Spectral Sequences and applications

Tuesday 14:00-16:00

Stefan Friedl, Gerrit Herrmann and Enrico Toffoli

1	10 April	Julian Hannes	Generalities on homotopy groups
2	17 April	Daniel Grünbaum	Fibrations and fiber bundles
3	24 April	Christoph Schießl	The path space construction
-	1 May	-	-
4	8 May	Stefan Wolf	Spectral sequences from exact couples
5	15 May	Dima Shvetsov	The Serre spectral sequence and first applications
-	22 May	-	-
6	29 May	Catherine-Leveke Lowzow	Construction of the sequence
7	5 June	Dannis Zumbil	The Hurewicz theorem and other applications
8	12 June		The Serre spectral sequence in cohomology
9	19 June	Constantin Benninger	More about the homotpy groups of spheres
10	26 June	Felix Rupprecht	Applications to group (co)homology
11	3 July	Jakob Schubert	Generalized cohomology theories
12	10 July	Daniel Fauser	The Atiyah-Hirzebruch spectral sequence

Detailed plan

1. **Generalities on homotopy groups**([Ha1, Section 4.1]) Recall the definition of homotopy groups and define relative homotopy groups. Discuss the group structure and when these are abelian groups. Moreover, give a proof of [Ha1, Theorem 4.3]. Define the Hurewicz map and discuss [Ha1, Proposition 4.36] and the commutativity of the diagram on [Ha1, p. 370].

2. **Fibrations and fiber bundles** ([Ha1, Section 4.2]) Give the definition of a fiber bundle, the homotopy lifting problem and fibrations. You should prove [Ha1, Theorem 4.41] and [Ha1, Proposition 4.48]. During your talk you should discuss several examples. Sketch the prove of [Ha1, Proposition 4.61]. What happens in the special case that the path in the prove is a loop and the fibration is a fiber bundle?

3. **The path space construction** Recall the defintion of a fibration. Recall the definition of the compact open topology and discuss [Ha1, Proposition A.14]. De-

fine the loop space and the path space and prove [Ha1, Proposition 4.64]. Then discuss the concept of a homotopy fiber. Discuss [Ha1, Example 4.16]. Introduce the notation of an Eilenberg-MacLane space of a group G and state without a proof [Ha1, Proposition 4.30]. If time permits give the definition of a Postinokov tower.

4. **Spectral sequences from exact couples** ([Ha2, Section 5.1]) Introduce the general concept of a spectral sequence. Define exact couples and discuss the corresponding derived couples. Show how an exact couple (in the form of a staircase diagram) naturally arises from a filtration of a topological space. Discuss in detail the associated spectral sequence and prove the main algebraical result (Proposition 5.2).

5. **The Serre spectral sequence and first applications** Define the Serre spectral sequence associated to a fibration, and state the main result about its convergence ([Ha2, Theorem 5.3]). Use the Serre spectral sequence in order to compute the homology of $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ ([Ha2, Example 5.4]) and ΩS^n ([Ha2, Example 5.4] or [DK, Section 9.2]). If the time permits it, as a last application compute the Euler characteristic of a fibration with trivial action of the base space ([DK, Theorem 9.16]).

6. **Construction of the sequence** ([Ha2, Section 5.1]) Recall the definition of the Serre spectral sequence and the statement of Theorem 5.3. Prove the theorem in the case of the base space being a CW complex. Give an idea of how the result can be extended to general spaces through a CW-approximation.

7. **The Hurewicz theorem and other applications** Using the Serre spectral sequence, prove the classical Hurewicz theorem ([BT, Theorem 17.21]: ignore their technical issue about the spaces needing to be CW complexes). As a consequence, discuss $\pi_k(S^n)$ for $k \leq n$. Use then the Serre spectral sequence to construct the Gysin sequence and prove its exactness ([DK, Theorem 9.17]). As an application, recover the homology of $\mathbb{C}P^n$. This last computation, as well as more details for the rest of the talk, can be found in [HP, Chapter 3].

8. **The Serre spectral sequence in cohomology** ([Ha2, Section 5.1]) Introduce the Serre spectral sequence in cohomology, by constructing the appropriate staircase diagram. State Theorem 5.15 without proof. Discuss in detail the multiplicative structure in the spectral sequence (pages 543-546). As an application, derive the cohomology ring of $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ (Example 5.16).

9. **More about the homotopy groups of spheres** Use the Serre spectral sequence and the Hurewicz theorem to show that $\pi_4(S^3) = \mathbb{Z}/2$ ([DK, Theorem 9.26]). State without proof the mod $\mathcal{C}$ Hurewicz theorem ([DK, Theorem 10.5, 1.], or [Ha2, Theorem 5.8]), at least for the classes of finitely generated abelian groups and finite abelian groups (the proof is a generalization of the one we gave for the classical version of the theorem). Discuss the immediate consequences about homotopy

groups of spheres ([DK, Corollary 10.7 and 10.9]). Use the Serre spectral sequence and the mod $\mathcal{C}$ Hurewicz theorem to compute all homotopy groups of spheres up to torsion ([Ha2, Theorem 5.22]).

10. **Applications to group (co)homology** Discuss the special case of an Eilenberg-MacLane space $K(G,1)$ and give several examples ([Ha1, Section 1.B]). Discuss the construction of BG [Ha1, Example 1.B7] and the functoriality. If time permits give the idea of the proof of [Ha1, Proposition 1B.9]. Define group (co)-homology [DK, Definition 9.26]. Then discuss in detail [Ha2, Example 5.6] especially the part that a short exact sequence yields a fibration of Eilenberg-MacLane spaces. If time permits solve [DK, Exercise 156].

11. **Generalized cohomology theories** Discuss the Eilenberg-Steenrod axioms for homology ([Ha1, p. 160–162]). Define the sets $\mathrm{Vect}(X)$ and $\mathrm{Vect}(X)_{\mathbb{C}}$ and the semigroup structure on them. Then define higher K-groups as in [At, Chapter 2.4] and discuss the axioms for homology for the higher K-groups.

12. **The Atiyah-Hirzebruch spectral sequence** State the Atiyah-Hirzebruch spectral sequence without proof. As an applications show that a generelized homology theory satisfying the dimension axiom is unique. Moreover, state Bott periodicity and calculate the K-theory of $\mathbb{C}P^n$ [DK, p. 255-256].

References

[At] M. Atiyah, *K-theory.* Addison-Wesley, lecture notes, 1964

[BT] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology.* Springer-Verlag, Graduate Texts in Mathematics, 1982

[DK] J. F. Davis, P. Kirk, *Lecture Notes in Algebraic Topology.* American Mathematical Society, 2001

[Ha1] A. Hatcher, *Algebraic Topology.* Cambridge University Press, 2002

[Ha2] A. Hatcher, *Spectral Sequences.* Chapter 5 of *Algebraic Topology*, preliminary version, available online

[HP] M. Holmberg-Péroux, *The Serre Spectral Sequence*, Bachelor Semester Project at the École Polytechnique Fédérale de Lausanne, available online