given two closed n-dimensional manifolds M and N we obtain a new closed n-dimensional manifold by taking the disjoint union $M \sqcup N$. This operation has a neutral element, given by the empty manifold, and it looks abelian and associative.[1266] Thus it looks that all closed n-dimensional manifolds together with the operation "$\sqcup$" form a monoid.

For the reader's convenience we recall the definition of a monoid.

Definition. A *monoid* is a set S together with a map

$$\begin{array}{rcl} S \times S & \to & S \\ (x, y) & \mapsto & x \cdot y \end{array}$$

such that the following two conditions are satisfied:[1267]

(1) there exists a neutral element $e \in S$, i.e. an element which satisfies $e \cdot x = x \cdot e = x$ for all $x \in S$,
(2) the operation "$\cdot$" is associative, i.e. for all $x, y, z \in S$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Careful comparison between the definition of a monoid and the discussion of manifolds above reveals a subtle problem: "all closed n-dimensional manifolds" do not form a set. As most mathematicians I have no intention of getting into the debate about what is a set. Instead we refer to [FPr] and [Halm] for details. Also, at this stage reading "Logicomix" [DoP], see Figure 944, is almost a must.

Fortunately, we are saved by the following lemma.

Lemma 76.1. *Let $n \in \mathbb{N}_0$. The diffeomorphism classes of n-dimensional manifolds form a set.*

Remark.

(1) Note that here it is essential that manifolds are assumed to be second-countable. If we did not make this assumption, then any set with the discrete topology would be a 0-dimensional manifold, but the bijection classes of sets do not form a set.
(2) In [CK] it is shown that in any dimension there exist only finitely many homeomorphism classes of compact topological manifolds. Thus homeomorphism classes of classes of compact topological manifolds form a set.

[1266] Very well-trained eyes will spot that purely set theoretically, $M \sqcup N$ and $N \sqcup M$ are not exactly the same, see the definition on page 83.

[1267] A *semi-group* is defined similarly, except that we do not demand the existence of a neutral element.

FIGURE 944.

Sketchy proof. In Proposition 65.11 and the Whitney Embedding Theorem 65.18 we saw that any n-dimensional manifold is diffeomorphic to a submanifold of $\mathbb{R}^{2n+1}$. Thus we can identify the "diffeomorphism classes of closed n-dimensional manifolds" with a subset of the power set of $\mathbb{R}^{2n+1}$, which is a set. $\square$

Definition.

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathcal{M}_n$ the set of all diffeomorphism classes of closed oriented n-dimensional manifolds.[1268][1269]
(2) Given $S, T \in \mathcal{M}_n$ we say that S and T are *cobordant* if there exists a *cobordism between S and T*, i.e. a compact oriented $(n+1)$-dimensional manifold W such that[1270] ∂W admits an orientation-preserving diffeomorphism to $S \cup -T$. If S and T are cobordant, then we write $S \sim T$.

Convention. In the following we will often be slightly sloppy in our notation, meaning that we will not distinguish between a manifold and its diffeomorphism class. For example, given oriented manifolds W, S and T we often write $\partial M = S \sqcup T$, even though strictly speaking we have $\partial M = S' \sqcup T'$ where S' and T' are oriented manifolds that admit an orientation-preserving diffeomorphism to S respectively T.

[1268]Recall that the empty set is a closed oriented n-dimensional manifold.

[1269]Also recall that an orientation for a 0-dimensional manifold M consists of a choice of a sign in $\{\pm 1\}$ for each point in M, see the definition on page 1585.

[1270]Recall that each component of ∂W inherits an orientation from the orientation of W.

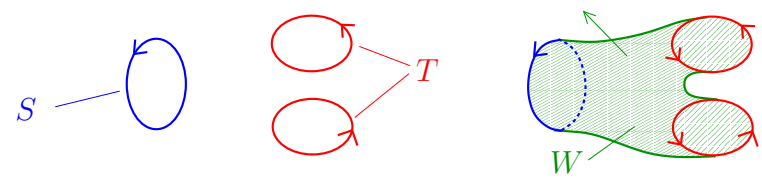


FIGURE 945.

Lemma 76.2. *Let $n \in \mathbb{N}_0$.*

(1) *The relation $\sim$ on $\mathcal{M}_n$ is an equivalence relation.*
(2) *The map*

$$\begin{array}{rcl} \mathcal{M}_n \times \mathcal{M}_n & \to & \mathcal{M}_n \\ ([S],[T]) & \mapsto & [S \sqcup T] \end{array}$$

is well-defined and it descends to an abelian group structure on the set $\mathcal{M}_n/\sim$[1271] *where the neutral element is given by the empty manifold.*

Proof.

(1) We first show that the relation $\sim$ on $\mathcal{M}_n$ is an equivalence relation.
   (a) Given $M \in \mathcal{M}_n$ we have $M \sim M$ since, by the discussion on page 1586, we have $\partial([0,1] \times M) = \{0\} \times (-M) \cup \{1\} \times M$.
   (b) Suppose that $S \sim T$, i.e. there exists a compact oriented $(n+1)$-dimensional manifold W with $\partial W = S \sqcup -T$. Then $\partial(-W) = -S \sqcup T = T \sqcup (-S)$ which shows that $T \sim S$.
   (c) Finally suppose that we have $R, S, T \in \mathcal{M}_n$ with $R \sim S$ and $S \sim T$. This means that there exist compact oriented $(n+1)$-dimensional manifolds X and Y with $\partial X = R \sqcup -S$ and $\partial Y = S \sqcup -T$. We set $Z := X \cup_S Y$. It follows from Proposition 10.12 that Z is a compact orientable $(n+1)$-dimensional manifold with $\partial Z = R \sqcup -T$. We refer to Figure 946 for an illustration.

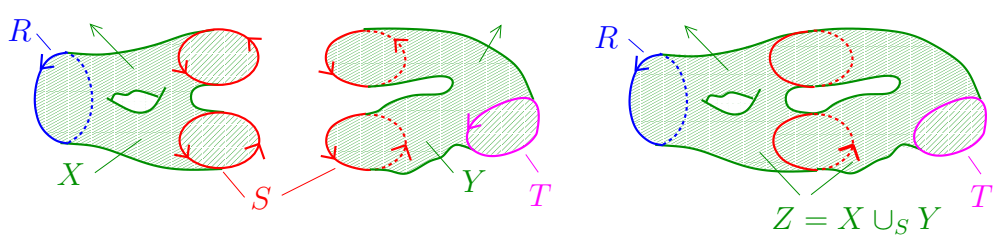


FIGURE 946.

(2) This statement follows from the following five observations:
   (a) One can easily verify that the definition does not depend on the choice of the representatives in $\mathcal{M}_n$.

[1271] By Lemma 76.1 we know that $\mathcal{M}_n$ and thus also $\mathcal{M}_n/\sim$ are a set.

(b) It is clear that the empty manifold is a neutral element for the above operation.
(c) We need to show that any $[M] \in \mathcal{M}_n$ has an inverse element. In fact this is represented by the oriented manifold $-M$, since by the discussion on page 1586, we have $\partial([0,1]\times M) = \{0\}\times(-M)\cup\{1\}\times M$, i.e. the manifold $[0,1]\times M$ defines a cobordism from $M \sqcup -M$ to the empty manifold. We refer to Figure 947 for an illustration.
(d) The product structure is commutative since $A \sqcup B$ and $B \sqcup A$ are naturally diffeomorphic, see Lemma 1.49 (4).[1272]
(e) The product structure is associative since $(A \sqcup B) \sqcup C$ and $A \sqcup (B \sqcup C)$ are naturally diffeomorphic, see Lemma 1.49 (5). □

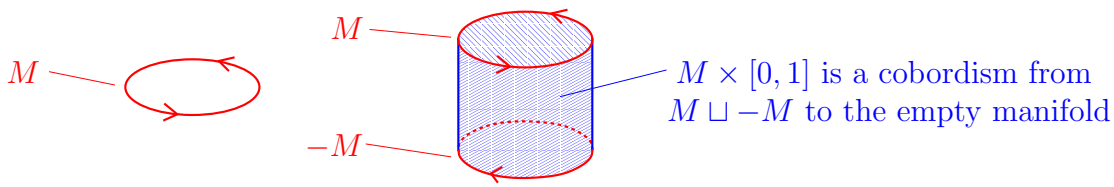


FIGURE 947.

Definition. Let $n \in \mathbb{N}_0$. We refer to

$$\Omega_n \ := \ (\mathcal{M}_n/\sim, \sqcup)$$

as the *n-th cobordism group.*

We have another natural operation on oriented connected manifolds, namely the connected sum operation. The following lemma says that up to cobordisms the connected sum operation is the same as the disjoint union.

Lemma 76.3. *Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional manifolds, then*

$$M\#N \ = \ M \sqcup N \ \in \ \Omega_n.$$

Sketch of proof. Let M and N be two closed oriented connected n-dimensional manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing.

We consider $[0,1]\times M$ and $[0,1]\times N$. Note that by the discussion on page 1586, we have $\partial([0,1]\times M) = (\{0\}\times(-M)) \cup (\{1\}\times M)$ and $\partial([0,1]\times N) = (\{0\}\times(-N)) \cup (\{1\}\times N)$. By a slight abuse of notation we denote by φ and ψ the obvious embeddings of $\overline{B}^n$ into $\{1\}\times M$ and $\{1\}\times N$. We use these two embeddings to form the boundary connected sum

$$W \ := \ ([0,1]\times M)\#_b([0,1]\times N) \ := \ ([0,1]\times M) \sqcup ([0,1]\times N)/\varphi(x) = \psi(x) \text{ for } x \in S^{n-1}$$

[1272]Strictly speaking Lemma 1.49 says that the two sides are naturally homeomorphic. But the same way we see that the two sides are diffeomorphic.

that we introduced on page 1690.

By Proposition 73.1 we know that W admits the structure of an $(n+1)$-dimensional orientable manifold such that

$$\partial W \;=\; \underbrace{(\{1\}\times M)\#(\{1\}\times N)}_{=M\#N} \sqcup \underbrace{(-\{0\}\times M)}_{=-M} \sqcup \underbrace{(-\{0\}\times N)}_{=-N}.$$

We have thus found the desired cobordism. □

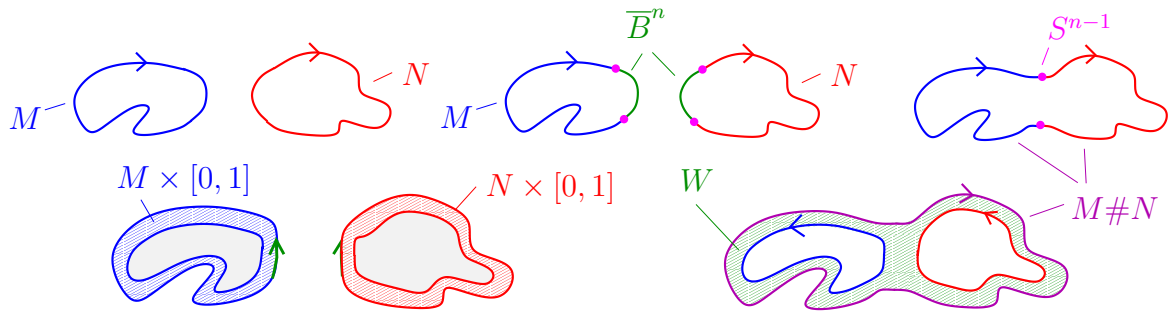


FIGURE 948.

Definition. If we use exactly the same definitions as before except that we drop the condition that the manifolds are orientable and we drop the condition that the cobordisms are orientable, then given $n\in\mathbb{N}_0$ we obtain the *unoriented cobordism group*

$$\mathcal{N}_n \;:=\; \{\text{closed } n\text{-dimensional manifolds}\}/\text{unoriented cobordisms}.$$

The following lemma summarizes a few basic facts about unoriented cobordism groups.

Lemma 76.4.

(1) *Let $n\in\mathbb{N}_0$. Every non-trivial element in $\mathcal{N}_n$ is of order two, i.e. given any $M\in\mathcal{N}_n$ we have $M\sqcup M=0\in\mathcal{N}_n$.*

(2) *Let $n\in\mathbb{N}$. Let M and N be two closed connected n-dimensional manifolds.*[1273] *Then*

$$M\#N \;=\; M\sqcup N \;\in\; \mathcal{N}_n.$$

Proof. Let $n\in\mathbb{N}_0$.

(1) Let $M\in\mathcal{N}_n$. We have

$$2\cdot M \;=\; M\sqcup M \;=\; (\{0\}\times M)\sqcup(\{1\}\times M) \;\underset{\uparrow}{=}\; \partial([0,1]\times M).$$

by Proposition 3.42 we have this equality of unoriented manifolds

[1273] If M and N are orientable, then the connected sum $M\#N$ is only well-defined once we pick an orientation for M and N. If M and N are orientable, then the equality is understood as holding for the definition of the connected sum for any choice of orientation for M and N.

(2) The proof of this statement is verbatim the same as in Lemma 76.3. □

In the following proposition we collect the computations of the low-dimensional (unoriented) cobordism groups.

Proposition 76.5. *We have the following table of isomorphism types of the (unoriented) cobordism groups for $n = 0, 1, 2, 3$:*

	$n=0$	$n=1$	$n=2$	$n=3$
Ω_n	$\mathbb{Z}$	0	0	0
$\mathcal{N}_n$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \cdot [\mathbb{R}\mathrm{P}^2]$	0

Proof.

($n = 0$) Using the classification of compact 1-dimensional manifolds, see Proposition 3.53, one can easily see that the maps

$$\begin{array}{rcl} \Omega_0 & \to & \mathbb{Z} \\ \bigsqcup_{i=1}^{k} \{(P_i, \epsilon_i)\} & \mapsto & \sum_{i=1}^{k} \epsilon_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{N}_0 & \to & \mathbb{Z}_2 \\ \bigsqcup_{i=1}^{k} \{P_i\} & \mapsto & k \bmod 2 \end{array}$$

are isomorphisms.

($n = 1$) In Proposition 3.53 we saw that every closed connected 1-dimensional manifold is diffeomorphic to S^1. Evidently $S^1 = \partial \overline{B}^2$[1274], hence every generator, i.e. every (oriented) compact 1-dimensional manifold represents the trivial element in Ω_1 and in $\mathcal{N}_1$. Therefore we see that $\Omega_1 = 0$ and $\mathcal{N}_1 = 0$.

($n = 2$) In Theorem 14.10 we saw that every closed orientable 2-dimensional manifold is diffeomorphic to the surface of genus g for some g. By Proposition 73.1 (3) the surface of genus g is the boundary of the genus g handlebody. Thus we see that $\Omega_2 = 0$.

We turn to the calculation of $\mathcal{N}_2$. We want to prove the following claim.

Claim. The map

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}_2 & \to & \mathcal{N}_2 \\ n & \mapsto & n \cdot \mathbb{R}\mathrm{P}^2 \end{array}$$

is an isomorphism.

We prove the claim in several steps:

(a) First we show that φ is a homomorphism. We denote by $\psi\colon \mathbb{Z} \to \mathcal{N}_2$ the map that is given by $n \mapsto n \cdot \mathbb{R}\mathrm{P}^2$. Since $\mathbb{Z}$ is a free group this is well-defined homomorphism. It remains to show that ψ factors through the projection $\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. But that is an immediate consequence of Lemma 76.4 (1).

[1274]This is an equality of oriented manifolds if we equip S^1 and $\overline{B}^2$ with the usual orientation and also if we equip both manifolds with the opposite orientation.

(b) Next we show that $\varphi\colon \mathbb{Z}_2 \to \mathcal{N}_2$ is a monomorphism. We only need to show that $\varphi(1) = \mathbb{R}\mathrm{P}^2$ represents a non-trivial element in $\mathcal{N}_2$. In Corollary 61.6 we already saw, using the Euler characteristic and Poincaré Duality, that there is no compact 3-dimensional manifold whose boundary is homeomorphic to $\mathbb{R}\mathrm{P}^2$. Put differently, $\mathbb{R}\mathrm{P}^2$ represents a non-trivial element in $\mathcal{N}_2$.

(c) It remains to show that φ is an epimorphism. So let M be a compact 2-dimensional manifold representing an element in $\mathcal{N}_2$. Lemma 76.4 (2) shows that without loss of generality we can assume that M is connected. If M is orientable, then we already saw above that M is the boundary of a compact 3-manifold, hence $[M] = 0 \in \mathcal{N}_2$.
Finally we consider the case that M is non-orientable. We have

$$\begin{array}{ccccc} [M] & = & [\text{connected sum of } n \text{ copies of } \mathbb{R}\mathrm{P}^2] & = & n\cdot[\mathbb{R}\mathrm{P}^2] & = & (n \bmod 2)\cdot[\mathbb{R}\mathrm{P}^2]. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by Theorem 14.10 that there exists an } n \in \mathbb{N}_0 & & \text{Lemma 76.4 (2)} & & \text{Lemma 76.4 (1)} & \\ & \text{such that } M \text{ is diffeomorphic to the} & & & & & \\ & \text{connected sum of } n \text{ copies of } \mathbb{R}\mathrm{P}^2 & & & & & \end{array}$$

This completes the proof that φ is an epimorphism.

($n = 3$) It was first proved by René Thom [Tho, Tho2] in 1952 that $\Omega_3 = 0$ and by Albrecht Dold [Dol] in 1956 that $\mathcal{N}_3 = 0$. A fairly elementary proof for both statements was later was given by William Lickorish [Lic, Theorem 3] and [Lic2]. A particularly short proof that $\Omega_3 = 0$ is also given by Colin Rourke [Rou]. □

We have thus shown that the (unoriented) cobordism groups for $n = 0, 1, 2, 3$ are not particularly exciting. Our next goal is to study Ω_4. By the discussion on page 966 we know that $\chi(\mathbb{C}\mathrm{P}^2) = 3$. Thus the argument of Corollary 61.6 shows that $\mathbb{C}\mathrm{P}^2$ is a non-trivial element[1275] in Ω_4, but from Corollary 61.6 we cannot conclude whether or not $\mathbb{C}\mathrm{P}^2$ is an element of infinite order in Ω_4.

The following proposition is the key to showing that $\mathbb{C}\mathrm{P}^2$ is in fact an element of infinite order in Ω_4.

Proposition 76.6. *Let M be a closed oriented $4k$-dimensional topological manifold with $k \geq 0$.*[1276] *If there exists a compact orientable topological manifold X with $\partial X = M$, then* $\operatorname{sign}(M) = 0$.[1277]

It will take us a while to prove Proposition 76.6, therefore we first provide the following corollary.

Corollary 76.7. *For any $k \in \mathbb{N}$ the map*

$$\begin{array}{rcl} \operatorname{sign}\colon \Omega_{4k} & \to & \mathbb{Z} \\ [M] & \mapsto & \operatorname{sign}(M) \end{array}$$

[1275] In fact Corollary 61.6 also implies that $\mathbb{C}\mathrm{P}^2$ is a non-trivial element in $\mathcal{N}_4$.

[1276] Does the statement also hold for 0-dimensional manifolds?

[1277] We refer to page 1674 for the definition of the signature of M.

is an epimorphism.

Proof of Corollary 76.7. First we need to show that the map is well-defined. Thus suppose we are given closed oriented $4k$-dimensional manifolds M and N with $[M] = [N] \in \Omega_{4k}$. By definition there exists a compact oriented manifold X with $\partial X = M \sqcup -N$. We have

$$\begin{array}{ccccccccc} 0 & = & \operatorname{sign}(\partial X) & = & \operatorname{sign}(M \sqcup (-N)) & = & \operatorname{sign}(M) + \operatorname{sign}(-N) & = & \operatorname{sign}(M) - \operatorname{sign}(N). \\ & \uparrow & & & & \uparrow & & \uparrow & \\ & \text{Proposition 76.6} & & & & \text{Lemma 72.11 (2)} & & \text{Lemma 72.11 (3)} & \end{array}$$

This shows that $\operatorname{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is well-defined. It follows from Lemma 72.11 (2) that this map is a homomorphism. Finally it follows immediately from Proposition 62.7 that $\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) = \pm 1$,[1278] which shows that $\operatorname{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is an epimorphism. □

The proof of Proposition 76.6 will rely on a purely algebraic lemma. Before we can formulate that lemma we have to introduce one more definition.

Definition. Given a form (V, g) and a subset $W \subset V$ we write

$$W^\perp := \{v \in V \mid g(v, w) = 0 \text{ for all } w \in W\}.$$

Example. We consider the form on $\mathbb{R}^2$ that is defined by the symmetric matrix

$$P := \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this case for $H = \mathbb{R} \cdot (1, 0)$ we have $H^\perp = H$.

Lemma 76.8. *Let (V, g) be a non-singular symmetric form over the reals. If there exists a subspace W with $g|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$, then $\operatorname{sign}(g) = 0$.*

Proof of Proposition 76.6. It follows from our hypothesis that $\dim(W) = \frac{1}{2}\dim(V)$ that we can find a basis $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ for V such that $\{a_1, \dots, a_k\}$ is a basis for W. Since $g|_W = 0$ and since g is symmetric we see that with respect to this basis the form g is given by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}$$

where Q is a symmetric matrix. It follows from the hypothesis that g is non-singular that $\det(P) \neq 0$, see e.g. Lemma 62.1. Now we consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix} \begin{pmatrix} (P^T)^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$

$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}Q & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\frac{1}{2}Q \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

[1278] In fact the more fancy Lemma 67.18 says that $\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) = +1$, but this is irrelevant for the proof.

It follows from Lemma 72.1 (1) that the three matrices to the right all define isometric forms. Considering the last matrix we obtain from Proposition 72.8 that the signature is zero. □

Lemma 76.9. *Let X be a compact orientable $(2n+1)$-dimensional topological manifold. We denote by $i\colon \partial X \to X$ the inclusion map. Then*

$$(1) \qquad \dim\big(\ker\{i_*\colon H_n(\partial X;\mathbb{R}) \to H_n(X;\mathbb{R})\}\big) \;=\; \tfrac{1}{2}\dim(H_n(\partial X;\mathbb{R}))$$

and

$$(2) \qquad \dim\big(\operatorname{im}\{i^*\colon H^n(X;\mathbb{R}) \to H^n(\partial X;\mathbb{R})\}\big) \;=\; \tfrac{1}{2}\dim(H^n(\partial X;\mathbb{R})).$$

The same statement also holds, regardless of whether or not X is orientable, if we replace $\mathbb{R}$-coefficients by $\mathbb{F}_2$-coefficients.

Remark. Given any homomorphism $\varphi\colon V \to W$ between finite-dimensional vector spaces the equality $\dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi)) = \dim(V)$ holds. Thus the first statement of the lemma can be reformulated as saying that

$$\dim\big(\ker\{i_*\colon H_n(\partial X;\mathbb{R}) \to H_n(X;\mathbb{R})\}\big) \;=\; \dim\big(\operatorname{im}\{i_*\colon H_n(\partial X;\mathbb{R}) \to H_n(X;\mathbb{R})\}\big).$$

This explains why the lemma is often referred to as the "half-live half-die lemma".

Examples.

(1) We consider the genus 3 handlebody X. In Figure 949 we show the usual basis $a_1, a_2, a_3, b_1, b_2, b_3$ for $H_1(\partial X;\mathbb{R})$. In this case the kernel of the inclusion induced map $H_1(\partial X;\mathbb{R}) \to H_1(X;\mathbb{R})$ is the three dimensional subspace generated by b_1, b_2 and b_3.

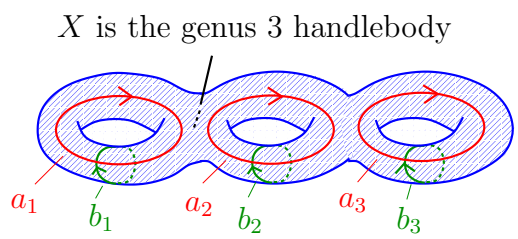


FIGURE 949.

(2) Let $K \subset S^3$ be a knot, we denote by X_K the knot exterior, see page 971. The boundary of X_K is a torus. By Lemma 70.7 (1) the meridian μ and the longitude λ form a basis for $H_1(X_K;\mathbb{R})$. have

$$\dim\big(\underbrace{\ker\{H_1(\partial X_K;\mathbb{R}) \to H_1(X_K;\mathbb{R})\}}_{=\,\mathbb{R}\cdot\lambda \text{ by Lemma 70.7 (1)}}\big) \;=1\;=\; \tfrac{1}{2}\cdot 2 \;=\; \tfrac{1}{2}\cdot\dim(H_1(\partial X_K;\mathbb{R})).$$

Proof of Lemma 76.9. Let X be a compact oriented $(2n+1)$-dimensional topological manifold. We denote by $i\colon \partial X \to X$ the inclusion map and we write $M = \partial X$. We first prove the first equality. Until further notice, we henceforth work with real coefficients for all (co-) homology groups. For space reasons we do not mention the real coefficients explicitly. We consider the long exact sequence in homology of the pair (X, M):

$$\to H_{n+1}M \to H_{n+1}X \to H_{n+1}(X,M) \xrightarrow{\partial} H_nM \xrightarrow{i_*} H_nX \to H_n(X,M) \to H_{n-1}M \to$$

The key observation is that this long exact sequence is "symmetric" around $H_n(M;\mathbb{R})$. More precisely, by the Poincaré Duality Theorem 61.3 (2)[1279] we have the following isomorphisms:

$$\to \overbrace{H_{n+1}M}^{\cong H_{n-1}M} \to \overbrace{H_{n+1}X}^{\cong H_n(X,M)} \to \overbrace{H_{n+1}(X,M)}^{\cong H_nX} \xrightarrow{\partial} H_nM \xrightarrow{i_*} H_nX \to H_n(X,M) \to H_{n-1}M \to$$

Therefore it suffices to prove the following purely algebraic claim.

Claim. If

$$0 \to A_k \xrightarrow{\beta_k} A_{k-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{k-1} \xrightarrow{\alpha_k} A_k \to 0$$

is an exact sequence of finite-dimensional real vector spaces,[1280] then

$$\dim(\ker\{\gamma\colon B \to A_1\}) \;=\; \frac{1}{2}\cdot \dim B.$$

We prove the claim by induction on k. The case $k = 0$ is clear, since in this case $B = 0$. Now consider a symmetric sequence

$$0 \to A_k \xrightarrow{\beta_k} A_{k-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{k-1} \xrightarrow{\alpha_k} A_k \to 0.$$

as above. It follows from Lemma 33.4 that the following sequence is also exact

$$0 \to A_{k-1}/\beta_k(A_k) \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to \ker\{A_{k-1} \xrightarrow{\alpha_k} A_k\} \to 0.$$

But we have

$$\dim(A_{k-1}/\beta_k(A_k)) \;\underset{\substack{\uparrow \\ \text{since } \beta_k \text{ is injective}}}{=}\; \dim(A_{k-1}) - \dim(A_k) \;\underset{\substack{\uparrow \\ \text{since } \alpha_k \text{ is surjective}}}{=}\; \dim(\ker\{A_{k-1} \xrightarrow{\alpha_k} A_k\}).$$

So the new shorter sequence is also symmetric. So we can apply the induction hypothesis to get the desired equality. This concludes the proof of the claim and thus of the first equality of the lemma.

The proof of the second equality of the lemma is almost identical to the proof of the first equality. This concludes the proof of the lemma regarding $\mathbb{R}$-valued (co-) homology groups.

[1279] Even though it is irrelevant we would like to stress that the isomorphisms $H_k(M;\mathbb{R}) \cong H_{2n-k}(M;\mathbb{R})$, $H_k(X;\mathbb{R}) \cong H_{2n+1-k}(X,M;\mathbb{R})$ and $H_k(X,M;\mathbb{R}) \cong H_{2n+1-k}(X;\mathbb{R})$ are not natural.

[1280] Recall that by Proposition 45.2 we know that the real homology groups of X, M and (X,M) are finite-dimensional vector spaces.

Now let X be any compact $(2n+1)$-dimensional topological manifold, without any assumptions on orientability. In this case verbatim the same argument as above works, we just need to replace $\mathbb{R}$-coefficients by $\mathbb{F}_2$-coefficients and we need to replace the Poincaré Duality Theorem 61.3 (2) by the Poincaré Duality Theorem 61.3 (1). $\square$

Finally we can turn to the actual proof of Proposition 76.6.

Proof of Proposition 76.6. Let X be a compact oriented $(4k+1)$-dimensional topological manifold with $k \geq 1$. We write $M = \partial X$. We want to show that $\operatorname{sign}(M) = 0$. If $M = \varnothing$, then there is nothing to show. So suppose that $M \neq \varnothing$. We denote by g the form

$$\begin{array}{rcl} H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) & \to & \mathbb{R} \\ (a,b) & \mapsto & \langle a \cup b, [M]\rangle_M. \end{array}$$

By Lemma 72.11 (1) we have $\operatorname{sign}(M) = \operatorname{sign}(g)$. We set

$$W \;:=\; \operatorname{im}\{i^*\colon H^{2k}(X;\mathbb{R}) \to H^{2k}(M;\mathbb{R})\}.$$

Note that by Proposition 71.5 we know that g is symmetric and non-singular. Therefore by Lemmas 76.8 and 76.9 (2) it suffices to show that $g|_W = 0$. Fortunately that is quite easy. Indeed, let $a, b \in H^{2k}(X;\mathbb{R})$. We have

$$\langle i^*(a) \cup i^*(b), [M]\rangle_M \;\underset{\substack{\uparrow \\ \text{Lemma 57.10}}}{=}\; \langle i^*(a \cup b), [M]\rangle_M \;\underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=}\; \langle a \cup b, i_*([M])\rangle_X \;\underset{\substack{\uparrow \\ \text{Corollary 46.27}}}{=}\; \langle a \cup b, 0\rangle_X \;=\; 0.$$

This concludes the proof that $g|_W \equiv 0$. $\square$

Before we continue with the determination of cobordism groups we prove the following elementary lemma.

Lemma 76.10. *For any $m, n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} \Omega_m \times \Omega_n & \to & \Omega_{m+n} \\ ([M],[N]) & \mapsto & [M \times N] \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} \mathcal{N}_m \times \mathcal{N}_n & \to & \mathcal{N}_{m+n} \\ ([M],[N]) & \mapsto & [M \times N] \end{array}$$

are well-defined.

Proof $(*)$. We prove the statement regarding the Ω_k's, the proof for the $\mathcal{N}_k$'s is verbatim the same. So suppose that we have $[M] = [M'] \in \Omega_m$ and $[N] \in \Omega_n$. This means that there exists a compact oriented $(m+1)$-dimensional manifold W with $\partial W = M \sqcup (-M')$. But then we also have

$$\partial(W \times N) \;\underset{\substack{\uparrow \\ \text{by Proposition 3.42 (3) since } N \text{ closed}}}{=}\; \partial W \times N \;=\; (M \sqcup (-M')) \times N \;\underset{\substack{\uparrow \\ \text{since } (-M') \times N = -(M' \times N)}}{=}\; (M \times N) \sqcup -(M' \times N).$$

This shows that $[M \times N] = [M' \times N]$. The same argument shows that the map does not depend on the representative of $[N] \in \Omega_n$. $\square$

Definition. We define

$$\Omega_* := \bigoplus_{n\in\mathbb{N}_0} \Omega_n \quad \text{and} \quad \mathcal{N}_* := \bigoplus_{n\in\mathbb{N}_0} \mathcal{N}_n.$$

Together with the maps defined in Lemma 76.10 we can view these groups as commutative rings. We refer to Ω_* as the *cobordism ring* and we refer to $\mathcal{N}_*$ as the *unoriented cobordism ring*.

The following theorem summarizes some of the most important results on the cobordism ring Ω_*.

Theorem 76.11. (Thom)

(1) *The ring homomorphism*[1281] [12821283]

$$\mathbb{Q}[\{x_m\}_{m\in\mathbb{N}}] \to \Omega_* \otimes \mathbb{Q}$$

that is determined by

$$x_m \mapsto \mathbb{C}\mathrm{P}^{2m}$$

is an isomorphism.

(2) *For small n the isomorphism types of Ω_n are given by*

n	0	1	2	3	4	5	6	7	8	9	10	11
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Hereby a generator for $\Omega_4 \cong \mathbb{Z}$ is given by $\mathbb{C}\mathrm{P}^2$ and a basis for $\Omega_8 \cong \mathbb{Z}^2$ is given by $\{\mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2\}$.[1284]

(3) *The group Ω_n is non-trivial for every $n \geq 8$.*

(4) *Every torsion element in Ω_n has order 2.*

Proof.

(1) The first statement was proved by René Thom [Tho2, Chapter IV], see also [Sto, p. 177].

(2) The calculation of the cobordism groups $\Omega_0, \dots, \Omega_3$ was given in Proposition 76.5, the values $\Omega_4, \dots, \Omega_7$ are given by René Thom in [Tho2, Theorem IV.13]. For $\Omega_8, \dots, \Omega_{11}$ we refer to [MiS, p. 203].

(3) This statement is proved in [Dol, p. 34].

(4) The last statement was proved by Terrence Wall [Wall4], see also [Sto, p. 182]. □

The following theorem, which is due to [Tho2, Theorem IV.12] and [Dol, p. 32], gives the complete calculation of the unoriented cobordism ring $\mathcal{N}_*$.

[1281] On the left-hand side $\mathbb{Q}[\{x_m\}_{m\in\mathbb{N}}]$ denotes the polynomial ring over $\mathbb{Q}$ with variables $\{x_m\}_{m\in\mathbb{N}}$.

[1282] Note that the tensor product of two commutative rings is again a commutative ring with the "naive multiplication". In particular the right-hand side $\Omega_* \otimes \mathbb{Q}$ is a commutative ring.

[1283] If we give each variable x_m the grading $4m$, and if we equip $\Omega_* \otimes \mathbb{Q}$ with the obvious grading, then this is an isomorphism of graded rings.

[1284] Note that in the ring homomorphism from (1) we have $x_1 \mapsto [\mathbb{C}\mathrm{P}^2]$ and $x_1^2 \mapsto [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]$ and also $x_2 \mapsto [\mathbb{C}\mathrm{P}^4]$.

Theorem 76.12. (Thom-Dold) *We let $I = \{n \in \mathbb{N} \,|\, n \neq 2^i - 1\} = \{2, 4, 5, 6, 8, 9, \dots\}$. There exists a ring isomorphism*

$$\mathbb{Z}_2[\{x_i\}_{i \in I}] \;\to\; \mathcal{N}_*$$

which is given by sending x_{2k} to $\mathbb{R}\mathrm{P}^k$ and given by sending x_i for odd i to an appropriate i-dimensional Dold manifold. Here a Dold manifold is a manifold of the form[1285]

$$P(m,n) \;:=\; (S^m \times \mathbb{C}\mathrm{P}^n)/(x,z) \sim (-x, \overline{z}) \qquad \text{for } x \in S^m \text{ and } z \in \mathbb{C}\mathrm{P}^n$$

for some $n, m \in \mathbb{N}_0$.

76.2. **The cobordism groups of rational homology spheres** $(*)$**.** First we recall the definition of an R-homology sphere from page 1711 and we introduce some, almost self-explanatory, related notions.

Definition. Let R be a commutative ring and let M be a compact orientable n-dimensional topological manifold.

(1) We say M is an *R-homology sphere*, if for each $k \in \mathbb{N}_0$ we have $H_k(M;R) \cong H_k(S^n;R)$.
(2) We say M is an *R-homology ball*, if for each $k \in \mathbb{N}_0$ we have $H_k(M;R) \cong H_k(\overline{B}^n;R)$.
(3) We say M is an *R-cylinder*, if for each $k \in \mathbb{N}_0$ we have $H_k(M;R) \cong H_k(S^{n-1} \times [0,1];R)$.

Instead of $\mathbb{Z}$-homology sphere we usually say *integral homology sphere* and instead of $\mathbb{Q}$-homology sphere we usually say *rational homology sphere*. The same convention holds also for balls and cylinders.

The following lemma will be proved in Exercise 76.4.

Lemma 76.13. *Let M be an n-dimensional topological manifold.*

(1) *If M is a rational homology n-ball, then ∂M is a rational homology $(n-1)$-sphere.*
(2) *Let M be a rational homology cylinder.*
   (a) *The boundary ∂M has precisely two components X and Y, each of which is a rational homology $(n-1)$-sphere.*
   (b) *The inclusion induced maps $H_*(X;\mathbb{Q}) \to H_*(M;\mathbb{Q})$ and $H_*(Y;\mathbb{Q}) \to H_*(M;\mathbb{Q})$ are isomorphisms.*
(3) *Let M be a rational homology ball. If $\varphi\colon \overline{B}^n \to M \setminus \partial M$ is an embedding, then $M \setminus \varphi(B^n)$ is a rational homology cylinder.*

The same statements also hold if we replace "rational" by "integral".

Now we can modify the cobordism groups Ω_n to obtain new and interesting groups.

[1285] It follows from Proposition 3.25 that each $P(m,n)$ is in fact a closed manifold.

Definition. Let $n \in \mathbb{N}$. We define

$$\Theta^n_{\mathbb{Q}} := \{\text{diffeomorphism classes of oriented } n\text{-dimensional rational homology spheres}\}/\sim_{\mathbb{Q}}$$

where we write $M \sim_{\mathbb{Q}} N$ if there exists an oriented cobordism W between M to N that is a rational homology cylinder. Similarly we define

$$\Theta^n_{\mathbb{Z}} := \{\text{diffeomorphism classes of oriented } n\text{-dimensional integral homology spheres}\}/\sim_{\mathbb{Z}}$$

where we write $M \sim_{\mathbb{Z}} N$ if there exists an oriented cobordism W between M and N that is an integral homology cylinder. Furthermore we define $\Theta^{\text{top},n}_{\mathbb{Q}}$ and $\Theta^{\text{top},n}_{\mathbb{Z}}$ the same way as above except that we "throw over board" all smooth structures. More precisely, we make the following two changes:

(1) We consider homeomorphism classes of oriented n-dimensional rational homology spheres.
(2) We no longer demand that the cobordism W is a manifold, we only assume that it is a topological manifold.

To simplify the discussion, and out of personal preference, henceforth we only consider the case $n = 3$ and we drop the decoration "3" from the notation. Recall that Theorem 28.24 says that in dimension *three* any topological manifold admits a smooth structure and two manifolds are homeomorphic if and only if they are diffeomorphic. Put differently, there is basically no difference between the study of topological 3-dimensional manifolds and the study of 3-dimensional manifolds. On the other hand, we saw in Theorem 72.27 that in dimension *four* there is a big difference between the theory of topological manifolds and the theory of manifolds. In particular the above notions of cobordisms might change depending on whether or not we consider topological 4-dimensional manifolds or 4-dimensional manifolds.

Proposition 76.14. *In all four cases the connected sum operation defines an abelian group structure where the neutral element is given by S^3 and where the inverse of a homology sphere Y is given by $-Y$.*

Example. Let Y be a 3-dimensional rational homology sphere. If Y is amphichiral, then we have the following equality in $\Theta_{\mathbb{Q}}$:

$$\begin{array}{ccccc} [Y]\#[Y] & \underset{\substack{\uparrow \\ \text{since } Y \text{ amphichiral}}}{=} & [Y]\#[-Y] & \underset{\substack{\uparrow \\ \text{Proposition 76.14}}}{=} & S^3 \in \Theta_{\mathbb{Q}} \end{array}$$

This implies that an amphichiral 3-dimensional rational sphere defines either the trivial element in $\Theta_{\mathbb{Q}}$ or an element of order 2.

Sketch of proof. In the following steps we verify all the statements that are implicit in the formulation of the proposition for $\Theta_{\mathbb{Q}}$.

(1) It follows from Proposition 39.15 that the connected sum of two rational homology spheres is again a rational homology sphere.

(2) We need to show that the connected sum operation descends to a well-defined operation on $\Theta_{\mathbb{Q}}$. Thus suppose that we are given oriented, rational homology spheres M and M' and also N and N' such that $M = M' \in \Theta_{\mathbb{Q}}$ and $N = N' \in \Theta_{\mathbb{Q}}$. We need to show that $M\#N = M'\#N' \in \Theta_{\mathbb{Q}}$. We pick compact oriented 4-dimensional manifolds X and Y with $\partial X = M \sqcup -M'$ and $\partial Y = N \sqcup -N'$ as in the definition of $\sim_{\mathbb{Q}}$.

By Proposition 65.27 we can pick an oriented proper connected 1-dimensional submanifold $[0,1] \cong \alpha \subset X$ which connects a point on M to a point on M'. Similarly we can pick a proper connected 1-dimensional submanifold $[0,1] \cong \beta \subset Y$ which connects a point on N to a point on N'. By the Tubular Neighborhood Theorem 10.19 we can thicken the submanifolds α and β to submanifolds $[0,1] \times \overline{B}^3 = \alpha \times \overline{B}^3$ and $[0,1] \times \overline{B}^3 = \beta \times \overline{B}^3$. We can and will pick the former map to be orientation-preserving and the latter orientation-reversing. We define

$$Z \ := \ \big((X \setminus (\alpha \times B^3)) \sqcup (Y \setminus (\beta \times B^3))\big)/\sim$$

where we identify a point on $\alpha \times S^2 = [0,1] \times S^2$ with the corresponding point on $\beta \times S^2 = [0,1] \times S^2$. After smoothing corners, see [Wall2, Proposition 2.6.2], we can view Z as a compact oriented manifold such that $\partial Z = (M\#N) \sqcup -(M'\#N')$. (This construction is illustrated in Figure 950.) Using that X and Y are rational homology cylinders one can fairly easily show that Z itself is also a rational homology cylinder.

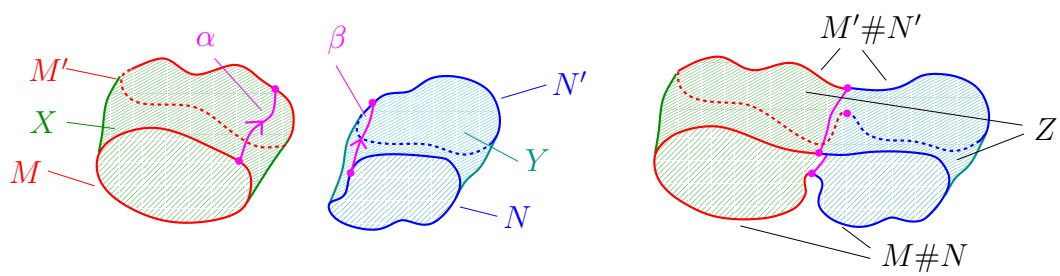


FIGURE 950.

(3) It follows from Lemma 10.30 that the connected sum operation is commutative and associative. Furthermore it follows from the discussion on page 355 that S^3 is the neutral element with respect to the connected sum operation.

(4) Let Y be a rational homology sphere. We have to show that $(Y\# - Y) \sim_{\mathbb{Q}} S^3$. We pick a closed 3-ball $\overline{B}^3$ in Y. We consider $W = ([0,1] \times Y) \setminus ([0,1] \times B^3)$. After smoothing corners we have $\partial W = Y\# - Y$. Finally we remove an open 4-ball from W and we obtain a 4-manifold Z with $\partial Z = (Y\# - Y) \sqcup S^3$. We leave it as an exercise to verify that Z is a rational homology cylinder. We illustrate this construction in Figure 951.

The proof that $\Theta_{\mathbb{Z}}$ forms a group is basically identical to the above proof. The proof that $\Theta_{\mathbb{Q}}^{\text{top}}$ and $\Theta_{\mathbb{Z}}^{\text{top}}$ form groups is in principle the same, except that we need to

replace the references dealing with manifolds by references that take care of topological manifolds. For example, we need to replace the Tubular Neighborhood Theorem 10.19 by the fact that compact 1-dimensional submanifolds of compact topological manifolds admit trivial tubular neighborhoods, see the discussion on page 1497.

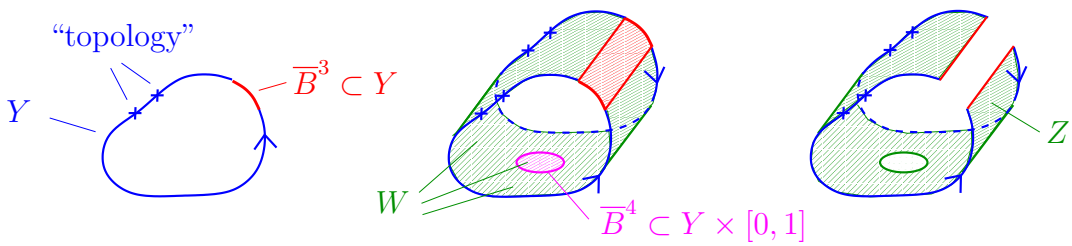


FIGURE 951.

□

The following lemma gives a characterization of manifolds representing the trivial element in the various groups that we just introduced.

Lemma 76.15. *Let Y be a 3-dimensional rational homology sphere. Then*

$$[Y] = 0 \in \Theta_{\mathbb{Q}} \quad \Longleftrightarrow \quad \begin{array}{l}\textit{there exists a 4-dimensional manifold } W \textit{ that} \\ \textit{is a rational homology ball } W \textit{ with } \partial W = Y.\end{array}$$

There are obvious analogues to this statement for the groups $\Theta_{\mathbb{Z}}$, $\Theta_{\mathbb{Q}}^{\text{top}}$ and $\Theta_{\mathbb{Z}}^{\text{top}}$.

Proof. Let Y be a 3-dimensional rational homology sphere.

"$\Rightarrow$" If Y represents the trivial element in $\Theta_{\mathbb{Q}}$, then by definition there exists a 4-dimensional rational homology cylinder Z with $\partial Z = Y \sqcup -S^3$. A straightforward argument, using Proposition 10.12, Lemma 76.13 and the Mayer–Vietoris Theorem for manifolds 33.10 shows that $W = Z \cup_{\partial Z = S^3} \overline{B}^4$ is a rational homology ball. Evidently we have $\partial W = Y$.

"$\Leftarrow$" Now suppose that there exists a 4-dimensional manifold W that is a rational homology ball W with $\partial W = Y$. We pick an embedding $\varphi\colon \overline{B}^4 \to W \setminus \partial W$. It follows from Lemma 76.13 (3) that $Z = Y \setminus \varphi(B^4)$ is a rational cylinder which then shows that $[Y] = [S^3] = 0 \in \Theta_{\mathbb{Q}}$. We refer to Figure 952 for an illustration. □

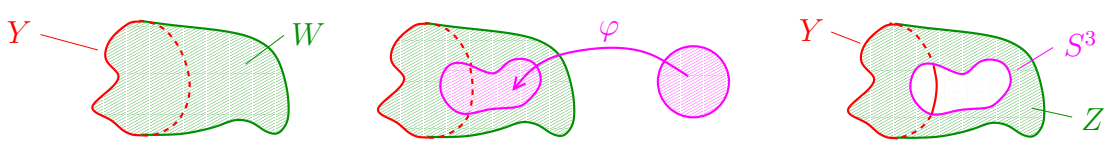


FIGURE 952.

We start out with the investigation of the group $\Theta_{\mathbb{Q}}^{\text{top}}$.

Proposition 76.16. *Let M be a 3-dimensional rational homology sphere. If $M = 0 \in \Theta^{\text{top}}_{\mathbb{Q}}$, then $|H_1(M;\mathbb{Z})|$ is a square.*

Proof. Let M be a 3-dimensional rational homology sphere which represents the trivial element in $\Theta^{\text{top}}_{\mathbb{Q}}$. By Lemma 76.15 there exists a 4-dimensional manifold W that is a rational homology ball W with $\partial W = M$. Thus the proposition follows from the following claim.

Claim. We have[1286]

$$|\ker\{H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})\}|^2 \;=\; |H_1(M;\mathbb{Z})|.$$

First we note that the fact that M is a rational homology sphere and that W is a rational homology ball implies, by the argument of Lemma 74.1, that the homology groups $H_1(M;\mathbb{Z})$, $H_2(M;\mathbb{Z})$ and also $H_1(W;\mathbb{Z})$, $H_2(W;\mathbb{Z})$ are finite.

We consider the long exact sequence in homology groups corresponding to the pair (W, M) with $\mathbb{Z}$-coefficients:

$$\begin{array}{l}
\text{isomorphic to } H^1(M;\mathbb{Z}) \cong \operatorname{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) \text{ by the Poincaré Duality Theorem 61.1,}\\
\text{and Proposition 51.17, therefore zero since } H_1(M;\mathbb{Z}) \text{ is finite}\\
\qquad\qquad\downarrow\\
\ldots \to H_2(M;\mathbb{Z}) \to H_2(W;\mathbb{Z}) \to H_2(W,M;\mathbb{Z})\\
\quad \to H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z}) \to H_1(W,M;\mathbb{Z}) \to H_0(M;\mathbb{Z}) \to H_0(W;\mathbb{Z}) \to 0.\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\uparrow\\
\qquad\qquad\qquad\qquad\qquad\qquad\text{isomorphism by the discussion on page 747}
\end{array}$$

By Lemma 33.4 we now get the following exact sequence:

$$0 \to H_2(W;\mathbb{Z}) \to H_2(W,M;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z}) \to H_1(W,M;\mathbb{Z}) \to 0.$$

We consider the group on the left. We have

$$H_2(W,M;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Poincaré Duality}\\ \text{Theorem 61.1}}}{\cong} H^2(W;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Universal Coefficient}\\ \text{Theorem 51.13}}}{\cong} \overbrace{\operatorname{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z})}^{=0 \text{ since } H_2(W;\mathbb{Z}) \text{ is finite}} \oplus \operatorname{Ext}(H_1(W;\mathbb{Z}),\mathbb{Z}) \underset{\substack{\uparrow\\ \text{by Lemma 51.5 (5)}\\ \text{since } H_1(W;\mathbb{Z}) \text{ is finite}}}{\cong} H_1(W;\mathbb{Z}).$$

Almost the same argument shows that $H_2(W;\mathbb{Z}) \cong H_1(W,M;\mathbb{Z})$. This shows that the above exact sequence is an exact sequence of finite abelian groups that is "symmetric" around $H_1(M;\mathbb{Z})$. Therefore it remains to prove the following claim.

Claim. If

$$0 \to A_k \to \ldots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \ldots \to A_k \to 0$$

[1286]Put differently, we can say that $|\ker\{H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})\}| = \sqrt{|H_1(M;\mathbb{Z})|}$. With this formulation the statement of the claim is quite similar to the statement of Lemma 76.9. In fact we will see that the proof of the claim is a variation of the proof of Lemma 76.9.

is an exact sequence of finite abelian groups, then

$$|\ker\{\gamma\colon B \to A_1\}|^2 \;=\; |B|.$$

The claim is proved by induction on k, exactly the same way as we proved the claim on page 1768. We leave the details as an exercise to the reader. □

Corollary 76.17. *Let M and N be two 3-dimensional rational homology spheres. If we have $M = N \in \Theta^{\text{top}}_{\mathbb{Q}}$, then $|H_1(M;\mathbb{Z})| \cdot |H_1(N;\mathbb{Z})|$ is a square.*

Proof. Let M and N be 3-dimensional rational homology spheres with $[M] = [N] \in \Theta^{\text{top}}_{\mathbb{Q}}$. From Proposition 76.14 we obtain that $[M\#(-N)] = [M] - [N] = 0 \in \Theta^{\text{top}}_{\mathbb{Q}}$. Now Proposition 76.16 implies that $|H_1(M\#(-N);\mathbb{Z})|$ is a square. The corollary now follows from Proposition 39.15 which states that $H_1(M\#(-N);\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \oplus H_1(N;\mathbb{Z})$. □

The following proposition shows that the groups $\Theta^{\text{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ are very much non-trivial.

Proposition 76.18.

(1) *The groups $\Theta^{\text{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ admit an epimorphism onto $\mathbb{Z}_2^\infty$.*
(2) *The groups $\Theta^{\text{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ admit a subsummand isomorphic to $\mathbb{Z}_2^\infty$.*[1287]

Remark. It follows from [KK, Theorems 5.2 and 6.1] and a generalization of the arguments used in the proof of Proposition 76.18 (1) that the group $\Theta^{\text{top}}_{\mathbb{Q}}$ admits an epimorphism onto $\mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_4^\infty$.

The proof of Proposition 76.18 (2) relies on the following theorem from number theory. We refer to [IR, p. 251] for a proof.

Theorem 76.19. (Dirichlet's Theorem on Arithmetic Progressions) *For any coprime $a, n \in \mathbb{N}$ there exist infinitely many primes of the form $a + kn$ with $k \in \mathbb{N}$.*

Proof of Proposition 76.18.

(1) We have an obvious epimorphism $\Theta_{\mathbb{Q}} \to \Theta^{\text{top}}_{\mathbb{Q}}$, so it suffices to show that the group $\Theta^{\text{top}}_{\mathbb{Q}}$ admits an epimorphism onto $\mathbb{Z}_2^\infty$. For each prime p we consider the map

$$\begin{array}{rcl} \alpha_p\colon \Theta^{\text{top}}_{\mathbb{Q}} & \to & \mathbb{Z}_2 \\ M & \mapsto & \max\{n \in \mathbb{N}_0 \,|\, p^n \text{ divides } |H_1(M;\mathbb{Z})|\} \mod 2. \end{array}$$

It follows easily from Corollary 76.17, together with the uniqueness of the prime power decomposition of natural numbers and the fact that p is a prime, that this map is well-defined. Furthermore it follows from Proposition 39.15 that α_p is a group homomorphism.

[1287] Here we say that an abelian group G is a subsummand of an abelian group H if there exists an abelian group A such that $H \cong G \oplus A$.

It remains to show that the homomorphism

$$\begin{array}{rcl} \alpha\colon \Theta_{\mathbb{Q}}^{\mathrm{top}} & \to & \bigoplus\limits_{\text{prime } p} \mathbb{Z}_2 \\ M & \mapsto & \bigoplus\limits_{\text{prime } p} \alpha_p(M) \end{array}$$

is an epimorphism. But this follows immediately from considering lens spaces. More precisely, let p be a prime. By Corollary 38.6 we have $H_1(L(p,1);\mathbb{Z}) \cong \mathbb{Z}_p$, i.e. we have $\alpha_p(L(p,1)) = 1$ and $\alpha_q(L(p,1)) = 0$ for every prime $q \neq p$.

(2) By Dirichlet's Theorem 76.19 there exist infinitely many primes $p_i, i \in \mathbb{N}$ such that $p_i \equiv 1 \bmod 4$. By the calculation of the Legendre symbol, see e.g. [IR, p. 52], given any such prime p_i there exists an $a_i \in \{1, \dots, p_i - 1\}$ such that $a_i^2 \equiv -1 \bmod p_i$. It follows from Lemma 47.7 that the lens spaces $L(p_i, a_i)$ are amphichiral. It follows from the discussion on page 1772 and from (1) that we have well-defined homomorphisms

$$\begin{array}{rcccl} \mathbb{Z}_2^\infty & \to & \Omega_{\mathbb{Q}}^{\mathrm{top}} & \to & \mathbb{Z}_2^\infty \\ & & M & \mapsto & \sum\limits_i \alpha_{p_i}(M) \\ (x_1, x_2, \dots) & \mapsto & \#\limits_{\substack{i \text{ with} \\ x_i = 1}} L(p_i, a_i), & & \end{array}$$

furthermore we see easily that the composition of these two maps is the identity. This shows that the right-hand map splits. Therefore it follows from Lemma 33.2 that $\mathbb{Z}_2^\infty$ is in fact a subsummand of $\Omega_{\mathbb{Q}}^{\mathrm{top}}$. Exactly the same argument applies to $\Omega_{\mathbb{Q}}$. □

By definition any 3-dimensional integral homology sphere lies in the kernel of the epimorphism $\Omega_{\mathbb{Q}}^{\mathrm{top}} \to \mathbb{Z}_2^\infty$ that we had just constructed in the proof of Proposition 76.18.

In fact it was proved by Michael Freedman [Fre, Theorem 1.4'] in 1981 that every integral homology 3-dimensional sphere bounds a simply connected integral homology 4-dimensional ball.[1288] This result implies in particular the following theorem.

Theorem 76.20. *The group $\Theta_{\mathbb{Z}}^{\mathrm{top}}$ is trivial.*

It turns out that the situation is quite different if we consider the group $\Theta_{\mathbb{Z}}$ instead of the group $\Theta_{\mathbb{Z}}^{\mathrm{top}}$. In fact we have the following proposition.

Proposition 76.21. *Given a 3-dimensional integral homology sphere M we define*

$$\mu(M) \ := \ \frac{1}{8}\operatorname{sign}(W) \mod 2$$

where W is a compact oriented 4-dimensional manifold with $\partial W = M$, with $H_1(W;\mathbb{Z}) = 0$ and with even intersection form. Then the following statements hold:

[1288] The methods that Michael Freedman used in the proof of this result are closely related to the methods in Freedman's proof of the topological Poincaré Conjecture in dimension 4, see Theorem 28.23.

(1) *The number $\mu(M)$ is well-defined and it lies in $\mathbb{Z}_2$.*
(2) *The map*

$$\begin{array}{rcl} \mu\colon \Theta_{\mathbb{Z}} & \to & \mathbb{Z}_2 \\ M & \mapsto & \mu(M), \end{array}$$

is a well-defined homomorphism, it is called the Rokhlin homomorphism.
(3) *The Rokhlin homomorphism $\mu\colon \Theta_{\mathbb{Z}} \to \mathbb{Z}_2$ is an epimorphism, in particular $\Theta_{\mathbb{Z}}$ is non-trivial.*

Remark. The Rokhlin homomorphism is the starting point for many developments in the study of 3-dimensional and 4-dimensional manifolds. For example Andrew Casson showed in the mid 1980s that there exists a map

$$\lambda\colon \left\{ \begin{array}{c} \text{diffeomorphism classes of oriented} \\ \text{3-dimensional integral homology spheres} \end{array} \right\} \to \mathbb{Z}$$

that has the following properties:

(1) It is additive in the sense that for two 3-dimensional integral homology spheres M and N we have $\lambda(M\#N) = \lambda(M) + \lambda(N)$.
(2) For every 3-dimensional integral homology sphere we have $\mu(M) = \lambda(M) \bmod 2$.

More information on the Casson invariant λ can be found in [Sav2, AM].

Sketch of the proof of Proposition 76.21.

(1) Let M be a 3-dimensional integral homology sphere. The following arguments show that $\mu(M)$ is indeed defined, that it is well-defined and that it lies in $\mathbb{Z}_2$.
   (a) By [Kap], see also [Sav2, p. 38], there exists a compact oriented 4-dimensional manifold with $\partial W = M$, with $H_1(W;\mathbb{Z}) = 0$ and with even intersection form.[1289]
   (b) Since ∂W is an integral homology sphere and since $H_1(W;\mathbb{Z}) = 0$ one can easily adjust the argument of the proof of Proposition 71.5 to show that the intersection form Q_W is non-singular.
   (c) Since the intersection form Q_W is even and non-singular it follows from Proposition 72.18 that $\operatorname{sign}(W) = \operatorname{sign}(Q_W)$ is divisible by 8, i.e. $\frac{1}{8}\operatorname{sign}(W) \in \mathbb{Z}$.
   (d) Now suppose that we have another compact oriented 4-dimensional manifold W' with $H_1(W';\mathbb{Z}) = 0$, with $\partial W' = M$ and with even intersection form. We need to show that $\operatorname{sign}(W) \equiv \operatorname{sign}(W')$.
   We consider the closed oriented 4-dimensional manifold[1290]

$$X \;:=\; W \cup_{\partial W = \partial W'} (-W')$$

   that we obtain from gluing W and $-W'$ along the common boundary given by M. Since $H_1(W;\mathbb{Z}) = H_1(W';\mathbb{Z}) = 0$ and since M is an integral homology sphere one can easily adapt the proof of Proposition 39.15 to show that $H_1(X;\mathbb{Z}) = 0$ and

[1289] In fact in [Kap] it is shown that W can be chosen to be simply connected, which implies by the Hurewicz Theorem 38.5 that the first homology group vanishes.

[1290] Here we use Proposition 10.12 to conclude that X is indeed a closed oriented 4-dimensional manifold.

one can modify the proof of Proposition 71.11 to conclude that $Q_X \cong Q_W \oplus -Q_{W'}$, in particular Q_X is even. Now we can apply Rokhlin's Theorem 72.24 to conclude that $\operatorname{sign}(X) \equiv 0 \bmod 16$. But by the isomorphism $Q_X \cong Q_W \oplus -Q_{W'}$ this implies that $\operatorname{sign}(W) - \operatorname{sign}(W') \equiv 0 \bmod 16$, which gives us the desired result.

(2) Now we have to show that μ defines a well-defined homomorphism $\Theta_{\mathbb{Z}} \to \mathbb{Z}_2$.

(a) First we show that μ is additive. So let M and N be 3-dimensional integral homology spheres. We pick 4-dimensional manifolds X and Y with $\partial X = M$ and $\partial Y = N$ and the extra properties that are required in the definition of $\mu(M)$ and $\mu(N)$. We denote by $Z = X \#_b Y$ the boundary connected sum that we introduced on page 1690. It follows from Proposition 73.1 that Z as a 4-dimensional manifold with $\partial Z = M \# N$ and such that $H_1(Z;\mathbb{Z}) = 0$. Furthermore a modest modification of the proof of Proposition 71.11 shows that $Q_Z \cong Q_X \oplus Q_Y$. This means that we can use Z to calculate $\mu(M \# N)$ and we obtain that

$$\mu(M \# N) \;=\; \tfrac{1}{8}\operatorname{sign}(Z) \;=\; \tfrac{1}{8}\operatorname{sign}(X) + \tfrac{1}{8}\operatorname{sign}(Y) \;=\; \mu(M) + \mu(N).$$

We refer to Figure 953 for an illustration of the argument.

(b) It follows easily from the definitions that for any integral homology sphere M we have $\mu(-M) = \mu(M)$.

(c) It remains to show that μ is well-defined. By (a) and (b) it suffices to show that if M represents the trivial element in $\Theta_{\mathbb{Z}}$, then $\mu(M) = 0$. Let such M be given. It follows from Lemma 76.15 that M is the boundary of an integral homology sphere W. Since $H_2(W;\mathbb{Z}) = 0$ the intersection form is zero, in particular even. Since $H_1(W;\mathbb{Z}) = 0$ we can use W to determine $\mu(M)$ and we get immediately that $\mu(M) = \frac{1}{8}\operatorname{sign}(W) = 0$.

(3) Finally we want to show that there exists a 3-dimensional integral homology sphere M with $\mu(M) = 1$. Let M be the Poincaré homology sphere that we introduced on page 993. In [Sav2, p. 38] it is shown that M is the boundary of a compact oriented simply connected 4-dimensional manifold W with intersection form isometric to $-E_8$. As we saw on page 1671, the form $-E_8$ of rank 8 is negative definite, hence its signature is -8. By the definition of the Rokhlin homomorphism this implies that $\mu(M) = 1$. □

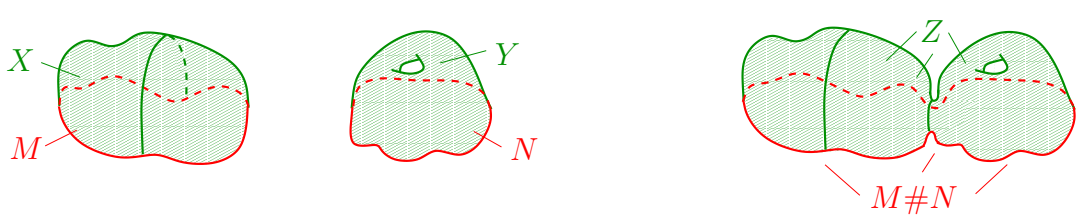


FIGURE 953.

In fact a much stronger statement than Proposition 76.21 holds. Building on the techniques developed by Simon Donaldson [Don, Don2] Mikio Furuta [Fu] proved in 1990 the following theorem.

Theorem 76.22. *The group $\Theta_{\mathbb{Z}}$ contains a subgroup that is isomorphic to $\mathbb{Z}^\infty$.*

Among the four groups that we have studied in this section, the trivial group $\Theta_{\mathbb{Z}}^{\text{top}}$ is the only one which has been fully calculated. At the moment we still know depressingly little about the remaining three groups. For example the following three questions are still open.

Question 76.23.

(1) *Is the group $\Theta_{\mathbb{Q}}^{\text{top}}$ isomorphic to $\mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_8^\infty$?*
(2) *Does the group $\Theta_{\mathbb{Q}}^{\text{top}}$ contain elements of infinite order?*
(3) *Let $n \neq 1, 2$ be a natural number. Does there exist an element in the groups $\Theta_{\mathbb{Z}}$ or $\Theta_{\mathbb{Q}}$ of order n?*

Exercises for Chapter 76.

Exercise 76.1. We consider the Klein bottle

$$K \;=\; (S^1 \times [0,1])/\sim \quad \text{where } (z,0) \;\sim\; (\overline{z},1) \text{ for all } z \in S^1.$$

On page 435 we showed that K is diffeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Thus it follows from Lemma 76.4 that $[K] = [\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2 \sqcup \mathbb{R}\mathrm{P}^2] = 2 \cdot [\mathbb{R}\mathrm{P}^2] = 0 \in \mathcal{N}_2$. Write down an explicit compact 3-dimensional manifold M with $\partial M = K$.

Exercise 76.2. Provide a proof for Lemma 76.8 that does not uses bases.[1291] Let us elaborate. Let (V, g) be a non-singular symmetric form over $\mathbb{R}$ and suppose that there exists a subspace W of V with $g|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$. Show that given any subspace A of V such that $g|_A$ is positive (respectively negative) definite there exists a subspace B of V such that $g|_B$ is negative (respectively positive) definite and which satisfies $\dim(A) = \dim(B)$. A proof of this statement implies that $\operatorname{sign}(g) = 0$.

Exercise 76.3. Let $L = L_1 \cup \cdots \cup L_m$ be an m-component link in S^3. We denote by X_L its exterior which is defined the same way as we defined the exterior of a knot on page 971. The boundary of X_L consists of m tori $T_1, \dots, T_m$. Let $k \in \{1, \dots, m\}$. We denote by $i\colon T_k \to X_L$ the inclusion map.

(a) Show that $\dim(\operatorname{im}\{i_*\colon H_1(T_k;\mathbb{R}) \to H_1(X_L;\mathbb{R})\}) \geq 1$.
(b) Do we necessarily have $\dim(\operatorname{im}\{i_*\colon H_1(T_k;\mathbb{R}) \to H_1(X_L;\mathbb{R})\}) = 1$?

Exercise 76.4. Provide a proof for Lemma 76.13.

[1291] As the reader surely knows, introducing bases is morally wrong and should be avoided at all costs.

Exercise 76.5. Let M be an oriented 3-dimensional rational homology sphere. We denote by $\lambda_M\colon H_1(M;\mathbb{Z})\times H_1(M;\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$ the linking pairing that we introduced on page 1720. Let W be a 4-dimensional rational homology ball with $\partial W = M$. We denote by $i\colon M\to W$ the inclusion and we write $K := \ker\{i_*\colon H_1(M;\mathbb{Z})\to H_1(W;\mathbb{Z})\}$. Show that λ_M vanishes on $K\times K$.

Exercise 76.6.

(a) Let M be an orientable n-dimensional manifold with $n\geq 4$ and let $C\cong S^1$ be a closed connected one-dimensional submanifold. By the Tubular Neighborhood Theorem 10.19 there exists a tubular neighborhood $\overline{B}^{n-1}\times C = \overline{B}^{n-1}\times S^1$. By Propositions 10.2 and 10.12

$$W \;:=\; (M\setminus(C\times B^{n-1}))\cup_{C\times S^{n-2}=S^1\times S^{n-2}}(\overline{B}^2\times S^{n-2})$$

is an n-dimensional manifold. Compute $\pi_1(W)$ in terms of $\pi_1(M)$ and C. (This approach to modifying the manifold M is sometimes called a *surgery*.)

(b) Let $n\geq 3$ and let X be a closed n-dimensional manifold. Suppose that there exists a compact orientable $(n+1)$-dimensional M with $\partial M = X$. Show that there exists also a compact orientable *simply connected* $(n+1)$-dimensional manifold N with $\partial N = X$.

(c) Does (b) also hold for $n=2$?

Exercise 76.7. Let Y_1 and Y_2 be two compact 3-dimensional manifolds with one torus boundary component. Let $Y = Y_1\cup Y_2$ be the result of gluing Y_1 and Y_2 along the boundary. Show that if $b_1(Y)=0$, then $b_1(Y_1)=1$ and $b_1(Y_2)=1$.

Exercise 76.8. Does there exist a compact simply-connected 3-dimensional manifold M such that ∂M has at least one boundary component that is a torus?

77. Vector bundles and Pontryagin classes

In this chapter we introduce vector bundles and their Pontryagin classes. This is only intended as a quick introduction. Therefore on many occasions we will only provide sketches of proofs or we will refer to the literature for the proofs. The best reference for this chapter is the classic book by Milnor-Stasheff [MiS] on characteristic classes. Other references are [Mi12, Hat2] and [Hus].

77.1. **Vector bundles.** In this section we introduce vector bundles and we give several constructions and examples. The definition of a vector bundle is fairly similar to the definition of a smooth $\overline{B}^m$-bundle that we gave on page 1492 and the definition of a linear $\overline{B}^m$-bundle that we gave on page 1493.

Loosely speaking, a vector bundle over a topological space X consists of a vector space for each $x \in X$, where the "vector spaces vary continuously with x". The following definition makes this idea precise.

Definition. An *n-dimensional vector bundle* over a topological space X is a topological space E together with a map $p\colon E \to X$ and a real vector space structure on each $p^{-1}(x)$, $x \in X$ such that for each $x \in X$ there exists an open neighborhood U of x and a homeomorphism $\varphi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ such that the following two conditions are satisfied:

(1) the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^n \arrow[dl, "(y,v)\mapsto y"] \\ & U & \end{tikzcd}$$

(2) for each $y \in U$ the map $\varphi\colon p^{-1}(y) \to \{y\} \times \mathbb{R}^n = \mathbb{R}^n$ is an isomorphism of vector spaces.

(The definition is illustrated in Figure 954.) A 1-dimensional vector bundle is often called a *line bundle*. We refer to the topological space E as the *total space* of the vector bundle and we refer to X as the *base space* of the vector bundle. Furthermore we refer to the map $p\colon E \to X$ as the *projection*.

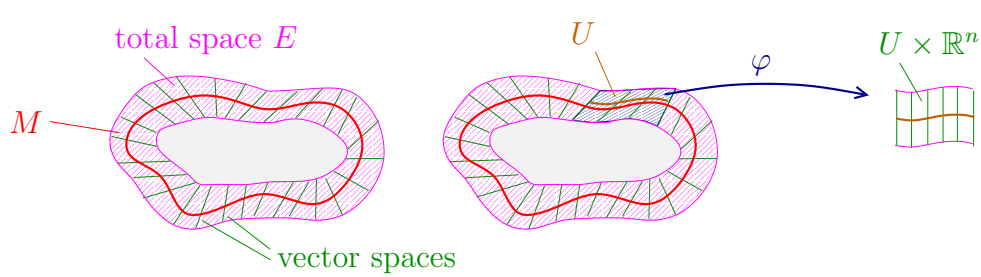


Figure 954.

Examples. In each of the following examples it is fairly elementary to verify that these are indeed vector bundles. For the sake of brevity[1292] we do not give all the details.

(1) For each topological space X the product $X \times \mathbb{R}^n$ together with the projection map $p\colon X \times \mathbb{R}^n \to X$ is an n-dimensional vector bundle over X. We refer to this vector bundle as the *trivial n-dimensional vector bundle over X*.

(2) Let M be an m-dimensional submanifold of $\mathbb{R}^k$.

(a) The map[1293]

$$\begin{array}{rcl} TM := \{(x,v) \in M \times \mathbb{R}^k \,|\, x \in M \text{ and } v \in V_xM\} & \to & M \\ (x,v) & \mapsto & x \end{array}$$

is an m-dimensional vector bundle over M. We refer to it as the *tangent bundle* of the submanifold M.

(b) The map

$$\begin{array}{rcl} NM := \{(x,v) \in M \times \mathbb{R}^k \,|\, x \in M \text{ and } v \in \mathbb{R}^k \text{ with } v \perp V_xM\} & \to & M \\ (x,v) & \mapsto & x \end{array}$$

is a $(k-m)$-dimensional vector bundle over M. We call it the *normal bundle* of M in $\mathbb{R}^k$.

The definition of the tangent bundle and the normal bundle are illustrated in Figure 955

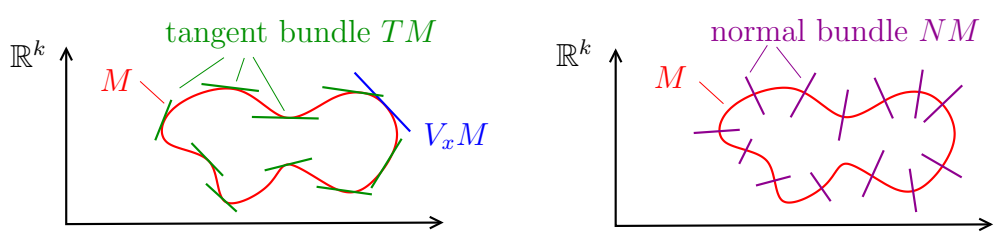


FIGURE 955.

(3) Recall that given $n \in \mathbb{N}_0$ we defined on page 104 the real projective space to be

$$\mathbb{R}\mathrm{P}^n \;:=\; (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

where $\sim$ denotes the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ given by multiplication by an element in $\mathbb{R} \setminus \{0\}$. The map

$$\begin{array}{rcl} X = \{([v], \lambda v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \,|\, [v] \in \mathbb{R}\mathrm{P}^n, \lambda \in \mathbb{R}\} & \to & \mathbb{R}\mathrm{P}^n \\ ([v], w) & \mapsto & [v] \end{array}$$

defines a one-dimensional vector bundle over $\mathbb{R}\mathrm{P}^n$. We refer to it as the *tautological line bundle* over $\mathbb{R}\mathrm{P}^n$.

[1292] No snickering please.

[1293] We refer to page 196 for the definition of the "visual tangent spaces" $V_xM \subset \mathbb{R}^k$ of a submanifold $M \subset \mathbb{R}^k$.

(4) The real projective space $\mathbb{R}\mathrm{P}^n$ admits an obvious bijection with the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$. This observation leads us to a generalization of the previous example. Given natural numbers $n, k \in \mathbb{N}$ we denote by $G(n,k)$ the set of all n-dimensional subspaces of $\mathbb{R}^{n+k}$. We have a surjection[1294]

$$\begin{array}{rcl} \overbrace{\left\{ \begin{array}{c} \text{all } k\text{-tuples } (v_1,\dots,v_k) \text{ of vectors in } \mathbb{R}^{n+k} \\ \text{that are linearly independent} \end{array} \right\}}^{\text{subset of } \mathbb{R}^{n(n+k)}} & \to & G(n,k) \\ (v_1,\dots,v_n) & \mapsto & \begin{array}{c} \text{subspace of } \mathbb{R}^{n+k} \\ \text{spanned by } v_1,\dots,v_n. \end{array} \end{array}$$

We equip $G(n,k)$ with the quotient topology. We refer to $G(n,k)$ as a *Grassmannian.*[1295][1296] With some modest effort one can show that

$$\begin{array}{rcl} F := \overbrace{\{(V,w) \mid [V] \in G(n,k) \text{ and } w \in V\}}^{\subset G(n,k)\times\mathbb{R}^n} & \to & G(n,k) \\ (V,w) & \mapsto & V \end{array}$$

is in fact an n-dimensional vector bundle. We point out that the obvious inclusions $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k} \times \{0\} \subset \mathbb{R}^{n+k+1}$ induce inclusion maps $G(n,k) \to G(n,k+1)$ and we refer to the direct limit as an *infinite Grassmannian* $G(n)$. The n-dimensional vector bundles over $G(n,k)$ give rise to an n-dimensional vector bundle over $G(n)$. We refer to this vector bundle as the *tautological vector bundle* over the Grassmannian.

Before we introduce more examples we give the following, basically self-evident definition.

Definition. We say that two vector bundles $p\colon E \to X$ and $q\colon F \to X$ over a topological space X are *isomorphic* if there exists a homeomorphism $\varphi\colon E \to F$ such that for each $x \in X$ the map φ restricts to an isomorphism $p^{-1}(x) \to q^{-1}(x)$ of vector spaces.

We continue with a few more examples of vector bundles.

Examples.

(1) The map

$$\begin{array}{rcl} ([0,1]\times\mathbb{R})/(0,v)\sim(1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & e^{2\pi i t} \end{array}$$

is a line bundle over S^1. The discussion on page 118 shows that the total space is homeomorphic to the open Möbius band. Therefore we refer to this bundle as the *Möbius bundle*. This bundle, and a bundle that is isomorphic to it, are illustrated in Figure 956.

[1294]The left-hand side is in fact an open subset of $\mathbb{R}^{n(n+k)}$. Why is that?

[1295]These objects are named after the German mathematician Hermann Grassmann (1809-1877).

[1296]It is shown in [MiS, p. 56] and [Hat2, p. 28 and 34] that one can equip $G(n,k)$ with a smooth atlas such that $G(n,k)$ becomes a closed nk-dimensional manifold. We will not make use of this fact.

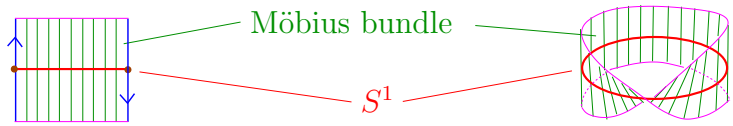


FIGURE 956.

(2) Let M be an n-dimensional manifold. We consider the map

$$\begin{array}{rcl} p\colon TM := \{(x,v)\,|\,x \in M \text{ and } v \in T_xM\} & \to & M \\ (x,v) & \mapsto & x. \end{array}$$

We would like to show that $p\colon TM \to M$ is an n-dimensional vector bundle. For each $x \in M$ the preimage $p^{-1}(x) = \{x\} \times T_xM$ admits an obvious vector space structure. Our next goal is to equip TM with a reasonable topology. For this we note that any chart $\Phi\colon U \to V$ for M induces a bijection

$$\begin{array}{rcl} \widetilde{\Phi}\colon p^{-1}(U) & \to & V \times \mathbb{R}^n \\ (x,v) & \mapsto & (\Phi(x), \underbrace{D\Phi_x(v)}_{\in T_{\Phi(x)}V = \mathbb{R}^n}). \end{array}$$

We say that a subset $W \subset TM$ is open if for every chart $\Phi\colon U \to V$ for M the image $\widetilde{\Phi}(W) \subset V \times \mathbb{R}^n$ is open. With this definition it is now reasonably easy to show that $p\colon TM \to M$ is an n-dimensional vector bundle over M. We refer to it as the *tangent bundle of* M.[1297]

If M is an n-dimensional submanifold of $\mathbb{R}^k$, then using the usual isomorphism $V_PM \cong T_PM$ from Proposition 3.32 one can show fairly easily that the two tangent bundles that we defined above are isomorphic. In practice we will not distinguish between these two notions.

(3) Let M be a connected manifold, let $x_0 \in M$ be a point and furthermore let $\alpha\colon \pi_1(M,x_0) \to \mathrm{GL}(n,\mathbb{R})$ be a representation. We denote by $p\colon \widetilde{M} \to M$ the universal covering. The group $\pi_1(M,x_0)$ acts on $\widetilde{M}$ via the action provided by Proposition 20.13. Furthermore $\pi_1(M,x_0)$ acts on $\widetilde{M} \times \mathbb{R}^n$ via $g \cdot (x,v) := (g \cdot x, \alpha(g) \cdot v)$. One can easily verify that the map

$$\begin{array}{rcl} (\widetilde{M} \times \mathbb{R}^n)/\pi_1(M,x_0) & \to & M \\ [(x,v)] & \mapsto & p(x) \end{array}$$

is an n-dimensional vector bundle over M. We refer to it as the *vector bundle defined by* α. One can show fairly easily that the bundle over S^1 corresponding to the representation $\pi_1(S^1,1) \to \mathrm{GL}(1,\mathbb{R})$ given by sending the standard generator of $\pi_1(S^1,1)$ to (-1) is isomorphic to the Möbius bundle.

[1297] It does not take too much effort to show that the total space TM of the tangent bundle is in fact a $2n$-dimensional manifold. We will not make use of this neat fact.

We throw in a few more definitions.

Definition.

(1) We say that an n-dimensional vector bundle $E \to X$ is *trivial*, if it is isomorphic to the trivial n-dimensional vector bundle $X \times \mathbb{R}^n \to X$.
(2) We say that a manifold M is *parallelizable* if its tangent bundle TM is trivial.
(3) A *section* of a vector bundle $p\colon E \to X$ is a continuous map $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$. We say a section is *non-vanishing* if $s(x) \neq 0$ for all $x \in X$.

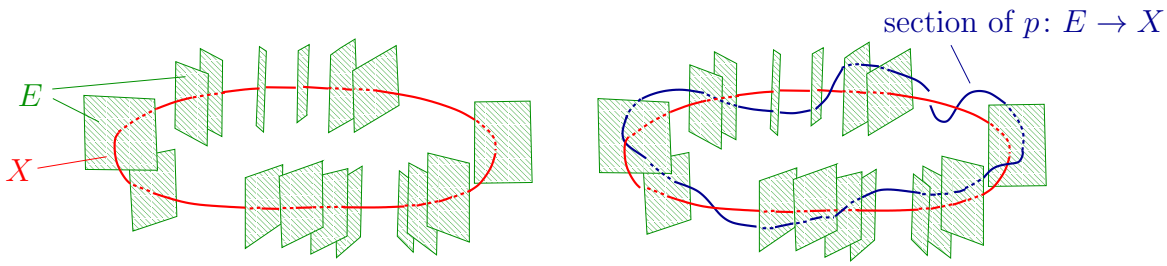


FIGURE 957.

Examples.

(1) Given a vector bundle $p\colon E \to X$ a section $s\colon X \to E$ is given by sending each $y \in E$ to the zero-element of the vector space $p^{-1}(y)$.[1298] This section is usually called the *zero-section* of p.
(2) Let M be a manifold. A *section* of the tangent bundle is usually called a *vector field* on M. If M is a submanifold of $\mathbb{R}^n$, then this is precisely the notion that we introduced on page 824.

Lemma 77.1.

(1) *Let $p\colon E \to X$ be an n-dimensional vector bundle. Then the following two statements are equivalent:*
   (a) *The vector bundle is trivial.*
   (b) *There exist n sections $s_1, \dots, s_n$ such that for every point $x \in X$ the vectors $s_1(x), \dots, s_n(x) \in p^{-1}(x)$ form a basis of the vector space $p^{-1}(x)$.*
(2) *A line bundle is trivial if and only if it admits a non-vanishing section.*
(3) *If M is an orientable 2-dimensional manifold, then its tangent bundle is trivial if and only if it admits a non-vanishing vector field.*

Sketch of proof.

[1298]It is a good exercise in practicing the definitions to convince oneself that this map $s\colon X \to E$ is indeed continuous.

(1) Let $p\colon E \to X$ be an n-dimensional vector bundle. If there exists an isomorphism $\varphi\colon X \times \mathbb{R}^n \to E$ of vector bundles, then the maps $s_i\colon x \mapsto \varphi(x, e_i)$, $i = 1, \dots, n$, define sections such that for each $x \in X$ the vectors $s_1(x), \dots, s_n(x) \in p^{-1}(x)$ form a basis of $p^{-1}(x)$. Conversely, if we are given such sections, then the map

$$\begin{array}{rcl} X \times \mathbb{R}^n & \to & E \\ (x, (\lambda_1, \dots, \lambda_n)) & \mapsto & \sum\limits_{i=1}^{n} s_i(x) \cdot \lambda_i \end{array}$$

is easily seen to be an isomorphism.

(2) This statement is an immediate consequence of (1).

(3) Let M be an orientable 2-dimensional manifold. By (1) we only have to show the following statement: if M admits a non-vanishing vector field v, then the tangent bundle is trivial. So suppose that v is a non-vanishing vector field on M. We pick an orientation for M. By [Le3, Proposition 13.3] the manifold M admits a Riemannian structure g.[1299] Since M is 2-dimensional and oriented we can now define

$$\begin{array}{rcl} w\colon M & \to & TM \\ x & \mapsto & \begin{array}{l} \text{unique vector } w(x) \in T_xM \text{ that has the following three properties:} \\ \text{(a)} \quad g(v(x), w(x)) = 0, \text{ i.e. } v(x) \text{ and } w(x) \text{ are orthogonal,} \\ \text{(b)} \quad g(w(x), w(x)) = 1, \text{ i.e. } w(x) \text{ has length one,} \\ \text{(c)} \quad v(x) \text{ and } w(x) \text{ form a positive basis for } T_xM \end{array} \end{array}$$

(We refer to Figure 958 for an illustration of the definition of $w(x)$.) It is fairly easy to see that this is a section of the tangent bundle. It has the property that for each $x \in M$ the vectors $v(x)$ and $w(x)$ form a basis for T_xM. Thus it follows from (1) that TM is a trivial bundle. □

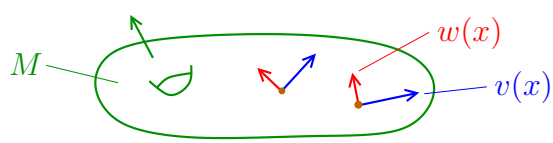


FIGURE 958. Illustration for the proof of Lemma 77.1.

Examples.

(1) By Lemma 77.1 (2) the normal bundle of $S^n \subset \mathbb{R}^{n+1}$ is trivial since it admits the non-vanishing section $P \mapsto (P, P)$. We illustrate this statement for $n = 1$ in Figure 959.

[1299] We refer to page 586 for the definition of a Riemannian structure.

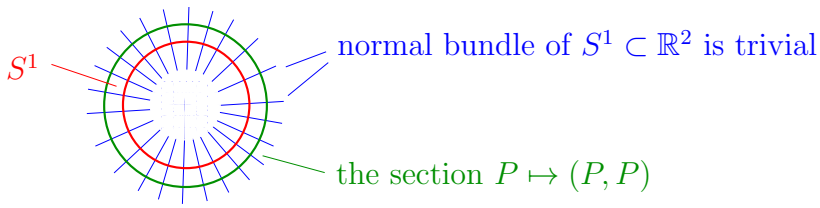


FIGURE 959.

(2) We can easily generalize the previous example. More precisely, let $n \in \mathbb{N}$ and let M be a closed oriented submanifold of $\mathbb{R}^n$ of codimension one. The map

$$\begin{array}{rcl} v\colon M & \to & NM \\ P & \mapsto & \begin{array}{c}\text{the unique vector } v(P) \in \mathbb{R}^n \text{ of length one that is normal to } V_PM \\ \text{and which, followed by a positive basis for } V_PM, \\ \text{forms a positive basis for } \mathbb{R}^n\end{array} \end{array}$$

is easily seen to be a section of the normal bundle. Since the normal bundle is one-dimensional we obtain from Lemma 77.1 (2) that the normal bundle of $M \subset \mathbb{R}^n$ is trivial. We refer to Figure 960 for an illustration.

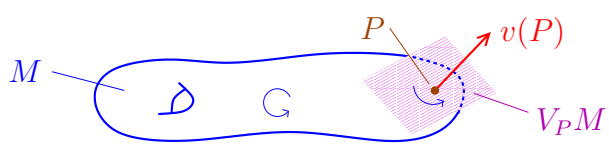


FIGURE 960.

(3) In Exercise 77.5 we will show that the Möbius bundle does not admit a non-vanishing section. It follows from Lemma 77.1 (2) that the Möbius bundle is non-trivial.
(4) It follows from the Hairy Ball Theorem 32.20 and Lemma 77.1 (1) that even-dimensional spheres are not parallelizable.
(5) For any $n \in \mathbb{N}$ the n-dimensional torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$ is parallelizable, in fact if we denote by $q\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection map, then an isomorphism of vector bundles is given by

$$\begin{array}{rcl} \varphi\colon (\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n & \mapsto & T(\mathbb{R}^n/\mathbb{Z}^n) \\ (q(x), v) & \mapsto & (q(x), Dq_x(v)) \\ & & \uparrow \\ & & Dq_x\colon \mathbb{R}^n = T_x\mathbb{R}^n \to T_{q(x)}(\mathbb{R}^n/\mathbb{Z}^n) \end{array}$$

(6) There are many proofs why for $g > 1$ the surface of genus g is not parallelizable, see for example [MiS, p. 313]. Note that together with Lemma 77.1 (3) this implies that such surfaces do not admit a nowhere vanishing vector field. We refer to Figure 961 for an illustration of a vector field on the surface of genus 2 that vanishes on precisely two points.

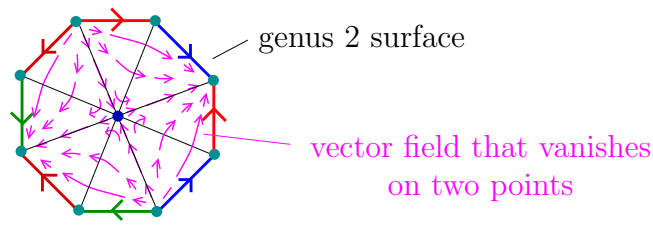


FIGURE 961.

(7) If a manifold is parallelizable, then evidently it is also orientable in the sense of Section 3.8. Put differently, a non-orientable manifold is not parallelizable.
(8) Every closed orientable 3-manifold is parallelizable. This was first proved by Eduard Stiefel [Stie] in 1935. An elementary proof for this statement is given in [BL]. Alternative proofs can be found in [DGGK] and [Kir4, p. 46].

77.2. **The pullback and the Whitney sum of vector bundles.** In this section we will introduce two constructions for obtaining new vector bundles out of given vector bundles.

Lemma 77.2. *Let $g\colon X \to Y$ be a map between topological spaces and let $q\colon F \to Y$ be an n-dimensional vector bundle. As in Lemma 16.8 (1) we consider the pullback*

$$g^*F \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(g(x))\} \;\subset\; X \times F,$$

which is equipped with the subspace topology. Then the following statements hold:

(1) *The map*

$$\begin{array}{rcl} p\colon g^*F \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(g(x))\} & \to & X \\ (x,v) & \mapsto & x \end{array}$$

is also an n-dimensional vector bundle where for each $x \in X$ the vector space structure on $p^{-1}(x) = \{x\} \times q^{-1}(g(x)) \subset X \times F$ is the vector space structure on the vector space $\{x\} \times q^{-1}(g(x)) = q^{-1}(g(x))$.

(2) *The following diagram commutes*

$$\begin{array}{ccc} g^*F & \xrightarrow{(x,v)\mapsto v} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{g} & Y. \end{array}$$

Proof.

(1) Let $x \in X$. If $V \subset Y$ is a neighborhood of $g(x)$ and $\varphi\colon q^{-1}(V) \to \mathbb{R}^n \times V$ has the desired properties (1) and (2) of the definition of a vector bundle, then the open

neighborhood $U := g^{-1}(V) \subset X$ of x together with the map

$$\begin{array}{rcl} \psi\colon p^{-1}(U) & \to & U \times \mathbb{R}^n \\ (z,v) & \mapsto & (p(z), \varphi(v)) \\ & & \uparrow \\ & & \text{note that } v \in q^{-1}(g(z)) \text{ and that } g(z) \in g(U) = V \end{array}$$

is easily seen to have the properties (1) and (2) of the definition of a vector bundle.

(2) It is straightforward to verify that the diagram commutes. Alternatively, the statement that the diagram commutes is part of the fact that g^*F is a pullback. □

Definition. Using the notation from Lemma 77.2 we refer to $g^*F \to X$ as the *pullback* of the vector bundle $q\colon F \to Y$ under the map $g\colon X \to Y$.

Example. Let $p\colon M \to S^1$ be the Möbius bundle. We denote by $g\colon S^1 \to S^1$ the map that is given by $g(z) = z^2$. In Exercise 77.8 we will see that the pullback of the Möbius bundle under g is a trivial bundle. We refer to Figure 962 for an illustration of the pullback of the Möbius bundle.[1300]

FIGURE 962.

The following proposition summarizes a few properties of the pullback of vector bundles.

Proposition 77.3.

(1) *The pullback of a trivial vector bundle is again trivial.*
(2) *If $q\colon Y\colon F \to Y$ is a vector bundle and if X is a subset of Y with inclusion map $i\colon X \to Y$, then i^*F is canonically isomorphic*[1301] *to the restriction of F to the vector bundle $q^{-1}(X) \to X$.*
(3) *Let $q\colon F \to Z$ be an n-dimensional vector bundle. The pullback $\mathrm{id}_Z^* F \to Z$ corresponding to the identity map id_Z is naturally isomorphic to $q\colon F \to Z$.*
(4) *Let $q\colon F \to Z$ be an n-dimensional vector bundle. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces, then the vector bundles $f^*(g^*F) \to X$ and $(g \circ f)^*F \to X$ are canonically isomorphic.*

Proof. All statements follow quite easily from the definitions. Thus we leave the verification of them to the industrious reader. □

[1300] The reader who is blessed with a good memory might recognize that this statement is quite similar to discussion on page 260 where we saw that the annulus is a 2-fold covering of the Möbius band.

[1301] One might initially expect that these two vector bundles are identical, but this is not the case. Nonetheless, a closer inspection of the definitions shows that there is an obvious isomorphism between these two vector bundles.

Later on we will need the following lemma.

Lemma 77.4.

(1) *If $f\colon M \to N$ is a local diffeomorphism between manifolds, then the map*

$$\begin{array}{rcl} TM & \to & f^*TN \\ (x,v) & \mapsto & (x, \underbrace{(f(x), Df_x(v))}_{\in TN}) \end{array}$$

is an isomorphism of vector bundles.

(2) *If $p\colon \widetilde{M} \to M$ is a covering of a manifold, then p^*TM is canonically isomorphic to $T\widetilde{M}$.*

Proof. We leave the straightforward verification of the first statement to the reader. The statement regarding the Pontryagin classes is now an immediate consequence of Theorem 77.9 (3). □

Proof. We rely on the never-ending energy of the reader to provide the proof of the first statement. The second statement is an immediate consequence of the first statement. □

Before we can formulate the next proposition we need to introduce the following definition.

Definition. A topological space X is called *paracompact* if it is Hausdorff and if given any open covering $\{U_i\}_{i\in I}$ there exists a partition of unity subordinate to the open covering.[1302]

Remark. In many textbooks, e.g. [Mun3, Chapter 41], [Kel, p. 156] or [Dug, Chapter VIII.2] an alternative definition for paracompactness is used. More precisely, a topological space X is often called paracompact if it is Hausdorff and if if each open covering $\{U_i\}_{i\in I}$ admits a locally finite open refinement, this means that there exists an open covering $\{V_j\}_{j\in J}$ such that each V_j is contained in some U_i and such that each $x \in X$ admits a neighborhood W that is contained in only finitely many V_j. We refer to [LW, Appendix I Proposition 6] or [Dug, Chapter VIII.4] for a proof that these two definitions of paracompactness are in fact equivalent. We will not make use of this alternative definition of paracompact.

We will never work directly with the definition of paracompact, instead we will use the following lemma to convince ourselves that most "reasonable" topological spaces are in fact paracompact.

Lemma 77.5.

(1) *Every topological manifold is paracompact.*
(2) *Every CW-complex is paracompact.*

Proof. The first statement is precisely the statement of Theorem 54.5. The second statement is [Hat2, Proposition 1.20]. □

[1302]See page 1276 for the definition of a partition of unity subordinate to a given open covering.

Example. Elementary topological psychology tells us that surely not every topological space is paracompact. Indeed, the "long line" from page 215 is not paracompact, we refer to [Kra, p. 66] for details.

Remark. Some books, see e.g. [Hat2, p. 77], make at some point the blanket assumption that all base spaces of vector bundles are paracompact.

Now we can formulate the proposition that we are actually interested in.

Proposition 77.6.

(1) *Let $q\colon F \to Y$ be a vector bundle, let X be a topological space and let $f_0, f_1\colon X \to Y$ be maps. If f_0 and f_1 are homotopic and if X is paracompact, then the vector bundles $f_0^*F \to X$ and $f_1^*F \to X$ are isomorphic.*
(2) *Let X and Y be two paracompact topological spaces. If $f\colon X \to Y$ is a homotopy equivalence, then the map f induces a bijection between the set of isomorphism classes of vector bundles over Y and the set of isomorphism classes of vector bundles over X.*
(3) *Let X be a topological space. If X is paracompact and contractible, then every vector bundle over X is trivial.*

Example. It follows immediately from Lemma 77.5 (1) and Proposition 77.6 (3) that every vector bundle over the topological spaces $\overline{B}^n, B^n$ and $\mathbb{R}^n$ is trivial.

Proof.

(1) This statement is [Hat2, Theorem 1.6].
(2) Let $g\colon Y \to X$ be a homotopy inverse to f. Recall that this means that $f \circ g$ is homotopic to id_Y and that $g \circ f$ is homotopic to id_X. Now the statement follows immediately from (1) and Proposition 77.3 (4).
(3) Let $F \to X$ be a vector bundle over a topological space that is paracompact and contractible. By definition of being contractible there exists a homotopy equivalence from the topological space X to the topological space $\{*\}$ consisting of a single point. Since every vector bundle over $\{*\}$ is trivial it follows from (2) and Proposition 77.3 (1) that every vector bundle over X is trivial. □

The following proposition says that every n-dimensional vector bundle is the pullback of the tautological bundle over the infinite Grassmannian $G(n)$ in an "essentially unique way".

Proposition 77.7. *Let $p\colon E \to X$ be an n-dimensional vector bundle over a paracompact topological space. We denote by $F \to G(n)$ the n-dimensional vector bundle over the infinite Grassmannian $G(n)$ that we introduced on page 1784. The following two statements hold:*

(1) *There exists a map $g\colon X \to G(n)$ such that g^*F is isomorphic to E.*
(2) *If $g_0, g_1\colon X \to G(n)$ are maps such that g_0^*F and g_1^*F are isomorphic to E, then g_0 and g_1 are homotopic.*

Sketch of proof. We only consider the case of the tangent bundle $p\colon TM \to M$ over some closed manifold M. By Proposition 65.11 we know that there exists a $k \in \mathbb{N}$ and an embedding $\varphi\colon M \to \mathbb{R}^{n+k}$.[1303] Now we consider the map

$$\begin{array}{rcl} g\colon M & \to & G(n, n+k) \subset G(n) \\ P & \mapsto & V_{\varphi(P)}(\varphi(M)). \end{array}$$

(We refer to Figure 963 for an illustration.) One can verify fairly easily that the pullback of the tautological vector bundle is indeed isomorphic to the original vector bundle. This concludes the proof of (1) for this particular case.

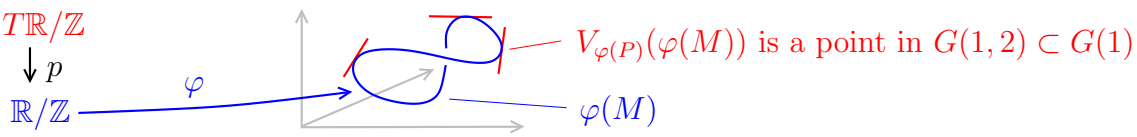


FIGURE 963. Illustration for the proof of Proposition 77.7.

Full details for the proofs of the two statements for any vector bundle over a paracompact topological space are provided by [MiS, Lemma 3.1 and Theorems 5.6 and 5.7]. □

We continue with our second construction of new vector bundles. The same way that we could obtain new vector spaces out of given vector spaces we can also obtain new vector bundles out of given vector bundles. For example we will now generalize the notion of the direct sum of two vector spaces to the notion of the direct sum of two vector bundles over a given topological space.

Definition. Let $p\colon E \to X$ be an m-dimensional vector bundle and let $q\colon F \to X$ be an n-dimensional vector bundle over a topological space X. The map

$$\begin{array}{rcl} E \oplus F := \{(v,w) \in E \times F \,|\, p(v) = q(w)\} & \to & X \\ (v,w) & \mapsto & p(v) \end{array}$$

is easily seen to be an $(m+n)$-dimensional vector bundle. It is called the *direct sum of the vector bundles E and F*.[1304]

Examples.

(1) Let $M \subset \mathbb{R}^n$ be a submanifold. Then for any point $P \in M$ we evidently have $V_P M \oplus (V_P M)^\perp = \mathbb{R}^n$. With this observation it follows that the map

$$\begin{array}{rcl} TM \oplus NM & \to & M \times \mathbb{R}^n \\ ((P,v),(P,w)) & \mapsto & (P, v+w) \end{array}$$

is an isomorphism of vector bundles. Put differently, the direct sum of the tangent bundle and the normal bundle is a trivial bundle.

[1303] Note that in the proof of that proposition we used implicitly that manifolds are paracompact.

[1304] The direct sum of two vector bundles is often also called the *Whitney sum*.

(2) We consider the sphere $S^2 \subset \mathbb{R}^3$. On page 1788 we saw that the tangent bundle TM is non-trivial and on page 1787 we saw that the normal bundle is trivial. Thus we see that the direct sum of a non-trivial vector bundle with a trivial vector bundle can be trivial.

We state the last example as a lemma.

Lemma 77.8. *Let M be an m-dimensional manifold and let N be an n-dimensional manifold. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projection maps. The map*[1305]

$$\begin{array}{rcl} T(M \times N) & \to & p^*(TM) \oplus q^*(TN) \\ ((x,y),v) & \mapsto & \big(((x,y),(x,p_*(v))),\ ((x,y),(y,q_*(v)))\big) \end{array}$$

is an isomorphism of $(m+n)$-dimensional vector bundles over $M \times N$.

Proof. The lemma is fairly elementary, a detailed proof is given in [Tu, p. 371]. □

77.3. **Pontryagin classes.** Given a topological space X and $i \in \mathbb{N}_0$ we define

$$\overline{H}^i(X;\mathbb{Z}) \ := \ H^i(X;\mathbb{Z})/\{a \in H^i(X;\mathbb{Z}) \,|\, 2a = 0\}.$$

It should be clear that these cohomology groups $\overline{H}$ are also contravariantly functorial and that all the results on usual cohomology groups have obvious analogues.

We can now formulate the following result that is one of the key theorems of this chapter. It was proved initially in about 1940 by Lev Pontryagin.[1306]

Theorem 77.9. *For each $i \in \mathbb{N}_0$ we can assign to every vector bundle $p\colon E \to X$ over a paracompact topological space a cohomology class $p_i(E) \in \overline{H}^{4i}(X;\mathbb{Z})$, called i-th* Pontryagin class*, such that the following conditions are satisfied:*

(1) *The 0-th Pontryagin class of any vector bundle over a topological space X is the unit element in $\overline{H}^0(X;\mathbb{Z})$.*
(2) *For any trivial vector bundle all Pontryagin classes p_i with $i \geq 1$ are zero.*
(3) *The Pontryagin class behaves well under pullbacks, i.e. if we are given a vector bundle $q\colon F \to Y$ and a map $g\colon X \to Y$ of topological spaces, then for any $i \in \mathbb{N}_0$ we have*

$$p_i(g^*F) \ = \ g^*(p_i(F)) \ \in \ \overline{H}^{4i}(X;\mathbb{Z}).$$

(4) *If $p\colon E \to X$ is an n-dimensional vector bundle, then $p_i(E) = 0$ for $i > \frac{n}{2}$.*

[1305]The map might appear slightly intimidating, but it is "the obvious" map. It looks somewhat complicated since the pullback vector bundles $p^*(TM)$ and $q^*(TM)$ are both defined to be pairs, where the first entry is a point in $M \times N$ and the second entry is also a pair.

[1306]Lev Pontryagin is also an old friend whom we had already encountered on page 1630.

(5) *We denote by* $x \in \overline{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ *the standard generator as introduced on page 1453 and in Proposition 67.17. Then for* $i = 0, \dots, n+1$ *we have*[1307]

$$p_i(T\mathbb{C}\mathrm{P}^n) \;=\; \binom{n+1}{i} \cdot x^{2i} \;\in\; \overline{H}^{4i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;=\; \mathbb{Z}[x]/(x^{n+1})$$

and all other Pontryagin classes are zero.

Given a vector bundle $p\colon E \to X$ *we refer to*

$$p(E) \;:=\; \underbrace{\bigoplus_{i=0}^{\infty} p_i(E)}_{\text{finite sum by Theorem 77.9 (4)}} \;\in\; \overline{H}^*(X;\mathbb{Z}) \;=\; \bigoplus_{i=0}^{\infty} \overline{H}^i(X;\mathbb{Z})$$

as the total Pontryagin class. *With this definition we can state the last property of Pontryagin classes:*

(6) *The total Pontryagin class is multiplicative with respect to the direct sum of vector bundles* up to 2-torsion. *More precisely, if* $e\colon E \to X$ *and* $f\colon F \to X$ *are two vector bundles, then*

$$p(E \oplus F) \;=\; p(E) \cup p(F) \;\in\; \overline{H}^*(X;\mathbb{Z}).$$

Remark.

(1) The above introduction to Pontryagin classes differs somewhat from the usual approach taken in [MiS, Hat2]. In particular Pontryagin classes are usually defined as elements in the usual cohomology groups.
(2) It is not entirely clear whether classes $p_i(E)$ are uniquely defined by the properties stated in Theorem 77.9. Similar uniqueness statements hold for "Stiefel-Whitney classes" and "Chern classes", see [Hat2, Theorems 3.1 and 3.2], but it is nonetheless not clear to me how to prove uniqueness for Pontryagin classes. Fortunately that is of no concern to us.
(3) For category aficionados one can squeeze some of the statements of Theorem 77.9 into the language of category theory:
   (a) Let $\mathcal{T}$ be the category of paracompact topological spaces, here the objects are paracompact topological spaces and the morphisms between two paracompact topological spaces X and Y are as usual all continuous maps from X to Y.
   (b) Let $\mathcal{S}$ be the category of sets, i.e. the objects are sets and the morphisms are just maps.
   (c) Given a paracompact topological space X we denote by $V_n(X)$ the *set* of isomorphism classes of n-dimensional vector bundles. This is indeed a set since by Proposition 77.7 we can view $V_n(X)$ as a subset of the set of homotopy classes of maps $X \to G(n)$. We write $V(X) = \bigcup_{n\in\mathbb{N}} V_n(X)$.

[1307]Note that for $i > \lfloor \frac{n}{2} \rfloor$ we have $x^{2i} = 0 \in H^{4i}(\mathbb{C}\mathrm{P}^n) = 0$. Nonetheless, for later it is convenient to write down this expression for all i up to n.

(d) For each $i \in \mathbb{N}_0$ we consider the following two contravariant functors:

$$\begin{array}{c} \mathcal{T} \to \mathcal{S} \\ X \mapsto H^{4i}(X;\mathbb{Z}) \\ \big(X \xrightarrow{f} Y\big) \mapsto \big(\overline{H}^{4i}(Y) \xrightarrow{f^*} \overline{H}^{4i}(X)\big) \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{T} \to \mathcal{S} \\ X \mapsto V(X) \\ \big(X \xrightarrow{f} Y\big) \mapsto \left(\begin{array}{c} V(Y) \to V(X) \\ [G \xrightarrow{p} Y] \mapsto [f^*G \to X] \end{array}\right). \end{array}$$

(e) In this language we can say that Pontryagin classes are a natural transformation between these two contravariant functors. This sentence now models property (3) of Pontryagin classes.

Sketchy sketch of a proof. Given $n \in \mathbb{N}$ we denote by $G(n)$ the infinite Grassmannian that we introduced on page 1784. In [Hat2, Proposition 3.16] it is shown that[1308][1309]

$$(\overline{H}^*(G(n);\mathbb{Z}), \cup) \;\cong\; \mathbb{Z}[q_1, \dots, q_k] \text{ for } n = 2k \text{ or } n = 2k+1$$

where each q_i has degree $4i$. Now let $p\colon E \to X$ be an n-dimensional vector bundle. We denote by $F \to G(n)$ the n-dimensional vector bundle over the infinite Grassmannian $G(n)$ that we introduced on page 1784. By Proposition 77.7 there exists a map $g\colon X \to G(n)$ such that g^*F is isomorphic to E. We define $p_i(E) := g^*(q_i) \in \overline{H}^{4i}(X;\mathbb{Z})$. It follows from Proposition 77.7 that this definition does not depend on the choice of the map g.

The proof that the Pontryagin classes have all the stated properties is given in [MiS, Chapter 15]. □

Example. Let $n \in \mathbb{N}$. Let M be a closed orientable submanifold of $\mathbb{R}^n$ of codimension one. For example we could take $M = S^{n-1} \subset \mathbb{R}^n$ or M could be the surface of some genus g which, by the discussion on page 210, we can view as a submanifold of $\mathbb{R}^3$.[1310] We want to determine the total Pontryagin class of the tangent bundle TM. As before we denote by NM the normal bundle of $M \subset \mathbb{R}^n$. Then

$$\begin{array}{ccccccc} p(TM) & = & p(TM) \cup p(NM) & = & p(TM \oplus NM) & = & p(\text{trivial bundle } M \times \mathbb{R}^n) \quad = \; 1. \\ & \uparrow & & \uparrow & & \uparrow & \qquad\qquad\qquad\qquad \uparrow \\ & \begin{array}{c}\text{by Theorem 77.9 (1) and (2)} \\ \text{since the vector bundle } NM \\ \text{is trivial, see page 1788}\end{array} & & \text{by Theorem 77.9 (6)} & & \text{see page 1793} & \qquad\qquad\qquad\qquad \text{Theorem 77.9 (2)} \end{array}$$

In the remainder of this chapter we are mostly interested in tangent bundles of manifolds, which leads us to the following definition.

[1308]The fact that the cohomology ring $(\overline{H}^*(G(n);\mathbb{Z}), \cup)$ is a polynomial algebra should ring a bell: When we introduced the (infinite) Grassmannian $G(n)$ we were saying that it should be viewed as a generalization of the (infinite) real projective spaces. Thus the calculation of the cup product mirrors our calculation that $(H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}), \cup) \cong \mathbb{Z}[x]$, see Proposition 62.12, and our calculation that $(H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2), \cup) \cong \mathbb{F}_2[x]$, see Proposition 62.16 (3).

[1309]The calculation of the cohomology ring over the usual cohomology groups $H^*(G(n);\mathbb{Z})$ is somewhat more involved, see [Hat2, Theorem 3.9] for more information.

[1310]The Pontryagin classes for surfaces are zero for the stupid reason that they live in dimensions $4i$, but that is besides the point for this example.

Definition. Given a manifold M we define its Pontryagin classes as the Pontryagin classes of its tangent bundle, i.e. for $i \in \mathbb{N}_0$ we write $p_i(M) := p_i(TM) \in \overline{H}^{4i}(M;\mathbb{Z})$.

Example. With these definitions the statement of Theorem 77.9 (5) is equivalent to

$$\begin{array}{ccccccc} p(\mathbb{C}\mathrm{P}^n) & = \sum_{i=0}^{n} \binom{n+1}{i} \cdot x^{2i} & = & \overbrace{(1+x^2)^{n+1}}^{\text{higher terms are zero in the ring}} & \in \overline{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) & = & \mathbb{Z}[x]/(x^{n+1}). \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

by Theorem 77.9 (2) and (5) since $T\mathbb{C}\mathrm{P}^n$ is $2n$-dimensional — general binomial formula, together with the fact that $x^j = 0$ for $j \geq n+1$ — see Proposition 67.17

For example we have

$$p(\mathbb{C}\mathrm{P}^1) = 1, \quad p(\mathbb{C}\mathrm{P}^2) = 1 + 3x^2, \quad p(\mathbb{C}\mathrm{P}^3) = 1 + 4x^2, \quad p(\mathbb{C}\mathrm{P}^4) = 1 + 5x^2 + 10x^4.$$

77.4. **Pontryagin classes as invariants of manifolds.** So far, throughout these lectures we have basically only studied invariants of topological spaces. Even the de Rham cohomology groups that we introduced in Section 54.2 were shown in Proposition 55.4 to be topological invariants, i.e. their isomorphism types depend only on the homeomorphism type of the underlying topological space.

The Pontryagin classes of a manifold M turn out to be more subtle invariants: they are defined using the tangent bundle of M. The definition of the tangent bundle in turn relies on the choice of the smooth structure. Put differently, a priori the Pontryagin classes $p_i(M) \in \overline{H}^{4i}(M;\mathbb{Z})$ depend on the smooth structure of the manifold.

The following lemma shows how diffeomorphisms interact with Pontryagin classes.

Lemma 77.10. *If $f\colon M \to N$ is a local diffeomorphism between two manifolds, then for any $i \in \mathbb{N}_0$ we have*

$$p_i(TM) = f^*(p_i(TN)) \in \overline{H}^{4i}(M;\mathbb{Z}).$$

Proof. The lemma is an immediate consequence of Lemma 77.4 (1) and Theorem 77.9 (3). □

In [Shar, Theorem 1.6] it is shown that the first Pontryagin class cannot detect the difference between different smooth structures. But the following theorem demonstrates that the higher Pontryagin classes contain information on smooth structures of manifolds.

Theorem 77.11. *For any $n \geq 2$ there exist two closed manifolds M and M' with the following properties:*

1. *the manifolds M and M' are homeomorphic,*
2. *the n-th Pontryagin class of M is trivial, whereas*
3. *the n-th Pontryagin class of M' is non-zero.*

In particular it follows from Lemma 77.10 that the manifolds M and M' are not diffeomorphic.

Proof. This theorem is proved in [KL, Theorem 4.8], see also [KL, Proof of Proposition 4.7]. Examples of non-compact manifolds M and M' with the same properties (1), (2) and (3) were given before by John Milnor [Mi8, p. 78] in 1964, see also [Rudy2, p. 87]. Furthermore, similar ideas had already been used before by John Milnor [Mi1] in the proof of Theorem 28.21 where he showed that there exists a manifold that is homeomorphic but not diffeomorphic to S^7. □

In fact the non-zero Pontryagin classes of the manifold M' in [Mi8, p. 78] and [KL, Theorem 4.8] happen to be torsion elements, so the Pontryagin classes of M and M' differ by some torsion element in cohomology. This leads us to the definition of the *rational Pontryagin* classes of a manifold.

Definition. Given a manifold M and $i \in \mathbb{N}_0$ we define the *i-th rational Pontryagin class* $p_i(M;\mathbb{Q})$ of M to be the image of $p_i(M)$ under the map $\overline{H}^{4i}(M;\mathbb{Z}) \to H^{4i}(M;\mathbb{Q})$.

The following theorem says that rational classes can *not* be used to find examples of manifolds that are homeomorphic but non-diffeomorphic. The theorem was proved by Sergei Novikov [No] in 1965.[1311]

Theorem 77.12. *Let $f\colon M \to M'$ be a homeomorphism between closed manifolds. We denote by $f^*\colon H^*(M';\mathbb{Q}) \to H^*(M;\mathbb{Q})$ the induced map on rational cohomology. Then for any $i \in \mathbb{N}_0$ we have*

$$p_i(M;\mathbb{Q}) \;=\; f^*(p_i(M';\mathbb{Q})).$$

77.5. **Pontryagin numbers and the signature.** We start out with the definition of the Pontryagin numbers of a manifold.

Definition.

(1) Let $n \in \mathbb{N}_0$. A *partition of n* is a set $I = \{i_1, \dots, i_r\}$ of numbers in $\mathbb{N}$ such that $n = i_1 + \cdots + i_r$. Often we just write, *$I = i_1 + \cdots + i_r$ is a partition of n.*[1312]

(2) Let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. Given a closed oriented $4n$-dimensional manifold we define its *I-Pontryagin class* as

$$p_I(M) \;:=\; p_{i_1}(M) \cup \cdots \cup p_{i_r}(M) \;\in\; \overline{H}^{4n}(M;\mathbb{Z})$$

and we define the corresponding *I-Pontryagin number* as

$$\widetilde{p}_I(M) \;:=\; \langle p_I(M), [M]\rangle \;\in\; \mathbb{Z}.$$

Remark. Let M and M' be two closed oriented $4n$-dimensional manifolds. If there exists an orientation-preserving diffeomorphism from M to M', then it follows easily from Lemma 77.10 that all of the Pontryagin numbers of M and M' agree.

The following proposition relates Pontryagin classes to the previous chapter on cobordisms. The proposition was first proved by Pontryagin.

[1311] Sergei Novikov (*1938) is a Russian mathematician who obtained a Fields medal in 1970.

[1312] Note that a partition is unordered, hence we view $4 = 3 + 1$ and $4 = 1 + 3$ as the same partition.

Proposition 77.13. *Let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n.*

(1) *Let M be a closed oriented $4n$-dimensional manifold. If M is the boundary of a compact orientable manifold W, then $\widetilde{p}_I(M) = 0$.*

(2) *The map*

$$\begin{array}{rcl} \widetilde{p}_I \colon \Omega_{4n} & \to & \mathbb{Z} \\ M & \mapsto & \widetilde{p}_I(M) \end{array}$$

is a well-defined homomorphism.

Proof. Let W be a compact oriented $(4n+1)$-dimensional manifold. We set $M = \partial W$ and we denote by $i \colon M \to W$ the inclusion map. The subsequent calculation relies on the following claim.

Claim. If we denote by $TM \oplus \mathbb{R}$ the direct sum of TM and the one-dimensional trivial bundle over M, then $i^*(TW) \cong TM \oplus \mathbb{R}$.

By the Collar Neighborhood Theorem 10.10 we can find a neighborhood $M \times \{0\} \subset W$ for $M = M \times [0,1]$. We have

$$\begin{array}{ccccccc} i^*(TW) & \cong & i^*(T(M \times [0,1])) & \cong & TM \oplus i^*(T[0,1]) & \cong & TM \oplus \mathbb{R}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 77.4 (1)} & & \text{Lemma 77.8} & & \text{Proposition 77.3 (1)} & \end{array}$$

This concludes the proof of the claim.

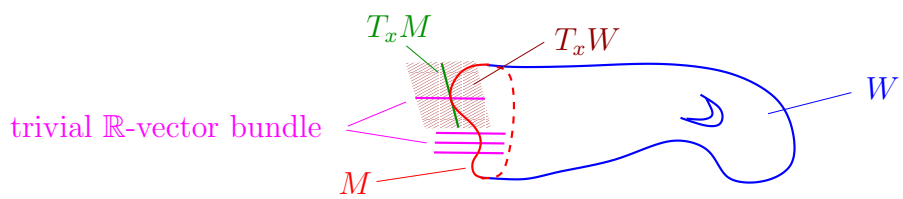


FIGURE 964. Illustration for the proof of Proposition 77.13

Let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. We turn to the proof of the first statement. We calculate that

$$\begin{array}{ccccccc} \widetilde{p}_I(M) & = & \langle p_I(TM), [M] \rangle & = & \langle p_I(TM \oplus \mathbb{R}), [M] \rangle & = & \langle p_I(i^*TW), [M] \rangle \\ & & & \uparrow & & \uparrow & \\ & & & \text{Theorem 77.9 (2) and (6)} & & \text{by the claim} & \\ & = & \langle i^* p_I(TW), [M] \rangle & = & \langle p_I(TW), i_*([M]) \rangle & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Theorem 77.9 (3)} & & \text{Lemma 50.6} & & \text{Corollary 46.27} & \end{array}$$

Finally the second statement is an immediate consequence of the first statement and the trivial[1313] observation that for two closed oriented $4n$-dimensional manifolds M and N we

[1313] Why is this "trivial"? How can one prove this "obvious" equality?

have

$$\widetilde{p}_I(M \sqcup N) \;=\; \widetilde{p}_I(M) + \widetilde{p}_I(N). \qquad \square$$

The following theorem gives the converse to Proposition 77.13.

Theorem 77.14. *Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n.*

(1) *The map*

$$\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I \colon \Omega_{4n} \otimes \mathbb{Q} \;\to\; \bigoplus_{i\in\mathcal{I}} \mathbb{Q}\cdot i$$

is an isomorphism.

(2) *Let M be a closed oriented $4n$-dimensional manifold. If all Pontryagin numbers of M vanish, then $[M] = 0 \in \Omega_{4n}/\text{torsion}$.*

Sketch of proof. We consider the following diagram of homomorphisms:

$$\begin{tikzcd}
\Omega_{4n}/\text{torsion} \arrow[r, "\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I"] \arrow[d, hook] & \bigoplus_{i\in\mathcal{I}} \mathbb{Z}\cdot i \arrow[d, hook] \\
\Omega_{4n}\otimes\mathbb{Q} \arrow[r, "\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I"] & \bigoplus_{i\in\mathcal{I}} \mathbb{Q}\cdot i. \\
\bigoplus_{i_1+\dots+i_r=n\in\mathcal{I}} \mathbb{Q}\cdot(\mathbb{CP}^{2i_1}\times\dots\times\mathbb{CP}^{2i_r}) \arrow[u, "="] \arrow[ur, dashed, "\Psi"'] &
\end{tikzcd}$$

The left upper vertical map is a monomorphism by the discussion on page 1004. The right upper vertical map is evidently a monomorphism. It is clear that the diagram commutes. Thus we see that statement (2) of the theorem is an immediate consequence of statement (1).

Put differently, it remains to show that the bottom horizontal map is an isomorphism. Note that the lower vertical map is the identity by Theorem 76.11. Thus it suffices to show that the diagonal map is an isomorphism. As a reality check, note that the dimensions agree, in both cases the dimension is just given by the cardinality of $\mathcal{I}$.

In the following claim we carry out the arguments for $n = 1$ and $n = 2$ "by hand". The general case needs more clever arguments, for which we refer to [Tho2, Corollary IV.16].

Claim.

(1) Let $n = 1$. With respect to the bases $\{\mathbb{CP}^2\}$ and $\{p_1\}$ the diagonal map Ψ is represented by the matrix (3).

(2) Let $n = 2$. With respect to the bases $\{\mathbb{CP}^4, \mathbb{CP}^2\times\mathbb{CP}^2\}$ and $\{p_2, p_{1+1}\}$ the diagonal map Ψ is represented by the matrix $\begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}$.

In particular in both cases the matrix is invertible.

The case $n = 1$ is easy: We make the usual identification $pH^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x]/(x^3)$, and we calculate that

$$\begin{array}{ccccccc} \widetilde{p}_1(\mathbb{C}\mathrm{P}^2) & = & \langle p_1(\mathbb{C}\mathrm{P}^2), [\mathbb{C}\mathrm{P}^2]\rangle & = & \langle 3x^2, [\mathbb{C}\mathrm{P}^2]\rangle & = & 3. \\ & & & \uparrow & & \uparrow & \\ & & & \text{Theorem 77.9 (5)} & & \text{Proposition 67.17} & \end{array}$$

The case $n = 2$ is slightly more involved. First note that by Theorem 77.9 (5) we have $p(\mathbb{C}\mathrm{P}^4) = 1 + 5x^2 + 10x^4 \in H^*(\mathbb{C}\mathrm{P}^4;\mathbb{Z}) = \mathbb{Z}[x]/(x^5)$. Thus as above we see that $\widetilde{p}_2(\mathbb{C}\mathrm{P}^4) = 10$. Furthermore, with similar arguments as above we have

$$\widetilde{p}_{1+1}(\mathbb{C}\mathrm{P}^4) \;=\; \langle p_1(\mathbb{C}\mathrm{P}^4) \cup p_1(\mathbb{C}\mathrm{P}^4), [\mathbb{C}\mathrm{P}^4]\rangle \;=\; \langle 5x^2 \cdot 5x^2, [\mathbb{C}\mathrm{P}^4]\rangle \;=\; 25.$$

This concludes the calculation of the first column of the matrix.

It remains to consider $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$. By Proposition 60.21 we can make the identification $H^*(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x,y]/(x^3,y^3)$ where x and y are the pullbacks of the standard generator of $H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ under the two obvious projection maps. Using Lemma 77.8 and Theorem 77.9 (3), (5), (6) we see that $p(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (1 + 3x^2) \cdot (1 + 3y^2)$. In particular $p_1(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 3x^2 + 3y^2$ and $p_2(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 9x^2y^2$. Furthermore note that this implies that $p_{1+1}(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (3x^2 + 3y^2)^2 = 18x^2y^2$. The claim for $n = 2$ now follows from the observation that Proposition 60.2 tells us that $\langle x^2 \cdot y^2, [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]\rangle = 1$. $\square$

Our next goal is to use Theorem 77.14 to prove the following proposition. It says that the signature of a closed oriented manifold can be determined from its Pontryagin numbers.

Proposition 77.15. *Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n. For each $I \in \mathcal{I}$ there exists $a_I \in \mathbb{Q}$ such that for any closed oriented $4n$-dimensional manifold M we have*

$$\mathrm{sign}(M) \;=\; \sum_{I \in \mathcal{I}} a_I \cdot \widetilde{p}_I(M).$$

Proof. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n. We consider the following diagram

$$\begin{tikzcd} \Omega_{4n} \arrow[r, "{[M]\mapsto[M]\otimes 1}"] \arrow[d, "\mathrm{sign}"'] & \Omega_{4n} \otimes \mathbb{Q} \arrow[r, "\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I", "\cong"'] \arrow[d] & \bigoplus_{i\in\mathcal{I}} \mathbb{Q} \cdot i \arrow[dl, dashed, "\Phi"] \\ \mathbb{Z} \arrow[r, hook] & \mathbb{Q} & \end{tikzcd}$$

$$[M] \otimes r \mapsto \mathrm{sign}(M) \cdot r$$

Here the left-hand map is a homomorphism by Corollary 76.7. By Proposition 77.13 we know that the top right map is an isomorphism. Therefore there exists a diagonal homomorphism Φ that makes the diagram commute. Given $i \in I$ we set $a_i \in \mathbb{Q}$ to be the value

of Φ at $1 = 1 \cdot i \in \mathbb{Q} \cdot i \subset \bigoplus_{i \in I} \mathbb{Q} \cdot i$. In other words, the map

$$\bigoplus_{i \in \mathcal{I}} \mathbb{Q} \cdot i \quad \to \quad \mathbb{Q}$$

is given by

$$\bigoplus_{i \in \mathcal{I}} x_i \cdot i \quad \mapsto \quad \sum_{i \in \mathcal{I}} x_i \cdot a_i.$$

The desired equality $\operatorname{sign}(M) = \sum_{j=1}^{r} a_j \cdot \widetilde{p}_{I_j}(M)$ now follows immediately from the commutativity of the previous diagram. □

The coefficients in Proposition 77.15 had been completely determined by Friedrich Hirzebruch [Hirz]. The low-dimensional terms are given in the following theorem.

Theorem 77.16. (Hirzebruch Signature Theorem) *Let $n \in \{1, 2, 3, 4\}$ and let M be a closed oriented $4n$-dimensional manifold. Given $i \in \mathbb{N}_0$ we write $p_i := p_i(M) \in \overline{H}^{4i}(M; \mathbb{Z})$. The signature of M can be determined as follows:*

n	$\operatorname{sign}(M)$ *equals the Kronecker pairing of* $[M]$ *with the cohomology class*
1	$\frac{1}{3} p_1$
2	$\frac{1}{45}(7p_2 - p_1^2)$
3	$\frac{1}{945}(62p_3 - 13p_2 \cup p_1 + 2p_1^3)$
4	$\frac{1}{14175}(381p_4 - 71p_3 \cup p_1 - 19p_2^2 + 22p_2 \cup p_1^2 - 3p_1^4)$.

Proof. We carry out the case $n = 2$. We adopt the notation from the proof of Proposition 77.15. By Theorem 76.11 we have $\Omega_8 \otimes \mathbb{Q} = \mathbb{Q} \cdot [\mathbb{CP}^4] \oplus \mathbb{Q} \cdot [\mathbb{CP}^2 \times \mathbb{CP}^2]$. Now let M be a closed oriented 8-dimensional manifold. By the proof of Proposition 77.15 we have $\operatorname{sign}(M) = a_2 \cdot \widetilde{p}_2(M) + a_{1+1} \cdot \widetilde{p}_{1+1}(M)$ where

$$\begin{pmatrix} a_2 \\ a_{1+1} \end{pmatrix} = \begin{matrix} \text{matrix of sign applied to} \\ \text{the basis } \mathbb{CP}^4 \text{ and } \mathbb{CP}^2 \times \mathbb{CP}^2 \end{matrix} \cdot \begin{matrix} \text{inverse of the matrix that expresses } p_2, p_{1+1} \\ \text{for the manifolds } \mathbb{CP}^4 \text{ and } \mathbb{CP}^2 \times \mathbb{CP}^2 \end{matrix}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}^{-1} = \tfrac{-1}{45} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 18 & -9 \\ -25 & 10 \end{pmatrix} = \tfrac{1}{45} \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

$\uparrow$

by Lemma 67.18 and Exercise 72.2 we know that the signatures of these two manifolds are $+1$
for the second matrix we use the calculation from the proof of Theorem 77.14

This gives precisely the desired result for $n = 2$. The case $n = 1$ is evidently even easier.

The higher-dimensional cases are dealt with, more cleverly, in [Hirz] or alternatively in [MiS, p. 224]. These references also contain a precise formula for all $n \in \mathbb{N}_0$. □

Remark.

(1) The expressions in Theorem 77.16 might seem random, but they do arise in a cogent way from the coefficients of the power series $\frac{\sqrt{t}}{\tanh(t)}$. This connection is explained in [MiS, Chapter 19].
(2) The signature of a manifold is of course an integer. Thus we obtain from Theorem 77.16 the slightly mysterious statement that for any closed oriented 4-dimensional manifold the integer $\langle p_1(M), [M]\rangle$ is divisible by 3.

In Proposition 26.25 we saw that the Euler characteristic of a finite CW-complex is multiplicative under finite covers. In fact we saw that this statement follows almost immediately from the definition of a CW-complex. The following theorem gives the analogous statement that the signature of a closed oriented $4n$-dimensional manifold is multiplicative under finite covers.

Theorem 77.17. *Let $n \in \mathbb{N}_0$. Let M be a closed oriented $4n$-dimensional manifold and let $\widetilde{M} \to M$ be a finite covering. Then*

$$\operatorname{sign}\left(\widetilde{M}\right) \;=\; \left[\widetilde{M} : M\right] \cdot \operatorname{sign}(M).$$

Remark.

(1) Perhaps it is initially difficult to appreciate the significance of Theorem 77.17. If we only consider (co-) homological invariants, then there are basically only two statements that relate the invariants of a (reasonable) topological space X and an n-fold covering $p\colon \widetilde{X} \to X$ thereof:
   (a) as pointed out above, if X is a finite CW-complex, then $\chi(\widetilde{X}) = n \cdot \chi(X)$,
   (b) Proposition 44.14 shows that $p_*\colon H_*(\widetilde{X};\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is an epimorphism.
   Besides these observations, (co-) homological invariants behave fairly erratically under finite covers. One example of such "random" behavior is given in Exercise 77.10.
(2) One might think that one should be able to prove Theorem 77.17 in a naive way. But this is not the case. This can be seen from the fact that our sketch of the proof of Theorem 77.17 works only for manifolds, the proof that we will sketch does not apply to topological manifolds.
(3) The statement of Theorem 77.17 does in fact hold for topological manifolds, but the proof is even harder, see e.g. [Schaf, Theorem 8].
(4) The fact that there is no proof of Theorem 77.17 that does not in an essential way make use of the fact that M is a (topological) manifold is highlighted by [Wall6, Corollary 5.4.1].
(5) For compact $4n$-dimensional manifolds with non-empty boundary the signature is in general *not* multiplicative under finite covers. We refer to [APS] for details.

Proof. Let $n \in \mathbb{N}_0$. By Proposition 77.15 there exist partitions $I_1, \ldots, I_r$ of $4n$ and rational numbers $a_1, \ldots, a_r$ such that for any closed oriented $4n$-dimensional manifold X the

following equality holds:

$$(*) \qquad \operatorname{sign}(X) \;=\; \sum_{j=1}^{r} a_j \cdot \langle p_{I_j}(X), [X]\rangle_X .$$

Now let M be a closed oriented $4n$-dimensional manifold and let $q\colon \widetilde{M} \to M$ be a finite covering. We have

$$\begin{aligned}
\operatorname{sign}\big(\widetilde{M}\big) &\underset{\substack{\uparrow \\ \text{by } (*)}}{=} \sum_{j=1}^{r} a_j \cdot \big\langle p_{I_j}\big(\widetilde{M}\big), \big[\widetilde{M}\big]\big\rangle_{\widetilde{M}} \underset{\substack{\uparrow \\ \text{by Theorem 77.9 (3), Lemma 57.10 and since } q^*TM = T\widetilde{M}\text{, see Lemma 77.4 (2)}}}{=} \sum_{j=1}^{r} a_j \cdot \big\langle q^*(p_{I_j}(M)), \big[\widetilde{M}\big]\big\rangle_{\widetilde{M}} \\
&\underset{\substack{\uparrow \\ \text{Lemma 50.6}}}{=} \sum_{j=1}^{r} a_j \cdot \big\langle p_{I_j}(M), q_*\big[\widetilde{M}\big]\big\rangle_M \underset{\substack{\uparrow \\ \text{Proposition 46.24}}}{=} \big[\widetilde{M}{:}M\big] \cdot \sum_{j=1}^{r} a_j \cdot \langle p_{I_j}(M), [M]\rangle_M \underset{\substack{\uparrow \\ \text{by } (*)}}{=} \big[\widetilde{M}{:}M\big] \cdot \operatorname{sign}(M).
\end{aligned}$$

□

We conclude this chapter with a short discussion of other characteristic classes.

(1) Loosely speaking, a *characteristic class* is a map that associates to a vector bundle a cohomology class in the base space. We had just discussed the Pontryagin classes in some detail. There are several other characteristic classes:
   (a) To a vector bundle $p\colon E \to X$ one can also associate the *Stiefel-Whitney classes* $w_i(E) \in \overline{H}^i(X;\mathbb{F}_2)$, $i \in \mathbb{N}_0$.
   (b) If $p\colon E \to X$ is an "oriented" n-dimensional vector bundle, then there exists the *Euler class* $e(E) \in \overline{H}^n(X;\mathbb{Z})$.
   (c) Finally, if $p\colon E \to X$ is a complex vector bundle, then the corresponding *Chern classes* $c_i(E) \in \overline{H}^{2i}(X;\mathbb{Z})$, $i \in \mathbb{N}_0$ are defined.

   The properties of Stiefel-Whitney classes and Chern classes are very similar to the properties stated in Theorem 77.9, in fact these characteristic classes are completely determined by the corresponding properties. Finally note that there are some relations between the various characteristic classes, for example given a vector bundle $p\colon E \to X$ the i-th Pontryagin class equals[1314]

$$p_i(E) \;:=\; c_{2i}(\text{the "complexification" } E \otimes \mathbb{C}) \;\in\; \overline{H}^{4i}(X;\mathbb{Z}).$$

   We refer to [MiS] and [Hat2] for more details on these characteristic classes.

(2) For complex vector bundles over a manifold M the images of the Chern classes in $\overline{H}^{2i}(M;\mathbb{R})$ can also be defined in the context of de Rham cohomology. Since Pontryagin classes are defined via Chern classes this shows in particular that the images of the Pontryagin classes in real cohomology can be defined via de Rham cohomology. We refer to [Ch, MiS] for details.

[1314] In fact, this usually the way Pontryagin classes are defined.

Exercises for Chapter 77.

Exercise 77.1. Does there exist a smooth vector field on S^2 with a single zero?

Exercise 77.2. Let W be a manifold and let M be a submanifold of W. We denote by $i\colon M \to W$ the inclusion map. Show that there exists a vector bundle $E \to M$ such that $TM \oplus E \cong i^*(TW)$.

Exercise 77.3. Show that S^3 and S^7 are parallelizable.

Exercise 77.4. Let $z \in \mathbb{R}\backslash\{0\}$. We denote by $\rho_z\colon \pi_1(S^1, 1) \to \mathrm{GL}(1, \mathbb{R})$ the representation that sends the standard generator of $\pi_1(S^1, 1) \cong \mathbb{Z}$ to the matrix (z). We denote by $E_z \to S^1$ the line bundle corresponding to this representation, see the definition on page 1785. For which z is this line bundle isomorphic to the trivial bundle and for which z is it isomorphic to the Möbius bundle?

Exercise 77.5. Show that the Möbius bundle does not admit a non-zero section.

Exercise 77.6. By Exercise 1.32 we can identify $\mathbb{R}\mathrm{P}^1$ with S^1. Let $p\colon E \to \mathbb{R}\mathrm{P}^1 = S^1$ be the tautological line bundle over $\mathbb{R}\mathrm{P}^1 = S^1$ from page 1783. Is this line bundle trivial? Is it isomorphic to the Möbius bundle?

Exercise 77.7. Let $g \geq 1$ and let Σ be the surface of genus g. Show that Σ admits a non-trivial line bundle.

Exercise 77.8. Let $p\colon M \to S^1$ be the Möbius bundle and let $g\colon S^1 \to S^1$ be the map that is given by $g(z) = z^2$. Show that the pullback of the Möbius bundle under g is a trivial bundle.

Exercise 77.9. Let M be a closed oriented manifold of dimension $4k + 3$. Show that $\mathrm{sign}(S^1 \times M) = 0$.

Exercise 77.10. Show that given any $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ there exists a CW-complex X with $H_k(X; \mathbb{Q}) = 0$ and that admits a 2-fold covering $\widetilde{X}$ with $H_k(\widetilde{X}; \mathbb{Q}) \cong \mathbb{Q}^n$.
Hint: You can take X to be a mapping torus of some high-dimensional torus. Use Proposition 33.20 to compute homology groups.

78. Steenrod operations

In this chapter we introduce the Steenrod operations. These are cohomology operations which satisfy several axioms. In the first three sections we will state the axioms and draw several interesting conclusions. Afterwards we will sketch a proof for the existence of the Steenrod operations.

78.1. **The axioms of Steenrod operations.**

Definition.

(1) Let A and B be abelian groups and let $m, n \in \mathbb{N}_0$. Any natural transformation from the contravariant functor

$$H^m(-;A)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

to the contravariant functor

$$H^n(-;B)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

is called a *cohomology operation of degree* $n-m$. In other words, a cohomology operation consists for any topological space X of a map $\Phi_X\colon H^m(X;A) \to H^n(X;B)$ of sets such that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{tikzcd} H^m(Y;A) \arrow[r, "f^*"] \arrow[d, "\Phi_Y"'] & H^m(X;A) \arrow[d, "\Phi_X"] \\ H^n(Y;B) \arrow[r, "f^*"] & H^n(X;B). \end{tikzcd}$$

(2) A cohomology operation is called *additive* if each map $H^m(X;A) \to H^n(X;B)$ is a homomorphism.

(3) A cohomology operation is called *non-trivial* if there exists a topological space X such that the map $H^m(X;A) \to H^n(X;B)$ is non-trivial in the sense that the image is not just the trivial element.

With the obvious modifications the same definitions also apply to relative cohomology viewed as functors from the category of pairs of topological spaces to the category of sets.

Perhaps somewhat surprisingly we have already assembled a substantial list of non-trivial cohomology operations.

Examples.

(1) Let $n \in \mathbb{N}_0$ and let G be an abelian group. Given a pair of topological spaces (X, A) we consider the connecting homomorphism $H^n(X, A; G) \to H^{n+1}(X; G)$. It follows from Lemma 48.15 (3) that these homomorphisms define an additive cohomology operation of degree one.

(2) Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups and let $n \in \mathbb{N}_0$. On page 1163 we saw that given any topological space X we obtain an induced

homomorphism $\varphi_*\colon H^n(X;A) \to H^n(X;B)$. By Lemma 48.9 we know that these maps define an additive cohomology operation of degree zero.

(3) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. On page 1714 we introduced the n-th Bockstein homomorphism $\beta_n\colon H^n(X;C) \to H^{n+1}(X;A)$. In Lemma 74.3 we saw that the n-th Bockstein homomorphisms define an additive cohomology operation of degree one.

(4) Let $k \in \mathbb{N}$. It follows from Proposition 58.3 (1) that for any $m \in \mathbb{N}_0$ and any commutative ring R the maps[1315]

$$\begin{array}{rcl} H^m(X,A;R) & \to & H^{km}(X,A;R) \\ \varphi & \mapsto & \varphi^k = \underbrace{\varphi \cup \cdots \cup \varphi}_{k \text{ times}} \end{array}$$

define a cohomology operation of degree $km-m$. Note that this type of a cohomology operation is in general not additive. For example, if $k \geq 2$, then it follows easily from the calculation of the cup product of $\mathbb{C}\mathrm{P}^k$, see Proposition 62.7, that the map $H^2(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \to H^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ given by $\varphi \mapsto \varphi^k$ is not a homomorphism.

(5) If R is a commutative ring with $1+1=0$, then the cohomology operation $\varphi \mapsto \varphi^2$ is in fact additive. Indeed for any $\varphi, \psi \in H^m(X,A;R)$ we have[1316]

$$\begin{array}{rcl} (\varphi+\psi)^2 & = & \varphi \cup \varphi + \varphi \cup \psi + \psi \cup \varphi + \psi \cup \psi \\ & = & \varphi \cup \varphi + (1+(-1)^{m^2}) \cdot \varphi \cup \psi + \psi \cup \psi \;=\; \varphi \cup \varphi + \psi \cup \psi \;=\; \varphi^2 + \psi^2. \\ & \uparrow & \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \end{array}$$

by Proposition 57.8 we have $\psi \cup \varphi = (-1)^{m^2} \cdot (\varphi \cup \psi)$ since $1-1=0=1+1$ in R

Note that all of the cohomology operations that we have introduced above are of non-negative degree. This raises the following amusing question.

Question 78.1. *Does there exist a non-trivial cohomology operation of negative degree?*

We will come back to this question at a later point.

The following theorem gives us new and interesting cohomology operations. The theorem was first proved by Norman Steenrod [Ste] in 1947.

Theorem 78.2. *There exists a family*

$$\mathrm{Sq}^i\colon H^n(X,A;\mathbb{Z}_2) \;\to\; H^{n+i}(X,A;\mathbb{Z}_2) \qquad \textit{with } i \in \mathbb{N}_0 \textit{ and } n \in \mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) *For any* $\varphi \in H^i(X,A;\mathbb{Z}_2)$ *we have* $\mathrm{Sq}^i(\varphi) = \varphi^2 = \varphi \cup \varphi \in H^{2i}(X,A;\mathbb{Z}_2)$.
(3) *For any* $\varphi \in H^j(X,A;\mathbb{Z}_2)$ *with* $i > j$ *we have* $\mathrm{Sq}^i(\varphi) = 0$.

[1315] We refer to page 1358 for the definition of the relative cup product.

[1316] This can be viewed as the cohomological "freshman's dream". Perhaps it should called the incoming PhD student's dream.

(4) *Let (X, A, B) be an excisive triad of topological spaces. For any $\varphi \in H^r(X, A; \mathbb{Z}_2)$ any $\psi \in H^s(X, B; \mathbb{Z}_2)$ and any $k \in \mathbb{N}_0$ we have*[1317]

$$\mathrm{Sq}^k(\varphi \cup \psi) \;=\; \sum_{i=0}^{k} \underbrace{\mathrm{Sq}^i(\varphi)}_{\in H^{r+i}(X,A;\mathbb{Z}_2)} \;\cup\; \underbrace{\mathrm{Sq}^{k-i}(\psi)}_{\in H^{s+i}(X,B;\mathbb{Z}_2)} \;\in\; H^{r+s+k}(X, A \cup B; \mathbb{Z}_2).$$

This equality is often called the Cartan formula.

In the next two sections we will discuss some properties and applications of these cohomology operations. Only after these two sections, once we are convinced that Steenrod operations are indeed useful and interesting, will we provide a proof for Theorem 78.2.

Remark.

(1) The first three axioms of Theorem 78.2 can be summarized as follows: Given any $\varphi \in H^k(X, A; \mathbb{Z}_2)$ we obtain the following table

$\mathrm{Sq}^0(\varphi)$	$\mathrm{Sq}^1(\varphi)$	$\dots$	$\mathrm{Sq}^{k-1}(\varphi)$	$\mathrm{Sq}^k(\varphi)$	$\mathrm{Sq}^{k+1}(\varphi)$	$\dots$
φ	?	$\dots$	?	φ^2	0	$\dots$

(2) Throughout the remainder of this chapter we will almost invariably consider cohomology with $\mathbb{Z}_2$-coefficients. We point out that by Proposition 57.8 the cup product on the $\mathbb{Z}_2$-cohomology $H^*(X, A; \mathbb{Z}_2)$ of a given pair of topological spaces (X, A) is actually *commutative*.

The following theorem is [SE, Theorem VIII.3.10].

Theorem 78.3. *Any two families of additive cohomology operations that satisfy Axioms* (1) *to* (4) *are identical.*

This theorem allows us to give a name to the above cohomology operations.

Definition. We refer to the cohomology operations from Theorem 78.2 as the *Steenrod operations.*[1318]

The following theorem shows that the first Steenrod operation Sq^1 is actually an old acquaintance of ours.

Theorem 78.4. *The first Steenrod operation* Sq^1 *is the Bockstein homomorphism corresponding to the short exact sequence* $0 \to \mathbb{Z}_2 \xrightarrow{[a]\mapsto[2a]} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Proof. We will not make use of this theorem, thus we decline to provide a proof. The theorem is proved in [Bre, p. 418] or alternatively in [MTa, p. 23] [Hat, Theorem 4.L.12]. □

[1317] We refer to page 1203 for the definition of an excisive triad. By Proposition 50.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A = B$, or that at least one of A or B is the empty set. We refer again to page 1358 for the definition of the relative cup product.

[1318] Sometimes they are also called Steenrod squares, and the term Steenrod operations is used for a more general concept.

Evidently our goal now is to compute Steenrod operations in specific examples. We start out with the following slightly disappointing lemma.

Lemma 78.5. *For any wedge of finitely many spheres all Steenrod operations in degrees ≥ 1 are zero.*

Proof. We will prove the lemma in Exercise 78.2. □

In remainder of this section we intend to compute the Steenrod operations for the real projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \in \mathbb{N}_0$. Before we consider the real projective spaces in detail it is convenient to reformulate the Cartan formula.

For the reformulation we need the following notation:

Notation. Let (X, A) be a pair of topological spaces. Given $\varphi \in H^k(X, A; \mathbb{Z}_2)$ we write

$$\mathrm{Sq}(\varphi) \ := \ \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \ \in \ H^*(X, A; \mathbb{Z}_2) = \bigoplus_{n \in \mathbb{N}_0} H^n(X, A; \mathbb{Z}_2).$$

Furthermore we extend the homomorphism Sq in the obvious way to a homomorphism $\mathrm{Sq}\colon H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$.

Remark. Let (X, A) be a pair of topological spaces and let $\varphi \in H^k(X, A; \mathbb{Z}_2)$. Recall that by Axiom (3) given $\varphi \in H^k(X, A; \mathbb{Z}_2)$ we have $\mathrm{Sq}^i(\varphi) = 0$ for $i > k$. Thus we can write

$$\mathrm{Sq}(\varphi) \ = \ \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \ = \ \sum_{i \in \mathbb{N}_0} \mathrm{Sq}^i(\varphi).$$

With this notation we can formulate the following lemma.

Lemma 78.6. *Let (X, A) be a pair of topological spaces.*

(1) *Given any $\varphi, \psi \in H^*(X, A; \mathbb{Z}_2)$ we have*

$$\mathrm{Sq}(\varphi \cup \psi) \ = \ \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi).$$

(2) *Given any $\varphi \in H^*(X, A; \mathbb{Z}_2)$ and given any $k \in \mathbb{N}_0$ we have $\mathrm{Sq}^k(\varphi) = \mathrm{Sq}(\varphi)^k$.*

Proof. The first statement follows easily from the Cartan formula. Indeed, we see that

$$\begin{aligned} \mathrm{Sq}(\varphi \cup \psi) = \sum_{k \in \mathbb{N}_0} \mathrm{Sq}^k(\varphi \cup \psi) \ &\overset{\text{Cartan formula, i.e. Axiom (4)}}{=} \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{k} \mathrm{Sq}^i(\varphi) \cup \mathrm{Sq}^{k-i}(\psi) \\ &= \sum_{r \in \mathbb{N}_0} \sum_{s \in \mathbb{N}_0} \mathrm{Sq}^r(\varphi) \cup \mathrm{Sq}^s(\psi) = \Big(\sum_{r \in \mathbb{N}_0} \mathrm{Sq}^r(\varphi) \Big) \cup \Big(\sum_{s \in \mathbb{N}_0} \mathrm{Sq}^s(\psi) \Big) = \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi). \end{aligned}$$

The second statement is an immediate consequences of the first statement. □

Before we state a corollary we introduce the following semi-obvious definition.

Definition. Let $r, s \in \mathbb{N}_0$. If $0 \leq r \leq s$, then we define $\binom{s}{r} = \frac{s!}{r!(s-r)!}$, otherwise we set $\binom{s}{r} = 0$.

Now we can state the promised corollary.

Corollary 78.7. *Let X be a topological space. For any cohomology class $\varphi \in H^1(X;\mathbb{Z}_2)$ of degree* one*, any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have*

$$\mathrm{Sq}^i(\varphi^k) \;=\; \binom{k}{i} \cdot \varphi^{k+i} \;\in\; H^{k+i}(X;\mathbb{Z}_2).$$

Examples.

(1) By Proposition 62.16 we know that given any $n \in \mathbb{N}_0$ we have a ring isomorphism $(H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2),\cup) \cong \mathbb{Z}_2[x]/(x^{n+1})$, where $x \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$, and we have a ring isomorphism $(H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2),\cup) \cong \mathbb{Z}_2[x]$, where $x \in H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. Since Steenrod operations are additive we see that the result of Corollary 78.7 gives us the promised calculations of all Steenrod operations for the real projective spaces.

(2) In Exercise 78.3 we will use a similar argument to calculate the Steenrod operations on $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$.

Proof. We calculate that

$$\mathrm{Sq}(\varphi^k) \underset{\substack{\uparrow \\ \text{Lemma 78.6 (2)}}}{=} \mathrm{Sq}(\varphi)^k \underset{\substack{\uparrow \\ \text{by Axiom (3)} \\ \text{since } \varphi \in H^1(X;\mathbb{Z}_2)}}{=} (\mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi))^k \underset{\substack{\uparrow \\ \text{Axioms (1) and (2)}}}{=} (\varphi + \varphi^2)^k \underset{\substack{\uparrow \\ \text{since the ring } (H^*(X;\mathbb{Z}_2),\cup) \\ \text{is commutative}}}{=} \varphi^k \cup (1+\varphi)^k \;=\; \sum_{j=0}^{k} \binom{k}{j} \varphi^{k+j}.$$

The desired conclusion follows from comparing the terms in degree $k+i$ on both sides. □

78.2. **Steenrod operations and connecting homomorphisms.** This section is completely dedicated to the proof of the following proposition. In the subsequent section we will see that the proposition has powerful applications.

Proposition 78.8. *If (X,A) is a pair of topological spaces, then the Steenrod operations commute with the connecting homomorphisms in cohomology. More precisely, given any $i,n \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
H^n(A;\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "\mathrm{Sq}^i"'] & H^{n+1}(X,A;\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] \\
H^{n+i}(A;\mathbb{Z}_2) \arrow[r, "\delta"] & H^{n+i+1}(X,A;\mathbb{Z}_2).
\end{tikzcd}$$

In the proof of Proposition 78.8 we will make use of the cross product on cohomology which we now introduce.

Definition. Let R be a commutative ring, let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi \in H^m(X;R)$ and $\psi \in H^n(Y,B;R)$. We denote

by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections. We define the *cross product*[1319]

$$\varphi \times \psi \;:=\; \underbrace{p^*(\varphi)}_{\substack{\uparrow \\ \text{lies in } H^m(X\times Y;R)}} \cup \underbrace{q^*(\psi)}_{\substack{\uparrow \\ \text{lies in } H^n(X\times Y, X\times B;R)}} \;\in\; H^{m+n}(X \times Y, X \times B; R).$$

The following proposition basically says that in Axiom (4), i.e. in the Cartan formula, we can replace the cup product by the cross product.

Proposition 78.9. *Let X be a topological space and let (Y, B) be a pair of topological spaces. For any $\varphi \in H^r(X; \mathbb{Z}_2)$, $\psi \in H^s(Y, B; \mathbb{Z}_2)$ and $n \in \mathbb{N}_0$ the following equality holds:*

$$\mathrm{Sq}^n(\varphi \times \psi) \;=\; \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi) \times \mathrm{Sq}^{n-i}(\psi) \;\in\; H^{r+s+n}(X \times Y, X \times B; \mathbb{Z}_2).$$

Proof $(*)$. Let X be a topological space and let (Y, B) be a pair of topological spaces. Furthermore let $\varphi \in H^r(X; \mathbb{Z}_2)$ and $\psi \in H^s(Y, B; \mathbb{Z}_2)$. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections. We start out with the following claim.

Claim. The following equalities hold in $H^{m+n}(X \times Y; \mathbb{Z}_2)$:[1320]

(a) $\quad p^*(\varphi) \;=\; \varphi \times 1_Y \quad$ and $\quad q^*(\psi) \;=\; 1_X \times \psi.$
(b) $\quad \varphi \times \psi \;=\; (\varphi \times 1_Y) \cup (1_X \times \psi).$

Let us provide the proof of the claim.

(a) We perform the following easy calculation:

$$\varphi \times 1_Y \;=\; p^*(\varphi) \cup q^*(1_Y) \;\underset{\substack{\uparrow \\ \text{by definition we have} \\ p^*(1_X) = q^*(1_Y) = 1_{X\times Y}}}{=}\; p^*(\varphi) \cup 1_{X\times Y} \;\underset{\substack{\uparrow \\ 1_{X\times Y} \text{ is multiplicatively neutral,} \\ \text{see Proposition 57.7}}}{=}\; p^*(\varphi).$$

Evidently the same argument also shows that $q^*(\psi) = 1_X \times \psi$.

(b) This statement follows immediately from (a) and the definition of the cross product.

This concludes the proof of the claim.

[1319]Recall that on page 1316, given two *homology* classes $\mu \in H_m(X; \mathbb{Z})$ and $\nu \in H_n(Y; \mathbb{Z})$ we defined the *cross product* $\mu \times \nu \in H_{m+n}(X \times Y; \mathbb{Z})$. It should usually be clear from the context whether in a given situation we mean the cross product on homology or on cohomology.

[1320]Recall that given a topological space W we denote by $1_Z \in H^0(W; \mathbb{Z}_2)$ the cohomology class that is represented by the cocycle that assigns to each singular 0-simplex the value $1 \in \mathbb{Z}_2$.

Now we turn to the proof of the actual statement of the proposition. We have the following equalities in $H^{r+s+n}(X \times Y; \mathbb{Z}_2)$:

$$\begin{aligned}
\mathrm{Sq}^n(\varphi \times \psi) &\overset{\text{part (b) of the claim}}{=} \mathrm{Sq}^n\big((\varphi \times 1_Y) \cup (1_X \times \psi)\big) &&\overset{\text{by the Cartan formula, i.e. by Axiom (4)}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi \times 1_Y) \cup \mathrm{Sq}^{n-i}(1_X \times \psi) \\
&\underset{\text{part (a) of the claim}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(p^*(\varphi)) \cup \mathrm{Sq}^{n-i}(q^*(\psi)) &&\underset{\text{since } \mathrm{Sq}^i \text{ and } \mathrm{Sq}^{n-2} \text{ are natural transformations}}{=} \sum_{i=0}^{n} p^*(\mathrm{Sq}^i(\varphi)) \cup q^*(\mathrm{Sq}^{n-i}(\psi)) \\
&\underset{\text{by definition of the cross product}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi) \times \mathrm{Sq}^{n-i}(\psi).
\end{aligned}$$

We have thus obtained the desired equality. □

The proof of Proposition 78.8 rests on the following lemma.

Lemma 78.10. *Let X be a topological space, let (Y, B) be a pair of topological spaces and let R be a commutative ring. If $\varphi \in H^m(X; R)$ and $\psi \in H^n(B; R)$, then*

$$\underset{\substack{\text{connecting homomorphism} \\ \text{in cohomology for the} \\ \text{pair } (X \times Y, X \times B)}}{\delta(\overbrace{\varphi \times \psi}^{\in H^{m+n}(X \times B; R)})} = (-1)^m \cdot \varphi \times \underset{\substack{\text{connecting homomorphism} \\ \text{in cohomology for the pair } (Y, B)}}{\overbrace{\delta\psi}^{\in H^{n+1}(Y, B; R)}} \in H^{m+n+1}(X \times Y, X \times B; R).$$

Proof of Lemma 78.10. This equality follows fairly easily from Lemma 57.2, the definition of the cross product via the cup product and the explicit description of the connecting homomorphism on page 1160. We leave it to the reader to fill in the details. □

Proof of Proposition 78.8. We start out with a warm-up case.

Claim. Let Z be a topological space and let $a < b$ be two real numbers. The conclusion of the proposition holds for the pair $(Z \times [a, b], Z \times \partial([a, b]))$ of topological spaces.

Let $\theta \in H^n(Z \times \partial([a, b]); \mathbb{Z}_2) = H^n(Z \times \{a, b\}; \mathbb{Z}_2)$. Recall that we need to show that $\mathrm{Sq}^i(\delta\theta) = \delta(\mathrm{Sq}^i(\theta))$. First note that θ can be written as $\theta = \varphi \times \psi$ for some $\varphi \in H^n(Z; \mathbb{Z}_2)$ and some $\psi \in H^0(\partial([a, b]); \mathbb{Z}_2) = H^0(\{a, b\}; \mathbb{Z}_2)$.[1321] Note that we have the following

[1321] This statement follows quite easily from the fact that $Z \times \partial([a, b]) = (Z \times \{a\}) \sqcup (Z \times \{b\})$, the observation that $H^0(\{a, b\}) = \mathbb{Z} \cdot 1_{\{a\}} \oplus \mathbb{Z} \cdot 1_{\{b\}}$, Proposition 57.7 and Lemma 59.9. We leave it to the assiduous reader to assemble these steps in the right logical order.

equalities in $H^{n+i+1}(Z \times [a,b], Z \times \partial([a,b]); \mathbb{Z}_2)$:

$$\begin{array}{ccccccc}
 & \text{since } \theta = \varphi \times \psi & & \text{by Lemma 78.10, we also use that } (-1)^n = 1 \in \mathbb{Z}_2 & & \text{Proposition 78.9} & \begin{array}{c}\text{since } \delta\psi \in H^1([a,b], \partial([a,b]); \mathbb{Z}_2) \\ \text{and since } H^i([a,b], \partial([a,b]); \mathbb{Z}_2) = 0 \\ \text{for every } i \geq 2 \text{ we know that} \\ \mathrm{Sq}^j(\delta\psi) = 0 \text{ for } j \geq 1\end{array} \\
 & \downarrow & & \downarrow & & \downarrow & \downarrow \\
\mathrm{Sq}^i(\delta\theta) & = \mathrm{Sq}^i(\delta(\varphi \times \psi)) & & = \mathrm{Sq}^i(\varphi \times \delta\psi) & & = \sum_{j=0}^{i} \mathrm{Sq}^{i-j}(\varphi) \times \mathrm{Sq}^j(\delta\psi) & = \mathrm{Sq}^i(\varphi) \times \mathrm{Sq}^0(\delta\psi) \\
 & = \mathrm{Sq}^i(\varphi) \times \delta\psi & & = \delta(\mathrm{Sq}^i(\varphi) \times \psi) & & = \delta(\mathrm{Sq}^i(\varphi \times \psi)) & = \delta(\mathrm{Sq}^i(\theta)). \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \begin{array}{c}\text{Axiom (1) says} \\ \text{that } \mathrm{Sq}^0 = \mathrm{id}\end{array} & & \text{Lemma 78.10} & & \begin{array}{c}\text{by Proposition 78.9 together with Axioms (1) and (3)} \\ \text{and the fact that } \psi \in H^0(\partial([a,b]); \mathbb{Z}_2)\end{array} &
\end{array}$$

This concludes the proof of the claim.

The remainder of the proof of the proposition consists in cleverly reducing the general case to the above special case. Thus let (X, A) be a pair of topological spaces. We denote by $i\colon A \to X$ the inclusion map. We set

$$\mathrm{Cyl}(i) = \begin{array}{c}\text{mapping cylinder of the} \\ \text{inclusion } i\colon A \to X\end{array} := ((A \times [0,1]) \sqcup X)/\sim \quad \text{where } (a,1) \sim a \text{ for all } a \in A.$$

We point out that by Exercise 15.5 and Lemma 15.5 we know that the obvious inclusions $A \times [0,1] \to \mathrm{Cyl}(i)$ and $X \to \mathrm{Cyl}(i)$ are homeomorphism onto their images. Thus we can view $A \times [0,1]$ and X as subsets of $\mathrm{Cyl}(i)$. Now we consider the following subsets of the mapping cylinder $\mathrm{Cyl}(i)$:

$$B := (A \times [0, \tfrac{1}{2}]) \cup X \quad \text{and} \quad C := A \times \{0\}.$$

We refer to Figure 965 for an illustration. The five statements in the following claim break

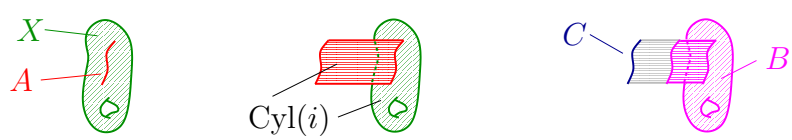


FIGURE 965.

the proof of the proposition into five manageable steps.

Claim.

(1) The conclusion of the proposition holds for $(A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}]))$.
(2) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), B \cup C)$.
(3) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), C)$.
(4) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), A \times [0,1])$.
(5) The conclusion of the proposition holds for (X, A).

The five statements of the claim are illustrated in Figure 966.

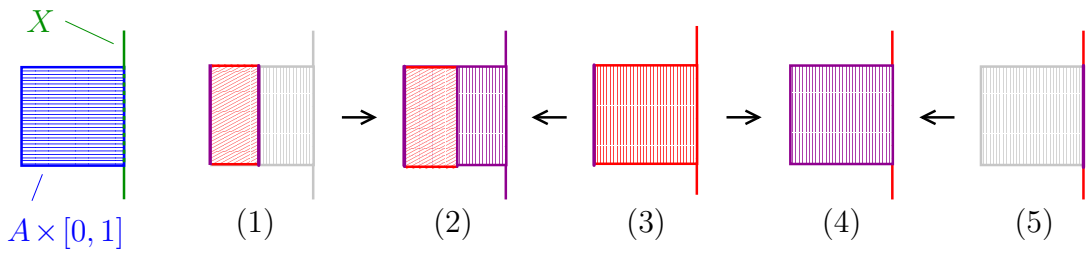


FIGURE 966.

Before we work our way through the five steps we need to make two related observations. Let $f\colon (X,A) \to (Y,B)$ be a map of pairs of topological spaces. We obtain the following cubical diagram:

$$\begin{tikzcd}
H^n(Y;\mathbb{Z}_2) \arrow[rr, "\delta"] \arrow[dr, "f^*"] \arrow[dd, "\mathrm{Sq}^i"'] & & H^{n+1}(Y,B;\mathbb{Z}_2) \arrow[dr, "f^*"] \arrow[dd, "\mathrm{Sq}^i"] & \\
& H^n(X;\mathbb{Z}_2) \arrow[rr, "\delta"] \arrow[dd, "\mathrm{Sq}^i"] & & H^{n+1}(X,A;\mathbb{Z}_2) \arrow[dd, "\mathrm{Sq}^i"] \\
H^{n+i}(B;\mathbb{Z}_2) \arrow[rr] \arrow[dr, "f^*"] & & H^{n+i+1}(Y,B;\mathbb{Z}_2) \arrow[dr, "f^*"] & \\
& H^{n+i}(A;\mathbb{Z}_2) \arrow[rr, "\delta"] & & H^{n+i+1}(X,A;\mathbb{Z}_2).
\end{tikzcd}$$

It follows from the naturality of the Steenrod operations that the parallelograms to the left and right commute. Furthermore, it follows from the naturality of the connecting homomorphism, see Lemma 48.15, that the top and bottom parallelograms commute. Using these facts we can make the following two observations:

(i) If the square in the front commutes and if the map f^* to the bottom right is an isomorphism, then it follows from a stroll through the cube, as indicated in Figure 967, that the square in the back also commutes.

(ii) If the square in the back commutes and if the map f^* to the top left is an isomorphism, then it follows as in (i) that the square in the front also commutes.

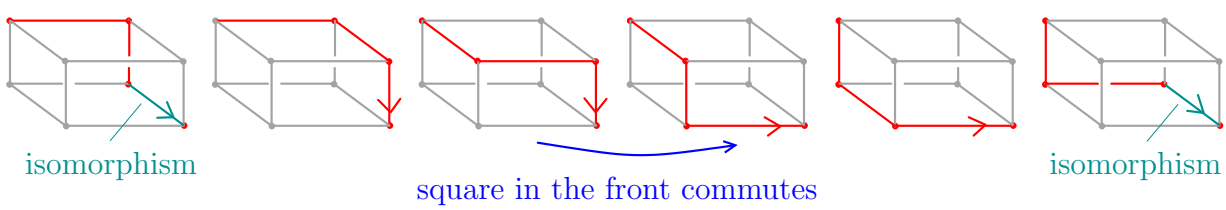


FIGURE 967.

Now we turn to the actual proof of the five steps in the claim.

(1) This statement follows immediately from the previous claim.
(2) We consider the inclusion map $f\colon (A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}])) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (1) we know that the square in the front commutes. We leave it to the reader to cleverly apply the Excision Theorem for Cohomology Groups 50.1 to show that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back commutes.
(3) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), C) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (2) we know that the square in the back commutes. Since on the first entry f is the identity we see that the top left map f^* is an isomorphism. Thus we obtain from (ii) that the square in the front also commutes.
(4) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), A \times \{0\}) \to (\mathrm{Cyl}(i), \overbrace{A \times [0,1]}^{=C})$. By (3) we know that the square in the front commutes. Since $A \times \{0\}$ is a deformation retract of $A \times [0,1]$ we obtain from Lemma 48.13 (4) that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back also commutes.
(5) We consider the obvious inclusion

$$\begin{array}{rcl} f\colon X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & [(x,1)]. \end{array}$$

It also defines a map $f\colon (X, A) \to (\mathrm{Cyl}(i), A \times [0,1])$ of pairs of topological spaces. We consider the corresponding cube. By (4) we know that the square in the back commutes. By Lemma 15.7 (1) the map $f\colon X \to \mathrm{Cyl}(i)$ is in fact a homotopy equivalence. Thus it follows from Lemma 48.13 that the top left f^* is an isomorphism. Therefore we obtain from (ii) that the square in the front commutes.

This concludes the proof of the claim. □

78.3. **Steenrod operations and suspensions.** In Proposition 78.8 we saw that Steenrod operations commute with connecting homomorphisms in cohomology. In this section we will use this fact to show that Steenrod operations behave well under taking suspensions. This fact will help us to obtain new information on homology groups of spheres.

First recall that in Proposition 32.14 we saw that taking the suspension $\Sigma(X)$ of a topological space shifts the reduced homology groups "upwards by one". The following proposition says that the same phenomenon occurs when we consider cohomology groups.

Proposition 78.11. *Let X be a non-empty topological space, let $n \in \mathbb{N}$[1322] and let G be an abelian group. As on page 818 we denote by C_-X and C_+X the "lower and upper half" of the suspension $\Sigma(X)$. We refer to Figure 968 for an illustration.*

[1322]Basically the same statement works for $n = 0$, the only nuisance is that in that case we need to work with reduced cohomology groups.

(1) *The following maps are isomorphisms:*[1323]

$$H^n(X;G) \xrightarrow{\delta} H^{n+1}(C_+X, X; G) \leftarrow H^{n+1}(\Sigma(X), C_-X; G) \to H^{n+1}(\Sigma(X); G).$$

Here the map to the left is the connecting homomorphism of the pair (C_+X, X) *from Lemma 48.15 The remaining two horizontal maps are induced by the obvious inclusion maps.*

(2) *The isomorphism*

$$\Sigma_X \colon H^n(X;G) \xrightarrow{\cong} H^{n+1}(\Sigma(X); G)$$

that is given by the maps in (1) *is in fact a natural isomorphism.*

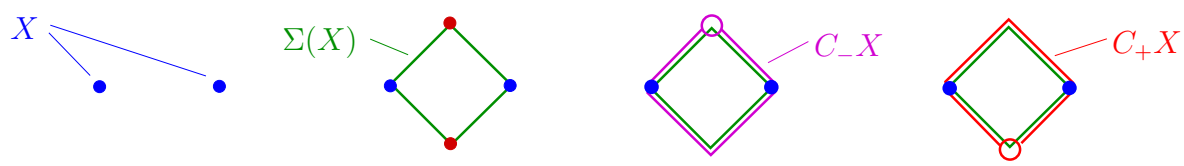


FIGURE 968. Illustration for Proposition 78.11.

Proof. The proof is a slight variation on the proof of Proposition 32.14. We leave it to the reader to fill in the details. □

The following proposition shows that Steenrod operations behave well under taking suspensions.

Proposition 78.12. *Given any topological space* X*, any* $n \in \mathbb{N}$ *and any* $i \in \mathbb{N}_0$ *the following diagram commutes:*

$$\begin{array}{ccc} H^n(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & H^{n+1}(\Sigma(X);\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ H^{n+i}(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & H^{n+i+1}(\Sigma(X);\mathbb{Z}_2). \end{array}$$

Remark. The proposition says in particular that if there exists a non-trivial Steenrod operation on a topological space X, then its suspension inherits a non-trivial Steenrod operation. This is in so far interesting as we know by Lemma 58.6 that the cup product vanishes on suspensions. The fact that suspensions can have non-trivial Steenrod operations will allow us to prove several hitherto inaccessible results.

[1323]Here we use Lemma 15.2 (3) to identify X with the subset $C_-X \cap C_+X$ of $\Sigma(X)$.

Proof. We consider the following diagram

$$\begin{tikzcd}
H^n(X;\mathbb{Z}_2) \arrow[r, "\delta"'] \arrow[d, "\mathrm{Sq}^i"] \arrow[rrr, bend left, "\Sigma_X"] & H^{n+1}(C_+X, X;\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] & H^{n+1}(\Sigma(X), C_-X;\mathbb{Z}_2) \arrow[l, "\cong"] \arrow[r, "\cong"'] \arrow[d, "\mathrm{Sq}^i"] & H^{n+1}(\Sigma(X);\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] \\
H^{n+i}(X;\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[rrr, bend right, "\Sigma_X"'] & H^{n+1+i}(C_+X, X;\mathbb{Z}_2) & H^{n+1+i}(\Sigma(X), C_-X;\mathbb{Z}_2) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^{n+1+i}(\Sigma(X);\mathbb{Z}_2).
\end{tikzcd}$$

The first square commutes by Proposition 78.8. The two other squares commute by the naturality of the Steenrod operation Sq^i. Thus we see that, as desired, the diagram commutes. $\square$

Next we recall that by the definitions on pages 726 and 1468 we have three different types of Hopf maps:

Definition. We consider the following maps

$$\underbrace{\begin{array}{rcl} H\colon S^3 & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in\mathbb{C}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{the Hopf map}} \quad \underbrace{\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 & \to & S^4 = \mathbb{H}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in\mathbb{H}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{quaternionic Hopf map}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} H_{\mathbb{O}}\colon S^{15} & \to & S^8 = \mathbb{O}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in\mathbb{O}^2} & \mapsto & [z_1 : z_2]. \end{array}}_{\text{octonionic Hopf map}}$$

In Theorem 63.3 we showed that the Hopf map $H\colon S^3 \to S^2$ is not null-homotopic. The key to doing so were the following two observations:

(1) The mapping cone $\mathrm{Cone}(H\colon S^3 \to S^2)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$.
(2) The cup product on $\mathbb{C}\mathrm{P}^2$ has the property that for a generator $\varphi \in H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ the square φ^2 is non-zero.

Since the Steenrod operations are a generalization of the mod 2 "cup product square" and since Steenrod operations behave well with respect to suspensions it seems reasonable that we can use Steenrod operations to show that the suspensions of the Hopf maps are not null-homotopic.

Indeed, we can prove the following theorem.

Theorem 78.13.

(1) *For any $n \geq 2$ the $(n-2)$-fold suspension*

$$\Sigma^{n-2}(H)\colon \underbrace{\Sigma^{n-2}(S^3)}_{=S^{n+1}} \to \underbrace{\Sigma^{n-2}(S^2)}_{=S^n}$$

of the Hopf map defines a non-trivial element in $\pi_{n+1}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+1}(S^n) \neq 0$.

(2) *For any $n \geq 4$ the $(n-4)$-fold suspension*

$$\Sigma^{n-4}(H_{\mathbb{H}})\colon \underbrace{\Sigma^{n-4}(S^7)}_{=S^{n+3}} \to \underbrace{\Sigma^{n-4}(S^4)}_{=S^n}$$

of the quaternionic Hopf map defines a non-trivial element in $\pi_{n+3}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+3}(S^n) \neq 0$.

(3) *For any $n \geq 8$ the $(n-8)$-fold suspension*

$$\Sigma^{n-8}(H_{\mathbb{O}})\colon \underbrace{\Sigma^{n-8}(S^{15})}_{=S^{n+7}} \to \underbrace{\Sigma^{n-8}(S^8)}_{=S^n}$$

of the octonionic Hopf map defines a non-trivial element in $\pi_{n+7}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+7}(S^n) \neq 0$.

Before we turn to the proof of Theorem 78.13 we recall, for the reader's convenience, the following, frequently overlooked, proposition which says that taking suspensions and taking mapping cones commutes.

Proposition 15.15. *If $f\colon X \to Y$ is a map between topological spaces, then there exists a homeomorphism*

$$g\colon \operatorname{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \xrightarrow{\cong} \Sigma(\operatorname{Cone}(f\colon X \to Y)).$$

We also state the following lemma which we are already dimly aware of.

Lemma 78.14. *If a map $\varphi\colon S^m \to S^n$ is homotopic to a constant map, then the corresponding mapping cone* $\operatorname{Cone}(\varphi\colon S^m \to S^n)$ *is homotopy equivalent to $S^{m+1} \vee S^n$.*

Proof. We have

$$\operatorname{Cone}(\varphi\colon S^m \to S^n) \underset{\substack{\uparrow \\ \text{by Lemma 15.11, since } \varphi \text{ is homotopic to a constant map}}}{\simeq} \operatorname{Cone}(\text{constant map}\colon S^m \to S^n) \underset{\substack{\uparrow \\ \text{Lemma 63.1}}}{\cong} S^{m+1} \vee S^n.$$

□

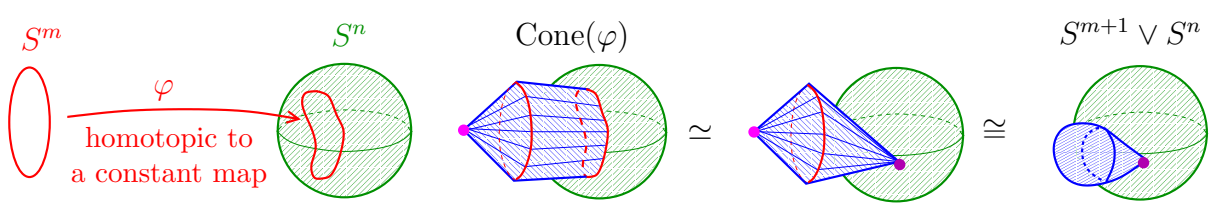


FIGURE 969. Illustration of Lemma 78.14.

Now we turn to the proof of Theorem 78.13. For clarity's sake we first provide a detailed proof of Theorem 78.13 (1). Afterwards we explain how to modify that proof to obtain statements (2) and (3) of Theorem 78.13.

Proof of Theorem 78.13 (1). In the following we will show that $\Sigma(H)\colon S^4 \to S^3$ is not homotopic to a constant map. The general statement regarding the $(n-2)$-fold suspension $\Sigma^{n-2}(H)$ follows easily from iterating the argument below.

For once we do a proof by contradiction. So let us assume that the map $\Sigma(H)\colon S^4 \to S^3$ is in fact homotopic to a constant map. We make the following preparations:

(1) To simplify the notation we denote the mapping cone of a map $f\colon X \to Y$ by $\mathrm{C}(f)$ instead of $\mathrm{Cone}(f)$. Furthermore we write $\mathbb{F} = \mathbb{Z}_2$.
(2) By Lemma 63.2 there exists a homeomorphism $f\colon\ \mathrm{C}(H\colon S^3 \to S^2) \xrightarrow{\cong} \mathbb{C}\mathrm{P}^2$.
(3) We denote by $g\colon \Sigma(\mathrm{C}(H)) \xrightarrow{\cong} \mathrm{C}(\Sigma(H))$ the homeomorphism from Proposition 15.15.
(4) Since we assume that the map H is homotopic to a constant map we obtain from Lemma 78.14 that there exists a homotopy equivalence $h\colon\ \mathrm{C}(\Sigma(H)) \to S^3 \vee S^5$.

Now we consider the following diagram:

$$\begin{tikzcd}
H^2(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] \arrow[d, "x\mapsto x\cup x"'] \arrow[d, shift left=6, "\mathrm{Sq}^2"] & H^2(\mathrm{C}(H);\mathbb{F}) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "\mathrm{Sq}^2"] & H^3(\Sigma(\mathrm{C}(H));\mathbb{F}) \arrow[r, "g^*", "\cong"'] \arrow[d, "\mathrm{Sq}^2"] & H^3(\mathrm{C}(\overbrace{\Sigma(H)}^{S^4\to S^3});\mathbb{F}) \arrow[d, "\mathrm{Sq}^2"] & H^3(S^5\vee S^3;\mathbb{F}) \arrow[l, "h^*"', "\cong"] \arrow[d, "\mathrm{Sq}^2"] \\
H^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] & H^4(C(H);\mathbb{F}) \arrow[r, "\Sigma", "\cong"'] & H^5(\Sigma(\mathrm{C}(H));\mathbb{F}) \arrow[r, "g^*", "\cong"'] & H^5(\mathrm{C}(\Sigma(H));\mathbb{F}) & H^5(S^5\vee S^3;\mathbb{F}). \arrow[l, "h^*"', "\cong"]
\end{tikzcd}$$

We make the following observations:

(1) The maps Σ are the suspension isomorphisms given by Proposition 78.11.
(2) It follows from Axiom (2) that the two vertical maps on the left are actually the same.
(3) It follows from the naturality of Sq^2 that the first, third and the fourth square commute.
(4) It follows from Proposition 78.12 that the second square commutes.
(5) Since f and g are homeomorphism we see that the induced maps f^* and g^* are isomorphisms. Furthermore, since h is a homotopy equivalence we obtain from Lemma 48.13 (3) that the horizontal maps h^* are isomorphisms.
(6) By Proposition 62.13 we know that there exists a unique non-trivial $x \in H^2(\mathbb{C}\mathrm{P}^2;\mathbb{F})$ and that this element satisfies $x \cup x \neq 0 \in H^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}$. In particular we see that the left vertical homomorphism is an isomorphism and that it is non-trivial.
(7) By Lemma 78.5 we know that the vertical homomorphism to the right is trivial.

In summary we have shown that the above diagram commutes, we have shown that all horizontal maps are isomorphisms, we have seen that the left vertical map is non-trivial and have pointed out that the right vertical map is trivial. But this is absurd. In other words, we have obtained a contradiction to our hypothesis that the map $\Sigma(H)\colon S^4 \to S^3$ is null-homotopic. $\square$

Proof of Theorem 78.13 (2) and (3). As we will see shortly, the proof for the quaternionic and octonionic Hopf maps is basically the same as the proof for the "classical" Hopf map.

Let us first consider the quaternionic Hopf. By Lemma 63.2 (2) there exists a homeomorphism $f\colon \mathrm{C}(H_{\mathbb{H}}) \xrightarrow{\cong} \mathbb{H}\mathrm{P}^2$. If $H_{\mathbb{H}}\colon S^7 \to S^4$ was null-homotopic we would, as in the proof of (1), obtain a commutative diagram of the following form:

$$\begin{tikzcd}
H^4(\mathbb{H}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] \arrow[d, "x\mapsto x\cup x"'] \arrow[d, shift left, "\mathrm{Sq}^4"] & H^4(\mathrm{C}(H_{\mathbb{H}});\mathbb{F}) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Sq}^4"] & H^5(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) \arrow[r, "g^*", "\cong"'] \arrow[d, "\mathrm{Sq}^4"] & H^5(\mathrm{C}(\overbrace{\Sigma(H_{\mathbb{H}})}^{S^8\to S^5});\mathbb{F}) \arrow[d, "\mathrm{Sq}^4"] & H^5(S^9\vee S^5;\mathbb{F}) \arrow[l, "h^*"', "\cong"] \arrow[d, "\mathrm{Sq}^4"] \\
H^8(\mathbb{H}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] & H^8(C(H_{\mathbb{H}});\mathbb{F}) \arrow[r, "\Sigma"] & H^9(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) \arrow[r, "g^*", "\cong"'] & H^9(\mathrm{C}(\Sigma(H_{\mathbb{H}}));\mathbb{F}) & H^9(S^9\vee S^5;\mathbb{F}) \arrow[l, "h^*"', "\cong"]
\end{tikzcd}$$

By Proposition 62.20 (1) the cup product on the left is non-trivial. Thus the same logic as in the proof of (1) leads to a contradiction.

The argument for the octonionic Hopf map is almost identical to the argument for the quaternionic Hopf map. We just need to work with Lemma 63.2 (3) and Proposition 62.20 (2). □

For the record we jot down the following proposition that we proved in the proof of Theorem 78.13. We will make use of it later on.

Proposition 78.15. $(*)$ *Given any $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{lrcl}
(1) & \mathrm{Sq}^2\colon H^{2+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) & \to & H^{4+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) \\
(2) & \mathrm{Sq}^4\colon H^{4+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) & \to & H^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) \\
(3) & \mathrm{Sq}^8\colon H^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2) & \to & H^{16+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2)
\end{array}$$

are isomorphisms.

78.4. **Proof of Theorem 78.2 I.** In the final two sections we will give a partial proof of Theorem 78.2. More precisely, we will construct the Steenrod operations and we will provide the proof of some of the properties. The proof is split into two parts:

(1) The goal of this section is to construct a certain interesting sequence of natural maps $\Delta_n\colon C_*(X)\otimes C_*(X)\otimes C_*(X)$, $n\in\mathbb{N}_0$, where the zeroth map Δ_0 is a diagonal approximation.
(2) In the subsequent section we will show how this sequence of maps can be used to define the Steenrod operations.

In preparation for the construction of the sequence Δ_n we recall some definitions and results from earlier on. First recall that on page 1041 we introduced the tensor product $A_*\otimes B_*$ of two chain complexes A_* and B_*. We will make use of the following harmless lemma.

Lemma 60.7. *Let A_* and B_* be chain complexes. The unique map*

$$T\colon A_* \otimes B_* \;\to\; B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \;\mapsto\; (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Since we will make use of this later on we point out that T is natural in A_* and B_* and we point out that $T^2 = \mathrm{id}$. Next we recall the following definition from page 1338.

Definition. A *diagonal approximation* consists of a natural chain map

$$\Phi\colon C_*(X) \;\to\; C_*(X) \otimes C_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(C_*(X) \otimes C_*(X))_0 = C_0(X) \otimes C_0(X)$:

$$\Phi(x) \;=\; x \otimes x.$$

Example. We will try to rise to the challenge of finding a suitable illustration for a diagonal approximation Φ. We start out with the following observation: if $\sigma\colon \Delta^1 \to X$ is a singular 1-simplex from a point P to a point Q, then $\Phi(\sigma) \in (C_*(X) \otimes C_*(X))_1$ needs to satisfy

$$\partial\Phi(\sigma) \;\underset{\substack{\uparrow \\ \text{since } \Phi \text{ is a chain map}}}{=}\; \Phi(\partial\sigma) \;=\; \Phi(P - Q) \;=\; \Phi(P) - \Phi(Q) \;\underset{\substack{\uparrow \\ \text{since } \Phi \text{ is determined on singular 0-simplices}}}{=}\; Q \otimes Q \;-\; P \otimes P.$$

For example $\Phi(\sigma) = (C_*(X) \otimes C_*(X))_1 = C_1(X) \otimes C_0(X) \oplus C_0(X) \otimes C_1(X)$ could be of the form $\alpha \otimes P + Q \otimes \beta$ where $\alpha\colon \Delta^1 \to X$ and $\beta\colon \Delta^1 \to X$ are singular simplices from P to Q. This situation is illustrated, with artistic liberties, in Figure 970.

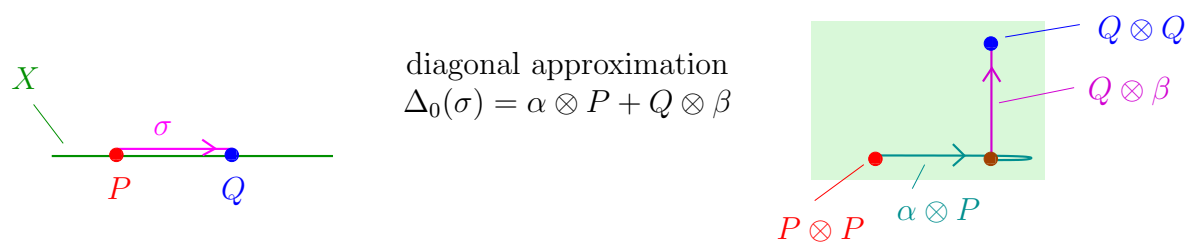


FIGURE 970.

The following proposition contains the two key results on diagonal approximations.

Proposition 57.6.

(1) *There exist diagonal approximations.*

(2) *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

Proof. We gave an explicit example of a diagonal approximation in Lemma 57.5. In Proposition 56.17 we gave a proof of (2) using the Acyclic Model Theorem 56.16 (2). □

The following theorem shows that any diagonal approximation can be extended to a sequence of maps.

Theorem 78.16.

(1) *Given any topological space X there exists a sequence of natural maps*

$$\Delta_n \colon C_*(X) \;\to\; C_*(X) \otimes C_*(X), \quad n \in \mathbb{N}_0$$

of degree n[1324] *such that the following two conditions are satisfied:*
(a) *The map Δ_0 is a diagonal approximation.*
(b) *For every $n \in \mathbb{N}_0$ the following equality holds:*[1325]

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \;=\; (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

(2) *Given two sequences Δ_n, Δ'_n, $n \in \mathbb{N}_0$, as in (1) there exists a sequence of natural maps*

$$H_n \colon C_*(X) \;\to\; C_*(X) \otimes C_*(X), \quad n \in \mathbb{N}_0$$

of degree n such that the following two conditions are satisfied:
(a) *H_0 is the zero map.*
(b) *For every $n \in \mathbb{N}_0$ the following equality holds:*[1326]

$$\partial \circ H_{n+1} + (-1)^n \cdot H_{n+1} \circ \partial \;=\; \Delta_n - \Delta'_n - (T + (-1)^n \cdot \mathrm{id}) \circ H_n.$$

In the proof of Theorem 78.16 we will need the following innocuous variation on the definition of a chain map that we had given on page 745.

Definition.

(1) A *chain map $f \colon C_* \to D_*$ of degree d* between two chain complexes consists of a family $\{f_n \colon C_n \to D_{n+d}\}_{n \in \mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \;=\; \partial_{n+d} \circ f_n.$$

(2) An *anti-chain map $f \colon C_* \to D_*$ of degree d* is basically the same data as a chain map of degree d, except that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \;=\; -\partial_{n+d} \circ f_n.$$

[1324] By a map of degree n between chain complexes C_* and D_* we mean a collection of maps $C_k \to D_{k+n}$, $k \in \mathbb{N}_0$.

[1325] Note that both sides are maps $C_*(X) \to C_*(X) \otimes C_*(X)$ of degree n.

[1326] In other words, the map H_{n+1} is a chain homotopy between Δ_n and Δ'_n up to the "error term" $-(T + (-1)^n \cdot \mathrm{id}) \circ H_n$.

Remark. Given an anti-chain map $\{f_n\}_{n\in\mathbb{N}_0}$ of degree d one can easily see that the maps $\{(-1)^n \cdot f_n\}_{n\in\mathbb{N}_0}$ form a chain map of degree d. This little trick makes it possible to translate all kinds of statements about chain maps to anti-chain maps. For example, as in Lemma 29.7 we see that an anti-chain map induces a map on homology groups.

In the proof of Theorem 78.16 we will need the following consequence of the Acyclic Model Theorem 56.13.

Theorem 78.17. (Acyclic Model Theorem) *Let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized*[1327] *chain complexes.*

(1) *Let $S\colon C_*(X) \to C_*(X) \otimes C_*(X)$ be a natural chain map of degree d. If we have $d \geq 1$, then for any topological space X and any $k \in \mathbb{N}$ there exists a natural map $P_X\colon C_k(X) \to (C_*(X) \otimes C_*(X))_{k+d+1}$ such that for any $k \in \mathbb{N}_0$ we have*

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon C_k(X) \;\to\; (C_*(X) \otimes C_*(X))_{k+d}.$$

(2) *For a natural anti-chain map $S\colon C_*(X) \to C_*(X) \otimes C_*(X)$ of degree d the analogous data exists except that now we have*

$$\partial \circ P_X - P_X \circ \partial \;=\; S_X\colon C_k(X) \;\to\; (C_*(X) \otimes C_*(X))_{k+d}.$$

Proof of the Acyclic Model Theorem 78.17.

(1) Given any topological space X we consider $G(X) := (C_*(X) \otimes C_*(X))[-d]$. Here we use the notation introduced on page 1392, namely given a chain complex D_* we denote by $D_*[-d]$ the chain complex D_* "shifted by d to the right". This shift turns $G(X)$ into a generalized chain complex. Furthermore, this shift turns $S\colon C_*(X) \to G(X)$ into a natural chain map of degree 0.

Next note that for any $k \in \mathbb{N}_0$ and any $i \geq 0$ we have

$$\begin{array}{rclcl} H_i(G(\Delta^k)) & = & H_i((C_*(X) \otimes C_*(\Delta^k))[-d]) & = & H_{i+d}(C_*(\Delta^k) \otimes C_*(\Delta^k)) \\ & = & \bigoplus\limits_{p+q=i+d} H_p(\Delta^k) \otimes H_q(\Delta^k) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{by the Künneth Theorem 43.7} & & \text{by the calculation of } H_*(\Delta^k) & \\ & \text{for chain complexes} & & \text{and since } d \geq 1 & \end{array}$$

Thus we see that all the hypothesis of the Acyclic Model Theorem 56.13 are satisfied. Therefore we see that for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X\colon C_k(X) \to G_{k+1}(X)$ such that for any $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon C_k(X) \;\to\; G_k(X).$$

(2) Suppose we are given a natural anti-chain map S of degree d. As in the remark preceding the theorem, we can turn S into a natural chain map T by defining $T_k = (-1)^k \cdot S_k$, $k \in \mathbb{N}_0$. We apply (1) and we obtain natural maps Q_X. We

[1327] Recall that a generalized chain complex is like a chain complex except that we allow chain groups in negative degrees.

now define $P_k := (-1)^k \cdot \widetilde{P}_k$, $k \in \mathbb{N}_0$. One easily verifies that the desired equality holds. $\square$

Now, finally, we can turn to the actual proof of Theorem 78.16.

Proof of Theorem 78.16.

(1) We construct the maps $\Delta_n \colon C_*(X) \to C_*(X) \otimes C_*(X)$, $n \in \mathbb{N}_0$ iteratively.

(i) First of all, we know by the Proposition 57.6 (1) above that there exists a diagonal approximation $\Delta_0 \colon C_*(X) \to C_*(X) \otimes C_*(X)$.

(ii) Note that $T \circ \Delta_0 \colon C_*(X) \to C_*(X) \otimes C_*(X)$ is another diagonal approximation. It follows from Proposition 57.6 (2) that there exists a natural chain homotopy $\Delta_1 \colon C_*(X) \to C_*(X) \otimes C_*(X)$ between Δ_0 and $T \circ \Delta_0$. In other words, there exists a natural map $\Delta_1 \colon C_*(X) \to C_*(X) \otimes C_*(X)$ of degree one such that

$$\partial \circ \Delta_1 + \Delta_1 \circ \partial = \underbrace{T \circ \Delta_0 - \Delta_0}_{=(T-\mathrm{id})\circ\Delta_0}.$$

Thus we are done with the construction of the map Δ_1. This step is illustrated in Figure 971.

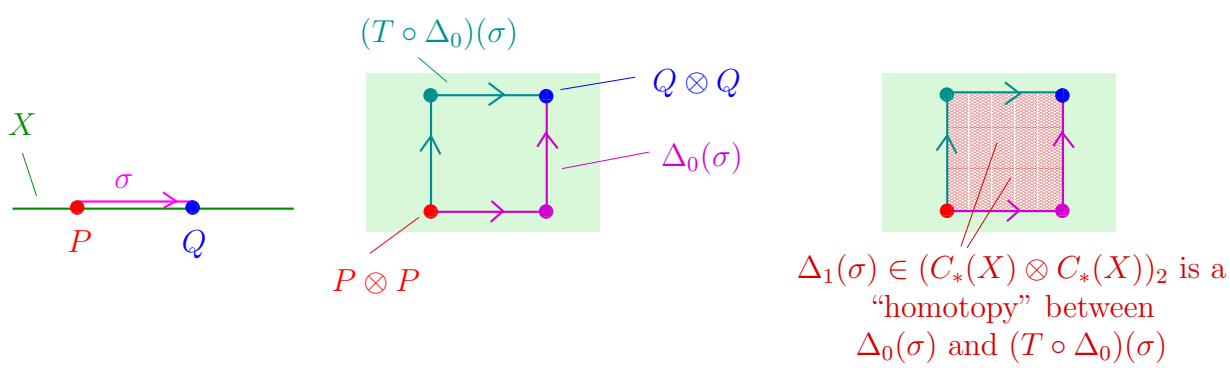


FIGURE 971.

(iii) The starting point for defining the next map Δ_2 is the following amusing little calculation:[1328]

$$\begin{aligned} 0 \overset{\text{since } T^2 = \mathrm{id}}{=} (T^2 - \mathrm{id}) \circ \Delta_0 &= (T + \mathrm{id}) \circ (T - \mathrm{id}) \circ \Delta_0 \overset{\text{by (ii)}}{=} (T + \mathrm{id}) \circ (\partial \circ \Delta_1 + \Delta_1 \circ \partial) \\ &\underset{\substack{\uparrow \\ \text{since } T \text{ and id are chain maps they commute with the boundary maps}}}{=} \partial \circ (T + \mathrm{id}) \circ \Delta_1 + (T + \mathrm{id}) \circ \Delta_1 \circ \partial. \end{aligned}$$

[1328]These are equalities of maps $C_*(X) \to C_*(X) \otimes C_*(X)$.

This shows that the map $(T+\mathrm{id})\circ\Delta_1\colon C_*(X)\to C_*(X)\otimes C_*(X)$ is a natural anti-chain map of degree one. By the Acyclic Model Theorem 78.17 (2) there exists a natural map $\Delta_2\colon C_*(X)\to C_*(X)\otimes C_*(X)$ of degree two such that

$$\partial\circ\Delta_2-\Delta_2\circ\partial \;=\; (T+\mathrm{id})\circ\Delta_1.$$

This conclude the construction of the map Δ_2.

(iv) Just for kicks we do the calculation for the next stage as well. We perform the following calculation:

$$\begin{aligned}
0 \;\overset{\substack{\text{since } T^2=\mathrm{id}\\ \downarrow}}{=}\; (T^2-\mathrm{id})\circ\Delta_1 \;&=\; (T-\mathrm{id})\circ(T+\mathrm{id})\circ\Delta_1 \;\overset{\substack{\text{by (iii)}\\ \downarrow}}{=}\; (T-\mathrm{id})\circ(\partial\circ\Delta_2-\Delta_2\circ\partial)\\
&\underset{\substack{\uparrow\\ \text{since } T \text{ and id are chain maps they commute with the boundary maps}}}{=}\; \partial\circ(T-\mathrm{id})\circ\Delta_2-(T-\mathrm{id})\circ\Delta_2\circ\partial.
\end{aligned}$$

This shows that the map $(T-\mathrm{id})\circ\Delta_2\colon C_*(X)\to C_*(X)\otimes C_*(X)$ is a natural chain map of degree two. By the Acyclic Model Theorem 78.17 (1) there exists a natural map $\Delta_3\colon C_*(X)\to C_*(X)\otimes C_*(X)$ of degree three such that

$$\partial\circ\Delta_3+\Delta_3\circ\partial \;=\; (T-\mathrm{id})\circ\Delta_2.$$

This conclude the construction of the map Δ_3.

(iv) We now iterate this procedure and we obtain the desired sequence of natural maps Δ_n.

(2) Given two sequences Δ_n,Δ_n' as in (1) we define the desired maps H_n, $n\in\mathbb{N}_0$ in a fashion that is quite similar to the approach taken in (1).

(i) We set $H_0=0$.

(ii) Nextwe point out that by definition Δ_0 and Δ_0' are diagonal approximations. Thus it follows from Proposition 57.6 (2) that there exist natural maps H_1 such that $\partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta_0'$.

(iii) We calculate that

$$\begin{aligned}
&\partial\circ(\Delta_1-\Delta_1'-(T-\mathrm{id})\circ H_1)+(\Delta_1-\Delta_1'-(T-\mathrm{id})\circ H_1)\circ\partial\\
&\quad=\; (\partial\circ\Delta_1+\Delta_1\circ\partial)-(\partial\circ\Delta_1'+\Delta_1'\circ\partial)-\partial\circ(T-\mathrm{id})\circ H_1-(T-\mathrm{id})\circ H_1\circ\partial\\
&\quad\underset{\substack{\uparrow\\ \text{since } \partial\circ\Delta_1+\Delta_1\circ\partial=(T-\mathrm{id})\circ\Delta_0 \text{ (same for } \Delta_0',\Delta_1')\\ \text{and since } T \text{ and id are chain maps}}}{=}\; (T-\mathrm{id})\circ(\Delta_0-\Delta_0')-(T-\mathrm{id})\circ(\partial\circ H_1+H_1\circ\partial) \;\underset{\substack{\uparrow\\ \text{since } \partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta_0'}}{=}\; 0.
\end{aligned}$$

This shows that $\Delta_1-\Delta_1'-(T-\mathrm{id})\circ H_1$ is an anti-chain map of degree one. By the Acyclic Model Theorem 78.17 (2) there exists a natural map H_2 with

$$H_2\circ\partial-\partial\circ H_2 \;=\; -(T-\mathrm{id})\circ H_1+\Delta_1-\Delta_1'.$$

(iv) Now we iterate this procedure to obtain the desired sequence of natural maps. One easily sees that we get the correct pattern for the signs involved. $\square$

Remark. If there did exist a diagonal approximation $\Delta_0\colon C_*(X) \to C_*(X) \otimes C_*(X)$ that is symmetric in the sense that $T \circ \Delta_0 = \Delta_0$, then we could take $\Delta_1 = 0$ and we could also take all higher Δ_n to be zero. Later on we will see that this is in general not the case. So even though diagonal approximations are symmetric on the degree 0 level, they cannot be chosen to be symmetric in higher degrees.

78.5. **Proof of Theorem 78.2 II.** In this section we will use the sequence of natural maps $\Delta_n\colon C_*(X) \to C_*(X) \otimes C_*(X)$, $n \in \mathbb{N}_0$, from Theorem 78.16 to define the Steenrod operations.

To get an inspiration for how to define the Steenrod operations we first recall the definition of the cup product. To do so we need the following lemma which is inspired by the definitions on from page 1332.

Lemma 78.18. *Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes and let R be a commutative ring. Given $p, q \in \mathbb{N}_0$ we consider the map*[1329]

$$\begin{aligned} \Xi_{p,q}\colon \operatorname{Hom}(\mathcal{C}_p, R) \otimes \operatorname{Hom}(\mathcal{D}_q, R) &\to \operatorname{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ \varphi \otimes \psi &\mapsto \left(\begin{array}{rcl} (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\text{projection}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi\otimes\psi} R \otimes R & \to & R \\ a \otimes b & \mapsto & a \cdot b \end{array} \right). \end{aligned}$$

The corresponding map

$$\Xi := \bigoplus_{p,q\in\mathbb{N}_0} \Xi_{p,q}\colon \operatorname{Hom}(\mathcal{C}_*, R) \otimes \operatorname{Hom}(\mathcal{D}_*, R) \quad \to \quad \operatorname{Hom}(\mathcal{C}_* \otimes \mathcal{D}_*, R)$$

is a cochain map.

Proof. The statement follows basically immediately from the definitions. □

The following lemma gives us a slight reformulation of the original definition of the cup product.

Lemma 78.19. *Let X be a topological space and let R be a commutative ring. Furthermore let $\Delta_0\colon C_*(X) \to C_*(X) \otimes C_*(X)$ be a diagonal approximation. We consider the cochain map*

$$\begin{tikzcd} \operatorname{Hom}(C_*(X), R) \otimes \operatorname{Hom}(C_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend right, "=:h_0"'] & \operatorname{Hom}(C_*(X) \otimes C_*(X), R) \arrow[r, "\Delta_0^*"] & \operatorname{Hom}(C_*(X), R). \end{tikzcd}$$

The corresponding map

$$\begin{array}{rcl} H^p(X;R) \times H^q(X;R) & \to & H^{p+q}(X;R) \\ ([\varphi],[\psi]) & \mapsto & [h_0(\varphi \otimes \psi)] \end{array}$$

equals the cup product as defined on page 1333.

[1329]The tensor product of cochain complexes is defined basically the same way as we defined the tensor product of chain complexes, see page 1041.

Proof $(*)$. Let $d\colon X \to X \times X$ be the diagonal map that is given by $d(x) = (x, x)$. Furthermore let $\Theta\colon C_*(X \times X) \to C_*(X) \otimes C_*(X)$ denote a Eilenberg-Zilber map from Theorem 56.1. For the purpose of this proof the only thing we need to know about the Eilenberg-Zilber map Θ is that, basically by definition, the map $\Theta \circ d\colon C_*(X) \to C_*(X \times X) \to C_*(X) \otimes C_*(X)$ is a diagonal approximation. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(C_*(X), R) \otimes \mathrm{Hom}(C_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend left, "=:h_0"] & \mathrm{Hom}(C_*(X) \otimes C_*(X), R) \arrow[r, "\Delta_0^*"] \arrow[d, "\Theta^*"] & \mathrm{Hom}(C_*(X), R) \\
& \mathrm{Hom}(C_*(X \times X), R). \arrow[ur, "d^*"'] &
\end{tikzcd}$$

The lower sequence of maps is at the heart of the definition of the cup product on page 1333. Since $\Theta \circ d$ is a diagonal approximation we obtain from Proposition 57.6 that Δ_0 and $\Theta \circ d$ are chain homotopic. But by Lemma 48.8 (4) this means that the two maps induce the same map on cohomology. The lemma follows almost immediately from the above observations. $\square$

Now we turn to the actual construction of the Steenrod operations. To do so we pick a sequence of natural maps

$$\Delta_n\colon C_*(X) \;\to\; C_*(X) \otimes C_*(X), \quad n \in \mathbb{N}_0$$

which satisfies the conditions set out in Theorem 78.16 (1). In light of Lemma 78.19 it is clear what we want to do next. Namely we want to replace Δ_0 by the "higher maps" Δ_n. This leads us straight to the following definition.

Definition. Let X be a topological space and let R be a commutative ring. Given $i \in \mathbb{N}_0$ we define

$$\begin{tikzcd}
\mathrm{Hom}(C_*(X), R) \otimes \mathrm{Hom}(C_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend right, "=:h_i"'] & \mathrm{Hom}(C_*(X) \otimes C_*(X), R) \arrow[r, "\Delta_i^*"] & \mathrm{Hom}(C_*(X), R).
\end{tikzcd}$$

Recall that Δ_i is a map of degree i. Since we dualize we see that Δ_i is a map of degree $-i$. For convenience we define $h_i := 0$ for $i \in \mathbb{Z}_{<0}$.

Now we can finally define the Steenrod operations.

Theorem 78.20. *Let X be a topological space, let $A \subset X$ be a subset and let R be a commutative ring. If $1 + 1 = 0$ in R, then for any $q \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl}
\mathrm{Sq}^i\colon H^q(X, A; R) & \to & H^{q+i}(X, A; R) \\
[\underbrace{\varphi}_{\in C^q(X;R)}] & \mapsto & [h_{q-i}(\underbrace{\varphi \otimes \varphi}_{\in C^q \otimes C^q})]
\end{array}$$

is a well-defined natural homomorphism.

Remark. The proof of Theorem 78.20 that we give below is a long and winding calculation, with few interesting view points. Perhaps more interesting than the question, why the definition works, is the question, why does the map

$$\begin{array}{rcl} C^q(X,A;R)\otimes C^r(X,A;R) & \to & C^{q+r+j}(X,A;R) \\ \varphi\otimes\psi & \mapsto & h_j(\underbrace{\varphi\otimes\psi}_{\in C^q\otimes C^r}) \end{array}$$

not descend to a map on cohomology?

The proof of Theorem 78.20 requires some preparations. First of all, note that the equalities in Theorem 78.16 translate into the following lemma.

Lemma 78.21. *Let X be a topological space and let R be a commutative ring with $1+1=0$. For any $\alpha,\beta\in C^*(X;R)$ we have the following equality:*

$$h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)) \;=\; h_n(\alpha\otimes\beta+\beta\otimes\alpha) \;\in\; C^*(X;R).$$

Proof. Recall, see Theorem 78.16, that one of the defining properties of the sequence Δ_n is that for every $n\in\mathbb{N}_0$ the following equality holds:

$$\partial\circ\Delta_{n+1}+(-1)^n\cdot\Delta_{n+1}\circ\partial \;=\; (T+(-1)^{n+1}\cdot\mathrm{id})\circ\Delta_n.$$

We dualize this equality, we use the equality $1+1=0$, we use the fact, shown in Lemma 78.18 and that Ξ is a cochain map. After performing all these steps one obtains the desired equality. We leave it to the reader to fill in the few remaining details. □

For elements of the form $\alpha\otimes\alpha$ the following lemma shows how the maps h_n relate to coboundaries.

Lemma 78.22. *Let X be a topological space and let R be a commutative ring. If $1+1=0$ in R, then for any $\alpha\in C^q(X;R)$ we have*

$$h_{n+1}(\delta\alpha\otimes\delta\alpha) \;=\; \delta h_{n+1}(\alpha\otimes\delta\alpha)+\delta h_n(\alpha\otimes\alpha) \;\in\; C^{n+1}(X;R).$$

Proof. Since we assume that $1+1=0$ in R we can ignore conveniently enough ignore signs throughout the proof. Thus we compute

$$\begin{array}{rcl} & \text{since } \delta(c\otimes d)=\delta c\otimes d+c\otimes\delta d & \text{Lemma 78.21} \\ & \downarrow & \downarrow \\ h_{n+1}(\delta\alpha\otimes\delta\alpha) & = & h_{n+1}(\delta(\alpha\otimes\delta\alpha)) \\ & = & \delta h_{n+1}(\alpha\otimes\delta\alpha)+h_n(\delta(\alpha\otimes\alpha)) \\ & \uparrow & \uparrow \\ & \text{definition of } \delta \text{ on tensor product} & \text{by Lemma 78.21 and since } h_{n-1}(\underbrace{a\otimes a+a\otimes a}_{=0 \text{ since } 1+1=0})=0 \end{array}$$

$$\begin{array}{rcl} & \text{Lemma 78.21} \\ & \downarrow \\ h_{n+1}(\delta\alpha\otimes\delta\alpha) & = & \delta h_{n+1}(\alpha\otimes\delta\alpha)+h_n(\alpha\otimes\delta\alpha+\delta a\otimes\alpha) \\ & = & \delta h_{n+1}(\alpha\otimes\delta\alpha)+\delta h_n(\alpha\otimes\alpha). \\ & \uparrow & \end{array}$$

□

In the proof of Theorem 78.20 we will need the following little lemma.

Lemma 78.23. *Let X be a topological space and let R be a commutative ring with $1+1=0$. For any $\alpha,\beta\in C^q(X;R)$ the following equality holds:*

$$h_n((\alpha+\beta)\otimes(\alpha+\beta))=h_n(\alpha\otimes\alpha)+h_n(\beta\otimes\beta)+h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

Proof of Lemma 78.23. This follows from "multiplying out", the linearity of h_n and the equality

$$h_n(\alpha\otimes\beta+\beta\otimes\alpha) \underset{\substack{\uparrow \\ \text{by Lemma 78.21}}}{=} h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

□

Now, after these lengthy preparations we can provide the proof of Theorem 78.20.

Proof of Theorem 78.20. Let X be a topological space, let $A\subset X$ be a subset and let R be a commutative ring. First we consider the case that $A=\varnothing$. Let $\varphi\in C^q(X;R)$ be a representative of some element in $H^q(X;R)$.

(1) Note that $\varphi\otimes\varphi$ is of degree $2q$. Thus $h_{q-i}(\varphi\otimes\varphi)$ is of degree $2q-(q-i)=q+i$.
(2) Let $\varphi\in C^q(X;R)$ be a cocycle. We calculate that

$$\delta h_{q-i}(\varphi\otimes\varphi) \underset{\substack{\uparrow \\ \text{Lemma 78.21}}}{=} h_{q-i}(\delta(\varphi\otimes\varphi))+h_{q-i-1}(\overbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}^{=0 \text{ since } 1+1=0}) \underset{\substack{\uparrow \\ \text{definition of } \delta \text{ on tensor product}}}{=} h_{q_i}(\delta\varphi\otimes\varphi+\varphi\otimes\delta\varphi) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is a coycle}}}{=} 0.$$

Thus we have shown that $h_{q-i}(\varphi\otimes\varphi)$ is a cocycle. In other words, we have shown that it defines an element in cohomology.
(3) Let $\psi\in C^{q-1}(X;R)$. We have the following equality in $H^{q+i}(X;R)$:

$$[h_{q-i}((\varphi+\delta\psi)\otimes(\varphi+\delta\psi))] \underset{\substack{\uparrow \\ \text{by Lemma 78.23, here we use that } \delta(\varphi\otimes\delta\psi)=\delta\varphi\otimes\delta\psi+\varphi\otimes\delta\delta\psi=0}}{=} [h_{q-i}(\varphi\otimes\varphi)+h_{q-i}(\delta\psi\otimes\delta\psi)] \underset{\substack{\uparrow \\ \text{Lemma 78.22}}}{=} [h_{q-i}(\varphi\otimes\varphi)].$$

This shows that the map Sq^i is well-defined, i.e. it does not depend on the choice of the representative of the cohomology class.
(4) It follows quite easily from Lemma 78.23 that Sq^i is in fact a homomorphism.

Finally we consider the case that $A\neq\varnothing$. Using the observation on page 1170 we make the identification

$$C^q(X,A;R) \;=\; \{\varphi\in C^q(X;R)=\mathrm{Hom}(C_q(X),R)\,|\,\varphi \text{ vanishes on } C_q(A)\subset C_q(X)\}.$$

It follows from the naturality of h_n that if $\varphi\in C_q(X)$ vanishes on $C_q(A)$, then $h_{q-i}(\varphi\otimes\varphi)$ vanishes on the subgroup $C_*(A)\otimes C_*(X)+C_*(X)\otimes C_*(A)$ of $C_*(X)\otimes C_*(X)$. Thus we see that the map

$$\begin{array}{rcl}\mathrm{Sq}^i\colon H^q(X,A;R) & \to & H^{q+i}(X,A;R)\\ [\underbrace{\varphi}_{\in C^q(X;R)}] & \mapsto & [h_{q-i}(\varphi\otimes\varphi)]\end{array}$$

is well-defined. □

We continue with the following proposition.

Proposition 78.24. *The definition of the maps* Sq^i *does not depend on the choice of the natural maps* Δ_n.

Proof. Suppose we are given another sequence of maps $\widetilde{\Delta}_n$ that have the properties described in Theorem 78.16 (1). We define $\widetilde{h}_i$ in the obvious way. By Theorem 78.16 (2) there exists a sequence of natural maps

$$H_n\colon C_*(X) \;\to\; C_*(X)\otimes C_*(X), \quad n\in\mathbb{N}_0$$

of degree n such that $H_0=0$ and such that for every $n\in\mathbb{N}_0$ the following equality holds:

$$(\,*\,) \qquad \partial\circ H_{n+1} + (-1)^n\cdot H_{n+1}\circ\partial \;=\; \Delta_n - \widetilde{\Delta}_n - (T+(-1)^n\cdot \mathrm{id})\circ H_n.$$

Given a cocycle $\varphi\in C^q(X,A;Z_2)$ and $i\in\mathbb{N}_0$ we see that we have the following equality in $H^{q+i}(X,A;R)$:

$$\begin{aligned}
[h_i(\varphi\otimes\varphi)] - [\widetilde{h}_i(\varphi\otimes\varphi)] \;&\overset{\text{definition of } h_i,\widetilde{h}_i}{=}\; [\Delta_i^*(\Xi(\varphi\otimes\varphi)) - \widetilde{\Delta}_i^*(\Xi(\varphi\otimes\varphi))]\\
&\underset{\substack{\text{by } (*) \text{ and since } 1+1=0\\ \text{since } \Xi \text{ is a cochain map and}\\ \text{since } \Xi \text{ commutes with } T}}{=}\; [H_{n+1}^*(\delta(\Xi(\varphi\otimes\varphi)) + \underbrace{\delta(H_{n+1}^*(\varphi\otimes\varphi))}_{\text{coboundary}} + H_n^*((T+\mathrm{id})^*(\Xi(\varphi\otimes\varphi)))]\\
&= [H_{n+1}^*(\Xi(\underbrace{\delta(\varphi\otimes\varphi)}_{=0 \text{ since } \varphi \text{ cocycle}})) + H_n^*(\Xi(\varphi\otimes\underbrace{\varphi+\varphi\otimes\varphi}_{=0}))] \;=\; 0.
\end{aligned}$$

We have thus shown that the two definitions using Δ_n and $\widetilde{\Delta}_n$ agree. □

By Proposition 78.24 the maps Sq^i are unambiguously defined. Thus we can make the following definition.

Definition. Given a pair of topological spaces (X,A) and $i\in\mathbb{N}_0$ we refer to each map

$$\mathrm{Sq}^i\colon H^q(X,A;\mathbb{Z}_2) \;\to\; H^{q+i}(X,A;\mathbb{Z}_2)$$

as the *i-th Steenrod operation.*

By Theorem 78.20 the Steenrod operations are additive cohomology operations. To complete the proof of Theorem 78.2 we need to verify that the maps Sq^i have all the other desired properties. In other words we need to prove the following theorem.

Theorem 78.25. *The Steenrod operations* Sq^i *have the following properties:*

(1) $\mathrm{Sq}^0=\mathrm{id}$.
(2) *For any* $\varphi\in H^i(X,A;\mathbb{Z}_2)$ *we have* $\mathrm{Sq}^i(\varphi)=\varphi^2$.
(3) *For any* $\varphi\in H^j(X,A;\mathbb{Z}_2)$ *with* $j<i$ *we have* $\mathrm{Sq}^i(\varphi)=0$.
(4) *For any* $\varphi\in H^r(X,A;\mathbb{Z}_2)$ *and* $\psi\in H^s(X,A;\mathbb{Z}_2)$ *we have*

$$\mathrm{Sq}^k(\varphi\cup\psi) \;=\; \sum_{i=0}^{k}\mathrm{Sq}^i(\varphi)\cup\mathrm{Sq}^{k-i}(\psi).$$

Proof.

(1) The notation might suggest that this statement is surely trivial, but that is not the case. For time and space reasons we will not give the, arguably not terribly enlightening proof, instead we refer to [Sp, p. 274] for details.

(2) First we deal with the absolute case. Thus let $\varphi \in C^i(X;R)$ be a cocycle. We calculate

$$\mathrm{Sq}^i(\varphi) \underset{\substack{\uparrow \\ \text{by definition}}}{=} [h_{i-i}(\varphi \otimes \varphi)] = [h_0(\varphi \otimes \varphi)] \underset{\substack{\uparrow \\ \text{Lemma 78.19}}}{=} [\varphi] \cup [\varphi].$$

The case that we are dealing with a cocycle $\varphi \in C^i(X,A;R)$ is proved basically the same way.

(3) Let $\varphi \in C^j(X,A;\mathbb{Z}_2)$ be a cocycle and suppose that $i > j$. We have

$$\mathrm{Sq}^i([\varphi]) \underset{\substack{\uparrow \\ \text{by definition}}}{=} [h_{j-i}(\varphi)] \underset{\substack{\uparrow \\ \text{since } j-i<0 \text{ we have } h_{j-i}=0}}{=} [0].$$

(4) This statement is proved in [Sp, p. 274]. □

Exercises for Chapter 78.

Exercise 78.1.

(a) Show that for any $k \in \mathbb{N}_{\geq 3}$ and $m \in \mathbb{N}$ the map $H^m(X;\mathbb{Z}_2) \to H^{mk}(X;\mathbb{Z}_2)$ given by $\varphi \mapsto \varphi^k$ is not an additive cohomology operation.

(b) Is the map $H^1(X;\mathbb{Z}_3) \to H^3(X;\mathbb{Z}_3)$ given by $\varphi \mapsto \varphi^3$ an additive cohomology operation?

Exercise 78.2. Let $X = S^m \vee S^n$ be the wedge of two spheres. Show that for $i \geq 1$ all Steenrod operations $\mathrm{Sq}^i\colon H^k(X;\mathbb{Z}_2) \to H^k(X;\mathbb{Z}_2)$ are zero.
Hint: Reread the proof of Lemma 57.11.

Exercise 78.3. Find a closed formula for the i-th Steenrod operation on $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}_2)$.
Hint: Use Proposition 62.13 (2).

79. The Adem relations

The following theorem was first proved by José Ádem [Ade, Ade2] in 1952.

Theorem 79.1. (Adem Relations) *For any $0 < a < 2b$ we have the following equality of cohomology operations:*[1330]

$$\mathrm{Sq}^a \circ \mathrm{Sq}^b \;=\; \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j \,.$$

These equalities are called the Adem relations.

Remark. As is remarked on [SE, p. 2], the Adem relations are a consequence of the properties of Steenrod operations that were stated in Theorem 78.2. This observation also follows from the uniqueness result stated in Theorem 78.3.

Proof. Since the proof would consume too much of our valuable time we will not provide it. Instead we refer to the original paper by Adem [Ade, Ade2] or alternatively to [BuM], [SE, Theorem VIII.1.5], [MTa, p. 23] [Hat, Theorem 4.L.13] for a proof. □

Even though we will not prove the Adem relations, we will shamelessly exploit them in this chapter. Afterwards the conscientious reader will surely feel compelled to read the proof of the Adem relations.

Just to get a feeling for the Adem relations we determine for small values of a what the relations really look like.

Example. For $a = 1$ and for any $b \in \mathbb{N}$ we have

$$\mathrm{Sq}^1 \circ \mathrm{Sq}^b \;=\; \underbrace{\sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{with } a=1} \;=\; \underbrace{\binom{b-1}{1} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^0}_{j=0} \;\underset{\substack{\uparrow \\ \text{since } \mathrm{Sq}^0 = \mathrm{id}}}{=}\; (b-1) \cdot \mathrm{Sq}^{1+b} \,.$$

Furthermore, for $a = 2$ and any $b \in \mathbb{N}_{\geq 2}$ we have

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^b \;=\; \underbrace{\binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} \circ \mathrm{Sq}^0}_{j=0} + \underbrace{\binom{b-2}{0} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1}_{j=1} \;\underset{\substack{\uparrow \\ \text{since } \mathrm{Sq}^0 = \mathrm{id}}}{=}\; \binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} + \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1 \,.$$

In particular setting $b = 2$ we see that

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^2 \;=\; \mathrm{Sq}^3 \circ \mathrm{Sq}^1 \,.$$

[1330] These are equalities of cohomology operations $H^n(X, A; \mathbb{Z}_2) \to H^{n+a+b}(X, A; \mathbb{Z}_2)$. In particular, since we work over $\mathbb{Z}_2$-coefficients, we only care about the binomial coefficient modulo 2. We will discuss the mod 2 reductions of binomial coefficients in Lemma 79.3.

79.1. **Application of the Adem relations to homotopy groups of spheres.** Recall that in Theorem 78.13 we used Steenrod operations to show that the three Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ and their suspensions define non-trivial elements in the corresponding homotopy groups of spheres.

In this section we will use the Adem relations to show that the 2-fold iteration of these Hopf maps also define non-trivial elements in the homotopy groups of spheres. More precisely, we have the following theorem.

Theorem 79.2.

(1) *For any $n \geq 2$ the map*

$$\Sigma^{n-2}\big(\underbrace{\overbrace{H}^{S^3\to S^2} \circ \overbrace{\Sigma(H)}^{S^4\to S^3}}_{\text{map } S^4 \to S^2}\big)\colon S^{n+2} \to S^n$$

defines a non-trivial element in $\pi_{n+2}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+2}(S^n) \neq 0$.

(2) *For any $n \geq 4$ the map*

$$\Sigma^{n-4}\big(\underbrace{\overbrace{H_{\mathbb{H}}}^{S^7\to S^4} \circ \overbrace{\Sigma^3(H_{\mathbb{H}})}^{S^{10}\to S^7}}_{\text{map } S^{10} \to S^4}\big)\colon S^{n+6} \to S^n$$

defines a non-trivial element in $\pi_{n+6}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+6}(S^n) \neq 0$.

(3) *For any $n \geq 8$ the map*

$$\Sigma^{n-8}\big(\underbrace{\overbrace{H_{\mathbb{O}}}^{S^{15}\to S^8} \circ \overbrace{\Sigma^7(H_{\mathbb{O}})}^{S^{22}\to S^{15}}}_{\text{map } S^{22} \to S^8}\big)\colon S^{n+14} \to S^n$$

defines a non-trivial element in $\pi_{n+14}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+14}(S^n) \neq 0$.

For the reader's convenience we recall the two lemmas which will play a key role in the proof of Theorem 79.2.

Lemma 63.4. *Let $f\colon A \to X$ be a map between two topological spaces. We denote by $j\colon X \to \mathrm{Cone}(f\colon A \to X)$ the obvious inclusion. Then there exists a natural long exact sequence of the form*

$$\ldots \to H^{k-1}(A;\mathbb{Z}_2) \xrightarrow{\delta} \widetilde{H}^k(\mathrm{Cone}(f\colon A \to X);\mathbb{Z}_2) \xrightarrow{j^*} H^k(X;\mathbb{Z}_2) \xrightarrow{f^*} H^k(A;\mathbb{Z}_2) \xrightarrow{\delta} \ldots$$

As an example for Lemma 63.4 let us consider the mapping cone of the iterated suspension of the Hopf map.

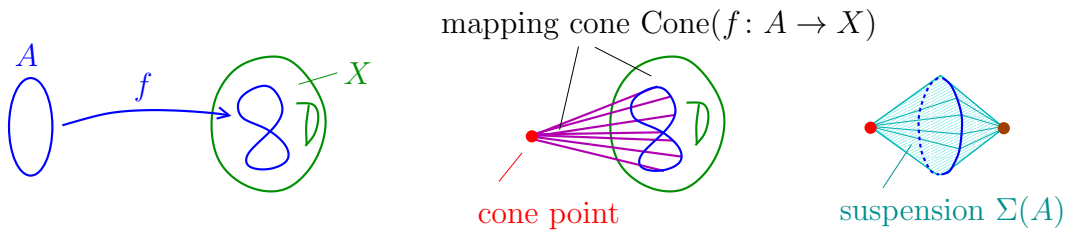


FIGURE 972.

Example. Let $k \in \mathbb{N}_0$. We consider $\Sigma^k(H)\colon S^{k+3} \to S^{k+2}$, i.e. we consider the k-fold suspension of the Hopf map $H\colon S^3 \to S^2$. As we saw in Lemma 63.5, it follows almost immediately from the long exact sequence of Lemma 63.4 that the maps

$$\widetilde{H}^{k+2}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2) \xrightarrow[\cong]{j^*} H^{k+2}(S^{k+2};\mathbb{Z}_2)$$

and

$$H^{k+3}(S^{k+3};\mathbb{Z}_2) \xrightarrow[\cong]{\delta} \widetilde{H}^{k+4}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2)$$

are isomorphisms, and that all the other reduced cohomology groups of the mapping cone vanish.

In the following we recall three statements of Lemma 15.8 which are at the moment relevant to us. It might be helpful to have a peek at Figure 972.

Lemma 15.8. *Let $f\colon A \to X$ be a map between topological spaces.*

(1) *The inclusion $X \to \mathrm{Cone}(f\colon A \to X)$ is a homeomorphism onto its image.*
(2) *The map*

$$\begin{array}{rcl} h\colon \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$

is a homeomorphism.

(4) *The subset X is a deformation retract of the open neighborhood that is given by the complement of the cone point.*

Now we turn to the proof of Theorem 79.2. As in the proof of Theorem 78.13 we first consider the case of the classical Hopf map, i.e. we first prove Statement (1).

Proof of Theorem 79.2 (1). Let $n \in \mathbb{N}_{\geq 2}$. We write

$$\Sigma^{n-2}(H \circ \Sigma(H)) \underset{\substack{\uparrow \\ \text{Lemma 15.4 (2)}}}{=} \overbrace{\underbrace{\Sigma^{n-2}(H)}_{=:g}}^{S^{n+1}\to S^n} \circ \overbrace{\underbrace{\Sigma^{n-1}(H)}_{=:f}}^{S^{n+2}\to S^{n+1}}\colon S^{n+2} \to S^n.$$

We need to show that $g \circ f\colon S^{n+2} \to S^n$ defines a non-trivial element in $\pi_{n+2}(S^n)$. As in the proof of Theorem 78.13 we do a proof by contradiction. Thus suppose that there exists a homotopy from $g \circ f\colon S^{n+2} \to S^n$ to a constant map. By Lemma 28.4 we can use this homotopy to construct a map[1331] $\varphi\colon \operatorname{Cone}(S^{n+2}) = \overline{B}^{n+3} \to S^n$ such that $\varphi|_{S^{n+2}} = g \circ f$. We consider the map[1332]

$$\begin{array}{rcl} \phi\colon \overbrace{(\operatorname{Cone}(S^{n+2}) \sqcup S^{n+1})/\sim}^{=\operatorname{Cone}(f\colon S^{n+2}\to S^{n+1})} & \to & S^n \\ [P] & \mapsto & \begin{cases} g(P), & \text{if } P \in S^{n+1}, \\ \varphi(P), & \text{if } P \in \operatorname{Cone}(S^{n+2}) = \overline{B}^{n+3}. \end{cases} \end{array}$$

Our main object of interest is now, perhaps rather confusingly, the mapping cone of ϕ. This mapping cone comes with two interesting maps:

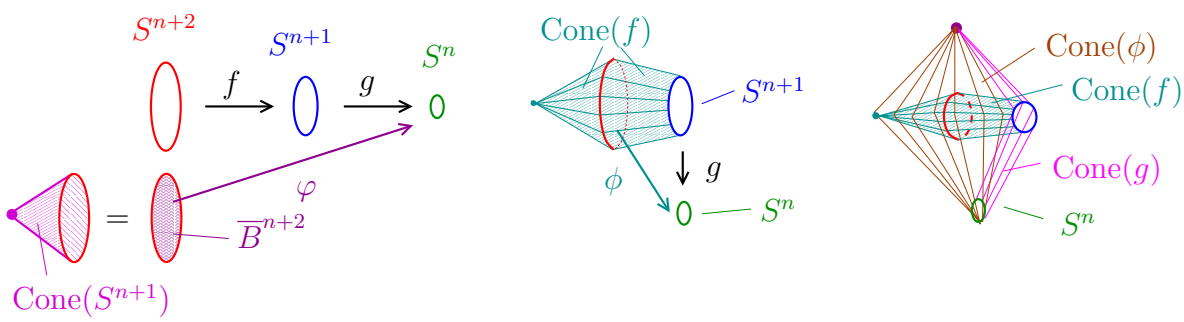


FIGURE 973. Illustration for the proof of Theorem 79.2.

(1) Since we have an inclusion $S^{n+1} \to \operatorname{Cone}(f)$ and since $\phi|_{S^{n+1}} = g$ we obtain, see Lemma 15.9 (3), also an inclusion map

$$i\colon \operatorname{Cone}(g\colon S^{n+1} \to S^n) \;\to\; \operatorname{Cone}(\phi\colon \operatorname{Cone}(f) \to S^n).$$

(2) We also consider the projection map

$$p\colon \operatorname{Cone}(\phi\colon \operatorname{Cone}(f) \to S^n) \;\to\; \operatorname{Cone}(\phi\colon \operatorname{Cone}(f) \to S^n)/S^n.$$

In the following two claims we will study the maps on cohomology induced by the inclusion i and the projection p.

Claim 1.

[1331]As we discussed on page 450, given any $m \in \mathbb{N}_0$ the map $\operatorname{Cone}(S^m) \to \overline{B}^{m+1}$ given by $[(v,t)] \mapsto v \cdot t$ is a homeomorphism. We will use this map to make the identification $\overline{B}^{m+1} = \operatorname{Cone}(S^m)$.

[1332]It follows from $\varphi|_{S^{n+2}} = g \circ f$ that this map is well-defined. Furthermore we obtain from Lemma 1.83 (3) that this map is continuous.

(a) The induced map

$$i^*\colon H^k(\mathrm{Cone}(\phi);\mathbb{Z}_2) \;\to\; H^k(\mathrm{Cone}(g);\mathbb{Z}_2)$$

is an isomorphism for $k \neq n+1$.

(b) We have $H^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2) = 0$.

We consider the following diagram:

$$\begin{tikzcd}
H^{k-2}(S^{n+2};\mathbb{Z}_2) \arrow[d] & & & & H^{k-1}(S^{n+2};\mathbb{Z}_2) \arrow[d] \\
\to H^{k-1}(\mathrm{Cone}(f);\mathbb{Z}_2) \arrow[r] \arrow[d] & H^k(\mathrm{Cone}(\phi\colon \mathrm{Cone}(f)\to S^n);\mathbb{Z}_2) \arrow[r] \arrow[d, "i^*"] & H^k(S^n;\mathbb{Z}_2) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{id}"] & H^k(\mathrm{Cone}(f);\mathbb{Z}_2) \to \arrow[d] \\
\longrightarrow H^{k-1}(S^{n+1};\mathbb{Z}_2) \arrow[r] \arrow[d, "f^*"] & H^k(\mathrm{Cone}(g\colon S^{n+1}\to S^n);\mathbb{Z}_2) \arrow[r] & H^k(S^n;\mathbb{Z}_2) \arrow[r, "f^*"] & H^k(S^{n+1};\mathbb{Z}_2) \longrightarrow \arrow[d, "f^*"] \\
H^{k-1}(S^{n+2};\mathbb{Z}_2) & & & H^k(S^{n+2};\mathbb{Z}_2)
\end{tikzcd}$$

Here the two horizontal and the two vertical sequences are obtained from Lemma 63.4. The diagram commutes by the naturality of the long exact sequence. The two statements of the claim follow easily from the above diagram (see also the discussion on page 1834) and the calculation of the cohomology groups of spheres with $\mathbb{Z}_2$-coefficients. This concludes the proof of Claim 1.

Claim 2. The map

$$p^*\colon H^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \;\to\; H^k(\mathrm{Cone}(\phi);\mathbb{Z}_2)$$

is an isomorphism for $k \neq n, n+1$.

We consider the following diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & H^{k-1}(S^n;\mathbb{Z}_2) \arrow[r, "\delta"] & H^k(\mathrm{Cone}(\phi), S^n;\mathbb{Z}_2) \arrow[r] \arrow[d] & H^k(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r] & H^k(S^n;\mathbb{Z}_2) \arrow[r] & \ldots \\
& & H^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[ur, "p^*"'] & & &
\end{tikzcd}$$

Here on top we see the long exact sequence in cohomology of the pair $(\mathrm{Cone}(\phi), S^n)$. Furthermore the vertical map is the natural map from Proposition 50.2 which has in particular the property that it makes the diagram commute. Furthermore, it follows from Proposition 50.2 (2) and Lemma 15.8 (4) that the vertical map is an isomorphism. The desired statement follows immediately from this discussion and the calculation of the cohomology groups of S^n. This concludes the proof of Claim 2.

Next we consider the following diagram.

$$
\begin{tikzcd}
H^n(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[d, "="] & H^{n+2}(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) \arrow[d, "="'] & \\
H^n(\mathrm{Cone}(g);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & H^{n+2}(\mathrm{Cone}(g);\mathbb{Z}_2) & \\
H^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[u, "i^*", "\cong"'] \arrow[r, "\mathrm{Sq}^2"] & H^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[u, "\cong", "i^*"'] \arrow[r, "\mathrm{Sq}^2"] & H^{n+4}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \\
& H^{n+2}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[u, "p^*", "\cong"'] \arrow[r, "\mathrm{Sq}^2"] & H^{n+4}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[u, "\cong", "p^*"'] \\
& H^{n+2}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) \arrow[u, "h^*", "\cong"'] \arrow[r, "\mathrm{Sq}^2", "\cong"'] & H^{n+4}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) \arrow[u, "\cong", "h^*"'] \\
& H^{n+2}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) \arrow[d, "\theta^*", "\cong"'] \arrow[r, "\mathrm{Sq}^2", "\cong"'] & H^{n+4}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) \arrow[d, "\cong", "\theta^*"'] \\
& H^{n+2}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & H^{n+4}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2). \arrow[d, "="] \\
& & {}
\end{tikzcd}
$$

We make the following clarifications and observations:

(1) At the top and bottom we use that by definition $f = \Sigma^{n-1}(H)\colon S^{n+2} \to S^{n+1}$ and $g = \Sigma^{n-2}(H)\colon S^{n+1} \to S^n$.
(2) We obtain from Proposition 78.15 (1) that the top and bottom horizontal maps Sq^2 are isomorphisms.
(3) It follows from the above two claims that the vertical maps i^* and p^* are isomorphisms.
(4) The map $h\colon \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n)/S^n \to \Sigma(\mathrm{Cone}(f))$ is the homeomorphism from Lemma 15.8 (2).
(5) The map $\theta\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \to \Sigma(\mathrm{Cone}(f\colon X \to Y))$ is the homeomorphism from Proposition 15.15.
(6) It follows from the naturality of the Steenrod operations that all the squares commute.
(7) The combination of the above arguments implies that both of the maps Sq^2 in the third row are isomorphisms.

Finally we consider the following diagram.

$$
\begin{tikzcd}
\overbrace{H^n(\mathrm{Cone}(\phi);\mathbb{Z}_2)}^{\neq 0} \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[dr, "\mathrm{Sq}^1"'] & H^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & H^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2). \\
& \underbrace{H^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)}_{=0} \arrow[ur, "\mathrm{Sq}^3"'] &
\end{tikzcd}
$$

We make the following observations:

(1) By the discussion preceding the diagram we know that the two maps on the top are isomorphisms.
(2) As we saw on page 1832, we have the Adem relation $\mathrm{Sq}^2 \circ \mathrm{Sq}^2 = \mathrm{Sq}^3 \circ \mathrm{Sq}^1$. This implies that the diagram commutes.
(3) We have

$$\begin{array}{ccccc} H^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) & \cong & H^n(\mathrm{Cone}(\overbrace{\Sigma^{n-2}(H)}^{=g});\mathbb{Z}_2) & \cong & H^n(S^n;\mathbb{Z}_2) \;\cong\; \mathbb{Z}_2. \\ & \uparrow & & \uparrow & \\ & \text{Claim 1 (a)} & & \text{see the example on page 1834} & \end{array}$$

(4) In Claim 1 (b) we had shown that the group $H^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)$ is trivial.

By (1) and (3) the two horizontal maps in the diagram give us an isomorphism of a non-trivial group. On the other hand, by (2) and (4) this isomorphism factors through a trivial group. This is absurd. We have thus obtained a contradiction. □

Now we turn to the proof of the remaining two statements of Theorem 79.2.

Proof of Theorem 79.2 (2) and (3). The proof for the quaternionic Hopf map $H_{\mathbb{H}}$ and the octonion Hopf map $H_{\mathbb{O}}$ is almost the same as the above proof that deals with the classical Hopf map H. We just need to replace Proposition 78.15 (1) by Proposition 78.15 (2) and (3) and we need to use the following Adem relations:

$$\begin{aligned} \mathrm{Sq}^4 \circ \mathrm{Sq}^4 &= \mathrm{Sq}^6 \circ \mathrm{Sq}^2 + \mathrm{Sq}^7 \circ \mathrm{Sq}^1 \\ \mathrm{Sq}^8 \circ \mathrm{Sq}^8 &= \mathrm{Sq}^{12} \circ \mathrm{Sq}^4 + \mathrm{Sq}^{14} \circ \mathrm{Sq}^2 + \mathrm{Sq}^{15} \circ \mathrm{Sq}^1 . \end{aligned}$$

These relations can be obtained easily by hand from Theorem 79.1. Alternatively one can use the Adem relations calculator:

`https://math.berkeley.edu/~kruckman/adem/`

As in the proof for the Hopf maps one sees that the initial Steenrod operations on the right hand side all end up in a cohomology group which is zero. Thus one obtains again a contradiction to the hypothesis that the maps are $\Sigma^{n-4}(H_{\mathbb{H}} \circ \Sigma^3(H_{\mathbb{H}})) \colon S^{n+6} \to S^n$ and $\Sigma^{n-8}(H_{\mathbb{O}} \circ \Sigma^7(H_{\mathbb{O}})) \colon S^{n+14} \to S^n$ are null-homotopic. □

79.2. **Decomposing Steenrod operations and the Hopf invariant revisited.** The key to extracting extra mileage out of Steenrod operations is the following innocuous lemma from elementary number theory.

Lemma 79.3. *Let $n, k \in \mathbb{N}_0$. If we write $n = \sum_{i\in\mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i\in\mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0,1\}$, then*

$$\binom{n}{k} = \prod_{i\in\mathbb{N}_0} \binom{n_i}{k_i} \mod 2.$$

Remark.

(1) Note that for $r, s \in \{0, 1\}$ it follows easily from the definition on page 1809 that

$$\binom{s}{r} = \begin{cases} 1, & \text{if } r \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, d\}$. We define n_i, k_i as above. It follows from Lemma 79.3 and (1) that $\binom{n}{k}$ is odd if and only if for each i we have $k_i \leq n_i$. If we draw the usual Pascal triangle, then by shading odd and even terms in different colors we obtain a Sierpinski triangle. We refer to Figure 974 for an illustration.[1333]

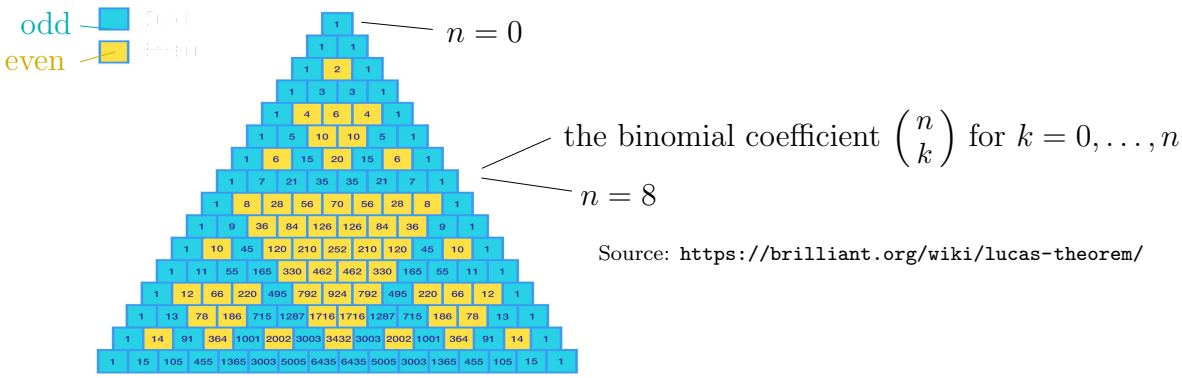


FIGURE 974.

(3) In fact the obvious analogue of Lemma 79.3 holds with the prime 2 replaced by any prime p. This more general statement is known as Lucas' Theorem, a proof is for example provided in [Fi].

Proof. Let $n, k \in \mathbb{N}_0$. We write $n = \sum_{i \in \mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i \in \mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0, 1\}$ In the polynomial ring $\mathbb{Z}_2[x]$ we have the following equalities[1334]

$$\sum_{i=0}^{n} \binom{n}{i} x^i \underset{\substack{\uparrow \\ \text{binomial formula}}}{=} (1+x)^n = (1+x)^{\sum n_i 2^i} = \prod_{i \in \mathbb{N}_0} (1+x)^{2^i n_i} \underset{\substack{\uparrow \\ \text{since in } \mathbb{Z}_2[x] \text{ we have } (a+b)^2 = a^2 + b^2}}{=} \prod_{i \in \mathbb{N}_0} (1+x^{2^i})^{n_i} \underset{\substack{\uparrow \\ \text{binomial formula}}}{=} \prod_{i \in \mathbb{N}_0} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} x^{j_i \cdot 2^i}.$$

[1333] A more elaborate version of this figure can be found here:

https://upload.wikimedia.org/wikipedia/commons/8/87/Sierpinski_Pascal_triangle.svg

[1334] The argument has some similarities with the proof of the claim on page 1465.

Now we compare the x^k-coefficients, we see that[1335]

$$\binom{n}{k} = x^k\text{-coefficient of } \sum_{i=0}^{n}\binom{n}{i}x^i \underset{\substack{\uparrow \\ \text{by the above equality}}}{=} x^k\text{-coefficient of } \prod_{i\in\mathbb{N}_0}\sum_{j_i=0}^{n_i}\binom{n_i}{j_i}x^{j_i\cdot 2^i} \underset{\substack{\uparrow \\ \text{by the uniqueness of the 2-adic description} \\ \text{of a natural number we only get } x^k \\ \text{if each variable } j_i \text{ equals } k_i}}{=} \prod_{i\in\mathbb{N}_0}\binom{n_i}{k_i}.$$

We have thus proved the desired equality. □

This lemma allows us to prove the following theorem.

Theorem 79.4. *Let $i \in \mathbb{N}_0$. If i is not a power of 2, then Sq^i is decomposable, in the sense that it can be written as the sum of compositions of Steenrod operations of smaller degree.*

Example. By the discussion on page 1832 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{Sq}^{2k+1} = \mathrm{Sq}^1 \circ \mathrm{Sq}^{2k} \quad \text{and} \quad \mathrm{Sq}^{4k+2} = \mathrm{Sq}^2 \circ \mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+1} \circ \mathrm{Sq}^1 .$$

In particular we see that

$$\mathrm{Sq}^3 = \mathrm{Sq}^1 \circ \mathrm{Sq}^2, \quad \mathrm{Sq}^5 = \mathrm{Sq}^1 \circ \mathrm{Sq}^4, \quad \mathrm{Sq}^6 = \mathrm{Sq}^2 \circ \mathrm{Sq}^4 + \mathrm{Sq}^5 \circ \mathrm{Sq}^1 \quad \text{and } \mathrm{Sq}^7 = \mathrm{Sq}^1 \circ \mathrm{Sq}^6 .$$

Proof. Suppose that we are given $i \in \mathbb{N}_0$ that is not a power of two. This means that we can write $i = a + b$ where $b = 2^k$ and $a \in \{1, \dots, 2^k - 1\}$. Since $0 < a < 2b$ we obtain an Adem relation for $\mathrm{Sq}^a \circ \mathrm{Sq}^b$ from Theorem 79.1. This relation can be rewritten as follows:

$$\binom{b-1}{a} \cdot \mathrm{Sq}^{a+b} = \underbrace{\mathrm{Sq}^a \circ \mathrm{Sq}^b + \sum_{j=1}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{compositions of Steenrod operations of degree} < a+b} .$$

It remains to show that $\binom{b-1}{a} = 1 \bmod 2$. We calculate

$$\binom{b-1}{a} \underset{\substack{\downarrow \\ \text{by Lemma 79.3 where } b-1 = \sum_{i\in\mathbb{N}_0} d_i \cdot 2^i \text{ and } 2a = \sum_{i\in\mathbb{N}_0} c_i \cdot 2^i \text{ with } c_i, d_i \in \{0,1\}}}{=} \prod_{i\in\mathbb{N}_0}\binom{d_i}{c_i} \underset{\substack{\uparrow \\ \text{since } b-1 = 2^k - 1 = 1 + 2 + \cdots + 2^{k-1} \text{ we have } d_0 = \cdots = d_{k-1} = 1 \text{ and } d_k = d_{k+1} = \cdots = 0 \\ \text{furthermore, since } a < 2^k - 1 \text{ we have } c_k = c_{k+1} = \cdots = 0}}{=} \prod_{i=0}^{k-1}\binom{1}{c_i} \cdot \prod_{i=k}^{\infty}\binom{0}{0} = 1 \mod 2.$$

□

Corollary 79.5. *Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the Steenrod operation*

$$\mathrm{Sq}^i \colon H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2).$$

If the cohomology groups inbetween vanish, i.e. if $H^{k+1}(X;\mathbb{Z}_2) = \ldots = H^{n+k-1}(X;\mathbb{Z}_2) = 0$, and if i is not a power of 2, then the Steenrod operation is the zero map.

[1335]Note that the equalities also make sense for $k > n$. In this setting one sees easily that all terms are in fact zero.

Proof. Since i is not a power of 2 we know by Theorem 79.4 that Sq^i can be written as the sum of compositions of Steenrod operations of smaller degree. But by our hypothesis for any $j \in \{1, \dots, n-1\}$ we have $H^{n+j}(X; \mathbb{Z}_2) = 0$. But this implies in particular that for any such j we have $\mathrm{Sq}^j \colon H^n(X; \mathbb{Z}_2) \to H^{n+j}(X; \mathbb{Z}_2) = 0$. $\square$

$$\begin{tikzcd}
H^n(X;\mathbb{Z}_2) \arrow[rrrrrrr, bend left, "\mathrm{Sq}^i"] \arrow[rrrr, bend right, "\mathrm{Sq}^j"'] & 0 & 0 & 0 & 0 & 0 & 0 & H^{n+i}(X;\mathbb{Z}_2)
\end{tikzcd}$$

FIGURE 975. Illustration for the proof of Corollary 79.5.

In Question 60.19 we asked whether all (reasonable) superalgebras can be realized as cohomology rings. We will now see that this is not the case. Before we can state the corresponding proposition we need to introduce the following definition.

Definition. Let $m \in \mathbb{N}$ be an even number and let $s \in \mathbb{N}$. We define $R_m(s)$ to be the superalgebra which is given by the groups[1336]

$$R_m(s) \;=\; \begin{cases} \mathbb{Z} \cdot 1, & \text{in degree } 0, \\ \mathbb{Z} \cdot x, & \text{in degree } m, \\ \mathbb{Z} \cdot y, & \text{in degree } 2m, \\ 0, & \text{in all other degrees} \end{cases}$$

and where the multiplication is given by $x \cdot x = s \cdot y$. In other words, the multiplication table is given as follows

	1	x	y
1	1	x	y
x	x	$s \cdot y$	0
y	y	0	0.

Example.

(1) For any m the superalgebra $R_m(1)$ is evidently isomorphic to $\mathbb{Z}[x]/(x^3)$ where x is of degree m.
(2) By (1) together with Proposition 62.7 and Proposition 62.20 we have isomorphisms $(H^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_2(1)$, $(H^*(\mathbb{H}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_4(1)$ and $(H^*(\mathbb{O}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_8(1)$.

We can now prove the following proposition which gives a negative answer to Questions 60.19 and Question 62.21.

Proposition 79.6. *Let m be an even number and let s be an odd number. If m is not a power of 2, then there is no topological space X such that $(H^*(X; \mathbb{Z}), \cup)$ is isomorphic to $R_m(s)$.*

[1336] It follows from the fact that m is even that this is indeed a superalgebra.

In the proof of Proposition 79.6 we will need the following lemma.

Lemma 79.7. *Let $m \in \mathbb{N}$. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_m$ the canonical map. Given any topological space X there exists a long exact sequence*

$$\ldots \to H^k(X;\mathbb{Z}) \xrightarrow{\cdot m} H^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} H^k(X;\mathbb{Z}_2) \xrightarrow{\delta} H^{k+1}(X;\mathbb{Z}) \to \ldots$$

Proof of Lemma 79.7. We denote by $\psi\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by m. We have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_m \to 0.$$

By the discussion on page 1714 we have the corresponding Bockstein sequence in cohomology which looks as follows:

$$\ldots \xrightarrow{\delta} H^k(X;\mathbb{Z}) \xrightarrow{\psi_*} H^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} H^k(X;\mathbb{Z}_2) \xrightarrow{\delta} H^{k+1}(X;\mathbb{Z}) \to \ldots$$

By the almost, but not entirely trivial Lemma 74.2 we know that φ_* is indeed multiplication by m. □

Proof of Proposition 79.6. Let m be an even number. Furthermore let X be a topological space such that $(H^*(X;\mathbb{Z}), \cup) = R_m(s)$ for some odd number s. We need to show that m is a power of 2. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_2$ the canonical map. We consider the following diagram

$$\begin{array}{ccc}
H^m(X;\mathbb{Z}) & \xrightarrow{x \mapsto x^2 = x \cup x = s \cdot y} & H^{2m}(X;\mathbb{Z}) \\
\downarrow{\scriptstyle \varphi_* \times \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\
H^m(X;\mathbb{Z}_2) & \xrightarrow{z \mapsto z^2 = z \cup z} & H^{2m}(X;\mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H^m(X;\mathbb{Z}_2) & \xrightarrow{z \mapsto \mathrm{Sq}^m(z)} & H^{2m}(X;\mathbb{Z}_2).
\end{array}$$

We make the following clarifications and observations:

(1) It follows from Proposition 58.3 (2) that the upper square commutes.
(2) It follows from Lemma 79.7 and our hypothesis on the topological space X that the map $\varphi_*\colon H^{2m}(X;\mathbb{Z}) \to H^{2m}(X;\mathbb{Z}_2)$ is an epimorphism and that $H^{2m}(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$. In particular we see that $\varphi_*(y) \in H^{2m}(X;\mathbb{Z}_2)$ is a non-trivial element of order 2.
(3) Since s is odd we obtain from (2) that $\varphi_*(s \cdot y) = s \cdot \varphi_*(y)$ is a non-trivial element of $H^{2m}(X;\mathbb{Z}_2)$.
(4) The lower square commutes by Axiom (2) of Steenrod operations, see Theorem 78.2 (2).
(5) By our hypothesis on X we know that the intermediate cohomology groups vanish, ie. we have $H^{m+1}(X;\mathbb{Z}_2) = \cdots = H^{2m-1}(X;\mathbb{Z}_2) = 0$.

It follows from (3) and (4) that the bottom horizontal map has non-trivial image. But by Corollary 79.5 together with (5) this is only possible if m is indeed a power of 2. □

As promised in the title of the section, we now return to the Hopf invariant. Let $f\colon S^{2n-1} \to S^n$ be a map where $n \geq 2$. Recall that on page 1473 we introduced the Hopf invariant $\mathrm{Hopf}(f) \in \mathbb{Z}$. The following lemma translates the Hopf invariant into the language of the above discussion. The lemma can also be viewed as an alternative definition of the Hopf invariant, at least up to sign.

Lemma 79.8. *Let $n \geq 2$. If $f\colon S^{2n-1} \to S^n$ is a map, then there exists an isomorphism*

$$(H^*(\mathrm{Cone}(f);\mathbb{Z}), \cup) \;\cong\; R_n(\mathrm{Hopf}(f))$$

of superalgebras.

Proof. This statement follows immediately from Lemma 63.5 and the definition of the Hopf invariant, see page 1473. □

The following proposition is the summary of our previous discussion of the Hopf invariant.

Proposition 79.9. *Let $n \geq 2$.*

(1) *The Hopf invariant* $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ *is a homomorphism.*
(2) *The Hopf invariant* $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ *is zero for odd n.*
(3) *If $\mathbb{R}^n$ can be equipped with the structure of a division algebra, then there exists a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$.
(4) *For $k = 1, 2, 4$ the Hopf maps $H, H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ give elements in $\pi_{4k-1}(S^{2k})$ with Hopf invariant* 1.
(5) *Given any $k \in \mathbb{N}$ there exists a map $f\colon S^{4k-1} \to S^{2k}$ with* $\mathrm{Hopf}(f) = 2$.

Proof. The proof follows from Proposition 63.7, Corollary 63.8, Lemma 63.11 and Propositions 63.13 and 63.12. □

We recall the following question that we asked a while ago.

Question 63.17. *For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$*?*

The following theorem is an immediate consequence of Proposition 79.6 and Lemma 79.8.

Theorem 79.10. *Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$*, then n is a power of* 2.

The combination of Proposition 79.9 (3) and Theorem 79.10 gives us an alternative proof of the following theorem.

Theorem 62.23. (Hopf 1940) *If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.*

It is an amusing mental exercise, left to the reader, to compare the two proofs of Theorem 62.23. The first one relies on the calculation of the cup product of $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ with

$\mathbb{Z}_2$-coefficients and a fact in elementary number theory that is proved on page 1465. The second proof uses Steenrod operations and a different input from elementary number theory, namely Lemma 79.3. Note that in some sense the former proof is more satisfactory since we could provide full details, whereas the latter proof relies on the Adem relations that we did not prove.

For completeness we remark that in 1960 Frank Adams [Adam] (see also [AdA], [May, Chapter 24.6] and [Hus, Theorem 15.4.3]) proved the following strengthening of Theorem 79.10.

Theorem 79.11. *Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with* $\operatorname{Hopf}(f) = 1$*, then $n = 2, 4$ or $n = 8$.*

The combination of Proposition 79.9 (3) and Theorem 79.11 gives us an alternative proof of the following beautiful theorem.

Theorem 62.25. *Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.*

79.3. **The Steenrod cyclic reduced powers** $(*)$**.** The Steenrod operations Sq^i are cohomology operations on cohomology groups with $\mathbb{Z}_2$-coefficients. It turns out that there exist also analogous cohomology operations for cohomology with $\mathbb{Z}_p$-coefficients. More precisely, the following theorem holds:

Theorem 79.12. *Let p be an odd prime. There exists a family*[1337]

$$P^i\colon H^n(X, A; \mathbb{Z}_p) \;\to\; H^{n+2i(p-1)}(X, A; \mathbb{Z}_p) \qquad \text{with } i \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $P^0 = \mathrm{id}$.
(2) *For any $\varphi \in H^{2i}(X, A; \mathbb{Z}_p)$ we have $P^i(\varphi) = \varphi^p \in H^{2pi}(X, A; \mathbb{Z}_p)$.*
(3) *For any $\varphi \in H^j(X, A; \mathbb{Z}_p)$ with $i > j$ we have $P^i(\varphi) = 0$.*
(4) *Let (X, A, B) be an excisive triad of topological spaces. For any $\varphi \in H^r(X, A; \mathbb{Z}_p)$ any $\psi \in H^s(X, B; \mathbb{Z}_p)$ and any $k \in \mathbb{N}_0$ we have*[1338]

$$P^k(\varphi \cup \psi) \;=\; \sum_{i=0}^{k} \underbrace{P^i(\varphi)}_{\in H^{r+2i(p-1)}(X,A;\mathbb{Z}_p)} \;\cup\; \underbrace{P^{k-i}(\psi)}_{\in H^{s+2i(p-1)}(X,B;\mathbb{Z}_p)} \;\in\; H^{r+s+2k(p-1)}(X, A \cup B; \mathbb{Z}_p).$$

This equality is often called the Cartan formula.

[1337]Note that if for argument's sake one plugs $p = 2$ into the statements of Theorem 79.12, then one does *not* get the statements of Theorem 78.2 since the degree formulas are different. In other words there is no easy way to combine Theorems 78.2 and 79.12 into a single statement.

[1338]We refer to page 1203 for the definition of an excisive triad. By Proposition 50.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A = B$, or that at least one of A or B is the empty set. We refer again to page 1358 for the definition of the relative cup product.

Proof. This theorem is proved in [SE, Chapter VI]. A very different proof is given in [Hat, Theorem 4.L.16]. □

The following theorem plays the role of Theorem 79.1.

Theorem 79.13. (Adem Relations) *Let p be an odd prime.*

(1) *For any $0 < a < pb$ we have the following equality of cohomology operations:*

$$P^a \circ P^b \;=\; \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)-1}{a-pj} \cdot P^{a+b-j} \circ P^j.$$

(2) *Given $i \in \mathbb{N}_0$ we denote by $\beta\colon H^i(X;\mathbb{Z}_p) \to H^{i+1}(X;\mathbb{Z}_p)$ the Bockstein homomorphism that corresponds to the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{P^2} \to \mathbb{Z}_p \to 0$. For any $0 < a \leq pb$ the following mystifying equality holds:*

$$\begin{aligned} P^a \circ \beta \circ P^b \;=\; & \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)}{a-pj} \cdot \beta \circ P^{a+b-j} \circ P^j \\ & - \sum_{j=0}^{\lfloor \frac{a-1}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)-1}{a-pj-1} \cdot P^{a+b-j} \circ \beta \circ P^j. \end{aligned}$$

These equalities are called the Adem relations.

Example. Let us consider the case of the prime $p = 3$. In this case the corresponding reduced Steenrod operations are additive cohomology operations of the form

$$P^i\colon H^n(X,A;\mathbb{Z}_3) \;\to\; H^{n+4i}(X,A;\mathbb{Z}_3).$$

For $a = 1$ and $b = 3$ we obtain the Adem relation

$$P^1 \circ P^3 \underset{\substack{\uparrow \\ \text{Theorem 79.13 (1)}}}{=} \sum_{j=0}^{\lfloor \frac{1}{3} \rfloor} (-1)^{1+j} \binom{2\cdot(3-j)-1}{1-3\cdot j} \cdot P^{4-j} \circ P^j \underset{\substack{\uparrow \\ \text{only one summand, namely for } j=0}}{=} (-1) \cdot \binom{5}{1} \cdot P^4 \;=\; -5 \cdot P^4 \underset{\substack{\uparrow \\ \text{we work in } \mathbb{Z}_3}}{=} P^4.$$

The Steenrod cyclic reduced powers can, not surprisingly, be used to give more information on homotopy groups of spheres. We refer to [SE, Chapter VI.5] and [Hat, Example 4.L.6] for details.

Instead of providing more results on homotopy groups of spheres we want to give a different application of the Steenrod cyclic reduced powers, namely we want to prove the following proposition which gives a negative answer to Question 62.22.

Proposition 79.14. *There is no topological space X such that $H^*(X;\mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in H^8(X;\mathbb{Z}_3)$.*

Proof. We provide a proof by contradiction. So suppose there exists a topological space X such that $H^*(X;\mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in H^8(X;\mathbb{Z}_3)$. We see that

$$\begin{array}{ccccccccccc} & & & \text{Theorem 79.12 (2)} & & & & & & & \\ & & & \downarrow & & & & & & & \\ 0 & \neq & \varphi^3 & = & P^4(\varphi) & = & (P^1 \circ P^3)(\varphi) & = & P^1(\overbrace{P^3(\varphi)}^{\in H^{8+3\cdot 4}(X;\mathbb{Z}_3)}) & = & 0. \\ & \uparrow & & & & \uparrow & & & & \uparrow & \\ & \text{since we assume that} & & & & \text{since } P^4 = P^1 \circ P^3 & & & & \text{since } H^{20}(X;\mathbb{Z}_3) = 0 & \\ & H^*(X;\mathbb{Z}_3) \cong \mathbb{Z}_3[\varphi]/(\varphi^4) & & & & & & & & & \end{array}$$

We have thus obtained a contradiction. □

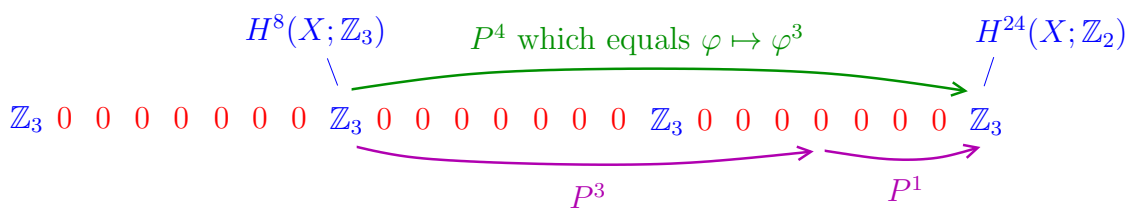


FIGURE 976. Illustration for the proof of Proposition 79.14.

Remark. In Exercise 79.2 we will use Proposition 79.14 to give a negative answer to Question 44.12. So with most reasonable interpretations it is fair to say that there is no manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ with $n \geq 3$.

Exercises for Chapter 79.

Exercise 79.1. Let X be a topological space and let $\varphi \in H^1(X;\mathbb{Z}_2)$ be a cohomology class of degree *one*. Show that for any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have

$$\mathrm{Sq}^i(x^{2k}) = \begin{cases} x^{2k}, & \text{if } i = 0, \\ x^{2^{k+1}}, & \text{if } i = 2^k, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 79.2. Let $n \geq 3$. Show that there is no closed orientable $8n$-dimensional manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \dots, 8n$.
Remark: This exercise gives a negative answer to Question 44.12.

80. The Hurewicz Theorem in higher dimensions

After a very long hiatus we now return to the study of higher homotopy groups that we had initiated in Chapter 28. The goal of this chapter is to formulate and prove the high-dimensional analogue of the original Hurewicz Theorem 38.5. Furthermore we will discuss several applications of the higher-dimensional Hurewicz Theorem.

80.1. **The group structure on $\pi_n(X, x_0)$.** Before we turn to the formulation of the high-dimensional Hurewicz Theorem we want to recall some conventions regarding higher homotopy groups and we want to introduce a useful approach to understanding the sum operation in the higher homotopy groups.

First we recall some conventions and notations that apply to any $n \in \mathbb{N}$.

(1) We write $* = (-1, 0, \ldots, 0) \in S^n$.
(2) We write $I = [0, 1]$.
(3) We denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(4) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(5) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 80.
(6) Let $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ be the map from page 99 that induces the usual identification $\overline{B}^n/S^{n-1} \to S^n$.
(7) As on page 715 we use φ and ψ to make the identifications

$$\pi_n(X, x_0) = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n, \partial I^n) \to (X, x_0)\\ \|\\ \text{set (of homotopy classes)}\\ \text{of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0).\end{array} = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n, *) \to (X, x_0)\end{array}$$

The maps φ and ψ are illustrated in Figure 977.

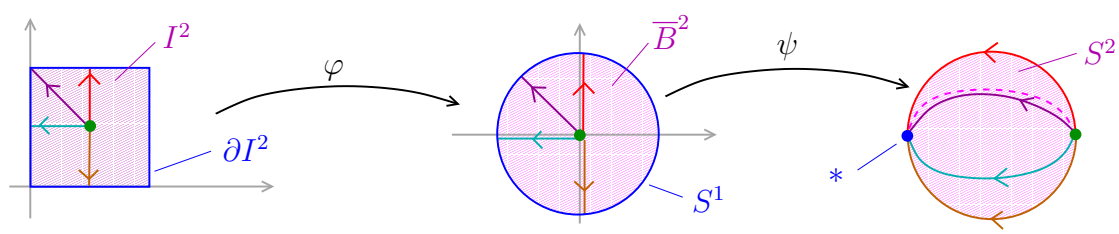


Figure 977.

Now let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Recall that on page 710 we defined the group structure on $\pi_n(X, x_0)$ by considering elements in the homotopy groups as homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. Under the above identifications it is at times difficult to see what the group structure actually looks like. We start out with considering inverses of homotopy classes.

Lemma 80.1. *Let $n \in \mathbb{N}$ and let (X, x_0) be a pointed topological space.*

(1) *Let* $f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ *be a map. If* $\rho\colon \overline{B}^n \to \overline{B}^n$ *is the reflection in some hyperplane of* $\mathbb{R}^n$*, then* $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.
(2) *Let* $f\colon (S^n, *) \to (X, x_0)$ *be a map. If* $\rho\colon S^n \to S^n$ *is the reflection in some hyperplane of* $\mathbb{R}^{n+1}$ *that contains the line* $\mathbb{R} \cdot (-1, 0, \dots, 0)$*, then we have the equality* $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.[1339]

Proof $(*)$.

(1) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 80 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. It follows from Proposition 28.1 (2) and from considering the definition of ρ that if ρ is the reflection in the $(x_2 = 0)$-hyperplane, then $-[f] = [f \circ \rho]$. If ψ is the reflection in some other hyperplane, then it follows from Lemma 9.4 that ψ and ρ are homotopic as maps $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, S^{n-1})$. But this observation implies $[f \circ \psi] = [f \circ \rho]$ and thus we see that $[f \circ \psi] = -[f]$.
(2) This statement follows easily from statement (1) and from considering the map $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ from page 99 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. □

To consider the group structure on the higher homotopy groups we need the following new and fairly harmless definition.

Definition. Let $n \in \mathbb{N}$ and let M be an n-dimensional topological manifold.

(1) We say topological embeddings[1340] $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ are (*weakly*) *disjoint* if the images (of the interiors) do not intersect.
(2) Let X be a topological space and let $x_0 \in X$. Given weakly disjoint topological embeddings $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ and maps $f_1, \dots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ we define[1341]
$$\begin{array}{rcl} \Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)\colon M & \to & X \\ y & \mapsto & \begin{cases} f_i(z), & \text{if } y = \alpha_i(z) \text{ with } z \in \overline{B}^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$
The definition of this map is illustrated in Figure 978.

The following proposition gives a versatile tool for dealing with the group structure.

Proposition 80.2. *Let* (X, x_0) *be a pointed topological space and let* $n \in \mathbb{N}_{\geq 2}$*. Suppose we are given weakly disjoint embeddings*[1342] $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to S^n$ *such that the images of the* B^n *lie in* $S^n \setminus \{*\}$*. Also suppose we are given maps* $f_1, \dots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$*.*

[1339] Note that $\rho(*) = *$, hence $(f \circ \rho)(*) = x_0$.

[1340] We refer to page 1511 for the definition of a topological embedding. As discussed on page 1512, an embedding of a manifold in the sense of the definition on page 330 is indeed a topological embedding.

[1341] It follows from Lemmas 1.37 and 1.18, Proposition 32.2 and Lemma 1.32 (2) that this map is indeed continuous.

[1342] Note that here we work on purpose with "embeddings" instead of "topological embeddings".

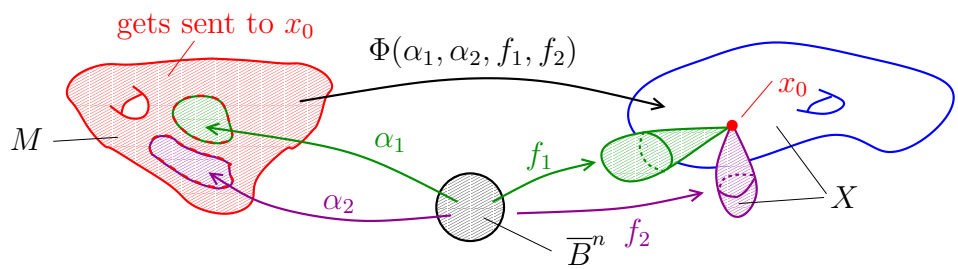


FIGURE 978.

Given $i \in \{1, \dots, m\}$ we write $\epsilon_i = 1$ if α_i is orientation-preserving otherwise we write $\epsilon_i = -1$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)\colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \epsilon_i \cdot \big[\underbrace{f_i\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Example. A slight modification of the map $\overline{B}^n \to S^n$ given on page 99 provides us with an orientation-preserving embedding $\alpha_1\colon \overline{B}^2 \to S^2$ that sends the disk to the "left hemisphere". Similarly we obtain an orientation-preserving embedding $\alpha_2\colon \overline{B}^2 \to S^2$ that sends the disk to the "right hemisphere". It follows from Proposition 80.2 that for any two maps $f_1, f_2\colon (\overline{B}^2, S^1) \to (X, x_0)$ we have

$$[\Phi(\alpha_1, \alpha_2, f_1, f_2)\colon (S^2, *) \to (X, x_0)] = [f_1\colon (\overline{B}^2, S^1) \to (X, x_0)] + [f_2\colon (\overline{B}^2, S^1) \to (X, x_0)].$$

This situation is illustrated in Figure 979. We obtain basically the same picture as we had already obtained on page 716.

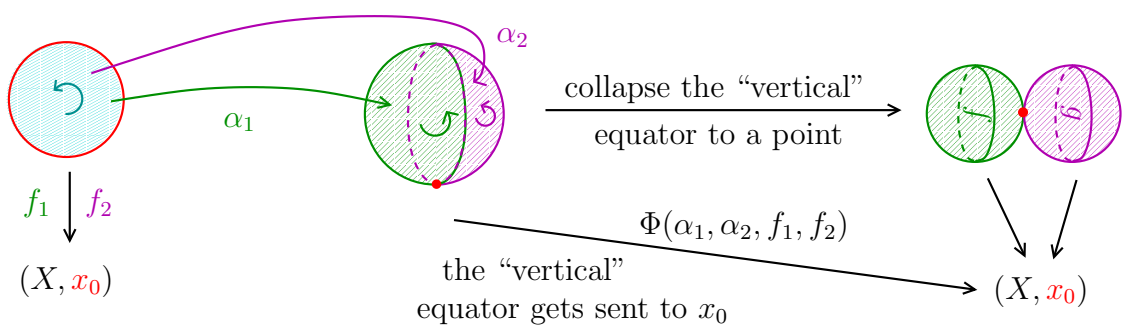


FIGURE 979.

Remark. The statement of Proposition 80.2 also holds for topological embeddings. But in that case, in the proof below, one needs to replace the mainstream Theorem 10.28 by the surprisingly difficult Theorem 71.9. In particular it is not clear to me what goes into the

proof of Theorem 71.9 (does it require the Hurewicz Theorem, are we in danger of using circular logic?). Therefore I prefer to stick to embeddings.[1343]

In the proof of Proposition 80.2 we need the following lemma which basically says the following: if we isotope the topological embeddings α_i, then the resulting Φ-maps are homotopic.

Lemma 80.3. $(*)$ *Let $n \in \mathbb{N}$, let M be an n-dimensional manifold and let Z be a subset of M. Furthermore suppose we are given a topological space X, a base point $x_0 \in X$ and maps $f_1, \dots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Finally suppose that for $i = 1, \dots, m$ we are given homotopies*

$$\Omega_i \colon \overline{B}^n \times [0,1] \ \to \ M \setminus Z$$

such that for each $t \in [0,1]$ the maps $\Omega_{1,t}, \dots, \Omega_{m,t} \colon \overline{B}^n \to M$ are weakly disjoint topological embeddings, then the maps

$$\Phi(\underbrace{\Omega_{1,0}, \dots, \Omega_{m,0}}_{\Omega \text{ at } t=0}, f_1, \dots, f_m), \Phi(\underbrace{\Omega_{1,1}, \dots, \Omega_{m,1}}_{\Omega \text{ at } t=1}, f_1, \dots, f_m) \colon (M, Z) \ \to \ (X, x_0)$$

are homotopic as maps of pairs of topological spaces.

Proof $(*)$. As before it follows from Lemmas 1.37 and 1.18, Proposition 32.2 and Lemma 1.32 (2) that the map

$$\begin{array}{rcl} H \colon M \times [0,1] & \to & X \\ (P,t) & \mapsto & \Phi(\Omega_{1,t}, \dots, \Omega_{m,t}, f_1, \dots, f_m)(P) \end{array}$$

is continuous. This map is then the desired homotopy. □

Sketch of a proof of Proposition 80.2 $(*)$. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Furthermore suppose we are given maps $f_1, \dots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Finally let $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to S^n$ be weakly disjoint embeddings such that the images of the B^n lie in $S^n \setminus \{*\}$. We start out with the following observations:

(1) Let $\rho \colon \overline{B}^n \to \overline{B}^n$ be the reflection in some hyperplane of $\mathbb{R}^n$. We replace every pair (α_i, f_i) for which α_i is orientation-reversing by the pair $(\alpha_i \circ \rho, f_i \circ \rho)$. It follows immediately from Lemma 80.1 that the desired equality holds for the new data if and only if it holds for the old data.

(2) If the α_i are not disjoint we make them disjoint by shrinking. More precisely, we consider the isotopies

$$\begin{array}{rcl} F_i \colon \overline{B}^n \times [0,1] & \to & M \\ (P,t) & \mapsto & \alpha_i(P \cdot (1 - \frac{t}{2})). \end{array}$$

Since the α_i were weakly disjoint we see that the resulting topological embeddings are now disjoint.

[1343] Recall that the absence of the word "topological" implies in our setup that embeddings are smooth maps.

(3) The map $(\overline{B}^n, S^{n-1}) \to (S^n, \{*\})$ that we gave on page 99 restricts to a diffeomorphism $B^n \to S^n \setminus \{*\}$.

Keeping the above three observations in mind we obtain from Lemma 80.3 that it suffices to prove the following statement.

Statement. For any disjoint orientation-preserving embeddings $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to B^n$ we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Perhaps the key observation in the proof is that Theorem 10.28 (where we use our hypothesis that $n \geq 2$ and here we use that work with embeddings not just topological embeddings of balls) together with Lemma 80.3 implies that we only need to prove the statement for a single choice of disjoint orientation-preserving embeddings $\alpha_1, \dots, \alpha_m$. Put differently, it remains to prove the following claim.

Claim. There exist disjoint orientation-preserving embeddings $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to B^n$ such that

$$\big[\underbrace{\Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

By induction it suffices prove the claim for $m = 2$. Even your famously verbose author does not feel like spelling out all the details. We leave it to the reader to piece together the argument using Figure 980 and the following observations:

(a) by Proposition 80.2 (or alternatively Lemma 28.2) we can "move cubes" in I^n,
(b) by Lemma 9.1 the map $\varphi^{-1} \colon \overline{B}^n \to I^n$ is isotopic to the identity,
(c) the restriction of the map $\varphi \colon I^n \to \overline{B}^n$ to each open cone on one of the sides of the cube I^n is smooth.

Using Lemma 80.3 these moves translate into homotopies of the corresponding maps. □

80.2. **The statement of the Hurewicz Theorem and applications.** We recall the following definition from page 932.

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We denote by $[S^n] \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ the fundamental class of the oriented manifold S^n.[1344] We refer to the map

$$\begin{array}{rcl} \Phi_{(X,x_0)} \colon \pi_n(X, x_0) & \to & H_n(X; \mathbb{Z}) \\ [f \colon (S^n, *) \to (X, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

as the *Hurewicz homomorphism.*

[1344]Earlier on we also called it the standard generator, see page 814.

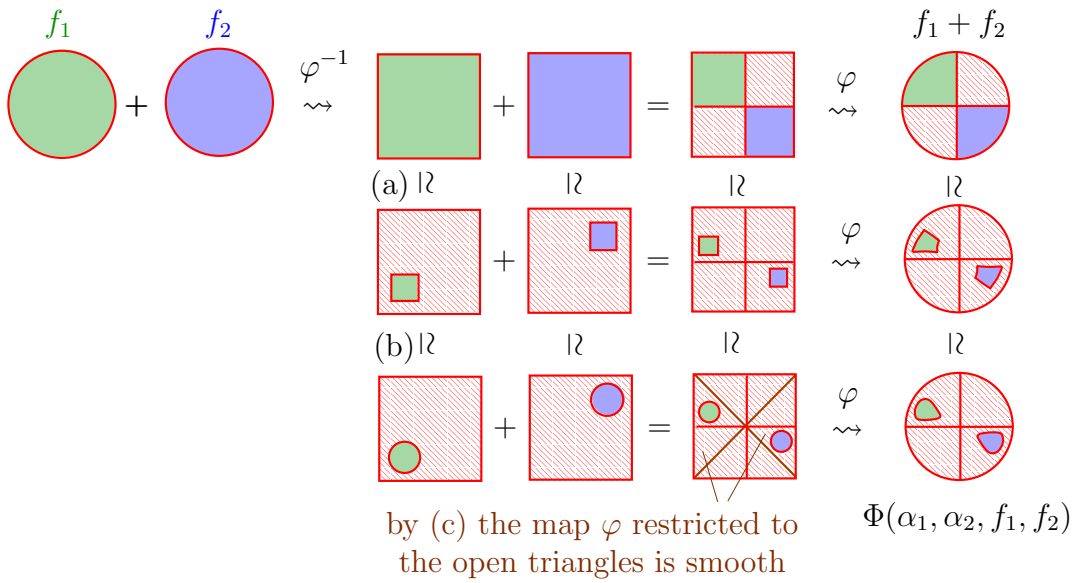


FIGURE 980. Illustration of the proof of Proposition 80.2.

Remark. Let (X, x_0) be a pointed topological space and let $n \geq 1$. If we think of elements of $\pi_n(X, x_0)$ as homotopy classes of maps $(I^n, \partial I^n) \to (X, \{x_0\})$, then it follows easily from the fact that the above diffeomorphism $I^n \setminus \partial I^n \to S^n \setminus \{*\}$ is orientation-preserving, that the Hurewicz homomorphism is also given by the map

$$\begin{array}{rcl} \Phi_{(X,x_0)} \colon \pi_n(X, x_0) & \to & H_n(X; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n) \to (X, x_0)] & \mapsto & f_*([I^n]) \end{array}$$

In the following proposition we collect same basic facts about the Hurewicz homomorphism. In particular we will justify why it is called the Hurewicz *homomorphism*.

Proposition 80.4. *Let $n \in \mathbb{N}$.*

(1) *For any pointed topological space (X, x_0) the Hurewicz homomorphism*

$$\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$$

is a homomorphism.

(2) *The Hurewicz homomorphisms define a natural transformation from n-th homotopy groups to n-th homology groups. In plain English this means that for a map $f \colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces we have a commutative diagram*

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d, "\Phi_{(X,x_0)}"'] & \pi_n(Y, y_0) \arrow[d, "\Phi_{(Y,y_0)}"] \\ H_n(X; \mathbb{Z}) \arrow[r, "f_*"] & H_n(Y; \mathbb{Z}). \end{tikzcd}$$

(3) *If X is path-connected, then the Hurewicz homomorphism induces an isomorphism*[1345]

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \;\to\; H_1(X;\mathbb{Z}).$$

Proof. These statements are precisely the contents of Lemma 38.9 and Hurewicz Theorem 38.5. □

The goal of this chapter is to prove the following theorem which can be viewed as the high-dimensional analogue of Proposition 80.4 (3), i.e. of the original Hurewicz Theorem 38.5.

Theorem 80.5. (Hurewicz Theorem) *Let (X,x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected*[1346]*, then the following two statements hold:*

(1) *we have $\widetilde{H}_k(X;\mathbb{Z}) = 0$ for $k = 0,\dots,n-1$, and*
(2) *the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to H_n(X;\mathbb{Z})$ is an isomorphism.*

Remark. On page 933 we already saw that in general the hypothesis in the Hurewicz Theorem 80.5 that X is $(n-1)$-connected cannot be dropped.

Using the Hurewicz Theorem 80.5 we can now determine some homotopy groups that had been eluding us for a long time.

Corollary 80.6.

(1) *For any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$, in fact $\pi_n(S^n,*) = \mathbb{Z}\cdot[\mathrm{id}_{S^n}]$.*
(2) *For any $n \in \mathbb{N}\cup\{\infty\}$ we have $\pi_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$. In fact, if we make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ from page 113 and if we use the convention from page 111 to view $\mathbb{C}\mathrm{P}^1 = S^2$ as a subset of $\mathbb{C}\mathrm{P}^n$, then $\pi_2(\mathbb{C}\mathrm{P}^n,*) = \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$.*

Proof.

(1) Let $n \in \mathbb{N}$. By Proposition 28.9 we know that S^n is $(n-1)$-connected. Thus the statement follows immediately from the Hurewicz Theorems 38.5 and 80.5 and the fact that $H_n(S^n;\mathbb{Z}) = \mathbb{Z}\cdot[S^n]$.
(2) Let $n \in \mathbb{N}\cup\{\infty\}$. We denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the obvious inclusion. From Proposition 80.4 (2) we obtain the following commutative diagram

$$\begin{tikzcd}
\pi_2(\mathbb{C}\mathrm{P}^1,*) \arrow[r, "i_*"] \arrow[d, "\Phi_{(\mathbb{C}\mathrm{P}^1,*)}"'] & \pi_2(\mathbb{C}\mathrm{P}^n,*) \arrow[d, "\Phi_{(\mathbb{C}\mathrm{P}^n,*)}"] \\
H_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) \arrow[r, "i_*"] & H_2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).
\end{tikzcd}$$

If $n \in \mathbb{N}$, then it follows immediately from Lemma 62.6 that the inclusion induced map $H_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) \to H_2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is an isomorphism. For $n = \infty$ we need to appeal to the discussion on page 885 to draw the same conclusion. Finally, since $[\mathbb{C}\mathrm{P}^1] = [S^2]$ is a generator of $H_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) = H_2(S^2;\mathbb{Z})$ we obtain that $\pi_2(\mathbb{C}\mathrm{P}^n,*) = \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$ from the above commutative diagram. □

[1345] Recall that given a group π we denote by π_{ab} its abelianization, as defined on page 415.

[1346] Recall that by the definition on page 728 this means that $\pi_i(X) = 0$ for $i = 0,\dots,n-1$.

On several occasions we will also need the following straightforward corollary to the Hurewicz Theorem 80.5.

Corollary 80.7. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is simply connected. If given every $k \in \{2, \dots, n\}$ we have $\pi_k(X, x_0) = 0$ or $H_k(X; \mathbb{Z}) = 0$, then the following two statements hold:*

1. *for $k = 1, \dots, n$ we have $\pi_k(X, x_0) = H_k(X; \mathbb{Z}) = 0$,*
2. *the Hurewicz homomorphism $\pi_{n+1}(X, x_0) \to H_{n+1}(X; \mathbb{Z})$ is an isomorphism.*

Proof. We prove the corollary by induction on n. First consider the case $n = 1$. Since X is simply connected we obtain from the Hurewicz Theorem 38.5 that $H_1(X; \mathbb{Z}) = 0$. Furthermore it follows from the Hurewicz Theorem 80.5 that $\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to H_2(X; \mathbb{Z})$ is an isomorphism.

Now suppose the statement holds for $n - 1$ with $n \geq 2$. Suppose that for every $k \in \{2, \dots, n\}$ we have $\pi_k(X) = 0$ or $H_k(X; \mathbb{Z}) = 0$. By induction we know that the homotopy and homology groups vanish up to degree $n - 1$. Thus we can apply the Hurewicz Theorem 80.5 which says that $\pi_n(X)$ and $H_n(X; \mathbb{Z})$ are isomorphic. Since one is zero, both are zero. Hence we can apply the Hurewicz Theorem 80.5 once more to see that the Hurewicz homomorphism $\pi_{n+1}(X, x_0) \to H_{n+1}(X; \mathbb{Z})$ is an isomorphism. □

Example. Let M be the Poincaré homology sphere that we introduced on page 993. Recall that M is a 3-dimensional homology sphere with non-trivial fundamental group. We denote by $\Sigma(M)$ its suspension as defined on page 452. By Lemma 15.3 we know that $\Sigma(M)$ is simply connected. Furthermore we have

$$\widetilde{H}_k(\Sigma(M); \mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 32.14}}}{\cong} \widetilde{H}_{k-1}(M; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{since } M \text{ is a 3-dimensional} \\ \text{homology sphere}}}{\cong} \begin{cases} 0, & \text{if } k \neq 4, \\ \mathbb{Z}, & \text{if } k = 4. \end{cases}$$

With this information we obtain from Corollary 80.7 that $\pi_2(\Sigma(M)) = \pi_3(\Sigma(M)) = 0$ and that $\pi_4(\Sigma(M)) \cong \mathbb{Z}$. As a side remark we point out that we saw in Exercise 28.7 that $\Sigma(M)$ is not a topological manifold.

We recall the following definition from page 728.

Definition. Let $n \in \mathbb{N}$. An n-dimensional topological manifold X that is closed and that is $\lfloor \frac{n}{2} \rfloor$-connected is called an *n-dimensional homotopy sphere.*

Example. A closed 3-dimensional topological manifold is by definition a 3-dimensional homotopy sphere if it is 1-connected, i.e. if it is simply connected.

We have now the following characterizations of n-dimensional homotopy spheres.

Proposition 80.8. *Let $n \geq 2$. If X is a closed connected n-dimensional topological manifold, then the following statements are equivalent:*

1. *X is a homotopy sphere,*

(2) *X is $(n-1)$-connected,*
(3) *$\pi_1(X)$ is trivial and $H_i(X;\mathbb{Z}) = 0$ for $i = 2, \ldots, n-1$,*
(4) *$\pi_1(X)$ is trivial and $H_i(X;\mathbb{Z}) = 0$ for $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$.*

Remark. The reader might wonder why we defined a homotopy sphere the way we did, after all the characterization of Proposition 80.8 (2) seems much more natural. The purely didactical reason is that, as pointed out in the example before the proposition, in the 3-dimensional setting we only have to take the fundamental group into account and not the second homotopy group.

Proof. We have (1) $\Rightarrow$ (4) by Corollary 80.7. We can easily show that (4) $\Rightarrow$ (3) by applying the Poincaré Duality Theorem 61.1 (here we use that X is orientable by Corollary 45.27) and the Universal Coefficient Theorem 51.13 for Cohomology Groups.[1347] The conclusion (3) $\Rightarrow$ (2) is again Corollary 80.7 and finally (2) $\Rightarrow$ (1) is trivial. $\square$

We conclude this section with the following proposition.

Proposition 80.9. *Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. Then the following equality holds in $\pi_n(S^n, *)$:*

$$[f] \;=\; \begin{cases} \;\;[\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-preserving,} \\ -[\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Remark. Let $n \in \mathbb{N}$. Proposition 80.9 says that every orientation-preserving self-homeomorphism f of S^n with $f(*) = *$ is *homotopic*, rel the base point $*$, to the identity. It is not clear to me whether f is *isotopic*, rel the base point $*$, to the identity. Put differently, can one find a homotopy F from f to id such that each $F_t\colon S^n \to S^n$ is also a homeomorphism?

Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. By the naturality of the Hurewicz homomorphism, see Proposition 80.4 (2), we have the following commutative diagram

$$\begin{tikzcd} \pi_n(S^n, *) \arrow[r, "f_*"] \arrow[d, "\Phi_{(S^n,*)}"'] & \pi_n(S^n, *) \arrow[d, "\Phi_{(S^n,*)}"] \\ H_n(S^n;\mathbb{Z}) \arrow[r, "f_*"] & H_n(S^n;\mathbb{Z}). \end{tikzcd}$$

By Proposition 28.9 we know that S^n is $(n-1)$-connected. Therefore the vertical maps are isomorphisms by the Hurewicz Theorems 38.5 and 80.5. The corollary now follows from Proposition 47.3 and the simple observation that $[f] = f_*([\mathrm{id}_{S^n}]) \in \pi_n(S^n, *)$. $\square$

Now that we are fully convinced of the value of the Hurewicz Theorem 80.5 we turn to the proof thereof. We will discuss more applications in the next chapter.

[1347] Note that the proof of (4) $\Rightarrow$ (3) is almost identical to the proof of Lemma 74.1.

80.3. **The proof of the Hurewicz Theorem 80.5.** The proof of the Hurewicz Theorem 80.5 requires some preparations.

Convention. Let $k \in \mathbb{N}_0$. We view Δ^k as a CW-complex in the following way. Given $m \in \{0, \dots, k\}$ the m-cells correspond precisely to all the maps

$$\begin{array}{rcl} \overline{B}^m = \Delta^m & \to & \Delta^k \\ (t_0, \dots, t_m) & \mapsto & \sum\limits_{i=0}^{m} t_i e_{\varphi(i)} \end{array}$$

where φ is a strictly increasing map $\varphi\colon \{0, \dots, m\} \to \{0, \dots, k\}$. Note that the $(k-1)$-skeleton of Δ^k is given by $\partial\Delta^k$.

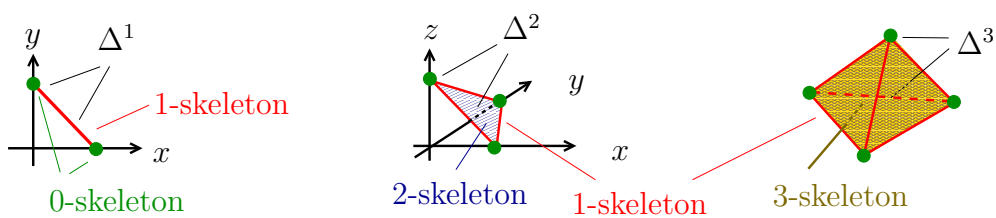


FIGURE 981.

Definition. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we define[1348]

$$C_k^{(n)}(X, x_0) \ := \ \begin{array}{c} \text{the subgroup of the relative chain group } C_k(X, \{x_0\}) \text{ generated} \\ \text{by all singular simplices } \sigma\colon \Delta^k \to X \text{ with } \sigma(n\text{-skeleton of } \Delta^k) \subset \{x_0\}. \end{array}$$

These groups define a subcomplex of $C_*(X, \{x_0\})$. In particular we obtain corresponding homology groups $H_k^{(n)}(X, x_0)$ and the inclusion $C_*^{(n)}(X, x_0) \to C_*(X, \{x_0\})$ defines natural maps $H_k^{(n)}(X, x_0) \to H_k(X, \{x_0\})$.

Example. In Figure 982 we show a pointed topological space (X, x_0) and we consider an element $\sigma \in C_1^{(0)}(X, x_0)$ and an element $\tau \in C_2^{(1)}(X, x_0)$. From the pictures it becomes apparent that elements in $C_n^{(n-1)}(X, x_0)$ are "close" to elements in $\pi_n(X, x_0)$. In particular it seems reasonable to expect that $H_n^{(n-1)}(X, x_0)$ is a group that "sits between" $H_n(X; \mathbb{Z})$ and $\pi_n(X, x_0)$.

In the following lemma we summarize a few basic facts about these new chain groups.

Lemma 80.10. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Furthermore let $k \in \{0, \dots, n\}$. The following statements hold:*

(1) *We have $C_k^{(n)}(X, x_0) = 0$,*
(2) *we have $H_k^{(n)}(X, x_0) = 0$,*

[1348]Note that the notation $C_k^{(n)}(X, x_0)$ instead of $C_k^{(n)}(X, \{x_0\})$ is on purpose.

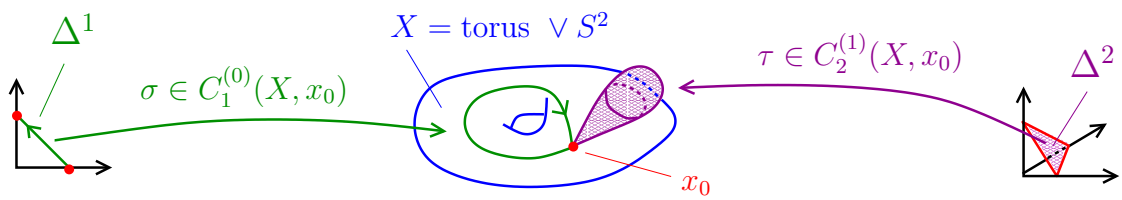


FIGURE 982.

(3) *any element in* $C_{k+1}^{(n)}(X, x_0)$ *is a cycle.*

Proof. Let $k \in \{0, \ldots, n\}$. Put differently, we have k with $n \geq k$. Note that this implies that Δ^k is the n-skeleton of Δ^k. In particular, by definition, $C_k^{(n)}(X, x_0)$ is the subgroup of $C_k(X, \{x_0\})$ generated by the constant maps $\Delta^k \to \{x_0\}$, but these are zero in the relative chain group $C_k(X, \{x_0\})$. This concludes the proof of (1). The remaining two statements are immediate consequences of (1). □

The following lemma is one of the key ingredients in the proof of the Hurewicz Theorem 80.5

Lemma 80.11. *Let* (X, x_0) *be a pointed topological space and let* $n \in \mathbb{N}$. *If* X *is* n*-connected, then the inclusion map*

$$C_*^{(n)}(X, x_0) \quad \to \quad C_*(X, \{x_0\})$$

is a chain homotopy equivalence. In particular for any $k \in \mathbb{N}_0$ *the induced map*

$$H_k^{(n)}(X, x_0) \quad \to \quad H_k(X, \{x_0\})$$

is an isomorphism.

Proof. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is n-connected. We recall the following notation from Lemma 30.6: given $t = 0$ and $t = 1$ and $k \in \mathbb{N}_0$ we denote by

$$\begin{aligned} \eta_t \colon \Delta^k &\to \Delta^k \times [0, 1] \\ x &\mapsto (x, t) \end{aligned}$$

the obvious inclusion map. We start out with the following claim which basically says that any singular simplices $\Delta^k \to X$ can be homotoped consistently to a singular simplices that lie in $C_k^{(n)}(X, x_0)$.

Claim. Given any $k \in \mathbb{N}_0$ and given any singular simplex $\sigma \colon \Delta^k \to X$ we can choose a map $P(\sigma) \colon \Delta^k \times [0, 1] \to X$ such that the following four statements are satisfied:

(a) we have $P(\sigma) \circ \eta_0 = \sigma \colon \Delta^k \to X$,
(b) the singular k-simplex $P(\sigma) \circ \eta_1 \colon \Delta^k \to X$ lies in $C_k^{(n)}(X, x_0)$,
(c) if $\sigma \in C_k^{(n)}(X, x_0)$, then $P(\sigma) = \sigma \circ \pi$ where $\pi \colon \Delta^k \times [0, 1] \to \Delta^k$ is the projection map, and

(d) for any $j \in \{0, \dots, k\}$ we have

$$P(\sigma) \circ \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})}_{\Delta^{k-1}\times[0,1]\to\Delta^k\times[0,1]} = P(\underbrace{\sigma \circ i_j^k}_{\Delta^{k-1}\to X}) \colon \Delta^{k-1} \times [0,1] \to X.$$

We turn to the proof of the claim. Property (d) already suggests that we should prove the claim by induction on k. It is convenient to view $k = -1$ as the (trivial) base case. Now let $k \in \mathbb{N}_0$ and suppose that we have proved the claim for all singular i-simplices with $i < k$. Now let $\sigma \colon \Delta^k \to X$ be a singular simplex. If $\sigma \in C_k^{(n)}(X, x_0)$, then we set $P(\sigma) = \sigma \circ \pi$. In this case it is straightforward to verify that the conditions (a), (b) and (d) are satisfied.

Now consider the case that $\sigma \notin C_k^{(n)}(X, x_0)$. We distinguish two cases.

(1) First we consider the case that $k \le n$. By induction the maps $P(\sigma \circ i_j^k)$ have been defined for all j. We set $\partial(\Delta^k \times [0,1]) := (\partial\Delta^k \times [0,1]) \cup (\Delta^k \times \{0,1\})$ and we consider the map

$$Q \colon \partial(\Delta^k \times [0,1]) \to X$$

which is defined via σ on $\Delta^k \times \{0\}$, via the maps $P(\sigma \circ i_j^k)$ on $\partial\Delta^k \times [0,1]$[1349] and which sends all points in $\Delta^k \times \{1\}$ to x_0.[1350] (We refer to Figure 983 for an illustration of this map.) It follows almost immediately from Proposition 1.47 (2) that the pair $(\Delta^k \times [0,1], \partial(\Delta^k \times [0,1]))$ is homeomorphic to $(\overline{B}^{k+1}, S^k)$. Since $k \le n$ and since X is n-connected we obtain from Lemma 28.4 that there exists an extension of Q to a map $\Delta^k \times [0,1] \to X$. We define $P(\sigma)$ to be any such extension. We obtain a map $P(\sigma)$ which is easily seen to have all the desired properties.

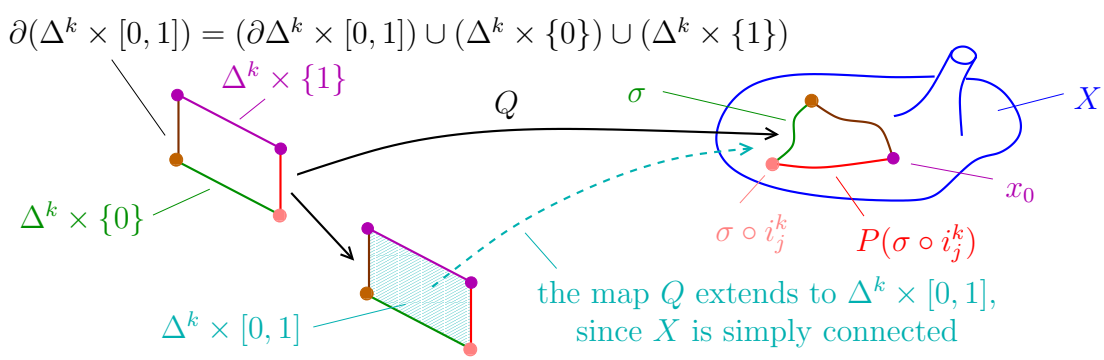


FIGURE 983.

(2) Now we consider the case that $k > n$. Note that in this case condition (b) is satisfied "for free" by any map $\Delta^k \times [0,1] \to X$ that satisfies (d). Thus we only have to find

[1349] Here we use that we assume that the maps $P(\mu)$ are defined for simplices of dimension $< k$.

[1350] It follows from Lemma 1.32 (2') that this map is indeed continuous.

a map $P(\sigma)$ that satisfies (a), (c) and (d). This is much easier than in (1). Indeed, similar to (1) we consider the map

$$G\colon (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times [0,1]) \;\to\; X$$

that is defined via σ on $\Delta^k \times \{0\}$ and via the maps $P(\sigma \circ i_j^k)$ on $\partial\Delta^k \times [0,1]$. (We refer to Figure 984 for an illustration.) Note that by Lemma 27.6 (together with the homeomorphism $\overline{B}^k \cong \Delta^k$) there exists a retraction r from $\Delta^k \times [0,1]$ to the domain of G. Thus combining this retraction r with Q we obtain an extension of Q to map $\Delta^k \times [0,1] \to X$. We define $P(\sigma)$ to be this extension. It follows basically by construction that all the statements are satisfied.

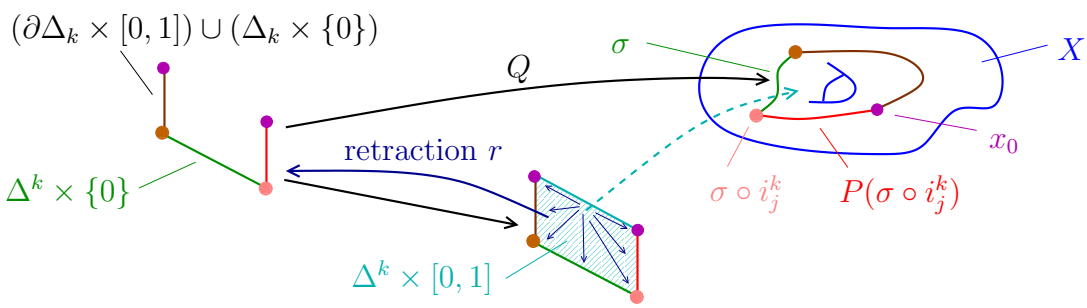


FIGURE 984.

This concludes the proof of the claim.

Now we turn to the actual proof of the statement that the inclusion map[1351]

$$\iota\colon C_*^{(n)}(X, x_0) \;\to\; C_*(X, \{x_0\})$$

is a chain homotopy equivalence. To do so we consider the map

$$\phi\colon C_*(X, \{x_0\}) \;\to\; C_*^{(n)}(X, x_0)$$

that is determined by

$$(\sigma\colon \Delta^n \to X) \;\mapsto\; \phi(\sigma) := \underbrace{P(\sigma) \circ \eta_1}_{\Delta^k \to \Delta^k \times [0,1] \to X}.$$

It follows from (b) that ϕ does indeed take values in $C_*^{(n)}(X, x_0)$ and it follows easily from (d) that ϕ is in fact a chain map. We want to show that ϕ is a chain homotopy inverse to ι. By (c) we have $\phi \circ \iota = \mathrm{id}$. Thus it remains to prove the following claim.

Claim. There exists a chain homotopy from $\iota \circ \phi$ to the identity of $C_*(X, \{x_0\})$.

[1351]Here and throughout the proof we do not distinguish in the notation between a singular chain in $C_k(X)$ and the element it represents in $C_k(X, \{x_0\})$.

Let $k \in \mathbb{N}_0$. We recall that on page 756 we introduced the singular $(k+1)$-chain

$$\Omega_k \;:=\; \sum_{j=0}^{k} (-1)^j \cdot \underbrace{[v_0, \dots, v_j, w_j, \dots, w_k]}_{\Delta^{k+1} \to \Delta^k \times [0,1]} \;\in\; C_{k+1}(\Delta^k \times [0,1])$$

where $v_0, \dots, v_k, w_0, \dots, w_k$ denote the vertices of $\Delta^k \times [0,1]$, see page 755 for details. The key property of these singular chains is that Lemma 30.6 says that for any $k \in \mathbb{N}_0$ we have

$$(*) \qquad \partial_{k+1}(\Omega_k) + \sum_{j=0}^{k} (-1)^j \cdot \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})_*}_{C_*(\Delta^{k-1} \times [0,1]) \to C_*(\Delta^k \times [0,1])} (\Omega_{k-1}) \;=\; \eta_1 - \eta_0 \;\in\; C_k(\Delta^k \times [0,1]).$$

For any $k \in \mathbb{N}_0$ we consider the map

$$D_k \colon C_k(X, \{x_0\}) \;\to\; C_{k+1}(X, \{x_0\})$$

that is determined by

$$(\sigma \colon \Delta^k \to X) \;\mapsto\; \underbrace{P(\,\sigma\,)_*}_{C_*(\Delta^k \times [0,1]) \to C_*(X, \{x_0\})} \;(\; \underbrace{\Omega_k}_{\in C_{k+1}(\Delta^k \times [0,1])}\;).$$

We claim that these maps define a chain homotopy from $\iota \circ \phi$ to the identity of $C_*(X, \{x_0\})$. Thus let $\sigma \colon \Delta^k \to X$ be a singular k-simplex. We need to show that $\partial D\sigma + D\partial\sigma = \phi(\sigma) - \sigma$. Indeed we have

$$\begin{aligned}
\partial_{k+1}(D_k(\sigma)) + D_{k-1}(\partial_k(\sigma)) \;&=\; \partial_{k+1}(D_k(\sigma)) + D_{k-1}\Big(\sum_{j=0}^{k} (-1)^j \cdot \sigma \circ i_j^k\Big) \\
&=\; \partial_{k+1}(P(\sigma)_*(\Omega_k)) + \sum_{j=0}^{k} (-1)^j \cdot P(\sigma \circ i_j^k)_*(\Omega_{k-1}) \\
&=\; P(\sigma)_*\Big(\partial_{k+1}(\Omega_k) + \sum_{j=0}^{k} (-1)^j \cdot (i_j^k \times \mathrm{id}_{[0,1]})_*(\Omega_{k-1})\Big) \\
&\quad\uparrow \\
&\quad \text{in the first summand we used that } P(\sigma)_* \text{ is a chain map and} \\
&\quad \text{in the second summand we used that by (d) we have } P(i_j^k \circ \sigma) = P(\sigma) \circ (i_j^k \times \mathrm{id}_{[0,1]}) \\
&=\; P(\sigma)_*(\eta_1 - \eta_0) \;=\; P(\sigma) \circ \eta_1 - P(\sigma) \circ \eta_0 \;=\; \phi(\sigma) - \sigma. \\
&\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
&\quad \text{by the above equality } (*) \qquad\qquad\qquad \text{by (a) and definition of } \phi
\end{aligned}$$

This concludes the proof of the claim. □

Before we can continue with the proof of the Hurewicz Theorem 80.5 we need to introduce another, slightly different point of view, regarding higher homotopy groups.

So let $n \in \mathbb{N}$. We consider the homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ that is given by the composition of the following three maps:[1352]

$$\begin{aligned} \eta_n \colon \Delta^n &\xrightarrow{\cong} \left\{ (x_1, \dots, x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^n x_i \in [0,1] \right\} \xrightarrow[\text{with } Q = (\frac{1}{n+1}, \dots, \frac{1}{n+1})]{\text{Proposition 1.48 (2)}} \overline{B}^n \xrightarrow[\text{by } A_n]{\text{multiplication}} \overline{B}^n. \\ (t_0, \dots, t_n) &\mapsto (t_0, \dots, t_{n-1}) \end{aligned}$$

where $A_n \in \mathrm{O}(n)$ is a matrix which is chosen in such a way that we have the equality $\eta_n((1,0,\dots,0)) = * := (-1,0,\dots,0)$.[13531354] We sketch the definition of η_n in Figure 985.

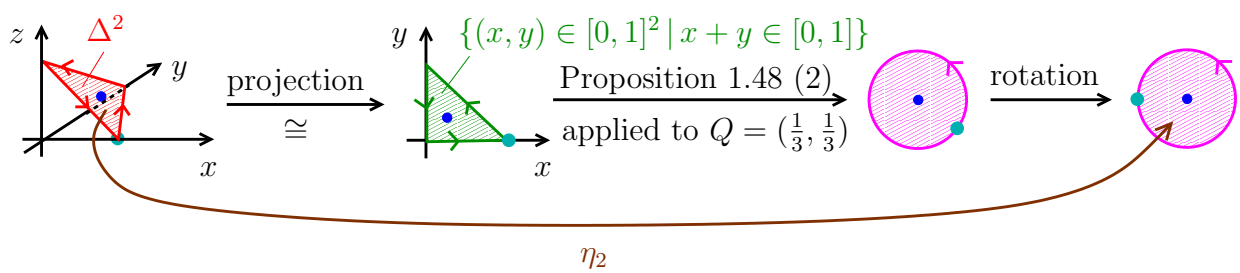


FIGURE 985.

We use the homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ to identify these two pairs of topological spaces. In particular we use it to make the identification $\partial\Delta^n = S^{n-1}$. By a slight abuse of notation we write write $* := (1,0,\dots,0) \in \Delta^n$. In particular for any $n \in \mathbb{N}_0$ we can extend the identifications from page 1847 to obtain the following identifications

$$\begin{array}{ccccc} \pi_n(X, x_0) & = & \begin{array}{c} \text{set (of homotopy classes)} \\ \text{of maps } (I^n, \partial I^n) \to (X, x_0) \\ \| \\ \text{set (of homotopy classes)} \\ \text{of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \| \\ \text{set (of homotopy classes)} \\ \text{of maps } (\Delta^n, \partial\Delta^n) \to (X, x_0). \end{array} & = & \begin{array}{c} \text{set (of homotopy classes)} \\ \text{of maps } (S^n, *) \to (X, x_0) \\ \| \\ \text{set (of homotopy classes)} \\ \text{of maps } (\partial\Delta^{n+1}, *) \to (X, x_0) \end{array} \end{array}$$

As above we will use these identifications to pick at each given moment our most convenient point of view.

As an illustration, under the above identifications Lemma 28.4 translates into the following lemma.

[1352]Note that the composition of the first two homeomorphisms is precisely the homeomorphism from Lemma 29.1.

[1353]It follows from Lemma 1.73 (3) that such a matrix exists.

[1354]The map η_n depends on the choice of the matrix A_n, this will not affect us at all.

Lemma 80.12. *Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f\colon (\partial\Delta^{n+1}, *) \to (X, x_0)$ be a map. Then f represents the trivial element in $\pi_n(X, x_0)$ if and only if there exists a map $F\colon \Delta^{n+1} \to X$ so that $F|_{\partial\Delta^{n+1}} = f$.*

The following lemma is the other key technical ingredient in the proof of the Hurewicz Theorem 80.5.

Lemma 80.13. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma\colon \partial\Delta^{n+1} \to X$ be a map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Then*[1355][1356]

$$[\sigma\colon (\partial\Delta^{n+1}, *) \to (X, x_0)] \;=\; \sum_{j=0}^{n+1} (-1)^j \cdot [\sigma \circ i_j^{n+1}\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)] \;\in\; \pi_n(X, x_0).$$

We refer to Figure 986 for an illustration for $n = 1$, even though this case is strictly speaking not allowed.

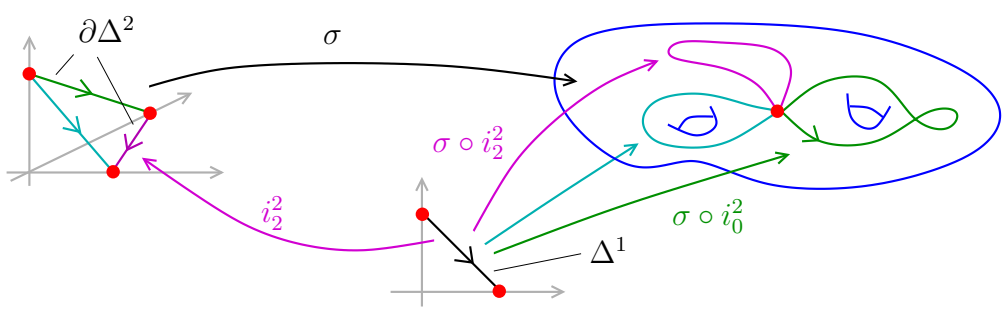


FIGURE 986. Illustration of Lemma 80.13.

Proof $(*)$. The reader might already have noticed that the statement of the lemma resembles the statement of Proposition 80.2. Indeed, after some massaging we will be able to deduce the desired statement from Proposition 80.2.

Given any $k \in \mathbb{N}_0$, let $\eta_k\colon (\Delta^k, \partial\Delta^k) \to (\overline{B}^k, S^{k-1})$ the homeomorphism from page 1861. Using our identifications the desired statement is now equivalent to the following statement.

Statement. Given $j \in \{0, \dots, n+1\}$ we write $\alpha_i = \eta_{n+1} \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to S^n = \partial\overline{B}^{n+1}$ and furthermore we write $f_j = \sigma \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to X$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_0, \dots, \alpha_n, f_0, \dots, f_n)\colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot \big[\underbrace{f_j\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

[1355]Note that our hypothesis that σ sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0 implies in particular that for all $j \in \{0, \dots, n+1\}$ we have $(\sigma \circ i_j^{n+1})(\partial\Delta^n) = \{x_0\}$.

[1356]Also note that by our base point $*$ is chosen in such a way that it lies on the $(n-1)$-skeleton of $\partial\Delta^{n+1}$.

We refer to Figure 987 for an illustration.

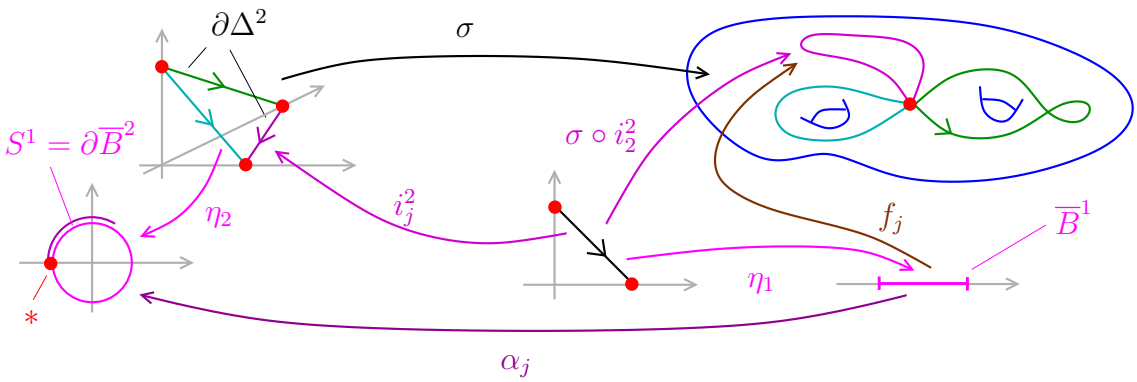


FIGURE 987. Illustration of the proof of Lemma 80.13.

First note that it follows from the discussion on page 1089 that the maps η_k is orientation-preserving if and only if k is even. Furthermore it is fairly straightforward to see that the maps i_j^{n+1} are orientation-preserving if and only if j is even. In summary, the topological embedding α_j is orientation-preserving if and only if j is odd.[1357]

The only reason why we cannot apply Proposition 80.2 immediately is that the maps α_j are topological embeddings but they are not smooth, so they are not embeddings. Keeping Lemma 80.1 in mind we see that it suffices to prove the following claim.

Claim. The maps $\alpha_0, \ldots, \alpha_{n+1} \colon \overline{B}^n \to S^n$ are isotopic, through weakly disjoint topological embeddings, to embeddings $\beta_0, \ldots, \beta_{n+1}$.

For $j = 0, \ldots, n+1$ we write $F_j = i_j^{n+1}(\mathring{\Delta}^n)$ and we write $\overline{F}_j = i_j^{n+1}(\Delta^n)$. Note that F_j is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, we endow it with the usual orientation where the "normal vector sticks outward". Now this claim can be proved using the following observations:

(1) similar to the discussion on page 1850 we can shrink the weakly disjoint topological embeddings $\alpha_j \colon \overline{B}^n \to \overline{F}_j$ to disjoint topological embeddings $\beta_j \colon \overline{B}^n \to F_j$,
(2) using Lemma 9.1 we can isotope these topological embeddings $\beta_j \colon \overline{B}^n \to F_j$ to embeddings[1358] $\gamma_i \colon \overline{B}^n \to F_j$
(3) for each $j \in \{0, \ldots, n+1\}$ the restriction of $\eta_{n+1} \colon \partial\Delta^{n+1} \to S^n$ to the map $\eta_{n+1} \colon F_j \to S^n$ is smooth and orientation-preserving.

Together with Figure 988 the reader should have no troubles assembling a full proof of the claim and the lemma using the above observations. □

Now we can turn to the proof of the Hurewicz Theorem 80.5.

[1357]Here we used the deep fact that precisely one out of the numbers n, $n+1$ is odd.
[1358]Recall that an embedding is by definition smooth.

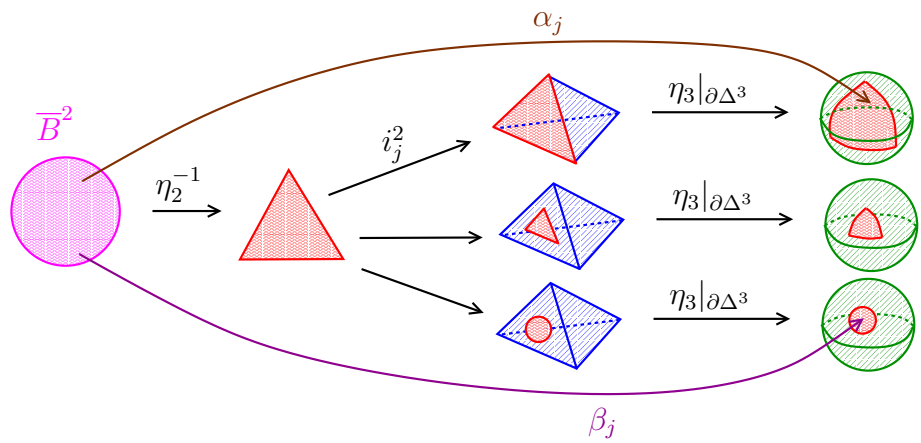


FIGURE 988. Illustration of the proof of Lemma 80.13.

Proof of the Hurewicz Theorem 80.5. Let $n \in \mathbb{N}_{\geq 2}$ and let (X, x_0) be a pointed topological space. We suppose that X is $(n-1)$-connected.

Statement (1) of the theorem can be proved easily. Indeed, let $k \in \{0, \dots, n-1\}$. We have

$$\begin{array}{ccccccc} \widetilde{H}_k(X;\mathbb{Z}) & \cong & H_k(X, \{x_0\};\mathbb{Z}) & = & H_k^{(n-1)}(X, x_0) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 31.12} & & \begin{array}{c}\text{by Lemma 80.11 since} \\ X \text{ is } (n-1)\text{-connected}\end{array} & & \begin{array}{c}\text{by Lemma 80.10 (2)} \\ \text{since } k \leq n-1\end{array} & \end{array}$$

We turn to the much more interesting proof of statement (2). As a reminder, we want to show that the map

$$\pi_n(X, x_0) \xrightarrow{\Phi_{(X,x_0)}} H_n(X;\mathbb{Z})$$

is an isomorphism. The goal again is to bring the homology groups $H_n^{(n-1)}(X, x_0)$ into the game. This leads us to the following claim.

Claim.

(1) The map

$$\begin{array}{rcl} \Psi_{(X,x_0)} \colon \pi_n(X, x_0) & \to & H_n^{(n-1)}(X, x_0) \\ [f \colon (\Delta^n, \partial\Delta^n) \to (X, x_0)] & \mapsto & [\underbrace{f \colon \Delta^n \to X}_{\in C_n^{(n-1)}(X,x_0)}] \end{array}$$

is well-defined and it is a homomorphism.

(2) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{tikzcd}
\pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}", "{[f]\mapsto f_*([S^n])}"'] \arrow[d, "\Psi_{(X,x_0)}"'] & H_n(X;\mathbb{Z}) \arrow[d, "\cong"] \\
H_n^{(n-1)}(X,x_0) \arrow[r, "\cong", "\iota"'] & H_n(X,\{x_0\};\mathbb{Z})
\end{tikzcd}$$

where ι is induced by the inclusion $C_*^{(n-1)}(X,x_0) \to C_n(X,\{x_0\})$.

(3) The right-vertical and the bottom-horizontal maps of the diagram in (2) are isomorphisms.

We prove the claim using a somewhat convoluted logic.[1359] We consider the following diagram

$$\begin{tikzcd}
\{\text{set of maps } (\Delta^n,\partial\Delta^n) \to (X,x_0)\} \arrow[rr, "f\mapsto f_*([\Delta^n])"] \arrow[dr, two heads] \arrow[dd, "f\mapsto[f]"'] & & H_n(X;\mathbb{Z}) \arrow[dd, "\cong"] \\
& \pi_n(X,x_0) \arrow[ur, "\Phi_{(X,x_0)}"'] \arrow[dl, dashed, "\Psi_{(X,x_0)}"'] & \\
H_n^{(n-1)}(X,x_0) \arrow[rr, "\cong", "\iota"'] & & H_n(X,\{x_0\};\mathbb{Z})
\end{tikzcd}$$

We make the following observations:

(a) It follows from $n \geq 2$ and from the long exact sequence of homology groups of the pair $(X,\{x_0\})$, see Corollary 31.14, that the right-hand vertical map of both diagrams is an isomorphism.

(b) It follows from Lemma 80.11 and our hypothesis that X is $(n-1)$-connected that the bottom horizontal map ι of both diagrams is an isomorphism.

(c) Let $f\colon (\Delta^n,\partial\Delta^n) \to (X,x_0)$ be a map. The $(n-1)$-skeleton of Δ^n equals $\partial\Delta^n$, thus the map $f\colon \Delta^n \to X$ does indeed define an element in $C_n^{(n-1)}(X,x_0)$. Furthermore by Lemma 80.10 (3) we know that every element in $C_n^{(n-1)}(X,x_0)$ is a cycle, thus f defines an element in $H_n^{(n-1)}(X,x_0)$. This shows that in the above diagram the left vertical map is well-defined.

(d) Let $f\colon (\Delta^n,\partial\Delta^n) \to (X,x_0)$ be a map. It follows from Lemma 32.12 that going down-right and going right-down leads to the same element in $H_n(X,\{x_0\};\mathbb{Z})$, up to the sign $(-1)^n$. Thus in the above diagram the rectangle commutes up to sign.

(e) Evidently the top small triangle in the above diagram commutes.

(f) We want to show that $\Psi_{(X,x_0)}$ is well-defined and that it is a homomorphism. By (b) we know that ι is an isomorphism. Thus it suffices to show these statements for $\iota \circ \Psi_{(X,x_0)}$. But these now follow from (d) and (e) and the fact, see Proposition 80.4 (1), that the Hurewicz homomorphism $\Phi_{(X,x_0)}$ is well-defined and a homomorphism.

[1359] The convoluted logic is forced upon us since proving (1) as a stand-alone statement is actually slightly tricky.

This concludes the proof of the claim.

By the claim it suffices to show that the homomorphism

$$\begin{array}{rcl} \Psi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & H_n^{(n-1)}(X,x_0) \\ [f\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)] & \mapsto & [f\colon \Delta^n\to X] \end{array}$$

is in fact an isomorphism. We will do so by defining an explicit inverse.

Claim.

(1) The map

$$\Xi_{(X,x_0)}\colon C_n^{(n-1)}(X,x_0) \ \to\ \pi_n(X,x_0)$$

that is induced by

$$(\sigma\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)) \ \mapsto\ [\sigma\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)]$$

is well-defined.

(2) The map $\Xi_{(X,x_0)}$ from (1) descends to a well-defined map

$$H_n^{(n-1)}(X,x_0) \ \to\ \pi_n(X,x_0).$$

(3) The map $\Xi_{(X,x_0)}$ is an inverse to $\Psi_{(X,x_0)}$.

Since the end is near we set out with renewed vigor to prove the claim.

(1) Recall that we know from Proposition 28.3 and our hypothesis $n\geq 2$ that the group on the right is abelian. Therefore it follows from Lemma 11.1 that the map is well-defined.[1360]

(2) Let $\sigma\colon \Delta^{n+1}\to X$ be a singular $(n+1)$-simplex that defines an element in the group $C_{n+1}^{(n-1)}(X,x_0)$. We need to show that $\Xi_{(X,X_0)}(\partial\sigma)=0$. We calculate that we have the following equality in $\pi_n(X,x_0)$:

$$\Xi_{(X,X_0)}(\partial\sigma)=\Xi_{(X,X_0)}\Big(\sum_{j=0}^{n+1}(-1)^j\cdot\sigma\circ i_j^n\Big)\underset{\substack{\uparrow\\ \text{by definition of }\Xi_{(X,x_0)}}}{=}\sum_{j=0}^{n+1}(-1)^j\cdot[\overbrace{\sigma\circ i_j^{n+1}}^{(\Delta^n,\partial\Delta^n)\to(X,x_0)}]\underset{\substack{\uparrow\\ \text{Lemma 80.13}}}{=}-[\overbrace{\sigma|_{\partial\Delta^{n+1}}}^{(\partial\Delta^{n+1},*)\to(X,x_0)}]\underset{\substack{\uparrow\\ \text{Lemma 80.12}}}{=}0.$$

(3) It follows immediately from the definitions that $\Xi_{(X,x_0)}\circ\Psi_{(X,x_0)}$ is the identity on $\pi_n(X,x_0)$. Furthermore it follows easily from Lemma 80.10 and the definitions that $\Psi_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is the identity on a generating set for $H_n^{(n-1)}(X,x_0)$. But by the claim and Proposition 80.4 (1) we know that $\Psi_{(X,x_0)}$ and $\Xi_{(X,x_0)}$ are both homomorphisms. Thus we see that $\Psi_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is actually the identity on all of $H_n^{(n-1)}(X,x_0)$.

This concludes the proof of the claim and thus we are done with the proof of the Hurewicz Theorem 80.5. □

[1360]Put differently, it would be a serious dereliction of duty to use the additive notation for a group structure, if the group was not known to be additive.

Remark. The most awkward aspect of the proof of the Hurewicz Theorem 80.5 is that one needs to go back and forth between the different points of view regarding elements in higher homotopy groups, in particular the fact that we need Proposition 80.2 and Lemma 80.13 is arguably aesthetically somewhat unpleasant.

In fact there is, in principle, an elegant way around this problem: One could prove the Hurewicz Theorem 80.5 using "cubical singular homology" instead of the usual singular homology. As the name suggests, in this theory the "building blocks" are singular cubes, i.e. maps from $[0,1]^n$ to a topological space X. (This theory is developed in [Mas2, Chapter VII.2].) Since cubes are also used in the definition of the product structure on higher homotopy groups I would expect that many of the technical issues we had to deal with vanish magically.

The downside is that one still needs to show that "cubical singular homology" is naturally isomorphic to the usual singular homology. This can be shown using the Acyclic Model Theorem 56.16.

Exercises for Chapter 80. In the following two exercises we need the following definition.

Definition. A topological space X is called *aspherical* if it is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

Exercise 80.1. Let X be a connected 2-dimensional CW-complex with $\pi_2(X) = 0$. Show that X is aspherical.

Exercise 80.2. Let X be a CW-complex and let U and V be path-connected subcomplexes of X. Suppose that $U \cap V$ is path-connected and furthermore suppose that the inclusion induced maps $\pi_1(U \cap V) \to \pi_1(U)$ and $\pi_1(U \cap V) \to \pi_1(V)$ are monomorphisms. Finally suppose that U, V and $U \cap V$ are aspherical. Show that X is also aspherical.
Hint: You might want to use Propositions 13.14, 20.15 and Proposition 50.12 (6).

Exercise 80.3. Let M and N be two closed oriented n-dimensional manifolds. Suppose there exists a $k \in \mathbb{N}$ such that M and N are k-connected. Show that the connected sum $M \# N$ is also k-connected.

Exercise 80.4. Let X be an n-dimensional CW-complex and let Y be a subcomplex of X that is homotopy equivalent to S^n. Show that the inclusion induced map $\pi_n(Y) \to \pi_n(X)$ is a monomorphism.

Exercise 80.5. We denote by $*$ suitable base points of the spheres. Now let $f\colon S^n \to S^n$ be a map with $f(*) = *$. We suppose that f is a map of degree $k \in \mathbb{Z}$, i.e. f is a map such that the induced map $f_*\colon H_n(S^n;\mathbb{Z}) \to H_n(S^n;\mathbb{Z})$ is given by multiplication by k.

(a) Let $g\colon S^n \to S^m$ be a map. Show that $[g \circ f] = k \cdot [g] \in \pi_n(S^m)$.
(b) Show that if $g\colon S^m \to S^n$ is a map, then it is not necessarily the case that the equality $[f \circ g] = k \cdot [g] \in \pi_m(S^n)$ holds.

Hint: We view S^3 as a subset of $\mathbb{C}^2$ and we make the usual identification $\mathbb{C}\mathrm{P}^1 = S^1$. Now consider the following commutative diagram:

$$\begin{tikzcd} S^3 \arrow[r, "(z,w)\to[(z,w)]"] \arrow[d, "(z,w)\mapsto(\overline{z},\overline{w})"'] & S^2 \arrow[d, "{[(z,w)]\mapsto[(\overline{z},\overline{w})]}"] \\ S^3 \arrow[r, "(z,w)\to[(z,w)]"] & S^2. \end{tikzcd}$$

81. Applications of the Hurewicz Theorem

In this chapter we want to spend some more time with the Hurewicz Theorem and we want to discuss some applications thereof.

81.1. **Paths acting on homotopy groups.** In Propositions 5.11 and 28.5 we already saw that for path-connected topological spaces the *isomorphism types* of the homotopy groups do not depend on the choice of the base point.

In this section we want to discuss this statement in greater detail. We recall the following definition which appeared already in the proof of Proposition 28.5.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. Let $\gamma\colon [0,1] \to X$ be a path from a point x_0 to a point x_1. Given a map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we define the map

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \dots, t_n) & \to & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1, \dots, t_n) \in \partial\big(\big[\frac{t}{4}, \frac{4-t}{4}\big]^n\big) \text{ for } t \in [0,1]. \end{cases} \end{array}$$

The definition of f^γ is illustrated in Figure 989.

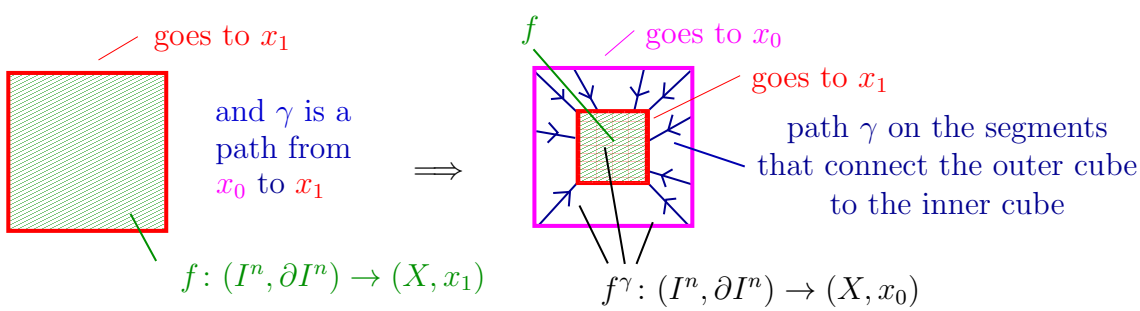


Figure 989.

Example. We consider the case $n = 1$ in the above definition. In this situation the map $f\colon (I, \partial I) \to (X, x_1)$ is a loop. It follows almost immediately from the definitions that the new loop $f^\gamma\colon (I, \partial I) \to (X, x_0)$ equals, up to an orientation-preserving reparametrization, the loop $\gamma * f * \overline{\gamma}$ that is given by the product of the three paths γ, f and $\overline{\gamma}$. We refer to Figure 990 for an illustration.

Figure 990.

Remark. Recall that on pages 715 and 1847 we saw that we can also interpret elements in a homotopy group $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. Let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1 and let $f\colon (S^n, *) \to (X, x_1)$ be a map. We define[1361]

$$\begin{array}{rcl} f^\gamma\colon S^n & \to & X \\ \underbrace{\cos(\varphi)\cdot(1,0,\dots,0)+\sin(\varphi)\cdot(0,v)}_{\varphi\in[0,\pi], v\in\mathbb{R}^{n-1}} & \mapsto & \begin{cases} f(\cos(2\varphi)\cdot(1,0,\dots,0)+\sin(2\varphi)\cdot(0,v)), & \text{if } \varphi\in[0,\frac{\pi}{2}] \\ \gamma(2-\frac{2}{\pi}t), & \text{if } \varphi\in[\frac{\pi}{2},\pi]. \end{cases} \end{array}$$

Of course one should only work with the formula if one is forced to at gun point. It is much more illuminating to look at Figure 991 where we try our best to visualize the map $f^\gamma\colon (S^n, *) \to (X, x_0)$. One can easily verify that, under the various identifications, the map f^γ corresponds to the construction on page 1869.

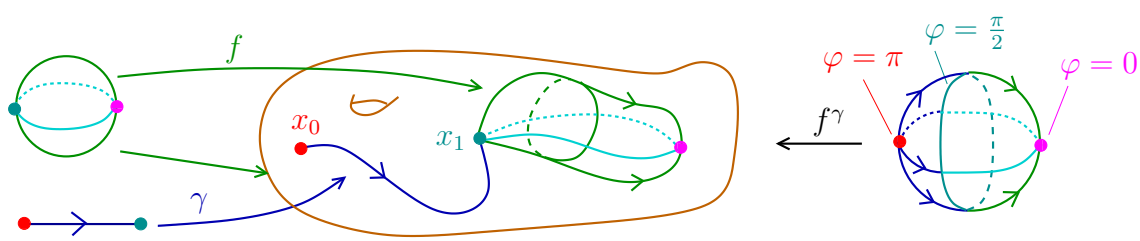


FIGURE 991.

Before we continue we introduce the following basically self-explanatory definition.

Definition. Let G be a group and let A be a group (or module or vector space). An *action of G on the group A* (respectively module or vector space) is a an action $G \times A \to A$ in the sense of the definition on page 102 such that for each $g \in G$ the map

$$\begin{array}{rcl} A & \to & A \\ a & \mapsto & g\cdot a \end{array}$$

is a group homomorphism (respectively module or vector space homomorphism).[1362]

The following proposition can be viewed as a more precise version of Proposition 28.5.

Proposition 81.1. *Let X be a topological space and let $n \in \mathbb{N}$.*

(1) *If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the map*

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is a well-defined isomorphism.

(2) *If x_0 and x_1 are two points in X which lie in the same path-component of X, then $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$.*

[1361]It follows easily from Lemma 1.32 (2') that this map is continuous.

[1362]In fact, since multiplication by g^{-1} is an inverse we see that these maps are actually isomorphisms.

(3) *Let $\varphi\colon X \to Y$ be a map between topological spaces.*
   (a) *If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the following diagram commutes*

$$\begin{tikzcd} \pi_n(X,x_1) \arrow[r, "\gamma_*"] \arrow[d, "\varphi_*"'] & \pi_n(X,x_0) \arrow[d, "\varphi_*"] \\ \pi_n(Y,\varphi(x_1)) \arrow[r, "(\varphi\circ\gamma)_*"] & \pi_n(Y,\varphi(x_0)). \end{tikzcd}$$

   (b) *If $\varphi_*\colon \pi_n(X,x_0) \to \pi_n(Y,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $\varphi_*\colon \pi_n(X,x_1) \to \pi_n(Y,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).*[1363]

(4) *Let γ and δ be two paths in X from x_0 to x_1. If γ and δ are path-homotopic in X, then the corresponding maps γ_* and δ_* from $\pi_n(X,x_1)$ to $\pi_n(X,x_0)$ agree.*

(5) *If γ is a path in X from x_0 to x_1 and if δ is a path in X from x_1 to x_2, then*[1364]

$$(\underbrace{\gamma * \delta}_{\text{path from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,x_2) \ \to\ \pi_n(X,x_0).$$

(6) *The map*

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] := [f^\gamma] \end{array}$$

defines an action (in the sense of the definition on page 102) of the group $\pi_1(X,x_0)$ on the group $\pi_n(X,x_0)$.

(7) *Let $f_0, f_1\colon S^n \to X$ be two maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$. The following statement holds:*

$$f_0 \text{ and } f_1 \text{ are homotopic} \quad\Longleftrightarrow\quad \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [f_1^\gamma] = [f_0] \in \pi_n(X,x_0).\end{array}$$

Remark. Let X be a topological space, let x_0, x_1 be two points in X and let γ and δ be two paths from x_0 to x_1. If X is simply connected, then Lemma 5.13 tells us that the two paths γ and δ are in fact homotopic. Therefore it follows from Proposition 81.1 (4) that $\gamma_* = \delta_*\colon \pi_n(X,x_0) \to \pi_n(X,x_1)$. In other words, if X is simply connected, then given any two base points x_0 and x_1 there exists in fact a natural[1365] isomorphism $\pi_n(X,x_0) \xrightarrow{\cong} \pi_n(X,x_1)$. In particular in this case it is almost always entirely safe to suppress the base point in the notation. By Proposition 5.14 this observation applies in particular to $X = S^k$ for $k \geq 2$.

Proof.

[1363] For $n = 1$ this is basically the content of Exercise 6.4.

[1364] Here it is worth remembering that by definition, see page 231, the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

[1365] As always, when the adjective "natural" gets used it is an amusing exercise to figure out what categories and functors are involved.

(0) Let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. It is straightforward to see that the map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined. It is also elementary, albeit potentially slightly painful, to see that γ_* is a group homomorphism. (See the corresponding discussion in the proof of Proposition 28.5.)

(4),(5) These two statements are elementary to verify. It is worth spending a second to convince oneself that in (5) we have the equality $(\gamma * \delta)_* = \gamma_* \circ \delta_*$ instead of $(\gamma * \delta)_* = \delta_* \circ \gamma_*$.

(1) Recall that in Proposition 5.6 (3) we saw that for any path δ the composition $\delta * \overline{\delta}$ is path-homotopic to the constant path. It follows easily from this observation together with (4) and (5) that γ_* is a bijection. Together with (0) we see that for any path $\gamma\colon [0,1] \to X$ from x_0 to x_1 the induced map $\gamma_*\colon \pi_n(X,x_1) \to \pi_n(X,x_0)$ is an isomorphism.

(2) This is an immediate consequence of (1)

(3) Statement (a) follows immediately from the definitions. Statement (b) is a consequence of the commutative diagram of (a) and the fact that the horizontal maps are isomorphisms by (1).

(6) It follows immediately from (4) and (5) that the map defines an action of $\pi_1(X,x_0)$ on the set $\pi_n(X,x_0)$. Furthermore (1) implies that this action is in fact an action on the group $\pi_n(X,x_0)$.

(7) The proof of this statement is a moderately complicated generalization of the proof of Proposition 9.22. We leave it to the reader to fill in the details. □

81.2. **Group rings.** Before we continue with our study of higher homotopy groups it is convenient to introduce the purely algebraic concept of a group ring.

Definition. Let G be a group and let R be a commutative ring. We denote the free R-module generated by the set G by $R[G]$. Put differently, with the notation from page 366 we have

$$R[G] \;=\; \Big\{ \sum_{i=1}^m r_i g_i \,\Big|\, r_1,\dots,r_m \in R \text{ and } g_1,\dots,g_m \in G \Big\}.$$

We equip $R[G]$ with the multiplication that is given by[1366]

$$\Big(\sum_{i=1}^m r_i g_i\Big) \cdot \Big(\sum_{j=1}^n s_j h_j\Big) \;:=\; \sum_{i=1}^m \sum_{j=1}^n \underbrace{r_i s_j}_{\in R} \cdot \underbrace{(g_i h_j)}_{\in G}.$$

We refer to $R[G]$ as the *group ring of G with R-coefficients.* (For $R = \mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as *the group ring of G.*)

The following lemma summarizes a few basic facts about group rings.

Lemma 81.2. *Let G be a group and let R be a commutative ring.*

[1366]The multiplication is basically given by naive "multiplying out".

(1) *The group ring $R[G]$ is an associative ring with a multiplicatively neutral element, namely $1 \cdot e$ where e denotes the trivial element in G.*
(2) *The group ring $R[G]$ is commutative if and only if G is an abelian group.*
(3) *If R is a field, then $R[G]$ is evidently an algebra over the field R.*
(4) *Given a unit $r \in R$ and an element $g \in G$ the product $r \cdot g$ is a unit in $R[G]$.*
(5) *If G has a torsion element, i.e. if there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$, then $R[G]$ has zero divisors.*

Proof. All the statements are verified easily. Perhaps the most interesting statement is (5). We will not rob the reader of the pleasure of proving this in Exercise 81.2. □

Examples. Let R be a commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map
$$\begin{array}{ccc} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum\limits_{i=r}^{i=s} a_i t^i & \mapsto & \sum\limits_{i=r}^{i=s} a_i t^i \end{array}$$
is a ring isomorphism.
(2) More generally, if H is a free abelian group of rank m, then any choice of a basis $t_1, \ldots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ to the group ring $R[H]$.

The group rings of free groups are particularly interesting. Therefore we dedicate a lemma to them.

Lemma 81.3. *Let $\langle T \rangle$ be the free group on a generating set T. Given any ring S and given any map $g\colon T \to S$ there exists a unique ring homomorphism $\varphi\colon \mathbb{Z}[\langle T \rangle] \to S$ with $\psi(t) = g(t)$ for all $t \in T$.*

Remark. Given a commutative ring R the group ring $R[\langle T \rangle]$ is sometimes called the *free ring on the set S over the ring R.*

Proof. It follows immediately from Lemma 11.13 that there exists a unique group homomorphism $\varphi\colon \langle T \rangle \to S^* := \{\text{units of } S\}$ that extends the given map g. It is now straightforward to verify that the map
$$\begin{array}{ccc} \mathbb{Z}[\langle T \rangle] & \to & S \\ \sum\limits_{i=1}^{m} r_i g_i & \mapsto & r_i \cdot \varphi(g_i) \end{array}$$
is a ring homomorphism that extends g and that it is the unique ring homomorphism with this desirable property. □

As a side remark we mention the following beautiful conjectures. These say in particular that in some sense the statements of Lemma 81.2 (4) and (5) are optimal.

Conjecture 81.4. (Kaplansky Conjectures) *Let G be a torsion-free group and let $\mathbb{F}$ be a field.*

(1) *The elements* 0 *and* 1 *are the only idempotents of* $\mathbb{F}[G]$*, i.e. they are the only elements in* $\mathbb{F}[G]$ *with* $x^2 = x$*.*
(2) *The group ring* $\mathbb{F}[G]$ *is a domain, i.e. it has no non-trivial zero divisors.*
(3) *The only units in* $\mathbb{F}[G]$ *are the elements of the form* $f \cdot g$ *with* $f \in \mathbb{F} \setminus \{0\}$ *and* $g \in G$*.*
(4) *The group ring* $\mathbb{C}[G]$ *can be embedded in a skew field.*

Remark.

(1) The conjectures are named after the Canadian mathematician Irving Kaplansky (1917-2006). The first conjecture is often also referred to as the *zero divisor conjecture*. They are formulated in [Lü2, Conjecture 10.14], [Lam2, p. 95] and [FG, p. 729]. As is pointed out in [Lam2, p. 95], Conjecture (3) implies Conjecture (2). It is elementary to to see that Conjecture (4) implies Conjecture (2) and that Conjecture (2) implies Conjecture (1).
(2) The Kaplansky Conjectures are very attractive since they are easy to state. But it is usually very very hard to prove non-trivial group-theoretic statements that are valid for all (torsion-free) groups. In particular all of the Kaplansky Conjectures are still wide open. We refer to [Lü5, Chapter 4.5] for a summary of what is known regarding these conjectures.

We continue with the following elementary lemma.

Lemma 81.5. *Let* G *be a group and let* A *be an abelian group (with additive notation for the group structure). Suppose we are given a* G*-action on the group* A*. Then the map*[1367]

$$\begin{array}{ccc} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum\limits_{i=1}^{m} r_i g_i, a \right) & \mapsto & \sum\limits_{i=1}^{m} r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural $\mathbb{Z}[G]$*-left module structure on* A*.*[1368]

Proof. The lemma is indeed elementary and trivial. We leave it to the reader to ponder what "natural" might mean in this context. □

Before we continue we introduce or recall the following convention:

Convention. If a homotopy group is abelian, then sometimes we use additive notation for the group multiplication instead of the multiplicative notation.

[1367]Here we use a somewhat perilous notation, namely we use the symbol "$\cdot$" with two different meanings. First of all, for $g \in G$ and $a \in A$ we denote by $g \cdot a \in A$ the element defined by the G-action on A. Secondly, for $r \in \mathbb{Z}$ and $b \in A$ we write, as usual, $r \cdot b = \underbrace{b + \cdots + b}_{r\text{-times}}$ for $r \geq 0$ and $r \cdot b = \underbrace{-b - \cdots - b}_{(-r)\text{-times}}$ for $r < 0$.

[1368]If R is a non-commutative ring, then one needs to be a little careful when one talks about "R-modules". More precisely, there are now two different concepts one can consider: Given an abelian group M one can demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = r \cdot (s \cdot m)$ in which case one says that M is an "R-left module". Alternatively one could demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = s \cdot (r \cdot m)$ in which case one says that M is an "R-right module". This discussion is the analogue of the discussion on page 102.

This convention, Lemma 81.5, Propositions 28.3 and Proposition 81.1 lead us to the following lemma.

Lemma 81.6. *Let (X, x_0) be a pointed topological space. If $n \geq 2$, then the map*

$$\begin{aligned} \mathbb{Z}[\pi_1(X, x_0)] \times \pi_n(X, x_0) &\to \pi_n(X, x_0) \\ ([\gamma], [f]) &\mapsto [f^\gamma] \end{aligned}$$

induced by

defines a natural $\mathbb{Z}[\pi_1(X, x_0)]$-left module structure on $\pi_n(X, x_0)$.

81.3. **Homotopy groups of $S^1 \vee S^2$.** In this section we want to consider the homotopy groups of $S^1 \vee S^2$ since they are quite instructive. On page 933 we already saw that $\pi_2(S^1 \vee S^2)$ is very big, more precisely, we saw that it admits an epimorphism onto $\mathbb{Z}^\infty$. We will now see that if we take the view of Lemma 81.6, i.e. if we view $\pi_2(S^1 \vee S^2)$ as a module over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$, then the apparently messy group $\pi_2(S^1 \vee S^2)$ suddenly gets a simple description.

We explicitly construct $X = S^1 \vee S^2$ by identifying the point $i \in S^1$ with the point $(0, 0, -1) \in S^2$. In the following we evidently use the wedge point as the base point of $X = S^1 \vee S^2$. We refer to it by x_0. Finally we denote by $\gamma\colon [0, 1] \to S^1 \subset X$ the obvious counterclockwise loop that starts at i. Furthermore we denote by $\mathrm{id}_{S^2}\colon S^2 \to S^2 \subset X$ the obvious inclusion map.

Now we can formulate the following proposition which delivers the promised concise description of $\pi_2(S^1 \vee S^2)$.

Proposition 81.7. *The ring homomorphism*[1369]

$$\begin{aligned} \Psi\colon \mathbb{Z}[t^{\pm 1}] &\to \pi_2(S^1 \vee S^2, x_0) \\ t &\mapsto \left[(\mathrm{id}_{S^2})^\gamma\right] \end{aligned}$$

that is uniquely determined by

is an isomorphism.

Proof. For aesthetic reasons for once we use the base point $* = (0, 0, -1)$ for S^2 and we use this base point to define the second homotopy groups of X and $\widetilde{X}$. As on page 568 we see that the universal covering of $X = S^1 \vee S^2$ is given by the map

$$\begin{aligned} p\colon \overbrace{\big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) / n \sim (n, (0, 0, -1))}^{=:\widetilde{X}} &\to S^1 \vee S^2 \\ P &\mapsto \begin{cases} e^{2\pi i (P + \frac{1}{4})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n, Q) \text{ with } n \in \mathbb{Z}, Q \in S^2. \end{cases} \end{aligned}$$

We pick the base points $x_0 = i \in S^1 \subset X$ and $\widetilde{x}_0 = 0 \in \mathbb{R} \subset \widetilde{X}$. Even though we have seen this covering map on numerous occasions we show it again in Figure 992. We consider the

[1369]It follows from Lemma 81.3 that there is indeed a unique such ring homomorphism.

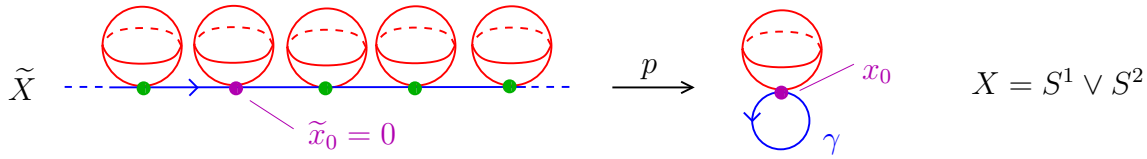


FIGURE 992. First illustration for the proof of Proposition 81.7.

following sequence of maps

$$(*) \qquad \mathbb{Z}[t^{\pm 1}] \xrightarrow{\Psi} \pi_2(X, x_0) \xrightarrow{\Xi} \pi_2(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\Phi_{(\widetilde{X},\widetilde{x}_0)}} H_2(\widetilde{X}; \mathbb{Z}) \xleftarrow{\Omega} \mathbb{Z}[t^{\pm 1}].$$

Here $\Phi_{(X,x_0)}$ denotes the Hurewicz homomorphism. Since $\widetilde{X}$ is simply connected we obtain from the Hurewicz Theorem 80.5 that it is an isomorphism. The reader will not have failed to notice that we did not define the two maps Ξ and Ω yet. We will do so now.

(1) First we recall that given a map $g\colon (S^2, *) \to (X, x_0)$ there exists by Proposition 20.2 a unique lift $\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)$. Now we consider the map

$$\begin{array}{rcl} \Xi\colon \pi_2(X, x_0) & \to & \pi_2(\widetilde{X}, \widetilde{x}_0) \\ [g\colon (S^2, *) \to (X, x_0)] & \to & \big[\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)\big]. \end{array}$$

By Proposition 28.13 we know that Ξ is well-defined and that it is an isomorphism.

(2) Given $i \in \mathbb{Z}$ we denote by $f_i\colon S^2 \to X$ the "obvious" i-th inclusion map. In Lemma 34.7 we saw that the map

$$\begin{array}{rcl} \Omega\colon \mathbb{Z}[t^{\pm 1}] & \to & H_2(\widetilde{X}; \mathbb{Z}) \\ \sum\limits_{i=r}^{s} a_i \cdot t^i & \mapsto & \sum\limits_{i=r}^{s} a_i \cdot f_i([S^2]). \end{array}$$

is an isomorphism.

We see in particular that in $(*)$ the three maps to the right are isomorphisms. Therefore it suffices to show that the maps $\Omega^{-1} \circ \Phi_{(X,x_0)} \circ \Xi \circ \Psi$ is the identity. In other words, it suffices to show the following claim.

Claim. For any $n \in \mathbb{Z}$ we have

$$(\Phi_{(X,x_0)} \circ \Xi \circ \Psi)(t^n) \;=\; \Omega(t^n) \;\in\; H_2(\widetilde{X}; \mathbb{Z}).$$

So let $n \in \mathbb{Z}$. It follows immediately from the definitions of the various maps that the desired statement is equivalent to the statement that

$$\underbrace{\big(\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}\big)}_{\text{map } S^2 \to \widetilde{X}}{}_*[S^2] \;=\; \underbrace{f_n}_{\text{map } S^2 \to \widetilde{X}}([S^2]) \;\in\; H_2(\widetilde{X}; \mathbb{Z}).$$

Thus by Proposition 30.5 it suffices to show that the two maps $(\mathrm{id}_{S^2})^{\gamma^n}$ and f_n are homotopic maps from S^2 to $\widetilde{X}$. For those readers who are still awake this should be basically obvious.[1370] As a digestion aid we illustrate both maps in Figure 992. $\square$

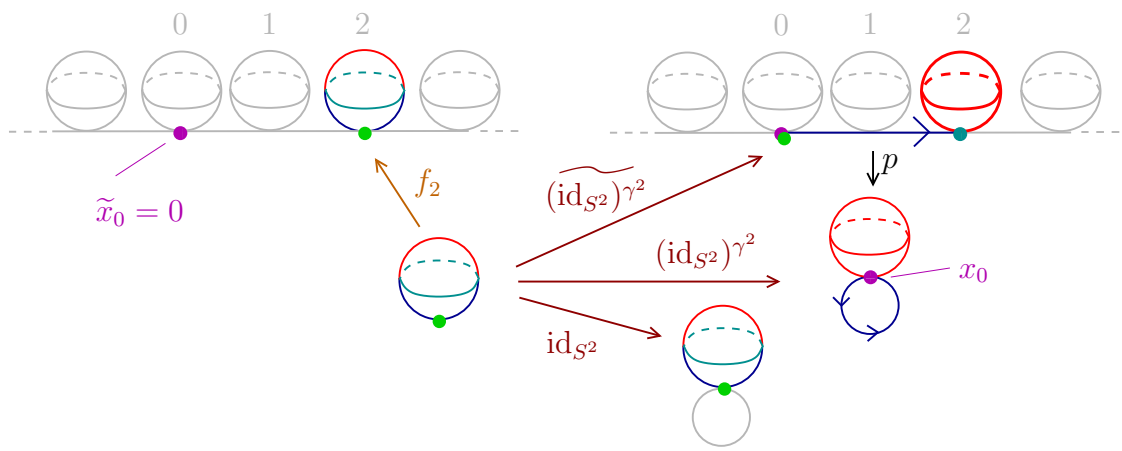


FIGURE 993. Second illustration for the proof of Proposition 81.7.

In Proposition 35.5 we saw that the homology groups of a finite CW-complex are finitely generated abelian groups. Since the discussion on page 933 we had known that the corresponding statement is not true for homotopy groups. But we now also know that one should really view the homotopy groups of a pointed topological space (X, x_0) as modules over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Thus the following question arises.

Question 81.8. *Let X be a finite CW-complex. Are the homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, finitely generated $\mathbb{Z}[\pi_1(X, x_0)]$-modules?*

It turns out that the answer is yes if X is simply connected. More precisely, Jean-Pierre Serre [Ser3][1371] proved the following theorem.

Theorem 81.9. (Serre) *If X is a simply-connected topological space such that all homology groups are finitely generated (for example X could be a compact topological manifold or a finite CW-complex), then all homotopy groups are also finitely generated abelian groups.*[1372]

Example. It follows in particular from Serre's Theorem 81.9 that the homotopy groups of spheres are finitely generated. Even this statement is highly non-trivial.

[1370] In fact this follows from the "$\Leftarrow$"-direction of Proposition 81.1 (7).

[1371] Jean-Pierre Serre (*1926) is a French mathematician. He was, and remains, the youngest person ever to be awarded the Fields Medal.

[1372] It follows immediately from the Hurewicz Theorem 80.5 that $\pi_2(X)$ is finitely generated. The fact that the higher homotopy groups are also finitely generated is much stronger.

Nonetheless, it turns out that the answer to Question 81.8 is negative. In fact, one does not need to wander off particularly far to find such an example:

Proposition 81.10. *The group $\pi_3(S^1 \vee S^2)$ is not finitely generated over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)] = \mathbb{Z}[t^{\pm 1}]$.*

A proof for the proposition is for example sketched in [Hat, Chapter 4.2, Exercise 38].

Remark. The Hilton-Milnor theorem, see [Hilt, Theorem A],[Mi11] or [Whda, Chapter XI], gives a calculation of the homotopy groups of a finite wedge of spheres, more precisely, these homotopy groups are computed in terms of the homotopy groups of the spheres involved. Unfortunately, at least at a first glance, it is not so clear how the module structure over the group ring can be read off from the Hilton-Milnor theorem.

81.4. **The Connected Sum Decomposition Theorem** $(*)$**.** We conclude this chapter with the study of connected sum decompositions of closed oriented topological manifolds.

Definition. Let M be a closed oriented n-dimensional topological manifold.

(1) A *connected sum decomposition of* M is a homeomorphism $M \cong X \# Y$ where X and Y are closed oriented n-dimensional topological manifolds. We say a connected sum decomposition is *non-trivial* if neither X nor Y is homeomorphic to S^n.
(2) We say M is *irreducible* if it does not admit a non-trivial connected sum decomposition.

Example. It follows easily from Proposition 71.10 and the classification of closed 2-dimensional topological manifolds, see Theorem 14.9, that the 2-sphere S^2 is irreducible. We will see shortly that in fact all spheres are irreducible.

The following theorem says that any closed oriented topological manifold can be decomposed into irreducible manifolds.

Theorem 81.11. (Connected Sum Decomposition Theorem) *Given any closed oriented connected n-dimensional topological manifold M there exists a homeomorphism $M \cong N_1 \# \dots \# N_k$ where $N_1, \dots, N_k$ are closed oriented irreducible n-dimensional topological manifolds.*

Remark. For 3-dimensional manifolds the statement of Theorem 81.11 was first proved in 1929 by Hellmuth Kneser [Kn]. In our proof of Theorem 81.11 we will make use of the proof of the Poincaré Conjecture which for 3-dimensional topological manifolds was settled by Grigori Perelman only in 2002. The proof of Hellmuth Kneser relied on a different complexity argument and did not make use of the Poincaré Conjecture.

The idea behind the proof of Theorem 81.11 is quite simple: We want to show that if a closed oriented topological manifold M admits a non-trivial connected sum decomposition $X \# Y$, then X and Y should both be "smaller" than M, e.g. they should have "smaller" algebraic invariants. So the process of decomposing a closed oriented topological manifold should stop after finitely many steps. To carry out this idea we need some group theoretic input.

Definition. Let A be a finitely generated abelian group.

(1) The *rank of* A is defined as $\operatorname{rank}(A) := \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$.[1373]
(2) As on page 372 we refer to
$$\operatorname{tor}(A) \;=\; \{a \in A \mid \text{ there exists an } n \in \mathbb{N} \text{ with } na = 0\}$$
as the *torsion subgroup of* A.[1374]
(3) We define $t(A) = \#\operatorname{tor}(A)$, i.e. $t(A)$ is the number of elements in the torsion subgroup $\operatorname{tor}(A)$.

We continue with the following elementary lemma.

Lemma 81.12. *If A and B are finitely generated abelian groups, then*
$$\operatorname{rank}(A \oplus B) \;=\; \operatorname{rank}(A) + \operatorname{rank}(B) \quad \textit{and} \quad t(A \oplus B) = t(A) \cdot t(B).$$

Proof. The first statement is an immediate consequence of the additivity of dimensions under the direct sum operation or alternatively it is a consequence of Lemma 11.6. The second statement follows from the observation that $\operatorname{tor}(A \oplus B) = \operatorname{tor}(A) \oplus \operatorname{tor}(B)$. □

We also recall the following definition from page 382.

Definition. Let G be a finitely generated group. We denote by $d(G) \in \mathbb{N}_0$ the minimal number of elements in a generating set for G.

Remark. It is an unfortunate fact of life that in the literature the number $d(G)$ is usually referred to as the rank of the group G. But for abelian groups the invariants $\operatorname{rank}(G)$ and $d(G)$ can differ quite dramatically. For example for $G = (\mathbb{Z}_2)^n$ we have $\operatorname{rank}(G) = 0$ but $d(G) = n$.

For the reader's convenience we also recall the Grushko-Neumann Theorem 11.18. It can be viewed as an analogue of Lemma 81.12.

Theorem 81.13. (Grushko-Neumann Theorem) *Given any two finitely generated groups A and B we have*
$$d(A * B) \;=\; d(A) + d(B).$$

Now we can provide the proof of Theorem 81.11.

Proof of Theorem 81.11. Let $n \in \mathbb{N}$. Let M be a closed oriented connected n-dimensional topological manifold. In Proposition 45.2 we saw that the fundamental group $\pi_1(M)$ and the homology groups $H_i(M;\mathbb{Z})$ are finitely generated groups. Thus we can define
$$c(M) \;:=\; d(\pi_1(M)) \,+\, \operatorname{rank}\Big(\bigoplus_{i=1}^{n-1} H_i(M;\mathbb{Z})\Big) \,+\, t\Big(\bigoplus_{i=1}^{n-1} H_i(M;\mathbb{Z})\Big) \,\in\, \mathbb{N}_0.$$
We make two observations:

[1373] As we had discussed on page 1006, this definition agrees with the earlier definition that we gave on page 371.

[1374] It follows from the classification of finitely generated abelian groups, see Theorem 11.4, that $\operatorname{tor}(A)$ is in fact a finite subgroup.

(1) It follows from the proof of the Poincaré Conjecture in all dimensions, see Section 28.6 together with Proposition 80.8 that $c(M) = 0$ if and only if M is homeomorphic to S^n.
(2) If $M = X\#Y$ is a non-trivial connected sum decomposition, then it follows immediately from Proposition 71.10, Lemma 81.12 and the Grushko-Neumann Theorem 81.13 that $c(X) < c(M)$ and $c(Y) < c(M)$.

Theorem 81.11 now follows from an elementary induction argument. □

Remark. Note that the argument used in the above proof of Theorem 81.11 also shows that spheres are irreducible.

In dimension 3 the decomposition of Theorem 81.11 is essentially unique. More precisely John Milnor [Mi7] (see also [Hem, Chapter 3] and [Schul, Proposition 5.5.3]) proved in 1962 the following theorem.

Theorem 81.14. *Let M be a closed oriented connected 3-dimensional topological manifold. If $M \cong N_1\#\dots\#N_r$ and $M \cong N_1'\#\dots\#N_s'$ where the topological 3-manifolds N_i and N_i' are oriented irreducible 3-manifolds and such that none of the N_i and N_i' are homeomorphic to S^3, then $r = s$ and (possibly after reordering) there exists an orientation-preserving homeomorphism $N_i \to N_i'$ for each i.*

Remark.

(1) In the 3-dimensional setting our language unfortunately diverges slightly from standard parlance. More precisely, what we call an irreducible 3-dimensional topological manifold is usually in the literature (see e.g. [Hem, p. 27]) referred to as a "prime 3-dimensional topological manifold". Furthermore in the literature there is a different notion of a 3-dimensional manifold being irreducible. Namely, in the literature (see e.g. [Hem, p. 28]) a 3-dimensional manifold is usually called irreducible if every embedded 2-dimensional sphere bounds an embedded 3-dimensional ball. Fortunately, as is shown in [Hem, Lemma 3.13] for orientable 3-dimensional topological manifolds the two notions of "irreducible" only differ for $S^1 \times S^2$.
(2) In Theorem 81.14 we cannot drop the condition that M is oriented. For example in Exercise 81.10 we will see that $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ is homeomorphic to $\mathbb{R}\mathrm{P}^3\#(S^1 \times S^2)$.

The following proposition says that the analogous statement does not hold in dimension four.

Proposition 81.15. *The closed oriented topological manifolds $\mathbb{C}\mathrm{P}^2, \overline{\mathbb{C}\mathrm{P}}^2$ and $S^2 \times S^2$ are irreducible and $(S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ is homeomorphic*[1375] *to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$.*

Proof. We use the notation from the proof of Theorem 81.11. The calculation of the invariants of $\mathbb{C}\mathrm{P}^2$, $\overline{\mathbb{C}\mathrm{P}}^2$ and $S^2 \times S^2$ shows that $c(\mathbb{C}\mathrm{P}^2) = c(\overline{\mathbb{C}\mathrm{P}^2}) = 1$ and $c(S^2 \times S^2) = 2$.

[1375] In fact the two topological manifolds are diffeomorphic if we equip them with the obvious smooth structure.

Thus the argument of the proof of Theorem 81.11 shows that $\mathbb{C}\mathrm{P}^2$ and $\overline{\mathbb{C}\mathrm{P}}^2$ are prime. Next recall that on page 1655 we showed that the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Using this piece of information and using Proposition 71.11 one can fairly easily show that $S^2 \times S^2$ is prime. The full details will be carried out in Exercise 81.9.

Finally note that on page 1677 we pointed out that the manifolds $(S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ are diffeomorphic, in particular they are homeomorphic. $\square$

We turn our gaze to all dimensions ≥ 4 and quote the following result from [ICFHLN], which extends earlier work in [KrS, Theorem 1.1].

Proposition 81.16. *Let $n \in \mathbb{N}_{\geq 4}$. We write $n = 2k + \epsilon$ with $\epsilon \in \{0, 1\}$. There exist closed oriented connected topological n-dimensional manifolds M and N that are not homotopy equivalent but such that $M\#(S^k \times S^{k+\epsilon})$ and $N\#(S^k \times S^{k+\epsilon})$ are diffeomorphic.*

Since the connected sum operation is well-defined up to homeomorphism we see that $M\#(S^k \times S^{k+\epsilon})$ admits two non-equivalent decompositions into irreducible topological manifolds.

Remark. In the above exposition we worked throughout with topological manifolds up to homeomorphisms. Of course one could also work with manifolds and diffeomorphisms. This leads us to the notion of a "smoothly prime" manifold. As we pointed out in Theorem 28.24, in the 3-dimensional setting there is essentially no difference between these two approaches. But in the higher-dimensional setting it is not even true any more that spheres are necessarily smoothly prime. More precisely, by [KeM, Theorem 1.2] there exists a closed oriented 7-dimensional manifold X that is not diffeomorphic to S^7 but such that $X\#X$ is diffeomorphic to S^7. As we remarked on on page 355, given any closed oriented n-dimensional manifold the connected sum $M\#S^n$ is diffeomorphic to M. The above discussion implies that no closed oriented connected 7-dimensional manifold is smoothly prime.

Exercises for Chapter 81.

Exercise 81.1. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We consider the Hurewicz homomorphism $\Phi\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$. Show that for any $g \in \pi_1(X, x_0)$ and any $\sigma \in \pi_n(X, x_0)$ we have

$$\Phi(g \cdot \sigma) \;=\; \Phi(\sigma) \in H_n(X; \mathbb{Z}).$$

Exercise 81.2. Let G be a group and let R be a commutative ring. Suppose there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$. Show that $R[G]$ has a zero divisor, i.e. show that there exist non-zero elements $p, q \in R[G]$ with $p \cdot q = 0$.
Hint: First consider the case that $G = \langle t \,|\, t^n \rangle$ is a finite cyclic group with $n \geq 2$ elements.

Exercise 81.3. Is the ring $M(3 \times 3, \mathbb{R})$ of real (3×3)-matrices isomorphic, as a ring, to the real group ring $\mathbb{R}[G]$ of some group G?

Exercise 81.4. Let G be a finite group. We view $\mathbb{Q}[G]$ as a $\mathbb{Q}[G]$-left module via left multiplication. Show that $\mathbb{Q}[G]$ admits a submodule $V \neq 0$ with $V \neq \mathbb{Q}[G]$ that is isomorphic to the trivial $\mathbb{Q}[G]$-module $\mathbb{Q}$ where every $g \in G$ acts via the identity on $\mathbb{Q}$.

Exercise 81.5. Let G be a finite group. Show that there exists a monomorphism from G to some unitary group $\mathrm{U}(n)$.

Exercise 81.6. We consider the ring $M(2, \mathbb{R})$. Does there exist a finite set T and an epimorphism from the free ring $\mathbb{R}[\langle T \rangle]$ onto $M(2, \mathbb{R})$?

Exercise 81.7. Let G be a group that acts on an abelian group A. Show that the set

$$\{g \cdot a - a \,|\, g \in G \text{ and } a \in A\} \;\subset\; A$$

is in fact a subgroup of A.

Exercise 81.8. Let X be a simply connected topological space and let $f\colon X \to X$ be a self-homeomorphism. Recall that on page 469 we introduced the mapping torus

$$M \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1).$$

Express the $\mathbb{Z}[\pi_1(M)]$-module $\pi_2(M)$ in terms of the homology groups of X and the action by f on the homology groups of X.

Exercise 81.9. Show that the topological manifold $S^2 \times S^2$ is prime.
Hint: Consider the proof of Theorem 81.11 and use Proposition 71.11. Also make use of the fact that on page 1655 we showed that the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Exercise 81.10. Show that $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ is homeomorphic to $\mathbb{R}\mathrm{P}^3 \#(S^1 \times S^2)$.

82. Relative homotopy groups

82.1. **The definition of relative homotopy groups.** While studying homology groups we saw at some point that it was convenient to introduce the relative homology groups $H_k(X, A)$ corresponding to a pair of topological spaces (X, A). In our latest attempt to understand homotopy groups we will now, perhaps not surprisingly, introduce relative homotopy groups. We start out with introducing the following notation.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Furthermore, given $n \in \mathbb{N}$ we use the following notation:

(1) we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$,
(2) we denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$,
(3) by a slight abuse of notation we make the identification $I^{n-1} = I^{n-1} \times \{0\}$,
(4) we write $J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$.

We refer to Figure 994 for an illustration.

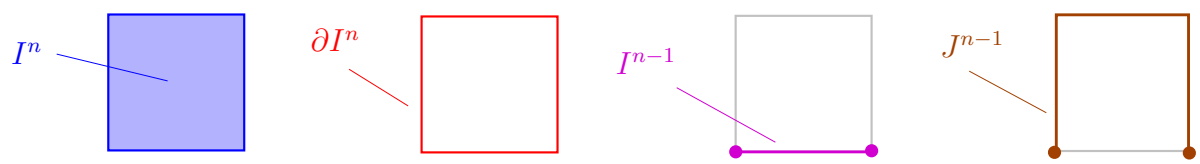


Figure 994.

The following definition is a variation on the notion of a pointed topological space that we introduced on page 242.

Definition. A *pointed pair of topological spaces* is a triple (X, A, x_0), where X is a topological space, A is a subset of X and x_0 is a point in A. A *morphism* between pointed pairs of topological spaces (X, A, x_0) and (Y, B, y_0) is a map $f\colon X \to Y$ with $f(A) \subset B$ and $f(x_0) = y_0$. We refer to the corresponding category as the *category of pointed pairs of topological spaces*.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces. Given $n \geq 1$ we define the *n-th relative homotopy group of* (X, A, x_0) to be[1376]

$$\pi_n(X, A, x_0) \ := \ \text{homotopy classes of maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\}).$$

The definition is illustrated in Figure 995.

Examples.

[1376]On page 641 we defined the notion of maps between pairs of topological spaces and we defined the notion of a homotopy between such maps. These notions extend in an obvious way to triples of topological spaces.

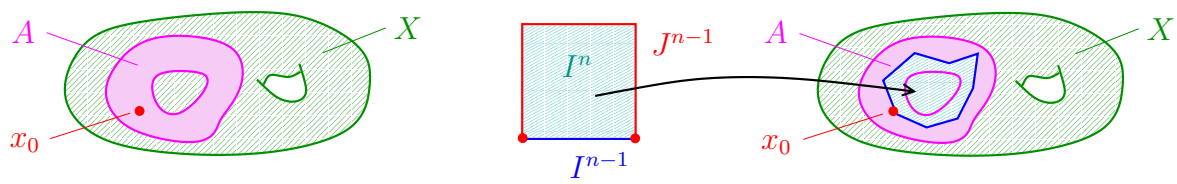


FIGURE 995.

(1) Let (X, A, x_0) be a pointed pair of topological spaces. We want to get a better understanding of elements in $\pi_1(X, A, x_0)$. For $n = 1$ we have $I^1 = [0, 1]$, $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. A map $(I^1, \partial I^1, J^0) \to (X, A, \{x_0\})$ is thus the same as a path in X that starts at a point in A and that ends in x_0. We refer to Figure 996 and 997 for an illustration.

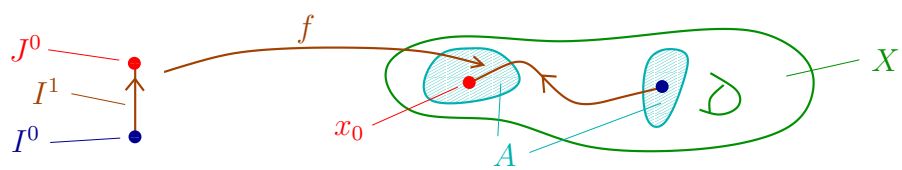


FIGURE 996.

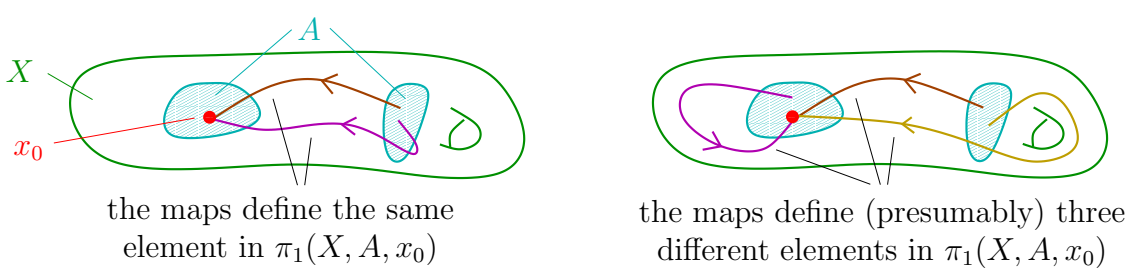


FIGURE 997.

(2) Let (X, x_0) be a pointed topological space. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, \{x_0\}, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \to & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, \{x_0\}, \{x_0\})] \end{array}$$

defines, basically by definition, a natural bijection. In the following we will use this natural bijection to make the identification $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$.

(3) Let X be a topological space, let A be a subset of X and let $x_0 \in A$ be a point. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, A, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \to & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] \end{array}$$

is a natural map from absolute homotopy groups to relative homotopy groups.

In the previous discussion we already referred to $\pi_n(X, A, x_0)$ as a "relative homotopy group". This was slightly premature, but now we will actually deliver the goods and provide a group structure.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 2$.[1377] Given two maps $f, g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ we define[1378]

$$\begin{array}{rcl} f * g\colon I^n & \to & X \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f(2t_1, t_2, \ldots, t_n), & \text{if } t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that $f * g$ is again a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ of triples. The definition of $f * g$ is illustrated in Figure 998.

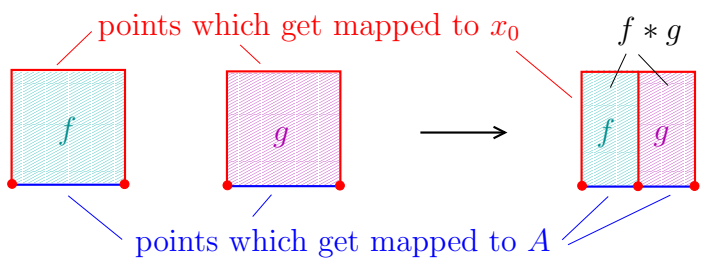


FIGURE 998.

The following proposition finally gives the justification for the name *relative homotopy group*.

Proposition 82.1. *Let (X, A, x_0) be a pointed pair of topological spaces.*

(1) *Let $n \geq 2$. The map*

$$\begin{array}{rcl} \pi_n(X, A, x_0) \times \pi_n(X, A, x_0) & \to & \pi_n(X, A, x_0) \\ ([f], [g]) & \mapsto & [f * g] \end{array}$$

is well-defined and it defines a group structure on $\pi_n(X, A, x_0)$. Furthermore the following two statements hold:

(a) *the trivial element in $\pi_n(X, A, x_0)$ is represented by the constant map $I^n \to X$ that sends every point in I^n to x_0,*

(b) *the inverse of the relative homotopy class $[f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})]$ is represented by the map*

$$\begin{array}{rcl} (I^n, \partial I^n, J^{n-1}) & \to & (X, A, \{x_0\}) \\ (t_1, t_2, \ldots, t_n) & \mapsto & (1 - t_1, t_2, \ldots, t_n). \end{array}$$

[1377] What, if anything, goes wrong for $n = 1$?

[1378] This map is continuous by Lemma 1.32 (2).

(2) *For $n \geq 2$ the above natural bijection $\pi_n(X, x_0) \xrightarrow{=} \pi_n(X, \{x_0\}, x_0)$ is in fact a group isomorphism.*
(3) *For $n \geq 2$ the above natural map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is a group homomorphism.*
(4) *For $n \geq 3$ the group $\pi_n(X, A, x_0)$ is abelian.*
(5) *Let $n \in \mathbb{N}$. Given a morphism $f\colon (X, A, x_0) \to (Y, B, y_0)$ of pointed pairs of topological spaces the map*

$$\begin{array}{rcl} f_*\colon \pi_n(X, A, x_0) & \to & \pi_n(Y, B, y_0) \\ [\varphi] & \mapsto & [f \circ \varphi] \end{array}$$

is well-defined. Furthermore the following four statements hold:
(a) *For $n \geq 2$ the induced map f_* is a group homomorphism.*
(b) *The maps f_* define a functor from the category of pointed pairs of topological spaces to the category of sets (respectively groups for $n \geq 2$).*
(c) *Let $n \geq 1$ and let $f, g\colon (X, A) \to (Y, B)$ be two maps between pairs of topological spaces. If $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$ is a bijection for all $x_0 \in A$ and if g is homotopic[1379] to f, then $g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, g(x_0))$ is also a bijection for all $x_0 \in A$.*
(d) *Let $f\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let $x_0 \in A$. If f is a homotopy equivalence, then the map*

$$f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$$

is a bijection (in particular it is an isomorphism if $n \geq 2$).

Sketch of proof.
(1) The proof of the first statement is quite similar to the proof of Proposition 28.1. We leave it to the reader to fill in the details.
(3) This statement requires just slightly more thought than one might think. More precisely, given two maps f and g from $(I^n, \partial I^n, J^{n-1})$ to $(X, \{x_0\}, \{x_0\})$, the "product maps" $f * g$ considered on page 710 for $\pi_n(X, \{x_0\})$ (and illustrated in Figure 448) and in the above proposition for $\pi_n(X, A, x_0)$ are not identical. Nonetheless, using Lemma 28.2 or Proposition 80.2 it is straightforward to show that they are homotopic. This observation shows that the above identification of sets is in fact an identification of groups.
(2) We already know that the map is a bijection. Thus it suffices to show that it is a group homomorphism. But this follows from (3) applied to $A = \{x_0\}$.
(4) The proof of this statement is an entertaining variation of the proof of Proposition 28.3. The proof is the content of Exercise 82.2.
(5) Statements (a) and (b) follow basically immediately from the definitions. Statement (c) and (d) are proved in a similar fashion to Proposition 28.7. □

Remark.

[1379] Here we mean of course, "homotopic as a map between pairs of topological spaces".

(1) We will see shortly that the relative homotopy group $\pi_2(X, A, x_0)$ is in general not abelian.
(2) For want of a better idea, given a pointed pair of topological spaces (X, A, x_0) we refer to $\pi_1(X, A, x_0)$ also as a *relative homotopy group* even though strictly speaking we did not define a group structure on $\pi_1(X, A, x_0)$.

In light of Proposition 82.1 we adopt the following convention:

Convention. For $n \geq 3$ we usually use additive notation for the group structure on the relative homotopy groups $\pi_n(X, A, x_0)$.

Let X be a topological space and let $A \subset B \subset X$ be two subsets. In Proposition 31.13 we saw that this data gives rise to a long exact sequence of homology groups. Shortly we will see that a similar statement holds for relative homotopy groups. We did not define a group structure on $\pi_1(X, A, x_0)$ and we did not define a group structure on $\pi_0(X, x_0)$. (In fact we will see in Exercise 82.3 that there is no natural way to turn these sets into groups.) Nonetheless we would like to include these objects at the lower end of the long exact sequence of (relative) homotopy groups. In fact, as it turns out, $\pi_1(X, A, x_0)$ and $\pi_0(X, x_0)$ have ever so slightly more structure than just the structure of a set.

Definition.

(1) A *pointed set* is a pair (S, s_0) where S is a set and $s_0 \in S$ is a point. We call s_0 the *trivial element* in S.
(2) A *morphism* from a pointed set (S, s_0) to a pointed set (T, t_0) is a map $f\colon S \to T$ with $f(s_0) = t_0$. Furthermore we adopt the following language:
   (a) Given a morphism $f\colon (S, s_0) \to (T, t_0)$ we refer to $\ker(f) := f^{-1}(t_0)$ as the *kernel of* f.
   (b) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is *trivial* if $\ker(f) = S$, put differently, if all elements in S get sent to t_0.
   (c) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is a *monomorphism* if $\ker(f) = \{s_0\}$.
   (d) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is an *epimorphism* if f is surjective.
(3) We say that a sequence

$$\ldots \xrightarrow{f_{n+2}} (S_{n+1}, s_{n+1}) \xrightarrow{f_{n+1}} (S_n, s_n) \xrightarrow{f_n} (S_{n-1}, s_{n-1}) \xrightarrow{f_{n-1}} \ldots$$

of morphisms of pointed sets is *exact* if for every $n \in \mathbb{N}_0$ we have the usual equality $\ker(f_n) = \operatorname{im}(f_{n+1})$.

Examples.

(1) Any group G gives rise to the pointed set $(G, \{e\})$ where $e \in G$ evidently denotes the trivial element. We will thus always view a group as a pointed set. Any group homomorphism is also a morphism of the corresponding pointed sets. Furthermore an exact sequence of homomorphisms of groups is also an exact sequence of pointed sets.

(2) If G is a group and H is a subgroup, then the set of right cosets G/H is a pointed set with trivial element given by $H \in G/H$. The obvious projection map $G \to G/H$ is a morphism of pointed sets.

The following little lemma will be proved in Exercise 82.4.

Lemma 82.2. *Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence*

$$\{e\} \;\to\; G \;\xrightarrow{\varphi}\; H \;\xrightarrow{\psi}\; (S, s_0) \;\to\; \{e\}$$

where ψ is a morphism of pointed sets that preserves the H-action. Then $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Example. The previous lemma might sound rather artificial. But we have seen it before in group theory: suppose that H is a group and G is a subgroup. As above we view the set of right cosets G/H as a pointed set. Left-multiplication by G defines an action on G/H. Evidently the projection $G \to G/H$ preserves the right action and we get a short exact sequence $\{e\} \to G \to H \to G/H \to \{e\}$ as in Lemma 82.2.

We continue with pointed sets that arise in topology.

Examples.

(1) Let (X, x_0) be a pointed topological space. On page 710 we defined $\pi_0(X, x_0)$ to be the set of homotopy classes of maps from the point I^0 to X. In other words, $\pi_0(X, x_0)$ is the set of path-components of X. We view $\pi_0(X, x_0)$ as a pointed set where the trivial element is given by the map that sends the unique point in I^0 to x_0. A map $f\colon X \to Y$ between topological spaces defines a morphism $f_*\colon \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ of pointed sets in the obvious way. Sometimes, by an unfortunate abuse of language, we refer to $\pi_0(X, x_0)$ as a homotopy group of X.

(2) Similarly to the above, given a pointed pair of topological spaces (X, A, x_0) we view $\pi_1(X, A, x_0)$ as pointed set where the trivial element is once again represented by the constant path $I \to X$ given by sending all points to x_0.

We conclude this section with the following elementary lemma that will be proved in Exercise 82.6.

Lemma 82.3. *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 1$. Let $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ be a map. If $f(I^n) \subset A$, then f represents the trivial element in $\pi_n(X, A, x_0)$.*[1380][1381]

Since it is much easier to convince oneself of a statement if one sees an explicit example, we show such a map with $f(I^n) \subset A$ in Figure 999.

[1380] Put differently, as for relative homology groups, all objects that "land in A" are in fact trivial in the relative homotopy group.

[1381] Note that by the above discussion of pointed sets the statement of Lemma 82.3 is also meaningful for $n = 1$.

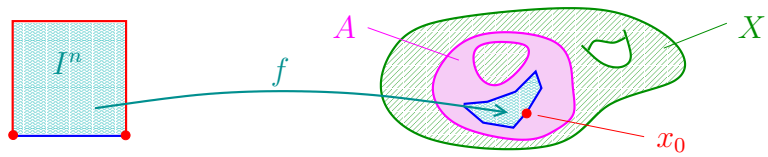


FIGURE 999.

82.2. **The long exact homotopy sequence of a pair.** In Proposition 31.13 we saw that we can associate to a pair (X, A) of topological spaces a long exact sequence of (relative) homology groups. The goal of this section is to do the same for homotopy groups. More precisely we will associate to a pointed pair (X, A, x_0) of topological spaces a long exact sequence of (relative) homotopy groups.

Proposition 82.4. *Let (X, A, x_0) be a pointed pair of topological spaces.*

(1) *For any $n \geq 1$ the map*

$$\begin{array}{rcl} \partial_n \colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \to & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}) & \to & (A, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array} \right] \end{array}$$

is well-defined. (See Figure 1000 for an illustration.) We refer to it as the connecting morphism*. It has the following two properties:*

(a) *The connecting morphism is a morphism of pointed sets and it is a group homomorphism for $n \geq 2$.*

(b) *The connecting morphism is a natural transformation from $\pi_n(X, A, x_0)$ to $\pi_{n-1}(A, x_0)$. In everyday language this just simply means that for a map $f \colon (X, A, x_0) \to (Y, B, y_0)$ between pointed pairs of topological spaces the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_n(Y, B, y_0) & \xrightarrow{\partial} & \pi_{n-1}(B, y_0). \end{array}$$

(2) *The following sequence is exact*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

Here $i \colon A \to X$ is the inclusion map and the map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is the natural map introduced on page 1885.

Examples.

(1) We consider $X = \mathbb{R}^2$ and we consider $A = A_0 \sqcup A_1$ which is the union of disjoint closed disks A_0 and A_1 in $X = \mathbb{R}^2$. We denote by $x_0 \in A_0$ and $x_1 \in A_1$ the centers of the two disks. Furthermore we denote by $f \colon I = [0, 1] \to X$ a map with $f(0) = x_1$

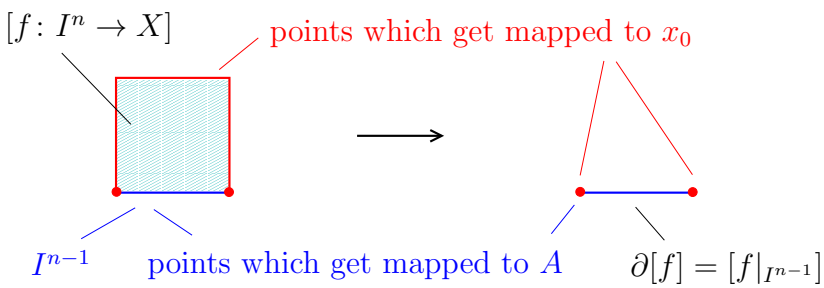


FIGURE 1000.

and $f(1) = x_0$. (We refer to Figure 1001 for an illustration.) Finally, for any $n \in \mathbb{N}_0$ and any $i \in \{0,1\}$ we denote by $c_{x_i}\colon I^n \to A_i$ the constant map that sends all points to x_i. It follows from Lemma 9.10 together with Lemma 9.9 and Propositions 9.12 (1) and 28.7 that all the homotopy groups of A and X are generated by $[c_{x_0}]$ except for $\pi_0(A, x_0)$ that equals the set $\{[c_{x_0}], [c_{x_1}]\}$.

As discussed on page 1884, f defines an element in $\pi_1(X, A, x_0)$. Using the definitions one sees almost immediately that $\partial[f] = [c_{x_1}] \in \pi_0(A, x_0)$. It follows easily from this data and Proposition 82.4 (2) that $\pi_1(X, A, x_0)$ consists of precisely two elements, namely $[c_{x_0}]$ and $[f]$.

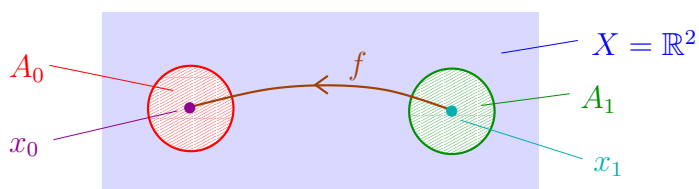


FIGURE 1001.

(2) Let $X = \mathbb{R}^2$. We denote by A the wedge of two circles, viewed as a subset of $X = \mathbb{R}^2$. Furthermore we denote by x_0 the wedge point. It follows from Proposition 82.4 (2) that $\pi_2(X, A, x_0) \xrightarrow{\cong} \pi_1(A, x_0)$. In Lemma 14.4 we saw that $\pi_1(A, x_0)$ is non-abelian, thus $\pi_2(X, A, x_0)$ is also non-abelian.

In fact we can identify two elements in $\pi_2(X, A, x_0)$ that do not commute. We consider the two maps $f, g\colon (I^2, \partial I^2, J^1) \to (X, A, \{x_0\})$ that are illustrated in Figure 1002. It follows easily from the definitions that $\partial[f], \partial[g]$ are the standard generators of the free group[1382] $\pi_1(S^1 \vee S^1, x_0)$. Since these do not commute we see that $[f]$ and $[g]$ do not commute either.

Proof. We start out with the proof of statement (1). In fact this statement follows mostly easily from the definitions. One only needs to think for a minute about the question why the connecting homomorphism is in fact a homomorphism.

[1382]We refer to page 390 for the calculation of the fundamental group of $S^1 \vee S^1$.

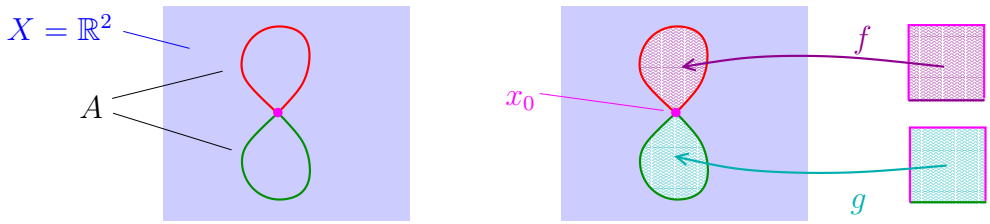


FIGURE 1002.

We turn to the proof of statement (2). In fact we prove a slight variation of the second statement. More precisely, let X be a topological space, let A be a subset of X and let B be a subset of A. Finally let $x_0 \in B$. For any $n \geq 2$ the map

$$\begin{array}{rcl} \partial_n \colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, B, x_0) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \to & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}, J^{n-2}) & \to & (A, B, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array} \right] \end{array}$$

is easily seen to be a well-defined morphism of pointed sets. Our goal is to prove the following claim.

Claim. The following sequence is exact[1383]

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, B, x_0) \to \ldots \to \pi_1(X, A, x_0).$$

where $i \colon (A, B) \to (X, B)$ and $j \colon (X, B) \to (X, A)$ are the obvious morphisms of pointed pairs of topological spaces.

Note that this claim, applied to $B = \{x_0\}$ together with Proposition 82.1 (2) almost implies that the sequence of the proposition is exact. The only problem is that the sequence of the claim stops slightly earlier, more precisely it stops at the subindex one instead of subindex zero. Thus the only piece missing is the exactness of the sequence of the proposition at the last three pointed sets. We leave the verification of exactness at these three spots as an elementary exercise to the conscientious reader.

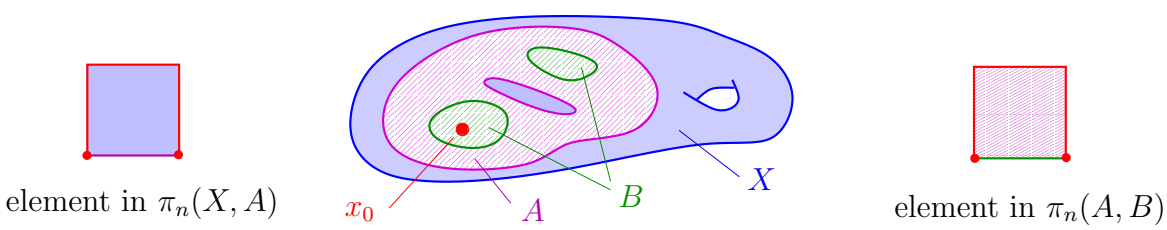


FIGURE 1003. First illustration for the proof of Proposition 82.4.

[1383]This sequence plays of course the role of the long exact sequence in homology groups of a triple of topological spaces, see Proposition 31.13.

Now we turn to the proof of the claim. Not surprisingly the proof of the claim breaks up into three separate proofs, namely we need to verify exactness at the three different groups.
Exactness at $\pi_n(X, B, x_0)$. It follows easily from Lemma 82.3 that $j_* \circ i_* = (j \circ i)_*$ is the trivial morphism. We leave it as an exercise to the reader to verify that $\ker(j_*) \subset \operatorname{im}(i_*)$.
Exactness at $\pi_n(X, A, x_0)$. It follows again easily from Lemma 82.3 and the definition of the connecting homomorphism ∂ that $\partial \circ j_*$ is the trivial morphism. Now suppose we are given $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ such that the restriction of f to $I^{n-1} \times \{0\}$ represents the trivial element in $\pi_{n-1}(A, B, x_0)$. This means that there exists a homotopy $H\colon I^{n-1} \times I \to A$ (with all the required restrictions) from $f|_{I^n \times \{0\}}$ to the constant map. The map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \dots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \dots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ H\big(x_1, \dots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big] \end{cases} \end{array}$$

is continuous by Lemma 5.3 and it defines a homotopy (again with all the desired extra properties) from f to a map that represents an element in $\pi_n(X, B, x_0)$. The definition of the homotopy is sketched, to the best of my abilities, in Figure 1004.

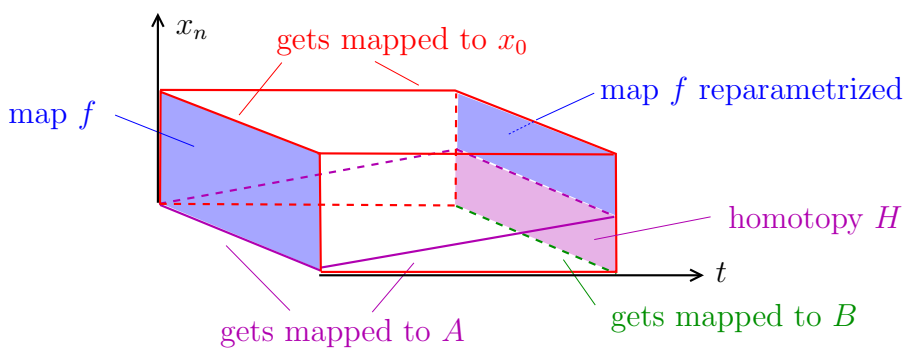


FIGURE 1004.

Exactness at $\pi_n(A, B, x_0)$. First note that $i_* \circ \partial$ is the trivial morphism. Indeed, if $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, \{x_0\})$ is a map, then $f\colon (I^n \times [0,1], \partial I^n \times [0,1]) \to (X, x_0)$ defines a homotopy from $\partial f = f|_{I^n \times \{0\}}$ to the constant map c_{x_0}. Thus $(i_* \circ \partial)[f] = [i_* \circ \partial f]$ is the trivial element in $\pi_n(X, B, x_0)$.

Now suppose that we are given a map $f\colon (I^n, \partial I^n, J^{n-1}) \to (A, B)$ such that $j \circ f$ represents the trivial element in $\pi_n(X, B, x_0)$. This means that there exists a homotopy H (with all the obvious conditions) from $j \circ f$ to the constant map c_{x_0}. We define

$$\begin{array}{rcl} g\colon I^n & \to & B \\ (x_1, \dots, x_n) & \mapsto & H(x_1, \dots, x_{n-1}, 0, x_n). \end{array}$$

We pick a homeomorphism $\Psi\colon I^2 \to I^2$ with the properties as illustrated in Figure 1005.[1384] (Here it is psychologically convenient to think of I^2 as the square in the plane with x_n and x_{n+1}-coordinates.) It follows immediately from the definitions that the map

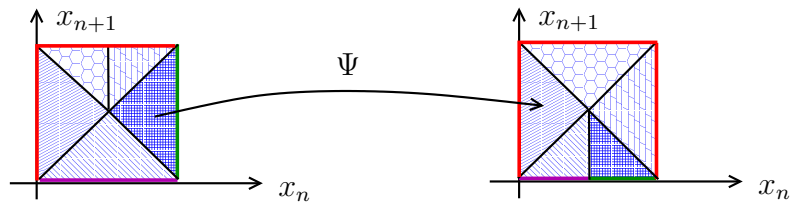


FIGURE 1005.

$$\begin{array}{rcl} I^{n+1} & \to & X \\ (x_1,\dots,x_n,x_{n+1}) & \to & H(x_1,\dots,x_{n-1},\Psi(x_n,x_{n+1})) \end{array}$$

defines an element in $\pi_{n+1}(X,A,x_0)$ such that the connecting morphism sends it to the element in $\pi_n(A,B,x_0)$ represented by the map[1385]

$$\begin{array}{rcl} f*g\colon I^n & \to & X \\ (t_1,\dots,t_n) & \mapsto & \begin{cases} f(t_1,\dots,t_{n-1},2t_n), & \text{if } t_n\in[0,\frac12], \\ g(t_1,\dots,t_{n-1},2t_n-1), & \text{if } t_n\in[\frac12,1] \end{cases} \end{array}$$

It remains to show that $f*g$ and f represent the same element in $\pi_n(A,B,x_0)$. This can be done in a fashion similar to the above proof of exactness. More precisely, a homotopy between f and $f*g$, again with all the desired properties, is given by

$$\begin{array}{rcl} I^n\times[0,1] & \to & X \\ (x_1,\dots,x_n,t) & \mapsto & \begin{cases} f\big(x_1,\dots,x_{n-1},\frac{t}{2}\cdot x_n+(1-x_n)\big), & \text{if } x_n\in\big[\frac{t}{2},1\big] \\ H\big(x_1,\dots,x_{n-1},2x_n-t\big), & \text{if } x_n\in\big[0,\frac{t}{2}\big] \end{cases} \end{array}$$

□

82.3. **Paths acting on relative homotopy groups.** Let (X,x_0) be a pointed topological space. In Section 81.1 we saw that the fundamental group $\pi_1(X,x_0)$ of X acts on any homotopy group $\pi_n(X,x_0)$. In this section we will discuss an analogous construction for relative homotopy groups. In particular, given a pointed pair (X,A,x_0) of topological spaces we will see that the fundamental group $\pi_1(A,x_0)$ of A (let us stress that here we write A instead of X) acts on any relative homotopy group $\pi_n(X,A,x_0)$.

The following definition is quite similar to the definition on page 1869.

[1384] The triangles in Figure 1005 indicate how one could write down an explicit homeomorphism.

[1385] Strictly speaking this definition of $f*g$ differs from the definition on page 1885 since we use a different coordinate to perform the product. But it is convenient to use the same notation.

Definition. Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.[1386] Furthermore let $\gamma\colon [0,1] \to A$ be a path in the subset A[1387] from a point x_0 to a point x_1. Finally let

$$f\colon (I^n, \partial I^n, J^{n-1}) \quad \to \quad (X, A, \{x_1\})$$

be a map. We define

$$f^\gamma\colon (I^n, \partial I^n, J^{n-1}) \quad \to \quad (X, A, \{x_0\})$$

to be the map that is illustrated in Figure 1006. For our particularly pedantic readers we now also give a super-rigorous definition of f^γ: First of all, given $t \in [0,1]$ we define $C_t = \left[\frac{t}{4}, \frac{4-t}{4}\right]^{n-1} \times [0, 1-\frac{t}{2}]$. Now we define

$$\begin{aligned} f^\gamma\colon (I^n, \partial I^n, J^{n-1}) &\to (X, A, x_0) \\ (x_1, \dots, x_n) &\to \begin{cases} f\big(2x_1 - \frac{1}{2}, \dots, 2x_n\big), & \text{if } (x_1, \dots, x_n) \in C_1 = \left[\frac{1}{4}, \frac{3}{4}\right]^{n-1} \times [0, \frac{1}{2}], \\ \gamma(t), & \text{if } (x_1, \dots, x_n) \in \partial C_t \text{ for } t \in [0,1]. \end{cases} \end{aligned}$$

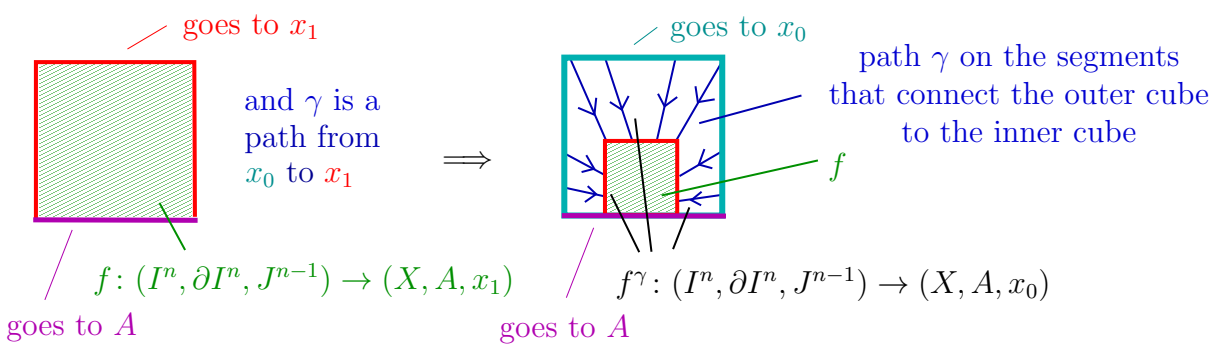


FIGURE 1006.

The following proposition can be viewed as the "relative analogue" of Proposition 28.5.

Proposition 82.5. *Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.*

(1) *If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the map*

$$\begin{aligned} \gamma_*\colon \pi_n(X, A, x_1) &\to \pi_n(X, A, x_0) \\ [f] &\mapsto [f^\gamma] \end{aligned}$$

is a well-defined map that is a group homomorphism.

(2) *If x_0 and x_1 are two points in A which lie in the same path-component of A, then $\pi_n(X, A, x_0)$ is isomorphic to $\pi_n(X, A, x_1)$.*

(3) *Let $\varphi\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces.*

[1386] What is wrong with $n = 1$?

[1387] Note that in contrast to the definition on page 1869 we only consider paths in A. At this point we do not allow paths in X.

(a) *If* $\gamma\colon [0,1] \to A$ *is a path from* x_0 *to* x_1*, then the following diagram commutes*

$$\begin{array}{ccc} \pi_n(X,A,x_1) & \xrightarrow{\gamma_*} & \pi_n(X,A,x_0) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \pi_n(Y,B,\varphi(x_1)) & \xrightarrow{(\varphi\circ\gamma)_*} & \pi_n(Y,B,\varphi(x_0)). \end{array}$$

(b) *If* $\varphi_*\colon \pi_n(X,A,x_0) \to \pi_n(Y,B,\varphi(x_0))$ *is an isomorphism (epimorphism, monomorphism) for some base point* $x_0 \in A$*, then for any other point* x_1 *in the same path component of* A *the map* $\varphi_*\colon \pi_n(X,A,x_1) \to \pi_n(Y,B,\varphi(x_1))$ *is also an isomorphism (epimorphism, monomorphism).*

(4) *Let* γ *and* δ *be two paths in* A *from* x_0 *to* x_1*. If* γ *and* δ *are path-homotopic in* A*, then the corresponding maps* γ_* *and* δ_* *from* $\pi_n(X,A,x_1)$ *to* $\pi_n(X,A,x_0)$ *agree.*

(5) *If* γ *is a path in* A *from* x_0 *to* x_1 *and if* δ *is a path in* A *from* x_1 *to* x_2*, then*[1388]

$$(\underbrace{\gamma * \delta}_{\text{path in } A \text{ from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,A,x_2) \ \to\ \pi_n(X,A,x_0).$$

(6) *The map*

$$\begin{array}{rcl} \pi_1(A,x_0) \times \pi_n(X,A,x_0) & \to & \pi_n(X,A,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] := [f^\gamma] \end{array}$$

defines an action (in the sense of the definition on page 102) of the group $\pi_1(A,x_0)$ *on the group* $\pi_n(X,A,x_0)$*.*

(7) *We denote by* $i\colon A \to X$ *the inclusion map and by* $\varphi\colon \pi_n(X,x_0) \to \pi_n(X,A,x_0)$ *the natural map. Then, with the convention from page 1360, the following diagram commutes*

$$\begin{array}{ccc} \pi_1(X,x_0) \times \pi_n(X,x_0) & \longrightarrow & \pi_n(X,x_0) \\ {\scriptstyle i_*}\uparrow \quad \downarrow{\scriptstyle \varphi_*} & & {\scriptstyle \varphi_*}\downarrow \\ \pi_1(A,x_0) \times \pi_n(X,A,x_0) & \longrightarrow & \pi_n(X,A,x_0). \end{array}$$

(8) *If* $\gamma\colon [0,1] \to A$ *is a path from* x_0 *to* x_1*, then the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X,A,x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_n(X,A,x_0) \\ \downarrow{\scriptstyle \partial} & & {\scriptstyle \partial}\downarrow \\ \pi_{n-1}(A,x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_{n-1}(A,x_0). \end{array}$$

Proof. The proof of the first six statements of the proposition is basically identical to the proof of Proposition 81.1. The last two statements follow easily from the definitions. To protect the trees we will not carry out the details. □

[1388]Here it is worth remembering that by definition, see page 231, the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

Remark. Let X be a topological space, let A be a non-empty subset and let $n \in \mathbb{N}_{\geq 2}$. If A is path-connected, then it follows from Proposition 82.5 (2) that the isomorphism type of the relative higher homotopy groups $\pi_n(X, A, x_0)$ does not depend on the choice of a base point in A. Thus if we are only interested in the isomorphism type of the n-th relative homotopy group of (X, A), then sometimes we just write $\pi_n(X, A)$, i.e. we suppress the base point from the notation.

Recall that for the relative homotopy groups $\pi_n(X, A, x_0)$ with $n \geq 3$ we use the additive notation for the group structure. Thus Lemma 81.5, Propositions 28.3, 82.1 (4) and 82.5 lead us to the following observation:

Observation. Let (X, A, x_0) be a pointed pair of topological spaces. If $n \geq 3$, then the abelian group $\pi_n(X, A, x_0)$ admits a natural $\mathbb{Z}[\pi_1(A, x_0)]$-left module structure.

Example. We consider the wedge $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0, 0, -1) \in S^2$. We denote by x_0 the wedge point. Furthermore let $A = S^1 \subset X$. We denote by $t \in \pi_1(S^1, i)$ the standard generator. (We refer to Figure 1007 for an illustration.) By Proposition 12.3 we know that the inclusion $A \to X$ induces an isomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$. We use this isomorphism to make the identification $\pi_1(A, x_0) = \pi_1(X, x_0) = \langle t \rangle$. It follows from Proposition 82.4 and Proposition 82.5 (7) that the natural map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules. In particular, by Proposition 81.7, we have

$$\pi_2(X, A, x_0) \overset{\cong}{\leftarrow} \pi_2(S^1 \vee S^2) \overset{\cong}{\rightarrow} \mathbb{Z}[t^{\pm 1}].$$

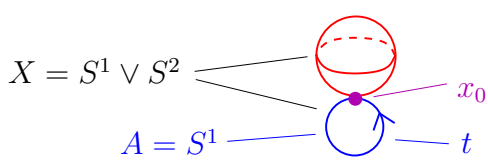


FIGURE 1007.

82.4. **Alternative description of relative homotopy groups.** Let (X, x_0) be a pointed topological space. Recall that on page 715 we had in particular introduced the following identifications:

$$\pi_n(X, x_0) \quad = \quad \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n, \partial I^n) \to (X, x_0)\end{array} \quad = \quad \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n, *) \to (X, x_0).\end{array}$$

These identifications allowed us to go back and forth between different view points regarding homotopy groups.

Before we continue with our discussion of relative homotopy groups we also want to introduce a second view point regarding relative homotopy groups. The key to doing so is the following elementary lemma.

Lemma 82.6. *Let $n \in \mathbb{N}$. There exists an explicit*[1389] *map $\varphi\colon I^n \to \overline{B}^n$ with the following three properties:*

(1) *we have $\varphi(J^{n-1}) = * := (-1, 0, 0, \ldots, 0)$ and $\varphi(I^{n-1}) = S^{n-1}$,*
(2) *the map φ descends to a homeomorphism $(I^n/J^{n-1}, I^{n-1}/\partial I^{n-1}) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$,*
(3) *the restriction of φ to the map $I^n \setminus \partial I^n \to B^n$ is an orientation-preserving diffeomorphism.*

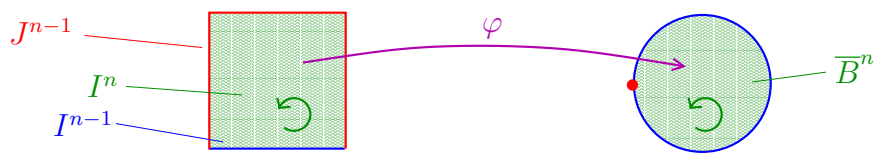


FIGURE 1008.

Sketch of a proof for Lemma 82.6 $(*)$. We write $Q = (\frac{1}{2}, \ldots, \frac{1}{2}, 0) \in I^n$. Given any point $x \in I^n \setminus \{Q\}$ we define

$$\rho(x) \ := \ \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in I^n\big\}.$$

We consider the map

$$\begin{array}{rcl} \alpha\colon I^n/J^{n-1} & \to & \overbrace{\{(x_1,\ldots,x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}}^{=H_n} \cup \{\infty\} \\ [x] & \mapsto & \begin{cases} \frac{x-Q}{\|x-Q\|} \cdot \tan\left(\frac{\pi}{2} \cdot \frac{\|x-Q\|}{\rho(x)}\right), & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

This map stretches each of the segments from Q to a point on J^{n-1} to an infinite ray emanating from the origin. In particular all points in J^{n-1} get sent to infinity, i.e. the map is well-defined. We refer to Figure 1009 for an illustration.

Similar to the proof of Proposition 1.48 we see that the map α is a homeomorphism that restricts to a homeomorphism $I^{n-1}/\partial I^{n-1} \to (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$. Let $\beta\colon \mathbb{R}^n \cup \{\infty\} \to S^n$ be the inverse of the stereographic projection that we introduced in Lemma 1.41. Note that $\beta(\infty) = (0, \ldots, 0, 1)$. Furthermore note that the homeomorphism β restricts to a homeomorphism[1390]

$$\beta\colon H^n \cup \{\infty\} \ \to \ S^n_{\geq 0} \ := \ \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_n \geq 0\}.$$

[1389] "Explicit" means that in the proof we explicitly write down such a map that does not depend on any choices. No reader will ever want to see the precise definition, but the fact that the map is explicitly given implies that there is no ambiguity involved when we write "let φ be the map from Lemma 82.6."

[1390] Note that the condition $x_n \geq 0$ is not a typo.

Finally we consider the following map:

$$\begin{array}{ccccccc} I^n & \to & I^n/J^{n-1} & \xrightarrow{\alpha} & H_n \cup \{\infty\} & \xrightarrow{\beta} & S^n_{\geq 0} & \to & \overline{B}^n \\ & & & & & & (x_1, \dots, x_n, x_{n+1}) & \mapsto & (-x_{n+1}, x_2, \dots, x_{n-1}, (-1)^n \cdot x_1). \end{array}$$

Using Exercise 3.5 one can easily verify that this map has all the desired properties.[1391] □

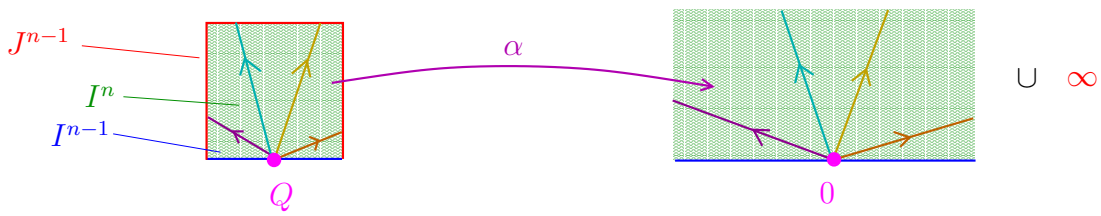


FIGURE 1009.

Convention. Using the map

$$\varphi\colon (I^n, \partial I^n, J^{n-1}) \to (\overline{B}^n, S^{n-1}, *)$$

from Lemma 82.6 we make the following identifications

$$\pi_n(X, A, x_0) = \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\end{array} = \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0).\end{array}$$

We refer to Figure 1010 for an illustration.

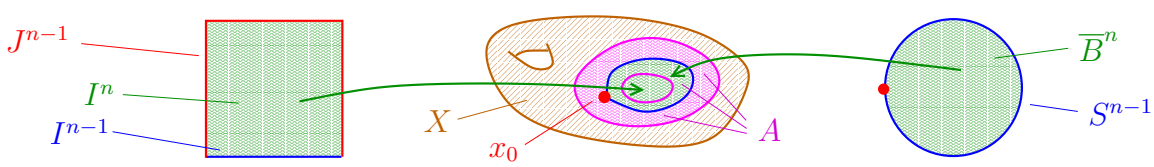


FIGURE 1010.

Remark. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. As above we make the identifications

$$\pi_n(X, A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of maps} \\ (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)\end{array} \right\} \text{ and } \pi_{n-1}(A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of maps} \\ (S^{n-1}, *) \to (A, x_0)\end{array} \right\}.$$

We leave it as an exercise to the ever-vigilant reader to verify that the connecting homomorphism $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ from Proposition 82.4 is given by the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)] & \mapsto & [f|_{S^{n-1}}\colon (S^{n-1}, *) \to (A, x_0)]. \end{array}$$

[1391] More precisely, the author is somewhat optimistic that the signs are right and that the map is indeed orientation-preserving.

82.5. **Connected (pairs of) topological spaces.** For convenience we recall the following definition from page 728.

Definition. Let $k \in \mathbb{N}_0$. We say a topological space X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \dots, k$.

This definition has the following obvious "relative" analogue.

Definition. Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. We say (X, A) *is k-connected* if every path-component of X contains a point of A and if for every $i \in \{1, \dots, k\}$ and every $x_0 \in A$ we have $\pi_i(X, A, x_0) = 0$.

The following lemma provides a convenient reformulation of the previous definition.

Lemma 82.7. *Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. The following two statements are equivalent:*

(1) *The pair (X, A) is k-connected.*
(2) *Every path-component of X contains a point of A, for every $a \in A$ and every $j \in \{1, \dots, k-1\}$ the inclusion induced map $\pi_j(A, a) \to \pi_j(X, a)$ is an isomorphism and the inclusion induced map $\pi_k(A, a) \to \pi_k(X, a)$ is an epimorphism.*

Example. Let X be a topological space and let $A \subset X$ be a deformation retract. It follows from Proposition 28.7 (1) that for any $x_0 \in A$ and any $i \in \mathbb{N}_0$ the inclusion induced map $\pi_i(A, x_0) \to \pi_i(X, x_0)$ is an isomorphism. It follows from Lemma 82.7 that (X, A) is k-connected for every $k \in \mathbb{N}$.

Proof. We denote by $i \colon A \to X$ the inclusion map. Let $a \in A$. By Proposition 82.4 we have the following long exact sequence of homotopy groups:

$$\dots \to \pi_k(A, a) \xrightarrow{i_*} \pi_k(X, a) \to \pi_k(X, A, a) \xrightarrow{\partial} \pi_{k-1}(A, a) \xrightarrow{i_*} \pi_{k-1}(X, a) \to \dots$$

The lemma is an immediate consequence of the existence of this long exact sequence. □

The following proposition gives a convenient characterization of the vanishing relative homotopy groups.

Proposition 82.8. *Let X be a topological space and let $A \subset X$. For any $n \in \mathbb{N}$ the following two statements are equivalent:*

(1) *Every map $(\overline{B}^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map $\overline{B}^n \to A$,*
(2) *for every $x_0 \in A$ we have $\pi_n(X, A, x_0) = 0$.*

Furthermore, for $n = 0$ the above statement (1) is equivalent to

(2') *every path-component of X contains a point of A.*

Example. We consider the infinite cylinder $X = S^1 \times \mathbb{R}$ and the annulus $A = S^1 \times [-1, 1]$. Evidently A is a deformation retract of X. Thus, by the previous example the pair (X, A) is k-connected for every $k \in \mathbb{N}_0$. In Figure 1011 we show a map $f \colon (\overline{B}^2, S^1) \to (X, A)$. Furthermore we show a map $g \colon \overline{B}^2 \to A$ that is homotopic, rel S^1, to f.

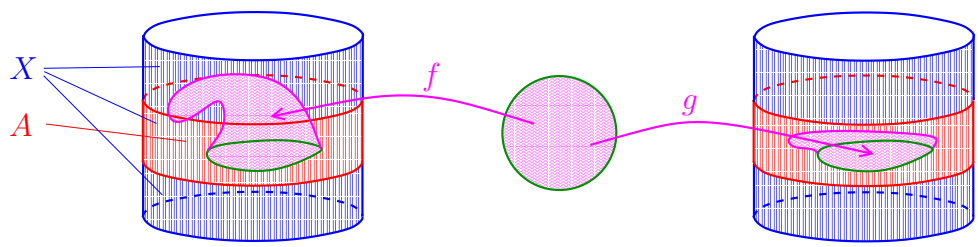


FIGURE 1011.

Proof of Proposition 82.8. First note that it is elementary to see that for $n = 0$ the statements (1) and (2') are equivalent. Now let $n \geq 1$. The implication (1) $\Rightarrow$ (2) is an immediate consequence of Lemma 82.3 and the by now well-known fact that there exists a homeomorphism $(I^n, \partial I^n) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$.

The most interesting statement of the proposition is the implication (2) $\Rightarrow$ (1). Thus we now assume that $\pi_n(X, A, x_0) = 0$ for every $x_0 \in A$. Let $f\colon (\overline{B}^n, S^{n-1}) \to (X, A)$ be a map. We need to show that f is homotopic rel S^{n-1} to a map that takes values only in A. Let $\varphi\colon I^n \to \overline{B}^n$ be the map from Lemma 82.6 with $\varphi(J^{n-1}) = (-1, 0, \dots, 0)$. We set $x_0 := f(-1, 0, \dots, 0)$. Note that the map $f \circ \varphi\colon I^n \to X$ is actually a map

$$f \circ \varphi\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (X, A, \{x_0\})$$

of triples. Since $\pi_n(X, A, x_0) = 0$ we see that there exists a homotopy

$$F\colon (I^n, \partial I^n, J^{n-1}) \times [0,1] \;\to\; (X, A, \{x_0\})$$

which satisfies $F_0 = f \circ \varphi$ and $F_1 = c_{x_0}$. Now we consider the map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \dots, x_n, t) & \mapsto & \begin{cases} F\big(x_1, \dots, x_{n-1}, \frac{1}{1-\frac{t}{2}} \cdot (x_n - \frac{t}{2}), t\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ F\big(x_1, \dots, x_{n-1}, 0, 2 \cdot x_n\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big]. \end{cases} \end{array}$$

The definition of the homotopy is sketched, with moderate success, in Figure 1012. This

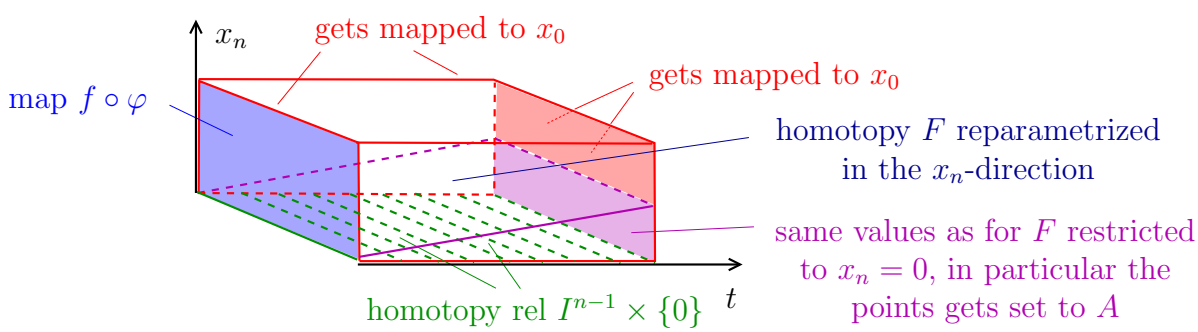


FIGURE 1012.

map has the following properties:

(0) the map is continuous by Lemma 5.3,
(1) the map is a homotopy rel $I^{n-1} \times \{0\}$,
(2) the map is a homotopy from $f \circ \varphi$ to a map that takes values in A,
(3) the map sends $J^{n-1} \times [0,1]$ to x_0.

It follows from Property (3), and using the above map $\varphi\colon (I^n, J^{n-1}) \to (\overline{B}^n, (-1, 0, \dots, 0))$, that F defines a homotopy $G\colon \overline{B}^n \times [0,1] \to X$. Property (1) implies that G is a homotopy rel S^{n-1}. Furthermore Property (2) says that G is a homotopy from f to a map that takes values in A. Thus G is the desired a homotopy. □

The following proposition gives us an important example of a k-connected pair of topological spaces.

Proposition 82.9. *Let X be a CW-complex and let $k \in \mathbb{N}_0$.*

(1) *If A is a subcomplex such that all cells in $X \setminus A$ have dimension greater than k, then the pair (X, A) is k-connected.*
(2) *The pair (X, X^k) is k-connected.*[1392]

Proof.

(1) Let X be a CW-complex, let A be a subcomplex and let $k \in \mathbb{N}_0$. Suppose that all cells in $X \setminus A$ have dimension greater than k. This implies in particular that every path-component of X contains at least one point in A. The remainder of the proof is almost the same as the proof of Proposition 28.10. The key difference is that we need to replace the Cellular Approximation Theorem 27.15 by the Cellular Approximation Theorem 27.16 for Pairs of CW-complexes and that now we use the (1) ⇒ (2) implication of Proposition 82.8. We leave it to the reader to carry out the modest modifications of the proof of Proposition 28.10.
(2) This statement is of course an immediate consequence of the first statement. □

The following corollary is an extremely useful consequence of Lemma 82.7 and Proposition 82.9 (2).

Corollary 82.10. *Let X be a CW-complex, let $k \in \mathbb{N}_0$ and let $x \in X^k$. The following two statements hold:*

(1) *For $j = 1, \dots, k-1$ the inclusion induced map $\pi_j(X^k, x) \to \pi_j(X, x)$ is an isomorphism.*
(2) *The inclusion induced map $\pi_k(X^k, x) \to \pi_k(X, x)$ is an epimorphism.*

82.6. **The relative Hurewicz Theorem.** In this section we state the "Relative Hurewicz Theorem" that generalizes the original Hurewicz Theorem 80.5 to the relative setting.

[1392]Recall that X^k denotes the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. We denote by $[I^n] \in H_n(I^n, \partial I^n; \mathbb{Z})$ the fundamental class of I^n.[1393] We refer to the map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & H_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

as the *relative Hurewicz homomorphism.*

The following lemma summarizes some elementary properties of the relative Hurewicz homomorphism.

Lemma 82.11.

(1) *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. The map*

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & H_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

is well-defined and it is a homomorphism.

(2) *Let (X, x_0) be a pointed topological space and let $n \geq 2$. The following diagram commutes*

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & H_n(X; \mathbb{Z}) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \pi_n(X, \{x_0\}, x_0) & \xrightarrow{\Phi_{(X,\{x_0\},x_0)}} & H_n(X, \{x_0\}; \mathbb{Z}). \end{array}$$

Here the left-hand vertical map is given by the identification from page 1884 and the right-hand vertical map is the obvious map from absolute to relative homology groups.

(3) *Let (X, A, x_0) be a pointed pair of topological spaces. The following diagram commutes*

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \longrightarrow \ldots \to & \pi_1(X, x_0) \\ & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \ldots \longrightarrow & H_n(A; \mathbb{Z}) & \longrightarrow & H_n(X; \mathbb{Z}) & \longrightarrow & H_n(X, A; \mathbb{Z}) & \xrightarrow{\partial} & H_{n-1}(A; \mathbb{Z}) & \longrightarrow \ldots \to & H_1(X; \mathbb{Z}). \end{array}$$

Here the horizontal sequences are the long exact sequences given by Corollary 31.14 and by Proposition 82.4 and the vertical maps are the Hurewicz homomorphisms.

Sketch of proof.

(1) The proof of the first statement is evidently just a straightforward variation of the proof of Lemma 38.9. We refer to [Bre, Lemma VII.10.2] or [Hat, Proposition 4.36] for details.

(3) We have to show that each of the three squares commutes.

  (a) The first square commutes basically by definition, see also Lemma 38.9 (2).

[1393] Here we endow $I^n \subset \mathbb{R}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(b) The fact that the second square commutes is an immediate consequence of the description of the Hurewicz homomorphism $\pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ that is given on page 1852.

(c) Finally it follows fairly easily from the aforementioned discussion on page 1852 together with Lemma 46.23 and Proposition 46.26 that the third square commutes.

(2) This is just a special case of the second square considered in (3). □

Again the following question arises: to what degree is the relative Hurewicz homomorphism in fact an isomorphism? One might hope that if a pair (X, A) is $(n-1)$-connected, then the Hurewicz homomorphism $\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \to H_n(X, A; \mathbb{Z})$ is an isomorphism. But the following example shows that this is *not* the case.

Example. We consider again $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. It follows easily from the long exact sequence of homotopy groups provided by Proposition 82.4 that (X, A) is 1-connected.[1394] We see that in this case the map

$$\Phi_{(X,A,x_0)} \colon \underbrace{\pi_2(X, A, x_0)}_{\cong \mathbb{Z}^\infty,\ \text{see page 1896}} \to \underbrace{H_2(X, A; \mathbb{Z})}_{\cong \mathbb{Z}}$$

is *not* an isomorphism.

The following lemma is the key to understanding why the relative Hurewicz homomorphism fails to be a monomorphism.

Lemma 82.12. *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma \in \pi_n(X, A, x_0)$. For any $g \in \pi_1(A, x_0)$ we have*

$$\Phi_{(X,A,x_0)}(g \cdot \sigma) \;=\; \Phi_{(X,A,x_0)}(\sigma).$$

Proof. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $[f] \in \pi_n(X, A, x_0)$ and let $[\gamma] \in \pi_1(A, x_0)$. As in Proposition 81.1 (7) we see that the maps $f \colon (I^n, \partial I^{n-1}) \to (X, A)$ and $f^\gamma \colon (I^n, \partial I^{n-1}) \to (X, A)$ are homotopic. We obtain that

$$\Phi_{(X,A,x_0)}([\gamma] \cdot [f]) \;=\; \Phi_{(X,A,x_0)}([f^\gamma]) \;=\; (f^\gamma)_*([I^n]) \;\underset{\substack{\uparrow \\ \text{by Proposition 31.15, since } f \text{ and } f^\gamma \text{ are homotopic as maps } (I^n, \partial I^n) \to (X, A)}}{=}\; f_*([I^n]) \;=\; \Phi_{(X,A,x_0)}([f]).$$

□

The above lemma leads us to the following definition.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces.[1395]

[1394] Alternatively we could view X as a CW-complex in the usual way. Then we obtain from Proposition 82.9 that the pair (X, A) is 1-connected.

[1395] Here we use the common, but sometimes confusing, convention that we write the group structure on the abelian group $\pi_n(X, A, x_0)$, $n \geq 3$, additively whereas we write the group structure on $\pi_2(X, A, x_0)$ multiplicatively. Evidently there is no rule without an exception, hence, if $\pi_2(X, A, x_0)$ happens to be abelian after all, then we also like to use additive notation for $\pi_2(X, A, x_0)$.

(1) For $n \in \mathbb{N}_{\geq 3}$ we define $\pi'_n(X, A, x_0)$ to be the quotient group[1396]

$$\pi'_n(X, A, x_0) \ := \ \pi_n(X, A, x_0)/\{g \cdot \sigma - \sigma \,|\, g \in \pi_1(A, x_0) \text{ and } \sigma \in \pi_n(X, A, x_0)\}.$$

(2) We define[1397]

$$\pi'_2(X, A, x_0) \ := \ \pi_2(X, A, x_0)/\langle\langle\{(g\cdot\sigma)\cdot\sigma^{-1} \,|\, g \in \pi_1(A, x_0), \sigma \in \pi_2(X, A, x_0)\}\rangle\rangle.$$

(3) By Lemma 82.12 the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \ \to \ H_n(X, A; \mathbb{Z})$$

descends to a homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi'_n(X, A, x_0) \ \to \ H_n(X, A; \mathbb{Z}).$$

We refer to this homomorphism again as the *Hurewicz homomorphism.*

Remark. Let (X, A, x_0) be a pointed pair of topological spaces. It is a fun exercise to show "by hand" that $\pi'_2(X, A, x_0)$ is an abelian group. We refer to [Hat, Lemma 4.39] for details.

Example. Once again we consider $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. As usual we write $\pi_1(A, x_0) = \langle t \rangle$. On page 1896 we already saw that $\pi_2(X, A, x_0) \cong \mathbb{Z}[t^{\pm 1}]$ as $\mathbb{Z}[\pi_1(A, x_0)] = \mathbb{Z}[t^{\pm 1}]$-modules. Thus we have an isomorphism

$$\pi'_2(X, A, x_0) \ \cong \ \mathbb{Z}[t^{\pm 1}]/\{(t^k - 1) \cdot f \,|\, f \in \mathbb{Z}[t^{\pm 1}] \text{ and } k \in \mathbb{Z}\}.$$

It is an amusing algebra exercise to show that the group on the right-hand side is isomorphic to $\mathbb{Z}$.

The following theorem is a generalization of the Hurewicz Theorem 80.5 to the relative case.

Theorem 82.13. (Relative Hurewicz Theorem) *Let (X, A, x_0) be a pointed pair of topological spaces such that X and A are path-connected. Let $n \geq 2$. If the pair (X, A) is $(n-1)$-connected, then the following two statements hold:*

(1) *We have $H_k(X, A; \mathbb{Z}) = 0$ for $k = 1, \dots, n-1$.*
(2) *The Hurewicz homomorphism*

$$\Phi_{(X,A,x_0)} \colon \pi'_n(X, A, x_0) \ \to \ H_n(X, A; \mathbb{Z})$$

is an isomorphism.

Proof. We will not provide a proof of the Relative Hurewicz Theorem. A full proof is given in [Bre, Theorem VII.10.7] or alternatively in [Hat, Theorem 4.37] and [Sp, Proposition 7.5.1]. If A is simply connected, then it is a great exercise to figure out to what degree the proof of the Hurewicz Theorem 80.5 can be modified to deal with the relative case. □

[1396] Let $n \in \mathbb{N}_{\geq 3}$. By Proposition 82.1 (4) we know that $\pi_n(X, A, x_0)$ is abelian. Therefore it follows from Exercise 81.7 that $\{g\cdot\sigma - \sigma \,|\, g \in \pi_1(A, x_0) \text{ and } \sigma \in \pi_n(X, A, x_0)\}$ is in fact a subgroup of $\pi_n(X, A, x_0)$.

[1397] Recall that given a group π and a subset $X \subset \pi$ we denote by $\langle\langle X \rangle\rangle$ the subgroup of π normally generated by X. We refer to page 407 for details.

The following theorem, which can be viewed as a corollary to the Relative Hurewicz Theorem 82.13, is a nice illustration of the usefulness of relative homotopy groups.

Theorem 82.14. *Let X and Y be two topological spaces and furthermore let $f\colon X \to Y$ be a map. We pick a base point $x_0 \in X$ and we write $y_0 = f(x_0)$.*

(1) *If for every $i \in \mathbb{N}_{\geq 2}$ the induced map*

$$f_*\colon \pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, y_0)$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon H_i(X;\mathbb{Z}) \xrightarrow{\cong} H_i(Y;\mathbb{Z})$$

is also an isomorphism.

(2) *If X and Y are simply connected, then the converse to (1) holds. More precisely, in this case, if for every $i \in \mathbb{N}_{\geq 2}$ the induced map*

$$f_*\colon H_i(X;\mathbb{Z}) \xrightarrow{\cong} H_i(Y;\mathbb{Z})$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, y_0)$$

is also an isomorphism.

Remark. In some textbooks, see e.g. [Bre, p. 481], [DK, p. 162] and [Sp, p. 399], this theorem is referred to as the "Whitehead Theorem". We will reserve the "Whitehead Theorem" for a later result.

Proof. The idea is to "replace" the map $f\colon X \to Y$ by an inclusion map so that we can apply the commutative diagram from Lemma 82.11. The key to doing so is the notion of the mapping cylinder that we introduced on page 455. Recall that the mapping cylinder corresponding to f is defined as the topological space

$$\mathrm{Cyl}(f) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \;\text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We consider the maps

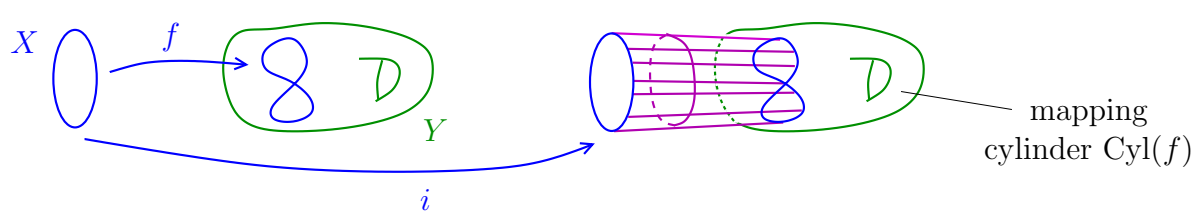


FIGURE 1013.

$$\begin{aligned} i\colon X &\to \mathrm{Cyl}(f) \\ x &\mapsto (x,0) \end{aligned} \quad \text{respectively} \quad \begin{aligned} r\colon \mathrm{Cyl}(f) &\to Y \\ P &\mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases} \end{aligned}$$

Note that we have $r \circ i = f\colon X \to Y$. Also note that in Lemma 15.7 (2b) we showed that the retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. We use the inclusion $i\colon X \to \mathrm{Cyl}(f)$ to identify X with $i(X)$, i.e. we view X as a subset of $\mathrm{Cyl}(f)$.

In the following we prove Statement (2) of the theorem. Thus we assume that X and Y are simply connected and we assume that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon H_i(X;\mathbb{Z}) \xrightarrow{\cong} H_i(Y;\mathbb{Z})$$

is an isomorphism. We need to show that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X,x_0) \xrightarrow{\cong} \pi_i(Y,y_0)$$

is also an isomorphism. We consider the following maps

$$\begin{tikzcd}
 & & & \pi_2(Y,y_0) & & \\
\dots \arrow[r] & \pi_3(\mathrm{Cyl}(f),X,x_0) \arrow[r, "\partial"] \arrow[d] & \pi_2(X,x_0) \arrow[r, "i_*"] \arrow[d] \arrow[ur, "f_*"] & \pi_2(\mathrm{Cyl}(f),x_0) \arrow[r] \arrow[d] \arrow[u, "r_*"'] & \pi_2(\mathrm{Cyl}(f),X,x_0) \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r] & H_3(\mathrm{Cyl}(f),X;\mathbb{Z}) \arrow[r, "\partial"] & H_2(X;\mathbb{Z}) \arrow[r, "i_*"] \arrow[dr, "f_*"'] & H_2(\mathrm{Cyl}(f);\mathbb{Z}) \arrow[r] \arrow[d, "r_*"] & H_2(\mathrm{Cyl}(f),X;\mathbb{Z}) \arrow[r] & 0 \\
 & & & H_2(Y;\mathbb{Z}) & &
\end{tikzcd}$$

where the vertical maps between the second and the third row are the Hurewicz homomorphisms. The desired statement follows from the following observations:

(a) The triangles commute since $f = r \circ i$. Furthermore the squares commute by Lemma 82.11.
(b) By our hypothesis the maps $f_*\colon H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ are isomorphisms. Furthermore, since r is a deformation retraction we obtain from Corollary 30.8 that the maps $r_*\colon H_*(\mathrm{Cyl}(f);\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ are isomorphisms.
(c) Since $f_* = (r \circ i)_* = r_* \circ i_*$ we obtain immediately from (b) that the maps $i_*\colon H_*(X;\mathbb{Z}) \to H_*(\mathrm{Cyl}(f);\mathbb{Z})$ are also isomorphisms.
(d) It follows from (c) and the bottom long exact sequence that all the relative homology groups $H_*(\mathrm{Cyl}(f),X;\mathbb{Z})$ vanishes.
(e) It follows easily from (d), the hypothesis that X is simply connected and the Relative Hurewicz Theorem 82.13 that all the relative homotopy groups $\pi_*(\mathrm{Cyl}(f),X,x_0)$ vanish.
(f) Using (e) we obtain immediately from the top long exact sequence that the maps $i_*\colon \pi_*(X,x_0) \to \pi_*(\mathrm{Cyl}(f),x_0)$ are isomorphisms.

(g) By Proposition 28.7 we know that the maps $r_*\colon \pi_*(\mathrm{Cyl}(f), x_0) \to \pi_*(Y, y_0)$ are isomorphisms.
(h) Finally it follows from $f_* = r_* \circ i_*$ together with (f) and (g) that the maps $f_*\colon \pi_*(X, x_0) \to \pi_*(Y, y_0)$ are indeed isomorphisms.

The proof of Statement (1) is very similar to the above argument. We leave it to fill in the details. □

We conclude this section with a discussion of homotopy spheres. Recall that by the definition on page 728 an n-dimensional homotopy sphere Σ is a closed n-dimensional topological manifold that is $\lfloor \frac{n}{2} \rfloor$-connected. In Proposition 80.8 we saw that the n-dimensional homotopy sphere Σ is in fact $(n-1)$-connected, i.e. the homotopy groups of Σ agree with the homotopy groups of S^n up to degree $n-1$.

The following pleasant corollary shows that in fact all homotopy groups of Σ are isomorphic to the corresponding homotopy groups of S^n.

Corollary 82.15. *Let $n \in \mathbb{N}_{\geq 2}$. If Σ is an n-dimensional homotopy sphere, then for every $i \in \mathbb{N}_0$ we have $\pi_i(\Sigma) \cong \pi_i(S^n)$.*

Proof. Let Σ be an n-dimensional homotopy sphere. By Theorem 82.14 (2) it suffices to find a map $f\colon S^n \to \Sigma$ such that for every $i \in \mathbb{N}_{i\geq 2}$ the induced map $f_*\colon H_i(S^n;\mathbb{Z}) \to H_i(\Sigma;\mathbb{Z})$ is an isomorphism.

So let us try to find such a map f. We pick a base point $x_0 \in \Sigma$. By the definition and Proposition 80.8 we know that Σ is a non-empty closed connected n-dimensional topological manifold with $H_i(\Sigma;\mathbb{Z}) \cong \pi_i(\Sigma, x_0) = 0$ for $i = 1, \dots, n-1$. It follows from the Hurewicz Theorem 80.5 that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi\colon \pi_n(\Sigma, x_0) & \to & H_n(\Sigma;\mathbb{Z}) \\ [f\colon (S^n, *) \to (\Sigma, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

is an isomorphism. Note that the fact that $\pi_1(\Sigma, x_0) = 0$ together with Corollary 45.27 implies that Σ is orientable. Therefore by Theorem 46.2 we know that $H_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$. Since the Hurewicz homomorphism is an isomorphism we can pick a map $f\colon S^n \to \Sigma$ such that $f_*([S^n]) \in H_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

We claim that f has the desired properties. By choice of f we know that the map $f_*\colon H_n(S^n;\mathbb{Z}) \to H_n(\Sigma;\mathbb{Z})$ is an epimorphism. But since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism. Since S^n and Σ are path-connected we obtain from the discussion on page 747 that $f_*\colon H_0(S^n;\mathbb{Z}) \to H_0(\Sigma;\mathbb{Z})$ is also an isomorphism. Furthermore for all other dimensions f_* induces an isomorphism of homology groups for the rather banal reasons that all other homology groups of S^n and Σ are zero. (Indeed, we saw above that the homology groups in dimension less than n vanish and it follows from Theorem 46.4 that the homology groups vanish in dimension greater than n.) □

82.7. **Excision for relative homotopy groups?** We have now seen that the (relative) homotopy groups and (relative) homology groups have some properties in common:

(1) both invariants are functorial,

(2) in both cases homotopic maps induce the same maps on the invariants,
(3) a (pointed) pair of topological spaces gives rise to a long exact sequence.

These formal similarities between (relative) homotopy and (relative) homology groups makes one wonder whether an analogue of the Excision Theorem 31.17 holds for relative homotopy groups.

For better or worse this is not the case. More precisely, the next lemma shows that a naive analogue of the Excision Theorem 31.17 fails even in very simple situations.

Lemma 82.16. *We consider the 2-dimensional sphere $X = S^2$ together with the subset $A := S^2_{\leq 0} = \{(x, y, z) \in S^2 \,|\, z \leq 0\}$ given by the lower hemisphere and together with the subset $Z = \{(0, 0, -1)\}$ that is given by the "South Pole". Then*[1398]

$$\pi_3(X, A) \ \not\cong \ \pi_3(X \setminus Z, A \setminus Z).$$

Figure 1014.

Proof. We start out with the following observations:

(1) On page 73 we saw that $A = S^2_{\leq 0}$ is homeomorphic to $\overline{B}^2$. In particular A is contractible which implies by Proposition 28.7 that $\pi_2(A) = \pi_3(A) = 0$.
(2) The above homeomorphism also shows that $A \setminus Z$ is homeomorphic to $\overline{B}^2 \setminus \{(0)\}$ which is homotopy equivalent to S^1. This implies that $\pi_2(A \setminus Z) = \pi_3(A \setminus Z) = 0$.
(3) Finally in Lemma 1.41 we saw that $X \setminus Z = S^2 \setminus \{(0, 0, -1)\}$ is homeomorphic to $\mathbb{R}^2$ which implies that $\pi_2(X \setminus Z) = \pi_3(X \setminus Z) = 0$.

It follows from these observations and the long exact sequence of homotopy groups, provided by Proposition 82.4, that $\pi_3(X \setminus Z, A \setminus Z) \cong \pi_2(A \setminus Z) \cong \pi_2(S^1) = 0$ and furthermore that $\pi_3(X, A) \cong \pi_3(S^2)$. But in Theorem 63.3 we saw that $\pi_3(S^2)$ is non-trivial. We have thus shown that $\pi_3(X, A)$ is not isomorphic to $\pi_3(X \setminus Z, A \setminus Z)$. □

Remark. We just used the fact that $\pi_3(S^2) \neq 0$ to show that relative homotopy groups do not satisfy excision. It is a good moment to remind ourselves of Theorem 42.29. The theorem suggests that if relative homotopy groups did satisfy excision, then it would follow from Proposition 82.4 that, at least morally, (abelianized) homotopy groups would agree with homology groups.[1399] But the calculations $\pi_3(S^2) \neq 0$ and $H_3(S^2) = 0$ show that this is not the case.

[1398] Here evidently the closure of Z is contained in the interior of A, i.e. the hypotheses for the Excision Theorem 31.17 for homology groups are satisfied.

[1399] To turn this logic into a proper argument might be a little tricky because of base point issues.

In much more restricted circumstances there is in fact an excision theorem. For the record we state the following theorem. We will not make use of it and therefore we will not provide the proof (which by all accounts seems to be somewhat messy.)

Theorem 82.17. (Blakers-Massey Theorem)

(1) *Let X be a topological space and let A and B be subsets such that $X = \mathring{A} \cup \mathring{B}$. If $(A, A \cap B)$ is n-connected and if $(B, A \cap B)$ is m-connected, then the inclusion induced map*

$$\pi_k(A, A \cap B) \ \to \ \pi_k(X, B)$$

is an isomorphism for $k < m + n$ and it is an epimorphism for $k = m + n$.

(2) *The conclusion of (1) also holds if X is a CW-complex and A and B are two subcomplexes of X with $X = A \cup B$ such that $A \cap B$ is non-empty and connected.*

Proof. The first statement was originally proved in 1952 by Albert Blakers and William Massey [BlM]. Proofs can also be found in [Gra, Corollary 16.27], [tD2, Theorem 6.4.1] or [tDKP, p. 211]. The second statement can presumably be deduced from the first statement. Alternatively see [Hat, Theorem 4.23]. □

Exercises for Chapter 82.

Exercise 82.1. Show that the three paths to the right of Figure 997 define three different elements in $\pi_1(X, A, x_0)$.

Exercise 82.2. Let (X, A, x_0) be a pointed pair of topological spaces. Show that if $n \geq 3$, then the relative homotopy group $\pi_n(X, A, x_0)$ is abelian.

Exercise 82.3. Let X be a topological space and $x_0 \in X$. We consider the set $\pi_0(X, x_0)$. We all know that $\pi_0(X, x_0)$ is "not a group". But why not? After all, every non-empty set can be equipped with a group structure (why is that?). So why is not possible to equip $\pi_0(X, x_0)$ with a group structure in a "meaningful way"? More precisely, if we could equip it with a group structure in a reasonable sense, what properties would we demand? Why can these not be satisfied?

Exercise 82.4. Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence

$$\{e\} \ \to \ G \ \xrightarrow{\varphi} \ H \ \xrightarrow{\psi} \ (S, s_0) \ \to \ \{e\}$$

where ψ is a morphism of pointed sets that preserves the H-action. Show that $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Exercise 82.5. Is a monomorphism of pointed sets necessarily injective?

Exercise 82.6. Provide a proof for Lemma 82.3.

Exercise 82.7. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. By the convention of page 1898 we can view elements in $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, \{x_0\})$. Using this point of view, how can we visualize the action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$?

Exercise 82.8. Give an example of a connected CW-complex X, a connected subcomplex $A \subset X$ and an $n \in \mathbb{N}$ such that the homotopy group $\pi_n(X/A)$ is not isomorphic to the relative homotopy group $\pi_n(X, A)$.

83. Fibrations

In this chapter we introduce fibrations and we show that fibrations give rise to a long exact sequence of homotopy groups. In the subsequent two chapters we will give many more examples of fibrations and we will see that fibrations allow us to compute various interesting higher homotopy groups.

83.1. **Definition and examples of fibrations.** The following is the key definition of this chapter.

Definition. Let $p\colon Y \to B$ be a map between topological spaces.

(1) Let X be a topological space. We say that the map $p\colon Y \to B$ has the *homotopy lifting property with respect to* X if given any map $F\colon X \times [0,1] \to B$ and given any map $\widetilde{f}\colon X \times \{0\} \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a map $\widetilde{F}\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}", "??"'] & B. \end{tikzcd}$$

Sometimes we refer to $\widetilde{F}\colon X \times [0,1] \to Y$ as a *lift* of $F\colon X \times [0,1] \to B$. (Note that we do *not* demand that the lift $\widetilde{F}$ is any way unique.) We refer to Figure 1015 for an illustration of the definition.

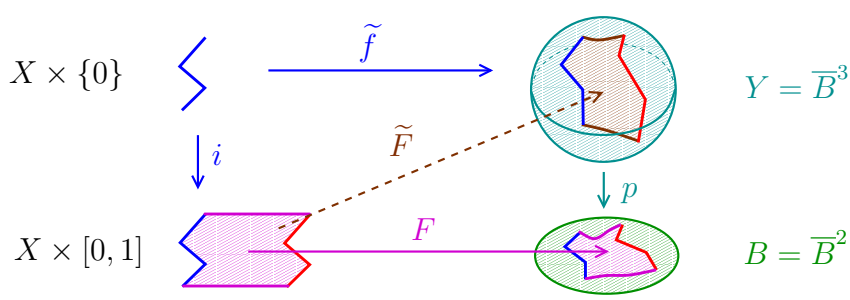


Figure 1015.

(2) We say that the map $p\colon Y \to B$ is a *fibration* if p has the homotopy lifting property with respect to all topological spaces.

(3) We say that the map $p\colon Y \to B$ is a *Serre fibration* if p has the homotopy lifting property with respect to all closed balls $\overline{B}^n$, $n \in \mathbb{N}_0$. So perhaps somewhat confusingly a Serre fibration is not necessarily a fibration.

Remark. Different textbooks mean different objects when they talk about "fibrations". Out definition coincides with the language used in [Hat, Rot, DK]. In [Bre], what we call a "fibration" is called a "Hurewicz fibration".

Examples.

(1) In Proposition 7.13 we already showed that every covering $p\colon Y \to B$ has the homotopy lifting property (in fact with a unique lift) with respect to every topological manifold, in particular with respect to any closed ball $\overline{B}^n$. This implies in particular that every covering is a Serre fibration.

(2) Let B and G be topological spaces. We denote by $p\colon B\times G \to B$ and $q\colon B\times G \to G$ the two obvious projections. In the following we show that the projection map $p\colon B \times G \to B$ is a fibration. Indeed, in the setting above the map

$$\begin{array}{rcl} \widetilde{F}\colon X \times [0,1] & \to & B \times G \\ (x,t) & \mapsto & \big(F(x,t), q\big(\underbrace{\widetilde{f}(x,0)}_{\in B\times G}\big)\big) \end{array}$$

makes the diagram commute. As is shown in Figure 1016, in contrast to the first example, the lift $\widetilde{F}$ is in general not unique.

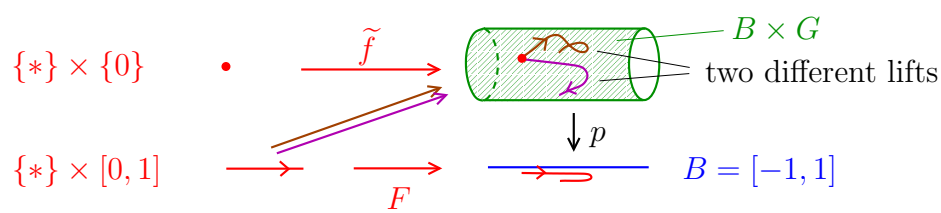


FIGURE 1016.

(3) For any $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n) \end{array}$$

is a Serre fibration. With some effort this statement can be proved "by hand". The case $n = 1$ is precisely the content of Exercise 83.1.

We conclude the list of examples with a "non-example". To make this example more readable we package it is as a lemma.

Lemma 83.1. *The map $p\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate is not a Serre fibration.*

Proof. We consider the topological space $X = \overline{B}^1 = [-1,1]$ and the maps $\widetilde{f}\colon X \times \{0\} \to S^1$ and $F\colon [-1,1] \times [0,1] \to [-1,1]$ that are illustrated in Figure 1017. We claim that there is no lifting $\widetilde{F}\colon X \times [0,1] \to S^1$. More precisely, we prove the following claim.

Claim. If $\widetilde{F}\colon X \times [0,1] \to S^1$ is a map with $p \circ \widetilde{F} = F$, then $\widetilde{F}|_{X\times\{0\}}$ does not agree with $\widetilde{f}$.

Thus let $\widetilde{F}\colon X\times[0,1] \to S^1$ be a map with $p\circ\widetilde{F} = F$. We make the following observations:

(1) The preimage $F^{-1}(0)$ is connected by the definition of F.
(2) We obtain from (1) and Lemma 1.88 that the image $\widetilde{F}(F^{-1}(0))$ is also connected.
(3) The fact that $p \circ \widetilde{F} = F$ implies that $\widetilde{F}(F^{-1}(0)) \subset p^{-1}(0)$.[1400]
(4) By definition of the map p we have $p^{-1}(0) = \{(0, \pm 1)\}$.
(5) It follows from (2), (3) and (4) that $\widetilde{F}$ restricted to $F^{-1}(0)$ is constant.
(6) The map $\widetilde{f}$ is not constant on $F^{-1}(0)$.

The combination of (5) and (6) proves our claim. □

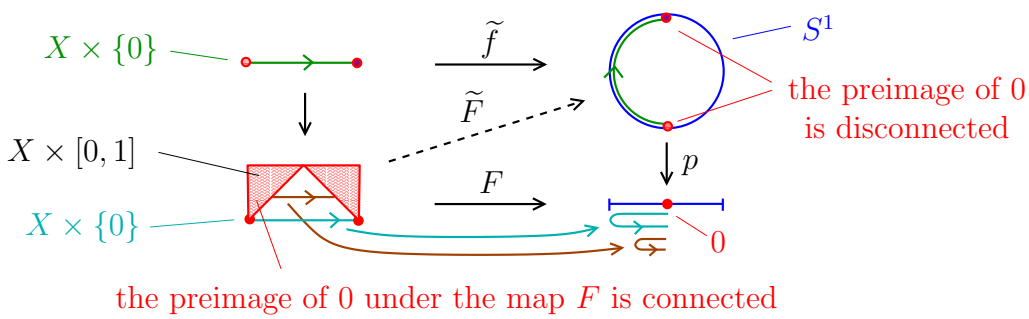


FIGURE 1017.

We continue with introducing some language and notation.

Notation.

(1) Given a fibration or a Serre fibration $p\colon Y \to B$ we sometimes refer to B as the *base space* and to Y as the *total space*.[1401]

Furthermore, given a point $* \in B$ we refer to $F := p^{-1}(*)$ as a *fiber*. The last definition is somewhat dangerous since, as we see in the last of the above examples, the homeomorphism type $p^{-1}(*)$ depends on the choice of $*$. We will shortly discuss "uniqueness" of the fiber. Finally throughout this chapter we denote by $i\colon F \to Y$ the inclusion map.

(2) Given $n \in \mathbb{N}_0$ we define I, I^n and J^n as on page 1883.

Before we can formulate the main theorem of this chapter we need to make some preparations.

Lemma 83.2.

[1400]Since I always find these arguments confusing let me give the argument: Let $y \in \widetilde{F}(F^{-1}(0))$. This means that there exists a $P \in F^{-1}(0)$ with $y = \widetilde{F}(P)$. We have $p(y) = p(\widetilde{F}(P)) = F(P)$, but $P \in F^{-1}(0)$, thus $p(y) = 0$. In other words, we have $y \in p^{-1}(0)$.

[1401]This mirrors the language that we introduced on page 1782 for vector bundles.

(1) *For each $n \in \mathbb{N}$ there exists an orientation-preserving homeomorphism*

$$\left(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]\right) \xrightarrow{\cong} \left(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\right)$$

of pairs of topological spaces. (*This statement is illustrated in Figure 1018 on the left.*)

(2) *For each $n \in \mathbb{N}$ there exists a homeomorphism*

$$(I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\}).$$

of pairs of topological spaces. (*This statement is illustrated in Figure 1018 on the right.*)

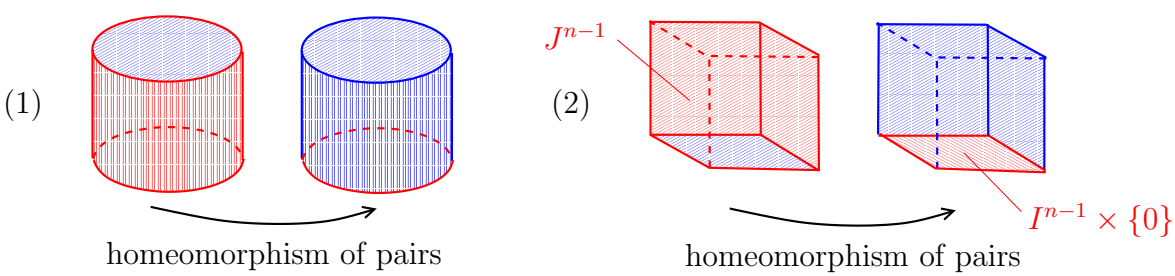


FIGURE 1018. Illustration of Lemma 83.2.

Sketch of proof (∗).

(1) For $n = 1$ such a homeomorphism is illustrated in Figure 1019. The general case is

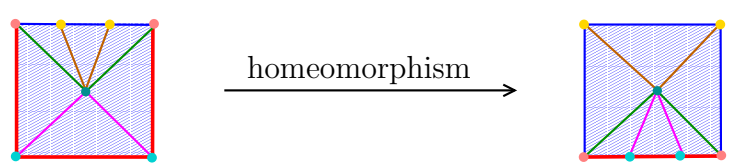


FIGURE 1019. Illustration for the proof of Lemma 83.2.

given by "rotating this homeomorphism around the $x_1 = 0$-axis in $\mathbb{R}^n$".

(2) Using the homeomorphism $\overline{B}^{n-1} \cong I^{n-1}$ from Proposition 1.48 (2) one can easily reduce the proof of (2) to the statement that we had just proved in (1). □

We will also soon need the following elementary lemma.

Lemma 83.3. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Furthermore let $n \in \mathbb{N}$. Given any map $\varphi\colon (I^n, \partial I^n) \to (B, \{b_0\})$ there exists a map*

$\widetilde{\varphi}\colon (I^n, J^{n-1}) \to (Y, \{y_0\})$ *such that the following diagram commutes:*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ I^n \arrow[ur, dashed, "\widetilde{\varphi} \text{ with } J^{n-1} \mapsto y_0"] \arrow[r, "\varphi \text{ with } \partial I^n \mapsto b_0"'] & B. \end{tikzcd}$$

We refer to Figure 1020 for an illustration.

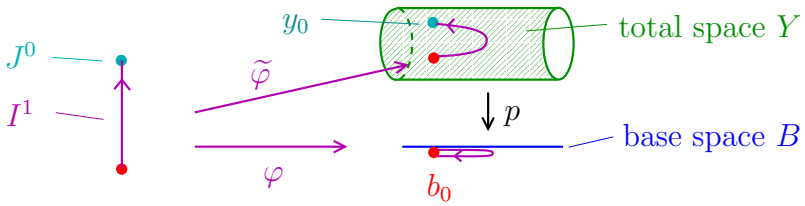


FIGURE 1020. Illustration of Lemma 83.3.

Proof. Let $\Psi\colon (I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\})$ be a homeomorphism as in Lemma 83.2 (2). We consider the diagram

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "\Psi^{-1}"] \arrow[d, hook] & J^{n-1} \arrow[r, "\text{constant map } y_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\ I^{n-1} \times [0,1] \arrow[r, "\Psi^{-1}"'] \arrow[urr, dashed, "G"] & I^n \arrow[r, "\varphi"'] & B. \end{tikzcd}$$

Note that the diagram without the red dashed arrow commutes. Since p is a Serre fibration and since I^{n-1} is homeomorphic to the $(n-1)$-ball, see Proposition 1.47 (2), it follows that a map $G\colon I^{n-1} \times I \to Y$ exists that makes the diagram commute. It is now straightforward to verify that $G \circ \Psi\colon I^n \to B$ has the desired properties. $\square$

83.2. **The homotopy fiber of a fibration.** First we introduce the following, arguably long overdue notation.

Notation. Given two topological spaces X and Y we define $[X, Y]$ to be the set of homotopy equivalence classes of maps from X to Y.[1402]

The following lemma is a reformulation of Lemma 9.5.

Lemma 83.4. *Let X, Y and Z be topological spaces. The map*

$$\begin{array}{rcl} [Y, Z] \times [X, Y] & \mapsto & [X, Z] \\ ([g], [f]) & \mapsto & [g \circ f] \end{array}$$

is well-defined.

[1402]Recall that in Lemma 9.3 we saw that "being homotopic" is an equivalence relation.

Now we can formulate the key technical proposition of this section.

Proposition 83.5. *Let $p\colon Y \to B$ be a fibration. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram*

$$\begin{tikzcd} F_{b_0} \times \{0\} \arrow[r, "(x,0)\mapsto x"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ F_{b_0} \times [0,1] \arrow[r, "(x,t)\mapsto \alpha(t)"'] \arrow[ur, "H(\alpha)"'] & B. \end{tikzcd}$$

Since $p\colon Y \to B$ is a fibration there exists a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$.[1403] *(We refer to Figure 1021 for an illustration.) We define $\Theta(\alpha) := H(\alpha)_1$. The following statements hold:*

(1) *The map $\Theta(\alpha)$ goes from F_{b_0} to F_{b_1}.*
(2) *The map $\Theta(\alpha)$, viewed as an element in $[F_{b_0}, F_{b_1}]$, is well-defined.*
(3) *The map $\Theta(\alpha)\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence.*
(4) *If two paths $\alpha, \beta\colon [0,1] \to B$ from b_0 to b_1 are path-homotopic, then $[\Theta(\alpha)] = [\Theta(\beta)]$.*
(5) *If c_b is the constant path at $b \in B$, then $[\Theta(c_b)] = [\mathrm{id}_{F_b}] \in [F_b, F_b]$.*
(6) *If α is a path in B from b_0 to b_1 and β is a bath in B from b_1 to b_2, then*

$$\underbrace{[\Theta(\alpha * \beta)]}_{\in [F_{b_0}, F_{b_2}]} = \underbrace{[\Theta(\beta)]}_{\in [F_{b_1}, F_{b_2}]} \circ \underbrace{[\Theta(\alpha)]}_{\in [F_{b_0}, F_{b_1}]}.$$

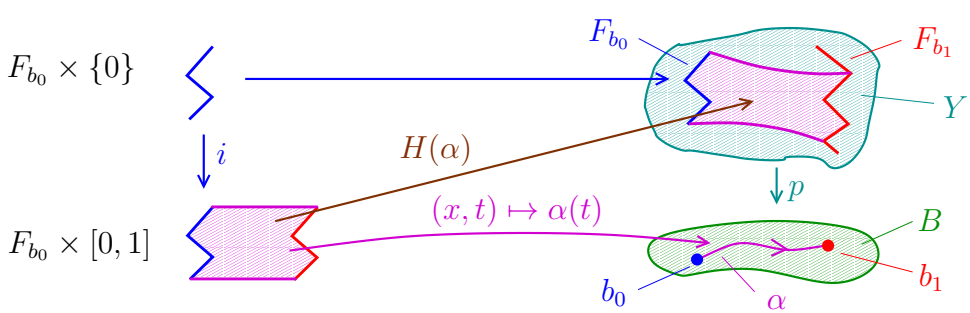


FIGURE 1021. Illustration of Proposition 83.5.

Remark. The picture behind the statement of Proposition 83.5 is that while walking along B we can drag the fiber along. Thus any path defines a map between the corresponding fibers.

[1403] Recall that in general $H(\alpha)$ is not uniquely determined by the data, i.e. there might be many such maps.

Proof $(*)$. Let $p\colon Y \to B$ be a fibration over a topological space B. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram

$$\begin{tikzcd} F_{b_0} \times \{0\} \arrow[r, "(p,0)\mapsto p"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ F_{b_0} \times [0,1] \arrow[r, "(x,t)\mapsto\alpha(t)"'] \arrow[ur, "H(\alpha)"] & B. \end{tikzcd}$$

Since $p\colon Y \to B$ is by hypothesis a fibration there exists in fact such a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$. Note that the map $H(\alpha)_0\colon F_{b_0} \to B$ is just the obvious inclusion. We turn to the proofs of the statements, albeit perhaps in an unexpected order:

(1) Since the above diagram commutes we see immediately that the map $H(\alpha)_1$ is a map from $F_{b_0} \to F_{b_1}$.

(4) Let $\alpha, \beta\colon [0,1] \to B$ be two paths from b_0 to b_1. We pick maps $H(\alpha)$ and $H(\beta)$ as above, keeping in mind that $H(\alpha)$ and $H(\beta)$ are not uniquely determined. Now we assume that α and β are path-homotopic. We need to show the following claim.

Claim. The maps $H(\alpha)_1, H(\beta)_1\colon F_{b_0} \to F_{b_1}$ are homotopic.

We start out the proof of the claim with the following observation. It follows immediately from Lemma 83.2 (2), applied to $n = 2$, that there exists a self-homeomorphism

$$\Psi\colon [0,1] \times [0,1] \quad \to \quad [0,1] \times [0,1]$$

that restricts to a homeomorphism

$$\underbrace{([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1])}_{=:U} \quad \to \quad [0,1] \times \{0\}.$$

For reassurance the construction of such a map is also sketched in Figure 1022. Now

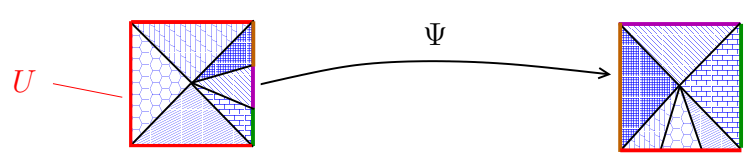


FIGURE 1022.

we pick a homotopy $G\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ from α to β. We consider the

following diagram

$$\begin{tikzcd}
F_{b_0}\times[0,1]\times\{0\} \arrow[r, "\mathrm{id}_{F_{b_0}}\times\Psi^{-1}"] \arrow[d, hook] & \overbrace{(F_{b_0}\times[0,1]\times\{0,1\})\cup(F_{b_0}\times\{0\})\times[0,1]}^{=F_{b_0}\times U} \arrow[r, "\Omega"] \arrow[d, hook] & Y \arrow[d, "p"] \\
F_{b_0}\times[0,1]\times[0,1] \arrow[r, "\mathrm{id}_{F_{b_0}}\times\Psi^{-1}"'] \arrow[urr, dashed, "\Xi"] & F_{b_0}\times[0,1]\times[0,1] \arrow[r, "(P,s,t)\mapsto G(s,t)"'] \arrow[ur, dashed, "\Xi\circ(\mathrm{id}_{F_{b_0}}\times\Psi)"'] \arrow[l, bend left, "\mathrm{id}_{F_{b_0}}\times\Psi"] & B.
\end{tikzcd}$$

We make the following clarifications and observations:

(a) The map Ω is defined as follows:[1404]
  (i) on $F_{b_0}\times[0,1]\times\{0\}=F_{b_0}\times[0,1]$ the map Ω is given by $H(\alpha)$,
  (ii) on $F_{b_0}\times[0,1]\times\{1\}=F_{b_0}\times[0,1]$ the map Ω is given by $H(\beta)$,
  (iii) on $F_{b_0}\times\{0\}\times[0,1]$ the map Ω is given by the projection onto F_{b_0} followed by the inclusion into Y.

(b) It follows immediately from the definitions that the diagram commutes.

(c) Since $p\colon Y\to B$ is a fibration we see that there exists a map Ξ that makes the outer diagram commute.

(d) The map $\Xi\circ(\mathrm{id}_{F_{b_0}}\times\Psi)\colon F_{b_0}\times[0,1]\times[0,1]\to Y$ is a homotopy between $H(\alpha)$ and $H(\beta)$. In particular the map $(\Xi\circ(\mathrm{id}_{F_{b_0}}\times\Psi))_1\colon F_{b_0}\times[0,1]\to Y$ is the desired homotopy between $H(\alpha)_1$ and $H(\beta)_1$.

This concludes the proof of the claim.

(5) This statement follows from the observation that for the constant path c_b at a point $b\in B$ we can choose $H(c_b)\colon F_b\times[0,1]\to F_b$ to be the projection. Evidently we obtain that $\Theta(c_b)=H(c_b)_1=\mathrm{id}_{F_b}$.

(6) Let α be a path in B from b_0 to b_1 and let β be a bath in B from b_1 to b_2. Almost the same way that we define the product $\alpha*\beta$ we combine $H(\alpha)$ and $H(\beta)$. More precisely, we consider the map[1405]

$$\begin{aligned} H(\alpha*\beta)\colon F_{b_0}\times[0,1] &\to X \\ (P,t) &\mapsto \begin{cases} H(\alpha)(P,2t), & \text{if } t\in[0,\tfrac12], \\ H(\beta)\big(\underbrace{H(\alpha(P),1)}_{\in F_{b_1}},2t-1\big), & \text{if } t\in[\tfrac12,1]. \end{cases}\end{aligned}$$

One can easily verify that this map has all the properties that we expect from a map that is called $H(\alpha*\beta)$. The desired statement is now an immediate consequence of the observation that $H(\alpha*\beta)_1=H(\beta)_1\circ H(\alpha)_1$.

[1404]As on so many other occasions we can use Lemma 1.32 (2) to show that the map Ω is indeed continuous.

[1405]It follows again from Lemma 5.3 that this map is continuous.

(3) Let $\alpha\colon [0,1] \to B$ be a path from b_0 to b_1. We pick a map $H(\alpha)\colon F_{b_0} \times [0,1] \to X$ as above. We want to show that the corresponding map $H(\alpha)_1\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence. We denote by $\overline{\alpha}\colon [0,1] \to B$ the inverse path of α, i.e. the path from b_1 to b_0 that is given by $\overline{\alpha}(t) = \alpha(1-t)$. We pick a map $H(\overline{\alpha})\colon F_{b_1} \times [0,1] \to X$ as above. It remains to prove the following claim.

Claim.
(a) The map $H(\overline{\alpha})_1 \circ H(\alpha)_1\colon F_{b_0} \to F_{b_0}$ is homotopic to the identity of F_{b_0}.
(b) The map $H(\alpha)_1 \circ H(\overline{\alpha})_1\colon F_{b_1} \to F_{b_1}$ is homotopic to the identity of F_{b_1}.

We prove the first statement. The proof of the second statement is verbatim the same. We have

$$\begin{array}{ccccccc} [\,H(\overline{\alpha})_1 \circ H(\alpha)_1\,] & = & [\,H(\alpha * \overline{\alpha})\,] & = & [\,H(c_{b_0})\,] & = & [\,\mathrm{id}_{F_{b_0}}\,]. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by (6)} & \text{by (4) and Proposition 5.6 (3)} & & & \text{by (5).} & \end{array}$$

This concludes the proof of the claim.

(2) We apply the argument from (4) to $\beta = \alpha$ together with two maps $H(\alpha)$ and $H(\beta)$ that make the diagram commute. We obtain by (3) that $H(\alpha)$ and $H(\beta)$ are homotopic, i.e. we see that $[H(\alpha)] \in [F_{b_0}, F_{b_1}]$ is well-defined. □

Using Proposition 83.5, and the proof thereof, we almost immediately obtain the following proposition.

Proposition 83.6. *Let $p\colon Y \to B$ be a map between topological spaces. We assume that B is path-connected.*

(1) *If $p\colon Y \to B$ is a fibration, then for any two points $b_0, b_1 \in B$ the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.*
(2) *If $p\colon Y \to B$ is a Serre fibration, then for any two points $b_0, b_1 \in B$ such that the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are discrete subsets, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.*

Proof.

(1) This statement follows immediately from Proposition 83.5 (3) and the hypothesis that B is path-connected.
(2) Let $p\colon Y \to B$ be a Serre fibration with B a path-connected topological space. Let $b_0, b_1 \in B$ such that the fibers $F_{b_0} := p^{-1}(b_0)$ and $F_{b_1} := p^{-1}(b_1)$ are discrete subsets. In the proof of Proposition 83.5 (3) we used the homotopy lifting property for the topological spaces $X = F_{b_i} \times [0,1]^n$ with $i \in \{0,1\}$ and $n \in \{1,2\}$. Since F_{b_0} and F_{b_1} are discrete we see that each component of $X = F_{b_i} \times [0,1]^n$ is homeomorphic to $[0,1]^n$, i.e. each component is homeomorphic to $\overline{B}^n$. Since we can do the lifting component-wise it suffices in our context that we are given a Serre fibration. We leave it to the reader to fill in the details. □

Proposition 83.6 (1) allows us to make the following definition.

Definition. Let $p\colon Y \to B$ be a fibration over a path-connected topological space B. We refer to the homotopy type of the fibers as the *homotopy fiber* of the fibration.

Remark. A more precise statement of Proposition 83.6 (1) holds: any path $\gamma\colon [0,1] \to B$ from b_0 to b_1 defines naturally a homotopy equivalence between the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$. We refer to [DK, Theorem 6.12] for the precise formulation and proof of this statement.

Examples.

(1) Let $p\colon \widetilde{X} \to X$ be a covering map over a path-connected topological space. As discussed on page 1912 such a map is a Serre fibration. It follows immediately from the definition of a covering map that for each $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\widetilde{X}$. Therefore it follows from Lemma 9.9 and Proposition 83.6 (2) that given any two points $x_0, x_1 \in X$ there exists a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. In other words, we have found a new proof for Lemma 7.2.

(2) Let us consider again the "non-example" from Lemma 83.1, i.e. we consider the map

$$\begin{array}{rcl} p\colon Y = S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2=1\} & \to & B = [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

In this case $p^{-1}(-1) = \{(-1,0)\}$ and $p^{-1}(0) = \{(0,\pm 1)\}$ are discrete subsets, but the number of components is different. From Lemma 9.9 we deduce that $p^{-1}(-1)$ is not homotopy equivalent to $p^{-1}(0)$. Thus Proposition 83.6 (2) gives a new proof (after the more hands-on proof provided by Lemma 83.1) that p is not a Serre fibration.

(3) We consider the map $p\colon Y \to B$ given by the total space

$$Y \;:=\; \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,\big|\, y = 1-x \text{ and } x \in [0,1]\} \cup \bigcup_{n\in\mathbb{N}} \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,|\, x \in [0,1] \text{ and } y = 1+\tfrac{1}{n}\}$$

the base space $B = [0,1]$ and the obvious projection $p\colon Y \to X$. (This example is illustrated in Figure 1023.) It is an amusing exercise to show that $p\colon Y \to B$ is a Serre fibration. We saw in Exercise 9.9 that the fibers $p^{-1}(0)$ and $p^{-1}(1)$ are not homotopy equivalent. It follows from Proposition 83.6 (1) that $p\colon Y \to B$ is not a fibration.[1406]

We can also show more directly that p is not a fibration: We consider $X = p^{-1}(1)$, the usual projection map $f\colon X \times [0,1] \to B = [0,1]$ together with the obvious inclusion $\widetilde{f}\colon X \times \{0\} \to Y$. It is once again an amusing elementary exercise to show that there is no map $\widetilde{F}\colon X \times [0,1] \to B$ with all the desired properties. This shows that $p\colon Y \to B$ is not a fibration.

[1406]The discussion of this example would be more intelligible if we followed the convention of [Bre, p. 450] and if we referred to "fibrations" as "Hurewicz fibrations" to better distinguish them from "Serre fibrations".

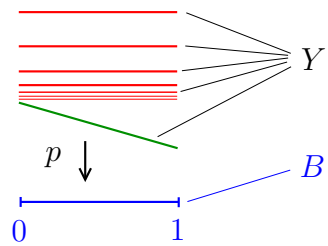


FIGURE 1023.

83.3. **The long exact homotopy sequence of a Serre fibration.** For us the most interesting aspect of fibrations is that they give rise to long exact sequences of homotopy groups. More precisely, we have the following theorem.

Theorem 83.7. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$.*

(1) *Given any $n \in \mathbb{N}$ the following map is well-defined:*

$$\begin{aligned} \partial_n\colon \pi_n(B, b_0) &\to \pi_{n-1}(F, y_0) \\ [\varphi\colon (I^n, \partial I^n) \to (B, b_0)] &\mapsto \left[\begin{array}{c} \textit{pick a map } \widetilde{\varphi}\colon (I^n, J^{n-1}) \to (Y, \{y_0\}) \textit{ with } p \circ \widetilde{\varphi} = \varphi \\ (\textit{we know from Lemma 83.3 that such a map exists}) \\ \textit{and then restrict the map } \widetilde{\varphi} \textit{ to } I^{n-1} = I^{n-1} \times \{0\} \end{array}\right]. \end{aligned}$$

(*The definition of ∂_n is moderately successfully illustrated in Figure 1024.*) *Furthermore, these maps ∂_n have the following properties:*

(a) *The maps are morphisms of groups (if $n \geq 2$) and of pointed sets (if $n = 1$).*

(b) *The maps are natural in the following two senses: Suppose we are given another Serre fibration $p'\colon Y' \to B'$ and suppose we are given a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "\Phi"'] & B \arrow[d, "\phi"] \\ Y' \arrow[r, "p'"] & B'. \end{tikzcd}$$

If we write $y_0' = \Phi(y_0), b_0' = \phi(b_0)$ and $F' = (p')^{-1}(b_0')$, then for any $n \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} \pi_n(B, b_0) \arrow[r, "\partial_n"] \arrow[d, "\phi_*"'] & \pi_{n-1}(F, y_0) \arrow[d, "(\Phi|_F)_*"] \\ \pi_n(B', b_0') \arrow[r, "\partial_n"] & \pi_{n-1}(F', y_0'). \end{tikzcd}$$

(c) *Let $\alpha\colon [0,1] \to B$ be a path from b_0 to some point b_1.*[1407] *We write $F_1 = p^{-1}(b_1)$. If $\Theta(\alpha)\colon F \to F_1$ is a map as in Proposition 83.5, then the following diagram*

[1407] Here we allow of course that $b_1 = b_0$.

commutes

$$\begin{array}{ccc} \pi_n(B,b_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F,y_0) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \Theta(\alpha)_*} \\ \pi_n(B,b_1) & \xrightarrow{\partial_n} & \pi_{n-1}(F_1,y_1) \end{array}$$

where $y_1 = \Theta(\alpha)(y_0)$.

(2) *The following sequence is exact:*[1408]

$$\begin{aligned} \ldots &\xrightarrow{\partial_{n+1}} \pi_n(F,y_0) \xrightarrow{i_*} \pi_n(Y,y_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial_n} \\ &\xrightarrow{\partial_n} \pi_{n-1}(F,y_0) \xrightarrow{i_*} \ldots\ldots\ldots \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial_1} \\ &\xrightarrow{\partial_1} \pi_0(F,y_0) \xrightarrow{i_*} \pi_0(Y,y_0) \xrightarrow{p_*} \pi_0(B,b_0). \end{aligned}$$

(3) *If the inclusion map* $i\colon F \to Y$ *is homotopic rel* y_0 *to the constant map* $c_{y_0}\colon F \to \{y_0\}$, *then for any* $n \in \mathbb{N}_{\geq 2}$ *the above long exact sequence breaks into a short exact sequence*

$$0 \;\to\; \pi_n(Y,y_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial_n} \pi_{n-1}(F,y_0) \;\to\; 0.$$

Furthermore the short exact sequence splits, which implies that for any $n \in \mathbb{N}_{\geq 2}$ *we have an isomorphism*

$$\pi_n(B,b_0) \;\cong\; \pi_n(Y,y_0) \oplus \pi_{n-1}(F,y_0).$$

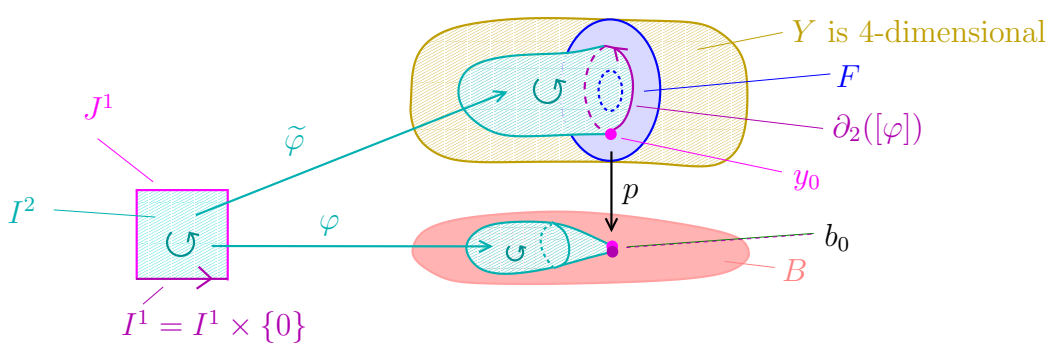


FIGURE 1024.

Before we head towards the proof of Theorem 83.7 we want to see what the theorem tells us in two types of fibrations that we had considered above, namely for coverings and products.

Example. Let $p\colon Y \to B$ be a covering of path-connected topological spaces. As mentioned above, by Proposition 7.13 we know that p is a Serre fibration. We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. It follows immediately from the definition of a covering

[1408]Here, at the lower end, we mean "exact" in the sense of the definition on page 1887.

map that F, equipped with the subspace topology coming from Y, is a discrete topological space, in particular $\#\pi_0(F,y_0) = \#F$ and $\pi_i(F,y_0) = 0$ for $i \geq 1$. From the long exact sequence provided by Theorem 83.7 we obtain the following two pieces of information:

(1) At the lower end the long exact sequence looks as follows:

$$\underbrace{\pi_1(F,y_0)}_{=\{e\}} \to \pi_1(Y,y_0) \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial} \underbrace{\pi_0(F,y_0)}_{\#\pi_0(F,y_0)=\#F} \to \underbrace{\pi_0(Y,y_0)}_{=\{e\}} \to \underbrace{\pi_0(B,b_0)}_{=\{e\}}.$$

In other words we obtain the short exact sequence

$$\{e\} \to \pi_1(Y,y_0) \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial} \pi_0(F,y_0) \to \{e\}.$$

We draw the following two conclusions:

(a) The map $p_*\colon \pi_1(Y,y_0) \to \pi_1(B,b_0)$ is a monomorphism.

(b) With a little bit of extra thought[1409] we obtain from the short exact sequence, Proposition 83.5, Theorem 83.7 (1c) and Lemma 82.2 that $p_*(\pi_1(Y,y_0))$ is a subgroup of $\pi_1(B,b_0)$ of index $\#\pi_0(F,y_0) = \#F = [Y:B]$.

These two statements are precisely the content of Lemma 6.4 and Lemma 7.16 (3).

(2) Since all other homotopy groups of F vanish we see that for any $n \geq 2$ the map $p_*\colon \pi_n(Y,x_0) \to \pi_n(B,b_0)$ is an isomorphism. This result recovers Proposition 28.13.

We move on to the next application. More precisely we want to apply Theorem 83.7 to the case of a Serre fibration of the form $X \times Y \to X$. This leads to a new proof of Propositions 7.21 and 28.8.

Proposition 28.8. *Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. We consider the maps*

$$\begin{array}{rcl} i\colon X & \to & X \times Y \\ x & \mapsto & (x,y_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & X \times Y \\ y & \mapsto & (x_0,y). \end{array}$$

For any $n \in \mathbb{N}$ the map[1410]

$$\begin{array}{rcl} \Phi\colon \pi_n(X,x_0) \times \pi_n(Y,y_0) & \to & \pi_n(X \times Y,(x_0,y_0)) \\ (\varphi,\psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is an isomorphism.

Proof. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the two obvious projections. On page 1912 we saw that both projections are fibrations. We concentrate on the first projection map $p\colon X \times Y \to X$. It is a fibration with fiber $p^{-1}(x_0) = \{x_0\} \times Y$. Using

[1409] We leave this extra thinking as a mental exercise to the reader.

[1410] Note that for $n \geq 2$ the map is a group homomorphism since in this case the group on the right-hand side is abelian. It is a priori not completely clear that the map is also a homomorphism for $n = 1$, but we will see in the proof that the map is indeed a homomorphism.

the obvious homeomorphism $Y \to \{x_0\} \times Y$ we obtain from Theorem 83.7 the long exact sequence

$$\ldots \;\to\; \pi_n(Y, y_0) \;\xrightarrow{j_*}\; \pi_n(X \times Y, (x_0, y_0)) \;\xrightarrow{p_*}\; \pi_n(X, x_0) \;\to\; \pi_{n-1}(Y, y_0)) \;\xrightarrow{j_*}\; \ldots .$$

Evidently we have $q \circ j = \mathrm{id}_Y$ which implies that $q_* \circ j_* = \mathrm{id}_{\pi_1(Y,y_0)}$. It follows that for every $n \in \mathbb{N}_0$ the map $j_* \colon \pi_n(F, (x_0, y_0)) \to \pi_n(X \times Y, (x_0, y_0))$ is a monomorphism. Thus for each $n \in \mathbb{N}$ we see that

$$0 \;\to\; \pi_n(Y, y_0) \;\xrightarrow{j_*}\; \pi_n(X \times Y, (x_0, y_0)) \;\xrightarrow{p_*}\; \pi_n(X, x_0) \;\to\; 0$$

is exact. Once again using that $q_* \circ j_* = \mathrm{id}$ we obtain from Lemma 33.2 (and the remark on page 838) that the given map

$$\begin{array}{rcl} \Phi \colon \pi_n(X, x_0) \times \pi_n(Y, y_0) & \to & \pi_n(X \times Y, (x_0, y_0)) \\ (\varphi, \psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is indeed an isomorphism. □

Even these modest applications should already convince the reader that it is worth proving Theorem 83.7. After the proof we will give many more examples of fibrations and we will find many more interesting applications.

83.4. **Proof of Theorem 83.7.** The proof of Theorem 83.7 requires some preparations, i.e. we first need to prove some propositions.

Proposition 83.8. *Let $p \colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes. We denote by $i \colon K \times \{0\} \cup L \times [0,1] \to K \times [0,1]$ the inclusion map. Given any map $F \colon K \times [0,1] \to B$ and given any map $\widetilde{f} \colon K \times \{0\} \cup L \times [0,1] \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a map $\widetilde{F} \colon K \times [0,1] \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} K \times \{0\} \cup L \times [0,1] \arrow[r, "\widetilde{f}"] \arrow[d, hook, "i"] & Y \arrow[d, "p"] \\ K \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}", "??"'] & B. \end{tikzcd}$$

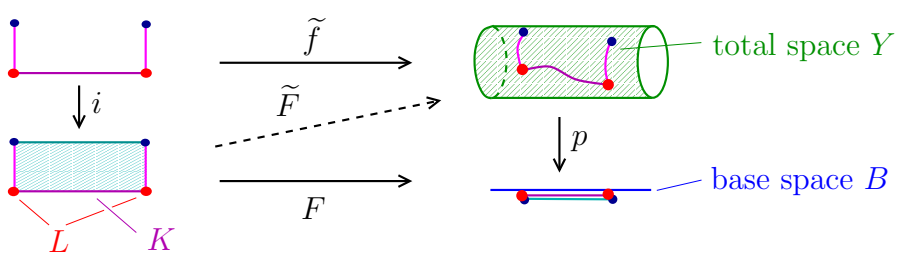


FIGURE 1025. Illustration of Proposition 83.8.

Proof. We start out with a reformulation of our task: Given $n = 0, 1, 2, 3, \dots$ we need to define maps

$$\widetilde{F}^n \colon (K \times \{0\}) \cup (K^n \cup L) \times [0,1] \;\to\; Y$$

that make the above diagram commute and which have the property that for $n < m$ the maps $\widetilde{F}^n$ and $\widetilde{F}^m$ agree on the common domain.

We set $\widetilde{F}^{-1} = \widetilde{f}$. Suppose we have already defined maps $\widetilde{F}^{-1}, \dots, \widetilde{F}^{n-1}$ as above. Let $\{\Phi_i \colon (\overline{B}^n, S^{n-1}) \to (K^n, L^{n-1})\}_{i \in I}$ be the characteristic maps of the n-cells of K that do not lie in L. Recall that by Lemma 83.2 (1) there exists a homeomorphism

$$\Psi \colon \big(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])\big) \;\overset{\cong}{\to}\; \big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\big).$$

For each $i \in I$ we consider the following diagram

$$\begin{tikzcd}
\overline{B}^n \times \{0\} \arrow[r, "\Psi^{-1}"] \arrow[d, hook] & \overline{B}^n \times \{0\} \cup S^{n-1} \times \{1\} \arrow[r, "\widetilde{F}^{n-1} \circ \Phi_i"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [0,1] \arrow[r, "\Psi^{-1}"'] \arrow[urr, dashed, "\Xi_i"] & \overline{B}^n \times [0,1] \arrow[r, "F \circ \Phi_i"'] & B.
\end{tikzcd}$$

One can easily verify that the two rectangles commute. Since $p \colon Y \to B$ is a Serre fibration there exists a map $\Xi_i \colon \overline{B}^n \times [0,1] \to Y$ that makes the diagram commute. These maps $\{\Xi_i\}_{i \in I}$ can now be used to extend $\widetilde{F}^{n-1}$ to the desired map $\widetilde{F}^n$. $\square$

Proposition 83.9. *Let $p \colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes such that L is a deformation retract of K. We denote by $i \colon L \to K$ the inclusion map. Given any map $f \colon K \to B$ and given any map $g \colon L \to Y$ with $f \circ i = p \circ g \colon L \to B$ there exists a map $\widetilde{f} \colon K \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
L \arrow[r, "g"] \arrow[d, hook, "i"'] & Y \arrow[d, "p"] \\
K \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{f}"] & B.
\end{tikzcd}$$

Example. Later on we will apply Proposition 83.9 to the special case that $K = \overline{B}^n$ and that $L = \{(-1, 0, \dots, 0)\}$. We refer to Figure 1026 for an illustration.

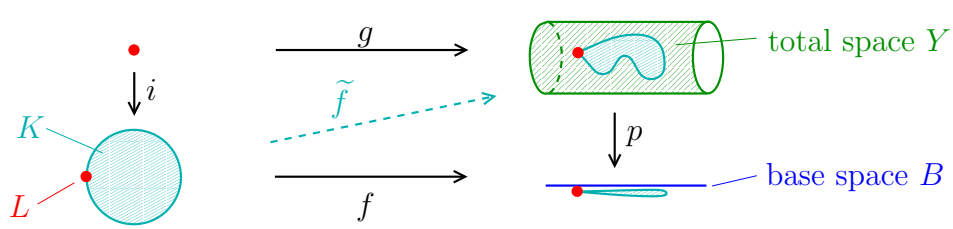


FIGURE 1026.

Proof. We make the following preparations:

(1) Since L is a deformation retract of K there exists by definition of a deformation retraction, see page 308, a homotopy $H\colon K \times [0,1] \to K$ rel L such that $H_0 = \mathrm{id}_K$ and $H_1(K) \subset L$. We denote by $r = H_1\colon K \to L$ the corresponding retraction.

(2) We denote by $\mathrm{proj}\colon L \times [0,1] \to L$ the obvious projection.

Now we consider the following commutative diagram

$$\begin{tikzcd}
L \times \{0\} \arrow[r, hook] \arrow[d, hook] & K \times \{1\} \cup L \times [0,1] \arrow[r, "r \cup \mathrm{proj}"] \arrow[d, hook] & L \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "p"] \\
K \times \{0\} \arrow[r, hook] & K \times [0,1] \arrow[r, "H"'] \arrow[urr, dashed, "\Phi"] & K \arrow[r, "f"'] & B.
\end{tikzcd}$$

To be somewhat more precise, the map $r \cup \mathrm{proj}\colon K \times \{1\} \cup L \times [0,1]$ is defined via the retraction r on $K = K \times \{1\}$ and the projection $L \times [0,1] \to L$. The right-hand square commutes by our hypothesis. One easily verifies that the middle square commutes. It follows from Proposition 83.8 that there exists a map $\Phi\colon K \times [0,1] \to Y$ that makes the diagram commute. For any $k \in K$ we have

$$(p \circ \Phi)(k,0) \;=\; f(H(k,0)) \;=\; f(H_0(k)) \;=\; f(k)$$

and for any $l \in L$ we have

$$\Phi(l,0) \;=\; (g \circ \mathrm{proj})(l,0) \;=\; g(\mathrm{proj}(l,0)) \;=\; g(l).$$

This shows that the restriction of Φ to $K = K \times \{0\}$ has the desired properties. □

Proposition 83.10. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Then*

$$p_*\colon \pi_n(Y, Y', y_0) \;\to\; \pi_n(B, B', b_0)$$

is an isomorphism for all $n \in \mathbb{N}$. (We refer to Figure 1027 for an illustration.)

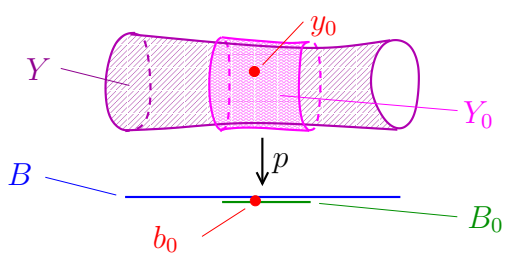


FIGURE 1027. Illustration of Proposition 83.10.

In the proof of Proposition 83.10 we will need the following elementary lemma.

Lemma 83.11. *Let $n \in \mathbb{N}$. If $x \in S^{n-1}$, then $(\overline{B}^n \times \partial[0,1]) \cup (\{x\} \times [0,1])$ is a deformation retract of $\overline{B}^n \times [0,1]$.*

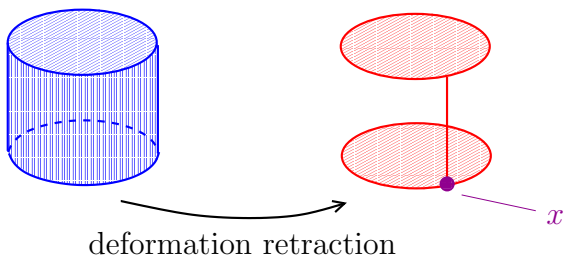


FIGURE 1028.

Proof of Lemma 83.11 $(*)$. As usual we write $I = [0,1]$. Given $k \in \{1, \ldots, n\}$ we consider

$$X_k \ := \ (I^{n-1} \times \partial I) \ \cup \ (\{\underbrace{(1,1,\ldots,1)}_{\in \mathbb{R}^{n-k}}\} \times I^k).$$

(We refer to Figure 1029 for an illustration.) Using Proposition 1.47 (2) and Lemma 1.73 (3) one can show easily that there exists a homeomorphism

$$(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0,1\}) \cup (\{x\} \times [0,1])) \ \stackrel{\cong}{\to} \ (X_n, X_1).$$

Thus it suffices to show that each X_i is a deformation retract of X_{i+1}. But such deformation retractions one can actually write down by hand. $\square$

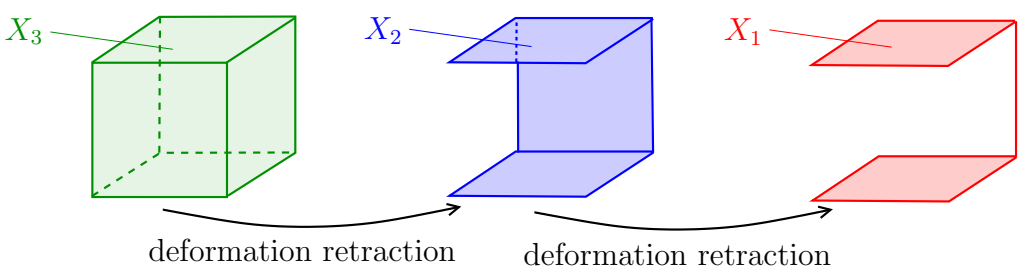


FIGURE 1029.

Now we can provide the proof of Proposition 83.10.

Proof of Proposition 83.10. Let $p\colon Y \to B$ be a Serre fibration and let $n \in \mathbb{N}$. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$. Finally let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Throughout this proof we use the identification from page 1898, i.e. we view elements in a relative homotopy group $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)$.

First we show that $p_*\colon \pi_n(Y,Y',y_0) \to \pi_n(B,B',b_0)$ is an epimorphism. Thus suppose we are given a map $f\colon (\overline{B}^n, S^{n-1}, *) \to (B,B',b_0)$. It follows from Proposition 83.9, applied to $(K,L) = (\overline{B}^n, *)$, that there exists a map $g\colon \overline{B}^n \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
\{*\} \arrow[r, "*\mapsto y_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \arrow[r, "f"'] \arrow[ur, dashed, "g"] & B.
\end{tikzcd}$$

Basically by definition g is a map $(\overline{B}^n, S^{n-1}, *) \to (Y,Y',y_0)$. Since $f = p \circ g$ we have $[f] = p_*([g])$.

Next we show that $p_*\colon \pi_n(Y,Y',y_0) \to \pi_n(B,B',b_0)$ is also a monomorphism. Therefore suppose that we are given two maps $g_0, g_1\colon (\overline{B}^n, S^{n-1}, *) \to (Y,Y',y_0)$ such that the equality $p_*([g_0]) = p_*([g_1]) \in \pi_n(B,B',b_0)$ holds. This equality means that $p \circ g_0$ is homotopic to $p \circ g_1$ via a homotopy $F\colon (\overline{B}^n, S^{n-1}, *) \times [0,1] \to (B,B',b_0)$. We consider the following commutative diagram (first without the dashed arrow):

$$\begin{tikzcd}
(\overline{B}^n \times \{0,1\}) \cup (\{*\} \times [0,1]) \arrow[r, "\text{via } g_0, g_1"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B.
\end{tikzcd}$$

By Proposition 83.9 and Lemma 83.11 there exists a map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes. This map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ is easily seen to be a homotopy between g_0 and g_1 which shows that $[g_0] = [g_1] \in \pi_n(Y,Y',y_0)$. □

With these preparations it is surprisingly easy to provide a proof of Theorem 83.7 (1) and (2).

Proof of Theorem 83.7 (1) and (2). Let $p\colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i\colon F \to Y$ the inclusion map. We consider the following diagram of maps

$$\begin{tikzcd}
\dots \arrow[r] & \pi_n(F,y_0) \arrow[r, "i_*"] & \pi_n(Y,y_0) \arrow[r] \arrow[dr, bend right, "p_*"'] & \pi_n(Y,F,y_0) \arrow[r, "\partial_n"] \arrow[d, "\cong"', "p_*"] & \pi_{n-1}(F,y_0) \arrow[r, "i_*"] & \dots \\
& & & \pi_n(B,\{b_0\},b_0) \arrow[d, "="] & & \\
& & & \pi_n(B,b_0). \arrow[uur, bend right, "\partial_n \circ p_*^{-1}"'] & &
\end{tikzcd}$$

We make the following comments:

(a) The top sequence of maps is the long exact sequence of homotopy groups of the pointed pair of topological spaces (Y,F,y_0), see Proposition 82.4 for details.

(b) The top vertical map is an isomorphism by Proposition 83.10.
(c) The bottom vertical map is the identification given on page 1884.
(d) It follows immediately from the definitions that the left-hand triangle commutes.
(e) The right-hand triangle commutes by definition of the diagonal map.

It follows immediately from the definition of p_* that given any $n \in \mathbb{N}$ the map

$$\partial_n \circ p_*^{-1} \colon \pi_n(B, b_0) \quad \to \quad \pi_{n-1}(F, y_0)$$

is given by

$$[\varphi \colon (I^n, \partial I^n) \to (B, b_0)] \quad \mapsto \quad \left[\begin{array}{c} \text{the restriction of } \widetilde{\varphi} \text{ to } I^{n-1} = I^{n-1} \times \{0\} \\ \text{where } \widetilde{\varphi} \text{ is a map } (I^n, J^{n-1}) \to (Y, \{y_0\}) \\ \text{with } p \circ \widetilde{\varphi} = \varphi \end{array} \right].$$

This observation shows that the map given in statement (1) of the theorem is well-defined and that it is a homomorphism (respectively a morphism of pointed sets). We leave it to the few readers who made it that far to verify that the maps ∂_n have the two properties stated in (1b) and (1c).

Furthermore, since the top sequence is exact and since the vertical maps are isomorphisms we see that the sequence given in statement (2) of the theorem is indeed exact. □

Proof of Theorem 83.7 (3). Let $p \colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i \colon F \to Y$ the inclusion map. Now we suppose that the inclusion map $i \colon F \to Y$ is homotopic rel y_0 to the constant map $c_{y_0} \colon F \to \{y_0\}$. Put differently, we suppose that there exists a homotopy $G \colon F \times [0,1] \to Y$ rel y_0 such that $G_0 = i$ and $G_1(f) = y_0$ for every $f \in F$.

It follows almost immediately from Lemmas 33.2 and 33.4 that it suffices to show that for any $n \in \mathbb{N}_{\geq 2}$ there exists a homomorphism $\lambda \colon \pi_{n-1}(F, y_0) \to \pi_n(B, b_0)$ such that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F, y_0)}$. We consider the map[1411]

$$\begin{array}{rcl} \lambda \colon \pi_{n-1}(F, y_0) & \to & \pi_n(Y, F, y_0) \xrightarrow{p_*} \pi_n(B, \{b_0\}, b_0) = \pi_n(B, b_0) \\ [\varphi \colon (I^{n-1}, \partial I^{n-1}) \to (F, y_0)] & \mapsto & \left[\begin{array}{c} (I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\}) \\ (x,t) \mapsto G(\varphi(x), t) \end{array} \right] \end{array}$$

It is elementary to verify that λ is in fact a homomorphism. It follows almost immediately from the definitions that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F, y_0)}$. □

83.5. **The path space fibration.** We recall the following, long dormant, definition from page 150.

Definition. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all maps[1412] from Y to X.

[1411] It takes a second to actually verify that the map $(x,t) \mapsto G(\varphi(x), t)$ defines in fact a map $(I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\})$.

[1412] Of course here we mean "all continuous maps".

(2) The *compact-open topology* on X^Y is the topology generated, in the sense of the definition on page 63, by all the sets of the form

$$M(K,U) \;:=\; \{f \in X^Y \mid f(K) \subset U\}$$

where $K \subset Y$ is compact and where $U \subset X$ is open.

In the following we state slightly simplified versions of two technical results which we proved a long time ago.

Proposition 2.4. *Let X be a topological space.*

(1) *The evaluation map*

$$\begin{array}{rcl} e\colon X^{[0,1]} \times [0,1] & \to & X \\ ((f\colon [0,1] \to X), y) & \mapsto & f(y) \end{array}$$

is continuous.

(2) *For each $y \in [0,1]$ the evaluation map*

$$\begin{array}{rcl} e\colon X^{[0,1]} & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(y) \end{array}$$

is continuous.

Proposition 2.6. *Let X and T be two topological spaces. Let $H\colon T \times [0,1] \to X$ be a map. Given $t \in T$ we denote by $H_t\colon [0,1] \to X$ the map defined by $H_t(y) = H(t,y)$. Then*

$$H\colon T \times [0,1] \to X \text{ is continuous } \Leftrightarrow \begin{array}{l} (1) \text{ each } H_t\colon [0,1] \to X \text{ is continuous, and} \\ (2) \text{ the map } T \to X^{[0,1]} \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

This reminder allows us to introduce our latest best friends.

Definition. Let X be a topological space.

(1) We define

$$\textit{free path space } X^{[0,1]} \;:=\; \text{set of all maps } [0,1] \to X$$

where $X^{[0,1]}$ is as usual equipped with the compact-open topology. Furthermore we introduce the

$$\textit{free loop space } \Omega(X) \;:=\; \{f \in X^{[0,1]} \mid f(0) = f(1)\}.$$

We equip the free loop space with the compact-open topology coming from $X^{[0,1]}$.

(2) Suppose we are also given a base point $x_0 \in X$. We introduce the

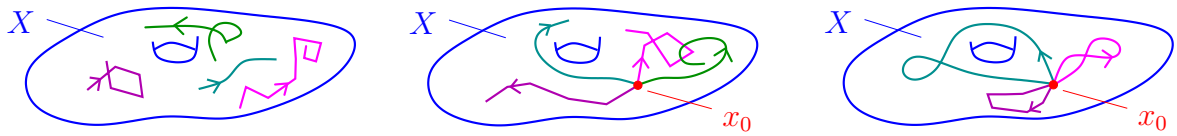
$$\textit{path space } P(X,x_0) \;:=\; \{f \in X^{[0,1]} \mid f(0) = x_0\}$$

and the

$$\textit{loop space } \Omega(X,x_0) \;:=\; \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}.$$

We equip both sets with the compact-open topology coming from $X^{[0,1]}$.

(3) We denote by $c_{x_0}\colon [0,1] \to X$ the constant path given by sending all t's to x_0.



elements in the free path space $X^{[0,1]}$ | elements in the path space $P(X, x_0)$ | elements in the loop space $\Omega(X, x_0)$

FIGURE 1030.

Examples.

(1) Let (X, x_0) be a pointed topological space and let $\alpha, \beta \in \Omega(X, x_0)$. It follows from Proposition 2.6 that a path in $\Omega(X, x_0)$ from α to β is basically the same as a homotopy between the maps α and β viewed as loops in (X, x_0). Thus we see that we have a natural identification

$$\pi_0(\Omega(X, x_0)) \;=\; \pi_1(X, x_0).$$

(2) Let (X, x_0) be a pointed topological space. It follows from Proposition 2.4 that the map

$$\begin{array}{rcl} e\colon P(X, x_0) & \to & X \\ f & \mapsto & f(1) \end{array}$$

is continuous. We can ask whether this map has a left-inverse. More precisely, the question is whether there exists a continuous map $s\colon X \to P(X, x_0)$ such that $e(s(x)) = x$ for all $x \in X$. This sounds like a rather random question, so let us reformulate it. The question is, can we *continuously* assign to each $x \in X$ a path from x_0 to x? Clearly that is possible say for $(X, x_0) = (\mathbb{R}^2, 0)$, we just assign to each $x \in \mathbb{R}^2$ the direct path $e(x)$ that is given by $t \mapsto t \cdot x$. But can we do so say for $X = S^1$ or $X = S^2$?[1413] We will answer this question in Exercise 83.2.

For the record we state the following lemma.

Lemma 83.12. *The maps*

$$\begin{array}{rcl} (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{array} \right) \end{array}$$

define a covariant functor from the category of pointed topological spaces to the category of pointed topological spaces. We refer to this functor as the loop space functor*.*

[1413]This question is somewhat similar to Question 37.7 and it also falls into the vaguely defined realm of "topological robotics".

Proof. Let $f\colon (X,x_0) \to (Y,y_0)$ be a map between pointed topological spaces. It follows from Lemma 2.5 (1) that the map

$$\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(Y,y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f\circ\gamma\colon [0,1]\to Y) \end{array}$$

is actually continuous. With this observation out of the way it is straightforward to prove the lemma. □

The following lemma says that in some sense (free) path spaces, in isolation, are not particularly interesting examples of topological spaces.

Lemma 83.13.

(1) *Let X be a topological space. We view X as a subset of the free path space $X^{[0,1]}$ via the inclusion $x \mapsto c_x$.*[1414] *Then X is a deformation retract of the free path space $X^{[0,1]}$.*

(2) *If (X,x_0) is a pointed topological space, then $\{c_{x_0}\}$ is a deformation retract of the path space $P(X,x_0)$, in particular the path space $P(X,x_0)$ is contractible.*

FIGURE 1031. Illustration of Lemma 83.13 (2).

Proof. We prove statement (2). The proof of statement (1) is basically identical. Thus let (X,x_0) be a pointed topological space. We consider the "obvious" deformation retraction given by

$$\begin{array}{rcl} F\colon P(X,x_0)\times[0,1] & \to & P(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & f(s\cdot(1-t)) \end{array}\right) \end{array}$$

The only reason why we are not already done is that we still need to verify that F is in fact continuous. As we will see this is slightly delicate. We consider the map

$$G\colon \underbrace{X^{[0,1]}\times[0,1]}_{=:T} \to X^{[0,1]}$$

that is defined in exactly the same way as F. Since $F = G|_{P(X,x_0)\times[0,1]}$ it suffices to show that G is continuous. By the "⇒"-direction of Proposition 2.6 applied to $T = X^{[0,1]}\times[0,1]$

[1414]Note that this inclusion $X \to X^{[0,1]}$ is continuous by Lemma 2.3.

it suffices to show that the map

$$\begin{array}{rcl} H\colon [0,1] \times \overbrace{(X^{[0,1]} \times [0,1])}^{=T} & \to & X \\ (s, ((g\colon [0,1] \to X), x)) & \mapsto & g(s \cdot x) \end{array}$$

is continuous. But the map H can be written as the composition

$$\begin{array}{rcccl} H\colon [0,1] \times X^{[0,1]} \times [0,1] & \to & X^{[0,1]} \times [0,1] & \to & X \\ (s, g, x) & \mapsto & (g, s \cdot (1-x)) & & \\ & & (g, t) & \mapsto & g(t). \end{array}$$

The first map is continuous by Lemma 1.53 and the second map is continuous by Proposition 2.4 (1). Thus we see that H itself is continuous. □

In contrast to the previous proposition the following theorem shows that (free) path spaces give rise to interesting *maps*.

Theorem 83.14.

(1) *Let X be a topological space. The evaluation map*

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(1) \end{array}$$

is a fibration. Furthermore, given any $x_0 \in X$ the fiber $p^{-1}(x_0)$ is the topological space of all paths $[0,1] \to X$ that end *in x_0.*

(2) *Let (X, x_0) be a pointed topological space. The evaluation map*

$$\begin{array}{rcl} p\colon P(X, x_0) & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(1) \end{array}$$

is a fibration with fiber $p^{-1}(x_0) = \Omega(X, x_0)$.

Theorem 83.14 leads us to the following definition.

Definition.

(1) Let X be a topological space. We refer to the above fibration $p\colon X^{[0,1]} \to X$ as the *free path space fibration of X*.

(2) Let (X, x_0) be a pointed topological space. In the following we refer to the above fibration $p\colon P(X, x_0) \to X$ as the *path space fibration of (X, x_0)*.

Proof. We prove statement (2). The proof of statement (1) is basically identical and is thus left to the reader. Thus let (X, x_0) be a pointed topological space. First note that the map $p\colon P(X, x_0) \to X$ is continuous by Proposition 2.4 (2). Next recall that we need to solve

the following type of extension problem for some topological space W and $c > 0$: [1415]

$$\begin{tikzcd}
W \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & P(X, x_0) \subset X^{[0,1]} \arrow[d, "p"', "\gamma \mapsto \gamma(1)"] \\
W \times [0, c] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X.
\end{tikzcd}$$

We refer to Figure 1032 for an illustration.

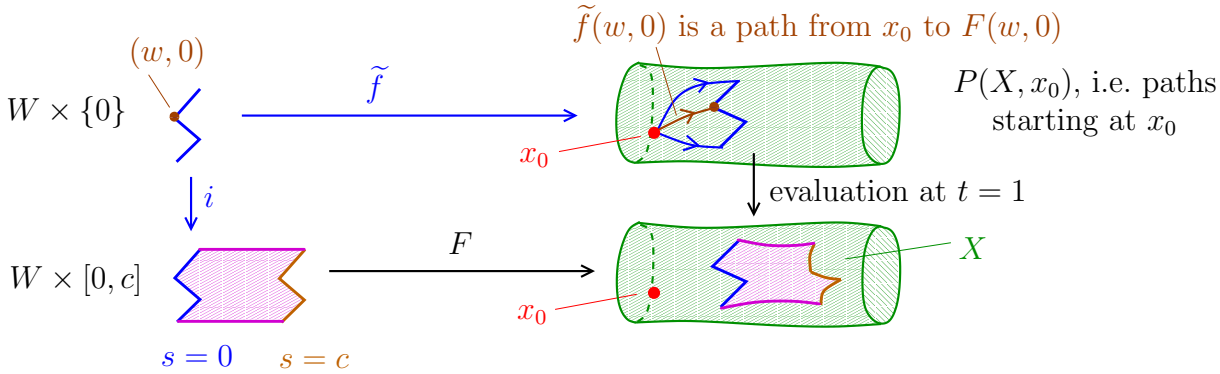


FIGURE 1032. Illustration for the proof of Theorem 83.14.

The idea for the proof is quite simple: Given $(w, s) \in W \times [0, c]$ we have to "systematically" find an element $\gamma \in P(X, x_0)$ with $p(\gamma) = F(w, s)$, i.e. we have to find a path $\gamma\colon [0, 1] \to X$ from x_0 to $F(w, s)$. But there is an obvious path, namely we first travel along $\widetilde{f}(w, 0)$ from x_0 to $F(w, 0)$ and then we use the path $t \mapsto F(w, t)$ to get from $F(w, 0)$ to $F(w, s)$. That is exactly what we will do, except that we need to be a little careful with our parametrizations.

Now we have to turn this idea into a proper proof. We consider the map

$$\begin{array}{rcl}
\widetilde{F}\colon W \times [0, c] & \to & P(X, x_0) \\
(w, s) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \begin{cases} \widetilde{f}(w,0)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ F(w, 2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c}, 1] \end{cases} \end{array} \right).
\end{array}$$

We claim that this map has all the desired properties:

(1) Let $(w, s) \in W \times [0, c]$. We make the following observations:
  (a) The given map $[0, 1] \to X$ starts at x_0.

[1415] In the diagram we write $W \times [0, c]$ instead of the more customary $W \times [0, 1]$ for purely psychological reasons, it makes it easier to distinguish the two types of intervals which arise throughout the proof and which play very different roles.

(b) It follows easily from Lemma 1.32 (2) that the given map $[0,1] \to X$ is indeed continuous.

In other words, the given map $[0,1] \to X$ does indeed define an element in $P(X, x_0)$.

(2) For each $(w,s) \in W \times [0,c]$ we have by construction that $\widetilde{F}(w,s)(t=1) = F(w,s)$. In other words, we have $p \circ \widetilde{F} = F$.

Thus it remains to prove the following claim:

Claim. The map $\widetilde{F}\colon W \times [0,c] \to P(X, x_0)$ is continuous.

We consider the map

$$\begin{array}{rcl} \widetilde{G}\colon W \times [0,c] \times [0,1] & \to & X \\ (w,s,t) & \mapsto & \widetilde{F}(w,s)(t) = \begin{cases} \widetilde{f}(w,0)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ F(w, 2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c}, 1]. \end{cases} \end{array}$$

It follows easily from the fact that F and $\widetilde{f}$ are continuous that the restrictions of $\widetilde{G}$ to the closed subsets

$$A_1 \;:=\; W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [0, 1-\tfrac{s}{2c}]\},$$

and

$$A_2 \;:=\; W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [1-\tfrac{s}{2c}, 1]\}$$

are continuous. Therefore it follows from Lemma 1.32 (2) that $\widetilde{G}$ is continuous. Finally we deduce from the "$\Rightarrow$"-direction of Proposition 2.6 applied to $T = W \times [0,c]$, $Y = [0,1]$ and $H = \widetilde{G}$ that $\widetilde{F}$ is also continuous. This concludes the proof of the claim. $\square$

We obtain the following amusing corollary which later on will turn out to play an important role.

Corollary 83.15. *Given any pointed topological space (X, x_0) and any $n \geq 1$ there exists a natural isomorphism*

$$\partial_n\colon \pi_n(X, x_0) \;\overset{\cong}{\to}\; \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

Remark. For $n = 1$ the statement of Corollary 83.15 recovers the result from page 1931 where we saw that there exists a natural identification

$$\pi_0(\Omega(X, x_0), c_{x_0}) \;=\; \pi_1(X, x_0).$$

Proof. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between pointed topological spaces. It is clear that we obtain a commutative diagram

$$\begin{array}{ccc} P(X, x_0) & \xrightarrow{\gamma \mapsto f \circ \gamma} & P(Y, y_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

It follows from Theorems 83.14 (1) and (2) and 83.7 that we have commutative diagram of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \to & \pi_n(P(X,x_0),c_{x_0}) & \to & \pi_n(X,x_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(X,x_0),c_{x_0}) & \to & \pi_{n-1}(P(X,x_0),c_{x_0}) & \to \dots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \pi_n(P(Y,y_0),c_{y_0}) & \to & \pi_n(Y,y_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(Y,y_0),c_{y_0}) & \to & \pi_{n-1}(P(Y,y_0),c_{y_0}) & \to \dots
\end{array}$$

We obtain from Lemma 83.13 and Proposition 28.7 that the homotopy groups of the path spaces $P(X,x_0)$ and $P(Y,y_0)$ are trivial. Thus the connecting homomorphisms are the desired *natural* isomorphisms. □

Example. We consider the loop space of the 2-dimensional sphere with the base point given by the North Pole $N=(0,0,1)$. We calculate that

$$\begin{array}{ccccc}
\pi_1(\Omega(S^2,N),c_N) & \xleftarrow{\cong} & \pi_2(S^2,N) & \xrightarrow{\cong} & \mathbb{Z}. \\
& \uparrow & & \uparrow & \\
& \text{Corollary 83.15} & & \text{Corollary 80.6 (1)} &
\end{array}$$

In Figure 1033 we see a loop $\{\gamma_z\}_{z\in S^1}$ of loops in the pointed topological space (S^2,N). Put differently, we see a loop in the pointed topological space $(\Omega(S^2,N),c_N)$. It is an amusing and instructive exercise to run through the above two isomorphisms to verify that our loop in the pointed topological space $(\Omega(S^2,N),c_N)$ is actually a generator of $\pi_1(\Omega(S^2,N),c_N)$.

So what does this example actually really mean? It means that even though each individual loop γ_z is null-homotopic in S^2, we cannot continuously deform all the loops γ_z altogether into the constant loop. It is a good idea to think about what it would mean for $[z\mapsto\gamma_z]\in\pi_1(\Omega(S^2,N),c_N)$ to be null-homotopic and why, pictorially, this is not the case in our situation.

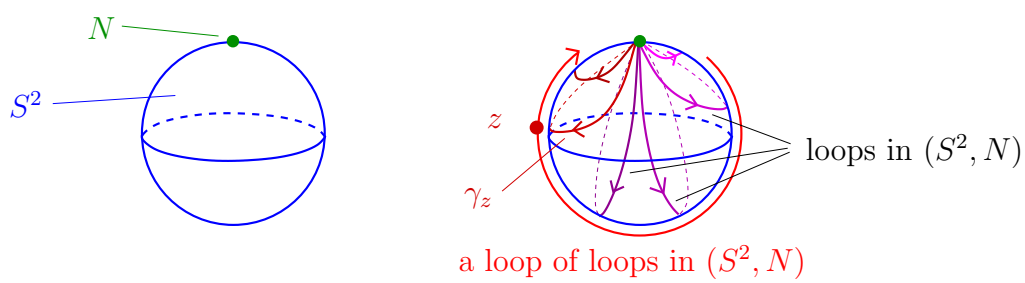


FIGURE 1033.

Note that the statement of Corollary 83.15 can be summarized as follows: The loop space functor

$$\begin{array}{rcl}
\Omega\colon \text{category of pointed topological spaces} & \to & \text{category of pointed topological spaces} \\
(X,x_0) & \mapsto & (\Omega(X,x_0),c_{x_0})
\end{array}$$

shifts homotopy groups down by one degree, i.e. for every pointed topological space (X, x_0) and every $n \in \mathbb{N}$ we have a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

This should be compared to the fact that in Proposition 32.14 we saw that the suspension functor

$$\begin{array}{rcl} \Sigma\colon \text{category of topological spaces} & \to & \text{category of topological spaces} \\ X & \mapsto & \Sigma(X) \end{array}$$

shifts reduced homology groups up by one degree, i.e. for every topological space X and any $n \in \mathbb{N}$ we have a natural isomorphism

$$\widetilde{H}_n(X) \xrightarrow{\cong} \widetilde{H}_{n+1}(\Sigma(X)).$$

In Exercise 83.7 we will discuss whether we can also shift homotopy groups *upwards* and whether we can shift reduced homology groups *downwards* in a functorial way.

83.6. **Every map can be replaced by a fibration.** In this section we will see that one can "replace" any map $f\colon X \to Y$ between topological spaces by a fibration $P_f \to Y$ that has very similar properties to the map f.

We start out with the notion of the pullback of a fibration. Perhaps not surprisingly the definition is almost identical to the definition of the pullback of a vector bundle that we gave on page 1790.

Lemma 83.16. *Let $f\colon A \to B$ be a map between topological spaces and let $q\colon Z \to B$ be a fibration. As in Lemma 16.8 (1) we consider the pullback*

$$f^*Z \;:=\; \{(a, z) \,|\, a \in A \text{ and } \overbrace{z \in q^{-1}(f(a))}^{\text{i.e. } q(z) = f(a)}\} \;\subset\; A \times Z.$$

Then the following statements hold:

(1) *The pullback map*

$$\begin{array}{rcl} p\colon f^*Z \;:=\; \{(a, z) \,|\, a \in A \text{ and } z \in q^{-1}(f(a))\} & \to & A \\ (a, z) & \mapsto & a \end{array}$$

is also a fibration. Furthermore for any $a \in A$ the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$.

(2) *The following diagram commutes*

$$\begin{tikzcd} f^*Z \arrow[r, "(a,z)\mapsto z"] \arrow[d, "p"] & Z \arrow[d, "q"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Proof.

(2) It is straightforward to verify that the diagram commutes.

(1) Let X be any topological space. We consider the following diagram where initially we ignore the diagonal arrows:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{g}"] \arrow[d, "i"] & f^*Z \arrow[r, "(a,z)\mapsto z"] \arrow[d, "p"] & Z \arrow[d, "q"] \\
X \times [0,1] \arrow[r, "G"'] \arrow[ur, dashed, "?\widetilde{G}?"] \arrow[urr, "\widetilde{f\circ G}"] & A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Here the maps $G\colon X \times [0,1] \to A$ and $\widetilde{g}\colon X \times \{0\} \to f^*Z$ are "the challenge" we have to deal with in the definition of a fibration. Since $q\colon Z \to B$ is a fibration there exists a map $\widetilde{f \circ G}\colon X \times [0,1] \to Z$ that makes the diagram commute. It is now straightforward to verify that the map

$$\begin{array}{rcl} X \times [0,1] & \to & f^*Z = \{(a,z) \,|\, a \in A \text{ and } z \in q^{-1}(f(a))\} \\ (x,t) & \mapsto & \big(G(x,t), \big(\widetilde{f \circ G}\big)(x,t)\big) \end{array}$$

makes the above diagram commute. We have thus shown that $p\colon f^*Z \to A$ is also a fibration.

Finally we have to prove the statement regarding the fibers. Thus let $a \in A$. It follows immediately from the definitions that $p^{-1}(a) = \{a\} \times q^{-1}(f(a)) \subset A \times Z$. It follows from Lemma 1.53 (3b) that the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$. □

Definition. Using the notation from Lemma 83.16 we refer to $f^*Z \to A$ as the *pullback* of the fibration $q\colon Z \to B$ under the map $f\colon A \to B$.

Now we turn to the actual problem that we had stated in the beginning of this section.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. Similar to the free path space fibration we now consider the fibration[1416]

$$\begin{array}{rcl} q\colon Y^{[0,1]} & \to & Y \\ \gamma & \mapsto & \gamma(0). \end{array}$$

We refer to the map

$$\begin{array}{rcl} p\colon P_f := f^*(Y^{[0,1]}) = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1) \end{array}$$

as the *mapping path fibration.*[1417]

The following proposition in particular justifies the name "mapping path *fibration*".

Proposition 83.17. *Let $f\colon X \to Y$ be a map between topological spaces. We denote by $p\colon P_f \to Y$ the corresponding mapping path fibration.*

[1416]This map is basically the same as the free path space fibration except that now we consider the evaluation $\gamma(0)$ instead of the evaluation $\gamma(1)$.

[1417]Note that it follows from Proposition 2.4 (2) that this map is in fact continuous.

(1) *The map*

$$\begin{array}{rcl} h\colon X & \to & P_f \\ x & \mapsto & (x, \text{constant path } c_{f(x)}) \end{array}$$

is a homotopy equivalence and it has the property that it makes the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & P_f \arrow[dl, "p"] \\ & Y. & \end{tikzcd}$$

(2) *The map $p\colon P_f \to Y$ is a fibration. Given any $y \in Y$ the fiber is given by*

$$\{(x,\gamma) \,|\, x \in X \text{ and } \gamma\colon [0,1] \to Y \text{ is a path from } f(x) \text{ to } y\}.$$

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We suppose that Y is path-connected. We refer to the homotopy type of the fiber of the corresponding mapping path fibration $P_f \to Y$[1418] as the *homotopy fiber of f*.

Remark. If $f\colon X \to Y$ is a fibration, then it is shown in [DK, Theorem 6.18] that the map $h\colon X \to P_f$ in Proposition 83.17 (1) is a "fiber homotopy equivalence", thus, loosely speaking, the original fibration $f\colon X \to Y$ and the new fibration $p\colon P_f \to Y$ are essentially the same. We will not make use of this fact.

Example. Proposition 83.17 thus says that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration. As a reality check let us consider again the "non-example" from Lemma 83.1, i.e. we consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. Lemma 83.1 says that f is not a fibration. Let us see what the above construction does in this case.

As pointed out in Proposition 83.17 (2), given any $y \in [-1,1]$ the preimage of y under the map $p\colon P_f \to Y = [-1,1]$ is given by

$$p^{-1}(y) \;=\; \{(x,\gamma) \,|\, x \in S^1 \text{ and } \gamma\colon [0,1] \to [-1,1] \text{ is a path from } f(x) \text{ to } y\}.$$

In Figure 1034 we strive to illustrate three points in the preimage of a point $y \in [-1,1]$. In Exercise 83.3 we will see that for each $y \in [-1,1]$ the preimage $p^{-1}(y)$ is homotopy equivalent to S^1. In particular this shows that the homotopy fiber of $f\colon S^1 \to [-1,1]$ is S^1. This exercise does not quite show yet that the map $p\colon P_f \to [-1,1]$ is a fibration, but at the very least it is consistent with Proposition 83.6 (1).

Proof of Proposition 83.17. Let $f\colon X \to Y$ be a map between topological spaces. We consider the map

$$\begin{array}{rcl} p\colon P_f = f^*(Y^{[0,1]}) = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1). \end{array}$$

We turn to the proof of the two statements:

[1418]Recall that Proposition 83.6 (1) says that the homotopy type of the fiber of a fibration over a path-connected topological space is well-defined.

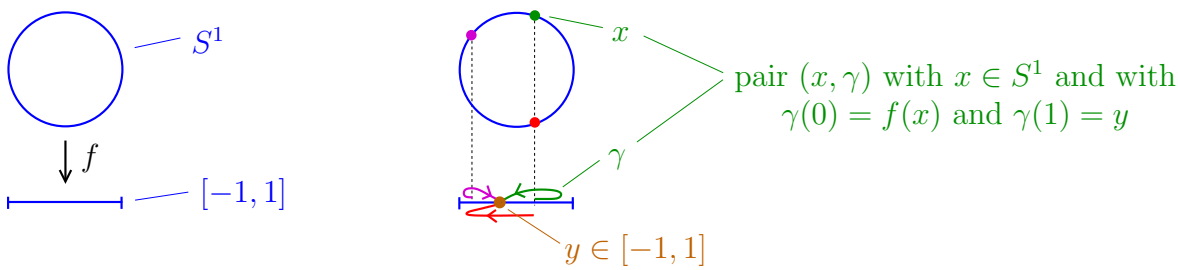


FIGURE 1034.

(1) Given $y \in Y$ we denote, as usual, by $c_y\colon [0,1] \to Y$ the constant path given by $c_y(t) = y$ for all $t \in [0,1]$. We consider the map $h\colon X \to P_f$ that is given by $x \mapsto (x, c_{f(x)})$.[1419] We claim it has the desired properties. It is obvious that $p \circ h = f$, i.e. the diagram given in the proposition commutes. So it remains to show that h is a homotopy equivalence. We consider the map $g\colon P_f \to X$ given by $g(x,\gamma) = x$. Evidently we have $g \circ h = \mathrm{id}_X$. We are done once we have shown that $h \circ g$ is homotopic to id_{P_f}. In fact such a homotopy is given by[1420]

$$\begin{array}{rcl} H\colon P_f \times [0,1] & \to & P_f \\ ((x,\gamma),t) & \mapsto & \left(x, \left(\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \gamma(s \cdot t) \end{array} \right)\right). \\ \uparrow & & \\ \gamma\colon [0,1] \to Y \text{ is a path with } \gamma(0) = f(x) & & \end{array}$$

One easily verifies that $H_0 = h \circ g$ and that $H_1 = \mathrm{id}_{P_f}$.

(2) We want to show that the map $p\colon P_f \to Y$ is a fibration. Thus let A be any topological space. Let $c > 0$. We consider the following "challenge diagram":[1421]

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\widetilde{g}"] \arrow[d, "i"] & P_f = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \mid \gamma(0) = f(x)\} \arrow[d, "p"', "(x,\gamma) \mapsto \gamma(1)"] \\ A \times [0,c] \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}", "??"'] & Y. \end{tikzcd}$$

To define $\widetilde{G}$ we first split up the map $\widetilde{g}$ into its components. More precisely, we define maps $\widetilde{g}_1\colon A \to X$ and $\widetilde{g}_2\colon A \to Y^{[0,1]}$ via the equality

$$\widetilde{g}(a,0) \;=\; \big(\widetilde{g}_1(a), \widetilde{g}_2(a)\big) \;\in\; P_f \subset X \times Y^{[0,1]} \quad \text{for all } a \in A.$$

[1419] It follows from Lemma 1.53 (3c) and Lemma 2.3 that this map is continuous.

[1420] An argument as in the proof of Lemma 83.13 (2) shows that the map H is indeed continuous.

[1421] As in the proof of Theorem 83.14 we prefer to work, for purely psychological reasons, with $A \times [0,c]$ instead of $A \times [0,1]$.

Note that by definition of P_f the path $\widetilde{g}_2(a) \in Y^{[0,1]}$ is a path from $f(\widetilde{g}_1(a))$ to $G(a,0)$. We need to define a suitable map $\widetilde{G}\colon A \times [0,c] \to P_f \subset X \times Y^{[0,1]}$. We consider the two factors separately:

(a) We already have the map $A \times [0,c] \to A \xrightarrow{\widetilde{g}_1} X$.

(b) To each $(a,s) \in A \times [0,c]$ we need to assign "systematically" a path $[0,1] \to Y$ from $f(x)$ to $G(a,s)$. The idea is quite simple: Given $(a,s) \in A \times [0,c]$ we consider the path $[0,1] \to Y$ that is given by first traveling along the path $\widetilde{g}_2(a)$ from $f(\widetilde{g}_1(a))$ to $G(a,0)$ and then we travel along the path $t \mapsto G(a,t)$ from $G(a,0)$ to $G(a,s)$. More precisely, similar to the proof of Theorem 83.14 we consider the map[1422]

$$\begin{array}{rcl} \widetilde{G}_2\colon A \times [0,c] & \to & Y^{[0,1]} \\ (a,s) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & \begin{cases} \widetilde{g}_2(a)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ G(a, 2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

We refer to Figure 1035 for an illustration of the definition of $\widetilde{G}_2$.

Finally we consider the following map:

$$\begin{array}{rcl} \widetilde{G}\colon A \times [0,c] & \to & P_f \\ (a,s) & \mapsto & \big(\widetilde{g}_1(a), \widetilde{G}_2(a,s)\big). \end{array}$$

We make the following observations:

(a) It follows easily from the definitions that $\widetilde{G}$ takes values in P_f.

(b) It follows from the above and Lemma 1.53 (3c) that $\widetilde{G}$ is continuous.

(c) It is straightforward to verify that $\widetilde{G}$ makes the diagram commute.

We have thus verified that $\widetilde{G}$ has all the desired properties. □

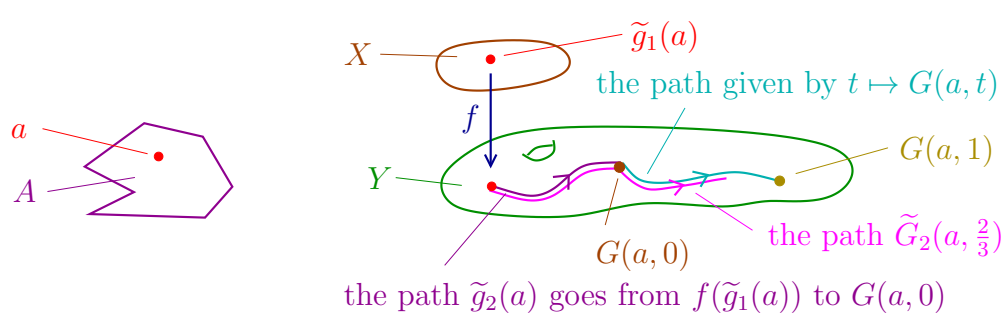


FIGURE 1035. Illustration for the proof of Proposition 83.17.

[1422]The argument in the proof of Theorem 83.14 also shows that the map $\widetilde{G}_2$ does indeed take values in $Y^{[0,1]}$ and that the map $\widetilde{G}_2\colon A \times [0,c] \to Y^{[0,1]}$ is continuous.

The statement of Proposition 83.17 that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration is quite surprising and perhaps also somewhat puzzling. Therefore we want to consider one more example.

Example. We denote by Σ the surface of genus 2 and we denote by T the torus. We consider the map $f\colon \Sigma \to T$ that is illustrated in Figure 1036. We denote by $p\colon P_f \to T$ the mapping path fibration and we denote by F the corresponding homotopy fiber. So what can we say about the mysterious homotopy fiber F?

We have the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_2(F) \arrow[r] & \pi_2(P_f) \arrow[r, "p_*"] & \pi_2(T) \arrow[r, "\partial"] & \pi_1(F) \arrow[r] & \pi_1(P_f) \arrow[r, "p_*"] & \pi_1(T) \arrow[r] & \dots \\
& & \pi_2(\Sigma) \arrow[u, "h_*"', "\cong"] \arrow[ur, "f_*"'] & & & \pi_1(\Sigma). \arrow[u, "h_*"', "\cong"] \arrow[ur, "f_*"'] & &
\end{tikzcd}$$

We make the following explanations and observations:

(1) The top sequence is the long exact sequence coming from Theorem 83.7. The vertical maps are induced by the homotopy equivalence $h\colon \Sigma \to P_f$ from Proposition 83.17 (1). In particular the vertical maps are isomorphisms by Proposition 28.7.
(2) Since $p \circ h = f$ we see that the diagram commutes.
(3) By the discussion on page 724 we know that $\pi_n(T) = \pi_n(\Sigma) = 0$ for $n \geq 2$.
(4) It is pretty clear that the map $f_*\colon \pi_1(\Sigma) \to \pi_1(T)$ is an epimorphism.
(5) We obtain from the above discussion and from Proposition 39.13 that F is path-connected, that $\pi_n(F) = 0$ for $n \geq 2$ and that $\pi_1(F)$ is an *infinitely generated* free group.

Thus we see that the homotopy groups of F are isomorphic to the homotopy groups of an infinite wedge of circles.

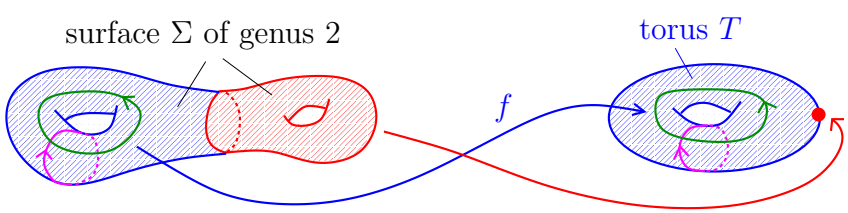


FIGURE 1036. A degree-one map from the surface of genus 2 to the torus.

Exercises for Chapter 83.

Exercise 83.1. Show that the projection map

$$\begin{array}{rcl} \overline{B}^2 & \to & \overline{B}^1 = [-1,1] \\ (x,y) & \mapsto & x \end{array}$$

is a Serre fibration.

Exercise 83.2. . Let $n \in \mathbb{N}_0$ and let x_0 be a point on S^n. It follows from Proposition 2.4 that the map

$$\begin{array}{rcl} e\colon P(S^n, x_0) & \to & S^n \\ \{f\colon [0,1] \to S^n \mid f(0) = x_0\} & \mapsto & f(1) \end{array}$$

is continuous. For which $n \in \mathbb{N}_0$ does e admit a left-inverse?

Exercise 83.3. We consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. We define the mapping path fibration $q\colon P_f \to [-1,1]$ as in Proposition 83.17. Show that for each $y \in [-1,1]$ the preimage $q^{-1}(y)$ admits a deformation retraction to a circle.
Remark: See Figure 1034 for an illustration of the mapping path fibration.

Exercise 83.4. Let (X, x_0) be a pointed topological space. In Corollary 83.15 we wrote that for any $n \geq 1$ there exists a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

What does this statement mean for $n = 1$? It seems to say that $\pi_0(\Omega(X, x_0), c_{x_0})$ admits a natural group structure. Can you define such a natural group structure without using the above isomorphism?

Exercise 83.5. Convince yourself, without using Corollary 83.15, that the loop in the pointed topological space $(\Omega(S^2, N), c_N)$ that is shown in Figure 1033 is not null-homotopic.

Exercise 83.6. Let X be a topological space and let $x_0 \in X$ be a base point. Recall that the loop space $\Omega(X, x_0)$ is defined as the set $\Omega(X, x_0) = \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}$ together with the compact-open topology. We denote by $\sim$ the equivalence relation on $\Omega(X, x_0)$ given by homotopy of paths. By definition we have $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$. We equip $\pi_1(X, x_0)$ with the quotient topology. We can thus view $\pi_1(X, x_0)$ as a topological space. Show that $\pi_1(S^1, 1)$ is a discrete topological space.
Hint: It is possibly easier to show that for any pointed topological space (X, x_0) that admits a universal covering the fundamental group $\pi_1(X, x_0)$ is a discrete topological space.
Remark: There are examples of pointed topological spaces for which $\pi_1(X, x_0)$ is not a discrete topological space. For example the fundamental group of the Hawaiian Earrings, introduced in Exercise 26.6, is not a discrete topological space. We refer to [Fab, Fab2] for details.

Exercise 83.7.

(a) Show that there is no functor

$$\Phi\colon \text{category of pointed topological spaces} \to \text{category of pointed topological spaces}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n+1}(\Phi(X, x_0)).$$

(b) Show that there is no functor

$$\Psi\colon \text{category of topological spaces} \;\to\; \text{category of topological spaces}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\widetilde{H}_n(X) \;\overset{\cong}{\to}\; \widetilde{H}_{n-1}(\Psi(X)).$$

Hint: In Exercise 31.2 (b) we proved the following statement. If X is a topological space with $\widetilde{H}_0(X) \cong \mathbb{Z}$, then given any map $f\colon X \to X$ the induced map f_* on $\widetilde{H}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

84. Cofibrations (∗)

In this chapter we introduce the notion of a cofibration. As of right now we will make little use of them later on. Nonetheless we intend to introduce them since they get used frequently in various branches of algebraic topology.

As the name suggests, the concept of a cofibration is connected to the notion of a fibration. Towards the end of this chapter we will explain the formal relationship between fibrations and cofibrations.

Throughout this chapter we will on numerous occasions make use of the notion of a mapping cylinder and of a mapping cone. We start out by recalling the definition of a mapping cylinder that we had initially given on page 455.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We define the corresponding *mapping cylinder* to be the topological space

$$\mathrm{Cyl}(f) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \;\text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We refer to the maps

$$\begin{array}{l} X \xrightarrow{i} \mathrm{Cyl}(f) \\ x \mapsto (x,0) \end{array} \quad \text{respectively} \quad \begin{array}{l} \mathrm{Cyl}(f) \xrightarrow{r} Y \\ P \mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases} \end{array}$$

as the *natural inclusion* respectively the *natural retraction*.

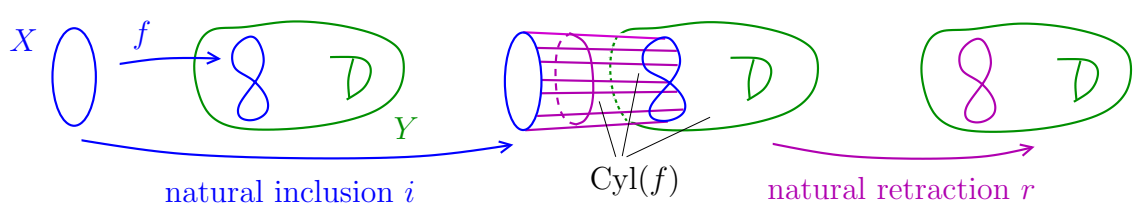


Figure 1037.

We also recall the definition of a mapping cone that we gave on page 458.

Definition. Given a map $f\colon A \to X$ between topological spaces the corresponding *mapping cone* is defined as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (A \times [0,1]) \sqcup X/\sim \qquad \text{where} \quad \begin{array}{l} (a,0) \sim (b,0) \text{ for all } a,b \in A \text{ and} \\ (a,1) \sim f(a) \text{ for all } a \in A. \end{array}$$

We refer to the point $[A \times \{0\}] \in \mathrm{Cone}(f)$ as the *cone point*. We illustrate the definition for an inclusion map $i\colon A \to X$ in Figure 1038.

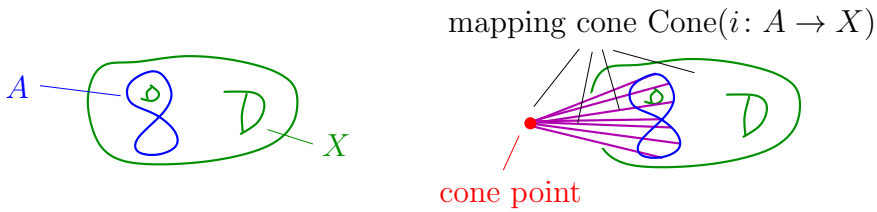


FIGURE 1038.

84.1. **Cofibrations: definition and examples.** Now we can finally introduce the leading contender for our attention in this chapter.

Definition. A map $i\colon A \to X$ between topological spaces is called a *cofibration* if it has the homotopy extension property, i.e. if given any topological space Y and any two maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, hook] \arrow[d, "i"'] & A \times [0,1] \arrow[dl, "G"'] \arrow[d, "i\times\mathrm{id}"] \\
X \times \{0\} \arrow[r, hook] \arrow[ur, "f"] & X \times [0,1]. \arrow[ul, dashed, "F"']
\end{tikzcd}$$

Remark. As we will see in a minute, in most cases of interest the map $i\colon A \to X$ is for all intents and purposes an inclusion. For such situations the definition of the homotopy extension property is illustrated in Figure 1039.

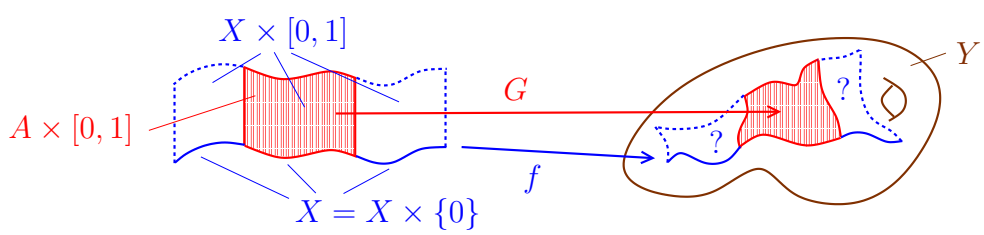


FIGURE 1039.

Example. Let X be a CW-complex and let A be a subcomplex. The Homotopy Extension Theorem 27.4 says that the inclusion $i\colon A \to X$ has the homotopy extension property, i.e. the map $i\colon A \to X$ is a cofibration.

The following lemma gives a useful characterization of cofibrations.

Lemma 84.1. *If $i\colon A \to X$ is a map between two topological spaces, then the following three statements are equivalent:*

(1) *The map* $i\colon A \to X$ *is a cofibration.*
(2) *The map*[1423]

$$\begin{array}{rcl} s\colon \mathrm{Cyl}(i) & \to & X \times [0,1] \\ P & \mapsto & \begin{cases} (i(a), 1-t), & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ (P, 0), & \text{if } P \in X \end{cases} \end{array}$$

admits a left-inverse, in other words, there exists a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ *such that* $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.
(3) *The set* $(X \times \{0\}) \cup (A \times [0,1])$ *is a retract of* $X \times [0,1]$.

Proof. First we prove the "(1) ⇒ (2)"-direction. Thus we suppose that the map $i\colon A \to X$ is a cofibration. We consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x,0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$

We obtain the following diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook] \arrow[d, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[d, hook, "i \times \mathrm{id}"] \\ & \mathrm{Cyl}(i) & \\ X \times \{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X \times [0,1]. \arrow[ul, dashed, "r"] \end{tikzcd}$$

One easily verifies that the black diagram commutes. Since $i\colon A \to X$ is a cofibration we obtain a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. It is straightforward to verify that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

The proof of the converse, i.e. the proof of the "(2) ⇒ (1)"-direction is also quite straightforward. We will provide the proof in Exercise 84.2.

Now we turn to the equivalence of (2) and (3). This turns out to be quite subtle. First note that the image of $\mathrm{Cyl}(i)$ in $X \times [0,1]$ under the map s is precisely $(X \times \{0\}) \cup (A \times [0,1])$. Thus we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Cyl}(i) \arrow[rr, "s"] \arrow[dr, "s"'] & & (X \times \{0\}) \cup (A \times [0,1]) \arrow[dl, hook] \\ & X \times [0,1]. & \end{tikzcd}$$

To show that (2) and (3) are equivalent it would suffice to show that the top horizontal map is a homeomorphism.

It is basically clear that the top horizontal map is a bijection. Furthermore it follows easily from Lemma 1.83 (3) that the map is continuous. Actually in Lemma 15.6 (1) we

[1423]It follows from Lemma 1.83 (3) that this map is continuous.

showed that the map is a homeomorphism if A is a *closed* subset. On the other hand we saw in Lemma 15.6 (3) that in general the map is *not* a homeomorphism.

Rather surprisingly it is nonetheless true that (2) and (3) are equivalent. This delicate argument is given in to [Strø, Lemma 4] or alternatively to [Hat, Proposition A.18]. □

Lemma 84.1 allows us to prove the following proposition which gives us many more examples of cofibrations.

Proposition 84.2. *Let M be a manifold. If N is a compact and proper submanifold, then the inclusion $i\colon N \to M$ is a cofibration.*

In the proof of Proposition 84.2 we will need the following elementary lemma.

Lemma 84.3. *There exists a retraction*

$$\rho\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $\rho(\{1\}\times[0,1]) = \{(1,0)\}$.

Proof. In Figure 1040 we show how such a retraction ρ can be constructed. □

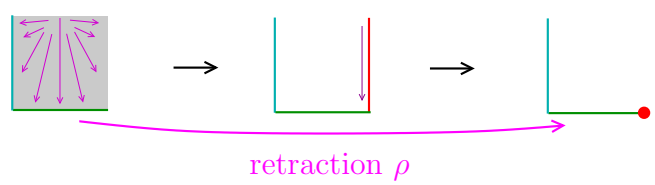


FIGURE 1040. Second illustration for the proof of Proposition 84.5.

Sketch of a proof of Proposition 84.2. Let M be an m-dimensional manifold and let N be a compact proper n-dimensional submanifold. We want to show that the inclusion $i\colon N \to M$ is a cofibration. By the General Tubular Neighborhood Theorem 64.7 the submanifold N admits a tubular neighborhood. To simplify the discussion we will only treat the case that N admits a trivial tubular neighborhood $\overline{B}^{m-n}\times N$. The general case is notationally more messy since we have to work with a trivialization of the tubular neighborhood, similar to the argument in the proof of Proposition 64.13. We leave it to the doubting Thomases among the readership to deal with the general case.

Now we recall that in Lemma 84.3 we saw that there exists a retraction

$$\varphi = (\varphi_1,\varphi_2)\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $\varphi(\{1\} \times [0,1]) = \{(1,0)\}$. We consider the map[1424]

$$\begin{aligned} r\colon M \times [0,1] \ &\to\ \mathrm{Cyl}(i) = ((N \times [0,1]) \sqcup M)/\sim \ \text{ where } (x,1) \sim x \text{ for } x \in N \\ (P,t) \ &\mapsto \begin{cases} (\underbrace{(z \cdot \varphi_1(|z|,t), Q)}_{\in \overline{B}^{m-n} \times N}, \varphi_2(|z|,t)), & \text{if } P = (z,Q) \text{ with } z \in \overline{B}^{m-n} \text{ and } Q \in N, \\ P, & \text{if } P \notin \overline{B}^{m-n} \times N. \end{cases} \end{aligned}$$

(We refer to Figure 1041 for an illustration of the map r.) One can easily verify that r is a left-inverse to the map $s\colon \mathrm{Cyl}(i) \to M \times [0,1]$ that we had considered in Lemma 84.1 (2). Thus it follows from Lemma 84.1 that $i\colon N \to M$ is a cofibration. □

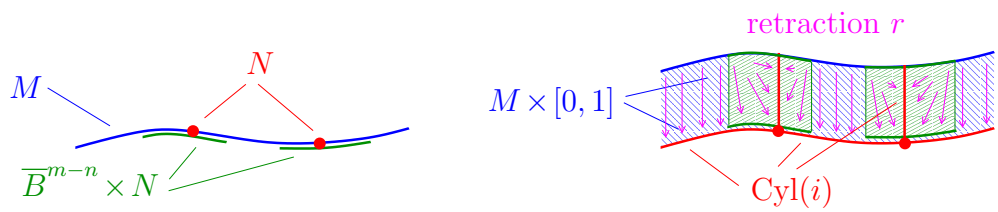


FIGURE 1041. Illustration of the proof of Proposition 84.2.

The next proposition gives another rich source of examples of cofibrations.

Proposition 84.4. *Let $f\colon X \to Y$ be a map between topological spaces. The natural inclusion $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration.*

Proof. So suppose we are given the following "challenge diagram":

$$\begin{tikzcd} X \times \{0\} \arrow[r, hook] \arrow[d, hook, "i"] & X \times [0,1] \arrow[dl, "H"'] \arrow[d, hook, "i \times \mathrm{id}"] \\ Z & \\ \mathrm{Cyl}(f) \times \{0\} \arrow[r, hook] \arrow[uur, "g"] & \mathrm{Cyl}(f) \times [0,1]. \arrow[ul, dashed, "G", "??"'] \end{tikzcd}$$

This situation is illustrated in Figure 1042.

The key to finding the desired map G is the following claim.

Claim. There is a retraction r from $\mathrm{Cyl}(f) \times [0,1]$ to its subset $(\mathrm{Cyl}(f) \times \{0\}) \cup (X \times [0,1])$.

In Figure 1043 we do our best to illustrate the objects involved. To construct the desired retraction r we recall that in Lemma 84.3 we saw that there exists a retraction

$$s\colon [0,1] \times [0,1] \ \to\ (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$$

[1424]It follows from the definition of a tubular neighborhood, see page 1495, together with Lemma 1.32 (2) and Lemma 64.1 that the map is indeed continuous.

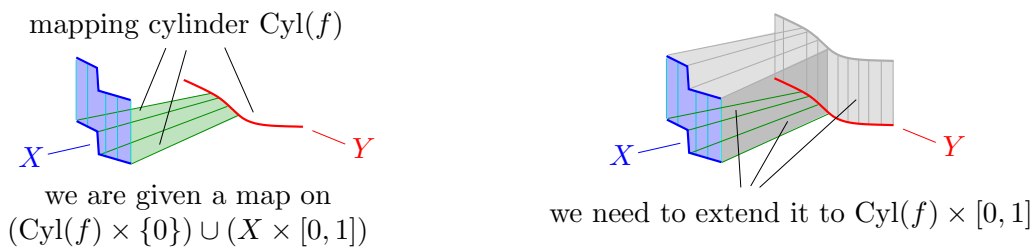


FIGURE 1042. Illustration for the proof of Proposition 84.4.

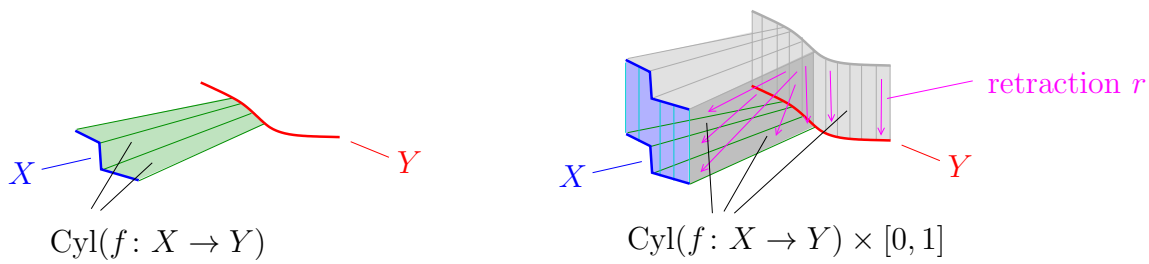


FIGURE 1043. Illustration for the proof of Proposition 84.4.

that has the property that $s(\{1\} \times [0,1]) = \{(1,0)\}$. Now the desired retraction is given by the map[1425]

$$\begin{array}{rcl} r\colon \operatorname{Cyl}(f) \times [0,1] & \to & (\operatorname{Cyl}(f) \times \{0\}) \cup (X \times [0,1]) \\ P & \mapsto & \begin{cases} [(x, s(t,u))], & \text{if } P = [(x,t,u)] \text{ with } x \in X \text{ and } t,u \in [0,1], \\ [(y,0)], & \text{if } P = [(y,u)] \text{ with } y \in Y \text{ and } u \in [0,1]. \end{cases} \end{array}$$

This concludes the proof of the claim.

Next we note that the fact that the above "challenge diagram" commutes, together with Lemma 1.32 (2), implies that the following map is well-defined and continuous:

$$\begin{array}{rcl} \Phi\colon (\operatorname{Cyl}(f) \times \{0\}) \cup_{X \times \{0\}} (X \times [0,1]) & \to & Z \\ P & \mapsto & \begin{cases} H(P), & \text{if } P \in X \times [0,1] \\ g(P), & \text{if } P \in \operatorname{Cyl}(f) \times \{0\}. \end{cases} \end{array}$$

It is now clear that the map $G := \Phi \circ r\colon \operatorname{Cyl}(f) \times [0,1] \to Z$ has the desired properties. □

In Proposition 83.17 we saw that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration $P_f \to Y$. The following proposition is the analogue of Proposition 83.17 for cofibrations.

Proposition 84.5. *Let $f\colon X \to Y$ be a map between topological spaces. We denote by $i\colon X \to \operatorname{Cyl}(f)$ the natural inclusion. The natural retraction $r\colon \operatorname{Cyl}(f) \to Y$ is a homotopy*

[1425]This map is continuous by Lemma 1.32 (2).

equivalence and the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathrm{Cyl}(f) \arrow[rr, "r"] & & Y. \end{tikzcd}$$

By Proposition 84.4 we know that the left diagonal map $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration.

Proof. We showed in Lemma 15.7 (2b) that the natural retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. It is clear that the given diagram commutes. Thus we are done by Proposition 84.4. That was quick. □

84.2. **Properties of cofibrations.** The following lemma summarizes a few key properties of cofibrations.

Lemma 84.6. *If $i\colon A \to X$ is a cofibration, then the following three statements hold:*
(1) *the map $i\colon A \to X$ is injective,*
(2) *the map $i\colon A \to X$ is a homeomorphism onto its image,*
(3) *if X is Hausdorff, then $i(A)$ is a closed subset of X.*

Remark. In Exercise 84.3 we will see that in Lemma 84.6 (3) we cannot drop the hypothesis that X is Hausdorff.

Proof. Let $i\colon A \to X$ be a cofibration.

(1) Similar to the proof of Lemma 84.1 we consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x, 0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a, s) & \mapsto & [(a, 1-s)]. \end{array}$$

We obtain the following diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "i \times \mathrm{id}"] \\ & \mathrm{Cyl}(i) & \\ X \times \{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X \times [0,1]. \arrow[ul, dashed, "F"'] \end{tikzcd}$$

One easily verifies that the black diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. Note that by design for any $t \in (0,1]$ the map $G\colon A \times \{t\} \to \mathrm{Cyl}(i)$ is injective. Since the diagram commutes this implies that $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is also injective, i.e. $i\colon A \to X$ is injective.

(2) We continue with the argument and the notation in (1). Let $t \in (0,1]$. The map $G|_{A \times \{t\}}\colon A \times \{t\} \to A \times \{1-t\}$ is evidently a homeomorphism. Thus we see that

$$(G|_{A \times \{t\}})^{-1} \circ F|_{X \times \{t\}}\colon i(A) \times \{t\} \ \to \ A \times \{t\}$$

is an inverse to the map $i \times \mathrm{id}_{\{t\}} \colon A \times \{t\} \to X \times \{t\}$. This implies that the map $i \times \mathrm{id}_{\{t\}} \colon A \times \{t\} \to X \times \{t\}$ is a homeomorphism onto its image, but this also implies that the map $i \colon A \to X$ is a homeomorphism onto its image.

(3) It follows from (1) that we can assume that A is a subset of X and that $i \colon A \to X$ is the inclusion map. Thus we can make the identification $\mathrm{Cyl}(i) = (A \times [0,1]) \cup (X \times \{0\})$ and we can view $\mathrm{Cyl}(i)$ as a subset of $X \times [0,1]$. The discussion in (1) shows that $F \colon X \times [0,1] \to \mathrm{Cyl}(i)$ is a retraction. We see that

$$\begin{array}{rcl} & \text{for } x \in A \text{ we have } F(x,0) = (x,0) \text{ by construction of } F\text{, furthermore} \\ & \text{for } x \notin A \text{ the point } (x,0) \text{ does not lie in } \mathrm{Cyl}(i)\text{, thus it does not lie in the image of } F \\ & \downarrow & \\ A & = & \{x \in X \mid F(x,0) = (x,0)\} \\ & = & \left\{ \begin{array}{c} \text{the points in } X \text{ on} \\ \text{which the maps} \end{array} \begin{array}{rcl} X & \to & X \times [0,1] \\ x & \mapsto & F(x,0) \end{array} \text{ and } \begin{array}{rcl} X & \to & X \times [0,1] \\ x & \mapsto & (x,0) \end{array} \text{ agree} \right\}. \end{array}$$

It follows from this description of A, our hypothesis that X is Hausdorff (which implies by Lemma 1.55 (1) that $X \times [0,1]$ is also Hausdorff) and Exercise 1.31 that A is indeed a closed subset of X. □

The following lemma shows that inclusion maps that are cofibrations are particularly nicely behaved.

Lemma 84.7. *Let X be a topological space and let $A \subset X$ be a subset. We denote by $i \colon A \to X$ the inclusion map. If i is a cofibration and if A is a closed subset, then the following three statements hold:*

(1) *The map*[1426]

$$\begin{array}{rcl} \varphi \colon \mathrm{Cone}(i \colon A \to X) & \to & X/A \\ P & \mapsto & \left\{ \begin{array}{ll} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{array} \right. \end{array}$$

is a homotopy equivalence.

(2) *If A is contractible, then the projection map $X \to X/A$ is a homotopy equivalence.*

(3) *If $a_0 \in A$ is a deformation retract of A, then the projection map $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.*

(4) *For any $n \in \mathbb{N}_0$ the map $H_n(X,A) \to \widetilde{H}_n(X/A)$ from page 783 is an isomorphism.*

Proof. Let X be a topological space and let $A \subset X$ be a subset.

(1) Similar to the proof of Lemma 84.6 (1) we consider the following two maps:

(a) First of all we consider the obvious inclusion map $j \colon X \times \{0\} \to \mathrm{Cone}(i)$ which is given by $j(x,0) = x$.

[1426]It follows from the ever popular Lemma 1.32 (2) and the fact that A is a closed subset of X that the map φ is continuous.

(b) We consider the map

$$\begin{array}{rcl} G\colon A\times[0,1] & \to & \mathrm{Cone}(i) \\ (a,s) & \mapsto & [(a,1-s)]. \end{array}$$

Note that G_1 sends all points in A to the cone point.
We obtain the following diagram:

$$\begin{tikzcd} A\times\{0\} \arrow[rr, hook] \arrow[d, hook, "i"'] & & A\times[0,1] \arrow[dl, "G"'] \arrow[d, hook, "i\times\mathrm{id}"] \\ & \mathrm{Cone}(i) & \\ X\times\{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X\times[0,1]. \arrow[ul, dashed, "F"'] \end{tikzcd}$$

One easily verifies that the diagram commutes. Since i is a cofibration we obtain a map $F\colon X\times[0,1]\to\mathrm{Cone}(i)$ which makes the diagram commute. Note that the map $F_1\colon X\to\mathrm{Cone}(i)$ sends all points in A to the cone point. Thus F_1 descends to a map $\psi\colon X/A\to\mathrm{Cone}(i)$. It remains to prove the following claim.

Claim. The maps $\varphi\colon\ \mathrm{Cone}(i)\to X/A$ and $\psi\colon X/A\to\mathrm{Cone}(i)$ are homotopy inverses of one another.

Note that $F(A\times[0,1])=G(A\times[0,1])\subset A$. Therefore it follows from Lemma 9.14 that the map

$$H\colon (X/A)\times[0,1] \xrightarrow{([x],t)\mapsto[(x,t)]} (X\times[0,1])/(A\times[0,1]) \xrightarrow{[(x,t)]\mapsto[F(x,t)]} X/A$$

is continuous. It follows immediately from the definitions that H is a homotopy between $\mathrm{id}_{X/A}$ and $\varphi\circ\psi$.

Conversely we consider the map

$$\begin{array}{rcl} H\colon\ \mathrm{Cone}(i)\times[0,1] & \to & \mathrm{Cone}(i) \\ (P,s) & \mapsto & \begin{cases} F(P,s), & \text{if } P\in X, \\ [a,(1-s)\cdot t], & \text{if } P=[(a,t)] \text{ with } a\in A \text{ and } t\in[0,1]. \end{cases} \end{array}$$

One easily verifies that this is a homotopy from $\mathrm{id}_{\mathrm{Cone}(i)}$ to $\psi\circ\varphi$. This concludes the proof of the claim.

(2) We consider the following diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] & & \mathrm{Cone}(i\colon A\to X) \arrow[dl, "\varphi"] \\ & X/A. & \end{tikzcd}$$

It follows immediately from the definition of φ that this diagram commutes. Now suppose that A is contractible. By Proposition 15.14 (1) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A\to X)$ to X. By statement (1) we know that φ is a

homotopy equivalence. Thus it follows from Lemma 9.8 (1) that the map $X \to X/A$ is a homotopy equivalence.

(3) Now assume that $a_0 \in A$ is a deformation retract. By Proposition 15.14 (2) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that for any $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$. Also note that the homotopy equivalence H from (1) has the property that $H(a_0,t) = [(a_0, 1-t)]$. If one combines these observations with the argument in (2) it is not difficult to see that the projection map is in fact a homotopy equivalence of the pointed topological spaces (X, a_0) and $(X/A, [a_0])$.

(4) Now we suppose that A is non-empty. Let $n \in \mathbb{N}_0$. We consider the following diagram

$$\begin{tikzcd} H_n(X,A) \arrow[rr] \arrow[dr, "\cong"'] & & \widetilde{H}_n(X/A) \\ & \widetilde{H}_n(\mathrm{Cone}(i)). \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Here the top map is the natural homomorphism that we introduced on page 783 and the left diagonal map is the natural *isomorphism* of Lemma 33.7. We leave it to the reader to verify that the diagram commutes. We had just seen in (1) that φ is a homotopy equivalence. Therefore we obtain from Corollary 30.8 that the right diagonal map is also an isomorphism. Putting everything together we see that the top map is indeed an isomorphism. □

The following corollary is quite useful.

Corollary 84.8. *Let X be a CW-complex and let $A \subset X$ be a subcomplex.*

(1) *If A is contractible, then the projection map $X \to X/A$ is a homotopy equivalence.*
(2) *If $a_0 \in A$ is a deformation retract of A, then the projection map $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.*

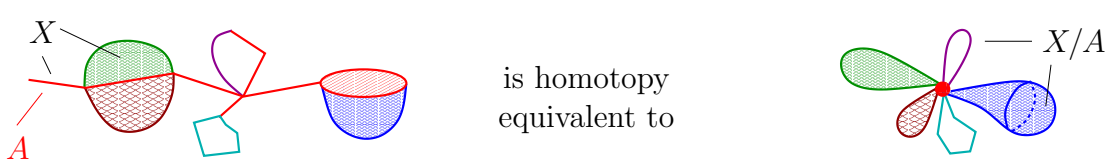


FIGURE 1044. Illustration of Corollary 84.8.

Proof. On page 1946 we pointed out that the inclusion $A \to X$ is a cofibration. The two statements now follow immediately from the corresponding statements in Lemma 84.7. □

84.3. **Adjoint functors.** In this section we will discuss the relationship between fibrations and cofibrations. First we point out a formal similarity between fibrations and cofibrations, namely they both give rise to long exact sequences:

(1) Let $p\colon Y \to B$ be a fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and we denote by $F = p^{-1}(b_0)$ the homotopy fiber of p. Let $i\colon F \to Y$ be the inclusion map. By Theorem 83.7 we obtain the following long exact sequence of homotopy groups:

$$\ldots \xrightarrow{\partial} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \ldots.$$

(2) Let $i\colon A \to X$ be a cofibration. We refer to the mapping cone $C := \mathrm{Cone}(i)$ as the *cofiber*. If $i(A)$ is a closed non-empty subset of X, then it follows easily from Corollary 31.14 and Lemma 84.7 (4) that we obtain a long exact sequence of reduced homology groups of the following form:

$$\ldots \xrightarrow{\partial} \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{p_*} \widetilde{H}_n(C) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \ldots.$$

This basically recovers the statement of Lemma 33.18.

Here is another similarity between fibrations and cofibrations:

(1) Let $p\colon Y \to B$ be a fibration and let $f\colon A \to B$ be a map. As in Lemma 16.8 (1) we consider the pullback

$$\begin{tikzcd} f^*Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $f^*Y = \{(a, y) \in A \times Y \mid f(a) = p(y)\}$. In Lemma 83.16 we saw that the pullback $f^*Y \to A$ is also a fibration.

(2) Now let $i\colon A \to X$ be a cofibration and let $f\colon A \to B$ be a map to another topological space. As on page 486 we consider the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\ X \arrow[r] & f_*X \end{tikzcd}$$

where $f_*X = (X \sqcup B)/i(a) \sim f(a)$ for all $a \in A$. In Exercise 84.6 we will see that the pushout $B \to f_*X$ is again a cofibration.

So, to summarize: *the pullback of a fibration is again a fibration* and *the pushout of a cofibration is again a cofibration.*

The main goal of this section though is to formulate an arguably more direct relationship between the notions of a fibration and a cofibration. The key to doing so is the following definition from category theory.

Definition. We say that two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ form an *adjoint pair* if given any $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\mathrm{Mor}_{\mathcal{D}}(F(c), d) \to \mathrm{Mor}_{\mathcal{C}}(c, G(d))$$

that is natural in c and d.[1427] We also say that the functor F is the *left adjoint of* G and that G is the *right adjoint of* F.

Examples.

(1) Let $\mathcal{S}$ be the category of sets and let $\mathcal{G}$ be the category of groups. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{S} & \to & \mathcal{G} \\ S & \mapsto & \langle S\rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{G} & \to & \mathcal{S} \\ (\pi,\ \cdot\ ) & \mapsto & \pi. \end{array}$$

In other words, F associates to a set S the free group $\langle S\rangle$ generated by S that we defined on page 377.[1428] Furthermore, G assigns to a group $(\pi,\ \cdot\ )$ the underlying set π.[1429] Using Lemma 11.13 one sees that for a set S and a group π the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{S}}(S,\ \overbrace{\pi}^{=G(\pi,\ \cdot\ )}\ ) & \to & \mathrm{Mor}_{\mathcal{G}}(\overbrace{\langle S\rangle}^{=F(S)}, (\pi,\ \cdot\ )) \\ (\text{map } h\colon S\to\pi \text{ of sets}) & \mapsto & \begin{array}{l}\text{the unique homomorphism } \langle S\rangle\to(\pi,\ \cdot\ ) \\ \text{that restricts to } h \text{ on the subset } S\subset\langle S\rangle\end{array} \end{array}$$

is a natural bijection. Thus we see that the functors F and G are adjoint.

(2) Let $\mathcal{A}$ be the category of abelian groups and let A be an abelian group. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{A} & \to & \mathcal{A} \\ B & \mapsto & A\otimes B \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{A} & \to & \mathcal{A} \\ B & \mapsto & \mathrm{Hom}(A,B). \end{array}$$

Now let $B, C\in\mathcal{A}$. We consider the map[1430]

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{A}}(\overbrace{A\otimes B}^{=F(B)}, C) & \to & \mathrm{Mor}_{\mathcal{A}}(B, \overbrace{\mathrm{Hom}(A,C)}^{=G(C)}) \\ (f\colon A\otimes B\to C) & \mapsto & \left(\begin{array}{rcl} B & \to & \mathrm{Hom}(A,C) \\ b & \mapsto & \left(\begin{array}{rcl} A & \to & Y \\ a & \mapsto & f(a\otimes b)\end{array}\right)\end{array}\right). \end{array}$$

Using Lemma 42.2 (3) one can easily show that this map is a bijection. This shows that F and G form an adjoint pair.

(3) Let $\mathcal{T}$ be the category of topological spaces. Rather similarly to the previous example we consider the two functors[1431]

$$\begin{array}{rcl} F\colon \mathcal{T} & \to & \mathcal{T} \\ X & \mapsto & X\times[0,1] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{T} & \to & \mathcal{T} \\ X & \mapsto & X^{[0,1]}. \end{array}$$

[1427] We leave it to our reader to figure out what "natural in c and d" means.

[1428] In what sense is this a functor?

[1429] Such a functor, where we "forget" some of the given structure is often called a *forgetful functor*.

[1430] For abelian groups X and Y we have by definition that $\mathrm{Mor}_{\mathcal{A}}(X,Y)=\mathrm{Hom}(X,Y)$.

[1431] Here we equip $X\times[0,1]$ with the product topology and we equip $X^{[0,1]}$ with the compact-open topology.

Now let $X, Y \in \mathrm{Ob}(\mathcal{T})$, i.e. let X and Y be topological spaces. We consider the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{T}}(\overbrace{X \times [0,1]}^{=F(X)}, Y) & \to & \mathrm{Mor}_{\mathcal{T}}(X, \overbrace{Y^{[0,1]}}^{=G(Y)}) \\ (f\colon X \times [0,1] \to Y) & \mapsto & \left(\begin{array}{rcl} X & \to & Y^{[0,1]} \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right). \end{array}$$

It follows from Proposition 2.6 that this map is well-defined and that it is a bijection.[1432] We have thus shown that F and G form an adjoint pair.

(4) The following example is a slight, but important modification of the previous example. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{T} & \to & \mathcal{T} \\ X & \mapsto & \Sigma(X) = \underbrace{(X \times [-1,1])/\sim}_{=\text{suspension of } X} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{T} & \to & \mathcal{T} \\ X & \mapsto & \underbrace{\Omega(X)}_{=\text{free loop space}}. \end{array}$$

Now let $X, Y \in \mathrm{Ob}(\mathcal{T})$. We consider the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{T}}(\overbrace{\Sigma(X)}^{=F(X)}, Y) & \to & \mathrm{Mor}_{\mathcal{T}}(X, \overbrace{\Omega(Y)}^{=G(Y)}) \\ \left(\begin{array}{rcl} f\colon \Sigma(X) & \to & Y \\ [(x,t)] & \mapsto & f([(x,t)]) \end{array} \right) & \mapsto & \left(\begin{array}{rcl} X & \to & \Omega(Y) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f([(x,2t-1)]) \end{array} \right) \end{array} \right). \end{array}$$

It follows quite easily from Proposition 2.6 and Lemma 1.68 that this map is well-defined and that it is a bijection. We have thus shown that F and G form an adjoint pair.

Now we can explain the formal relationship between fibrations and cofibrations. Let E and B be topological spaces and let $p\colon E \to B$ be a map. Recall that by definition $p\colon E \to B$ is a fibration if we can always complete the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, hook, "y \mapsto (y,0)"'] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B. \end{tikzcd}$$

We now "dualize" this diagram as follows:

(1) we reverse the orientation of each arrow,
(2) we replace monomorphisms by epimorphisms,

[1432]More precisely, if we consider this map in the category of sets, then it is obviously a bijection. Proposition 2.6 shows that the map f on the left is continuous if and only if the corresponding map on the right is continuous. This shows that we also have a bijection between the sets of continuous maps, i.e. between the sets of morphisms in the category of topological spaces.

(3) we replace $Y \times [0,1]$ by $Y^{[0,1]}$,
(4) for purely psychological reasons we replace $p\colon E \to B$ by $i\colon A \to X$.

We obtain the following diagram:

$$\begin{tikzcd}
Y & & X \arrow[ll, "\widetilde{f}"'] \arrow[dll, dashed, "\widetilde{F}"'] \\
Y^{[0,1]} \arrow[u, two heads, "f \mapsto f(0)"] & & A. \arrow[ll, "f"'] \arrow[u, "i"']
\end{tikzcd}$$

Finally recall that we had just shown in Example (3) that given two topological spaces A and B we have a natural identification

$$\mathrm{Mor}(A \times [0,1], B) \;=\; \mathrm{Mor}(A, B^{[0,1]}).$$

Thus (with a few other more or less mysterious changes) the previous challenge diagram translates into the following challenge diagram:

$$\begin{tikzcd}
X \arrow[r, "x \mapsto (x,0)"] \arrow[d, "\widetilde{f}"'] & X \times [0,1] \arrow[dl, dashed, "\widetilde{F}"'] \\
Y & A \times [0,1]. \arrow[l, "f"'] \arrow[u, "i"']
\end{tikzcd}$$

But this challenge diagram looks familiar: it is precisely the diagram that appears in the definition of a cofibration.

We conclude this chapter with three quotes:

(1) "In the mathematical disciplines of algebraic topology and homotopy theory, Eckmann–Hilton duality in its most basic form, consists of taking a given diagram for a particular concept and reversing the direction of all arrows [...]."[1433] [1434]
(2) "Eckmann–Hilton duality can be extremely helpful as an organizational principle, reducing significantly what one has to remember, and providing valuable hints on how to proceed in various situations." [Hat, p. 464]
(3) "In mathematics [...] currying[1435] is the technique of translating the evaluation of a function that takes multiple arguments into evaluating a sequence of functions, each with a single argument. For example, a function that takes two arguments, one from X and one from Y, and produces outputs in Z, by currying is translated into a function that takes a single argument from X and produces as outputs functions from Y to Z."[1436]

[1433] `https://en.wikipedia.org/wiki/Eckmann-Hilton_duality`

[1434] Note that the notion of pullback and pushout are related by Eckmann-Hilton duality.

[1435] This notion is named after the American mathematician Haskell Brooks Curry (1900-1982) who was an American mathematician and logician. There are three programming languages named after him, namely Haskell, Brook and Curry.

[1436] `https://en.wikipedia.org/wiki/Currying#Algebraic_topology`

So we see that in the above discussion we performed Eckmann-Hilton duality combined with some currying.

Exercises for Chapter 84.

Exercise 84.1. Solve the following problems "by hand", i.e. without making use of any of above results. For example you should not use Lemma 84.6.

(a) Let $A = [a, b]$ be a compact interval and let $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is a cofibration.
(b) Let $A = \mathbb{R} \setminus \{0\}$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.
(c) Let $A = (-1, 1)$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.

Exercise 84.2. Prove the "(2) $\Rightarrow$ (1)"-direction of Lemma 84.1.

Exercise 84.3. Let $X = \{x, y\}$ be a set with two elements and the trivial topology furthermore let $A = \{x\}$. Show that the inclusion $A \to X$ is a cofibration.

Exercise 84.4. Let M be a manifold. Show that the inclusion $i\colon \partial M \to M$ is a cofibration. Hint: The proof is not that different from the proof of Proposition 84.2.

Exercise 84.5. Let X be a topological space and let $x_0 \in X$. We suppose that the inclusion map $\{x_0\} \to X$ is a cofibration. Show that there exists an open neighborhood U of x_0 and a homotopy $H\colon U \times [0, 1] \to X$ rel $\{x_0\}$ from id_U to the constant map c_{x_0}.
Remark: This statement almost implies that the point $x_0 \in X$ is good in the sense of the definition on page 387.

Exercise 84.6. Let $i\colon A \to X$ be a cofibration and let $f\colon A \to B$ be a map to another topological space B. Show that the map

$$\begin{array}{rcl} B & \to & f_*X = (X \sqcup B)/\sim \ \text{ with } i(a) \sim f(a) \text{ for all } a \in A \\ b & \mapsto & [b] \end{array}$$

is a cofibration.

85. Fiber bundles

In this chapter we will introduce fiber bundles. For us they will be the most important source of fibrations. In this and the following chapter one of our favorite games will be to find more and more fiber bundles to obtain lots of non-trivial information on homotopy groups of various topological spaces.

85.1. **The definition of fiber bundles.**

The following proposition says that being a Serre fibration is a "local property".

Proposition 85.1. *Let $p\colon Y \to B$ be a map between topological spaces. If for every $b \in B$ there exists an open neighborhood U of b such that $p\colon p^{-1}(U) \to U$ is a Serre fibration, then $p\colon Y \to B$ itself is a Serre fibration.*

Proof. Let $p\colon Y \to B$ be a map between topological spaces. In this proof we say $U \subset B$ is *small* if $p\colon p^{-1}(U) \to U$ is a Serre fibration.

Now assume that $p\colon Y \to B$ has the property that B can be covered by small sets. We want to show that this hypothesis implies that the map $p\colon Y \to B$ is a Serre fibration. Therefore let $n \in \mathbb{N}_0$, let $F\colon \overline{B}^n \times [0,1] \to B$ and let $\widetilde{f}\colon \overline{B}^n \times \{0\} \to Y$ be maps such that the rectangle in the following diagram commutes:

$$\begin{tikzcd}
\overline{B}^n \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B.
\end{tikzcd}$$

We need to show that there exists a diagonal map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes.

Claim. There exists a CW-structure X on $\overline{B}^n$ and real numbers $0 = s_0 < s_1 < \cdots < s_k = 1$ such for each cell e of X and each $i \in \{0, \dots, k-1\}$ the image $F(e \times [s_i, s_{i+1}])$ is small.

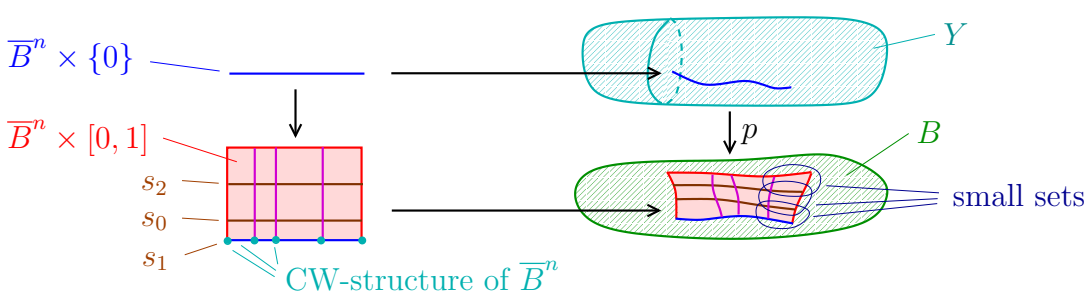


Figure 1045.

By Lemma 29.1 we can identify $\overline{B}^n$ with $[0,1]^n$. Since B is by hypothesis covered by small open subsets and since any subset of a small set is again small we obtain from Corollary 1.106 that there exists a $k \in \mathbb{N}$ such that each

$$F\Big(\underbrace{\Big[\tfrac{a_1}{k}, \tfrac{a_1+1}{k}\Big] \times \ldots \Big[\tfrac{a_n}{k}, \tfrac{a_n+1}{k}\Big]}_{\subset [0,1]^n = \overline{B}^n} \times \underbrace{\Big[\tfrac{a_{n+1}}{k}, \tfrac{a_{n+1}+1}{k}\Big]}_{\subset [0,1]}\Big)$$

is small for all $a_1, \ldots, a_{n+1} \in \{0, \ldots, k-1\}$. We equip $[0,1]^n = \overline{B}^n$ with the CW-structure defined on page 670 given by the "subcubes" of side length $\frac{1}{k}$. Furthermore we set $s_i = \frac{i}{k}$, $i = 0, \ldots, k$. By design this CW-structure and this choice of s_i has the desired properties. This concludes the proof of the claim.

We continue with the following claim.

Claim. Let $i \in \{0, \ldots, k-1\}$. Given a map $\widetilde{F}_i \colon \overline{B}^n \times \{s_i\} \to Y$ with $p \circ \widetilde{F}_i = F|_{\overline{B}^n \times \{s_i\}}$ there exists a map $\widetilde{F}_{i+1} \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{B}^n \times \{s_i\} \arrow[r, "\widetilde{F}_i"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [s_i, s_{i+1}] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}_{i+1}"'] & B.
\end{tikzcd}$$

In fact the claim follows immediately from the argument of the proof of Proposition 83.8 applied to the CW-complex $K := X = \overline{B}^n$ and the subcomplex $L = \varnothing$. Indeed, in the proof of Proposition 83.8 we did not need to assume that $p \colon Y \to B$ is a Serre fibration, all we ever used is that for every cell e of B the image $F(e)$ is small in the above sense. But in our present setting this is guaranteed by our choice of the CW-structure on $\overline{B}^n$. This concludes the proof of the claim.

We set $\widetilde{F}_0 = \widetilde{f}_0 \colon \overline{B}^n \times \{0\} \to Y$. Iteratively applying the previous claim we obtain maps $\widetilde{F}_i \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ that agree on the overlaps of the domains. By Lemma 5.3 all these maps together define a continuous map $\widetilde{F} \colon \overline{B}^n \times [0,1] \to Y$ which by construction has the desired properties. $\square$

As we saw on page 1912, every projection $p \colon B \times F \to B$ is a (Serre) fibration. This observation, together with Proposition 85.1 leads us naturally to the following definition.

Definition. Let F be a topological space.

(1) A map $p \colon Y \to B$ between two topological spaces is called an *F-bundle over B* if given any $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\Phi \colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "(x,f) \mapsto x"] \\
& U. &
\end{tikzcd}$$

Alternatively we call such a map $p\colon Y \to B$ a *bundle projection* with *fiber* F. As for fibrations we sometimes refer to B as the *base space* and to Y as the *total space*. Furthermore we refer to the map $p\colon Y \to B$ as the *projection*.

(2) We say that two F-bundles $p\colon Y \to B$ and $q\colon Z \to B$ are *equivalent* if there exists a homeomorphism $f\colon Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

The following proposition will be the incarnation of Theorem 83.7 that we will use most often.

Proposition 85.2. *Let $p\colon Y \to B$ be a fiber bundle with fiber F.*

(1) *The map $p\colon Y \to B$ is a Serre fibration.*

(2) *We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$. We pick an identification $F \xrightarrow{\cong} p^{-1}(b_0)$. We denote by $i\colon F = p^{-1}(b_0) \to Y$ the inclusion. We use the fact from (1) that $p\colon Y \to B$ is a Serre fibration to define maps $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$, $n \in \mathbb{N}$, as in Theorem 83.7. The following sequence is exact:*

$$\begin{aligned} \dots &\xrightarrow{\partial_{n+1}} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \\ &\xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \dots\dots\dots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \\ &\xrightarrow{\partial_1} \pi_0(F, y_0) \xrightarrow{i_*} \pi_0(Y, y_0) \xrightarrow{p_*} \pi_0(B, b_0). \end{aligned}$$

(3) *If we are given a commutative diagram*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

where $p'\colon Y' \to B'$ is a fiber bundle with fiber F', then given any base point $y_0' \in Y'$ the obvious diagram of the long exact sequences of homotopy groups corresponding to the fiber bundles $p\colon Y \to B$ and $p'\colon Y' \to B'$ is commutative.

(4) *If the fiber F and the base space B are path-connected, then the total space Y is also path-connected.*

Proof.

(1) This part of the proposition is an immediate consequence of the definitions, Proposition 85.1 and the discussion on page 1912.

(2) The second statement follows immediately from (1) and Theorem 83.7 (2).

(3) This statement follows easily from Theorem 83.7 (1b).

(4) This statement follows immediately from (2). □

Proposition 85.2 sparks our interest in fiber bundles. Fortunately, in the following discussion we will see that fiber bundles are ubiquitous and can be found throughout the notes.

Examples.

(1) For any two topological spaces B and F the projection map $p\colon B \times F \to B$ onto the first factor is evidently a fiber bundle with fiber F. Any F-bundle that is equivalent to such a bundle is called *trivial*, otherwise it is called non-trivial.

(2) The "obvious projection map" from the Möbius band to the circle S^1, i.e. the map

$$\begin{array}{rcl} Y = ([0,1] \times [-1,1])/(0,y) \sim (1,-y) & \to & S^1 \\ (x,y) & \mapsto & e^{2\pi i x} \end{array}$$

is easily seen to be a fiber bundle with fiber $[-1,1]$. Since the Möbius band is not homeomorphic to the trivial bundle $S^1 \times [-1,1]$ we see that the bundle is non-trivial.

(3) The same argument as in (2) shows that the Klein bottle is a non-trivial S^1-bundle over S^1. On page 435 we saw that the connected sum $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle. We have thus shown that $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is a non-trivial S^1-bundle over S^1. Amusingly an analogous statement holds in dimension four, namely the connected sum $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$ is a non-trivial S^2-bundle over S^2. It is a challenging exercise to prove this fact by hand, alternatively a proof can be found in [GS, p. 107] or [Sco, p. 124]. Similarly, one can show that the connected sum $\mathbb{H}\mathrm{P}^2 \# \overline{\mathbb{H}\mathrm{P}^2}$ is a non-trivial S^4-bundle over S^4.

(4) An n-dimensional vector bundle $p\colon W \to B$, in the sense of the definition on page 1782, is by definition an $\mathbb{R}^n$-bundle over B.

We consider the next example in greater detail.

Lemma 85.3.

(1) *Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. The natural projection*

$$\begin{array}{rcl} \overbrace{M(X,f) = ([0,1] \times X)/(0,x) \sim (1,f(x))}^{\text{basically the mapping torus of } (X,f)} & \to & [0,1]/0 \sim 1 = S^1 \\ [(t,x)] & \mapsto & [t] = e^{2\pi i t} \end{array}$$

is an X-bundle over S^1.

(2) *Let X be a topological space and let $f_0, f_1\colon X \to X$ be two homeomorphisms. If f_0 and f_1 are isotopic, then the two X-bundles $M(X,f_0) \to S^1$ and $M(X,f_1) \to S^1$ are equivalent.*

(3) *Let V be an n-dimensional real vector space and let $\varphi\colon V \to V$ be an isomorphism of vector spaces. We equip V with the topology given on page 73. For each $t \in [0,1)$ we equip $\{t\} \times V \subset M(V,\varphi)$ with the obvious vector space structure. This turns the projection map $M(V,\varphi) \to S^1$ into an n-dimensional vector bundle.*

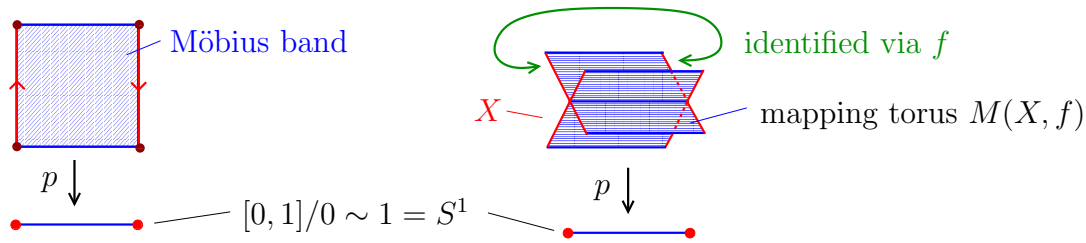


FIGURE 1046.

Proof.

(1) We denote by $\pi\colon [0,1] \to [0,1]/0 \sim 1$ the obvious projection map. We consider the two open sets $U_1 := \pi((\frac{1}{4}, \frac{3}{4}))$ and $U_2 := \pi([0, \frac{1}{3}) \cup (\frac{2}{3}, 1])$ in $[0,1]/0 \sim 1$. Furthermore we consider the maps

$$\begin{array}{ccc} p^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1 \times X \\ [(t,x)] & \mapsto & (t,x) \end{array} \quad \text{and} \quad \begin{array}{ccc} p^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2 \times X \\ [(t,x)] & \mapsto & \begin{cases} (t, f(x)), & \text{if } t \in [0, \frac{1}{3}), \\ (t,x) & \text{if } t \in (\frac{2}{3}, 1]. \end{cases} \end{array}$$

Note that the map on the right is indeed well-defined. Using Lemmas 1.66 and 5.3 one can show quite easily that both maps are homeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$, we have shown that $p\colon M(X,f) \to [0,1]/0 \sim 1$ is indeed an X-bundle.

(2) This statement will be proved in Exercise 85.3.

(3) The proof of this statement is basically the same as the proof of (1). We only have to make the extra observation that for each $i \in \{1,2\}$ and each $t \in U_i$ the map

$$\Phi_i\colon V = [\{t\} \times V] = p^{-1}(t) \ \to \ \{t\} \times V = V$$

is either given by id or ϕ. Thus in either case the map is given by an *isomorphism* of vector spaces. □

Remark. The previous lemma shows that any self-homeomorphism of $\mathbb{R}^n$ gives rise to an $\mathbb{R}^n$-bundle over S^1, whereas any isomorphism of $\mathbb{R}^n$ gives rise to an n-dimensional vector bundle over S^1. The question arises whether every $\mathbb{R}^n$-bundle is secretly a vector bundle. More precisely, the question arises whether every $\mathbb{R}^n$-bundle is equivalent to an n-dimensional vector bundle. In Exercise 85.4 we will answer this question in the affirmative if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.[1437]

It follows from the discussion on page 471 and Lemma 85.3 that the complement of the trefoil knot and the complement of the figure-8 knot are fiber bundles over S^1. Of course there are many other knots, for example consider the stevedore knot shown in Figure 1047. The following question arises:

[1437] Frankly, I am not sure whether the same statement holds if f is a homeomorphism.

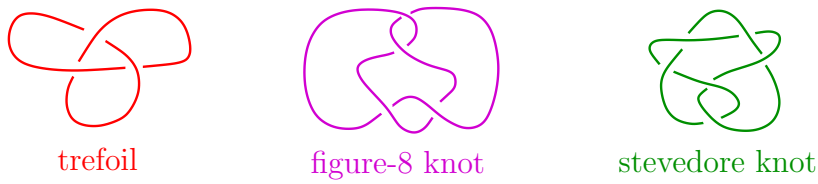


FIGURE 1047.

Question 85.4. *Is the complement of the stevedore knot a fiber bundle over S^1?*

We continue with more examples of fiber bundles.

Examples.

(1) As we pointed out above, any n-dimensional vector bundle $p\colon W \to B$ is by definition an $\mathbb{R}^n$-bundle over B. In fact p gives rise to two other bundles which are much more interesting topological objects:

(a) We denote by $s\colon B \to W$ the zero-section, i.e. the map given by sending each $b \in B$ to the zero-element of the vector space $p^{-1}(y)$. We write $W^* := W \backslash s(B)$. It follows easily from the definitions that $p\colon W^* \to B$ is a fiber bundle with fiber $\mathbb{R}^n \setminus \{0\}$.

(b) Let $\lambda \in \mathbb{R}$. By definition of a vector bundle we have a well-defined multiplication by λ on each fiber. We define

$$SW \;:=\; W^*/\sim \qquad \text{where } v \sim w \text{ if } p(v) = p(w) \text{ and if there exists a } \lambda > 0 \text{ with } \lambda v = w.$$

The map $p\colon W^* \to B$ descends to a map $p\colon SW \to B$. We leave it as an entertaining exercise to verify that $p\colon SW \to B$ is an S^{n-1}-bundle.

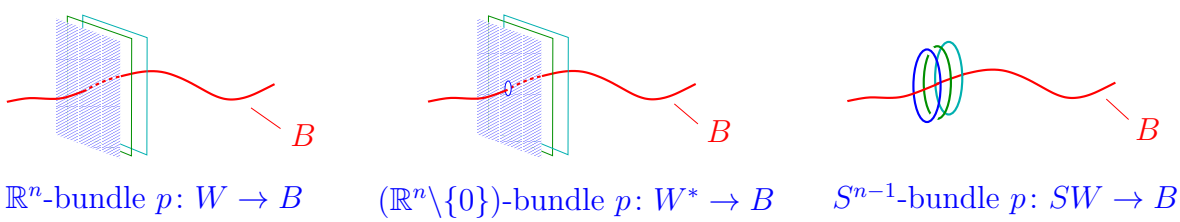


FIGURE 1048.

(2) A smooth $\overline{B}^n$-bundle $p\colon W \to M$ over a manifold M, in the sense of the definition on page 1492, is by definition an $\overline{B}^n$-bundle in the above sense. Slightly more interesting, if M has no boundary and if we restrict p to a map $\partial W \to M$, then it follows easily from Proposition 56.9 (1) that the restriction of p to ∂W defines an S^{n-1}-bundle $p\colon \partial W \to M$.

The following lemma is almost verbatim the same as Lemma 77.2.

Lemma 85.5. *Let $g\colon C \to B$ be a map between topological spaces and let $p\colon Y \to B$ be a fiber bundle with fiber F. As in Lemma 16.8 (1) we consider the pullback*

$$g^*Y \;:=\; \{(c,f)\,|\,c \in C \text{ and } f \in p^{-1}(g(c))\} \;\subset\; C \times Y,$$

which is equipped with the subspace topology. Then the map

$$\begin{array}{rcl} q\colon g^*Y \;:=\; \{(c,f)\,|\,c \in C \text{ and } f \in p^{-1}(g(c))\} & \to & C \\ (c,f) & \mapsto & c \end{array}$$

is also an F-bundle.

Proof. The proof is, not surprisingly, also almost identical to the proof of Lemma 77.2. □

Remark. The pullback of bundles satisfies basically the same statements as the pullback of vector bundles, see Proposition 77.3. There is surely no need to spell out these properties again.

The following lemma classifies bundles over compact intervals $[a,b]$ and also over the circle S^1.

Lemma 85.6.

(1) *Every bundle over a compact interval $[a,b]$ is trivial.*
(2) *If $p\colon Y \to S^1$ is a fiber bundle with fiber X, then there exists a homeomorphism $f\colon X \to X$ such that p is equivalent to the above fiber bundle $M(X,f) \to S^1$.*

Sketch of proof.

(1) This statement will be proved in Exercise 85.2.
(2) Suppose we are given a fiber bundle $p\colon Y \to S^1$ with fiber X. Let $\varphi\colon [0,2\pi] \to S^1$ be the map that is given by $\varphi(t) = e^{it}$. It follows from (1) that the pullback bundle

$$q\colon \varphi^*Y = \{(t,v) \in [0,2\pi] \times Y \,|\, v \in p^{-1}(\varphi(t))\} \;\to\; [0,2\pi]$$

is trivial, i.e. there exists a homeomorphism $\Phi\colon [0,2\pi] \times X \to \varphi^*Y$ such that the following diagram commutes

$$\begin{tikzcd} {[0,2\pi] \times X} \arrow[rr, "\Phi"] \arrow[dr, "(t,x)\mapsto t"'] & & \varphi^*Y \arrow[dl, "q"] \\ & {[0,2\pi].} & \end{tikzcd}$$

We define $f\colon X \to X$ to be the homeomorphism that is given by

$$\begin{array}{ccccccccccc} & & & & \varphi(0) = 1 = \varphi(2\pi) & & & & & & \\ & & & & \downarrow & & & & & & \\ X & \xrightarrow{x\mapsto(0,x)} & \{0\}\times X & \xrightarrow{\Phi} & q^{-1}(0) & \to & p^{-1}(1) & \leftarrow & q^{-1}(2\pi) & \xleftarrow{\Phi} & \{2\pi\}\times X \xrightarrow{(2\pi,x)\mapsto x} X. \\ & & & & (0,v) & \mapsto & v & & & & \\ & & & & & & v & \leftarrow\!\shortmid & (2\pi,v). & & \end{array}$$

We leave it as a mildly interesting exercise to the reader to verify that $p\colon Y \to S^1$ is equivalent to the fiber bundle $M(X,f) \to S^1$. □

For the record we mention the following generalization of Lemma 85.6 (1).

Proposition 85.7. *Let X be a topological space. If X is paracompact and contractible, then every bundle over X is trivial.*

Proof. The statement of the proposition is almost the same as the statement of Proposition 77.6 (3) for vector bundles. The reference we gave for vector bundles, namely [Hat2, Theorem 1.6], also applies to general bundles. □

We conclude this section with an example that shows how the above results can be used in practice. Thus let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the manifold S^n. We calculate that for $i = 1, \dots, n-k-2$ we have

$$\begin{array}{ccccc} H_i(S^n \setminus K; \mathbb{Z}) & \cong & H^{n-i-1}(K; \mathbb{Z}) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{Alexander Duality Theorem 66.1} & & \text{by Proposition 51.14 (5) and the fact that } n-i-1 > k = \dim(K) & \end{array}$$

The following proposition shows that not only do the first $n-k-2$ homology vanish, but the first $n-k-2$ homotopy groups also vanish.

Proposition 85.8. $(*)$ *Let $n \in \mathbb{N}$ and let K be a closed k-dimensional submanifold of the manifold S^n. For $i = 1, \dots, n-k-2$ we have $\pi_i(S^n \setminus K) = 0$.*

Example. Let K be the trefoil in $\mathbb{R}^3$ and let $\mu \subset \mathbb{R}^3$ be a meridian as defined on page 362. (See also Figure 1049 to the left.) In Lemma 33.15 we saw that μ represents a non-zero element in $H_1(S^3 \setminus K)$. It follows from the naturality of the Hurewicz homomorphism, see Proposition 38.2 (4), that μ also represents a non-trivial element in $\pi_1(S^3 \setminus K)$. Via the usual embedding $\mathbb{R}^3 \to \mathbb{R}^3 \times \{0\} \to \mathbb{R}^4$ we can view K and μ as submanifolds of $S^4 = \mathbb{R}^4 \cup \{\infty\}$. By Proposition 85.8 we have $\pi_1(S^4 \setminus K) = 0$. In particular we obtain that μ is trivial in $\pi_1(S^4 \setminus K)$. In fact in Figure 1049 we show an embedded disk D in $\mathbb{R}^4 \subset S^4$ whose boundary is given by μ and that does not intersect K. Here we use the convention from page 1517, i.e. "pinkness" of a point measures how large is the fourth coordinate.

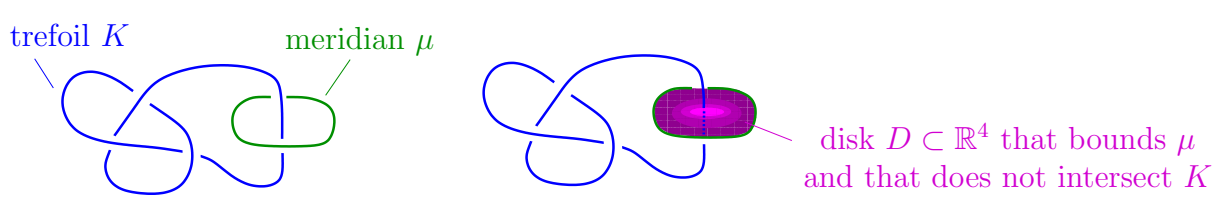


FIGURE 1049.

Proof $(*)$. Let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the manifold S^n. If $n-k-2 \leq 0$, then there is nothing to prove. Thus we assume that $n-k-2 \geq 1$, i.e. we assume that $n \geq k+3$.

We want to show that $\pi_i(S^n \setminus K) = 0$ for $i = 1, \dots, n-k-2$. We already know that the corresponding homology groups vanish. Thus by Corollary 80.7 it suffices to show that $\pi_1(S^n \setminus K) = 0$. That is precisely what we will do in the remainder of this proof.

In the following we consider the case that K is connected. We leave it to the reader to modify the proof to deal with the case that K is disconnected. By the General Tubular Neighborhood Theorem 64.7 the closed submanifold K admits a tubular neighborhood $p\colon W \to K$. Recall that W is a compact n-dimensional submanifold of S^n and that p is in particular a $\overline{B}^{n-k}$-bundle over K. As we pointed out on page 1965, the projection $p\colon \partial W \to K$ is an S^{n-k-1}-bundle.

We write $X = S^n \setminus \mathring{W}$. By Proposition 64.13 and Lemma 64.3 we know that $\pi_1(S^n \setminus K)$ is isomorphic to $\pi_1(X)$. Thus it suffices to show that X is simply connected.

By Proposition 3.24 we know that X is a compact n-dimensional submanifold of S^n with $W \cap X = \partial W = \partial X$. From Proposition 85.2 (4) we know that ∂W is connected. We pick a base point $x_0 \in \partial W$. By the Seifert-van Kampen Theorem 14.2 for manifolds we know that the inclusion induced map

$$\pi_1(X, x_0) *_{\pi_1(\partial W, x_0)} \pi_1(W, x_0) \ \to \ \underbrace{\pi_1(S^n, x_0)}_{=0}$$

is an isomorphism. We recall that Lemma 13.12 says that $A *_G G \cong A$. Thus, if we want to show that $\pi_1(X, x_0) = 0$ it follows from Lemma 13.12 that it suffices to prove the following claim.

Claim. The inclusion induced map $\pi_1(\partial W, x_0) \to \pi_1(W, x_0)$ is an isomorphism.

We denote both of the inclusions $\partial W \to W$ and $S^{n-k-1} \to \overline{B}^{n-k}$ by i. We have a commutative diagram

$$\begin{tikzcd}
\partial W \arrow[rr, "i"] \arrow[dr] & & W \arrow[dl] \\
& K &
\end{tikzcd}$$

of fiber bundles with fibers S^{n-k-1} and $\overline{B}^{n-k}$. From Proposition 85.2 (3) we obtain the following commutative diagram of long exact sequences of morphisms between pointed sets:

$$\begin{tikzcd}
\dots \arrow[r] & \pi_2(K) \arrow[r, "\partial"] \arrow[d, "="] & \pi_1(S^{n-k-1}) \arrow[r] \arrow[d, "i_*"] & \pi_1(\partial W) \arrow[r] \arrow[d, "i_*"] & \pi_1(K) \arrow[r, "\partial"] \arrow[d, "="] & \pi_0(S^{n-k-1}) \arrow[r] \arrow[d, "i_*"] & \dots \\
\dots \arrow[r] & \pi_2(K) \arrow[r, "\partial"] & \pi_1(\overline{B}^{n-k}) \arrow[r] & \pi_1(W) \arrow[r] & \pi_1(K) \arrow[r, "\partial"] & \pi_0(\overline{B}^{n-k}) \arrow[r] & \dots
\end{tikzcd}$$

Since $n-k \geq 3$ we have $n-k-1 \geq 2$, thus $\pi_1(S^{n-k-1}) = 0$ by Proposition 5.14. It follows from a slight generalization[1438] of the Five Lemma 31.11 that the middle vertical map $i_*\colon \pi_1(\partial W) \to \pi_1(W)$ is an isomorphism. This concludes the proof of the claim. $\square$

Remark. We want to sketch an alternative approach to proving Proposition 85.8. Thus let $n \in \mathbb{N}$, let K be a closed k-dimensional submanifold of the manifold S^n and finally let $i \in \{1, \dots, n-k-2\}$. We want to show that $\pi_i(S^n \setminus K, x_0) = 0$ for some base point $x_0 \in S^n \setminus K$. Let $[\varphi\colon (S^i, *) \to (S^n \setminus K, x_0)] \in \pi_i(S^n \setminus K, x_0)$ be a random element. We make the following observations:

(1) By the Whitney Approximation Theorem 65.6 we can assume that φ is smooth.
(2) Since $i < n$ we know that $\pi_i(S^n, x_0) = 0$. It follows from Lemma 28.4 that there exists a map $\psi\colon \overline{B}^{i+1} \to S^n$ with $\psi|_{S^i} = \varphi$.
(3) Once again, by the Whitney Approximation Theorem 65.6 we can assume that ψ is also smooth.
(4) Since $k+i+1 < n$ one would like to argue, similar to the Transversality Theorem 67.2, that we can homotop ψ rel S^i to a map that does not intersect K, i.e. we would like to homotop ψ rel S^i to a map $\overline{B}^{i+1} \to S^n \setminus K$. By Lemma 28.4 this would then imply that $[\varphi] = 0 \in \pi_i(S^n \setminus K, x_0)$.

This approach can be turned into a proper proof, but we will not do so.

85.2. **Hopf fibrations arising from complex and quaternionic projective spaces.** In this section we will see that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the quaternionic projective spaces[1439] $\mathbb{H}\mathrm{P}^n$ give rise to interesting fiber bundles. In particular we will see that the Hopf map $S^3 \to S^2$ that we introduced on page 726 is in fact an S^1-bundle.

In the following we introduce two straightforward higher-dimensional analogues of the original Hopf map.

Definition. Let $n \in \mathbb{N}$. We refer to the maps

$$\begin{aligned} p\colon S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \dots + |z_n|^2 = 1\} &\to \mathbb{C}\mathrm{P}^n \\ (z_0, \dots, z_n) &\mapsto [z_0 : \dots : z_n] \end{aligned}$$

and[1440]

$$\begin{aligned} q\colon S^{4n+3} = \{(h_0, \dots, h_n) \in \mathbb{H}^{n+1} \mid |h_0|^2 + \dots + |h_n|^2 = 1\} &\to \mathbb{H}\mathrm{P}^n \\ (h_0, \dots, h_n) &\mapsto [h_0 : \dots : h_n] \end{aligned}$$

as *Hopf maps*.

[1438] More precisely, if one goes through the proof of the Five Lemma 31.11, then one realizes that it suffices that the right most vertical map is an injective morphism of pointed sets, i.e. the right most vertical map does not need to be a homomorphism of groups.

[1439] We refer to page 1062 for the slightly subtle definition of the quaternionic projective spaces.

[1440] Recall that on page 1056 we defined the norm of a quaternion $h = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ as $|h| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

The following proposition, together with Proposition 85.2, explains our interest in the Hopf maps.[1441]

Proposition 85.9. *Let $n \in \mathbb{N}$.*

(1) *The Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle.*
(2) *The Hopf map $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle.*

Proof. We only provide the proof of (2). The proof of (1) is a straightforward modification of the proof of (2). Given $j \in \{0, \dots, n\}$ we define

$$V_j \ := \ \{[h_0 : \cdots : h_n] \in \mathbb{H}\mathrm{P}^n \mid h_j \neq 0\}.$$

The same argument as in the proof of Lemma 21.6 shows that V_j is an open subset of $\mathbb{H}\mathrm{P}^n$.

Claim. The map

$$\begin{aligned} \Phi_j\colon q^{-1}(V_j) = \Big\{(h_0, \dots, h_n) \in \mathbb{H}^{n+1} \,\Big|\, \textstyle\sum_{i=0}^n |h_i|^2 = 1 \text{ and } h_j \neq 0\Big\} \ &\to \ V_j \times S^3 \\ (h_0, \dots, h_n) \ &\mapsto \ \big([h_0 : \cdots : h_n], \tfrac{1}{|h_j|} \cdot h_j\big) \end{aligned}$$

is a homeomorphism.

We consider the map

$$\begin{aligned} \Psi_j\colon V_j \times S^3 \ &\to \ q^{-1}(V_j) \\ ([h_0 : \cdots : \underbrace{h_j}_{\neq 0} : \cdots : h_n], u) \ &\mapsto \ \begin{array}{c}\text{the normalization to length one of the non-zero vector} \\ \big(u \cdot h_j^{-1} \cdot h_0, \dots, \underbrace{u \cdot h_j^{-1} \cdot h_j}_{=u}, \dots, u \cdot h_j^{-1} \cdot h_n\big) \in \mathbb{H}^{n+1}.\end{array} \end{aligned}$$

This map Ψ_j is well-defined. Indeed, a short calculation shows that if we pick a different representative for $[h_0 : \cdots : h_n]$, i.e. if we plug in $[\lambda \cdot h_0 : \cdots : \lambda \cdot h_n]$ for some $\lambda \in \mathbb{H} \setminus \{0\}$ then we obtain the same result.[1442] Using Lemma 1.66 one can show that the map Ψ_j is continuous. One easily verifies that Ψ_j and Φ_j are inverses of one another. This concludes the proof of the claim.

It is clear that the open sets V_j cover all of $\mathbb{H}\mathrm{P}^n$ and that the triangles

$$\begin{tikzcd} q^{-1}(V_j) \arrow[rr, "\Phi_j"] \arrow[dr, "q"'] & & V_j \times S^3 \arrow[dl, "(v,h)\mapsto h"] \\ & S^3 & \end{tikzcd}$$

commute. This shows, according to the definition on page 1961, that $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle. $\square$

Proposition 85.9, combined with Proposition 85.2 gives us two interesting theorems.

[1441]For mysterious reasons the Hopf maps are almost invariably called "Hopf fibrations", even though the name "Hopf fiber bundle" would be more precise.

[1442]Note though that the order of the factors in the definition of Ψ_j is important since $\mathbb{H}$ is not commutative.

Theorem 85.10.

(1) *For all $n \geq 3$ we have $\pi_n(S^3) \cong \pi_n(S^2)$.*
(2) *We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2)$ is generated by the Hopf map $S^3 \to S^2$.*

The second statement gives in particular an answer to Question 63.15 (1).

Proof.

(1) On page 113 we gave an identification of $\mathbb{C}\mathrm{P}^1$ with S^2. Thus Proposition 85.9 (1), applied to $n = 1$, tells us that the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle. Proposition 85.2 gives us the long exact sequence[1443]

$$\ldots \to \pi_n(S^1) \to \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\partial} \pi_{n-1}(S^1) \to \pi_{n-1}(S^3) \to \ldots$$

The desired statement now follows from the example on page 723 where we saw that $\pi_k(S^1) = 0$ for $k \geq 2$.

(2) In Corollary 80.6 (1) we saw that $\pi_3(S^3) = \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$. By (1) the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ induces an isomorphism $p_*\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$. The image of $[\mathrm{id}_{S^3}]$ under p_* is by definition precisely the class in $\pi_3(S^2)$ represented by the Hopf map. Thus we see that $\pi_3(S^2) \cong \mathbb{Z}$ and that $\pi_3(S^2)$ is generated by the Hopf map. □

Theorem 85.11.

(1) *For all $n \geq 2$ we have $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$.*
(2) *For $n \in \{1, \ldots, 6\}$ we have $\pi_n(S^4) \cong \pi_{n-1}(S^3)$.*

Remark. From Corollary 80.6 (1) we know that $\pi_7(S^7) \cong \mathbb{Z}$. Thus it follows from Theorem 85.11 (1) that $\pi_7(S^4) \cong \pi_7(S^7) \oplus \pi_6(S^3)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 63.8 we already saw that the Hopf invariant is an epimorphism $\pi_7(S^4) \to \mathbb{Z}$.

Proof.

(1) By Lemma 44.10 (2) we can identify the quaternionic projective space $\mathbb{H}\mathrm{P}^1$ with S^4. Therefore if we apply Proposition 85.9 (2) with $n = 1$ we obtain an S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$. We pick $y_0 \in S^7$ and we write $b_0 = q(y_0)$. We pick an identification $S^3 = q^{-1}(b_0)$ and we denote by $i\colon S^3 \to S^7$ the map that is given by the inclusion of the fiber $F = S^3$ into S^7. From Proposition 85.2 we obtain the following long exact sequence

$$\cdots \to \pi_n(S^3, y_0) \xrightarrow{i_*} \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \xrightarrow{i_*} \pi_{n-1}(S^7, y_0) \to \ldots.$$

By Proposition 28.9 we know that $\pi_3(S^7, y_0) = 0$. By definition this implies that the map $i\colon S^3 \to S^7$ is homotopic rel y_0 to the constant map $c_{y_0}\colon S^3 \to \{y_0\}$. By

[1443] It is actually worth having a glance at the lower end of this sequence. We have the exact sequence

$$\pi_2(S^3) \xrightarrow{p_*} \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \to \pi_1(S^3).$$

From Proposition 28.9 we know that $\pi_1(S^3) = \pi_2(S^3) = 0$ and Proposition 7.18 we know that $\pi_1(S^1) \cong \mathbb{Z}$. Thus we obtain that $\pi_2(S^2) \cong \mathbb{Z}$. In Exercise 85.11 we will see that using this exact sequence we can obtain a new proof that id_{S^2} is a generator of $\pi_2(S^2)$.

Theorem 83.7 (3) this implies that the above long exact sequence splits into short exact sequences

$$0 \to \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \to 0.$$

Furthermore Theorem 83.7 (3) says that these short exact sequences split, therefore we obtain the promised isomorphisms $\pi_n(S^4, b_0) \cong \pi_n(S^7, y_0) \oplus \pi_{n-1}(S^3, y_0)$.

(2) The statement holds trivially Proposition 28.9 for $n = 1$. For $n \geq 2$ this statement follows from (1) and Proposition 28.9 which says in this context that $\pi_n(S^7) = 0$ for $n = 1, \dots, 6$. □

As we will see, Proposition 85.2 is a goose that just cannot stop laying golden eggs. For example we have the following theorem.

Theorem 85.12. *We use the homeomorphism from page 113 to make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ and as on page 111 we view $\mathbb{C}\mathrm{P}^1$ as a subset of each $\mathbb{C}\mathrm{P}^n$ and of $\mathbb{C}\mathrm{P}^\infty$.*

(1) *Let $n \in \mathbb{N}$. We have*

$$\pi_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \dots, 2n, \\ \mathbb{Z}, & \text{if } k = 2n + 1. \end{cases}$$

(2) *We have*

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, 6, \dots. \end{cases}$$

Remark. Let $k < l$. In Question 38.12 we had asked whether the complex projective space $\mathbb{C}\mathrm{P}^k$ is a retract of $\mathbb{C}\mathrm{P}^l$. In Proposition 62.14 we used the calculation of the cup product to give a negative answer. Now we can also use homotopy groups for the same purpose. More precisely, from Theorem 85.12, we obtain that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^k) \cong \mathbb{Z}$ but that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^l) = 0$. The usual argument, see Lemma 6.2 implies that once again we obtain a negative answer to Question 38.12, i.e. $\mathbb{C}\mathrm{P}^k$ is not a retract of $\mathbb{C}\mathrm{P}^l$.

Proof.

(1) By Proposition 85.9 (1) we know that for every $n \in \mathbb{N}$ the Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle. Thus we obtain from Proposition 85.2 the following long exact sequence of homotopy groups:

$$\dots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \xrightarrow{p_*} \pi_k(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_{k-1}(S^1) \to \dots$$

We remind ourselves of the facts that $\pi_k(S^{2n+1}) = 0$ for $k = 1, \dots, 2n$, that $\pi_{2n+1}(S^{2n+1}) \cong \mathbb{Z}$ and that $\pi_k(S^1) = 0$ for $k \geq 2$. Putting everything together implies immediately that $\pi_k(\mathbb{C}\mathrm{P}^n) = 0$ for $k = 3, \dots, 2n$ and that $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$. Only the lower end of the long exact sequence requires slightly more thought. More

precisely, we have the exact sequence

$$\underbrace{\pi_2(S^{2n+1})}_{=0} \to \pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_1(S^1) \to \underbrace{\pi_1(S^{2n+1})}_{=0} \to \pi_1(\mathbb{C}\mathrm{P}^n) \to \pi_0(S^1) \to \pi_0(S^{2n+1}).$$

Since S^1 and S^{2n+1} are path-connected we see that the map on the right is an isomorphism of pointed sets. It follows that $\pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. (We had of course known the latter statement for a long time, see e.g. page 697.) The only thing which is not entirely clearly from this argument is why $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$. One could either heroically fight one's way through the above isomorphism, in the same vein as in Exercise 85.11, and use that we know explicitly the generator of $\pi_1(S^1)$.

But fortunately we do not have to go down that route since in Corollary 80.6 (2) we already saw that $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$.

(2) We calculate that

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) \;=\; \pi_k\Big(\bigcup_{k\in\mathbb{N}} \mathbb{C}\mathrm{P}^n\Big) \;\underset{\substack{\uparrow \\ \text{by Proposition 28.11 since, see Lemma 26.7,} \\ \text{we can view } \mathbb{C}\mathrm{P}^\infty \text{ as a CW-complex where} \\ \text{each } k\text{-skeleton is given by } \mathbb{C}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^\infty}}{=} \lim_{\longrightarrow} \pi_k(\mathbb{C}\mathrm{P}^n) \;\underset{\substack{\uparrow \\ \text{by (1)}}}{=} \begin{cases} \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k=2, \\ 0, & \text{if } k = 1,3,4,5,\ldots. \end{cases}$$

□

85.3. **The Hopf fibration arising from the octonions** (∗)**.** In Proposition 85.9 we had in particular seen that the Hopf map $p\colon S^3 \to \mathbb{C}\mathrm{P}^1 = S^2$ is an S^1-bundle and that the Hopf map $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ is an S^3-bundle. We exploited these fiber bundles in Theorems 85.10 and 85.11 to obtain non-trivial information on higher homotopy groups of spheres.

In our discussion of division algebras in Section 44.2 we gave the quaternions as a first example of a non-commutative division algebra. The algebra structure on the quaternions is still nonetheless quite reasonable, in particular the algebra structure is associative which was enough to define the quaternionic projective spaces on page 1062.

Next on the list of ever weirder division algebras were the octonions $\mathbb{O}$ that we introduced on page 1057. Recall that the underlying vector space of the octonions is $\mathbb{R}^8$ but that this time the algebra structure is neither commutative nor associative. The only redeeming feature is that $\mathbb{O}$ is still a division algebra.

Since the octonions are no longer associative one cannot just copy-paste the definition of the quaternionic projective spaces on page 1062 to define octonionic projective spaces in any obvious way. But we can make a prediction: If there was a first octonionic projective space, then it would presumably give rise to a Hopf fibration $r\colon S^{15} \to \mathbb{O}P^1 = S^8$ with fiber S^7. Alas, octonionic projective spaces do not exist. But as we will see, we can still use octonions to produce such a fiber bundle. More precisely, our goal in this section is to prove the following proposition.

Proposition 85.13. *There exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7.*

The proof of Proposition 85.13 is rather longish, thus we will first point out that it allows us to prove the following theorem.

Theorem 85.14.

(1) *For all* $n \geq 2$ *we have* $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.
(2) *For* $n \in \{1, \dots, 14\}$ *we have* $\pi_n(S^8) \cong \pi_{n-1}(S^7)$.

Proof. The proof is basically the same as the proof of Theorem 85.11. We just need to replace Proposition 85.9 (2) by Proposition 85.13. □

We can copy-paste the remark after Theorem 85.11:

Remark. From Corollary 80.6 (1) we know that $\pi_{15}(S^{15}) \cong \mathbb{Z}$. Thus it follows from Theorem 85.11 (1) that $\pi_{15}(S^8)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 63.8 we already saw that the Hopf invariant is an epimorphism $\pi_{15}(S^8) \to \mathbb{Z}$.

The remainder of this chapter is occupied with providing a proof of Proposition 85.13. The proof requires several preparations. In particular we need the notion of the join of two topological spaces X and Y. Loosely speaking the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval. The following definition makes this idea precise.

Definition. Let X and Y be two topological spaces. We define the *join* $X * Y$ to be the topological space obtained from $X \times Y \times [0, 1]$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times Y \times \{0\}$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{y\} \times \{1\}$ to a single point.

Remark. The join $X * Y$, viewed as a set, can be equipped with another reasonable topology, namely the "initial topology", see [BrownR2, Chapter 5.7]. By [BrownR2, Corollary 5.7.2] these two topologies agree if X and Y are compact and Hausdorff. When dealing with joins it seems that it is safest to stick to the case that X and Y are compact and Hausdorff. For example, if X, Y and Z are topological spaces that are compact and Hausdorff, then the join is "associative", i.e. there exists a natural homeomorphism $(X * Y) * Z \cong X * (Y * Z)$. This statement is not true for all topological spaces. We refer to [Cohe3], [BrownR2, Chapter 5.7] and [FF, Chapter 2.4] for details.[1444]

Before we deal with a few examples we give a criterion for showing that a map of the form $f\colon X * Y \to Z$ is a homeomorphism.

Lemma 85.15. *If* X *and* Y *are compact topological spaces, then the following two statements hold:*

(1) *The join* $X * Y$ *is compact.*

[1444] As we saw in Lemma 2.13, products and quotients of topological spaces do not mix well. It is easy to write down "continuous looking" maps that are not continuous.

(2) *Let $f\colon X \times Y \times [0,1] \to Z$ be a map to a topological space Z such that for any $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ we have the equalities $f(x, y_1, 0) = f(x, y_2, 0)$ and $f(x_1, y, 1) = f(x_2, y, 1)$. If Z is Hausdorff and if the induced map*[1445]

$$\begin{array}{rcl} \Psi\colon X * Y & \to & Z \\ [(x,y,t)] & \mapsto & f(x,y,t) \end{array}$$

is a bijection, then the map Ψ is a homeomorphism.

Proof.

(1) It follows immediately from our hypothesis and Lemmas 1.37 and 1.55 that the join $X * Y = (X \times Y \times [0,1])/\sim$ is compact.
(2) The map $\Psi\colon X * Y \to Z$ is continuous by Lemma 1.66. Thus we obtain from our hypotheses on Ψ and Z together with Proposition 1.40 that Ψ is indeed a homeomorphism. □

In the following examples we will see that the join operation sometimes produces well-known topological spaces.

Examples.

(1) We start out with the join of the two intervals $X = [0,1]$ and $Y = [0,1]$. First let us consider Figure 1050. The figure suggests that the join $X * Y = [0,1] * [0,1]$ should be homeomorphic to the 3-dimensional standard simplex Δ^3. In fact let us consider the map

$$\begin{array}{rcl} \overbrace{([0,1] \times [0,1] \times [0,1])/\sim}^{=[0,1]*[0,1]} & \to & \Delta^3 = \{(t_0,t_1,t_2,t_3) \in (\mathbb{R}_{\geq 0})^4 \,|\, t_0 + t_1 +_2 + t_3 = 1\} \\ [(x,y,t)] & \mapsto & \frac{1}{2} \cdot \big(x \cdot t, y \cdot (1-t), t, 2 - (x \cdot t + y \cdot (1-t) + t)\big). \end{array}$$

It follows easily from Lemma 85.15 (2) that this map is a homeomorphism.

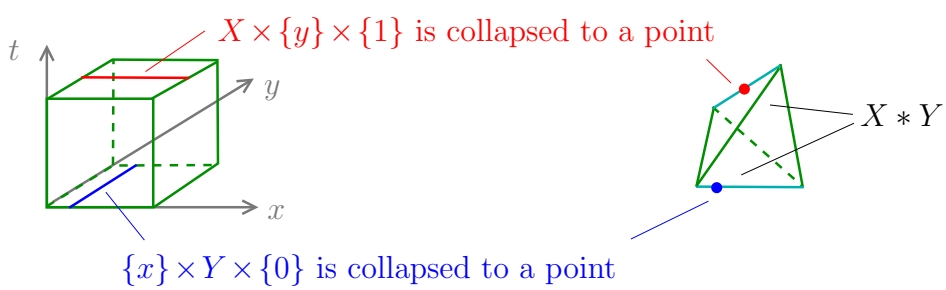


FIGURE 1050.

[1445]It follows from our hypothesis on f that the map f descends to a map on $X * Y = (X \times Y \times [0,1])/\sim$.

(2) Let X be a topological space that is compact and that is Hausdorff. Furthermore let $Y = \{-1, 1\}$ be the topological space given by the set with two elements ± 1 and the discrete topology. As usual we denote by $\Sigma(X)$ the suspension of X. For the reader's convenience we recall that it is defined as

$$\Sigma(X) \ := \ (X \times [-1, 1])/\sim$$

where we squash all points in $X \times \{-1\}$ to a single point and where we squash all points in $X \times \{1\}$ to a single point. We consider the map

$$\begin{array}{rcl} X * \{-1, 1\} = (X \times \{-1, 1\} \times [0, 1])/\sim & \mapsto & \Sigma(X) = (X \times [-1, 1])/\sim \\ [(x, \epsilon, t)] & \mapsto & [(x, \epsilon \cdot t)]. \end{array}$$

By Lemma 15.2 (2) and our hypothesis that X is Hausdorff we know that $\Sigma(X)$ is Hausdorff. Now it follows easily from Lemma 85.15 (2) that the map is in fact a homeomorphism. This homeomorphism is sketched in Figure 1051 for $X = S^1$.[1446]

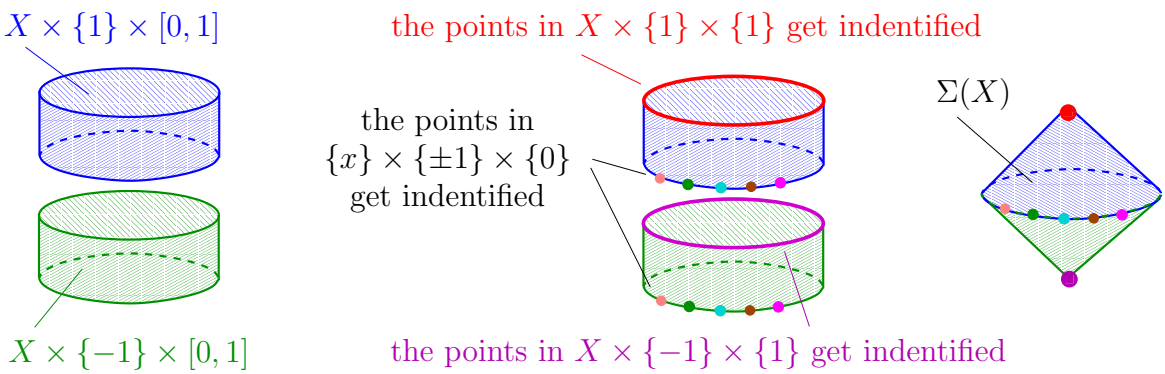


FIGURE 1051.

We record the next example of joins as a lemma.

Lemma 85.16. *Given any $m, n \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x, y, t)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

Proof. Using Lemma 15.2 (2) one can fairly easily verify that the given map is a homeomorphism. □

By now the reader can be excused for thinking that we veered off-topic. But the following proposition explains our sudden interest in the join of two topological spaces.

[1446] I do not know whether it is true that given any topological space the join $X * \{-1, 1\}$ is homeomorphic to the suspension $\Sigma(X)$.

Proposition 85.17. *Let X be a topological space that is compact and Hausdorff. Furthermore let $\varphi\colon X \times X \to X$ be a map. Given $x \in X$ we consider the maps*[1447]

$$\begin{array}{rcl} L_x\colon X & \to & X \\ a & \mapsto & \varphi(x,a) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} R_x\colon X & \to & X \\ a & \mapsto & \varphi(a,x). \end{array}$$

If each map $L_x\colon X \to X$ and each map $R_x\colon X \to X$ is a homeomorphism, then the map

$$\begin{array}{rcl} h_\varphi\colon X * X & \to & \Sigma(X) \\ [(x,y,t)] & \mapsto & [(\varphi(x,y), 2t-1)] \end{array}$$

is a fiber bundle with fiber X.[1448]

Example. We write $S^3 = \{z \in \mathbb{H} \,|\, |z| = 1\}$. By Lemma 44.1 we know that for any $w, z \in \mathbb{H}$ we have $|w \cdot z| = |w| \cdot |z|$. Thus multiplication of quaternions defines a map $\varphi\colon S^3 \times S^3 \to S^3$. Given any $z \in S^3$ the map L_z is a homeomorphism. In fact an inverse is given by $L_{z^{-1}}$ since for any $x \in S^3$ we have

$$(L_{z^{-1}} \circ L_z)(x) \;=\; z^{-1} \cdot (z \cdot x) \;\underset{\substack{\uparrow \\ \text{since } \mathbb{H} \text{ is associative}}}{=}\; (z^{-1} \cdot z) \cdot x \;=\; x,$$

and similarly we have $L_z \circ L_{z^{-1}} = \mathrm{id}_{S^3}$. The same argument shows that R_z is also a homeomorphism. Thus we obtain from Proposition 85.17 and Lemma 85.16 a fiber bundle $p\colon S^7 \to S^4$ with fiber S^3. We leave it to the heroically inclined reader to verify that this S^3-bundle is equivalent to the S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ that we had constructed in the proof of Proposition 85.9.

Proof. We consider the following two subsets of $\Sigma(X)$:

$$C \;:=\; (X \times [-1, \tfrac{1}{2}]) \,/\, X \times \{-1\} \quad \text{and} \quad D \;:=\; (X \times [-\tfrac{1}{2}, 1]) \,/\, X \times \{1\}.$$

We refer to Figure 1052 for an illustration. Since the interiors of these two sets cover $\Sigma(X)$ it suffices to find trivializations of the map $h_\varphi\colon X * X \to \Sigma(X)$ with fiber X over C and over D.

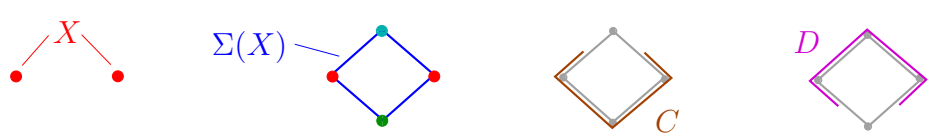


FIGURE 1052.

We start out with finding a trivialization over C.

[1447]Here the mnemonic device is that "L" should suggest "left-multiplication" and "R" should suggest "right-multiplication".

[1448]It is a good exercise to first verify that the map is actually well-defined.

Claim. The map

$$\begin{array}{rcl} \Psi\colon X\times C & \to & X*X \\ (x,[(y,t)]) & \mapsto & [(x,L_x^{-1}(y),\tfrac{1}{2}(t+1))] \end{array}$$

is continuous and it restricts to a homeomorphism onto $h_\varphi^{-1}(C)$.

By now the reader is surely aware of the fact that whenever we mix quotients with products we need to tread carefully. Thus let us first consider the maps

$$\begin{array}{rcl} \alpha\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,\varphi(x,y))=(x,L_x(y)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,L_x^{-1}(y)). \end{array}$$

Note that these two maps are inverses of one another. Also note that α is continuous by Lemma 1.53 (3). Furthermore note that α is a bijection, in fact the inverse is given by β. Since X is compact and Hausdorff we obtain from Lemma 1.55 and Proposition 1.40 that α is a homeomorphism. In particular $\beta=\alpha^{-1}$ is also continuous.

Next we consider the map

$$\begin{array}{rcl} \psi\colon X\times(X\times[-1,\tfrac{1}{2}]) & \to & X\times X\times[0,1] \\ (x,y,t) & \mapsto & (x,L_x^{-1}(y),\tfrac{1}{2}(t+1))=(\beta(x,y),\tfrac{1}{2}(t+1)). \end{array}$$

Since β is continuous we see that ψ is also continuous. Finally we consider the following commutative diagram

$$\begin{tikzcd} X\times X\times[-1,\tfrac{1}{2}] \arrow[r, "(x,y,t)\mapsto(x,[(y,t)])"] \arrow[d, "\psi"'] \arrow[dr, "f:=p\circ\psi"] & \overbrace{X\times\big((X\times[-1,\tfrac{1}{2}])\,/\,(X\times\{-1\})\big)}^{=X\times C} \arrow[d, "\Psi"] \\ X\times X\times[0,1] \arrow[r, two heads, "p(x,y,t):=[(x,y,t)]"'] & X*X. \end{tikzcd}$$

Claim. The map Ψ is continuous.

We had just seen that $f=p\circ\psi$ is continuous. Note that by Lemma 2.14 (1) the projection map $q\colon X\times[-1,\tfrac{1}{2}]\to(X\times[-1,\tfrac{1}{2}])\,/\,(X\times\{-1\})$ is a quotient map. Since X is compact and Hausdorff we know by Lemma 1.103 that X is locally compact.[1449] Thus it follows from Theorem 2.15 that the top horizontal map $\mathrm{id}_X\times q$ is a quotient map. Thus it follows from Lemma 2.16 (2) that f is continuous. This concludes the proof of the claim.

One easily verifies that Ψ defines a bijection

$$\Psi\colon \underbrace{X\times C}_{\subset X*X} \to \underbrace{h_\varphi^{-1}(C)}_{\subset\Sigma(X)}.$$

Since X is compact we know by Lemma 85.15 that the left-hand side is compact. By Lemma 15.2 (2) and our hypothesis that X is Hausdorff we know that the right-hand

[1449]If the reader is unhappy about using Lemma 1.103 we point out that in our application of the proposition the topological space will be $X=S^n$ and thus obviously locally compact.

side is Hausdorff. Therefore it follows from Proposition 1.40 that the continuous bijection $\Psi\colon X \times C \to h_\varphi^{-1}(C)$ is indeed a homeomorphism. This concludes the proof of the claim.

We consider the following diagram

$$\begin{tikzcd} X \times C \arrow[rr, "\Psi"] \arrow[dr, "(x,c)\mapsto c"'] & & h_\varphi^{-1}(C) \arrow[dl, "h_\varphi"] \\ & C. & \end{tikzcd}$$

It is straightforward to verify that this diagram commutes. Since the top map is homeomorphism by the previous claim we have found a trivialization of h_φ over C. We leave it as an exercise to the reader to make the minor modifications needed to find a trivialization of h_φ over D.[1450] □

Evidently the idea now is to replace the quaternions in the example on page 1977 by the octonions. In the following we recall the most salient bits from the discussion of octonions on page 1057. The octonions $\mathbb{O}$ are $\mathbb{R}^8$ together with an algebra structure over $\mathbb{R}$ that has the following properties:

(1) The octonions $\mathbb{O}$ are a division algebra, i.e. for all $a \neq 0$ and $b \in \mathbb{O}$ there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$.
(2) The algebra structure is neither commutative nor associative.
(3) Multiplication on $\mathbb{O}$ respects the euclidean norm, more precisely, given $a, b \in \mathbb{O}$ we have $\|a \cdot b\| = \|a\| \cdot \|b\|$.

It follows from (3) that the algebra multiplication induces a map $\varphi\colon S^7 \times S^7 \to S^7$.

Lemma 85.18. *Given any $z \in S^7$ the maps $L_z\colon S^7 \to S^7$ and $R_z\colon S^7 \to S^7$, as defined in Proposition 85.17, are homeomorphisms.*

Proof. First note that we can *not* recycle the argument from page 1977 since there we used that the quaternions are associative.

By Corollary 36.8 (2) it suffices to show that the maps L_z and R_z are injective. First we show that given any $z \in S^7$ the map $L_z\colon S^7 \to S^7$ is injective. So suppose that we have $a, b \in S^7$ with $L_z(a) = L_z(b)$. We obtain that

$$\begin{array}{ccccccccc} \|a-b\| & = & \|z\| \cdot \|a-b\| & = & \|z \cdot (a-b)\| & = & \|z \cdot a - z \cdot b\| & = & \|L_z(a) - L_z(b)\| & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } z \in S^7 & & \text{by (3)} & & \text{since } \mathbb{O} \text{ is an algebra} & & \text{by definition of } L_z & & \text{since } L_z(a) = L_z(b) & \end{array}$$

Thus we see that $a = b$. Almost the same argument shows that R_z is injective.[1451] □

Now we are basically done with the proof of Proposition 85.13.

[1450] Note that in this second part we will need to use that the maps $R_x\colon X \to X$ are homeomorphisms.

[1451] At first glance it might appear odd that we never used that the octonions are a division algebra, we only used the innocuous looking third property. But it is clear that (3) already implies that $\mathbb{O}$ has no zero-divisors. By Lemma 44.6 this forces $\mathbb{O}$ to be a division algebra.

Proof of Proposition 85.13. It follows immediately from Lemma 85.18, Proposition 85.17 and Lemma 85.16 that there exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7. $\square$

Exercises for Chapter 85.

Exercise 85.1. Let F be a topological space and let $p\colon Y \to B$ be an F-bundle. Suppose that the fiber F and the base space B are path-connected. Show directly, without making use of Proposition 85.2 (2), that the total space Y is also path-connected.

Exercise 85.2. Let $p\colon Y \to [0,1]$ be a bundle with fiber F. Show with your bare hands (i.e. without using Proposition 85.7) that p is trivial, i.e. show that p is equivalent to the trivial bundle $F \times [0,1] \to [0,1]$.

Exercise 85.3. Let X be a topological space and let $f_0, f_1\colon X \to X$ be homeomorphisms. Suppose that f_0 and f_1 are isotopic. Show that the two X-bundles $M(X, f_0) \to S^1$ and $M(X, f_1) \to S^1$ are equivalent.

Exercise 85.4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Show that the corresponding $\mathbb{R}^n$-bundle $M(\mathbb{R}^n, f) \to S^1$ is equivalent to an n-dimensional vector bundle.
Hint: Use Lemma 32.25 and Lemma 85.3 (2), the latter is a. k. a. Exercise 85.3.

Exercise 85.5. The Ehresmann Fibration Theorem says the following:
Let $f\colon M \to N$ be a smooth map between closed manifolds. We suppose that N is connected. If f is a submersion, i.e. if given any $P \in M$ the map $Df_P\colon T_PM \to T_{f(P)}N$ is an epimorphism, then f is a fiber bundle. Provide a proof using the Submersion Theorem 3.46.

Exercise 85.6. Use the Ehresmann Fibration Theorem, see Exercise 85.5, to give an alternative proof of the fact that the Hopf map $S^3 \to \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle over $\mathbb{C}\mathrm{P}^1 = S^2$.

Exercise 85.7. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon\ \mathrm{U}(n) = \{A \in M(n \times n, \mathbb{C}) \,|\, A \cdot \overline{A}^T = \mathrm{id}\} & \to & \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A). \end{array}$$

(a) Show that φ takes values in S^1.
(b) Show that the map $\varphi\colon \mathrm{U}(n) \to S^1$ is a fiber bundle.
Hint: You can use the Ehresmann Fibration Theorem from Exercise 85.5.

Exercise 85.8. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. By the discussion on page 1963 and Proposition 85.2 we know that the map

$$\begin{array}{rcl} M(X,f) = X \times [0,1]/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & e^{2\pi i t} \end{array}$$

is a fibration. We make the obvious identification $X = X \times \{0\} \subset M(X, f)$. Let $k \in \mathbb{N}_0$ and let $\alpha\colon [0,1] \to S^1$ be the usual loop given by $\alpha(t) = e^{2\pi i t}$. By Proposition 83.5 and Lemma 30.2 we get a well-defined map

$$\Theta(\alpha)_*\colon H_k(X;\mathbb{Z}) \ \to \ H_k(X;\mathbb{Z}).$$

Show that $\Theta(\alpha)_* = (f_*)^{-1}$.

Exercise 85.9. Let W be an n-dimensional manifold and let K be a compact proper connected k-dimensional submanifold with $k \leq n-3$. Show that the inclusion induced map $\pi_1(W \setminus K) \to \pi_1(W)$ is an isomorphism.
Hint: Use Lemma 13.12 (3).

Exercise 85.10. The same way that we defined the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$, see page 655, we define the infinite quaternionic projective space $\mathbb{H}\mathrm{P}^\infty$ as the direct limit of the finite dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^k$. Show that for any $i \in \mathbb{N}_{\geq 2}$ we have an isomorphism $\pi_i(\mathbb{H}\mathrm{P}^\infty) \cong \pi_{i-1}(S^3)$.

Exercise 85.11. Let $p\colon S^3 \to S^2$ the Hopf fibration. In Proposition 85.9 we saw that it is an S^1-fiber bundle. We consider the connecting homomorphism $\partial\colon \pi_2(S^2) \to \pi_1(S^1)$ of the long exact sequence of homotopy groups from Proposition 85.2 (2). Show, just using the definition of the connecting homomorphism, that $\partial([\mathrm{id}_{S^2}]) = [\mathrm{id}_{S^1}]$.

Exercise 85.12. Show that for any choice of $m, n \in \mathbb{N}_0$ there exists a homeomorphism $\overline{B}^m * \overline{B}^n \xrightarrow{\cong} \overline{B}^{m+n+1}$.

Exercise 85.13. We let $X = S^1$ be the standard circle in the xy-plane of $\mathbb{R}^3$ and we let Y be the circle in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is given by the z-axis together with the point $\{\infty\}$. (See Figure 1053 for a simple illustration.) Show that we can "continuously" connect every point on X to every point Y such that the paths fill out S^3. In other words, try to visualize the fact that $S^1 * S^1 \cong X * Y$ is homeomorphic to S^3.

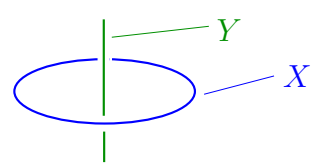


FIGURE 1053.

Exercise 85.14. Show that the homotopy groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are isomorphic.

86. Homotopy groups of Lie groups

In this chapter our focus will lie on the study of the homotopy groups of the matrix groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. Later on in the section we will see that we should also define a third type of groups, namely the groups $\mathrm{Sp}(n)$ which can be viewed as the quaternion analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$.

Since these groups are Lie groups and since this structure is helpful in our study of $\mathrm{O}(n)$ and $\mathrm{U}(n)$ we first give a short introduction to Lie groups.

86.1. **Topological groups and Lie groups.** First we recall the following definition from page 139.

Definition. A *topological group* is a topological space X together with a group structure such that the two maps

$$\begin{array}{rcl} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. A *morphism between topological groups* X and Y is a homomorphism $f\colon X\to Y$ that is furthermore continuous.

Examples.

(1) On page 140 we already saw that for any $n\in\mathbb{N}$ the matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

(2) Let X be a topological space and let $x_0\in X$ be a base point. We consider the loop space $\Omega(X,x_0)$, i.e. the set

$$\Omega(X,x_0) \;=\; \{\gamma\colon [0,1]\to X \mid \gamma(0)=\gamma(1)=x_0\}/\text{homotopy}$$

equipped with the compact-open topology. As in Exercise 83.6 we equip the set $\pi_1(X,x_0)=\Omega(X,x_0)/\sim$ with the quotient topology, i.e. we view $\pi_1(X,x_0)$ as a topological space. Evidently $\pi_1(X,x_0)$ admits a group structure, but somewhat surprisingly, the multiplication map $\pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X,x_0)$ is in general *not* continuous. In fact such an example is given by the Hawaiian Earrings, introduced in Exercise 26.6. We refer to [Fab2] for details.[1452]

We leave the elementary proof of the following lemma to the reader.

Lemma 86.1. *Let G be a topological group and let H be a subgroup of G. The map*

$$\begin{array}{rcl} G\times H & \to & G \\ (g,h) & \mapsto & g\cdot h \end{array}$$

defines a right-action of H on G that is free and continuous.

To avoid any misunderstanding we introduce the following convention.

[1452]In fact, as is pointed out in [Fab2], the failure of $\pi_1(X,x_0)$ to be a topological group is another instance of the phenomenon, discussed in Section 2.2, that in general product and quotient topologies do not mix well.

Convention. Let G be a topological group and let H be a subgroup. We denote by G/H the topological space that is given by the equivalence relation[1453]

$$g_1 \sim g_2 \quad \Longleftrightarrow \quad \text{there exists an } h \in H \text{ with } g_1 \cdot h = g_2.$$

The following definition of a Lie group is quite similar to the above definition of a topological group.

Definition. A *Lie group* is a manifold X with empty boundary together with a group structure such that the two maps[1454]

$$\begin{array}{rcl} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth.

Remark.

(1) It follows immediately from Lemma 3.15 (2) that a Lie group is also a topological group.
(2) As we will see in Exercise 86.2, strictly speaking one does not need to demand that the map $x \mapsto x^{-1}$ is smooth, since it follows already from the other properties.

Examples.

(1) It is straightforward to see that the manifold $\mathbb{R}$ with the group structure given by addition and the manifold S^1 with the group structure given by multiplication are Lie groups.
(2) The product $G_1 \times G_2$ of two Lie groups G_1 and G_2 admits a canonical manifold structure by Proposition 3.42 and the group structures on G_1 and G_2 define a group structure on $G_1 \times G_2$. One can easily verify that $G_1 \times G_2$ is again a Lie group. In particular the torus $S^1 \times S^1$ is a Lie group.

As mentioned in the beginning of this chapter, we are now interested in the homotopy groups of the groups

$$\begin{aligned} \mathrm{O}(n) &= \{A \in M(n \times n, \mathbb{R}) \,|\, A^T A = \mathrm{id}\} \\ \mathrm{U}(n) &= \{A \in M(n \times n, \mathbb{C}) \,|\, \overline{A}^T A = \mathrm{id}\}. \end{aligned}$$

As we will see later on, even if we are only interested in these groups it makes sense to extend our gaze and to consider the quaternion analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$:

Definition. Let $n \in \mathbb{N}$. We define.[1455]

$$\mathrm{Sp}(n) = \{A \in M(n \times n, \mathbb{H}) \,|\, \overline{A}^T A = \mathrm{id}\}.$$

[1453]The point of this convention is that in principle G/H could mean the quotient of the topological space G by the subset H.

[1454]Here we equip $X \times X$ with the product manifold structure introduced in Proposition 3.42.

[1455]Recall that the conjugate of a quaternion $h = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ is defined as $\overline{h} = a \cdot 1 - b \cdot i - c \cdot j - d \cdot k$. Given a matrix $A = (a_{ij})$ over $\mathbb{H}$ we define $\overline{A} = (\overline{a}_{ij})$.

We will first discuss $\mathrm{Sp}(n)$ a little bit before we settle on a name for it:

Lemma 86.2.

(1) *We have* $\mathrm{Sp}(1) = S^3$.
(2) *Let* $n \in \mathbb{N}$. *Matrix multiplication turns* $\mathrm{Sp}(n)$ *into a group.*
(3) *Let* $n \in \mathbb{N}$. *We write* $J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$. *The map*

$$\begin{array}{rcl} \Phi\colon \mathrm{Sp}(n) & \to & \{X \in \mathrm{U}(2n) \mid JXJ^{-1} = \overline{X}\} \\ P & \mapsto & \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \quad \textit{where } A, B \in M(n \times n; \mathbb{C}) \textit{ with}^{1456} P = A + Bj \end{array}$$

is a group isomorphism.

Proof.

(1) In Lemma 44.1 we saw that for any quaternion $h \in \mathbb{H}$ we have $h \cdot \overline{h} = \|h\|^2$. It follows immediately that $\mathrm{Sp}(1) = S^3$.
(2) We start out with the following observation: A straightforward calculation[1457] shows that for any two matrices $A, B \in M(n \times n, \mathbb{H})$ we have $\overline{(AB)}^T = \overline{B}^T \overline{A}^T$. Now we turn to the proof that $\mathrm{Sp}(n)$ is a group:
   (a) The fact that $\mathbb{H}$ is associative implies that the multiplication of matrices in $M(n \times n, R)$ is associative.
   (b) The identity matrix is as usual a neutral element.
   (c) Given $A \in \mathrm{Sp}(n)$ we have by definition that $\overline{A}^T A = \mathrm{id}$, i.e. $\overline{A}^T$ is a left-inverse. The above observation implies that $\overline{A}^T$ is also a right-inverse.
   (d) It follows from the above observation that $\mathrm{Sp}(n)$ is closed under matrix multiplication and taking inverses.

   We have thus shown that $\mathrm{Sp}(n)$ is indeed a group.
(3) We will only use this isomorphism to justify the name for $\mathrm{Sp}(n)$, otherwise it will not play a role. Therefore we will not provide the fairly elementary proof of this statement, instead we refer to [Bre, p. 103] for the proof. □

Not surprisingly we will now see that $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are Lie groups.

Lemma 86.3.

(1) *Let* $n \in \mathbb{N}$.
   (a) *The set* $\mathrm{O}(n)$ *is a submanifold of* $M(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$ *of dimension* $\frac{1}{2}n(n-1)$.
   (b) *The set* $\mathrm{U}(n)$ *is a submanifold of* $M(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ *of dimension* n^2.

[1456] It follows from $i \cdot j = k$ that any quaternion $a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ can be uniquely written as $(a \cdot 1 + b \cdot i) + (c \cdot 1 + d \cdot i) \cdot j$ where $a \cdot 1 + b \cdot i$ and $c \cdot 1 + d \cdot i$ are complex numbers. The same statement applies to matrices.

[1457] In the calculation we need to use the fact, mentioned in Lemma 44.1 (3), that for quaternions $w, z \in \mathbb{H}$ we have $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$.

(c) *The set* $\mathrm{Sp}(n)$ *is a submanifold of* $M(n \times n, \mathbb{H}) = \mathbb{H}^{n^2} = \mathbb{R}^{4n^2}$ *of dimension* $2n^2 + n$.

Furthermore, if we view $\mathrm{O}(n)$ *as a subspace of* $\mathrm{O}(n+1)$ *in the obvious way, then* $\mathrm{O}(n)$ *is a submanifold of* $\mathrm{O}(n+1)$. *The same statements apply to* $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$.

(2) $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$, *viewed as manifolds in their own right, are closed, in particular they are compact.*

(3) *The usual matrix multiplication turns* $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *into Lie groups.*

Remark.

(1) Recall that on page 1707 we defined the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ as follows:

$$\mathrm{Sp}(2g, \mathbb{Z}) \;=\; \{A \in \mathrm{GL}(2g, \mathbb{Z}) \,|\, A^T J A = A\} \quad \text{where} \quad J \;=\; \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}.$$

This definition, together with Lemmas 86.2 and 86.3, leads us to calling $\mathrm{Sp}(n)$ the *n-th compact symplectic group.*

(2) The statements regarding the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are mostly already contained in Lemma 3.45.

Proof. In Lemma 3.45 we had basically proved statements (1) and (2) for the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. The only aspect that we did not mention explicitly so far is that $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. We leave it to the reader to verify that this statement holds. The proof of (1) and (2) for $\mathrm{Sp}(n)$ is basically the same.

In Lemma 1.108 we showed that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are topological groups. Almost the same argument given in that proof shows that matrix multiplication and taking inverses are smooth maps. This implies that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are Lie groups. Furthermore it is clear that multiplication in $\mathrm{Sp}(n)$ is smooth. It remains to show that the map

$$\begin{array}{rcl} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is smooth. We will show this fact in Exercise 86.4. □

Basically by definition we have $\mathrm{O}(1) = S^0$ and $\mathrm{U}(1) = S^1$, furthermore by Lemma 86.2 (1) we have $\mathrm{Sp}(1) = S^3$. Thus it follows from Lemma 86.3 that these three spheres can be viewed as Lie groups. The question arises, whether perhaps other (all?) spheres can be equipped with the structure of a Lie group. The following lemma gives an easy way to rule out the existence of Lie group structures on some manifolds.

Lemma 86.4. *Every Lie group is parallelizable.*[1458]

Examples. Recall that on page 1788 we saw that $2n$-dimensional spheres with $n \geq 1$ are not parallelizable. Furthermore t on page 1788 we showed that for $g \neq 1$ the surface of genus g is not parallelizable. Thus it follows from Lemma 86.4 that these spheres and surfaces cannot be equipped with a Lie group structure.

[1458]Recall that according to the definition on page 1786 a manifold is called parallelizable if its tangent bundle is trivial.

Sketch of proof. Let G be an n-dimensional Lie group with trivial element e. We denote by $R_g\colon G \to G$ the map that is given by right-multiplication by g. Since G is a Lie group this map is in fact a diffeomorphism. The map

$$\begin{array}{rcl} G \times T_eG & \to & TG = \{(g,h) \,|\, g \in G \text{ and } w \in T_hG\} \\ (g,v) & \mapsto & (g,(R_g)_*(v)) \end{array}$$

is an isomorphism of vector bundles. (This statement is basically obvious, except that perhaps it is not completely clear why the map is continuous, we refer to [Le2, Corollary 8.39] for more details.) But this shows that G is parallelizable. □

The following lemma gives us more examples of Lie groups.

Lemma 86.5. *Let G be a Lie group and let K be a finite normal subgroup. Then the quotient G/K admits a unique manifold structure such that the projection $G \to G/K$ is a local diffeomorphism. Furthermore the group structure on the quotient group G/K defines a Lie group structure on G/K.*

Example. It follows from Lemma 86.5 that for any $n \in \mathbb{N}$ the quotients $\mathrm{SU}(n)/\{\pm \mathrm{id}\}$ and $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\{\pm(\mathrm{id},\mathrm{id})\}$ are Lie groups.

Proof. In Lemma 86.1 we already saw that the action of K on G is free and continuous. Since G is a Lie group one also sees that the action is smooth. Finally, since G is finite we get for free that the action is proper. Thus we obtain from Proposition 3.25 that G/K admits a unique manifold structure (without boundary) such that the projection $G \to G/K$ is a local diffeomorphism. We leave it to the reader to verify that the induced multiplication map $G/K \times G/K \to G/K$ is again smooth. □

Above we have seen that subgroups of matrix groups often give rise to Lie groups. Perhaps somewhat surprisingly, in a sense, all compact Lie groups arise that way. More precisely, the following theorem holds:[1459]

Theorem 86.6. *Any compact Lie group is isomorphic (in the obvious sense) to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$.*

The theorem is a consequence of the so-called Peter-Weyl Theorem. We only stated the theorem for "general culture". We will not make use of it and thus we refer to [BtD, Theorem III.4.1] or alternatively to [Sep, Theorem 3.28] for a proof.

86.2. **The homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.** In this chapter we want to see what we can say about the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Convention.

[1459] One could view Theorem 86.6 as an analogue of Proposition 65.11 or of the Whitney Embedding Theorem 65.18.

(1) Given $n \in \mathbb{N}$ we have the obvious monomorphism

$$\begin{array}{rcl} \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Often we will use it to identify $\mathrm{O}(n)$ with its image in $\mathrm{O}(n+1)$. Similarly we view $\mathrm{U}(n)$ as a subgroup of $\mathrm{U}(n+1)$ and we view $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{Sp}(n+1)$.

(2) We will view any subgroup of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{H})$ as a topological space that is equipped with the base point given by the identity matrix. Thus in the remainder of this chapter we will suppress the base point in the discussion of fundamental groups.

The following lemma evidently uses the convention from page 1983.

Lemma 86.7. *Let $n \in \mathbb{N}$. The maps*

$$\begin{array}{r} \mathrm{O}(n+1)/\mathrm{O}(n) \xrightarrow{p} S^n \\ [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} \end{array} \qquad \begin{array}{r} \mathrm{U}(n+1)/\mathrm{U}(n) \xrightarrow{q} S^{2n+1} \\ [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} \end{array} \quad \textit{and} \quad \begin{array}{r} \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \xrightarrow{r} S^{4n+3} \\ [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} \end{array}$$

are well-defined and they are homeomorphisms.

Proof.

(1) First we show that the given map $p\colon \mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ is well-defined. To do so we consider the map[1460]

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ V := (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} = V \cdot e_{n+1}. \end{array}$$

Note that if we multiply V on the right by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then the multiplication with e_{n+1} is not affected. Therefore the above map $\mathrm{O}(n+1) \to S^n$ descends to a map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ which is exactly the map p we consider in the lemma. This shows that p is well-defined.

Also note that basic linear algebra tells us that given any point $v_{n+1} \in S^n$ there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $v_1, \ldots, v_n, v_{n+1}$ form an orthonormal basis $\mathbb{R}^{n+1}$. This shows that the given map p is a surjection. Similarly, elementary linear algebra shows that p is injective. Finally note that S^n is Hausdorff and note that Lemmas 86.3 and 1.37 imply that $\mathrm{O}(n+1)/\mathrm{O}(n)$ is compact. Therefore it follows from Proposition 1.40 that the map p is indeed a homeomorphism.

(2),(3) The proof for (2) and (3) is basically identical to the proof in (1), except that the linear algebra aspects require slightly more thought. □

The following proposition gives us new interesting examples of fiber bundles.

[1460] It follows easily from $A^T A = \mathrm{id}$ that every column v of A satisfies $\|v\| = 1$, i.e. every column lies in S^n.

Proposition 86.8. *Let $n \in \mathbb{N}$. The maps*

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1) & \to & S^{4n+3} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

are bundle projections with fibers given by $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *respectively.*

Remark. Without too much effort one can show that the bundle projections $\mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$ and $\mathrm{U}(n+1) \to S^{2n+1}$ with fiber $\mathrm{U}(n)$ from the proposition restrict to bundle projections $\mathrm{SO}(n+1) \to S^n$ with fiber $\mathrm{SO}(n)$ and $\mathrm{SU}(n+1) \to S^{2n+1}$ with fiber $\mathrm{SU}(n)$.

We will outline two approaches to proving Proposition 86.8. The first approach relies on the following proposition that shows more generally that Lie groups (and sometimes topological groups) give rise to fiber bundles.

Proposition 86.9. $(*)$

(1) *Let G be a Lie group. If H is a closed subgroup of G, then the projection $G \to G/H$ is a bundle projection with fiber H.*
(2) *Let G be a topological group and H be a closed subgroup of G. If H admits the structure of a Lie group, then the projection $G \to G/H$ is a bundle projection with fiber H.*

Remark. One might hope that Proposition 86.9 (1) generalizes to all topological groups and subgroups. But as shown in [Karu, p. 347], there exists a topological group G and a closed subgroup H such that the projection $G \to G/H$ is not a bundle projection.

Proof of Proposition 86.9 $(*)$.

(1) This statement is proved in [Kiri, Theorem 2.11]. Alternatively, if G is a closed subgroup of some $\mathrm{GL}(n, \mathbb{C})$, then the proposition is proved in [Bre, p. 110].
(2) This statement is proved in [Pal4, Corollary 4.1]. □

First proof of Proposition 86.8 $(*)$. These three statements are an immediate consequence of Lemma 86.7 and Proposition 86.9 (1). □

Second proof of Proposition 86.8. In the following we give an outline of a "bare hands only" proof that the first projection map $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a bundle projection with fiber $\mathrm{O}(n)$. We leave it to the reader to modify the proof to deal with the two other cases.[1461]

Thus let $n \in \mathbb{N}$. As usual we equip $\mathbb{R}^{n+1}$ with the euclidean inner product $\langle\, , \rangle$. In this proof we identify $\mathrm{O}(n+1)$ with the group of isometries of $(\mathbb{R}^{n+1}, \langle\, , \rangle)$. More precisely, we do not distinguish in the notation between an isometry of $(\mathbb{R}^{n+1}, \langle\, , \rangle)$ and the matrix it represents with respect to the standard basis of $\mathbb{R}^{n+1}$.

We consider the two compact subsets

$$V \;=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \leq \tfrac{1}{2}\} \quad \text{and } W \;=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq -\tfrac{1}{2}\}.$$

[1461] Alternatively the reader can look up the proof of Proposition 86.9 (1).

We continue with the following claim.

Claim. Let $P \in V$. We write $W(P) := \operatorname{span}\{e_{n+1}, P\}$. There exists a unique orientation-preserving isometry $\alpha(P)$ of $\mathbb{R}^{n+1} = W(P) \oplus W(P)^\perp$ that satisfies the following two conditions:

(1) the restriction of $\alpha(P)$ to $W(P)$ is an orientation-preserving isometry of $W(P)$ that sends e_{n+1} to P,
(2) it equals the identity on $W(P)^\perp$.

We illustrate the definition of $\alpha(P)$ in Figure 1054. In the 3-dimensional setting the isometry $\alpha(P)$ is the unique rotation around "the $W(P)^\perp$-axis" that sends e_{n+1} to P.

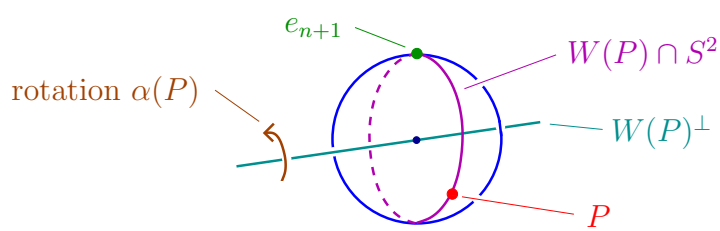


FIGURE 1054.

Note that $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are both positive-definite and symmetric forms. Thus it follows from Exercise 72.1 that both forms are also non-singular. Therefore we obtain from Sylvester's Theorem 72.10 that the two forms $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are isometric. In the latter case the orientation-preserving isometries are given by $\mathrm{SO}(2)$. Since $\mathrm{SO}(2) = S^1$ acts freely and transitively on S^1 we see that given any two points $X, Y \in \mathbb{R}^2$ with $\|X\| = \|Y\|$ there exists a unique orientation-preserving isometry β of $(\mathbb{R}^2, \langle\,,\,\rangle)$ with $\beta(X) = Y$. Since $(W(P), \langle\,,\,\rangle)$ is isometric to $(\mathbb{R}^2, \langle\,,\,\rangle)$ the same uniqueness statement applies to $W(P)$ and the two points $e_{n+1}, P \in W(P)$. The claim immediately follows from this observation.

We leave it as an exercise to verify that the map $\alpha\colon V \to \mathrm{O}(n+1)$ that we had just defined is indeed continuous. Next we consider the map

$$\begin{aligned} f\colon V \times \mathrm{O}(n) &\to \mathrm{O}(n+1) \\ (P, A) &\mapsto \alpha(P) \circ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

An entertaining exercise in linear algebra shows that the map f restricts to a bijection $V \times \mathrm{O}(n) \to p^{-1}(V)$. Since V and $\mathrm{O}(n)$ are compact we obtain from Proposition 1.40 that f defines a homeomorphism $V \times \mathrm{O}(n) \to p^{-1}(V)$.

Now recall that for any $P \in V$ we have, by design, that $\alpha(P)(e_{n+1}) = P$. Using this observation we see almost immediately that the following diagram commutes:

$$\begin{tikzcd} V \times \mathrm{O}(n) \arrow[rr, "f", "\cong"'] \arrow[dr, "(P,A)\mapsto P"'] & & p^{-1}(V) \arrow[dl, "p", "B\mapsto B\cdot e_{n+1}"'] \\ & V. & \end{tikzcd}$$

This shows that we have found a trivialization of p over V. With very minor modifications one can also find a trivialization of p over W. Since the interiors of V and W cover all of S^n we see that these two trivializations show that $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a fiber bundle with fiber $\mathrm{O}(n)$. □

Now we can continue with the actual study of the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Proposition 86.10. *Let $n \in \mathbb{N}$.*

(1) *The inclusion induced map*[1462]

$$\pi_k(\mathrm{O}(n)) \;\to\; \pi_k(\mathrm{O}(n+1)) \quad \textit{is an} \quad \begin{cases} \textit{isomorphism for } k < n-1, \\ \textit{epimorphism for } k = n-1. \end{cases}$$

(2) *The inclusion induced map*

$$\pi_k(\mathrm{U}(n)) \;\to\; \pi_k(\mathrm{U}(n+1)) \quad \textit{is an} \quad \begin{cases} \textit{isomorphism for } k < 2n, \\ \textit{epimorphism for } k = 2n. \end{cases}$$

(3) *The inclusion induced map*

$$\pi_k(\mathrm{Sp}(n)) \;\to\; \pi_k(\mathrm{Sp}(n+1)) \quad \textit{is an} \quad \begin{cases} \textit{isomorphism for } k < 4n+2, \\ \textit{epimorphism for } k = 4n+2. \end{cases}$$

Example.

(1) We know that $\mathrm{U}(1) = S^1$. Thus we obtain from Proposition 86.10 (2) that all unitary groups $\mathrm{U}(n)$ are path-connected.
(2) Recall that in Lemma 86.2 (1) we pointed out that $\mathrm{Sp}(1) = S^3$. Thus we obtain from Proposition 86.10 (3) that all compact symplectic groups $\mathrm{Sp}(n)$ are simply connected.

Proof. We prove the first statement. The proofs of the remaining two statements are almost identical.[1463] By Proposition 86.8 we have a bundle projection $p\colon \mathrm{O}(n+1) \to S^n$ with fiber

[1462]Recall that in Lemma 1.96 (1) we saw that $\mathrm{O}(n)$ is disconnected. Thus in principle it is not completely clear what we mean by $\pi_k(\mathrm{O}(n))$. But, as we pointed out on page 1987, we always consider $\mathrm{O}(n)$ as equipped with the base point that is given by the identity. Also note that it follows easily from Lemma 1.110 that for a topological group the isomorphism type of the homotopy groups does not depend on the choice of a base point.

[1463]The only catch is that one needs to be careful about the degrees involved.

$\mathrm{O}(n)$. From Proposition 85.2 we obtain a long exact sequence of the form

$$\ldots \to \pi_{k+1}(S^n) \xrightarrow{\partial} \pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1)) \to \pi_k(S^n) \xrightarrow{\partial} \ldots$$

The desired statement now follows from the familiar fact that by Proposition 28.9 we have $\pi_i(S^k) = 0$ for $i < k$. □

As a corollary to Proposition 86.10 (1) we can now finally answer Question 37.5. More precisely, we have the following proposition.

Proposition 86.11.

(1) *For any $n \in \mathbb{N}$ the inclusions $i\colon \mathrm{SO}(n) \to \mathrm{SL}(n,\mathbb{R})$ and $j\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}_+(n,\mathbb{R})$*[1464] *are homotopy equivalences.*

(2) *For any $n \geq 3$ we have isomorphisms*

$$\mathbb{Z}_2 \cong \pi_1(\mathrm{SO}(n)) \xrightarrow[i_*]{\cong} \pi_1(\mathrm{SL}(n,\mathbb{R})) \xrightarrow[j_*]{\cong} \pi_1(\mathrm{GL}_+(n,\mathbb{R})).$$

(3) *In each of the three fundamental groups of (2) the unique non-trivial element is represented by the loop*

$$\begin{array}{rcl} S^1 & \to & \mathrm{SO}(n) \\ e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Proof.

(1) We proved this statement in Lemma 10.7.

(2) Let $n \geq 3$. We have isomorphisms

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \cong & \pi_1(\mathrm{SO}(3),\mathrm{id}) & \cong & \pi_1(\mathrm{SO}(n),\mathrm{id}) & \xrightarrow[i_*]{\cong} & \pi_1(\mathrm{SL}(n,\mathbb{R}),\mathrm{id}) \xrightarrow[j_*]{\cong} \pi_1(\mathrm{GL}_+(n,\mathbb{R}),\mathrm{id}). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Corollary 37.4} & & \text{by Proposition 86.10 and the fact that by Lemma 1.96 (1) SO}(k) \text{ is the path-component of O}(k) \text{ that contains id} & & & & \text{by (1) and Proposition 9.12 (1)} \end{array}$$

(3) We consider the loop

$$\begin{array}{rcl} \gamma\colon S^1 & \to & \mathrm{SO}(2) \\ e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \end{array}$$

Since $\mathrm{SO}(2) = S^1$ we know by Proposition 7.18 that γ represents a generator of $\pi_1(\mathrm{SO}(2))$. By Proposition 86.10 we know that the inclusion $\mathrm{SO}(2) \to \mathrm{SO}(n)$ induces an epimorphisms of fundamental groups. It follows immediately that the loop given in (3) is indeed a generator of $\pi_1(\mathrm{SO}(n))$. Since the inclusions i and j induce

[1464] Recall that we write $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$.

isomorphisms of fundamental groups we see that the given loop also represents a generator of $\pi_1(\mathrm{SL}(n,\mathbb{R}))$ and $\pi_1(\mathrm{GL}(n,\mathbb{R}))$. $\square$

86.3. **Spin groups.** Let $n \geq 3$. Recall that in Lemma 1.96 (1) we showed that $\mathrm{SO}(n)$ is path-connected and in Corollary 86.11 (2) we had just seen that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$. By Lemma 7.16 this implies that the universal covering, as defined on page 567, of $\mathrm{SO}(n)$ is a 2-fold covering of $\mathrm{SO}(n)$.[1465] For example in Theorem 37.2 we showed that the 2-fold covering of $\mathrm{SO}(3)$ is diffeomorphic to $\mathrm{SU}(2)$.

Thus the following question arises:

Question 86.12. *Let $n \geq 3$.*

(1) *Is the universal covering of $\mathrm{SO}(n)$ a group?*
(2) *If yes, is it a group "we already know"?*

It turns out that the answer to Question 86.12 (1) is yes. In fact, the following, initially perhaps somewhat surprising proposition holds. The proof turns out to be almost disappointingly easy.

Proposition 86.13. *Let G be a connected Lie group and let $p\colon \widetilde{G} \to G$ be the universal covering. We denote by 1_G the neutral element in G. We pick a point $1_{\widetilde{G}} \in p^{-1}(1_G)$. Then $\widetilde{G}$ admits a unique Lie group structure such that $p\colon \widetilde{G} \to G$ is a smooth map and a homomorphism and such that $1_{\widetilde{G}}$ is the neutral element in $\widetilde{G}$.*

Remark. Let $k \in \mathbb{N}_{\geq 2}$. By Proposition 86.13 the universal covering $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ of the Lie group $\mathrm{SL}(k,\mathbb{R})$ is again Lie group. In [HallB, Theorem C.12] it is shown that $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ is *not* isomorphic to any subgroup of $\mathrm{GL}(n,\mathbb{R})$ for any $n \in \mathbb{N}$. This shows that Theorem 86.6 does not extend to non-compact Lie groups.

Sketch of the proof. First note that we saw in Proposition 8.1 that $\widetilde{G}$ is a topological manifold that admits a unique smooth structure such that $p\colon \widetilde{G} \to G$ is a smooth map. Now we turn towards defining the Lie group structure on the manifold $\widetilde{G}$. In fact we will give two equivalent definitions of the group structure on $\widetilde{G}$:

(1) First note that by Proposition 7.21 we know that $\widetilde{G} \times \widetilde{G}$ is also simply connected. We consider the maps

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ \widetilde{G} \times \widetilde{G} \arrow[r, "(x,y)\mapsto p(x)\cdot p(y)"'] & G. \end{tikzcd}$$

It follows from Proposition 20.2 that there exists a unique lift

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ \widetilde{G} \times \widetilde{G} \arrow[r, "(x,y)\mapsto p(x)\cdot p(y)"'] \arrow[ur, dashed, "(1_{\widetilde{G}},1_{\widetilde{G}})\mapsto 1_{\widetilde{G}}"] & G. \end{tikzcd}$$

[1465]Note that here we use that $\mathrm{SO}(n)$ is a manifold, which implies by the discussion on page 567 that the universal covering does indeed exist.

(2) By Proposition 20.8 and Corollary 20.11 we can assume that

$$\widetilde{G} \;=\; \{\text{paths in } G \text{ starting at } 1_G\}/\text{ homotopies that fix the starting point and the endpoint}$$

With that explicit description we define

$$\begin{array}{rcl} \widetilde{G}\times\widetilde{G} & \to & \widetilde{G} \\ ([a(t)],[b(t)]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & G \\ t & \mapsto & a(t)\cdot b(t) \end{array}\right]. \end{array}$$

Using the fact that the multiplication map $\widetilde{G}\times\widetilde{G}\to\widetilde{G}$ is continuous one can show fairly easily that this map is well-defined.

We leave it as an exercise to show that the two given definitions of the group structure agree and that in both cases we defined a Lie group structure on $\widetilde{G}$.[1466]

It remains to show that the group structure on $\widetilde{G}$ is the unique Lie group structure on the manifold $\widetilde{G}$ for which $1_{\widetilde{G}}$ is the neutral element and that turns p into a homomorphism. Suppose we are given a smooth map $\varphi\colon \widetilde{G}\times\widetilde{G}\to\widetilde{G}$ that has the same properties. We obtain the following commutative diagram:

$$\begin{tikzcd} \widetilde{G}\times\widetilde{G} \arrow[r, "\varphi"] \arrow[d, "p\times p"'] \arrow[dr, "(x,y)\mapsto p(x)\cdot p(y)"] & \widetilde{G} \arrow[d, "p"] \\ G\times G \arrow[r, "(x,y)\mapsto x\cdot y"'] & G. \end{tikzcd}$$

Since the diagram commutes we see that $\varphi\colon \widetilde{G}\times\widetilde{G}\to\widetilde{G}$ is precisely a lift of the blue map $\widetilde{G}\times\widetilde{G}\to G$ that sends $(1_{\widetilde{G}},1_{\widetilde{G}})$ to $1_{\widetilde{G}}$. Thus by the uniqueness of the lift we see that the Lie group structure on $\widetilde{G}$ is uniquely determined by our wish list. □

This proposition leads us to the following definition.

Definition. Let $n\geq 3$. We refer to the universal covering of $\mathrm{SO}(n)$ as the *spin group* $\mathrm{Spin}(n)$.

The combination of Propositions 86.11 (2) and 86.13 and Corollary 20.11 shows that $\mathrm{Spin}(n)$ is indeed a Lie group and that $p\colon \mathrm{Spin}(n)\to\mathrm{SO}(n)$ is a 2-fold covering that is a group homomorphism.

Remark. By Corollary 20.11 the universal covering of a manifold, even if it is equipped with a base point, is only well-defined up to "equivalence". This slightly awkward issue can be circumvented in two different ways:

(1) either we use the explicit model of the universal covering of a based manifold in terms of equivalence classes of paths, or

[1466]The smoothness of the group multiplication is perhaps easier to prove with the first point of view, the fact that it is a group structure is possibly less confusing to verify with the second point of view.

(2) in this particular situation there is a more elegant solution, one can give a purely algebraic explicit description of $\mathrm{Spin}(n)$. We refer to [Mei, Chapter 3.1] for details.

It is harder to give an answer to Question 86.12 (2). For reasonably low dimensions the following proposition gives a positive answer.

Proposition 86.14.

(1) *There exist Lie group isomorphisms*[1467]

$$\begin{array}{lrcl} \text{(a)} & \mathrm{SO}(3) & \cong & \mathrm{SU}(2)/\{\pm\,\mathrm{id}\} \\ \text{(b)} & \mathrm{SO}(4) & \cong & (\mathrm{SU}(2)\times\mathrm{SU}(2))/\{\pm(\mathrm{id},\mathrm{id})\} \\ \text{(c)} & \mathrm{SO}(5) & \cong & \mathrm{Sp}(2)/\{\pm\,\mathrm{id}\}. \end{array}$$

(2) *There exist Lie group isomorphisms*

$$\begin{array}{lrcl} \text{(a)} & \mathrm{Spin}(3) & \cong & \mathrm{SU}(2) \\ \text{(b)} & \mathrm{Spin}(4) & \cong & \mathrm{SU}(2)\times\mathrm{SU}(2) \\ \text{(c)} & \mathrm{Spin}(5) & \cong & \mathrm{Sp}(2) \\ \text{(d)} & \mathrm{Spin}(6) & \cong & \mathrm{SU}(4). \end{array}$$

Proposition 86.14 (b) thus says that for $n = 3, 4, 5, 6$ the spin group $\mathrm{Spin}(n)$ is isomorphic to Lie groups that we are already familiar with. But according to [Mei, p. 53] "for $n \geq 7$, there are no further accidental isomorphisms of this type." In other words, in general the answer to Question 86.12 (2) is negative.

Proof.

(1) (a) This statement follows immediately from Lemma 37.1 and Theorem 37.2.

(b) In Proposition 44.2 we saw that the quaternion multiplication defines a group structure on $S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$ where the trivial element is given by $1 \in \mathbb{H}$. Furthermore in Proposition 44.2 we had also seen that the Lie groups SU(2) and S^3 are isomorphic. Thus it suffices to show that the Lie groups $(S^3 \times S^3)/\{\pm(1,1)\}$ and SO(4) are diffeomorphic. We consider the following map:

$$\begin{array}{rcl} \Phi\colon S^3\times S^3 & \to & \mathrm{SO}(4) \\ (q_1,q_2) & \mapsto & \left(\begin{array}{rcl} \mathbb{R}^4=\mathbb{H} & \to & \mathbb{R}^4=\mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1} \end{array}\right). \end{array}$$

We make the following observations:

(i) In Exercise 44.1 we showed that the map $z \mapsto q_1 z q_2^{-1}$ does indeed define an element in SO(4).

(ii) One can easily verify that the map Φ is smooth and that it is a homomorphism.

(iii) We leave it to the energetic reader to verify that the differential of Φ at each point is invertible.

[1467] Note that it follows from Lemma 86.5 that the groups on the right-hand side are indeed Lie groups.

(iv) We note that the kernel of this homomorphism is given by $\{\pm(1,1)\}$. Indeed, the kernel of this homomorphism is given by the set of all $(q_1, q_2) \in S^3 \times S^3$ such that $q_1 z q_2^{-1} = z$ for all $z \in \mathbb{H}$. By setting $z = 1$ we obtain that $q_1 = q_2$. Furthermore, since $q_1 z q_1^{-1} = z$ for all $z \in S^3$ we see that $q_1 = q_2$ lies in the center of the group S^3. An elementary calculation shows that the center of S^3 is given by ± 1. Thus the kernel of Φ contains precisely the two elements $(1,1)$ and $(-1,-1)$.

It follows from the above, Lemmas 1.66 and 86.5 and elementary group theory[1468] that the above map Φ descends to a smooth and injective map

$$\overline{\Phi}\colon (S^3 \times S^3)/\{\pm(1,1)\} \;\to\; \mathrm{SO}(4)$$

such that the differential at each point is invertible. By Lemmas 86.3 and 86.5 both sides are closed connected 6-dimensional manifolds. Thus it follows from Corollary 36.8 (2) that the map $\overline{\Phi}$ is also surjective. It follows from the above results and the Inverse Mapping Theorem 3.33 (2) that $\overline{\Phi}$ is in fact a diffeomorphism.

(c) Let V be the real vector space of quaternionic Hermitian 2×2-matrices with trace 0. In other words we consider

$$V \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2 \times 2, \mathbb{H}) \,\middle|\, a + d = 0, a, d \in \mathbb{R} \text{ and } b = \overline{c} \right\}.$$

Note that V is a 5-dimensional real vector space. The group $\mathrm{Sp}(2)$ acts by conjugation on V.[1469] One easily verifies that the kernel is given by $\{\pm\,\mathrm{id}\}$. Furthermore by [Pro, Chapter 4.5.3] there exists a positive-definite symmetric form on V that is preserved by this action. Thus, by Sylvester's Theorem 72.10 this action defines a homomorphism $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. Similar to (b) we obtain a diffeomorphism $\mathrm{Sp}(2)/\{\pm\,\mathrm{id}\} \xrightarrow{\cong} \mathrm{SO}(5)$.

(2) (a) By (2a) we can make the identification $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm\,\mathrm{id}\}$ of Lie groups. By Theorem 37.2 we know that $\mathrm{SU}(2)$ is homeomorphic to S^3, in particular we see that $\mathrm{SU}(2)$ is simply connected. By Lemma 86.5 the quotient map $\mathrm{SU}(2) \to \mathrm{SU}(2)/\{\pm 1\}$ is a covering map, it is smooth and it is a homomorphism. Thus we see that $\mathrm{SU}(2)$ is in fact the universal covering of $\mathrm{SU}(2)/\{\pm\,\mathrm{id}\} = \mathrm{SO}(3)$ and we deduce from Proposition 86.13 that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

(b) By the same argument as in (2a) this statement can be deduced from (1b).

(c) By the same argument as in (2a) this statement can be deduced from (1c).

(d) We will not make use of this results, hence we will not provide a proof, instead we refer to [Mei, p. 84] for details. □

[1468] More precisely, we use the elementary fact that if $\alpha\colon G \to H$ is a group homomorphism, then the induced map $G/\ker(\alpha) \to H$ is injective.

[1469] The group $\mathrm{Sp}(2)$ acts evidently by conjugation on all quaternionic 2×2-matrices, in principle one now needs to verify that this action restricts to an action on V.

Remark. Initially the spin groups might appear as amusing curiosities. In fact they arise naturally in many parts of mathematics and physics. We refer to [LM, Mor, Mei] for more details.

86.4. **The Bott Periodicity Theorem.** We continue with the following definition.

Definition. We define[1470]

$$\mathrm{O} \;=\; \varinjlim \mathrm{O}(n) \;=\; \left\{ A = (a_{ij})_{i,j\in\mathbb{N}} \;\middle|\; \begin{array}{l}\text{there exists an } n\in\mathbb{N}\\ \text{and a } B\in \mathrm{O}(n) \text{ with}\end{array} A = \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \right\}$$

where O has the obvious group structure and where O has the topology that is defined by the condition that subset $U \subset \mathrm{O}$ is open if and only if for each n the intersection $U \cap \mathrm{O}(n)$ is open. Similarly we define

$$\mathrm{U} \;=\; \varinjlim \mathrm{U}(n), \quad \mathrm{Sp} \;=\; \varinjlim \mathrm{Sp}(n)$$

In Proposition 16.1 we saw that every direct system $\{G_i\}_{i\in I}$ of groups admits a direct limit $\varinjlim G_i$. In fact for the above direct limit $\mathrm{O} = \varinjlim \mathrm{O}(n)$ the group structure is the "obvious" one given above. The following shows that these group structure are continuous, i.e. O, U and Sp are topological groups.

Lemma 86.15. *The above group structures on* O*,* U *and* Sp *are continuous.*

Remark. Initially one might be tempted to dismiss Lemma 86.15 as obvious. But in general there is a subtle issue. If $\{G_i\}_{i\in I}$ is a direct system of topological groups, then the above multiplication on the direct limit $\varinjlim G_i$ might not be continuous. Such examples are for example provided in [TSH]. If one thinks about it for a while one realizes that this failure for the multiplication map to be continuous is another instance of the observation made in Section 2.2, that mixing product and quotient topologies is fraught with dangers.

Proof. This lemma can surely be proved, with a little effort, "by hand". Alternatively one can use [TSH, Theorem 2.7] which shows that for a direct system $\{X_i\}_{i\in\mathbb{N}}$ of compact topological groups the above multiplication on $\varinjlim X_i$ is indeed continuous. □

For later usages of O, U and Sp we have to get the following technical lemma out of the way.

Lemma 86.16. $(*)$ *The topological spaces* O*,* U *and* Sp *admit CW-structures such that each* $\mathrm{O}(n) \subset \mathrm{O}$*,* $\mathrm{U}(n) \subset \mathrm{U}$ *and* $\mathrm{Sp}(n) \subset \mathrm{Sp}$ *is a subcomplex.*

[1470]When we talk about the direct limit we have to say what category we consider. Here we want to consider the category of groups and of topological spaces at the same time. Since we have obvious inclusions maps $\mathrm{O}(n) \to \mathrm{O}(n+1)$ we obtain from the discussion on page 488 that in both settings we obtain the same underlying set, which we can then equip with a group structure and a topology.

Proof. In [Whd9] it is shown that we can equip the $\mathrm{O}(n)$ with CW-structures such that for each n the subset $\mathrm{O}(n) \subset \mathrm{O}(n+1)$ is actually a subcomplex. Analogous statements for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are proved in [Yo, Theorem 5.3 and 5.5] or alternatively in [Stro, Chapter 9.3.3]. It follows from Lemma 26.22 that O is a CW-complex. □

Now we turn to the study of the homotopy groups of O, U and Sp.

Lemma 86.17. *Given any $k \in \mathbb{N}$ we have*

$$\begin{array}{llllll}
(1) & \pi_k(\mathrm{O}) & = & \varinjlim \pi_k(\mathrm{O}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{O}(N)) \quad \textit{for any } N \in \mathbb{N} \textit{ with } N > k+1, \\
(2) & \pi_k(\mathrm{U}) & = & \varinjlim \pi_k(\mathrm{U}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{U}(N)) \quad \textit{for any } N \in \mathbb{N} \textit{ with } 2N > k, \\
(3) & \pi_k(\mathrm{Sp}) & = & \varinjlim \pi_k(\mathrm{Sp}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{Sp}(N)) \quad \textit{for any } N \in \mathbb{N} \textit{ with } 4N > k-2.
\end{array}$$

Proof $(*)$. We prove (1). The proofs of (2) and (3) are almost identical. We start out with the following claim.

Claim. Given any compact subset K of O there exists an $n \in \mathbb{N}$ with $K \subset \mathrm{O}(n)$.

Let K be a compact subset of O. Given $n \in \mathbb{N}$ we write $X_n := \{1, \dots, n\} \times \{1, \dots, n\}$. First recall that $\mathrm{O}(n)$ is equipped with the subspace topology coming from the inclusion map $g_n \colon \mathrm{O}(n) \subset M(n \times n, \mathbb{R}) = \mathbb{R}^{X_n} = \mathbb{R}^{n^2}$. Given $n \in \mathbb{N}$ we denote by $f_n \colon \mathbb{R}^{X_n} \to \mathbb{R}^{X_{n+1}}$ the obvious inclusion map. These have the property that for any $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{O}(n) \arrow[r, "g_n"] \arrow[d, hook] & \mathbb{R}^{X_n} \arrow[d, hook, "f_n"] \\
\mathrm{O}(n+1) \arrow[r, "g_{n+1}"] & \mathbb{R}^{X_{n+1}}.
\end{tikzcd}$$

The above inclusion maps $g_n \colon \mathrm{O}(n) \to \mathbb{R}^{X_n}$ induce a map $g \colon \mathrm{O} = \varinjlim \mathrm{O}(n) \hookrightarrow \varinjlim \mathbb{R}^{X_n}$. By our hypothesis that K is compact and Lemma 1.37 we know that $g(K) \subset \varinjlim \mathbb{R}^{X_n}$ is also compact. Basically the same argument as in the proof of Lemma 26.5 shows that there exists an $n \in \mathbb{N}$ with $g(K) \subset \mathbb{R}^{X_n}$. But this implies that $K \subset \mathrm{O}(n)$. This concludes the proof of the claim.

Now let $N \in \mathbb{N}$ with $N > k+1$. We have

$$\begin{array}{ccccc}
\pi_k(\mathrm{O}) & = & \varinjlim \pi_k(\mathrm{O}(n)) & = & \pi_k(\mathrm{O}(N)). \\
 & \uparrow & & \uparrow & \\
 & \text{by Proposition 28.11 and the claim} & & \text{Proposition 86.10 (1) and Lemma 16.2} &
\end{array}$$

□

We will now state two versions of the Bott Periodicity Theorem which implies in particular that the homotopy groups of O, U and Sp are periodic.

The first theorem requires just a little bit of preparation.

(1) As above we equip U, Sp and O with the base point given by the infinite identity matrix. We use these base points to form the loop spaces ΩU, $\Omega\,\mathrm{Sp}$ and ΩO.

(2) Given a point x in a topological space X we denote by c_x the corresponding constant path.
(3) We use notation (2) to equip each loop space $\Omega(X, x_0)$ with the base c_{x_0} and we can thus iterate the loop space construction.
(4) The obvious isomorphism $\mathbb{H} \xrightarrow{\cong} \mathbb{R}^4$ of real vector spaces defines inclusion maps $\mathrm{Sp}(n) \to \mathrm{O}(4n)$ and $\mathrm{Sp} \to \mathrm{O}$.
(5) The inclusion $\mathbb{R} \to \mathbb{H}$ defines inclusions $\mathrm{O}(n) \to \mathrm{Sp}(n)$ and thus $\mathrm{O} \to \mathrm{Sp}$.

Theorem 86.18. (Bott Periodicity Theorem) *There exist homotopy equivalences*

$$(1) \quad \mathrm{U} \to \Omega^2 \mathrm{U} \qquad (2) \quad \mathrm{Sp} \to \Omega^4 \mathrm{O} \qquad \textit{and} \qquad (3) \quad \mathrm{O} \to \Omega^4 \mathrm{Sp}\,.$$

Before we talk about the proof of the Bott Periodicity Theorem 86.18 let us move on to the following direct consequence.

Theorem 86.19. (Bott Periodicity Theorem) *For any $i \in \mathbb{N}_0$ we have*[1471]

$$\begin{aligned} &(1) & \pi_{i+2}(\mathrm{U}) &\cong \pi_i(\mathrm{U}), \\ &(2) & \pi_{i+4}(\mathrm{Sp}) &\cong \pi_i(\mathrm{O}), \\ &(3) & \pi_{i+4}(\mathrm{O}) &\cong \pi_i(\mathrm{Sp}). \end{aligned}$$

In particular, combining (2) *and* (3) *we have for any $i \in \mathbb{N}_0$ that*

$$\begin{aligned} &(4) & \pi_{i+8}(\mathrm{O}) &\cong \pi_i(\mathrm{O}), \\ &(5) & \pi_{i+8}(\mathrm{Sp}) &\cong \pi_i(\mathrm{Sp}). \end{aligned}$$

Proof of the Bott Periodicity Theorems 86.18 and 86.19. First note that Theorem 86.19 follows immediately from Theorem 86.18 together with Corollary 83.15. All the known proofs are non-trivial and go well-beyond what we can do in these modest notes.

Theorem 86.18 was first proved by Raoul Bott [Bot, Bot2, Bot3] in 1957. Another proof is given in [Mi4, p. 129, 142]. There explicit maps that give these homotopy equivalences, they are described well in [DL]. Nonetheless the description is too long for us to write down at this point.

There are many proofs of Theorem 86.19 which use a slightly different language, in particular it is at first glance not entirely clear whether they also imply Theorem 86.18. We refer to [Stro, Chapter 32.6], [AGP, Chapter 9.5], [APr] and [Hat2] for a discussions and proofs of Theorem 86.19. □

Using the Bott Periodicity Theorem 86.19 we can completely determine the homotopy groups of O, U and Sp.

Proposition 86.20. *The homotopy groups of* O, U *and* Sp *are as follows:*

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$.

[1471] For $i = 0$ we actually have an isomorphism of pointed sets.

Proof. The homotopy groups of U are straightforward to determine. More precisely, let $i \in \mathbb{N}_0$. We have

$$\begin{array}{ccccccccc} \pi_i(\mathrm{U}) & = & \pi_{i \bmod 2}(\mathrm{U}) & \overset{\cong}{\leftarrow} & \pi_{i \bmod 2}(\mathrm{U}(1)) & = & \pi_{i \bmod 2}(S^1) & \cong & \begin{cases} 0, & \text{if } i = 0 \bmod 2, \\ \mathbb{Z}, & \text{if } i = 1 \bmod 2. \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{Bott Periodicity Theorem 86.19} & & \text{Lemma 86.17 (2)} & & \text{since } \mathrm{U}(1) = S^1 & & \begin{array}{c}\text{see Lemma 1.96} \\ \text{Proposition 7.18}\end{array} & \end{array}$$

Now we consider the homotopy groups of O and Sp. By the Bott Periodicity Theorem 86.19 it suffices to compute the homotopy groups $\pi_i(\mathrm{O})$ and $\pi_i(\mathrm{Sp})$ for $i = 0, 1, 2, 3$.[1472] So let $i \in \{0, 1, 2, 3\}$. We have

$$\begin{array}{ccccccc} \pi_i(\mathrm{Sp}) & \overset{\cong}{\leftarrow} & \pi_i(\mathrm{Sp}(1)) & = & \pi_i(S^3) & \cong & \begin{cases} 0, & \text{if } i = 0, 1, 2, \\ \mathbb{Z}, & \text{if } i = 3. \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 86.17 (3)} & & \text{since } \mathrm{Sp}(1) = S^3 & & \text{Proposition 28.9 and Corollary 80.6 (1)} & \end{array}$$

We turn to the calculation of the homotopy groups of O. First we note that for any $i \geq 1$ we have

$$\begin{array}{ccccc} \pi_i(\mathrm{O}) & \overset{\cong}{\leftarrow} & \pi_i(\mathrm{O}(i+2)) & \cong & \pi_i(\mathrm{SO}(i+2)). \\ & \uparrow & & \uparrow & \\ & \text{Lemma 86.17 (3)} & & \text{Lemmas 1.96 and 1.110} & \end{array}$$

Thus it remains to prove the following claim.

Claim.

(a)	$\pi_0(\mathrm{O}(2))$	$\cong$	$\mathbb{Z}_2$	(c)	$\pi_2(\mathrm{SO}(4))$	$\cong$	0
(b)	$\pi_1(\mathrm{SO}(3))$	$\cong$	$\mathbb{Z}_2$	(d)	$\pi_3(\mathrm{SO}(5))$	$\cong$	$\mathbb{Z}$.

We turn to the proof of these four separate claims.

(a) By Lemma 1.96 (1) we know that O(2) consists of two path components, thus $\pi_0(\mathrm{O}(2))$ is by definition a pointed set with two elements.
(b) By Proposition 86.11 we know that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.
(c) We have the following isomorphisms:

$$\begin{array}{ccccccccc} \pi_2(\mathrm{SO}(4)) & \cong & \pi_2((S^3 \times S^3)/\mathbb{Z}_2) & \cong & \pi_2(S^3 \times S^3) & \cong & \pi_2(S^3) \oplus \pi_2(S^3) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \begin{array}{c}\text{by Proposition 86.14 (1b)} \\ \text{and Proposition 44.2}\end{array} & & \begin{array}{c}\text{by Lemma 86.5} \\ \text{and Proposition 28.13}\end{array} & & \text{Proposition 28.8} & & \text{Proposition 28.9} & \end{array}$$

[1472] We will see that $\pi_0(\mathrm{O})$ is the pointed set with two elements. It follows from the Bott Periodicity Theorem 86.19 that $\pi_{4+8k}(\mathrm{Sp})$ and $\pi_{8k}(\mathrm{O})$, $k \geq 1$, are also pointed sets with two elements. But these pointed sets are also groups. Since there is only one group structure on a pointed set with two elements we see that all these groups are isomorphic to $\mathbb{Z}_2$.

(d) We have the following isomorphisms:

$$\begin{array}{ccccccccc} \pi_3(\mathrm{SO}(5)) & \cong & \pi_3(\mathrm{Sp}(2)/\{\pm\,\mathrm{id}\}) & \cong & \pi_3(\mathrm{Sp}(2)) & \cong & \pi_3(\mathrm{Sp}) & \cong & \mathbb{Z}. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 86.14 (1c)} & & \text{by Lemma 86.5 and Proposition 28.13} & & \text{Proposition 86.10} & & \text{see above} & \end{array}$$

This concludes the proof of the claim. □

86.5. **H-spaces** (∗)**.** In this final section of this chapter we want to state two general statements about homotopy groups of Lie groups. In fact these statements hold for a much wider class of topological spaces that we introduce now:

Definition. We say a topological space X is an *H-space*[1473] if there exists a map[1474]

$$\mu\colon X \times X \;\to\; X$$

and an element $e \in X$ such that $\mu(e,e) = e$ and such that the maps

$$\begin{array}{rcl} (X,e) & \mapsto & (X,e) \\ x & \mapsto & \mu(x,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} (X,e) & \mapsto & (X,e) \\ x & \mapsto & \mu(e,x) \end{array}$$

of pairs of topological spaces are homotopic to the identity.

Examples.

(1) Evidently Lie groups and more generally, topological groups are H-spaces, where μ is given by the group structure and e is the trivial element of the group.

(2) The topological space S^7 together with the multiplication given by the octonion structure on $\mathbb{O} = \mathbb{R}^8$ and with $e = 1 = (1,0,\dots,0) \in S^7$ is an H-space.[1475] Since octonion multiplication is not associative we see that, at least with this multiplication, S^7 is not a topological group.

(3) Given a non-zero polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{C}[t]$ we define

$$[p(t)] \;:=\; [a_0 : a_1 : \dots : a_n : 0 : \dots] \;\in\; \mathbb{C}\mathrm{P}^\infty.$$

Evidently every element in $\mathbb{C}\mathrm{P}^\infty$ is of that form. We consider the "multiplication" map

$$\begin{array}{rcl} \mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty & \to & \mathbb{C}\mathrm{P}^\infty \\ ([p(t)],[q(t)]) & \mapsto & [\underbrace{p(t)\cdot q(t)}_{\in\mathbb{C}[t]}]. \end{array}$$

We leave it to the reader to verify that μ is continuous. It is clear that left and right "multiplication" by $e = [1] = [1 : 0 : \dots]$ is the identity. This shows that $\mathbb{C}\mathrm{P}^\infty$

[1473] According to [Hub, p. 747] the term H-space was coined by Jean-Pierre Serre to honor the work of Heinz Hopf.

[1474] As usual, whenever we deal with topological spaces, all maps are understood to be continuous.

[1475] Recall that by the discussion on page 1057 we know that for octonions $w, z \in \mathbb{O} = \mathbb{R}^8$ we have $\|w \cdot z\| = \|w\| \cdot \|z\|$, in particular the multiplication restricts to a map on S^7. Also note that multiplication by $e = 1$ is the identity.

together with μ and e is an H-space. Note that this multiplication μ is commutative and associative, but it does not have inverses[1476], thus μ does not define a group structure on $\mathbb{C}\mathrm{P}^\infty$.

We formulate the last example as a lemma.

Lemma 86.21. *Let (X, x_0) be a pointed topological space. We denote by $\Omega(X, x_0)$ the loop space as defined on page 1930.*

(1) *The map*

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f\colon [0,1]\to X, g\colon [0,1]\to X) & \mapsto & (\underbrace{f * g}_{\substack{\text{multiplication of loops}\\ \text{as defined on page 231}}} : [0,1]\to X) \end{array}$$

is continuous.

(2) *The map $\mu\colon \Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ from (1) together with the constant loop given by $e(t)\equiv x_0$, turns $\Omega(X,x_0)$ into an H-space.*

Remark.

(1) Note that for the H-space $\Omega(X,x_0)$ "multiplication" by e is not the identity, it is only homotopy equivalent to the identity.
(2) We had hinted at the existence of a multiplication map on $\Omega(X,x_0)$ in Exercise 83.4.

Proof $(*)$. Let (X,x_0) be a pointed topological space. To simplify the notation we write ΩX instead of $\Omega(X,x_0)$.

(1) We need to show that the map $\mu\colon \Omega X\times\Omega X\to\Omega X$ is in fact continuous. By the "$\Rightarrow$"-direction of Proposition 2.6, applied to $T=\Omega X\times\Omega X$ and the locally compact topological space $Y=[0,1]$, it suffices to show that the following map is continuous:

$$\Phi\colon (\Omega X\times\Omega X)\times[0,1] \xrightarrow{\mu\times\mathrm{id}} \Omega X\times[0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

On the subset $(\Omega X\times\Omega X)\times[0,\frac{1}{2}]$ this map is equal to

$$(\Omega X\times\Omega X)\times[0,\tfrac{1}{2}] \xrightarrow{((f,g)\mapsto f)\times(t\mapsto 2t))} \Omega X\times[0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

The first map is evidently continuous. Furthermore, the second map is continuous by Proposition 2.4. Thus we have shown that the restriction of Φ to the subset $(\Omega X\times\Omega X)\times[0,\frac{1}{2}]$ is continuous. Basically the same argument shows that the restriction of Φ to $(\Omega X\times\Omega X)\times[\frac{1}{2},1]$ is continuous. it follows from Lemma 5.3 that Φ itself is continuous.

[1476] Why are there elements in $\mathbb{C}\mathrm{P}^\infty$ which do not have multiplicative inverses?

(2) Let $e \in \Omega(X, x_0)$ be the constant loop given by $e(t) \equiv x_0$. It is clear that $\mu(e, e) = e$. Thus it remains to show that the maps

$$\begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(f,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(e,f) \end{array}$$

of pairs of topological spaces are homotopic to the identity. We consider the map

$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{2s}{t+1}), & \text{if } s\in[0,\frac{t+1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array} \right) \end{array}$$

We leave it to the underchallenged reader to show, using the "$\Rightarrow$"-direction of Proposition 2.6, that this map H is in fact continuous. Note that we have $H(e,t) = e$ for all $t \in [0,1]$. With these observations it is basically clear that it is the desired homotopy between the map $f \mapsto \mu(f,e)$ and the identity.

Almost exactly the same way we deal with the second map $f \mapsto \mu(e,f)$. □

The following lemma is the only statement on H-spaces that we can prove with our limited space resources.

Lemma 86.22. *If X is a path-connected H-space, then $\pi_1(X)$ is abelian.*

Remark. Let (X, x_0) be a pointed topological space. As we pointed out above the loop space $\Omega(X, x_0)$ is an H-space which implies by Lemma 86.22 that $\pi_1(\Omega(X, x_0))$ is abelian. Now recall that in Corollary 83.15 we saw that there exists an isomorphism $\pi_1(\Omega(X, x_0)) \xrightarrow{\cong} \pi_2(X)$, which is abelian by Proposition 28.3. So we now have two different (?) proofs for the statement that $\pi_1(\Omega(X, x_0))$ is abelian.

Proof $(*)$. We denote by $\mu\colon X \times X \to X$ and $e \in X$ the objects in the definition of an H-space. Let $f, g\colon [0,1] \to X$ be loops in e. We need to show that the path $f * g$ is path-homotopic to the path $g * f$.

We denote by by $\mu(f, e)$ the path $t \mapsto \mu(f(t), e)$ and similarly we define $\mu(e, g)$. We consider the map

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \mu(f(s),g(t)). \end{array}$$

Note that this map is continuous by Lemma 1.53 (3) and the fact that μ is continuous. Furthermore we consider the following path-homotopy:[1477]

$$\begin{array}{rcl} H\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \begin{cases} F(2s\cdot(1-t,t)), & \text{if } s\le\frac12 \\ F((2-2s)\cdot(1-t,t)+(2s-1)\cdot(1,1)), & \text{if } s>\frac12. \end{cases} \end{array}$$

By carefully plugging in $t = 0$ and $t = 1$ we see that H is a path-homotopy from the loop $\mu(f, e) * \mu(e, g)$ to the loop $\mu(e, g) * \mu(f, e)$.

[1477] It follows easily from Lemma 1.32 (2') that this map is continuous.

Next we recall that by definition of an H-space there exists a homotopy $A\colon X\times[0,1]\to X$ rel $\{e\}$ with $A_0=\mathrm{id}$ and $A_1=\mu(-,e)$ and there exists a homotopy $B\colon X\times[0,1]\to X$ rel $\{e\}$ with $B_0=\mathrm{id}$ and $B_1=\mu(e,-)$. We consider the maps

$$\begin{array}{rcl} A_f\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & A(\underbrace{f(s)}_{\in X},t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B_g\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & B(\underbrace{g(s)}_{\in X},t). \end{array}$$

Note that A_f is a path-homotopy from the loop f to the loop $\mu(f,e)$ and B_g is a path-homotopy from the loop g to the loop $\mu(e,g)$. The path-homotopy between the loops $f*g$ and $g*f$ is given by combining H with A_f and B_g as indicated in Figure 1055. We leave it to the reader to conscientiously write down the precise homotopy. □

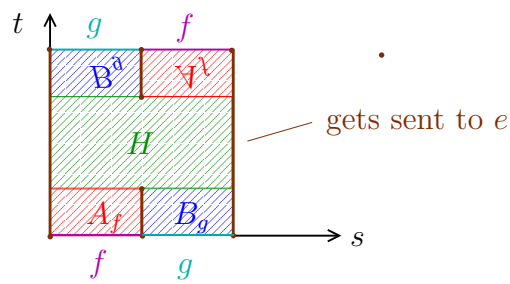


FIGURE 1055. Illustration for the proof of Lemma 86.22.

The following theorem was proved by William Browder [Browd, Theorem 2] [Browd2, Theorem 6.11] in 1961.

Theorem 86.23. *Let X be a path-connected H-space. If the group $\bigoplus_{i=0}^{\infty} H_i(X;\mathbb{Z})$ is finitely generated, then*

$$\min\{m\in\mathbb{N}_{\geq 2}\,|\,\pi_m(X)\neq 0\}$$

is odd. In particular we have $\pi_2(X)=0$.

Example. On page 2000 we saw that $\mathbb{C}\mathrm{P}^\infty$ is an H-space. Furthermore in Corollary 80.6 (2) we saw that $\pi_2(\mathbb{C}\mathrm{P}^\infty)\neq 0$. Thus we see that in the statement of Theorem 86.23 we cannot drop the hypothesis that $\bigoplus_{i=0}^{\infty} H_i(X;\mathbb{Z})$ is a finitely generated group.

We conclude this chapter with a corresponding theorem for Lie groups.

Theorem 86.24. *If G is a path-connected Lie group, then*

$$\min\{m\in\mathbb{N}_{\geq 2}\,|\,\pi_m(G)\neq 0\}$$

is odd. In particular $\pi_2(G)=0$.

Remark. There are many direct proofs in the literature for the statement that given a path-connected Lie group G the second homotopy group $\pi_2(G)$ vanishes, see e.g. [Cart] or [Mi4, Theorem 21.7] or [BtD, Proposition V.7.5].

Note that Theorem 86.24 has no conditions on the Lie group except that it needs to be path-connected. The reason why we can get away with so few conditions is given by the following proposition.

Proposition 86.25. *Let G be a Lie group. There exists a compact Lie subgroup K*[1478] *such that G is homeomorphic to $K \times \mathbb{R}^n$ for some $n \in \mathbb{N}_0$.*

Proof of Proposition 86.25. Like many other foundational results on Lie groups this proposition has its origins in the work of Henri Cartan [Cart]. A full proof is given in [Mal, Theorem 11] or [Mos, Theorem 2]. A sketch of the proof is provided in [Sam, Chapter 7]. □

Example. We consider the group $\mathrm{Isom}(\mathbb{R}^n)$ of isometries of the vector space $\mathbb{R}^n$ equipped with the usual inner product. It is an elementary, but not totally trivial fact, see [Ree, p. 11], that

$$\mathrm{Isom}(\mathbb{R}^n) \;=\; \{\text{all maps of the form } x \mapsto Ax + v \text{ where } A \in \mathrm{O}(n) \text{ and } v \in \mathbb{R}^n\}.$$

The maps

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathbb{R}^n & \to & \mathrm{Isom}(\mathbb{R}^n) \\ (A, v) & \mapsto & (x \mapsto Ax + v) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Isom}(\mathbb{R}^n) & \to & \mathrm{O}(n) \times \mathbb{R}^n \\ f & \mapsto & (\underbrace{(x \mapsto (f(x) - f(0))}_{\text{represented by matrix in } \mathrm{O}(n)}, f(0)) \end{array}$$

are inverses of one another. Thus we can use the manifold structure on $\mathrm{O}(n) \times \mathbb{R}^n$ to turn $\mathrm{Isom}(\mathbb{R}^n)$ into a manifold and it is quasi-obvious to see that this turns $\mathrm{Isom}(\mathbb{R}^n)$ into a Lie group. We can view $\mathbb{R}^n$ and $\mathrm{O}(n)$ as subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ in an obvious way. But one easily verifies that the map $\mathrm{O}(n) \times \mathbb{R}^n \to \mathrm{Isom}(\mathbb{R}^n)$ is not an isomorphism. In other words, as predicted by Proposition 86.25 the Lie group $\mathrm{Isom}(\mathbb{R}^n)$ splits topologically into a product. But the Lie group does not split as a Lie group.

Proof of Theorem 86.24. Let G be a path-connected Lie group. By Proposition 86.25 we know that G is homotopy equivalent to a compact path-connected Lie group K. By Proposition 28.7 it suffices to prove the desired statement for K. Since K is in particular a compact manifold we obtain from Proposition 45.2 that $\bigoplus_{i=0}^{\infty} H_i(X; \mathbb{Z})$ is a finitely generated abelian group. Thus we can conclude the proof by appealing to Theorem 86.23. □

[1478] In fact a Lie group contains a maximal compact Lie subgroup which is furthermore unique up to conjugation. We will not make use of this fact.

Exercises for Chapter 86.

Exercise 86.1. Let X be the torus minus one open disk. Does X admit the structure of a topological group?

Exercise 86.2. Let M be manifold without boundary and let $g\colon M \times M \to M$ be a map that is smooth and that defines a group structure on M. Show that (M, g) is a Lie group. In other words, show that the map $M \to M$ given by $x \mapsto x^{-1}$ is smooth.
Hint: Use the Implicit Function Theorem, see e.g. [Le2, p. 661].

Exercise 86.3. Let $n \in \mathbb{N}$. We define

$$\mathrm{GL}(n, \mathbb{H}) \;=\; \left\{ A \in M(n \times n, \mathbb{H}) \;\middle|\; \begin{array}{c} \text{there exist matrices } X, Y \in M(n \times n, \mathbb{H}) \\ \text{such that } AX = YA = \mathrm{id} \end{array} \right\}.$$

Is $\mathrm{GL}(n, \mathbb{H})$ an open subset of $M(n \times n, \mathbb{H})$?

Exercise 86.4. Let $n \in \mathbb{N}$. In the proof of Lemma 86.3 we already saw that $\mathrm{Sp}(n)$ is a manifold and that matrix multiplication turns $\mathrm{Sp}(n)$ into a group. Show that the map

$$\begin{array}{rcl} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is a smooth map.
Note: The argument in the proof of Lemma 1.108 can be used to show that taking inverses in $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ is smooth. Does this help us in any way?
Remark: In principle one could use Exercise 86.2. But what do readers do who are still traumatized by the Implicit Function Theorem?

Exercise 86.5. Use the Ehresmann Fibration Theorem that we had stated in Exercise 85.5 to give a third proof that the map $p\colon\ \mathrm{O}(n+1) \to S^n$ from Proposition 86.8 is a fiber bundle.

Exercise 86.6. Determine all closed non-orientable 2-dimensional manifolds that admit a Lie group structure.

Exercise 86.7. We define $\mathrm{SU} = \varinjlim \mathrm{SU}(n)$ the same way as we defined U. Determine the homotopy groups of SU.
Remark: You might want to use Exercise 85.7.

87. The Thom-Pontryagin Theory

In this chapter we will introduce the Thom-Pontryagin approach to calculating homotopy groups of spheres. Furthermore we will see that it allows us to reprove several earlier results. In the subsequent chapter we will use the Thom-Pontryagin approach to obtain new results.

Before we start with discussing the Thom-Pontryagin theory we fix some notations and conventions.

Notation and Convention. Let $n \in \mathbb{N}_0$.

(1) We denote by $*$ the point in $\overline{B}^n/S^{n-1}$ that is given by $[S^{n-1}]$.
(2) We denote by $*$ the "North Pole" $(0,\dots,0,1) \in S^n$.

We make the following identifications:

(a) We use the homeomorphism $S^n \to \mathbb{R}^n \cup \{\infty\}$ from Lemma 1.41, which is given by stereographic projection, that sends the North Pole $*$ to ∞ to make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$.
(b) On page 99 we wrote down an explicit homeomorphism $\overline{B}^n/S^{n-1} \to S^n$. This homeomorphism sends $*$ to $(-1,0,\dots,0)$. Thus we postcompose it with the diffeomorphism of S^n that is given by[1479] $(x_1,\dots,x_{n+1}) \mapsto (x_{n+1},x_2,\dots,x_n,-x_1)$. This way we obtain a homeomorphism $\overline{B}^n/S^{n-1} \to S^n$ that sends $*$ to $(0,\dots,0,1)$.
(c) Putting the two homeomorphisms from (a) and (b) together we obtain a homeomorphism
$$\varphi\colon \overline{B}^n/S^{n-1} \quad \to \quad \mathbb{R}^n \cup \{\infty\}.$$
It follows fairly easily from the definitions of the two maps in (a) and (b) that φ restricts to a diffeomorphism $B^n \to \mathbb{R}^n$ and that φ is of the form
$$x \quad \mapsto \quad \begin{cases} \eta(\|x\|)\cdot x, & \text{if } x \in B^n, \\ \infty, & \text{if } x = *, \end{cases}$$
where $\eta\colon [0,1) \to [0,\infty)$ is a strictly monotonously increasing function with $\eta(0)=0$ and $\lim\limits_{t\to 1}\eta(t)=\infty$.

Summarizing we have the identifications $S^n = \mathbb{R}^n \cup \{\infty\} = \overline{B}^n/S^{n-1}$ and under these identifications the points $* \in S^n$, ∞ and $* \in \overline{B}^n/S^{n-1}$ get identified. We refer to Figure 1056 for an illustration.

87.1. **The generalized Thom-Pontryagin construction.** On page 1618 we saw that a proper oriented codimension one submanifold of a compact oriented manifold M gives rise, via the Thom-Pontryagin construction, to a map $M \to S^1$. In this section we will generalize this construction.

The key objects in the more general construction of Thom-Pontryagin maps are "fattened" submanifolds:

[1479] Note that this map is given by multiplication by a matrix in $\mathrm{SO}(n+1)$. In Lemma 10.4 we showed that such a diffeomorphism is diffeotopic to the identity. In particular it is orientation-preserving.

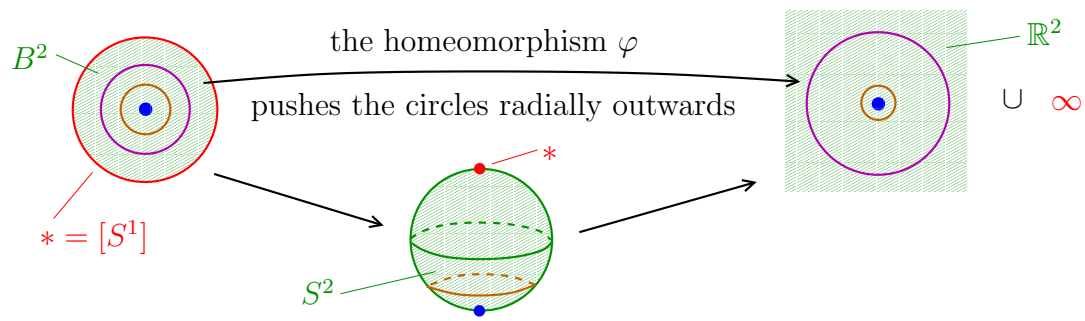


FIGURE 1056.

Definition. Let M be an orientable n-dimensional manifold.

(1) Let N be a compact orientable proper k-dimensional submanifold of M. A *fattening of* N is an injective map $g\colon \overline{B}^{n-k} \times N \to M$ with the following properties:[1480][1481][1482]
   (a) for all $P \in N$ we have $g(0, P) = P$,
   (b) the image $g(\overline{B}^{n-k} \times N)$ is a submanifold with corner of M,
   (c) we have $g(\overline{B}^{n-k} \times N) \cap \partial M = g(\overline{B}^{n-k} \times \partial N)$.
   We refer to Figure 1057 for an illustration.

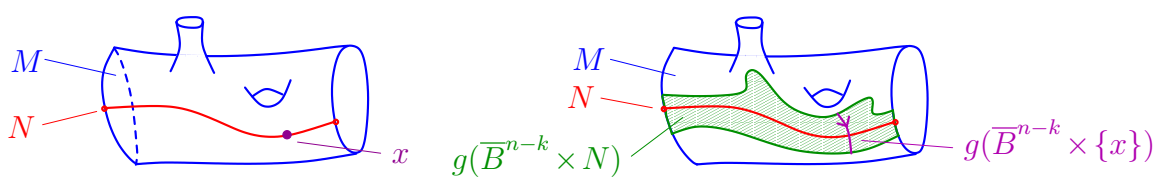


FIGURE 1057.

(2) A *fattened submanifold* of M is a pair (N, g) where N is a compact orientable proper[1483] submanifold of M and g is a fattening for N.
(3) Let $\varphi\colon M \to W$ be an embedding of M into a manifold W of the same dimension. Given a fattened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N)$ of $M \setminus \partial M$ we equip

[1480]Here one really would like to say that $g\colon \overline{B}^{n-k} \times N \to M$ is an "embedding", but if N has boundary, then it is not entirely clear in what sense $\overline{B}^{n-k} \times N$ is a manifold and what "embedding" is supposed to mean.

[1481]The definition looks very similar to the definition of a tubular neighborhood. In fact a fattening is the same data as a trivial tubular neighborhood, it is just phrased in slightly different words.

[1482]On page 1626 we introduced the fattening of an *oriented* knot. For knots the present definition is the same as before, except that we no longer demand that the knot is oriented.

[1483]Note that here, in the definition of a fattened submanifold, we sneaked in the adjectives "compact", "orientable" and "proper", so that we will not have to carry them around all the time.

$\varphi(N)$ with the "obvious fattening", namely with the fattening that is given by

$$\begin{array}{rcl} \varphi_*(g)\colon \overline{B}^{n-k} \times \varphi(N) & \to & W \\ (v,P) & \mapsto & g(v,\varphi^{-1}(P)). \end{array}$$

Evidently $(\varphi(N), \varphi_*(g))$ is a fattened submanifold of W.

The following definition can be viewed as a generalization of the Thom-Pontryagin map introduced on page 1618.

Definition. Let M be an orientable n-dimensional manifold. Given a fattened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N \to M)$ we refer to the map[1484]

$$\begin{array}{rcl} \rho_{(N,g)}\colon M & \to & S^{n-k} = \overline{B}^{n-k}/S^{n-k-1} \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^{n-k} \times N, \\ *, & \text{otherwise} \end{cases} \end{array}$$

as the *Thom-Pontryagin map.* (We refer to Figure 1058 for an illustration.) Sometimes we refer to this approach of assigning to a fattened k-dimensional submanifold a map $M \to S^{n-k}$ as the *Thom-Pontryagin construction.*

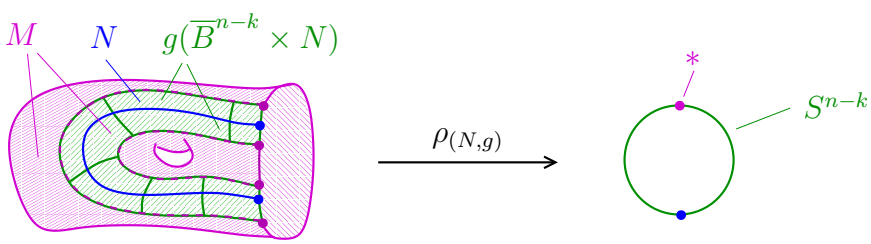


FIGURE 1058.

Remark.

(1) Let N be a compact *oriented* proper codimension one submanifold of an *oriented* manifold M. It follows immediately from Proposition 64.11 that N admits a fattening $[-1,1]\times N$ such that the inclusion $(-1,1)\times N \to M$ is orientation-preserving.[1485] The map $\rho_N\colon M \to S^1$ that we defined on page 1618 is basically the same as the above map $\rho_{(N,g)}\colon M \to S^1$. The only difference is that the two maps differ by multiplication by $i \in S^1$.

(2) We saw in Proposition 67.20 that the submanifold $\mathbb{C}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^2$ does *not* admit a trivial tubular neighborhood, in particular it does not admit a fattening.

[1484] It follows easily from Lemma 64.1 and Lemma 1.32 (2) that this map is indeed continuous.

[1485] Furthermore we saw in Proposition 64.9 that this fattening is essentially unique.

Example. Let $n \in \mathbb{N}$ and let M be a closed orientable n-dimensional manifold together with a 0-dimensional submanifold $N = \{P_1, \ldots, P_m\}$. Suppose N is equipped with a fattening

$$g = g_1 \sqcup \cdots \sqcup g_m \colon (\overline{B}^n \times \{P_1\}) \sqcup \cdots \sqcup (\overline{B}^n \times \{P_m\}) \ \to \ M.$$

From the Thom-Pontryagin construction we obtain a map $\rho_{(N,g)} \colon M \to S^n$. It might not be clear what this is good for, but it is kind of fun. This example is illustrated in Figure 1059.

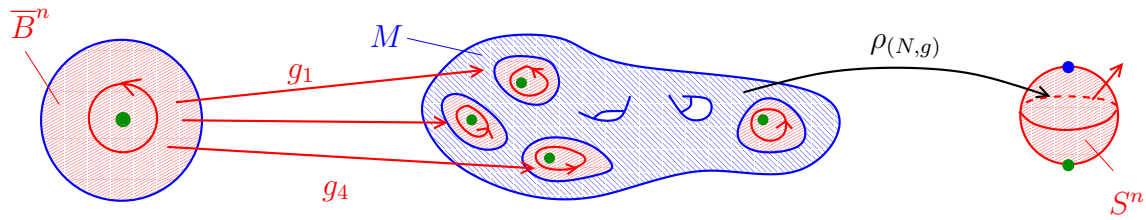


FIGURE 1059.

The following lemma gives us a first interesting example of a Thom-Pontryagin map.

Lemma 87.1. *Let $n \in \mathbb{N}$ and let $P \in B^n$. We equip the 0-dimensional manifold $\{P\}$ with any fattening $g \colon \overline{B}^n \times \{P\} \to B^n$ that is orientation-preserving*[1486]*. The corresponding map*

$$\rho_{(\{P\},g)} \colon \overline{B}^n / S^{n-1} \ \to \ \overline{B}^n / S^{n-1}$$

is homotopic rel $*$ to the identity map of $\overline{B}^n / S^{n-1}$.

The proof of Lemma 87.1 relies on the following lemma which we will use on several occasions.

Lemma 87.2. *Let $m \in \mathbb{N}$. Given any real number $r \in (0,1)$ there exists a homotopy $F \colon \overline{B}^m / S^{m-1} \times [0,1] \to \overline{B}^m / S^{m-1}$ rel $*$ with the following three properties:*

(1) *$F_0 = \mathrm{id}$,*
(2) *the restriction of F_1 to B^m_r is given by multiplication by $\frac{1}{r}$,*
(3) *for any point $x \notin B^m_r$ we have $F_1(x) = *$.*

Proof. Let $r \in (0,1)$. We consider the map

$$\begin{array}{rcl} (\overline{B}^m / S^{m-1}) \times [0,1] & \to & \overline{B}^m / S^{m-1} \\ ([x], t) & \mapsto & \begin{cases} [x \cdot (1 - t + t \cdot \frac{1}{r})], & \text{if } x \in B^m \text{ and } t < \frac{1 - \|x\|}{(\frac{1}{r} - 1) \cdot \|x\|} \\ *, & \text{otherwise.} \end{cases} \end{array}$$

We leave as an exercise to the engaged reader to verify that this map is indeed continuous. It is clear that F has the desired properties. $\square$

Now we can easily prove Lemma 87.1.

[1486] We equip $\overline{B}^n$ and B^n with the usual orientation.

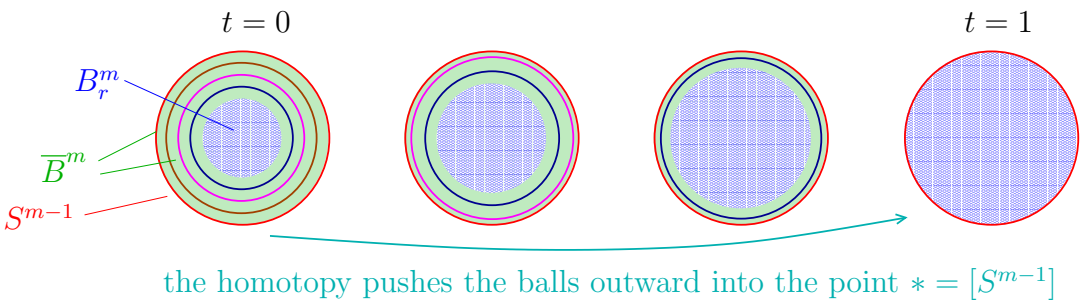


FIGURE 1060. Illustration of Lemma 87.2 (1).

Sketch of proof of Lemma 87.1. It follows quite easily[1487] from Theorem 10.28 that we can assume without loss of generality that $P = 0$ and that we can assume that g is the fattening $g\colon \overline{B}^n \times \{0\} \to B^n$ given by $g(v,0) = \frac{1}{2}v$. By Lemma 87.2 there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to $B^n_{\frac{1}{2}}$ is given by the map $x \mapsto 2 \cdot x$,
(3) for any point $x \notin B^n_{\frac{1}{2}}$ we have $F_1(x) = *$.

If one thinks about it for a second, then one realizes that F_1 equals $\rho_{(0,g)}$ "on the nose". Thus we have found the desired homotopy. □

87.2. **The Thom-Pontryagin construction and homotopy groups.** The following is the key observation which justifies our renewed interest in the Thom-Pontryagin construction.

Observation. Let $n, k \in \mathbb{N}_0$ and let $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ be a fattened k-dimensional submanifold of $\mathbb{R}^{n+k}$. We can view $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ also as a fattened submanifold of $\mathbb{R}^{n+k} \cup \{\infty\}$.[1488] Note that the corresponding Thom-Pontryagin map

$$\begin{aligned} \rho_{(N,g)}\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} &\to \overline{B}^n/S^{n-1} = S^n \\ P &\mapsto \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^n \times N, \\ *, & \text{otherwise,} \end{cases} \end{aligned}$$

sends ∞ to $*$. In particular $\rho_{(N,g)}$ defines an element in $\pi_{n+k}(S^n, *)$. It is worth repeating this observation:

fattened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to elements in $\pi_{n+k}(S^n, *)$.

The following questions immediately arise:

Question 87.3. *Let $n, k \in \mathbb{N}_0$.*

[1487] The slightly hesitant "quite easily" refers to the fact that, if one thinks about it, we need that the diffeotopy of $\mathbb{R}^{m+k}$ given by Theorem 10.28 has compact support.

[1488] Why can we do so?

(1) *Which elements in $\pi_{n+k}(S^n, *)$ can be realized by Thom-Pontryagin maps?*
(2) *When do two fattened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to the same element in $\pi_{n+K}(S^n, *)$.*

Before we address this question we intend to discuss two examples in greater detail:

(1) the case of fattened 0-dimensional submanifolds of some $\mathbb{R}^n$, and
(2) the case of fattened 1-dimensional submanifolds of $\mathbb{R}^3$.

We start out with the first case. Dealing with this case requires one definition.

Definition. Let $P \in \mathbb{R}^n$ be a point and let $g\colon \{P\} \times \overline{B}^n \to \mathbb{R}^n$ be a fattening of P. We equip $\{P\} \times \overline{B}^n = \overline{B}^n$ and $\mathbb{R}^n$ with the standard orientations. We define

$$\operatorname{sign}(g) \ := \ \begin{cases} +1, & \text{if } g \text{ is orientation-preserving,} \\ -1, & \text{if } g \text{ is orientation-reversing.} \end{cases}$$

Now we can discuss the case of fattened 0-dimensional submanifolds.

Example. Let $N = \{P_1, \ldots, P_m\}$ be a 0-dimensional submanifold of some $\mathbb{R}^n$ and suppose N is equipped with a fattening

$$g = g_1 \sqcup \cdots \sqcup g_m \colon (\overline{B}^n \times P_1) \sqcup \cdots \sqcup (\overline{B}^n \times P_m) \ \to \ \mathbb{R}^n.$$

It follows almost immediately from Proposition 80.2 that

$$[\rho_{(N,g)}] \ = \ \sum_{i=1}^{m} \operatorname{sign}(g_i) \cdot [\mathrm{id}_{S^n}] \ \in \ \pi_n(S^n, *).$$

We continue our discussion of examples of the Thom-Pontryagin construction with knots and links. By the above discussion every fattened knot or link in $\mathbb{R}^3$, i.e. every fattened 1-dimensional submanifold of $\mathbb{R}^3$, gives rise to an element in $\pi_3(S^2, *)$. For the trefoil together with some fattening we illustrate the corresponding map $S^3 \to S^2$ in Figure 1061.

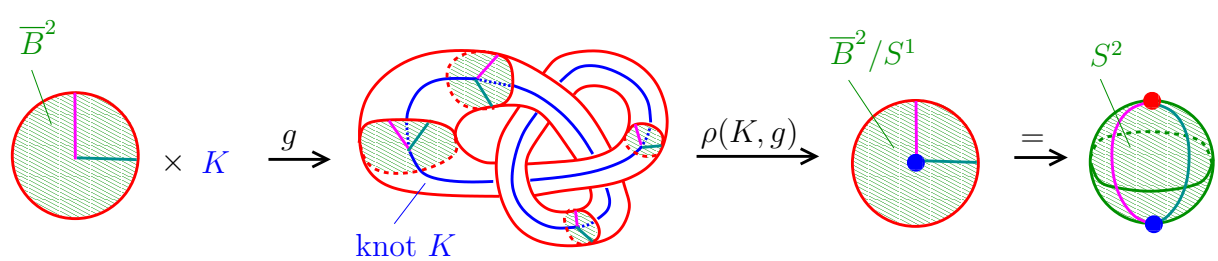


FIGURE 1061.

Before we continue we need to introduce the following slight variation on a definition from page 1626.

Definition. Let K be a knot and let $g\colon \overline{B}^2 \times K \to S^3$ be a fattening. We pick an orientation for K and we equip $g(\{1\} \times K)$ with the corresponding definition. We define[1489]

$$\text{self-linking number of } g \ := \ \mathrm{lk}(K, g(\{1\} \times K)).$$

It follows from Proposition 70.13 (2) that this definition does not depend on the choice of the orientation of K.

The following lemma deals with two basic, but important examples. Later on both examples will follow (at least up to sign) from a more general result, but for the geometrically inclined person it can be fun to consider the more direct arguments provided in the proof of the lemma.

Lemma 87.4. *Let K be the trivial knot and let $g\colon \overline{B}^2 \times K \to \mathbb{R}^3$ be a fattening.*

(1) *If g is a fattening with self-linking number zero, then $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$.*
(2) *If g is a fattening with self-linking number one, then*[1490] *$[\rho_{(K,g)}] = [H] \in \pi_3(S^2, *)$ where $H\colon (S^3, *) \to (S^2, *)$ denotes the Hopf map, that we defined on page 726.*

Sketch of proof $(*)$. It follows easily from Lemma 70.6[1491] that any two fattenings with the same self-linking number give rise to the same element in $\pi_3(S^2, *)$. Furthermore it follows almost immediately from Lemma 18.1 that we can make the identification

$$S^3 \ = \ (\overline{B}^2 \times S^1) \sqcup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$$

such that $K = \{0\} \times S^1$. Furthermore it follows from Lemma 18.2 that the obvious inclusion of $\overline{B}^2 \times S^1$ into the above decomposition of S^3 is a fattening of the trivial knot $K = S^1$ with fattening number zero.

(1) Let g be the fattening of the trivial knot K with self-linking number zero. We want to show that $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$. By Proposition 81.1 it suffices to show that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is homotopic to a constant map. (More precisely, Proposition 81.1 (7) says that in this context we do not need to worry about base points since S^2 is path-connected.) Using the above preparations we will prove in Exercise 87.1 that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is indeed homotopic to a constant map.

[1489] We refer to page 1635 for the definition of the linking number $\mathrm{lk}(K, J)$ of two disjoint oriented knots K and J.

[1490] The sign should be correct. Hopefully. Perhaps.

[1491] Note that here we need that in the statement of Lemma 70.6 for $\mathbb{R}^3$ we stated that there exists a diffeotopy of $\mathbb{R}^3$ with compact support.

(2) First we note that under the above identification of S^3 the Hopf map is given as follows:[1492]

$$\begin{array}{rcl} H\colon \overbrace{(\overline{B}^2 \times S^1) \sqcup_{S^1\times S^1} (S^1 \times \overline{B}^2)}^{=S^3} & \to & \mathbb{C}\mathrm{P}^1 \\ [(z,w)] & \mapsto & \begin{cases} [1 : \frac{z}{w}], & \text{if } z \in \overline{B}^2 \text{ and } w \in S^1, \\ [\frac{w}{z} : 1], & \text{if } z \in S^1 \text{ and } w \in \overline{B}^2. \end{cases} \end{array}$$

Now note that it follows from Lemma 70.6 (1) that the map

$$\begin{array}{rcl} g\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \subset S^3 = (\overline{B}^2 \times S^1) \sqcup_{S^1\times S^1} (S^1 \times \overline{B}^2) \\ (z,w) & \mapsto & (z\cdot w, w) \end{array}$$

is a fattening of the trivial knot of self-linking number one. Furthermore by definition the corresponding map $\rho_{(K,g)}$ is easily seen to be the map[1493]

$$\begin{array}{rcl} \rho_{(K,g)}\colon (\overline{B}^2 \times S^1) \sqcup_{S^1\times S^1} (S^1 \times \overline{B}^2) & \to & \overline{B}^2/S^1 \\ [(z,w)] & \mapsto & \begin{cases} [\frac{z}{w}], & \text{if } z \in \overline{B}^2 \text{ and } w \in S^1, \\ \infty = [S^1], & \text{if } z \in S^1 \text{ and } w \in \overline{B}^2. \end{cases} \end{array}$$

Comparing the definitions of the Hopf map H and the map $\rho_{(K,g)}$ on $\overline{B}^2 \times S^1$ we see that we are perhaps not that far off.

Next we need to relate the targets $\mathbb{C}\mathrm{P}^1$ and $\overline{B}^2/S^1$. To do so we consider the map $\Theta\colon \mathbb{C}\mathrm{P}^1 \to \overline{B}^2/S^1$ that is given by $\Theta([1:z]) = \frac{z}{1+|z|}$ and $\Theta([0:1]) = [S^1] = * \in \overline{B}^2/S^1$. It follows easily from Proposition 1.40 that Θ is a homeomorphism. The particularly audacious reader can verify that this map is homotopic to the "official" identification $\mathbb{C}\mathrm{P}^1 = \overline{B}^2/S^1$ which is given by the identification $\mathbb{C}\mathrm{P}^1 = S^2$ from page 113 and the above identification $S^2 = \overline{B}^2/S^1$. Note that the image of the first solid torus $\overline{B}^2 \times S^1$ under the map $\Theta \circ H$ equals the disk of radius $\frac{1}{2}$, i.e. it equals $\overline{B}^2_{\frac{1}{2}} \subset \overline{B}^2/S^1$.

We apply Lemma 87.2 to obtain a homotopy F of $\overline{B}^2/S^1$ with the following properties:

(a) $F_0 = \mathrm{id}$,
(b) the restriction of F_1 to $B^2_{\frac{1}{2}}$ is given by multiplication by 2,
(c) for any point $x \notin B^2_{\frac{1}{2}}$ we have $F_1(x) = *$.

The map $F_0 \circ \Theta \circ H = \Theta \circ H$ is homotopic rel $*$ to $F_1 \circ \Theta \circ H$. It is now straightforward to show that $F_1 \circ \Theta \circ H$ is homotopic rel $*$ to $\rho_{(K,g)}$. We leave it to the reader to fill in the details. □

[1492] As a reality check one can verify that the map is well-defined, i.e. for $[(z,w)]$ with $(z,w) \in S^1 \times S^1$ we get consistent answers.

[1493] Why does it say $[\frac{z}{w}]$ instead of $[z \cdot w]$, which might look more natural?

We conclude this list of basic examples of the Thom-Pontryagin construction with the following lemma.

Lemma 87.5. *Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. Let (N_i, g_i), $i = 1, \dots, m$ be fattened k-dimensional submanifolds of $\mathbb{R}^{n+k}$. If the images $g_i(\overline{B}^n \times N_i)$ are contained in m disjoint closed smooth $(n+k)$-dimensional balls*[1494]*, then*

$$\rho_{(N_1 \sqcup \dots \sqcup N_m, g_1 \sqcup \dots \sqcup g_m)} \;=\; \rho_{(N_1, g_1)} + \dots + \rho_{(N_m, g_m)} \;\in\; \pi_{n+k}(S^n, *).$$

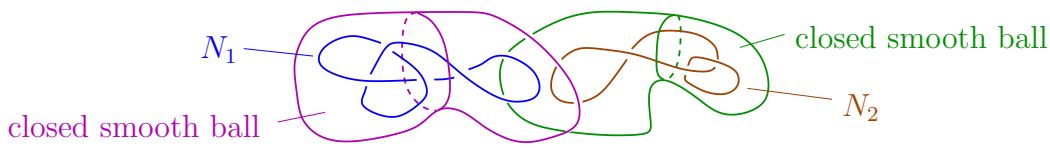


FIGURE 1062.

Proof. This lemma follows easily from Proposition 80.2. We leave it to the reader to fill in the details. □

The following proposition gives an affirmative answer to Question 87.3.

Proposition 87.6. *Let $k, n \in \mathbb{N}_0$. Given any $\varphi \in \pi_{n+k}(S^n, *)$ there exists a fattened k-dimensional submanifold (N, g) of $\mathbb{R}^{n+k}$ with $[\rho_{(N,g)}] = \varphi$.*

Proof. Throughout this proof we find it convenient to view each m-dimensional sphere as the quotient $\overline{B}^m / S^{m-1}$. So suppose we are given a map $\varphi \colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$ with $\varphi(*) = *$. As we pointed out in Corollary 65.7, it is a consequence of the Whitney Approximation Theorem 65.6 that we can arrange that φ is in fact a smooth map. It follows immediately from Sard's Theorem 3.48 and Proposition 3.47 (2) that there exists a regular value $z \in \overline{B}^n/S^{n-1}$ such that $\varphi^{-1}(z)$ does not contain the point $*$. By Proposition 10.23 (2)[1495] we can without loss of generality assume that the origin 0 is a regular value. By the Regular Value Theorem 65.1 we know that the preimage $N := \varphi^{-1}(0)$ is a closed orientable k-dimensional submanifold of B^{n+k}. Furthermore by statement (4) of the Regular Value Theorem 65.1 we know that there exists an $\epsilon > 0$ and a fattening $g \colon \overline{B}^n \times N \to B^{n+k}$ such

[1494]Here a "closed smooth $(n+k)$-dimensional ball" is shorthand for the image of an embedding of $\overline{B}^{n+k}$ in $\mathbb{R}^{n+k}$.

[1495]To be more precise, we apply Proposition 10.23 to the manifold $\overline{B}^n/S^{n-1} = \mathbb{R}^n \cup \{\infty\}$ with $P_1 = z$, $P_2 = \infty$ and $Q_1 = 0$ and $Q_2 = \infty$.

that $\varphi^{-1}(\overline{B}^n_\epsilon) = g(\overline{B}^n \times N)$ and such that the following diagram commutes[1496][1497]

$$\begin{tikzcd} \overline{B}^n \times N \arrow[r, "g"] \arrow[d, "(p,x)\mapsto \epsilon\cdot p"] & B^{n+k} \arrow[d, "\varphi"] \\ \overline{B}^n_\epsilon \arrow[r, hook] & \overline{B}^n/S^{n-1}. \end{tikzcd}$$

This fattening g allows us to consider the map $\rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$. Note that in general φ and $\rho_{(N,g)}$ are quite different maps. (We refer to Figure 1063 for an illustration of the two maps φ and $\rho_{(N,g)}$.) But that does not bother us too much, since it suffices to prove the following claim.

Claim. The maps φ and $\rho_{(N,g)}$ from $\overline{B}^{n+k}/S^{n+k-1}$ to $\overline{B}^n/S^{n-1}$ are homotopic rel $*$.

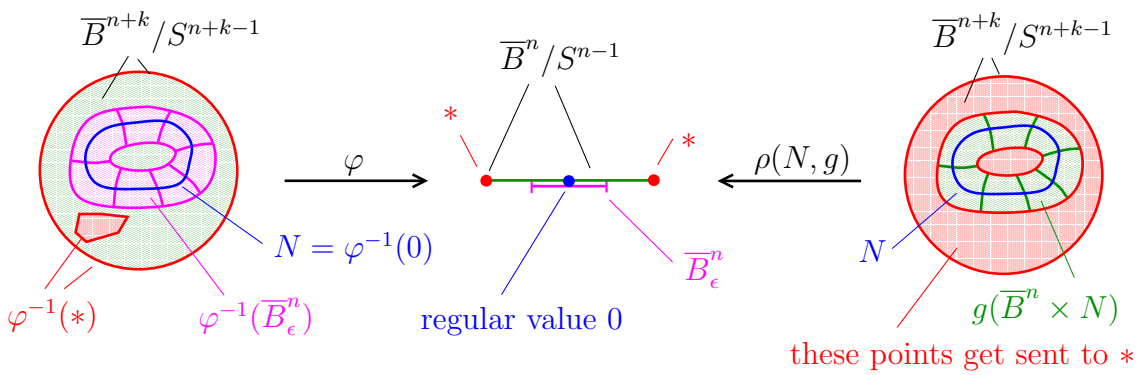


FIGURE 1063.

First we remind the reader that in Lemma 87.2 (1) we showed that there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to B^n_ϵ is given by multiplication by $\frac{1}{\epsilon}$,
(3) for any point $x \notin B^n_\epsilon$ we have $F_1(x) = *$.

We define $\psi = F_1 \circ \varphi$. It follows from (1) that ψ is homotopic rel $*$ to φ. Therefore we are done once we have shown that ψ in fact equals $\rho_{(N,g)}$. As a reminder, see the definition on

[1496]Note the discretely hidden multiplication by ϵ in the left-vertical map.
[1497]Note that the above information implies in particular that $\varphi^{-1}(B^n_\epsilon) = g(B^n \times N)$.

page 2008, the map $\rho_{(N,g)}$ is given as follows:[1498]

$$\begin{array}{rcl} \rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} & \to & S^n = \overline{B}^n/S^{n-1} \\ P & \mapsto & \begin{cases} x, & \text{if } P = g(x,y) \text{ with } (x,y) \in B^n \times N, \\ *, & \text{otherwise.} \end{cases} \end{array}$$

We now verify that ψ does indeed agree with $\rho_{(N,g)}$. Thus first suppose we are given a point $P = g(x,y)$ with $(x,y) \in B^n \times N$. We have

$$\psi(P) \underset{\substack{\uparrow \\ \text{definition of } \psi}}{=} F_1(\varphi(g(x,y))) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{commutative diagram}}}{=} F_1(\epsilon \cdot x) \underset{\substack{\uparrow \\ \text{by Property (2) of } F_1 \\ \text{and since } \epsilon \cdot x \in B^n_\epsilon}}{=} \tfrac{1}{\epsilon} \cdot \epsilon \cdot x = x = \rho_{(N,g)}(g(x,y)) = \rho_{(N,g)}(P).$$

Finally suppose that we are given a point $P \in \overline{B}^{n+k}/S^{n+k-1}$ that does not lie in the subset $g(B^n \times N)$. In this case we calculate that

$$\psi(P) \underset{\substack{\uparrow \\ \text{definition of } \psi}}{=} F_1(\varphi(P)) \underset{\substack{\uparrow \\ \text{since } \varphi^{-1}(B^n_\epsilon) = g(B^n \times N) \\ \text{and by choice of } P}}{=} F_1(\text{point that does not lie in } B^n_\epsilon) \underset{\substack{\uparrow \\ \text{by Property (3) of } F_1}}{=} * = \rho_{(N,g)}(P).$$

This concludes the proof of the claim. □

87.3. **The fattened cobordism groups and the Thom-Pontryagin Theorem.** Recall that the Thom-Pontryagin construction from page 2008 makes the assignment

$$\text{fattened } k\text{-dimensional submanifold of } \mathbb{R}^{n+k} \quad \rightsquigarrow \quad \text{map } (S^{n+k}, *) \to (S^n, *).$$

We now would like to answer Question 87.3 (2), i.e. we would like to determine when do two fattened k-dimensional submanifolds give rise to the same homotopy class of maps $(S^{n+k}, *) \to (S^n, *)$. Loosely speaking we would like to fill in the left-hand side of

$$?? \quad \rightsquigarrow \quad \text{map } (S^{n+k} \times [0,1], * \times [0,1]) \to (S^n, *).$$

A quick look at the Thom-Pontryagin construction suggests that the left-hand side should be filled in with a suitable fattened $(k+1)$-dimensional submanifold of $\mathbb{R}^{n+k} \times [0,1]$.

This makes sense, except that in Proposition 3.42 we saw that for products of two manifolds it is advisable to have the manifold with boundary as a first factor. Thus we will consider submanifolds of $[0,1] \times \mathbb{R}^{n+k}$.[1499]

Before we continue we introduce the following straightforward convention.

Convention. Given a topological space X and an interval I and some $t \in I$ we make at times the identification $X = \{t\} \times X$ via the obvious bijection.

[1498] Eyes which profited from a carrot-rich diet might notice that the description is not verbatim the same as on page 2008. But with either description of the map the points $P = g(x,y)$ with $x \in S^{n-1}$ and $y \in N$ get sent to $*$. So we really do consider the same map as before.

[1499] Perhaps in hindsight we should have defined homotopies as maps $[0,1] \times X \to Y$ instead of defining them as maps $X \times [0,1] \to Y$?

The above observation leads us to the following definition which looks awfully similar to the definition of the notion of a cobordism of manifolds that we introduced on page 1760.

Definition. Let $k, m \in \mathbb{N}_0$.

(1) Let N_0 and N_1 be two closed orientable k-dimensional submanifolds of $\mathbb{R}^m$. A *cobordism between* N_0 *and* N_1 is a compact orientable proper $(k+1)$-dimensional submanifold W of $[0,1] \times \mathbb{R}^m$ with $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$.[1500]

(2) Let (N_0, g_0) and (N_1, g_1) be two fattened k-dimensional submanifolds of $\mathbb{R}^m$. A *fattened cobordism between* (N_0, g_0) *and* (N_1, g_1) is a fattened $(k+1)$-dimensional submanifold (W, G) of $[0,1] \times \mathbb{R}^m$ where W is a cobordism between N_0 and N_1 and G is a fattening of W with the following two properties:[1501]
   (a) $G|_{\{0\}\times\overline{B}^m} = g_0$ under the identification $\{0\} \times \overline{B}^m = \overline{B}^m$,
   (b) $G|_{\{1\}\times\overline{B}^m} = g_1$ under the identification $\{1\} \times \overline{B}^m = \overline{B}^m$.
   If such a fattened cobordism exists, then we say that (N_0, g_0) and (N_1, g_1) are *cobordant*.[1502] If (N, g) is cobordant to the empty submanifold, then we say N is a *fattened null-cobordism* and we say that (N, g) is *null-cobordant*. The definition of a fattened cobordism is illustrated in Figure 1064.

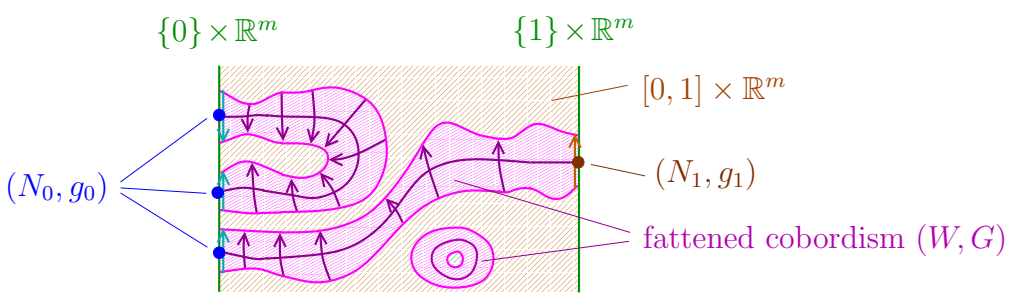


FIGURE 1064.

The following lemma shows that diffeotopies give rise to fattened cobordisms.

Lemma 87.7. *Let (M, g) be a fattened submanifold of $\mathbb{R}^m$ and let φ be a diffeomorphism of $\mathbb{R}^m$. If φ is diffeotopic to the identity, then $(\varphi(M), \varphi_*(g))$ is cobordant to (M, g).*

Example. Let $m \in \mathbb{N}$. Any map $\mathbb{R}^m \to \mathbb{R}^m$ of the form $x \mapsto r \cdot x + P$ for some $r > 0$ and $P \in \mathbb{R}^m$ is clearly diffeotopic to the identity. Thus Lemma 87.7 says that given a fattened submanifold (M, g) we can translate and shrink it without changing the fattened cobordism class.

[1500] Note that we work with submanifolds that are orientable, but we do not assume that they are equipped with an orientation.

[1501] Note that it follows from property (d) of a fattened submanifold, see page 2007, that we know that in fact $G|_{\{0\}\times\overline{B}^m} \subset \{0\} \times \mathbb{R}^m$ and $G|_{\{1\}\times\overline{B}^m} \subset \{1\} \times \mathbb{R}^m$.

[1502] We spare the reader the even more logical name "fattened cobordant".

Proof. Let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a fattened k-dimensional submanifold of $\mathbb{R}^m$ and furthermore let $F\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ be a diffeotopy from the identity to a diffeomorphism φ. It is straightforward to verify that the map

$$\begin{array}{rcl} [0,1] \times M & \to & [0,1] \times \mathbb{R}^m \\ (t, P) & \mapsto & (t, F(P,t)) \end{array}$$

is an embedding. We write $W := F([0,1] \times M)$. It is straightforward to verify that W together with the fattening given by

$$\begin{array}{rcl} \overline{B}^{m-k} \times W & \to & [0,1] \times \mathbb{R}^m \\ (v, (t, F(P,t))) & \mapsto & (t, F(g(v,P), t)) \end{array}$$

is a fattened cobordism between (M, g) and $(\varphi(M), \varphi_*(g))$. □

Lemma 87.8. *"Being cobordant" is an equivalence relation on the set of fattened k-dimensional submanifolds of $\mathbb{R}^m$.*

Sketch of proof. To simplify the discussion we ignore throughout the proof the fattenings. The reader should have no major troubles modifying the proof to also include the fattenings.

It is straightforward to see that "being cobordant" is reflexive and symmetric: Indeed, given a closed k-dimensional submanifold N a cobordism from N to itself is given by $[0,1] \times N \subset [0,1] \times \mathbb{R}^m$. Furthermore, suppose there exists a cobordism W between N_0 and N_1. "Flipping" W along $\mathbb{R}^m \times \{\frac{1}{2}\}$ we obtain a cobordism between N_1 and N_0.

The proof of transitivity takes more efforts. Initially the idea is quite simple. Given a cobordism V from N_0 to N_1 and another cobordism W from N_1 to N_2 one would like to stick them together to obtain a cobordism from N_0 to N_2. Unfortunately, as we had discussed on page 342, the resulting object is in general not a (smooth) submanifold of the manifold $\mathbb{R}^m$.

Fortunately we had already developed all the tools to deal with the situation. More precisely, in Proposition 10.16 we saw that we can modify the cobordisms V and W to obtain cobordisms V' and W' that are products on $[0, \epsilon] \times \mathbb{R}^m$ and $[1-\epsilon, 1] \times \mathbb{R}^m$ for a suitable $\epsilon \in (0, \frac{1}{2})$. Now we can stick V' and W' together to obtain the desired cobordism X. We refer to Figure 1065 for an illustration. □

The previous lemma allows us to make the following definition.

Definition. We denote by $\Omega_k^{\text{fat}}(\mathbb{R}^m)$ the set of fattened cobordism classes of fattened k-dimensional submanifolds of $\mathbb{R}^m$. In light of the subsequent Proposition 87.9 we refer to $\Omega_k^{\text{fat}}(\mathbb{R}^m)$ as a *fattened cobordism group.*

The definition of $\Omega_k^{\text{fat}}(\mathbb{R}^m)$ is reminiscent of the definition of the cobordism group Ω_k that we gave on page 1762. The group structure on Ω_k that we had introduced in Lemma 76.2 is given by "disjoint union". We will now see that $\Omega_k^{\text{fat}}(\mathbb{R}^m)$ also admits a group structure that is defined quite similarly.

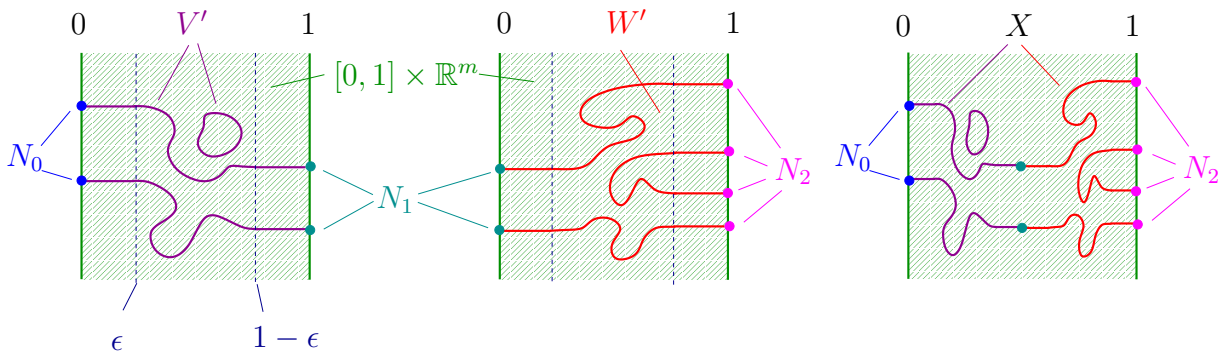


FIGURE 1065. Illustration for the proof of Lemma 1065.

Proposition 87.9. *Let $k \in \mathbb{N}_0$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, g) and (N, h) be two fattened k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $g(\overline{B}^{m-k} \times M) \subset B^m_r(0)$ and $h(\overline{B}^{m-k} \times N) \subset B^m_s(0)$. Furthermore we pick two orientation-preserving embeddings $\varphi\colon \overline{B}^m_r(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}^m_s(0) \to \mathbb{R}^m$ with disjoint images. We define*

$$[(M, g)] + [(N, h)] \;:=\; [(\varphi(M) \sqcup \psi(N), \varphi_*(g) \sqcup \psi_*(h))] \;\in\; \Omega^{\text{fat}}_k(\mathbb{R}^m).$$

(The definition of the operation "+" is illustrated in Figure 1066.)

(1) *The definition of "+" on $\Omega^{\text{fat}}_k(\mathbb{R}^m)$ is well-defined and commutative.*
(2) *The operation "+" defines a group structure on $\Omega^{\text{fat}}_k(\mathbb{R}^m)$. Here the following two statements hold:*
   (a) *The neutral element is represented by the empty submanifold.*[1503]
   (b) *Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, g)]$ is given by $[(\rho(M), \rho_*(g))]$.*

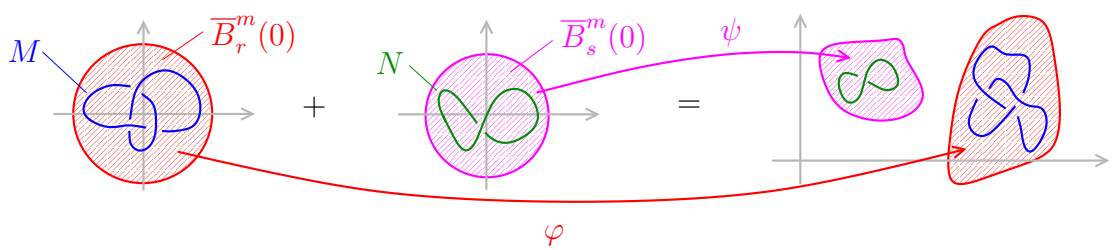


FIGURE 1066.

Proof.

(1) It follows from Theorem 10.28 and Lemma 87.7 that "+" is well-defined and commutative. Note that here we use that $m \geq 2$.

[1503]It is worth remembering that the empty set is a submanifold of any dimension.

(2) It is basically obvious that "+" is associative and that the empty submanifold is a neutral element. Thus it remains to prove the statement regarding the inverse. It follows from Lemmas 87.7 and 9.4 that without loss of generality we can assume that the reflection $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ is the reflection in the $(x_m = 0)$-hyperplane. So let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a fattened k-dimensional submanifold of $\mathbb{R}^m$. We set $\epsilon = \frac{1}{4}$. It follows immediately from the discussion on page 2017 that we can assume that $g(\overline{B}^{m-k} \times M)$ is contained in the ball $\overline{B}^m_\epsilon(\underbrace{(0,\dots,0,2\epsilon)}_{\in\mathbb{R}^m})$. We consider the map

$$\begin{array}{rcl} \varphi\colon \overline{B}^{m-k} \times (M \times [0,\pi]) & \to & [0,1] \times \mathbb{R}^m \\ (v,(P,\gamma)) & \mapsto & \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & \mathrm{id}_{m-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ g(v,P) \end{pmatrix}}_{\in\mathbb{R}^{m+1}}. \end{array}$$

Using that $g(\overline{B}^{m-k} \times M)$ is contained $\overline{B}^m_\epsilon((0,\dots,0,2\epsilon))$ we can make the following three observations:

(a) The image of φ does indeed lie in $[0,1] \times \mathbb{R}^m$.
(b) The restriction to $\{0\} \times M \times [0,\pi]$ is a proper embedding. Therefore we can now consider the compact orientable proper $(k+1)$-dimensional submanifold $W = \varphi(\{0\} \times M \times [0,\pi])$ of $[0,1] \times \mathbb{R}^m$.
(c) The map φ gives rise to a fattening for W.

It is now basically clear that we have constructed a fattened cobordism from the fattened submanifold $(M \sqcup \rho(M), g \sqcup \rho_*(g))$ to the empty submanifold. (We refer to Figure 1067 for an illustration.) But by the definition of "+" this implies that $(\rho(M), \rho_*(g))$ represents the inverse of (M, g). □

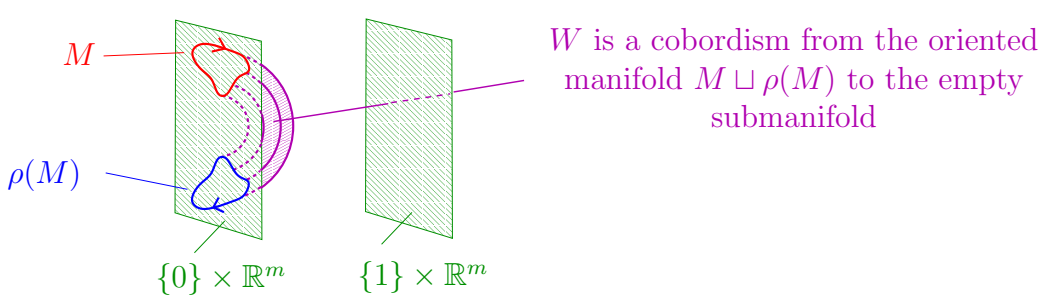


FIGURE 1067.

Evidently our challenge now is to compute the groups $\Omega_k^{\mathrm{fat}}(\mathbb{R}^m)$ for suitable values of m and k. In general this might be quite difficult, but the following proposition gives a reasonably amusing calculation.

Proposition 87.10. *Let $m \geq 2$. The map*[1504]

$$\begin{array}{rcl} \varphi\colon \Omega_0^{\mathrm{fat}}(\mathbb{R}^m) & \to & \mathbb{Z} \\ [(\{P_1\}, g_1) \sqcup \cdots \sqcup (\{P_k\}, g_k)] & \mapsto & \sum_{i=1}^{k} \operatorname{sign}(g_i) \end{array}$$

is well-defined and it is an isomorphism.

Sketch of proof. First we need to show that $\varphi\colon \Omega_0^{\mathrm{fat}}(\mathbb{R}^m) \to \mathbb{Z}$ is well-defined. It suffices to show that for any *connected* fattened 1-dimensional cobordism $(Z, V) \subset [0,1] \times \mathbb{R}^m$ between two 0-dimensional fattened submanifolds (N_0, g_0) and (N_1, g_1) we have the equality $\varphi([(N_0, g_0)]) = \varphi([(N_1, g_1)])$. This statement follows from the classification of compact 1-dimensional manifolds, see Proposition 3.53, and some orientation considerations. Since writing down the precise arguments regarding the orientations will only obfuscate the argument we refer to Figure 1068 for convincing pictures, and we leave it to the reader to fill in the details.

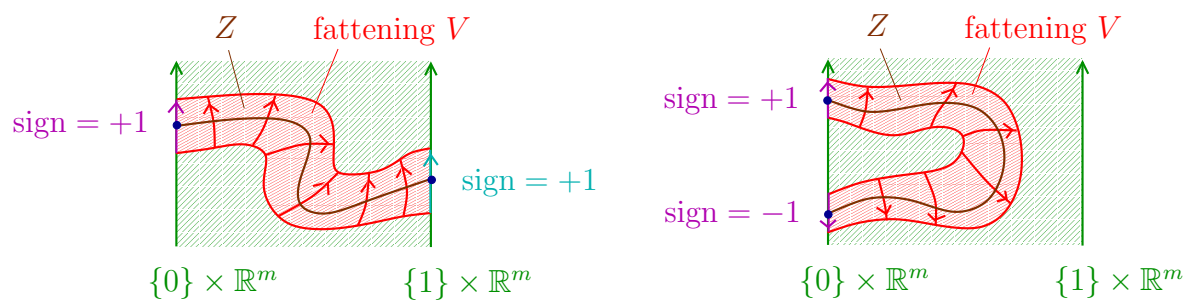


FIGURE 1068. Illustration for the proof of Proposition 87.10.

Next note that the map $\varphi\colon \Omega_0^{\mathrm{fat}}(\mathbb{R}^m) \to \mathbb{Z}$ is, basically by definition, a homomorphism. Finally note that it is clear that φ is an epimorphism. Thus it remains to show that φ is a monomorphism. A straightforward induction argument shows that it suffices to prove the following claim.

Claim. For points $P, Q \in \mathbb{R}^m$ and fattenings g and h of P and Q with $\operatorname{sign}(g) = -\operatorname{sign}(h)$ we have $[(P, g)] + [(Q, h)] = 0 \in \Omega_0^{\mathrm{fat}}(\mathbb{R}^m)$.

In the following let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. By the discussion on page 2017 we can assume, after possibly applying a translation, that $Q = \rho(P)$. Since ρ is orientation-reversing we see that $\operatorname{sign}(\rho_*(g)) = -\operatorname{sign}(g) = \operatorname{sign}(h)$. Now we calculate that

$$\begin{array}{ccccccc} [(P,g)] + [(Q,h)] & = & [(P,g)] + [(\rho(P),h)] & \underset{\uparrow}{=} & [(P,g)] + [(\rho(P),\rho_*(g))] & \underset{\uparrow}{=} & 0 \in \Omega_0^{\mathrm{fat}}(\mathbb{R}^m). \\ & & & \text{by } \operatorname{sign}(h) = \operatorname{sign}(\rho_*(g)), \text{ Theorem 10.28 and Lemma 87.7} & & \text{by Proposition 87.9 (2b)} & \end{array}$$

This concludes the proof of the claim. □

[1504]We refer to page 2011 for the definition of the sign of a fattening of a point.

Considering the discussion on page 2016, the following theorem does not come as a big surprise. It justifies our interest in the groups $\Omega_k^{\text{fat}}(\mathbb{R}^m)$.

Theorem 87.11. (Thom-Pontryagin Theorem) *Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. The map*

$$\begin{array}{rcl} \Psi\colon \Omega_k^{\text{fat}}(\mathbb{R}^{n+k}) & \to & \pi_{n+k}(S^n, *) \\ [(N, g)] & \mapsto & [\rho_{(N,g)}] \end{array}$$

is well-defined and it is an isomorphism.

Sketch of proof. Fortunately we have already done basically all the work:

(1) First we argue that Ψ is well-defined. Thus suppose that we are given fattened k-dimensional submanifolds (M, g) and (N, h) of $\mathbb{R}^{n+k}$ that are cobordant. We pick a fattened cobordism $(W, F\colon \overline{B}^n \times W \to [0, 1] \times \mathbb{R}^{n+k})$ in $[0, 1] \times \mathbb{R}^{n+k}$. The corresponding map

$$\rho_{(W,F)}\colon [0, 1] \times \mathbb{R}^{n+k} \to S^n$$

has the property that it sends a neighborhood of $[0, 1] \times \{\infty\}$ to $*$. Therefore it follows easily from Lemma 1.32 (2) that the following map is continuous:

$$\begin{array}{rcl} S^{n+k} \times [0, 1] = (\mathbb{R}^{n+k} \cup \{\infty\}) \times [0, 1] & \to & S^n \\ (P, t) & \mapsto & \begin{cases} \rho_{(W,F)}(t, P), & \text{if } P \in \mathbb{R}^{n+k}, \\ *, & \text{if } P = \infty. \end{cases} \end{array}$$

It follows from the above that this map is in fact a homotopy rel $*$ between the map $\rho_{(M,g)}\colon (S^{n+k}, *) \to (S^n, *)$ and the map $\rho_{(N,h)}\colon (S^{n+k}, *) \to (S^n, *)$. In particular we have $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n, *)$.

(2) It follows immediately from Lemma 87.5 that Ψ is a homomorphism.

(3) The statement of Proposition 87.6 is equivalent to saying that Ψ is an epimorphism.

(4) Finally it remains to show that Ψ is in fact a monomorphism. So suppose that we are given two fattened k-dimensional submanifolds (M, g) and (N, h) of $\mathbb{R}^{n+k}$ such that $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n, *)$, i.e. such that there exists a homotopy $F\colon S^{n+k} \times [0, 1] \to S^n$ rel $*$ from $\rho_{(M,g)}$ to $\rho_{(N,h)}$.

The proof of Proposition 87.6 can be modified without too many problems to show that the homotopy F gives rise to a fattened cobordism between (M, g) and (N, h). We leave it to the reader to fill in the details. See also [Kos, Theorem 9.5.5] or [Bre, Chapter 16] for similar claims stated with more authority. □

From our above results we obtain the following corollary. Of course we proved this result earlier in Corollary 80.6, using very different methods.

Corollary 87.12. *For any $m \geq 2$ the map*

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_m(S^m, *) \\ n & \mapsto & n \cdot [\mathrm{id}_{S^m}] \end{array}$$

is an isomorphism.

Proof. Let $P \in \mathbb{R}^m$ and let g be a fattening for the submanifold $\{P\}$. We consider the two maps

$$\begin{array}{rcl} \mathbb{Z} & \to & \Omega_0^{\mathrm{fat}}(\mathbb{R}^m) \\ n & \mapsto & n \cdot [(\{0\}, g \text{ with } \mathrm{sign}(g) = 1)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Omega_0^{\mathrm{fat}}(\mathbb{R}^m) & \to & \pi_m(S^m, *) \\ [(N, g)] & \mapsto & [\rho_{(N,g)}]. \end{array}$$

The first map is an isomorphism by Proposition 87.10 and the second map is an isomorphism by the Thom-Pontryagin Theorem 87.11, applied to $k = 0$ and $m \geq 2$. Thus the composition of the two maps is an isomorphism. Finally it follows from Lemma 87.1 that the composition of these two maps sends n to $n \cdot [\mathrm{id}_{S^m}]$. □

87.4. **The calculation of the fattened cobordism group $\Omega_1^{\mathrm{fat}}(\mathbb{R}^3)$.** Surely the most interesting calculation of a higher homotopy group that we have performed so far is that in Theorem 85.10 (2) we saw $\pi_3(S^2) \cong \mathbb{Z}$. Our goal now is to give a new proof of this fact using the Thom-Pontryagin Theorem 87.11. Thus we need to show that $\Omega_1^{\mathrm{fat}}(\mathbb{R}^3) \cong \mathbb{Z}$.

We start out with the following slightly subtle definition.

Definition. Let $m \in \mathbb{N}$.

(1) Let $(M, v\colon \overline{B}^{m-k} \times \mathbb{R}^m)$ be a fattened $(m-k)$-dimensional submanifold of $\mathbb{R}^m$. The *natural orientation of M* is the unique orientation that turns v into an orientation-preserving map.[1505]
(2) Similarly we define the natural orientation of a fattened cobordism.

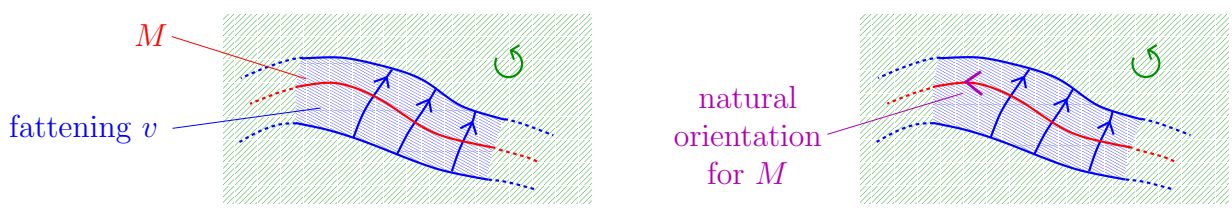


FIGURE 1069.

This definition leads us to the following convention.

Convention. In this section we equip any fattened submanifold and any fattened cobordism with the natural orientation.

Before we write down an explicit isomorphism we introduce the following definition which generalizes the definition of the self-linking number of a fattened knot that we introduced on page 1626.

Definition.

[1505]Recall that M is assumed to be orientable, but a priori it does not come with an orientation. It is not difficult to see that there exists a unique orientation on M such that, if we equip $\overline{B}^{m-k} \times M$ with the product orientation, v becomes orientation-preserving.

(1) Let $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ be two disjoint oriented links. We define the *linking number of K and L* as
$$\mathrm{lk}(K, L) \;=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{lk}(K_i, L_j).$$

(2) Let (L, g) be a fattened link. We write $L' = g(\{1\} \times L)$. We define the *self-linking number of* (L, g) as $\mathrm{slk}(L, g) := \mathrm{lk}(L, L')$.[1506]

The following elementary lemma is at times useful for determining self-linking numbers of fattened links.

Lemma 87.13. *If $(L, g) = (L_1 \sqcup \cdots \sqcup L_m, g_1 \sqcup \cdots \sqcup g_m)$ is a fattened link, then*
$$\mathrm{slk}(L, g) \;:=\; \sum_{i \neq j} \mathrm{lk}(L_i, L_j) \;+\; \sum_{i=1}^{m} \mathrm{slk}(L_i, g_i).$$

Proof. We will prove the lemma in Exercise 87.2. □

Now we can give the promised isomorphism.

Theorem 87.14. *The map*
$$\begin{array}{rcl} \Psi\colon \Omega_1^{\mathrm{fat}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ [(L, g)] & \mapsto & \textit{self-linking number } \mathrm{slk}(L, g) \end{array}$$
is well-defined and it is an isomorphism. In particular we have $\pi_3(S^2) \cong \mathbb{Z}$.

Examples.

(1) We consider the trivial knot K with a fattening g of self-linking number $+1$. By Theorem 87.14 the fattened knot (K, g) defines a generator of $\Omega_1^{\mathrm{fat}}(\mathbb{R}^3)$. It follows from the isomorphism provided by the Thom-Pontryagin Theorem 87.11 that $\rho_{(K,g)}\colon S^3 \to S^2$ represents a generator of $\pi_3(S^2, *)$. In Lemma 87.4 we saw that $\rho_{(K,g)}$ represents the same element of $\pi_3(S^2, *)$ as the Hopf map $H\colon S^3 \to S^2$. Thus we see once again that the Hopf map represents a generator of $\pi_3(S^2)$.

(2) We consider the Hopf link $L = L_1 \sqcup L_2$ with the orientations shown in Figure 1070. We equip $L = L_1 \sqcup L_2$ with a "trivial fattening" $g_1 \sqcup g_2$, i.e. where the fattening g_i of each component L_i has self-linking number 0. We see that
$$\mathrm{slk}(L_1 \sqcup L_2, g_1 \sqcup g_2) \underset{\substack{\uparrow \\ \text{Lemma 87.13}}}{=} \mathrm{lk}(L_1, L_2) + \mathrm{lk}(L_2, L_1) + \overbrace{\mathrm{slk}(L_1, g_1)}^{=0} + \overbrace{\mathrm{slk}(L_2, g_2)}^{=0} \underset{\substack{\uparrow \\ \text{pick your favorite method from Section 70.4}}}{=} 1 + 1 = 2.$$
This shows that the Hopf link, perhaps slightly counter-intuitively, gives rise to *twice* the generator of $\pi_3(S^2, *)$.

[1506] Here we use the above convention to view L as an *oriented* submanifold, and we equip $L' = g(\{1\} \times L)$ with the obvious corresponding orientation.

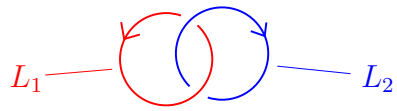


FIGURE 1070.

Since we already have an independent proof of Theorem 87.14 we will take the liberty to cut a few corners in the new proof of the theorem. The new proof of Theorem 87.14 will build on the following four lemmas. The first two lemmas generalize Proposition 70.8 and Lemma 70.17 from knots to links. They are perhaps interesting in their own right.

Lemma 87.15. *Given any oriented link $L \subset S^3$ there exists a compact oriented connected surface $F \subset S^3$ such that $\partial F = L$.*[1507]

Proof of Lemma 87.15. Let $L \subset S^3$ be an oriented link. A modest generalization of Proposition 70.8 shows that there exists a compact oriented surface $G \subset S^3$ with $\partial G = L$. In Exercise 87.5 we will show that we can turn G into a compact oriented *connected* surface F with $\partial F = L$.[1508] □

Lemma 87.16. *If $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ are two oriented links, then the following statements hold:*

(1) *There exist compact oriented proper 2-dimensional submanifolds S and T of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$ such that S and T are transverse.*
(2) *Let S, T be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$. If S and T are transverse, then*[1509]

$$\mathrm{lk}(K, L) \;=\; S \cdot_{\overline{B}^4} T.$$

Example. As on page 965 we consider the Hopf link that is illustrated in Figure 1071 on the left. On the right of Figure 1071 we show a compact oriented connected surface whose boundary equals the Hopf link.

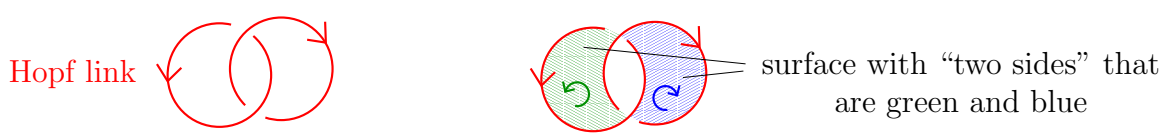


FIGURE 1071.

[1507]Here the equality $\partial F = L$ is understood to be an equality of *oriented* submanifolds.

[1508]Frankly it can be slightly annoying to give a completely rigorous proof that such a surface F exists. This is one of the main reasons why we made the caveat above that we are going to cut a few corners in the argument.

[1509]Here $S \cdot_{\overline{B}^4} T$ denotes the algebraic intersection number of the oriented transverse surfaces S and T in the oriented manifold $\overline{B}^4$. We refer to page 1560 for the definition of the algebraic intersection number.

Proof of Lemma 87.16.

(1) The proof of this statement is virtually the same as the proof of Lemma 70.17, we just need to replace Proposition 70.8 by Lemma 87.15.
(2) We leave it to the reader to verify that the proof of Lemma 70.17 (2) can be modified to deal not only with knots but also with links. □

The following two lemmas that we need for the proof of Theorem 87.14 are arguably more technical.

Lemma 87.17. *Let K be an oriented* knot *in $\mathbb{R}^3$ and let g and h be two fattenings for K in $\mathbb{R}^3$. If $\operatorname{slk}(K,g)=\operatorname{slk}(K,h)$, then $[(K,g)]=[(K,h)]\in\Omega_1^{\text{fat}}(\mathbb{R}^3)$.*

Proof of Lemma 87.17(∗). Since $\operatorname{slk}(K,g)=\operatorname{slk}(K,h)$ it follows from Lemma 70.6 that there exists a smooth isotopy $F\colon(\overline{B}^2\times K)\times[0,1]\to\mathbb{R}^3$ between g and h which has compact support and which has the property that $F(0,k,t)=k$ for all $k\in K$ and all $t\in[0,1]$. We consider $W=[0,1]\times K$ with the fattening given by

$$\begin{array}{rcl} j\colon\overline{B}^2\times([0,1]\times K) & \to & [0,1]\times\mathbb{R}^3\\ (v,(t,x)) & \mapsto & (t,F((v,x),t)).\end{array}$$

The fattened manifold (W,j) is a fattened cobordism between (K,g) and (K,h). □

Lemma 87.18. *Let W be a compact oriented proper connected 2-dimensional submanifold of $\mathbb{R}^3\times[0,1)$ with boundary components $L_1,\dots,L_m$. Given $s_1,\dots,s_{m-1}\in\mathbb{Z}$ there exists a fattening h for W such that for $i=1,\dots,m-1$ we have $\operatorname{slk}(L_i,h|_{L_i})=s_i$.*

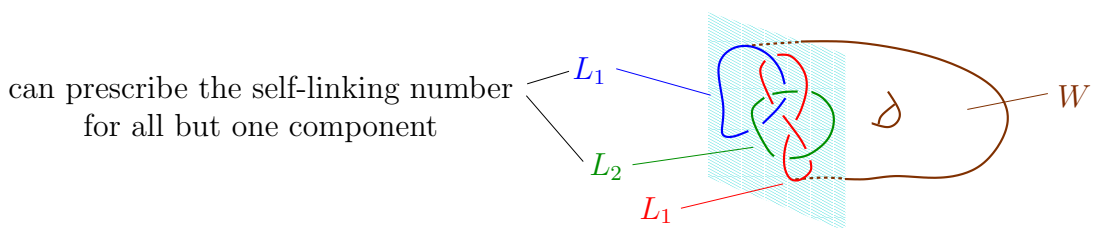


FIGURE 1072.

Proof. Evidently we can assume that $m\geq 1$. In this case W is a compact oriented proper connected 2-dimensional submanifold with non-empty boundary. Thus by Proposition 64.8 there exists a fattening $f\colon\overline{B}^2\times W\to\mathbb{R}^3\times[0,1)$ for W. For $i=1,\dots,m-1$ we write $r_i:=\operatorname{slk}(L_i,g|_{L_i})$.

By Lemma 14.20 there exists a map $\varphi\colon W\to L_1\vee\dots\vee L_{m-1}$[1510] that is the identity on each L_i. We pick a map $\psi\colon L_1\vee\dots\vee L_{m-1}\to\mathrm{SO}(2)=S^1$ with the following property:

[1510]Here, as a reminder, $L_1\vee\dots\vee L_{m-1}$ denotes the wedge of the topological spaces $L_1,\dots,L_{m-1}$.

under a orientation-preserving identification $L_i = S^1$ the restriction of ψ to L_i is given by $z \mapsto z^{s_i - r_i}$. Now we consider the fattening

$$\begin{aligned} h\colon \overline{B}^2 \times W &\to W \\ (v, P) &\mapsto f(\underbrace{(\overbrace{\psi \circ \varphi)(P)}^{\in \mathrm{SO}(2)} \cdot v, P)}_{\in \overline{B}^2 \times W}). \end{aligned}$$

By Lemma 70.6 (1) we know that for any $i \in \{1, \dots, m-1\}$ we added $s_i - r_i$ to the self-linking number of L_i. Thus the fattening h has all the desired properties. $\square$

Now we are in a position to provide the proof of Theorem 87.14.

Sketch of proof of Theorem 87.14.

(1) First we need to show that $\Psi\colon \Omega_1^{\mathrm{fat}}(\mathbb{R}^3) \to \mathbb{Z}$ is well-defined. In the proof of this fact we borrow some ideas from the proof of Lemma 70.17 (2). So let (W, F) be a fattened cobordism between (K, g) and (L, h). We write $W' = F(\{1\} \times W)$, $K' = g(\{1\} \times K)$ and $L' = h(\{1\} \times L)$. By Lemma 87.16 (1) there exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse. Similarly there exist T, T' for L and L'.

We can make the identification $S^4 = (-\overline{B}^4) \cup [0,1] \times S^3 \cup \overline{B}^4$[1511] (See Figure 1073 for an illustration.) Using Proposition 10.17 we can arrange that both $(-S) \cup W \cup T$ and $(-S') \cup W' \cup T'$ are oriented submanifolds of the manifold S^4. We calculate that

$$\begin{aligned} -\operatorname{slk}(K,g) + \operatorname{slk}(L,h) &\overset{\substack{\text{by definition of slk} \\ \downarrow}}{=} -\operatorname{lk}(K,K') + \operatorname{lk}(L,L') \overset{\substack{\text{Lemma 87.16 (2)} \\ \downarrow}}{=} (-S) \cdot_{\overline{B}^4} S' + T \cdot_{\overline{B}^4} T' \\ = (-S) \cdot_{-\overline{B}^4} (-S') + T \cdot_{\overline{B}^4} T' &\underset{\substack{\uparrow \\ \text{since the only intersections appear} \\ \text{between } S, S' \text{ and } T, T', \text{ here we use that} \\ W \text{ and } W' \text{ are disjoint}}}{=} (-S \cup W \cup T) \cdot_{S^4} (-S' \cup W' \cup T') \underset{\substack{\uparrow \\ \text{by Theorem 68.11 and} \\ \text{since } H_2(S^4;\mathbb{Z}) = 0}}{=} 0. \end{aligned}$$

We have thus shown that the self-linking numbers of (K, g) and (L, h) agree. This shows that the map $\Psi\colon \Omega_1^{\mathrm{fat}}(\mathbb{R}^3) \to \mathbb{Z}$ is indeed well-defined.

(2) It follows easily from Exercise 70.6 that Ψ is a homomorphism.

(3) By Lemma 70.6 there exists a knot in $\mathbb{R}^3$ with a fattening in $\mathbb{R}^3$ with self-linking number one, thus Ψ is an epimorphism.

(4) It remains to show that $\Psi\colon \Omega_1^{\mathrm{fat}}(\mathbb{R}^3) \to \mathbb{Z}$ is a monomorphism. So suppose we are given a fattened m-component link $(L, g) = \bigsqcup_{i=1}^{m} (L_i, g_i)$ with $\Psi([(L, g)]) = 0$.

[1511] If the reader does not feel like making this identification, then we point out that we only need that the right-hand side is a manifold and that the second homology vanishes.

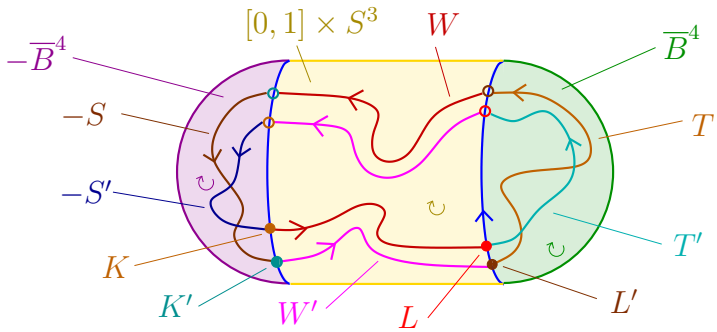


FIGURE 1073. Illustration for the proof of Theorem 87.14.

By Lemma 87.15 there exists a compact oriented connected surface W in $\mathbb{R}^3\times[0,1)$ whose boundary equals $L\subset\mathbb{R}^3=\mathbb{R}^3\times\{0\}$. By Lemma 87.18 we can equip W with a fattening h such that for $i=1,\dots,m-1$ we have $\operatorname{slk}(L_i,h|_{L_i})=\operatorname{slk}(L_i,g_i)$.

Claim. We also have $\operatorname{slk}(L_m,h|_{L_m})=\operatorname{slk}(L_m,g_m)$.

Note that (W,h) is a fattened cobordism between $(L,h|_L)$ and the empty link. In particular we obtain from (1) that $\operatorname{slk}(L,h|_L)=0$. By our hypothesis on (L,g) we also have $\operatorname{slk}(L,g)=0$. We had already arranged that for $i=1,\dots,m-1$ we have $\operatorname{slk}(L_i,h|_{L_i})=\operatorname{slk}(L_i,g_i)$. It now follows immediately from Lemma 87.13 that $\operatorname{slk}(L_m,h|_{L_m})=\operatorname{slk}(L_m,g_m)$. This concludes the proof of the claim.

Now we are done, since

$$\begin{array}{ccccccc}[(L,g)] & = & [(L,h|_g)] & = & 0 & \in & \Omega_1^{\text{fat}}(\mathbb{R}^3).\\ & \uparrow & & \uparrow & & & \\ & \text{Lemma 87.17} & & \text{since } (W,h) \text{ is a null-cobordism for } (L,h|_g) & & & \end{array}$$

□

87.5. **The fattened cobordism group $\Omega_{2n-1}^{\text{fat}}(\mathbb{R}^{4n-1})$.** In Corollary 63.14 we had sketched a proof that given any $n\in\mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n})\to\mathbb{Z}$. By the Thom-Pontryagin Theorem 87.11 this statement is equivalent to saying that there exists an epimorphism $\Omega_{2n-1}^{\text{fat}}(\mathbb{R}^{4n-1})\to\mathbb{Z}$.

In this section we will give an alternative proof that such an epimorphism exists. Since we already have a proof for this statement we will only prove the most salient statements. In other words, this section is stuffed with entertaining exercises for the reader.

First of all we work with the same convention as in the last section.

Convention. In this section we equip any fattened submanifold and any fattened cobordism with the natural orientation that we introduced on page 2023.

We continue with the following notation that is germane to this section.

Notation. Let $n\in\mathbb{N}$.

(1) Given a closed oriented m-dimensional submanifold K of S^{2m+1} we denote by
$$\beta_K\colon H_m(K;\mathbb{Z}) \;\to\; H^m(S^{2m+1}\setminus K;\mathbb{Z})$$
the inverse of the isomorphism given by the Alexander Duality Theorem 66.5.

(2) Given a compact oriented proper $(m+1)$-dimensional submanifold W of $[0,1]\times S^{2m+1}$ we denote by
$$\beta_W\colon H_{m+1}(W,\partial W;\mathbb{Z}) \;\to\; H^m(([0,1]\times S^{2m+1})\setminus W;\mathbb{Z})$$
the inverse of the isomorphism given by the Alexander Duality Theorem 66.5.

With this notation we can introduce the following definition.

Definition. Let $m\in\mathbb{N}$ and let K and L be two disjoint closed oriented m-dimensional submanifolds of S^{2m+1}. We define the *linking number of K and L* as
$$\mathrm{lk}(K,L) \;=\; \langle\beta_K([K]),[L]\rangle \;\in\; \mathbb{Z}$$
where we view the fundamental class $[L]$ of L as a homology class in $H_m(S^{2m+1}\setminus K;\mathbb{Z})$.

Remark. The vigilant reader will not have failed to notice that in the 3-dimensional setting it is a priori not clear that the above definition of the linking number agrees with the definition given on page 1635. In Exercise 87.6 we will prove that in the 3-dimensional setting the two competing notions of linking numbers agree.

Proposition 87.19. *Let $n\in\mathbb{N}$ and let $*\in\overline{B}^{2n}$. The map*
$$\begin{aligned}\Psi\colon \Omega^{\mathrm{fat}}_{2n-1}(\mathbb{R}^{4n-1}) &\;\to\; \mathbb{Z}\\ [(N,g\colon \overline{B}^{2n}\times N\to\mathbb{R}^{4n-1})] &\;\mapsto\; \mathrm{lk}(N,g(*\times N))\end{aligned}$$
is a well-defined homomorphism.

Remark. Let $n\in\mathbb{N}$. We recall that on page 1473 we used the cup product to define the Hopf invariant
$$\mathrm{Hopf}\colon \pi_{4n-1}(S^{2n},*) \;\to\; \mathbb{Z}$$
which is, as shown in Proposition 63.7, in fact a homomorphism. The audacious reader can attempt to show that, under the above isomorphism $\Omega^{\mathrm{fat}}_{2n-1}(\mathbb{R}^{4n-1})\xrightarrow{\cong}\pi_{4n-1}(S^{2n},*)$, the Hopf invariant and the homomorphism $\Psi\colon\Omega^{\mathrm{fat}}_{2n-1}(\mathbb{R}^{4n-1})\to\mathbb{Z}$ from Proposition 87.19 agree up to sign.[1512]

Proof. In Exercise 87.7 we will verify that Ψ is a homomorphism. Now let us actually show that Ψ is well-defined. Thus suppose that we are given fattened $(2n-1)$-dimensional submanifolds (N_0,g_0) and (N_1,g_1) of $\mathbb{R}^{4n-1}$ that are cobordant via a fattened cobordism (W,G). We denote by $i_0\colon\{0\}\times S^{4n-1}\to[0,1]\times S^{4n-1}$ the inclusion. Similarly we define i_1. We also use i_0,i_1 in a rather obvious way for restrictions of the above inclusion maps

[1512]Attempting to determine the correct sign with confidence might have negative effects on one's mental health.

to suitable subsets. It follows immediately from the Alexander Duality Theorem 66.5 that we have the following commutative diagram

$$\begin{array}{ccc}
H_{2n}(W,\partial W;\mathbb{Z}) & \xrightarrow[\beta_W]{\cong} & H^{2n-1}(([0,1]\times S^{4n-1})\setminus W;\mathbb{Z}) \\
\Big\downarrow \partial & & \begin{matrix} i_0^* \\ i_1^* \end{matrix} \Big\downarrow \\
\begin{matrix} H_{2n-1}(\{0\}\times N_0;\mathbb{Z}) \\ H_{2n-1}(\{1\}\times N_1;\mathbb{Z}) \end{matrix} & \xrightarrow[\cong]{\begin{pmatrix} \beta_{N_0} & 0 \\ 0 & \beta_{N_1}\end{pmatrix}} & \begin{matrix} H^{2n-1}(\{0\}\times (S^{4n-1}\setminus N_0);\mathbb{Z}) \\ H^{2n-1}(\{1\}\times (S^{4n-1}\setminus N_1);\mathbb{Z}) \end{matrix}
\end{array}$$

where all horizontal maps are isomorphisms and where the left-hand vertical map is the connecting homomorphism of the long exact sequence in homology of the pair $(W,\partial W)$.

It follows that

$$\begin{array}{rcl}
\mathrm{lk}(N_0, g_0(*\times N_0)) & = & \langle \beta_{N_0}([N_0]), [g_0(*\times N_0)]\rangle \\
& = & \langle \beta_W([W]), (i_0)_*([g_0(*\times N_0)])\rangle \\
& \uparrow & \\
& \text{Lemma 50.6} & \text{it follows from } \partial(G(*\times W)) = g_0(*\times N_0)\cup g_1(*\times N_1) \text{ together with} \\
& & \text{Corollary 46.27 (2) that } (i_0)_*([g_0(*\times N_0)]) = (i_1)_*([g_1(*\times N_1)]) \\
& = & \mathrm{lk}(N_1, g_1(*\times N_1)). \\
& \uparrow & \\
& \text{same argument backwards} &
\end{array}$$

$$\begin{array}{rcl}
& & \text{follows from the above diagram and the fact} \\
& & \text{that } [\partial W] = [\{0\}\times N_0] + [\{1\}\times N_1], \text{ see Proposition 46.26} \\
& & \downarrow \\
& = & \langle i_0^*(\beta_W([W])), [g_0(*\times N_0)]\rangle \\
& = & \langle \beta_W([W]), (i_1)_*([g_1(*\times N_1)])\rangle \\
& & \uparrow
\end{array}$$

We have thus shown that Ψ is indeed well-defined. □

Of course an invariant is only interesting if we can show that it assumes non-trivial values. If $n=1$, then we already know[1513] that the trivial knot K together with the fattening g of self-linking number one satisfies $\Psi([(N,g)]) = 1$. In particular we see that $\Psi\colon \Omega_1^{\mathrm{fat}}(\mathbb{R}^3)\to\mathbb{Z}$ is an epimorphism.

Recall that the combination of Propositions 63.7 and 63.13 shows that that given any $n\in\mathbb{N}$ there exists a map $f\colon S^{4n-1}\to S^{2n}$ such that the Hopf invariant equals 2. The following proposition that can be viewed as an analogue of this fact.

Proposition 87.20. *For any $n\in\mathbb{N}$ there exists a fattened $(2n-1)$-dimensional submanifold (N,g) in $\mathbb{R}^{4n-1}$ with $\Psi([(N,g)]) = 2$.*

Sketch of proof. We consider the "trivial $(2n-1)$-dimensional knot"

$$K \;=\; S^{2n-1}\times\{0\} \;=\; \{(z_1,\dots,z_{2n},0,\dots,0)\in\mathbb{R}^{4n-1} \mid z_1^2+\dots+z_{2n}^2 = 1\}$$

[1513]Here we assume for convenience that we already know that the two different definitions of the linking pairing in the 3-dimensional setting agree.

together with the map[1514]

$$\begin{array}{rcl} g\colon \overline{B}^{2n} \times K & \to & \mathbb{R}^{4n-1} \\ \big((x, \underbrace{(y_1, \dots, y_{2n-1})}_{=:y\in\mathbb{R}^{2n-1}}), \underbrace{(z,0)}_{\in K}\big) & \mapsto & (z + \frac{1}{4}x \cdot z, y). \end{array}$$

One can easily verify that, with a suitable orientation for K, the pair (K, g) is a fattened $(2n-1)$-dimensional submanifold of $\mathbb{R}^{4n-1}$. It is not particularly difficult to show that $[g(* \times K)] = 0 \in H_{2n-1}(\mathbb{R}^{4n-1} \setminus K)$. Unfortunately this implies that $\operatorname{lk}(K, g(* \times K)) = 0$.

The first idea would be to modify the fattening g in a similar vein as in Lemma 70.6. This might work in principle, but at least your author does not know how to make this work in general.[1515] Instead we will recycle the example from page 2024 where we saw that the Hopf link gives rise to a non-trivial element in $\Omega_1^{\text{fat}}(\mathbb{R}^3)$. More precisely, we consider the diffeomorphism

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^{4n-1} & \to & \mathbb{R}^{4n-1} \\ (\underbrace{x_1, \dots, x_{2n}}_{=x}, \underbrace{y_1, \dots, y_{2n-1}}_{=y}) & \mapsto & (y, x) + (0, \dots, 0, \underset{\substack{\uparrow \\ 2n\text{-th coordinate}}}{1}, 0, \dots, 0). \end{array}$$

Similar to the discussion on page 2024 one can now fairly easily show (voluntary exercise!) that $\Psi([K \sqcup \Theta(K), g \sqcup \Theta_*(g)]) = \pm 2$. Since Ψ is a homomorphism we see that there exists a fattened $(2n-1)$-dimensional submanifold (N, g) in $\mathbb{R}^{4n-1}$ with $\Psi([(N, g)]) = 2$. $\square$

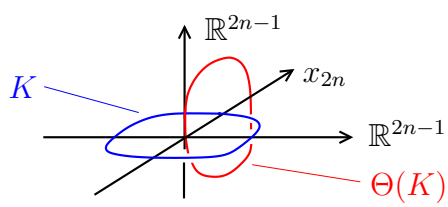


FIGURE 1074.

Remark. In Theorem 79.11 we saw that, unless $n = 2, 4, 8$, there does not exist a map $f\colon S^{2n-1} \to S^n$ with $\operatorname{Hopf}(f) = 1$. If we believe for a second that the Hopf invariant corresponds to the above homomorphisms Ψ, then this shows that Proposition 87.20 is optimal for $n \notin \{1, 2, 4\}$.

In summary, the Thom-Pontryagin Theorem 87.11 together with Propositions 87.19 and 87.20 give us the following result, which is precisely the content of Corollary 63.14.

Proposition 87.21. *Given any* even $n \in \mathbb{N}$ *there exists an epimorphism* $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

[1514] This map g is basically just the "obvious" fattening of $S^{2n-1} \subset \mathbb{R}^{2n}$ extended to higher dimensions.

[1515] Also the remark after the proof might temper one's optimism.

Exercises for Chapter 87.

Exercise 87.1. We consider the map

$$\begin{array}{rcl} f\colon (\overline{B}^2 \times S^1) \sqcup_{S^1\times S^1} (S^1 \times \overline{B}^2) & \to & \overline{B}^2/S^1 \\ P & \mapsto & \begin{cases} [Q], & \text{if } P = [(Q,z)] \text{ with } Q \in \overline{B}^2 \text{ and } z \in S^1, \\ [S^1], & \text{if } P = [(z,Q)] \text{ with } z \in S^1 \text{ and } Q \in \overline{B}^2. \end{cases} \end{array}$$

Show that f is null-homotopic.

Exercise 87.2. Let $(L = L_1 \sqcup \cdots \sqcup L_m, g = g_1 \sqcup \cdots \sqcup g_m)$ be a fattened link. Show that

$$\operatorname{slk}(L,g) \ := \ \sum_{i\neq j} \operatorname{lk}(L_i, L_j) \ + \ \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

Exercise 87.3. Let $m \geq 2$. The Thom-Pontryagin Theorem 87.11 implies in particular that $\Omega^{\text{fat}}_{m-1}(\mathbb{R}^m) \cong \pi_1(S^m) = 0$. This implies that every fattened submanifold of $\mathbb{R}^m$ of codimension one is null-cobordant. Prove this statement without making use of the Thom-Pontryagin Theorem 87.11.
Hint: Use Lemma 69.3.

Exercise 87.4. We consider the Hopf link H as shown in Figure 1075. Draw an explicit oriented connected compact surface whose boundary equals the oriented link H.
Remark: Note that the orientation of one of the components is different from the Hopf link that we had considered above in Figure 1071.

FIGURE 1075.

Exercise 87.5. Let $L \subset S^3$ be an oriented link and let $G \subset S^3$ be a compact oriented surface with $\partial G = L$. We assume that G has no closed components.

(a) Show that $S^3 \setminus G$ is path-connected.
(b) Give an outline of an argument why there exists a compact oriented *connected* surface F with $\partial F = L$.

Exercise 87.6. Let K and J be two oriented disjoint knots in S^3. Show that the two definitions of the linking number $\operatorname{lk}(K,J)$ given on pages 1635 and 2029 agree.

Exercise 87.7. Show that the map

$$\begin{array}{rcl} \Psi\colon \Omega^{\text{fat}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z} \\ [(N, g\colon \overline{B}^{2n} \times N \to \mathbb{R}^{4n-1})] & \mapsto & \operatorname{lk}(N, g(* \times N)) \end{array}$$

considered in Proposition 87.19 is a homomorphism.

Exercise 87.8. In Section 87.5 we had just seen that given any even $n \in \mathbb{N}$ there exists an epimorphism $\Omega_{n-1}^{\text{fat}}(\mathbb{R}^{2n-1}) \to \mathbb{Z}$. What about the group $\Omega_{n-1}^{\text{fat}}(\mathbb{R}^{2n-1})$ if n is odd? Can we use linking numbers to define a homomorphism to $\mathbb{Z}$? Can we use them to find an epimorphism from $\Omega_{n-1}^{\text{fat}}(\mathbb{R}^{2n-1})$ to $\mathbb{Z}$?

88. Thom-Pontryagin Theory II: The Freudenthal Suspension Theorem

88.1. **The framed cobordism group.** In the previous chapter we introduced the fattened cobordism groups $\Omega_k^{\text{fat}}(\mathbb{R}^{n+k})$ and we showed that they are isomorphic to $\pi_{n+k}(S^n, *)$. Furthermore we computed that $\Omega_0^{\text{fat}}(\mathbb{R}^n) \cong \mathbb{Z}$ and $\Omega_1^{\text{fat}}(\mathbb{R}^3) \cong \mathbb{Z}$. Finally we showed that given any $n \in \mathbb{N}$ there exists an epimorphism $\Omega_{2n-1}^{\text{fat}}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$. These calculations gave us new proofs for Corollary 80.6, Theorem 85.10 (2) and Corollary 63.14.

Even though these new proofs were very pleasant, it is now time for some new results. In this section we will prove the Freudenthal Suspension Theorem and we will calculate the homotopy groups $\pi_{n+1}(S^n)$ for $n \geq 3$. For the subsequent calculations it is better though to work with the "framed cobordism groups" $\Omega_k^{\text{fr}}(\mathbb{R}^n)$ that we introduce in this section. As we will see, the framed cobordism groups are isomorphic to the fattened cobordism group, but at times they are easier to calculate since they allow us to reduce many issues to linear algebra.

First we introduce the following definitions in linear algebra.

Definition. Given $m, k \in \mathbb{N}_0$ we write

$$\begin{aligned} \mathrm{GL}(m,k) &:= \text{ set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^m \\ \mathrm{O}(m,k) &:= \text{ set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^m. \end{aligned}$$

We adopt the following conventions:

(1) Given $v \in \mathrm{GL}(m,k)$ we denote by $v_1, \dots, v_k$ the corresponding column vectors.
(2) We view $\mathrm{GL}(m,k)$ and $\mathrm{O}(m,k)$ as subsets of the manifold $M(m \times k, \mathbb{R}) = \mathbb{R}^{mk}$.

Now we can give the topological definition we are really interested in.

Definition. Let $m \in \mathbb{N}_0$ and let N be an m-dimensional submanifold of $\mathbb{R}^m$. (Later on we will only be interested in $N = \mathbb{R}^m$, $N = [0,1] \times \mathbb{R}^{m-1}$ and the case that $N = \overline{B}_r^m(P)$ is a closed m-dimensional ball.)

(1) Let M be a compact orientable proper k-dimensional submanifold of N.
   (a) Let $P \in M$. We say $v = (v_1, \dots, v_{m-k}) \in \mathrm{GL}(m, m-k)$ is a *frame* of M at P if the vectors $v_1, \dots, v_{m-k}$, together with $T_P M$[1516] spans $\mathbb{R}^m$.
   (b) A *framing for M* is a smooth[1517] map $v\colon M \to \mathrm{GL}(m, m-k)$ such that for each $P \in M$ the vectors $v_1(P), \dots, v_{m-k}(P)$ are a frame of M at P.
   (c) We say that a framing $v = (v_1, \dots, v_{m-k})$ is *perpendicular* if for each $P \in M$ the vectors $v_1(P), \dots, v_{m-k}(P)$ lie in the orthogonal complement of $T_P M$, i.e. if they lie in the vector space

$$(T_P M)^\perp \;=\; \{v \in \mathbb{R}^m \mid \langle v, w\rangle = 0 \text{ for all } w \in T_P M\}.$$

[1516]Here we use the isomorphism from Proposition 3.32 to identify $T_P M$ with the subvector space $V_P M$ of $\mathbb{R}^m$.

[1517]Since we view $\mathrm{GL}(m, m-k)$ as a subset of the manifold $M(m \times (m-k), \mathbb{R}) = \mathbb{R}^{m\cdot(m-k)}$ it makes sense to say that a map $M \to \mathrm{GL}(m, m-k)$ is smooth.

If the framing is perpendicular and if all the vectors are orthonormal, then we that the framing is *orthonormal*.

(2) A *framed k-dimensional submanifold* of N is a pair (M, v) where M is a compact orientable proper k-dimensional submanifold of N and v is a framing for M.

(3) Let (M, v) be a framed k-dimensional submanifold of N and let $\varphi\colon N \to W$ be an embedding of N into a manifold of the same dimension. We consider the map

$$\begin{array}{rcl} \varphi_*(v)\colon \varphi(M) & \to & \mathrm{GL}(m, m-k) \\ P & \mapsto & (\varphi_*(v_1(\varphi^{-1}(P))), \ldots, \varphi_*(v_{m-k}(\varphi^{-1}(P)))). \end{array}$$

The pair $(\varphi(M), \varphi_*(v))$ is a framed submanifold of W.

Some of the above definitions are illustrated in Figure 1076.

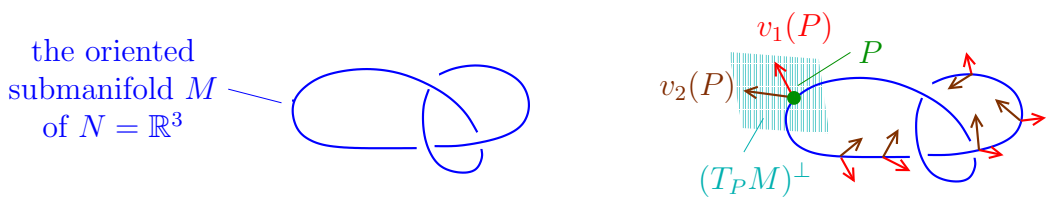


FIGURE 1076.

Definition. Let $k, m \in \mathbb{N}_0$ and let (N_0, v_0) and (N_1, v_1) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. A *framed cobordism between* (N_0, v_0) *and* (N_1, v_1) is a framed $(k+1)$-dimensional submanifold (W, V) of $[0, 1] \times \mathbb{R}^m$ such that the following three conditions are satisfied:

(1) $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$,
(2) $V|_{\{0\}\times N_0} = v_0$ under the identification $\{0\} \times \mathbb{R}^m = \mathbb{R}^m$,
(3) $V|_{\{1\}\times N_1} = v_1$ under the identification $\{1\} \times \mathbb{R}^m = \mathbb{R}^m$,

If such a framed cobordism exists, then we say that (N_0, v_0) and (N_1, v_1) are *cobordant*. The definition of a framed cobordism is illustrated in Figure 1077.

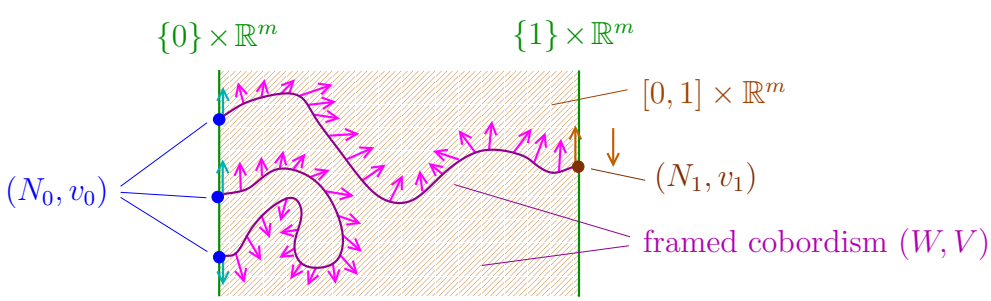


FIGURE 1077.

Basically the same proof as in Lemma 87.8 gives us the following lemma.

Lemma 88.1. *The property of "being cobordant" is an equivalence relation on the set of framed k-dimensional submanifolds of $\mathbb{R}^m$.*

This lemma leads us, not surprisingly, to the following definition.

Definition. We denote by $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^m$. We refer to $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ as a *framed cobordism group.*

The following proposition, which is the analogue of Proposition 87.9, justifies the name "framed cobordism group".

Proposition 88.2. *Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $M \subset B_r^m(0)$ and $N \subset B_s^m(0)$. Furthermore we pick two orientation-preserving embeddings $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ with disjoint images. We define*

$$[(M, v)] + [(N, w)] := [(\varphi(M) \sqcup \psi(N), \varphi_*(v) \sqcup \psi_*(w))] \in \Omega_k^{\mathrm{fr}}(\mathbb{R}^m).$$

This operation "+" has the following properties:

1. *The definition of "+" on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ is well-defined and commutative.*
2. *The operation "+" defines a group structure on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The following two statements hold:*
   (a) *The neutral element is represented by the empty submanifold.*
   (b) *Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, v)]$ is given by $[(\rho(M), \rho_*(v))]$.*

Proof. The proof is basically identical to the proof of Proposition 87.9. We leave it to the reader to carry out the minor adjustments that are necessary to make the transition from fattened cobordisms groups to framed cobordism groups. □

Lemma 88.3. *Let $m \in \mathbb{N}_{\geq 2}$ and $k \in \{0, \dots, m\}$.*

1. *Every framed k-dimensional submanifold (M, v) of $\mathbb{R}^m$ is cobordant to (M, w) where w is an orthonormal framing.*
2. *Let (W, V) be a framed cobordism between two framed k-dimensional submanifolds (M_0, v_0) and (M_1, v_1) of $\mathbb{R}^m$. If v_0 and v_1 are orthonormal, then there exist an orthonormal framing V' for W such that (W, V') is a framed cobordism between (M_0, v_0) and (M_1, v_1).*

Proof $(*)$. We prove statement (1) of the lemma. The proof of statement (2) is a mild variation on the proof of statement (1). Thus let (M, u) be a framed k-dimensional submanifold of $\mathbb{R}^m$. Using the fact that being cobordant is transitive, the proof of the lemma can be broken up into two stages:

1. First we show that there exists a perpendicular framing v such that (M, u) is cobordant to (M, v).
2. Then we show that there exists an orthonormal framing w such that (M, v) is cobordant to (M, w).

We carry out the two steps:

(1) Let $(M, u\colon M \to \mathrm{GL}(m, m-k))$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. Given $P \in M$ we write $U_P M := \mathrm{span}\{u_1(P), \dots, u_k(P)\}$ and furthermore we write $Z_P M := (T_P M)^\perp$. For each $P \in M$ the decomposition $\mathbb{R}^m = T_P M \oplus Z_P M$ defines a projection of $\mathbb{R}^m$ onto $Z_P M$. We denote the restriction of this projection to $U_P M$ by $\rho_P\colon U_P M \to Z_P M$. By definition of a framing this map is an isomorphism. We consider the product cobordism $W = [0,1] \times M$ equipped with the framing[1518]

$$\begin{array}{rcl} V\colon W = [0,1] \times M & \to & \mathrm{GL}(m, m-k) \\ (t, P) & \mapsto & (u(P) \cdot (1-t) + \rho_P(u(P)) \cdot t). \end{array}$$

Basically by definition this is a framed cobordism from (M, u) to (M, v) where $v = V_1$ is a perpendicular framing.

(2) In Lemma 10.7 (1) we showed that there exists a smooth deformation retraction $R\colon \mathrm{GL}(m, m-k) \times [0,1] \to \mathrm{O}(m, m-k)$ from $\mathrm{GL}(m, m-k)$ to $\mathrm{O}(m, m-k)$ which has the following extra property:

($*$) For any $v = (v_1, \dots, v_{m-k}) \in \mathrm{GL}(m, m-k)$ and any $t \in [0,1]$ we have

$$\text{span of } R_t(v_1), \dots, R_t(v_{m-k}) \;=\; \text{span of } v_1, \dots, v_{m-k}.$$

Now let (M, v) be a framed k-dimensional submanifold. We consider the product cobordism $W = [0,1] \times M$ equipped with the framing

$$\begin{array}{rcl} V\colon W = [0,1] \times M & \to & \mathrm{GL}(m, m-k) \\ (t, P) & \mapsto & R_t(v(P)). \end{array}$$

It follows from the fact that R is smooth and that ($*$) holds that this is a framed cobordism from (M, v) to (M, w) where $w = V_1$ is an orthonormal framing. □

The following proposition shows that the two types of cobordism groups that we introduced in this and the last chapter are in fact isomorphic.

Proposition 88.4. *Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \dots, m\}$. The map*

$$\begin{array}{rcl} \Phi\colon \Omega_k^{\mathrm{fat}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \\ [(N, g\colon \overline{B}^{m-k} \times N)] & \to & \left[\begin{array}{l} N \textit{ equipped with the framing that associates to } P \in N \\ \textit{the frame } Dg_{(0,P)}(e_1), \dots, Dg_{(0,P)}(e_{m-k}) \end{array}\right] \end{array}$$

is well-defined and it is an isomorphism.

Convention. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. We use the isomorphisms from the Thom-Pontryagin Theorem 87.11 and Proposition 88.4 to make the identifications

$$\pi_{n+k}(S^n, *) \;=\; \Omega_k^{\mathrm{fat}}(\mathbb{R}^{n+k}) \;=\; \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}).$$

Sketch of proof.

[1518] It takes a little bit of thought to show that this is indeed a smooth map.

(1) It follows basically from the definitions that the framing associated to a fattened submanifold is indeed a framing. Applying the same procedure to a fattened cobordism between fattened submanifolds we obtain a framed cobordism between framed submanifolds. This shows that Φ is well-defined.
(2) It follows immediately from the definitions of the group structures that Φ is a homomorphism.
(3) We sketch the construction of the inverse to Φ. Let $(N, \{v_1, \dots, v_{m-k}\})$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. We consider the map

$$\begin{array}{rcl} g(v)\colon \mathbb{R}^{m-k} \times N & \to & \mathbb{R}^m \\ ((t_1, \dots, t_{m-k}), P) & \mapsto & P + t_1 \cdot v_1(P) + \dots + t_{m-k} \cdot v_{m-k}(P). \end{array}$$

The same argument as in the proof of Theorem 10.19 shows that there exists an $\epsilon > 0$ such that $g(v)$ restricts to an embedding $\overline{B}^k_\epsilon(0) \times N \to \mathbb{R}^m$. After rescaling the ball we thus get a fattening $\overline{B}^k \times N \to \mathbb{R}^m$ of N. It is straightforward to see that the class $[(N, g(v)] \in \Omega^{\text{fat}}_k(\mathbb{R}^m)$ does not depend on the choice of $\epsilon > 0$. Since the same approach also works for cobordisms we obtain a well-defined map

$$\Psi\colon \Omega^{\text{fr}}_k(\mathbb{R}^m) \ \to \ \Omega^{\text{fat}}_k(\mathbb{R}^m).$$

It remains to show that Φ and Ψ are inverses of one another. It follows basically immediately from the definitions and Lemma 88.3 (1) that $\Phi \circ \Psi$ is the identity on $\Omega^{\text{fr}}_k(\mathbb{R}^m)$. The fact that $\Psi \circ \Phi$ is the identity on $\Omega^{\text{fat}}_k(\mathbb{R}^m)$ follows fairly immediately from the uniqueness of tubular neighborhoods, see Proposition 64.9, together with Lemma 64.6. □

Definition. Let N be an orientable k-dimensional submanifold of $\mathbb{R}^m$ and let $n \geq m$.

(1) We write $N_n := N \times \{0\} \subset \mathbb{R}^n$.

Now suppose we are given a framing $v\colon N \to \mathrm{GL}(m, m-k)$ for N.

(2) We define

$$\begin{array}{rcl} v_n\colon N_n & \to & \mathrm{GL}(n, n-k) \\ (P, 0) & \mapsto & \begin{pmatrix} v(P) & 0 \\ 0 & \mathrm{id}_{n-m} \end{pmatrix}. \end{array}$$

It is clear that v_n is a framing for N_n. Note that $N_m = N$ and $v_m = v$.
(3) We refer to (N_n, v_n) as the $(n-m)$*-fold suspension of* (N, v). In particular we refer to (N_{m+1}, v_{m+1}) as *the suspension of* (N, v).

The same notation applies in an obvious way to (framed) submanifolds of $[0, 1] \times \mathbb{R}^m$.

The following lemma justifies the name "suspension" that we had used in the above definition.

Lemma 88.5. *Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \dots, m\}$.*

(1) *The map*

$$\begin{array}{rcl} \Omega^{\text{fr}}_k(\mathbb{R}^m) & \to & \Omega^{\text{fr}}_k(\mathbb{R}^{m+1}) \\ [(N, v)] & \mapsto & [\textit{the suspension of } (N, v)] \end{array}$$

is well-defined and it is a homomorphism. We refer to it as the suspension homomorphism.

(2) *Under the identification from the convention on page 2037 the suspension homomorphisms on $\pi_{n+k}(S^n, *)$ and $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k})$ agree up to sign,*[1519] *in other words, the following diagram commutes up to sign:*

$$\begin{tikzcd} \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \arrow[r, "\textit{suspension}"] \arrow[d, "\cong"] & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1}) \arrow[d, "\cong"] \\ \pi_{n+k}(S^n, *) \arrow[r, "\textit{suspension}"] & \pi_{n+k+1}(S^{n+1}, *). \end{tikzcd}$$

Proof.

(1) The first statement is almost obvious.
(2) We leave it to the reader to go through all the definitions to verify this statement. □

88.2. **The Freudenthal Suspension Theorem.** In this section we will prove the following theorem that was first proved in 1937 by Hans Freudenthal [Freu, p. 300].

Theorem 88.6. (Freudenthal Suspension Theorem) *Let $n, k \in \mathbb{N}_0$. We suppose that $n + k \geq 2$. The suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ from Lemma 28.18 is an isomorphism if $n > k + 1$ and it is an epimorphism if $n = k + 1$.*

Remark.

(1) The Freudenthal Suspension Theorem 88.6 can be reformulated as follows: given $n \in \mathbb{N}$ the suspension homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$ is an isomorphism if $m < 2n-1$ and that it is an epimorphism if $m = 2n-1$. We consider two situations in slightly more detail:
   (a) The Freudenthal Suspension Theorem 88.6 says that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is an epimorphism and it says that for any $n \geq 3$ the suspension homomorphism $\pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$ is an isomorphism. This special case will play an important role shortly.
   (b) The Freudenthal Suspension Theorem 88.6 almost recovers Theorems 85.11 (2) and 85.14 (2). These theorems stated that for $m \in \{0, \ldots, 5\}$ we have $\pi_m(S^3) \cong \pi_{m+1}(S^4)$ and that for $m \in \{0, \ldots, 13\}$ we have $\pi_m(S^7) \cong \pi_{m+1}(S^8)$.
(2) Let $n \in \mathbb{N}_0$. Recall that on page 453 we saw that the suspension $\Sigma(S^n)$ is homeomorphic to S^{n+1}. Thus we see that the Freudenthal Suspension Theorem 88.6 is a consequence of the following stronger statement: Let X be an $(n-1)$-connected topological space and let $* \in X$ be a good base point.[1520] Then the suspension homomorphism $\pi_{n+k}(X, *) \to \pi_{n+k+1}(\Sigma(X), *)$ is an isomorphism for all $n > k + 1$

[1519] To be more precise, they might agree on the nose, but we will not spend valuable time on figuring out the sign.

[1520] We refer to page 387 for the definition of a good point.

and it is an epimorphism for $n = k+1$. This statement is proved in most textbooks on homotopy theory, see e.g. [Gra, Theorem 16.34], [Whda, Theorem VII.7.13] or [DK, Theorem 8.7 and Chapter 10.3]. If X is a CW-complex, then the statement is also proved in [Hat, Corollary 4.24]. The proof of [Hat, Corollary 4.24] is actually a corollary to the Blakers-Massey Theorem 82.17 that we had stated but not proved.

It follows from Lemma 88.5 that the Freudenthal Suspension Theorem 88.6 is equivalent to the following theorem.

Theorem 88.7. *Let $n, k \in \mathbb{N}$.*

(1) *The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism whenever $n \geq k+1$.*
(2) *The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism whenever $n > k+1$.*

The basic behind the proof of Theorem 88.7 (1) is quite simple: given a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$ we want to use the "room provided by a large enough n" to show that it is cobordant to a framed submanifold coming from $\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\}$. This idea is for the most part carried out in the following two lemmas.

Lemma 88.8. *Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{n+k+1}$ with $N \subset \mathbb{R}^{n+k} \times \{0\}$.*

Proof. Let $(M, v\colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. We point out that by Lemma 88.3 (1) we can assume that v is an orthonormal framing.

It follows from $n \geq k+1$ and Lemma 65.12 that there exists a vector $\nu \in \mathbb{R}^{n+k+1}$ which satisfies the following condition:

($*$) the vector ν does not lie in any tangent space $T_P M$[1521] and ν is not secant to M, i.e. for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to ν.

By Lemma 1.73 (4) there exists a matrix $A \in \mathrm{SO}(n+k+1)$ with $A\nu = e_{n+k+1}$. By Lemma 10.4 there exists a diffeotopy of $\mathbb{R}^{n+k+1}$ from $\mathrm{id}_{\mathbb{R}^{n+k+1}}$ to the diffeomorphism given by multiplying by A. Therefore, by the obvious analogue of Lemma 87.7 for the framed cobordism group, we can assume without loss of generality that $\nu = e_{n+k+1}$.

Claim. The map

$$\begin{array}{rcl} \Psi\colon [0,1] \times M & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, \underbrace{(x_1, \dots, x_{n+k+1})}_{\in M}) & \mapsto & (t, \underbrace{x_1, \dots, x_{n+k}, (1-t) \cdot x_{n+k+1}}_{\in \mathbb{R}^{n+k+1}}) \end{array}$$

is a proper embedding. The definition of Ψ is illustrated in Figure 1078.

[1521]Here and throughout this proof we use Proposition 3.32 to view the tangent spaces of submanifolds of some $\mathbb{R}^m$ as subvector spaces of $\mathbb{R}^m$.

It follows from the fact that for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to $\nu = e_{n+k+1}$ that the map is injective. Furthermore note that the map Ψ is the composition of the inclusion $i\colon [0,1] \times M \to [0,1] \times \mathbb{R}^{n+k+1}$ with the map

$$\begin{array}{rcl} \Theta\colon [0,1] \times \mathbb{R}^{n+k+1} & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, (x_1, \ldots, x_{n+k+1})) & \mapsto & (t, x_1, \ldots, x_{n+k}, (1-t) \cdot x_{n+k+1}). \end{array}$$

A straightforward calculation shows that

$$D\Theta_{(t,x_1,\ldots,x_{n+k+1})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix}.$$

It follows immediately from this calculation and the fact that e_{n+k+1} is not contained in any tangent space $T_P M$ that $\Psi = \Theta \circ i$ is an immersion. Together with the injectivity of Θ, the fact that M is compact and the ever-popular Proposition 1.40 we obtain that $\Psi = \Theta \circ i$ is in fact an embedding. It is also clear that the embedding is proper. We are thus done with the proof of the claim.

We set $W := \Psi([0,1] \times M)$. By the claim together with Proposition 10.1 (2) we know that W is a proper $(k+1)$-dimensional submanifold of $[0,1] \times \mathbb{R}^{n+k+1}$. Basically by definition W is a cobordism between $M \subset \mathbb{R}^{n+k+1}$ and some submanifold $N \subset \mathbb{R}^{n+k} \times \{0\}$.

It remains to extend the orthonormal framing $v\colon M \to \mathrm{GL}(n+k+1, n+1)$ of M to a framing of W. In fact, we obtain a framing of $W = \Psi([0,1] \times M)$ by using the "same vectors" as for M. More precisely, we consider the map

$$\begin{array}{rcl} V\colon W = \Psi([0,1] \times M) & \to & \mathrm{GL}(n+k+2, n+1) \\ \Psi(t, P) & \mapsto & \begin{pmatrix} 0 \\ v(P) \end{pmatrix}. \end{array}$$

Evidently under the identification $\{0\} \times \mathbb{R}^{n+k+1} = \mathbb{R}^{n+k+1}$ we have $V|_M = v$. Thus it remains to show that V is indeed a framing. A short moment's thought shows that V is smooth. The definition of W and V are illustrated in Figure 1078. Therefore it suffices to

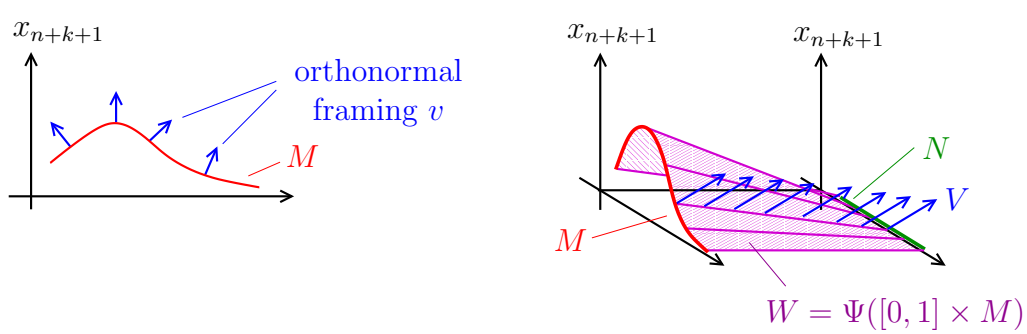


FIGURE 1078. Illustration for the proof of Lemma 88.8.

prove the following claim.

Claim. For any $t \in [0,1]$ and any $P \in M$ the matrix $V(\Psi(t,P)) \in \mathrm{GL}(n+k+2, n+1)$ is a frame for $W = \Psi([0,1] \times M)$ at $\Psi(t,P)$.

Let $t \in [0,1]$ and let $P \in M$. Note that the columns of $V(\Psi(t,P))$ span an $(n+1)$-dimensional subvector space of $\mathbb{R}^{n+k+2}$. Since $T_{\Psi(t,P)}(W)$ is a subvector space of the complementary dimension $k+1$ it suffices to show that the intersection of these two subvector spaces is trivial.

We note that

$$\underbrace{T_{\Psi(t,P)}(W) \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset \mathbb{R}^{1+n+k+1}} \overset{\substack{\text{by definition of } W = \Psi([0,1]\times M) = (\Theta \circ i)([0,1]\times M) \text{ and the calculation of } D\Theta \\ \downarrow}}{=} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix} \cdot \begin{pmatrix} \mathbb{R} \\ \oplus \\ T_P(M) \end{pmatrix} \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset \mathbb{R}^{1+n+k+1}}$$

$$\underset{\substack{\uparrow \\ \text{since the first row of } V \text{ is zero we can} \\ \text{“ignore” the first coordinate}}}{\cong} \underbrace{\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot T_P(M) \cap \begin{array}{c}\text{span of the}\\ \text{columns of } v(P)\end{array}}_{\subset \mathbb{R}^{n+k+1}} \underset{\substack{\uparrow \\ \text{since } v \text{ is an orthonormal framing}}}{=} \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot T_P(M) \cap (T_P M)^\perp.$$

Now let $u \in T_P M$ be a non-zero vector. We write $u = \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}$ with $u' \in \mathbb{R}^{n+k}$ and $u_{n+k+1} \in \mathbb{R}$. Note that u' is non-zero since we had arranged that $e_{n+k+1} \notin T_P M$. Now we calculate that

$$\left\langle \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}, \underbrace{\begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}}_{=u \in T_P M} \right\rangle = \langle u', u' \rangle + (1-t) \cdot u_{n+k+1}^2 \underset{\substack{\uparrow \\ \text{since } u' \neq 0 \text{ and } (1-t) \geq 0}}{>} 0.$$

But this shows that $\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot u$ is not orthogonal to $T_P M$, in other words, this vector does not lie in $(T_P M)^\perp$. This concludes the proof of the claim. □

Lemma 88.9. *Let (M, v) be a framed k-dimensional submanifold in $\mathbb{R}^{n+k} \times \{0\} \subset \mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism to $(M, w = (w_1, \dots, w_{n+1}))$ where we have $w_{n+1}(P) = e_{n+k+1}$ for all $P \in M$.*

Proof. Let $(M, u = (u_1, \dots, u_{n+1}) \colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold that is contained in $\mathbb{R}^{n+k} \times \{0\} \subset \mathbb{R}^{n+k+1}$. By Lemma 88.3 (1) and Lemma 88.1 we can assume that u is orthonormal. We write $e := e_{n+k+1}$.

Claim. There exists an orthonormal framing $v = (v_1, \dots, v_{n+1})$ for M such that (M, u) is cobordant to (M, v) and such that for every $P \in M$ we have $v_{n+1}(P) \neq -e$.

Let $x \in M$. Since M is contained in $\mathbb{R}^{n+k} \times \{0\}$ we know that e is orthogonal to $T_x M$. Therefore we can uniquely write $e = \sum_{i=1}^{n+1} \lambda_i(x) \cdot u_i(P)$. Since $\|e\| = 1$ and since u is an

orthonormal framing we actually have $\sum_{i=1}^{n+1} \lambda_i^2(P) = 1$. Thus we obtain a map

$$\begin{array}{rcl} \lambda\colon M & \to & S^n \\ x & \mapsto & (\lambda_1(x), \dots, \lambda_{n+1}(x)) \end{array}$$

which is fairly easily seen to be smooth. If $(0, \dots, 0, -1) \notin \lambda(M)$, then we can just take $u = v$ and we are done.[1522] If not we proceed as follows. Since $\dim(M) = k < n$ it follows from Proposition 3.47 that there exists a $\nu \in S^n$ with $\nu \notin \lambda(M)$. By Lemma 1.73 (4) there exists a matrix $A \in \mathrm{SO}(n+1)$ with $\nu = A \cdot (-e) \in S^n$. By Lemma 1.96 and Exercise 10.11 there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(n+1)$ from id to A. The desired framed cobordism is now given by the manifold $[0,1] \times M$ together with the framing[1523]

$$\begin{array}{rcl} [0,1] \times M & \to & \mathrm{GL}(n+k+2, n+1) \\ (t,x) & \mapsto & \underbrace{\begin{pmatrix} 0 \\ u(x) \end{pmatrix}}_{\in \mathrm{GL}(n+k+2,n+1)} \cdot \gamma(t). \end{array}$$

This concludes the proof of the claim.

As a final step we want to "rotate v_{n+1} into e". To do this smoothly for every point we need to introduce two maps. Given $u \in \mathbb{R}^{n+1} \setminus \{0\}$ we write

$$\underbrace{\begin{array}{rcl} T_u\colon \mathbb{R}^{n+k+1} & \mapsto & \mathbb{R}^{n+k+1} \\ y & \to & 2 \cdot \langle y, u \rangle \cdot u - y \end{array}}_{\text{reflection in the line } \mathbb{R} \cdot u} \quad \text{and} \quad \underbrace{\begin{array}{rcl} R_u\colon \mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1} \\ y & \mapsto & T_e(T_u(y)). \end{array}}_{\text{orientation-preserving diffeomorphism}}$$

We will need the following elementary observation from linear algebra.

Claim. Let $w \in \mathbb{R}^{n+k+1}$ with $\|w\| = 1$. Let $u = \frac{w+e}{\|w+e\|}$ be the normalized midpoint between w and e. Then $R_u(w) = e$.

We have

$$R_u(w) \;=\; T_e(T_u(w)) \;=\; T_e\Big(2 \cdot \Big\langle w, \tfrac{w+e}{\|w+e\|} \Big\rangle \cdot \tfrac{w+e}{\|w+e\|} - w\Big) \;\underset{\substack{\uparrow \\ \text{short calculation using } \langle w,w\rangle = \langle e,e\rangle}}{=}\; T_e(e) \;=\; e.$$

This concludes the proof of the claim.

[1522] Indeed, if $v_{n+1}(x) = -e$, then $e = -v_{n+1}(x)$, i.e. $\lambda(x) = (0, \dots, 0, -1)$.

[1523] That this framing has the correct properties can be seen as follows. Let v be the new framing at $\{1\} \times M$. If we denote by $\lambda_u, \lambda_v\colon M \to S^n$ the corresponding maps, then one easily sees that for each $x \in M$ we have $\lambda_v(x) = A^{-1} \cdot \lambda_u(x)$. Thus the fact that ν does not lie in the image of λ_u implies that $-e = A^{-1}\nu$ does not lie in the image of λ_v.

The above claim now allows us to come to the denouement of the argument. More precisely, given $t \in [0,1]$ and $x \in M$ we write

$$z(x) \;:=\; \underbrace{\frac{v_{n+1}(x)+e}{\|v_{n+1}(x)+e\|}}_{\text{note that } v_{n+1}(x)+e \neq 0} \quad \text{and} \quad z_t(x) \;=\; \underbrace{\frac{(1-t)\cdot e+t\cdot z(x)}{\|(1-t)\cdot e+t\cdot z(x)\|}}_{\text{note that } t\cdot e+(1\text{-}t)\cdot z(x) \neq 0}.$$

Note that $z_0 = e$ and $R_{z_0} = \mathrm{id}$. Furthermore note that $z_1 = z(x)$. Thus we obtain from the above claim that $R_{z_1(x)}(v_{n+1}(x)) = e$. Finally we consider the framed cobordism that is given by the submanifold $Z = [0,1] \times M$ of $[0,1] \times \mathbb{R}^{n+k+1}$ together with the framing

$$\begin{aligned} W\colon Z = [0,1]\times M &\to \mathrm{GL}(n+k+2, n+1) \\ (t,x) &\mapsto \begin{pmatrix} 0 \\ R_{z_t(x)} \circ v(x) \end{pmatrix}. \end{aligned}$$

We leave it to the reader to verify that the map W is indeed a framing for the cobordism $Z = [0,1] \times M \subset [0,1] \times \mathbb{R}^{n+k+1}$. The discussion above shows that $(Z = [0,1] \times M, W)$ is a framed cobordism from (M, v) to a framed manifold $(M, w = w_1, \dots, w_n, e)$. $\square$

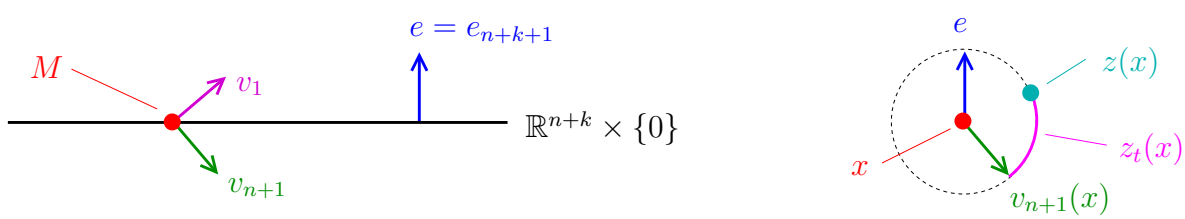


FIGURE 1079. Illustration for the proof of Lemma 88.9.

Now we turn to the actual proof of Theorem 88.7 and thus of the Freudenthal Suspension Theorem 88.6.

Proof. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$.

(1) Suppose that $n \geq k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism. Thus let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. It follows immediately from Lemmas 88.8 and 88.9 that (M, v) is cobordant to a k-dimensional framed submanifold (N, w) with $N \subset \mathbb{R}^{n+k} \times \{0\}$ and such that $w_{n+1}(P) = e_{n+k+1}$ for all $P \in N$. But this means that $[(M, v])] = [(N, w)]$ lies in the image of the suspension homomorphism. Thus the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism.

(2) Suppose that $n > k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an monomorphism. Thus let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$. Suppose there exists a framed cobordism (W, V) in $[0,1] \times \mathbb{R}^{n+k+1}$ between the suspensions of (M, v) and (N, w). Since $n > k+1$, then an argument as in Lemmas 88.8 and 88.9 shows that we can

turn (W, V) into a cobordism (W', V') with $W' \subset [0,1] \times \mathbb{R}^{n+k} \times \{0\}$ and such that $V'(P) = e_{n+k+1}$ for all $P \in W'$. But this shows that (M, v) and (N, w) are already cobordant. Thus we have shown that for $n > k+1$ the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. □

88.3. **The calculation of the framed cobordism groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$.** In Theorem 85.10 (2) (and alternatively in Theorem 87.14) we saw that $\pi_3(S^2) \cong \mathbb{Z}$ and we saw that the Hopf map represents a generator of $\pi_3(S^2)$. Furthermore, by the Freudenthal Suspension Theorem 88.6 we know that for every $n \geq 3$ the suspension homomorphism $\pi_3(S^2) \to \pi_{n+1}(S^n)$ is an epimorphism. Thus we know that $\pi_{n+1}(S^n)$ is isomorphic to a quotient of $\mathbb{Z}$ and that $\pi_{n+1}(S^n)$ is generated by the $(n-2)$-fold suspension of the Hopf map. The goal of this section is to prove the following theorem which determines this quotient.

Theorem 88.10. *For every $n \geq 3$ there exists an isomorphism $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$.*

Remark.

(1) In Theorem 78.13 we already saw that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \neq 0$. The proof of Theorem 88.10 below is independent of Theorem 78.13. In particular we will not make use of Steenrod operations.
(2) By the Thom-Pontryagin Theorem 87.11 and Proposition 88.4 the theorem is equivalent to showing that for every $n \geq 3$ we have an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. This is exactly what we will do below. The approach that we take below to calculate $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ differs, to the best of my knowledge, from the other proofs in the literature. We refer to [Po3, Chapter 14] for Pontryagin's original proof, which makes for challenging reading for the modern reader. A third, quite readable approach is presented in [Put].
(3) There are various alternative proofs of Theorem 88.10. The first proof was given by Hans Freudenthal [Freu, p. 301] in 1937. Another proof is given in [Hat6, p. 549] using "spectral sequences".
(4) The discussion preceding the theorem shows that the suspension of the Hopf map defines a non-trivial element in $\pi_4(S^3)$. This gives in particular an affirmative answer to Question 28.19.

As we had just mentioned above, proving Theorem 88.10 is equivalent to showing that for every $n \geq 3$ there exists an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. We will perform this calculation in the remainder of this section.

We start out our calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$ with the following definition.

Definition.

(1) We refer to $C := S^1$ as the *standard circle in* $\mathbb{R}^2$.

(2) We refer to the framing of $C \subset \mathbb{R}^2$ that is given by

$$\begin{array}{rcl} t\colon C & \to & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

as the *trivial framing* t *of* C.

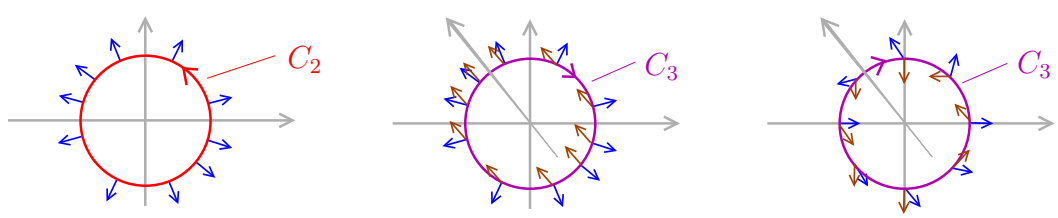
(3) Let $n \in \mathbb{N}_{\geq 2}$. We adopt the convention from page 2038 to define the *standard standard circle* C_n in $\mathbb{R}^n$. Furthermore, using the definition from page 2038 we define the *trivial framing* t_n *for* C_n.

(4) We refer to the framing of $C_3 \subset \mathbb{R}^3$ that is given by the map[1524]

$$\begin{array}{rcl} h\colon C_3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} & \mapsto & \underset{\substack{\uparrow \\ \text{don't overlook the } t_3}}{t_3} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \end{array}$$

as the *Hopf framing* h *of* C_3.

(5) Let $n \in \mathbb{N}_{\geq 3}$. As in (3) we use the convention from page 2038 to define the *Hopf framing* h_n *of* C_n. These definitions are illustrated in Figure 1080.



trivial framing t_2 trivial framing t_3 Hopf framing h_3

FIGURE 1080.

Throughout this section we also use the following notation:

Notation.

(1) Given $g \in \mathbb{N}_0$ we denote by $\Sigma_{g,1}$ the surface of genus g minus one open disk. We pick once and for all an orientation for $\Sigma_{g,1}$.

(2) For any $n \in \mathbb{N}_{\geq 2}$ we denote by

$$\begin{array}{rcl} \rho\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, x_3, \dots, x_n) & \mapsto & (x_1, -x_2, x_3, \dots, x_n) \end{array}$$

the reflection in the $(x_2 = 0)$-hyperplane.

(3) Given $n \in \mathbb{N}_0$ we make the identification $\mathbb{R}^n = \{0\} \times \mathbb{R}^n \subset [0,1] \times \mathbb{R}^n$.

[1524] We leave the task of showing that this map is smooth to the every-ready reader.

The following two lemmas justify the names "trivial framing" and "Hopf framing".

Lemma 88.11.

(1) *Let $g \in \mathbb{N}_0$. Given any $g \in \mathbb{N}$ there exists a proper embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ and a framing w for $\psi(\Sigma_{g,1})$ such that $\partial\psi(\Sigma_{g,1}) = C_2$ and such that the restriction of w to C_2 equals t.*

(2) *For any $n \geq 2$ we have $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.*

Sketch of proof.

(1) By now the seasoned reader should be fairly convinced that there exists a proper embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ with $\psi(\partial\Sigma_{g,1}) = C_2$ and such that the embedding is a "product near the boundary". More precisely, such that for some $\epsilon \in (0,1)$ we have the equality $\psi(\Sigma_{g,1}) \cap ([0,\epsilon] \times \mathbb{R}^2) = [0,\epsilon] \times C_2$. Since $\psi(\Sigma_{g,1})$ is a oriented connected submanifold of $[0,1] \times \mathbb{R}^2$ of codimension two we see that $\psi(\Sigma_{g,1})$ has precisely two normal vector fields of length one. We refer to Figure 1081 for an illustration.

It follows almost immediately from the fact that $\psi(\Sigma_{g,1})$ is a product near $\{0\} \times \mathbb{R}^2$ that one of the above two normal vectors fields agrees with t on $\{0\} \times S^1$.

(2) It follows from (1) that $[(C_2, t_2)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^2)$. From Lemma 88.5 we obtain that $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for any $n \in \mathbb{N}_{\geq 2}$. □

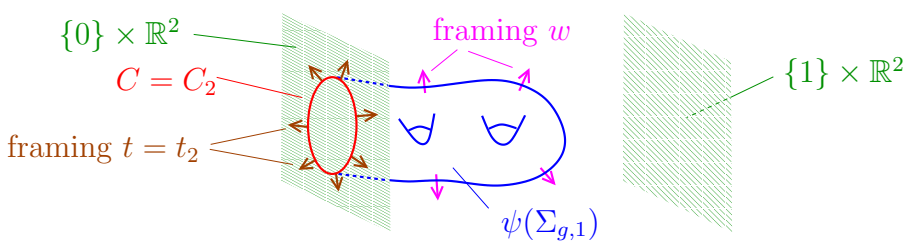


FIGURE 1081. Illustration of Lemma 88.11.

Lemma 88.12.

(1) *Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^3) \cong \pi_3(S^2, *)$ the element $[(C_3, h_3)]$ corresponds, up to a sign, to the element represented by the Hopf map $S^3 \to S^2$.*

(2) *Let $n \geq 4$. Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1}, *)$ the element $[(C_n, h_n)]$ corresponds, up to a sign, to the element represented by the $(n-3)$-fold suspension of the Hopf map $S^3 \to S^2$.*

Proof. The first statement follows easily from Lemma 87.4. The second statement is a straightforward consequence of Lemma 88.5 (2). □

The following proposition is the first key ingredient in the calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for $n \geq 4$.

Proposition 88.13. *Let $n \in \mathbb{N}_{\geq 4}$. The map*

$$\begin{array}{rcl} \mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is well-defined and it is an epimorphism.

Our proof of Proposition 88.13 requires one definition and two lemmas.

Definition. Let $n \in \mathbb{N}_{\geq 2}$, let $\gamma\colon S^1 \to \mathbb{R}^n$ be an embedding and let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$. Given a smooth map $\alpha\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ we define

$$\begin{array}{rcl} f(\alpha)\colon \gamma(S^1) & \to & \mathrm{GL}(n, n-1) \\ \gamma(z) & \mapsto & \underbrace{f(\gamma(z))}_{\in \mathrm{GL}(n,n-1)} \cdot \underbrace{\alpha(z)}_{\in \mathrm{GL}(n-1,\mathbb{R})}. \end{array}$$

It follows easily from the fact that α takes values in $\mathrm{GL}_+(n-1, \mathbb{R})$ that $f(\alpha)$ is again a framing of $\gamma(S^1)$.

The first lemma that goes into the proof of Proposition 88.13 is really just a crucial example masquerading as a lemma.

Lemma 88.14. *Let $n \in \mathbb{N}_{\geq 3}$ and let $\gamma\colon S^1 \to C_n$ be the obvious diffeomorphism. We consider the loop*

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1, \mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

As above let t_n be the trivial framing for C_n and let h_n be the Hopf framing for C_n. We have the following two equalities of framings of C_n:

1. *We have $t_n(\alpha) = h_n$.*
2. *We have $t_n(\overline{\alpha}) = \rho_*(h_n)$.*[1525]

Proof $(*)$. The first statement follows immediately from the definitions. We turn to the proof of the second statement. As we will see, the proof of the second statement is given by a slightly tedious calculation. To simplify the notation we only deal with the case $n = 3$.

[1525]Here $\overline{\alpha}$ is the "inverse of the loop α", i.e. it is the loop given by $\overline{\alpha}(e^{i\varphi}) = \alpha(e^{-i\varphi})$. As above $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the $(x_2 = 0)$-hyperplane. Finally the framing $\rho_*(h_n)$ is defined as on page 2035.

At a given point $P = \gamma(e^{i\varphi}) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \in C_3$ we have

$$\begin{aligned} t_n(\overline{\alpha}) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} &= t_n \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \underbrace{\alpha(e^{-i\varphi})}_{=\overline{\alpha}(e^{i\varphi})} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} \\ &\underset{\substack{\uparrow \\ \text{since } \rho \text{ is reflection in the second coordinate} \\ \text{together with the symmetries of sin and cos}}}{=} \rho \left(\begin{pmatrix} \cos(-\varphi) & 0 \\ \sin(-\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} \right) \underset{\substack{\uparrow \\ \text{definition of } h_n, \text{ see page 2046}}}{=} \rho \left(h_n \begin{pmatrix} \cos(-\varphi) \\ \sin(-\varphi) \\ 0 \end{pmatrix} \right) \\ &\underset{\substack{\uparrow \\ \text{definition of } \rho}}{=} \rho \left(h_n \left(\rho^{-1} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \right) \right) \underset{\substack{\uparrow \\ \rho_*(h_n) \text{ as defined on page 2035}}}{=} \rho_*(h_n) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}. \end{aligned}$$

This concludes the mind-numbing verification of the second statement. □

Lemma 88.15. *Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be an embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$. Finally let $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be two smooth maps. If α and β are smoothly homotopic, then*

$$[(\gamma(S^1), f(\alpha))] \;=\; [(\gamma(S^1), f(\beta))] \;\in\; \Omega_1^{\mathrm{fr}}(\mathbb{R}^n).$$

Proof of Lemma 88.15 (∗). We write $M = \gamma(S^1)$. Since $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ are smoothly homotopic we can pick a smooth homotopy $H\colon S^1 \times [0,1] \to \mathrm{GL}_+(n-1, \mathbb{R})$ with $H_0 = \alpha$ and $H_1 = \beta$. We consider the product cobordism $[0,1] \times M$ equipped with the framing

$$\begin{aligned} V\colon [0,1] \times M &\to \mathrm{GL}(n+1, n) \\ (t, \gamma(z)) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & f(\gamma(z)) \cdot H(z,t) \end{pmatrix} \end{aligned}$$

It is clear that $([0,1] \times M, V)$ is a framed cobordism between $(\gamma(S^1), f(\alpha))$ and $(\gamma(S^1), f(\beta))$. □

Now we can provide the proof of Proposition 88.13.

Proof of Proposition 88.13. Let $n \in \mathbb{N}_{\geq 4}$. We consider the following diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "k \mapsto k \cdot [(C_3, h_3)]", "\cong"'] \arrow[d, two heads] \arrow[dr, two heads, "k \mapsto k \cdot [(C_n, h_n)]"'] & \Omega_1^{\mathrm{fr}}(\mathbb{R}^3) \arrow[d, two heads, "(n\text{-}3)\text{-fold suspension homomorphism}"] \\ \mathbb{Z}_2 \arrow[r, dashed] & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n). \end{tikzcd}$$

We make the following observations:

(1) It follows quite easily from Theorem 87.14, Lemma 88.12 and Proposition 88.4 that the top horizontal map is an isomorphism.
(2) The right vertical map is given by the $(n-3)$-fold application of the suspension homomorphism defined on page 2038. Note that the right vertical map is an epimorphism by the Freudenthal Suspension Theorem 88.7.
(3) The top right triangle commutes by definition of the suspension homomorphism and by definition of (C_n, h_n).

The above diagram and the above discussion show that it remains to prove the following claim.

Claim. The element $2 \cdot [(C_n, h_n)]$ is trivial in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1,\mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

We have the following equalities in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$:

$$\begin{array}{rcl} 2 \cdot [(C_n, h_n)] & \underset{\uparrow}{=} & [(C_n, h_n)] + [(C_n, t_n(\alpha))] \underset{\uparrow}{=} [(C_n, h_n)] + [(C_n, t_n(\overline{\alpha}))] \\ & & \text{Lemma 88.14 (1)} \quad \begin{array}{l} \text{since } n \geq 4 \text{ we know by Proposition 86.11 (2) that } \pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}_2, \\ \text{therefore we see that } [\alpha] \cdot [\alpha] = 0 \in \pi_1(\mathrm{GL}_+(n-1,\mathbb{R})), \\ \text{this implies by Proposition 5.7 that } \alpha \text{ is path-homotopic to } \overline{\alpha}, \\ \text{in fact, by the Whitney Approximation Theorem 65.6} \\ \text{we see that } \alpha \text{ is actually smoothly homotopic to } \overline{\alpha}, \\ \text{by Lemma 88.15 this gives us the equality } [(C_n, t_n(\alpha))] = [(C_n, t_n(\overline{\alpha}))] \end{array} \\ & \underset{\uparrow}{=} & [(C_n, h_n)] + [(\rho_*(C_n), \rho_*(h_n))] \underset{\uparrow}{=} [(C_n, h_n)] + (-[(C_n, h_n)]) = 0. \\ & & \text{Lemma 88.14 (2)} \qquad\qquad \text{Proposition 88.2} \end{array}$$

This concludes the proof of the claim. □

Let $n \geq 4$. By Proposition 88.13 we now know that there exists an epimorphism $\mathbb{Z}_2 \to \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1})$. In Theorem 78.13 we showed that $\pi_{n+1}(S^n)$ is non-trivial. Therefore it follows that $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$, which means we have proved Theorem 88.10. The readers who studied Steenrod operations *and* who happen to be tired of Thom-Pontryagin theory can thus move on to next section.

For all others we provide a proof of Theorem 88.10 that does not rely on Theorem 78.13. More precisely, we prove the following proposition which is the main technical result of this section.

Proposition 88.16. *For any $n \in \mathbb{N}_{\geq 4}$ the map*

$$\begin{array}{rcl} \mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is an isomorphism.

As we had mentioned many times, Theorem 88.10 is an immediate consequence of the Thom-Pontryagin Theorem 87.11 and Proposition 88.4 together with Proposition 88.16.

Proof. Let $n \in \mathbb{N}_{\geq 4}$. By Proposition 88.13 it remains to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We start out with a more general discussion. Given any $m \in \mathbb{N}_{\geq 2}$ we make the identification $C_m = \{0\} \times C_m \subset [0,1] \times \mathbb{R}^m$. The following claim applies to all dimensions.

Claim. Let $m \in \mathbb{N}_{\geq 2}$. Let F be a compact orientable connected proper 2-dimensional manifold in $[0,1] \times \mathbb{R}^m$ with $\partial F = C_m$. At most one of the two framings t_m and h_m can be extended to a framing of F.

Suppose there exist framings T and H on F with $T|_{C_m} = t_m$ and $H|_{C_m} = h_m$. It follows from Lemma 88.3 that we can assume that T and H are orthonormal. We consider the map[1526]

$$\begin{array}{rcl} \Psi\colon F & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ P & \mapsto & \text{the unique matrix } \Psi(P) \in \mathrm{GL}(m-1,\mathbb{R}) \text{ with } T(P)\cdot\Psi(P) = H(P). \end{array}$$

We denote by $i\colon C_m \to F$ the inclusion map and we denote by $\gamma\colon S^1 \to C_m$ the usual diffeomorphism given by $\gamma(e^{i\varphi}) = (\cos(\varphi), \sin(\varphi), 0, \dots, 0)$. We make the following key observation: it follows immediately from the definitions and Lemma 88.14 that the map $\Psi \circ i \circ \gamma\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ is precisely, on the nose, given by the map

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{m-3} \end{pmatrix}. \end{array}$$

Now we consider the following diagram

$$\begin{tikzcd} H_1(\partial F;\mathbb{Z}) \arrow[d, "i_*"'] & H_1(S^1;\mathbb{Z}) \arrow[l, "\gamma_*"'] \arrow[d, "\alpha_*"'] & \pi_1(S^1,1) \arrow[l, "\text{Hurewicz isom.}", "\cong"'] \arrow[d, "\alpha_*"'] \\ H_1(F;\mathbb{Z}) \arrow[r, "\Psi_*"] & H_1(\mathrm{GL}(m-1,\mathbb{R})) & \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id}). \arrow[l, "\cong", "\text{Hurewicz isom.}"'] \end{tikzcd}$$

We make the following observations:

(1) The left-hand square commutes by the above equality $\alpha = \Psi \circ i \circ \gamma$.
(2) The two horizontal maps to the right are the Hurewicz homomorphisms that we introduced on page 924. The Hurewicz homomorphisms are isomorphisms by the Hurewicz Theorem 38.5 together with the fact that the fundamental groups on the right are both abelian.

[1526]Let $P \in F$. The fact that T and H are orthonormal means that the columns of $T(P)$ and $H(P)$ are bases for the same vector space, namely $(T_P F)^\perp$. Therefore there exists a matrix $\Psi(P) \in \mathrm{GL}(m-1,\mathbb{R})$ such that $T(P)\cdot\Psi(P) = H(P)$. The fact that $P \mapsto \Psi(P)$ is continuous is left as an exercise.

(3) The right-hand square commutes by the naturality of the Hurewicz homomorphisms, see Proposition 38.2 (4).
(4) By Proposition 86.11 (2) and (3) we know that the loop $\alpha\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ represents a non-trivial element in $\pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$. This implies that the right vertical map $\alpha_*\colon \pi_1(S^1,1) \to \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$ is non-trivial.
(5) It follows from (1) – (4) that the left vertical map is also non-trivial. But this contradicts Lemma 35.9 (1).

This concludes the proof of the claim.

Now we start out with the actual proof of the theorem. Thus let $n \in \mathbb{N}_{\geq 4}$. As we pointed out above, we need to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. Suppose that that (C_n, h_n) represents in fact the trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. This means that there exists a framed cobordism $(F, v) = (F_n, v_n)$ in $[0,1] \times \mathbb{R}^n$ from (C_n, h_n) to the empty manifold. In other words, $F = F_n$ is a compact orientable proper 2-dimensional submanifold of $[0,1] \times \mathbb{R}^n$ with $\partial F_n = C_n$ and $v = v_n$ is a framing with $v|_{C_n} = h_n$. By throwing away all closed components of F_n we can assume that F_n is connected.

By the classification of compact orientable connected 2-dimensional manifolds, see Theorem 14.14, we know that there exists a $g \in \mathbb{N}_0$ such that F is diffeomorphic to $\Sigma := \Sigma_{g,1}$. In other words, there exists a proper embedding $\varphi\colon \Sigma \to [0,1] \times \mathbb{R}^n$ with $\varphi(\Sigma) = F_n$. Given any $m \geq n$ we denote by $\varphi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the embedding that is given by the composition of the embedding φ with the obvious embedding $[0,1] \times \mathbb{R}^n \to [0,1] \times \mathbb{R}^m$.

By Lemma 88.11 (1) we know that there exists a proper embedding $\psi\colon \Sigma \to [0,1] \times \mathbb{R}^2$ and a framing $w_2 = w$ for $G_2 = G := \psi(\Sigma)$ such that $\partial\psi(\Sigma) = C_2$ and such that the restriction of $w_2 = w$ to C_2 equals t_2. Given $m \geq 2$ we denote by $\psi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the embedding that is given by the composition of the embedding ψ with the obvious embedding $[0,1] \times \mathbb{R}^2 \to [0,1] \times \mathbb{R}^m$.

Now we consider the two proper embeddings $\psi_n, \varphi_n\colon \Sigma \to [0,1] \times \mathbb{R}^n$. We can arrange[1527] that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

It follows from Proposition 65.16 (2) and the Isotopy Extension Theorem 10.21 that there exists an $m \geq n$ and a diffeomorphism Ψ of the manifold $[0,1] \times \mathbb{R}^m$ such that $\Psi(F_m) = G_m$ and such that the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity. We refer to Figure 1082 for an illustration.

Since (F_m, v_m) is an extension of (C_m, h_m) and since the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity we see that $(\Psi(F_m), \Psi_*(v_m)) = (G_m, \Psi_*(v_m))$ is an extension of (C_m, h_m). On the other hand we saw (G_m, w_m) is an extension of (C_m, t_m). Thus both h_m and t_m can be extended over G_m. But this contradicts the above claim. □

[1527]This sentence contains a slight technical nuisance which we cleverly hide in a footnote. First of all, it should be pretty clear by now that Σ admits an orientation-reversing self-diffeomorphism. Thus, after possibly precomposing φ_n with such a diffeomorphism we can assume that $\psi_n^{-1} \circ \varphi_n$ is orientation-preserving. Using Corollary 20.6 and using the Collar Neighborhood Theorem 10.10 we can then arrange, after possibly precomposing φ_n by another diffeomorphism, that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

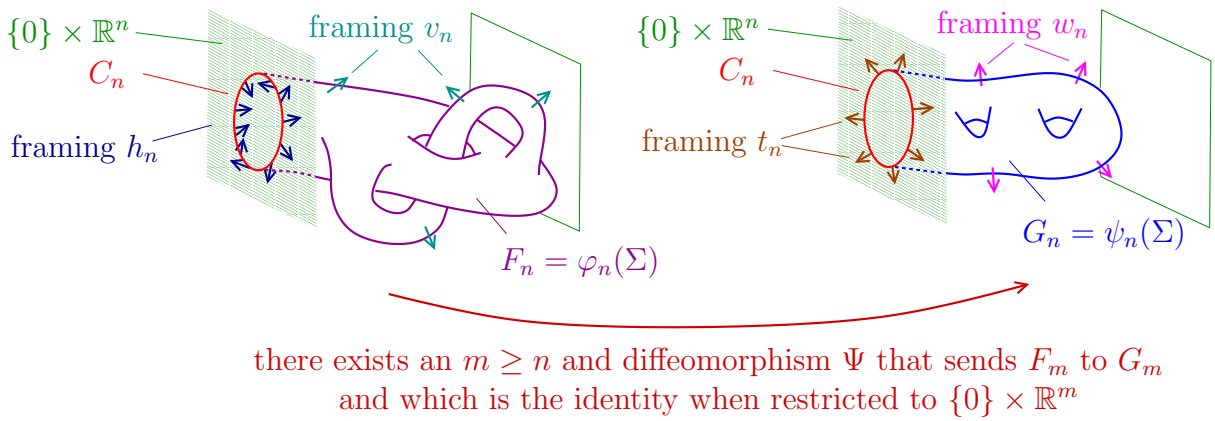


FIGURE 1082. Illustration for the proof of Proposition 88.16.

88.4. **Further results obtained using the Thom-Pontryagin approach.** In this final section we summarize a few more results that have been proved using the Thom-Pontryagin approach. First of all, using the Thom-Pontryagin method Pontryagin [Po2] proved in 1950 the following theorem.

Theorem 88.17. *For any $n \geq 2$ we have $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$.*

Remark.

(1) By Theorem 85.10 (1) we know that for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. Furthermore we know by Theorem 88.10 that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. Putting these two results together shows that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(2) In Theorem 79.2 we used Steenrod operations to show that $\pi_{n+2}(S^n) \neq 0$ for all $n \geq 2$.
(3) By the Freudenthal Suspension Theorem 88.6 we know that the suspension homomorphism $\pi_5(S^3) \to \pi_6(S^4)$ is an epimorphism and we know that for $n > k+1$ the suspension homomorphism $\pi_{n+2}(S^n) \to \pi_{n+3}(S^{n+1})$ if an isomorphism for $n \geq 5$.
(4) Unfortunately (1), (2) and (3) are *not* strong enough to give an alternative proof of Theorem 88.17.

Proof.

(1) As mentioned before, the theorem was proved by Lev Pontryagin [Po2, Po3]. See also [Sco, p. 233] and [WX2, Chapter 2.5] for outlines of the proof. The most readable account of the Thom-Pontryagin approach to proving the theorem is surely [Put].
(2) At the same time as Pontryagin two very different proofs that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 2$ were given by George W. Whitehead [Whda2] and Jean-Pierre Serre [Ser3, Proposition 10]. We also refer to [Hat6, Theorem 1.40 (a)] for a proof using spectral sequences and Steenrod operations. Alternatively see also [Hu2, Theorem XI.15.2].

□

Finally Rokhlin [Rok, Rok2] used the Thom-Pontryagin approach to prove the following theorem.

Theorem 88.18.

(1) *We have*[1528] $\pi_5(S^3) \cong \mathbb{Z}_2$, $\pi_6(S^3) \cong \mathbb{Z}_{12}$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$.
(2) *For any* $n \geq 5$ *we have* $\pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$.

Remark.

(1) In Theorem 78.13 we used Steenrod operations to show the weaker statement that $\pi_{n+3}(S^n) \neq 0$ for all $n \geq 2$.
(2) Note that Theorem 88.18 (1) gives a negative answer to Question 63.16.

Proof.

(a) The theorem was proved by Vladimir Rokhlin [Rok][Rok2, p. 21] in 1952, using the above Thom-Pontryagin approach.[1529]
(b) A somewhat different proof is sketched in [WX2, Chapter 2.6]. Some of the results can also be found in [Ser3, Proposition 10]. Other proofs are given in [Hat6, Theorem 1.40 (b)] and [Hu2, Chapter XI.16]. □

The results of the last two chapters summarize most (or all?) of the calculations of homotopy groups of spheres that have been obtained using the Thom-Pontryagin approach.

Exercises for Chapter 88.

Exercise 88.1. In the proof of Proposition 88.13 we used the Freudenthal Suspension Theorem 88.6 to show that the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^4) \\ [k] & \mapsto & k \cdot [(C_4, h_4)] \end{array}$$

is an epimorphism. Use Exercise 65.5 to give an alternative argument, which does not rely on the Freudenthal Suspension Theorem 88.6.

Exercise 88.2. Let $n \in \mathbb{N}_{\geq 2}$. We consider the loops

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{\alpha}\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} \end{array}.$$

Recall that in the proof of Lemma 88.15 we used the fact, proved in Proposition 86.11 (2), that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}_2$, to show that α is path-homotopic to $\overline{\alpha}$. Give a direct proof that the loop α is path-homotopic to the loop $\overline{\alpha}$.

[1528] In fact the first isomorphism $\pi_5(S^3) \cong \mathbb{Z}_2$ is a consequence of Theorem 85.10 and Theorem 88.17.

[1529] In fact Rokhlin [Rok3] initially thought that for $n \geq 5$ the groups $\pi_{n+3}(S^n)$ are isomorphic to $\mathbb{Z}_{12}$. The key to fixing the mistake is Rokhlin's Theorem 72.24 regarding signatures of certain 4-manifolds.

89. Summary of homotopy groups of spheres

The results from last chapter conclude our calculations of homotopy groups of spheres. Throughout the course we employed a long and varied list of tools to compute homotopy groups of spheres:

(1) Covering theory, see page 723.
(2) The Cellular Approximation Theorem 27.15, see Proposition 28.9.
(3) Homology theory and the Hurewicz Theorem 80.5, see Corollary 80.6.
(4) The cup product, see Corollary 63.14.
(5) Steenrod operations, see Theorems 78.13 and 79.2.
(6) The long exact sequence of a fibration, see Theorem 85.10.
(7) The Thom-Pontryagin Theory, see the Freudenthal Suspension Theorem 88.6 and Theorem 88.10.

In the following we summarize the results we have obtained on homotopy groups of spheres:

(1) On page 723 we saw that $\pi_n(S^1) = 0$ for $n \geq 2$.
(2) Proposition 28.9 says that for any $n \in \mathbb{N}$ we have $\pi_1(S^n) = \cdots = \pi_{n-1}(S^n) = 0$.
(3) Corollary 80.6 (or alternatively by Corollary 87.12) we know that for any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$.
(4) By Theorem 85.10 (1) we know that for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. In particular, together with (3) we know that $\pi_3(S^2) \cong \mathbb{Z}$. An alternative proof of the fact that $\pi_3(S^2) \cong \mathbb{Z}$ is given by the combination of the Thom-Pontryagin Theorem 87.11 and Theorem 87.14.
(5) In Corollary 63.14, and alternatively in Proposition 87.21 we saw that given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.
(6) By Theorem 85.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$. Furthermore, by Theorem 85.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.
(7) By the Freudenthal Suspension Theorem 88.6 we know that for $n = k+1$ we have an epimorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$ and we know that for $n > k+1$ we have an isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$.
(8) By Theorem 85.11 (2) we know that $\pi_6(S^4) \cong \pi_5(S^3)$ and by Theorem 85.14 (3) we know that $\pi_{14}(S^8) \cong \pi_{13}(S^7)$.
(9) In Theorem 78.13 we proved the following three statements:
    (a) for any $n \geq 2$ the group $\pi_{n+1}(S^n)$ is non-trivial,
    (b) for any $n \geq 4$ the group $\pi_{n+3}(S^n)$ is non-trivial,
    (c) for any $n \geq 8$ the group $\pi_{n+7}(S^n)$ is non-trivial.
(10) By Theorem 88.10 we know that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. This is a refinement of statement (9a). Note that it follows from this fact together with (4) that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(11) In Theorem 79.2 we proved the following three statements:
    (a) for any $n \geq 2$ the group $\pi_{n+2}(S^n)$ is non-trivial,

(b) for any $n \geq 4$ the group $\pi_{n+6}(S^n)$ is non-trivial,
(c) for any $n \geq 8$ the group $\pi_{n+14}(S^n)$ is non-trivial.

We recall that Serre's Theorem 81.9 says that all homotopy groups of spheres are finitely generated abelian groups. In the discussion above we saw that some of the homotopy groups of spheres are infinite.

The following theorem can be summarized as saying that we have picked up all elements of infinite order. In particular the theorem gives a complete answer to Question 63.16. This theorem can also be viewed as a strengthening of Serre's Theorem 81.9. In fact, the strengthening is also due to Jean-Pierre Serre.

Theorem 89.1.

(1) *If n is odd, then $\pi_k(S^n)$ is finite for $k \geq n+1$.*
(2) *If n is even, then the following two statements hold:*
  (a) *The group $\pi_k(S^n)$ is finite for $k \geq n+1$ and $k \neq 2n-1$.*
  (b) *The group $\pi_{2n-1}(S^n)$ is isomorphic to the direct sum of $\mathbb{Z}$ with a finite group.*

Proof. The theorem was first proved by Serre [Ser3] using "spectral sequences". Expositions of the proof are also given in [Hat6, Theorem 1.21] and [Sp, p. 515f]. □

The following table, obtained from [Tod, Chapter XIV], shows the low-dimensional homotopy groups of low-dimensional spheres. The entries colored in blue have been proved in these lecture notes.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$

Note that we showed in Theorem 85.10 (1) that the second and the third row, starting with π_3, are the same.

Recall that the Freudenthal Suspension Theorem 88.6 says that for any $n, k \in \mathbb{N}_0$ with $n > k+1$ the suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism. Thus we see that the groups $\pi_{n+k}(S^n)$ "stabilize" as $n \to \infty$. It is therefore perhaps more instructive to shift in the above diagram the i-th row leftward by $i-1$ terms. This way the diagonal lines become vertical. In other words we consider the following diagram where

π_{n+k} is short for $\pi_{n+k}(S^n)$.

n	π_{n+0}	π_{n+1}	π_{n+2}	π_{n+3}	π_{n+4}	π_{n+5}	π_{n+6}	π_{n+7}	π_{n+8}	π_{n+9}	π_{n+10}
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
6	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_9$
7	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_2$
8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_8^2\times\mathbb{Z}_2\times\mathbb{Z}_3^2$
9	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_6$
10	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2^3$	$\mathbb{Z}_4\times\mathbb{Z}_6$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}_3$

Here for convenience we colored the "diagonal" $\pi_n(S^{2n})$ in red, since the Freudenthal Suspension Theorem 88.6 implies that below the diagonal the groups on each column are the same.

We conclude this discussion of the precise homotopy groups of spheres with the following sobering fact:

Fact 89.2. *There is no sphere S^n with $n \geq 2$ for which we*[1530] *know the isomorphism types of all homotopy groups.*

As we had mentioned above, the Freudenthal Suspension Theorem 88.6 says that homotopy groups of spheres "stabilize". This leads us to the following definition.

Definition. Let $k \in \mathbb{N}_0$. We refer to[1531]

$$\pi_k^s \;:=\; \varinjlim \pi_{n+k}(S^n)$$

as the *k-th stable homotopy group of spheres.*[1532]

Note that it follows from Theorem 89.1 that for any $k \geq 1$ the k-th stable homotopy group π_k^s is finite. The following table obtained from [Wiki] gives the stable homotopy groups up to $k = 63$. It seems like 64 is the first dimension for which we do not know the

[1530]Here "we" is defined as members of our species.

[1531]Given $k \in \mathbb{N}_0$ the groups $\{\pi_{n+k}(S^n)\}_{n\in\mathbb{N}}$ together with the suspension homomorphisms form a direct system. Thus it makes sense to consider the direct limit.

[1532]Note that for any n, k with $n > k + 1$ the Freudenthal Suspension Theorem 88.6, together with Lemma 52.2 (4) and Lemma 52.3, says that the natural map $\pi_{n+k}(S^n) \to \pi_k^s$ is an isomorphism.

corresponding stable homotopy groups.

n	0	1	2	3	4	5	6	7
π^s_{0+n}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{8\cdot 3}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16\cdot 3\cdot 5}$
π^s_{8+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_{8\cdot 9\cdot 7}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{32\cdot 3\cdot 5}$
π^s_{16+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\,\mathbb{Z}_2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3\cdot 11}$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3}\,\mathbb{Z}_{16\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{24+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_{64\cdot 3\cdot 5\cdot 17}$
π^s_{32+n}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_4\,\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 27\cdot 7\cdot 19}$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_{2\cdot 3\cdot 5}$	$\mathbb{Z}_2^5\,\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 25\cdot 11}$
π^s_{40+n}	$\mathbb{Z}_2^5\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_{8\cdot 3\cdot 23}$	$\mathbb{Z}_8$	$\mathbb{Z}_2^3\,\mathbb{Z}_{16\cdot 9\cdot 5}$	$\mathbb{Z}_2^4\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_4\,\mathbb{Z}_3\,\mathbb{Z}_{32\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{48+n}	$\mathbb{Z}_4\,\mathbb{Z}_2^4$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4\,\mathbb{Z}_2$	$\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 5\cdot 29}$
π^s_{56+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 9\cdot 7\cdot 11\cdot 31}$	$\mathbb{Z}_4$	0	$\mathbb{Z}_2^2\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_{128\cdot 3\cdot 5\cdot 17}$

This concludes our discussion of the homotopy groups of spheres. We refer to [May, WX2, Wiki] and [Hat, Chapter 4] for a more detailed discussion.

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