AN INTRODUCTION TO 3-MANIFOLDS AND THEIR FUNDAMENTAL GROUPS

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Introduction

In these lecture notes we will give a quick introduction to 3-manifolds, with a special emphasis on their fundamental groups.

In the first section we will show that given $k \geq 4$ any finitely presented group is the fundamental group of a closed, oriented k–dimensional manifold. This is not the case for 3-manifolds. For example we will see that $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathbb{Z}^3$ are the only abelian groups which arise as fundamental groups of closed 3-manifolds. In the second section we recall the classification of surfaces via their geometry and outline the proofs for several basic properties of surface groups. Furthermore we will summarize the Thurston classification of diffeomorphisms of surfaces.

Then we will shift our attention to 3-manifolds. In the third section we will first introduce various examples of 3-manifolds, e.g. lens spaces, Seifert fibered spaces, fibered 3-manifolds and exteriors of knots and links. Furthermore we will see that new examples can be constructed by connected sum, by gluing along boundary tori and by taking finite covers. The goal in the remainder of the lecture notes will then be to bring some order into the world of 3-manifolds. The Prime Decomposition Theorem of Kneser and Milnor stated in Section 4.1 will allow us to restrict ourselves to prime 3-manifolds. In Section 4.2 we will state Dehn's Lemma and the Sphere Theorem, the combination of these two theorems shows that most prime 3-manifolds are aspherical and that most of their topology is controlled by the fundamental group.

In Section 2 we had seen that 'most' surfaces are hyperbolic, in Section 5 we will therefore study basic properties hyperbolic 3-manifolds. The justification for studying hyperbolic 3-manifolds comes from the Geometrization Theorem conjectured by Thurston and proved by Perelman. The theorem says that any prime manifold can be constructed by gluing Seifert fibered spaces and hyperbolic manifolds along incompressible tori. In Section 7 we will report on the recent resolution of Thurston's virtual fibering conjecture due to Agol and Wise. In Sections 8 and 9 we will state several other consequences of the work of Agol, Wise and many others.

Caveat. These are lecture notes and surely they still contain inaccuracies. For precise statements we refer to the references. Most theorems are stated precisely in [AFW15].

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Contents

1. Finitely presented groups and high dimensional manifolds

1.1. **Finitely presented groups.** We start out with several basic definitions in combinatorial group theory.

Definition. Let $x_1, \dots, x_n$ be symbols, then we denote by

$$\langle x_1, \dots, x_n \rangle$$

the free group with generators $x_1, \dots, x_n$. If $r_1, \dots, r_m$ are words in $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$, then we denote by

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$$

the quotient of $\langle x_1, \dots, x_n \rangle$ by the normal closure of $r_1, \dots, r_m$, i.e. the quotient by the smallest normal subgroup which contains $r_1, \dots, r_m$. We call $x_1, \dots, x_n$ *generators* and $r_1, \dots, r_m$ *relators*.

Definition. If G is isomorphic to a group of the form $\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$, then we say that G is *finitely presented*, and we call

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$$

a *presentation of* G. We call $n - m$ the deficiency of the presentation, and we define the *deficiency of* G to be the maximal deficiency of any presentation of G.

Example.

(1) The free abelian group $\mathbb{Z}^3$ is isomorphic to

$$\langle x_1, x_2, x_3 \,|\, [x_1, x_2], [x_1, x_3], [x_2, x_3] \rangle,$$

the deficiency of this presentation is zero, and one can show, see e.g. [CZi93, Section 5], that the deficiency of $\mathbb{Z}^3$ is indeed zero.

(2) The free abelian group $\mathbb{Z}^4$ is isomorphic to

$$\langle x_1, x_2, x_3, x_4 \,|\, [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4] \rangle,$$

the deficiency of this presentation is -2, and one can show, see again [CZi93, Section 5], that the deficiency of $\mathbb{Z}^4$ is indeed -2. Similarly, the deficiency of any free abelian group of rank greater than three is negative.

1.2. **Fundamental groups of high dimensional manifolds.** Let M be a manifold. (Here, and throughout these lectures, manifold will almost always mean a smooth, compact, connected, orientable manifold, we will not assume though that manifolds are closed.) Any manifold has a CW structure with one 0–cell and finitely many 1–cells and 2–cells. This decomposition gives rise to a presentation for $\pi = \pi_1(M)$, where the generators correspond to the 1–cells and the relators correspond to the 2–cells. We thus see that $\pi_1(M)$ is finitely presented. The following question naturally arises:

Question 1.1. *Which finitely presented groups can arise as fundamental groups of manifolds?*

Already by looking at dimensions 1 and 2 it is clear that the answer depends on the dimension. It turns out that the question has a simple answer once we go to manifolds of dimension greater than three.

Theorem 1.2. *Let G be a finitely presented group and let $k \geq 4$. Then there exists a closed k-dimensional manifold M with $\pi_1(M) = G$.*

Proof. We pick a finite presentation

$$G = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle.$$

We consider the connected sum of n copies of $S^1 \times S^{k-1}$. Its fundamental group is canonically isomorphic to $\langle x_1, \ldots, x_n \rangle$. We represent $r_1, \ldots, r_m$ by disjoint embedded closed curves $c_1, \ldots, c_m$. We consider the inclusion map

$$\begin{aligned} X &:= (S^1 \times S^{k-1} \# \ldots \# S^1 \times S^{k-1}) \setminus (c_1 \times D^{k-1} \cup \cdots \cup c_m \times D^{k-1}) \\ &\qquad \downarrow \iota \\ Y &:= S^1 \times S^{k-1} \# \ldots \# S^1 \times S^{k-1}. \end{aligned}$$

This map induces an epimorphism of fundamental groups. Indeed, this follows from the observation that by general position any closed curve can be pushed off the curves $c_1, \ldots, c_m$. But this map also induces a monomorphism. Indeed, if a curve $c \subset X$ bounds a disk $D \subset Y$ in $S^1 \times S^{k-1} \# \ldots \# S^1 \times S^{k-1}$, then again by a general position argument we can push the disk off the curves $c_1, \ldots, c_m$ (here we used that $n \geq 4 > 2 + 1$). Thus the curve c already bounds a disk in X, i.e. it is null homotopic in X.

Finally we consider the closed manifold

$$(S^1 \times S^{k-1} \# \ldots \# S^1 \times S^{k-1}) \setminus \bigcup_{i=1}^{m} c_i \times D^{k-1} \ \cup \bigcup_{i=1}^{m} D^2 \times S^{k-2},$$

where we glue a disk to each curve c_i. It follows from the van Kampen theorem, that this closed manifold has the desired fundamental group. □

Adyan [Ad55] and Rabin [Rab58] showed that the isomorphism problem for finitely presented groups is not solvable. This deep fact allowed Markov [Mav58, Mav60] to prove the following corollary to Theorem 1.2.

Corollary 1.3. *Let $k \geq 4$. Then there is no algorithm which can decide whether or not two k-dimensional manifolds are diffeomorphic.*

Now we will see that the statement of Theorem 1.2 does not hold in dimension 3:

Proposition 1.4. *Let N be a closed 3-manifold, then $\pi_1(N)$ admits a presentation of deficiency zero.*

Remark.

(1) It is well-known that 'most' abelian groups have negative deficiency, see e.g. [CZi93, Section 5]. In fact the only abelian groups which admit a presentation of deficiency zero are $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n$ or $\mathbb{Z} \oplus \mathbb{Z}/n$. Later in Proposition 4.9 we will get a few more restrictions on fundamental groups of 3-manifolds which will allow us to completely determine the abelian groups that appear as fundamental groups of closed 3-manifolds.
(2) Epstein [Ep61] showed that the deficiency of the fundamental group of a closed prime 3-manifold is in fact zero.

Proof. We pick a triangulation of N and we denote by H a closed tubular neighborhood of the 1–skeleton. Note that H and $K := \overline{N \setminus H}$ are handlebodies of the same genus, say g. Thus we obtain N by starting out with the handlebody H and gluing on the handlebody K. Put differently, we start out with the handlebody H of genus g, then we glue in g disks and finally we glue in one 3-ball. It follows from the Seifert van-Kampen Theorem that $\pi_1(N)$ admits a presentation with g generators and g relators. □

2. Surfaces and their fundamental groups

2.1. **The classification of surfaces.** Now we turn to the study of surfaces and their fundamental groups. (Here, unless we say explicitly otherwise, by a surface we mean a connected, smooth, orientable, compact 2-dimensional manifold.) Surfaces and their fundamental groups are for the most part well understood and many have nice properties, which will be guiding us later in the study of 3-manifold groups. Surfaces will also play a key rôle in the study of 3-manifolds. For the most part we will in the following also include the case of non–orientable surfaces.

Surfaces have been completely classified, more precisely the following theorem was already proved in the 19th century. We refer to [CKK] for a modern proof.

Theorem 2.1. *Two surfaces are diffeomorphic if and only if they have the same Euler characteristic, the same number of components and the same orientability.*[1]

In the study of surfaces it is helpful to take a geometric point of view. In particular, it follows from the Gauss–Bonnet Theorem that if a closed surface Σ admits a Riemannian metric of area A and constant curvature K, then

$$K \cdot A \;=\; 2\pi\chi(\Sigma),$$

in particular the Euler characteristic gives an obstruction to what type of constant curvature metric a surface can possibly admit.

The uniformization theorem says, that a constant curvature metric which is allowed by the Gauss–Bonnet theorem, will actually occur. More precisely, we have the following table

[1] In fact the same conclusion holds with 'diffeomorphic' replaced by 'homeomorphic'. This is a considerably deeper result due to Radó [Rad25], see also [Hat13].

for closed surfaces, where for once we also allow non-orientable surfaces:

$\chi(\Sigma)$	> 0	$= 0$	< 0
type of surface	S^2 or $\mathbb{R}\mathrm{P}^2$	torus or Klein bottle	everything else
Σ admits metric of constant curvature	$\equiv 1$	$\equiv 0$	$\equiv -1$
universal cover	S^2	$(\mathbb{R}^2, \text{Euclidean metric})$	$\mathbb{H}^2$

Here we think of S^2 and $\mathbb{R}^2$ as equipped with the usual metrics of constant curvature $+1$ respectively 0, and we denote by

$$\mathbb{H}^2 = \{(x, y) \,|\, y > 0\}$$

the upper half plane together with the complete metric of curvature -1 given by

$$\frac{1}{y} \cdot \text{standard metric on } \mathbb{R}^2.$$

The action of $\pi_1(\Sigma)$ on the universal cover $\widetilde{\Sigma}$ shows that $\pi_1(\Sigma)$ is a discrete subgroup of $\mathrm{Isom}(\widetilde{\Sigma})$ which acts on $\widetilde{\Sigma}$ cocompactly and without fixed points. For closed orientable surfaces we thus obtain the following table:

$\chi(\Sigma)$	> 0	$= 0$	< 0
$\mathrm{Isom}(\widetilde{\Sigma})$	$O(3)$	$O(2) \ltimes \mathbb{R}^2$	$PS^*L(2,\mathbb{R})$
$\pi_1(\Sigma)$	0 or $\mathbb{Z}/2$	$\mathbb{Z}^2$ or $\langle a, b \,|\, abab^{-1}\rangle$	torsion-free Fuchsian group

Here

$$\begin{aligned} S^*L(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \,|\, \det(A) = \pm 1\}, \\ PS^*L(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \,|\, \det(A) = \pm 1\}/\pm \mathrm{id} \end{aligned}$$

acts on $\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

Remark.

(1) The fact that every surface supports a complete metric of constant curvature is often referred to as the 'Uniformization Theorem'.
(2) The hyperbolic structure on a closed surface is not necessarily unique, in fact the space of hyperbolic structures on a closed surface of genus g (up to isotopy) is $(6g - 6)$–dimensional. [2]
(3) The fundamental group of an orientable hyperbolic surface is a discrete subgroup of

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{id}\}.$$

[2]Given a fixed surface we can associate to each hyperbolic structure a vector in $\mathbb{R}^{6g-6}$ by taking the lengths of certain fixed $6g - 6$ curves. This defines a homeomorphism. See e.g. `www.math.sunysb.edu/~jabehr/GeomandTeich.ps` for details.

(4) Surfaces with boundary can be classified in a very similar fashion. More precisely, if Σ is a surface with boundary, then we refer to $\Sigma \setminus \partial\Sigma$ as the *interior of* Σ. The Uniformization Theorem for surfaces with boundary says that the interior of $\Sigma \neq D^2$ supports a complete metric of constant curvature, where the sign of the curvature is once again given by the sign of the Euler characteristic.

We obtain the following corollary to the uniformization theorem:

Lemma 2.2. *Let $\Sigma \neq S^2, \mathbb{R}P^2, D^2$ be a (possibly non-orientable) surface. The the following hold:*

(1) *Σ is aspherical, in particular Σ is an Eilenberg–Maclane space for π.*
(2) *$\pi_1(\Sigma)$ is torsion-free.*

Proof.

(1) We denote by $\widetilde{\Sigma}$ the universal cover of Σ. For any $k \geq 2$ we have $\pi_k(\Sigma) \cong \pi_k(\widetilde{\Sigma})$, but the latter groups are zero since $\widetilde{\Sigma} = \mathbb{R}^2$ or $\widetilde{\Sigma} = \mathbb{H}^2$ by the uniformization theorem.
(2) This will follow from (1) and the following more general claim:

Claim. Let π be a group which admits a finite dimensional $K(\pi, 1)$, then π is torsion free.

Let X be a finite dimensional Eilenberg–Maclane space for π. Let $G \subset \pi$ be a non-trivial cyclic subgroup. We have to show that G is infinite cyclic. We denote by $\widehat{X}$ the cover corresponding to $G \subset \pi = \pi_1(X)$. Then $\widehat{X}$ is an Eilenberg–Maclane space for G. Since $\widehat{X}$ is finite dimensional it follows that $H_i(G; \mathbb{Z}) = 0$ for all but finitely many dimensions. Since finite cyclic groups have non–trivial homology in all odd dimensions it follows that G is infinite cyclic. See [Hat02, Proposition 2.45].

□

2.2. **Fundamental groups of surfaces.** Given a space X with fundamental group π, we want to answer the following questions:

(1) Given a ring R, is π *linear over the ring* R, i.e. does there exist a monomorphism $\pi \to \mathrm{GL}(n, R)$ for a sufficiently large n?
(2) Does π have 'many finite index quotients', i.e. does X have many finite covers?
(3) Does π admit finite index subgroups with large homology?

Positive answers are useful for various reasons:

(1) Linear groups are reasonably well understood and have many good properties, e.g. they are residually finite (see Proposition 2.5) and they satisfy the Tits alternative [3] (see [Ti72]),

[3] The Tits alternative says that a finitely generated linear group either contains a non–abelian free group or it admits a finite index subgroup which is solvable.

(2) The existence of 'many finite covers' allows us to study X through its finite covers, for example if N is a smooth 4-manifold, then the Seiberg–Witten invariants of its finite covers will in general contain more information then the Seiberg–Witten invariants of N alone,
(3) 'Large homology groups' means that a space has 'lots of interesting submanifolds'. For example, if N is a closed n–manifold, then $H_{n-1}(N;\mathbb{Z}) = H^1(N;\mathbb{Z}) \neq 0$ implies that N admits codimension-one submanifolds along which we can decompose N into hopefully easier pieces.

In the following we will see that surface groups have, perhaps not surprisingly, very good properties. In particular we will get 'best possible' answers to the above questions.

Proposition 2.3. *The fundamental group of any surface is linear over* $\mathbb{R}$*.*

Proof. We only consider the case that Σ is orientable with $\chi(\Sigma) < 0$. The other cases are left as an exercise. By the Uniformization Theorem we know that $\pi_1(\Sigma)$ is a subgroup of $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$, but the latter group is isomorphic to $\mathrm{SO}^+(1,2) \subset \mathrm{SL}(3,\mathbb{R})$. Here $\mathrm{SO}^+(1,2)$ □

In fact the following stronger statement holds:

Proposition 2.4. *Let* Σ *be a surface, then* $\pi := \pi_1(\Sigma)$ *is linear over* $\mathbb{Z}$*.*

Proof. A general principle says that a 'generic' pair of matrices $A, B \in \mathrm{GL}(n,\mathbb{Z})$ will generate a free group on two generators. For example, one can use the 'ping-pong lemma' to show that

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group.[4] In particular $\mathrm{SL}(2,\mathbb{Z})$ contains a free group on two generators, and it thus contains any free group, in particular it contains the fundamental group of any surface with boundary.

If Σ is a closed surface, then Newman [Ne85] has shown that there exists an embedding $\pi_1(\Sigma) \to \mathrm{SL}(8,\mathbb{Z})$. [5] □

Definition. Let P be a property of groups. We say that a group π is *residually* P if given any non–trivial $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a group G which has property π. [6]

Remark.

(1) Any finitely generated abelian group is residually finite.
(2) The group $(\mathbb{Q}, +)$ is not residually finite, in fact it has no finite quotients at all.

[4]For a proof see:
`http://en.wikipedia.org/wiki/Ping-pong_lemma`

[5]Alternatively, it follows from [Sco78, Section 3] and [Bou81, Chapitre V, §4, Section 4] that $\pi_1(\Sigma)$ an be embedded into $\mathrm{SL}(5,\mathbb{Z})$. I do not know whether dimension 5 is optimal.

[6]Put differently, a group π is residually P if we can detect any non trivial element in a P–quotient.

(3) Let p be a prime. Any finitely generated free abelian group is residually a p–group, i.e. residually a group of p–power order.
(4) If a finitely presented group is residually finite, then it has solvable word problem. This means that there exists an algorithm which can decide whether or not a given word in the generators represents the trivial element. We refer to [Moa66] for details. [7]

Proposition 2.5. *Let Σ be a surface, then $\pi := \pi_1(\Sigma)$ is residually finite.*

Proof. We pick a monomorphism $\alpha\colon \pi \to \mathrm{GL}(n, \mathbb{Z})$. Let $g \in \pi$ be non–trivial. Pick $k \in \mathbb{N}$ such that k is larger than all the absolute values of the entries of $\alpha(g)$. Then the image of g under the map

$$\alpha\colon \pi \to \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/k)$$

is non–trivial. In fact, with some extra effort one can show that a stronger statement holds: for any prime p the group π is residually p. We refer to [Weh73] for details. □

Definition. We say that a group π is *subgroup separable* if for any finitely generated subgroup A and any $g \in \pi \setminus A$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group such that $\alpha(g) \notin \alpha(A)$. [8]

Remark.
(1) A subgroup separable group is in particular residually finite. Indeed, this follows immediately from applying the definition to $A = \{e\}$.
(2) Any finitely generated abelian group is subgroup separable. Indeed, given $A \subset \pi$ the group π/A is again finitely generated, in particular residually finite.
(3) If a finitely presented group is subgroup separable, then the extended word problem is solvable, i.e. it can be decided whether or not a given finitely generated subgroup contains a given element. We refer again to [Moa66] for details.

The following theorem was proved by Scott [Sco78] in 1978:

Theorem 2.6. (Scott's theorem) *The fundamental group of any surface is subgroup separable.*

Given a space X and a ring R we write

$$vb_1(X; R) := \sup\{b_1(X'; R) \mid X' \to X \text{ finite covering}\} \in \mathbb{N} \cup \{\infty\}.$$

Put differently, $vb_1(X; R) = \infty$ if X admits finite covers with arbitrarily large first R–Betti numbers.

Lemma 2.7. *Let Σ be a hyperbolic surface, then $vb_1(X; R) = \infty$ for any ring.*

[7]See also
`www.math.umbc.edu/~campbell/CombGpThy/RF_Thesis/1_Decision_Problems.html`

[8]Put differently, a group π is subgroup separable if given any finitely generated group A and $g \notin A$ we can tell that $g \notin A$ by going to a finite quotient.

Proof. We consider the case that Σ is closed, the bounded case is proved the same way. Let Σ' be an n–fold cover of Σ. Then it follows from the multiplicativity of the Euler characteristic under finite covers that

$$b_1(\Sigma') - 2 = -\chi(\Sigma') = -n\chi(\Sigma) \geq n.$$

□

2.3. **The mapping class group and Dehn twists.** Let Σ be an orientable surface. We want to study

$$\text{Diff}(\Sigma) := \{\text{orientation preserving diffeomorphisms of } \Sigma\}$$

and the *mapping class group*

$$\text{MCG}(\Sigma) := \text{Diff}(\Sigma)/\text{homotopy} = \pi_0(\text{Diff}(\Sigma)).$$

Definition. Let $c \subset \Sigma$ be an oriented simple closed curve. The *Dehn twist along* c is defined to be the diffeomorphism

$$\begin{array}{rcl} \Sigma & \to & \Sigma \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \Sigma \setminus c \times [0,1] \\ (e^{2\pi i t} z, t), & \text{if } x = (z,t) \in c \times [0,1]. \end{cases} \end{array}$$

The following theorem was proved by Lickorish [Li62]:

Theorem 2.8. (Lickorish's theorem) *Let Σ be a surface, then any element in* $\text{MCG}(\Sigma)$ *is the composition of finitely many Dehn twists.*

In fact the mapping class group is generated by Dehn twists on certain $3g + 1$–curves and one can give a finite presentation for the mapping class group.

2.4. **Classification of diffeomorphisms.** Up to diffeotopy S^2 and $\mathbb{R}P^2$ admit no orientation preserving diffeomorphisms, i.e.

$$\text{MCG}(S^2) = 0.$$

So let us turn to the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. Any diffeomorphism of T lifts to an diffeomorphism of $\mathbb{R}^2$ which preserves $\mathbb{Z}^2$ as a set. One can show that the diffeotopy class of the diffeomorphism is determined by the restriction to $\mathbb{Z}^2$, i.e.

$$\text{MCG}(T) = \text{SL}(2,\mathbb{Z}).$$

Given $A \in \text{SL}(2,\mathbb{Z})$ an elementary exercise in linear algebra shows there are precisely three cases we have to distinguish:

(1) A is not diagonalizable. This implies 1 or -1 is an eigenvalue, i.e. A fixes a line or reverses a line.
(2) A is diagonalizable with complex eigenvalues, in fact the only possible complex eigenvalues are $\pm i$ and $\pm e^{\pi i/3}$, i.e. A has finite order.
(3) A has two real eigenvalues λ and λ^{-1}.

Exercise 2.9. *Prove the above assertion that only these three cases can occur and prove the statement about the eigenvalues in the complex diagonalizable case.*

In terms of diffeomorphisms for tori this means that given any $\varphi \in \mathrm{MCG}(T)$ one of the following happens:

(1) φ is reducible, i.e. φ fixes an essential curve as a set, [9] this happens for example if φ is the Dehn twist along one curve,
(2) φ is periodic, i.e. φ has finite order,
(3) φ is *Anosov*, i.e. there exists a transverse pair of geodesic curves c and d on T and a $\lambda > 1$ such that c gets 'stretched' by $\lambda > 1$ and d gets 'compressed' by the factor $\lambda^{-1} < 1$.

Exercise 2.10. *The 'generic' diffeomorphism of a torus is of Anosov type. Turn this into a precise statement and prove it.*

Thurston in the late 1970's showed that a complete analogue holds for hyperbolic surfaces. More precisely he proved that if Σ is a hyperbolic surface and $\varphi \in \mathrm{MCG}(\Sigma)$, then one of the following occurs:

(1) φ is reducible, i.e. there exists a collection of essential curves which is preserved by φ up to isotopy, or
(2) φ is periodic, i.e. there exists an $n \in \mathbb{N}$ such that φ^n is isotopic to the identity,
(3) φ is *pseudo–Anosov*, i.e. there exists a transverse pair of measured foliations c and d and a $\lambda > 1$ such that the foliations are preserved by φ and their transverse measures are multiplied by λ and λ^{-1}.

For a prove of this statement and for the precise meaning of (3) we refer to [Th88, CB88] and [FaM12]. Maher [Mah11] and Rivin [Riv08] showed that the third case is again, in a precise sense, the generic case.

3. Examples and constructions of 3-manifolds

For the remainder of this lecture course we will study 3-manifolds. In dimension three we do not have to distinguish between the categories of topological, smooth and PL manifolds: by Moise's Theorem [Moi52], [Moi77, p. 252 and 253] any topological 3-manifold also admits a unique PL and a unique smooth structure. [10]

In the following, by a 3-manifold we always mean an orientable, compact, connected 3-manifold. Furthermore, unless we say explicitly something else we will restrict ourselves to 3-manifolds which are either closed or which have toroidal boundary.

3.1. Examples of 3-manifolds.

Before we start out with the theory of 3-manifolds we first want to collect examples of 3-manifolds.

[9] We call a simple closed curve *essential* if it does not bound a disk and if it is not boundary parallel

[10] The analogous statement does of course not hold in dimension four, for example it follows from the work of Freedman and Donaldson that $\mathbb{R}^4$ admits uncountably many smooth structures.

3.1.1. *Spherical 3-manifolds.* The most basic example of a 3-manifold is of course the 3–sphere:

$$S^3 \;=\; \{x \in \mathbb{R}^4 \,|\, \|x\| = 1\} \;=\; \{(w,z) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1\}.$$

If Γ is a group that acts freely and isometrically on S^3, then the quotient S^3/Γ is again a 3-manifold which inherits a Riemannian metric of constant curvature equal to 1. The resulting manifolds are called *spherical 3-manifolds.* The finite groups that can act freely and isometrically on S^3 were completely classified by Hopf [Ho26, §2].

The quotients of S^3 by cyclic groups form a particularly interesting class of manifolds. More precisely, let $p, q \in \mathbb{N}$ be coprime and let $\xi = e^{2\pi i/p}$. Then we consider the *lens space*

$$L(p,q) := S^3/\sim \text{ where } (z,w) \sim (\xi z, \xi^q w),$$

Thus we can view $L(p,q)$ as the quotient of S^3 by a free action of $\mathbb{Z}/p$. It follows that

$$\pi_1(L(p,q)) = \mathbb{Z}/p.$$

One of the first deep results in 3-manifold topology was the classification of lens spaces by Reidemeister [Rer35].

Theorem 3.1. (Reidemeister 1935) *Let $L(p,q)$ and $L(p,q')$ be two lens spaces.*

(1) *$L(p,q)$ and $L(p,q')$ are homotopy equivalent if and only if*

$$q' \equiv \pm a^2 q \bmod p \textit{ for some } a.$$

(2) *$L(p,q)$ and $L(p,q')$ are diffeomorphic if and only if*

$$q' \equiv \pm q^{\pm 1} \bmod p.$$

Proof. The 'if' directions can be shown directly, the first 'only if' direction follows from studying the linking forms on the first homology, the second 'only if' direction follows from considering Reidemeister torsion. We refer to [Hat, Section 2.1] for details. □

In particular we see that in general the fundamental group of a 3-manifold does not determine the homotopy type, and the homotopy type does not determine the diffeomorphism type of a lens space.

Exercise 3.2. *Show that S^3 and also any lens space can be written as a union of two solid tori.*

3.1.2. *Hyperbolic 3-manifolds.* Similar to spherical 3-manifolds we can also introduce hyperbolic 3-manifolds. More precisely, let

$$\mathbb{H}^3 = \{(x,y,z) \,|\, z > 0\}$$

be the upper half space together equipped with the metric

$$\frac{1}{z} \cdot \text{standard metric on } \mathbb{R}^3.$$

If Γ acts freely, isometrically and cocompactly on $\mathbb{H}^3$ then we refer to the quotient $\mathbb{H}^3/\Gamma$ as a *hyperbolic 3-manifold.* Slightly more generally, if Γ acts freely and isometrically such that

$\mathbb{H}^3/\Gamma$ has finite volume, then $\mathbb{H}^3/\Gamma$ is the interior of a compact 3-manifold with (possibly empty) toroidal boundary. We refer to such a manifold as hyperbolic as well.

In contrast to the case of surfaces it is actually quite difficult to explicitly give examples of hyperbolic 3-manifolds. In the first decades of the 20th century a few examples of hyperbolic 3-manifolds were explicitly constructed by Gieseking [Gi12], Löbell [Lo31] and Seifert–Weber [SW33][11], but in the following 40 years no new examples of hyperbolic 3-manifolds were found. Therefore we move on to other constructions of 3-manifolds.

3.1.3. *Complements of knots and links.* Given any knot or link L in S^3 we can consider its exterior

$$X(L) := S^3 \setminus \nu L$$

where νL denotes an open tubular neighborhood of L. Important examples are given by

(1) the unknot, the trefoil knot and the figure-8 knot,
(2) torus knots and links, i.e. knots and links which lie on the standard torus in S^3. Note that the trefoil knot is a torus knot, whereas the figure-8 knot is not, see [BZH14] for details.

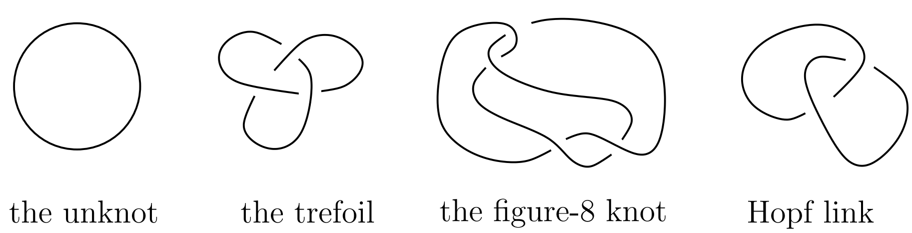

Figure 1.

3.1.4. *Mapping tori.* We can build 3-manifolds out of lower dimensional manifolds. For example, if Σ is a surface, then the product $S^1 \times \Sigma$ is a 3-manifold. More generally, if $\varphi\colon \Sigma \to \Sigma$ is a self-diffeomorphism, then the mapping torus

$$M(\Sigma, \varphi) \in ([0,1] \times \Sigma) \,/\, (0,x) \sim (1, \varphi(x))$$

is a 3-manifold. If a given 3-manifold is diffeormorphic to such a mapping torus, then we say that N *fibers over* S^1 or shorter, that N *is fibered*. For example the exteriors of torus knots and of the figure-8 knot fiber over S^1, see e.g. [BZH14] for details. On the other hand 'most' knot exteriors do not fiber over S^1.

[11]Or alternatively see: `http://en.wikipedia.org/wiki/Seifert-Weber_space`

3.1.5. *Seifert fibered spaces.*

Definition. A *Seifert fibered 3-manifold* is a 3-manifold N together with a decomposition into disjoint simple closed curves (called *fibers*) such that each fiber has a tubular neighborhood that forms a standard fibered torus. A *standard fibered torus* corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the diffeomorphism of a disk given by rotation by an angle of $2\pi b/a$, with the natural fibering by circles. If $a > 1$, then the central fiber is called *singular*.

There are various different equivalent ways to think about Seifert fibered manifolds, we refer to [Sco83] for details.

(1) Seifert fibered manifolds are S^1–bundles over a surface with isolated 'singular fibers',
(2) Seifert fibered manifolds are S^1–bundles over 2–dimensional orbifolds,
(3) A Seifert fibered 3-manifold is finitely covered by an S^1–bundle over a surface.[12]

Example. The following are Seifert fibered spaces:

(1) Any product $S^1 \times \Sigma$ is a Seifert fibered spaces, in particular the three torus.
(2) Any non-trivial S^1-bundle over a surface is also a Seifert fibered space. In particular S^3 is Seifert fibered, since the map

$$\begin{array}{rcl} S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & \mathbb{C}P^1 = S^2 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

is a fibration with fiber S^1.
(3) If Σ is surface and $\varphi \colon \Sigma \to \Sigma$ is a self-diffeomorphism of finite order, i.e. such that for some n the power φ^n is the identity, then the intervals $[0, 1] \times z$, $z \in \Sigma$ glue together to form a Seifert fibered structure on the mapping torus

$$M(\Sigma, \varphi) \in ([0, 1] \times \Sigma) \,/\, (0, x) \sim (1, \varphi(x)).$$

Exercise 3.3.

(1) *Show that any closed 3-manifold that is obtained from gluing two solid along their boundary is Seifert fibered.*
(2) *Show that lens spaces are Seifert fibered spaces. Hint: show that lens spaces can be described as in* (1)*.*
(3) *Show that the exterior of any torus link is a Seifert space. Hint: start out with an appropriate Seifert fibered structure for* S^3*.*

Seifert fibered spaces are well understood and completely classified. We refer to [Sei33, Or72, He76, Ja80, Br93] for further information and for the classification of Seifert fibered 3-manifolds. For future reference we record the following lemma:

[12]In fact it is a consequence of the Geometrization Theorem of Perelman that the converse holds, i.e. a 3-manifold is Seifert fibered if and only if it is finitely covered by an S^1–bundle over a surface. We refer to [AFW15] for details.

Lemma 3.4. *Let N be a Seifert fibered 3-manifold with infinite fundamental group. Then a regular fiber of the Seifert fibration generates an infinite cyclic normal subgroup of* $\pi_1(N)$.

Since Seifert fibered spaces are finitely covered by circle bundles over surfaces it is not difficult to show that fundamental groups of Seifert fibered spaces share many of the nice properties of surface groups, more precisely we have the following proposition:

Proposition 3.5. *Let N be a Seifert fibered space. Then the following hold:*
(1) $\pi_1(N)$ *is linear over* $\mathbb{Z}$,
(2) $\pi_1(N)$ *is residually finite,*
(3) $\pi_1(N)$ *is subgroup separable.*

The first statement is proved in [AFW15], the third statement was proved by Scott [Sco78] and the second statement is a consequence of the third statement.

3.2. **Constructions of more 3-manifolds.** In the previous section we gave many examples of 3-manifolds. We can produce more examples using several operations.

(1) Given two oriented 3-manifolds N_1 and N_2 we can consider the connected sum
$$N_1 \# N_2 = (N_1 \setminus \text{open 3-ball}) \cup (N_2 \setminus \text{open 3-ball}),$$
where we identify the two boundary spheres using an orientation reversing diffeomorphism. In general the diffeomorphism type of the connected sum depends on the orientation, i.e. in general
$$N_1 \# N_2 \ncong N_1 \# - N_2.$$
(2) Given two 3-manifolds N_1 and N_2 with toroidal boundary, we can create a new manifold by gluing N_1 to N_2 along a boundary torus. Similarly, if N has two boundary tori, then we obtain a new manifold by gluing one boundary torus to the other.
(3) Finally we get more examples of 3-manifolds by taking finite covers.

In fact with these constructions one can obtain all 3-manifolds. More precisely, Lickorish [Li62, Theorem 2] showed that all 3-manifolds can be obtained by gluing solid tori to exteriors of links. (This result can also be obtained from Rokhlin's theorem from 1951 that any closed 3-manifold is the boundary of an orientable 4-manifold.)

In the following we refer to 3-manifolds that are obtained by gluing several Seifert fibered manifolds along boundary tori as *graph manifolds.*

Exercise 3.6. *Show that the exteriors of connected sums of iterated torus knots are graph manifolds.*

4. 3-manifolds up to 1973

4.1. **The Prime Decomposition Theorem.** A 3-manifold N is called *prime* if N can not be written as a non-trivial connected sum of two manifolds, i.e. if $N \cong N_1 \# N_2$, then $N_1 = S^3$ or $N_2 = S^3$. It follows from the Schoenflies Theorem that S^3 is prime.

Furthermore N is called *irreducible* if every embedded S^2 bounds a 3-ball. An irreducible 3-manifold is evidently prime. Conversely, if N is a prime 3-manifold, then either N is irreducible or $N \cong S^1 \times S^2$, see by [He76, Lemma 3.13] for details. The following theorem says that we can 'undo' the connected sum operation.

Theorem 4.1. (Prime decomposition theorem) *Let N be an oriented 3-manifold.*

(1) *There exists a decomposition $N \cong N_1 \# \dots \# N_r$, where the 3-manifolds $N_1, \dots, N_r$ are oriented prime 3-manifolds.*
(2) *If $N \cong N_1 \# \dots \# N_r$ and $N \cong N_1' \# \dots \# N_s'$ where the 3-manifolds N_i and N_i' are oriented prime 3-manifolds, then $r = s$ and (possibly after reordering) there exists an orientation preserving diffeomorphism $N_i \to N_i'$.*

Remark.

(1) The first part of the theorem was proved by Kneser [Kn29]. The difficulty of course lies in showing that the process of decomposing a 3-manifold as a connected sum will end after finitely many steps. The second statement was proved by Milnor [Mi62]. We also refer to [He76, Chapter 3] for a proof.
(2) The theorem says in particular, that any 3-manifold group can be written as the free product of fundamental groups of prime 3-manifolds. In fact the converse holds: if $\pi_1(N^3)$ is isomorphic to a free product $A * B$, then there exist 3-manifolds X and Y with $\pi_1(X) = A$, $\pi_1(Y) = B$ and $N = A \# B$. This statement is referred to as the 'Kneser conjecture' and was first proved by Stallings [St59].
(3) Schubert [Sch49] proved that a similar theorem holds for knots: an oriented knot can be uniquely written as the connect sum of finitely many oriented prime knots.
(4) By work of Jaco–Rubinstein [JR03] and Jaco–Tollefson [JT95, Algorithm 7.1] the prime decomposition of a given 3-manifold can actually be algorithmically computed, see also [AFW14] for the precise statement.

The Prime Decomposition Theorem says in particular that for all intents and purposes we can restrict ourselves henceforth to prime 3-manifolds.

Exercise 4.2. *Given a group G we denote by* $\operatorname{rank}(G)$ *the minimal number of generators of G. Grushko's Theorem (see e.g.* [Ly65]*) says that for any finitely generated groups G and H the equality* $\operatorname{rank}(G * H) = \operatorname{rank}(G) + \operatorname{rank}(H)$ *holds. Show that the existence of the prime decomposition also follows from Grushko's Theorem and the resolution of the Poincaré Conjecture 6.3.*

The argument of the exercise that a prime decomposition exists (in the topological category) applies also to all dimensions greater than four, since in these dimensions the Poincaré Conjecture has been solved by Smale [Sm61] and Freedman [Fr82]. It seems to be unknown whether in higher dimensions the prime decomposition is unique.

4.2. **Dehn's Lemma and the Sphere Theorem.** The following theorem was first formulated by Dehn in 1910. In 1927 Kneser showed that Dehn's proof had a gap, and a correct proof was finally given by Papakyriakopoulos [Pa57] [13] in 1957:

Theorem 4.3. **(Dehn's Lemma)** *Let N be a 3-manifold, let T be a boundary component, and suppose that*

$$K := \operatorname{Ker}\{\pi_1(T) \to \pi_1(N)\}$$

is non–trivial. Then there exists a properly embedded disk $D \subset N$ such that ∂D represents a non–trivial element in K.

Remark. This formulation of Dehn's Lemma is also known as the Loop Theorem.

We obtain immediately the following corollary:

Corollary 4.4. *Let $K \subset S^3$ be a knot. Then K is trivial if and only if $\pi_1(S^3 \setminus \nu K) = \mathbb{Z}$.*

Proof. If K is trivial, then $\pi_1(S^3 \setminus \nu K) = \pi_1(S^1 \times D^2) = \mathbb{Z}$. Conversely, if $\pi_1(S^3 \setminus K) = \mathbb{Z}$, then the longitude of K represents an element in

$$\operatorname{Ker}\{\pi_1(\partial(S^3 \setminus \nu K)) \to \pi_1(S^3 \setminus \nu K)\}.$$

But by Dehn's Lemma the longitude bounds a disk, i.e. the knot is trivial. □

Exercise 4.5. *An embedded torus T in a 3-manifold N is called* incompressible *if the inclusion induced map $\pi_1(T) \to \pi_1(N)$ is injective. Otherwise T is called compressible. Now suppose that N is irreducible. Show that the torus T is compressible if and only if T bounds a solid torus in N.*

Papakyriakopoulos [Pa57] also proved the following theorem:

Theorem 4.6. **(Sphere Theorem)** *Let N be a 3-manifold such that $\pi_2(N) \neq 0$. Then there exists an embedded essential [14] sphere in N, in particular N is reducible.*

Remark.

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[13] The following citation is taken from the wikipedia article about Papakyriakopoulos: 'The following limerick was composed by John Milnor, shortly after learning of several graduate students' frustration at completing a project where the work of every Princeton mathematics faculty member was to be summarized in a limerick:

The perfidious lemma of Dehn
Was every topologist's bane
'Til Christos Papa-
kyriakopou-
los proved it without any strain.

The phrase 'without any strain' is not meant to indicate that Papa did not expend much energy in his efforts. Rather, it refers to Papa's 'tower construction', which quite nicely circumvents much of the difficulty in the cut-and-paste efforts that preceded Papa's proof.'

[14] An embedded sphere is called essential if it does not bound a 3–ball

(1) We can summarize Dehn's Lemma and the Sphere Theorem slightly sloppily as follows: 'Dehn's Lemma says, that if there exists a singular disk, then it can be replaced by an embedded disk', and the Sphere Theorem says, morally speaking, 'a singular sphere can be replaced by an embedded sphere'. Another way of saying this is that for 3-manifolds information from homotopy groups can be translated back into topology.
(2) If N is a manifold of dimension $n \geq 5$, then it follows from a general position argument, that any embedded curve $g \subset N$ that represents the trivial element in $\pi_1(N)$ is the boundary of an embedded disk in N. The analogous statement does not hold in dimension four. This can be viewed as the source of all the troubles (or exciting phenomena, depending on your point of view) in dimension four.

Corollary 4.7. *Let N be an irreducible 3-manifold with infinite fundamental group. Then N is aspherical.*

Proof. It follows from the Sphere Theorem, that $\pi_2(N) = 0$. We denote by $\widetilde{N}$ the universal cover of N. It follows from the Hurewicz Theorem that

$$\pi_3(N) = \pi_3(\widetilde{N}) = H_3(\widetilde{N}).$$

Our assumption that $\pi_1(N)$ is infinite implies that $\widetilde{N}$ is not compact, i.e. that $H_3(\widetilde{N}) = 0$. By induction we can now show that in fact $\pi_k(N) = 0$ for all $k \geq 3$. □

This corollary applies in particular to knot complements. By the argument of Lemma 2.2 we now obtain the following corollary:

Corollary 4.8. *Let N be an irreducible 3-manifold with infinite fundamental group. Then $\pi_1(N)$ is torsion–free.*

We can now give a complete answer to the question, which abelian groups can arise as fundamental groups of 3-manifolds:

Proposition 4.9. *The only abelian groups which appear as fundamental groups of closed 3-manifolds are* [15]

$$\mathbb{Z} = \pi_1(S^1 \times S^2), \mathbb{Z}/n = \pi_1(\textit{lens space}) \textit{ and } \mathbb{Z}^3 = \pi_1(\textit{3-torus}).$$

Proof. Let N be a 3-manifold with abelian fundamental group. By the remark after Proposition 1.4 we already know that $\pi_1(N)$ is isomorphic to one of $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n$ or $\mathbb{Z} \oplus \mathbb{Z}/n$. If N is not prime, then $\pi_1(N)$ is a free product of groups. The free product of non-trivial groups is not abelian. Thus we see that all but one components in the prime decomposition has trivial fundamental group. Thus we can assume that N is in fact prime. If N has infinite, non–cyclic fundamental group, then it follows from Corollary 4.7 that N is

[15] If we allow non–orientable manifolds, then we have to add

$$\mathbb{Z} \times \mathbb{Z}/2 = \pi_1(S^1 \times \mathbb{R}P^2).$$

aspherical, i.e. $N = K(\pi_1, 1)$. In Corollary 4.8 we showed that this implies that $\pi_1(N)$ is torsion–free, we can thus exclude the possibility that $\pi_1(N) = \mathbb{Z} \oplus \mathbb{Z}/n$. Finally note that $H_3(\mathbb{Z}^2) \neq H_3(N) = \mathbb{Z}$, so we can also exclude the case that $\pi_1(N) = \mathbb{Z}^2$. □

4.3. **Haken manifolds.** Before we continue we introduce the notion of Haken manifolds. These manifolds play an important rôle in 3-manifold topology.

Definition.

(1) A properly embedded surface $\Sigma \subset N$ is called *incompressible* if $\pi_1(\Sigma) \to \pi_1(N)$ is injective. Otherwise Σ is called *compressible*. In Exercise 4.5 we showed that a torus in an irreducible 3-manifold is compressible if and only if it bounds a solid torus.
(2) A 3-manifold N is called *Haken* if N is irreducible and if N admits a non-simply connected embedded incompressible surface $\Sigma \subset N$.

The following lemma shows in particular that the exteriors of non-trivial knots are Haken:

Lemma 4.10. *Let $N \neq S^1 \times D^2$ be any irreducible 3-manifold (for once with no restrictions on the boundary) with $b_1(N) \geq 1$, then N is Haken.*

Proof. Let $\Sigma \subset N$ be a properly embedded surface of 'minimal complexity' [16] representing a given non–trivial element in $H^1(N; \mathbb{Z}) = H_2(N, \partial N; \mathbb{Z})$.

Claim. The surface Σ is incompressible.

If Σ is not incompressible, then applying Dehn's Lemma to $N \setminus \nu\Sigma$ we can find an embedded essential curve c on Σ which bounds an embedded disk D in N. We can thus do surgery on Σ along c and we obtain a surface of smaller complexity which represents the same homology class. This concludes the proof of the claim.

Our assumption that $N \neq S^1 \times D^2$ and that N is irreducible implies that $\Sigma \neq D^2$ and $\Sigma \neq S^2$. We thus found an incompressible non–simply connected surface in N. □

The basic idea for the study of Haken manifolds is very simple: given an incompressible surface $\Sigma \subset N$ we can cut N along Σ, we obtain a (possibly disconnected) 3-manifold such that each component has positive first Betti number, i.e. each component is Haken again. Haken [Hak62] associated a complexity to Haken manifolds (inspired by the complexity used by Kneser [Kn29]) and showed that this complexity goes down when one cuts a Haken manifold along an incompressible 3-manifold. Put differently, we iterate the process of cutting a Haken manifold along surface and reduce the complexity of the 3-manifold along the way till we obtain 3–balls.

[16] The complexity of a surface Σ with connected components $\Sigma_1, \dots, \Sigma_k$ is defined as

$$\chi_-(\Sigma) = \sum_{i=1}^{k} \max(0, -\chi_i(\Sigma)).$$

This turns out to be the correct generalization of the concept of 'minimal genus' to disconnected surfaces. We refer to [Th86a] for details.

Waldhausen [Wa68, Corollary 6.5] used this approach to prove the following theorem:

Theorem 4.11. *Let N and N' be two closed Haken manifolds with $\pi_1(N) \cong \pi_1(N')$, then N and N' are diffomorphic.*

Remark. If N is a Haken manifold with non–trivial toroidal boundary, then it is in general not true, that its diffeomorphism type is determined by its fundamental group. In fact if K denotes the trefoil knot, then

$$S^3 \setminus \nu(K\#K) \text{ and } S^3 \setminus \nu(K\# -K)$$

are not diffeomorphic, even though the fundamental groups are isomorphic.

The problem that Haken manifolds with boundary are no longer determined by their fundamental group can be dealt with as follows: Let N be a 3-manifold with incompressible boundary. Then we refer to the fundamental group of $\pi_1(N)$ together with the conjugacy classes of subgroups determined by the boundary components as the *peripheral structure of N*. Waldhausen showed that Haken 3-manifolds with non–spherical boundary are determined by their 'peripheral structure'. We refer to [Wa68] for details. Alternatively, any two Haken 3-manifolds with the same fundamental group are related by a finite number of 'Dehn flips', see e.g. [Jo79, Theorem 29.1 and Corollary 29.3] or [AFW15, Theorem 2.2.1] for details.

5. Interlude: More on hyperbolic 3-manifolds

The following proposition gives several equivalent characterizations of hyperbolic 3-manifolds.

Proposition 5.1. *Let N be a 3-manifold. The following statements are equivalent:*

(1) *N is hyperbolic,*
(2) *the interior of N admits a complete finite-volume metric of constant curvature -1,*
(3) *there exists a discrete and faithful representation $\alpha\colon \pi_1(N) \to \operatorname{Isom}^+(\mathbb{H}^3)$ such that $\alpha(\pi_1(N)) \subset \operatorname{Isom}^+(\mathbb{H}^3)$ is torsion–free.*

Proof. A hyperbolic 3-manifold has finite volume if and only if it is either closed or has toroidal boundary and it is not diffeomorphic to $T^2 \times I$. (See [Th79, Theorem 5.11.1] or [Bon02, Theorem 2.9], [BP92, Proposition D.3.18].)

The equivalence of (1) and (3) follows from the general relationship between actions on the universal cover and properties of the fundamental group of a manifold.

It is clear that (1) implies (2). Finally, if the interior of N admits a complete finite-volume metric of constant curvature -1, then the universal cover $\widetilde{N}$ of N is a simply connected 3-manifold that admits a complete metric of constant curvature -1. It follows from [Lan95, Theorem IX.3.12] that $\widetilde{N}$ is isometric to $\mathbb{H}^3$. □

By [Bon09, Theorem 9.8] we can identify $\operatorname{Isom}^+(\mathbb{H}^3)$ with $\operatorname{PSL}(2,\mathbb{C})$. In particular, if N is a hyperbolic 3-manifold, then there exists a discrete and faithful representation $\alpha\colon \pi_1(N) \to$

PSL$(2,\mathbb{C})$. Thurston (see e.g. [Sh02, Section 1.6]) showed that this representation lifts to a representation to SL$(2,\mathbb{C})$. Thus we obtain the following corollary

Corollary 5.2. *If N is hyperbolic, then there exists a discrete and faithful representation $\alpha\colon \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$.*

In the early 1970's Riley [Ri75a, Ri75b] used the relationship between hyperbolic structures and PSL$(2,\mathbb{C})$ representations to show that the figure-8 knot [17] is hyperbolic. But in most cases it is very difficult to directly show that a 3-manifold is hyperbolic.

In the following we say that a 3-manifold N is *atoroidal* if any incompressible torus is boundary parallel, i.e. can be isotoped into the boundary. The following lemma gives an obstruction to a 3-manifold being hyperbolic:

Lemma 5.3. *Let N be a hyperbolic 3-manifold and let $A \subset N$ be a subgroup isomorphic to $\mathbb{Z}^2$. Then there exists a boundary torus T such that $A = \pi_1(T)$ as subgroups in $\pi_1(N)$.*

It follows directly from the lemma that a closed hyperbolic 3-manifold does not contain any incompressible tori.

Proof. We can view $\pi_1(N)$ as a discrete subgroup of SL$(2,\mathbb{C})$. A basic argument shows that any discrete subgroups of SL$(2,\mathbb{C})$ isomorphic to $\mathbb{Z}^2$ is of the form

$$\begin{pmatrix} \varepsilon & \mathbb{Z}+\lambda\mathbb{Z} \\ 0 & \varepsilon \end{pmatrix}$$

where $\varepsilon = \pm 1$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By [Bon02, Theorem 2.9] any such subgroup corresponds to a boundary torus. □

The following Rigidity Theorem is one of the key results in the study of hyperbolic 3-manifolds. It was proved by Mostow [Mob68, Theorem 12.1] in the closed case and independently by Prasad [Pr73, Theorem B] and Marden [Ma74] in the case of non-empty boundary.

Theorem 5.4. (Rigidity Theorem) *Every isomorphism $\pi_1(M) \to \pi_1(N)$ between fundamental groups of hyperbolic 3-manifolds is induced by a unique isometry $M \to N$.*

In the case of surfaces we noted that any surface Σ with $\chi(\Sigma) < 0$ is hyperbolic, but the hyperbolic structure is by no means unique. Amazingly the situation is completely different in dimension three. Indeed, we can apply the Rigidity Theorem to two hyperbolic

[17] In this case one can write down the representation to PSL$(2,\mathbb{C})$ explicitly: The fundamental group of the figure-8 knot complement has the following presentation

$$\langle x, y \,|\, xyx^{-1}y^{-1}x = yxy^{-1}x^{-1}y\rangle$$

and a discrete and faithful representation is given by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } y \mapsto \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

where $z = e^{\pm\pi i/3}$.

3-manifolds M and N with the same underlying topological manifold and we obtain the following corollary.

Corollary 5.5. *Let N be a hyperbolic 3-manifold, then the hyperbolic structure of N is unique up to isometry.*

The Rigidity Theorem implies in particular that if two hyperbolic 3-manifolds are homotopy equivalent, then they are already isometric. This shows that all geometric invariants of hyperbolic 3-manifolds (e.g. volume) are in fact topological invariants.

It also follows, that a hyperbolic 3-manifold admits a unique (up to conjugation) discrete and faithful representation $\alpha\colon \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$.

Remark.

(1) Rigidity also holds in fact for closed hyperbolic manifolds of any dimension greater than two.
(2) One can use rigidity to show that the discrete and faithful representation $\alpha\colon \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ is conjugate to a representation $\pi_1(N) \to \mathrm{SL}(2,\overline{\mathbb{Q}})$ over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ (see [MR03, Corollary 3.2.4]).

We conclude this section with the following variation on Proposition 2.5.

Proposition 5.6. *Let N be a hyperbolic 3-manifold, then $\pi_1(N)$ is residually finite.*

Proof. We can view $\pi_1(N)$ as a subgroup of $\mathrm{SL}(2,\mathbb{C})$. It thus suffices to prove the following claim:

Claim. Any finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$ is residually finite.

Let $\pi \subset \mathrm{GL}(n,\mathbb{C})$ be a subgroup which is generated by $g_1,\dots,g_k$. We denote by R the ring which is generated by the entries of $\alpha(g_1),\dots,\alpha(g_k)$. The ring R is finitely generated over $\mathbb{Z}$. Since R is finitely generated over $\mathbb{Z}$ we can find for any $r \neq 0 \in R$ a maximal ideal $m \subset R$ such that r is non–trivial in R/m. The field R/m has a prime characteristic and it is finitely generated, hence R/m is in fact a finite field. Given a matrix $A \neq \mathrm{id} \subset \pi$ can now find a maximal ideal $m \subset R$ such that $A - \mathrm{id}$ represents a non–zero matrix over the field R/m. Put differently, A represents a non–trivial element in the finite group $\mathrm{GL}(n, R/m)$. □

We thus see that hyperbolic 3-manifolds have interesting properties, but as we pointed out earlier, in contrast to surfaces it is very difficult to construct examples of hyperbolic 3-manifolds 'by hand'. A few examples were explicitly constructed by Gieseking [Gi12], Löbell [Lo31] and Seifert–Weber [SW33]. In the 1970s events suddenly took a dramatic turn. First, to everybody's surprise Riley [Ri75a, Ri75b] showed that many knot complements, and in particular the complement of the figure-8 knot, admit a hyperbolic structure [18].

[18] Riley [Ri13] points out that the complement of the Figure-8 knot is in fact the 2-fold cover of Gieseking's example. Gieseking was killed in France in 1915, shortly after his work on hyperbolic 3-manifold. It is conceivable that hyperbolic structures on knot complements would have been discovered much earlier if it had not been for World War I.

6. The JSJ and the geometric decomposition

6.1. **The statement of the theorems.** In Section 4.1 we saw that 3-manifolds have a unique prime decomposition, i.e. they have a unique decomposition along spheres. In this section we will see that 3-manifolds also have a canonical decomposition along incompressible tori. After this second stage of decompositions we will finally end up with pieces which are either hyperbolic or Seifert fibered.

The following theorem was proved independently by Jaco and Shalen [JS79] and Johannson [Jo79].

Theorem 6.1. (JSJ Decomposition Theorem) *Let N be an irreducible 3-manifold. Then there exists a collection of disjointly embedded incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is atoroidal or Seifert fibered. Furthermore any such collection with a minimal number of components is unique up to isotopy.*

In the following we will refer to the tori $T_1, \dots, T_k$ as the *JSJ tori* and we will refer to the components of N cut along $\cup_{i=1}^k T_i$ as the *JSJ components*.

Now the goal is to determine which 3-manifolds are atoroidal. In the following we say that a closed 3-manifold is *spherical* if it admits a complete metric of curvature $+1$. The universal cover of a spherical 3-manifold is necessarily S^3. It is clear that spherical 3-manifolds have finite fundamental groups, in particular they are atoroidal. By the discussion of the previous section we also know that hyperbolic 3-manifolds are atoroidal. Thurston [Th82a] conjectured that these are all examples of atoroidal 3-manifolds.

This conjecture was proved by Thurston for Haken manifolds. The proof for the general case was first given by Perelman in his seminal papers [Pe02, Pe03a, Pe03b], we refer to [MT07, MT14, CZ06a, CZ06b, KL08, FoM10, BBBMP10] for full details. More precisely, Perelman proved the following theorem:

Theorem 6.2. (Perelman) *Let N be an irreducible atoroidal 3-manifold. Then either N is spherical or N is hyperbolic.*

A spherical 3-manifold by definition is finitely covered by S^3. Hence the only finitely connected spherical 3-manifold is S^3. We thus obtain the following theorem:

Theorem 6.3. (Poincaré Conjecture) *The 3–sphere S^3 is the only simply connected, closed 3-manifold.*

Note that spherical 3-manifolds are in fact Seifert fibered (see [Bon02, Theorem 2.8]). Thus we obtain the following theorem:

Theorem 6.4. (Geometrization Theorem) *Let N be an irreducible 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is hyperbolic or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.*

6.2. **Geometric structures on 3-manifolds.** We had seen that any surface admits a metric of constant curvature. For 3-manifolds the situation is more complicated: we first have to decompose a given manifold along embedded spheres and incompressible tori. The resulting pieces are then either hyperbolic or Seifert fibered. The Seifert fibered pieces with finite fundamental groups are spherical, and the Seifert fibered pieces with infinite abelian fundamental groups are Euclidean, i.e. they have a metric of constant curve zero. Seifert fibered pieces with infinite, non–abelian fundamental groups do not admit a metric of constant curvature. But they do carry a unique geometry if we expand our definition of 'geometry of a manifold'. The five extra geometries needed are referred to as

$$\text{Sol}, \widetilde{\text{SL}(2,\mathbb{R})}, \text{Nil}, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}.$$

Even though the geometric point of view is very pretty, in practice these geometries are studied very little, since they all correspond to Seifert fibered spaces or torus bundles, which are well understood anyway. The only geometric 3-manifolds which are not well understood are the most important manifolds: hyperbolic 3-manifolds.

For completeness' sake we also describe how the geometry of a Seifert fibered space N can be determined from the topology of N: As we mentioned earlier, any Seifert fibered space is finitely covered by an S^1–bundle over a surface Σ. We denote by e the Euler class of the S^1–bundle. Then $\chi(\Sigma)$ and e determine the geometry of N:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\text{SL}(2,\mathbb{R})}$.

The Sol–geometry appears only for torus bundles or for gluings of twisted I-bundles over Klein bottles, which are not Seifert fibered spaces. Their JSJ decomposition is given by cutting the torus bundle along a fiber. We refer to [Th82a], [Sco83] and [Bon02] for more information on the eight geometries.

The above discussion shows that one can reformulate the Geometrization Theorem in such a way, that the name 'Geometrization Theorem' is fully justified:

Theorem 6.5. (Geometrization Theorem) *Let N be an irreducible 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is geometric. Furthermore any such collection with a minimal number of components is unique up to isotopy.* [19]

Finally, Thurston's eight geometries also inspired fashion designer Issey Miyake:
`http://www.youtube.com/watch?v=lMneAQsAZUA`

[19]The two decompositions in the two formulations of the Geometrization Theorem are identical with one exception: if N is a torus bundle over S^1 with Sol geometry, then in the second formulation we do not need any tori, whereas in the first formulation the decomposition will consist of one fiber, which then cuts N into two manifolds of the form $T \times [0,1]$, which are both Seifert fibered spaces.

6.3. **Examples of the decompositions.** Now we want to see what the Geometrization Theorem does for several 'real life' examples.

Let N_1 and N_2 be two hyperbolic 3-manifolds such that $T_i = \partial N_i, i = 1, 2$ consists of one torus each. Using Dehn's Lemma one can easily show that the boundary tori are incompressible. Then we glue N_1 and N_2 along their boundary tori using any diffeomorphism. The resulting 3-manifold N contains an incompressible torus. Therefore by the discussion of Section 5 the manifold N can not be hyperbolic. One can also show that N is not Seifert fibered. [20] Thus we need at least one torus to cut N into a union of Seifert fibered spaces and hyperbolic pieces. The torus T of course does the trick.

In general, if we glue two 3-manifolds N_1 and N_2 along a torus boundary component, then in almost all cases [21] the JSJ decomposition of the resulting manifold N is given by the JSJ tori of N_1, together with the JSJ tori of N_2 and the gluing torus. We refer to [AFW15] for details.

Remark. It follows from the above arguments that a knot $K \subset S^3$ is either a torus knot, a hyperbolic knot, or a satellite knot, i.e. a knot which is given by wrapping a non–trivial knot in a solid torus around another non–trivial knot.

Exercise 6.6. *Let K_1 and K_2 be two hyperbolic knots. What is the JSJ decomposition of $S^3 \setminus (K_1 \cup K_2)$?*

In Section 4.4 we obtained new 3-manifolds by the connect sum operation and by gluing along boundary tori. The Prime Decomposition Theorem and the Geometrization Theorem now unravel these two operations: the Prime Decomposition Theorem detects the prime components and the Geometrization Theorem detects 'almost all' the gluing tori.

As a final example we discuss fibered 3-manifolds. The following theorem was proved by Thurston [Th86b, Th86c] in the late 1970's, see also [Ot96, Ot98].

Theorem 6.7. *Let Σ be a surface with $\chi(\Sigma) < 0$ and let $\varphi \in \mathrm{MCG}(\Sigma)$. We denote by N the mapping torus of φ.*

1. *If φ is reducible, then N admits an incompressible torus,*
2. *if φ is periodic, then N is Seifert fibered,*
3. *if φ is pseudo–Anosov, then N is hyperbolic.*

6.4. **Applications.** First we will prove the following lemma:

Lemma 6.8. *Let N be a 3-manifold which contains a subgroup isomorphic to $\mathbb{Z}^2$, then N contains an incompressible torus.*

Proof. If N is hyperbolic, then we saw at the end of Section 5 that any subgroup isomorphic to $\mathbb{Z}^2$ comes from a boundary torus. On the other hand, if N is Seifert fibered, then the

[20] For example incompressible tori in Seifert fibered spaces are well-understood, in particular the complement of an incompressible torus in a Seifert fibered space has to be Seifert fibered again.

[21] The only exception is the case that the JSJ components abutting the gluing torus are both Seifert fibered, and if the Seifert fibrations match up.

statement follows easily from the classification of Seifert fibered spaces. Now suppose that N is neither hyperbolic nor Seifert fibered. It then follows from geometrization that N has a non–trivial JSJ decomposition, in particular it contains an incompressible torus. □

Theorem 6.9. *Let N and N' be two closed, prime 3-manifolds with $\pi_1(N) \cong \pi_1(N')$. Then either N and N' are diffeomorphic, or N and N' are both lens spaces.*

Recall that a 3-manifold N is aspherical if and only if N is irreducible and if $\pi_1(N)$ is infinite. The theorem thus in particular proves the Borel conjecture for 3-manifolds.[22]

Proof. Let N and N' be two closed, prime 3-manifolds with $\pi_1(N) \cong \pi_1(N')$. We first assume that N contains an incompressible torus. It follows from Lemma 6.8 that N' also contains an incompressible torus. In particular N and N' are both Haken, and the statement follows from Theorem 4.11.

We now consider the case that neither N nor N' contain an incompressible torus. It follows from geometrization that N and N' are either Seifert fibered or hyperbolic. First suppose that N is Seifert fibered. It follows from Lemma 3.4 that $\pi_1(N)$ contains an infinite cyclic normal subgroup. An argument similar to the proof of Lemma 5.3 shows that fundamental groups of hyperbolic 3-manifolds do not contain infinite cyclic normal subgroups. We thus see that N' is also Seifert fibered. It follows from the classification of Seifert fibered spaces (see [Sco83, Theorem 3.1] and [Or72, p. 113]) that N and N' are diffeomorphic.

Finally consider the case that N and N' are hyperbolic. In this case it follows from the Rigidity Theorem 5.4 that N and N' are diffeomorphic. □

If N is an irreducible 3-manifold we can write $\pi_1(N)$ as an iterated HNN extension and amalgam of fundamental groups of Seifert fibered spaces and hyperbolic 3-manifolds along torus groups. We already saw that fundamental groups of hyperbolic 3-manifolds and Seifert fibered spaces are residually finite. Hempel [He87], building on ideas of Thurston [Th82a], and assuming/using the Geometrization Theorem showed the following theorem:

Theorem 6.10. *The fundamental group of any 3-manifold is residually finite.*

7. Hyperbolic 3-manifolds

In the previous section we saw that the key to understanding hyperbolic 3-manifolds lies in understanding hyperbolic 3-manifolds. Before we state some of the most important questions and results regarding hyperbolic 3-manifolds we introduce one more definition: Let P be a property of 3-manifolds, then we say that a 3-manifold N is *virtually* P if N admits a finite cover which has Property P.

Thurston [Th82b, Questions 16 to 18] asked the following 'challenge questions' regarding closed hyperbolic 3-manifolds:

[22] The Borel Conjecture says that if N, N' are two aspherical closed n-manifolds with isomorphic fundamental groups, then they are homeomorphic. This conjecture is still open in dimension greater than three.

T(1) Is every closed hyperbolic 3-manifold virtually Haken, i.e. does every hyperbolic 3-manifold admit a finite cover which is Haken?
T(2) Does every closed hyperbolic 3-manifold admit a finite cover with positive first Betti number?
T(3) Is every closed hyperbolic 3-manifold finitely covered by a fibered 3-manifold? [23]

We also add one more question:

T(0) Does the fundamental group of every closed hyperbolic 3-manifold contain the fundamental group of a closed surface of negative Euler characteristic?

Exercise 7.1. *Show that the questions become increasingly difficult. Put differently, show that a positive answer to* $T(i)$ *implies a positive answer to* $T(i-1)$.

In the subsequent subsections we will report on the work of many mathematicians which eventually led to a complete answer to these questions. We will proceed in a mostly chronological order.

1981: Right-angled Artin groups. Let G be a finite graph with vertex set V, then it gives rise to a group presentation as follows:

$$A_\Gamma = \langle \{g_v\}_{v\in V} \,|\, [g_u, g_v] = e \text{ if } u \text{ and } v \text{ are connected by an edge}\rangle.$$

Any group which is isomorphic to such a group is called a *right angled Artin group* (*RAAG*). Right-angled Artin groups were introduced by Baudisch [Ba81] in 1981 under the name *semi-free groups*, but they are also sometimes referred to as *graph groups* or *free partially commutative groups*. We refer to [Ch07] for a very readable survey on RAAGs.

Example. If the graph has no vertices, then the resulting RAAG is a free group. If the graph is complete, i.e. if any two vertices are connected by a vertex, then the resulting RAAG is a free abelian group.

Exercise 7.2. *Let G be the graph with vertex set $V = \{1, 2, 3, 4\}$ that is illustrated in Figure 7.2. What is a concise description of the corresponding RAAG?*

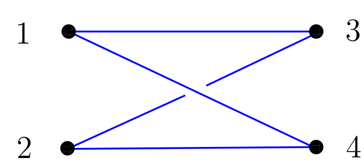

FIGURE 2.

Remark. Right angled Artin groups are commensurable with the perhaps more familiar right angled Coxeter groups (see [DJ00]), which correspond to reflections in orthogonal hyperplanes.

[23]Thurston comments the question with 'This dubious-sounding question seems to have a definite chance for a positive answer.'

This looks like a nice theory, but it is not clear what it has to do with Thurston's questions. So let's move on.

1995: Non-positively curved cube complexes. In 1995 Sageev [Sa95] introduced non-positively curved cube complexes. Loosely speaking a cube complex is a complex build out of cubes of possibly different dimensions. It is is non-positively curved if it has no piece which looks 'locally like a piece of a higher-dimensional sphere'. These notions are illustrated in Figure 7.

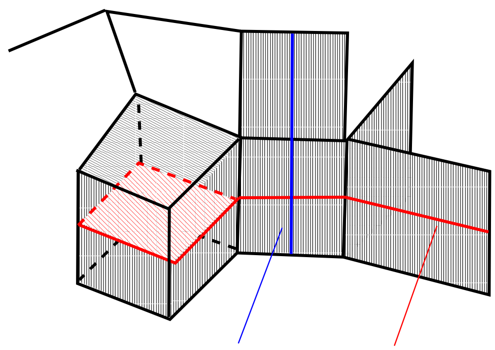

FIGURE 3.

In the following we give the precise definitions. But these can easily be skipped at a first reading.

(1) An *n-cube* is a copy of $[-1,1]^n$ and a 0-cube is a single point.
(2) A *cube complex* is a cell complex formed from cubes, such that the attaching map of each cube is combinatorial in the sense that it sends cubes homeomorphically to cubes by a map modelled on a combinatorial isometry of n-cubes.
(3) The *link* of a 0-cube v is the complex whose 0-simplices correspond to ends of 1-cubes adjacent to v, and these 0-simplices are joined up by n–simplices for each corner of an $(n+1)$–cube adjacent to v.
(4) A *flag complex* is a simplicial complex with the property that any finite pairwise adjacent collection of vertices spans a simplex.
(5) A cube complex C is *non–positively curved* if $\text{link}(v)$ is a flag complex for each 0-cube $v \in C^0$.

Sageev [Sa95, Sa97] proved the following result, which despite its vague formulation we record as a theorem for future reference.

Theorem 7.3. (Sageev) *If a group π has 'lots of codimension-one subgroups', then π is the fundamental group of a non-positively curved cube complex.*

Again, it is not clear what this has to do with Thurston's questions. Therefore we move forward to 2007.

2007: The virtual fibering theorem of Agol. Following Agol [Ag08] we say that a group π is *residually finite rationally solvable* (*RFRS*) if there exists a descending sequence

$$\pi = \pi_1 \supset \pi_2 \supset \pi_3 \supset \dots$$

of subgroups such that the following conditions hold:

(1) each subgroup π_n is normal and of finite index in π,
(2) the intersection $\cap \pi_n$ is trivial,
(3) for each n the quotient π_n/π_{n+1} is abelian,
(4) for each n the surjection $\pi_n \to \pi_n/\pi_{n+1}$ factors through $\pi_n \to H_1(\pi_n;\mathbb{Z})/\text{torsion}$.

Exercise 7.4.

(1) *Show that a group admits a sequence π_n that satisfies* (1) *and* (2) *if and only if π is residually finite.*
(2) *Show that a group admits a sequence π_n that satisfies* (1) *to* (3) *if and only if π is residually finite solvable.*

It is straightforward to see that free abelian groups are RFRS. It is also well-known that free groups are residually finite solvable, i.e. there exists a sequence of groups that satisfies (1) to (3). But subgroups of free groups are free, hence $H_1(\pi_n;\mathbb{Z})$ is torsion-free. It follows that (4) is also satisfied. Put differently, free groups are RFRS.

RAAGs negotiate between free abelian groups and free groups. Very often, statements that hold for free abelian groups and free groups also hold for RAAGs. Fortunately this is basically also the case in our situation. More precisely, Agol [Ag08] proved the following theorem.

Theorem 7.5. *Any subgroup of a RAAG is virtually RFRS.*

Agol [Ag08] also proved the following much deeper theorem. (An alternative account of the proof is given in [FK14].)

Theorem 7.6. (Agol) *Let N be an irreducible 3-manifold such that $\pi_1(N)$ is virtually a subgroup of a RAAG. If $\pi_1(N)$ is infinite, then N is virtually fibered.*

Theorem 7.6 gave the first general criterion for virtual fiberedness. Nonetheless the condition that $\pi_1(N)$ is virtually RFRS is a priori rather stringent. In fact at the time of the writing of [Ag08] only very few hyperbolic 3-manifold groups were known to have this property, e.g. arithmetic hyperbolic groups defined by a quadratic form.

We conclude the discussion of Agol's virtual fibering theorem with a more precise formulation of Theorem 7.6. This precise formulation has been particularly helpful in many applications, see e.g. [FV12, FV14]. In the following, given a 3-manifold N we say that $\phi \in H^1(N;\mathbb{Q}) = \text{Hom}(\pi_1(N),\mathbb{Q})$ is *fibered* if there exists a surface bundle map $p\colon N \to S^1$ and an $r \in Q \setminus \{0\}$ such that $\phi = rp_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$. The following theorem is the promised more precise version of Theorem 7.6. It was again proved by Agol [Ag08, Theorem 5.1].

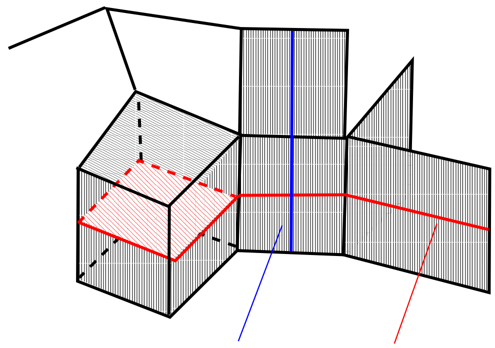

FIGURE 4.

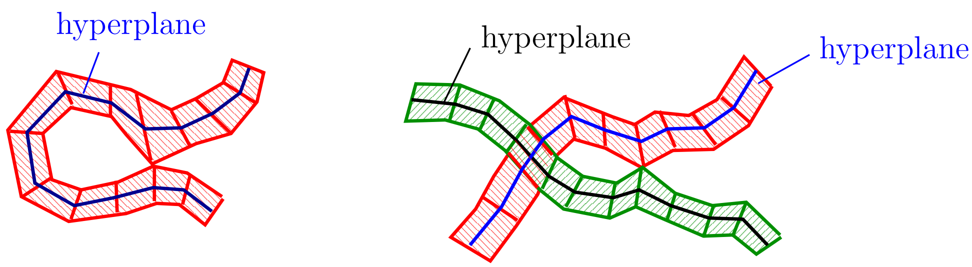

FIGURE 5. Directly self-osculating and inter-osculating hyperplanes.

Theorem 7.7. (Agol) *Let N be an irreducible 3-manifold such that $\pi_1(N)$ is virtually a subgroup of a RAAG. Let $\phi \in H^1(N;\mathbb{Q})$ be a non-fibered non–trivial class, then there exists a finite cover $p\colon N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Q})$ is the limit of fibered classes in $H^1(N';\mathbb{Q})$.*

It is a consequence of Stallings' fibering theorem [St62] that the pull–back of a non–fibered class can not be fibered. Theorem 7.7 thus says, loosely speaking, that provided that $\pi_1(N)$ is virtually RFRS any non–fibered class can be made 'as fibered as possible' in a finite cover.

2008: Special cube complexes. A cube complex has a natural notion of 'hyperplanes'. Loosely speaking these are given by joining 'midcubes' as long as possible. Examples of hyperplanes are illustrated in Figure 7.

In 2008 Haglund–Wise [HW08] introduced the notion of a 'special' cube complex. Loosely speaking cube complex is called *special* if the following conditions hold:

(1) all hyperplanes are embedded and two-sided,
(2) there are no 'directly self-osculating' and no 'inter-osculating' hyperplanes, as illustrated in Figure 7.

We refer to [HW08] for a precise definition.

Definition. A group π is (*compact*) *special* if π is the fundamental group of a non-positively curved special (compact) cube complex X.

The following theorem of Haglund and Wise [HW08] gives a purely group theoretic reformulation of the property of being virtually (compact) special.

Theorem 7.8. (Haglund–Wise) *A group π is virtually* (*compact*) *special if and only if π admits a subgroup of finite index which is a* (*quasi-convex*)[24] *subgroup of a RAAG.*

The connection between being special and being a subgroup of a RAAG comes through the 'nice hyperplanes' in special cube complexes (which necessarily meet 'orthogonally') and the orthogonal hyperplanes in the definition of a right angled Coxeter groups, which are in turn commensurable with right angled Artin groups.

Summarizing, it follows from Theorems 7.5, 7.6 and 7.8 that an irreducible 3-manifold N is virtually fibered if $\pi_1(N)$ is virtually special, i.e. if $\pi_1(N)$ is the fundamental group of special non-positively curved cube complex. But why would $\pi_1(N)$ be the fundamental group of a non-positively curved cube complex, let alone one, which is virtually special?

2009: The surface subgroup theorem of Kahn–Markovic. Recall that above we formulated the following question:

T(0) Does the fundamental group of every closed hyperbolic 3-manifold contain the fundamental group of a closed surface of negative Euler characteristic?

This question was completely answered by Kahn and Markovic [KM12].

Theorem 7.9. (Kahn-Markovic) *Let N be a closed hyperbolic 3-manifold, then N admits a π_1–injective immersion $\iota\colon \Sigma \to N$ of a connected surface such that $\iota_*(\pi_1(\Sigma))$ is quasi-Fuchsian*[25] *surface.*

We refer to [Ka09] for an outline of the argument and we refer to [Ber12] for an exposition of the proof by Kahn–Markovic [KM12]. This answers T(0), but it is not clear how it helps us with the other questions. Initially the hope was that one represents a surface group by an immersed surface, which, with luck, one can promote to an embedded surface in a finite cover to answer Question T(1). But this process is very difficult.

The route to questions T(1) to T(3) turned out to be somewhat different. The key is that Kahn–Markovic provide not just one surface group, but in a sense, 'lots of surface subgroups'. We will now make this precise. We fix an identification of $\pi_1(N)$ with a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. We say that *$N$ contains a dense set of quasi-Fuchsian surface groups* if for each great circle C of $\partial\mathbb{H}^3 = S^2$ there exists a sequence of π_1-injective immersions $\iota_i\colon \Sigma_i \to N$ of surfaces Σ_i such that the following hold:

(1) for each i the group $(\iota_i)_*(\pi_1(\Sigma_i))$ is a quasi-Fuchsian surface group,
(2) the sequence $(\partial\Sigma_i)$ converges to C in the Hausdorff metric on $\partial\mathbb{H}^3$.

---

[24] We give the definition of 'quasi-convex' on page 38.

[25] See [KAG86, p. 4] for the definition of a *quasi-Fuchsian surface group.*

The following is now a more precise statement of the theorem of Kahn–Markovic [KM12]. (This particular formulation is [Ber12, Théorème 5.3].)

Theorem 7.10. (Kahn-Markovic) *Every closed hyperbolic* 3*-manifold contains a dense set of quasi-Fuchsian surface groups.*

Now remember that in Theorem 7.3 we hinted at a result of Sageev [Sa95, Sa97] which says that if a group has enough 'codimension-one subgroups', then it is the fundamental group of a non-positively curved cube complex. Surface subgroups in the fundamental group of a 3-dimensional manifold sounds like they should be 'codimension-one'. In fact the following theorem was proved by Bergeron–Wise [BW12] building on the aforementioned work of Sageev.

Theorem 7.11. (Sageev, Bergeron–Wise) *Let N be a closed hyperbolic* 3*-manifold which contains a dense set of quasi-Fuchsian surface groups. Then $\pi_1(N)$ is also the fundamental group of a compact non-positively curved cube complex.*

Now the situation looks much better. If we start out with a closed hyperbolic 3-manifold, then the combination of Theorems 7.10 and 7.11 shows that $\pi_1(N)$ is the fundamental group of a non-positively curved cube complex. Nonetheless, we are not done yet, since we really need a special cube complex.

2012: Agol's Theorem. In March 2012 Agol stunned the world of 3-manifold topology by announcing a proof of the following theorem, which had been initially conjectured by Wise [Wi12a, Conjecture 19.5].

Theorem 7.12. *Let π be a word-hyperbolic group which is also the fundamental group of a compact non-positively curved cube complex. Then π is virtually compact special.*

Remark.

(1) The precise meaning of 'word-hyperbolic group' is of no concern to us. What matters is that fundamental groups of closed hyperbolic 3-manifolds are word-hyperbolic.
(2) The proof of Theorem 7.12 relies heavily on results in the appendix to [Ag13], which are due to Agol, Groves and Manning. The results of this appendix extend the techniques of [AGM09] to word-hyperbolic groups with torsion, and combine them with the Malnormal Special Quotient Theorem of Wise [Wi12a, Theorem 12.3]. (An alternative proof of this theorem is in [AGM14, Corollary 2.8].) We refer to [Bes14, Section 8] for a summary of Agol's proof and to [Ber14] for another account of this proof.

Summarizing, the combination of Theorems 7.10, 7.11 and 7.12 implies the following theorem.

Theorem 7.13. *The fundamental group of any closed hyperbolic 3-manifold is virtually compact special.*

N is a closed hyperbolic 3-manifold

↓ Kahn-Markovic [KM12]

$\pi_1(N)$ contains 'lots of surface groups'

↓ Sageev [Sa95] and Bergeron-Wise [BW12]

$\pi_1(N)$ is the fundamental group of a non-positively curved cube complex

↓ Agol [Ag13] building on Wise [Wi12a, Wi12b]

$\pi_1(N)$ is virtually the fundamental group of a non-positively curved *special* cube complex

↓ Haglund-Wise [HW08]

$\pi_1(N)$ is virtually the subgroup of a RAAG

↓ Agol [Ag08]

$\pi_1(N)$ is virtually RFRS

↓ Agol [Ag08]

N is virtually fibered

Figure 6. The key steps in the virtual fibering theorem for closed hyperbolic 3-manifolds.

Furthermore, combining this theorem with Theorems 7.8, 7.5 and 7.6 we obtain the following theorem.

Theorem 7.14. *The fundamental group of any closed hyperbolic 3-manifold is virtually the subgroup of a RAAG, in particular N is virtually fibered.*

Thus this theorem gives a positive answer to all the aforementioned questions of Thurston. For convenience we summarized all the steps in Figure 7. This theorem is of course fantastic, but what about hyperbolic 3-manifolds with boundary, and what about 3-manifolds that are not hyperbolic?

2009: The work of Wise. The revolution in 3-manifold topology owes a big debt to the ideas and work of Wise [Wi09, Wi12a, Wi12b]. In particular he proved the following theorem.

Theorem 7.15. (Wise 2009) *The fundamental group of any closed hyperbolic 3-manifold which contains a geometrically finite surface is virtually compact special.*

We do not need to be too concerned about the definition of geometrically finite surface. It suffices to know that this theorem applies to 'most' closed hyperbolic 3-manifolds that are Haken. The theorem, together with work of Bonahon–Thurston [Bon86] implies for example that all closed Haken hyperbolic 3-manifolds are virtually fibered.

Furthermore Wise also proved the following theorem which perfectly complements Theorem 7.13.

Theorem 7.16. (Wise 2009) *The fundamental group of any hyperbolic 3-manifold with non–trivial boundary is virtually compact special.*

As we mentioned before, we can not expect that these theorems for hyperbolic 3-manifolds extend to all reasonable (say aspherical) 3-manifolds since there exist 3-manifolds that are not virtually fibered. For example, it is relatively straightforward to see that any non-trivial S^1-bundle over a surface is not virtually fibered. More interestingly, there also exist graph manifolds with non-trivial JSJ-decomposition which are not virtually fibered (see, e.g., [LW93, p. 86] and [Ne96, Theorem D]). In particular in both cases the fundamental group can not be virtually special.

Nonetheless it turns out that graph manifolds are the only problematic cases. More precisely, the following theorem was proved by Przytycki–Wise [PW12].

Theorem 7.17. (Przytycki–Wise 2012) *Let N be an irreducible 3-manifold which is not a graph manifold. Then $\pi_1(N)$ is virtually special.*

Furthermore Liu [Liu13] (see also the work of Przytycki and Wise [PW14]) determined for which graph manifolds the fundamental group is virtually special. Combining Theorems 7.16 and 7.17 with the work of Liu [Liu13] and Leeb [Leb95] one obtains the following beautiful theorem.

Theorem 7.18. *Let N be an aspherical 3-manifold. Then $\pi_1(N)$ is virtually special if and only if N supports a non-positively curved metric.*

8. Consequences of being virtually special

In the previous section we saw that the work of Agol, Kahn–Markovic, Przytycki–Wise, Wise and many other authors implies the following theorem.

Theorem 8.1. *If N is an irreducible 3-manifold that is not a closed graph manifold, then $\pi_1(N)$ is virtually special. In particular $\pi_1(N)$ is virtually a subgroup of a RAAG.*

We already mentioned that this theorem, together with Agol's virtual fibering theorem implies that these manifolds are virtually fibered. But being virtually a subgroup of a RAAG has many other useful consequences.

More precisely, we have the following corollary.

Corollary 8.2. *If π is the fundamental group of an irreducible 3-manifold that is not a closed graph manifold, then the following hold:*

(1) *π is large, i.e. π admits a finite-index subgroup that surjects onto a non-cyclic free group,*
(2) *$vb_1(N) = \infty$, i.e. N admits finite covers with arbitrarily large first Betti numbers,*
(3) *π admits a finite index subgroup which is residually torsion–free nilpotent,*
(4) *π admits a finite index subgroup which is residually p for any prime p,*
(5) *π admits a finite index subgroup which is biorderable,*
(6) *π is linear over $\mathbb{Z}$, i.e. $\pi \subset \mathrm{GL}(n, \mathbb{Z})$ for some $n \in \mathbb{N}$.*

The first statement follows from Antolín–Minasyan [AM15, Corollary 1.6]. The second is an immediate consequence of the first statement. The third statement is shown in [DK92]. The fourth statement is a consequence of the third statement (see [Gru57, Theorem 2.1]) and the fifth statement is a consequence of the fourth statement (see [Rh73]). Finally recall that a RAAG is commensurable with a right angled Coxeter group, which in turn is easily seen to be linear over $\mathbb{Z}$ (see [HsW99] for details). Many more consequences of being virtually special are outlined in [AFW15, Flowchart 4].

Now we will see that groups which are virtually *compact* special (or equivalently, groups which are virtually a quasi–convex subgroup of a RAAG) are even better behaved. The reason is the following theorem of Haglund [Hag08, Theorem F]:

Theorem 8.3. (Haglund) *Let Γ be a quasi-convex subgroup of a RAAG* [26] *A, then Γ is a virtual retract of A, i.e. there exists a finite index subgroup A' of A which contains Γ and a homomorphism $\varphi\colon A' \to \Gamma$ such that $\varphi(g) = g$ for all $g \in \Gamma$.*

Note that the conclusion of the theorem trivially holds for all finitely generated subgroups of abelian groups and it is a classical theorem that it also holds for finitely generated subgroups of free groups. Haglund's result is therefore a generalization of these two classical results.

Recall that a group π is called *conjugacy separable* if for any two non-conjugate elements $g, h \in \pi$ there exists an epimorphism $\alpha\colon \pi \to G$ onto a finite group G such that $\alpha(g)$ and $\alpha(h)$ are not conjugate. Minasyan [Min12] showed that finite index subgroups of RAAGs are conjugacy separable. Using the fact that retracts of conjugacy separable groups are again conjugacy separable one can now easily prove the following theorem:

Theorem 8.4. (Minasyan) *Let N be a 3-manifold such that $\pi = \pi_1(N)$ is virtually compact special, then π is conjugacy separable.*

This theorem is a key ingredient in the proof of Hamilton–Wilton–Zalesskii [HWZ13] that the fundamental group of any closed, orientable, irreducible 3-manifold is conjugacy separable.

9. Supgroup separability

Thurston [Th82b, Question 15] also asked the following question regarding hyperbolic 3-manifolds:

[26] We refer to page 38 for a definition of quasi-convex subgroup. Here we mean 'quasi-convex' with respect to a canonical generating set of a RAAG, as in the definition of a RAAG.

Question 9.1. *Is the fundamental group of a hyperbolic 3-manifold subgroup separable?*

Here recall that a group π is called *subgroup separable* if given any finitely generated subgroup $A \subset \pi$ and any $g \notin A$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group such that $\alpha(g) \notin \alpha(A)$.

Some positive evidence towards a positive answer is given by Scott's Theorem 2.6 which says that the fundamental group of any surface is subgroup separable. On the other hand Niblo–Wise [NW01, Theorem 4.2] showed that fundamental groups of 'most' 3-dimensional graph manifolds are not subgroup separable.

9.1. **The Tameness Theorem of Agol and Calegari–Gabai.** Agol [Ag07] and Calegari–Gabai [CG06] (see also [Ca08, Corollary 8.1] and [Bow10] for further details) proved independently in 2004 the following theorem, which was first conjectured by Marden in 1974:

Theorem 9.2. (Tameness Theorem) *Let N be a hyperbolic 3-manifold with finitely generated fundamental group. Then N is topologically tame, i.e. N is homeomorphic to the interior of a compact 3-manifold.*

The Tameness Theorem together with Canary's covering theorem (see [Ca94, Section 4] and [Ca96]) implies the following dichotomy:

Theorem 9.3. (Dichotomy Theorem) *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup of infinite index. Then either*

(1) Γ *is a virtual surface fiber group, i.e. there exists a finite cover $\tilde{N} \to N$ and a fiber surface $\Sigma \subset \tilde{N}$ of a fibration $\tilde{N} \to S^1$ such that $\Gamma = \pi_1(\Sigma)$, or*
(2) Γ *is geometrically finite.*

We will not give the definition of a geometrically finite subgroup (see [KAG86, p. 10] for details). Instead below we will rephrase the dichotomy in various alternative ways.

In order to state one possible reformulation we need the notion of the commensurator of a subgroup Γ of a group π, which is defined as

$$\mathrm{Comm}_\pi(\Gamma) := \{g \in \pi \,|\, \Gamma \cap g\Gamma g^{-1} \text{ has finite index in } \Gamma\}.$$

Then the above dichotomy can be phrased as follows: If $\Gamma \subset \pi_1(N)$ is a finitely generated subgroup of infinite index of the fundamental group of a hyperbolic 3-manifold, then either

(1) $\mathrm{Comm}_\pi(\Gamma)$ is a finite index subgroup of π, or
(2) $\mathrm{Comm}_\pi(\Gamma)$ is a finite index supergroup of Γ.

We refer to [Ca08, Theorem 8.7] for a proof.

Loosely speaking this version says that a finitely generated subgroup of the fundamental group of a hyperbolic 3-manifold is either 'almost normal' or 'very non-normal'. Another way of phrasing this dichotomy is in terms of the 'width' of a subgroups, which is a different measure of 'normality' respectively 'non-normality' of a subgroup. We refer to [GMRS98], [AGM09] and [Wi12a, Definition 12.7] for details.

In order to give one more formulation of the dichotomy we will need a few more definitions:

Definition.

(1) Let X be a geodesic metric space. A subspace Y is said to be *quasi-convex* if there exists $\kappa \geq 0$ such that any geodesic in X with endpoints in Y is contained within the κ-neighbourhood of Y.
(2) Let π be a group with a fixed generating set S. A subgroup $H \subseteq \pi$ is said to be *quasi-convex* if it is a quasi-convex subspace of $\mathrm{Cay}_S(\pi)$, the Cayley graph of π with respect to the generating set S. In general quasi-convexity depends on the choice of generating set S. However, if π is word-hyperbolic, then the quasi-convexity of a subgroup H does not depend on the choice of generating set.

Let N be a hyperbolic 3-manifold. A subgroup of $\pi_1(N)$ is geometrically finite if and only if it is quasi-convex (see for example [Hr10, Corollary 1.3] for a reference and see [KS96, Theorem 2]). We thus obtain the following reformulation of the above dichotomy theorem:

Theorem 9.4. (Dichotomy Theorem) *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup of infinite index. Then one of the following occurs:*

(1) Γ *is a virtual surface fiber group, or*
(2) Γ *is a quasi–convex subgroup of* π.

9.2. **The subgroup separability theorem for hyperbolic 3-manifolds.** Let N be a hyperbolic 3-manifold. Recall that by the dichotomy theorem a finitely generated subgroup $\Gamma \subset \pi = \pi_1(N)$ of infinite index is either a virtual surface fiber group, or it is a quasi–convex subgroup of π. Using Haglund's theorem and using the philosophy [27] that if $\Gamma \subset \pi$ is quasi–convex and if $\pi \subset A$ is quasi–convex, then $\Gamma \subset A$ should be quasi–convex one can prove the following result:

Theorem 9.5. *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup which is not a virtual surface fiber group. Then Γ is a virtual retract of $\pi_1(N)$.*

It is relatively easy to see that a virtual surface fiber group of a hyperbolic 3-manifold N can not be a virtual retract of $\pi_1(N)$. We thus obtain the following reformulation of the dichotomy theorem: If N is a hyperbolic 3-manifold and if Γ is a finitely generated subgroup of $\pi_1(N)$, then

(1) Γ is either a virtual surface fiber group, or
(2) Γ is a virtual retract of $\pi_1(N)$.

An elementary argument shows that virtual surface fiber groups are separable in $\pi_1(N)$. Furthermore, a virtual retract of a group π is also separable in π (see e.g. [Hag08, Section 3.4]), we thus obtain the following result:

Theorem 9.6. *The fundamental group of any hyperbolic 3-manifold is subgroup separable.*

This answers Thurston's question in the affirmative.

[27]This statement does not hold in general, but one can apply this philosophy to prove the subsequent theorem.

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