Simplicial volume of one-relator groups and stable commutator length

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Abstract

A one-relator group is a group G_r that admits a presentation $\langle S \mid r\rangle$ with a single relation r. One-relator groups form a rich classically studied class of groups in Geometric Group Theory. If $r \in F(S)'$, we introduce a *simplicial volume* $\|G_r\|$ *for one-relator groups*. We relate this invariant to the *stable commutator length* $\operatorname{scl}_S(r)$ of the element $r \in F(S)$. We show that often (though not always) the linear relationship $\|G_r\| = 4\cdot \operatorname{scl}_S(r) - 2$ holds and that every rational number modulo 1 is the simplicial volume of a one-relator group.

Moreover, we show that this relationship holds approximately for proper powers and for elements satisfying the small cancellation condition $C'(1/N)$, with a multiplicative error of $O(1/N)$. This allows us to prove for random elements of $F(S)'$ of length n that $\|G_r\|$ is $2\log(2|S|-1)/3 \cdot n/\log(n) + o(n/\log(n))$ with high probability, using an analogous result of Calegari–Walker for stable commutator length.

1 Introduction

A *one-relator group* is a group G_r that admits a presentation $\langle S \mid r\rangle$ with a single relation $r \in F(S)$. This rich and well studied class of groups in Geometric Group Theory generalises surface groups and shares many properties with them.

A common theme is to relate the geometric properties of a classifying space of G_r to the algebraic properties of the relator $r \in F(S)$. For example, $r \in F(S)' \setminus \{e\}$ if and only if $H_2(G_r;\mathbb{Z}) \ncong 0$. In this case $H_2(G_r;\mathbb{Z})$ is infinite cyclic and generated by the *fundamental class* $\alpha_r \in H_2(G_r;\mathbb{Z})$. We define the *simplicial volume* of G_r as $\|G_r\| := \|\alpha_{r,\mathbb{R}}\|_1$, the l^1-semi-norm of the fundamental class α_r (Section 3.1).

For every element $w \in F(S)'$, the *commutator length* $\operatorname{cl}_S(w)$ *of* w *in* $F(S)$ is defined via

$$\operatorname{cl}_S(w) := \min\bigl\{n \in \mathbb{N} \bigm| \exists_{g_1,\dots,g_n,h_1,\dots,h_n \in F(S)} \quad w = [g_1,h_1]\cdots[g_n,h_n]\bigr\}$$

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and the *stable commutator length* is the limit

$$\mathrm{scl}_S(w) := \lim_{n\to\infty} \frac{\mathrm{cl}_S(w^n)}{n}.$$

The study of stable commutator length has seen much progress in recent years by Calegari and others [Cal09a, Cal11, CF10]. Calegari showed that in a non-abelian free group, stable commutator length is always rational and computable in polynomial time with respect to the word length [Cal09b]. Moreover, it is known that there is a *gap* of $1/2$ in the stable commutator length, i.e., if $w \in F(S)' \setminus \{e\}$, then $\mathrm{scl}_S(w) \geq 1/2$ [DH91].

The theme of this paper is to connect the (topological) invariant $\|G_r\|$ to the (algebraic) invariant $\mathrm{scl}_S(r)$. The motivating example is the following:

Key Example (surface groups)**.** Let $g \in \mathbb{N}_{>0}$, set $S_g := \{a_1, \dots, a_g, b_1, \dots, b_g\}$ and let $r_g := [a_1, b_1] \cdot \dots \cdot [a_g, b_g]$ in $F(S_g)$. Then $\|\langle S_g \mid r_g\rangle\| = 4g - 4$ [Gro82, p. 9] and $\mathrm{scl}_{S_g}(r_g) = g - \frac{1}{2}$ [Cal09a, Theorem 2.93, Theorem 2.101]. Therefore in this case, stable commutator length and simplicial volume are related by the formula

$$\big\|\langle S_g \mid r_g\rangle\big\| = 4 \cdot \Big(\mathrm{scl}_{S_g}(r_g) - \frac{1}{2}\Big).$$

We see that the relationship $\|G_r\| = 4 \cdot \mathrm{scl}_S(r) - 2$ holds in many instances, though *not* always:

Example 1.1 (Example 6.14)**.** The element $v = \mathtt{aaaabABAbaBAAbAB} \in F(\{\mathtt{a}, \mathtt{b}\})$, where $\mathtt{A} = \mathtt{a}^{-1}$ and $\mathtt{B} = \mathtt{b}^{-1}$, satisfies that $\mathrm{scl}_{\{\mathtt{a},\mathtt{b}\}}(v) = 5/8$, but $\|G_v\| = 0$.

Thus we ask:

Question 1.2. Let S be a set and let $r \in F(S)'$ be non-trivial. When is it true that

$$\|G_r\| = 4 \cdot \Big(\mathrm{scl}_S(r) - \frac{1}{2}\Big) \ ?$$

Observe that the right-hand side is always non-negative because of the $1/2$-gap of stable commutator length in free groups [DH91].

As seen in Example 1.1 there are elements $r \in F(S)'$ where $\|G_r\| \geq 4 \cdot \mathrm{scl}_S(r) - 2$ fails to hold. We do not know if the other inequality always holds. We are only able to obtain a weaker strict inequality $\|G_r\| < 4 \cdot \mathrm{scl}_S(r)$ (Corollary 3.12).

In this article, we find a positive answer to Question 1.2 in various instances. There are several ways to compute stable commutator length. In order to prove the results of this article we will make these interpretations available also for the simplicial volume of one-relator groups. These will be:

- *topologically*, in terms of surfaces (Proposition 3.11),
- *algebraically*, in terms of commutator lengths (Corollary 3.13),
- *dually*, in terms of quasimorphisms (Proposition 3.15).
- *combinatorially/algorithmically*, in terms of van Kampen diagrams on surfaces (Proposition 4.2),

Decomposable relators

Theorem A (decomposable relators; Section 3.3). *The answer to Question 1.2 is positive in the following cases:*

1. $S = S_1 \cup S_2$ *with* $S_1 \cap S_2 = \emptyset$, *and* $r = r_1 r_2$, *where* $r_1 \in F(S_1)', r_2 \in F(S_2)'$ *are non-trivial;*
2. $S = S' \cup t$ *and* $r = r_1 t r_2 t^{-1}$ *with* $t \notin S$, $r_1, r_2 \in F(S')' \setminus \{e\}$.

In previous work, we combined similar calculations over more general base groups with known values of stable commutator length, to manufacture closed 4-manifolds with arbitrary rational simplicial volume [HL20a] or with arbitrarily small transcendental simplicial volume [HL20c].

Using results of Calegari [Cal11], Theorem A implies:

Corollary B. For every rational number $0 \leq q < 1$ there is one-relator group G_r with $\|G_r\| \equiv q \mod 1$.

We do not know if there are one-relator groups with irrational simplicial volume.

Hyperbolic one-relator groups

It is well-known that one-relator groups are hyperbolic if the relator r is a proper power r'^N or if the relator satisfies a small cancellation condition. We obtain an affirmative answer to Question 1.2 in those cases, up to multiplicative constants of size $O(N^{-1})$.

Theorem C (small cancellation elements, Theorem 4.7). *Let* $r \in F(S)'$ *be an element that satisfies the small cancellation condition* $C'(1/N)$ *for some* $N \geq 6$. *Then*

$$4 \cdot \mathrm{scl}_S(r) > \|G_r\| \geq \left(1 - \frac{6}{N}\right) \cdot 4 \cdot \mathrm{scl}_S(r).$$

Theorem D (proper powers, Theorem 4.9). *If* $r = r'^N$ *for some* $r' \in F(S)' \setminus \{e\}$ *and* $N > 6$, *then*

$$4 \cdot \mathrm{scl}_S(r) > \|G_r\| \geq \left(1 - \frac{6}{N}\right) \cdot 4 \cdot \mathrm{scl}_S(r).$$

In particular, we have that

$$\lim_{N \to \infty} \frac{\|G_{r^N}\|}{N} = 4 \cdot \mathrm{scl}_S(r).$$

Using Theorem C and a result by Calegari–Walker [CW13] we are able to compute the distribution of the simplicial volume of random one-relator groups:

Theorem E (Theorem 5.1). *Fix a set* S *and let* $r \in F(S)$ *be a random reduced element of even length* n, *conditioned to lie in the commutator subgroup* $F(S)'$. *Then for every* $\epsilon > 0$ *and* $C > 1$,

$$\left| \|G_r\| \cdot \frac{\log(n)}{n} - \frac{2 \cdot \log(2|S| - 1)}{3} \right| \leq \epsilon$$

with probability $1 - O(n^{-C})$.

Simplicial volume via linear programming

Calegari showed that $\mathrm{scl}_S(r)$ may be computed in polynomial time in $|r|$ by reducing it to a linear programming problem [Cal09b]. This revealed that $\mathrm{scl}_S(r)$ is in particular rational. The corresponding algorithm (`scallop`) has been implemented and is open-source available [WC12]. We are not able to reduce the computation of $\|G_r\|$ to a similar programming problem. However, we introduce a new invariant $\mathtt{lallop}(r)$ (Definition 6.3) to bound $\|G_r\|$ from below. We show that `lallop` can be computed by reducing it to a linear programming problem, we implemented this algorithm [HL19, HL20b], and used this lower bound effectively.

Theorem F (`lallop`; Theorem 6.1)**.** *Let S be a set and $r \in F(S)' \setminus \{e\}$. Then*

$$\mathtt{lallop}(r) \leq \|G_r\|,$$
$$\mathtt{lallop}(r) \leq 4 \cdot \Big(\mathrm{scl}_S(r) - \frac{1}{2}\Big),$$

and there is an algorithm to compute $\mathtt{lallop}(r)$ that is polynomial in the word length of r over S. Moreover, $\mathtt{lallop}(r) \in \mathbb{Q}$.

In this way we may estimate $\|G_r\|$ explicitly, which sometimes allows us to compute $\|G_r\|$ also for non-decomposable relators.

Example G (Proposition 6.13)**.** Let $m \in \mathbb{N}_{\geq 2}$ and $r_m = [\mathtt{a}, \mathtt{b}][\mathtt{a}, \mathtt{b}^{-m}] \in F(\{\mathtt{a}, \mathtt{b}\})$. Then

$$\|G_{r_m}\| \leq \frac{2m-4}{m-1} = 4 \cdot \Big(\mathrm{scl}_{\{\mathtt{a},\mathtt{b}\}}(r_m) - \frac{1}{2}\Big).$$

For $m \in \{2, 3, 4\}$ we compute that $\mathtt{lallop}(r_2) = 0$, $\mathtt{lallop}(r_3) = 1$ and $\mathtt{lallop}(r_4) = \frac{4}{3}$. Thus

$$\forall_{m \in \{2,3,4\}} \quad \|G_{r_m}\| = \frac{2m-4}{m-1} = 4 \cdot \Big(\mathrm{scl}_{\{\mathtt{a},\mathtt{b}\}}(r_m) - \frac{1}{2}\Big).$$

Follow-up questions

Combining Question 1.2 with known properties of stable commutator length and simplicial volume raises these follow-up questions:

Question 1.3.

1. Let S, S' be sets and let $r \in F(S)' \setminus \{e\}$, $r' \in F(S')' \setminus \{e\}$ be relators with $\langle S \mid r \rangle \cong \langle S' \mid r' \rangle$. Does this imply that $\mathrm{scl}_S\, r = \mathrm{scl}_{S'}\, r'$?
2. Is the simplicial volume of one-relator groups computable?
3. Is there a gap $C > 0$ such that for every set S and every relator $r \in F(S)' \setminus \{e\}$ either $\|G_r\| = 0$ or $\|G_r\| \geq C$?
4. Louder and Wilton [LW18a] showed that much of the geometry of one-relator groups with defining relation r may be controlled by the *primitivity rank*, denoted by $\pi(r)$. From their computations it is apparent that if $\pi(r) > 2$ then $\mathrm{scl}_S(r) > 1/2$. Is there a similar connection to the simplicial volume?

Organisation of this article

We first recall simplicial volume of manifolds as well as stable commutator length (Section 2). We then introduce simplicial volume of one-relator groups (Section 3) and establish some basic properties (Theorem A). In Section 4, we describe simplicial volume of one-relator groups in terms of van Kampen diagrams, leading to a proof of Proposition C and Theorem D. The analysis of simplicial volume of random one-relator groups is carried out in Section 5. In Section 6, we introduce the computational invariant `lallop` and prove Theorem F; moreover, we include a sample computation (Example G).

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2 Preliminaries

We summarise notation and basic properties of simplicial volume and stable commutator length.

2.1 Simplicial volume

We quickly recall the notion of simplicial volume of manifolds, which is based on the l^1-semi-norm on singular homology. Let X be a topological space and let $d \in \mathbb{N}$. Then the l^1*-semi-norm* on $H_d(X;\mathbb{R})$ is defined as

$$\begin{aligned} \|\cdot\|_1 \colon H_d(X;\mathbb{R}) &\to \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf\bigl\{|c|_1 \bigm| c \in C_d(X;\mathbb{R}),\ \partial c = 0,\ [c] = \alpha\bigr\}; \end{aligned}$$

here, $C_d(X;\mathbb{R})$ is the singular chain module of X in degree d with $\mathbb{R}$-coefficients and $|\cdot|_1$ denotes the l^1-norm on $C_d(X;\mathbb{R})$ associated with the basis of singular simplices.

Definition 2.1 (simplicial volume [Gro82]). Let M be an oriented closed connected d-dimensional manifold. Then the *simplicial volume of M* is

$$\|M\| := \bigl\|[M]_{\mathbb{R}}\bigr\|_1,$$

where $[M]_{\mathbb{R}} \in H_d(M;\mathbb{R})$ is the $\mathbb{R}$-fundamental class of M.

On the one hand, simplicial volume is a homotopy invariant of (oriented) compact manifolds that is compatible with mapping degrees: If $f \colon M_1 \to M_2$ is a continuous map between oriented closed connected manifolds of the same dimension, then

$$|\deg f| \cdot \|M_2\| \leq \|M_1\|.$$

On the other hand, simplicial volume is related in a non-trivial way to Riemannian volume, e.g., in the presence of enough negative curvature [Gro82, IY82, Thu79, LS06, CW18, Löh11]. A very different source of manifolds with non-zero simplicial volumes are our constructions via stable commutator length [HL20a].

Dually, we can describe the l^1-semi-norm (and whence simplicial volume) in terms of bounded cohomology $H_b^*(\,\cdot\,;\mathbb{R})$:

Proposition 2.2 (duality principle for the l^1-semi-norm [Gro82, p. 6/7][Fri17, Lemma 6.1])**.** *Let X be a topological space, let $d \in \mathbb{N}$, and let $\alpha \in H_d(X;\mathbb{R})$. Then*

$$\|\alpha\|_1 = \sup\bigl\{\langle \beta, \alpha\rangle \bigm| \beta \in H_b^d(X,\mathbb{R}), \|\beta\|_\infty \leq 1\bigr\}.$$

Corollary 2.3 (duality principle for simplicial volume [Gro82, p. 7])**.** *Let M be an oriented closed connected d-manifold. Then*

$$\|M\| = \frac{1}{\|\varphi\|_\infty},$$

where $\varphi \in H^d(M;\mathbb{R})$ is the singular cohomology class satisfying $\langle \varphi, [M]_\mathbb{R}\rangle = 1$.

2.2 Stable commutator length

In this section we give a very brief introduction to stable commutator length. The main reference is Calegari's book [Cal09a]. For a group G let G' be its commutator subgroup and let $g \in G'$. We define the *commutator length* $\mathrm{cl}_G(g)$ of an element $g \in G'$ via

$$\mathrm{cl}_G(g) := \min\bigl\{n \in \mathbb{N} \bigm| \exists_{x_1,\dots,x_n,y_1,\dots,y_n \in G} \quad g = [x_1,y_1]\cdots[x_n,y_n]\bigr\}.$$

It is easy to see that commutator length is invariant under automorphisms, in particular conjugations.

It will be convenient to extend the notion of commutator length to "sums" of group elements. If $m \in \mathbb{N}$ and $g_1,\dots,g_m \in G$ with $g_1\cdots g_m \in G'$, then one writes

$$\mathrm{cl}_G(g_1 + \cdots + g_m) := \min_{t_1,\dots,t_m \in G} \mathrm{cl}_G(t_1 g_1 t_1^{-1} \cdots t_m g_m t_m^{-1}).$$

It is not hard to see that, as the notation suggests, the value $\mathrm{cl}_G(g_1 + \cdots + g_m)$ is independent of the order of $g_1,\dots,g_m$.

Definition 2.4 (stable commutator length)**.** Let G be a group, let $m \in \mathbb{N}$, and let $g_1,\dots,g_m \in G$ with $g_1\cdots g_m \in G'$. The *stable commutator length* of the tuple $(g_1,\dots,g_m)$ is defined via

$$\mathrm{scl}_G(g_1 + \cdots + g_m) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g_1^n + \cdots + g_m^n)}{n}.$$

This limit indeed exists and *stable* commutator length has the following additive behaviour [Cal09a, Chapter 2.6]: For all $n \in \mathbb{N}_{>0}$ and all $g \in G'$, we have

$$\mathrm{scl}_G(n \cdot g) \quad = \quad \mathrm{scl}_G(g^n);$$

For all $g \in G$, $m \in \mathbb{N}$, and all $g_1,\dots,g_m \in G$ with $g_1\cdots g_m \in G'$, we have

$$\mathrm{scl}_G\biggl(g + g^{-1} + \sum_{i=1}^m g_i\biggr) \quad = \quad \mathrm{scl}_G\biggl(\sum_{i=1}^m g_i\biggr).$$

2.2.1 (Stable) Commutator length in free groups via surfaces

Commutator length and stable commutator length have a geometric interpretation. For what follows, we will restrict our attention to (stable) commutator length of the free group $F(S)$ with generating set S, even though every result in this section holds for general groups.

Let $m \in \mathbb{N}$ and let $g_1, \dots, g_m \in F(S)$ be elements such that $g_1 \cdots g_m \in F(S)'$. Let B_S be a bouquet of $|S|$ circles labelled by the elements of S; we identify $F(S)$ with $\pi_1(B_S)$ in the canonical way. Moreover, let $\gamma_1, \dots, \gamma_m \colon S^1 \to X$ be based loops in B_S such that $[\gamma_i]_* = g_i$ in $F(S)$.

Definition 2.5 (cl- and scl-admissible maps)**.** Let Σ be an orientable surface with boundary $\partial\Sigma$, with genus at least 1 and with the inclusion map $\iota \colon \partial\Sigma \to \Sigma$. Moreover, let $f \colon \Sigma \to B_S$ be a map from Σ to B_S and let $\partial f \colon \partial\Sigma \to \coprod_{i=1}^m S^1$ be the restriction of f to the boundary such that the diagram

$$\begin{tikzcd} \partial\Sigma \arrow[r, "\iota"] \arrow[d, "\partial f"] & \Sigma \arrow[d, "f"] \\ \coprod_{i=1}^m S^1 \arrow[r, "\gamma_1,\dots,\gamma_m"] & X \end{tikzcd}$$

commutes. We say that the pair (f, Σ) is

- cl-*admissible to* $g_1 + \cdots + g_m$, if ∂f is a degree 1 map on all components and
- scl-*admissible to* $g_1 + \cdots + g_m$, if there is an integer $n(\Sigma, f) \in \mathbb{Z}$, called the *degree of* (Σ, f), such that $H_1(\partial f; \mathbb{Z})[\partial\Sigma] = n(\Sigma, f) \cdot [\coprod_{i=1}^m S^1]$ in $H_1(\coprod_{i=1}^m S^1; \mathbb{Z})$.

The "set" of all cl- and scl-admissible pairs (f, Σ) to the formal sum $g_1 + \cdots + g_m$ will be denoted by $\Sigma_\partial^{\mathrm{cl}}(g_1 + \cdots + g_m)$ and $\Sigma_\partial(g_1 + \cdots + g_m)$, respectively (strictly speaking, this set is a class, but we could fix models for each homeomorphism type of surfaces to turn this into an actual set).

Proposition 2.6 ((stable) commutator length via surfaces [Cal09a, Proposition 2.74])**.** *Let S be a set, let $m \in \mathbb{N}$, and let $g_1, \dots, g_m \in F(S)$ with $g_1 \cdots g_m \in F(S)'$. Then*

$$\begin{aligned} \mathrm{cl}_{F(S)}(g_1 + \cdots + g_m) &= \min_{(f,\Sigma) \in \Sigma_\partial^{\mathrm{cl}}(g_1 + \cdots + g_m)} \mathrm{genus}(\Sigma), \textit{ and} \\ \mathrm{scl}_{F(S)}(g_1 + \cdots + g_m) &= \inf_{(f,\Sigma) \in \Sigma_\partial(g_1 + \cdots + g_m)} \frac{-\chi^-(\Sigma)}{2 \cdot n(f, \Sigma)}. \end{aligned}$$

Here, χ^- denotes the Euler characteristic that ignores spheres and disks. I.e., if $\Sigma = \bigsqcup_{i=1}^n \Sigma_i$ with connected components Σ_i, then we define

$$\chi^-(\Sigma) = \sum_{i=1}^n \min\bigl(0, \chi(\Sigma_i)\bigr).$$

To shorten notation we will frequently simply write cl_S and scl_S instead of $\mathrm{cl}_{F(S)}$ and $\mathrm{scl}_{F(S)}$.

2.2.2 Stable commutator length via quasimorphisms

Let G be a group. A map $\phi\colon G \to \mathbb{R}$ is called a *quasimorphism* if there is a constant $C > 0$ such that

$$\sup_{g,h \in G} |\phi(g) + \phi(h) - \phi(gh)| \leq C.$$

The smallest such bound C is called the *defect of* ϕ and is denoted by $D(\phi)$. If ϕ is a linear combination of a bounded function and a homomorphism, then ϕ is called a *trivial* quasimorphism. Quasimorphisms are intimately related to $H^2_b(G;\mathbb{R})$, the bounded cohomology of G in degree 2 with trivial real coefficients: The boundary of a quasimorphism $\delta^1\phi$ defines a non-trivial class in $H^2_b(G,\mathbb{R})$ if and only if ϕ is non-trivial. Moreover, all exact classes in $H^2_b(G,\mathbb{R})$ arise in this way [Cal09a, Theorem 2.50].

A quasimorphism $\phi\colon G \to \mathbb{R}$ is called *homogeneous*, if for all $g \in G$, $n \in \mathbb{Z}$ we have that $\phi(g^n) = n \cdot \phi(g)$. The set of all homogeneous quasimorphisms on G is denoted by $Q^h(G)$. Stable commutator length may be computed via quasimorphisms using *Bavard's duality theorem* proved by Bavard and generalised by Calegari:

Theorem 2.7 (Bavard duality [Bav91][Cal09a, Theorem 2.79])**.** *Let G be a group, let $m \in \mathbb{N}$, and let $g_1, \dots, g_m \in G$ such that $g_1 \cdots g_m \in G'$. Then*

$$\mathrm{scl}_G(g_1 + \dots + g_m) = \sup_{\phi \in Q^h(G)} \frac{\sum_{i=1}^m \phi(g_i)}{2 \cdot D(\phi)}.$$

3 Simplicial volume of one-relator groups

We introduce the simplicial volume of one-relator presentations and one-relator groups and establish basic properties as well as alternative descriptions (via surfaces, commutator length, and quasimorphisms).

3.1 Setup and notation

Setup 3.1. Let $F(S)$ be the free group on some alphabet S, let $r \in F(S)'$ be a non-trivial element in the commutator subgroup, and let $G_r := \langle S \mid r \rangle$ be the one-relator group defined by the presentation (S, r).

We write P_r for the presentation complex of G_r associated with the presentation (S, r) and X_r for a model of the classifying space of G_r obtained by attaching higher-dimensional cells to P_r. Let $c_r\colon P_r \to X_r$ be the inclusion map. Because r is in the commutator subgroup, the 2-cell of P_r defines a homology class $\widetilde{\alpha}_r \in H_2(P_r;\mathbb{Z})$.

Definition 3.2 (fundamental class, simplicial volume of a one-relator presentaion)**.** In the situation of Setup 3.1, we define:

- The *fundamental class of* (S, r):

$$\alpha_r := H_2(c_r;\mathbb{Z})(\widetilde{\alpha}_r) \in H_2(G_r;\mathbb{Z}).$$

- The $\mathbb{R}$-*fundamental class* $\alpha_{r,\mathbb{R}} \in H_2(G_r;\mathbb{R})$ *of* (S, r) as the image of α_r under the change of coefficients map $H_2(G_r;\mathbb{Z}) \to H_2(G_r;\mathbb{R})$.

- The *simplicial volume of* (S, r):

$$\|(S,r)\| := \|\alpha_{r,\mathbb{R}}\|_1 \in \mathbb{R}_{\geq 0}.$$

Here, $\|\cdot\|_1$ denotes the l^1-semi-norm on singular homology $H_*(\,\cdot\,;\mathbb{R})$.

Remark 3.3 (simplicial volume of one-relator groups)**.** In the situation of Setup 3.1, the Hopf formula [Bro94, Theorem II.5.3] shows that $H_2(G_r;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and that α_r is a generator of $H_2(G_r;\mathbb{Z})$. In particular: If (S', r') is another one-relator presentation of G_r with $r' \in F(S')'$, then $\alpha_{r'} \in \{\alpha_r, -\alpha_r\}$. Hence, the simplicial volume $\|(S,r)\| = \|(S',r')\|$ depends only on the group and not on the chosen presentation. Therefore, we also write

$$\|G_r\| := \|(S,r)\|$$

for the *simplicial volume of the one-relator group* G_r.

Because $c_r \colon P_r \to X_r$ is a π_1-isomorphism, the mapping theorem in bounded cohomology [Gro82, p. 40][Iva85][Fri17, Theorem 5.9] shows that

$$\|G_r\| = \|\alpha_{r,\mathbb{R}}\|_1 = \|\widetilde{\alpha}_{r,\mathbb{R}}\|_1.$$

If r is not a proper power, then G_r is torsion-free [KMS60] and the presentation complex P_r already is a model of the classifying space of G_r [Coc54].

Example 3.4 (hyperbolic groups and proper powers)**.** If, in the situation of Setup 3.1, G_r is hyperbolic, then because the class $\alpha_{r,\mathbb{R}}$ is non-zero, it follows from Mineyev's non-vanishing result for bounded cohomology of hyperbolic groups [Min01, Theorem 15] and the duality principle (Proposition 2.2) that

$$\|G_r\| = \|\alpha_{r,\mathbb{R}}\|_1 > 0.$$

For instance, whenever the relator r is a proper power, then G_r is a word-hyperbolic group. Newman's spelling theorem [New68] shows that Dehn's algorithm works in such groups.

Example 3.5 (amenable case)**.** In the situation of Setup 3.1, the group G_r is amenable if and only if $G_r \cong \mathbb{Z}^2$ [CSG97]. Clearly, in this case, $P_r \simeq S^1 \times S^1$ and $\|G_r\| = 0$.

3.2 Mapping degrees

The simplicial volume of one-relator groups has the following simple functoriality property with respect to group homomorphisms:

Definition 3.6 (degree)**.** Let S_1, S_2 be sets and let $r_1 \in F(S_1)' \setminus \{e\}$, $r_2 \in F(S_2)' \setminus \{e\}$. If $f \colon G_{r_1} = \langle S_1 \mid r_1 \rangle \to \langle S_2 \mid r_2 \rangle = G_{r_2}$ is a group homomorphism, then there is a unique integer $\deg f$, the *degree of* f, with

$$H_2(f;\mathbb{Z})(\alpha_{r_1}) = \deg f \cdot \alpha_{r_2} \in H_2(G_{r_2};\mathbb{Z}).$$

This notion of degree is a generalisation of the notion of degree for maps between manifolds or for l^1-admissible maps in the sense of Definition 3.10. Strictly speaking, the sign of the degree depends on the chosen one-relator presentation (and not only on the one-relator group), but this will not cause any trouble.

Proposition 3.7 (functoriality)**.** *Let S_1, S_2 be sets, let $r_1 \in F(S_1)' \setminus \{e\}$, $r_2 \in F(S_2)' \setminus \{e\}$, and let $f\colon G_{r_1} \to G_{r_2}$ be a group homomorphism. Then*

$$\left|\deg f\right| \cdot \|G_{r_2}\| \leq \|G_{r_1}\|.$$

Proof. We have $H_2(f;\mathbb{R})(\alpha_{r_1,\mathbb{R}}) = \deg f \cdot \alpha_{r_2,\mathbb{R}}$. Because $H_2(f;\mathbb{R})$ does not increase $\|\cdot\|_1$, the claim follows. □

Example 3.8. Let S be a set, let $r \in F(S)' \setminus \{e\}$, and let $N \in \mathbb{N}_{>0}$. Then the canonical homomorphism $\langle S \mid r^N \rangle \to \langle S \mid r \rangle$ has degree N, and we obtain

$$\|G_r\| \leq \frac{1}{N} \cdot \|G_{r^N}\|.$$

Moreover, we will see that the limit $\lim_{N\to\infty} 1/N \cdot \|G_{r^N}\|$ is equal to $\operatorname{scl}_S r$ (Theorem 4.9).

Example 3.9. Let $S \subset \widetilde{S}$ be sets, let $r \in F(S)' \setminus \{e\}$, and let $\widetilde{r} \in F(\widetilde{S})'$ be the corresponding element of $F(\widetilde{S})$. Then the two canonical group homomorphisms $\langle S \mid r \rangle \to \langle \widetilde{S} \mid \widetilde{r} \rangle$ (given by the inclusion of S into $\widetilde{S}$) and $\langle \widetilde{S} \mid \widetilde{r} \rangle \to \langle S \mid R \rangle$ (given by projecting $\widetilde{S} \setminus S$ to the neutral element) both have degree 1. Hence,

$$\|G_r\| = \|G_{\widetilde{r}}\|.$$

In particular, omitting the generating set S in the notation $\|G_r\|$ is no real loss of information.

One-relator groups that satisfy the property in Question 1.2 might inherit interesting properties for the stable commutator length of the relator from the mapping degree functoriality of simplicial volume (Proposition 3.7).

3.3 Decomposable relators

We will now compute the simplicial volume of one-relator groups with decomposable relators, using the computation of the l^1-semi-norm in degree 2 in these cases via the filling view and the calculation of stable commutator length of decomposable relators [HL20a, Section 6.3]. We only need to verify that our current situation fits into that context.

Proof of Theorem A. For the *first part*, we let $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$ and $r = r_1 r_2$ with $r_1 \in F(S_1)' \setminus \{e\}$, $r_2 \in F(S_2)' \setminus \{e\}$, and we note that

$$G_r = \langle S \mid r \rangle = (F(S_1) * F(S_2))/\langle r_1 \cdot r_2 \rangle^{\triangleleft} \cong F(S_1) *_{\mathbb{Z}} F(S_2),$$

where the amalgamation homomorphisms $\mathbb{Z} \to F(S_1)$ and $\mathbb{Z} \to F(S_2)$ are given by r_1 and r_2, respectively. In order to use the previous computations for decomposable relators [HL20a, Section 6.3], we consider the double mapping cylinder

$$P := Z_1 \cup_{(z,1)\sim(\overline{z},1)} Z_2$$

constructed by gluing the cylinders

$$Z_1 := \Bigl(\bigvee_{S_1} S^1\Bigr) \cup_{r_1 \text{ on } S^1 \times \{0\}} \bigl(S^1 \times [0,1]\bigr)$$

$$Z_2 := \Bigl(\bigvee_{S_2} S^1\Bigr) \cup_{r_2 \text{ on } S^1 \times \{0\}} \bigl(S^1 \times [0,1]\bigr)$$

Let $\widetilde{\alpha} \in H_2(P;\mathbb{Z})$ be the canonical class constructed by gluing generators of $H_2(Z_1, S^1 \times \{1\};\mathbb{Z}) \cong \mathbb{Z}$ and $H_2(Z_2, S^1 \times \{1\};\mathbb{Z}) \cong \mathbb{Z}$ and let $c\colon P \to BG_r$ be the classifying map. Then $H_2(c;\mathbb{Z})(\widetilde{\alpha})$ is a generator of $H_2(G_r;\mathbb{Z})$ and thus

$$H_2(c;\mathbb{Z})(\widetilde{\alpha}) = \pm\alpha_r \in H_2(G_r;\mathbb{Z}).$$

Therefore, the $\mathbb{R}$-version $\alpha_{\mathbb{R}} \in H_2(P;\mathbb{R})$ of $H_2(c;\mathbb{Z})(\widetilde{\alpha})$ satisfies

$$\begin{aligned} \|G_r\| &= \|\alpha_{r,\mathbb{R}}\|_1 = \|\alpha_{\mathbb{R}}\|_1 && \text{(Remark 3.3)} \\ &= 4 \cdot \Big(\operatorname{scl}_{S_1 \cup S_2}(r_1 \cdot r_2) - \frac{1}{2}\Big) && \text{[HL20a, Theorem 6.14]} \\ &= 4 \cdot \Big(\operatorname{scl}_S r - \frac{1}{2}\Big). \end{aligned}$$

For the *second part*, we can argue similarly: Let $S = S' \cup \{t\}$ and $r = r_1 t r_2 t^{-1}$ with $t \notin S'$ and $r_1, r_2 \in F(S') \setminus \{e\}$. The canonical class in the second homology of

$$\Big(\bigvee_S S^1\Big) \cup_{r_1,r_2} \big(S^1 \times [0,1] \sqcup S^1 \times [0,1]\big)$$

maps under the classifying map to the fundamental class $\pm\alpha_r$. Hence, we obtain from the filling view [HL20a, Theorem 6.14]

$$\|G_r\| = \|\alpha_{r,\mathbb{R}}\|_1 = 4 \cdot \Big(\operatorname{scl}_{S' \cup \{t\}}(r_1 \cdot t \cdot r_2 \cdot t^{-1}) - \frac{1}{2}\Big) = 4 \cdot \Big(\operatorname{scl}_{F(S)} r - \frac{1}{2}\Big),$$

as claimed. □

3.4 Simplicial volume via surfaces

Analogously to Proposition 2.6 we will compute $\|G_r\|$ using admissible surfaces.

Definition 3.10 (l^1-admissible map)**.** In the situation of Setup 3.1, an *l^1-admissible map for* (S, r) is a pair (f, Σ), consisting of an oriented *closed* connected surface Σ of genus at least 1 and a continuous map $f\colon \Sigma \to X_r$. The unique integer $n(f, \Sigma)$ satisfying

$$H_2(f;\mathbb{Z})[\Sigma]_{\mathbb{Z}} = n(f, \Sigma) \cdot \alpha_r \in H_2(G_r;\mathbb{Z})$$

is the *degree* of (f, Σ). We write $\Sigma(r)$ for the "set" of all l^1-admissible maps for r.

Proposition 3.11 (simplicial volume via surfaces)**.** *In the situation of Setup 3.1, we have*

$$\|G_r\| = \inf_{(f,\Sigma) \in \Sigma(r)} \frac{-2 \cdot \chi(\Sigma)}{\big|n(f, \Sigma)\big|}.$$

Proof. This is a special case of the fact that the l^1-semi-norm in degree 2 coincides with the surface semi-norm [BG88][CL15, Proposition 2.4]. □

In the following, we will mainly use this surface description of the simplicial volume. For example, Proposition 3.11 implies a weak upper bound for simplicial volume of one-relator groups and leads to a straightforward proof of a description of simplicial volume of one-relator groups in terms of commutator lengths:

Corollary 3.12 (weak upper bound)**.** *In the situation of Setup 3.1, we have*

$$\|G_r\| < 4 \cdot \mathrm{scl}_S\, r.$$

Proof. Let $(f, \Sigma) \in \Sigma_\partial(r)$ be an extremal scl-admissible surface for r; such a surface is known to exist [Cal09a, Theorem 4.24], satisfies

$$\mathrm{scl}_S\, r = \frac{-\chi(\Sigma)}{2 \cdot n(f, \Sigma)}$$

and has positive degree on every boundary. We then consider the oriented closed connected surface $\overline{\Sigma}$ obtained by gluing disks to the boundary components of Σ. This adds at most $n(f, \Sigma)$ many disks to the surface Σ, and thus $-\chi(\overline{\Sigma}) \geq -\chi(\Sigma) - n(f, \Sigma)$. Since $\mathrm{scl}_S(r) \geq \frac{1}{2}$ [DH91], we see that $-\chi(\overline{\Sigma}) \geq 0$, which shows that $\overline{\Sigma}$ has genus at least 1.

Then f extends to an l^1-admissible map $\overline{f} \colon \overline{\Sigma} \to X_r$, since the boundary loops of f are trivial in X_r. The degree of this map satisfies

$$n(\overline{f}, \overline{\Sigma}) = n(f, \Sigma).$$

By construction, $\chi(\overline{\Sigma}) > \chi(\Sigma)$, and from Proposition 3.11 we obtain

$$\|G_r\| \leq \frac{-2 \cdot \chi(\overline{\Sigma})}{\bigl|n(\overline{f}, \overline{\Sigma})\bigr|} < \frac{-2 \cdot \chi(\Sigma)}{\bigl|n(f, \Sigma)\bigr|} = 4 \cdot \mathrm{scl}_S\, r. \qquad \square$$

Corollary 3.13 (algebraic description of simplicial volume)**.** *In the situation of Setup 3.1, we have*

$$\|G_r\| = \inf_{(n,\epsilon) \in E} 4 \cdot \frac{\mathrm{cl}_S(r^{\epsilon_1} + \cdots + r^{\epsilon_n}) - 1}{|\epsilon_1 + \cdots + \epsilon_n|},$$

where $E := \bigl\{(n, \epsilon) \bigm| n \in \mathbb{N}_{>0},\ \epsilon \in \{-1, 1\}^n,\ \epsilon_1 + \cdots + \epsilon_n \neq 0\bigr\}$.

Proof. During this proof, we will abbreviate the right hand side of the claimed equality by $c(r)$. We will first show that $\|G_r\| \leq c(r)$: Let $n \in \mathbb{N}_{>0}$, let $t_1, \ldots, t_n \in F(S)$, let $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ with $\sum_{j=1}^n \epsilon_j \neq 0$, and let

$$N := \mathrm{cl}_S(t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}) \in \mathbb{N}.$$

It should be noted that $\epsilon_1 + \cdots + \epsilon_n \neq 0$ implies that $N > 0$ (because we work in the free group $F(S)$). Then there exist $a_1, \ldots, a_N, b_1, \ldots, b_N \in F(S)$ such that

$$t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r^{\epsilon_n} \cdot t_n^{-1} = [a_1, b_1] \cdot \cdots \cdot [a_N, b_N] \tag{1}$$

holds in $F(S)$. In particular, $[a_1, b_1] \cdot \cdots \cdot [a_N, b_N]$ lies in the normal subgroup of $F(S)$ generated by r and we obtain a corresponding, well-defined, group homomorphism

$$\varphi \colon \langle a_1, \ldots, a_N, b_1, \ldots, b_N \mid [a_1, b_1] \cdot \cdots \cdot [a_N, b_N] \rangle \to G_r$$

(given by mapping the generators to the corresponding elements in G_r). Passing to classifying spaces, we find a continuous map $f \colon \Sigma_N \to P_r$ with $\pi_1(f) = \varphi$;

more concretely, we can construct f as the cellular map that wraps the 2-cell of the standard CW-model of Σ_N around the 2-cell of X_r according to the relation in Equation (1). By construction, (f, Σ_N) is an l^1-admissible map for r with

$$n(f, \Sigma) = \epsilon_1 + \cdots + \epsilon_N.$$

Applying Proposition 3.11 shows that

$$\begin{aligned}\|G_r\| &\leq \frac{-2 \cdot \chi(\Sigma_N)}{|\epsilon_1 + \cdots + \epsilon_n|} = 4 \cdot \frac{N-1}{|\epsilon_1 + \cdots + \epsilon_n|} \\ &= 4 \cdot \frac{\mathrm{cl}_S(t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}) - 1}{|\epsilon_1 + \cdots + \epsilon_n|}.\end{aligned}$$

Taking the infimum over the right hand side shows that $\|G_r\| \leq c(r)$.

It remains to prove the converse inequality $\|G_r\| \geq c(r)$: Again, we use the description of $\|G_r\|$ in terms of l^1-admissible maps (Proposition 3.11). Let $(f, \Sigma) \in \Sigma(r)$ with $n(f, \Sigma) \neq 0$ and let N denote the genus of Σ. Without loss of generality we may assume that f is cellular. Following the map induced by f on the 1-skeleta, we lift $\pi_1(f)\colon \pi_1(\Sigma) \to G_r$ to a homomorphism $\psi\colon F(a_1, \ldots, a_N, b_1, \ldots, b_N) \to F(S)$. In particular, $\psi([a_1, b_1] \cdot \cdots \cdot [a_N, b_N])$ lies in the normal subgroup of $F(S)$ generated by r; hence, there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in F(S)$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ with

$$\mathrm{cl}_S(t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}) \leq \mathrm{cl}_S\bigl(\psi([a_1, b_1] \cdot \cdots \cdot [b_1, b_N])\bigr) \leq N.$$

This shows that

$$4 \cdot \bigl(\mathrm{cl}_S(t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}) - 1\bigr) \leq 4 \cdot (N-1) = -2 \cdot \chi(\Sigma).$$

Furthermore, the same arguments as above imply that $n(f, \Sigma) = \epsilon_1 + \cdots + \epsilon_n$; in particular, $\epsilon_1 + \cdots + \epsilon_n \neq 0$ and $n > 0$. Therefore, we obtain

$$c(r) \leq \frac{-2 \cdot \chi(\Sigma)}{\bigl|n(f, \Sigma)\bigr|}.$$

By Proposition 3.11, taking the infimum over all l^1-admissible maps shows that

$$c(r) \leq \|G_r\|,$$

as claimed. □

Proposition 3.14 (weak lower bound)**.** *In the situation of Setup 3.1, we have*

$$\inf_{n \in \mathbb{N}_{>0}} \frac{\mathrm{cl}_S(n \cdot r) - 1}{n} \geq \mathrm{scl}_S(r) - \frac{1}{2}.$$

Proof. Let $n \in \mathbb{N}_{>0}$, let $t_1, \ldots, t_n \in F(S)$, and let

$$N := \mathrm{cl}_S(t_1 \cdot r \cdot t_1^{-1} \cdot \cdots \cdot t_n \cdot r \cdot t_n^{-1});$$

then $N > 0$ and we can geometrically implement this by an scl-admissible map $(f, \Sigma) \in \Sigma_\partial(r)$ with

$$n(f, \Sigma) = n \quad \text{and} \quad \chi(\Sigma) = 2 - 2 \cdot N - n.$$

Using the description of scl in terms of surfaces (Proposition 2.6), we obtain

$$\mathrm{scl}_S\, r \leq \frac{-\chi(\Sigma)}{2 \cdot n(f,\Sigma)} = \frac{\mathrm{cl}_S(t_1 \cdot r \cdot t_1^{-1} \cdot \dots \cdot t_n \cdot r \cdot t_n^{-1}) - 1}{n} + \frac{1}{2}.$$

Taking the infimum over all $n \in \mathbb{N}_{>0}$ and all $t_1, \dots, t_n \in F(S)$ proves the claim. □

3.5 Simplicial volume via quasimorphisms

Stable commutator length in the free group can be computed using quasimorphisms via Bavard's duality theorem (Theorem 2.7). We obtain a similar result for the simplicial volume of one-relator groups:

Proposition 3.15 (simplicial volume via quasimorphisms)**.** *Let S be a set and $r \in F(S)' \setminus \{e\}$. Then*

$$\|G_r\| = \sup_{\phi \in Q(r)} \frac{\phi(r)}{D(\phi)},$$

where $Q(r)$ is the space of all quasimorphisms $\phi\colon F(S) \to \mathbb{R}$ satisfying that for all $g, h \in F(S)$ we have $\phi(g \cdot hrh^{-1}) = \phi(g) + \phi(hrh^{-1})$.

Proof. In view of the duality principle (Proposition 2.2), it suffices to look at $H_b^2(G_r;\mathbb{R})$ to compute $\|G_r\|$. Let $\omega \in C_b^2(G_r;\mathbb{R})$ be a bounded (bar) cocycle on G_r that is dual to the fundamental class $\alpha_{r,\mathbb{R}} \in H_2(G_r;\mathbb{R})$, i.e., such that $\langle [\omega], \alpha_{r,\mathbb{R}} \rangle = \|G_r\|$. We may assume that ω is alternating and thus that $\omega(g,e) = 0$ for all $g \in G_r$.

Let $\widetilde{\omega} \in C_b^2(F(S);\mathbb{R})$ denote the pullback of ω via the canonical projection $F(S) \to G_r$. Then, because of $H^2(F(S);\mathbb{R}) \cong 0$, there exists a quasimorphism $\phi\colon F(S) \to \mathbb{R}$ on $F(S)$ such that $\delta^1\phi = \widetilde{\omega}$ and $D(\phi) = \|\widetilde{\omega}\|_\infty = \|\omega\|_\infty$.

For all $h \in F(S)$, the conjugate $h \cdot r \cdot h^{-1}$ represents the neutral element in G_r. Therefore, using that ω is alternating, we see that

$$\delta^1\phi(g, h \cdot r \cdot h^{-1}) = \widetilde{\omega}(g, h \cdot r \cdot h^{-1}) = \omega([g], e) = 0$$

for all $g, h \in F(S)$. Therefore, $\phi(g) + \phi(h \cdot r \cdot h^{-1}) = \phi(g \cdot h \cdot r \cdot h^{-1})$ for all $g, h \in F(S)$, as claimed. □

Moreover, we have the following relationship between scl-extremal and l^1-extremal quasimorphisms:

Proposition 3.16. *Let S be a set, let $r \in F(S)'$, and for $N \in \mathbb{N}$ let ϕ_N be an l^1-extremal quasimorphism to r^N (i.e., $\|G_{r^N}\| = \phi_N(r^N)$) with defect 1. Further, let Ω be a non-principal ultrafilter on $\mathbb{N}$ and let*

$$\begin{aligned} \psi\colon F(S) &\to \mathbb{R} \\ g &\mapsto \frac{1}{4} \cdot \lim_{N \in \Omega} \frac{\phi_N(g)}{N}, \end{aligned}$$

where $\lim_{N\in\Omega}$ denotes the ultralimit along Ω. Then $\overline{\psi}$, the homogenisation of ψ, is an scl-extremal quasimorphism for r, i.e., $\mathrm{scl}_S(r) = \overline{\psi}(r)/D(\overline{\psi})$.

Proof. Using the properties of ultralimits we may estimate for all $g, h \in F(S)$,

$$1 \geq \lim_{N \in \Omega} \left|\phi_N(g) + \phi_N(h) - \phi_N(g \cdot h)\right| = \left|\psi(g) + \psi(h) - \psi(g \cdot h)\right|$$

and hence ψ is a quasimorphism with defect $D(\psi) \leq 1$. Therefore, the homogenisation $\overline{\psi}\colon r \mapsto \lim_{N\to\infty} \psi(r^N)/N$ satisfies $D(\overline{\psi}) \leq 2$ and (where "$\oplus C$" means up to error at most $\pm C$)

$$\begin{aligned}
\overline{\psi}(r) &= \lim_{N\to\infty} \lim_{K \in \Omega} \frac{\phi_K(r^{N\cdot K})}{N \cdot K} && \text{(definition of } \psi \text{ and } \overline{\psi}\text{)} \\
&= \lim_{N\to\infty} \lim_{K \in \Omega} \frac{N \cdot \phi_K(r^K) \oplus N \cdot 1}{N \cdot K} && (\phi_K \in Q(F(S)) \text{ and } D(\varphi_K) = 1) \\
&= \lim_{K \in \Omega} \frac{\|G_{r^K}\|}{K} && \text{(by } l^1\text{-extremality)} \\
&= 4 \cdot \mathrm{scl}_S(r). && \text{(Theorem 4.9)}
\end{aligned}$$

From Bavard duality (Theorem 2.7), we obtain

$$\mathrm{scl}_S(r) \geq \frac{\overline{\psi}(r)}{2 \cdot D(\overline{\psi})} \geq \frac{4 \cdot \mathrm{scl}_S(r)}{4}$$

and hence $\overline{\psi}$ is scl-extremal with defect $D(\overline{\psi}) = 2$. □

4 Van Kampen diagrams on surfaces

We recall van Kampen diagrams on surfaces, which we will use to encode the l^1-admissible maps of Proposition 3.11. This allows us to use combinatorial methods to estimate and sometimes compute the simplicial volume of one-relator groups. The main result of this section is the estimate for powers of elements; see Section 4.3.

Parts of this section are an adaptation of corresponding work on scl [Heu19, Section 4]. We will estimate the Euler characteristic of van Kampen diagrams by defining a combinatorial curvature $\kappa(D)$ for the disks D of a van Kampen diagram in Section 4.2. For the theorem on powers (Theorem 4.9), we will then estimate $\kappa(D)$, using *branch vertices* in Section 4.2. In Section 4.3, we will prove the theorem estimating the simplicial volume of one relator groups where the relation is a proper power.

4.1 l^1-Admissible surfaces via van Kampen diagrams

Van Kampen diagrams on surfaces have been introduced by Olshanskii to study homomorphisms from surface groups to a group with a given presentation [Ols89, CSS07].

Definition 4.1 (van Kampen diagram)**.** Let $r \in F(S)' \setminus \{e\}$ and let P_r be the presentation complex of $G_r = \langle S \mid r\rangle$ as in Setup 3.1; furthermore, let Σ be an oriented closed surface. A *van Kampen diagram $\mathcal{D}$ for the presentation r on Σ* is a decomposition of Σ into finitely many polygons, also called *disks*, where the edges are labelled by words over $S^{\pm}$ such that the boundary of each disk is labelled counterclockwise (i.e., orientation-preservingly with respect to

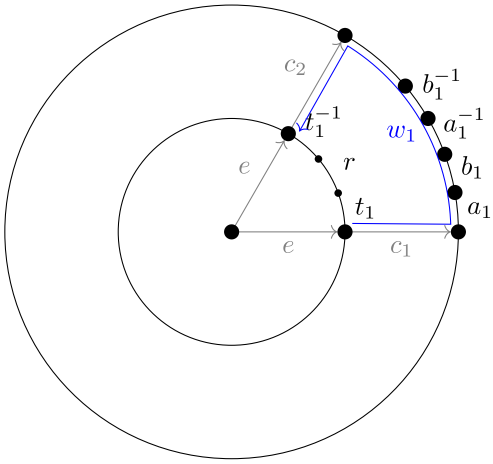


Figure 1: From Equation 2 to a van Kampen diagram

the orientation induced from Σ) in a reduced way by r^+ or r^-. Moreover, the labels of edges of adjacent disks are required to be compatible, i.e., if an edge is adjacent to two disks, then the label of one edge is $w \in F(S)$ and the label of the other one is w^{-1}. The underlying surface of $\mathcal{D}$ is denoted by $\Sigma_{\mathcal{D}}$. For a disk D in a van Kampen diagram labelled by r^ϵ we call ϵ the sign of D. The *total degree* of the van Kampen diagram is defined as $\sum_{D \in \mathcal{D}} n(D)$ where the sum runs over all disks of $\mathcal{D}$.

We write $\Delta(r)$ for the "set" of all van Kampen diagrams for r.

Every van Kampen diagram $\mathcal{D}$ for r induces a continuous map $f_{\mathcal{D}} \colon \Sigma_{\mathcal{D}} \to P_r$ to the presentation complex of G_r by mapping the labelled edges to the edges in the 1-skeleton of P_r and mapping the disks to the 2-cell of P_r. Every such map is l^1-admissible in the sense of Definition 3.10; the degree of this map is the difference of the number of positive and negative disks. Conversely, we may replace every l^1-admissible map by a map induced by a van Kampen diagram; thus van Kampen diagrams may be used to compute $\|G_r\|$:

Proposition 4.2 (simplicial volume via van Kampen diagrams)**.** *In the situation of Setup 3.1, if r is cyclically reduced, we have*

$$\|G_r\| = \inf_{\mathcal{D} \in \Delta(r)} \frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}})}{\left|n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}})\right|}.$$

Here, χ^- denotes the Euler characteristic that ignores spherical components, i.e., for a surface $\Sigma = \bigsqcup_i^n \Sigma_i$ with connected components Σ_i we have that

$$\chi^-(\Sigma) := \sum_{i=1}^n \min\bigl(0, \chi(\Sigma_i)\bigr).$$

Proof. Because van Kampen diagrams induce l^1-admissible maps, the inequality "$\leq$" holds. For the converse estimate, we use the description of $\|G_r\|$ from

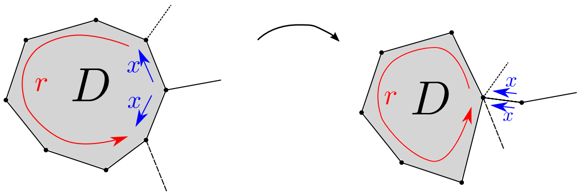


Figure 2: If the boundary word of D has backtracking we may glue up the backtracking and replace it by a disk with shorter boundary word.

Corollary 3.13. Let $n \in \mathbb{N}_{>0}$, let $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$, let $t_1, \dots, t_n \in F(S)$, and let

$$N := \operatorname{cl}_S(t_1 \cdot r^{\epsilon_1} \cdot t_1 \cdots t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}) > 0.$$

It then suffices to construct a van Kampen diagram for r with n disks with the signs $\epsilon_1, \dots, \epsilon_n$ on an oriented closed connected surface of genus N (the degree of the associated map will be $\epsilon_1 + \cdots + \epsilon_n$ and the Euler characteristic of the surface will be $2 - 2 \cdot N$).

By definition of N, there exist $a_1, \dots, a_N, b_1, \dots, b_N \in F(S)$ with

$$t_1 \cdot r^{\epsilon_1} \cdot t_n \cdots t_n \cdot r^{\epsilon_n} \cdot t_n^{-1} = [a_1, b_1] \cdots [a_N, b_N]. \tag{2}$$

We now consider a $4N$-gon, whose edges are labelled by $a_1, \dots, b_N$; inside, we put an n-gon, whose edges are labelled by $t_1 \cdot r^{\epsilon_1} \cdot t_1^{-1}, \dots, t_n \cdot r^{\epsilon_n} \cdot t_n^{-1}$ (Figure 1). Because of Equation (2), the corresponding annulus admits a continuous map f to $\bigvee_S S^1$ (where the circles are labelled by the elements of S) that is compatible with the labels of the edges. We now connect the vertices of the inner disks radially (and without crossings) with vertices of the outer disk (Figure 1); we label these radial sectors $c_1, \dots, c_n$ by the elements of $F(S)$ represented by the corresponding loop in $\bigvee_S S^1$ via f. For $j \in \{1, \dots, n\}$, let $w_j \in F(S)$ be the element obtained by following c_j, then walking on the outer disk until the endpoint of c_{j+1}, and then following the inverse $\bar{c}_{j+1}$ of c_{j+1}. By construction, w_j is conjugate to r^{ϵ_j} in $F(S)$.

As next step, we fill in the inner disk by n radial sectors $d_1, \dots, d_n$, all labelled by e. We now subdivide all edges according to reduced representations over $S^{\pm}$ (or e) of their labels. In this way, we obtain an oriented closed connected surface Σ of genus N that is decomposed into n compatibly edge-labelled disks, each of which is labelled by a conjugate of $r^{\pm}$.

It remains to reduce the words labelling the boundaries of the disks. We first contract all edges labelled by e to points; this leads to a homeomorphic surface (no pathologies can occur because $N > 0$). If the label of the boundary of a disk is *not* reduced, we may reduce it by gluing the corresponding edges (Figure 2); this reduces the number of unreduced positions in the label of this disk and leaves all other labels unchanged. Therefore, inductively, we obtain a decomposition of Σ into n disks with cyclically reduced boundary labels that are conjugate to $r^{\epsilon_1}, \dots, r^{\epsilon_n}$. Because r is cyclically reduced, this means that each disk is labelled by (a cyclic shift of) $r^{\pm}$ [LS01, Theorem IV.1.4]. Therefore, we obtain the desired van Kampen diagram for r. $\square$

4.2 Combinatorial Gauß-Bonnet

Let $\mathcal{D}$ be a van Kampen diagram and let D be a disk of $\mathcal{D}$ (we will also abbreviate this by writing "$D \in \mathcal{D}$"). Recall that every van Kampen diagram has an associated surface $\Sigma_{\mathcal{D}}$ such that the disks in $\mathcal{D}$ decompose $\Sigma_{\mathcal{D}}$ into finitely many polygons glued together along their edges. A *vertex* v of $\mathcal{D}$ is a vertex of those disks and $\deg(v)$ denotes the *degree* of v, i.e., the number of edges adjacent to v in $\Sigma_{\mathcal{D}}$. Morever, we write V_D for the set of all vertices and E_D be the set of all edges of D.

Definition 4.3 (curvature in a van Kampen diagram)**.** Let $\mathcal{D}$ be a van Kampen diagram. Then the *curvature* of disks of $\mathcal{D}$ is define by

$$\kappa(D) := \sum_{v \in V_D} \Big(\frac{1}{\deg(v)} - \frac{1}{2} \Big) + 1.$$

Proposition 4.4 (combinatorial Gauß-Bonnet)**.** *Let $\mathcal{D}$ be a van Kampen diagram on a surface $\Sigma_{\mathcal{D}}$. Then*

$$\chi(\Sigma_{\mathcal{D}}) = \sum_{D \in \mathcal{D}} \kappa(D).$$

Proof. Every vertex in $\Sigma_{\mathcal{D}}$ is adjacent to $\deg(v)$ many disks. Thus the total number of vertices in $\Sigma_{\mathcal{D}}$ equals $\sum_{D\in\mathcal{D}} \sum_{v \in V_D} \frac{1}{\deg(v)}$. Similarly, $\sum_{D\in\mathcal{D}} \sum_{e\in E_D} \frac{1}{2}$ is the total number of edges as every edge is counted twice in the two adjacent disks; and $\sum_{D\in\mathcal{D}} 1$ is the total number of disks. Hence,

$$\sum_{D\in\mathcal{D}} \kappa(P) = \#\text{vertices} - \#\text{edges} + \#\text{faces} = \chi(\Sigma_{\mathcal{D}}). \qquad \square$$

If D is a disk in a van Kampen diagram, we may estimate $\kappa(D)$ in terms of the number of vertices of degree at least 3, so-called *branch vertices*.

Proposition 4.5. *Let $\mathcal{D}$ be a van Kampen diagram, let D be a disk of $\mathcal{D}$, and let $\beta(D)$ be the number of branch vertices of D. Then*

$$\kappa(D) \leq \frac{6 - \beta(D)}{6}.$$

Proof. Every vertex in the disk D has degree at least 2. Thus we compute

$$\begin{aligned} \kappa(D) &= \sum_{v \in V_D} \Big(\frac{1}{\deg(v)} - \frac{1}{2} \Big) + 1 \\ &\leq \sum_{\deg(v) \geq 3} \Big(\frac{1}{3} - \frac{1}{2} \Big) + 1 \\ &\leq 1 - \frac{\beta(D)}{6} = \frac{6 - \beta(D)}{6}. \end{aligned} \qquad \square$$

4.3 Strong bounds from hyperbolicity

It is generally not known which one-relator groups are hyperbolic [CH20, LW18b]. However, there are two types of elements in $F(S)$ for which hyperbolicity is well-known: proper powers and small cancellation elements. In both cases, we obtain strong lower bounds for the simplicial volume in terms of stable commutator length. The key insight is the following lemma:

Lemma 4.6. *In the situation of Setup 3.1, suppose that there is an $N \geq 7$ such that the infimum in Proposition 4.2 is achieved as the infimum over van Kampen diagrams $\mathcal{D}$ such that $\beta(D) \geq N$ for every $D \in \mathcal{D}$. Then*

$$\left(1 - \frac{6}{N}\right) \cdot 4 \cdot \mathrm{scl}_S(r) \leq \|G_r\|.$$

Proof. Let $\epsilon > 0$. Choose a van Kampen diagram $\mathcal{D}$ on a surface $\Sigma_{\mathcal{D}}$ such that $\beta(D) \geq N$ for every $D \in \mathcal{D}$ and such that

$$\|G_r\| \geq \frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}})}{n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}})} - \epsilon.$$

By removing spherical components we may assume that $\chi^-(\Sigma_{\mathcal{D}}) = \chi(\Sigma_{\mathcal{D}})$. Let m^+ be the number of positive disks and let m^- be the number of negative disks of $\mathcal{D}$. Then the degree is $n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}}) = m^+ - m^-$ and the total number of disks is $m^+ + m^-$. Using the combinatorial Gauß-Bonnet Theorem (Proposition 4.4) and Proposition 4.5, we see that

$$\chi^-(\Sigma_{\mathcal{D}}) = \chi(\Sigma_{\mathcal{D}}) = \sum_{D \in \mathcal{D}} \kappa(D) \leq \frac{6-N}{6} \cdot (m^+ + m^-)$$

and hence

$$\|G_r\| + \epsilon \geq \frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}})}{n(\Sigma_{\mathcal{D}}, f_{\mathcal{D}})} \geq 2 \cdot \frac{N-6}{6} \cdot \frac{m^+ + m^-}{m^+ - m^-}.$$

We conclude that

$$\frac{1}{2} \cdot \frac{6}{N-6} \cdot \left(\|G_r\| + \epsilon\right) \geq \frac{m^+ + m^-}{m^+ - m^-}.$$

Let Σ_∂ be the surface obtained by removing the $(m^+ + m^-)$ disks of $\Sigma_{\mathcal{D}}$. Then Σ_∂ contracts to the 1-skeleton of P_r via f_∂ and every boundary word of Σ_∂ maps to a word labelled by $r^\pm$. Thus $(f_\partial, \Sigma_\partial)$ is scl-admissible for r; see Definition 2.5. We see that

$$\chi^-(\Sigma_\partial) = \chi^-(\Sigma_{\mathcal{D}}) - (m^+ + m^-)$$

and observe that the scl-degree of $(f_\partial, \Sigma_\partial)$ is $(m^+ - m^-)$.

This leads to the estimate

$$\begin{aligned} \mathrm{scl}_S(r) &\leq \frac{-\chi^-(\Sigma_\partial)}{2 \cdot (m^+ - m^-)} \\ &= \frac{-\chi^-(\Sigma_{\mathcal{D}})}{2 \cdot (m^+ - m^-)} + \frac{m^+ + m^-}{2 \cdot (m^+ - m^-)} \\ &\leq \frac{1}{4} \cdot \left(\|G_r\| + \epsilon\right) \cdot \left(1 + \frac{6}{N-6}\right). \end{aligned}$$

As this inequality holds for every ϵ we conclude that

$$\mathrm{scl}_S(r) \leq \frac{1}{4} \cdot \|G_r\| \cdot \left(1 + \frac{6}{N-6}\right)$$

and by rearranging terms that

$$\left(1 - \frac{6}{N}\right) \cdot 4 \cdot \mathrm{scl}_S(r) \leq \|G_r\|. \qquad \square$$

We apply Lemma 4.6 to the case of small cancellation elements. Recall that two elements $w, v \in F(S)$ are said to *overlap* in the word x, if x is a prefix of both w and v, i.e. if we may write $w = x \cdot w'$ and $v = x \cdot v'$ as reduced words, for an adequate choice of w', v'. An element $r \in F(S)$ is said to satisfy small cancellation condition $C'(1/N)$, if whenever it overlaps in x with a cyclic conjugate of r or r^{-1} that is not equal to r, then $|x| \leq \frac{1}{N} \cdot |r|$.

Theorem 4.7 (small cancellation elements)**.** *In the situation of Setup 3.1, if the relator $r \in F(S)' \setminus \{e\}$ satisfies the small cancellation condition $C'(1/N)$ for some $N \geq 7$, then*

$$\Big(1 - \frac{6}{N}\Big) \cdot 4 \cdot \mathrm{scl}_S(r) \leq \|G_r\| < 4 \cdot \mathrm{scl}_S(r).$$

.

Proof. If r satisfies the small cancellation condition $C'(1/N)$, then for every van Kampen diagram we have that all disks D satisfy $\beta(D) \geq N$. Thus, the first inequality follows from Lemma 4.6. The second inequality holds generally; see Corollary 3.12. □

Remark 4.8. We note that an element r that satisfies small cancellation condition $C'(1/N)$ satisfies $\mathrm{scl}_S(r) \geq \frac{N-6}{12}$. This can be seen by considering the branch edges, similarly to Proposition 4.5 and computing the Euler characteristic for the corresponding surface with boundary. Thus, using Lemma 4.6, we see that

$$\|G_r\| \geq \Big(1 - \frac{6}{N}\Big) \cdot 4 \cdot \frac{N-6}{12} = \frac{N}{3} - 1$$

for all r which satisfy small cancellation condition $C'(1/N)$ with $N > 6$.

On the other hand, we see that for sufficiently large powers, we get a strong connection between stable commutator length and the simplicial volume of one-relator groups.

Theorem 4.9 (proper powers)**.** *In the situation of Setup 3.1, we have for all $N > 6$:*

$$\Big(1 - \frac{6}{N}\Big) \cdot 4 \cdot \mathrm{scl}(r^N) \leq \|G_{r^N}\| < 4 \cdot \mathrm{scl}_S(r^N).$$

In particular, we obtain

$$\lim_{N \to \infty} \frac{\|G_{r^N}\|}{N} = 4 \cdot \mathrm{scl}_S(r).$$

Proof. The second inequality follows from the weak upper bound for simplicial volume (Corollary 3.12). To see the other inequality we will use Lemma 4.6 and show that $\|G_{r^N}\|$ may be approximated by van Kampen diagrams $\mathcal{D}$ that satisfy $\beta(D) \geq N$ for all $D \in \mathcal{D}$.

Let $r = \mathtt{x}_0 \cdots \mathtt{x}_{n-1}$ with $\mathtt{x}_i \in S$ be the reduced word representing r; we may assume that r is cyclically reduced and not a proper power.

Claim 4.10. *Let $\mathcal{D}$ be a van Kampen diagram on a surface $\Sigma_{\mathcal{D}}$ over r^N such that for every van Kampen diagram $\mathcal{D}'$ on a surface $\Sigma_{\mathcal{D}'}$ over r^N with fewer disks than $\mathcal{D}$ we have that*

$$\frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}})}{n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}})} < \frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}'})}{n(f_{\mathcal{D}'}, \Sigma_{\mathcal{D}'})}. \tag{3}$$

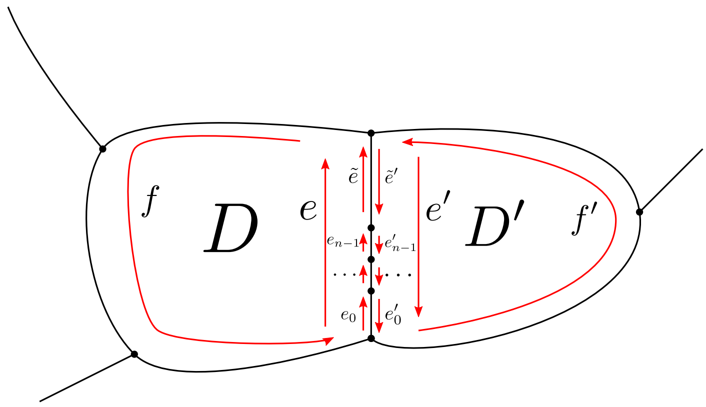


Figure 3: The disks D and D' share the subpaths e and e' in $\Sigma_{\mathcal{D}}$.

Let $D \in \mathcal{D}$ be a disk and let $e \subset \partial D$ be a connected subpath of the boundary of D such that e has no branch vertices in the interior.

Then the label of e has word length strictly less than $|r| = n$.

Proof. Without loss of generality we assume that D is positive, i.e., its boundary is labelled by r^N. Thus the label $w \in F(S)$ of e is a reduced subword of r^N (cyclically written). *Assume* for a contradiction that $|w| \geq n$. Then, by cyclically relabelling r we may assume that $w = \mathtt{x}_0 \cdots \mathtt{x}_{n-1}\tilde{r}$.

Because e has no branch vertices in its interior, there is a disk $D' \in \mathcal{D}$ that is adjacent to e; let e' be the subpath of the boundary of D' that corresponds to e in D. Then the label of e' is w^{-1}. We consider two different cases:

- The disk D' is positive. Then w^{-1} is a subword of r^N (cyclically written) as e' is an edge of D' and the boundary of D' is labelled by the word r^N. Suppose that the word w^{-1} ends in $\mathtt{x}_i$. Then we see that $\mathtt{x}_0^{-1} = \mathtt{x}_i$. Similarly, we see that $\mathtt{x}_1^{-1} = \mathtt{x}_{i-1}$ and $\mathtt{x}_k^{-1} = \mathtt{x}_{i-k}$ for every $k < i$. If i is even, then this implies that $\mathtt{x}_{i/2}^{-1} = \mathtt{x}_{i/2}$, which is a contradiction; if i is odd, this implies that $\mathtt{x}_{i/2-1/2}^{-1} = \mathtt{x}_{i/2+1/2}$, which contradicts that r is a reduced word.

- The disk D' is negative. In this case w^{-1} is a subword of r^{-N}. By adding degree 2 vertices to the disks of the van Kampen diagram $\mathcal{D}$, we may assume that every edge is labelled by a single letter in $S^{\pm}$. Suppose that the boundary of D is $e \cdot f$ and that the boundary of D' is $f' \cdot e'$. Here, $a \cdot b$ denotes the concatenation of two paths a and b.

  Then e may be written as $e = e_0 \cdot \cdots \cdot e_{n-1} \cdot \tilde{e}$ where e_i is labelled by $\mathtt{x}_i$ for all $i \in \{0, \ldots, n-1\}$ and $\tilde{e}$ is labelled by $\tilde{r}$. Similarly, e' may be written as $e' = \tilde{e}' \cdot e'_{n-1} \cdots e'_0$, where e'_i is labelled by $\mathtt{x}_i^{-1}$ for all $i \in \{0, \ldots, n-1\}$.

  The boundary labels of both D and D' are n-periodic, i.e., after n segments the labels repeat. Thus the first edge of $\tilde{e}$ has to be labelled by $\mathtt{x}_0$ and the last edge of $\tilde{e}'$ has to be labelled by $\mathtt{x}_0^{-1}$. If we continue comparing the labels of the edges in this way we see that the label for $\tilde{e}$ is inverse to the label for $\tilde{e}'$ and that the label for f is inverse to the label for f' (see Figure 3).

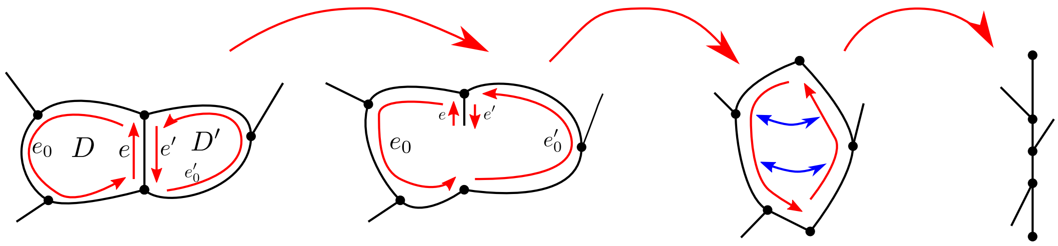


Figure 4: Gluing up D and D'.

Now we may glue both D and D' together along the boundaries as in Figure 4.

This procedure does not change the surface $\Sigma_{\mathcal{D}}$ up to homotopy equivalence. The result is a van Kampen diagram on a surface with the same Euler characteristic. The resulting van Kampen diagram also has the same degree as $\mathcal{D}$ as the degree of both D and D' cancelled. This contradicts the minimality of Equation 3.

In both cases we contradicted that the label had word length at least $|r|$. This proves Claim 4.10. $\square$

Claim 4.11. *We may approximate $\|G_{r^N}\|$ by a sequence of van Kampen diagrams $\mathcal{D}$ that satisfy that $\beta(D) \geq N$ for all $D \in \mathcal{D}$.*

Proof. Let $\epsilon > 0$ and let $\Sigma_{\mathcal{D}}$ be a surface with van Kampen diagram $\mathcal{D}$ with the least number of disks such that

$$\|G_{r^N}\| \geq \frac{-2\chi^-(\Sigma_{\mathcal{D}})}{n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}})} - \epsilon.$$

Let $D \in \mathcal{D}$ be a disk in $\mathcal{D}$. Claim 4.10 implies that each connected subpath of the boundary of D without branch vertices has length less than $|r|$. Thus there are at least $|r^N|/|r| = N$ such subpaths and branch vertices, i.e., $\beta(D) \geq N$. $\square$

Conclusion of the proof of Theorem 4.9. Applying Claim 4.11 to Lemma 4.6, we see that

$$\left(1 - \frac{6}{N}\right) \cdot 4 \cdot \mathrm{scl}_S(r^N) \leq \|G_{r^N}\|.$$

Using the multiplicativity of stable commutator length we conclude that

$$\lim_{N \to \infty} \frac{\|G_{r^N}\|}{N} = 4 \cdot \mathrm{scl}_S(r). \qquad \square$$

5 Random one-relator groups

In this section, we describe the large scale distribution of $\|G_r\|$ for random elements $r \in F(S)'$. This is an application of Theorem 4.7 and a result by Calegari and Walker on the random distribution of stable commutator lengths in free groups [CW13]. The aim of this section is to show the following result.

Theorem 5.1. *Let S be a finite set, let $\epsilon > 0$ and $C > 1$. Then, for every random reduced element $r \in F(S)$ of even length n, conditioned to lie in the commutator subgroup $F(S)'$, we have*

$$\left| \|G_r\| \cdot \frac{\log(n)}{n} - \frac{2 \cdot \log(2|S|-1)}{3} \right| \leq \epsilon$$

with probability $1 - O(n^{-C})$.

We derive this result by relating the simplicial volume to the stable commutator length of random elements using the small cancellation estimate (Theorem 4.7) and that random elements are small cancellation. The result is then a direct application of the corresponding result for stable commutator length by Calegari and Walker:

Theorem 5.2 (Calegari-Walker [CW13, Theorem 4.1])**.** *Let S be a finite set, let $\epsilon > 0$ and $C > 1$. Then, for every random reduced element $r \in F(S)$ of even length n, conditioned to lie in the commutator subgroup $F(S)'$, we have*

$$\left| \mathrm{scl}_S(r) \cdot \frac{\log(n)}{n} - \frac{\log(2|S|-1)}{6} \right| \leq \epsilon$$

with probability $1 - O(n^{-C})$.

5.1 Random elements of the commutator subgroup

We recall well-known results about random elements of the free group and introduce notation.

Setup 5.3. Let S be a finite set. We set $k := |S|$ and let $F := F(S)$ be the free group over S. Furthermore:

- F_n denotes the set of all words of length n.
- F'_n denotes the set of all words of length n that lie in the commutator subgroup of F. Here, n is supposed to be even.
- A_n^N denotes the set of all elements (not necessarily cyclically reduced) of length n that do *not* satisfy the small cancellation condition $C'(1/N)$.

In this situation, $|F_n| = 2k \cdot (2k-1)^{n-1}$. We recall the following Theorem of Sharp, estimating the size of $|F'_n|$ relative to $|F_n|$.

Theorem 5.4 (Sharp [Sha01, Theorem 1][CW13, Theorem 2.1])**.** *In the situation of Setup 5.3, we have: If n is odd then F'_n is empty. Moreover, there is an explicit constant σ^k (which depends only on k) such that*

$$\lim_{n\to\infty} \left| \sigma^k \cdot n^{k/2} \cdot \frac{|F'_n|}{|F_n|} - \frac{2}{(2\pi)^{k/2}} \right| = 0,$$

where the limit is taken over all even *natural numbers n.*

We may crudely estimate the exceptional set $|A_n^N|$ as follows:

Proposition 5.5. *In the situation of Setup 5.3, we have for all natural numbers $n, N \geq 1$:*

$$|A_n^N| \leq 3n^3 \cdot (2k)^2 \cdot (2k-1)^{n-\frac{n}{N}}$$

Proof. Suppose that w does not satisfy the small cancellation condition $C'(1/N)$. We consider the following cases:

1. $w \in A_n^N$ overlaps with a cyclic conjugate of w^{-1} in a piece larger than n/N. Then a cyclic conjugate of w may be written as $v = rv_1r^{-1}v_2$ such that $|r| = \lceil \frac{n}{N} \rceil$ and $|v| = n$ and where there is at most one cancellation either in v_1 or in v_2, if w was not cyclically reduced. Here and throughout this section we say that a word *has a cancellation* if there is a subword $\mathtt{x} \cdot \mathtt{x}^{-1}$, where $\mathtt{x}$ is a letter in the alphabet.

   We will estimate the possible choices for v. We have $2k \cdot (2k-1)^{\lceil \frac{n}{N} \rceil - 1}$ choices for r. If $|v_1| = n_1$ and $|v_2| = n_2$, then there are at most $n \cdot 2k \cdot (2k-1)^{n_1+n_2-1}$ choices for v_1 and v_2: For any letter there are $(2k-1)$ choices in order to avoid cancellation with the previous letter, apart from one time where we allow the letter to be an inverse of the previous letter. There are at most n possiblities where such an inverse may occur, and thus we get a total of $n \cdot 2k \cdot (2k-1)^{n_1+n_2-1}$.

   Thus we estimate that in this case there are a total of

   $$n \cdot (2k)^2 \cdot (2k-1)^{\lceil \frac{n}{N} \rceil - 1} \cdot (2k-1)^{n_1+n_2-1}$$

   choices for $v = rv_1r^{-1}v_2$ with $|v_1| = n_1$ and $|v_2| = n_2$. We note that $n_1 + n_2 + \lceil \frac{n}{N} \rceil = n$, and thus we may crudely estimate that there are

   $$n^2 \cdot (2k)^2 \cdot (2k-1)^{\lceil \frac{n}{N} \rceil - 1} \cdot (2k-1)^{n-2 \cdot \lceil \frac{n}{N} \rceil}$$

   many choices for $v = rv_1r^{-1}v_2$, such that $r = \lceil \frac{n}{N} \rceil$ and where there is at most one cancellation either in v_1 or in v_2. Finally, there are n elements that are cyclically conjugate to such a v.

   Thus, an upper bound for the total number of words $w \in A_N^n$ that overlap with an inverse may be bounded by

   $$n^3 \cdot (2k)^2 \cdot (2k-1)^{n - \lceil \frac{n}{N} \rceil}.$$

2. $w \in A_n^N$ overlaps with a cyclic conjugate of w and w overlaps with itself in a piece r with $|r| = \lceil \frac{n}{N} \rceil$ that does not overlap with itself. Then a cyclic conjugate of w may be written as $v = rv_1rv_2$ with the conditions on r, v_1 and v_2 as in case (1.). We may deduce the same bound by replacing r^{-1} by r, if appropriate.

   Thus we see that in this case, there are again at most

   $$n^3 \cdot (2k)^2 \cdot (2k-1)^{n - \lceil \frac{n}{N} \rceil}.$$

   such elements $w \in A_n^N$.

3. $w \in A_n^N$ overlaps with a cyclic conjugate of w and w overlaps with itself in a piece r with $|r| = \lceil \frac{n}{N} \rceil$ that overlaps with itself. Then a cyclic conjugate

of w may be written as $v = rv_1$ and $v = v_2 r v_2'$ for $|r| = \lceil \frac{n}{N} \rceil$, and $|v_2| < |r|$. where there is at most one cancellation in v_1.

Thus, in particular, we have that $v = rv_1'$ and $v = v_2 r$ for v_1' the prefix of v_1 such that $|v_1'| = |v_2|$.

Claim 5.6. *There are at most $2k \cdot (2k-1)^{M-1}$ elements v_2, r, v_1', such that $rv_1' = v_2 r$, $|v_2| = |v_1'| = M$, $|r| = \lceil \frac{n}{N} \rceil$ and $M < |r|$.*

Proof. Indeed, we will see that for any choice of v_2, the elements r and v_1' are fully determined: By comparing the first M letters of the equality $rv_1' = v_2 r$ we see that the first $|v_2|$ letters of r are v_2. By continuing this way we see that r has to be a prefix of a power of v_2. We know its length, and thus r is determined. We may recover v_1' by evaluating $v_1' = r^{-1} v_2 r$. There are $2k \cdot (2k-1)^{M-1}$ choices of v_2, and thus this shows the claim. $\square$

We write $v_1 = v_1' \cdot v_1''$, for v_1 and v_1' as above with $|v_1'| = M$. Note that there is at most one cancellation in v_1 and thus at most one cancellation in v_1''. Thus, there are at most $n \cdot 2k \cdot (2k-1)^{n - \frac{n}{N} - M - 1}$ many choices for v_1'', following the estimate of case (1.) for words that contain exactly one cancellation. Together with Claim 5.6 we see that for $|v_1'| = M$ there are a total of at most

$$n \cdot (2k)^2 \cdot (2k-1)^{n - \frac{n}{N} - 2}$$

choices for such v. As $M \leq n$ we see that there are at most a total of

$$n^2 \cdot (2k)^2 \cdot (2k-1)^{n - \frac{n}{N} - 2}$$

choices for v without any condition on v_1'.

As at most n words in A_n^N are cyclically conjugate to such a v we may estimate the total of words in A_n^N with overlap with the same power in itself by

$$n^3 \cdot (2k)^2 \cdot (2k-1)^{n - \frac{n}{N}}.$$

By putting the estimates from the cases (1.), (2.), and (3.) together we deduce that

$$|A_n^N| \leq 3n^3 \cdot (2k)^2 \cdot (2k-1)^{n - \frac{n}{N}}.$$

This finishes the proof of Proposition 5.5. $\square$

Corollary 5.7. *In the situation of Setup 5.3, let q_n be the probability that a random element of F_n' does not satisfy the small cancellation condition $C'(1/\sqrt{n})$. Then*

$$q_n = o\big((2k-1)^{-\sqrt{n}/2}\big).$$

Proof. Let $B_n \subset F_n$ be the set of elements that do not satisfy the small cancellation condition $C'(1/\sqrt{n})$. By Proposition 5.5 we see that

$$|B_n| \leq 3 \cdot n^3 \cdot (2k)^2 \cdot (2k-1)^{n - \lceil \sqrt{n} \rceil} = 3 \cdot n^3 \cdot (2k) \cdot (2k-1)^{1 - \lceil \sqrt{n} \rceil} \cdot |F_n|.$$

Thus, we may estimate that the probability q_n of a random element in F'_n to also lie in B_n to be

$$q_n = \frac{|B_n|}{|F'_n|} \leq 3 \cdot n^3 \cdot 2k \cdot (2k-1)^{1-\lceil\sqrt{n}\rceil} \cdot \frac{|F_n|}{|F'_n|}.$$

Using Sharp's result (Theorem 5.4) we can estimate that $q_n = o((2k-1)^{-\sqrt{n}/2})$ as $n \to \infty$. □

5.2 Proof of Theorem 5.1

We now give the argument for Theorem 5.1.

Proof. Let $k := |S|$. By Theorem 5.2, we see that for every $C > 1$, $\epsilon > 0$ the probability of a random element in F'_n to satisfy that

$$\left| \operatorname{scl}(r) \cdot \frac{\log(n)}{n} - \frac{\log(2k-1)}{6} \right| \leq \epsilon \tag{4}$$

is $1 - O(n^{-C})$. By Corollary 5.7, the probability that a random element in F'_n satisfies the small cancellation condition $C'(1/\sqrt{n})$ is $1 - o((2k-1)^{-\sqrt{n}/2})$. Thus, the probability that a random element in F'_n satisfies Equation (4) may be bounded by $1 - O(n^{-C}) - o((2k-1)^{-\sqrt{n}/2}) = 1 - O(n^{-C})$.

In the following, let $n \geq 49$. Then, Theorem 4.7 implies that

$$4 \cdot \operatorname{scl}(r) \cdot \left(1 - \frac{6}{\sqrt{n}}\right)^{-1} \leq \|G_r\| \leq 4 \cdot \operatorname{scl}(r).$$

Putting things together we see that if r satisfies Equation (4), then

$$\begin{aligned} \|G_r\| \cdot \frac{\log(n)}{n} &\leq 4 \cdot \operatorname{scl}(r) \cdot \frac{\log(n)}{n} \\ &\leq \frac{2\log(2k-1)}{3} + 4\epsilon \end{aligned}$$

and

$$\begin{aligned} \|G_r\| \cdot \frac{\log(n)}{n} &\geq 4 \cdot \operatorname{scl}(r) \cdot \frac{\log(n)}{n} \cdot \left(1 - \frac{6}{\sqrt{n}}\right)^{-1} \\ &\geq \frac{2\log(2k-1)}{3} \cdot \left(1 - \frac{6}{\sqrt{n}}\right)^{-1} - 4\epsilon \cdot \left(1 - \frac{6}{\sqrt{n}}\right)^{-1} \\ &\geq \frac{2\log(2k-1)}{3} - 2 \cdot 4\epsilon. \qquad \text{(because } n \geq 36\text{)} \end{aligned}$$

By relabelling ϵ and C we obtain that the probability that

$$\left| \|G_r\| \cdot \frac{\log(n)}{n} - \frac{2\log(2k-1)}{3} \right| \leq \epsilon$$

may be estimated by $1 - O(n^{-C})$. □

6 Computational bounds: `lallop`

In this section we describe an invariant for elements in $F(S)'$ called `lallop`. This invariant will give a lower bound to the simplicial volume of one-relator groups and is computable in polynomial time.

We briefly describe the motivation for the definition of `lallop`. For this, recall that any element in $F(S)'$ may be written up to cyclic conjugation as r^M, where r is root-free and cyclically reduced and $M \in \mathbb{N}_{>0}$. Throughout this section, we will write $\texttt{lallop}(r^M)$, to indicate that `lallop` is evaluated on a power of size M, even if the element is root-free, i.e. if $M = 1$.

Recall that by Proposition 4.2 we have that

$$\|G_{r^M}\| = \inf_{\mathcal{D} \in \Delta(r^M)} \frac{-2 \cdot \chi^-(\Sigma_{\mathcal{D}})}{\left|n(f_{\mathcal{D}}, \Sigma_{\mathcal{D}})\right|},$$

where the infimum ranges over all `lallop`-admissible van Kampen diagrams. Similar to the algorithm `scallop` [WC12] that computes stable commutator length of elements in free group, we wish to compute $\|G_{r^M}\|$ using a linear programming problem.

Crudely, `scallop` associates to any scl-admissible surface a vector in a finite dimensional vector space, spanned by the finitely many reasonable configurations around the vertices of scl-admissible surfaces. Then both the Euler characteristic and the degree of the original van Kampen diagram may be computed via linear functions of the associated vector, and thus scl becomes the solution of a finite dimensional linear programming problem.

When adapting this algorithm for the simplicial volume of one-relator groups one runs into the problem that the Euler characteristic for admissible van Kampen diagrams may *not* be computed by the information around the vertices alone. However, we remedy this by adding an extra term to the Euler characteristic; see Definition 6.3. This only gives a lower bound of $\|G_{r^M}\|$ but allows us to do exact computations. To control this term we will need to restrict to certain van Kampen diagrams, which we call reduced `lallop`-admissible van Kampen diagrams (see Remark 6.4).

The aim of this section is to show:

Theorem 6.1 (`lallop`)**.** *Let S be a set and let $r \in F(S)'$. Then*

1. $\texttt{lallop}(r) \leq \|G_r\|$,
2. $\texttt{lallop}(r) \leq 4 \cdot \mathrm{scl}_S(r) - 2$, *and*
3. *there is an algorithm to compute* $\texttt{lallop}(r)$ *that is polynomial in $|r|$, the word length of r. Moreover,* $\texttt{lallop}(r) \in \mathbb{Q}$.

We will prove the first two items of Theorem 6.1 in Section 6.2. The proof of the third part will be developed in Sections 6.3 and 6.4.

6.1 `lallop`

We now define reduced `lallop`-admissible van Kampen diagrams, which we will use to define `lallop` in Definition 6.3.

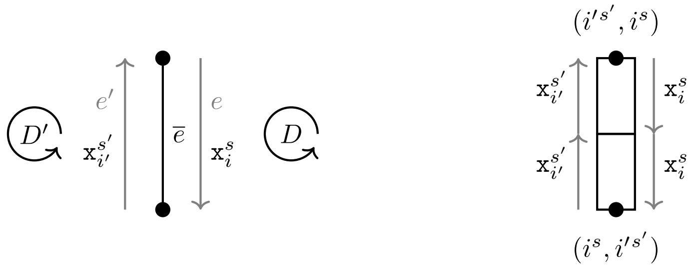


Figure 5: From edges to corresponding rectangles

Definition 6.2 (`lallop`-admissible van Kampen diagram)**.** Let r^M be a cyclically reduced word with root $r = \mathtt{x}_1 \cdots \mathtt{x}_n \in F(S)' \setminus \{e\}$ and $M \in \mathbb{N}_{>0}$. We say that a van Kampen diagram $\mathcal{D}$ to r on a surface $\Sigma_{\mathcal{D}}$ is `lallop`-*admissible to* r^M, if every edge is labelled by a single letter in S such that the counterclockwise label around every disk $D \in \mathcal{D}$ is cyclically labelled by $r^{n \cdot M}$, where n is a non-zero integer $n \in \mathbb{Z}$, called the *degree of* D and denoted by $n(D)$.

Let e be an oriented edge in the van Kampen diagram. This edge is adjacent to two disks D and D', where the edge is in counterclockwise (positive) orientation for one of the disks and in clockwise (negative) orientation in the other. Thus, if the label of the edge e is $\mathtt{x}$, then $\mathtt{x}$ labels a subletter of the label of D and a subletter of the inverse of the label of D'. For D we have a *position* $i \in \{1, \ldots, n\}$ and a *sign* $\epsilon \in \{+1, -1\}$ corresponding to the letter $\mathtt{x}_i$ labelled by that edge in the disk and the sign of the degree of the disk. We will write i^ϵ as a shorthand for the position/sign of the edge at a disk and note that $\mathtt{x}_i^\epsilon = \mathtt{x}$. Similarly we have a position and sign for D' which we denote by $i'^{\epsilon'}$ and note that $\mathtt{x}_{i'}^{\epsilon'} = \mathtt{x}^{-1}$.

Note that this way, every oriented edge e in the van Kampen diagram has two positions and signs $(i^\epsilon, i'^{\epsilon'})$ corresponding to the two disks the edge is adjacent to. In this case, $\mathtt{x}_i^\epsilon = \mathtt{x}_{i'}^{-\epsilon'}$. In analogy to `scallop` [Cal09a], we call $(i^\epsilon, i'^{\epsilon'})$ the *rectangle* associated to the edge e (Figure 5).

We say that a `lallop`-admissible van Kampen diagram is *reduced* if there are no rectangles $(i^\epsilon, i'^{\epsilon'})$ with $i = i'$. We denote the set of reduced `lallop`-admissible van Kampen diagrams to r^M by $\Delta_1(r^M)$.

We will see that we may always replace a `lallop`-admissible van Kampen diagram by a reduced `lallop`-admissible van Kampen diagram; see Proposition 6.5.

Definition 6.3 (`lallop`)**.** Let $s \in F(S)'$ be an element in the commutator subgroup of the free group $F(\mathcal{S})$ and let s be conjugate to r^M where r is cylically reduced, root-free and $M \in \mathbb{N}_{>0}$. Then, we define

$$\mathtt{lallop}(r^M) := \inf_{\mathcal{D} \in \Delta_1(r^M)} \frac{-2\chi(\Sigma_{\mathcal{D}}) + 2\sum_{D \in \mathcal{D}}(1 - |n(D)|)}{n(\Sigma_{\mathcal{D}})},$$

where the infimum is taken over all reduced `lallop`-admissible van Kampen diagrams for r^M.

As stated in the introduction, the (de)nominator of the terms in the definition of `lallop` are carefully chosen in such a way that they can be computed using a linear programming problem similar to `scallop`.

Remark 6.4. One may wonder why we needed to define reduced `lallop`-admissible van Kampen diagrams in the first place and didn't simply take the infimum in Definition 6.3 over *all* van Kampen diagrams.

To see that this is necessary, note that for every word $r \in F(S)'$ and any natural number $N \in \mathbb{N}$ we may glue a disk labelled by r^N to a disk labelled by r^{-N} by identifying corresponding letters. Topologically this is a sphere. We may add this sphere to any van Kampen diagram over r to obtain a new van Kampen diagram. While this does not change the total degree of the van Kampen diagram, and only changes the Euler characteristic by 1, it changes the term $2\sum_{D\in\mathcal{D}}(1-|n(D)|)$ by $2-2\cdot N$. Thus, an infimum as in Definition 6.3 over *all* `lallop`-admissible van Kampen diagrams would not exist.

Similarly, if the word we consider is not root-free, say it is of the form r^M for some $M \in \mathbb{N}_{\geq 2}$, we may glue up a disk labelled by $r^{N\cdot M}$ with a disk labled by $r^{-N\cdot M}$ by gluing up corresponding letters shifted by r. Topologically, this again is a sphere that we may add to any van Kampen diagram.

6.2 From van Kampen diagrams to reduced `lallop`-admissible van Kampen diagrams

In this section, we show how an arbitrary van Kampen diagram may be replaced by a reduced `lallop`-admissible van Kampen diagram:

Proposition 6.5. *Let $r^M \in F(S)$ be a cyclically reduced element where $M \geq 1$ and r is root-free. Let $\mathcal{D}$ be a van Kampen diagram for r^M. Then, there is a reduced* `lallop`*-admissible van Kampen diagram $\mathcal{D}'$ with the same degree, such that $-\chi(\Sigma_{\mathcal{D}}) \geq -\chi(\Sigma_{\mathcal{D}'})$.*

Moreover, we have for every reduced `lallop`*-admissible van Kampen diagram $\mathcal{D}$ that $\chi(\Sigma_{\mathcal{D}}) = \chi^-(\Sigma_{\mathcal{D}})$.*

Proof. We may assume that every edge of $\mathcal{D}$ is labelled by some element in $F(S)$ by possibly shrinking the edges that are not labelled by any word. By subdividing the edges, we may further assume that every edge is labelled by a single letter in S. We know that every disk is cyclically labelled by a power of r^M. By recording which letter of r corresponds to which edge, we may construct the rectangles.

Suppose that $\mathcal{D}$ is not reduced. Then there is a rectangle $(i^\epsilon, i'^{\epsilon'})$ with $i = i'$. Since $\mathtt{x}_i^\epsilon = \mathtt{x}_{i'}^{-\epsilon'}$, we deduce that $\epsilon = -\epsilon'$, in other words, the two disks adjacent to this rectangle have opposite signs. We may then cut up the two disks at the edge and glue the boundaries together analogously to Figure 4. This does not change the degree and only increases the Euler characteristic.

We are left to show that χ^- and χ agree for reduced `lallop`-admissible van Kampen diagrams. If not, there is a reduced spherical `lallop`-admissible van Kampen diagram $\mathcal{D}^S$ for r^M, such that $\Sigma_{\mathcal{D}^S}$ is a sphere. Note that this would also be a van Kampen diagram for r. This would then define a non-trivial spherical map to the presentation complex. However, the presentation complex of root-free words is aspherical by a result of Cockcroft [Coc54]. $\square$

Using the last proposition, we may prove items 1 and 2 of Theorem 6.1.

Proof of Theorem 6.1 items 1 and 2. The fact that $\texttt{lallop}(r) \leq \|G_r\|$ is a consequence of the description of $\|G_r\|$ in terms of van Kampen diagrams (Proposition 4.2) and that χ and χ^- agree for reduced `lallop`-admissible van Kampen diagrams. By Proposition 6.5, we may replace van Kampen diagrams by `lallop`-admissible van Kampen diagrams.

To see item 2, let Σ be an scl-admissible surface to r^M with one boundary component. We may assume that we just have one boundary component with positive degree N. By gluing in a disk to the boundary we obtain a `lallop`-admissible van Kampen diagram on a surface Σ' with $\chi(\Sigma) = \chi(\Sigma') - 1$. We may estimate

$$\texttt{lallop}(r^M) \leq \frac{-2\chi(\Sigma') + 2(1-N)}{N} = \frac{-2\chi(\Sigma) - 2N}{N} = 4 \cdot \frac{-2\chi(\Sigma)}{N} - 2$$

Taking the infimum over all scl-admissible surfaces Σ to r^M, shows with the help of the right-hand side that $\texttt{lallop}(r^M) \leq 4 \cdot \mathrm{scl}(r^M) - 2$. $\square$

6.3 From reduced `lallop`-admissible van Kampen diagrams to linear programming

Recall that throughout this section we will write the relator of our one-relator group as r^M, where r is cyclically reduced and root-free and $M \in \mathbb{N}_{>0}$. Note that any element in the free group may be conjugated to an element that can be written in this way.

Proposition 6.6. *Let S be a set and let $r^M \in F(S)' \setminus \{e\}$ be cyclically reduced such that r is root-free. Then $\texttt{lallop}(r^M)$ can be computed via the information around the vertices as follows:*

$$\texttt{lallop}(r^M) \quad = \quad \inf_{\mathcal{D} \in \Delta_1(r^M)} \frac{\sum_{v \in V_{\mathcal{D}}} \big(\deg(v) - 2\big) - 2 \cdot \sum_{D \in \mathcal{D}} |n(D)|}{\sum_{D \in \mathcal{D}} n(D)}$$

Here, we write $V_{\mathcal{D}}$ for the set of vertices of a van Kampen diagram $\mathcal{D}$.

Proof. Observe that $\sum_{v \in V_{\mathcal{D}}} \big(\deg(v) - 2\big)$ is equal to $-2 \cdot (\#\text{vertices} - \#\text{edges})$. Similarly, $2 \cdot \sum_{D \in \mathcal{D}} |n(D)| = -2\#\text{faces} + 2\sum_{D \in \mathcal{D}}(1 - |n(D)|)$. Putting things together we see that

$$\sum_{v \in V_{\mathcal{D}}} \big(\deg(v) - 2\big) - 2 \cdot \sum_{D \in \mathcal{D}} |n(D)| = -2 \cdot \chi(\Sigma_{\mathcal{D}}) + 2 \cdot \sum_{D \in \mathcal{D}} (1 - |n(D)|).$$

Thus, the result follows from the definition of `lallop`-admissible van Kampen diagrams (Definition 6.2). $\square$

A key observation will be that $\texttt{lallop}(r^M)$ may be computed "locally" by computing the degrees of the vertices in the van Kampen diagram. In contrast, it is impossible to compute the Euler characteristic of the underlying surface in this way because the information of how large the disks are cannot be encoded in the vertices.

In a first step, we will associate to a reduced `lallop`-admissible van Kampen diagram a vector in an infinite dimensional vector space by encoding the local compatibility conditions around the vertices. We can then compute `lallop` as an affine function on this vector space. Moreover, we will characterise all vectors that arise in this correspondence (Lemma 6.8).

Let $r^M \in F(S)' \setminus \{e\}$ be such that r is cyclically reduced and root-free with $M \in \mathbb{N}_{>0}$ and write $r = \mathtt{x}_0 \cdots \mathtt{x}_{n-1}$. Let $\mathcal{D} \in \Delta_1(r^M)$ be a reduced `lallop`-admissible van Kampen diagram.

Recall (Definition 6.2) that for every oriented edge $\bar{e}$ in the reduced `lallop`-admissible van Kampen diagram $\mathcal{D}$ we associate a pair $(i^\epsilon, {i'}^{\epsilon'})$, called rectangle, as follows: The integers i, i' correspond to the letters $\mathtt{x}_i$ and $\mathtt{x}_{i'}$ that label the sides e and e' of $\bar{e}$ and the signs $\epsilon, \epsilon' \in \{+1, -1\}$ correspond to the disks adjacent to $\bar{e}$. We denote this rectangle by $\mathrm{R}(\bar{e})$. As $\mathcal{D}$ is reduced we know that $i \neq i'$.

By abuse of notation we denote by $\mathrm{R}(r^M)$ the set

$$\mathrm{R}(r^M) := \left\{ (i^s, {i'}^{s'}) \;\middle|\; i \neq i' \in \{0, \ldots, n-1\},\ s, s' \in \{+, -\},\ \mathtt{x}_i^s = \mathtt{x}_{i'}^{-s'} \right\}$$

of all possible rectangles.

Observe that if $\bar{e}'$ is the inverse of the oriented edge $\bar{e}$ and if $\mathrm{R}(\bar{e}) = (i^\epsilon, {i'}^{\epsilon'})$, then $\mathrm{R}(\bar{e}') = ({i'}^{\epsilon'}, i^\epsilon)$. This defines an involution $\iota\colon \mathrm{R}(r^M) \to \mathrm{R}(r^M)$ on the set of rectangles and we think of ι as flipping the orientation of the edge (Figure 5).

We now turn to the structure around a vertex: Let v be a vertex of $\mathcal{D}$ and let $\bar{e}_1, \ldots, \bar{e}_k$ be the edges in $\mathcal{D}$ pointing towards v and ordered clockwise around v. We associate the tuple

$$\mathrm{V}(v) := \left[\mathrm{R}(\bar{e}_1), \ldots, \mathrm{R}(\bar{e}_k) \right].$$

to v. The tuples of rectangles arising in this way are not arbitrary, as they have to be compatible with the labelling of the disks in $\mathcal{D}$. More precisely: Let $\bar{e}_1$ have the sides e_1 and e'_1 with rectangle $\mathrm{R}(\bar{e}_1) = (i_1^{\epsilon_1}, {i'}_1^{\epsilon'_1})$ and let $\bar{e}_2$ have the sides e_2 and e'_2 with rectangle $\mathrm{R}(\bar{e}_2) = (i_2^{\epsilon_2}, {i'}_2^{\epsilon'_2})$; see Figure 6. Then the disk D adjacent to the edges $\bar{e}_1$ and $\bar{e}_2$ has to have the same sign, i.e., $\epsilon'_1 = \epsilon_2$.

Suppose that $\epsilon'_1 = \epsilon_2 = +$. The labels of the disk D thus read cyclically a positive power of r. Hence, if the label of $\bar{e}_1$ for D is $\mathtt{x}_{i'_1}$, the label of $\bar{e}_2$ is $\mathtt{x}_{i_2}$ and $i_2 = i'_1 + 1$. Similarly, if the sign of D were negative we would have $i_2 = i'_1 - 1$, where all indices are taken modulo n; see Figure 6.

This motivates the following definition: We say that *a rectangle* $({i_1}^{s_1}, {i'_1}^{s'_1}) \in \mathrm{R}(r)$ *follows the rectangle* $({i_2}^{s_2}, {i'_2}^{s'_2}) \in \mathrm{R}(r)$ if

- $i'_1 = i_2 + 1$ and $s'_1 = + = s_2$, or
- $i'_1 = i_2 - 1$ and $s'_1 = - = s_2$.

A tuple $[R_1, \ldots, R_k]$ of rectangles $R_1, \ldots, R_k \in \mathrm{R}(r)$ is a *k-pod* if R_{i+1} follows R_i for all $i \in \{1, \ldots, k-1\}$ and R_1 follows R_k. We then define

$$\mathrm{V}(r^M) = \left\{ [R_1, \ldots, R_k] \;\middle|\; k \in \mathbb{N}_{\geq 2},\ R_1, \ldots, R_k \in \mathrm{R}(r),\ [R_1, \ldots, R_k] \text{ is a } k\text{-pod} \right\}.$$

By construction, if v is a vertex of $\mathcal{D}$, then $\mathrm{V}(v) \in \mathrm{V}(r^M)$. The set $\mathrm{V}(r^M)$ is infinite and should be thought of as the set of all possible labels around a vertex in a `lallop`-admissible van Kampen diagram to r^M.

We illustrate this by the following example:

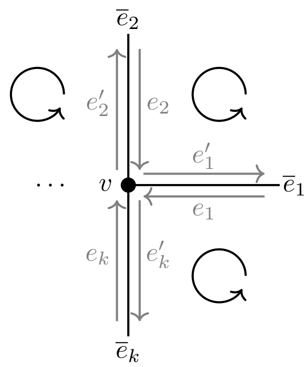


Figure 6: The local structure around a vertex

Example 6.7. Let $r = \mathtt{aba}^{-1}\mathtt{b}^{-1}$ be the commutator of $\mathtt{a}$ and $\mathtt{b}$ in $F(\mathtt{a}, \mathtt{b})$. In the above setting, $\mathtt{x}_1 = \mathtt{a}$, $\mathtt{x}_2 = \mathtt{b}$, $\mathtt{x}_3 = \mathtt{a}^{-1}$ and $\mathtt{x}_4 = \mathtt{b}^{-1}$. Then

$$\begin{aligned}\mathrm{R}(r) = \big\{&(1^+, 3^+), (3^+, 1^+), (1^-, 3^-), (3^-, 1^-),\\ &(2^+, 4^+), (4^+, 2^+), (2^-, 4^-), (4^-, 3^-)\big\}.\end{aligned}$$

Examples of 4-pods are

$$\begin{aligned}&[(1^+, 3^+), (4^+, 2^+), (3^+, 1^+), (2^+, 4^+)] \quad \text{or}\\ &[(1^-, 3^-), (2^-, 4^-), (3^-, 1^-), (4^-, 2^-)].\end{aligned}$$

Let $\mathbb{Z}\,\mathrm{V}(r^M)$ be the $\mathbb{Z}$-module freely generated by $\mathrm{V}(r^M)$. We will now define a map $\Phi\colon\ \Delta_1(r) \to \mathbb{Z}\,\mathrm{V}(r^M)$ encoding the local structure of van Kampen diagrams: For a `lallop`-admissible van Kampen diagram $\mathcal{D}$ on a surface $\Sigma_{\mathcal{D}}$ with vertex set $V_{\mathcal{D}}$, we set

$$\Phi(\mathcal{D}) := \sum_{v \in V_{\mathcal{D}}} \mathrm{V}(v) \in \mathbb{Z}\,\mathrm{V}(r^M).$$

Let $\mathrm{A}(r^M) \subset \mathbb{Z}V(r^M)$ be the set of all elements in $\mathbb{N}\,\mathrm{V}(r^M) \subset \mathbb{Z}\,\mathrm{V}(r^M)$ such that for each rectangle $R \in \mathrm{R}(r^M)$, the number of occurrences of R coincides with the number of occurrences of the flipped rectangle $\iota(R)$. The set $\mathrm{A}(r^M) \subset \mathbb{Z}\,\mathrm{V}(r^M)$ is defined by a finite set of integral linear equations and inequalities. Furthermore, we will consider the corresponding rational version

$$\mathrm{A}^{\mathbb{Q}}(r^M) \subset \mathbb{Q}\,\mathrm{V}(r^M),$$

which is defined by the corresponding (in)equalities.

Lemma 6.8. *We have $\Phi(\Delta_1(r^M)) \subset \mathrm{A}(r^M)$. For every $a \in \mathrm{A}(r^M)$, there is a van Kampen diagram $\mathcal{D}$ such that $\Phi(\mathcal{D}) = 2 \cdot M \cdot a$.*

For the proof we need the following result, originally found in Neumann's article [Neu01].

Lemma 6.9 ([Neu01, Lemma 3.2]). *Let Σ be an oriented surface of positive genus. Let $N \in \mathbb{N}$ be an integer and suppose that for every boundary component of Σ there is a collection of degrees summing to N. Then there is a connected N-fold covering Σ' of Σ with prescribed degrees on the boundary components over each boundary component of Σ if and only if the prescribed number of boundary components of the cover has the same parity as $N \cdot \chi(\Sigma)$.*

Proof of Lemma 6.8. By construction, $\Phi(\Delta_1(r^M)) \subset \mathrm{A}(r^M)$. To prove that for every $a \in \mathrm{A}(r^M)$ we have that $2 \cdot M \cdot a \in \Phi(\Delta_1(r^m))$, we first assume that $M = 1$. In this case, every element of $\mathrm{A}(r)$ gives rise to a `lallop`-admissible van Kampen diagram over r as follows: Represent pods geometrically by stars. We then choose a matching for the rectangles related by flipping and use this to construct the 1-skeleton by gluing the corresponding rectangles of the pods with opposite orientations. We now use the ordering of the rectangles in the pods to glue in 2-disks (whose labels will be non-trivial powers of r^M because the rectangles in the pods are following each other). The resulting 2-dimensional CW-complex is homeomorphic to an orientable closed connected surface [MT01, p. 87] with a `lallop`-admissible van Kampen diagram $\mathcal{D}$ coming from the labels of the disks. It is easy to see that $\Phi(\mathcal{D}) = a$. By taking two copies of $\mathcal{D}$ we conclude the proposition for $M = 1$.

Now suppose that $M > 1$ and let $a \in \mathrm{A}(r^M)$. As $\mathrm{A}(r^M) = \mathrm{A}(r)$, we can first construct a reduced `lallop`-admissible van-Kampen diagram $\mathcal{D}$ for $\mathrm{A}(r)$. Let $\Sigma_{\mathcal{D}}$ be the associated surface to $\mathcal{D}$ and let $\Sigma_{\mathcal{D}}^{\partial}$ be the surface $\Sigma_{\mathcal{D}}$ with the van Kampen diagrams removed. Thus $\Sigma_{\mathcal{D}}^{\partial}$ is a surface with $|\mathcal{D}|$ many boundary components. We will use the following claim:

Claim 6.10 (coverings of surfaces). *Let Σ be a surface with p boundary components and let $M \in \mathbb{N}$. Then there is a covering Σ' of Σ, such that each boundary component has degree precisely M.*

Proof. We use Lemma 6.9. Suppose that Σ has p boundary components. Choose for every boundary of Σ two degree M boundaries as in the statement of Lemma 6.9. The resulting map has degree $N = 2 \cdot M$ and it has $2 \cdot p$ boundary components. This is an even number, just as $2 \cdot N \cdot \chi(\Sigma)$. Thus, there is a $2 \cdot M$-covering Σ' of Σ with the desired property. □

Using Claim 4.11, we see that there is a $2 \cdot M$-covering Σ'^{∂} of $\Sigma_{\mathcal{D}}^{\partial}$, where each of the $2 \cdot p$ boundaries maps with degree M. We may now fill in all the boundaries of Σ'^{∂} with disks and pull back the labels from $\Sigma_{\mathcal{D}}$. This describes a van Kampen diagram $\mathcal{D}'$ on a surface Σ' with $2|\mathcal{D}|$ disks over r^M. We see that every vertex of $\mathcal{D}$ corresponds to $2 \cdot M$ vertices of $\mathcal{D}'$ under the covering and that the labels around the vertices are identical. Thus, $\Phi(\mathcal{D}') = 2M \cdot a$, as claimed in the proposition. □

We will now express the (de)nominators in the computation of $\mathtt{lallop}(r^M)$ in Proposition 6.6 by suitable linear maps on $\mathbb{Z}\,\mathrm{V}(r^M)$. For a rectangle R, let $s_1(R), s_2(R) \in \{\pm 1\}$ denote the signs of the first and second component, respectively. We define the following $\mathbb{Z}$-linear maps:

$$
\begin{aligned}
&\nu\colon \mathbb{Z}\,\mathrm{V}(r^M) \to \mathbb{Q}\\
&[R_1,\dots,R_k] \mapsto \frac{1}{M\cdot|r|}\cdot\sum_{i=1}^{k}\bigl(s_1(R_i)+s_2(R_i)\bigr)\\
&\bar{\nu}\colon \mathbb{Z}\,\mathrm{V}(r^M) \to \mathbb{Q}\\
&[R_1,\dots,R_k] \mapsto \frac{1}{M\cdot|r|}\cdot 2\cdot k\\
&\lambda\colon \mathbb{Z}\,\mathrm{V}(r^M) \to \mathbb{Q}\\
&[R_1,\dots,R_k] \mapsto \frac{1}{2}\cdot(k-2)
\end{aligned}
$$

Lemma 6.11. *If $\mathcal{D}\in\Delta_1(r)$, then*

$$
\begin{aligned}
\nu\bigl(\Phi(\mathcal{D})\bigr) &= \sum_{D\in\mathcal{D}} n(D)\\
\bar{\nu}\bigl(\Phi(\mathcal{D})\bigr) &= \sum_{D\in\mathcal{D}} |n(D)|\\
\lambda\bigl(\Phi(\mathcal{D})\bigr) &= \frac{1}{2}\cdot\sum_{v\in V_{\mathcal{D}}}\bigl(\deg(v)-2\bigr).
\end{aligned}
$$

Proof. For ν and $\bar{\nu}$ we only need to note that every occurrence of r will be counted $|r|$ times when counting the two edges of all rectangles (with or without signs). As vertices of degree k in $\mathcal{D}$ are modelled by k-pods, the claim for λ follows. ☐

Proposition 6.12. *Let S be a set and let $r\in F(S)'\setminus\{e\}$. Then `lallop`(r) is the solution of an infinite linear programming problem that is defined over $\mathbb{Q}$.*

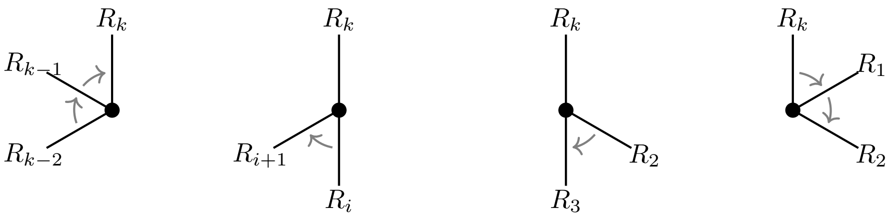
Proof. Using Proposition 6.6 and Lemma 6.8, we see that

$$
\begin{aligned}
\texttt{lallop}(r) &= \inf_{\mathcal{D}\in\Delta_1(r)} \frac{\sum_{v\in V_{\mathcal{D}}}\bigl(\deg(v)-2\bigr)-2\cdot\sum_{D\in\mathcal{D}}|n(D)|}{\sum_{D\in\mathcal{D}} n(D)}\\
&= \inf_{a\in\mathrm{A}(r)} 2\cdot\frac{\lambda(a)-\bar{\nu}(a)}{\nu(a)} = \inf_{\substack{a\in\mathrm{A}(r)\\ \nu(a)\geq 1}} 2\cdot\frac{\lambda(a)-\bar{\nu}(a)}{\nu(a)}
\end{aligned}
$$

The function on the right-hand side is invariant under scaling. Because the (de)nominator is Lipschitz continuous, we conclude that

$$
\texttt{lallop}(r) = \inf_{\substack{a\in\mathrm{A}^{\mathbb{Q}}(r)\\ \nu^{\mathbb{Q}}(a)\geq 1}} 2\cdot\frac{\lambda^{\mathbb{Q}}(a)-\bar{\nu}^{\mathbb{Q}}(a)}{\nu^{\mathbb{Q}}(a)},
$$

where $\nu^{\mathbb{Q}}$, $\bar{\nu}^{\mathbb{Q}}$, and $\lambda^{\mathbb{Q}}$ are the rational extensions of the corresponding functions on $\mathrm{A}(r)$. Hence, `lallop`(r) is the solution of an infinite fractional linear programming problem that is defined over $\mathbb{Q}$. Applying the Charnes-Cooper transformation, shows that `lallop`(r) is also the solution of a corresponding infinite linear programming problem that is defined over $\mathbb{Q}$. ☐



$[(R_k, R_{k-2}), R_{k-1}]$ $[\![(R_k, R_i), (R_k, R_{i+1})]\!]$ $[\![(R_k, R_2), (R_k, R_3)]\!]$ $[R_1, (R_k, R_2)]$

Figure 7: Breaking up a k-pod into (doubly) open tripods

6.4 Breaking up the pods: A polynomial algorithm

Finally, we reduce the linear programming problem of Proposition 6.12 to a finite linear programming problem (defined over $\mathbb{Q}$), which allows to compute `lallop` in polynomial time. This will be achieved by "breaking up" the elements in $\mathrm{V}(r^M)$ into finitely many types of pod-like configurations with two or three edges, which in turn are related by linear equations.

For this we first define abstract *pairs* of rectangles:

$$\mathrm{RP}(r^M) := \{(R_1, R_2) \mid R_1, R_2 \in \mathrm{R}(r^M)\}.$$

These rectangles will represent "open" parts in pod fragments. Furthemore, we define the following sets (Figure 7):

- $\mathrm{BP}(r^M) \subset \mathrm{V}(r^M)$, the set of all 2-pods, called *bipods*,
- $\mathrm{TP}(r^M) \subset \mathrm{V}(r^M)$, the set of all 3-pods, called *tripods*,
- $\mathrm{OTP}(r^M) := \mathrm{OTP}_1(r^M) \sqcup \mathrm{OTP}_2(r^M)$, where

  $\mathrm{OTP}_1(r^M) := \{[R, (R_1, R_2)] \mid R, R_1, R_2 \in \mathrm{R}(r^M),\ R_2 \text{ follows } R \text{ and } R \text{ follows } R_1\}$,
  $\mathrm{OTP}_2(r^M) := \{[(R_2, R_1), R] \mid R, R_1, R_2 \in \mathrm{R}(r^M),\ R_2 \text{ follows } R \text{ and } R \text{ follows } R_1\}$,

  the set of *open tripods* (they are open "between R_1 and R_2"),
- $\mathrm{DOTP}(r^M) = \{[\![(R, R_1), (R, R_2)]\!] \mid R, R_1, R_2 \in \mathrm{R}(r^M),\ R_2 \text{ follows } R_1\}$, the set of *doubly open tripods* (they are open "between R and R_1" and "between R_2 and R").

We will now break up pods into these building blocks (Figure 7). Let $\mathrm{B}(r^M)$ be the free $\mathbb{Z}$-module freely generated by the disjoint union

$$\mathrm{BP}(r^M) \sqcup \mathrm{TP}(r^M) \sqcup \mathrm{OTP}(r^M) \sqcup \mathrm{DOTP}(r^M)$$

and let $\mathrm{B}^{\mathbb{Q}}(r^M) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{B}(r^M)$. Clearly, $\mathrm{B}^{\mathbb{Q}}(r^M)$ is finite dimensional. We then consider the $\mathbb{Q}$-linear decomposition map

$$\begin{aligned} \Phi_0 \colon \mathbb{Q}\,\mathrm{V}(r^M) &\to \mathrm{B}^{\mathbb{Q}}(r^M) \\ [R_1, \dots, R_k] &\mapsto \begin{cases} [R_1, \dots, R_k] & \text{if } k \in \{2,3\} \\ [R_1, (R_k, R_2)] + \sum_{i=2}^{k-3} [\![(R_k, R_i), (R_k, R_{i+1})]\!] + [(R_k, R_{k-2}), R_{k-1}] & \text{if } k \geq 4. \end{cases} \end{aligned}$$

If $x \in \mathrm{V}(r^M)$ and $(R_1, R_2) \in \mathrm{RP}(r^M)$, then the number of occurrences of the pair (R_1, R_2) in $\Phi_0(x)$ in the first component of (doubly) open tripods coincides with the number of occurrences of (R_1, R_2) in the second component. We define the subset $\mathrm{A}_0(r^M) \subset \mathrm{B}(r^M)$ as the set of all elements such that

1. all coefficients are non-negative and
2. for every $R \in \mathrm{R}(r^M)$, the number of occurrences of R equals the number of occurrences of $\iota(R)$ and
3. for every $P \in \mathrm{RP}(r^M)$, the number of occurences of P in the first component equals the number of occurrences of P in the second component.

Furthermore, we consider the corresponding rational version $\mathrm{A}_0^{\mathbb{Q}}(r^M) \subset \mathrm{B}^{\mathbb{Q}}(r^M)$. By construction, $\Phi_0(\mathrm{A}^{\mathbb{Q}}(r^M)) \subset \mathrm{A}_0^{\mathbb{Q}}(r^M)$. Conversely, by matching up rectangle pairs in the first/second component, we see that $\Phi_0(\mathrm{A}^{\mathbb{Q}}(r^M)) = \mathrm{A}_0^{\mathbb{Q}}(r^M)$.

The functions λ, ν, and $\bar{\nu}$ can be translated to functions $\lambda_0, \nu_0, \bar{\nu}_0 \colon \mathrm{B}^{\mathbb{Q}}(r^M) \to \mathbb{Q}$ as follows: On elements of $\mathrm{BP}(r^M) \sqcup \mathrm{TP}(r^M)$, we define them as before.

- If $v = [R, (R_1, R_2)] \in \mathrm{OTP}_1(r^M)$ or $v = [(R_2, R_1), R] \in \mathrm{OTP}_2(r^M)$, then
$$\lambda_0(v) := \frac{1}{2}, \quad \bar{\nu}_0(v) := \frac{4}{|r| \cdot M}, \quad \nu_0(v) := \frac{s_1(R_1) + s_2(R) + s_1(R) + s_2(R_2)}{|r| \cdot M}.$$
- If $v = [\![(R, R_1), (R, R_2)]\!] \in \mathrm{DOTP}(r^M)$, then
$$\lambda_0(v) := \frac{1}{2}, \quad \bar{\nu}_0(v) := \frac{2}{|r| \cdot M}, \quad \nu_0(v) := \frac{s_1(R_1) + s_2(R_2)}{|r| \cdot M}.$$

A straightforward computation shows that
$$\lambda_0 \circ \Phi_0 = \lambda^{\mathbb{Q}}, \quad \bar{\nu}_0 \circ \Phi_0 = \bar{\nu}^{\mathbb{Q}}, \quad \nu_0 \circ \Phi_0 = \nu^{\mathbb{Q}}.$$

We can now complete the *proof of Theorem 6.1.3*: By Proposition 6.12 and the previous considerations, we have
$$\mathtt{lallop}(r^M) = \inf_{\substack{a \in \mathrm{A}_0^{\mathbb{Q}}(r^M) \\ \nu_0(a) \geq 1}} 2 \cdot \frac{\lambda_0(a) - \bar{\nu}_0(a)}{\nu_0(a)}.$$

Thus it suffices to solve the (fractional) linear programming problem on $\mathrm{A}_0^{\mathbb{Q}}(r^M)$. The linear cone $\mathrm{A}_0^{\mathbb{Q}}(r^M)$ has only polynomial dimension (namely of order $\mathcal{O}(|r|^5)$) and via the Charnes-Cooper transform this corresponds to a linear programming problem in the same order of dimension. In particular, $\mathtt{lallop}(r^M) \in \mathbb{Q}$, because everything is defined over $\mathbb{Q}$. There are now several available methods to compute the exact value of a linear programming problem, for example, the algorithm by Karmarkar [Kar84]. Thus, there is an algorithm that determines $\mathtt{lallop}(r^M)$ in polynomial time in $|r| \cdot M$). This finishes the proof of Theorem 6.1.

6.5 Examples

We implemented the algorithm skeleton `lallop` described in the previous section in MATLAB [HL19] and in Haskell [HL20b]. Thus, we have a polynomial time algorithm to compute lower bounds for $\|G_r\|$. Upper bounds, on the other hand, may be computed by finding an explicit van Kampen diagram on a surface for this relator r.

We will illustrate this by an example, whose stable commutator length was studied by Calegari [Cal09a, Section 4.3.5][Cal11]: For all $m \in \mathbb{N}_{\geq 2}$, we have

$$\operatorname{scl}_{\{\mathtt{a},\mathtt{b}\}}(r_m) = \frac{2m-3}{2m-2},$$

where $r_m := [\mathtt{a}, \mathtt{b}][\mathtt{a}, \mathtt{b}^{-m}]$.

An upper bound for $\|G_{r_m}\|$. Calegari [Cal09a, Section 4.3.5] described a van Kampen diagram $\mathcal{D}_m$ on a surface Σ_m of genus $m-1$ with $2m-2$ positive disks that are labelled by the word $r_m = [\mathtt{a}, \mathtt{b}][\mathtt{a}, \mathtt{b}^{-m}]$. Thus, using Proposition 4.2, we see that

$$\|G_{r_m}\| \leq \frac{-2 \cdot \chi(\Sigma_m)}{n(f_{\mathcal{D}_m}, \Sigma_{\mathcal{D}_m})} = \frac{2m-4}{m-1} = 4 \cdot \Big(\operatorname{scl}_{\{\mathtt{a},\mathtt{b}\}}(r) - \frac{1}{2}\Big).$$

We now describe the explicit van Kampen diagram for the case of $r_3 = \mathtt{a}\mathtt{b}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}\mathtt{b}^{-3}\mathtt{a}^{-1}\mathtt{b}^{-3}$; the resulting surface Σ_3 will have genus 2 and the van Kampen diagram will consist of four disks. Let us consider Figure 8, where x_i is glued to X_i for all $i \in \{1, \dots, 13\}$. We may check that the result is a surface of genus 2. We will label the edges by group elements. For an oriented edge x we will denote the label by $\omega(x)$. If X is the inverse of x then we require that $\omega(X) = \omega(x)^{-1}$. We set:

$$\begin{pmatrix} \omega(x_1) = \mathtt{bA} & \omega(x_2) = \mathtt{b} & \omega(x_3) = \mathtt{bb} & \omega(x_4) = \mathtt{AB} \\ \omega(x_5) = \mathtt{B} & \omega(x_6) = \mathtt{a} & \omega(x_7) = \mathtt{A} & \omega(x_8) = \mathtt{BA} \\ \omega(x_9) = \mathtt{Ab} & \omega(x_{10}) = \mathtt{Ab} & \omega(x_{11}) = \mathtt{BA} & \omega(x_{12}) = \mathtt{bbb} \\ \omega(x_{13}) = \mathtt{bbb} & & & \end{pmatrix}.$$

We see that this indeed describes an l^1-admissible van Kampen diagram for r_3. All of the disks D_1, D_2, D_3 and D_4 are cyclically labelled by r. For example the boundary of D_1 is (anticlockwise) $x_{10}, x_2, X_4, X_5, X_6, X_9, X_{13}$, where capitalization of letters corresponds to the inverse of the lower case label. Thus the boundary label is $\mathtt{Ab} \cdot \mathtt{b} \cdot \mathtt{ba} \cdot \mathtt{b} \cdot \mathtt{A} \cdot \mathtt{Ba} \cdot \mathtt{BBB} = \mathtt{AbbbabABaBBB}$, which is a cyclic conjugate of r. The result is a van Kampen diagram on a surface of genus 2.

A lower bound for $\|G_{r_m}\|$. On the other hand, we may compute lower bounds of $\|G_{r_m}\|$ using the algorithm described in the previous section [HL19, HL20b]. In this way, we obtained the values $\mathtt{lallop}(r_2) = 0$, $\mathtt{lallop}(r_3) = 1$, and $\mathtt{lallop}(r_4) = \frac{4}{3}$. We were not able to compute $\mathtt{lallop}(r_i)$ for larger i since the linear programming problem involved in the solution of `lallop` becomes too large. Using Theorem 6.1 and the upper bounds described above, we deduce that $\|G_{r_2}\| = 0$, $\|G_{r_3}\| = 1$ and $\|G_{r_4}\| = \frac{4}{3}$.

We summarise these computations in the following proposition:

Proposition 6.13. *Let $m \in \mathbb{N}_{\geq 2}$ and $r_m := [\mathtt{a}, \mathtt{b}][\mathtt{a}, \mathtt{b}^{-m}]$. Then*

$$\|G_{r_m}\| \leq \frac{2m-4}{m-1} = 4 \cdot \Big(\operatorname{scl}_{\{\mathtt{a},\mathtt{b}\}}(r) - \frac{1}{2}\Big).$$

For $m \in \{2, 3, 4\}$, we have equality, i.e., $\|G_{r_2}\| = 0, \|G_{r_3}\| = 1$, and $\|G_{r_4}\| = \frac{4}{3}$.

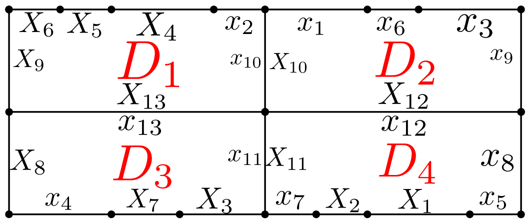


Figure 8: An l^1-admissible van Kampen diagram on a surface $\Sigma_{\mathcal{D}}$ with $\chi(\Sigma_{\mathcal{D}}) = -2$ and degree 4.

6.6 A counterexample to Question 1.2

Computing `lallop`(r) is polynomial in the length of r, yet it requires lots of time, even for small words. A much more feasable linear programming problem can be built by only considering 2-, 3-, and 4-pods. This gives an upper bound of `lallop` and drastically speeds up computation. Using such computations, we were able to find a counterexample to the Main Question 1.2:

Example 6.14 (counterexample)**.** Consider $v = \texttt{aaaabABAbaBAAbAB}$. This element satisfies $\mathrm{scl}_{\{\texttt{a},\texttt{b}\}}(v) = 5/8$. However, $\|G_v\| = 0$. To see this we will show that

$$\mathrm{cl}_{\{\texttt{a},\texttt{b}\}}(v^{-1} + v + v) = 1.$$

Indeed, we will compute that

$$\mathrm{cl}_{\{\texttt{a},\texttt{b}\}}(v^{-1} \cdot (t_1 v t_1^{-1}) \cdot (t_2 v t_2^{-1})) = 1$$

where $t_1 = \texttt{baBAAAA}$ and $t_2 = \texttt{baBaabABB}$. In fact, we see that

$$v^{-1} \cdot (t_1 v t_1^{-1}) \cdot (t_2 v t_2^{-1}) = d \cdot [g, h] \cdot d^{-1}$$

where

$$\begin{aligned} d &= \texttt{baBaabABaBaa}, \\ g &= \texttt{AAbbaBAAAAAbaBAA}, \text{ and} \\ h &= \texttt{bABaaaaaabABBaa}. \end{aligned}$$

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