De Rham cohomology

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Wednesday 14ct in M102

	Office hours
Stefan Friedl	Thursday 14-15
Gerrit Herrmann	Tuesday 10-11
Enrico Toffoli	Tuesday 14-15

1	19 October		fällt aus wegen Baulärm
2	26 October	Stefan Friedl	Differential forms and the de Rham complex
3	2 November	Enrico Toffoli	The Poincaré lemma
4	9 November	Andreas Wicher	The Mayer-Vietoris sequence
5	16 November	Oliver Trinkaus	Homotopy properties
6	23 November	Georg Lipp	The de Rham cohomology of surfaces
7	30 November	E. T. and G. H.	Approximation by smooth maps & Sard's Theorem
8	7 December	Christian Plankl	The Brouwer-Jordan separation theorem
9	14 December		cancelled b/c of Windberg
10	21 December	Gerrit Herrmann	The de Rham cohomology in top dimension
11	11 January	Felix Rupprecht	The degree of a continuous map
12	18 January	Maximilian Blüml	The linking number of submanifolds
13	25 January	Stefan Wolf	The "ham sandwich" theorem
14	1 February	Christian Lotter	De Rham cohomology with compact support
15	8 February	Alexander Neumann	The Poincaré Duality

Detailed plan

For the topics listed without a specific reference, it is to be intended that the material can be found in the main reference next to the title.

1. Cancelled because of construction work

2. **Differential forms and the de Rham complex** ([MT97], Chapter 2 and 3) State and prove Theorem 3.7 and 3.12.

3. **The Poincaré lemma** ([MT97], Chapter 1 and 3) Give the examples from Chapter 1. Then prove Theorem 3.15.

4. **The Mayer-Vietoris sequence** ([Tu11], Chapter 7, §26). Recall the smooth partitions of unity (Theorem 9.11, Friedl Analysis IV: Satz 9.18), without proof. Recall the snake lemma and prove the Mayer-Vietoris sequence for smooth manifolds. Use Mayer-Vietoris for computing the de Rham cohomology of S^1.

5. **Homotopy properties** ([MT97], Chapters 4 and 6). Define homotopy of continuous maps and prove basic properties (from Definition 6.1 to Example 6.5). Define chain homotopy and prove that chain-homotopic maps induce the same map in cohomology (Lemma 4.11). Discuss Remark 4.12. Prove that homotopic smooth maps on open sets induce chain homotopic maps on the de Rham complex (Theorem 6.7: assume that the homotopy is also smooth). Use Lemma 4.11 to conclude that homotopic smooth maps between open sets induce the same map in cohomology.

6. **The de Rham cohomology of surfaces** ([Tu11], Chapter 7, §28).

7. **Approximation by smooth maps** ([MT97], Chapter 6 and 9, Appendix A). Prove the approximation results of continuous maps by smooth maps: Lemma A.9 and Lemma 6.6. Remark that these theorems allow us to to loose the hypotheses of the Theorem like 6.7 (in the previous talk, we had to suppose that the two maps were "smoothly homotopic"). Define the map induced on de Rham cohomology by a continuous (non-smooth) map between open sets. Prove Theorem 6.8. Prove topological invariance (Corollary 6.9) and triviality for contractible sets (Corollary 6.9).

8. **The Brouwer-Jordan separation theorem** ([MT97], Chapter 7) State the classical Jordan-curve theorem as a motivation. Aim of the talk is to understand and prove Theorem 7.10 and Proposition 7.16. In order to achieve this your talk has to cover Proposition 6.11 and page 48-51 in [MT97].

9. **The de Rham cohomology in top dimension** ([MT97], Chapter 10). Prove Theorem 10.13, i.e. $H^n(M) = \mathbb{R}$ for a n-dimensional manifold M.

10. **Sard's theorem.** Show that for every smooth map $f : M^m \to N^n$ the set of regular values is dense in N^n. You can either use [MT97, 11.5-11.7] or [Br93, Chapter 6 & Appendix C] as a reference. Choose some standard applications of Sard's Theorem e.g. there is no smooth space filling curve $c\colon [0,1] \to [0,1]^2$ or the fundamental theorem of algebra.

11. **The degree of a continuous map** ([MT97], Chapters 11 and 7). Define the degree of a map, state the basic properties and prove them. Show that the degree is always an integer (Theorem 11.9). As an application, prove Lemma 7.2 using the degree and deduce from this The Brouwer fixed-point theorem (Lemma 7.1).

12. **The linking number of submanifolds** ([MT97], Chapter 11). Define the linking number (Definition 11.12) and prove its basic properties and alternative definitions

(Proposition 11.13, Theorem 11.14). Show how one can compute the linking number combinatorially in the case of two knots in $\mathbb{R}^3$ (Remark 11.15).

13. **The "ham sandwich" theorem.** ([Fu95, Chapter 4c& d]) Cover the material of the main reference, with a special regard to the Propositions 4.22 and 4.32. You may use the definition of degree as defined in [Fu95].

14. **De Rham cohomology with compact support** ([MT97], Chapter 13). Define the compactly supported de Rham cohomology and prove its basic properties. Show that a Mayer-Vietoris sequence exists in this framework. Define the bilinear pairing between $H^p(M^n)$ and $H_c^{n-p}(M^n)$, and the induced map D_M^p. Discuss the five-lemma, which will be needed in the last talk (Exercise 4.1).

15. **The Poincaré Duality** ([MT97], Chapter 13). Recall the map D_M^p defined previously. State and prove Theorem 13.5, through the series of lemmas which appear in the chapter.

References

[Br93] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, Springer New York (1993)

[Fu95] W. Fulton, *Algebraic Topology: A First Course*, Graduate Texts in Mathematics, Springer Science (1995)

[MT97] I. Madsen und J. Tornehave, *From calculus to cohomology, de Rham cohomology and characteristic classes*, Cambridge University Press (1997)

[Tu11] L. Tu, *An Introduction to Manifolds*, Graduate Texts in Mathematics, Springer Science (2011)