CLARIFICATION TO ‘NEW TOPOLOGICALLY SLICE KNOTS’

Abstract. In [FT05] we claimed that the three figures of [FT05, Figure 7.1] represent the Stevedore knot $6_1$. In fact the middle knot is $9_{46}$. In this note we clarify the situation and the ensuing examples.

Consider the knot $K(n)$ in Figure 1. The left most band is twisted by $n$ twists.

![Figure 1](image)

**Figure 1.** The knot $K(n)$.

We summarize the properties of the knots $K(n)$:

**Lemma 1.**

1. $K(n)$ is a ribbon knot with a ribbon disk $D$ such that $\pi_1(D) \cong SR = \mathbb{Z} \times \mathbb{Z}[1/2]$.
2. A Seifert matrix of $K(n)$ is given by
   \[
   \begin{pmatrix}
   n & 2 \\
   1 & 0
   \end{pmatrix}.
   \]
3. The knot $K(-2)$ is $6_1$.
4. The knot $K(0)$ is $9_{46}$.

**Proof.** Figure 2 shows that the knot $K(n)$ is formed by band connected sum of two trivial knots. In particular $K(n)$ is a ribbon knot. We refer to [GS99, p. 210–212] for the computation of the fundamental group of a ribbon disk complement. The argument in [GS99, p. 210–212] also shows immediately that the fundamental group is independent of $n$.

Now consider the Seifert surface for $K(n)$ given in Figure 3 with the curves $a, b$ representing a basis for $H_1$. It is clear that with respect to this choice the Seifert matrix is given by

\[
\begin{pmatrix}
   n & 2 \\
   1 & 0
\end{pmatrix}.
\]
Now consider the isotopies given in Figure 4. Clearly for $n = -2$ the resulting knot equals the Stevedore knot 6_1 given in Figure 5.

Finally we turn to $K(0)$. Note that $K(0)$ has a diagram with 12 crossings. A direct computation shows that the Alexander polynomial equals $2t^2 - 5t + 2$ and that the Jones polynomial equals $t^6 - t^5 + t^4 - 2t^3 + t^2 - t^1 + 2$. The knot tables show that the only knot with 12 crossings or less with these polynomials is 9_{46}. □

In [FT05, Section 7] we incorrectly thought that $K(0) = 6_1$. On pages 2153 and 2155 it should therefore say $K(0)$ instead of 6_1. The proof of [FT05, Proposition 7.7] is written for $K(0)$.

In fact, as we will show now, a version of [FT05, Proposition 7.7] holds for all knots $K(n)$, in particular for $K(-2) = 6_1$.

Indeed, consider the knot $K(n)$ together with curves $a, b$ as in Figure 3. For given knots $C_\alpha, C_\beta$ consider the knot $S = S(K(n), \alpha, \beta, C_\alpha, C_\beta)$ which is the result of tieing the knots $C_\alpha$ and $C_\beta$ into the bands $\alpha$ and $\beta$.

**Proposition 2.** If one of the following holds:

1. $\Delta_{C_\alpha}(t) \neq 1$ and $\Delta_{C_\beta}(t) \neq 1$ or
2. $\Delta_{C_\beta}(t) \neq 1$ and $n \neq 0$,

then $S$ has no h–ribbon with fundamental group $SR$.

**Proof.** Let $S = S(K(n), \alpha, \beta, C_\alpha, C_\beta)$ be such a satellite knot for which (1) or (2) holds. Assume that $S$ has in fact a h–ribbon $D$ with fundamental group $G :=$
SR = Z × Λ/(t − 2). We denote the 0–framed surgery on S by \(M_S\) and we write \(\Lambda := Z[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}].\) We also write \(K = K(n).\) We write \(N_D = M_S \setminus \nu D.\) Then \(\text{Ker}\{H_1(M_S; \mathbb{Z}[\mathbb{Z}]) \rightarrow H_1(N_D; \mathbb{Z}[\mathbb{Z}])\}\) is a metabolizer for \(B\ell(\mathbb{Z})\) (cf. e.g. [Fr04]).

Note that \(\alpha, \beta\) in Figure 3 lift to elements \(\tilde{\alpha}, \tilde{\beta}\) in \(H_1(M_S; \Lambda),\) in fact

\[
H_1(M_S; \Lambda) \cong (\Lambda \tilde{\alpha} \oplus \Lambda \tilde{\beta})/(At - A^t).
\]

Furthermore the Blanchfield pairing \(B\ell(\mathbb{Z})\) with respect to the generators \(\tilde{\alpha}\) and \(\tilde{\beta}\) is given by the matrix \((t − 1)(At - A^t)^{-1}.\)
First assume that $n = 3k$ for some $k$. Then for $P = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ we have

$$P^t A P = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = A'.$$

Note that $A'$ is also a Seifert matrix for $K$. We get a commutative diagram

$$B\ell(\mathbb{Z}) : \quad H_1(M_S; \Lambda) \times H_1(M_S; \Lambda) \to \mathbb{Q}(t)/\Lambda$$

$$(t - 1)(At - A')^{-1} : \quad \Lambda^2/(At - A') \times \Lambda^2/(At - A') \to \mathbb{Q}(t)/\Lambda$$

$$(t - 1)(A't - A'^n) : \quad \Lambda^2/(A't - A'^n) \times \Lambda^2/(A't - A'^n) \to \mathbb{Q}(t)/\Lambda.$$

Here the top vertical map is given by $(1, 0) \to \tilde{\alpha}, (0, 1) \to \tilde{\beta}$ and the bottom vertical map is given by $w \mapsto P^t w$.

We see immediately that $B\ell(\mathbb{Z})$ has two metabolizers, which are generated by $\tilde{\alpha}' = \tilde{\alpha}$ and $\tilde{\beta}' = \tilde{\beta} + k\tilde{\alpha}$.

In particular the map $\pi := \pi_1(M_S) \to \pi_1(N_D)$ is up to automorphism of $G$ either of the form

$$\varphi_{\tilde{\alpha}'} : \pi_1(M_S) \to \pi/\pi^{(2)} \cong \mathbb{Z} \times H_1(M_S; \Lambda) \to \mathbb{Z} \times (H_1(M_S; \Lambda)/\tilde{\alpha}'\Lambda) \cong SR$$

or it is of the same form with $\tilde{\alpha}'$ replaced by $\tilde{\beta}'$. We denote this homomorphism by $\varphi_{\tilde{\beta}'}$. By Theorem [FT05, Theorem 1.3] we get $\text{Ext}_{\mathbb{Z}[G]}(H_1(M_S; \mathbb{Z}[G]), \mathbb{Z}[G]) = 0$ with $G$-coefficients induced by either $\varphi_{\tilde{\alpha}'}$ or by $\varphi_{\tilde{\beta}'}$. Now consider coefficients induced by $\varphi_{\tilde{\alpha}'}$. Note that $\varphi_{\tilde{\alpha}'}(\alpha) = 0$ and $\varphi_{\tilde{\alpha}'}(\beta) \neq 0$. It therefore follows from [FT05, Lemma 6.2] that

$$H_1(M_S; \mathbb{Z}[G]) \cong H_1(M_K; \mathbb{Z}[G]) \oplus H_1(M_{C, \beta}; \mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G].$$
We compute
\[
\text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_S; \mathbb{Z}[G]), \mathbb{Z}[G]) \\
\cong \text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K; \mathbb{Z}[G]), \mathbb{Z}[G]) \oplus H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G], \mathbb{Z}[G]) \\
\cong \text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K; \mathbb{Z}[G]), \mathbb{Z}[G]) \oplus \text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G], \mathbb{Z}[G]) \\
\cong \text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K; \mathbb{Z}[G]), \mathbb{Z}[G]) \oplus \text{Ext}^1_{\mathbb{Z}[\mathbb{Z}]}(H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}]), \mathbb{Z}[\mathbb{Z}]).
\]
Note that 
\[ H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}]) \cong H_1(S^3 \setminus C_\beta; \mathbb{Z}[\mathbb{Z}]), \] in particular it is \( \mathbb{Z} \)-torsion free. It follows from [Le77, Theorem 3.4] that \( \text{Ext}^1_{\mathbb{Z}[\mathbb{Z}]}(H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}]), \mathbb{Z}[\mathbb{Z}]) \cong H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}]), \) which is not possible since by assumption \( \Delta_{C^\beta}(t) \neq 1 \). The only other possibility is therefore that \( \text{Ext}^1_{\mathbb{Z}[G]}(H_1(M_S; \mathbb{Z}[G]), \mathbb{Z}[G]) = 0 \) with \( G \)-coefficients induced by \( \varphi_{\bar{\alpha}} \).
If \( n = 0 \), we then have \( \varphi_{\bar{\alpha}'}(\alpha) \neq 0 \) and \( \varphi_{\bar{\alpha}'}(\beta) = 0 \) and
\[ H_1(M_S; \mathbb{Z}[G]) \cong H_1(M_K; \mathbb{Z}[G]) \oplus H_1(M_{C_\alpha}; \mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G]. \]
If \( n \neq 0 \), then we have \( \varphi_{\bar{\alpha}'}(\alpha) \neq 0 \) and \( \varphi_{\bar{\alpha}'}(\beta) \neq 0 \) and
\[ H_1(M_S; \mathbb{Z}[G]) \cong H_1(M_K; \mathbb{Z}[G]) \oplus H_1(M_{C_\alpha}; \mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G] \oplus H_1(M_{C^\beta}; \mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G]. \]
But in both cases the same calculation as above shows that we get a contradiction to either \( n \neq 0 \) or \( \Delta_{C_\alpha}(t) \neq 0 \).

Now assume that \( n \neq 0 (3) \). We claim that \( \Lambda^2/(At - A^t) \) is cyclic. Indeed, using simultaneous row and column operations the presentation matrix \( At - A^t \) can be turned into
\[
\begin{pmatrix}
    k(t - 1) & 2t - 1 \\
    1 - 2t & 0
\end{pmatrix}
\]
where \( k \in \{1, 2\} \) and \( k \equiv n (3) \). In the case \( k = 1 \) we can do the following row and column operations
\[
\begin{pmatrix}
    t - 1 & 2t - 1 \\
    t - 2 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
    -t & 2t - 1 \\
    t - 2 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
    -t & 2t - 1 \\
    0 & (2t - 1)(1 - 2t^{-1})
\end{pmatrix} \Rightarrow \begin{pmatrix}
    t & 0 \\
    0 & (2t - 1)(1 - 2t^{-1})
\end{pmatrix}.
\]
This shows that \( \Lambda^2/(At - A^t) \) is cyclic. A similar sequence of row and column operations proves the claim for \( k = 2 \). This shows that the Blanchfield form has a unique metabolizer. It is clear that this metabolizer is generated by \( \bar{\alpha} \). We can now conclude the proof as in the case \( n \equiv 0 (3) \).

\[ \square \]

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References