THURSTON'S VISION AND THE VIRTUAL FIBERING THEOREM FOR 3-MANIFOLDS

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Abstract. The vision and results of William Thurston (1946-2012) have shaped the theory of 3-dimensional manifolds for the last four decades. The high point was Perelman's proof of Thurston's Geometrization Conjecture which reduced 3-manifold topology for the most part to the study of hyperbolic 3-manifolds. In 1982 Thurston gave a list of 24 questions and challenges on hyperbolic 3-manifolds. The most daring one came to be known as the Virtual Fibering Conjecture. We will give some background for the conjecture and we will explain its precise content. We will then report on the recent proof of the conjecture by Ian Agol and Dani Wise.

1. Introduction

The development of the theory of 3-dimensional manifolds, henceforth referred to as 3-manifolds, does not start out with a theorem, but with the formulation of the following conjecture by Henri Poincaré [Po1904, Vo1996] in 1904: The 3-dimensional sphere is the only simply connected, closed 3-manifold. We will give a definition of 'simply connected' in Section 3.1, but at this stage it suffices that in essence the Poincaré Conjecture gives an elegant, intrinsic and purely topological characterization of the 3-dimensional sphere.

Progress towards resolving the Poincaré Conjecture was virtually non-existent for over 50 years. In fact, the first genuinely non-trivial result in 3-manifold topology was proved only in 1957 by Christos 'Papa' Papakyriakopoulos [Pa1957]. His proof of 'Dehn's Lemma'[1] removed a major stumbling block which had held back the development of 3-manifold topology for many decades. This result was in particular instrumental in the work of Wolfgang Haken [Ha1962] and Friedhelm Waldhausen [Wa1968] who resolved many classification problems for what became known as 'Haken manifolds'. We will give a definition of a 'Haken manifold' in Section 5.2. For the time being it is enough to know that many natural examples of 3-manifolds, e.g. complements of non-trivial knots in S^3, are Haken manifolds.

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[1] Dehn's Lemma says that 'if c is an embedded curve on the boundary of a 3-manifold N such that c bounds an immersed disk in N, then it already bounds an embedded disk in ∂N'. This statement goes back to Max Dehn [De1910] in 1910, but Hellmuth Kneser [Kn1929, p. 260] found a gap in the proof provided by Dehn. It then took another 30 years to find a correct proof.

The modern theory of 3-manifolds is for the most part due to the results and the vision of William Thurston [Th1979, Th1982]. In the 1970s Thurston developed the point of view that 3-manifolds should be viewed as geometric objects. In particular he formulated the Geometrization Conjecture which loosely speaking states that for the most part the study of 3-manifolds can be reduced to the study of hyperbolic 3-manifolds. As we will see in Section 6, the Geometrization Conjecture can be viewed as a far reaching generalization of the aforementioned Poincaré Conjecture. In an amazing tour de force Thurston proved the Geometrization Conjecture for all Haken manifolds. This proof was announced in 1979, but due to the complexity of the argument it took another 20 years before all details had finally been established rigorously and had appeared in print.

The full Geometrization Conjecture, and thus in particular the Poincaré Conjecture, was finally proved by Grisha Perelman [Pe2002, Pe2003a, Pe2003b, MT2007]. The question thus became, what can we say about hyperbolic 3-manifolds? What do hyperbolic 3-manifolds look like? To this effect Thurston [Th1982] posed 24 questions and challenges which have been guiding 3-manifold topologists over the last 30 years.

Arguably the most famous of these questions is the following that we quote verbatim:

> *'Does every hyperbolic* 3*-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer.'*

We will give a definition of 'fibers over the circle' in Section 4.2 and we will give a definition of 'finite-sheeted cover' in Section 7. In a slightly simplified formulation the question asks whether every hyperbolic 3-manifolds can be given a particularly simple description after doing a certain basic modification. According to [Ga1986] the 'question was upgraded in 1984' to the 'Virtual Fibering Conjecture', i.e. it was conjectured by Thurston that the question should be answered in the affirmative.

The recent article by Otal [Ot2014] discussing Thurston's famous article [Th1982] shows in particular that many of the aforementioned 24 questions and challenges were answered in the years after Thurston formulated them, and that each time his vision was vindicated. But for many years there had been only very scant evidence towards the Virtual Fibering Conjecture. It seems fair to say that nobody really had an idea for how to address the conjecture. The situation changed dramatically within the last couple of years with the revolutionary work of Ian Agol [Ag2008, Ag2013] and Dani Wise [Wis2009, Wis2012a, Wis2012b]. As a consequence, in April 2012, just before William Thurston's untimely death, the Virtual Fibering Conjecture was finally proved. The work of Agol and Wise is easily the greatest step forward in 3-manifold topology since Perelman's proof of the Geometrization Theorem. In some sense it is arguably an even more astounding achievement: everybody 'knew' that the Geometrization Conjecture just had to be true, but researchers had rather mixed opinions on Thurston's Virtual Fibering Conjecture.

Prerequisites and further reading. Our goal has been to write a paper which on the one hand is accessible for mathematicians who might have only a modest background in topology, but which on the other hand is also interesting for researchers in the field. This dual goal creates some unavoidable tensions. For example, our attempt at making the paper as accessible as possible leads at times to consciously vague and imprecise formulations. We refer the reader to the indicated references for precise statements. On the other hand at times we need to use technical terms, but hopefully one can follow the flow of the story, even if one treats some terms as black boxes.

We refer to [Ag2014], [Ber2014] and [Bes2014] for more technical and detailed accounts of the work of Agol and Wise,

Organization. This paper is organized as follows. In Section 2 we will first give an introduction to manifolds and we will revisit the classification of surfaces, i.e. of 2-manifolds, and of geometric structures on surfaces. In Section 3 we will introduce the notion of a simply connected space and of the fundamental group of a space, and we will have a quick peek at manifolds of dimension greater than three. Afterwards we finally settle for the 3-dimensional case. In Section 4 we will provide ourselves with some examples of 3-manifolds to work with and in Section 5 we will introduce two special types of 3-manifolds. In Section 6 we will discuss the Geometrization Theorem, its relation to the Poincaré Conjecture, the partial proof by Thurston and the full proof by Perelman. In Section 7 we will explain in detail the statement of the Virtual Fibering Conjecture and we will report on its proof by Ian Agol and Dani Wise. We conclude this note with an exposition in Section 8 of the last of Thurston's challenges that is still open.

Conventions. On several occasions we will use a mathematical term in quotes. This means that we will not give a definition, and the term can be treated as a black box. In some cases, e.g. when we use the term 'non-positively curved cube complex', the name will hopefully give at least a vague feeling of what it stands for.

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2. Surfaces

2.1. **The definition of a manifold.** Loosely speaking, an *n-dimensional manifold* is an object which at any given point looks like we are in $\mathbb{R}^n$. Some low-dimensional examples are given as follows:

(1) a point is a 0-dimensional manifold,

(2) a curve is a 1-dimensional manifold,
(3) a surface, e.g. the surface of a ball, of a donut, of a pretzel or of the earth, is a 2-dimensional manifold,
(4) the physical universe, as we personally experience it, is a 3-dimensional manifold, and
(5) spacetime is a 4-dimensional manifold.

Following the usual terminology we will subsequently refer to an n-dimensional manifold as an *n-manifold*. Furthermore, to be on the safe side, throughout this note we will make the technical assumption that all manifolds are connected, orientable, differentiable and 'compact', unless we say explicitly otherwise. We will not attempt to give the definition of 'compact'. Loosely speaking it means that we are only interested in 'finite' manifolds; for example the euclidean space $\mathbb{R}^n$ is not compact.

We are interested in the study of the intrinsic shapes of manifolds. If two manifolds have the same intrinsic shape then they are called *homeomorphic*. For example, let us consider the 1-manifolds sketched in Figure 1. The 1-manifolds (a), (c) and (e) have

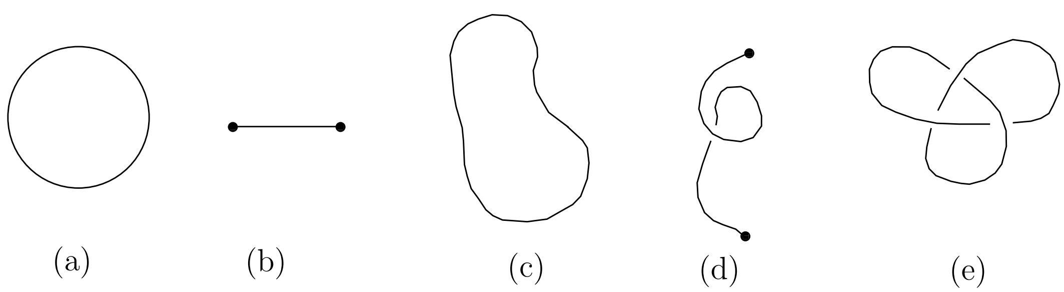

FIGURE 1. Examples of 1-manifolds.

the property that 'walking along the 1-manifold one eventually ends up at the starting point'. On the other hand (b) and (d) behave qualitatively very differently: walking along either of the 1-manifolds we eventually come to an end. The mathematically precise way of saying this is that the 1-manifolds (a), (c) and (e) are homeomorphic to each other and that the 1-manifolds (b) and (d) are homeomorphic to each other, but none of the former manifolds is homeomorphic to either of the latter.

In the following we say that a manifold is *closed* if it has no boundary, i.e. walking on the manifold one never reaches an end. For example, the manifolds (a), (c) and (e) are closed, the other ones are not. This notion also makes sense in other dimensions, for example the surface of the earth is closed. On the other hand an annulus or the Möbius band are not closed.

With this notion we can now state the classification of 1-manifolds: Any closed 1-manifold is homeomorphic to (a) and any non-closed 1-manifold is homeomorphic to (b)[2].

2.2. **The classification of 2-manifolds.** After stating the classification of 1-manifolds we now move up one dimension. As usual we will refer to a 2-manifold as a surface. Just like for curves, the classification of surfaces up to homeomorphism is quite straightforward to state. First of all, any closed surface is homeomorphic to the standard surface of genus g that is shown in Figure 2. Furthermore, g is uniquely determined. For instance, the surface of a pretzel is a surface of genus three.

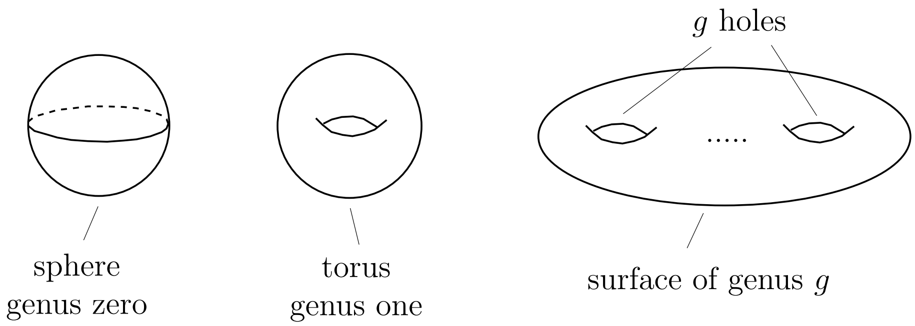

Figure 2. Examples of surfaces.

The classification of surfaces with boundary is only slightly more complicated: any surface with boundary is homeomorphic to the result of removing k disjoint 'open' disks from a surface of genus g. Again g and k are uniquely determined. For example, a disk is homeomorphic to the surface one obtains from deleting one 'open' disk from the 2-sphere. Furthermore, one obtains the annulus by deleting two 'open' disks from the 2-sphere.

This classification of (orientable) surfaces seems obvious. After all, what other (orientable) surfaces should there be? But a quick look at non-orientable surfaces, such as the Möbius band, shows that perhaps we can easily become a victim of our intuition. Even among more experienced mathematicians, how many readers feel comfortable in stating the classification of non-orientable surfaces?

2.3. **Geometric structures on surfaces.** Before we proceed to other dimensions we want to consider surfaces as geometric objects[3]. This is a fascinating story in its own right and it will also help us later on in our study of 3-manifolds.

The most familiar geometries are of course euclidean geometry and spherical geometry, but it has been known since the early 1800s that these two geometries are

[2]Here recall that throughout the paper we restrict ourselves to compact manifolds; that is why $\mathbb{R}$ is missing from our list of 1-manifolds.

[3]This topic was also discussed by Klaus Ecker [Ec2008] in an earlier Jahresbericht.

naturally complemented by hyperbolic geometry. In Figure 3 we show the 2-sphere, the euclidean plane and the Poincaré disk model for hyperbolic geometry[4]. In each picture we sketch a triangle formed by geodesics. In the euclidean plane the angle

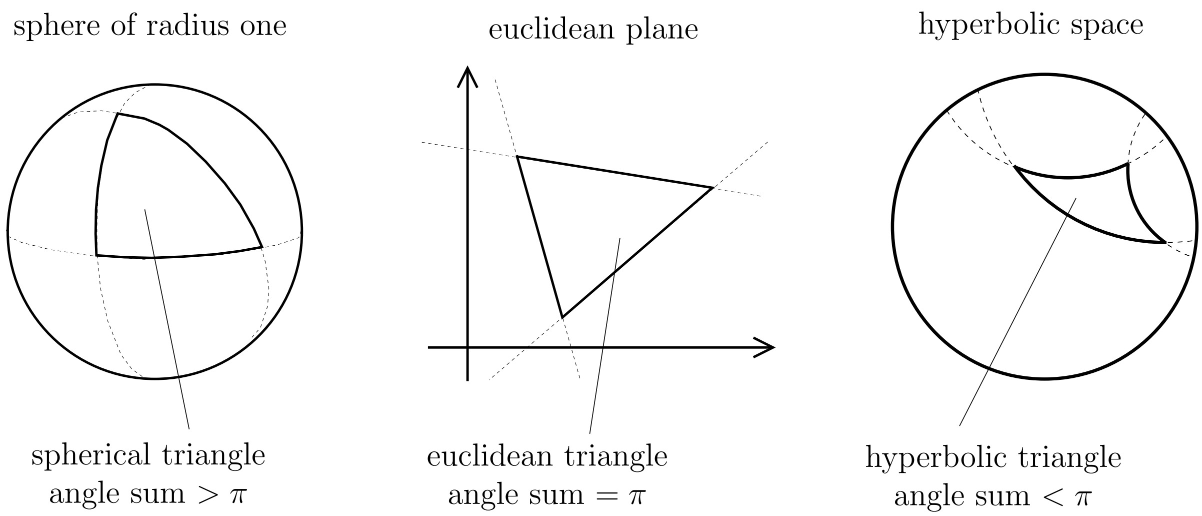

FIGURE 3. The spherical, euclidean and hyperbolic geometry.

sum of a triangle is of course equal to π. The fact that the sphere is positively curved implies that the angle sum of a triangle is always greater than π. On the other hand the angle sum of a triangle in the hyperbolic space is always less than π, reflecting the fact that hyperbolic space is negatively curved.

We can now state one of the most beautiful findings of 19th century mathematics: Every surface is either spherical, euclidean or hyperbolic. This means that every surface can be equipped with a 'metric' such that at each point the surface looks like either the 2-sphere, or the euclidean plane or hyperbolic space.

In the following we will quickly outline where these metrics come from. Almost by definition the 2-sphere has a spherical metric. Moving on, the fact that the annulus admits a euclidean metric follows from the observation that we can roll a piece of paper into a tube (which is homeomorphic to an annulus) without stretching or wrinkling. Now we turn to the 2-dimensional torus. We will in fact present not just one, but three arguments that the torus supports a euclidean metric.

(1) The first approach generalizes what we did for the annulus. The torus can be built out of the tube by gluing the two ends together. If we do the bending and gluing in $\mathbb{R}^3$, then we have to deform the tube, and the resulting torus

[4]In the Poincaré disk model for hyperbolic geometry the space is given by an open disk in $\mathbb{R}^2$ and the geodesics are given by the segments of Euclidean circles which intersect the given disk orthogonally. This model of hyperbolic geometry inspired M. C. Escher to create his famous woodcuts Circle Limit I, II, III and IV.

is no longer euclidean. On the other hand, if we do the bending and gluing in $\mathbb{R}^4$, then this can be done without stretching and the resulting torus has a euclidean metric.

(2) As we illustrate in Figure 4, we can build the torus out of the humble unit square in $\mathbb{R}^2$ by gluing the opposite sides together. While gluing we have to ensure that the metrics match up. The gluing can be performed via isometries and a moment's thought shows that the result is indeed a euclidean metric on the torus.

(3) Finally, the most concise but also the most abstract way of seeing an euclidean structure on the torus is by realizing the torus as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.

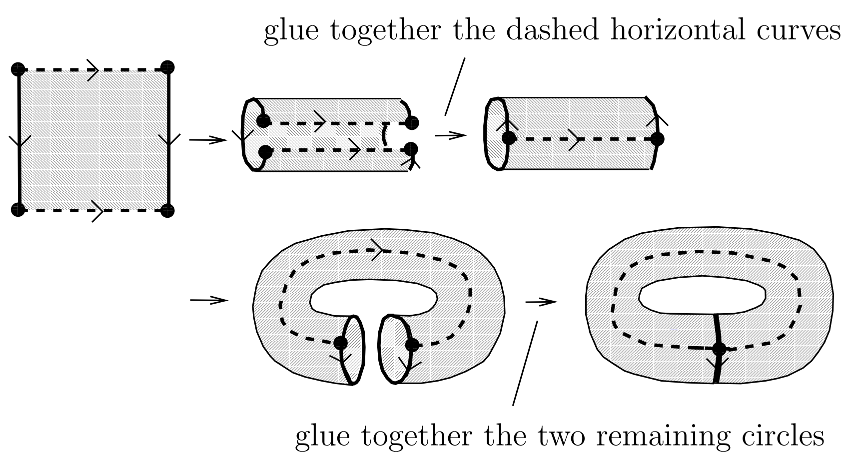

FIGURE 4. By gluing together opposing sides of a square we obtain a torus.

Now we turn to a surface of genus $g \geq 2$. We will try to see how far we can get with the second approach that we took for the torus. We obtained the torus by gluing the opposite sides of a square, i.e. of a regular 4-gon. Similarly one can obtain the surface of genus g by gluing the sides of a regular $4g$-gon in an appropriate way. In order to simplify our discussion and our pictures henceforth we restrict ourselves to the case $g = 2$. We consider the regular octagon on the left of Figure 5 and we glue the sides with the same symbol to each other in such a way that the orientations match. With very well-trained eyes one can spot that we just obtained a surface of genus 2. (For less well-trained eyes an illustration is given in [Fra2007, Chapter 7.1].)

Again we can do the gluing via isometries, but now there is an obstacle. In order to see the problem, note that all the eight vertices of the octagon get glued together and form one point on the surface. The eight interior angles of the octagon form one full angle around the new point. But the interior angles add up to $8 \cdot \frac{3\pi}{4} = 6\pi$, which is much more than a full circle. Thus we see that this attempt of finding a euclidean metric on the surface of genus 2 has failed. In fact there is a deeper reason why this approach does not work: it is an immediate consequence of the Gauss-Bonnet theorem that the surface of genus 2 cannot admit a euclidean metric.

We will now modify the approach. Instead of a euclidean octagon we will use a hyperbolic octagon. As we already pointed out, the salient feature of hyperbolic geometry is that the angle sum of n-gons is smaller than for euclidean n-gons. In

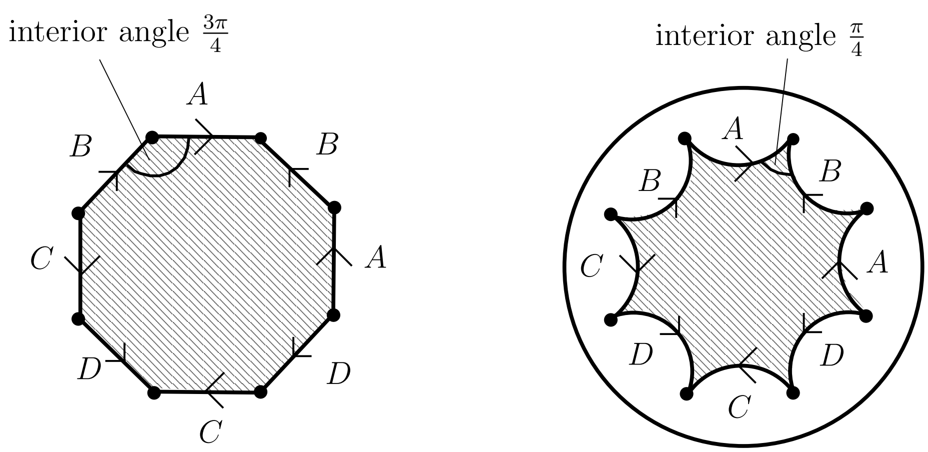

FIGURE 5. Construction of a hyperbolic metric on a surface of genus 2.

fact there exists a regular hyperbolic octagon such that the interior angle at each vertex is $\frac{\pi}{4}$, see Figure 5 on the right. Now we use reflections and translations in the hyperbolic plane to perform the same type of gluings as before. Again we obtain a surface of genus 2, but this time the interior angles add up to $8 \cdot \frac{\pi}{4} = 2\pi$, so we obtain a hyperbolic metric on the surface.

In fact, playing with the construction, using irregular hyperbolic octagons, one can produce many more hyperbolic metrics which are pairwise non-isometric. (For example they can be distinguished by the length of the shortest closed geodesic.) We refer to [St1992] for a much more detailed discussion of metrics on surfaces.

3. Manifolds of dimension greater than three

Now we want to have a quick peek at manifolds of dimension greater than three. At this point it is helpful to introduce the notion of a 'simply connected' manifold, which we already used in the formulation of the Poincaré Conjecture.

3.1. **Simply connected manifolds and the fundamental group.** Loosely speaking, a space is said to be *simply connected* if every lasso in the space can be pulled tight. A little more precisely, a space is called simply connected if every loop in the space can be contracted to a point. In Figure 6 we show on the left that the equator on the sphere can be contracted to a point. In fact the 2-sphere, and also all spheres of dimension greater than two are simply connected.

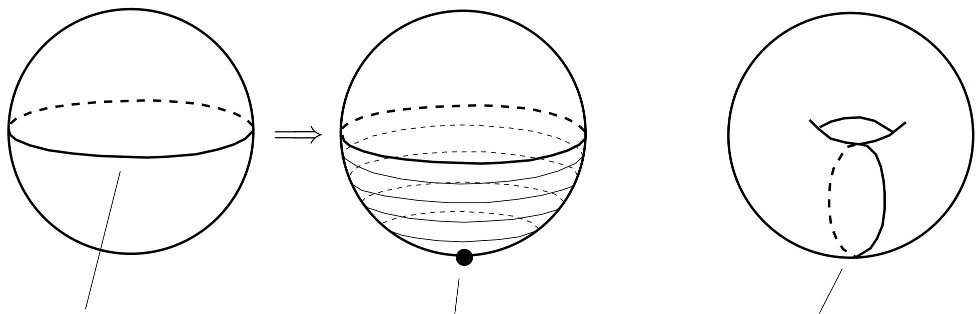

FIGURE 6. Loops on the sphere and on the torus.

In Figure 6 on the right we show a loop on the 2-dimensional torus which cannot be contracted to a point. The torus, and in fact any surface of genus greater than zero, is not simply connected. Together with the classification of closed surfaces in Section 2.2 this gives us the Poincaré Conjecture in dimension 2: the 2-sphere is the only simply connected, closed 2-manifold.

Given a connected space X, the *fundamental group* $\pi_1(X)$ of a space is a group which 'measures' how far X is from being simply connected. More precisely, the fundamental group $\pi_1(X)$ is trivial if and only if the space X is simply connected. For example, the fundamental group of the torus is $\mathbb{Z}^2$ and the fundamental group of a surface of genus greater than one is an infinite non-abelian group. In fact the fundamental group of any hyperbolic manifold, of any dimension, is infinite and non-abelian.

3.2. **Manifolds of dimension greater than three.** Now we will have a quick look at manifolds whose dimensions are greater than four. Somewhat surprisingly, the 'extra room' one has in the high-dimensional setup makes some of the classification problems much easier. For example the natural generalization of the Poincaré Conjecture to higher dimensions was proved in dimensions greater than four by Stephen Smale in the early 1960s. This predates Perelman's proof of the original Poincaré Conjecture by a wide margin [5].

Despite this success, once one considers non-simply connected manifolds, a very different picture emerges. It is relatively easy to show, see e.g. [CZ1993, Theorem 5.1.1], that given any $n \geq 4$ one has enough flexibility in constructing n-manifolds to realize

[5]The Poincaré Conjecture in dimension four was proved by Michael Freedman [Fre1982] in 1982. More precisely, he showed that any simply connected closed, topological 4-manifold is *homeomorphic* to S^4. It is not known, whether any simply connected, closed, differential 4-manifold is *diffeomorphic* to S^4. Resolving that question is often considered as the hardest problem in low-dimensional topology.

any 'finitely presented' group as the fundamental group of a closed n-manifold. This saddles all problems from group theory onto topology. For example, Sergei Adyan [Ad1955] showed that 'finitely presented' groups cannot be classified, which then implies that it is impossible to classify closed n-manifolds. Here 'impossible to classify' is meant in the strongest terms: not only are we at this moment not able to classify those manifolds, in fact there cannot exist an algorithm which determines whether or not two given closed n-manifolds are homeomorphic. We refer to [St1993, Section 9.4] for a detailed discussion.

4. Examples of 3-manifolds

Before we delve into the theory of 3-manifolds it is convenient to equip ourselves with some examples of 3-manifolds. In order to avoid pathologies we henceforth only consider 3-manifolds which are either closed or such that the boundary consists of a union of tori.

4.1. **Knot complements.** The easiest example of a 3-manifold is of course the 3-sphere S^3, which is by definition the sphere of radius one in $\mathbb{R}^4$. We obtain many more examples of 3-manifolds by taking the complement of a knot in the 3-sphere. Here, loosely speaking, a knot is a tied up piece of rope as shown in Figure 7. (More technically speaking, in this paper a knot is an embedded open solid torus in S^3.) As

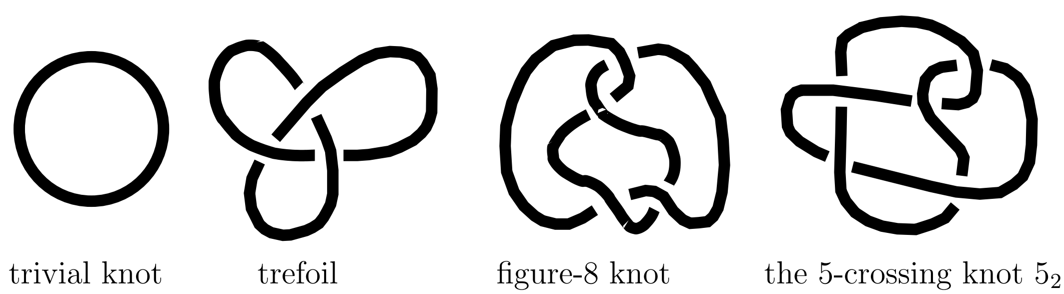

Figure 7. Examples of knots.

we will see later on, this deceptively easy way of constructing 3-manifolds is in fact a surprisingly rich source of examples.

It is an amusing visual exercise to convince oneself that the complement of the trivial knot in S^3 is a solid torus.

4.2. **Fibered manifolds.** Now we turn to a general three-step procedure for building an $(n+1)$-manifold out of an n-manifold:

(1) pick an n-manifold F,
(2) consider the product $F \times [-1, 1]$,
(3) glue the manifold $F \times \{-1\}$ on the left to the manifold $F \times \{1\}$ on the right.

The result is an $(n+1)$-dimensional manifold. This new manifold can be viewed as a disjoint union of copies of the manifold F. In fact there is a 'circle's worth of copies' of F. We therefore say that the resulting $(n+1)$-manifold *fibers over the circle* and we refer to each copy of F as a *fiber*. We sometimes simplify the language and we just say that the $(n+1)$-manifold is *fibered*.

Let us look at several low-dimensional examples to get a feeling for the definition of a fibered manifold. If we take F to be a 0-dimensional manifold, i.e. a point, then the resulting fibered 1-manifold is a circle; see Figure 8.

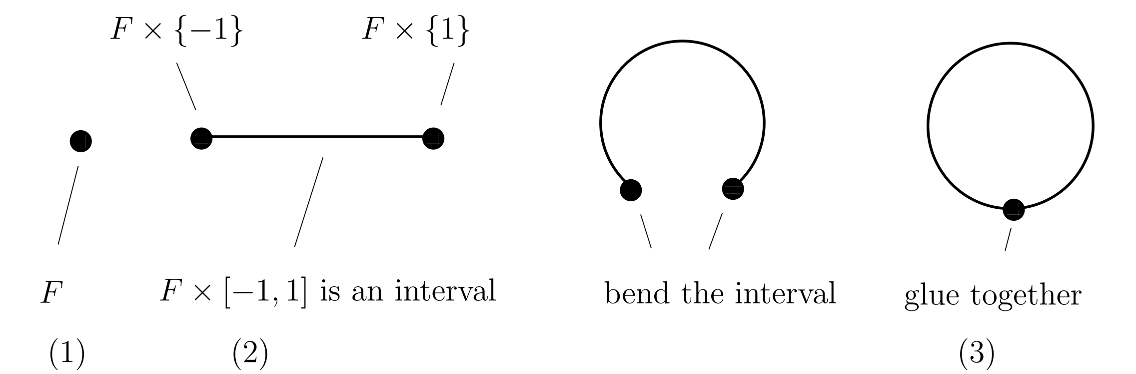

FIGURE 8. Building a circle as a fibered manifold.

We move on to the next dimension and we take F to be an interval. In Figure 9

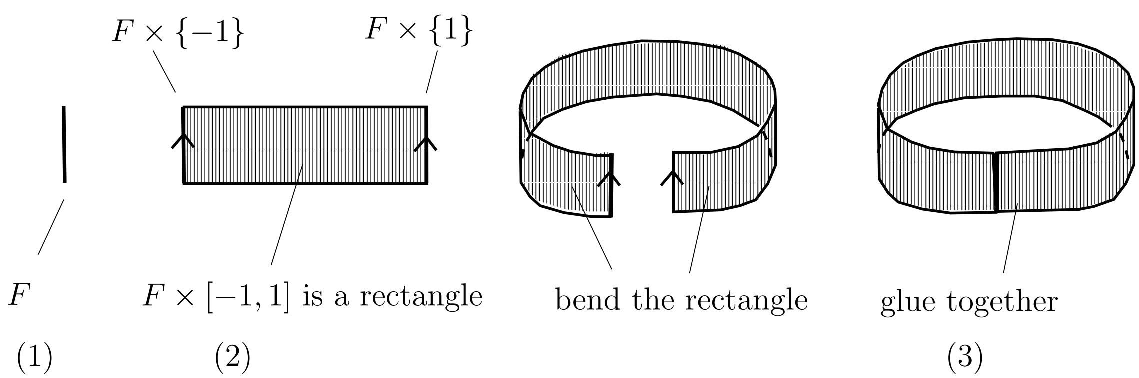

FIGURE 9. Building an annulus as a fibered manifold.

we see that we obtain the annulus by gluing the interval on the left to the interval on the right in the 'obvious way'.

While gluing the 'left interval' to the 'right interval' we notice that we can also glue in a different way: instead of gluing as in Figure 9 we can also first perform a twist and then glue. As we can see in Figure 10 the result is a Möbius band.

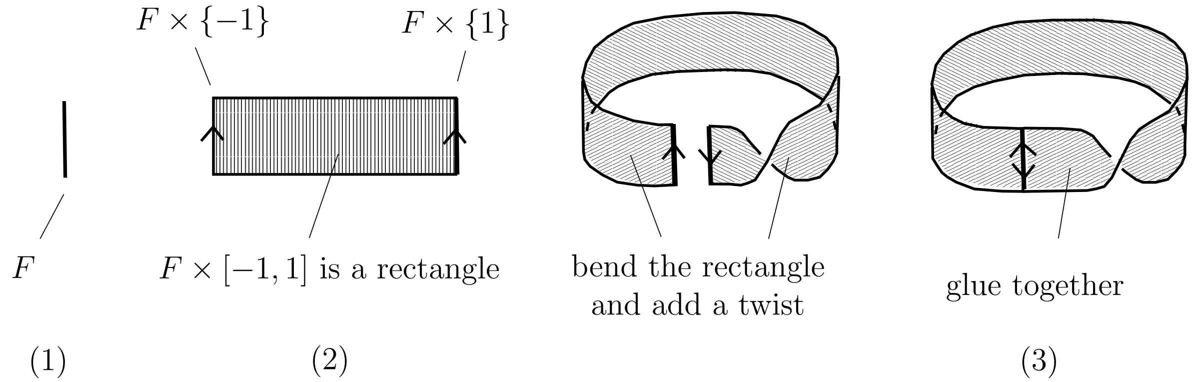

FIGURE 10. Building a Möbius band as a fibered manifold.

Thus we see that in Step (3) above we have a choice for how to glue the left to the right. This can be formalized as follows: given a homeomorphism $f\colon F \to F$ the mapping torus

$$F \times [-1,1] \;/\; (x,-1) \sim (f(x),1)$$

is an $(n+1)$-manifold. For example, if $F = S^1 := \{z \in \mathbb{C} : |z| = 1\}$ and $f = \mathrm{id}$, then the corresponding fibered 2-manifold is the torus. On the other hand, if $F = S^1$ and $f(z) := \overline{z}$ where $z \in S^1$, then the resulting fibered 2-manifold is the Klein bottle. It is an entertaining exercise to try to visualize the latter construction.

Now we used up all 1-manifolds and all self-homeomorphisms of 1-manifolds. This implies, that the list of fibered 2-manifolds we just constructed is complete. Summarizing, even if we do not demand orientability, there are only four fibered 2-manifolds, namely the annulus, the Möbius band, the torus, and the Klein bottle.

The picture is quite different once we increase the dimension by one. In fact every surface of genus greater than zero has many 'different' self-homeomorphisms, and thus gives rise to infinitely many distinct fibered 3-manifolds.

Returning to the knot complements, very well-trained eyes can spot that the complements of the first three knots of Figure 6 are fibered (see e.g. [Ro1990, Chapter 10.I] for the trefoil), whereas the complement of the knot 5_2 is not fibered. Even though in this sample most knot complements are fibered, as so often, 'small' examples give the wrong impression. In fact the complement of a 'generic' knot is not fibered. More generally, Joseph Maher [Mah2010] showed that a 'generic' 3-manifold is not fibered.

4.3. **Gluing handlebodies.** In the previous section we constructed 3-manifolds by a gluing construction. We will now present a somewhat different gluing construction which was introduced by Poul Heegaard at the beginning of the 20th century. We start out with two handlebodies H and H' of the same genus. (A *handlebody of genus* g is the 3-manifold that is bounded by the standard genus g surface in $\mathbb{R}^3$ that is shown in Figure 2.) We obtain a closed 3-manifold by gluing H to H' along the respective boundaries. For example, if we take H and H' to be handlebodies of genus zero, i.e. H and H' are copies of the 3-ball, then this construction gives rise to S^3.

This construction is of particular interest since one can show, surprisingly easily, that any closed 3-manifold can be obtained from two handlebodies, using a suitable gluing.

5. Special types of 3-manifolds

In this section we will introduce Seifert manifolds and Haken manifolds. Both classes of 3-manifolds will play a rôle in the subsequent sections. Nonetheless, if the reader is already drowning in new definitions, then this section can safely be skipped at a first reading. It suffices to know that both types of manifolds have 'enough topology' to be amenable to classical methods. In particular Seifert manifolds have been classified and Haken manifolds are relatively accessible.

5.1. **Seifert manifolds.** A *Seifert manifold*, or alternatively *Seifert fibered manifold*, is defined as a 'singular circle bundle over a surface'. Some of the 3-manifolds we are already familiar with are Seifert manifolds. For example the solid torus, which can be viewed as a disjoint union of circles, is a Seifert manifold. Also, the 'Hopf fibration' shows that the 3-sphere is a Seifert manifold. With some practice one can also detect that the complement of the trefoil is a Seifert manifold. But most knot complements, e.g. the complements of the figure-8 knot and the knot 5_2, and in fact most 3-manifolds are not Seifert.

For the remainder of the paper we are not concerned with the precise definition of a Seifert manifold. What matters to us is that Seifert manifolds were completely classified by Herbert Seifert, see [Se1933], in 1933. In particular it follows fairly quickly from the definitions that the 3-sphere is the only simply connected Seifert manifold.

5.2. **Haken manifolds.** A 3-manifold M is said to be *prime* if it cannot be written as a 'connected sum' of two manifolds $M_1, M_2 \neq S^3$. A 3-manifold M is called *Haken* if it is prime and if it admits an incompressible surface, i.e. a surface F of genus ≥ 1 such that the inclusion induced map $\pi_1(F) \to \pi_1(M)$ is injective.

For example most fibered 3-manifolds are Haken. Indeed, let F be a surface of genus ≥ 1 and let $f\colon F \to F$ be a self-homeomorphism, then F is an incompressible surface in the corresponding fibered 3-manifold. More interestingly perhaps, basically every prime 3-manifold with boundary is Haken. In particular the complement of any non-trivial knot is Haken. On the other hand, 3-manifolds with finite fundamental groups are non-Haken. There are also many examples of 3-manifolds with infinite fundamental groups that are non-Haken, see e.g. [AFW2012] for references.

The reason Haken manifolds play such an important rôle in 3-manifold topology is the fact that they always admit a *hierarchy*. This means that given a Haken manifold M there exists a finite sequence of manifolds $M = M_1, \dots, M_k$ such that each M_i is obtained from the previous manifold M_{i-1} by cutting along an incompressible surface, and such that the final 3-manifold M_k is a union of 3-dimensional balls. Many theorems for Haken manifolds have been proved by induction on the minimal length of such a sequence.

6. The Geometrization Theorem

In the previous sections we have seen all kinds of 3-manifolds, and at first glance 3-manifolds seem to form a rather confusing zoo. The question thus arises whether one can restore some order by finding a classification scheme or a unifying theme. Up to the mid 1970s the only 3-manifolds that were somewhat understood were Seifert manifolds and Haken manifolds. Both types of 3-manifolds are amenable to purely topological methods. But for the remaining manifolds one had no topological tools to work with, and one had absolutely no idea how to study them.

Since purely topological methods failed to deliver, it is (in hindsight!) natural to ask whether perhaps geometric methods can be brought to bear. For example, we had previously seen that every closed surface admits a geometric structure, and that for all but the two simplest closed surfaces one can exhibit hyperbolic metrics using a fairly straightforward gluing construction. Is the situation similar for 3-manifolds?

For a long time it looked like the question should be answered in the negative. In the first decades of the 20th century a few examples of hyperbolic 3-manifolds were explicitly constructed by Hugo Gieseking [Gi1912], Frank Löbell [Lo1931] and Herbert Seifert–Constantin Weber [SW1933], but in the following 40 years no new examples of hyperbolic 3-manifolds were found. In the 1970s events suddenly took a dramatic turn. First, to everybody's surprise Robert Riley [Ri1975] showed that many knot complements, and in particular the complement of the figure-8 knot, admit a hyperbolic structure [6].

Shortly afterwards William Thurston [Th1979, Th1982] formulated the Geometrization Conjecture, which in a slightly simplified form can be formulated as follows [7].

Geometrization Conjecture. *Every 3-manifold admits a canonical decomposition along a (possibly empty) collection of spheres and incompressible tori, such that each of the resulting 3-manifolds is either a Seifert manifold or hyperbolic.*

To get a better understanding of the Geometrization Conjecture let us look at several examples which were already known by the time it was formulated. We have

[6] Riley [Ri2013] points out that the complement of the Figure-8 knot is in fact the 2-fold cover of Gieseking's example. Hugo Gieseking was killed in France in 1915, shortly after his work on hyperbolic 3-manifold. It is conceivable that hyperbolic structures on knot complements would have been discovered much earlier if it had not been for World War I.

[7] For knots the conjecture was foreshadowed by Riley, see [Ri2013] for Riley's account.

already seen several examples of Seifert manifolds and hyperbolic manifolds. In these cases one evidently does not need to decompose any further to obtain the desired result. More interestingly, if we glue the complement of the trefoil to the complement of the figure-8 knot along the boundary tori, then the resulting manifold is neither a Seifert manifold nor hyperbolic. But if we cut this manifold along the gluing torus, then the two resulting components are of course the complement of the trefoil and the complement of the figure-8 knot. Put differently, decomposing along a torus we obtain two 3-manifolds, one of which is a Seifert manifold and one of which is hyperbolic.

As we mentioned in the introduction, the Geometrization Conjecture implies the Poincaré Conjecture. It is a custom in mathematics talks to provide at least one proof. We will do the same here, and we will quickly outline why the Geometrization Conjecture implies the Poincaré Conjecture: Let M be a simply connected closed 3-manifold and suppose the Geometrization Conjecture holds. Some basic algebraic topology quickly implies that the decomposition of M provided by the Geometrization Conjecture has to be trivial. Put differently, M is either already a Seifert manifold or a hyperbolic manifold. In Section 3.1 we already pointed out that the fundamental group of a hyperbolic manifold is infinite. It remains to deal with the former case. But as we already mentioned in Section 5.1, the classification of Seifert manifolds readily implies that M is indeed the 3-sphere.

The first major step towards a proof of the Geometrization Conjecture was Thurston's 'Monster Theorem' from the late 1970s, namely the proof of the Geometrization Theorem for Haken manifolds. As we hinted at in the previous section, the proof uses an induction argument on hierarchies. But along the way Thurston also introduced a wealth of new concepts and ideas, many of which developed into major fields of study in their own right. William Thurston was awarded the Fields medal in 1983[8], but it took about 20 years and the efforts of many authors for all details to be written down rigorously. It is worth reading Thurston's interesting argument [Th1994] why he did not provide the detailed proof himself.

The full proof of the Geometrization Conjecture was finally given by Perelman [Pe2002, Pe2003a, Pe2003b] in 2003 using the Ricci flow on Riemannian metrics, building on ideas pioneered by Richard Hamilton [Ha1982]. A detailed exposition of Perelman's proof is provided by John Morgan and Gang Tian [MT2007], also an accessible outline of the ideas is given by Klaus Ecker [Ec2008] in an earlier Jahresbericht. Perelman declined the Fields medal which was awarded to him in 2006. He also declined the $1,000,000 prize offered to him by the Clay Institute for solving one of the seven Millenium Prize Problems.

[8]The ICM took place in 1983 in Warsaw. It was of course supposed to take place in 1982 but it was postponed by one year because of martial law in Poland which was in effect from December 1981 to July 1983

It is impossible to overstate the importance of the Geometrization Theorem to 3-manifold topology. Not only does it resolve the Poincaré Conjecture, but it underpins almost every deep result on 3-manifolds. For example, it lies at the heart of the algorithm which can determine whether or not two given closed 3-manifolds are homeomorphic. (We refer to [AFW2013] for precise references.)

7. The Virtual Fibering Theorem

The Geometrization Theorem says that any 3-manifold can be canonically decomposed into Seifert manifolds and hyperbolic manifolds. In fact, in a precise sense a 'generic' 3-manifold does not need to be decomposed: it is already hyperbolic. (We refer to [AFW2012, Section 1] for references)[9]. Thus we see that hyperbolic 3-manifolds lie at the heart of 3-manifold topology. The question thus arises, what can we say about the topology of hyperbolic 3-manifolds? As we mentioned before, the generic hyperbolic 3-manifold is not fibered and very many hyperbolic 3-manifolds are non-Haken. This is bad news for us 3-manifold topologists, since this means that there is little 'topology to work with' in the manifold. For example, the absence of an incompressible surface means that we cannot do our favorite trick of cutting 3-manifolds along surfaces into 'smaller' pieces.

Thurston [Th1982] and Waldhausen [Wa1968] speculated that perhaps the picture is very different, once we are allowed to consider finite-sheeted covers. Here, loosely speaking, a manifold $\widetilde{M}$ is an n-fold sheeted cover of another manifold M if there exists a continuous map $p\colon \widetilde{M} \to M$ such that the preimage of each point in M consists of precisely n points in $\widetilde{M}$. In Figure 11 we see that a twice-twisted band (which is nothing but the annulus) is a 2-fold cover of the Möbius band. The notion of a finite-sheeted cover admittedly takes a while to get used to. Suffice it to say, once one gets one's head around it, it is a very natural and central concept in topology and geometry.

In the following we say that a manifold *virtually* has a certain property, if it admits a finite-sheeted cover which has this property. For instance, as we saw above, the Möbius band is virtually an annulus. Similarly we say that a group virtually has a given property, if it admits a finite index subgroup which has that property. For example, every finitely generated abelian group is virtually torsion-free.

With this definition we can now restate Thurston's question from the introduction as follows:

Is every hyperbolic 3-manifold virtually fibered?

The formulation of this and closely related questions in [Th1982] led to many decades of intense research, an overview of the results is given in [La2011], [AFW2012, Section 5.9] and [Ot2014]. But it seems fair to say that progress was limited. In fact

[9] Interestingly the phenomenon that 'most objects are hyperbolic' also occurs in the context of group theory. Mikhail Gromov [Gr1987] showed that in a precise sense a generic finitely presented group is 'word hyperbolic'.

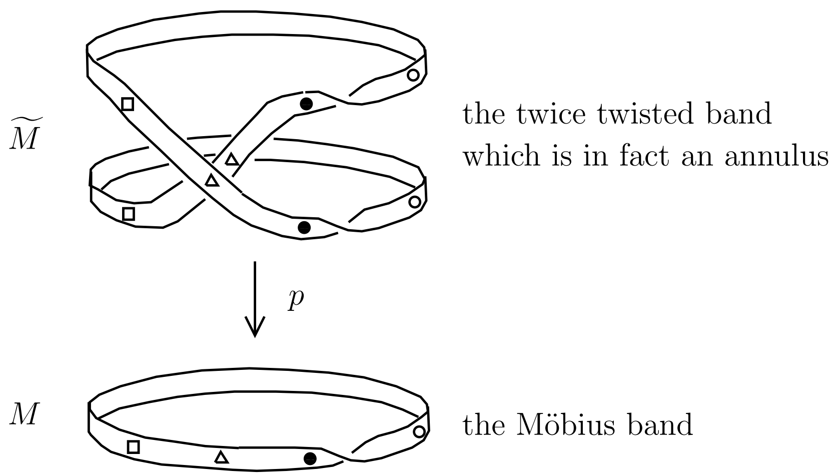

FIGURE 11. The twice twisted band is a 2-fold cover of the Möbius band.

there did not even emerge a consensus on whether one expects an affirmative or a negative answer to the above question. The first major step forward finally happened in 2007 when Ian Agol [Ag2008] proved the following theorem.

Theorem A. *If M is a prime 3-manifold such that its fundamental group $\pi_1(M)$ is infinite and virtually 'RFRS', then M is virtually fibered.*

Here the acronym 'RFRS' stands for 'residually finite-rationally solvable', which, in all likelihood, for most readers is not particularly enlightening. In fact the precise definition of RFRS is of no concern to us. But suffice it to say that being RFRS is a very strong condition on the fundamental group of M. In fact it is so strong that at least the author of this article initially thought that the theorem would apply to a minuscule number of 3-manifolds.

It quickly turned out that this assessment was far of the mark. In 2009 Dani Wise [Wis2009, Wis2012a, Wis2012b] announced a proof, which eventually turned out to be nearly 200 pages long, that the fundamental groups of 'most' Haken hyperbolic 3-manifolds are virtually the fundamental group of a 'special cube complex'. Frédéric Haglund and Dani Wise [HdW2008] and Ian Agol [Ag2008] in turn showed that fundamental groups of special cube complexes are virtually RFRS. The proof given by Wise, once again, used a particularly intricate argument based on hierarchies. Wise's proof was a tremendous achievement, but non-Haken manifolds still seemed intractable.

At the same time, in a completely independent development, Jeremy Kahn and Vlad Markovic [KaM2012] used dynamics on hyperbolic 3-manifolds to show that fundamental groups of closed hyperbolic 3-manifolds have 'lots' of surface subgroups. By work of Nicolas Bergeron and Dani Wise [BW2012] and Michah Sageev [Sa1997] this implies that the fundamental group of any closed hyperbolic 3-manifold is the fundamental group of a 'non-positively curved cube complex'. In light of the aforementioned result of Haglund and Wise [HdW2008] the challenge now became to promote a 'non-positively curved cube complex' to a 'special cube complex'.

This challenge could be formulated as a problem in geometric group theory, which a priori has nothing to do with 3-manifold topology. It was finally once again Agol [Ag2013], building on deep theorems of Wise [Wis2012a, Wis2012b], who rose to the challenge in 2012[10]. Putting the results of Agol [Ag2013] and Wise [Wis2009, Wis2012a, Wis2012b] together gives us the following theorem.

Theorem B. *The fundamental group of any hyperbolic 3-manifold is virtually RFRS.*

Finally, the combination of Theorems A and B gives us the desired affirmative answer to Thurston's question.

The Virtual Fibering Theorem. *Every hyperbolic 3-manifold is virtually fibered.*

The results of Agol and Wise were rounded off by Piotr Przytycki and Dani Wise [PW2012] who showed that Theorem B in fact holds for 'most' non-hyperbolic 3-manifolds as well.

As happens so often when an important long-standing conjecture is finally proved, the eventual proof of the Virtual Fibering Theorem delivered much more than just an answer to the initial question. In a recent book by Matthias Aschenbrenner, the author and Henry Wilton [AFW2012] it takes very dense 13 pages to just list all of the immediate consequences of the work of Agol, Przytycki and Wise. Among them, the author's favorite implication is that the fundamental group of any hyperbolic 3-manifold M is linear over the integers, i.e. $\pi_1(M)$ embeds into $\mathrm{GL}(n,\mathbb{Z})$ for a suitable n. This result is totally unexpected; nobody had even dared to conjecture it before it was proved.

The results of Agol and Wise have produced a seismic shift in our understanding of 3-manifolds and related fields. For example, besides direct applications to 3-manifolds [Su2013, FSW2013] there have already been applications to the Cannon Conjecture [Mac2013], free-by-cyclic groups [HnW2014], and 4-manifolds with a fixed-point free circle action [FV2013, FV2014, Bo2014]. Certainly there will be many more applications in the near future, and it will surely take several years before the full impact of the work of Agol and Wise has been absorbed.

[10] It is characteristic of Ian Agol's unassuming character that the first time he publicly mentioned this result was towards the end of an introductory lecture for graduate students in Paris. Thanks to the digital camera of the author and a blog the news of Agol's theorem spread across the world of 3-manifold topologists within a few hours [Wil2012].

8. Thurston's last challenge

The field of 3-manifold topology has now undoubtedly developed a certain maturity. Nonetheless there are still many basic questions that are wide open. Already the subfield of knot theory bursts with easy-to-state but depressingly hard-to-answer questions. For example, it is still unknown whether the Jones polynomial, first introduced by Vaughan Jones [Jo1987] in 1985, detects the trivial knot. A weaker version of this question was recently answered by Peter Kronheimer and Tom Mrowka [KrM2011] in a major tour de force using Instanton Floer Homology. Many more 'elementary' knot theoretic questions and conjectures are given in a recent survey by Marc Lackenby [La2014].

We want to conclude this article with the formulation of the one challenge of Thurston's that is still open. In order to do so we first note that a hyperbolic metric gives us naturally a notion of a volume. A priori this volume depends of course on the choice of the hyperbolic metric. In Section 2.3 we had hinted at the fact that most surfaces admit many non-pairwise isometric hyperbolic metrics. Amazingly the situation is radically different in dimension 3: George Mostow [Mo1968] and Gopal Prasad [Pr1973] showed that all hyperbolic structures on a given 3-manifold are isometric. In particular the volume of a hyperbolic 3-manifold is independent of the choice of the hyperbolic structure.

Now we can finally quote Thurston's remaining challenge:

Show that volumes of hyperbolic 3*-manifolds are not all rationally related.*

Put differently, the challenge is to find two hyperbolic 3-manifolds N and M such that the ratio of the volumes is not a rational number. This challenge is related to very hard number theoretic problems, which explains why it has not been answered yet. In fact we know so little about volumes of hyperbolic 3-manifolds that it is still unknown whether there exists a hyperbolic 3-manifold such that the volume is rational (or irrational). A short discussion of this challenge can be found in the aforementioned article by Jean–Pierre Otal [Ot2014].

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