# WHICH FINITELY GENERATED ABELIAN GROUPS ADMIT EQUAL GROWTH FUNCTIONS?

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ABSTRACT. We show that finitely generated Abelian groups admit equal growth functions with respect to symmetric generating sets if and only if they have the same rank and the torsion parts have the same parity. In contrast, finitely generated Abelian groups admit equal growth functions with respect to monoid generating sets if and only if they have same rank. Moreover, we show that the size of the torsion part is in fact determined by the set of *all* growth functions of a finitely generated Abelian group.

## 1. INTRODUCTION

Cayley graphs of finitely generated Abelian groups are rather rigid: Isomorphisms between Cayley graphs of finitely generated Abelian groups are almost affine [2, Theorem 1.3] and hence finitely generated Abelian groups admit isomorphic Cayley graphs if and only if they have the same rank and the torsion parts have the same size [2, Corollary 1.4]. Moreover, it is known that the rank coincides with the growth rate [1, Chapter VI] and also that the parity of the torsion part is encoded in the growth function of any Cayley graph [4, 2]. Thus it is a natural question whether also the exact size of the torsion part can be read off the growth functions of finitely generated Abelian groups [2, Problem 4.1].

In the following, we show that finitely generated Abelian groups (of non-zero rank) admit equal growth functions with respect to *symmetric* generating sets if and only if they have the same rank and the torsion parts have the same parity (Theorem 1.3). In contrast, finitely generated Abelian groups (of non-zero rank) admit equal growth functions with respect to *monoid* generating sets if and only if they have same rank (Theorem 1.4). However, we show that the size of the torsion part is in fact determined by the set of *all* growth functions of a finitely generated Abelian group (Corollary 1.8).

We now describe the results in more detail. For the sake of completeness, let us briefly recall some basic notation: A subset $A \subset G$ of a group $G$ is called *symmetric* if for all $g \in A$ also $g^{-1} \in A$.

**Definition 1.1** (Word metric, (spherical) growth function)**.** Let $G$ be a finitely generated group and let $S$ be a (not necessarily symmetric) finite monoid generating set of $G$.

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– The *word metric on $G$ with respect to $S$* is defined by

$$\begin{aligned} d_S \colon G \times G &\longrightarrow \mathbb{R}_{\geq 0} \\ (g,h) &\longmapsto \min\bigl\{n \in \mathbb{N} \bigm| \exists_{s_1,\dots,s_n \in S} \quad h^{-1} \cdot g = s_1 \cdot \dots \cdot s_n\bigr\}. \end{aligned}$$

(Notice that the "metric" $d_S$ in general will *not* be symmetric if $S$ is not symmetric.)

– For $r \in \mathbb{N}$ we write $B_{G,S}(r) := \{g \in G \mid d_S(g,e) \leq r\}$ for the *ball of radius $r$* around the neutral element $e$ in $G$ with respect to the word metric $d_S$.

– The *spherical growth function of $G$ with respect to $S$* is given by

$$\begin{aligned} \sigma_{G,S} \colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \bigl|\{g \in G \mid d_S(g,e) = r\}\bigr| = \bigl|B_{G,S}(r) \setminus B_{G,S}(r-1)\bigr|. \end{aligned}$$

– The *growth function of $G$ with respect to $S$* is given by

$$\begin{aligned} \beta_{G,S} \colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \bigl|B_{G,S}(r)\bigr| = \sum_{t=0}^{r} \sigma_{G,S}(t). \end{aligned}$$

Via the Švarc-Milnor lemma, growth functions of groups are related to volume growth functions of Riemannian manifolds [1, Theorem IV.23, Proposition VI.36]. Furthermore, growth functions of groups contain valuable large-scale geometric information that plays an important role in geometric group theory [1, Chapters VI–VIII].

**Definition 1.2.** Two finitely generated groups $G$ and $G'$ *admit equal growth functions* if there exist finite monoid generating sets $S \subset G$ and $S' \subset G'$ of $G$ and $G'$ respectively such that the corresponding growth functions coincide, i.e., such that $\beta_{G,S} = \beta_{G',S'}$. Analogously, $G$ and $G'$ *admit equal growth functions with respect to symmetric generating sets* if there exist finite symmetric generating sets $S \subset G$ and $S' \subset G'$ of $G$ and $G'$ respectively with $\beta_{G,S} = \beta_{G',S'}$.

If $G$ is a finitely generated Abelian group, then the *torsion subgroup* $\operatorname{tors} G$ of $G$, i.e., the subgroup of all elements of $G$ of finite order, is a finite group. Moreover, the quotient $G/\operatorname{tors} G$ is a finitely generated free Abelian group and the rank of $G/\operatorname{tors} G$ is called the *rank* $\operatorname{rk} G$ *of* $G$. In this situation, one has $G \cong G/\operatorname{tors} G \times \operatorname{tors} G \cong \mathbb{Z}^{\operatorname{rk} G} \times \operatorname{tors} G$.

**Theorem 1.3** (The case of symmetric generating systems)**.** *Two finitely generated Abelian groups of non-zero rank admit equal growth functions with respect to symmetric generating sets if and only if they have the same rank and the torsion parts have the same parity.*

**Theorem 1.4** (The case of monoid generating systems)**.** *Two finitely generated Abelian groups of non-zero rank admit equal growth functions (with respect to monoid generating sets) if and only if they have the same rank.*

Moreover, we observe that the size of the torsion part is a lower bound for the ratio between any growth function and the standard growth functions of the corresponding free part:

**Definition 1.5** (Standard growth functions)**.** Let $d \in \mathbb{N}$, let $E_d \subset \mathbb{Z}^d$ be the standard basis, and let $v_d := (-1, \dots, -1) \in \mathbb{Z}^d$. Then $E_d \cup (-E_d)$ is a finite symmetric generating set of $\mathbb{Z}^d$ and $E_d \cup \{v_d\}$ is a finite monoid generating set of $\mathbb{Z}^d$. We write

$$\beta_d := \beta_{\mathbb{Z}^d, E_d \cup (-E_d)} \quad \text{and} \quad \beta_d^+ := \beta_{\mathbb{Z}^d, E_d \cup \{v_d\}}.$$

**Proposition 1.6** (Minimal growth)**.** *Let $G$ be a finitely generated Abelian group.*

(1) *If $S$ is a finite symmetric generating set of $G$, then*

$$\limsup_{r \to \infty} \frac{\beta_{G,S}(r)}{\beta_{\operatorname{rk} G}(r)} \geq \bigl|\operatorname{tors}(G)\bigr|.$$

(2) *If $S$ is a finite monoid generating set of $G$, then*

$$\limsup_{r \to \infty} \frac{\beta_{G,S}(r)}{\beta^+_{\operatorname{rk} G}(r)} \geq \bigl|\operatorname{tors}(G)\bigr|.$$

As a consequence we obtain that only finitely many isomorphism types of finitely generated Abelian groups can share a single growth function and that the size of the torsion part can be recovered from the set of *all* growth functions of a finitely generated Abelian group:

**Corollary 1.7** (Finite ambiguity)**.** *Let $\beta \colon \mathbb{N} \longrightarrow \mathbb{N}$ be a function. Then there are at most finitely many isomorphism types of finitely generated Abelian groups $G$ that have a finite monoid generating set $S$ with $\beta_{G,S} = \beta$.*

**Corollary 1.8** (Recognising the size of the torsion part from the set of growth functions)**.** *Let $G$ and $G'$ be finitely generated Abelian groups.*

(1) *If the sets of all growth functions of $G$ and $G'$ with respect to symmetric generating sets coincide, i.e.,*

$$\begin{aligned} &\{\beta_{G,S} \mid S \text{ is a finite symmetric generating set of } G\} \\ &= \{\beta_{G',S'} \mid S' \text{ is a finite symmetric generating set of } G'\}, \end{aligned}$$

*then* $\operatorname{rk} G = \operatorname{rk} G'$ *and* $|\operatorname{tors} G| = |\operatorname{tors} G'|$.

(2) *If the sets of all growth functions of $G$ and $G'$ coincide, i.e.,*

$$\begin{aligned} &\{\beta_{G,S} \mid S \text{ is a finite monoid generating set of } G\} \\ &= \{\beta_{G',S'} \mid S' \text{ is a finite monoid generating set of } G'\}, \end{aligned}$$

*then* $\operatorname{rk} G = \operatorname{rk} G'$ *and* $|\operatorname{tors} G| = |\operatorname{tors} G'|$.

However, the converse of Corollary 1.8 does *not* hold (Example 5.6).

In fact, we will prove the above results for the slightly larger class of groups of type $\mathbb{Z}^d \times F$, where $d \in \mathbb{N}$ and $F$ is a finite group.

This paper is organised as follows: In Section 2, we recall some basics about growth functions. In Section 3, we deduce that the conditions given in Theorem 1.3 and 1.4 are necessary; conversely, in Section 4, we present examples that show that these conditions are also sufficient. In Section 5, we discuss minimal growth of finitely generated groups and prove Proposition 1.6, as well as its consequences Corollary 1.7 and Corollary 1.8.

## 2. Preliminaries on growth functions

For the sake of completeness, we collect some basic facts about growth functions of groups [1, Chapter VI–VIII], in particular, of groups that are products of finitely generated free Abelian groups and a finite group.

**Proposition 2.1** (Changing the generating set)**.** *Let $G$ be a finitely generated group, and let $S, T \subset G$ be finite monoid generating sets of $G$. Then there exists $C \in \mathbb{N}_{>0}$ such that for all $r \in \mathbb{N}$ we have*

$$\beta_{G,T}(r) \leq \beta_{G,S}(C \cdot r) \quad \textit{and} \quad \beta_{G,S}(r) \leq \beta_{G,T}(C \cdot r).$$

*Proof.* Because $S$ and $T$ are finite, the sets $\{d_S(t,e) \mid t \in T\}$ and $\{d_T(s,e) \mid s \in S\}$ are finite and so have finite upper bounds. Rewriting minimal length representations in one generating system in terms of the other one shows that there is a constant $C \in \mathbb{N}_{>0}$ such that for all $g \in G$ we have

$$d_S(g,e) \leq C \cdot d_T(g,e) \quad \text{and} \quad d_T(g,e) \leq C \cdot d_S(g,e),$$

from which the claim follows. □

**Proposition 2.2** (Polynomial growth rate and rank)**.** *Let $d \in \mathbb{N}_{>0}$, let $F$ be a finite group, and let $G \cong \mathbb{Z}^d \times F$. Let $S \subset G$ be a finite monoid generating set of $G$. Then for all $r \in \mathbb{N}$ we have*

$$\frac{1}{C} \cdot r^d \leq \beta_{G,S}(r) \leq C \cdot r^d.$$

*Consequently,*

$$\limsup_{r \to \infty} \frac{\beta_{G,S}(r-R)}{\beta_{G,S}(r)} = 1.$$

*Proof.* The finite set $T := \{g \in \mathbb{Z}^d \mid |g|_\infty = 1\}$ generates the additive monoid $\mathbb{Z}^d$ and $d_T$ coincides with the $\infty$-metric. Now a simple counting argument shows the first part for $\beta_{\mathbb{Z}^d,T}$. For the finite monoid generating set $T \cup F$ of $G$ we clearly have

$$\beta_{\mathbb{Z}^d,T} \leq \beta_{G,T\cup F} \leq \beta_{\mathbb{Z}^d,T} \cdot |F|,$$

so the first part holds also for $\beta_{G,T\cup F}$. Proposition 2.1 translates this into corresponding estimates for the monoid generating set $S$ of $G$.

It follows from the first part in particular that the limes superior in the second part indeed exists. Because $\beta_{G,S}$ is monotonically increasing, the limes superior is at most 1; if the limes superior were stricly less than 1, then $\beta_{G,S}$ would be growing exponentially, contradicting the first part. □

**Proposition 2.3** (Growth in product groups)**.** *Let $G_1$ and $G_2$ be finitely generated groups, and let $S_1 \subset G_1$ and $S_2 \subset G_2$ be finite monoid generating sets of $G_1$ and $G_2$ respectively. Then $S := (S_1 \times \{e\}) \cup (\{e\} \cup S_2)$ is a finite monoid generating set of $G := G_1 \times G_2$ and for all $r \in \mathbb{N}$ we have*

$$\sigma_{G,S}(r) = \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \sigma_{G_1,S_1}(r_1) \cdot \sigma_{G_2,S_2}(r_2).$$

*Proof.* By definition of the word metric, for all $(g_1, g_2) \in G$ we have

$$d_S\big((g_1,g_2),e\big) = d_{S_1}(g_1,e) + d_{S_2}(g_2,e),$$

which readily implies the stated decomposition of $\sigma_{G,s}$. □

Moreover, we will use the following version of an observation by Hainke and Scheele [4, 2]:

**Proposition 2.4** (Growth and elements of order 2). *Let $G$ be a finitely generated group and let $I \subset G$ be the set of elements of order at most 2. If $I$ is finite and $S \subset G$ is a finite symmetric generating set of $G$, then for all $r \in \mathbb{N}$ with $r > \operatorname{diam}_{d_S} I$ we have*

$$\beta_{G,S}(r) \equiv |I| \mod 2.$$

*Proof.* The inversion map $i\colon G \longrightarrow G$ satisfies $i(B_{G,S}(r)) \subset B_{G,S}(r)$ (and hence $i(B_{G,S}(r)) = B_{G,S}(r)$) for all $r \in \mathbb{N}$ because $S$ is symmetric. Because $i \circ i = \mathrm{id}_G$ and because the fixed points of $i$ are precisely the elements of order at most 2 the claim follows. □

## 3. Necessary conditions

The conditions given in Theorem 1.3 and Theorem 1.4 are necessary:

**Proposition 3.1** (Necessary conditions). *Let $d, d' \in \mathbb{N}$, let $F, F'$ be finite groups, and let $G \cong \mathbb{Z}^d \times F$, $G' \cong \mathbb{Z}^{d'} \times F'$. Moreover, let $S \subset G$ and $S' \subset G'$ be finite monoid generating sets with $\beta_{G,S} = \beta_{G',S'}$.*

(1) *Then $d = d'$.*
(2) *If $S$ and $S'$ are symmetric, then $|F| \equiv |F'| \mod 2$.*

*Proof.* It suffices to show that the rank and (in the symmetric case) the parity of the torsion part are encoded suitably in any growth function. The first part immediately follows from (the first part of) Proposition 2.2.

The second part follows from Proposition 2.4: On the one hand, applying Proposition 2.4 to $F$ shows that the parity of $|F|$ equals the parity of number of elements of order at most 2 in $F$ (and thus in $\mathbb{Z}^d \times F$). On the other hand, applying Proposition 2.4 to $\mathbb{Z}^d \times F$ shows that this number (and hence the parity of $|F|$) is determined by the long-time behaviour of $\beta_{G,S}$. □

## 4. Sufficient conditions

In view of the previous section, in order to prove Theorems 1.3 and 1.4 it remains to give examples of finite generating sets in the groups in question that witness that the corresponding groups admit equal growth functions.

### 4.1. The symmetric case.

In the symmetric case, the following two examples will be at the heart of our arguments:

*Example* 4.1. Let $F$ be a finite group, let $G := \mathbb{Z} \times F$, let $k \in \mathbb{N}_{>0}$, and let

$$S := \bigl(\{0\} \times F\bigr) \cup \bigl(\{-k, \dots, k\} \times \{e\}\bigr) \subset G.$$

Clearly, $S$ is a symmetric generating set of $G$, and Proposition 2.3 shows that

$$\begin{aligned} \sigma_{G,S}\colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \begin{cases} 1 & \text{if } r = 0 \\ |F| - 1 + 2 \cdot k & \text{if } r = 1 \\ (|F| - 1) \cdot 2 \cdot k + 2 \cdot k = |F| \cdot 2 \cdot k & \text{if } r > 1 \end{cases} \end{aligned}$$

(see also Figure 1). Notice that these terms are symmetric in $|F|$ and $2 \cdot k$.

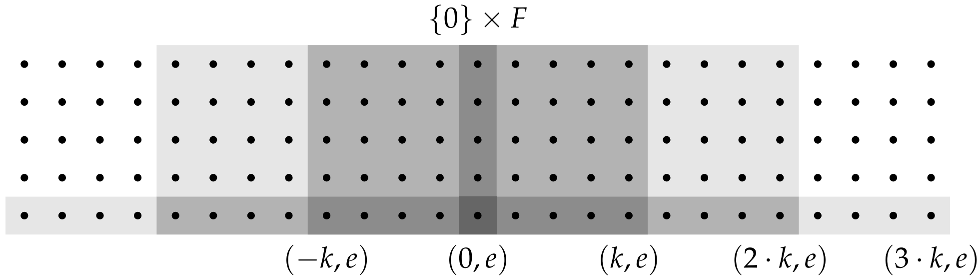


FIGURE 1. Small balls in Example 4.1

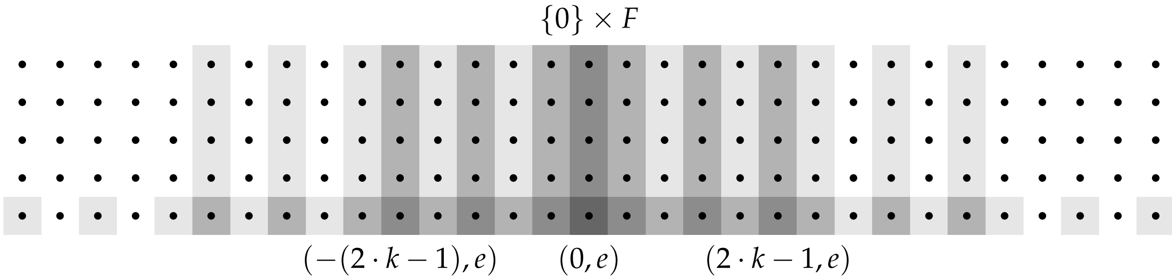


FIGURE 2. Small balls in Example 4.2

*Example* 4.2. Let $F$ be a finite group, let $G := \mathbb{Z} \times F$, let $k \in \mathbb{N}_{>0}$, and let

$$S := \bigl(\{0\} \times F\bigr) \cup \bigl(\{2 \cdot j + 1 \mid j \in \{-k, \dots, k-1\}\} \times \{e\}\bigr) \subset G.$$

Clearly, $S$ is a symmetric generating set of $G$, and Proposition 2.3 shows that

$$\begin{aligned} \sigma_{G,S} \colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \begin{cases} 1 & \text{if } r = 0 \\ |F| - 1 + 2 \cdot k & \text{if } r = 1 \\ (|F| - 1) \cdot 2 \cdot k + 2 \cdot (k-1) + 2 \cdot k & \text{if } r = 2 \\ (|F| - 1) \cdot 2 \cdot (k-1) + (|F| - 1) \cdot 2 \cdot k + 2 \cdot (k-1) + 2 \cdot k & \text{if } r > 2. \end{cases} \\ &= \begin{cases} 1 & \text{if } r = 0 \\ |F| + 2 \cdot k - 1 & \text{if } r = 1 \\ |F| \cdot (2 \cdot k - 1) + |F| + 2 \cdot k - 1 - 1 & \text{if } r = 2 \\ 2 \cdot |F| \cdot (2 \cdot k - 1) & \text{if } r > 2 \end{cases} \end{aligned}$$

(see also Figure 2). Notice that these terms are symmetric in $|F|$ and $2 \cdot k - 1$.

**Proposition 4.3** (Witnesses in the symmetric case). *Let $d \in \mathbb{N}_{>0}$, and let $F_1$ and $F_2$ be finite groups. If $|F_1|$ and $|F_2|$ have the same parity, then there exist finite symmetric generating sets $S_1$ and $S_2$ of $G_1 := \mathbb{Z}^d \times F_1$ and $G_2 := \mathbb{Z}^d \times F_2$ respectively satisfying $\sigma_{G_1,S_1} = \sigma_{G_2,S_2}$ and thus also*

$$\beta_{G_1,S_1} = \beta_{G_2,S_2}.$$

*Proof.* It suffices to consider the case $d = 1$: If $d > 1$, we can just extend symmetric generating sets for $\mathbb{Z} \times F_1$ and $\mathbb{Z} \times F_2$ that witness that $\mathbb{Z} \times F_1$ and $\mathbb{Z} \times F_2$ admit equal growth functions by a finite symmetric generating set of $\mathbb{Z}^{d-1}$ and apply Proposition 2.3 to produce finite symmetric generating sets for $G_1$ and $G_2$ that witness that $G_1 = \mathbb{Z}^{d-1} \times \mathbb{Z} \times F_1$ and $G_2 = \mathbb{Z}^{d-1} \times \mathbb{Z} \times F_2$ admit equal growth functions.

We begin with the *even case*: So, let $|F_1|$ and $|F_2|$ be even, say $|F_1| = 2 \cdot k_1$ and $|F_2| = 2 \cdot k_2$ for certain $k_1, k_2 \in \mathbb{N}_{>0}$. Then

$$S_1 := \big(\{0\} \times F_1\big) \cup \big(\{-k_2, \dots, k_2\} \times \{e\}\big) \subset \mathbb{Z} \times F_1,$$
$$S_2 := \big(\{0\} \times F_2\big) \cup \big(\{-k_1, \dots, k_1\} \times \{e\}\big) \subset \mathbb{Z} \times F_2$$

are finite symmetric generating sets of $\mathbb{Z} \times F_1$ and $\mathbb{Z} \times F_2$ respectively, and Example 4.1 shows that $\sigma_{\mathbb{Z} \times F_1, S_1} = \sigma_{\mathbb{Z} \times F_2, S_2}$, as desired.

It remains to deal with the *odd case*: So, let $|F_1|$ and $|F_2|$ be odd, say $|F_1| = 2 \cdot k_1 - 1$ and $|F_2| = 2 \cdot k_2 - 1$ for certain $k_1, k_2 \in \mathbb{N}_{>0}$. Then

$$S_1 := \big(\{0\} \times F_1\big) \cup \big(\{2 \cdot j + 1 \mid j \in \{-k_2, \dots, k_2 - 1\}\} \times \{e\}\big) \subset \mathbb{Z} \times F_1$$
$$S_2 := \big(\{0\} \times F_2\big) \cup \big(\{2 \cdot j + 1 \mid j \in \{-k_1, \dots, k_1 - 1\}\} \times \{e\}\big) \subset \mathbb{Z} \times F_2$$

are finite symmetric generating sets of $\mathbb{Z} \times F_1$ and $\mathbb{Z} \times F_2$ respectively, and Example 4.2 shows that $\sigma_{\mathbb{Z} \times F_1, S_1} = \sigma_{\mathbb{Z} \times F_2, S_2}$, as desired. $\square$

This finishes the proof of Theorem 1.3.

*Remark* 4.4. The construction in Example 4.1 and 4.2 does not produce witnesses for other non-trivial cases: The system

$$x_1 + y_1 = x_2 + y_2, \tag{1}$$
$$x_1 \cdot y_1 = x_2 \cdot y_2 \tag{2}$$

(corresponding to the constraints for radius 1 and larger radii, respectively) with $x_1, x_2, y_1, y_2 \in \mathbb{N}$ has only the two solutions where $x_1 = x_2, y_1 = y_2$ or $x_1 = y_1, x_2 = y_2$.

This can be easily seen as follows: Solving Equation (1) for $x_2$ and using Equation (2) yields

$$\begin{aligned}(x_1 - y_2) \cdot (y_1 - y_2) &= x_1 \cdot y_1 - y_2 \cdot (y_1 + x_1 - y_2) \\ &= x_1 \cdot y_1 - y_2 \cdot x_2 \\ &= 0,\end{aligned}$$

which implies $x_1 = y_2$ (and hence $x_2 = y_2$) or $y_1 = y_2$ (and hence $x_1 = x_2$).

4.2. **The monoid case.** Similarly to the symmetric case, we start with the corresponding key example:

*Example* 4.5 ([3]). Let $F$ be a finite group, let $G := \mathbb{Z} \times F$, let $k \in \mathbb{N}_{>0}$, and let

$$S := \big\{(-1, e)\big\} \cup \big(\{0, \dots, k-1\} \times F\big).$$

Then $S$ is a finite monoid generating set of $G$ (which except for trivial cases is not symmetric), and a straightforward induction over the radius of balls shows that

$$B_{G,S}(r) = \big\{(-r, e)\big\} \cup \big(\{-(r-1), \dots, r \cdot (k-1)\} \times F\big)$$

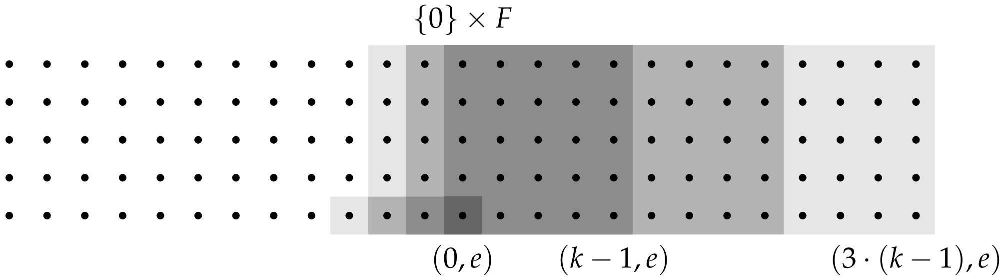


FIGURE 3. Small balls in Example 4.5

for all $r \in \mathbb{N}_{>0}$. Hence, we obtain (see also Figure 3)

$$\begin{aligned} \sigma_{G,S} \colon \mathbb{N} &\longrightarrow \mathbb{N} \\ r &\longmapsto \begin{cases} 1 & \text{if } r = 0 \\ |F| \cdot (k-1) + |F| - 1 + 1 = |F| \cdot k & \text{if } r > 0. \end{cases} \end{aligned}$$

**Proposition 4.6** (Witnesses in the monoid case [3]). *Let $m \in \mathbb{N}_{>0}$ and let $F_1, \dots, F_m$ be finite groups. Moreover, let $d \in \mathbb{N}_{>0}$ and let $G_j := \mathbb{Z}^d \times F_j$ for all $j \in \{1, \dots, m\}$. Then for every $j \in \{1, \dots, m\}$ there exists a finite monoid generating set $S_j \subset G_j$ of $G_j$ such that $\sigma_{G_1,S_1} = \dots = \sigma_{G_m,S_m}$ and thus also*

$$\beta_{G_1,S_1} = \dots = \beta_{G_m,S_m}.$$

*Proof.* As in the symmetric case, in view of Proposition 2.3 it suffices to consider the case $d = 1$. So, let $d = 1$ and let $K \in \mathbb{N}_{>0}$ be some common multiple of $|F_1|, \dots, |F_m|$. For $j \in \{1, \dots, m\}$ we then consider the finite monoid generating set

$$S_j := \{(-1, e)\} \cup \bigl(\{0, \dots, K/|F_j| - 1\} \times F_j\bigr)$$

of $G_j$. By Example 4.5 we have $\sigma_{G_j,S_j}(0) = 1$ and

$$\sigma_{G_j,S_j}(r) = |F_j| \cdot K/|F_j| = K$$

for all $r \in \mathbb{N}_{>0}$, which is independent of $j$. Hence, $\sigma_{G_1,S_1} = \dots = \sigma_{G_m,S_m}$, as desired. □

This completes the proof of Theorem 1.4.

## 5. MINIMAL GROWTH OF FINITELY GENERATED ABELIAN GROUPS

In the following, we prove Proposition 1.6 and its consequences from the introduction.

5.1. **Minimal growth.** We start with the proof of the following version of Proposition 1.6:

**Proposition 5.1** (Minimal growth). *Let $G \cong \mathbb{Z}^d \times F$, where $d \in \mathbb{N}$ and $F$ is a finite group.*

(1) *If $S$ is a finite symmetric generating set of $G$, then*

$$\limsup_{r \to \infty} \frac{\beta_{G,S}(r)}{\beta_d(r)} \geq |F|.$$

(2) *If $S$ is a finite monoid generating set of $G$, then*

$$\limsup_{r\to\infty} \frac{\beta_{G,S}(r)}{\beta_d^+(r)} \geq |F|.$$

We first reduce to the free Abelian case (Lemma 5.2), and then compare growth functions in the free Abelian case with the standard growth functions (Lemma 5.3).

**Lemma 5.2** (Reduction to the free Abelian case)**.** *Let $G \cong \mathbb{Z}^d \times F$, where $d \in \mathbb{N}$ and $F$ is a finite group, let $S \subset G$ be a finite monoid generating set of $G$, let $\pi\colon G \cong \mathbb{Z}^d \times F \longrightarrow \mathbb{Z}^d$ be the canonical projection, and let $R := \operatorname{diam}_{d_S} F$. Then*

$$\beta_{G,S}(r) \geq |F| \cdot \beta_{\mathbb{Z}^d,\pi(S)}(r-R)$$

*for all $r \in \mathbb{N}_{\geq R}$.*

*Proof.* Because $\pi$ is surjective, $\pi(S)$ is indeed a generating set of $\mathbb{Z}^d$; moreover, $F$ is finite, and so $R$ is well-defined. Let $r \in \mathbb{N}_{\geq R}$. Then, by definition of the word metric,

$$\pi\bigl(B_{G,S}(r-R)\bigr) = B_{\pi(G),\pi(S)}(r-R)$$

and, by definition of $R$, we have

$$B_{G,S}(r) \supset B_{G,S}(r-R) + F.$$

Hence, we obtain

$$\begin{aligned} \beta_{G,S}(r) &\geq \bigl|B_{G,S}(r-R) + F\bigr| = \bigl|\pi^{-1}\bigl(\pi(B_{G,S}(r-R)\bigr)\bigr| \\ &= |F| \cdot \bigl|\pi\bigl(B_{G,S}(r-R)\bigr)\bigr| = |F| \cdot \bigl|B_{\pi(G),\pi(S)}(r-R)\bigr| \\ &= |F| \cdot \beta_{\mathbb{Z}^d,\pi(S)}(r-R). \end{aligned}$$

□

**Lemma 5.3** (Minimal growth of free Abelian groups)**.** *Let $d \in \mathbb{N}$.*

(1) *Let $S \subset \mathbb{Z}^d$ be a finite symmetric generating set. Then*

$$\beta_{\mathbb{Z}^d,S} \geq \beta_d.$$

(2) *Let $S \subset \mathbb{Z}^d$ be a finite monoid generating set. Then*

$$\beta_{\mathbb{Z}^d,S} \geq \beta_d^+.$$

*Proof.* Let $S \subset \mathbb{Z}^d$ be a finite monoid generating set. Looking at $\mathbb{Z}^d \otimes \mathbb{Q}$ shows that $S$ contains a $d$-element subset $E$ that is linearly independent over $\mathbb{Q}$. In particular, the submonoid $N$ of $\mathbb{Z}^d$ generated by $E$ is isomorphic to $\mathbb{N}^d$, and the subgroup $Z$ of $\mathbb{Z}^d$ generated by $E$ is isomorphic to $\mathbb{Z}^d$; in both cases, $E$ is a free generating set of the corresponding submonoid or subgroup, respectively.

We now prove the first part of the lemma: If $S$ is symmetric, then also $-E \subset S$, and we obtain

$$\beta_{\mathbb{Z}^d,S} \geq \beta_{Z,E\cup(-E)} = \beta_{\mathbb{Z}^d,E_d\cup(-E_d)} = \beta_d.$$

For the second part, the combinatorics is slightly more complicated because not every finite monoid generating set of $\mathbb{Z}^d$ contains a generating set corresponding to $E_d \cup \{v_d\}$.

In order to prove the second part, it suffices to construct an injective map $\varphi\colon \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ that maps $E_d \cup \{v_d\}$-balls into $S$-balls of the same radius. We will now give the construction of such a map:

We choose an order $(e'_1, \dots, e'_d)$ on $E$ and denote by $\pi_1, \dots, \pi_d \colon \mathbb{Q}^d \longrightarrow \mathbb{Q}$ the coordinate maps corresponding to the (ordered) basis $E$ of $\mathbb{Z}^d \otimes \mathbb{Q}$. Because $S$ is a monoid generating set of $\mathbb{Z}^d$, for each $j \in \{1, \dots, d\}$ there exists $v'_j \in S$ with $\pi_j(v'_j) < 0$; we choose $v'_j$ in such a way that it minimises $\pi_j$ on $S$. We denote the set of minimal $E_d$-coordinates of an element $x \in \mathbb{Z}^d$ by

$$M(x) := \bigl\{ j \in \{1, \dots, d\} \bigm| x_j = \min_{k \in \{1,\dots,d\}} x_k \bigr\},$$

and the set of minimal rescaled $E$-coordinates of an element $x \in \mathbb{Q}^d$ by

$$M'(x) := \Bigl\{ j \in \{1, \dots, d\} \Bigm| \frac{\pi_j(x)}{|\pi_j(v'_j)|} = \min_{k \in \{1,\dots,d\}} \frac{\pi_k(x)}{|\pi_k(v'_k)|} \Bigr\};$$

moreover, in this situation, we write

$$m(x) := \min M(x) \quad \text{and} \quad m'(x) := \min M'(x),$$

respectively.

We now define the map $\varphi\colon \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ as follows: Let $x \in \mathbb{Z}^d$. Then one easily sees that $x$ has a unique minimal representation

$$x = \sum_{j=1}^d x'_j \cdot e_j + x' \cdot v_d$$

with $x'_1, \dots, x'_d, x' \in \mathbb{N}$ with respect to the word metric $d_{E_d \cup \{v_d\}}$. We set

$$\varphi(x) := \sum_{j=1}^d x'_j \cdot e'_j + x' \cdot v'_{m(x)}.$$

By construction, we have $\varphi(B_{\mathbb{Z}^d, E_d \cup \{v_d\}}(r)) \subset B_{\mathbb{Z}^d,S}(r)$ for all $r \in \mathbb{N}$. It remains to show that $\varphi$ is injective: Clearly, $\varphi|_{\mathbb{N}^d}$ is injective, and, by construction, $\pi_j(\varphi(x)) \geq 0$ for all $j \in \{1, \dots, d\}$ if and only if $x \in \mathbb{N}^d$.

In case $x \in \mathbb{Z}^d \setminus \mathbb{N}^d$ (which is equivalent to $x' > 0$) we have

$$m'\bigl(\varphi(x)\bigr) = m(x).$$

Hence, we can reconstruct $x'$ from the value $\varphi(x)$ as the minimal natural number $a$ such that all $E$-coordinates of $\varphi(x) - a \cdot v'_{m(x)}$ are non-negative; because $E$ is free, we can then also read off $x'_1, \dots, x'_d$ from $\varphi(x)$. Thus, $x$ is determined uniquely by $\varphi(x)$, and so $\varphi$ is injective. This finishes the proof of Lemma 5.3. □

We can now combine these two steps to complete the proof of Proposition 5.1:

*Proof of Proposition 5.1.* Let $R := \operatorname{diam}_{d_S} \operatorname{tors} G$. We begin with the *symmetric case*: In view of Lemma 5.2 and 5.3, we have $\beta_{G,S}(r) \geq |F| \cdot \beta_d(r - R)$ for all $r \in \mathbb{N}_{\geq R}$. Therefore,

$$\limsup_{r \to \infty} \frac{\beta_{G,S}(r)}{\beta_d(r)} \geq |F| \cdot \limsup_{r \to \infty} \frac{\beta_d(r-R)}{\beta_d(r)}.$$

The limes superior on the right hand side is equal to 1 by Proposition 2.2, which gives the desired estimate.

Using the corresponding cases for monoid generating sets allows to prove the *monoid case* by the same arguments. □

5.2. **Consequences of minimal growth: Finite ambiguity.** We now prove the finiteness statement Corollary 1.7:

**Corollary 5.4** (Finite ambiguity). *Let $\beta\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a function. Then there is at most one $d \in \mathbb{N}$ and at most finitely many isomorphism types of finite groups $F$ such that $\mathbb{Z}^d \times F$ has a finite monoid generating set $S$ with $\beta_{\mathbb{Z}^d \times F,S} = \beta$.*

*Proof.* Suppose $d \in \mathbb{N}$ and $F$ is a finite group such that there exists a finite monoid generating set $S$ of $G := \mathbb{Z}^d \times F$ with $\beta_{G,S} = \beta$. Then $d$ is determined by the growth rate of $\beta$ (Proposition 2.2), the limes superior

$$\limsup_{r\to\infty} \frac{\beta(r)}{\beta_d^+(r)} = \limsup_{r\to\infty} \frac{\beta_{G,S}(r)}{\beta_d^+(r)}$$

is finite (Proposition 2.1 and 2.2), and by Proposition 5.1 we have

$$|F| \le \limsup_{r\to\infty} \frac{\beta_{G,S}(r)}{\beta_d^+(r)} = \limsup_{r\to\infty} \frac{\beta(r)}{\beta_d^+(r)} < \infty.$$

Hence, $|F|$ is bounded in terms of $\beta$. As there are only finitely many isomorphism types of groups of a given finite order there are only finitely many different candidates of isomorphism types for $F$. □

5.3. **Consequences of minimal growth: Recognising the size of torsion from growth sets.** In this section, we show that the set of all growth functions encodes the size of the torsion part (Corollary 1.8):

**Corollary 5.5** (Recognising the size of the torsion part from the set of growth functions). *Let $d, d' \in \mathbb{N}$, let $F$ and $F'$ be finite groups, and let $G \cong \mathbb{Z}^d \times F$ as well as $G' \cong \mathbb{Z}^{d'} \times F'$.*

(1) *If*

$$\begin{aligned}&\{\beta_{G,S} \mid S \text{ is a finite symmetric generating set of } G\}\\ = &\{\beta_{G',S'} \mid S' \text{ is a finite symmetric generating set of } G'\},\end{aligned}$$

*then $d = d'$ and $|F| = |F'|$.*

(2) *If*

$$\begin{aligned}&\{\beta_{G,S} \mid S \text{ is a finite monoid generating set of } G\}\\ = &\{\beta_{G',S'} \mid S' \text{ is a finite monoid generating set of } G'\},\end{aligned}$$

*then $d = d'$ and $|F| = |F'|$.*

*Proof.* In view of Proposition 2.2 it suffices to show that the sizes of the torsion parts must be equal if the sets of growth functions coincide.

We begin with the *symmetric case*: We consider the finite symmetric generating set (see Definition 1.5 for the definition of $E_d$)

$$S := E_d \cup (-E_d) \cup F$$

of $G$ (viewing $\mathbb{Z}^{\operatorname{rk} G}$ and $F$ as subsets of $G$). For all $r \in \mathbb{N}$ we have

$$\beta_{G,S}(r) = \left|B_{G,S}(r)\right| \leq \left|B_{\mathbb{Z}^d, E_d \cup (-E_d)}(r) + F\right| = \beta_d(r) \cdot |F|,$$

and hence

$$\limsup_{r\to\infty} \frac{\beta_{G,S}(r)}{\beta_d(r)} \leq \limsup_{r\to\infty} \frac{\beta_d(r)\cdot|F|}{\beta_d(r)} = |F|.$$

Because we assumed that the growth sets of $G$ and $G'$ coincide, $\beta_{G,S}$ is also a growth function of $G'$ with respect to some finite symmetric generating set of $G'$ and $d = d'$. In combination with Proposition 5.1 we therefore obtain

$$|F'| \leq \limsup_{r\to\infty} \frac{\beta_{G,S}(r)}{\beta_d(r)} \leq |F|.$$

Hence, by symmetry of the setup, $|F'| = |F|$, as claimed.

In the *monoid case* the same argument applies with respect to the monoid generating set $E_d \cup \{v_d\} \cup F$ of $\mathbb{Z}^d \times F \cong G$. □

However, the converse does not hold in general:

*Example* 5.6. Let $d \in \mathbb{N}$ and let $F$ be an Abelian finite group that cannot be generated by $d+2$ elements, e.g., $F = (\mathbb{Z}/2)^{d+3}$. We then consider the groups

$$G := \mathbb{Z}^d \times \mathbb{Z}/|F| \quad \text{and} \quad G' := \mathbb{Z}^d \times F.$$

Clearly, $S := (E_d \times \{0\}) \cup \{(0,1),(0,-1)\}$ is a finite symmetric generating set of $G$. So, $\beta_{G,S}(1) = \left|S \cup \{0,e\}\right| = d+3$. On the other hand, if $S' \subset G'$ is a finite monoid generating set of $G'$, then the canonical projection of $S'$ to $F$ must generate $F$ as a group; hence, $|S'| > d+2$, and so

$$\beta_{G',S'}(1) = \left|S' \cup \{0,e\}\right| > d+3 = \beta_{G,S}(1).$$

In particular, $\beta_{G,S}$ cannot be realised as a growth function of $G'$. However, by construction, $\operatorname{rk} G = d = \operatorname{rk} G'$ and $|\operatorname{tors} G| = \left|\mathbb{Z}/|F|\right| = |F| = |\operatorname{tors} G'|$.

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