A SURVEY OF THE FOUNDATIONS OF FOUR-MANIFOLD THEORY IN THE TOPOLOGICAL CATEGORY

STEFAN FRIEDL, MATTHIAS NAGEL, PATRICK ORSON, AND MARK POWELL

Abstract. The goal of this survey is to state some foundational theorems on 4-manifolds, especially in the topological category, give precise references, and provide indications of the strategies employed in the proofs. Where appropriate we give statements for manifolds of all dimensions.

1. Introduction

Here and throughout the paper "manifold" refers to what is often called a "topological manifold"; see Section 2 for a precise definition. Here are some of the statements discussed in this article.

(1) Existence and uniqueness of collar neighbourhoods (Theorem 2.5).
(2) The Isotopy Extension theorem (Theorem 2.10).
(3) Existence of CW structures (Theorem 4.5).
(4) Multiplicativity of the Euler characteristic under finite covers (Corollary 4.8).
(5) The Annulus theorem 5.1 and the Stable Homeomorphism Theorem 5.3.
(6) Connected sum of two oriented connected 4-manifolds is well-defined (Theorem 5.11).
(7) The intersection form of the connected sum of two 4-manifolds is the sum of the intersection forms of the summands (Proposition 5.15).
(8) Existence and uniqueness of tubular neighbourhoods of submanifolds (Theorems 6.8 and 6.9).
(9) Noncompact connected 4-manifolds admit a smooth structure (Theorem 8.1).
(10) When the Kirby-Siebenmann invariant of a 4-manifold vanishes, both connected sum with copies of $S^2 \times S^2$ and taking the product with $\mathbb{R}$ yield smoothable manifolds (Theorem 8.6).
(11) Transversality for submanifolds and for maps (Theorems 10.3 and 10.8).
(12) Codimension one and two homology classes can be represented by submanifolds (Theorem 10.9).
(13) Classification of 4-manifolds up to homeomorphism with trivial and cyclic fundamental groups (Section 11).
(14) Compact orientable manifolds that are homeomorphic are stably diffeomorphic (Theorem 12.2 and Corollary 12.5).
(15) Multiplicativity of signatures under finite covers (Theorem 13.1).

(16) The definition of Reidemeister torsion for compact manifolds and some of its key technical properties (Section 14.3).
(17) Obstructions to concordance of knots and links (Theorem 15.2).
(18) Poincaré duality for compact manifolds with twisted coefficients (Theorem A.15 and Theorem A.16).

Many of these results are essential tools for the geometric topologist. Our hope is that with the statements from this note the "working topologist" will be equipped to handle most situations. For many of the topics discussed in this paper the corresponding statements for 4-manifolds with a smooth atlas are basic results in differential topology. However for general 4-manifolds, it can be difficult to find precise references. We aim to rectify this situation to some extent. Throughout the paper we make absolutely no claims of originality.

Conventions.

(1) Given a subset A of a topological space X we denote the interior by $\operatorname{Int} A$.
(2) For $n \in \mathbb{N}_0$ we write $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ for the closed unit ball in $\mathbb{R}^n$. We refer to $\operatorname{Int} D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ as the open n-ball.
(3) Unless indicated otherwise I denotes the interval $I = [0, 1]$.
(4) On several occasions we use cup and cap products and we cite several results from [Dol95, Bre97, Hat02, Fri19]. Different books on algebraic topology often have different sign conventions for cup and cap products, but in all statements that we give, the sign conventions are irrelevant, so it is not a problem to mix results from different sources.

Acknowledgments. Thanks to Steve Boyer, Anthony Conway, Jim Davis, Fabian Hebestreit, Jonathan Hillman, Min Hoon Kim, Alexander Kupers, Markus Land, Tye Lidman, Erik Pedersen, George Raptis, Arunima Ray, Eamonn Tweedy, and Chuck Livingston for very helpful conversations. We are particularly grateful to Gerrit Herrmann for providing us with detailed notes which form the basis of the Appendix A. SF was supported by the SFB 1085 'Higher Invariants' at the University of Regensburg, funded by the Deutsche Forschungsgemeinschaft (DFG). MP was supported by an NSERC Discovery Grant. SF wishes to thank the Université du Québec à Montréal and Durham University for warm hospitality.

Contents

2. Manifolds

In the literature the notion of a "manifold" gets defined differently, depending on the preferences of the authors. Thus we state in the following what we mean by a manifold.

Definition 2.1. Let X be a topological space.

(1) We say that X is *second countable* if there exists a countable basis for the topology.
(2) An *n-dimensional chart for X at a point* $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighbourhood of x and
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $E_{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

  In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).
(3) We say that X is an *n-dimensional manifold* if X is second countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.
(4) We say that a point x on a manifold is a *boundary point* if x admits a chart of type (ii). (A point cannot admit charts of both types [Hat02, Theorem 2B.3].) We denote the set of all boundary points of X by ∂X.
(5) An *atlas* for a manifold X consists of a family of charts such that the domains cover all of X. An atlas is *smooth* if all transition maps are smooth. A *smooth manifold* is a manifold together with a smooth atlas. Usually one suppresses the choice of a smooth atlas from the notation.

To avoid misunderstandings we want to stress once again that what we call a "manifold" is often referred to as a "topological manifold".

Definition 2.2. An *orientation* of an n-manifold M is a choice of generators $\alpha_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$ for each $x \in M \setminus \partial M$ such that for every $x \in M \setminus \partial M$ there exists an open neighborhood $U \subset M \setminus \partial M$ and a class $\beta \in H_n(M, M \setminus U, \mathbb{Z})$ such that β projects to α_y for each $y \in U$.

Using the cross product one can show that the product of two oriented manifolds admits a natural orientation. Furthermore, the boundary of an oriented manifold also comes with a natural orientation. The proof of the latter statement is slightly delicate; we refer to [GH81, Chapter 28] or to [Fri19, Chapter 45.9] for details.

2.1. Collar neighbourhoods.

We discuss the existence of a collar neighbourhood of the boundary. First we recall the definition of a neighbourhood.

Definition 2.3 (Neighbourhood)**.** Let X be a space. A *neighbourhood* of a subset $A \subseteq X$ is a set $U \subseteq X$ for which there is an open set V satisfying $A \subseteq V \subseteq U$.

Next we give our definition of a collar neighbourhood.

Definition 2.4 (Collar neighbourhood)**.** Let M be a manifold and let B be a compact submanifold of ∂M. A *collar neighbourhood* is a map $\Phi\colon B \times [0, r] \to M$ for some $r > 0$ with the following three properties:

(1) Φ is a homeomorphism onto its image,
(2) for all $P \in B$ we have $\Phi(P, 0) = P$,
(3) we have $\Phi^{-1}(B \times [0, r]) \cap \partial M = B$.

Often, by a slight abuse of language, we identify $B \times [0, r]$ with its image $\Phi(B \times [0, r])$ and we refer to $B \times [0, r]$ also as a collar neighbourhood.

It is a consequence of the invariance of domain theorem that a collar neighbourhood of ∂M is in fact a neighbourhood of ∂M. Now we can state the collar neighbourhood theorem in the formulation of [Arm70, Theorem 1]. The existence of collars is originally due to Brown [Bro62].

Theorem 2.5 (Collar neighbourhood theorem)**.** *Let M be an n-manifold. Let C be a compact $(n-1)$-dimensional submanifold of ∂M. (In most cases we take $C = \varnothing$.) Given any collar neighbourhood $C \times [0, 2]$, the restriction to $C \times [0, 1]$ can be extended to a collar neighbourhood $\partial M \times [0, 1]$.*

To formulate a uniqueness result for collar neighbourhoods it helps to introduce the following definition.

Definition 2.6. Let $f, g\colon X \to Y$ be two maps between topological spaces and let Z be a subset of Y. We say f and g are *ambiently isotopic rel.* Z if there exists an isotopy $H = \{H_t\}_{t\in[0,1]}\colon Y \times [0, 1] \to Y$ such that $H_0 = \mathrm{Id}$, $H_t|_Z = \mathrm{Id}_Z$ for all t and such that $H_1 \circ f = g$.

Theorem 2.7. *Let M be a compact manifold. Any two collar neighbourhoods $\Phi\colon \partial M \times [0, 1] \to M$ and $\Psi\colon \partial M \times [0, 1] \to M$ are ambiently isotopic rel. $\partial M \times \{0\}$.*

Proof. The theorem is a consequence of [KS77, Essay I, Theorem A.2]. $\square$

The following corollary is a straightforward consequence of the Collar neighbourhood theorem 2.5. The corollary often makes it possible to reduce arguments about manifolds with boundary to the case of closed manifolds.

Corollary 2.8.

(1) *Let N be an n-manifold, possibly disconnected, and let $f\colon A \to B$ be a homeomorphism between disjoint collections of boundary components of N. Then the quotient $N/\sim$ under the relation $a \sim f(a)$ is an n-manifold with boundary $\partial(N/\sim) = \partial N \setminus (A \cup B)$.*
(2) *Let M be an n-manifold. Its double $DM := M \cup_{\partial M = \partial M} M$ is an n-manifold with empty boundary.*

2.2. **The isotopy extension theorem.**

Definition 2.9. Let X be a k-dimensional manifold and let M be an m-dimensional manifold. Let $h\colon X \times [0,1] \to M$ be a homotopy.

(1) We say h is *locally flat* if for every $(x,t) \in X \times [0,1]$ there exists a neighbourhood $[t_0,t_1]$ of t and if there are level-preserving embeddings $\alpha\colon D^k \times [t_0,t_1] \to X \times [0,1]$ and $\beta\colon D^k \times D^{m-k} \times [t_0,t_1] \to M \times [0,1]$ to neighbourhoods of (x,t) and $(h_t(x),t)$ respectively, such that the following diagram commutes:

$$\begin{tikzcd} D^k \times \{0\} \times [t_0,t_1] \arrow[r, hook] \arrow[d, "\alpha"] & D^k \times D^{m-k} \times [t_0,t_1] \arrow[d, "\beta"] \\ X \times [0,1] \arrow[r, "(x,t)\mapsto(h_t(x),t)"] & M \times [0,1]. \end{tikzcd}$$

(2) We say h is *proper* if for every $t \in [0,1]$ we have $h_t(X) \cap \partial M = h_t(\partial X)$.

This definition allows us to formulate the following useful theorem [EK71, Corollary 1.4].

Theorem 2.10. (Isotopy Extension Theorem) *Let $h\colon X \times [0,1] \to M$ be a locally flat proper isotopy of a compact manifold X into a manifold M. Then h can be covered by an ambient isotopy of M, i.e. there exists an isotopy $H\colon M \times [0,1] \to M$ such that $H_0 = \mathrm{Id}$ and $h_t = H_t \circ h_0$ for all $t \in [0,1]$.*

As an application of the isotopy extension theorem we will prove the following theorem. We encourage the reader to find a more direct proof.

Proposition 2.11. *Let M be a connected n–manifold. Then for any two points x, y, there exists a chart $\phi\colon U \to \mathbb{R}^n$ with $x, y \in U$. Especially, the points x, y are connected by a locally flat embedded arc.*

Proof. Since M is path-connected, there exist points $x = x_0, x_1, \dots, x_{k+1} = y$ such that there are charts (U_i, ψ_i) for $i = 0, \dots, k$ and both x_i, x_{i+1} are contained in U_i. Additionally, arrange that $y \notin U_i$ for $i = 0, \dots, k-1$.

Now we construct a locally flat isotopy h moving x to x_k. Pick a function $\beta\colon [0,1] \to [0,1]$ with $\beta(t) = 0$ for $t \in [0,1/5]$ and $\beta(t) = 1$ for $t \in [4/5,1]$. Connect x_i to x_{i+1} by the linear path in the chart U_i, but pass along it with speed determined by β. That is, define

$$h_i(t) := \psi_i^{-1}\Big((1-\beta(t)) \cdot \psi_i(x_i) + \beta(t) \cdot \psi_i(x_{i+1})\Big).$$

Consider the composition (of isotopies) $h = h_0 * \cdots * h_k$, defined so that $h|_{[i/k,i+1/k]}$ corresponds to h_i. Each h_i is locally flat via the discs $D_i(t) = \psi_i^{-1}(B_1(\psi_i(h_i(t))))$, say. Also, h is locally flat at the times i/k, since it is the constant isotopy of a point in a neighbourhood of time i/k. Since being locally flat is a local condition, we deduce that h is a locally flat isotopy from x to x_k. Since $y \notin U_i$ for $i = 0, \dots, k-1$, the image of h is disjoint from y. Consequently, we can upgrade h to a locally flat isotopy $S^0 \times I \to M$ by declaring $(-1,t) \mapsto y$ and $(1,t) \mapsto h(t)$. The Isotopy Extension Theorem 2.10 yields an ambient isotopy $H_t\colon M \to M$ with $H_t(x) = h(t)$ and $H_t(y) = y$ for all $t \in I$.

Now we stretch the last chart U_k all the way back to $x = x_0$. Define $U = (H_1)^{-1}(U_k)$ and note that U contains both x and y. The homeomorphism $\phi = \psi_k \circ H_1 \colon U \to \mathbb{R}^n$ defines the required chart on M.

To obtain a locally flat arc connecting x and y, connect the points x and y by a straight line in the chart (U, ϕ). □

A related result [DV09, Proposition 4.5.1] shows that one can approximate a map of a compact polyhedra into a manifold by a (non-locally flat) embedding.

3. Smooth 4-manifolds and their intersection forms

In this chapter we consider some of the most famous 4-manifolds. Recall that not all symmetric unimodular pairings over $\mathbb{Z}$ can be realised as the intersection forms of closed, smooth 4-manifolds. We discuss some of these limitations below.

We start out with the definition of the intersection form.

Definition 3.1.

(1) Given a finitely generated abelian group H we write $\mathrm{F}H := H/\text{torsion subgroup}$.
(2) Given an oriented compact 4-manifold M we refer to the map

$$\begin{array}{rcl} Q_M \colon \mathrm{F}H_2(M;\mathbb{Z}) \times \mathrm{F}H_2(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (a,b) & \mapsto & Q_M(a,b) := \langle \mathrm{PD}_M^{-1}(a) \cup \mathrm{PD}_M^{-1}(b), [M] \rangle \end{array}$$

as the *intersection form*.

Let E_8 denote the even 8×8 Cartan matrix of the eponymous exceptional Lie algebra; that is,

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Note that this is a symmetric integral matrix with determinant one.

Example 3.2. Here are some important closed, smooth 4-manifolds.

(1) The 4-sphere S^4. This is simply connected and has $H_2(S^4;\mathbb{Z}) = \{0\}$.
(2) The complex projective plane $\mathbb{C}\mathrm{P}^2$, which comes with a canonical orientation. The same underlying manifold with the opposite orientation is $\overline{\mathbb{C}\mathrm{P}^2}$. They are simply-connected manifolds with $H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z}$. The intersection form of $\mathbb{C}\mathrm{P}^2$ is (1) and the intersection form of $\overline{\mathbb{C}\mathrm{P}^2}$ is (-1).
(3) The real projective space $\mathbb{RP}^4$, which is non-orientable and not simply-connected.

(4) The products $S^2 \times S^2$ and $S^1 \times S^3$. The manifold $S^2 \times S^2$ is simply-connected and $H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The intersection form of $S^2 \times S^2$ is represented by the standard hyperbolic form $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(5) The $K3$ surface or Kummer surface
$$K3 \; := \; \left\{ [z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathrm{P}^3 \,\middle|\, z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0 \right\}$$
This is a simply connected, smooth, spin, closed 4-manifold with $H_2(K3; \mathbb{Z}) \cong \mathbb{Z}^{22}$. As is shown in [GS99, Theorem 1.3.8] or alternatively [MS17, p. 176], the intersection form of K3 is isometric to $E_8 \oplus E_8 \oplus H \oplus H \oplus H$.

In Theorem 11.2 we will see that any unimodular symmetric form occurs as the intersection form of a closed oriented 4-manifold. In the following we survey results on intersection forms of closed oriented smooth 4-manifolds. As we will see, the results in the smooth setting differ dramatically from the results in the topological setting.

Proposition 3.3. *Let M be a closed, oriented, spin 4-manifold. Then the signature* $\operatorname{sign}(M)$ *is divisible by 8.*

Proof. Write Sq^k for the kth Steenrod square operation. The kth Wu class $v_k \in H^k(M; \mathbb{Z}/2)$ satisfies $\mathrm{Sq}^k(a) = v_k \cup a$, for every class $a \in H^{4-k}(M; \mathbb{Z}/2)$. Hence if $a \in H^2(M; \mathbb{Z}/2)$ then $v_2 \cup a = \mathrm{Sq}^2(a) = a \cup a$. But the nth Stiefel-Whitney class of M is given by $w_n = \sum_i \mathrm{Sq}^i(v_{n-i})$ (see [MS74, Theorem 11.14]). Since M is oriented and spin, we have $0 = w_1 = \mathrm{Sq}^0(v_1) = v_1$, and $0 = w_2 = \mathrm{Sq}^0(v_2) + \mathrm{Sq}^1(v_1) = v_2$. So for any $a \in H^2(M; \mathbb{Z}/2)$, we have $a \cup a = 0 \cup a = 0 \in \mathbb{Z}/2$. But this implies that for any $x \in FH_2(M; \mathbb{Z})$ we have that $Q_M(x, x) = \langle \mathrm{PD}^{-1}(x) \cup \mathrm{PD}^{-1}(x), [M] \rangle \equiv 0 \pmod 2$. In other words Q_M is an even form. It is then an algebraic fact, see e.g. [MH73, Theorem 5.1], that for any symmetric nonsingular bilinear even form Q the signature is divisible by 8. □

Rochlin's Theorem [Roh52] gives an extra restriction on the signatures of intersection forms of spin 4-manifolds that admit a smooth structure.

Theorem 3.4 (Rochlin)**.** *Let M be a closed, oriented, spin, smooth 4-manifold. Then the signature* $\operatorname{sign}(M)$ *is divisible by 16.*

Remark 3.5. Let M be a closed oriented 4-manifold with an even intersection form and such that $H_1(M; \mathbb{Z})$ has no 2-torsion. This implies $H^2(M; \mathbb{Z}/2) \cong \operatorname{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}/2)$ and that the mod 2 reduction of Q_M is isomorphic to the pairing $(a, b) = \langle a \cup b, [M] \rangle$ on $H^2(M; \mathbb{Z}/2)$. As Q_M is even, this implies that $(a, a) = 0 \in \mathbb{Z}/2$ for any $a \in H^2(M; \mathbb{Z}/2)$. But we saw above that $a \cup a = v_2 \cup a$, so we must have that $v_2 = 0$ as this pairing is nondegenerate. We also saw above that $v_2 = w_2$ when M is oriented, so in fact $w_2 = 0$ and M admits a spin structure.

It is not true that simply having an even intersection form implies M is spin. Indeed, it is possible to construct a closed oriented 4-manifold M that has $Q_M = 0$ (which is in particular an even form), but has non-vanishing w_2 [GS99, Exercise 5.7.7(a)]. In a similar spirit, by [Hab82, FS84] there exists a closed oriented 4-dimensional smooth manifold M with an even intersection form Q_M that satisfies $\operatorname{sign}(M) = 8$. Hence this must also fail to be spin, now by Rochlin's theorem.

Corollary 3.6. *There exists a closed orientable 4-manifold that does not admit a smooth structure.*

Proof. By Theorem 11.2 there exists a simply connected closed orientable 4-manifold M with $Q_M \cong E_8$. By Rochlin's Theorem 3.4 this manifold does not admit a smooth structure. □

In a remarkable twist, shortly after Freedman proved Theorem 11.2, Donaldson [Don83, Theorem A] [Don87, Theorem 1], proved the following result regarding intersection forms of smooth 4-manifolds.

Theorem 3.7 (Donaldson)**.** *Let M be a closed oriented smooth 4-manifold. If Q_M is positive-definite, then Q_M can be represented by the identity matrix.*

To understand the significance of Donaldson's Theorem it is helpful consider the following table from [MH73, p. 28], which basically says that there are lots of isometry types of nonsingular positive definite forms.

Dimension:	8	16	24	32	40
Number of isometry types of nonsingular positive definite even symmetric forms:	1	2	24	$\geq 10^7$	$\geq 10^{51}$

It follows from [MH73, Theorem II.5.3] that every nonsingular indefinite *odd* symmetric form is isometric to $k \cdot (1) \oplus \ell \cdot (-1)$. These are realised by $k \cdot \mathbb{C}\mathrm{P}^2 \# \ell \cdot \overline{\mathbb{C}\mathrm{P}}^2$. Therefore we only need to discuss the realisability of nonsingular indefinite *even* symmetric forms. Again by [MH73, Theorem II.5.3], every nonsingular *even* indefinite symmetric form is isometric to $n \cdot E_8 \oplus m \cdot H$ for some $(m, n) \in \mathbb{N}_0 \times \mathbb{Z} \setminus \{(0, 0)\}$. The following theorem, proved by Furuta [Fur01], gives some restrictions on the possible values of m and n.

Theorem 3.8 (Furuta's 10/8 Theorem)**.** *If M is a closed oriented connected smooth 4-manifold with* indefinite *even intersection form, then*

$$b_2(M) \;\geq\; \frac{10}{8} \cdot |\operatorname{sign}(M)| + 2.$$

In particular $Q_M \cong n \cdot E_8 \oplus m \cdot H$ for some $n \in 2\mathbb{Z}$ and $m \in \mathbb{N}$ with $m \geq |n| + 1$.

Furuta's 10/8 Theorem is not yet optimal since it does not quite close the gap between the forms we can realise by smooth manifolds and the forms we can exclude. More precisely, it follows from the calculation of the intersection form of the K3-surface and of $S^2 \times S^2$ that for any $n = 2p \in \mathbb{Z}$ and every $m \geq 3|p|$ there exists a closed oriented simply connected 4-dimensional smooth manifold with intersection form isometric to $n \cdot E_8 \oplus m \cdot H$. In other words, we have

$$\text{intersection form of } p \cdot \mathrm{K3} \,\#\, (m - 3|p|) \cdot (S^2 \times S^2) \;\cong\; 2p \cdot E_8 \oplus m \cdot H.$$

The following conjecture predicts that this result is optimal.

Conjecture 3.9 (11/8-Conjecture)**.** *If M is a closed oriented smooth 4-manifold with indefinite even intersection form, then*

$$b_2(M) \ \geq \ \frac{11}{8} \cdot |\operatorname{sign}(M)|.$$

Equivalently, if $Q_M \cong 2p \cdot E_8 \oplus m \cdot H$ with $p \neq 0$, then $m \geq 3|p|$.

Remark 3.10.

(1) A proof of the 11/8-Conjecture would imply, by Freedman's Theorem 11.2, that any closed oriented simply connected smooth 4-manifold is *homeomorphic* to either a connected sum of the form $k \cdot \mathbb{C}\mathrm{P}^2 \# \ell \cdot \overline{\mathbb{C}\mathrm{P}}^2$ or to a connected sum of the form $n \cdot \mathrm{K3} \# m \cdot (S^2 \times S^2)$.

(2) Currently the best known result in the direction of the 11/8-Conjecture is [HLSX18, Corollary 1.13], which says that if M is a closed oriented simply-connected 4-manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite and even, then $b_2(M) \geq \frac{10}{8} \cdot |\operatorname{sign}(M)| + 4$.

4. CW structures

In this chapter we will discuss the existence of CW-structures on manifolds.

Definition 4.1 (CW complex)**.** A *CW complex* is a topological space X together with a filtration

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

with $X = \varinjlim X_n$, such that for each $n \geq 0$, the space X_n arises as a pushout

$$\begin{array}{ccc} \coprod_{j \in \mathcal{J}_n} S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{j \in \mathcal{J}_n} D^n & \longrightarrow & X_n \end{array}$$

where $\mathcal{J}_n$ indexes the discs D^n. It is implicit in the statement $X = \varinjlim X_n$ that the topology of X agrees with the weak topology induced from the discs D^n. The interiors $\operatorname{Int} D^n$ of the discs are called the *n-cells*. For $n \geq 0$, a CW complex X is said to be *n-dimensional* if $X_n \setminus X_{n-1} \neq \varnothing$ and $X_i = X_{i+1}$ for all $i \geq n$. A manifold M *admits a CW-structure* if M is homeomorphic to a CW complex.

First we discuss the case of smooth manifolds. In [Mun66, Theorem 10.6] and [Whi57, Chapter IV.12] it is shown that every smooth manifold admits a simplicial structure, in particular it admits a CW-structure. Alternatively, it is shown in [Mil63, Section 3] and [Hir94, Section 6.4] that every compact smooth manifold admits a handle decomposition which implies by [Mil63, Theorem 3.5] that every compact smooth manifold is homotopy equivalent to a compact CW-complex.

It is natural to ask whether a similar result holds if we drop the smoothness hypothesis. The next theorem summarises what seems to be the state of the art.

Theorem 4.2.

(1) *For $n \leq 3$ every compact n-manifold admits the structure of a finite n-dimensional CW complex.*
(2) *Let $n \geq 5$ and let M be a compact n-manifold. Then M is homeomorphic to the mapping cylinder of some map $f \colon \partial M \to X$, where X is a finite CW complex.*
(3) *For $n \geq 5$, every* closed *n-manifold admits the structure of a finite n-dimensional CW complex.*

Proof. Radó [Rad26] showed in 1926 that every compact 2-manifold has a simplicial structure and so in particular has a CW structure. Hatcher's exposition [Hat13] is well worth reading. Moise [Moi52, Moi77] proved the analogous result for 3-manifolds. See also [Ham76]. Since a CW complex is finite if and only if it is compact we have shown (1).

For M an n-manifold with boundary, Kirby-Siebenmann [KS77, Essay III.2, Theorem 2.1] showed for $n \geq 6$ that M has a topological handlebody structure rel. $\partial M \times I$. Quinn [Qui82, Theorem 2.3.1] extended this result to $n = 5$. Kirby-Siebenmann [KS77, Essay III.2, Theorem 2.2] then says that M is homeomorphic to the mapping cylinder of some map $f \colon \partial M \to X$, where X is a finite CW complex. Thus (2) holds.

In particular, if M is closed, it admits the structure of a finite n-dimensional CW complex, which shows (3). □

It is not clear to us whether Theorem 4.2 suffices to show that every compact high-dimensional manifold admits a CW structure. Put differently, to the best of our knowledge the following question is still open for manifolds with nonempty boundary.

Question 4.3. *Let $n \geq 5$. Does every compact n-manifold have a CW structure?*

Casson [AM90, p. xvi] showed in the 1980s that there exist closed 4-manifolds that do not have a simplicial structure. It is now known that in every dimension $n \geq 5$, there exists a closed n-manifold that does not admit a simplicial structure. This question was reduced to a problem about homology 3-spheres [Mat78, GS80], which was then solved by Manolescu [Man16]. Note that a simplicial structure is not necessarily a PL structure; an n-dimensional PL structure satisfies the additional condition that the link of an m-simplex be homeomorphic to an $(n-m-1)$-dimensional sphere. To the best of our knowledge the following question is still open.

Question 4.4. *Does every compact 4-manifold have a CW structure?*

We have the following theorem regarding CW structures and manifolds. Given Theorem 4.2, the first statement is only of interest in dimension 4. We are not sure whether the second statement in dimension other than 4 has previously appeared explicitly in the literature.

Theorem 4.5.

(1) *Every compact n-manifold is homotopy equivalent to an n-dimensional finite CW complex.*

(2) *Every connected compact n-manifold with nonempty boundary is homotopy equivalent to an $(n-1)$-dimensional finite CW complex.*

In the proof of Theorem 4.5 we will use the following theorem proved by Wall [Wal66, Corollary 5.1].

Theorem 4.6. *Let X be a finite connected CW complex. Suppose that there is an integer $n \geq 3$ such that $H^i(X;\mathbb{Z}[\pi]) = 0$ for all $i > n$. Then X is homotopy equivalent to an n-dimensional finite CW complex.*

The proof of Theorem 4.5 also makes use of the following definition. Recall that a space is metrisable if it admits the structure of a metric space inducing the given topology.

Definition 4.7 (Absolute Neighbourhood Retract (ANR))**.** A space X is called an *absolute neighbourhood retract* if X is metrisable and if whenever $X \subseteq Y$ is a closed subset of a metrisable space Y, then X is a neighbourhood retract of Y. That is, there is an open neighbourhood $U \subseteq Y$ containing X, with a map $r\colon U \to X$ such that the composition $X \to U \xrightarrow{r} X$ is equal to the identity on X.

Proof of Theorem 4.5. Hanner [Han51, Theorem 3.3] showed that every manifold is an ANR, and West [Wes77] showed that every compact ANR is homotopy equivalent to a finite CW complex. (An alternative proof that every compact manifold has the homotopy type of a finite CW complex is given in Kirby-Siebenmann [KS69, Section 1 (III)].)

Let M be a compact n-manifold. By the results above, M is homotopy equivalent to a finite CW complex X. By Theorem 4.2 we need to complete the proof of (1) only in the case $n = 4$. But since the subsequent argument works for all $n \geq 4$ we also state it for all $n \geq 4$. Since M is n-dimensional it follows from Universal Poincaré duality (Theorem A.15) that for any $k > n$ and any $\mathbb{Z}[\pi_1(X)]$-module Λ we have

$$H^k(X;\Lambda) \;\cong\; H_{n-k}(X;\Lambda) \;=\; 0.$$

By Theorem 4.6, X is homotopy equivalent to an n-dimensional finite CW complex. Note that to apply Theorem 4.6 we have used that $n \geq 3$.

Now we turn to the proof of (2). We start with $n = 1, 2$ or 3. We saw in the proof of Theorem 4.2 that a compact connected n-manifold admits a simplicial structure. It is well known that a compact connected n-manifold with nonempty boundary and a simplicial structure is homotopy equivalent to an $(n-1)$-dimensional simplicial complex: collapse top dimensional simplices starting with those that have a face on the boundary. In particular such a manifold is homotopy equivalent to an $(n-1)$-dimensional finite CW complex.

Now suppose that $n \geq 4$. By (1) we know that if M is a connected n-manifold with nonempty boundary, then it admits the structure of a finite CW complex. Let $k \geq n$ and let Λ be a $\mathbb{Z}[\pi_1(X)]$-module. By Poincaré-Lefschetz duality (Section A) we have

$$H^k(X;\Lambda) \;\cong\; H_{n-k}(X,\partial X;\Lambda) \;=\; 0.$$

Here the last conclusion is obvious for $k > n$. For $k = n$ the conclusion follows from the fact that $\partial X \neq \varnothing$, that X is connected and the explicit calculation of 0-th twisted homology groups as given in [HS97, Chapter VI.3]. It follows from Theorem 4.6 that X is homotopy equivalent to an $(n-1)$-dimensional finite CW complex. Note here that to apply Theorem 4.6 we used $n \geq 4$. □

Theorem 4.5 is strong enough to recover many familiar statements.

Corollary 4.8. *Let M be a compact connected manifold.*

(1) *The group $\pi_1(M)$ is finitely presented.*
(2) *All homology groups $H_k(M;\mathbb{Z})$ are finitely generated abelian groups, in particular it makes sense to define the Euler characteristic*
$$\chi(M) \;:=\; \sum_n (-1)^n \cdot b_n(M).$$
(3) *Let $p\colon \widetilde{M} \to M$ be a finite covering. Then*
$$\chi(\widetilde{M}) \;=\; [\widetilde{M} : M] \cdot \chi(M).$$

Remark 4.9. Let M be a compact, connected manifold. Borsuk's theorem that M is a Euclidean Neighbourhood Retract shows that M is a retract of a finite CW complex; see [Hat02, Appendix A, Corollary A.9], [Bre97, Appendix E], and [Fri19, Proposition 65.22]. This fact is nontrivial but it is much easier to prove than Theorem 4.5. Borsuk's Theorem implies immediately that the homology groups of M are finitely generated and that the fundamental group of M is finitely generated. In fact using a group theoretic lemma as in [Wal65, Lemma 1.3] or alternatively [FR01, Theorem 3.1], one actually obtains that $\pi_1(M)$ is finitely presented. But it is not clear how Borsuk's Theorem can be used to prove Corollary 4.8 (3).

Proof. The first two statements in the corollary are an immediate consequence of Theorem 4.5 and standard results on fundamental groups and homology groups of finite CW complexes. We turn to the final statement. Let X be a finite CW complex homotopy equivalent to M. Use the fact that the Euler characteristic is multiplicative for finite covers of finite CW complexes and use that a k-fold cover $\widetilde{M}$ of M induces a k-fold cover $\widetilde{X}$ of X such that $\widetilde{M}$ and $\widetilde{X}$ are homotopy equivalent, to deduce the result. □

Remark 4.10. As pointed out above, every compact smooth manifold admits the structure of a finite CW complex. One can combine this fact with Theorem 8.6 below to obtain an alternative proof of Corollary 4.8. Theorem 8.6 says that for any 4-manifold M there is a 4-manifold N such that the connected sum $M\#N$ admits a smooth structure.

5. The annulus theorem

The annulus theorem is a fundamental result in the development of the theory of manifolds. In high dimensions, it underpins the product structure theorem [KS77, Essay I, Theorem 5.1], which itself underpins all the results of [KS77]. We state the product structure theorem in Section 5.4. In dimension four, the annulus theorem is needed for the proofs of smoothing theorems (Section 8), existence and uniqueness of normal bundles (Section 6), and transversality (Section 10). We discuss these developments in later sections. Later in this section (Section 5.2), we will discuss a more immediate application: showing that connected sum is a well-defined operation on connected topological manifolds. Here is the annulus theorem.

Theorem 5.1 (Annulus theorem)**.** *Let $n \in \mathbb{N}$ and let $f, g\colon D^n \to \mathbb{R}^n$ be two orientation-preserving locally flat embeddings. If $f(D^n) \subset \operatorname{Int}(g(D^n))$, then $g(D^n) \setminus \operatorname{Int}(f(D^n))$ is homeomorphic to $S^{n-1} \times [0,1]$.*

For $n = 2, 3$ the annulus theorem follows from the work of Radó [Rad26] and Moise [Moi52, Moi77] (see also [Edw84, p. 247]). The annulus theorem was proved for dimensions $\neq 4$ by Kirby [Kir69] and in dimension 4 by Quinn [Qui82, p. 506]; see also [Edw84, p. 247].

The known proofs of the annulus theorem deduce it from the stable homeomorphism theorem. In the next section we will state the stable homeomorphism theorem 5.3 and we will show how the annulus theorem can be deduced from that theorem.

5.1. **The stable homeomorphism theorem.** We reduce the annulus theorem to the stable homeomorphisms theorem stated in Theorem 5.3. This follows from work of Brown and Gluck [BG64b], but since it requires some work to find this deduction in [BG64b], we give the details here.

Definition 5.2. A homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be *stable* if there is a sequence of homeomorphisms $f_1, \dots, f_n\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f_n \circ \cdots \circ f_1 = f$ and such that for each i, the homeomorphism f_i is *somewhere the identity*, which means that there is an open nonempty set $U \subseteq \mathbb{R}^n$ such that $f_i|_U$ is the identity on U.

The key ingredient to the subsequent discussion is the following theorem.

Theorem 5.3 (Stable homeomorphism theorem)**.** *Let $n \in \mathbb{N}$. Every orientation preserving homeomorphism from $\mathbb{R}^n$ to itself is stable.*

For $n \geq 5$ this was proved by Kirby [Kir69, p. 575]. Slightly more precisely, Kirby [Kir69] showed that the stable homeomorphism theorem in dimensions at least five is a consequence of the surgery theoretic classification of PL homotopy tori which had been worked out by Wall [Wal70],[Wal99, Section 15A] and independently by Hsiang-Shaneson [HS69, p. 688], both proofs building on the work of Browder, Novikov and Wall. For $n = 4$ the stable homeomorphism theorem was proven by Quinn, see also [Edw84, p. 247].

Before we discuss consequences of the Stable homeomorphism theorem 5.3 we recall the two versions of the Alexander trick.

Lemma 5.4 (Alexander trick)**.**

(1) *Every homeomorphism of S^{n-1} can be extended radially to a homeomorphism of D^n that sends 0 to 0.*
(2) *Let f and g be two homeomorphisms of D^n. If the restrictions of f and g to S^{n-1} are isotopic, then f and g are isotopic homeomorphisms of D^n.*

The extension in the first statement can be obtained by coning: $f(t \cdot x) = t \cdot f(x)$. The second one is an amusing exercise; see [Han89, Lemma 5.6] for a proof. The same idea extends to show that the topological group $\mathrm{Homeo}_\partial(D^n)$ of homeomorphisms of D^n fixing the boundary pointwise is contractible.

We can now prove the following almost immediate consequence of the Stable homeomorphism theorem 5.3.

Corollary 5.5. *Every orientation preserving self-homeomorphism of S^n is isotopic to the identity.*

Proof. We identify S^n with $\mathbb{R}^n \cup \{\infty\}$. Let h be a self-homeomorphism of $S^n = \mathbb{R}^n \cup \{\infty\}$. After an isotopy (using Theorem 2.10) we can assume that $h(\infty) = \infty$. By the Stable homeomorphism theorem 5.3 we know that h is stable. Thus we only have to consider the case that h fixes an open subset of $\mathbb{R}^n \cup \{\infty\} = S^n$. After an isotopy we can assume that h fixes an open neighbourhood of ∞, so in particular there exists $C > 0$ such that h is the identity on $\{x \in \mathbb{R}^n \mid \|x\| \geq C\}$. It follows from Lemma 5.4 (2) that h is isotopic to the identity. $\square$

Denote the set of locally flat embeddings of D^n into $\mathbb{R}^n$ by $\mathrm{Emb}(D^n, \mathbb{R}^n)$; see Definition 6.2 for the definition of locally flat.

Definition 5.6. We say that two elements $f_0, f_1 \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ are *intertwined* if there exists an $h \in \mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n)$ with $h \circ f_0 = f_1$.

We will need the following straightforward technical lemma.

Lemma 5.7. *Let M be an n-dimensional manifold and let $f\colon D^n \to M$ be a locally flat embedding into $\operatorname{Int} M = M \setminus \partial M$. Then there exists a locally flat embedding $F\colon \mathbb{R}^n \to M$ such that the restriction of F to D^n equals f.*

Proof. Let $f\colon D^n \to M$ be a locally flat embedding. By definition $f(D^n)$ is a submanifold of M. It is straightforward to see that $W := M \setminus f(\operatorname{Int} D^n)$ is also a submanifold of M. By the collar neighbourhood theorem 2.5 there exists a collar $f(S^{n-1}) \times [0,1]$. The map

$$\begin{array}{rcl} F\colon \mathbb{R}^n & \to & M \\ x & \mapsto & \begin{cases} f(x), & \text{if } x \in D^n, \\ \big(f(y), \frac{2}{\pi}\arctan(t-1)\big) & \text{if } x = t \cdot y \text{ with } t \in [1,\infty) \text{ and } y \in S^{n-1}, \end{cases} \end{array}$$

is easily seen to be a locally flat embedding. $\square$

Lemma 5.8. *Any two elements $f_0, f_1 \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ are intertwined.*

Proof. It suffices to show that any $f \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ is intertwined with the standard embedding $D^n \subset \mathbb{R}^n$. So let $f \in \mathrm{Emb}(D^n, \mathbb{R}^n)$. Apply Lemma 5.7 to extend f to a locally flat embedding $F\colon D^n(\frac{3}{2}) \to \mathbb{R}^n$. Note that F restricts to a locally flat embedding of $S^{n-1} \times [\frac{1}{2}, \frac{3}{2}]$ into $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $\widetilde{D}^n$ be another copy of D^n. By the generalised Schoenflies theorem [Bro60, Theorem 5] there exists a homeomorphism $g\colon \widetilde{D}^n \to S^n \setminus f(\operatorname{Int} \widetilde{D}^n)$. Since the homeomorphisms of D^n act transitively on the interior of $\widetilde{D}^n$, arrange that $g(0) = \infty$.

Note that $g^{-1} \circ f \colon S^{n-1} \to S^{n-1}$ is a homeomorphism. By Lemma 5.4 (1) this homeomorphism extends to a homeomorphism ϕ of D^n. Replace g by $g \circ \phi$ if necessary to obtain that $f = g \colon S^{n-1} \to f(S^{n-1})$. Identify $S^n = \mathbb{R}^n \cup \{\infty\} = D^n \cup \widetilde{D}^n$ in such a way that $0 \in \widetilde{D}^n$ corresponds precisely to ∞. Consider the map

$$\begin{aligned} F \colon S^n = D^n \cup \widetilde{D}^n &\to S^n \\ x &\mapsto \begin{cases} f(x) & x \in D^n \\ g(x) & x \in \widetilde{D}^n. \end{cases} \end{aligned}$$

The maps f and g agree on the overlap, so the map is well-defined and is a homeomorphism. Note that F restricts to a homeomorphism of $\mathbb{R}^n$ which has the property that the restriction to D^n equals f. This shows that $F \circ \mathrm{Id} = f$ so Id and f are intertwined. □

Definition 5.9. Let $f_0, f_1 \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ with $f_0(D^n) \subset \mathrm{Int}\, f_1(D^n)$. We say f_0 and f_1 are *strictly annularly equivalent* if there exists a map $F \colon S^{n-1} \times I \to M$ that is a homeomorphism onto its image such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in S^{n-1}$.

Theorem 5.10. *Let $f_0, f_1 \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ with $f_0(D^n) \subset \mathrm{Int}\, f_1(D^n)$. If f_0 and f_1 are orientation preserving, then they are strictly annularly equivalent if and only if they are intertwined.*

Proof. If two such elements are strictly annularly equivalent, then they are intertwined by [BG64b, Theorem 5.2].

Now suppose that f_0 and f_1 are intertwined, that is there exists an $h \in \mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n)$ with $h \circ f_0 = f_1$. By the Stable homeomorphism theorem 5.3 we know that h is stable. Thus we know from [BG64b, Theorem 5.4] that the embeddings are annularly equivalent, i.e. there exist $h_0, \dots, h_k \in \mathrm{Emb}(D^n, \mathbb{R}^n)$ such that $h_0 = f_0, h_k = f_1$ and for each i the maps h_i and h_{i+1} are strictly annularly equivalent. Since $f_0(D^n) \subset \mathrm{Int}\, f_1(D^n)$, the embeddings of the boundary spheres $f_0(\partial D^n)$ and $f_1(\partial D^n)$ are disjoint. Therefore it follows from [BG64a, Theorem 3.5] that f_0 and f_1 are not only annularly equivalent, but are moreover strictly annularly equivalent. □

Now we can easily prove the Annulus theorem 5.1.

Proof of the Annulus theorem 5.1. Let $f_0, f_1 \colon D^n \to \mathbb{R}^n$ be two orientation-preserving locally flat embeddings with $f_0(D^n) \subset \mathrm{Int}(f_1(D^n))$. By Lemma 5.8 and Theorem 5.10 the two maps f_0 and f_1 are strictly annularly equivalent. But this immediately implies that $f_1(D^n) \setminus \mathrm{Int}(f_0(D^n))$ is homeomorphic to $S^{n-1} \times [0, 1]$. □

5.2. **The connected sum operation.** We recall the construction of the connected sum of two connected oriented n-manifolds M and N. Pick two orientation preserving locally flat embeddings of n-balls $\Phi_M \colon D^n \to M$ and $\Phi_N \colon D^n \to N$. Define the *connected sum* $M \# N$ of M and N by

$$M \# N := (M \setminus \Phi_M(\mathrm{Int}(D^n))) \cup_{\Phi_M(S^{n-1}) = \Phi_N(S^{n-1})} (N \setminus \Phi_N(\mathrm{Int}(D^n)))$$

where we glue the left hand side to the right hand side via the map

$$\Phi_N \circ \Phi_M^{-1} \colon \Phi_M(S^{n-1}) \xrightarrow{\cong} \Phi_N(S^{n-1}).$$

It follows from the Collar neighbourhood theorem 2.5 that the topological space $M\#N$ inherits the structure of an n-manifold; see [Lee11, Proposition 6.6] for details. Furthermore $M\#N$ can be oriented in such a way that M and N are oriented submanifolds.

Theorem 5.11. *The connected sum $M\#N$ of two connected oriented n-manifolds M and N is independent of the choice of embeddings of the n-balls.*

Remark 5.12. The manifolds $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ and $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ have different intersection forms, so they are not homeomorphic; see Proposition 5.15. Thus connected sum is not well-defined on orientable 4-manifolds, rather it depends on the choice of orientation. On the other hand for nonorientable manifolds, the connected sum is well-defined. As discussed in [BCF$^+$19], in contrast to the case of orientable 3-dimensional manifolds, orientable 4-dimensional topological manifolds do not admit a unique decomposition as a connected sum of irreducible 4-manifolds.

The proof of Theorem 5.11 relies on the following two lemmas. The elementary proof of the first lemma is left to the reader.

Lemma 5.13. *Let $D^n_r(x)$ and $D^n_s(y)$ be two Euclidean balls in $\mathbb{R}^n$. There exists a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with $f(D^n_r(x)) = D^n_s(y)$ such that f is the identity outside of some compact set.*

The next lemma is a consequence of the annulus theorem 5.1.

Lemma 5.14. *Let $\varphi, \psi\colon D^n \to \mathbb{R}^n$ be two orientation-preserving locally flat embeddings. If $\varphi(D^n) \subset \operatorname{Int}(\psi(D^n))$, then there exists a homeomorphism f of $\mathbb{R}^n$ with $f(\varphi(D^n)) = \psi(D^n)$ such that f is the identity outside of some compact set.*

Proof. By the annulus theorem 5.1 and the collar neighbourhood theorem 2.5 we can find a locally flat embedding $\theta\colon S^{n-1} \times [-1, 2]$ such that $\theta(S^{n-1} \times [-1, 0])$ is a collar for $\varphi(D^n)$, such that $\theta(S^{n-1} \times [0, 1]) = \psi(D^n) \setminus \varphi(\operatorname{Int} D^n)$ and such that $\theta(S^{n-1} \times [1, 2])$ is a collar for $\mathbb{R}^n \setminus \psi(\operatorname{Int} D^n)$. It is now obvious that we can find a homeomorphism f with $f(\varphi(D^n)) = \psi(D^n)$ which is the identity outside of $\theta(S^{n-1} \times [-1, 2])$. $\square$

The subsequent proof is partly based on the sketch given in [Rol90, p. 42].

Proof of Theorem 5.11. We have to show that the connected sum is independent of the choice of $\Phi_M\colon D^n \to M$ and $\Psi_N\colon D^n \to N$. By symmetry and transitivity it suffices to show that the connected sum is independent of the choice of Φ_M. So suppose we are given two orientation-preserving embeddings $\Phi_1\colon D^n \to M$ and $\Phi_2\colon D^n \to M$ and suppose we are given an orientation-preserving embedding $\Psi\colon D^n \to N$. We introduce the following notation.

(1) Write $D_i := \Phi_i(D^n)$.

(2) Let $X_i := M \setminus \Phi_i(\operatorname{Int} D^n)$ and let $Y := N \setminus \Psi(\operatorname{Int} D^n)$,
(3) Denote the restriction of Φ_i to S^{n-1} by φ_i and denote the restriction of Ψ to S^{n-1} by ψ.

Figure 1 hopefully makes it easier for the reader to internalise the notation. We have to show that there exists a homeomorphism

$$(X_1 \cup Y)/\varphi_1(x) \sim \psi(x) \;\to\; (X_2 \cup Y)/\varphi_2(x) \sim \psi(x)$$

where the gluing on both sides is given by taking $x \in S^{n-1}$.

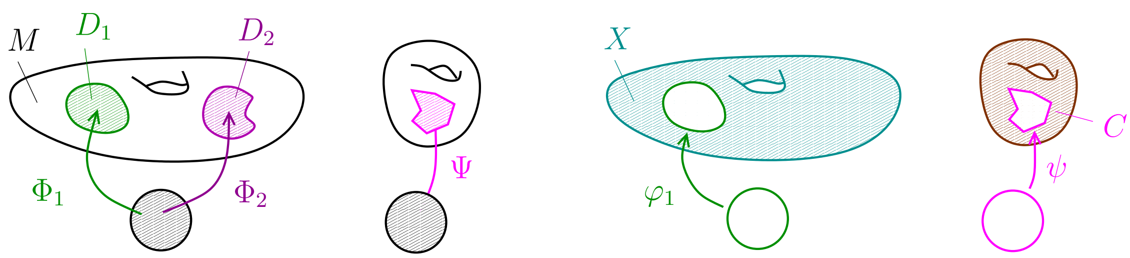

FIGURE 1.

Claim. There exists a homeomorphism h of M so that $h(D_1) = D_2$.

First note that it follows from Lemma 5.7 and Lemma 5.13, together with our hypothesis that M is path connected, that there exists a homeomorphism μ of M such that $\mu(D_1) \subset \operatorname{Int} D_2$. Then apply Lemma 5.7 and Lemma 5.14 to find a homeomorphism ν of M such that $\nu(\mu(D_1)) = D_2$. This concludes the proof of the claim.

It follows from the claim that $\varphi_2^{-1} \circ \varphi_1$ is a homeomorphism of S^{n-1}. By Corollary 5.5 we know that there exists an isotopy $H\colon S^{n-1} \times [0,1] \to S^{n-1}$ from $\varphi_2^{-1} \circ \varphi_1$ to the identity.

We write $C := \Phi(S^{n-1})$. By the collar neighbourhood theorem 2.5 we can pick a collar $C \times [0,1]$ for Y. It is straightforward to verify that

$$\begin{array}{rcl} (X_1 \cup Y)/\varphi_1(x) \sim \psi(x) & \to & (X_2 \cup Y)/\varphi_2(x) \sim \psi(x) \\ P & \mapsto & \begin{cases} h(P), & \text{if } P \in X_1, \\ \psi(H(\psi^{-1}(Q), t)) & \text{if } P = (Q,t) \in C \times [0,1] \\ P, & \text{if } Y \setminus P \in C \times [0,1] \end{cases} \end{array}$$

is a well-defined map and is a homeomorphism. This shows that the connected sums defined using Φ_1 and Φ_2 give rise to homeomorphic manifolds. □

5.3. **Further results on connected sums.** The definition of the intersection form was given in Definition 3.1. The next proposition shows that the intersection form is well behaved under the connected sum operation.

Proposition 5.15. *Let M and N be two oriented compact 4-manifolds. Then $Q_{M\#N}$ is isometric to $Q_M \oplus Q_N$.*

Proof. In the smooth case this statement follows immediately from the fact that any class in second homology can be represented by an embedded oriented submanifold [GS99, Proposition 1.2.3] and the fact that one can calculate the intersection form in terms of algebraic intersection numbers of embedded oriented surfaces [Bre97, Theorem VI.11.9]. To apply this approach to general manifolds, one needs to use topological transversality, which holds, as discussed in Section 10.

On the other hand the statement for general manifolds (and thus also for smooth manifolds) can be proved directly with the usual tools of algebraic topology, namely functoriality of the cup and cap products [Bre97, Theorem VI.5.2.(4)] for maps between pairs of topological spaces, a Mayer-Vietoris argument and the excision theorem. Full details are provided in [Fri19, Proposition 72.11]. □

5.4. **The product structure theorem.** The product structure theorem [KS77, Essay I, Theorem 5.1], is a key result for the development of topological manifold theory in high dimensions. It is a consequence of the stable homeomorphism theorem 5.3, and in turn is used in [KS77] to deduce handle structures, transversality, smoothing theory, and the existence of a canonical simple homotopy type, for high dimensional ($n \geq 6$) manifolds. We will give some examples of the use of the product structure theorem below. Even though it is only for high dimensional manifolds, it still appears in the development of the theory of 4-manifolds.

The product structure theorem will be stated for upgrading to either a smooth or PL structure. A *concordance* of (smooth, PL) structures Σ, Σ' on a manifold N is a (smooth, PL) structure Ω on $N \times I$ that restricts to Σ on $N \times \{0\}$ and restricts to Σ' on $N \times \{1\}$.

Theorem 5.16 (Product structure theorem)**.** *Let M be a manifold of dimension $n \geq 5$. Let Σ be a (smooth, PL) structure on $M \times \mathbb{R}^s$, with $s \geq 1$. Let U be an open subset of M with a (smooth, PL) structure ρ on U such that $\rho \times \mathbb{R}^s = \Sigma|_{U \times \mathbb{R}^s}$. If $n = 5$ then suppose that $\partial M \subset U$.*

Then there is a (smooth, PL) structure σ on M extending ρ, together with a concordance of (smooth,PL) structures from Σ to $\sigma \times \mathbb{R}^s$, that is a product concordance in some neighbourhood of $U \times \mathbb{R}^s$ and that is a product near $M \times \mathbb{R}^s \times \{i\}$ for $i = 0, 1$.

Remark 5.17. The statement of the product structure theorem was modelled on the Cairns-Hirsch theorem [KS77, Essay I, Theorem 5.3], which was proven in the early 1960s, and provided the analogous upgrade from PL structures to smooth structures. See [HM74] for a comprehensive treatment of smoothing theory for PL manifolds. The Cairns-Hirsch theorem tells us that if M already has a PL structure ϖ, such that $\varpi \times \mathbb{R}^s$ is *Whitehead compatible* (see the discussion below [KS77, Essay I, Theorem 5.3] for details) with a smooth structure Σ on $M \times \mathbb{R}^s$, then the smooth structure σ on M produced by Theorem 5.16 is Whitehead compatible with ϖ.

In Section 14.1 on the simple homotopy type of a manifold we make use of the following stronger local version.

Theorem 5.18 (Local product structure theorem)**.** *Let M be a manifold of dimension $n \geq 5$.*

(i) *Let W be an open neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}^s$, for some $s \geq 1$.*
(ii) *Let Σ be a (smooth, PL) structure on W.*
(iii) *Let $C \subseteq M \times \{0\}$ be a closed subset such that there is a neighbourhood $N(C)$ of C on which the (smooth,PL) structure Σ is a product $\Sigma|_{N(C)} = \sigma \times \mathbb{R}^s$ for some (smooth,PL) structure σ on M. If $n = 5$ then suppose that $\partial M \subseteq C$.*
(iv) *Let $D \subset M \times \{0\}$ be another closed subset.*
(v) *Let $V \subseteq W$ be an open neighbourhood of $D \setminus C$.*

Then we have the following.

(1) *A (smooth, PL) structure Σ' on W that equals Σ on $(W \setminus V) \cup ((C \times \mathbb{R}^s) \cap W)$ and is a product (smooth,PL) structure $\rho \times \mathbb{R}^s$ on $(N(D) \times \mathbb{R}^s) \cap W$ for some neighbourhood $N(D)$ of D and for some (smooth, PL) structure ρ on $N(D)$.*
(2) *A concordance of (smooth, PL) structures from Σ to Σ', that is a product concordance on some neighbourhood of $(W \setminus V) \cup ((C \times \mathbb{R}^s) \cap W)$ and that is a product near $W \times \{i\}$ for $i = 0, 1$.*

Note that the concordance implies isotopy theorem [KS77, Essay I, Theorem 4.1] means that the concordances in Theorems 5.16 and 5.18 can be upgraded to isotopies of (smooth, PL) structures under the same hypotheses on dimensions, that is $n \geq 6$ or $n = 5$ and the structures already agree on ∂M.

6. Tubular neighbourhoods

6.1. **Submanifolds.** Every smooth submanifold of a smooth manifold admits a normal vector bundle and, by the smooth Tubular neighbourhood theorem, also admits a tubular neighbourhood [Hir94, Sections 5 & 6][Wal16, Chapter 2.5]. However in the topological category n-manifolds may not admit normal vector bundles, a general problem we discuss further below and in Section 7 once we have developed the necessary language. Curiously, in the special case of 4-manifolds these general problems do not exist, and familiar smooth results hold true using an appropriate notion of normal vector bundles (Definition 6.15).

Before discussing tubular neighbourhoods and normal vector bundles in the topological category, we give our convention for submanifolds. Recall that $E_{n-1} \subset \mathbb{R}^n$ is the hyperplane $\{x_n = 0\}$.

Definition 6.1. Let M be an n-dimensional manifold. We say a subset $X \subset M$ is a *k-dimensional submanifold* if given any $P \in X$ one of the holds:

(1) there exists a chart $\Phi \colon U \to V$ of type (i) for M and P such that
$$\Phi(U \cap X) \ \subset \ \{(0, \ldots, 0, x_1, \ldots, x_k) \mid x_i \in \mathbb{R}\},$$
(2) there exists a chart $\Phi \colon U \to V$ of type (ii) for M and P such that $\Phi(P)$ lies in E_{n-1} and
$$\Phi(U \cap X) \ \subset \ \{(0, \ldots, 0, x_1, \ldots, x_k) \in \mathbb{R}^n \mid x_k \geq 0\},$$

(3) or there exists a chart $\Phi\colon U \to V$ of type (i) for M and P such that $\Phi(P)$ lies in E_{n-1} and

$$\Phi(U \cap X) \ \subset \ \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \mid x_k \geq 0\}.$$

If for every $P \in X$ we can find charts as in (1) and (2), then we call M a *proper* submanifold.

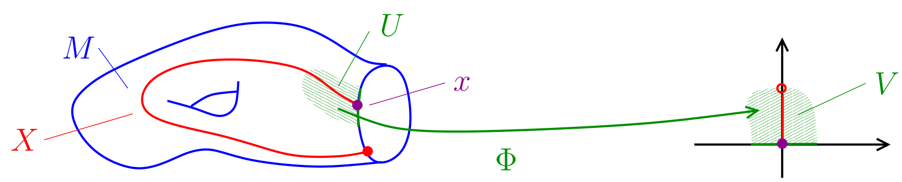

FIGURE 2.

Definition 6.2. A map $f\colon X \to M$ from a k-manifold to an m-manifold M is called a (proper) *locally flat embedding* if f is a homeomorphism onto its image and if the image is a (proper) submanifold of M.

Given any submanifold X of M, the inclusion map $X \to M$ is a locally flat embedding. Conversely, if $f\colon X \to M$ is a locally flat embedding, then the image $f(X)$ is a submanifold.

Remark 6.3.

(1) Note that if M is a k-manifold and U is an open subset of $\mathbb{R}^k$, then it follows from the invariance of domain that the image any injective map $f\colon U \to M$ is an open subset of M. In particular $f(U)$ is a submanifold of M. Put differently, f is locally flat.

(2) In point set topology, one often defines a *topological embedding* to be a map $f\colon X \to Y$ of topological spaces that is a homeomorphism to its image. The image of a topological embedding is not necessarily a submanifold and such an image is sometimes called *wild* due to the bizarre properties that such objects can exhibit. For example, the famous Alexander horned sphere [Ale24] is not a submanifold of S^3 under Definition 6.1, but it is the image of a wild topological embedding $S^2 \to S^3$.

(3) In the literature a compact subset F of 4-manifold is often called a *locally flat surface* if F is homeomorphic to a compact 2-dimensional manifold with $\partial F = F \cap \partial M$ and if F has the following properties.

   (a) Given any $P \in F \setminus \partial F$ there exists a topological embedding $\varphi\colon D^2 \times D^2 \to M \setminus \partial M$ with $\varphi(D^2 \times D^2) \cap F = \varphi(D^2 \times \{0\})$ and with $P \in \varphi(D^2 \times \{0\})$.

   (b) Given any $P \in \partial F$ there exists a topological embedding $\varphi\colon D^2_{\geq 0} \times D^2 \to M$ such that $\varphi(D^2_{\geq 0} \times D^2) \cap F = \varphi(D^2_{\geq 0} \times \{0\})$, $\varphi(D^2_{\geq 0} \times D^2) \cap \partial M = \varphi(\partial_{y=0} D^2_{\geq 0} \times D^2)$, and with $P \in \varphi(D^2_{\geq 0} \times \{0\})$. Here, we used the following abbreviations $D^2_{\geq 0} = \{(x,y) \in D^2 \,|\, y \geq 0\}$ and $\partial_{y=0} D^2 = \{(x,0) \in D^2\}$.

   It follows easily from the definitions that $F \subset M$ is a locally flat surface if and only if F is proper submanifold of M.

The following is a common justification for requiring slice discs to be locally flat in the theory of knot concordance.

Proposition 6.4. *Given a knot $K \subset S^3$ the corresponding cone*

$$\mathrm{Cone}(K) := \{r \cdot Q \mid Q \in K \text{ and } r \in [0,1]\} \ \subset \ D^4.$$

is locally flat if and only if K is the unknot.

Proof. Consider the specific unknot U that is the equator of the equator $U = S^1 \subset S^2 \subset S^3 = \partial D^4$. Taking the cone radially inwards to the origin of D^4 exhibits cone(U) as a locally flatly (properly) embedded disc. Any other unknotted $K \subset S^3$ is related to U by a homeomorphism of S^3. By the Alexander trick 5.4(1), this homeomorphism extends radially inwards to a homeomorphism of D^4 fixing the origin. Thus the cone on any other unknot K is locally flatly embedded, as we obtain a chart as in Definition 6.1(1) at the origin of D^4.

Conversely, suppose $K \subset S^3$ is a knot such that $C := \mathrm{Cone}(K)$ is locally flat. This implies that there is a chart $\Phi\colon D^4 \to D^4$ such that $\Phi(0) = P$ and such that $\Phi(D^2 \times \{0\}) = \Phi(D^4) \cap C$. We introduce the following notation.

(i) Given $I \subset [0,1]$ we write $D_I := \{v \in D^4 \mid \|v\| \in I\}$.
(ii) Given $I \subset [0,1]$ we write $N_I := \Phi(D_I)$.

An elementary argument shows that there exist $s_1 < t_1 < s_2 < t_2 < s_3$ such that $D_{[0,s_1]} \subset N_{[0,t_1]} \subset D_{[0,s_2]} \subset N_{[0,t_2]} \subset D_{[0,s_3]}$. We make the following observations:

(1) For any $I \subset [0,1]$ we have homeomorphisms $D_I \setminus C \cong (S^3 \setminus K) \times I$ and $N_I \setminus C \xrightarrow{\Phi^{-1}} D_I \setminus U \cong (S^3 \setminus U) \times I$.
(2) For any inclusion $I \subset J$ of intervals the inclusion induced maps $D_I \to D_J$ and $N_I \to N_J$ are homotopy equivalences.

We consider the following commutative diagram where all maps are induced by inclusions

$$\begin{tikzcd}
\pi_1(S^3 \setminus K) \cong \pi_1(D_{\{t_2\}} \setminus C) \arrow[rr] \arrow[dr, "\cong"'] & & \pi_1(N_{[t_1,t_2]} \setminus C) \cong \mathbb{Z} \arrow[dl] \\
& \pi_1(D_{[t_1,t_3]} \setminus C). &
\end{tikzcd}$$

Since the inclusion $D_{\{t_2\}} \setminus C \to D_{[t_1,t_3]} \setminus C$ is a homotopy equivalence we see that the left diagonal map is an isomorphism. Thus we see that we have an automorphism of $\pi_1(S^3 \setminus K)$ that factors through $\mathbb{Z}$. Since the abelianisation of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$ we see that $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$. It follows from the Loop Theorem that K is in fact the unknot [Rol90, Theorem 4.B.1]. □

In some applications one needs the following refinement of the Collar neighbourhood theorem 2.5.

Theorem 6.5 (Collar neighbourhood theorem for proper submanifolds)**.** *Let M be a manifold and let $X \subset M$ be a proper submanifold. There exists a collar neighbourhood $\partial M \times [0,1]$ such that $(\partial M \times [0,1]) \cap X$ is a collar neighbourhood for $\partial X \subset X$.*

Proof. By the earlier Collar neighbourhood theorem 2.5 we can pick a collar neighbourhood $\partial M \times [0,2]$ for ∂M and we can also pick a collar neighbourhood $\partial X \times [0,2]$ for ∂X. Given $t \in [0,1]$ we consider the obvious homeomorphisms

$$f_t\colon M = (M \setminus (\partial M \times [0,2))) \cup (\partial M \times [0,2]) \to (M \setminus (\partial M \times [0,2))) \cup (\partial M \times [t,2])$$

and

$$g_t\colon X = (X \setminus (\partial X \times [0,2))) \cup (\partial X \times [0,2]) \to (X \setminus (\partial X \times [0,2))) \cup (\partial X \times [t,2]).$$

Next we consider the following proper locally flat isotopy:

$$\begin{array}{rcl} h\colon X \times [0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} (y,s) \in \partial M \times [0,t], & \text{if } x = (y,s) \text{ with } y \in \partial X, s \in [0,t], \\ f_t(g_t^{-1}(x)), & \text{otherwise.} \end{cases} \end{array}$$

Note that the collar neighborhood $\partial M \times [0,1]$ is of the desired form for the proper submanifold $h_1(X)$. By the Isotopy Extension Theorem 2.10 we can extend h to a isotopy H of M. Thus $H_1^{-1}(\partial M \times [0,1])$ is the desired collar neighborhood for M. $\square$

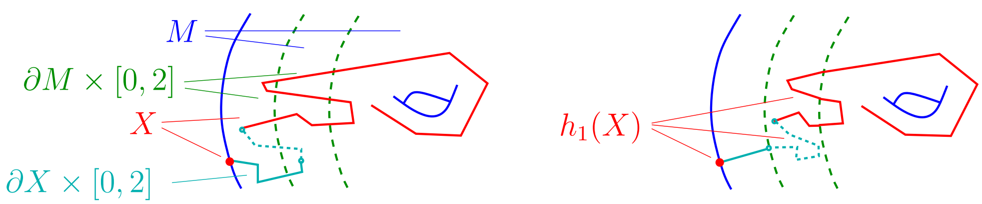

Figure 3. Illustration of the proof of the Collar neighbourhood theorem 6.5.

6.2. **Tubular neighbourhoods: existence and uniqueness.** In the literature one can find many different definitions of tubular neighbourhoods for smooth manifolds. We will give a definition for manifolds that is modelled on the definition provided by Wall [Wal16] for smooth manifolds. To do so we first need one extra definition.

Definition 6.6. Let M be an n-dimensional manifold. We say a subset $W \subset M$ is a *k-dimensional submanifold with corners* if given any $P \in W$ there exists a chart of the type (1), (2) or (3) as in Definition 6.1 above, or if

(4) there exists a chart $\Phi\colon U \to V$ of type (ii) for M such that

$$\Phi(U \cap W) \subset \{(0,\dots,0,x_1,\dots,x_k) \mid x_i \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$

and with $\Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0) \mid x_i \in \mathbb{R}\}$.

If W is an n-dimensional submanifold with corners we write

$$\partial_0 W := W \cap \overline{M \setminus W}, \quad \partial_1 W := W \cap \partial M, \qquad \text{and we note that} \qquad \operatorname{Int} W = W \setminus \partial_0 W.$$

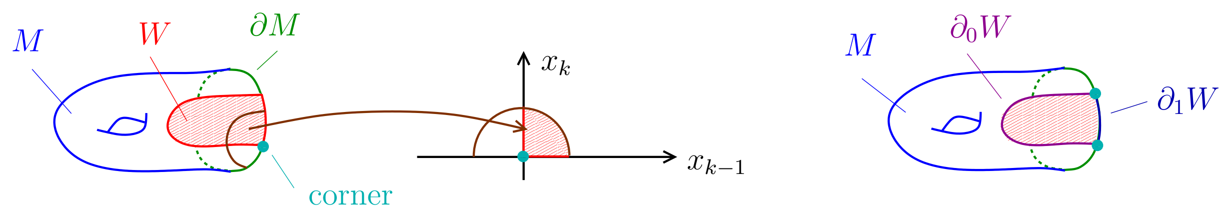

FIGURE 4.

Definition 6.7. Let M be an n-manifold and let X be a compact proper k-dimensional submanifold. A *tubular neighbourhood* for X is a pair $(N, p\colon N \to X)$ with the following properties:

1. N is a codimension zero submanifold with corners.
2. The map $p\colon N \to X$ is a linear D^{n-k}-bundle such that $p(x) = x$ for all $x \in X$.
3. We have $\partial_1 N = p^{-1}(\partial X)$.

Here linear means that there exists an atlas of trivialisations such that the transition maps take values in $O(n-k)$ instead of $\operatorname{Homeo}(D^{n-k})$.

In the topological category, tubular neighbourhoods do not always exist. Indeed it is shown in [Hir68, Theorem 4] that there exists a 4-dimensional submanifold of S^7 that does not admit a tubular neighbourhood.

Fortunately, for submanifolds of 4-manifolds, tubular neighbourhoods exist and they are unique in the appropriate sense.

Theorem 6.8 (Tubular neighbourhood theorem)**.** *Every compact proper submanifold X of a 4-manifold M admits a tubular neighbourhood.*

Theorem 6.9. (Uniqueness of tubular neighbourhoods) *Let M be a 4-manifold and let X be a compact proper k-dimensional submanifold. Furthermore let $p_i\colon N_i \to X$, $i = 1, 2$ be two tubular neighbourhoods of X, with inclusion maps $\iota_i\colon N_i \to M$. Then there exists an isomorphism $\Psi\colon N_1 \to N_2$ of linear disc bundles such that $\iota_2 \circ \Psi\colon N_1 \to M$ and $\iota_1\colon N_2 \to M$ are ambiently isotopic rel X.*

The proofs of the above two theorems rely on the existence and uniqueness results for normal vector bundles in [FQ90, Section 9], which we discuss further in Section 6.3. Thus we postpone the proofs of the above two theorems to Section 6.4.

Right now, let us first observe some nice consequences of the existence and uniqueness of tubular neighbourhoods.

Remark 6.10. Let X be a compact proper submanifold of a 4-manifold M. By Theorem 6.8 we can pick a tubular neighbourhood $p\colon N \to X$. We refer to $E_X := M \setminus \operatorname{Int} N$ as the *exterior of* X. By Theorem 6.9 the homeomorphism type of the exterior is well-defined.

Lemma 6.11. *Let X be a compact proper submanifold of a 4-manifold M. The exterior E_X of X is a deformation retract of the complement $M \setminus X$.*

Proof. Let $p\colon N \to X$ be a tubular neighbourhood for X. Using the fact that p is a *linear* bundle, introduce compatible radial coordinates in the fibres and isotope radially outwards. This implies that $\partial_0 N$ is a deformation retract of $N \setminus X$. But this also implies that the exterior $E_X = M \setminus N$ is a deformation retract of $M \setminus X$. □

Corollary 6.12. *Let X be a submanifold of a compact 4-manifold M. If X is compact, then $\pi_1(M \setminus X)$ and $H_*(M \setminus X)$ are finitely generated.*

Proof. It follows from Lemma 6.11 that $M \setminus X$ is homotopy equivalent to the exterior of X which in our case is a compact 4-manifold since we assume that M is compact. The corollary is now a consequence of Corollary 4.8. □

Proposition 6.13. *Let $X \subset M$ be a 2-dimensional orientable submanifold of a compact orientable 4-manifold M, such that each connected component of X has nonempty boundary. Then the tubular neighbourhood of Theorem 6.8 is homeomorphic to $X \times D^2$.*

Proof. A linear D^n-bundle is the unit disc bundle of a vector bundle. Connected surfaces with boundary are homotopy equivalent to wedges of circles. Every orientable vector space bundle over a wedge of circles is trivial, and so are their unit disc bundles. □

6.3. **Normal vector bundles.** The reader will be familiar with the definition of a normal vector bundle when working in the smooth category: if $X \subset M$ is a smooth submanifold of a smooth manifold, then the normal vector bundle is the quotient vector bundle $TM|_X/TX$. This definition uses the smooth structure to ensure the existence of tangent vector bundles, and vector bundles are a strong enough bundle technology to ensure the existence the perpendicular subspaces required to form the quotient bundle. While some (weaker) canonical tangential structures do exist in the topological category (see Section 7), the idea of 'quotient bundle' no longer makes sense for them.

In the topological category, following [FQ90, Section 9], we will use a definition of normal vector bundle that is much closer to the geometry of tubular neighbourhoods. We begin with a definition that is almost what we need but suffers from a slight technical problem, which we then remedy.

Definition 6.14. Let M be a n-manifold and let X be a proper k-dimensional submanifold. An *internal linear bundle over* X is a pair $(E, p\colon E \to X)$ with the following properties:

1. E is a codimension zero submanifold of M.
2. The map $p\colon E \to X$ is an $(n-k)$-dimensional vector bundle such that $p(x) = x$ for all $x \in X$.
3. We have $\partial E = p^{-1}(\partial X)$.

An internal linear bundle $(E, p\colon E \to X)$ is intended to recover, from the smooth category, an *open* tubular neighbourhood of X. As such, the definition as stands suffers from the potential technical problem that the closure of E in M, which should be a closed tubular neighbourhood, may be an immersion; see Figure 5. As in [FQ90, p. 137], we use the following additional idea to rule out this problem.

Definition 6.15. Let M be an n-manifold, let X be a proper k-dimensional submanifold, and let $(E, p\colon E \to X)$ be an internal linear bundle over X. Suppose that given any $(n-k)$-dimensional vector bundle $(F, q\colon F \to X)$, any radial homeomorphism from an open convex disc bundle of F to E can be extended to a homeomorphism from the whole of F to a neighbourhood of E. Then we say $(E, p\colon E \to X)$ is *extendable*.

In Figure 5 we illustrate an example of a non-extendable internal linear bundle.

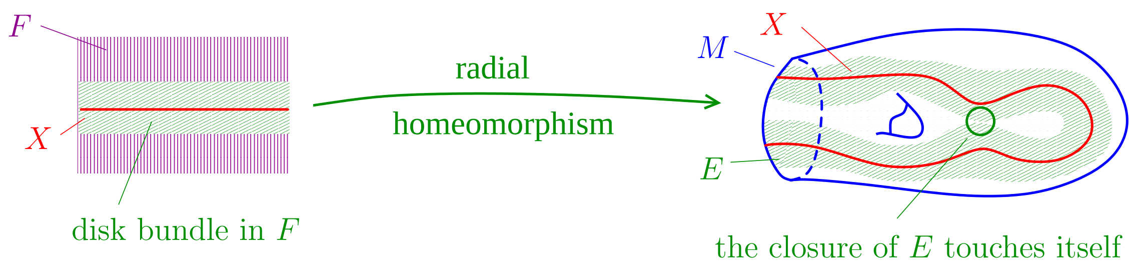

FIGURE 5.

Now we can define the notion of a normal vector bundle.

Definition 6.16 ([FQ90, p. 137])**.** Let M be a n-manifold and let X be a proper k-dimensional submanifold. A *normal vector bundle* for X is an internal linear bundle that is extendable.

Theorem 6.17 (Existence of normal vector bundles)**.** *Every proper submanifold of a compact 4-manifold admits a normal vector bundle.*

Remark 6.18. Generally, the existence of normal vector bundles is peculiar to when the submanifolds has low dimension or low codimension. We refer the reader to [FQ90, Section 9.4] for a discussion of the other known situations where these objects always exist. Here is a summary of the known cases. A submanifold of dimension at most 3 in a closed manifold of dimension at least 5 has a normal bundle, and codimension one submanifolds have normal bundles [Bro62]. That every codimension two submanifold of a manifold of dimension not equal to four have normal bundles was shown in [KS75], and this was extended to include dimension four in [FQ90, Section 9.3]. It is striking that, while among smooth manifolds dimension 4 exhibits worse than usual behaviour, in the topological category the existence of normal vector bundles seems to show it is among the better behaved of the dimensions.

For the proof of Theorem 6.17 we will essentially appeal to [FQ90, Theorem 9.3A] and [FQ90, Theorem 9.3D]; the former deals with existence while the latter deals with uniqueness. We reproduce this theorem here for the benefit of the reader.

Theorem 6.19. *Let N be a submanifold of a 4-manifold M, with a closed subset $K \subseteq N \setminus \partial N$ and a normal bundle over some neighbourhood of K in N. Then there is a normal bundle over N that agrees with the given one over the neighbourhood of K. Moreover this extension is unique up to ambient isotopy relative to some neighbourhood of K.*

Proof. Let X be a proper submanifold of a compact 4-manifold M. The case that X has no boundary is dealt with in [FQ90, Theorem 9.3A]. The case that X has nonempty boundary follows also from [FQ90, Theorem 9.3A] if we apply more care. We sketch the argument.

First, in dimension three the topological and the smooth category are the same. Thus we can view the submanifold $\partial X \subset \partial M$ as a smooth submanifold. Hence it has a smooth normal vector bundle e.g. [Kos93, Chapter III.2] or [Lan02b, Section IV.5].

Next use the Collar neighbourhood theorem 6.5 to obtain a collar $\partial M \times [0,1] \subset M$ that restricts to a collar $\partial X \times [0,1]$ for the boundary of X. Extend the smooth tubular neighbourhood of $\partial X \subset \partial M$ into the collar by taking a product with $[0,1]$.

Finally, consider the 4-manifold without boundary $M' := M \setminus (\partial M \times [0,\frac{1}{2}])$. What remains of X is a submanifold $N := X \setminus (\partial X \times [0,\frac{1}{2}])$. The submanifold N already has a preferred normal vector bundle on the closed subset $K := \partial X \times (1/2, 1]$. Now apply [FQ90, Theorem 9.3A] to the triple (M', N, K) to obtain a normal vector bundle $E \to N$ agreeing with the given one on K. The normal vector bundles over N and $\partial X \times [0,1]$ agree on the overlap K. Thus they define a normal vector bundle on all of X. □

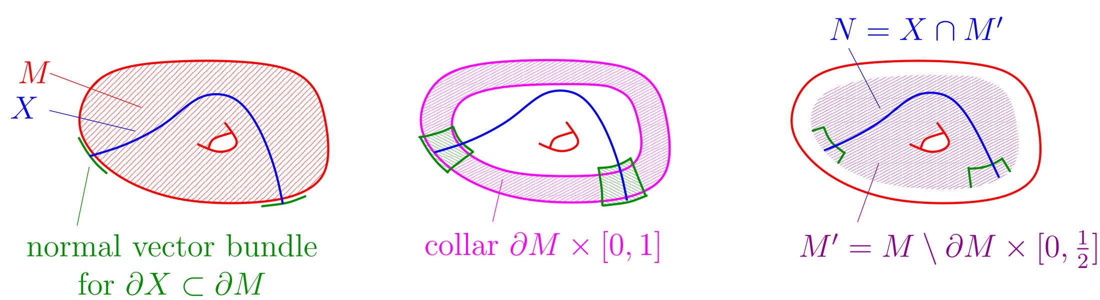

FIGURE 6. Illustration of the proof of Theorem 6.17.

Next we turn to the uniqueness of normal vector bundles.

Theorem 6.20 (Uniqueness of normal vector bundles)**.** *Let M be a compact 4-manifold and let X be a proper submanifold of M. Suppose we are given two normal vector bundles $p_i \colon E_i \to X$, $i = 1, 2$ for X. For $i = 1, 2$ let $\iota_i \colon E_i \to M$ be the inclusion map. Then there exists a bundle isomorphism $f \colon E_1 \xrightarrow{\cong} E_2$ such that $\iota_2 \circ f$ and ι_1 are ambiently isotopic rel. X.*

Proof. If X has no boundary, then the theorem is an immediate consequence of [FQ90, Theorem 9.3D]. Now suppose that X has nonempty boundary.

First we claim that any normal vector bundle of X is obtained by the construction outlined in the proof of Theorem 6.17. To see this, let $p\colon E \to X$ be a normal vector bundle. Pick a collar neighbourhood $\partial X \times [0,2] \subset X$. Since p is extendable, we can view p as the interior of a disc bundle $q\colon F \to X$ in M. Write $C := q^{-1}(\partial X) \subseteq \partial M$. The disc bundle $q\colon q^{-1}(\partial X \times [0,2]) \to \partial X \times [0,2]$ defines a collar neighbourhood $C \times [0,2]$ for the compact submanifold C of ∂M. By the collar neighbourhood theorem 2.5 we can extend the collar neighbourhood $C \times [0,1]$ of C to a collar neighbourhood $\partial M \times [0,1]$. With this choice of collar neighbourhood of ∂M, the construction in the proof of Theorem 6.17, with further appropriate choices, gives rise to the normal vector bundle $p\colon E \to X$. This completes the proof of the claim.

After this long preamble it suffices to prove the theorem for any two normal vector bundles obtained as in the proof of Theorem 6.17. Uniqueness follows by arguing that each step in the proof of existence of normal vector bundles was essentially unique. The proofs of uniqueness in the three steps make use of the following ingredients.

First, apply the uniqueness statement for normal vector bundles of submanifolds of smooth manifolds to $\partial X \subseteq \partial M$ e.g. [Kos93, Chapter III.2] or [Lan02b, Section IV.5].

Next use the uniqueness of collar neighbourhoods as formulated in Theorem 2.7, applied to the two collar neighbourhoods of ∂M subordinate to the given normal bundles of X.

Finally apply the full relative version of [FQ90, Theorem 9.3D] to extend the normal vector bundle uniquely over the rest of X. □

6.4. **Tubular neighbourhoods: proofs.** Now we will use the results from the previous section to prove the existence and uniqueness of tubular neighbourhoods. First we show how one can obtain tubular neighbourhoods from normal vector bundles.

Definition 6.21. Let $p\colon E \to X$ be a vector bundle. Given $x \in X$, write $E_x := p^{-1}(x)$. A *positive definite form* $g = \{g_x\}_{x\in X}$ consists of a positive definite form g_x for every E_x such that g_x changes continuously with x.

The following lemma follows from standard techniques, so we leave it to the reader to fill in the details.

Lemma 6.22. *Let X be a compact manifold and let $p\colon E \to X$ be an n-dimensional vector bundle. Then the space of positive definite forms on E is nonempty and convex. Furthermore, let $g = \{g_x\}_{x\in X}$ be a positive definite form on E and consider the map*

$$p\colon E(g) := \bigcup_{x\in X} \{v \in E_x \mid g_x(v,v) \leq 1\} \to X.$$

This map has the following properties:

(1) *The map $p\colon E(g) \to X$ is a linear D^n-bundle.*
(2) *Given two different positive definite forms g and h on E there exists an isotopy of the vector bundle E sending $E(g)$ to $E(h)$, and restricting to the identity on the 0-section and outside some compact subset of E.*

Let X be a compact manifold and let $p\colon E \to X$ be a vector bundle. Given any positive definite form g we refer to

$$p\colon E' \;:=\; \bigcup_{x\in X} \{v \in E_x \mid g_x(v,v) \leq 1\} \;\to\; X$$

as a *corresponding disc bundle*. It follows from Lemma 6.22 that for most purposes the precise choice of g is irrelevant.

We can now prove the existence of tubular neighbourhoods.

Proof of the tubular neighbourhood theorem 6.8. Let X be a compact proper submanifold of a 4-manifold M. By Theorem 6.17 there exists a normal vector bundle $p\colon N \to X$ for X. A choice of corresponding disc bundle is easily seen to be a tubular neighbourhood. □

The uniqueness proof for tubular neighbourhoods also requires us to associate a normal vector bundle to a tubular neighbourhood.

Lemma 6.23. *Let M be a compact 4-manifold and let X be a compact proper k-dimensional submanifold. Let $p\colon N \to X$ be a tubular neighbourhood for X. There exists a normal vector bundle $q\colon E \to X$ and a positive definite form g such that $N = E(g)$ and $p\colon N \to X$ equals $q\colon E(g) \to X$.*

Proof. Let $p\colon N \to X$ be a tubular neighbourhood for X. Recall that we have $\operatorname{Int} N = N \setminus \partial_0 N$. Consider $W := M \setminus \operatorname{Int} N$. This is a compact 4-manifold. Pick a collar neighbourhood $\partial W \times [0,1]$ and set $E := N \cup \partial_0 N \times [0, \frac{1}{2})$. We have an obvious projection map $q\colon E \to X$ turning q into a bundle map where the fibre is given by the open $(4-k)$-ball of radius $\frac{3}{2}$. We leave it to the reader to turn $q\colon N \to X$ into an internal linear bundle, to show that it is in fact extendable (at this point one has to use that in the definition of E we only used "half" of the collar neighbourhood $\partial_0 N \times [0,1]$), and to equip N with a positive definite form g such that $N = E(g)$. □

We conclude the section with the proof of the uniqueness theorem for tubular neighbourhoods.

Proof of Theorem 6.9. Let M be a 4-manifold and let X be a compact proper k-dimensional submanifold. Furthermore let $p_i\colon N_i \to X$, $i = 1,2$ be two tubular neighbourhoods of X. For $i = 1,2$, let $q_i\colon E_i \to X$ be two corresponding normal vector bundles and let g_i be the positive definite forms provided by Lemma 6.23. It follows from Theorem 6.20 that there exists a bundle isomorphism $f\colon E_1 \xrightarrow{\cong} E_2$ such that $\iota_2 \circ f$ and ι_1 are ambiently isotopic rel X. It follows from the definitions that N_2 is equivalent to the disc bundle defined by $f^* g_2$ on E_1. But Lemma 6.22 (3) implies that $f^* g_2$ and g_1 define equivalent tubular neighbourhoods. □

7. Background on bundle structures

In this section we recall the bundle technologies we will need to use in later sections. First we recall the three standard types of fibre bundle with fibre $\mathbb{R}^n$: O, PL and TOP.

Definition 7.1. Let $\mathrm{TOP}(n)$ be the subgroup of homeomorphisms of $\mathbb{R}^n$ that fix the origin, topologised using the compact open topology. A principal $\mathrm{TOP}(n)$-bundle has an associated fibre bundle with fibre $\mathbb{R}^n$ and a preferred 0-section. Call such a bundle a *topological* $\mathbb{R}^n$*-bundle*. Let TOP be the colimit $\varinjlim \mathrm{TOP}(n)$ under the inclusions $-\times \mathrm{Id}_{\mathbb{R}}\colon \mathrm{TOP}(n) \to \mathrm{TOP}(n+1)$. Write $\mathrm{BTOP}(n)$ and BTOP for the corresponding classifying spaces.

Let $\mathrm{O}(n)$ be the orthogonal homeomorphisms of $\mathbb{R}^n$ that fix the origin. Similarly to the case of homeomorphisms, define $\mathrm{BO}(n)$, O, and BO.

The definition of the analogous spaces for PL is a little more involved, using semi-simplicial groups. For a gentle introduction to simplicial sets, see [Fri12]. The canonical reference for classifying spaces constructed using simplicial groups is [May75].

Definition 7.2. A homeomorphism $f\colon K \to L$ between two simplicial complexes K and L is a PL-*homeomorphism* if there are subdivisions K' of K and L' of L such that $f\colon K' \to L'$ is a simplicial map.

For background on piecewise linear topology see [RS72]. The next definition comes from [RS68a]. Let Δ^k be the standard k-simplex.

Definition 7.3. Let $\mathrm{PL}(n)_\bullet$ be the semi-simplicial group defined as follows.

(i) The group $\mathrm{PL}(n)_k$ assigned to the k-simplex is the group of PL homeomorphisms $f\colon \mathbb{R}^n \times \Delta^k \to \mathbb{R}^n \times \Delta^k$ over Δ^k, such that $f|_{\mathbb{R}^n\times\{t\}}$ fixes the origin in $\mathbb{R}^n$ for every $t \in \Delta^k$. That is, with $p\colon \mathbb{R}^n \times \Delta^k \to \Delta^k$ the projection, the diagram

$$\begin{tikzcd} \mathbb{R}^n \times \Delta^k \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{R}^n \times \Delta^k \arrow[dl, "p"] \\ & \Delta^k & \end{tikzcd}$$

commutes.

(ii) The ith face map is given by restricting to the ith face of Δ^k.

Now define $\mathrm{BPL}(n)$ by first using the level-wise bar construction to obtain a semi-simplicial space, and then geometrically realising to obtain a space $\mathrm{BPL}(n)$. Define PL and BPL as colimits analogously to Definition 7.1.

Remark 7.4. It is interesting that we do not define $\mathrm{PL}(n)$ using the subspace topology from $\mathrm{TOP}(n)$. Note that we also do not define $\mathrm{Diff}(n) = \mathrm{Diff}(\mathbb{R}^n)$ in this way. But for defining $\mathrm{Diff}(n)$ as a topological group, and using this topology to define $\mathrm{BDiff}(n)$, we have the bespoke Whitney topology. In the absence of an analogous topology for $\mathrm{PL}(n)$, we use the simplicial strategy. In fact this simplicial method could be used to define all three of $\mathrm{TOP}(n)$, $\mathrm{PL}(n)$ and $\mathrm{Diff}(n)$, giving a uniform treatment. But only in the PL case do we really know of no other method that works.

All smooth manifolds have tangent vector bundles and all smooth submanifolds have normal vector bundles. This is one reason that vector bundles, corresponding to the structure group $\mathrm{O}(n)$, are the de facto bundle technology in the smooth category. A general difficulty we will face when talking about manifold transversality in Section 10 is that we will need to use some well-defined notion of normal structure for a submanifold and, outside of the smooth category, submanifolds do not necessarily admit normal vector bundles. However, various weaker bundle technologies have been developed, which replace this crucial concept in the topological category.

The rest of this section is devoted to a discussion of *microbundles* [Mil64]. The existence and uniqueness of microbundles leads to the existence and uniqueness of tangent and (stable) normal $\mathbb{R}^n$-bundles for $\mathrm{TOP}(n)$, as discussed below. Source material on microbundles is not hard to find in the literature, but has been included here for the convenience of the reader, in order for this survey to be more self-contained.

The interaction between the weaker fibre automorphism groups $\mathrm{PL}(n)$ and $\mathrm{TOP}(n)$ for tangent and (stable) normal $\mathbb{R}^n$-bundles, and the topological/piecewise linear/smooth structures on the manifold itself are the topic of *smoothing theory*, to which we turn in Section 8.

Definition 7.5. An n-dimensional *microbundle* ξ consists of a base space B and a total space E sitting in a diagram

$$B \xrightarrow{i} E \xrightarrow{r} B,$$

such that $r \circ i = \mathrm{Id}_B$, and that is *locally trivial* in the following sense: for every point $b \in B$, there exists an open neighbourhood U, an open neighbourhood V of $i(b)$ and a homeomorphism $\phi_b \colon V \to U \times \mathbb{R}^n$ such that

$$\begin{tikzcd} & V \arrow[dr, "r"] \arrow[dd, "\phi_b"] & \\ U \arrow[ur, "i"] \arrow[dr, "\times\{0\}"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "pr_1"'] & \end{tikzcd}$$

commutes.

Note that we only require a neighbourhood of $i(b)$ to be trivial, and not all of the fibre $r^{-1}(b)$. In fact, we only care about neighbourhoods $i(B) \subset E$, and declare two microbundles $B \to E \to B$ and $B \to E' \to B$ to be *equivalent*, if $i(B)$ and $i'(B)$ have homeomorphic neighbourhoods such that the homeomorphism commutes with both the inclusion map and the restriction of the retraction map.

Definition 7.6. Let $r \colon E \to B$ be a microbundle ξ and let $f \colon A \to B$ be a map. The *pullback* of ξ under f is the microbundle $f^*\xi$ with total space

$$f^*E = \big\{(a, e) \in A \times E \mid f(a) = r(e)\big\},$$

retraction $(f^*r)(a, e) = a$, and section $(f^*s)(a) = \big(a, s(f(a))\big)$. In the case that f is an inclusion, also consider the microbundle $\xi|_A$, which has total space $r^{-1}(A) \subset E$, and retraction $r_A\colon r^{-1}(A) \to A$ and section $s_A\colon A \to r^{-1}(A)$ are both the restrictions of r, s. In this case, the map of total spaces $(a,e) \mapsto e$ gives a preferred isomorphism $f^*\xi$ to $\xi|_A$.

A topological $\mathbb{R}^n$-bundle clearly has an underlying microbundle. Kister and Mazur proved independently the surprising result that *every* microbundle is equivalent to such an underlying microbundle [Kis64, Theorem 2].

Theorem 7.7 (Kister-Mazur)**.** *Let B be a manifold and $B \xrightarrow{i} E \xrightarrow{r} B$ be an n-dimensional microbundle ξ. Then there exists an open set $F \subset E$ containing $i(B)$ such that $r\colon F \to B$ is the projection map of a topological $\mathbb{R}^n$-bundle, whose 0-section is i and whose underlying microbundle is ξ. Moreover, if F_1 and F_2 are any two topological $\mathbb{R}^n$-bundles over B such that the underlying microbundles are isomorphic, then F_1 and F_2 are isomorphic as topological $\mathbb{R}^n$-bundles.*

Every manifold admits a tangent microbundle.

Definition 7.8 (Tangent microbundle)**.** The *tangent microbundle* of an n-dimensional manifold M is the microbundle $M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_1} M$ where Δ is the diagonal map. The Kister-Mazur theorem implies this corresponds to a unique *topological tangent bundle* $\tau_M\colon M \to \mathrm{BTOP}(n)$, with corresponding *stable topological tangent bundle* $\tau_M\colon M \to \mathrm{BTOP}$.

More subtle is the concept of a normal microbundle.

Definition 7.9 (Normal microbundle)**.** A *normal microbundle* of a submanifold S of a manifold M is a microbundle $S \to E \to S$ such that E is a neighbourhood of S in M.

It is immediate from the definition of normal microbundle that the local flatness in the definition of a submanifold S is a necessary condition for the existence of a normal microbundle. Wild submanifolds that are not submanifolds do not admit normal microbundles. Indeed, it is generally far from straightforward to prove the existence of normal microbundles at all. Here is an existence and uniqueness result due to Stern [Ste75, Theorem'4.5]. See also [Hir66],[Hir68, p. 65], and [KS77, Essay IV, Appendix A].

Theorem 7.10. *Let M^{n+q} be a manifold, and let $N^n \subset M^{n+q}$ be a proper submanifold of codimension q. Suppose that $n \leq q+1+j$ and $q \geq 5+j$ for some $j = 0,1,2$. Then N admits a normal microbundle restricting to a normal microbundle of $\partial N \subset \partial M$.*

If in addition $n \leq q+j$, then this normal microbundle is unique up to isotopy.

Remark 7.11 (Unique up to isotopy). For a submanifold $N \subset M$ we say a normal microbundle $\nu(N)$ is unique up to isotopy if whenever there is another normal microbundle $\nu'(N)$, there exists a microbundle equivalence f between $\nu(N)$ and $\nu'(N)$ such that $\pi' \circ f$ is isotopic to π relative to N.

We exploit these theorems to define a stable normal structure on any manifold, that will play an important role in Section 8. Consider that any closed n-manifold M can be embedded as a submanifold $M \subset \mathbb{R}^m$ for large m [Hat02, Corollary A.9]. For large enough

m, any two such embeddings are isotopic. For large enough m, Theorem 7.10 implies there is a normal microbundle ξ. After possibly increasing m further, the last sentence of Theorem 7.10 implies this normal microbundle ξ is unique. By the Kister-Mazur Theorem this defines a unique topological $\mathbb{R}^{m-n}$-bundle. We remove the dependence on m by passing to the stable bundle $\mathrm{TOP}(m-n) \subset \mathrm{TOP}$. Thus the process described gives a well-defined classifying map $\nu_M \colon M \to \mathrm{BTOP}$. Summarising, we have the following.

Definition 7.12. Given any closed n-manifold, the topological $\mathbb{R}^\infty$-bundle $\nu_M \colon M \to \mathrm{BTOP}$, described above, is called the *stable topological normal bundle*. It is well-defined and unique.

The next example shows that outside the hypotheses of Theorem 7.10, we should expect that normal microbundles can be very badly behaved.

Example 7.13. Normal microbundles do not necessarily exist. Rourke and Sanderson [RS67, Example 2] construct S^{19} as a submanifold of a certain 28-dimensional PL manifold M in such a way that it does not admit a topological normal microbundle. The embedding is even piecewise linear.

Even when topological normal microbundles exist, they are not always unique: Rourke and Sanderson consider the smooth standard embedding $S^{18} \subset S^{27}$ [RS68b, Theorem 3.12] and construct a certain normal microbundle ξ of $S^{18} \subset S^{27}$. The construction of ξ is such that if ξ were concordant to the trivial normal microbundle, this concordance would induce a normal microbundle structure back on the embedding $S^{19} \subset M^{28}$ of the previous example. As this is not possible, ξ is non-trivial. Note that the smooth normal bundle νS^{18} of the standard embedding is trivial, so $S^{18} \subset S^{27}$ admits at least two different normal microbundles.

The following theorem ensures the issues of the previous example are not seen in dimension 4.

Theorem 7.14. *Let X be a codimension n proper submanifold of a 4-manifold M. Then X admits a normal microbundle. Moreover, if ξ is a normal microbundle of X, it is the underlying microbundle to a normal vector bundle.*

Proof. The existence of normal microbundles in ambient dimension 4 is an immediate consequence of the existence of normal vector bundles (Theorem 6.17).

Given a normal microbundle ξ, we apply the Kister-Mazur theorem 7.7 to obtain an embedded $\mathbb{R}^n$-bundle with underlying microbundle ξ. For $n \leq 2$, the homotopy fibre $\mathrm{TOP}(n)/\mathrm{O}(n)$ for the forgetful map $\mathrm{BO}(n) \to \mathrm{BTOP}(n)$ is contractible, and for $k \leq 4$ we have $\pi_k(\mathrm{TOP}(3)/\mathrm{O}(3)) = 0$ [KS77, Essay V, Theorems 5.8 and 5.9]. Using these facts, and checking the obstructions in each of the cases $n = 0, 1, 2, 3, 4$, we see in each case the embedded topological $\mathbb{R}^n$-bundle can be upgraded to an embedded vector bundle. Choose such a vector bundle refinement. By restricting to an open disc bundle and rescaling we can ensure this internal linear bundle is extendable and thus a normal vector bundle in the sense of Definition 6.15. ∎

We will make use of our discussion of normal microbundles in Section 10 on topological transversality.

8. Smoothing 4-manifolds

We present three theorems which associate a smooth manifold to a given 4-manifold. Often these theorems can be used to reduce proofs about 4-manifolds to the case of smooth 4-manifolds, where the standard tools of differential topology are available.

8.1. **Smoothing non-compact 4-manifolds.**

The first of our smoothing theorems [Qui82, Corollary 2.2.3], [FQ90, p. 116] says that noncompact connected 4-manifolds admit a smooth structure.

Theorem 8.1. *Every connected, noncompact 4-manifold is smoothable. Thus every 4-manifold M has a smooth structure in the complement of any closed set that has at least one point in each compact component of M.*

There are some related statements in the literature on smoothing 4-manifolds in the complement of a point, that appeared prior to Freedman's work [Fre82] and prior to [Qui82]. We discuss them briefly here. For the case of PL structures on noncompact 4-manifolds, given a lift of the (unstable) tangent microbundle classifying map $M \to \mathrm{BTOP}(4)$ to the PL category $M \to \mathrm{BPL}(4)$ (see Section 7), the proof can be found in [Las70, p. 54] and [KS77, Essay V, Addendum 1.4.1]. This result was stated for smooth bundle structures and smooth structures on manifolds in [Las71]. Alternatively, [HM74], [FQ90, Theorem 8.3B] apply to improve a PL structure to a smooth structure, unique up to isotopy, for any manifold of dimension at most six. Again, in [Las71] Lashof assumes a lift of the (unstable) tangent microbundle classifying map $M \to \mathrm{BTOP}(4)$ to a map $M \to \mathrm{BO}(4)$. For noncompact connected 4-manifolds, such a lift always exists, as was later shown by Quinn [Qui82, Qui84], [FQ90, p. 116] using the full disc embedding theorem [Fre82], and giving rise to Theorem 8.1.

Due to the seminal nature of Freedman's Field's medal winning paper [Fre82], it is well worth clarifying the details of some citations therein. In the proof of Corollary 1.2, in the proof of Theorem 1.5 on page 369, in the proof of Theorem 1.6, and at the start of Section 10, Freedman uses that smoothing theory is available for noncompact 4-manifolds. In particular, smoothing for noncompact contractible 4-manifolds plays a vital rôle in Freedman's proof of the topological 4-dimensional Poincaré conjecture [Fre82, Theorem 1.6]. Freedman cites [KS77] for this fact, however [KS77, Essay V, Remarks 1.6 (A)] specifically excludes smooth structures (but for a stronger result). Nevertheless, as mentioned above, Lashof [Las71, p. 156] proved the smooth version of [KS77, Essay V, Addendum 1.4.1], or one can use PL smoothing theory [HM74], [FQ90, Theorem 8.3B] to improve a PL structure from [KS77, Essay V, Addendum 1.4.1] to a smooth structure, essentially uniquely.

Freedman only applies smoothing theory in cases, such as for contractible M, that he can ensure the existence of a lift of $\tau_M\colon M \to \mathrm{BTOP}(4)$ to $\mathrm{BO}(4)$. Later, Quinn [Qui82, Corollary 2.2.3] showed that such a lift always exists for connected noncompact 4-manifolds. In fact, he showed that the map $\mathrm{TOP}(4)/\mathrm{O}(4) \to \mathrm{TOP}/\mathrm{O} \simeq K(\mathbb{Z}/2,3)$ is 5-connected [FQ90, Theorem 8.7A], where only 3-connected is needed for Theorem 8.1. In other words, it was shown prior to Freedman's work that homotopy 4-spheres admit a smooth structure in the complement of a point, so the results that Freedman required were indeed known. However, smoothing in the complement of a point was not known for general connected 4-manifolds until after the work of Quinn in 1982. Further discussion can also be found in Quinn [Qui84] and Lashof-Taylor [LT84].

Below we will give applications of Theorem 8.1, see e.g. the proof of Theorem 10.9.

8.2. **The Kirby-Siebenmann invariant and stable smoothing of 4-manifolds.** The formulation of the other two statements on smoothing 4-manifolds make use of the Kirby-Siebenmann invariant. The Kirby-Siebenmann invariant $\mathrm{ks}(M) \in \mathbb{Z}/2$ of a compact 4-manifold is defined in [FQ90, Section 10.2B], or alternatively by [KS77, p. 318] or [Rud16, Definition 3.4.2], and we describe the construction now.

The homotopy fibre TOP/PL of the forgetful map $\mathrm{BPL} \to \mathrm{BTOP}$ has the homotopy type of a $K(\mathbb{Z}/2,3)$ and has the structure of a loop space, permitting the definition of the delooping $\mathrm{B}(\mathrm{TOP}/\mathrm{PL})$ [BV68, Theorem C], [BV73] which is an Eilenberg-Maclane space of type $K(\mathbb{Z}/2,4)$. A connected topological 4-manifold has a unique smooth structure on its boundary. Using the homotopy fibre sequence

$$\mathrm{TOP}/\mathrm{PL} \to \mathrm{BPL} \to \mathrm{BTOP} \to \mathrm{B}(\mathrm{TOP}/\mathrm{PL}),$$

the unique obstruction to a lift of the classifying map $\tau_M\colon M \to \mathrm{BTOP}$ of the stable tangent microbundle to BPL is therefore a homotopy class in

$$[(M,\partial M),(\underbrace{\mathrm{B}(\mathrm{TOP}/\mathrm{PL})}_{=K(\mathbb{Z}/2,4)},*)] \cong H^4(M,\partial M;\mathbb{Z}/2) = \mathbb{Z}/2.$$

Here, we used again that 4-manifolds have the homotopy type of a CW-complex. We refer to the corresponding element of $\mathbb{Z}/2$ as the Kirby-Siebenmann $\mathrm{ks}(M)$ invariant of the compact, connected manifold M. For disconnected compact 4-manifolds, $M = \bigsqcup_{i=1}^n M_i$, define

$$\mathrm{ks}(M) = \sum_{i=1}^n \mathrm{ks}(M_i) \in \mathbb{Z}/2.$$

In the following theorem we summarise some key properties of the Kirby-Siebenmann invariant.

Theorem 8.2. *Let M and N be compact 4-manifolds.*

(1) *If $M \times \mathbb{R}$ admits a smooth structure (e.g. if M admits a smooth structure), then* $\mathrm{ks}(M) = 0$.
(2) *The Kirby-Siebenmann invariant is additive under the connected sum operation.*
(3) *If there exists a compact 5-manifold with $\partial W = M \cup_{\partial M = \partial N} N$, then* $\mathrm{ks}(M) = \mathrm{ks}(N)$.
(4) *The Kirby-Siebenmann invariant gives rise to a homomorphism $\Omega_4^{\mathrm{TOP}} \to \mathbb{Z}/2$.*

(5) *If* $S \subset \partial M$ *and* $T \subset \partial N$ *are compact codimension zero submanifolds with* $S \cong T$*, then*

$$\mathrm{ks}(M \cup_{S \cong T} N) = \mathrm{ks}(M) + \mathrm{ks}(N).$$

(6) *The forgetful map* $\Omega_4^{\mathrm{Spin}} \to \Omega_4^{\mathrm{TOPSpin}}$ *fits into a short exact sequence*

$$0 \to \Omega_4^{\mathrm{Spin}} \to \Omega_4^{\mathrm{TOPSpin}} \xrightarrow{\mathrm{ks}=\sigma/8} \mathbb{Z}/2 \to 0$$

with the last map given by the signature divided by 8, modulo 2, which equals the Kirby-Siebenmann invariant $\Omega_4^{\mathrm{TOPSpin}} \to \Omega_4^{\mathrm{TOP}} \xrightarrow{\mathrm{ks}} \mathbb{Z}/2$*. This sequence does not split, so* $\Omega_4^{\mathrm{TOPSpin}} \cong \mathbb{Z}$*.*

We could not find explicit proofs of these facts in the literature, so we give some details.

Proof. Let us prove (1). The tangent bundle of $M \times \mathbb{R}$ is isomorphic to $\tau_M \oplus \varepsilon$, where τ_M is the tangent microbundle of M and ε denotes a rank one trivial vector bundle over M. If $M \times \mathbb{R}$ admits a smooth structure, then there is a lift $\tau^{\mathrm{Diff}}_{M\times\mathbb{R}} \colon M \to \mathrm{BO}(5)$, the smooth tangent bundle to $M \times \mathbb{R}$. Let $p\colon \mathrm{BO}(5) \to \mathrm{BTOP}(5)$ be the canonical map. Then $\tau_M \oplus \varepsilon = p \circ \tau^{\mathrm{Diff}}_{M\times\mathbb{R}}$. Passing to the stable classifying spaces, we obtain a lift $M \to \mathrm{BO}$ whose composition with the canonical map $\mathrm{BO} \to \mathrm{BTOP}$ agrees with $\tau_M \oplus \varepsilon^\infty$, the stable tangent microbundle of M. Since the map $\mathrm{BO} \to \mathrm{BTOP}$ factors through $\mathrm{BPL} \to \mathrm{BTOP}$, we have a stable lift of τ_M and so $\mathrm{ks}(M) = 0$. This completes the proof of (1).

Next, (2) is a consequence of (5). However since (5) follows from (3), we also remark that (2) follows from (3) more easily. To see this observe that a disjoint union $M \sqcup N$ is cobordant to $M\#N$ via the cobordism

$$(M \times I \sqcup N \times I) \cup_{S^0 \times D^4} (D^1 \times D^4),$$

with $\{-1\} \times D^4$ embedded in the interior of $M \times \{1\}$, and $\{1\} \times D^4$ embedded in the interior of $N \times \{1\}$. Then by (3) the Kirby-Siebenmann invariant vanishes on $M\#N \sqcup M \sqcup N$ and therefore $\mathrm{ks}(M\#N) = \mathrm{ks}(M) + \mathrm{ks}(N) \in \mathbb{Z}/2$.

Let us prove (3). Suppose that a compact 4-manifold M is the boundary of a compact 5-manifold W'. Perform 0 and 1-surgeries on W' to obtain a path connected, simply connected, compact 5-manifold with $\partial W = M$. Note that

$$H^4(W, \partial W; \mathbb{Z}/2) \cong H_1(W; \mathbb{Z}/2) = 0$$

since W is simply connected, so

$$[(W, \partial W), (B(\mathrm{TOP}/\mathrm{PL}), *)] \cong H^4(W, \partial W; \mathbb{Z}/2) = 0.$$

It follows from the homotopy fibre sequence associated to $\mathrm{BTOP} \to \mathrm{BPL}$ that the classifying map in $[W, \mathrm{BTOP}]$ of the stable tangent microbundle lifts to a map in $[W, \mathrm{BPL}]$. Therefore there is a lift $W \to \mathrm{BPL}$ of the stable tangent microbundle. The restriction $M \to W \to \mathrm{BTOP}$ equals the stable tangent microbundle of M, since M has a collar $M \times [0,1] \subset W$. It follows that the composition $M \to W \to \mathrm{BPL}$ is a stable lift of τ_M, so that $\mathrm{ks}(M) = 0$. This completes the proof of (3).

Now (4) follows easily. First note that the addition on Ω_4^{TOP} is by disjoint union, so ks is additive by definition. The map $\mathrm{ks}\colon \Omega_4^{\mathrm{TOP}} \to \mathbb{Z}/2$ is well-defined by (3). Therefore $\mathrm{ks}\colon \Omega_4^{\mathrm{TOP}} \to \mathbb{Z}/2$ is a homomorphism as desired.

To prove (5), it was suggested by Jim Davis to consider the exact sequence

$$\Omega_4^{\mathrm{O}} \to \Omega_4^{\mathrm{TOP}} \to \Omega_4^{\{\mathrm{O}\to\mathrm{TOP}\}} \to \Omega_3^{\mathrm{O}} = 0.$$

Here $\Omega_4^{\{\mathrm{O}\to\mathrm{TOP}\}}$ is represented by compact topological 4-manifolds with smooth boundary, up to 5-dimensional cobordism relative to a smooth cobordism on the boundary. That is, 4-manifolds with boundary $(M, \partial M)$ and $(N, \partial N)$ are equivalent if there is a compact 5-manifold with boundary

$$\partial W = M \cup_{\partial M} \partial_{\mathrm{vert}} W \cup_{\partial N} N,$$

for some smooth 4-dimensional cobordism $\partial_{\mathrm{vert}} W$ with boundary $\partial M \sqcup \partial N$.

By the exact sequence, $\Omega_4^{\{\mathrm{O}\to\mathrm{TOP}\}}$ is the cokernel of $\Omega_4^{\mathrm{O}} \to \Omega_4^{\mathrm{TOP}}$ and is therefore isomorphic to $\mathbb{Z}/2$ via the Kirby-Siebenmann invariant. We therefore need to show that the disjoint union $M \sqcup N$ is bordant to $M \cup_{S=T} N$, where $S \subset \partial M$ and $T \subset \partial N$ are compact codimension zero submanifolds with $S \cong T$. Here is a construction of such a bordism. For $I = [0, 1]$, take

$$(M \times I) \sqcup (S \times I \times [1/2, 1]) \sqcup (N \times I),$$

identify

$$S \times \{0\} \times [1/2, 1] \ \sim \ S \times [1/2, 1] \ \subseteq \ (M \times [1/2, 1]),$$

and, using the identification $S \cong T$, identify

$$S \times \{1\} \times [1/2, 1] \ \sim \ T \times [1/2, 1] \subseteq N \times [1/2, 1].$$

Let W be the result of this gluing and some rounding of corners. The boundary of W is

$$(M \sqcup N) \cup_{\partial M \sqcup \partial N} \partial_{\mathrm{vert}} W \cup_{\partial(M \cup_{S=T} N)} M \cup_{S=T} N,$$

where

$$\begin{aligned} \partial_{\mathrm{vert}} W \ &= (\partial M \times [0, 1/2]) \cup (\overline{\partial M \setminus S} \times [1/2, 1]) \\ &\cup (S \times I \times \{1/2\}) \cup (\partial S \times I \times [1/2, 1]) \\ &\ (\partial N \times [0, 1/2]) \cup (\overline{\partial N \setminus T} \times [1/2, 1]). \end{aligned}$$

This shows that $M \sqcup N$ and $M \cup_{S=T} N$ are equal in $\Omega_4^{\{\mathrm{O}\to\mathrm{TOP}\}}$, and therefore have the same Kirby-Siebenmann invariants. Since $\mathrm{ks}(M \sqcup N) = \mathrm{ks}(M) + \mathrm{ks}(N)$, this completes the proof of (5).

To prove (6), we consider the following diagram. The maps between bordism groups are structure forgetting maps, so the diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \Omega_4^{\mathrm{Spin}} \arrow[r] \arrow[d, "\cdot 16"] & \Omega_4^{\mathrm{TOPSpin}} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \Omega_4^{\mathrm{SO}} \arrow[r] & \Omega_4^{\mathrm{STOP}} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Recall that $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$ given by the signature and generated by $\mathbb{CP}^2$. The signature provides a splitting homomorphism, so $\Omega_4^{\mathrm{STOP}} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Also $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$ given by the signature divided by 16 and generated by the $K3$ surface.

Both sequences are exact: a smooth manifold has vanishing ks invariant, and vanishing $\mathrm{ks}(M)$ implies smoothable after adding copies of $S^2 \times S^2$ by Theorem 8.6 below. Since $M \# S^2 \times S^2$ is (spin) bordant to M, the sequences are exact at their middle terms. Finally, a topological null bordism of a compact smooth 4-manifold can be smoothed by high dimensional smoothing theory, so the left hand maps are injective.

We claim that the sequence in the upper row does not split. Consider the $K3$ surface generating $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$. By the down-then-left route, $[K3]$ maps to $(16, 0) \in \mathbb{Z} \oplus \mathbb{Z}/2 \cong \Omega_4^{\mathrm{STOP}}$. On the other hand the E_8-manifold, whose existence was established by Freedman [Fre82, Theorem 1.7] as a key step in the proof of the classification theorem 11.2, represents a class in $\Omega_4^{\mathrm{TOPSpin}}$ and maps to $(8, 1) \in \mathbb{Z} \oplus \mathbb{Z}/2 \cong \Omega_4^{\mathrm{STOP}}$.

We note for later that the E_8 manifold has $\mathrm{ks}(E_8) = 1$: E_8 cannot be smoothed, even after adding copies of $S^2 \times S^2$, by Rochlin's theorem (Theorem 3.4) that every closed spin smooth 4-manifold has signature divisible by 16, whereas if $\mathrm{ks}(E_8) = 0$, then E_8 would be stably smoothable by Theorem 8.6.

It follows that $2 \cdot [E_8]$ maps to $0 \in \mathbb{Z}/2$ so has trivial ks invariant, and therefore by exactness of the top row lies in the image of Ω_4^{Spin}. Let N be a closed spin smooth 4-manifold TOPSpin-bordant to $E_8 \# E_8$. Since $\sigma(E_8 \# E_8) = 16 = \sigma(K3)$, we have $[N] = [K3] \in \Omega_4^{\mathrm{Spin}}$. It follows that $K3$, the generator of $\Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$, maps to $2 \cdot [E_8] \in \Omega_4^{\mathrm{TOPSpin}}$. Thus we have a diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "1 \mapsto [E_8]"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1 \mapsto 2 \cdot [E_8]"] & \Omega_4^{\mathrm{TOPSpin}} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{ks}(E_8) = 1$, the diagram commutes. Then by the five lemma, $\Omega_4^{\mathrm{TOPSpin}} \cong \mathbb{Z}$, generated by E_8, and the sequence does not split, as claimed. The diagram

$$\begin{tikzcd} \Omega_4^{\mathrm{TOPSpin}} \arrow[r, "\cong", "\sigma/8"'] \arrow[d] & \mathbb{Z} \arrow[d, "1\mapsto(8,1)"'] \arrow[dr] & \\ \Omega_4^{\mathrm{STOP}} \arrow[r, "\cong"] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r, "\mathrm{ks}"] & \mathbb{Z}/2, \end{tikzcd}$$

which commutes by computing on the generator E_8 of $\Omega_4^{\mathrm{TOPSpin}} \cong \mathbb{Z}$, shows that $\mathrm{ks}(M) = \sigma(M)/8 \in \mathbb{Z}/2$ for TOPSpin manifolds M.

This completes the proof of (6) and therefore of Theorem 8.2. □

The following theorem says that the converse to Theorem 8.2 (1) holds.

Theorem 8.3. *If M is a compact, connected 4-manifold with vanishing Kirby-Siebenmann invariant, then $M \times \mathbb{R}$ admits a smooth structure.*

Proof. The vanishing of the Kirby-Siebenmann invariant implies that there is a lift of $\tau_M \colon M \to \mathrm{BTOP}$ to a map $M \to \mathrm{BPL}$. Since PL/O is 6-connected [FQ90, Theorem 8.3B], [HM74, Proof of 4.13], there is in fact a lift $\widetilde{\tau}_M \colon M \to \mathrm{BO}$. This corresponds to a lift $\widetilde{\tau}_M \oplus \varepsilon^n \colon M \to \mathrm{BO}(4+n)$, for some n. This in turn corresponds to a lift

$$\widetilde{\tau}_{M\times\mathbb{R}^n} \colon M \times \mathbb{R}^n \to \mathrm{BO}(4+n)$$

of the tangent microbundle $\tau_{M\times\mathbb{R}^n} \colon M \times \mathbb{R}^n \to \mathrm{BTOP}$. By [KS77, Essay V, Theorem 1.4], there exists a corresponding smooth structure on $M \times \mathbb{R}^n$. Then apply the Product Structure Theorem 5.16 [KS77, Essay I, Theorem 5.1], to deduce the existence of a smooth structure on $M \times \mathbb{R}$, using that the dimension of $M \times \mathbb{R}$ is at least five. □

Example 8.4. Here is an application of Theorem 8.3. By the classification of simply connected, closed 4-manifolds [FQ90, Section 10.1] (see also our Theorem 11.2), there is a simply connected, closed spin 4-manifold N with intersection form $E_8 \oplus E_8$. Since this form is not diagonalisable over $\mathbb{Z}$, by Donaldson's theorem [Don83] (Theorem 3.7) this 4-manifold does not admit a smooth structure. However the Kirby-Siebenmann invariant of N vanishes, since for a closed 4-manifold M with even intersection form, the Kirby-Siebenmann invariant $\mathrm{ks}(M)$ coincides with $\sigma(M)/8 \mod 2$, and $E_8 \oplus E_8$ is rank 16 and positive definite, with signature 16. Therefore $N \times \mathbb{R}$ admits a smooth structure by Theorem 8.3, even though N does not.

Construction 8.5. Here is a construction of the Chern manifold $*\mathbb{C}\mathrm{P}^2$. This manifold was first constructed in [Fre82, p. 370]. Attach a 2-handle $D^2 \times D^2$ to D^4 by identifying $S^1 \times D^2$ with a $+1$-framed trefoil in $\partial D^4 = S^3$. The boundary of the resulting manifold is an integral homology sphere. Freedman proved that every integral homology sphere bounds a contractible 4-manifold [Fre82, Theorem 1.4′], [FQ90, Corollary 9.3C]. Cap off $D^4 \cup D^2 \times D^2$ with this contractible 4-manifold, to obtain the closed 4-manifold $*\mathbb{C}\mathrm{P}^2$. By the Rochlin invariant, every compact, smooth, spin 4-manifold with boundary $+1$-surgery on the trefoil has $\sigma/8$ odd. Therefore the contractible 4-manifold (with signature zero) has Kirby-Siebenman invariant 1. So by Theorem 8.2 (5), we have $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2) = 1$. The Chern manifold $*\mathbb{C}\mathrm{P}^2$ is homotopy equivalent to $\mathbb{C}\mathrm{P}^2$ but is not homeomorphic. For further discussion of the star construction, see [FQ90, Section 10.4] and [Tei97].

The following theorem says in particular that given any compact 4-manifold M there exists a closed orientable simply-connected 4-manifold N such $M\#N$ is smoothable.

Theorem 8.6. *Let M be compact 4-manifold. There exists a closed, orientable, simply connected 4-manifold N such $M\#N$ admits a smooth structure. If moreover the Kirby-Siebenmann invariant of M is zero, then there exists a $k \in \mathbb{N}_0$ such that $M\#_k S^2 \times S^2$ admits a smooth structure.*

Proof. Let M be compact 4-manifold. Perform the connected sum with an appropriate number of copies of $*\mathbb{C}\mathrm{P}^2$, the closed oriented simply-connected 4-manifold with nontrivial Kirby-Siebenmann invariant, homotopy equivalent to $\mathbb{C}\mathrm{P}^2$, constructed on [FQ90, p. 167], in order to obtain a manifold with every connected component having zero Kirby-Siebenmann invariant. It follows from the discussion on [FQ90, p. 164] and the Sum-stable smoothing theorem [FQ90, p. 125], that performing the connected sum with enough copies of $S^2 \times S^2$ produces a manifold that admits a smooth structure. □

Remark 8.7. Given a lift of the classifying map of the (unstable) tangent microbundle of M to $\mathrm{BO}(4)$, Lashof-Shaneson [LS71] showed that there exists a $k \in \mathbb{N}_0$ such that $M\#_k S^2 \times S^2$ admits a smooth structure. The result quoted in the previous proof extended this to a lift of the corresponding *stable* maps.

9. Tubing of surfaces

As an example of the use of the technology we have discussed thus far, we show that one can tube together two locally flat embedded surfaces in a 4-manifold, to obtain an embedding of the connected sum. This operation is standard in the smooth category, but as ever in the topological category one should take some care.

The following situation is by no means the most general such result possible. We wish to illustrate two things. First, that operations on surfaces that can be performed in the smooth category can usually also be performed in general 4-manifolds with locally flat surfaces (although performing these operations in a parametrised way seems to be beyond current knowledge). Second, we want to show the level of detail required to demonstrate that such operations work.

Proposition 9.1 (Tubing)**.** *Let S and T be 2-dimensional proper submanifolds of a connected 4-manifold M, that is S and T are locally flat embedded surfaces. Pick a point $P \in S \setminus \partial S$ and $Q \in T \setminus \partial T$. Let $[\gamma] \in H_1(M, \{P, Q\}; \mathbb{Z})$ be a relative homology class. There is a locally flat embedded arc C joining P and Q, satisfying the following.*

(i) *We have $[C] = [\gamma] \in H_1(M, \{P, Q\}; \mathbb{Z})$.*
(ii) *The interior of C is disjoint from $S \cup T$.*

(iii) *The arc C extends to a neighbourhood $C \times D^2$ embedded in M such that $E_S := \{P\} \times D^2 \subseteq S$ and $E_T := \{Q\} \times D^2 \subseteq T$.*
(iv) *We have $(C \times D^2) \setminus (E_S \cup E_T) \subseteq M \setminus (S \cup T)$.*
(v) *The intersection of $C \times D^2$ with a normal disc bundle $D(S)$ of S is such that for every d, $(C \times \{d\}) \cap D(S)$ is a ray in a single fibre of $D(S)$, and similarly for T. Moreover there is a trivialisation of the normal bundle over E_S as $E_S \times D^2$ such that for every $c \in C$ with $(\{c\} \times D^2) \cap D(S) \neq \varnothing$, we have that $\{c\} \times D^2 = E_S \times \{e\}$ for some $e \in D^2$, and all such e that arise this way lie on a fixed ray from the origin of D^2.*

These data allow us to perform tubing of surfaces ambiently.

Proposition 9.2. *Given data S, T, $C \times D^2$, E_S and E_T as in Proposition 9.1, the subset*

$$(S \setminus \frac{1}{2}E_S) \cup (T \setminus \frac{1}{2}E_T) \cup C \times \frac{1}{2}S^1$$

is a 2-dimensional submanifold abstractly homeomorphic to $S\#T$.

Proof. The surfaces and the tube are locally flat by assumption, or by construction from Proposition 9.1. The circles where the tube is glued to the surface are locally flat points. To see this observe that we have arranged a coordinate system in which this gluing is a completely standard attachment at angle $\pi/2$. □

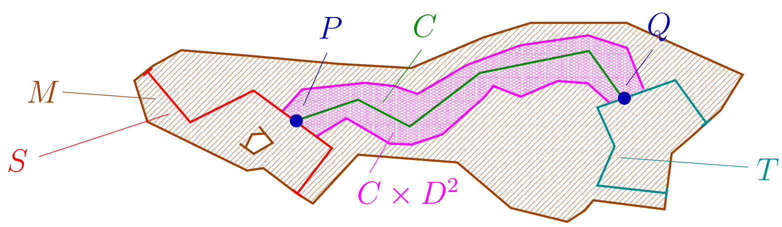

FIGURE 7. Illustration of Proposition 9.1.

Proof of Proposition 9.1. Since S and T are proper submanifolds, they have normal bundles by Theorem 6.17. Pick normal disc bundles $D(S)$ and $D(T)$, and remove the interiors of $\frac{1}{2}D(S)$ and $\frac{1}{2}D(T)$ i.e. smaller disc bundles inside the normal disc bundles. We obtain a manifold with boundary

$$X := M \setminus \left(\operatorname{Int} \tfrac{1}{2}D(S) \cup \operatorname{Int} \tfrac{1}{2}D(T)\right)$$

together with a collar neighbourhood of the boundary arising from $D(S) \setminus \operatorname{Int} \frac{1}{2}D(S)$, and the same with T replacing S, extended using Theorem 2.5 to a collar neighbourhood for all of ∂X. Choose a closed disc neighbourhood E_S of P in S. We write $\partial_S X$ for the fibrewise boundary of $\frac{1}{2}D(S)$, $\partial_T X$ for the fibrewise boundary of $\frac{1}{2}D(T)$, and $\partial_1 X$ for $\partial_S X \cup \partial_T X = \overline{\partial X \setminus \partial M}$.

Choose a trivialisation of the normal bundle νS in a neighbourhood $N(E_S)$ of E_S, as $N(E_S) \times D^2$. A ray in D^2 from the origin to the boundary determines an embedding $E_S \times [0,1] \subset \frac{1}{2}D(S)$. We obtain in particular a disc $E_S \times \{1\} \in N(E_S) \times \{\mathrm{pt}\} \subset N(E_S) \times S^1$. Choose a smooth structure on ∂X (which we may do since ∂X is a 3-manifold), and choose a smoothly embedded neighbourhood $F_S \cong D^3$ in $\partial_S X$ that contains $E_S \times \{1\}$ in its interior.

Make the analogous set of choices and constructions for T, to obtain E_T, $N(E_T)$, $E_T \times [0,1] \subset \frac{1}{2}D(T)$, and $F_T \cong D^3$ in $\partial_T X$ that contains $E_T \times \{1\}$ in its interior.

Remove a point r from X, and using Theorem 8.1 choose a smooth structure on $X \setminus \{r\}$ extending the chosen smooth structure on ∂X. Choose a smoothly embedded path $C_X \subset X$ between the centres of $E_S \times \{1\}$ and $E_T \times \{1\}$, such that C_X extends along the previously chosen rays inside the normal bundles to a path C between P and Q such that $[C] = [\gamma] \in H_1(M, \{P, Q\}; \mathbb{Z})$. Extend C_X to a codimension zero submanifold $N(C_X)$ homeomorphic to $I \times D^3$, with $I \times \{0\} \subset I \times D^3$ mapping to C_X, and such that $\{0\} \times D^3$ maps to $F_S \subseteq \partial_S X$ and $\{1\} \times D^3$ maps to $F_T \subset \partial_T X$.

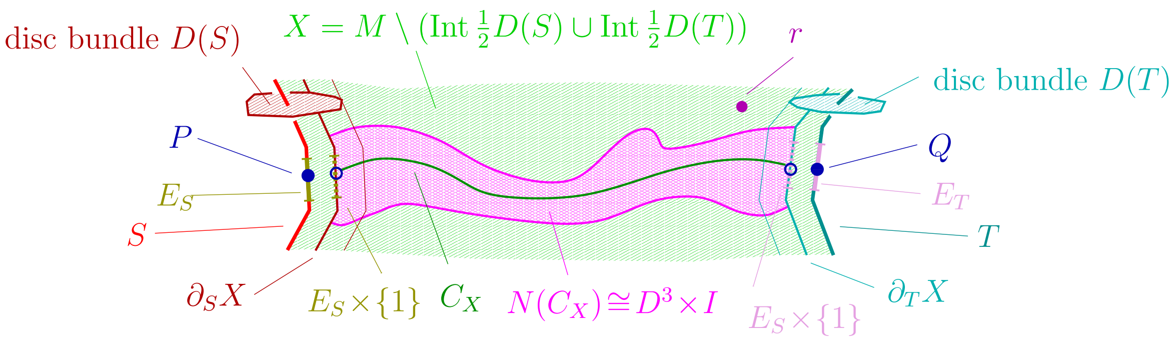

FIGURE 8. Illustration for the proof of Proposition 9.1.

Now, for small ε, $[0, \varepsilon] \times D^3$ and $[1-\varepsilon, 1] \times D^3$ give rise to collar neighbourhoods of the closed subsets F_S and F_T of $\partial_1 X$. Use Theorem 2.5 to extend this collar neighbourhood to a collar neighbourhood over all of ∂X.

We now have two collar neighbourhoods of ∂X, the collar $\Psi_1 \colon \partial X \times [0,1] \hookrightarrow X$ we have just constructed which is compatible with $N(C_X)$, and the collar neighbourhood $\Psi_2 \colon \partial X \times [0,1] \hookrightarrow X$ constructed above from $D(S) \setminus \mathrm{Int}\,\frac{1}{2}D(S)$ and $D(T) \setminus \mathrm{Int}\,\frac{1}{2}D(T)$. By Theorem 2.7, there is an isotopy $H_t \colon M \to M$ starting from the identity, such that $H_1 \circ \Psi_1 = \Psi_2$, i.e. sends the first collar to the second.

We now obtain a codimension zero submanifold $C_X \times D^3$ homeomorphic to $I \times D^3$ such that, with respect to the collar neighbourhood Ψ_2, we have:

- For all $c \in C_X$ such that $\{c\} \times D^3 \cap \Psi_2(\partial X \times [0,1]) \neq \varnothing$, we have that $\{c\} \times D^3 \subset \Psi_2(\partial X \times \{t\})$ for some $t \in [0,1]$.

- For every $d \in D^3$, $(C \times \{d\}) \cap (\partial X \times [0,1]) = \Psi_2(\{x\} \times [0,1])$ for some x in either F_S or F_T.

In addition, above we constructed two discs $E_S \subset F_S$ and $E_T \subset F_T$. Any two embedded discs in a 3-ball are ambiently isotopic: place this isotopy inside $C_X \times D^3$ to obtain a locally flat embedding $C_X \times D^2 \cong I \times D^2 \subset C_X \times D^3$.

Now consider $X \subset M$ and take the union

$$(E_S \times [0,1]) \cup (C_X \times D^2) \cup (E_T \times [0,1]) \;\subseteq\; M$$

to obtain an embedding $C \times D^2 \cong I \times D^2$ whose intersection with S equals E_S and whose intersection with T equals E_T. The core $C = C \times \{0\}$ is of course a locally flat embedded path in M from P to Q with interior in $M \setminus (S \cup T)$ and with the correct relative homology class in $H_1(M, \{P,Q\}; \mathbb{Z})$. We may then perform the tubing $S\#T := (S \setminus E_S) \cup (T \setminus E_T) \cup C \times S^1$ as promised. $\square$

10. Topological transversality

We turn to the subject of transversality in the topological category. Some discussion of this concept is in order. There are two important contexts for transversality: submanifold transversality and map transversality. In this article, map transversality will be deduced from submanifold transversality. Submanifold transversality when none of the manifolds involved has dimension 4 is due to to Marin [Mar77]; cf. [KS77, Essay III, Section 1]. Transversality in the remaining cases is due to Quinn [Qui82, Qui88]; see also [FQ90, Section 9.5].

A naive definition of submanifold transversality in the topological category is that manifolds are *locally transverse* if around any intersection point there is a chart in which the submanifolds appear as perpendicular planes. On the other hand, there are examples (in the relative setting) of submanifolds which cannot be made locally transverse via ambient isotopy; see Remark 10.4. Thus one cannot generally use this definition.

In light of this, in order to make general statements, one passes to some notion of *global transversality*. Global transversality means that transversality statements are made with respect to a given choice of normal structure on one of the submanifolds involved. Of course, this forces one to engage with the question of existence and uniqueness of whatever normal structure is used, and the 'correct' choice of normal structure is still not fully settled in the topological category. We refer the reader to [FQ90, Sections 9.4, 9.6C] for a brief discussion of the competitors.

The most general statement of transversality [Qui88, Theorem] uses microbundles to describe normal structure, and this is the technology we will use. As discussed in Section 7, for general manifolds, tangent microbundles always exist but normal microbundles do not (see Example 7.13). The case of dimension 4 is special, since here the normal vector bundles of Section 6.3, which are a stronger notion than normal microbundles, always exist. In fact, the results obtained for these normal vector bundles in dimension 4 are a strong enough to ensure that submanifold transversality holds in ambient dimension 4 with the

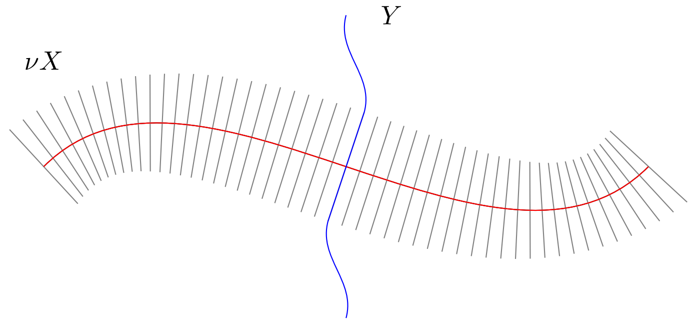

FIGURE 9. Sketch of a transverse intersection of Y to νX.

naive, local transversality definition discussed above. The reader may therefore wonder why we even introduce normal microbundles into a discussion of 4-manifold transversality. The answer is that the 'submanifold transversality implies map transversality' argument of Section 10.2 requires a bundle technology that works in all dimensions, and microbundles appear to be the most convenient.

10.1. **Transversality for submanifolds.**

Definition 10.1. Consider proper submanifolds X, Y of an ambient manifold and a normal microbundle νX for X, with retraction $r_X \colon E(\nu X) \to X$. The proper submanifold Y is *transverse to* νX if there exists a neighbourhood $U \subset E(\nu X)$ of X such that $Y \cap U = r_X^{-1}(X \cap Y) \cap U$.

Lemma 10.2. *Let X, Y submanifolds of M. Let Y be transverse to a normal microbundle νX. Then $X \cap Y$ is a submanifold of Y with normal microbundle $(\nu X)|_{X \cap Y}$.*

Proof. Once we have established that $(\nu X)|_{X\cap Y}$ is a normal microbundle of $X \cap Y$ in Y, the subspace $X \cap Y$ will automatically be a submanifold since the trivialisation of the microbundle $(\nu X)|_{X\cap Y}$ gives the required charts for $X \cap Y$.

At least after shrinking the total space of $E(\nu X)$, each fibre $r^{-1}(x)$ for $x \in X \cap Y$ will be contained in Y by the definition of transversality. That is $E((\nu X)|_{X\cap Y})$ is a subset of Y and neighbourhood of $X \cap Y$. This shows that $(\nu X)|_{X\cap Y}$ is a normal microbundle of $X \cap Y \subset Y$. $\square$

Transversality in high dimensions is due to to Marin [Mar77], cf. [KS77, Essay III, Section 1]. The formulation below is from Quinn [Qui88]. Recall Definition 6.1 of a proper submanifold. Note that in the next theorem there is no restriction on dimensions. The manifold M is allowed to be noncompact and have nonempty boundary.

Theorem 10.3 (Transversality for submanifolds)**.** *Let X and Y be proper submanifolds of a compact manifold M. Let νY be a normal microbundle for Y. Let C be a closed subset such that X is transverse to νY in a neighbourhood of C. Let U be a neighbourhood of* *the set* $\big(M \setminus C\big) \cap X \cap Y$. *Then there exists an isotopy of* X *supported in* U *to a proper submanifold* X' *such that* X' *is transverse to* νY.

Proof. See Quinn [Qui88] for all cases but $\dim M = 4$, $\dim X = 2$ and $\dim Y = 2$. For the remaining case, first establish local transversality using [FQ90, Section 9.5]. Note that $X \cap Y$ is a discrete collection of points. Therefore, the coordinate chart, witnessing local transversality, defines a normal neighbourhood of Y near $X \cap Y$. This normal vector bundle can be extended to a normal vector bundle $\nu Y'$ on all of Y by [FQ90, Theorem 9.3A]. The submanifold X is now transverse to $\nu Y'$, but (possibly) not to νY. By Theorem 7.14, our microbundle νY comes from a normal vector bundle. By uniqueness of normal vector bundles (Theorem 6.20), there is an isotopy from $\nu Y'$ to νY. Apply this isotopy to X. Now X is transverse to νY. □

Remark 10.4. The analogous statement to Theorem 10.3 is false for local transversality. Examples of this failure even exist in the PL category: Hudson [Hud69] constructs, for certain large n, closed PL submanifolds $X, Y \subset \mathbb{R}^n$, that are topologically unknotted Euclidean spaces of codimension ≥ 3, in such a way that X and Y are PL locally transverse near a closed neighbourhood K of infinity but also so that it is impossible to move X and Y by isotopy relative to K to make them PL locally transverse everywhere.

Although transversality for submanifolds (Theorem 10.3) is only stated for a pair of submanifolds, it can be used to make collections of submanifolds transverse.

Lemma 10.5. *Let* M *be an* $2m$*-dimensional manifold for* $m \geq 1$, *and let* $X_1, \dots, X_n$ *be* m*-dimensional compact submanifolds with normal microbundles* νX_i. *Then the submanifolds* X_i *can be isotoped such that there are no triple intersection points and the submanifolds intersect* (*pairwise*) *transversely.*

Proof. We give a proof by induction. When $n = 1$, there is nothing to show, since every submanifold is embedded. For the inductive step, denote X_n by Y. The inductive hypothesis states that we can isotope any $n-1$ submanifolds $X_1, \dots, X_{n-1}$ so that there are no triple points and that they intersect pairwise transversely. We will prove that the submanifolds $X_1, \dots, X_{n-1}$ can be further isotoped so that they are transverse to νY and Y is free of triple points. Note that having no triple points on Y implies that there exists an open set U_Y such that: $X_i \cap X_j \subset U_Y$ for $1 \leq i < j \leq n-1$, and $M \setminus U_Y$ is a neighbourhood of Y. To obtain the lemma apply the inductive hypothesis, picking all further isotopies to be supported in U_Y.

We proceed by showing the inductive step: we can isotope every X_i to be transverse to νY such that no triple points lie on Y. For each $i = 1, \dots, n-1$, apply Theorem 10.3 to arrange that Y and X_i intersect transversely. By compactness of the submanifolds, the subset

$$T_Y = Y \cap (X_1 \cup \cdots \cup X_{n-1})$$

is compact. Pick disjoint open neighbourhoods $V_y \subset Y$ around each point $y \in T_Y$. Pick a chart ϕ of Y around $\phi(0) = y$ contained in V_y, and a microbundle chart around $y \in Y$. In

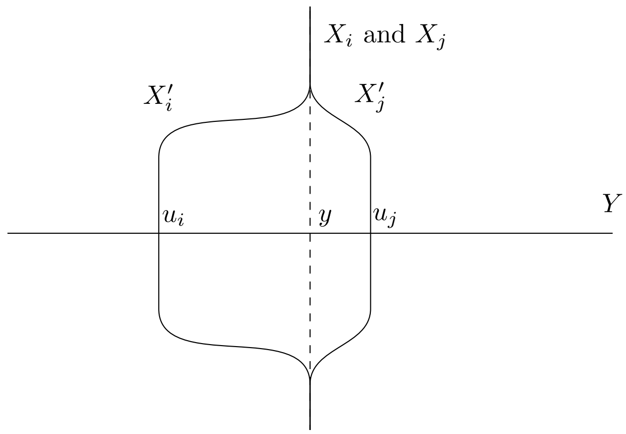

FIGURE 10. Displacing a triple point y in a microbundle chart.

the local model, Y corresponds to $\mathbb{R}^m \times \{0\}$ and the X_i that intersect Y in y will be mapped to $0 \times \mathbb{R}^m$. For those X_i, pick disjoint points $u_i \in \mathbb{R}^m$ (here we use $m > 0$), and pick a continuous function $\eta\colon \mathbb{R}_{\geq 0} \to [0,1]$ with $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. Replace X_i in the chart with the image of

$$\begin{aligned} \mathbb{R}^m &\to \mathbb{R}^m \times \mathbb{R}^m \\ v &\mapsto \big(\eta\big(\|v\|\big)u_i, v\big). \end{aligned}$$

Call this new submanifold X'_i. It agrees with X_i outside the ball of radius 2, and is isotopic to X_i. In V_y, the submanifold X'_i intersects Y only in $\phi(u_i)$ and there it intersects Y transversely with respect to νY. The collection $\{X'_i\}$ has no triple intersection points in the set V_y anymore. □

Here is another result on submanifold transversality. It might often happen that one can find a continuous map of, for example, a disc D^2 into a 4-manifold M, perhaps if fundamental group computations yield a null homotopy of a circle. Then this disc can be perturbed to locally flat immersion. If M were smooth, this would be a consequence of standard differential topology, an observation that we leverage.

Theorem 10.6. *Let Σ be a connected 1 or 2 dimensional manifold and let $f\colon \Sigma \to X$ be a continuous map of Σ into a connected 4-manifold X. Let C be a closed subset of Σ such that $f|_C$ is a locally flat immersion. Then there is a perturbation of f to a locally flat immersion $f'\colon \Sigma \to X$ with $f|_C = f'|_C$.*

Proof. Perturb f to a map that misses a point. Then smooth X in the complement of that point and C using Theorem 8.1. Now by [Hir94, Theorem 2.2.7] we can perturb f to a smooth immersion, which we can then perturb to be self-transverse f' by [Hir94, Theorem 4.2.1], [Wal16, Theorem 4.6.6], with no triple points by general position [Wal16, Theorem 4.7.7]. Now add back in the point: f' is a locally flat immersion. □

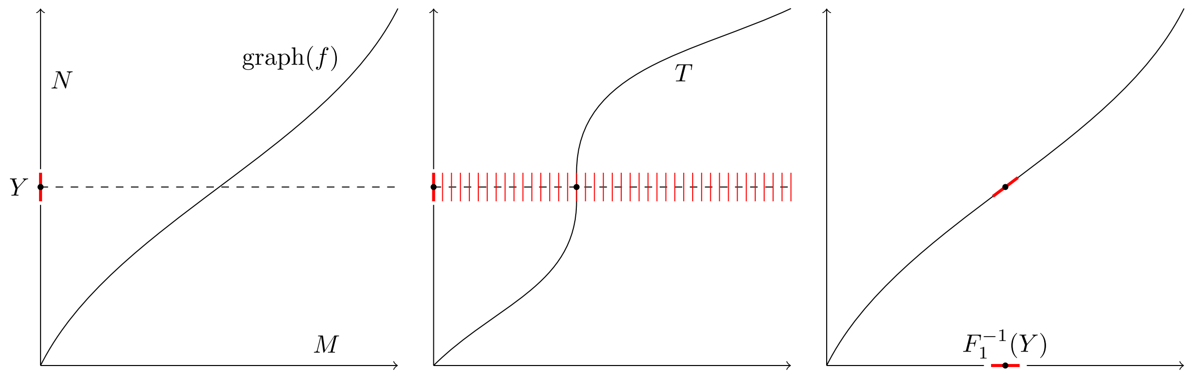

FIGURE 11. Transversality for maps from transversality for submanifolds.

We are not sure how to give a purely topological proof of this.

10.2. **Transversality for maps.**

Definition 10.7. Let $f\colon M \to N$ be a continuous map and let X be a submanifold of N with normal microbundle νX. The map f is said to be *transverse* to νX if $f^{-1}(X)$ is a submanifold admitting a normal microbundle $\nu f^{-1}(X)$ and

$$\begin{aligned} f\colon \nu f^{-1}(X) &\to f^*\nu X \\ m &\mapsto (r(m), f(m)) \end{aligned}$$

is an isomorphism of microbundles.

In the next theorem, we show how to reduce transversality for maps to transversality for submanifolds. Again, there are restrictions neither on dimensions nor codimensions.

Theorem 10.8. *Let $Y \subset N$ be a proper submanifold with normal microbundle νY. Let $f\colon M \to N$ be a map, and let U be a neighbourhood of the set*

$$\operatorname{graph} f \cap (M \times Y) \subset M \times N.$$

Then there exists a homotopy $F\colon M \times I \to N$ such that

1. *$F(m, 0) = f(m)$ for all $m \in M$;*
2. *$F_1\colon m \mapsto F(m, 1)$ is transverse to νY; and*
3. *for $m \in M$ either*
   - *(a) $(m, f(m)) \notin U$, in which case $F(m, t) = f(m)$ for all $t \in I$, or*
   - *(b) $(m, f(m)) \in U$, in which case $(m, F(m, t)) \in U$ for all $t \in I$.*

Proof. Note that $M \times Y \subset M \times N$ is a proper submanifold with normal microbundle $M \times \nu Y = \mathrm{pr}_Y^* \nu Y$. Also $\operatorname{graph} f$ is a proper submanifold of $M \times N$. By Theorem 10.3, there exists an isotopy

$$G\colon \operatorname{graph} f \times I \to M \times N,$$

supported in U, of the submanifold $\operatorname{graph} f$ to a submanifold $T \subset M \times N$ such that T is transverse to $M \times \nu Y$ over $M \times Y$.

Define the map F as the composition

$$F\colon M \times I \to \operatorname{graph} f \times I \xrightarrow{G} M \times N \xrightarrow{\mathrm{pr}_N} N.$$

Since the isotopy G is supported in U, statement (3) holds. By construction, $F(x, 0) = \mathrm{pr}_Y(x, f(x)) = f(x)$, which proves statement (1).

Now we prove statement (2). Let $F_1\colon M \to N$ be the map that sends $x \mapsto F(x, 1)$. We keep track of the preimages through the maps of the composition that defines F_1; see Figure 11. By transversality of T to $M \times \nu Y$, we see that $Z = T \cap (M \times Y) = \mathrm{pr}^{-1}(Y)$ is a submanifold of T with normal bundle $M \times \nu Y|_Z$, and that the projection to N induces a microbundle isomorphism $M \times \nu Y|_Z \xrightarrow{\sim} \mathrm{pr}_N^* \nu Y$. By definition, $\mathrm{pr}_N\colon T \to N$ is transverse to νY.

We transport the submanifold Z back to M. Consider the commutative diagram

$$\begin{tikzcd}
 & & N \\
M \arrow[r, "\cong"] \arrow[urr, bend left, "F_1"] \arrow[rr, bend right, "q"'] & \operatorname{graph} f \times \{1\} \arrow[r, "\cong"] & T \arrow[u, "\mathrm{pr}_N"']
\end{tikzcd},$$

where q is the composition, which is a homeomorphism. Now $F_1^{-1}(Y) = q^{-1}(Z)$ is a submanifold with normal microbundle

$$q^*\big(M \times \nu Y|_Z\big) = q^* \mathrm{pr}_N^* \nu Y = F_1^* \nu Y,$$

that is $F_1\colon M \to N$ is transverse to νY. $\square$

10.3. **Representing homology classes by submanifolds.** Our goal in this section is to prove the following theorem.

Theorem 10.9. *Let X be a compact orientable 4-manifold and let A be a union of components of ∂X. Let $k = 2$ or $k = 3$ and let $\sigma \in H_k(X, A; \mathbb{Z})$.*

1. *The class σ can be represented by a k-dimensional submanifold Y with $\partial Y \subset A$.*
2. *In the case $k = 3$, the boundary of Y can be specified: if $B \subset A$ is an oriented closed 2-dimensional smooth submanifold contained in A such that $\partial(\sigma) = [B] \in H_2(A)$, then σ can be represented by an oriented compact 3-dimensional submanifold Y with $\partial Y = B$.*

The submanifold B can be assumed to be smooth, since ∂X is a 3-manifold and so has a unique smooth structure by [Moi52], [Moi77, p. 252–253].

Note that Theorem 10.9 also holds for $k = 0$ and $k = 1$. This is trivial for $k = 0$. To see this for $k = 1$, remove a point from each connected component to get a smooth 4-manifold by Theorem 8.1. Note that $H_1(X \setminus \{\mathrm{pt}\}, A; \mathbb{Z}) \cong H_1(X, A; \mathbb{Z})$. Then by smooth approximation and general position, every 1-dimensional homology class can be represented by a 1-dimensional submanifold of X.

Example 10.10.

(1) If we apply the theorem to $A = \partial X$, we see that any homology class in $H_2(X, \partial X; \mathbb{Z})$ and $H_3(X, \partial X; \mathbb{Z})$ can be represented by a properly embedded submanifold. For $A = \varnothing$ we obtain the analogous statement for absolute homology groups.

(2) Let F be a properly embedded 2-dimensional submanifold of D^4 and let S be a surface in $\partial D^4 = S^3$ with $\partial S = \partial F$. Consider the 4-manifold $X := D^4 \setminus \nu F$. In the boundary of X we have the surface $B = S \cup F \times \{1\}$. It follows from the long exact sequence of the pair $(X, \partial X)$ and Poincaré duality that the map $H_3(X, \partial X; \mathbb{Z}) \to H_2(\partial X; \mathbb{Z})$ is an epimorphism. It follows from Theorem 10.9, applied to $A = \partial X$, that there exists a 3-dimensional submanifold Y of $D^4 \setminus \nu F$ with $\partial Y = B$. This statement is folklore, and a proof using topological transversality for maps was recently written down by Lewark-McCoy [LM15].

We will provide two proofs for Theorem 10.9. The first one use Theorem 8.1 to reduce the statement to the smooth case, and the second one uses the topological transversality arguments from Section 10.

For $n = 4$ the statement of the following theorem is precisely the statement of Theorem 10.9 in the smooth category.

Proposition 10.11. *Let X be a compact orientable smooth n-manifold and let A be a union of components of ∂X. Let $\ell = 1$ or $\ell = 2$ and let $\sigma \in H_{n-\ell}(X, A; \mathbb{Z})$. Then the following holds.*

(1) *The class σ is represented by an $(n - \ell)$-dimensional smooth orientable submanifold Y with $\partial Y \subset A$.*

(2) *Suppose $\ell = 1$ and suppose we are given a oriented closed $(n - 2)$-dimensional smooth submanifold of A such that $\partial(\sigma) = [B] \in H_{n-2}(A)$, then σ is represented by an oriented compact $(n - 1)$-dimensional smooth submanifold Y with $\partial Y = B$.*

Example 10.12. Let $K \subset S^3$ be a knot. We write $X = S^3 \setminus \nu K$. Let $\lambda \subset \partial X$ be a longitude of K. There exists a homology class $\sigma \in H_2(X, \partial X; \mathbb{Z})$ with $\partial(\sigma) = [\lambda]$. It follows from Proposition 10.11 that there exists an orientable surface F in X with $\partial F = K$.

Proof. Let X be a compact orientable smooth n-manifold and let A be a union of components of ∂X. First we deal with the case $\ell = 1$. Let $\sigma \in H_{n-1}(X, A; \mathbb{Z})$. Write $\widetilde{A} = \partial X \setminus A$. Let PD: $H^1(X, \widetilde{A}; \mathbb{Z}) \to H_{n-1}(X, A; \mathbb{Z})$ be the Poincaré duality isomorphism. We have $H^1(X, \widetilde{A}; \mathbb{Z}) \cong [X/\widetilde{A}, S^1]$ and any such class can be represented by a continuous map $\varphi\colon X \to S^1$ that is constant on $\widetilde{A}$, and uniquely determined up to homotopy rel. $\widetilde{A}$. We can and shall homotope φ to a smooth map. Furthermore, arrange that $-1 \in S^1$ is a regular value of φ. Then $\varphi^{-1}(-1)$ is an $(n - 1)$-dimensional submanifold whose boundary lies on $\partial X \setminus \widetilde{A}$, that is the boundary lies on A. This manifold has the desired property.

Now suppose that we are given an oriented closed $(n-2)$-dimensional submanifold B of A such that $\partial(\sigma) = [B] \in H_{n-1}(A)$. Pick a collar neighbourhood $\partial X \times [0,1]$ and choose a continuous map $\varphi\colon X \setminus \big(\partial X \times [0,1)\big) \to S^1$ as above. Also choose a tubular neighbourhood $B \times [-1/2, 1/2]$ of B in A. Consider the map sending (b,t) to $e^{\pi i(t-1)}$, $b \in B, t \in [-1/2, 1/2]$ and extend it to a smooth map $\psi\colon A \to S^1$ by sending all other points into $\{e^{\pi i t} \in S^1 \colon t \in [-2/3, 2/3]\}$. Since $\partial(\sigma) = [B] \in H_{n-2}(A) \cong H^1(A;\mathbb{Z})$, we see that the restriction of φ to $A \times \{1\} = A$ is homotopic to $\psi\colon A \to S^1$. Therefore, using this homotopy in the interval $[\frac{1}{2}, 1]$, we can extend φ to a function on X that restricts to ψ on each $A \times \{s\}$ with $s \in [0, \frac{1}{2}]$. Finally, smoothen φ without changing it on $A \times [0, \frac{1}{4}]$ to obtain a smooth map $X \to S^1$ in the same homotopy class. This is possible since the original φ was already smooth on $A \times [0, \frac{1}{2}]$. Put differently, the new smooth map $\varphi\colon X \to S^1$ restricts to ψ on $A = A \times \{0\}$.

Note that -1 is a regular value of ψ, and by changing φ outside $A \times [0, \frac{1}{4}]$, we can also arrange -1 to be a regular value of φ. The manifold $Y = \varphi^{-1}(-1)$ satisfies $[Y] = \sigma$ (this follows from Lemma 10.17 below, as explained in the second proof of Theorem 10.9) and $\partial Y = B \times \{0\} = B$.

For $\ell = 2$ the argument is similar: we have to replace the argument using S^1 by the argument of [GS99, Proposition 1.2.3]. Recall from Theorem 4.5 that X is homotopy equivalent to finite CW complex. Therefore, we represent a codimension 2 homology class $\sigma \in H_{n-2}(X, \partial X; \mathbb{Z}) \cong H^2(X;\mathbb{Z})$ by a map $X \to \mathbb{CP}^\infty$, and homotope into the k-skeleton to a map $f\colon X \to \mathbb{CP}^k$ for $k \geq 2$. Now arrange f to be transverse to the codimension 2 submanifold $\mathbb{CP}^{k-1} \subset \mathbb{CP}^k$. The desired submanifold is the preimage $Y = f^{-1}(\mathbb{CP}^{k-1})$. We leave further details to the reader. □

Lemma 10.13. *Let W be a smooth n-manifold and let C be a compact subset. There exists a compact smooth n-dimensional submanifold X of W that contains C.*

Proof. By the Whitney embedding theorem (see e.g. [Lee11, Theorem 6.15]), there exists a proper embedding $f\colon W \to \mathbb{R}^{2n+1}$. Recall that in this context proper means that the preimage of a compact set is compact. Pick a point $P \in \mathbb{R}^{2n+1}$ that does not lie in the image of f. Denote the Euclidean distance to the point P by $d\colon \mathbb{R}^{2n+1} \to \mathbb{R}_{\geq 0}$. This map is smooth outside P, so in particular $d \circ f\colon W \to \mathbb{R}_{\geq 0}$ is smooth. Since C is compact, there exists an $r \in \mathbb{R}_{\geq 0}$ such that $(d \circ f)(C) \subset [0, r]$. By Sard's theorem, there exists a regular value $x > r$. Then $X := (d \circ f)^{-1}([0, x])$ has the desired properties. □

First proof of Theorem 10.9. Let M be a compact orientable connected 4-manifold and let A be a union of components of ∂M. Let $k = 2$ or $k = 3$ and let $\sigma \in H_k(X, A; \mathbb{Z})$.

Pick a point $P \in M \setminus \partial M$ and pick an open ball $B \subset M \setminus \partial M$ containing P. It follows from a Mayer-Vietoris argument applied to $M = (M \setminus \{P\}) \cup B$ that the inclusion induced map $H_k(M \setminus \{P\}, A) \to H_k(M, A)$ is an isomorphism for $k = 2, 3$.

Now let $\sigma \in H_k(M, A)$. By the previous paragraph we can view σ as an element in $H_k(M \setminus \{P\}, A)$. By Theorem 8.1 the manifold $M \setminus \{P\}$ is smooth. There exists a compact subset K of $M \setminus \{P\}$ such that σ lies in the image of $H_k(K, A) \to H_k(M \setminus \{P\}, A)$, since one can take the union of the images of the singular simplices in a singular chain representing σ. By Lemma 10.13, there exists a compact 4-dimensional smooth submanifold X of $M \setminus \{P\}$ that contains the compact set $K \cup \partial M$. Note that A is again a union of components of ∂X. The desired statement of Theorem 10.9 is now an immediate consequence of Proposition 10.11 (1), with σ the image of $\sigma \in H_k(K, A)$ under the inclusion induced map to $H_k(X, A)$. □

Now we collect the tools to conduct the proof of Theorem 10.9 in the topological category. We use transversality for submanifolds as a black box, but we endeavour to provide all the other details.

Definition 10.14. Let $\xi = S \xrightarrow{i} E \xrightarrow{\pi} S$ be a k-dimensional microbundle over S. A *Thom class* of ξ is class $\tau(\xi) \in H^k(E, E \setminus i(S); \mathbb{Z})$ that restricts to generator $H^k(E_x, E_x \setminus i(x); \mathbb{Z}) \cong H_k(\mathbb{R}^k; \mathbb{R}^k \setminus \{0\}; \mathbb{Z}) = \mathbb{Z}$ for all $x \in S$. The microbundle ξ together with a Thom class is an *oriented microbundle.*

Remark 10.15. As in the smooth case, consider the orientation bundle $\pi\colon \operatorname{Or}(\xi) \to S$ with fibre over $x \in S$ the discrete set

$$\operatorname{Or}(\xi)_x = \big\{\text{primitive classes of } H_k(E_x, E_x \setminus i(x); \mathbb{Z})\big\}.$$

This is a $\mathbb{Z}/2$-principal bundle, and a Thom class $\tau(\xi)$ determines a global section $s \in \Gamma(\operatorname{Or}(\xi))$ by enforcing $\langle \tau(\xi), s(x)\rangle = 1$ for every $x \in S$. By the same equation, a global section $\Gamma(\operatorname{Or}(\xi))$ determines a Thom class.

Remark 10.16. Let X be an oriented manifold. Let S be a submanifold with normal microbundle νS. An orientation of S determines a unique Thom class of νS compatible with the ambient orientation and vice versa.

To prove Theorem 10.9, we will consider a map $f\colon X \to Y$ that is transverse to a submanifold S. By the Remark 10.16, νS is an oriented microbundle and carrying the Thom class τ. Note as $f\colon \nu f^{-1}(S) \to f^*\nu S$ is an isomorphic, also $f^*\tau$ is a Thom class of $\nu f^{-1}(S)$ and we orient $f^{-1}(S)$ accordingly. Before we proceed with the proof, we recall the following compatibility between Thom classes and Poincaré duality [Bre97, Definition 11.1, Corollary 11.6], interpreted for microbundles.

Lemma 10.17. *Let X be a compact oriented manifold with fundamental class $[X]$, and let $i\colon S \to X$ be an oriented k-dimensional submanifold of X with normal microbundle νS. The composition*

$$H^{n-k}\big(\nu S, \nu S \setminus i(S)\big) \to H^{n-k}(X, X \setminus S) \to H^{n-k}(X) \xrightarrow{\mathrm{PD}_X} H_k(X)$$

maps the Thom class τ of νS to the fundamental class $i[S]$.

Proof. Recall that the Poincaré duality map PD_X is just $\cap[X]$, capping with the fundamental class. The composition of the last two maps factors through $i\colon H_k(S;\mathbb{Z}) \to H_k(X;\mathbb{Z})$ by Poincaré-Lefschetz duality [Bre97, Corollary 8.4]. Recall that the fundamental class $[S] \in$

$H_k(S,\partial S;\mathbb{Z})$ of the submanifold S is characterised by the property that for all $x \in S$ the class $[S]$ is sent to the positive generator of $H_k(S, S\setminus\{x\};\mathbb{Z}) \cong \mathbb{Z}$.

Pick an $x \in S$ and a k-disc $U \subseteq S$ containing x such that the microbundle νS determines a trivialisation $E(U) = \nu S|_U = U \times D^{n-k}$. The fundamental class $[X]$ restricts to a fundamental class $[E(U)] \in H_n(E(U), \partial E(U))$, and Poincaré duality can be computed locally:

$$\begin{tikzcd}
H^{n-k}(X, X\setminus S) \arrow[r, "\cap[X]"] \arrow[dr] \arrow[dd, "\mathrm{res}"] & H_k(X) \\
& H_k(\nu S) \arrow[u] \arrow[d] \\
H^{n-k}(E(U), E(U)\setminus S) \arrow[r, "\cap[E(U)]"] \arrow[dr, "\phi"] & H_k(E(U), (\partial U)\times D^{n-k}) \arrow[d, "\cong"] \\
& H_k(U, \partial U).
\end{tikzcd}$$

We have to show that ϕ maps the Thom class τ_U to the positive generator. For this, we claim the following compatibility of the Künneth isomorphism with Poincaré duality of products.

Claim. The diagram below commutes:

$$\begin{tikzcd}
H^{n-k}(U\times D^{n-k}, U\times \partial D^{n-k}) \arrow[d, "\mathrm{PD}_{E(D_x)}"] & H^0(U)\otimes H^{n-k}(D^{n-k}, \partial D^{n-k}) \arrow[l] \arrow[d, "\mathrm{PD}_U \otimes \mathrm{PD}_{D^{n-k}}"] \\
H_k(U\times D^{n-k}, \partial U \times D^{n-k})) & H_k(U,\partial U)\otimes H_0(D^{n-k}) \arrow[l]
\end{tikzcd}.$$

We rewrite the diagram in terms of cap products and fundamental classes. Note that the fundamental class $[U\times D^{n-k}] = [U]\times[D^{n-k}]$ is a cross product. Using compatibility of the cross product with the cap product [Dol95, Section VII.12.17], we deduce that for arbitrary $\alpha \in H^0(U;\mathbb{Z})$, and $\beta \in H^{n-k}(D^{n-k}, \partial D^{n-k};\mathbb{Z})$ the equality

$$\big(\alpha\times\beta\big)\cap\big([U]\times[D^{n-k}]\big) = (-1)^{(n-k)\cdot|\alpha|}\Big(\big(\alpha\cap[U]\big)\times\big(\beta\cap[D^{n-k}]\big)\Big)$$

holds, where $\alpha\times\beta = \mathrm{pr}_U^*\,\alpha \cup \mathrm{pr}_{D^{n-k}}^*\,\beta$ denotes the external cohomological product. The horizontal arrows in the diagram above are exactly given by the cross product in homology, and the commutativity is exactly the formula above. This shows the claim.

We deduce that $\mathrm{PD}_{E(U)}\,\tau_U = [U]\otimes\mathrm{pt}$ since $\tau_U = \mathrm{pt}\otimes\vartheta$, where $\vartheta \in H^{n-k}(D^{n-k}, \partial D^{n-k})$ is the generator that evaluates positively on the orientation class. This implies that $\phi(\tau_U) = [U] \in H_k(U,\partial U;\mathbb{Z})$ is the positive generator, as desired. $\square$

We give a proof of the main theorem above in the case of the codimension 1 and for a compact oriented manifold X, with $A = \partial X$.

Second proof of Theorem 10.9. Let $\alpha \in H^1(X;\mathbb{Z})$ be the Poincaré dual to $\sigma \in H_3(X,\partial X)$. Recall the following correspondence between homotopy classes of maps to Eilenberg-Maclane spaces and cohomology classes of X,

$$\begin{aligned}[X,S^1]=[X,K(\mathbb{Z},1)] &\xrightarrow{\cong} H^1(X;\mathbb{Z})\\ f &\mapsto f^*\theta,\end{aligned}$$

where θ is the Hom dual of the fundamental class of S^1. Note that we used here that X is homotopy equivalent to a CW complex. Pick an arbitrary point $\mathrm{pt} \in S^1$ and denote a tubular neighbourhood by $\nu(\mathrm{pt})$. Note that the Thom class τ_{pt} for $\nu(\mathrm{pt})$ is mapped under $H^1(S^1,S^1\setminus \mathrm{pt}) \to H^1(S^1)$ to θ. Let $f\colon X\to S^1$ be a map corresponding to α, so $f^*\theta=\alpha$. Make f transverse to a tubular neighbourhood of $\mathrm{pt}\in S^1$ using Theorem 10.8. Consequently, $S:=f^{-1}(\mathrm{pt})$ is an $(n-1)$-dimensional submanifold of X. By definition, f induces an isomorphism $f\colon \nu S\to f^*\nu(\mathrm{pt})$. We have, as elements in $H^1(X;\mathbb{Z})$, that

$$\alpha = f^*\theta = f^*\,\mathrm{PD}_X^{-1}[\mathrm{pt}] = f^*\tau_{\mathrm{pt}}.$$

Since $f^*\tau_{\mathrm{pt}}$ is the Thom class of νS, apply Lemma 10.17 and obtain

$$\alpha = f^*\tau_{\mathrm{pt}} = \tau_S = \mathrm{PD}_X^{-1}[S].$$ □

Remark 10.18. We have seen that $f^![\mathrm{pt}] = \mathrm{PD}_X \circ f^* \circ \mathrm{PD}_{S^1}^{-1}[\mathrm{pt}] = [f^{-1}(\mathrm{pt})]$, when f is transverse to pt.

11. Classification results for 4-manifolds

It is well-known (e.g. [CZ90, Theorem 5.1.1]) that any finitely presented group is the fundamental group of a closed orientable smooth 4-manifold. Markov [Mar58] used this fact to show that closed 4-manifolds cannot be classified up to homeomorphism. To circumvent this group theoretic issue one aims to classify 4-manifolds with a given isomorphism type of a fundamental group.

In this section we present the known 4-manifold classification results that have been obtained by the techniques of classical surgery theory in the topological category and using Freedman's disc embedding theorem [Fre82]. The use of this theorem requires the fundamental group of the 4-manifold be "good" [FQ90, Part II, Introduction], a condition that has a precise geometric description using the "π_1-null disc property". We will not reproduce that description here, but will instead note which groups are currently known to be good. Freedman showed that the infinite cyclic group and finite groups are good [Fre82, pp. 658-659] (see also [FQ90, Section 5.1]). In addition, by [FT95, Lemma 1.2] the class of good groups is closed under extensions and direct limits. It follows that solvable groups are good. Furthermore in [FT95, Theorem 0.1] and [FT95, KQ00] it was shown that groups with subexponential growth are good.

11.1. **Simply connected 4-manifolds.** The following theorem was the first noteworthy result towards a classification of 4-dimensional manifolds.

Theorem 11.1. *Suppose M and N are two closed oriented simply-connected 4-dimensional manifolds. If the intersection forms are isometric, then M and N are homotopy equivalent.*

Proof. This theorem was proved for smooth manifolds by Milnor [Mil58, Theorem 3], building on work of Whitehead [Whi49]. Furthermore [GS99, Theorem 1.2.25] sketches a proof that works for the general case. □

We state Freedman's classification for closed, simply connected 4-manifolds [Fre82, Theorem 1.5]. Note that a special case is the 4-dimensional topological Poincaré conjecture that every homotopy 4-sphere is homeomorphic to S^4. We give the statement as in [FQ90, Theorem 10.1]. The last sentence comes from [Qui86].

Theorem 11.2. *Fix a triple (F, θ, k), where F is a finitely generated free abelian group, θ is a symmetric, nonsingular, bilinear form $\theta\colon F \times F \to \mathbb{Z}$, and $k \in \mathbb{Z}/2$. If θ is even, that is $\theta(x, x) \in \mathbb{Z}$ is even for every $x \in F$, then suppose that $\sigma(\theta)/8 \equiv k \in \mathbb{Z}/2$.*

Then there exists a closed, simply connected, oriented 4-manifold M with $H_2(M; \mathbb{Z}) \cong F$, with intersection form isometric to θ and with Kirby-Siebenmann invariant equal to k.

Let M and M' be two closed, simply connected, oriented 4-manifolds and let $\phi\colon H_2(M; \mathbb{Z}) \xrightarrow{\cong} H_2(M'; \mathbb{Z})$ be an isometry of the intersection forms. Suppose that $\mathrm{ks}(M) = \mathrm{ks}(M')$. Then there is an orientation preserving homeomorphism $M \xrightarrow{\cong} M'$ inducing ϕ on second homology. This homeomorphism is unique up to isotopy.

In other words, every even, symmetric, integral matrix with determinant ± 1 is realised as the intersection form of a unique closed, simply connected, oriented 4-manifold. For such matrices which are odd instead, there are precisely two closed, simply connected, oriented 4-manifolds up to homeomorphism, exactly one of which has vanishing Kirby-Siebenmann invariant and is therefore stably smoothable. These two manifolds are homotopy equivalent by Theorem 11.1.

In particular, the last paragraph with $M = M'$ implies that every automorphism of the intersection form of a closed, simply connected, oriented 4-manifold is realised by a self-homeomorphism of M.

The following special case, when $F = 0$, is worth pointing out explicitly.

Corollary 11.3 (4-dimensional Poincaré conjecture)**.** *Every homotopy 4-sphere is homeomorphic to S^4.*

Proof. Let N be a homotopy 4-sphere. Then N and S^4 are closed, simply connected and oriented. Furthermore, $H_2(N; \mathbb{Z}) = 0$ and the zero map $H_2(N; \mathbb{Z}) \to H_2(S^4; \mathbb{Z})$ is an isometry (between zero forms). By the last paragraph of Theorem 11.2, there is a homeomorphism $N \cong S^4$ realising this isometry. Note that since N has trivial and therefore even intersection form, $\mathrm{ks}(N) = \sigma(N)/8 = 0$ by Theorem 8.2 (6). □

11.2. **Non simply connected 4-manifolds.** First, we present a classification result [FQ90, Theorem 10.7A] for closed, oriented 4-manifolds with fundamental group $\mathbb{Z}$ which is quite similar to Theorem 11.2. To state the theorem we need some extra definitions.

Definition 11.4. For a finitely generated free $\mathbb{Z}[\mathbb{Z}]$ module F, a hermitian sesquilinear form $\theta\colon F \times F \to \mathbb{Z}[\mathbb{Z}]$ is called *even* if there is a left $\mathbb{Z}[\mathbb{Z}]$-module homomorphism $q\colon F \to \overline{\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}]}(F, \mathbb{Z}[\mathbb{Z}])}$ with $q + q^*\colon F \to \overline{\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}]}(F, \mathbb{Z}[\mathbb{Z}])}$ equal to the adjoint of θ. Otherwise we call the form *odd*.

Definition 11.5. Two homeomorphisms $h_0, h_1\colon M \to N$ are *pseudo-isotopic* if there is a homeomorphism $H\colon M \times I \to N \times I$ with $H|_{M\times\{i\}} = h_i\colon M \times \{i\} \to N \times \{i\}$ for $i = 0, 1$.

An isotopy of homeomorphisms gives rise to a pseudo-isotopy. Quinn proves that the converse holds for compact simply connected 4-manifolds [Qui86] (see also [Per86]), but it is open whether pseudo-isotopy implies isotopy for homeomorphisms between 4-manifolds with nontrivial fundamental groups.

Theorem 11.6. *Fix a triple (F, θ, k), where F is a finitely generated free $\mathbb{Z}[\mathbb{Z}]$-module, θ is a hermitian, nonsingular, sesquilinear form $\theta\colon F \times F \to \mathbb{Z}[\mathbb{Z}]$, and $k \in \mathbb{Z}/2$. If θ is even, then suppose that $\sigma(\mathbb{R} \otimes \theta)/8 \equiv k \in \mathbb{Z}/2$.*

Then there exists a closed, oriented 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$, with $H_2(M; \mathbb{Z}[\mathbb{Z}])$ isomorphic to F, whose equivariant intersection form

$$\lambda_M\colon H_2(M; \mathbb{Z}[\mathbb{Z}]) \times H_2(M; \mathbb{Z}[\mathbb{Z}]) \to \mathbb{Z}[\mathbb{Z}]$$

is isometric to θ, and with $\operatorname{ks}(M) = k$.

Let M and M' be two closed, oriented 4-manifolds with $\pi_1(M) \cong \mathbb{Z} \cong \pi_1(M')$ and let $\phi\colon H_2(M; \mathbb{Z}[\mathbb{Z}]) \xrightarrow{\cong} H_2(M'; \mathbb{Z}[\mathbb{Z}])$ be an isometry of the equivariant intersection forms. Suppose that $\operatorname{ks}(M) = \operatorname{ks}(M')$. Then there is an orientation preserving homeomorphism $M \xrightarrow{\cong} M'$ inducing ϕ on $\mathbb{Z}[\mathbb{Z}]$ coefficient second homology. This homeomorphism is unique up to pseudo-isotopy if θ is even, and if θ is odd there are exactly two pseudo-isotopy classes.

The last sentence of this theorem is a correction to [FQ90, Theorem 10.7A] by Stong and Wang [SW00].

Here is another family of groups for which a complete classification of closed orientable 4-manifolds up to homeomorphism is known. This is the family of solvable Baumslag-Solitar groups

$$B(k) := \langle a, b \mid aba^{-1}b^{-k} \rangle.$$

Note that $B(0) = \mathbb{Z}$ and $B(1) = \mathbb{Z}^2$. Baumslag-Solitar groups are solvable and, as we pointed out above, solvable groups are good. The next classification result was worked out by Hambleton, Kreck and Teichner in [HKT09].

Definition 11.7. The *w_2-type* of a closed, oriented 4-manifold M is type I, II, III, as follows: (I) $w_2(\widetilde{M}) \neq 0$; (II) $w_2(M) = 0$; and (III) $w_2(M) \neq 0$ but $w_2(\widetilde{M}) = 0$.

Theorem 11.8. *Let $B(k)$ be a solvable Baumslag-Solitar group and let M and N be closed, oriented 4-manifolds with fundamental group isomorphic to $B(k)$. Suppose that there is an isomorphism $\phi\colon H_2(M;\mathbb{Z}[B(k)]) \to H_2(N;\mathbb{Z}[B(k)])$ of $\mathbb{Z}[B(k)]$-modules such that:*

1. *The map ϕ induces an isometry between the intersection form $\lambda\colon H_2(M;\mathbb{Z}[B(k)]) \times H_2(M;\mathbb{Z}[B(k)]) \to \mathbb{Z}[B(k)]$ and the corresponding intersection form on $H_2(N;\mathbb{Z}[B(k)])$.*
2. *The Kirby-Siebenmann invariants agree $\operatorname{ks}(M) = \operatorname{ks}(N)$.*
3. *The w_2-types of M and N coincide.*

Then M and N are homeomorphic via an orientation preserving homeomorphism that induces $\phi\colon H_2(M;\mathbb{Z}[B(k)]) \to H_2(N;\mathbb{Z}[B(k)])$.

There is also a precise realisation result for these invariants [HKT09, Theorem B] and 4-manifolds with fundamental group $B(k)$.

Next, 4-manifolds with finite cyclic fundamental groups were studied by Hambleton and Kreck in [HK88, HK93]. Given a finitely generated abelian group G, let TG be its torsion subgroup and let $FG := G/TG$.

Theorem 11.9. *Let G be a finite cyclic group and let M and N be closed, oriented 4-manifolds with fundamental group isomorphic to G. Suppose that there is an isomorphism $\phi\colon \mathrm{F}H_2(M;\mathbb{Z}) \to \mathrm{F}H_2(N;\mathbb{Z})$ such that the following hold.*

1. *The map ϕ induces an isometry between the intersection form $\lambda_M\colon \mathrm{F}H_2(M;\mathbb{Z}) \times \mathrm{F}H_2(M;\mathbb{Z}) \to \mathbb{Z}$ and the intersection form $\lambda_N\colon \mathrm{F}H_2(N;\mathbb{Z}) \times \mathrm{F}H_2(N;\mathbb{Z}) \to \mathbb{Z}$.*
2. *The Kirby-Siebenmann invariants agree $\operatorname{ks}(M) = \operatorname{ks}(N)$.*
3. *The w_2-types of M and N coincide.*

Then M and N are homeomorphic via an orientation preserving homeomorphism that induces $\phi\colon \mathrm{F}H_2(M;\mathbb{Z}) \to \mathrm{F}H_2(N;\mathbb{Z})$.

To state a full realisation result for the invariants in Theorem 11.9 would require taking account of the interdependency between these invariants. However, a partial realisation result is given by the following outlined construction. For every finite cyclic group G, [HK93, Proposition 4.1] produces rational homology spheres with w_2-type II and III, and with fundamental group G. In w_2-type II, a rational homology sphere must have Kirby-Siebenmann invariant zero, since for spin manifolds M $\operatorname{ks}(M) \equiv \sigma(M)/8 \in \mathbb{Z}/2$, and the signature of a rational homology sphere vanishes. In w_2-type III, [HK93, Proposition 4.1] gives two rational homology spheres, one with vanishing Kirby-Siebenmann invariant, and one with nonvanishing Kirby-Siebenmann invariant.

1. By taking the connected sum with a closed, spin simply connected manifold, we can realise any even, nonsingular, symmetric, bilinear form as the intersection form $\lambda_M\colon \mathrm{F}H_2(M;\mathbb{Z}) \times \mathrm{F}H_2(M;\mathbb{Z}) \to \mathbb{Z}$ of a closed, oriented 4-manifold M with fundamental group G and with w_2 type II. In this case $\operatorname{ks}(M)$ is determined by the signature of λ_M.
2. Likewise we can realise every even λ_M as the intersection form of a closed, oriented 4-manifold M with fundamental group G and with w_2 type III, with prescribed Kirby-Siebenmann invariant.

(3) Finally, by taking connected sum with a closed, oriented, simply connected 4-manifold, we can realise any odd, nonsingular, symmetric, bilinear form as the intersection form $\lambda_M \colon \mathrm{F}H_2(M;\mathbb{Z}) \times \mathrm{F}H_2(M;\mathbb{Z}) \to \mathbb{Z}$ of a closed, oriented 4-manifold M with fundamental group G and with w_2 type I, with prescribed Kirby-Siebenmann invariant.

Some partial results towards a classification for 4-manifolds whose fundamental groups are good and have cohomological dimension 3 appear in [HH18].

For nonorientable closed 4-manifolds, the homeomorphism classification results we are aware of are for fundamental group $\mathbb{Z}/2$ in [HKT94] and for fundamental group $\mathbb{Z}$ in [Wan95]. For nonorientable closed 4-manifolds with fundamental group $\mathbb{Z}/2$, the paper [HKT94] gives a complete list of invariants for distinguishing such manifolds up to homeomorphism [HKT94, Theorem 2], and gives a list of the possible manifolds [HKT94, Theorem 3].

Finally, simply-connected compact 4-manifolds with a fixed 3-manifold as boundary were classified by Boyer in [Boy86, Boy93] and independently by Stong [Sto93].

12. Stable smoothing of homeomorphisms

Wall [Wal64] proved that simply connected, closed, smooth 4-manifolds with isometric intersection forms are stably diffeomorphic. It follows that every pair of simply connected, closed, homeomorphic smooth 4-manifolds are stably diffeomorphic. We shall discuss the analogous statement without the simply connected hypothesis.

Definition 12.1.

(1) Let M and N be connected, smooth 4-manifolds. We say that M and N are *stably diffeomorphic* if there is an integer k such that the connected sums $M\#^k S^2 \times S^2$ and $N\#^k S^2 \times S^2$ are diffeomorphic.
(2) Let M and N be connected 4-manifolds. We say that M and N are *stably homeomorphic* if there is an integer k such that the connected sums $M\#^k S^2 \times S^2$ and $N\#^k S^2 \times S^2$ are homeomorphic.

The next theorem is due to Gompf [Gom84].

Theorem 12.2. *Every homeomorphic pair of compact, connected, orientable, smooth 4-manifolds with diffeomorphic boundaries are stably diffeomorphic.*

One might imagine a stronger statement, that given a homeomorphism $f \colon M \to N$ we can smoothen it stably. However such a statement is only known, given a lift of the stable tangent microbundle classifying map to BO, for simply connected 4-manifolds [FQ90, Chapter 8].

The proof of Theorem 12.2 that we shall give using Kreck's modified surgery [Kre99] was outlined in Teichner's thesis [Tei92, Theorem 5.1.1].

Gompf also proved that for every pair of compact, connected, nonorientable, smooth 4-manifolds M and N that are homeomorphic, $M\#S^2\widetilde{\times}S^2$ and $N\#S^2\widetilde{\times}S^2$ are stably diffeomorphic. We shall slightly improve on this statement.

Theorem 12.3. *Let M and N be compact, connected, nonorientable, smooth 4-manifolds M and N. Suppose that M and N are homeomorphic via a homeomorphism restricting to a diffeomorphism $\partial M \cong \partial N$. If $w_2(\widetilde{M}) \neq 0 \neq w_2(\widetilde{N})$, that is the universal covers of M and N are not spin, then M and N are stably diffeomorphic.*

The hypothesis that $w_2(\widetilde{M}) \neq 0 \neq w_2(\widetilde{N})$ cannot be dropped in general. Cappell and Shaneson found an example of a smooth 4-manifold R that is homotopy equivalent to $\mathbb{RP}^4$ but that is not stably diffeomorphic to $\mathbb{RP}^4$ [CS71, CS76]. When these papers were published, it was not possible to prove that the fake $\mathbb{RP}^4$ manifold R is homeomorphic to $\mathbb{RP}^4$, but this was later established [Rub84, p. 221] as a consequence of the work of Freedman and Quinn [FQ90], and the fact that the Whitehead group of $\mathbb{Z}/2$ is trivial.

Later, Kreck [Kre84] showed a much more general statement in this direction. Let $K3$ denote the Kummer surface, a closed, smooth, spin 4-manifold with signature 16 and $b_2(K3) = 22$. Here is Kreck's result from [Kre84].

Theorem 12.4. *Let π be a finitely presented group with a surjective homomorphism $w\colon \pi \to \mathbb{Z}/2$. Then there exists a closed, smooth, connected 4-manifold W with fundamental group π and orientation character w, with the property that $W\#K3$ and $W\#^{11}S^2 \times S^2$ are homeomorphic 4-manifolds that are not stably diffeomorphic.*

One part of this is easy to see: if W is nonorientable then there are homeomorphisms

$$\begin{aligned} W\#K3 \;&\cong\; W\#E_8\#E_8\#^3S^2 \times S^2 \;\cong\; W\#E_8\#\overline{E_8}\#^3S^2 \times S^2 \\ &\cong\; W\#^8S^2 \times S^2\#^3S^2 \times S^2 \;\cong\; W\#^{11}S^2 \times S^2. \end{aligned}$$

Here we used Theorem 11.2 that simply connected closed 4-manifolds with Kirby-Siebenmann invariant vanishing are determined by their intersection forms, and we used that the connected sum $M\#N$ of an oriented manifold M with a nonorientable manifold N is homeomorphic to $\overline{M}\#N$.

Gompf's statement for the nonorientable case, given in the next corollary, follows easily from Theorem 12.3. However note that Theorem 12.3 shows that for many nonorientable 4-manifolds, the extra summand given by the twisted bundle $S^2\widetilde{\times}S^2$ is not necessary.

Corollary 12.5. *Let M and N be compact, connected, nonorientable, smooth 4-manifolds M and N. Suppose that M and N are homeomorphic via a homeomorphism restricting to a diffeomorphism $\partial M \cong \partial N$. Then $M\#S^2\widetilde{\times}S^2$ and $N\#S^2\widetilde{\times}S^2$ are stably diffeomorphic.*

Proof. Taking the connected sum of any 4-manifold with $S^2\widetilde{\times}S^2 \cong \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ gives rise to a 4-manifold whose universal cover is not spin. The corollary therefore follows from Theorem 12.3. □

In the following three sections we will prove Theorem 12.2 and Theorem 12.3. To keep the notation manageable we will only provide a proof for closed manifolds.

12.1. **Kreck's modified surgery.** Below we will state a theorem due to Kreck that relates stable diffeomorphisms of 4-manifolds with bordism theory. This came as a corollary of Kreck's modified surgery theory [Kre99]. First we need some definitions from [Kre99].

Definition 12.6. A *normal 1-type* of a closed, connected, smooth 4-manifold M is a 2-coconnected fibration $\xi\colon B \to \mathrm{BO}$ for which there is a 2-connected lift $\widetilde{\nu}_M\colon M \to B$ of the stable normal bundle $\nu_M\colon M \to \mathrm{BO}$ such that $\xi \circ \widetilde{\nu}_M\colon M \to \mathrm{BO}$ is homotopic to ν_M. We call such a choice of lift $\widetilde{\nu}_M\colon M \to B$ a *normal 1-smoothing*.

Remark 12.7.

(1) Here by definition a 2-coconnected map induces an isomorphism on homotopy groups π_i for $i \geq 3$ and induces an injection for $i = 2$. A 2-connected map induces a surjection on π_2 and an isomorphism on π_1 and π_0.
(2) The data of a normal 1-type is $\xi\colon B \to \mathrm{BO}$. The existence of $\widetilde{\nu}_M$ is a condition on that data.

Definition 12.8. A *normal 1-type* of a closed, oriented, connected 4-manifold M is a 2-coconnected fibration $\xi\colon B \to \mathrm{BTOP}$ for which there is a 2-connected lift $\widetilde{\nu}_M\colon M \to B$ of the stable normal microbundle $\nu_M\colon M \to \mathrm{BTOP}$ (Definition 7.12) such that $\xi \circ \widetilde{\nu}_M\colon M \to \mathrm{BTOP}$ is homotopic to ν_M. We call such a choice of lift $\widetilde{\nu}_M\colon M \to B$ a *normal* TOP *1-smoothing*.

Normal 1-types $\xi\colon B \to \mathrm{BO}$ of a closed, connected smooth 4-manifold are fibre homotopy equivalent over BO, and thus we may speak of *the* normal 1-type of a smooth 4-manifold, and similarly for the topological version. Here are some of the key examples in the oriented case. We will give the details of the nonorientable case in Section 12.3.

Write $\pi = \pi_1(M)$ and let $w_2 \in H^2(M;\mathbb{Z}/2)$ be the second Stiefel-Whitney class of M. There are three main cases for the normal 1-types of oriented, closed smooth 4-manifolds. For more details, see [KLPT17, Sections 2 and 3].

Lemma 12.9. *Let M be a closed, oriented, connected, smooth 4-manifold.*

(1) *Suppose that we have $w_2(\widetilde{M}) \neq 0$. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSO} \to \mathrm{BO}$ is the normal 1-type of M, with the map ξ given by projection to* BSO *followed by the canonical map* $\mathrm{BSO} \to \mathrm{BO}$.
(2) *Suppose that M is spin. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSpin} \to \mathrm{BO}$ is the normal 1-type of M, with the map ξ given by projection to* BSpin *followed by the canonical map* $\mathrm{BSpin} \to \mathrm{BO}$.
(3) *Suppose that we have $w_2(M) \neq 0$ but $w_2(\widetilde{M}) = 0$. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSpin} \to \mathrm{BSO}$ is the normal 1-type of M, with B as in the spin case, but the map ξ twisted using a complex line bundle on* $\mathrm{B}\pi$ *defined in terms of $w_2(M)$.*

Similarly there are three main cases for the normal 1-types of closed, oriented, connected 4-manifolds. Let $\mathrm{STOP}(n)$ be the group of orientation preserving homeomorphisms of $\mathbb{R}^n$ fixing the origin and let STOP be the corresponding colimit of $\mathrm{STOP}(n)$. Let TOPSpin be the universal (2-fold) cover of STOP. Here $\pi_1(\mathrm{STOP}) \cong \pi_1(\mathrm{TOP}) \cong \pi_1(\mathrm{O}) \cong \mathbb{Z}/2$. To see that $\pi_1(\mathrm{TOP}) \cong \pi_1(\mathrm{O})$ we use that there is a 6-connected map $\mathrm{TOP}/\mathrm{O} \to K(\mathbb{Z}/2, 3)$ [KS77, Essay V, Section 5].

Lemma 12.10. *Let M be a closed, oriented, connected 4-manifold.*

(1) *Suppose that we have $w_2(\widetilde{M}) \neq 0$. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSTOP} \to \mathrm{BTOP}$ is the normal 1-type of M, with the map given by projection to* BSTOP *followed by the canonical map* $\mathrm{BSTOP} \to \mathrm{BTOP}$.
(2) *Suppose that M is spin. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BTOPSpin} \to \mathrm{BTOP}$ is the normal 1-type of M, with the map given by projection to* BTOPSpin *followed by the canonical map* $\mathrm{BTOPSpin} \to \mathrm{BTOP}$.
(3) *Suppose that we have $w_2(M) \neq 0$ but $w_2(\widetilde{M}) = 0$. Then $\xi\colon B = \mathrm{B}\pi \times \mathrm{BTOPSpin} \to \mathrm{BTOP}$ is the normal 1-type of M, with B as in the spin case, but the map ξ twisted using a complex line bundle on $\mathrm{B}\pi$ defined in terms of $w_2(M)$.*

For more details on these assertions, see [KLPT17, Sections 2 and 3]. Here is the relevant theorem of Kreck, which relates bordism over the normal 1-type to stable diffeomorphism.

Theorem 12.11. *Two closed, connected, smooth 4-manifolds M and N with $\chi(M) = \chi(N)$ and fibre homotopy equivalent normal 1-types are stably diffeomorphic if and only if*

$$[(M, \widetilde{\nu}_M)] = [(N, \widetilde{\nu}_N)] \in \Omega_4(B, \xi)$$

for some choices of normal 1-smoothings $\widetilde{\nu}_M$ and $\widetilde{\nu}_N$.

Sketch of the proof. One direction is quite easy: one has to check that M and $M \# S^2 \times S^2$ are bordant over the normal 1-type of M.

For the other direction, start with a 5-dimensional bordism W over (B, ξ) and perform surgery on W below the middle dimension to make the map to B 1-connected. Now represent the elements of $\ker(\pi_2(W) \to \pi_2(B))$ by framed embedded spheres, and remove thickenings of these spheres. Also remove tubes of these to either M or N, tubing enough 2-spheres to either side so as to preserve the Euler characteristic equality. This operation of removing copies of $S^2 \times D^3$, tubed to the boundary, has the effect of adding copies of $S^2 \times S^2$ to M and N giving rise to M' and N' respectively, and it converts W to an s-cobordism W'. That $(W'; M', N')$ is an s-cobordism means by definition that the inclusion maps $M' \to W'$ and $N' \to W'$ are simple homotopy equivalences. The stable s-cobordism theorem [Qui83] states that every 5-dimensional s-cobordism becomes diffeomorphic to a product after adding copies of $S^2 \times S^2 \times I$ along a smoothly embedded interval $I \subset W'$ with one endpoint on each of M and N. This completes the sketch proof of Theorem 12.11. □

The proof of the topological version is similar.

Theorem 12.12. *Two closed, topological 4-manifolds M and N with $\chi(M) = \chi(N)$ and fibre homotopy equivalent normal 1-types are stably homeomorphic if and only if*

$$[(M, \widetilde{\nu}_M)] = [(N, \widetilde{\nu}_N)] \in \Omega_4^{\mathrm{TOP}}(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}})$$

for some choices of normal 1-smoothings $\widetilde{\nu}_M$ and $\widetilde{\nu}_N$.

From now on, to ease notation, we will abbreviate $\Omega_4^{\mathrm{TOP}}(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}})$ with $\Omega_4^{\mathrm{TOP}}(B, \xi)$.

Proof. One direction is again quite easy: we need that homeomorphic manifolds are bordant over B, and that M and $M \# S^2 \times S^2$ are bordant in $\Omega_4^{\mathrm{TOP}}(B, \xi)$. For the other direction, apply the same argument as above to improve a cobordism W to an s-cobordism. The stable s-cobordism theorem applies to topological s-cobordisms as well as to smooth s-cobordisms. This is not written in [Qui83], but the same proof applies, with the following additions (see the Exercise on [FQ90, p. 107]). First, 5-dimensional cobordisms admit a topological handlebody structure [FQ90, Theorem 9.1]. The proof of [Qui83] consists of simplifying a handle decomposition, and tubing surfaces in 4-manifolds around and into parallel copies of one another to remove intersections. This is possible in the topological category by using transversality (Theorem 10.3) to arrange that intersections between surfaces are isolated points, and the existence of normal bundles (Theorem 6.17) to take parallel copies using sections. □

12.2. **Stable diffeomorphism of homeomorphic orientable 4-manifolds.** Now we will explain the proof of Theorem 12.2. For the convenience of the reader, we recall the statement.

Theorem 12.13. *Every homeomorphic pair of closed, connected, orientable, smooth 4-manifolds are stably diffeomorphic.*

The proof will rest on the following proposition.

Proposition 12.14. *Let (B, ξ) be one of the oriented smooth normal 1-types from Lemma 12.9, and let $(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}})$ be the corresponding topological normal 1-type from Lemma 12.10 obtained by replacing* BSO *with* BSTOP *or* BSpin *with* BTOPSpin *as appropriate. The forgetful map*
$$F\colon \Omega_4(B, \xi) \to \Omega_4^{\mathrm{TOP}}(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}}) = \Omega_4^{\mathrm{TOP}}(B, \xi)$$
is injective.

The combination of this proposition with Theorem 12.11 and Theorem 12.12 implies the following corollary, which is the closed version of Theorem 12.2, with a slightly more precise statement concerning orientations.

Corollary 12.15. *Every pair of smooth, closed, connected, oriented 4-manifolds that are homeomorphic via an orientation preserving homeomorphism are stably diffeomorphic via an orientation preserving diffeomorphism.*

Proof. We prove the corollary assuming Proposition 12.14. Homeomorphic 4-manifolds are in particular stably homeomorphic and have the same normal 1-types. Therefore two homeomorphic smooth 4-manifolds as in the statement of the corollary are bordant over the normal 1-type, so give rise to equal elements in $\Omega_4^{\mathrm{TOP}}(B, \xi)$. By Proposition 12.14, they give rise to equal elements of $\Omega_4(B, \xi)$. Then by Theorem 12.11, the two 4-manifolds are stably diffeomorphic, as asserted. □

Proof of Proposition 12.14. Let S be SO in case (1) of the smooth list of 1-types above, and let S denote Spin in cases (2) and (3).

Let ST be STOP in case (1) of the topological list of 1-types above, and let ST denote TOPSpin in cases (2) and (3).

The James spectral sequence [Tei92, Theorem 3.1.1],[KLPT17, Section 3] is of the form:

$$E^2_{p,q} = H_p(\mathrm{B}\pi; \Omega^S_q) \Rightarrow \Omega_{p+q}(B, \xi).$$

We have that $\Omega^S_4 \cong \mathbb{Z}$, detected by the signature. Indeed, the signature is a $\mathbb{Z}$-valued invariant for stable diffeomorphisms. This arises on the E^2 page as $H_0(\mathrm{B}\pi; \Omega^S_4) \cong \mathbb{Z}$.

Claim. This term $H_0(\mathrm{B}\pi; \Omega^S_4)$ survives to the E^∞ page. That is, all differentials with this as codomain are trivial.

Let us prove the claim. Since Ω^S_q is torsion for $q = 1, 2, 3$, no terms from those q-lines can map to $H_0(\mathrm{B}\pi; \Omega^S_4)$ under a differential.

Aside from $H_0(\mathrm{B}\pi; \Omega^S_4)$, there is one other potentially infinite term on the E^∞ page, namely the subgroup of $H_4(\mathrm{B}\pi; \Omega^S_0)$ arising as the kernel of relevant differentials. The image of a 4-manifold in here is the image $c_*([M])$ of the fundamental class under the classifying map $c_* \colon H_4(M; \mathbb{Z}) \to H_4(\mathrm{B}\pi; \mathbb{Z}) \cong H_4(\mathrm{B}\pi; \Omega^S_0)$.

There could be a differential $H_5(\mathrm{B}\pi; \Omega^S_0) \to H_0(\mathrm{B}\pi; \Omega^S_4)$. However if there were a nonzero differential, then only finitely many signatures would occur for 4-manifolds with normal 1-type B and fixed invariant in $H_4(\mathrm{B}\pi; \Omega^S_0)$. But we can add copies of either $\mathbb{CP}^2$ or the $K3$-surface to keep the normal 1-type and $c_*([M])$ the same, but change the signature by 1 or 16, for each copy of $\mathbb{CP}^2$ or $K3$, respectively. This completes the proof of the claim that the term $H_0(\mathrm{B}\pi; \Omega^S_4)$ survives to the E^∞ page.

We note that in case (3), the entry in Ω^{Spin}_4 is *not* necessarily the signature of the manifold; this entry could be a multiple of the signature.

Since $H_0(\mathrm{B}\pi; \Omega^S_4)$ survives to the E^∞ term, we have a short exact sequence:

$$0 \to \Omega^S_4 \to \Omega_4(B, \xi) \to \widetilde{\Omega}_4(B, \xi) \to 0,$$

where $\widetilde{\Omega}_4(B, \xi)$ denotes the quotient. That is, there is a filtration with iterated graded quotients given by the E^∞ page:

$$0 \subseteq E^\infty_{4,0} = \Omega^S_4 \subseteq \cdots \subseteq \Omega_4(B, \xi),$$

and it is the quotient by the $E^\infty_{4,0}$ subgroup that we denote $\widetilde{\Omega}_4(B, \xi)$.

Similarly, for the topological case, we have

$$0 \to \Omega^{ST}_4 \to \Omega^{\mathrm{TOP}}_4(B, \xi) \to \widetilde{\Omega}^{\mathrm{TOP}}_4(B, \xi) \to 0.$$

The only difference in the proof from the smooth case is that we use E_8 in place of $K3$ in the argument that no differential has nontrivial image in $H_0(\mathrm{B}\pi; \Omega^{ST}_4)$, and we also have to argue that the Kirby-Siebenmann invariant $\mathbb{Z}/2 \subset \Omega^{ST}_4$ survives to the E^∞ page. But the Kirby-Siebenmann invariant is additive, and realised on simply connected manifolds, either by the Chern manifold $*\mathbb{CP}^2$ (Construction 8.5) or the E_8 manifold depending on the normal 1-type. Thus there exist bordism classes (i.e. stable homeomorphism classes) realising both trivial and nontrivial Kirby-Siebenmann invariants within a normal 1-type, and so this $\mathbb{Z}/2$ cannot be killed by a differential.

Since the structure forgetting map $\Omega_q^S \to \Omega_q^{ST}$ is an isomorphism for $0 \leq q \leq 3$, we have an isomorphism $\widetilde{\Omega}_4(B,\xi) \cong \widetilde{\Omega}_4^{\mathrm{TOP}}(B,\xi)$. This uses that the differentials agree, by naturality of the James spectral sequence with respect to homology theories. Indeed, note that the differentials depend only on the classifying space $\mathrm{B}\pi$, and on the complex line bundle $E \to \mathrm{B}\pi$ in case (3). Both are category independent.

Then there is a map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_4^S \arrow[r] \arrow[d] & \Omega_4(B,\xi) \arrow[r] \arrow[d] & \widetilde{\Omega}_4(B,\xi) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \Omega_4^{ST} \arrow[r] & \Omega_4^{\mathrm{TOP}}(B,\xi) \arrow[r] & \widetilde{\Omega}_4^{\mathrm{TOP}}(B,\xi) \arrow[r] & 0.
\end{tikzcd}$$

The left vertical map is injective, either inclusion into the first summand $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2$ for non-spin or $16\mathbb{Z} \to 8\mathbb{Z}$ in the spin case. Since the left and right vertical maps are injective, it follows from a diagram chase that the central vertical map is also injective, as required. □

12.3. **Non-orientable 4-manifolds and stable diffeomorphism.** For the convenience of the reader, we recall the statement of Theorem 12.3.

Theorem 12.16. *Let M and N be closed, connected, nonorientable, smooth 4-manifolds M and N. Suppose that M and N are homeomorphic. If $w_2(\widetilde{M}) \neq 0 \neq w_2(\widetilde{N})$, that is the universal covers of M and N are not spin, then M and N are stably diffeomorphic.*

Here is the normal 1-type for nonorientable manifolds with a certain w_2-type [Tei92, Chapter 2].

Lemma 12.17. *Let M be a nonorientable closed, connected smooth 4-manifold with $w_2(\widetilde{M}) \neq 0$. Then the normal 1-type of M is $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSO} \to \mathrm{BO}$ with the map $\xi = w_1 \oplus Bi$ given by the Whitney sum of a bundle on $\mathrm{B}\pi$ determined by $w_1\colon \pi \to \mathbb{Z}/2$ and the canonical map $Bi\colon \mathrm{BSO} \to \mathrm{BO}$ induced by the inclusion $i\colon \mathrm{SO} \to \mathrm{O}$.*

Lemma 12.18. *Let M be a nonorientable closed, connected 4-manifold with $w_2(\widetilde{M}) \neq 0$. Then the normal 1-type of M is $\xi\colon B = \mathrm{B}\pi \times \mathrm{BSTOP} \to \mathrm{BTOP}$ with the map $\xi = w_1 \oplus Bi$ given by the Whitney sum of a bundle on $\mathrm{B}\pi$ determined by the orientation character $w_1\colon \pi \to \mathbb{Z}/2$ and the canonical map $Bi\colon \mathrm{BSTOP} \to \mathrm{BTOP}$ induced by the inclusion $i\colon \mathrm{STOP} \to \mathrm{TOP}$.*

These normal 1-types gives rise to a James spectral sequence governing the bordism groups of (B,ξ)

$$E^2_{p,q} = H_p(\mathrm{B}\pi; \Omega_q^{w_1}) \Rightarrow \Omega_{p+q}(B,\xi).$$

Note that the coefficients are twisted using $\mathbb{Z}^{w_1} \otimes \Omega_q$, where by definition, $g \in \pi$ acts on $\mathbb{Z}^{w_1}$ by multiplication by $(-1)^{w_1(g)}$. The corresponding topological James spectral sequence is:

$$E^2_{p,q} = H_p(\mathrm{B}\pi; (\Omega_q^{\mathrm{STOP}})^{w_1}) \Rightarrow \Omega_{p+q}^{\mathrm{TOP}}(B, \xi).$$

As in the previous section, here we abbreviate $\Omega_4^{\mathrm{TOP}}(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}})$ with $\Omega_4^{\mathrm{TOP}}(B, \xi)$.

By Kreck's theorem (Theorem 12.11) and the argument in the proof of Theorem 12.2, in order to prove Theorem 12.3 it suffices to prove the next injectivity statement.

Proposition 12.19. *Let (B, ξ) be one of the normal 1-types in Lemma 12.17 and let $(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}})$ be the corresponding topological normal 1-type over* BTOP. *The forgetful map*

$$F \colon \Omega_4(B, \xi) \to \Omega_4^{\mathrm{TOP}}(B^{\mathrm{TOP}}, \xi^{\mathrm{TOP}}) = \Omega^{\mathrm{TOP}}(B, \xi)$$

is injective.

Proof of Theorem 12.3 assuming Proposition 12.19. Since homeomorphic 4-manifolds have the same normal 1-types and so are trivially TOP bordant over this normal 1-type, injectivity of F implies that homeomorphic nonorientable, closed, connected, smooth 4-manifolds are smoothly bordant over their normal 1-type, and therefore by Theorem 12.11 are stably diffeomorphic. □

Proof of Proposition 12.19. The structure of the proof is very similar to that of the proof of Proposition 12.14. This proof is therefore somewhat terse. In the smooth James spectral sequence computing $\Omega_4(B, \xi)$, we consider the term on the E_2 page $H_0(\mathrm{B}\pi; \Omega_4^{w_1}) \cong \mathbb{Z}/2$. This is detected by the Euler characteristic of the manifold modulo two.

Since we can add a copy of $\mathbb{CP}^2$ (note that for nonorientable manifolds connected sum with $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ is the same), both mod 2 Euler characteristics are realised by bordism classes over (B, ξ). Also note that adding $\mathbb{CP}^2$ does not change the normal 1-type when $w_2(\widetilde{M}) \neq 0$. Therefore $H_0(\mathrm{B}\pi; \Omega_4^{w_1}) \cong \mathbb{Z}/2$ survives to the E^∞ page.

In the topological case, the corresponding term in the James spectral sequence computing $\Omega_4^{\mathrm{TOP}}(B, \xi)$ is

$$H_0(\mathrm{B}\pi; (\Omega_4^{\mathrm{STOP}})^{w_1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

We can add copies of $\mathbb{CP}^2$ and $*\mathbb{CP}^2$ to show that this term survives to the E^∞ page. The structure forgetting map $\mathbb{Z}/2 \cong H_0(\mathrm{B}\pi; \Omega_4^{w_1}) \to H_0(\mathrm{B}\pi; (\Omega_4^{\mathrm{STOP}})^{w_1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is injective.

Therefore the filtrations of $\Omega_4(B, \xi)$ and $\Omega_4^{\mathrm{TOP}}(B, \xi)$ arising from the spectral sequence give rise to short exact sequences, that form the rows of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \Omega_4(B,\xi) \arrow[r] \arrow[d] & \widetilde{\Omega}_4(B,\xi) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & \Omega_4^{\mathrm{TOP}}(B,\xi) \arrow[r] & \widetilde{\Omega}_4^{\mathrm{TOP}}(B,\xi) \arrow[r] & 0.
\end{tikzcd}$$

We noted above that the left vertical map is injective. Since $\Omega_q \to \Omega_q^{\mathrm{STOP}}$ is an isomorphism for $0 \leq q \leq 3$, the right vertical map is an isomorphism and is therefore injective. It follows from a diagram chase that the central vertical map is also injective, as required. $\square$

13. Twisted intersection forms and twisted signatures

Let M be a compact, orientable, connected $4m$-dimensional manifold. We write $\pi := \pi_1(M)$. Let $\alpha\colon \pi \to U(k)$ be a unitary representation. We view the elements of $\mathbb{C}^k$ as row vectors. Given $g \in \pi$ and $v \in \mathbb{C}^k$, define $v \cdot g := v \cdot \alpha(g)$. Thus we can view $\mathbb{C}^k$ as a right $\mathbb{Z}[\pi]$-module. Denote this module by $\mathbb{C}^k_\alpha$. Define the *twisted intersection form* of (M, α) to be the form

$$\begin{array}{rcc} Q_M\colon H_{2m}(M;\mathbb{C}^k_\alpha) \times H_{2m}(M;\mathbb{C}^k_\alpha) & \xrightarrow{\mathrm{PD}^{-1}\times \mathrm{PD}^{-1}} & H^{2m}(M,\partial M;\mathbb{C}^k_\alpha) \times H^{2m}(M,\partial M;\mathbb{C}^k_\alpha) \\ & & \downarrow \cup \\ & & H^{4m}(M,\partial M;\mathbb{C}^k_\alpha \otimes \mathbb{C}^k_\alpha) \\ & & \downarrow \langle\, ,\, \rangle \\ & & H^{4m}(M,\partial M;\mathbb{C}) \\ & & \downarrow \mathrm{PD} \\ & & H_0(M;\mathbb{C}) = \mathbb{C}. \end{array}$$

Here the first and the last map are given by the isomorphisms from the Poincaré duality theorem A.15 and the second map is given by Lemma A.11. Note that in the bottom we view $\mathbb{C}$ as a trivial $\mathbb{Z}[\pi]$-module. The third map is induced by the following homomorphism of right $\mathbb{Z}[\pi]$-modules:

$$\begin{array}{rcl} \mathbb{C}^k_\alpha \otimes C^k_\alpha & \to & \mathbb{C} \\ (v,w) & \mapsto & \langle v,w\rangle = \overline{v}w^T. \end{array}$$

It follows easily from the definitions that Q_M is sesquilinear, namely $\mathbb{C}$-conjugate linear in the first entry and $\mathbb{C}$-linear in the second entry. The usual proof for the (anti-) symmetry of the cup product e.g. [Hat02, Theorem 3.14], can be modified to show that the above pairing is hermitian, that is for every $v, w \in H^{2m}(M;\mathbb{C}^k_\alpha)$ we have $Q_M(v,w) = \overline{Q_M(w,v)}$. Since Q_M is hermitian, its signature is defined as the difference in the number of positive and negative eigenvalues. We refer to the signature of Q_M as the *twisted signature* $\sigma(M,\alpha)$.

Finally, for a group homomorphism $\gamma\colon \pi_1(M) \to \Gamma$, denote the corresponding L^2-signature by $\sigma^{(2)}(M,\gamma)$, as defined in say [Ati76, Lüc02] and [COT03, Chapter 5].

Theorem 13.1. *Let M be a closed oriented 4-manifold.*

1. *For every finite cover $p\colon \widetilde{M} \to M$ we have $\sigma(\widetilde{M}) = [\widetilde{M} : M] \cdot \sigma(M)$.*
2. *For every unitary representation $\alpha\colon \pi_1(M) \to U(k)$ we have $\sigma(M,\alpha) = k \cdot \sigma(M)$.*
3. *For every group homomorphism $\gamma\colon \pi_1(M) \to \Gamma$ we have $\sigma^{(2)}(M,\gamma) = \sigma(M)$.*

Remark 13.2.

(1) The same statement does not hold for 4-dimensional Poincaré complexes in general. More precisely, Wall [Wal67, Corollary 5.4.1] gave examples of 4-dimensional Poincaré complexes for which the signature is not multiplicative under finite covers.
(2) Alternative proofs for the first statement and the third statement are provided by Schafer [Sch70, Theorem 8] and Lück-Schick [LS03, Theorem 0.2]. In fact these papers are also valid for manifolds of any dimension $4m$.

Proof. For smooth manifolds, the first statement is a consequence of the Hirzebruch signature theorem (see e.g. [MS74]) while the second statement was proven in [APS75] (in fact the second statement contains the first statement as a special case). The third statement was proven in [Ati76, p. 44].

We will prove the second statement of the theorem. The other statements can be proved in a similar fashion. We refer to [COT03, Lemma 5.9] for a proof of (3).

So let M be a closed oriented 4-manifold and let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation. By Theorem 8.6, there exists a closed orientable simply-connected 4-manifold N such that $M\#N$ is smooth. We have $\pi_1(M\#N) = \pi_1(M) * \pi_1(N) \cong \pi_1(M)$ since $\pi_1(N) = \{1\}$. Let $\beta\colon \pi_1(N) \to U(k)$ be the trivial representation. We also write $\alpha * \beta\colon \pi_1(M\#N) = \pi_1(M) \to U(k)$ for the representation uniquely determined by α on $\pi_1(M)$.

By Proposition 5.15, we have $\sigma(M\#N) = \sigma(M) + \sigma(N)$. Furthermore a slight generalisation of Proposition 5.15 shows that $\sigma(M\#N, \alpha * \beta) = \sigma(M, \alpha) + \sigma(N, \beta)$. Finally, we have $\sigma(N, \beta) = k \cdot \sigma(N)$. The desired statement follows from these equalities and from the formula for twisted signatures of the closed smooth manifold $M\#N$. □

14. Reidemeister torsion in the topological category

14.1. **The simple homotopy type of a manifold.** In the following we need the notion of a *simple* homotopy equivalence. We will not give a definition, instead we refer to [Tur01, p. 40] for details. Roughly, a simple homotopy equivalence between CW complexes is a sequence of elementary expansions and collapses of pairs of cells whose dimension differs by one.

The following definition allows us to define a simple homotopy type even for topological spaces which are not homeomorphic to a CW complex.

Definition 14.1. Let (W, V) be a pair of topological spaces. Consider tuples (W, V, f, X, Y), where (X, Y) is a finite CW complex pair with $Y \subseteq X$, and $f\colon W \to X$ and $f|_V\colon V \to Y$ are homotopy equivalences. Two such tuples (W, V, f, X, Y) and (W, V, f', X', Y') with (X', Y') another finite CW pair and $f'\colon (W, V) \to (X', Y')$ are *equivalent* if there exists a simple homotopy equivalence of pairs $s\colon (X, Y) \to (X', Y')$ such that $s \circ f$ is homotopic to f' and $s|_Y \circ f|_V\colon V \to Y'$ is homotopic to $f'|_V$. Such an equivalence class of (W, V, f, X, Y) is called a *simple homotopy type* of (W, V). In particular, a simple homotopy type of $(W, \varnothing)$ is called a *simple homotopy type* of W.

Consider now a compact connected n-manifold M. If M admits a smooth structure, then M admits in particular a CW structure [Hir94, Section 6.4], and we equip M with the simply homotopy type given by (M, Id). By Chapman's theorem [Cha74, p. 488] below, this simple homotopy type is independent of the choice of CW structure on M.

Theorem 14.2 (Chapman)**.** *Let W be a compact topological space. Any two CW structures on W are simple homotopy equivalent.*

As we pointed out in Section 4, it is unknown whether every compact manifold admits a CW structure. In the remainder of this section, we will nonetheless introduce the simple homotopy type of a compact manifold M following [KS77, Essay III, Section 4]. The first step is to construct a disc bundle $D(M) \to M$ together with a PL structure on the total space $D(M)$. We will work with a compact m-dimensional manifold M with boundary ∂M, and seek to construct the simple homotopy type of $(M, \partial M)$.

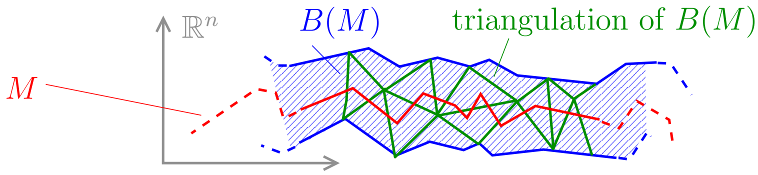

FIGURE 12.

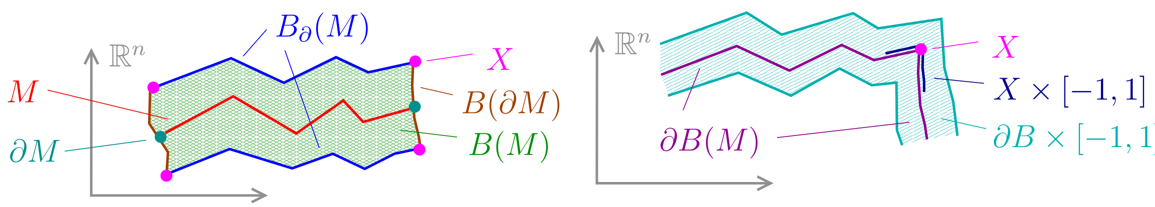

FIGURE 13.

Construction 14.3. We deal with the case $\partial M = \varnothing$ first, and then later address the additional complications arising from having nonempty boundary.

As a first step to constructing the disc bundle $D(M) \subset \mathbb{R}^n$, we need an embedding of M into $\mathbb{R}^{n-1}$ for some large integer $n - 1 > 2m + 5$. For a closed m–manifold M such an embedding is readily available [Hat02, Corollary A.9]. It follows from Theorem 7.10 that for $n - 1 > 2m + 5$ all such embeddings of M are isotopic, and that they admit a normal microbundle $\nu_{\mathbb{R}^{n-1}}(M)$ that is unique up to isotopy. By Theorem 7.7 this normal microbundle $\nu_{\mathbb{R}^{n-1}}(M)$ can be upgraded to a topological $\mathbb{R}^{n-1-m}$–bundle. By taking the product with $\mathbb{R}$, construct an embedding $M \subset \mathbb{R}^n$ whose normal microbundle is $\nu(M) = \nu_{\mathbb{R}^{n-1}}(M) \times \mathbb{R}$. Since we stabilised once, the normal microbundle $\nu(M)$ contains a normal disc bundle $B(M)$ [KS77, Essay III, Proposition 4.4].

The next big step will be to upgrade $B(M) \subset \mathbb{R}^n$ from a submanifold to a PL submanifold. Since the interior is codimension 0, the interior of $B(M)$ is automatically also a PL submanifold. However, we have to arrange $\partial B(M)$ to be a PL submanifold of $\mathbb{R}^n$ itself. In the next paragraphs, we modify the PL structure on $\mathbb{R}^n$ such that $\partial B(M)$ becomes a PL submanifold and then isotope this new PL structure on $\mathbb{R}^n$ back to the standard PL structure.

Using the Collar Neighbourhood Theorem 2.5, pick a collar $W_\partial = \partial B(M) \times (-1, 1)$ and $D_\partial = \partial B(M) \times [-\frac{1}{2}, \frac{1}{2}]$. The local product structure theorem 5.18 [KS77, Essay I, Theorem 5.2] gives a PL structure σ_∂ on $\mathbb{R}^n$ such that $\partial B(M)$ is a PL submanifold and σ_∂ is concordant to the standard PL structure σ_{std}.

Now we will isotope the pair $\partial B(M) \subset B(M)$ so that they become PL submanifolds of $(\mathbb{R}^n, \sigma_{\text{std}})$. The PL structure σ_∂ is concordant to σ_{std}. Since concordance implies isotopy [KS77, Essay I, Theorem 4.1] in dimension $m \geq 6$, there is an isotopy $\phi_t \in \text{Homeo}(\mathbb{R}^n)$ such that $\phi_0 = \text{Id}$, and $\phi_1^* \sigma_{\text{std}} = \sigma_\partial$. Consequently, $D(M) := \phi_1(B(M))$ and $D(\partial M) := \phi_1(B(\partial M))$ are PL submanifolds of $(\mathbb{R}^n, \sigma_{\text{std}})$, which defines a simple type of M.

Having finished the case $\partial M \neq \varnothing$, next we discuss the procedure for a manifold M with non-empty boundary.

Take the union of M with an external open collar $\partial M \times [0, 1)$ of its boundary. Write $M' := M \cup_{\partial M} \partial M \times [0, 1)$. Embed M' into $\mathbb{R}^n$ as in the closed case [Hat02, Corollary A.9]. Note that M' has empty boundary and so it is properly embedded. As in the closed case, obtain a disc bundle $B(M')$ and let $B(M)$ be the restriction of this disc bundle to M.

Now we have to take much more care. Note that $\partial B(M)$ decomposes as $\partial B(M) = B(\partial M) \cup_X B_\partial(M)$. Here $B_\partial(M)$ denotes the fibrewise boundary and $X = \partial\big(B(\partial M)\big)$ denotes the intersection of $B(\partial M)$ and $B_\partial(M)$. As above, we will find a PL structure σ_∂ of $\mathbb{R}^n$ such that each subset $B(\partial M)$, X and $B_\partial(M)$ is PL–submanifold of $\mathbb{R}^n$.

Our first goal is to modify the PL structure on $\mathbb{R}^n$ so that the corners X become a PL submanifold of $\mathbb{R}^n$. Denote the standard PL structure on $\mathbb{R}^n$ by σ_{std}. Pick a bicollar $\partial B(M) \times [-1, 1] \subset \mathbb{R}^n$ of the boundary of the codimension 0 submanifold $B(M)$. Again by the Collar Neighbourhood Theorem 2.5, we can pick a bicollar $X \times [-1, 1] \subset \partial B(M)$. We consider the open set $W_X := X \times (-1, 1)^2 \subset \partial B(M) \times (-1, 1) \subset \mathbb{R}^n$, and $D_X = X \times [-\frac{1}{2}, \frac{1}{2}]^2$. The local product structure theorem 5.18 [KS77, Essay I, Theorem 5.2] gives a PL structure on X and a PL structure σ_X on $\mathbb{R}^n$, which is concordant to σ_{std} rel. $\mathbb{R}^n \setminus \big(X \times (-\frac{2}{3}, \frac{2}{3})^2\big)$. This PL structure σ_X has the property that it agrees with the product PL structure on $X \times (-1, 1)^2$ in a neighbourhood of D_X. Thus X is a PL submanifold of $(\mathbb{R}^n, \sigma_X)$.

Now we arrange the next stratum $\partial B(M) \supset X$ to be a PL submanifold of $\mathbb{R}^n$. Near $D_X = X \times [-\frac{1}{2}, \frac{1}{2}]$, the PL structure σ_X is the product PL structure, and therefore $\partial B(M) \cap \operatorname{Int} D_X = X \times (-\frac{1}{2}, \frac{1}{2}) \times \{0\}$ is already a PL submanifold of $(\mathbb{R}^n, \sigma_X)$. Furthermore, σ_X is a product along $(-\frac{1}{2}, \frac{1}{2})$ near $X \times [-\frac{1}{3}, \frac{1}{3}]$. Pick $W_\partial = \partial B(M) \times (-1, 1)$, $C_\partial = X \times [-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}]$ and $D_\partial = \partial B(M) \times [-\frac{1}{2}, \frac{1}{2}]$. As above, the local structure theorem 5.18 [KS77, Essay I, Theorem 5.2] gives a PL structure σ_∂ on $\mathbb{R}^n$ such that $\partial B(M)$

is a PL submanifold and σ_∂ is concordant to σ_X rel. $\big(\mathbb{R}^n \setminus \partial B(M) \times (-\frac{2}{3}, \frac{2}{3})\big) \cup C_\partial$. Since $X \subset C_\partial$ the submanifold X is still a PL submanifold of $(\mathbb{R}^n, \sigma_\partial)$.

As in the closed case, use a concordance from σ_∂ to σ_{std} to obtain an isotopy $\phi_t \in \mathrm{Homeo}(\mathbb{R}^n)$ such that $\phi_0 = \mathrm{Id}$, and $\phi_1^*\sigma_{\mathrm{std}} = \sigma_\partial$. Define $D(M) := \phi_1(B(M))$ and $D(\partial M) := \phi_1(B(\partial M))$, which are both PL submanifolds of $(\mathbb{R}^n, \sigma_{\mathrm{std}})$. This finishes the case where M has nonempty boundary.

In both cases, ∂M empty and non-empty, our construction involved many choices. Let $D'(\partial M) \subset D'(M)$ be obtained by other choices. Following the discussion [KS77, p. 123], we can suitably stabilise the bundles and find a commutative diagram of PL maps:

$$\begin{tikzcd} D(M) \times D^s \arrow[r, "\cong"] & D'(M) \times D^r \\ D(\partial M) \times D^s \arrow[u] \arrow[r, "\cong"] & D'(\partial M) \times D^r, \arrow[u] \end{tikzcd}$$

where D^k denotes the disc with its standard PL structure and the horizontal maps are PL isomorphisms that preserve the zero sections up to homotopy.

Definition 14.4. The *simple homotopy type* of a compact connected n-manifold M is given by (M, s), where $s\colon M \to D(M)$ is the inclusion of the 0-section. The simple homotopy type of the pair $(M, \partial M)$ is given by the square

$$\begin{tikzcd} M \arrow[r, "s"] & D(M) \\ \partial M \arrow[u] \arrow[r, "s|_{\partial M}"] & D(\partial M), \arrow[u] \end{tikzcd}$$

where $D(\partial M) \subset D(M)$ are the disc bundles from Construction 14.3, with CW structures arising from a choice of PL triangulations corresponding to the PL structures.

By the commutative square at the end of Construction 14.3, the simple homotopy type of $(M, \partial M)$ is well-defined. Here we use that PL isomorphisms are simple: for any choice of triangulations underpinning the PL structures, the resulting homeomorphism is a simple homotopy equivalence. Also stabilising by D^s does not change the simple homotopy type, since as PL manifolds $D^s \cong D^{s-1} \times [-1, 1]$, and $D^{s-1} \times \{0\} \to D^{s-1} \times [-1, 1]$ is a simple equivalence.

Remark 14.5. Why is the simply homotopy type of ∂M obtained in this way the same as that obtained by applying Construction 14.3 with ∂M considered as a manifold without boundary?

For suitably high n, we may assume that the embedding of $(M, \partial M)$ into $\mathbb{R}^n$ is isotopic, and thus by Theorem 2.10 ambiently isotopic, to an embedding with $i\colon \partial M \hookrightarrow \{\vec{x} \in \mathbb{R}^n \mid x_1 = 0\} \cong \mathbb{R}^{n-1}$ and an (interior) collar $\partial M \times [0, 1]$ embedded as a product in $\{\vec{x} \in \mathbb{R}^n \mid 0 \leq x_1 \leq 1\}$ with $(x, t) \mapsto (i(x), t)$, as in Theorem 6.5. Such an isotopy does not affect the simple homotopy type obtained, by the argument sketched above, which can also be found on [KS77, p. 123]. The simple homotopy type of ∂M obtained from Construction 14.3, via an embedding of ∂M into $\mathbb{R}^{n-1}$, uses a disc bundle $D(\partial M)$ that stabilises using the x_1 direction to a disc bundle $D'(\partial M)$, with fibre a disc of one dimension higher, for ∂M embedded in $\mathbb{R}^n$. This latter disc bundle gives rise to the canonical simply homotopy type of ∂M from Definition 14.4.

Remark 14.6. If M is a smooth manifold, then M has an underlying PL structure, and with a bit more care in Construction 14.3, we can arrange that the bundle $D(M)$ is a PL bundle. Note that this is stronger than just a PL structure on the total space. For PL bundles, the bundle projection $D(M) \to M$ is a simple homotopy equivalence. Indeed, for trivial bundles this is discussed above, and in general the projection is an α-equivalence (a notion defined in loc. cit.) for any cover α of M and so is simple [Fer77, Corollary 3.2]. It follows that the simple homotopy type defined by (M, Id) agrees with the one of (M, s), and the same holds for the relative simple homotopy type of the pair $(M, \partial M)$.

According to [KS77, Essay III, Theorem 5.11], if a manifold has a triangulation, then the simple homotopy type of the manifold agrees with the simple homotopy type of that triangulation. It is not clear to us whether the analogous statement holds if M has a CW-structure not coming from a triangulation.

14.2. **The cellular chain complex and Poincaré triads.** Throughout this section let M be a compact connected n-manifold. Furthermore assume that we are given a decomposition $\partial M = R_- \cup R_+$ into codimension zero submanifolds such that $\partial R_- = R_- \cap R_+ = \partial R_+$.

The following proposition follows from the argument of Construction 14.3, applied with even more iterations to deal with corners of corners. See also the proof of [KS77, Essay III, Theorem 5.13].

Proposition 14.7. *There exists a finite CW-complex triad (X, X_-, X_+) and a homotopy equivalence of triads $f\colon (M, R_-, R_+) \to (X, X_-, X_+)$ such that the following two statements hold:*

1. *The restrictions of f to M, $R_\pm$ and $R_- \cap R_+$ give the simple homotopy types of these manifolds, as defined in the previous section.*
2. *The restrictions of f to the pairs $(M, \partial M)$, $(\partial M, R_\pm)$ and $(R_\pm, R_- \cap R_+)$ give the simple homotopy types of these pairs of manifolds, as defined in the previous section.*

We continue with a general definition regarding CW complexes.

Definition 14.8. Let (X, Y) be a pair of CW-complexes such that X is connected. We write $\pi = \pi_1(X)$ and we denote by $p\colon \widetilde{X} \to X$ the universal covering. We define

$$\begin{aligned} C_*^{\mathrm{cell}}(X, Y; \mathbb{Z}[\pi]) &:= \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi]} \underline{C_*^{\mathrm{cell}}(\widetilde{X}, p^{-1}(Y))} \\ C^*_{\mathrm{cell}}(X, Y; \mathbb{Z}[\pi]) &:= \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\underline{C_*^{\mathrm{cell}}(\widetilde{X}, p^{-1}(Y))}, \mathbb{Z}[\pi]). \end{aligned}$$

The group π acts freely on the left on the cells of the CW-complex $(\widetilde{X}, p^{-1}(Y))$. For each cell in X, pick a lift to $\widetilde{X}$. This turns $C_*^{\mathrm{cell}}(X, Y; \mathbb{Z}[\pi])$ and $C^*_{\mathrm{cell}}(X, Y; \mathbb{Z}[\pi])$ into based $\mathbb{Z}[\pi]$-module (co-) chain complexes.

Now we can state the main theorem of this section.

Theorem 14.9. *The finite CW-complex triad (X, X_-, X_+) is a simple Poincaré triad, meaning that there is a chain level representative $\sigma \in C_n^{\text{cell}}(X, X_- \cup X_+)$ of the fundamental class $[X] \in H_n(X, X_+ \cup X_-; \mathbb{Z}) = H_n(M, \partial M; \mathbb{Z})$ such that*

$$- \cap \sigma \colon C^{n-r}_{\text{cell}}(X, X_-; \mathbb{Z}[\pi_1(X)]) \;\to\; C_r^{\text{cell}}(X, X_+; \mathbb{Z}[\pi_1(X)])$$

is a simple chain homotopy equivalence.

The theorem is proved in [KS77, Essay III, Theorem 5.13]. In the Universal Poincaré Duality Theorem A.16 we will prove that there exists a homotopy equivalence between the two chain complexes. But we will not prove that there exists a *simple* homotopy equivalence; for that the reader will need to consult [KS77].

14.3. **Reidemeister torsion.** In this section we introduce Reidemeister torsion invariants for compact manifolds and discuss some of these key properties of these invariants.

Let M be a compact connected n-manifold and write $\pi = \pi_1(M)$. Let R_- be a compact codimension 0 submanifold of ∂M. In many applications $R_- = \varnothing$ or $R_- = \partial M$. We write $R_+ = \overline{\partial M \setminus R_-}$. Let F be a field and let $\alpha \colon \pi \to \mathrm{GL}(d, F)$ be a representation of the fundamental group of M. With respect to this representation, we consider the twisted homology $H_k(M, R_-; F^d)$, as defined in Section A.1.

Assumption 14.10. *Suppose that $H_k(M, R_-; F^d) = 0$ for all k.*

Pick a homotopy equivalence of triads $f \colon (M, R_-, R_+) \to (X, X_-, X_+)$ as in Proposition 14.7. We use the homotopy equivalence f to make the identification $\pi_1(X) = \pi$. By a serious abuse of notation, we refer to the cellular chain complex $C_*^{\text{cell}}(X, X_-; \mathbb{Z}[\pi])$ of (X, X_-) as the *cellular chain complex* $C_*^{\text{cell}}(M, R_-; \mathbb{Z}[\pi])$ *of* (M, R_-). As in Section 14.2 we view $C_*^{\text{cell}}(M, R_-; \mathbb{Z}[\pi])$ as a based left $\mathbb{Z}[\pi]$-module chain complex. Equip the F-module chain complex $C_*^{\text{cell}}(M, R_-; F^d) = F^d \otimes_{\mathbb{Z}[\pi]} C_*^{\text{cell}}(M, R_-; \mathbb{Z}[\pi])$ with the basing given by the tensor products of the $\mathbb{Z}[\pi]$-bases of $C_*^{\text{cell}}(M, R_-; \mathbb{Z}[\pi])$ and the canonical F-basis for F^d.

We write $\sim_\alpha$ for the equivalence relation on $F^\times := F \setminus \{0\}$ that is given by the subgroup $\{\pm \det(\alpha g) \mid g \in \pi_1(M)\} \subset F^\times$. We define $\tau(M, R_-, \alpha) \in F^\times / \sim_\alpha$ to be the Reidemeister torsion of the above acyclic, based F-module chain complex. We refer to [Tur01, Section 6] for the definition of the Reidemeister torsion of an acyclic, based F-module chain complex. It follows from a slight generalisation of [Tur01, Theorem 9.1] that $\tau(M, R_-, \alpha) \in F^\times / \sim_\alpha$ is well-defined, in that it is independent of the choice of the representative of the simple homotopy type of (X, X_-, X_+) and it is independent of the choice of the lifts of the cells.

The following two theorems give the two arguably most important properties of torsion.

Theorem 14.11. *Let M be a compact connected n-manifold and let R_- be a compact codimension 0 submanifold of ∂M. Let $\alpha \colon \pi_1(M) \to \mathrm{GL}(d, F)$ be such that $H_*(R_-; F^d) = 0 = H_*(M; F^d)$. By abuse of notation we also write α for the composition $\alpha \colon \pi_1(\partial M) \to \pi_1(M) \to \mathrm{GL}(d, F)$ defined for each connected component of ∂M using the basing paths as* *described above. Then we have*

$$\tau(M,\alpha) \;=\; \tau(R_-,\alpha)\cdot\tau(M,R_-,\alpha) \;\in\; F^\times/\sim_\alpha .$$

Proof. We have the following short exact sequence of chain complexes with compatible bases:

$$0 \;\to\; C_*^{\mathrm{cell}}(X_-;F^d) \;\to\; C_*^{\mathrm{cell}}(X;F^d) \;\to\; C_*^{\mathrm{cell}}(X,X_-;F^d) \;\to\; 0.$$

Given such a short exact sequence, the multiplicativity of the torsion is proven in [Tur01, Theorem 3.4]. □

Definition 14.12. Let F be a field with (possibly trivial) involution. Given a representation $\alpha\colon \pi\to \mathrm{GL}(d,F)$ we denote the representation $g\mapsto \overline{\alpha(g^{-1})}^T$ by $\alpha^\dagger$. We say that α is *unitary* if $\alpha=\alpha^\dagger$.

Example 14.13. Let $\phi\colon \pi\to\mathbb{Z}$ be a group homomorphism. Equip $\mathbb{Q}(t)$ with the usual involution given by $\bar{t}=t^{-1}$. The representation $\alpha\colon\pi\to\mathrm{GL}(1,\mathbb{Q}(t))$ given by $g\mapsto t^{\phi(g)}$ is unitary.

Theorem 14.14. *Let M be a compact n-manifold with (possibly empty) boundary. Assume that we are given a decomposition $\partial M = R_-\cup R_+$ into codimension zero submanifolds such that $\partial R_- = R_-\cap R_+ = \partial R_+$. Furthermore let F be a field with (possibly trivial) involution. Let $\alpha\colon \pi_1(M)\to\mathrm{GL}(d,F)$ be a representation such that $H_*(\partial M;F^d)=0=H_*(M;F^d)$. Then*

$$\tau(M,R_-,\alpha) \;=\; \overline{\tau(M,R_+,\alpha^\dagger)}^{(-1)^{n+1}} \text{ in } F^\times/\sim_\alpha .$$

In particular, if α is unitary we have

$$\tau(M,R_-,\alpha) \;=\; \overline{\tau(M,R_+,\alpha)}^{(-1)^{n+1}} \text{ in } F^\times/\sim_\alpha .$$

Proof. We write $\pi=\pi_1(M)$. Write $C_*^\pm = C_*^{\mathrm{cell}}(M,R_\pm;\mathbb{Z}[\pi])$, recalling the convention described below Assumption 14.10.

It follows from Theorem 14.9 that the torsion of the based F-module chain complex $F^d\otimes_{\mathbb{Z}[\pi]} C_*^-$ agrees with the torsion of the based F-module chain complex

$$F^d\otimes_{\mathbb{Z}[\pi]}\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C^+_{n-*}},\mathbb{Z}[\pi]).$$

Consider the following isomorphism of based left F-module chain complexes

$$\begin{array}{rcl} F^d_\alpha\otimes_{\mathbb{Z}[\pi]}\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C^+_{n-*}},\mathbb{Z}[\pi]) & \to & \overline{\mathrm{Hom}_{\text{left-}F}(F^d_{\alpha^\dagger}\otimes_{\mathbb{Z}[\pi]} C^+_{n-*},F)} \\ v\otimes\varphi & \mapsto & \left(\begin{array}{rcl} F^d_{\alpha^\dagger}\otimes_{\mathbb{Z}[\pi]}\overline{C^+_{n-*}} & \to & F \\ (w\otimes\sigma) & \mapsto & \overline{v\alpha(\varphi(\sigma))\overline{w}^T}\end{array}\right)\end{array}$$

Using this isomorphism $\tau(M,R_-,\alpha)$ also equals the torsion of the chain complex on the right hand side. It follows from algebraic duality for torsions [Tur01, Theorem 1.9] that the torsion of the based chain complex on the right hand side equals $\overline{\tau(M,R_+,\alpha^\dagger)}^{(-1)^{n+1}}$. □

15. Obstructions to being topologically slice

15.1. **The Fox-Milnor Theorem.** In this section we provide an example of the use of many of the theorems described above by applying them to obtain an obstructions for a knot to be topologically slice.

Definition 15.1. Let Y be a homology 3-sphere that is the boundary of an integral homology 4-ball X.

(1) We say a knot K in Y is *topologically slice in* X if K bounds a *slice disc*, that is a proper submanifold of X homeomorphic to a disc.
(2) Suppose X is equipped with a smooth structure, e.g. $X = D^4$. We say a knot K in Y is *smoothly slice in* X if K bounds a *smooth slice disc*, that is a proper smooth submanifold of X diffeomorphic to a disc.

There are many classical obstructions to a knot being smoothly slice. For example, there are obstructions based on the Alexander polynomial [FM66] and the Levine-Tristram signatures [Tri69, Lev69] and there are the more subtle Casson-Gordon [CG78, CG86] obstructions. Even though these results, having appeared prior to the work of Freedman and Quinn, were formulated as obstructions to being smoothly slice, it has been understood for many years that the original proofs can be modified to prove that these are in fact obstructions to being topologically slice.

In this section we will prove the following sample theorem on the Alexander polynomial of a topologically slice knot.

Theorem 15.2 (Fox-Milnor)**.** *Suppose that K is a knot in a homology 3-sphere Y that bounds an integral homology 4-ball X. If K is slice in X, then the Alexander polynomial $\Delta_K(t)$ of K factors as $\Delta_K(t) = \pm t^k f(t)f(t^{-1})$ for some $k \in \mathbb{Z}$ and for some $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $f(1) = \pm 1$.*

Even though this result is very well known we want to provide a detailed proof. In particular we want to highlight where some of the results discussed in this article are used. The reader is encouraged to go through the other papers mentioned above and to modify the proofs to deal with topologically slice knots.

15.2. **A proof of the Fox-Milnor Theorem.** For the proof of the Fox-Milnor Theorem 15.2 we adopt the following notation.

(1) Let Y be a homology 3-sphere Y bounding some integral homology 4-ball X.
(2) Given a knot K in Y, denote its zero framed surgery by N_K.
(3) Given an oriented knot K let μ_K be an oriented meridian.
(4) For a slice disc D in X, let $N(D)$ be a tubular neighbourhood provided by Theorem 6.8. We refer to $W_D = \overline{X \setminus N(D)}$ as the *exterior of* D.
(5) The ring of integral Laurent polynomials in one variable is denoted $\mathbb{Z}[t, t^{-1}]$ or $\mathbb{Z}[t^{\pm 1}]$.

Many topological slicing obstructions, such as knot signatures [Tri69], the Fox-Milnor condition [FM66], the Blanchfield form [Kea75], Casson-Gordon invariants [CG78, CG86], L^2-signature defects [COT03] and $L^{(2)}$-von Neumann ρ-invariants [CT07], rely implicitly and explicitly on the next three propositions or slight variations thereof.

Proposition 15.3. *Let K be an oriented knot in Y and let D be a slice disc in Y.*

(1) *We have $\partial W_D = N_K$.*
(2) *The inclusion map $\mu_K \to W_D$ induces a $\mathbb{Z}$-homology equivalence.*

In the remainder of this section, given an oriented knot, we use $\phi\colon \pi_1(N_K) \to \langle t\rangle$ and $\phi\colon \pi_1(W_D) \to \langle t\rangle$ to denote the unique homomorphisms that send the oriented meridian to t. These homomorphisms allow us to view $\mathbb{Z}[t^{\pm 1}]$ and $\mathbb{Q}(t)$ as a $\mathbb{Z}[\pi_1(N_K)]$-module and a $\mathbb{Z}[\pi_1(W_D)]$-module.

Proof. For the first statement, we have to check that the framing of K induced by the unique trivialisation $\nu D \cong D^2 \times \mathbb{R}^2$ is the 0–framing. Consider S^4 with an equatorial S^3, which contains K and splits S^4 into two 4–balls. Let D be contained in the one 4–ball, and push a Seifert surface Σ into the other 4–ball. Pick a normal bundle $\nu S^3 = S^3 \times I$, and arrange using Theorem 6.5 that $D \cap \nu S^3 = K \times [-1, 0]$, and $\Sigma \cap \nu S^3 = K \times [0, 1]$. Let $F = \Sigma \cup -D \subset S^4$. We compute the Euler number $e(F) \in \mathbb{Z}$ in two ways. First, note that $e(F) = [F] \cdot [F] = 0$, since $H_2(S^4; \mathbb{Z}) = 0$. On the other hand, the number $e(F)$ is also the difference between the induced framings of $\nu\Sigma|_K$ and $\nu D|_K$. Consequently, the two framings agree and νD induces the 0–framing, which by definition is the framing induced by $\nu\Sigma|_K$.

We turn to the proof of the second statement. By Proposition 6.13 the tubular neighbourhood of D is trivial, thus we can identify it with $D \times D^2$. Let $\mu_K \to W_D$ be the inclusion of the meridian μ_K of K. Then we have $H_*(W_D, * \times \mu_K; \mathbb{Z}) = H_*(W_D, D \times S^1; \mathbb{Z}) = H_*(X, D \times D^2) = 0$ by excision and the hypothesis that X be a homology 4-ball. By the homology long exact sequence for the pair (W_D, μ_K), the meridional map $\mu_K \to W_D$ induces a homology equivalence, so W_D is a homology circle. □

Proposition 15.4. *The exterior W_D of a slice disc D is homotopy equivalent to a finite 3-dimensional CW complex. In particular the homology groups*

$$H_*(W_D; \mathbb{Z}[t^{\pm 1}]),\ H_*(W_D, N_K; \mathbb{Z}[t^{\pm 1}]) \text{ and } H_*(N_K; \mathbb{Z}[t^{\pm 1}])$$

are all finitely generated.

Proof. Note that W_D is a compact 4-manifold with nonempty boundary. It follows from Theorem 4.5 that W_D is homotopy equivalent to a 3-dimensional CW complex. The statements regarding the homology groups follow from Proposition A.9. □

Proposition 15.5.

(1) *For any knot K in a homology 3-sphere the modules $H_*(N_K; \mathbb{Z}[t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$-torsion.*
(2) *If D is a slice disc, then all the modules $H_*(W_D; \mathbb{Z}[t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$-torsion.*

Proof. We start out with the proof of the second statement. Let $P \subset \mathbb{Z}[t^{\pm 1}]$ be the multiplicative subset of Laurent polynomials that augment to ± 1, that is $p(1) = \pm 1$ if and only if $p \in P$. We shall prove the slightly stronger statement, that $H_k(W_D; \mathbb{Z}[t^{\pm 1}])$ is P-torsion for $k > 0$. Since $H_0(W_D; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/(t-1)$ is $\mathbb{Z}[t^{\pm 1}]$-torsion, the result will follow. We write $\pi = \pi_1(W_D)$. Let $Q := P^{-1}\mathbb{Z}[t^{\pm 1}]$ be the result of inverting the polynomials in P. By Proposition A.5 there exists a chain complex C_* of finite length consisting of finitely generated free left $\mathbb{Z}[\pi]$-modules such that for any ring R and any $(R, \mathbb{Z}[\pi])$-bimodule A we have

$$H_k(W_D, \mu_K; A) \;\cong\; H_k(A \otimes_{\mathbb{Z}[\pi]} C_*).$$

By Proposition 15.3 we know that $H_k(\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*) = H_k(W_D, \mu_K; \mathbb{Z}) = 0$. Since C_* is a chain complex of finite length consisting of finitely generated free left $\mathbb{Z}[\pi]$-modules we obtain from chain homotopy lifting [COT03, Proposition 2.10], see also [NP17, Lemma 3.1], that $H_k(Q \otimes_{\mathbb{Z}[\pi]} C_*) = 0$. A straightforward calculation shows that $H_*(S^1, \mathrm{pt}; Q) = 0$. It follows that $H_*(W_D, \mathrm{pt}; Q) = 0$, so that $H_k(W_D; \mathbb{Z}[t^{\pm 1}])$ is P-torsion for $k > 0$.

The first statement is very well known. One of the many proofs would be to use the above argument and the fact that $S^1 \to Y \setminus \nu K$ is a homology equivalence to show that the modules $H_*(Y \setminus \nu K; \mathbb{Z}[t^{\pm 1}])$ are torsion. A basic Mayer-Vietoris argument then shows that the modules $H_*(N_K; \mathbb{Z}[t^{\pm 1}])$ are also torsion. $\square$

We want to recall the definition of the Alexander polynomial of a knot. To do so we need the notion of the order of a module.

Definition 15.6. Let H be a finitely generated free abelian group and let M be a finitely generated $\mathbb{Z}[H]$-module. By [Lan02a, Corollary IV.9.5] the ring $\mathbb{Z}[H]$ is Noetherian which implies that M admits a free resolution

$$\mathbb{Z}[H]^r \;\xrightarrow{\cdot A}\; \mathbb{Z}[H]^s \;\to\; M \;\to\; 0.$$

Without loss of generality we can assume that $r > s$. Since $\mathbb{Z}[H]$ is unique factorisation domain, see [Lan02a, Lemma IV.2.3], the order $\mathrm{ord}(M)$ is defined as the greatest common divisor of the $s \times s$-minors of A. By [Tur01, Lemma 4.4] the order is well-defined, i.e. independent of the choice of the free resolution, up to multiplication by a unit in $\mathbb{Z}[H]$.

In the proof of the Fox-Milnor theorem we will need the following lemma, collecting basic facts about orders of finitely generated $\mathbb{Z}[H]$-modules.

Lemma 15.7. *Let H be a finitely generated free abelian group.*

(1) *If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated $\mathbb{Z}[H]$-modules, then* $\mathrm{ord}(B) = \mathrm{ord}(A) \cdot \mathrm{ord}(C)$.
(2) *If $0 \to C_k \to \cdots \to C_0 \to 0$ is an exact sequence of finitely generated $\mathbb{Z}[H]$-torsion modules, then the alternating product of the orders is a unit in $\mathbb{Z}[H]$.*
(3) *For any finitely generated $\mathbb{Z}[H]$-module A we have* $\mathrm{ord}(\overline{A}) = \overline{\mathrm{ord}(A)}$.
(4) *For any finitely generated torsion $\mathbb{Z}[H]$-module A we have* $\mathrm{ord}(\mathrm{Ext}^1_{\mathbb{Z}[H]}(A, \mathbb{Z}[H])) = \mathrm{ord}(A)$.

Proof. Statement (1) is proved for $H \cong \mathbb{Z}$ in [Lev67, Lemma 5]. The general case follows from [Hil12, Theorem 3.12]. Note that (2) is an immediate consequence of (1), by separating the long exact sequence into short exact sequences such at $0 \to \operatorname{Im} C_j \to C_{j-1} \to \operatorname{Im} C_{j-1} \to 0$, applying (1), and performing substitutions using the resulting equations involving orders.

Next (3) follows immediately from the definition. Finally (4) is well-known to the experts, but we could not find a reference, therefore we sketch the key ingredients in the proof. We introduce the following notation.

(a) Given any prime ideal $\mathfrak{p}$ of $\mathbb{Z}[H]$, let $\mathbb{Z}[H]_{\mathfrak{p}}$ be the localisation at $\mathfrak{p}$, that is we invert all elements that do not lie in $\mathfrak{p}$. We view $\mathbb{Z}[H]$ as a subring of $\mathbb{Z}[H]_{\mathfrak{p}}$.
(b) Given a ring R and $f, g \in R$ we write $f \doteq_R g$ if f and g differ by multiplication by a unit in R.

Now we sketch the proof of (4). We will use the following five observations.

(i) First note that since $\mathbb{Z}[H]$ is a unique factorisation domain, for any prime element $p \in \mathbb{Z}[H]$ the ideal (p) is a prime ideal.
(ii) Note that being unique factorisation domain and being Noetherian is preserved under localisation, see [Pes96, Theorem 7.53] and [Row06, Corollary 8.8']. In particular each $\mathbb{Z}[H]_{\mathfrak{p}}$ is a noetherian unique factorisation domain. This allows us, by the same definitions as above, to define the order of a finitely generated module over $\mathbb{Z}[H]_{\mathfrak{p}}$.
(iii) Localisation is flat [Lan02a, Proposition XVI.3.2]. It follows that for any finitely generated $\mathbb{Z}[H]$-module M and any prime element $p \in \mathbb{Z}[H]$ one has $\operatorname{ord}(M) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} M)$ and $\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} \operatorname{Ext}^1_{\mathbb{Z}[H]}(M, \mathbb{Z}[H]) \cong \operatorname{Ext}^1_{\mathbb{Z}[H]_{(p)}}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} M, \mathbb{Z}[H]_{(p)})$ as $\mathbb{Z}[H]_{(p)}$-modules.
(iv) By [Osb00, Corollary A.14] any commutative ring that has the property that each prime ideal is principal, is a PID. It follows easily that for each prime element p, the localisation $\mathbb{Z}[H]_{(p)}$ is a PID.
(v) Let L be a torsion $\mathbb{Z}[H]_{(p)}$-module. Since $\mathbb{Z}[H]_{(p)}$ is a PID every two elements have a greatest common divisor. We can therefore perform row and column operations to find a resolution for L such that the presentation matrix is diagonal. From this observation one easily deduces that $L \cong \operatorname{Ext}^1_{\mathbb{Z}[H]_{(p)}}(L, \mathbb{Z}[H]_{(p)})$ as left $\mathbb{Z}[H]_{(p)}$-modules. Since L is torsion the presentation matrix is injective and so its transpose presents the Ext group. To convert the Ext group to a left module, we use the trivial involution, which we may do since $\mathbb{Z}[H]_{(p)}$ is a commutative ring.
(vi) Suppose that f and g are in $\mathbb{Z}[H]$. If $f \doteq_{\mathbb{Z}[H]_{(p)}} g$ for all prime elements $p \in \mathbb{Z}[H]$, then since $\mathbb{Z}[H]$ is a unique factorisation domain we must have $f \doteq_{\mathbb{Z}[H]} g$.

Now with $L = \mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A$ a finitely generated $\mathbb{Z}[H]$-torsion module, we have

$$\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A \cong \operatorname{Ext}^1_{\mathbb{Z}[H]_{(p)}}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A, \mathbb{Z}[H]_{(p)})$$

for every prime element p, by (iv). On the other hand, again for each prime element p, we have

$$\operatorname{ord}(A) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A)$$

by (iii). Combining these two observations yields

$$\operatorname{ord}(A) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\operatorname{Ext}^1_{\mathbb{Z}[H]_{(p)}}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A, \mathbb{Z}[H]_{(p)})).$$

By the second part of (iii) we have that

$$\operatorname{ord}(\operatorname{Ext}^1_{\mathbb{Z}[H]_{(p)}}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} A, \mathbb{Z}[H]_{(p)})) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} \operatorname{Ext}^1_{\mathbb{Z}[H]}(A, \mathbb{Z}[H])).$$

By the first part of (iii) again we have

$$\operatorname{ord}(\mathbb{Z}[H]_{(p)} \otimes_{\mathbb{Z}[H]} \operatorname{Ext}^1_{\mathbb{Z}[H]}(A, \mathbb{Z}[H])) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\operatorname{Ext}^1_{\mathbb{Z}[H]}(A, \mathbb{Z}[H])).$$

Thus combining the last three equalities we have

$$\operatorname{ord}(A) \doteq_{\mathbb{Z}[H]_{(p)}} \operatorname{ord}(\operatorname{Ext}^1_{\mathbb{Z}[H]}(A, \mathbb{Z}[H]))$$

for all prime elements p. Now (4) follows by applying (vi). □

We use the notion of order to define the Alexander polynomial of a knot in a homology 3-sphere.

Definition 15.8. The *Alexander polynomial* $\Delta_K(t)$ of a knot K is defined as the order of the Alexander module $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Note that this polynomial is only well-defined up to units in $\mathbb{Z}[t^{\pm}]$.

After these preparations we turn to the actual proof of the Fox-Milnor Theorem 15.2. We need the following elementary lemma.

Lemma 15.9. *Let π be a group, let C_* be a chain complex of left free $\mathbb{Z}[\pi]$-modules and let $\phi\colon \pi \to \langle t\rangle$ be a homomorphism. The map*

$$\begin{array}{rcl} \operatorname{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*}, \mathbb{Z}[t^{\pm 1}]) & \to & \overline{\operatorname{Hom}_{\text{left-}\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[\pi]} C_*, \mathbb{Z}[t^{\pm 1}])} \\ f & \mapsto & (p \otimes \sigma \mapsto p \cdot \overline{f(\sigma)}) \end{array}$$

is well-defined and is an isomorphism of left $\mathbb{Z}[t^{\pm 1}]$-cochain complexes. □

First proof of the Fox-Milnor theorem 15.2. In this proof we abbreviate $\Lambda := \mathbb{Z}[t^{\pm 1}]$. We start out with the following three observations.

(a) We have $H_0(W_D; \Lambda) \cong H_0(N_K; \Lambda) \cong \Lambda/(t-1)$.
(b) We have $H_0(W_D, N_K; \Lambda) = 0$,
(c) By Proposition 15.5 and Proposition 15.4 we know that for all k

$$\operatorname{Ext}^0_\Lambda(H_k(W_D, N_K; \Lambda), \Lambda) \;=\; \operatorname{Hom}_\Lambda(H_k(W_D, N_K; \Lambda), \Lambda) \;=\; 0.$$

Claim. For any $i \in \mathbb{N}$ we have

$$\begin{array}{rcl} H_i(N_K; \Lambda) & \cong & \overline{\operatorname{Ext}^1_\Lambda(H_{2-i}(N_K; \Lambda), \Lambda)} \\ H_i(W_D; \Lambda) & \cong & \overline{\operatorname{Ext}^1_\Lambda(H_{3-i}(W_D, N_K; \Lambda), \Lambda)}. \end{array}$$

We prove the second statement of the claim. The proof of the first statement is almost identical. By the Poincaré duality theorem A.15 we have an isomorphism $H_i(W_D;\Lambda) \cong H^{4-i}(W_D,N_K;\Lambda)$ of Λ-modules. By Lemma 15.9, applied to $C_* = C_*(W_D,N_K;\mathbb{Z}[\pi])$, we know that

$$H^{4-i}(W_D,N_K;\Lambda) \;\cong\; \overline{H_{4-i}(\operatorname{Hom}_\Lambda(\Lambda\otimes_{\mathbb{Z}[\pi]} C_*(W_D,N_K;\mathbb{Z}[\pi]),\Lambda))}.$$

Finally we apply the universal coefficient spectral sequence [Lev77, Theorem 2.3] to the Λ-module chain complex $C_*(W_D,N_K;\Lambda)$. It follows from the above observations (b) and (c) that the spectral sequence collapses and that we have an isomorphism

$$H_{4-i}\big(\operatorname{Hom}_\Lambda(\Lambda\otimes_{\mathbb{Z}[\pi]} C_*(W_D,N_K;\Lambda)\big) \;\cong\; \operatorname{Ext}^1_\Lambda(H_{3-i}(W_D,N_K;\Lambda),\Lambda).$$

This concludes the proof of the claim.

Next we consider the long exact sequence of the pair (W_D,N_K) of twisted homology with Λ-coefficients:

$$\cdots\to H_2(W_D;\Lambda)\to H_2(W_D,N_K;\Lambda)\to H_1(N_K;\Lambda)\to H_1(W_D;\Lambda)\to H_1(W_D,N_K;\Lambda)\to\dots$$

It follows from Propositions 15.4 that all the above modules are finitely generated modules. Thus it makes sense to consider their orders. Also note that in Proposition 15.5 we saw that the modules for N_K and W_D are all Λ-torsion. It follows from the long exact sequence that the relative homology groups $H_*(W_D,N_K;\Lambda)$ are also Λ-torsion. By Lemma 15.7 (3) the alternating product of the orders equals $\pm t^k$.

By the above claim and Lemma 15.7 (3) and (4) the orders are anti-symmetric around $H_1(N_K;\Lambda)$. More precisely, we have

$$\operatorname{ord}(H_2(W_D,N_K;\Lambda)) \;=\; \operatorname{ord}\big(\overline{\operatorname{Ext}^1_\Lambda(H_1(W_D;\Lambda),\Lambda)}\big) \;=\; \overline{\operatorname{ord}(H_1(W_D;\Lambda))},$$

and the same type of relation holds as we progress further from the middle term $H_1(N_K;\Lambda)$ in the above long exact sequence. But this implies that there exist non-zero polynomials $f,g\in\Lambda$ with $f\cdot\overline{f} = \Delta_K(t)\cdot g\cdot\overline{g}$. By considering irreducible factors, we obtain the desired result. $\square$

We conclude with an alternative argument for the Fox-Milnor theorem in the topological category using Reidemeister torsion. The advantage of the Reidemeister torsion invariant is that proofs are often easier, and it has in general a smaller indeterminacy than the order of homology, although this will not manifest itself in the upcoming proof.

Second proof of Theorem 15.2. We continue with the notation introduced above. As before we have a homomorphism $\alpha\colon \pi_1(W_D)\to H_1(W_D;\mathbb{Z})\xrightarrow{\cong}\mathbb{Z}$, sending an oriented meridian of K to $1\in\mathbb{Z}$. As usual $\mathbb{Q}(t)$ denotes the field of fractions of the Laurent polynomial ring $\mathbb{Z}[t,t^{-1}]$. We take $d=1$, and so obtain a representation $\phi\colon \pi_1(W_D)\to \mathrm{GL}(1,\mathbb{Q}(t))$, that sends $g\mapsto (t^{\alpha(g)})$. In the previous proof we had already seen that the modules $H_*(N_K;\mathbb{Z}[t^{\pm1}])$, $H_*(W_D;\mathbb{Z}[t^{\pm1}])$ and $H_*(N_K;\mathbb{Z}[t^{\pm1}])$ are $\mathbb{Z}[t^{\pm1}]$-torsion. Since $\mathbb{Q}(t)$ is flat over $\mathbb{Z}[t^{\pm1}]$ it follows that the corresponding twisted homology groups with $\mathbb{Q}(t)$-coefficients are zero.

By the discussion in Section 14.3 we can consider the Reidemeister torsions $\tau(W_D,\phi)$, $\tau(N_K,\phi)$ and $\tau(W_D,N_K,\phi)$. By Theorem 14.14, $\tau(W_D,N_K,\phi)=\overline{\tau(W_D,\phi)}^{(-1)^5}=\overline{\tau(W_D,\phi)}^{-1}$. Since the torsion is multiplicative in short exact sequences by Theorem 14.11, we have that

$$\tau(W_D,\phi)=\tau(N_K,\phi)\cdot\tau(W_D,N_K,\phi)=\tau(M_K,\phi)\cdot\overline{\tau(W_D,\phi)}^{-1}.$$

By [Tur01, Theorem 14.12] the torsion of the zero surgery of a knot is equal to $\Delta_K(t)/((t-1)(t^{-1}-1))$. It follows that $\Delta_K(t)$ is a norm as claimed. □

Remark 15.10. The two proofs presented above avoid the use of the smooth category, and so are in keeping with the spirit of this article. However, one can give a further alternative proof by allowing smooth techniques. First one can use Theorem 8.6 to find a simply connected 4-manifold W' such that $W:=V\#W'$ is smoothable. Then one can triangulate W and apply Reidemeister torsion machinery without appealing to [KS77, Essay III]. The disadvantage of this approach is that typically $H_2(W';\mathbb{Z})$ will be nontrivial, so that W is not acyclic over $\mathbb{Q}(t)$. One can proceed by choosing a self-dual basis for homology, so that one can still obtain a torsion invariant that is well-defined up to norms. Apply [CF13, Theorem 2.4], and argue that since the intersection form of W is nonsingular, the contribution of W' to the torsion is a norm.

Appendix A. Poincaré Duality with twisted coefficients

Surveying the literature, we felt it would be of benefit to have a more detailed proof of Poincaré duality with twisted coefficients for manifolds with boundary, but without a smooth or PL structure, so we offer one in this appendix. One can find other proofs of Poincaré duality for some subsets of these conditions.

A.1. **Twisted homology and cohomology groups.** We start out with the following notation.

Notation A.1. Given a group π and a left $\mathbb{Z}[\pi]$-module A, write $\overline{A}$ for the right $\mathbb{Z}[\pi_1(M)]$-module that has the same underlying abelian group but for which the right action of π is defined by $a\cdot g:=g^{-1}\cdot a$ for $a\in A$ and $g\in\pi$. The same notation is also used with the rôles of left and right reversed and $g\cdot a:=a\cdot g^{-1}$.

We recall the definition of twisted homology and cohomology groups.

Definition A.2. Let X be a connected topological space that admits a universal cover $p\colon\widetilde{X}\to X$. Write $\pi=\pi_1(X)$. Let Y be a subset of X, let A be a right $\mathbb{Z}[\pi]$-module. Let π act on $\widetilde{X}$ by deck transformations, which is naturally a left action. Thus, the singular chain complex $C_*(\widetilde{X},p^{-1}(Y))$ becomes a left $\mathbb{Z}[\pi]$-module chain complex. Define the *twisted chain complex*

$$C_*(X,Y;A)\ :=\ \big(A\otimes_{\mathbb{Z}[\pi]}C_*(\widetilde{X},p^{-1}(Y)),\mathrm{Id}\otimes\partial_*\big).$$

The corresponding *twisted homology groups* are $H_k(X,Y;A)$. With $\delta^k = \operatorname{Hom}(\partial_k, \operatorname{Id})$ define the *twisted cochain complex* to be

$$C^*(X,Y;A) \;:=\; \big(\operatorname{Hom}_{\text{right-}\mathbb{Z}[\pi]}\big(\overline{C_*(\widetilde{X},p^{-1}(Y))},A\big),\delta^*\big).$$

The corresponding *twisted cohomology groups* are $H^k(X,Y;A)$.

Note that if R is some ring (not necessarily commutative) and if A is an $(R,\mathbb{Z}[\pi])$-bimodule, then the above twisted homology and cohomology groups are naturally left R-modules.

Given a CW complex one can similarly define twisted cellular (co-) chain complexes and twisted cellular (co-) homology groups. The following proposition implies that twisted singular (co-) homology groups are isomorphic to twisted cellular (co-) homology groups.

Proposition A.3. *Let (X,Y) be a CW complex pair and write $\pi = \pi_1(X)$. The singular chain complexes $C_*^{\text{sing}}(X,Y;\mathbb{Z}[\pi])$ and $C_*^{\text{cell}}(X,Y;\mathbb{Z}[\pi])$ are chain equivalent as chain complexes of left $\mathbb{Z}[\pi]$-modules.*

The proof of Proposition A.3 relies on the following very useful lemma.

Lemma A.4. *Let $f\colon C_* \to D_*$ be a chain map of free $\mathbb{Z}[\pi]$-module chain complexes (here chain complexes are understood to start in degree 0) that induces an isomorphism on homology. Then f is a chain equivalence.*

Proof. Since f induces an isomorphism of homology groups we know that the $\operatorname{cone}(f)_*$ is acyclic. By assumption C_* and D_* are free $\mathbb{Z}[\pi]$-modules and so is $\operatorname{cone}(f)_*$. But this guarantees the existence of a chain homotopy $\operatorname{Id}_{\operatorname{cone} f_*} \simeq_P 0$, since we can view $\operatorname{cone}(f)_*$ as a free resolution of 0 and any two such resolutions are chain homotopic. Recall that chain homotopy means

$$\partial^{\operatorname{cone} f_*} \circ P + P \circ \partial^{\operatorname{cone} f_*} = \operatorname{Id}_{\operatorname{cone} f_*} \tag{A.1}$$

If we write P as a matrix

$$P_n = \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} : \begin{array}{c} C_{n-1} \\ D_n \end{array} \to \begin{array}{c} C_n \\ D_{n+1} \end{array}$$

then one easily verifies using Equation (A.1), that $P_*^{12}\colon D_* \to C_*$ is a chain homotopy inverse of f_*, where the chain homotopies are given by P_*^{11} and P_*^{22}. □

Proof of Proposition A.3. In [Sch68, p. 303] (see also [Lüc98, Lemma 4.2]) it is shown that there exists a natural chain homotopy equivalence $C_*^{\text{sing}}(\widetilde{X},p^{-1}(Y)) \to C_*^{\text{cell}}(\widetilde{X},p^{-1}(Y))$ of $\mathbb{Z}$-modules, where $p\colon \widetilde{X} \to X$ denotes the universal cover. Since the chain homotopy is natural it is in particular π-equivariant. In other words, it is a chain homotopy equivalence of $\mathbb{Z}[\pi]$-modules. Now the proposition follows from Lemma A.4. □

Proposition A.5. *Let M be a compact n-manifold and let $N \subset M$ be a subspace that is a compact manifold in its own right. Write $\pi = \pi_1(M)$. There exists a chain complex C_* of* *finite length consisting of finitely generated free left $\mathbb{Z}[\pi]$-modules such that for any ring R and for any $(R,\mathbb{Z}[\pi])$-bimodule A we have left R-module isomorphisms*

$$H_k(M,N;A) \;\cong\; H_k(A\otimes_{\mathbb{Z}[\pi]} C_*)$$

and

$$H^k(M,N;A) \;\cong\; H_k(\operatorname{Hom}_{\mathbb{Z}[\pi]}(\overline{C_*},A)).$$

Remark A.6. Note that we do *not* demand that N be a submanifold of M. For example N could be a union of boundary components of M, or N could be a submanifold of the boundary. Evidently N could also be the empty set.

Proof. By Theorem 4.5 the manifolds M and N are homotopy equivalent to finite CW complexes X and Y respectively. Let $i\colon N\to M$ be the inclusion map. By the cellular approximation theorem there exists a cellular map $j\colon X\to Y$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "\simeq"'] & M \arrow[d, "\simeq"] \\ Y \arrow[r, "j"] & X. \end{tikzcd}$$

Next we replace M and X by the mapping cylinders of i and j respectively, to create cofibrations. Given a map $f\colon U\to V$ between topological spaces let $\operatorname{cyl}(f)$ be the mapping cylinder. We view U as a subset of $\operatorname{cyl}(f)$ in the obvious way. With this notation we have

$$H_k(M,N;A)\cong H_k(\operatorname{cyl}(i\colon N\to M),N;A)\cong H_k(\operatorname{cyl}(j\colon Y\to X),Y;A).$$

The mapping cylinder $Z:=\operatorname{cyl}(j\colon Y\to X)$ admits the structure of a finite CW complex such that Y is a subcomplex. Thus we can compute the twisted homology groups $H_k(\operatorname{cyl}(j\colon X\to Y);A)$ using the relative twisted cellular chain complex, and similarly for cohomology. Put differently, $C_*=C_*^{\text{cell}}(Z,Y;\mathbb{Z}[\pi])$ has the desired properties. □

In order to give a criterion for twisted homology modules to be finitely generated, we need the notion of a Noetherian ring.

Definition A.7. A ring R is said to be *left Noetherian* if for any descending chain

$$R \;\supseteq\; I_1 \;\supseteq\; I_2 \;\supseteq\; I_3 \;\supseteq\; \dots$$

of left R-ideals the inclusions eventually become equality.

Example A.8. The following rings are left Noetherian:

(1) The ring $\mathbb{Z}$ is Noetherian.
(2) Any (skew) field is left Noetherian.
(3) If A is a commutative Noetherian ring, then the multivariable Laurent polynomial ring $A[t_1^{\pm1},\dots,t_k^{\pm1}]$ is also Noetherian [Lan02a, Corollary IV.9.5].

The following theorem is often implicitly used.

Proposition A.9. *Let M be a compact n-manifold, let $N \subset M$ be a subspace that is a compact manifold in its own right, let R be a ring and let A be an $(R, \mathbb{Z}[\pi])$-bimodule. If R is Noetherian and if A is a finitely generated R-module, then all the twisted homology modules $H_*(M, N; A)$ are finitely generated left R-modules.*

In the proof of Proposition A.9 we will need the following lemma; cf. [Lam91, Proposition 1.21] or [Lan02a, Proposition X.1.4].

Lemma A.10. *Let R be a Noetherian ring. If P is a finitely generated left R-module, then any submodule of P is also a finitely generated left R-module.*

Proof of Proposition A.9. By Proposition A.5, there exists a chain complex C_* of finite length consisting of finitely generated free left $\mathbb{Z}[\pi]$-modules such that

$$H_k(M, N; A) \;\cong\; H_k(A \otimes_{\mathbb{Z}[\pi]} C_*).$$

Given $k \in \mathbb{N}_0$ we denote the rank of C_k as a free left $\mathbb{Z}[\pi]$-module by r_k. Then we have $A \otimes_{\mathbb{Z}[\pi]} C_k \cong A \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi]^{r_k} \cong A^{r_k}$. In particular $H_k(M, N; A)$ is isomorphic to a quotient of a submodule of a finitely generated R-module. The desired statement follows from Lemma A.10. $\square$

A.2. **Cup and cap products on twisted (co-) chain complexes.** Throughout this section let X be a connected topological space admitting a universal cover, and write $\pi = \pi_1(X)$. We want to introduce the cup product and the cap product on twisted (co-) chain complexes. Given an n-simplex σ, define the p-simplices $\sigma\lfloor_p$ and $\sigma\rfloor_p$ by

$$\begin{aligned} \sigma\rfloor_p(t_0, \dots, t_p) &:= \sigma(t_0, \dots, t_p, 0, \dots, 0), \\ \sigma\lfloor_p(t_0, \dots, t_p) &:= \sigma(0, \dots, 0, t_0, \dots, t_p). \end{aligned}$$

Given two right $\mathbb{Z}[\pi]$-modules A and B we view $A \otimes_{\mathbb{Z}} B$ as a right $\mathbb{Z}[\pi]$-module via the diagonal action of π.

First we introduce the cup product on twisted cohomology. The following lemma can be verified easily by hand, say along the lines of the proof of [Hat02, Lemma 3.6].

Lemma A.11. *Let Y be a subset of X. We consider the map*

$$\begin{aligned} \cup\colon C^p(X; A) \times C^q(X; B) &\longrightarrow C^{p+q}(X; A \otimes_{\mathbb{Z}} B) \\ (\phi, \psi) &\longmapsto \big(\sigma \mapsto \varphi(\sigma\lfloor_p) \otimes_{\mathbb{Z}} \psi(\sigma\rfloor_{k-p})\big). \end{aligned}$$

(*Note that the right-hand side is indeed $\mathbb{Z}[\pi]$-homomorphism, i.e. it defines an element $C^{p+q}(X; A \otimes_{\mathbb{Z}} B)$.) Furthermore the map descends to a well defined map*

$$\cup\colon H^p(X, Y; A) \times H^q(X, Y; B) \;\longrightarrow\; H^{p+q}(X, Y; A \otimes_{\mathbb{Z}} B).$$

We refer to this map as the cup product.

Next we introduce the cap product. As with cup product, first we define it on the chain level.

Lemma A.12. *Let X be a topological space and let $S,T\subset X$ be subsets. We write $C_k(X,\{S,T\})=C_k(X)/(C_k(S)+C_k(T))$. The map*

$$\begin{aligned}\cap\colon C^p(X,S;A)\times C_k(X,\{S,T\};A) &\longrightarrow C_{k-p}(X,T;A\otimes_{\mathbb{Z}}B)\\ (\psi, b\otimes_{\mathbb{Z}[\pi]}\sigma) &\longmapsto (\psi(\sigma\lfloor_p)\otimes_{\mathbb{Z}} b)\otimes_{\mathbb{Z}[\pi]}\sigma\rfloor_{k-p}.\end{aligned}$$

is well-defined. We refer to this map as the cap product.

Proof. We verify that the given map respects the tensor product. Thus let $\psi\in C^p(X;A)$, $\sigma\in C_k(\widetilde{X})$, $\gamma\in\pi$ and $b\in B$. We calculate that

$$\begin{array}{rcl} \psi\cap b\otimes_{\mathbb{Z}[\pi]}\gamma\sigma &=& (\psi(\gamma\sigma\lfloor_p)\otimes_{\mathbb{Z}} b)\otimes_{\mathbb{Z}[\pi]}\gamma\sigma\rfloor_{k-p} \;=\; (\gamma\psi(\sigma\lfloor_p)\otimes_{\mathbb{Z}} b)\cdot\gamma\otimes_{\mathbb{Z}[\pi]}\sigma\rfloor_{k-p}\\ &=& (\gamma^{-1}\gamma\psi(\sigma\lfloor_p)\otimes_{\mathbb{Z}} b\gamma)\otimes_{\mathbb{Z}[\pi]}\sigma\rfloor_{k-p} \;=\; \psi\cap b\gamma\otimes_{\mathbb{Z}[\pi]}\sigma.\end{array}$$

It follows easily from the definitions that the cap product descends to the given quotient (co-) chain complexes. ☐

Lemma A.13. *Let $f\in C^p(X;A)$ and let $c\in C_k(X;B)$. We have*

$$\partial(f\cap c) \;=\; (-1)^p\cdot(-\delta(f)\cap c+f\cap\partial c)\in C_{k-1}(X;A\otimes_{\mathbb{Z}}B).$$

Proof. The lemma follows from a calculation using the definition of the cap product and the boundary maps, see e.g. [Fri19, Lemma 59.1] for details. Note that the precise signs differ from similar formulas in some textbooks in algebraic topology since there are many different sign conventions in usage. ☐

Corollary A.14. *Let X be a connected topological space, let $S,T\subset X$ be subsets, let R be a ring and let A be an $(R,\mathbb{Z}[\pi_1(M)])$-bimodule. For any cycle $\sigma\in C_n(X,\{S,T\};\mathbb{Z})$ the cap product*

$$\begin{array}{rcl}\cap[\sigma]\colon H^k(X,S;A) &\to& H_{n-k}(X,T;A)=H_{n-k}(X,T;A\otimes_{\mathbb{Z}}\mathbb{Z})\\ {[\varphi]} &\mapsto& [\varphi\cap\sigma]\end{array}$$

is well-defined. Furthermore this map only depends on the homology class $[\sigma]\in H_n(X,S\cup T;\mathbb{Z})$.

A.3. **The Poincaré duality theorem.** The following theorem is a generalisation of the familiar Poincaré duality for untwisted coefficients to the case of twisted coefficients.

Theorem A.15 (Twisted Poincaré duality)**.** *Let M an compact, oriented, connected n-dimensional manifold. Let S and T be codimension 0 compact submanifolds of ∂M such that $\partial S=\partial T=S\cap T$ and $\partial M=S\cup T$. Let $[M]\in H_n(M,\partial M;\mathbb{Z})$ be the fundamental class of M. If R is a ring and if A is an $(R,\mathbb{Z}[\pi_1(M)])$-bimodule, then the map*

$$-\cap[M]\colon H^k(M,S;A)\;\to\;H_{n-k}(M,T;A)$$

defined by Lemma A.13 is an isomorphism of left R-modules.

We also have the following Poincaré Duality statement on the (co-) chain level.

Theorem A.16 (Universal Poincaré duality)**.** *Let M an compact, oriented, connected n-dimensional manifold. Let S and T be codimension 0 compact submanifolds of ∂M such that $\partial S = \partial T = S \cap T$ and $\partial M = S \cup T$. Let $\sigma \in C_n(M, \{S, T\}; \mathbb{Z})$ be a representative of the fundamental class of M. If R is a ring and if A is an $(R, \mathbb{Z}[\pi_1(M)])$-bimodule, then the map*

$$- \cap \sigma \colon C^k(M, \partial M; \mathbb{Z}[\pi_1(M)]) \to C_{n-k}(M; \mathbb{Z}[\pi_1(M)])$$

defined by Lemma A.13 is a chain homotopy equivalence of left R-chain complexes.

Note that Theorem 14.9 can be used to give an alternative proof that the chain complexes of the theorem are chain homotopy equivalent.

Proof. The Universal Poincaré Duality Theorem A.16 follows immediately from the Twisted Poincaré Duality Theorem A.15 together with Lemma A.4. □

Below we will provide a proof of the Twisted Poincaré Duality Theorem A.15. But just for fun we would like to show that the Universal Poincaré Duality Theorem A.16 also implies the Twisted Poincaré Duality Theorem A.15.

Proof of Theorem A.15 using Theorem A.16. Let M be an compact, oriented, connected n-dimensional manifold. We pick a representative σ for $[M]$ and we write $\pi = \pi_1(M)$. Let A be an $(R, \mathbb{Z}[\pi])$-bimodule. Given a chain complex D_* of right $\mathbb{Z}[\pi]$-modules we consider the cochain map

$$\begin{array}{rcl} \Xi \colon A \otimes_{\mathbb{Z}[\pi]} \operatorname{Hom}_{\text{right-}\mathbb{Z}[\pi]}(D_*; \mathbb{Z}[\pi])) & \to & \operatorname{Hom}_{\text{right-}\mathbb{Z}[\pi]}(D_*, A) \\ a \otimes f & \mapsto & (\sigma \mapsto a \cdot f(\sigma)). \end{array}$$

Note that Ξ is an isomorphism if each D_k is a finitely generated free $\mathbb{Z}[\pi]$-module. But in general Ξ is not an isomorphism.

Furthermore we consider the following diagram

$$\begin{tikzcd} H^k(A \otimes_{\mathbb{Z}[\pi]} C^*(M, \partial M; \mathbb{Z}[\pi])) \arrow[r, "\mathrm{Id}_A \otimes (\cap \sigma)"] \arrow[d, "\Xi_*"] & H_{n-k}(A \otimes_{\mathbb{Z}[\pi]} C_*(M; \mathbb{Z}[\pi])) = H_{n-k}(M; A) \\ H^k(C^*(M, \partial M; A)) = H^k(M, \partial M; A). \arrow[ur, "\cap [M]"'] & \end{tikzcd}$$

One easily verifies that the diagram commutes. The top horizontal map is an isomorphism by the Universal Poincaré Duality Theorem A.16. It remains to show that the vertical map is an isomorphism. As we had pointed out above, on the chain level Ξ is in general not an isomorphism.

As in the proof of Proposition A.5 we can use Theorem 4.5 to find a pair (X, Y) of finite CW-complexes and a homotopy equivalence $f \colon (X, Y) \to (M, \partial M)$. By Proposition A.3 there exists a homotopy equivalence $\Theta \colon C_*^{\text{cell}}(X, Y; \mathbb{Z}[\pi]) \to C_*(X, Y; \mathbb{Z}[\pi])$ of $\mathbb{Z}[\pi]$-chain complexes. We consider the following diagram where all tensor products and homomorphism are over $\mathbb{Z}[\pi]$:

$$\begin{tikzcd}
A \otimes C^*_{\mathrm{cell}}(X,Y;\mathbb{Z}[\pi]) \arrow[r, "\Theta^*"] \arrow[d, "\Xi_*"] & A \otimes C^*(X,Y;\mathbb{Z}[\pi]) \arrow[r, "f^*"] \arrow[d, "\Xi_*"] & A \otimes C^*(M,\partial M;\mathbb{Z}[\pi]) \arrow[d, "\Xi_*"] \\
C^*_{\mathrm{cell}}(X,Y;A) \arrow[r, "\Theta^*"] & C^*(X,Y;A) \arrow[r, "f^*"] & C^*(M,\partial M;A).
\end{tikzcd}$$

One easily verifies that the diagram commutes. As pointed out above, the horizontal maps are chain homotopy equivalences over $\mathbb{Z}[\pi]$. Since X is a finite CW-complex we see that each $C_k^{\mathrm{cell}}(X,Y;\mathbb{Z}[\pi])$ is a finitely generated free $\mathbb{Z}[\pi]$-module. Thus we obtain from the above that the left vertical map is an isomorphism. Therefore we see that the right vertical map is a chain homotopy equivalence. In particular it induces an isomorphism of homology groups. □

The remainder of this appendix is dedicated to the proof of the Twisted Poincaré Duality Theorem A.15. Even though the theorem is well-known and often used, there are not many satisfactory proofs in the literature. The proof which is closest to ours in spirit is the proof of Sun [Sun17]. For closed manifolds Kwasik-Sun [KS18] provide a proof by using the work of Kirby-Siebenmann to reduce the proof to the case of triangulated manifolds.

The proof of the Twisted Poincaré Duality Theorem A.15 is modelled on the proof of untwisted Poincaré Duality that is given in Bredon's book [Bre97, Chapter VI Section 8]. The logic of his proof is unchanged, but some arguments and definitions have to be adjusted for the twisted setting.

A.4. **Preparations for the proof of the Twisted Poincaré Duality Theorem A.15.** We fix some notation that we will use for the remainder of this appendix. Let M be a *connected* manifold and denote by $\pi := \pi_1(M, x_0)$ the fundamental group. Finally let R be a ring and let A be an $(R,\mathbb{Z}[\pi])$-bimodule.

We write $p\colon \widetilde{M} \to M$ for the universal cover of M. For a subset $X \subset M$ (not necessarily connected) we consider the (co)-homology of X with respect to the coefficient system coming from M by setting

$$C_*(X;A) := A \otimes_{\mathbb{Z}[\pi]} C_*(p^{-1}(X);\mathbb{Z}),$$
$$C^*(X;A) := \mathrm{Hom}_{\mathbb{Z}[\pi]}\left(\overline{C_*(p^{-1}(X);\mathbb{Z})}, A\right),$$

with generalisation to pairs $Y \subset X \subset M$ by

$$C_*(X,Y;A) := A \otimes_{\mathbb{Z}[\pi]} C_*(p^{-1}(X), p^{-1}(Y);\mathbb{Z}),$$
$$C^*(X,Y;A) := \mathrm{Hom}_{\mathbb{Z}[\pi]}\left(\overline{C_*(p^{-1}(X), p^{-1}(Y);\mathbb{Z})}, A\right).$$

We summarise the basic properties of twisted coefficients in the following theorem, which should be compared to the untwisted case.

Theorem A.17. *Let M be a connected manifold with fundamental group π, and let A be an $(R, \mathbb{Z}[\pi])$-bimodule.*

(1) *Given $Y \subset X \subset M$ there is a long exact sequence of pairs in homology*

$$\cdots \longrightarrow H_k(Y;A) \longrightarrow H_k(X;A) \longrightarrow H_k(X,Y;A) \longrightarrow H_{k-1}(Y;A) \longrightarrow \cdots$$

and cohomology

$$\cdots \longrightarrow H^k(X,Y;A) \longrightarrow H^k(X;A) \longrightarrow H^k(Y;A) \longrightarrow H^{k+1}(X,Y;A) \longrightarrow \cdots .$$

(2) *Suppose we have a chain of subspaces $Z \subset Y \subset X \subset M$ such that the closure of Z is contained in the interior of Y. Then the inclusion $(X \setminus Z, Y \setminus Z) \to (X,Y)$ induces an isomorphism in homology and cohomology i.e.*

$$H_k(X \setminus Z, Y \setminus Z; A) \xrightarrow{\simeq} H_k(X,Y;A) \quad \text{and} \quad H^k(X \setminus Z, Y \setminus Z; A) \xleftarrow{\simeq} H^k(X,Y;A)$$

(3) *If $U_1 \subset U_2 \subset M$ and $V_1 \subset V_2 \subset M$ are open subsets in M, then there are long exact sequences in homology*

$$\begin{array}{ccccc} \ldots \longrightarrow H_k(U_1 \cap V_1, U_2 \cap V_2; A) \longrightarrow & \begin{array}{c} H_k(U_1,U_2;A) \\ \oplus \\ H_k(V_1,V_2;A) \end{array} & \to H_k(U_1 \cup V_2, U_2 \cup V_2; A) \\ H_{k-1}(U_1 \cap V_1, U_2 \cap V_2; A) \longrightarrow \ldots & & \end{array}$$

and cohomology

$$\begin{array}{ccccc} \ldots \longrightarrow H_k(U_1 \cup V_1, U_2 \cup V_2; A) \longrightarrow & \begin{array}{c} H_k(U_1,U_2;A) \\ \oplus \\ H_k(V_1,V_2;A) \end{array} & \to H_k(U_1 \cap V_2, U_2 \cap V_2; A) \\ H_{k-1}(U_1 \cup V_1, U_2 \cup V_2; A) \longrightarrow \ldots & & \end{array}$$

(4) *Suppose the inclusion $Y \to X$ is a homotopy equivalence, then the inclusion induced isomorphisms*

$$H_k(Y;A) \xrightarrow{\simeq} H_k(X;A) \quad \text{and} \quad H^k(Y;A) \xleftarrow{\simeq} H^k(X;A).$$

(5) *Let $U_1 \subset U_2 \subset \ldots$ be a sequence of open sets in M and let $U = \bigcup_{i \in \mathbb{N}} U_i$, then inclusions induce an isomorphism*

$$\varinjlim_{i \in \mathbb{N}} C_*(U_i;A) \xrightarrow{\cong} C_*(U;A).$$

The proofs are essentially the same as in the classical case. Therefore we will only sketch the arguments and focus on what is different. We also warn the reader that we give the "wrong proof" of statement (4). This is due to the fact that we developed the theory of twisted coefficients only for inclusions and hence a homotopy inverse does not fit in our theory. Therefore statement (4) will be deduced in a slightly round-about way using the following elementary lemma [Bre97, Chapter III Theorem 3.4 & remark after proof].

Lemma A.18 (The Covering Homotopy Theorem). *Given a covering $p\colon \widetilde{X} \to X$, a homotopy $H\colon Y \times I \to X$, and a lift $\widetilde{h}\colon Y \to \widetilde{X}$ of $H(-,0)$, there exists a unique lift $\widetilde{H}\colon Y \times I \to \widetilde{X}$ of H with $\widetilde{h} = \widetilde{H}(-,0)$.*

Proof of Theorem A.17. Recall that $p\colon \widetilde{M} \to M$ denotes the universal cover.

For statement (1) we consider the short exact sequence $0 \to C_*(Y;\mathbb{Z}[\pi]) \to C_*(X;\mathbb{Z}[\pi]) \to C_*(X,Y;\mathbb{Z}[\pi]) \to 0$ of *free* $\mathbb{Z}[\pi]$-modules. Since the modules are free the sequence stays exact after applying the functors $A \otimes_{\mathbb{Z}[\pi]} -$ and $\operatorname{Hom}_{\mathbb{Z}[\pi]}(-,A)$.

Recall the proof of statement (2) and (3) in the classical case as it is done for example in Bredon's book [Bre97, Chapter IV Section 17]. The main ingredient is to show that the inclusion of chain complexes $C_*^{\mathcal{U}}(X;\mathbb{Z}[\pi]) \to C^*(X;\mathbb{Z}[\pi])$ induces an isomorphism on homology [Bre97, Theorem 17.7]. Here $\mathcal{U}$ is an open cover of X and $C_*^{\mathcal{U}}(X;\mathbb{Z}[\pi])$ is the free abelian group generated by simplices σ for which there is a $U \in \mathcal{U}$ such that $\sigma\colon \Delta^* \to p^{-1}(U)$. This is done by defining the barycentric subdivision $\Upsilon_*\colon C_*(\widetilde{X};\mathbb{Z}) \to C_*(\widetilde{X};\mathbb{Z})$ and a chain homotopy T between Υ_* and the identity [Bre97, Lemma 17.1]. The important thing for us to observe is that both maps are natural [Bre97, Claim (1) in proof of Lemma 17.1]. Hence for a twisted chain $\Upsilon(e \otimes_{\mathbb{Z}[\pi]} \sigma) := e \otimes_{\mathbb{Z}[\pi]} \Upsilon(\sigma)$ is well-defined, because

$$\begin{aligned}\Upsilon(e \otimes_{\mathbb{Z}[\pi]} \gamma\sigma) &= e \otimes_{\mathbb{Z}[\pi]} \Upsilon(\gamma\sigma)\\ &= e \otimes_{\mathbb{Z}[\pi]} \gamma\Upsilon(\sigma) \quad \text{(naturality of } \Upsilon)\\ &= e\gamma \otimes_{\mathbb{Z}[\pi]} \Upsilon(\sigma) = \Upsilon(e\gamma \otimes_{\mathbb{Z}[\pi]} \sigma)\end{aligned}$$

The same holds for T and from now on one can follow the classical proofs. Alternatively, one could invoke Lemma A.4.

Next we prove statement (4). Let $f\colon X \to Y \subset X$ be a homotopy inverse of the inclusion and $H\colon X \times I \to X$ a homotopy between Id_X and f. Since $p\colon \widetilde{X} \to X$ is a covering and $\mathrm{Id}_{\widetilde{X}}$ is a lift of $H(p(-),0)$, we get by Lemma A.4 a lift $\widetilde{H}\colon \widetilde{X} \times I \to \widetilde{X}$ of the homotopy H. One easily verifies that the inclusion $\widetilde{Y} \to \widetilde{X}$ induces a homotopy equivalence where a homotopy inverse is given by $\widetilde{H}(-,1)$. Hence the inclusion induced map $H_k(C_*(\widetilde{Y};\mathbb{Z})) \to H_k(C_*(\widetilde{X};\mathbb{Z}))$ is an isomorphism for every k. Thus the claim follows from Lemma A.4.

The proof of Statement (5) is almost verbatim the same proof as in the classical case. □

A.5. **The main technical theorem regarding Poincaré Duality.** Given a group π we can view $\mathbb{Z}$ as a $\mathbb{Z}[\pi]$-module with trivial π-action. We denote this module by $\mathbb{Z}^{\mathrm{triv}}$. Let $p\colon \widetilde{M} \to M$ be the covering projection. We have the following useful lemma, concerning the chain map $C_*(X;\mathbb{Z}^{\mathrm{triv}}) \to C_*(X;\mathbb{Z})$ defined by $k \otimes_{\mathbb{Z}[\pi]} \widetilde{\sigma} \mapsto k \cdot p(\sigma)$.

Lemma A.19. *Given any subset $X \subset M$ the chain map above is an isomorphism between $C_*(X;\mathbb{Z}^{\mathrm{triv}})$ and $C_*(X;\mathbb{Z})$, and induces one between $C^*(X;\mathbb{Z})$ and $C^*(X;\mathbb{Z}^{\mathrm{triv}})$, where $C_*(X;\mathbb{Z})$ and $C^*(X;\mathbb{Z})$ are the untwisted singular chain complexes.*

Proof. The isomorphism is given by lifting a simplex, which is always possible since a simplex is simply connected. If one has two different choices of lifts, then they differ by an element in π. But the action of $\mathbb{Z}[\pi]$ on $\mathbb{Z}$ is trivial and hence this indeterminacy vanishes. $\square$

We will keep the notational difference between $C_*(X;\mathbb{Z})$ and $C_*(X;\mathbb{Z}^{\mathrm{triv}})$ to emphasise where our simplices live.

As above let R be a ring and let A is an $(R,\mathbb{Z}[\pi])$-bimodule. Let $K \subset M$ be a compact subset of M. We define the (twisted) Cech cohomology groups

$$\check{H}^p(K;A) := \varinjlim_{K \subset U \subset M} H^p(U;A),$$

where the direct limit runs over all open sets in M containing K. Since cohomology is contravariant, we define the order on open sets in the reversed way i.e. $U \leq V$ if $V \subset U$.

Now we assume that M is oriented. Being oriented gives us for any closed subset $Z \subset M$ a preferred element $\theta_Z \in H_n(M, M \setminus Z;\mathbb{Z}^{\mathrm{triv}}) \cong H_n(M, M \setminus Z;\mathbb{Z})$, which restricts for all $x \in Z$ to the generator in $H_n(M, M \setminus \{x\};\mathbb{Z}^{\mathrm{triv}})$.

For any open set $U \subset M$ containing K let $\mathrm{ex}_U\colon H_n(M, M\setminus K;\mathbb{Z}^{\mathrm{triv}}) \to H_n(U, U\setminus K;\mathbb{Z}^{\mathrm{triv}})$ be the inverse of the inclusion given by the excision isomorphism i.e. if $j\colon U \to M$ is the inclusion then $j_* \circ \mathrm{ex}_U = \mathrm{Id}$. We then obtain a map

$$\begin{aligned} D_U\colon H^p(U;A) &\longrightarrow H_n(M, M \setminus K;A) \\ \phi &\longmapsto j_*(\phi \cap \mathrm{ex}_U(\theta_K)). \end{aligned}$$

Given another open set $V \subset U$ denote by $i\colon V \to U$ the inclusion. Then one easily calculates:

$$\mathrm{PD}_V(i^*\phi) = j_* i_*(i^*\phi \cap \mathrm{ex}_V(\theta_K)) = j_*(\phi \cap i_*\,\mathrm{ex}_V(\theta_K)) = j_*(\phi \cap \mathrm{ex}_U(\theta_K)) = \mathrm{PD}_U(\phi).$$

Or, with other words, the following diagram commutes:

$$\begin{tikzcd} H^p(U;A) \arrow[dr, "\mathrm{PD}_U"] \arrow[dd, "i^*"'] & \\ & H_{n-p}(M, M \setminus K;A) \\ H^p(V;A) \arrow[ur, "\mathrm{PD}_V"] & \end{tikzcd}$$

By the universal property of the direct limit we obtain the *dualising* map $\mathrm{PD}_K\colon \check{H}^p(K;A) \to H_{n-p}(M, M \setminus K;A)$.

In the remainder of this section we will prove the following theorem.

Theorem A.20 (Poincaré duality)**.** *The map $\mathrm{PD}_K\colon \check{H}^p(K;A) \to H_{n-p}(M, M \setminus K;A)$ is a left R-module isomorphism for all compact subsets $K \subset M$.*

Here, as above, A is an $(R,\mathbb{Z}[\pi])$-bimodule. In the subsequent section we will see that the Twisted Poincaré Duality Theorem A.15 is a reasonably straightforward consequence of Theorem A.20.

The proof of Theorem A.20 will be an application of the following lemma.

Lemma A.21 (Bootstrap lemma). *Let $P_M(K)$ be a statement about compact sets K in M. If $P_M(\cdot)$ satisfies the following three conditions:*

1. *$P_M(K)$ holds true for all compact subsets $K\subset M$ with the property that for all $x\in K$ the inclusions $\{x\}\to K$ and $M\setminus K\to M\setminus\{x\}$ are deformation retracts,*
2. *If $P_M(K_1), P_M(K_2)$ and $P_M(K_1\cap K_2)$ is true, then $P_M(K_1\cup K_2)$ is true,*
3. *If $\cdots\subset K_2\subset K_1$ and $P_M(K_i)$ is true for all $i\in\mathbb{N}$, then $P_M(\bigcap_{i\in\mathbb{N}}K_i)$ is true.*

Then $P_M(K)$ is true for all $K\subset M$.

Proof. See [Bre97, Chapter VI Lemma 7.9]. □

The idea is to apply the bootstrap lemma to the statements that the conclusion of Theorem A.20 holds for a given compact set K. It turns out that condition (3) is the easiest to verify. It follows from formal properties about direct limits. For the verification of condition (1) we have do to one explicit calculation. This is the content of the next lemma.

Lemma A.22. *Let $x\in M$ be a point. The map $\mathrm{PD}_{\{x\}}\colon \check{H}^0(\{x\};A)\to H_n(M,M\setminus\{x\};A)$ is an R-module isomorphism.*

Proof. Let $p\colon\widetilde{M}\to M$ be the universal cover. Since x is a point in a manifold we can calculate the dualising map $\mathrm{PD}_{\{x\}}$ by taking the limit over open neighbourhoods U of x with the following two properties:

1. U is contractible,
2. for any connected component $\overline{U}\subset p^{-1}(U)$ the map $p|_{\overline{U}}$ is a homeomorphism.

This can be done, since any neighbourhood of x contains a neighbourhood with these two properties. Let U be such a neighbourhood of x and $\overline{U}\subset p^{-1}(U)$ a fixed connected component. This choice of connected component gives us an isomorphism $H^0(U;A)\cong A$ as follows. Let $f\in H^0(U;A)$ be arbitrary and $\overline{x}\in\overline{U}$ be a point in our connected component, then we get an element in A by evaluating $f([\overline{x}])$. Conversely, given an element $e\in A$ we can construct a function in $H^0(U;A)$ by setting $f([\overline{x}])=e$ for all $\overline{x}\in\overline{U}$. Note that there is a unique way to extend f equivariantly to $C_0(p^{-1}(U);\mathbb{Z})$.

We are now going to construct a representative of the orientation class $\theta_K\in H_n(M,M\setminus\{x\};\mathbb{Z}^{\mathrm{triv}})$ for which it is very simple to calculate the dualising map. Let $\overline{x}$ be the preimage of x in $\overline{U}$. Now take a cycle $\sum_{i=1}^d k_i\sigma_i$ which generates $H_n(\overline{U},\overline{U}\setminus\{\overline{x}\};\mathbb{Z})$. By excision and Lemma A.19 one easily sees that $1\otimes_{\mathbb{Z}[\pi]}\sum_{i=1}^d k_i\sigma_i$ is a generator of $H_n(M,M\setminus\{x\};\mathbb{Z}^{\mathrm{triv}})$.

Using the isomorphism $H^0(U;A)\cong A$ from above the dualising map becomes $\mathrm{PD}_{\{x\}}\colon A\to H_n(M,M\setminus\{x\};A), e\mapsto e\otimes_{\mathbb{Z}[\pi]}\sum_{i=1}^d k_i\sigma_i$. This is clearly an isomorphism, since on the chain level we have:

$$C_*(U, U \setminus \{x\}\,; A) = A \otimes_{\mathbb{Z}[\pi]} \bigoplus_{\gamma \in \pi} C_*(\gamma\overline{U}, \gamma\overline{U} \setminus \{\gamma\overline{x}\}\,; \mathbb{Z}) \cong A \otimes_{\mathbb{Z}} C_*(\overline{U}, \overline{U} \setminus \{\overline{x}\}\,; \mathbb{Z}). \qquad \square$$

In order to verify condition (2) of the bootstrap lemma we will need the following lemma (compare [Bre97, Lemma 8.2]).

Lemma A.23. *If K and L are two compact subsets of M, then the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K \cup L; A) \arrow[r] \arrow[d, "\mathrm{PD}_{K\cup L}"] & \begin{matrix}\check{H}^p(K;A)\\ \oplus \\ \check{H}^p(L;A)\end{matrix} \arrow[r] \arrow[d, "\mathrm{PD}_K \oplus \mathrm{PD}_L"] & \check{H}^p(K \cap L; A) \arrow[r] \arrow[d, "\mathrm{PD}_{K\cap L}"] & \check{H}^{p+1}(K \cup L; A) \arrow[r] \arrow[d, "\mathrm{PD}_{K\cup L}"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M, M \setminus (L \cup K); A) \arrow[r] & \begin{matrix}H_{n-p}(M, M\setminus K; A)\\ \oplus \\ H_{n-p}(M, M \setminus L; A)\end{matrix} \arrow[r] & H_{n-p}(M, M\setminus (L \cap K); A) \arrow[r] & H_{n-p-1}(M, M \setminus (L \cup K); A) \arrow[r] & \cdots
\end{tikzcd}$$

has exact rows and it commutes up to a sign depending only on p.

Proof. The rows are exact by Mayer-Vietoris and the fact that direct limit is an exact functor. The commutativity of the squares is clear except for the last one involving the boundary map. This will be a painful diagram chase. Let $U \supset K$ and $V \supset L$ be open neighbourhoods containg K resp. L. The sequence in the top row comes from the short exact sequence ($\mathcal{U} = \{U, V\}$):

$$0 \longrightarrow C^*_{\mathcal{U}}(U \cup V; A) \longrightarrow C^*(U; A) \oplus C^*(V; A) \longrightarrow C^*(U \cap V; A) \longrightarrow 0.$$

An element $\phi \in \check{H}^p(K \cap L; A)$ will already be represented by same element $f \in C^p(U \cap V; A)$ for some U and V as above. We can extend f to an element $\overline{f} \in C^p(M; A)$ by

$$\overline{f}(\sigma) = \begin{cases} f(\sigma) & \text{if } \operatorname{Im}\sigma \subset \widetilde{U} \cap \widetilde{V} \\ 0 & \text{else.} \end{cases}$$

Note that $\overline{f} \in C^p(M; A)$ since $p^{-1}(U \cap V)$ is an equivariant subspace and hence $\overline{f}$ is equivariant. If we consider $\overline{f}$ as an element in $C^p(U; A)$ then the cohomology class $\delta(\phi)$ is represented by the cochain $h \in C^{p+1}(U \cup V; A)$ which is given by

$$h(\sigma) = \begin{cases} \delta(\overline{f})(\sigma) & \text{if } \operatorname{Im}\sigma \subset \widetilde{U} \\ 0 & \text{else.} \end{cases}$$

Since ϕ is a cocycle we have $\delta(f)(\sigma) = 0$ for $\sigma \in C_*(U \cap V; A)$. It follows in particular that if σ is a simplex whose image is completely contained in $\widetilde{V}$, then $h(\sigma) = 0$. We can represent our orientation class $\theta \in H_n(M, M \setminus (K \cup L))$ by a cycle

$$\begin{aligned} a = b + c + d + e \quad \text{with} \quad & b \in C_n(U \cap V; \mathbb{Z}^{\text{triv}}) \quad c \in C_n(U \setminus (U \cap L); \mathbb{Z}^{\text{triv}}) \\ & d \in C_n(V \setminus (V \cap K); \mathbb{Z}^{\text{triv}}), \quad e \in C_n(M \setminus (K \cup L); \mathbb{Z}^{\text{triv}}). \end{aligned}$$

Obviously e does not play a role since we kill it in the end. With these representatives one computes that $\delta(\phi)(\theta)$ is represented by

$$h \cap (b + c + d) = \delta(\overline{f}) \cap c + h \cap d + \delta(f) \cap b = \delta(\overline{f}) \cap c.$$

The pairing of h with b is zero since f was a cocycle in $C^*(U \cap V; A)$ and the pairing of h with d is zero since d consist of simplices with image in $\widetilde{V}$.

The lower sequence comes from the short exact sequence:

$$0 \to C_*(M, M \setminus (K \cup L); A) \to \begin{array}{c} C_*(M, M \setminus K; A) \\ \oplus \\ C_*(M, M \setminus L); A) \end{array} \to C_*(M, M \setminus (K \cap L); A) \to 0.$$

Before we compute the other side $\partial(\phi \cap \mathrm{ex}_{U\cap V}(\theta))$ we want to recall that the cap product is natural on the chain complex level i.e. the following diagram commutes:

$$\begin{array}{ccc} C^p(U; A) \times C_n(U, U \setminus K; \mathbb{Z}^{\mathrm{triv}}) & \longrightarrow & C_*(U, U \setminus K; A) \\ \uparrow \qquad\qquad \downarrow & & \downarrow \\ C^p(M; A) \times C_n(M, M \setminus K; \mathbb{Z}^{\mathrm{triv}}) & \longrightarrow & C_*(M, M \setminus K; A). \end{array}$$

Therefore we use the representatives from above. To construct the boundary map ∂, we take as the preimage of $\overline{f} \cap a \in C_*(M, M \setminus (K \cap L); A)$ the element $(\overline{f} \cap a, 0) \in C_*(M, M \setminus K; A) \oplus C_*(M, M \setminus L; A)$. Then one computes in $C_*(M, M \setminus K; A)$

$$\begin{aligned} \partial(\overline{f} \cap a) &= (-1)^{p+1} \cdot \delta(\overline{f}) \cap a \pm f \cap \partial a \quad \text{(by Lemma A.13)} \\ &= (-1)^{p+1} \cdot \delta(\overline{f}) \cap a \quad (\text{since } f \cap \partial a \in C_{n-p-1}(M \setminus (K \cup L); A)) \\ &= (-1)^{p+1} \cdot \delta(\overline{f}) \cap b + c + d + e \\ &= (-1)^{p+1} \cdot \delta(\overline{f}) \cap (c + d) \quad \text{(same reason as above)} \\ &= (-1)^{p+1} \cdot \delta(\overline{f}) \cap c \quad (\text{since } d \in C_{n-p}(V \setminus (K \cap V); A)) \end{aligned}$$

Therefore the element $\partial(\phi \cap \mathrm{ex}_{U\cap V}(\theta))$ is also represented by $(-1)^{p+1} \cdot \delta(\overline{f}) \cap c \in C_{n-p-1}(M, M \setminus (K \cup L); A)$. □

Proof of Theorem A.20. Let $P_M(K)$ be the statement that the map PD_K is an isomorphism. Then it is sufficient to verify condition (1),(2) and (3) of the bootstrap lemma. We start with verifying (1). In the case that $K = \{x\}$ is just a point we have already seen in Lemma A.22 that the statement holds true. For a general compact K with the property of (1) the statement follows from the following commutative diagram:

$$\begin{array}{ccc} \check{H}^p(K; A) & \longrightarrow & H_{n-p}(M, M \setminus K; A) \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \check{H}^p(\{x\}; A) & \xrightarrow{\simeq} & H_{n-p}(M, M \setminus \{x\}; A), \end{array}$$

where the vertical maps are isomorphisms by the homotopy invariance and the bottom row by the observation above. Hence condition (1) is verified.

Condition (2) follows immediately from the five-lemma and Lemma A.23.

Let K_i be a sequence of compact subsets such that $P_M(K_i)$ holds for all $i \in \mathbb{N}$. We set $K = \bigcap_{i\in\mathbb{N}} K_i$. It is an exercise in pointset topology of manifolds that each K_i has a fundamental system $U_{i,j}$ of open neighbourhoods. Fundamental system means that $U_{i,j} \subset U_{i,k}$ if $j < k$ and that for each open set U containing K_i there is a j such that $U_{i,j} \subset U$. Another exercise in point set topology of manifolds shows, that one can construct these sets such that $U_{1,j} \supset U_{2,j} \supset U_{3,j} \supset \ldots$ for all $j \in \mathbb{N}$. Then $U_{i,j}$ is a fundamental system of open neighbourhoods of K with the order $(i,j) \le (k,l) \Leftrightarrow i \le k \ \wedge j \le l$. One has the natural isomorphism [Bre97, Appendix D5]:

$$\varinjlim_{i\in\mathbb{N}} \check{H}^p(K_i;A) = \varinjlim_{i\in\mathbb{N}} \varinjlim_{j\in\mathbb{N}} H^p(U_{i,j};A) \xrightarrow{\simeq} \varinjlim_{i,j\in\mathbb{N}} H^p(U_{i,j};A) \cong \check{H}^p(K;A).$$

And hence the theorem follows from the commutativity of the diagram:

$$\begin{tikzcd}
\varinjlim_{i\in\mathbb{N}} \check{H}^p(K_i;A) \arrow[r] \arrow[d] & \varinjlim_{i\in\mathbb{N}} H_{n-p}(M, M \setminus K_i;A) \arrow[d] \\
\check{H}^p(K;A) \arrow[r] & H_{n-p}(M, M\setminus K;A).
\end{tikzcd}$$

□

A.6. **Proof of the Twisted Poincaré Duality Theorem A.15.** For the reader's convenience we recall the main theorem from the last section. Here, as above, R is a ring and A is an $(R, \mathbb{Z}[\pi])$-bimodule.

Theorem A.20. *Let M be a compact, oriented, connected n-dimensional manifold. The map $\mathrm{PD}_K\colon \check{H}^p(K;A) \to H_{n-p}(M, M \setminus K;A)$ is an isomorphism of left R-modules for all compact subsets $K \subset M$.*

Furthermore, we also recall that we need to prove the following theorem.

Theorem A.15. *Let M an compact, oriented, connected n-dimensional manifold. Let S and T be codimension 0 compact submanifolds of ∂M such that $\partial S = \partial T = S \cap T$ and $\partial M = S \cup T$. Let $[M] \in H_n(M, \partial M;\mathbb{Z})$ be the fundamental class of M. The map*

$$- \cap [M] \colon H^k(M,S;A) \ \to \ H_{n-k}(M,T;A)$$

defined by Lemma A.13 is an isomorphism of left R-modules.

In the remainder of this appendix we will explain how to deduce Theorem A.15 from Theorem A.20. First note that if M is a closed manifold, then we can set $K = M$ in

Theorem A.20. Evidently we have $\check{H}^p(M;A) = H(M;A)$. Thus we obtain precisely the statement of Theorem A.15 in the closed case.

Next let M be a compact oriented manifold with non-empty boundary. First we consider the case $R = \varnothing$ and $S = \partial M$. By the Collar neighbourhood theorem 2.5 there exists a collar $\partial M \times [0,2] \subset M$ of the boundary such that $\partial M = \partial M \times \{0\}$. We obtain the following chain of isomorphisms:

$$\begin{aligned} H^p(M;A) &\cong H^p(M \setminus (\partial M \times [0,1));A) \quad \text{(homotopy)} \\ &\cong \check{H}^p(M \setminus (\partial M \times [0,1));A) \quad \text{(follows from considering the open} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{neighborhoods } M \setminus (\partial M \times [0,1-\tfrac{1}{n}])) \\ &\cong H_{n-p}(M \setminus \partial M, \partial M \times (0,1);A) \quad (\text{duality } K = M \setminus (\partial M \times [0,1))) \\ &\cong H_{n-p}(M, \partial M \times [0,1);A) \quad (\text{excision } U = \partial M) \\ &\cong H_{n-p}(M, \partial M;A), \end{aligned}$$

It follows from the definition of the dualising map and naturality of cap product that these isomorphisms are given by capping with a generator $[M] \in H_n(M, \partial M; \mathbb{Z}^{\text{triv}}) \cong H_n(M, \partial M; \mathbb{Z})$ as in the classical case.

The proof of the general case of Theorem A.15 relies on the following lemma.

Lemma A.24. *Let M a compact, oriented, connected n-dimensional manifold. Let R and S be compact codimension 0 submanifolds of ∂M such that $\partial R = \partial S = R \cap S$ and $\partial M = R \cup S$. Then the following diagram commutes up to a sign:*

$$\begin{tikzcd}
\ldots \arrow[r] & H^p(M,R;A) \arrow[r] \arrow[dd, "\cap[M]"] & H^p(M;A) \arrow[r] \arrow[dd, "\cap[M]"] & H^{p+1}(R;A) \arrow[r] \arrow[d, "\cap[R]"] & H^{p+1}(M,R;A) \arrow[r] \arrow[dd, "\cap[M]"] & \ldots \\
& & & H_{n-p-1}(R,\partial R;A) \arrow[d, "\text{excision isom.}"] & & \\
\ldots \arrow[r] & H_{n-p}(M,S;A) \arrow[r] & H_{n-p}(M,\partial M;A) \arrow[r] & H_{n-p-1}(R\cup S,S;A) \arrow[r] & H_{n-p-1}(M,S;A) \arrow[r] & \ldots
\end{tikzcd}$$

Proof. The commutativity is a more or less direct consequence of Lemma A.13 and the observation that $\partial_*[M] = [\partial M]$. More precisely, the proof in the untwisted case is given in detail in [Fri19, Proof of Theorem 61.1]. The proof in the twisted case is basically the same. $\square$

The proof of the general case of Theorem A.15 follows from the previously discussed Poincaré Duality isomorphisms $\cap[M]\colon H^p(M;A) \xrightarrow{\cong} H_{n-p}(M,\partial M;A)$, $\cap[R]\colon H^{p+1}(R,A) \xrightarrow{\cong} H_{n-p-1}(R,\partial R;A)$ together with Lemma A.24 and the five lemma.

References

[Ale24] James W. Alexander. An example of a simply connected surface bounding a region which is not simply connected. *Proceedings of the National Academy of Sciences*, 10(1):8–10, 1924.

[AM90] Selman Akbulut and John D. McCarthy. *Casson's invariant for oriented homology* 3*-spheres*, volume 36 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1990.

[APS75] Michael F. Atiyah, Vijay K. Patodi, and Isadore M. Singer. Spectral asymmetry and Riemannian geometry. II. *Math. Proc. Cambridge Philos. Soc.*, 78(3):405–432, 1975.

[Arm70] Mark A. Armstrong. Collars and concordances of topological manifolds. *Comment. Math. Helv.*, 45:119–128, 1970.

[Ati76] Michael F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.

[BCF$^+$19] Imre Bokor, Diarmuid Crowley, Stefan Friedl, Fabian Hebestreit, Daniel Kasprowski, Markus Land, and Johnny Nicholson. Connected sum decompositions of high-dimensional manifolds. *arXiv e-prints*, September 2019.

[BG64a] Morton Brown and Herman Gluck. Stable structures on manifolds. I. Homeomorphisms of S^n. *Ann. of Math. (2)*, 79:1–17, 1964.

[BG64b] Morton Brown and Herman Gluck. Stable structures on manifolds. II. Stable manifolds. *Ann. of Math. (2)*, 79:18–44, 1964.

[Boy86] Steven Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Amer. Math. Soc.*, 298(1):331–357, 1986.

[Boy93] Steven Boyer. Realization of simply-connected 4-manifolds with a given boundary. *Comment. Math. Helv.*, 68(1):20–47, 1993.

[Bre97] Glen Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[Bro60] Morton Brown. A proof of the generalized Schoenflies theorem. *Bull. Am. Math. Soc.*, 66:74–76, 1960.

[Bro62] Morton Brown. Locally flat imbeddings of topological manifolds. *Ann. of Math. (2)*, 75:331–341, 1962.

[BV68] J. Michael Boardman and Rainer M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74:1117–1122, 1968.

[BV73] J. Michael Boardman and Rainer M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[CF13] Jae Choon Cha and Stefan Friedl. Twisted torsion invariants and link concordance. *Forum Math.*, 25(3):471–504, 2013.

[CG78] Andrew Casson and Cameron Gordon. On slice knots in dimension three. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, pages 39–53. Amer. Math. Soc., Providence, R.I., 1978.

[CG86] Andrew Casson and Cameron Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986. With an appendix by P. M. Gilmer.

[Cha74] Thomas A. Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math.*, 96:488–497, 1974.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and L^2-signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.

[CS71] Sylvain E. Cappell and Julius L. Shaneson. On four dimensional surgery and applications. *Comment. Math. Helv.*, 46:500–528, 1971.

[CS76] Sylvain E. Cappell and Julius L. Shaneson. Some new four-manifolds. *Ann. of Math. (2)*, 104(1):61–72, 1976.

[CT07] Tim D. Cochran and Peter Teichner. Knot concordance and von Neumann ρ-invariants. *Duke Math. J.*, 137(2):337–379, 2007.

[CZ90] Donald J. Collins and Heiner Zieschang. Combinatorial group theory and fundamental groups. In *Algebra VII. Combinatorial group theory. Applications to geometry. Transl. from the Russian.* Berlin: Springer-Verlag, 1990.

[Dol95] Albrecht Dold. *Lectures on algebraic topology.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[Don83] Simon K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[Don87] Simon K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[DV09] Robert Daverman and Gerard Venema. *Embeddings in manifolds*, volume 106 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[Edw84] Robert D. Edwards. The solution of the 4-dimensional annulus conjecture (after Frank Quinn). In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 211–264. Amer. Math. Soc., Providence, RI, 1984.

[EK71] Richard D. Edwards and Robion C. Kirby. Deformations of spaces of imbeddings. *Ann. Math. (2)*, 93:63–88, 1971.

[Fer77] Steve Ferry. The homeomorphism group of a compact Hilbert cube manifold is an ANR. *Ann. of Math. (2)*, 106(1):101–119, 1977.

[FM66] Ralph H. Fox and John W. Milnor. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka J. Math.*, 3:257–267, 1966.

[FQ90] Michael H. Freedman and Frank Quinn. *Topology of* 4*-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[FR01] Steve Ferry and Andrew Ranicki. A survey of Wall's finiteness obstruction. In *Surveys on surgery theory. Vol. 2: Papers dedicated to C. T. C. Wall on the occasion of his 60th birthday*, pages 63–79. Princeton, NJ: Princeton University Press, 2001.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[Fri12] Greg Friedman. An elementary illustrated introduction to simplicial sets. *Rocky Mountain J. Math.*, 42:353–424, 2012.

[Fri19] Stefan Friedl. Algebraic topology I-IV. `https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/2019_algebraic-topology-iv.5`, 2019.

[FS84] Ronald Fintushel and Ronald J. Stern. A μ-invariant one homology 3-sphere that bounds an orientable rational ball. In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 265–268. Amer. Math. Soc., Providence, RI, 1984.

[FT95] Michael H. Freedman and Peter Teichner. 4-manifold topology. I. Subexponential groups. *Invent. Math.*, 122(3):509–529, 1995.

[Fur01] M. Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. *Math. Res. Lett.*, 8(3):279–291, 2001.

[GH81] Marvin J. Greenberg and John R. Harper. Algebraic topology: a first course. Mathematics Lecture Note Series, 58. Reading, Massachusetts, 1981.

[Gom84] Robert E. Gompf. Stable diffeomorphism of compact 4-manifolds. *Topology Appl.*, 18:115–120, 1984.

[GS80] David E. Galewski and Ronald J. Stern. Classification of simplicial triangulations of topological manifolds. *Ann. of Math. (2)*, 111(1):1–34, 1980.

[GS99] Robert E. Gompf and András Stipsicz. 4*-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[Hab82] Nathan Habegger. Une variété de dimension 4 avec forme d'intersection paire et signature -8. *Comment. Math. Helv.*, 57(1):22–24, 1982.

[Ham76] Andrew J. S. Hamilton. The triangulation of 3-manifolds. *Quart. J. Math. Oxford Ser. (2)*, 27(105):63–70, 1976.

[Han51] Olof Hanner. Some theorems on absolute neighborhood retracts. *Ark. Mat.*, 1:389–408, 1951.

[Han89] Vagn Lundsgaard Hansen. *Braids and coverings: selected topics.* Cambridge etc.: Cambridge University Press, 1989.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Hat13] Allen Hatcher. The Kirby torus trick for surfaces. Available at arXiv:1312.3518, 2013.

[HH18] Ian Hambleton and Alyson Hildum. Topological 4-manifolds with right-angled artin fundamental groups. *Journal of Topology and Analysis*, 2018.

[Hil12] Jonathan Hillman. *Algebraic invariants of links. 2nd ed.* Singapore: World Scientific, 2nd ed. edition, 2012.

[Hir66] Morris W. Hirsch. On normal microbundles. *Topology*, 5:229–240, 1966.

[Hir68] Morris W. Hirsch. On tubular neighborhoods of piecewise linear and topological manifolds. In *Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967)*, pages 63–80. Prindle, Weber & Schmidt, Boston, Mass., 1968.

[Hir94] Morris Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[HK88] Ian Hambleton and Matthias Kreck. On the classification of topological 4-manifolds with finite fundamental group. *Math. Ann.*, 280(1):85–104, 1988.

[HK93] Ian Hambleton and Matthias Kreck. Cancellation, elliptic surfaces and the topology of certain four-manifolds. *J. Reine Angew. Math.*, 444:79–100, 1993.

[HKT94] Ian Hambleton, Matthias Kreck, and Peter Teichner. Nonorientable 4-manifolds with fundamental group of order 2. *Trans. Amer. Math. Soc.*, 344(2):649–665, 1994.

[HKT09] Ian Hambleton, Matthias Kreck, and Peter Teichner. Topological 4-manifolds with geometrically two-dimensional fundamental groups. *J. Topol. Anal.*, 1(2):123–151, 2009.

[HLSX18] Michael J. Hopkins, Jianfeng Lin, XiaoLin Danny Shi, and Zhouli Xu. Intersection forms of spin 4-manifolds and the pin(2)-equivariant mahowald invariant, 2018.

[HM74] Morris W. Hirsch and Barry Mazur. *Smoothings of piecewise linear manifolds*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 80.

[HS69] Wu-Chung Hsiang and Julius .L. Shaneson. Fake tori, the annulus conjecture, and the conjectures of Kirby. *Proc. Natl. Acad. Sci. USA*, 62:687–691, 1969.

[HS97] Peter J. Hilton and Urs Stammbach. *A course in homological algebra. 2nd ed.* New York, NY: Springer, 2nd ed. edition, 1997.

[Hud69] John F. P. Hudson. On transversality. *Proc. Cambridge Philos. Soc.*, 66:17–20, 1969.

[Kea75] Cherry Kearton. Cobordism of knots and Blanchfield duality. *J. London Math. Soc. (2)*, 10(4):406–408, 1975.

[Kir69] Robion C. Kirby. Stable homeomorphisms and the annulus conjecture. *Ann. of Math. (2)*, 89:575–582, 1969.

[Kis64] James M. Kister. Microbundles are fibre bundles. *Ann. of Math. (2)*, 80:190–199, 1964.

[KLPT17] Daniel Kasprowski, Markus Land, Mark Powell, and Peter Teichner. Stable classification of 4-manifolds with 3-manifold fundamental groups. *J. Topol.*, 10(3):827–881, 2017.

[Kos93] Antoni A. Kosinski. *Differential manifolds*, volume 138 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1993.

[KQ00] Vyacheslav S. Krushkal and Frank Quinn. Subexponential groups in 4-manifold topology. *Geom. Topol.*, 4:407–430, 2000.

[Kre84] Matthias Kreck. Some closed 4-manifolds with exotic differentiable structure. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 246–262. Springer, Berlin, 1984.

[Kre99] Matthias Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.

[KS69] Robion C. Kirby and Laurence C. Siebenmann. On the triangulation of manifolds and the Hauptvermutung. *Bull. Amer. Math. Soc.*, 75:742–749, 1969.

[KS75] Robion C. Kirby and Laurence C. Siebenmann. Normal bundles for codimension 2 locally flat imbeddings. pages 310–324. Lecture Notes in Math., Vol. 438, 1975.

[KS77] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[KS18] Slawomir Kwasik and Fang Sun. Local Coefficients Revisited. *arXiv e-prints*, page arXiv:1801.01148, Jan 2018.

[Lam91] Tsit Y. Lam. *A first course in noncommutative rings.* New York etc.: Springer-Verlag, 1991.

[Lan02a] Serge Lang. *Algebra. 3rd revised ed.* New York, NY: Springer, 3rd revised ed. edition, 2002.

[Lan02b] Serge Lang. *Introduction to differentiable manifolds.* Universitext. Springer-Verlag, New York, second edition, 2002.

[Las70] Richard Lashof. The immersion approach to triangulation. In *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969)*, pages 52–56. Markham, Chicago, Ill., 1970.

[Las71] Richard Lashof. The immersion approach to triangulation and smoothing. In *Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970)*, pages 131–164. Amer. Math. Soc., Providence, R.I., 1971.

[Lee11] John Lee. *Introduction to topological manifolds*, volume 202 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2011.

[Lev67] Jerome Levine. A method for generating link polynomials. *Am. J. Math.*, 89:69–84, 1967.

[Lev69] J. Levine. Knot cobordism groups in codimension two. *Comment. Math. Helv.*, 44:229–244, 1969.

[Lev77] Jerome Levine. Knot modules. I. *Trans. Am. Math. Soc.*, 229:1–50, 1977.

[LM15] Lukas Lewark and Duncan McCoy. On calculating the slice genera of 11- and 12-crossing knots. arXiv:1508.01098, 2015.

[LS71] Richard Lashof and Julius L. Shaneson. Smoothing 4-manifolds. *Invent. Math.*, 14:197–210, 1971.

[LS03] Wolfgang Lück and Thomas Schick. Various L^2-signatures and a topological L^2-signature theorem. In *High-dimensional manifold topology. Proceedings of the school, ICTP, Trieste, Italy, May 21–June 8, 2001*, pages 362–399. River Edge, NJ: World Scientific, 2003.

[LT84] Richard Lashof and Laurence Taylor. Smoothing theory and Freedman's work on four-manifolds. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 271–292. Springer, Berlin, 1984.

[Lüc98] Wolfgang Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I: Foundations. *J. Reine Angew. Math.*, 495:135–162, 1998.

[Lüc02] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 2002.

[Man16] Ciprian Manolescu. Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture. *J. Amer. Math. Soc.*, 29(1):147–176, 2016.

[Mar58] Andrei Markov. Unsolvability of certain problems in topology. *Dokl. Akad. Nauk SSSR*, 123:978–980, 1958.

[Mar77] Alexis Marin. La transversalité topologique. *Ann. of Math. (2)*, 106(2):269–293, 1977.

[Mat78] Takao Matumoto. Triangulation of manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 3–6. Amer. Math. Soc., Providence, R.I., 1978.

[May75] J. Peter May. Classifying spaces and fibrations. *Mem. Amer. Math. Soc.*, 1(1, 155):xiii+98, 1975.

[MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms.* Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[Mil58] John Milnor. On simply connected 4-manifolds. In *Symposium internacional de topología algebraica International symposi um on algebraic topology*, pages 122–128. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[Mil63] John Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[Mil64] John W. Milnor. Microbundles: Part I. *Topology*, 3(1):53–80, 1964.

[Moi52] Edwin Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)*, 56:96–114, 1952.

[Moi77] Edwin Moise. Geometric topology in dimensions 2 and 3. Graduate Texts in Mathematics. 47., 1977.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.

[Mun66] James R. Munkres. *Elementary differential topology*, volume 1961 of *Lectures given at Massachusetts Institute of Technology, Fall.* Princeton University Press, Princeton, N.J., 1966.

[NP17] Matthias Nagel and Mark Powell. Concordance invariance of Levine-Tristram signatures of links. *Doc. Math.*, 22:25–43, 2017.

[Osb00] M.Scott Osborne. *Basic homological algebra.* New York, NY: Springer, 2000.

[Per86] B. Perron. Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique. *Topology*, 25(4):381–397, 1986.

[Pes96] Christian Peskine. *An algebraic introduction to complex projective geometry. 1: Commutative algebra.*, volume 47. Cambridge: Cambridge Univ. Press, 1996.

[Qui82] Frank Quinn. Ends of maps III: Dimensions 4 and 5. *J. Differential Geom.*, 17(3):503–521, 1982.

[Qui83] Frank Quinn. The stable topology of 4-manifolds. *Topology Appl.*, 15(1):71–77, 1983.

[Qui84] Frank Quinn. Smooth structures on 4-manifolds. In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 473–479. Amer. Math. Soc., Providence, RI, 1984.

[Qui86] Frank Quinn. Isotopy of 4-manifolds. *J. Differential Geom.*, 24(3):343–372, 1986.

[Qui88] Frank Quinn. Topological transversality holds in all dimensions. *Bull. Amer. Math. Soc. (N.S.)*, 18(2):145–148, 1988.

[Rad26] Tibor Radó. Über den Begriff der Riemannschen Fläche. *Acta Szeged*, 2:101–121, 1926.

[Roh52] V. A. Rohlin. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR (N.S.)*, 84:221–224, 1952.

[Rol90] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series.* Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[Row06] Louis Halle Rowen. *Graduate algebra: commutative view.*, volume 73. Providence, RI: American Mathematical Society (AMS), 2006.

[RS67] Colin P. Rourke and Brian J. Sanderson. An embedding without a normal microbundle. *Invent. Math.*, 3:293–299, 1967.

[RS68a] C. P. Rourke and B. J. Sanderson. Block bundles. III. Homotopy theory. *Ann. of Math. (2)*, 87:431–483, 1968.

[RS68b] Colin P. Rourke and Brian J. Sanderson. Block bundles. III. Homotopy theory. *Ann. of Math. (2)*, 87:431–483, 1968.

[RS72] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69.

[Rub84] Daniel Ruberman. Invariant knots of free involutions of S^4. *Topology Appl.*, 18(2-3):217–224, 1984.

[Rud16] Yuli Rudyak. *Piecewise linear structures on topological manifolds*. Hackensack, NJ: World Scientific, 2016.

[Sch68] Horst Schubert. Topology. Boston: Allyn and Bacon, Inc. 1968, 358 p. (1968)., 1968.

[Sch70] James A. Schafer. Topological Pontrjagin classes. *Comment. Math. Helv.*, 45:315–332, 1970.

[Ste75] Ronald J. Stern. On topological and piecewise linear vector fields. *Topology*, 14:257–269, 1975.

[Sto93] Richard Stong. Simply-connected 4-manifolds with a given boundary. *Topology Appl.*, 52(2):161–167, 1993.

[Sun17] Fang Sun. An elementary proof for Poincare Duality with local coefficients. *arXiv e-prints*, page arXiv:1709.00569, Sep 2017.

[SW00] Richard Stong and Zhenghan Wang. Self-homeomorphisms of 4-manifolds with fundamental group **z**. *Topology Appl.*, 106(1):49–56, 2000.

[Tei92] Peter Teichner. *Topological 4-manifolds with finite fundamental group*. PhD thesis, University of Mainz, Germany, 1992. Shaker Verlag, ISBN 3-86111-182-9.

[Tei97] Peter Teichner. On the star-construction for topological 4-manifolds. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 300–312. Amer. Math. Soc., Providence, RI, 1997.

[Tri69] Andrew G. Tristram. Some cobordism invariants for links. *Proc. Cambridge Philos. Soc.*, 66:251–264, 1969.

[Tur01] Vladimir Turaev. *Introduction to combinatorial torsions*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001. Notes taken by Felix Schlenk.

[Wal64] C. Terence C. Wall. On simply-connected 4-manifolds. *J. London Math. Soc.*, 39:141–149, 1964.

[Wal65] C. Terence C. Wall. Finiteness conditions for CW-complexes. I, II. *Ann. Math. (2)*, 81:56–69, 1965.

[Wal66] C. Terence C. Wall. Finiteness conditions for CW complexes. II. *Proc. Roy. Soc. Ser. A*, 295:129–139, 1966.

[Wal67] C. Terence C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.

[Wal70] C. Terence C. Wall. Surgery on compact manifolds. London Mathematical Society Monographs. No.1. London-New York: Academic Press X, 280 p. (1970)., 1970.

[Wal99] C. Terence C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[Wal16] C. Terence C. Wall. *Differential topology*. Cambridge: Cambridge University Press, 2016.

[Wan95] Zhenghan Wang. Classification of closed nonorientable 4-manifolds with infinite cyclic fundamental group. *Mathematical Research Letters*, 2:339–344, 01 1995.

[Wes77] James West. Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk. *Ann. of Math. (2)*, 106(1):1–18, 1977.

[Whi49] J. H. C. Whitehead. On simply connected, 4-dimensional polyhedra. *Comment. Math. Helv.*, 22:48–92, 1949.

[Whi57] Hassler Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.

Fakultät für Mathematik, Universität Regensburg, Germany
Email address: sfriedl@gmail.com

Department of Mathematics, ETH Zürich, Switzerland
Email address: matthias.nagel@math.ethz.ch

Mathematics Department, Boston College, USA
Email address: patrick.orson@edu.bc

Department of Mathematical Sciences, Durham University, UK
Email address: mark.a.powell@durham.ac.uk