# SIMPLICIAL VOLUME VIA NORMALISED CYCLES

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Abstract. We show that the Connes-Consani semi-norm on singular homology with real coefficients, defined via s-modules, coincides with the ordinary $\ell^1$-semi-norm on singular homology in all dimensions.

## 1. Introduction

Connes and Consani introduced a semi-norm on singular homology via s-modules and established that this semi-norm is equivalent to the $\ell^1$-semi-norm defined by Gromov [CC20]. Moreover, they proved that their semi-norm is equal to the $\ell^1$-semi-norm in the case of surfaces [CC20, Theorem 1.4], using a delicate construction specific to surfaces. In this note, we show that the two semi-norms agree in *all* dimensions, thereby confirming and extending a conjecture of Connes and Consani [CC20, p. 4]:

**Theorem 1.1.** *Let $X$ be a topological space, let $n \in \mathbb{N}$, let $\alpha \in H_n(X;\mathbb{R})$, and let $\lambda \in \mathbb{R}_{>0}$. Then $\|\alpha\|_1 < \lambda$ if and only if $\alpha$ lies in the image of the canonical map $H_n(X; \|H\mathbb{R}\|_\lambda) \to H_n(X;\mathbb{R})$.*

In particular, the simplicial volume of closed manifolds can also be expressed in terms of homology of s-modules.

As explained by Connes and Consani, in order to show Theorem 1.1 it suffices to prove that the $\ell^1$-semi-norm on singular homology can be computed via *normalised* singular cycles (see Section 2.3):

**Proposition 1.2.** *Let $X$ be a topological space and let $n \in \mathbb{N}$. Then, for all $\alpha \in H_n(X;\mathbb{R})$, we have*

$$\|\alpha\|_1 = \|\alpha\|_1^{\text{norm}}.$$

In Section 2, we recall basic definitions and notation. The proof of Proposition 1.2 is given in Section 3, based on a symmetrisation construction.

## 2. The (normalised) $\ell^1$-semi-norm

2.1. **The singular chain complex.** Let $n \in \mathbb{N}$ and let $\Delta^n$ be the standard $n$-simplex. For $j \in \{0, \cdots, n\}$, we denote by $\iota_j^n \colon \Delta^{n-1} \to \Delta^n$ the affine inclusion of the $j$-th facet of $\Delta^n$.

Given a topological space $X$, we consider the singular simplicial set $S(X)$: For $n \in \mathbb{N}$, we have $S_n(X) := \operatorname{map}(\Delta^n, X)$ and for $j \in \{0, \dots, n\}$, the face

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maps $\partial_j \colon S_n(X) \to S_{n-1}(X)$ are given by

$$\partial_j(\sigma) \coloneqq \sigma \circ \iota_j^n$$

for all $\sigma \in \operatorname{map}(\Delta^n, X)$. The singular chain complex $C_\bullet(X;\mathbb{R})$ with real coefficients is the free $\mathbb{R}$-chain complex associated with $S(X)$.

Furthermore, we have the *Moore normalisation* $NC_\bullet(X;\mathbb{R})$ of $C_\bullet(X;\mathbb{R})$, given by the submodules

$$NC_n(X;\mathbb{R}) \coloneqq \bigcap_{j=0}^{n-1} \ker(\partial_j) \subseteq C_n(X;\mathbb{R})$$

and the boundary maps $d \coloneqq \partial_n \colon NC_n(X;\mathbb{R}) \to NC_{n-1}(X;\mathbb{R})$.

**Definition 2.1** (normalised chain)**.** A singular chain $c \in C_n(X;\mathbb{R})$ is *normalised* if it lies in the submodule $NC_n(X;\mathbb{R})$.

2.2. **The $\ell^1$-semi-norms.** We briefly recall Gromov's $\ell^1$-semi-norm on singular homology [Gro82]: The $\ell^1$-norm $\|\cdot\|_1$ on $C_n(X;\mathbb{R})$ associated with the basis $S_n(X)$ induces a semi-norm on $H_n(X;\mathbb{R})$, the *$\ell^1$-semi-norm*, which we will also denote by $\|\cdot\|_1$.

Following Connes and Consani [CC20], one can also endow $H_n(X;\mathbb{R})$ with the semi-norm induced by the $\ell^1$-norm on the normalised complex $NC_\bullet(X;\mathbb{R})$: For $\alpha \in H_n(X;\mathbb{R})$ one sets

$$\|\alpha\|_1^{\mathrm{norm}} \coloneqq \inf\bigl\{\|c\|_1 \bigm| c \in C_n(X;\mathbb{R}) \text{ is a normalised cycle representing } \alpha\bigr\}.$$

Connes and Consani prove that the two semi-norms are equivalent [CC20, Lemma 3.4], namely, for every $\alpha \in H_n(X;\mathbb{R})$, we have

$$\|\alpha\|_1 \leq \|\alpha\|_1^{\mathrm{norm}} \leq \max(1, 2^{n-1}) \cdot \|\alpha\|_1.$$

Proposition 1.2 states that they are in fact equal.

2.3. **Deriving Theorem 1.1 from Proposition 1.2.** Connes and Consani introduce a filtration of the s-module $H\mathbb{R}$ by a family $(\|H\mathbb{R}\|_\lambda)_{\lambda\in\mathbb{R}_{>0}}$ of sub-s-modules and, for topological spaces $X$, associated singular homology objects $(H_n(X;\|H\mathbb{R}\|_\lambda)_{\lambda\in\mathbb{R}_{>0}}$ [CC20]. Moreover, these come with canonical maps

$$\varrho_{n,\lambda} \colon H_n(X;\|H\mathbb{R}\|_\lambda) \to H_n(X;\mathbb{R})$$

to the singular homology of $X$ [CC20, Section 3.4]. This filtration defines a semi-norm on $H_n(X;\mathbb{R})$ that is equivalent to $\|\cdot\|_1$ [CC20, Corollary 3.6]. More precisely, the image of $\varrho_{n,\lambda}$ coincides with the set of elements $\alpha \in H_n(X;\mathbb{R})$ with $\|\alpha\|_1^{\mathrm{norm}} < \lambda$ [CC20, Theorem 3.5]. Therefore, Theorem 1.1 is a direct consequence of Proposition 1.2.

## 3. Proof of Proposition 1.2

3.1. **Symmetrisation of chains.** We recall the *symmetrisation map* on singular chains, which is given by averaging singular simplices over all vertex-permutations of the standard simplex: In the following, let $X$ be a topological space and $n \in \mathbb{N}$. Let $\Sigma_{n+1}$ denote the symmetric group on $\{0,\dots,n\}$ and $\operatorname{sgn}\colon \Sigma_{n+1} \to \{\pm 1\}$ the sign function. For a map $\pi\colon \{0,\dots,k\} \to \{0,\dots,n\}$, we write $\Delta(\pi) \coloneqq \bigl[\pi(0),\dots,\pi(k)\bigr] \colon \Delta^k \to \Delta^n$ for the affine map that extends the map $\pi$ on the vertices.

**Definition 3.1** (symmetrisation map)**.** The *symmetrisation map*

$$\operatorname{symm}_n \colon C_n(X;\mathbb{R}) \to C_n(X;\mathbb{R})$$

is the $\mathbb{R}$-linear map defined on each singular $n$-simplex $\sigma$ as

$$\operatorname{symm}_n(\sigma) := \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \operatorname{sgn}(\pi) \cdot \sigma \circ \Delta(\pi).$$

**Lemma 3.2** ([FM11, Lemma 2.6])**.** *The symmetrisation map* $\operatorname{symm}_\bullet$ *is a chain map* $C_\bullet(X;\mathbb{R}) \to C_\bullet(X;\mathbb{R})$ *that is chain homotopic to the identity. Moreover, for all* $c \in C_n(X;\mathbb{R})$*, we have*

$$\bigl\|\operatorname{symm}_n(c)\bigr\|_1 \leq \|c\|_1.$$

For us, the key observation is that symmetrisation enforces normalisation on cycles:

**Lemma 3.3** (normalisation via symmetrisation)**.**

*(1) For all* $j \in \{0,\dots,n\}$*, we have*

$$\partial_j \circ \operatorname{symm}_n = (-1)^j \cdot \partial_0 \circ \operatorname{symm}_n .$$

*(2) In particular: If* $c \in C_n(X;\mathbb{R})$ *is a cycle, then* $\partial_j(\operatorname{symm}_n(c)) = 0$ *for all* $j \in \{0,\dots,n\}$*.*

*Proof. Ad 1.* Using the cyclic permutation $\tau_j := (j\ j-1\ \dots\ 1\ 0) \in \Sigma_{n+1}$, we can re-write $\partial_j \circ \operatorname{symm}_n$ as follows: Each permutation $\pi \in \Sigma_{n+1}$ satisfies

$$\begin{aligned}\Delta(\pi) \circ \iota_j^n &= \bigl[\pi(0),\dots,\pi(j-1),\pi(j+1),\dots,\pi(n)\bigr]\\ &= \bigl[\pi \circ \tau_j(1),\dots,\pi \circ \tau_j(j),\pi \circ \tau_j(j+1),\dots,\pi \circ \tau_j(n)\bigr]\\ &= \Delta(\pi \circ \tau_j) \circ \iota_0^n.\end{aligned}$$

Therefore, for all singular $n$-simplices $\sigma$ on $X$ we have

$$\begin{aligned}\partial_j \circ \operatorname{symm}_n(\sigma) &= \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \operatorname{sgn}(\pi) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n\\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \operatorname{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi) \circ \iota_j^n\\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\pi \in \Sigma_{n+1}} \operatorname{sgn}(\pi \circ \tau_j) \cdot \sigma \circ \Delta(\pi \circ \tau_j) \circ \iota_0^n\\ &= (-1)^j \cdot \frac{1}{(n+1)!} \sum_{\eta \in \Sigma_{n+1}} \operatorname{sgn}(\eta) \cdot \sigma \circ \Delta(\eta) \circ \iota_0^n\\ &= (-1)^j \cdot \partial_0 \circ \operatorname{symm}_n(\sigma).\end{aligned}$$

*Ad 2.* As $\operatorname{symm}_\bullet$ is a chain map (Lemma 3.2), if $c \in C_n(X;\mathbb{R})$ is a cycle, then $\operatorname{symm}_n(c)$ is a cycle, and in combination with the first part we see that

$$\begin{aligned}0 = \partial\bigl(\operatorname{symm}_n(c)\bigr) &= \sum_{j=0}^{n} (-1)^j \cdot \partial_j\bigl(\operatorname{symm}_n(c)\bigr) = \sum_{j=0}^{n} (-1)^{2j} \cdot \partial_0\bigl(\operatorname{symm}_n(c)\bigr)\\ &= (n+1) \cdot \partial_0\bigl(\operatorname{symm}_n(c)\bigr).\end{aligned}$$

Therefore, $\partial_0(\operatorname{symm}_n(c)) = 0$. Applying the first part once more shows that $\partial_j(\operatorname{symm}_n(c)) = 0$ for all $j \in \{0,\dots,n\}$. ☐

3.2. **Proof of Proposition 1.2.** We already know that $\|\alpha\|_1 \leq \|\alpha\|_1^{\mathrm{norm}}$ for every $\alpha \in H_n(X;\mathbb{R})$. Let us prove the opposite inequality. Let $c \in C_n(X;\mathbb{R})$ be a cycle representing $\alpha \in H_n(X;\mathbb{R})$. Then, we can consider $\mathrm{symm}_n(c) \in C_n(X;\mathbb{R})$. By Lemma 3.2 we know that $\mathrm{symm}_n(c)$ is homologous to $c$ and satisfies $\|\mathrm{symm}_n(c)\|_1 \leq \|c\|_1$. Moreover, Lemma 3.3 implies that $\mathrm{symm}_n(c)$ is normalised. This shows that

$$\|\alpha\|_1^{\mathrm{norm}} \leq \|\mathrm{symm}_n(c)\|_1 \leq \|c\|_1.$$

Taking the infimum over all cycles representing $\alpha$ completes the proof.

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