CUBICAL SIMPLICIAL VOLUME OF 3-MANIFOLDS

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ABSTRACT. We prove that cubical simplicial volume of oriented closed 3-manifolds is equal to one fifth of ordinary simplicial volume.

1. INTRODUCTION

Simplicial volumes describe how difficult it is to represent the fundamental class of a given manifold in terms of cycles. Classically, simplicial volume was introduced by Gromov in terms of singular homology (defined via singular simplices) [14, 21]. Similarly, cubical simplicial volume is defined in terms of cubical singular homology.

Gromov asked whether simplicial volume and cubical simplicial volume are proportional in every dimension [15, 5.40]. This clearly holds in dimension 0 and 1, and is known to be true for surfaces [22].

We will prove that cubical simplicial volume $\| \cdot \|^{\square}$ and ordinary simplicial volume $\| \cdot \|$ are proportional in dimension 3:

Theorem 1.1. *Let M be an oriented closed 3-manifold. Then*

$$\|M\|^{\square} = \frac{1}{5} \cdot \|M\|.$$

In higher dimensions, the corresponding question is still an open problem.

As for ordinary simplicial volume, the computation of cubical simplicial volume in dimension 3 is based on decomposing the manifold in question into hyperbolic and Seifert fibred pieces. Therefore, the main steps of the proof are as follows:

- We establish the general lower bound $\|M\|^{\square} \geq 1/5 \cdot \|M\|$ by subdividing cubes into five simplices (Section 4.3).
- We show that Seifert fibred pieces have cubical simplicial volume equal to 0 (Section 5).
- We show that hyperbolic pieces have cubical simplicial volume equal to one fifth of ordinary simplicial volume (Section 6).
- We prove a suitable (sub-)additivity for cubical simplicial volume under gluings along tori (Section 7).

More precisely, we will prove more general versions of all these steps:

We develop a framework that allows for a convenient translation from ordinary simplicial volume to other, generalised, simplicial volumes, formulated in terms of so-called normed models of the singular chain complex (see Definition 3.1). All normed models lead to simplicial volumes that only differ by a multiplicative gap (Proposition 3.9). In particular, we discuss the inheritance of vanishing of generalised simplicial volumes, which yields the vanishing for Seifert fibred pieces as a special case (Proposition 5.2, Corollary 5.5). Furthermore, we show that simplicial volumes associated with so-called geometric normed models satisfy a sub-additivity with respect to gluings along UBC boundaries (Theorem 7.1); this includes, in particular, manifolds whose boundary components are tori. Hence, for all geometric normed models we obtain an upper bound for the corresponding generalised simplicial volume of 3-manifolds in terms of their hyperbolic pieces (Theorem 8.5).

Because there exist cubical normed models (Proposition 4.3), we can apply these observations to cubical simplicial volume. In addition, we investigate more efficient cubical normed models in dimension 3, which give the desired lower bounds in dimension 3 (Corollary 4.8).

Concerning the hyperbolic pieces, we calculate cubical simplicial volume for oriented hyperbolic manifolds of finite volume in dimension 3 (Corollary 6.14) and give general, geometric lower and upper bounds for the proportionality constant in all dimensions (Section 6).

Organisation of this article. In Section 2, we recall the definition of classical simplicial volume and cubical simplicial volume. The language of normed models is developed in Section 3 and cubical models are studied in Section 4. Section 5 treats the Seifert case and Section 6 treats the hyperbolic case. In Section 7, we prove sub-additivity of generalised simplicial volumes under certain gluings, and Section 8 contains the proof of Theorem 1.1 and its generalisation to geometric normed models.

2. SIMPLICIAL VOLUME AND CUBICAL SIMPLICIAL VOLUME

We start with a review of Gromov's simplicial volume [14, 21] and some basic properties, as well as the definition of cubical simplicial volume.

2.1. **Classical simplicial volume.** We denote the singular chain complex with $\mathbb{R}$-coefficients, based on singular simplices, by $C_*(\, \cdot \, ; \mathbb{R})$ and the corresponding singular homology by $H_*(\, \cdot \, ; \mathbb{R})$.

Let M be an oriented compact n-manifold (with possibly empty boundary ∂M). The corresponding $\mathbb{R}$-fundamental class in $H_n(M, \partial M; \mathbb{R})$ is denoted by $[M, \partial M]_{\mathbb{R}}$. If $\partial M = \varnothing$, then we write $[M]_{\mathbb{R}}$ instead.

Definition 2.1 (simplicial volume, absolute case)**.** The *simplicial volume* of an oriented closed n-manifold is defined as

$$\|M\| := \inf\big\{ |c|_1 \,\big|\, c \in C_n(M; \mathbb{R}), \partial c = 0, [c] = [M]_{\mathbb{R}} \in H_n(M; \mathbb{R}) \big\},$$

where $|\cdot|_1$ denotes the ℓ^1-norm on $C_n(M;\mathbb{R})$ with respect to the basis of singular n-simplices in M.

In general, the explicit computation of simplicial volume is rather hard. Known results include the calculation of simplicial volume of hyperbolic manifolds [14, 31] and for manifolds that are locally isometric to the product of two hyperbolic planes [4].

2.2. **Additivity of simplicial volume.** In dimension 3, simplicial volume can be computed by cutting the manifold into smaller, well understood, pieces. To this end, it is necessary to consider the simplicial volume of manifolds with boundary and of open manifolds and to prove additivity under certain gluings. We now recall this calculation in more detail:

Definition 2.2 (simplicial volume, relative case). The *(relative) simplicial volume* of an oriented compact manifold M is defined as

$$\|M, \partial M\| := \inf\{|c|_1 \mid c \in C_n(M, \partial M; \mathbb{R}), \partial c = 0, [c] = [M, \partial M]_{\mathbb{R}}\},$$

where $|\cdot|_1$ denotes the quotient norm on $C_*(M, \partial M; \mathbb{R})$ induced by the ℓ^1-norm on $C_*(M; \mathbb{R})$.

If M is an oriented manifold without boundary, the fundamental class and the simplicial volume of M admit analogous definitions in the context of locally finite homology [14, 20].

Definition 2.3 (simplicial volume, non-compact case). The *(locally finite) simplicial volume* of an open n-manifold M is

$$\|M\|_{\mathrm{lf}} := \inf\{|c|_1 \mid c \in C_n^{\mathrm{lf}}(M; \mathbb{R}), \partial c = 0, [c] = [M]_{\mathbb{R}}^{\mathrm{lf}} \in H_n^{\mathrm{lf}}(M; \mathbb{R})\} \in [0, \infty],$$

where $[M]_{\mathbb{R}}^{\mathrm{lf}}$ denotes the fundamental class of M in locally finite homology $H_n^{\mathrm{lf}}(M; \mathbb{R})$ and $|\cdot|_1$ denotes the ℓ^1-norm on locally finite chains $C_n^{\mathrm{lf}}(M; \mathbb{R})$.

For example, an analysis of boundaries [or restrictions] of relative [or locally finite] fundamental cycles gives the following restrictions on simplicial volume of the boundary [14, 20, 19]:

Proposition 2.4 (simplicial volume of the boundary). *Let M be an oriented compact n-manifold and let $M^\circ := M \setminus \partial M$.*

(1) *Then $\|\partial M\| \le (n+1) \cdot \|M\|$.*
(2) *If $\|M^\circ\|_{\mathrm{lf}} < \infty$, then $\|\partial M\| = 0$.*

Remark 2.5. Actually, in this situation, the following stronger inequality $\|\partial M\| \le (n-1) \cdot \|M\|$ holds [7, Lemma 2.3, Proposition 3.1].

However, in general it is *not* true that $\|M^\circ\|_{\mathrm{lf}} = \|M, \partial M\|$, as can be seen by looking at $M := [0, 1]$ [14, 20].

In the case with boundary, the exact value of simplicial volume was computed for products of surfaces with the interval and for compact 3-manifolds obtained by adding 1-handles to Seifert manifolds [7]. Further computations can be obtained by taking connected sums or gluing along π_1-injective boundary components with amenable fundamental groups, relying on the following result.

Theorem 2.6 (additivity of simplicial volume under amenable gluings). *Let $k, n \in \mathbb{N}_{\geq 2}$, let $M_1, \ldots, M_k$ be oriented compact n-manifolds, and suppose that the fundamental group of every boundary component of every M_j is amenable. Let M be a manifold obtained by gluing $M_1, \ldots, M_k$ along (some of) their boundary components. Then*

$$\|M, \partial M\| \leq \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

In addition, if the gluings defining M are compatible, then

$$\|M, \partial M\| = \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

Here, a gluing $f : S_1 \to S_2$ of two boundary components $S_i \subset \partial M_{j_i}$ is called *compatible* if $\pi_1(f)(K_1) = K_2$ where K_i is the kernel of the map $\pi_1(S_i) \to \pi_1(M_{j_i})$ induced by the inclusion.

This statement is originally due to Gromov [14, p. 58], and is also discussed by Kuessner [18]. A complete proof was given in terms of a more algebraic framework [6, Theorem 3].

In particular, additivity allows to compute simplicial volume of closed 3-manifolds with help of geometrization: In view of additivity, the simplicial volume of a closed 3-manifold equals the sum of the simplicial volumes of its hyperbolic pieces [30].

As last step, one needs to calculate the simplicial volume of the hyperbolic pieces [14, 31, 11, 12, 13, 5]:

Theorem 2.7 (simplicial volume of hyperbolic manifolds). *Let M be an oriented compact n-manifold whose interior M° admits a complete hyperbolic metric of finite volume. Then*

$$\|M^\circ\|_{\mathrm{lf}} = \|M, \partial M\| = \frac{\mathrm{vol}(M^\circ)}{v_n^\triangle},$$

where $v_n^\triangle$ is the (finite!) volume of ideal, regular n-simplices in $\overline{\mathbb{H}^n}$.

Hence, in total we obtain [30]:

Theorem 2.8 (simplicial volume of 3-manifolds). *Let M be an oriented closed 3-manifold. Then*

$$\|M\| = \sum_{j=1}^{k} \frac{\mathrm{vol}\,(N_j^\circ)}{v_3^\triangle},$$

where $N_1, \ldots, N_k$ are the hyperbolic pieces of M (see Theorem 8.4).

2.3. **Cubical simplicial volume.** We will now recall the definition of cubical singular homology [25, 10] and of cubical simplicial volume. Cubical singular homology is defined in terms of standard cubes instead of standard simplices: For $n \in \mathbb{N}$ let $\square^n := [0, 1]^n$ be the *standard n-cube*. If X is a topological space, then continuous maps of type $\square^n \longrightarrow X$ are called *singular n-cubes of X*. The geometric/combinatorial boundary of $\square^n$ consists of $2 \cdot n$ cubical faces. More precisely, for $j \in \{1, \ldots, n\}$ and $i \in \{0, 1\}$ we define the *(j, i)-face of* $\square^n$ by

$$\square_{(j,i)}^n : \square^{n-1} \longrightarrow \square^n$$

$$x \longmapsto (x_1, \ldots, x_{j-1}, i, x_j, \ldots, x_{n-1}).$$

Correspondingly, for a singular n-cube $c\colon \square^n \longrightarrow X$ in a space X we define the cubical boundary by

$$\partial c := \sum_{j=1}^{n} (-1)^j \cdot \left(c \circ \square^n_{(j,0)} - c \circ \square^n_{(j,1)} \right).$$

This leads to a chain complex $Q_*(X; \mathbb{R})$ of cubical singular chains with $\mathbb{R}$-coefficients. A singular n-cube c is *degenerate* if it is independent of one of the coordinates, i.e., if there exists a $j \in \{1, \ldots, n\}$ such that for all $t \in [0,1]$ and all $x \in \square^{n-1}$ we have

$$c(x_1, \ldots, x_{j-1}, t, x_j, \ldots, x_{n-1}) = c(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{n-1}).$$

Dividing out the subcomplex $D_*(X; \mathbb{R})$ generated by degenerate singular cubes leads to the *cubical chain complex*

$$C_*^{\square}(X; \mathbb{R}) := Q_*(X; \mathbb{R}) / D_*(X; \mathbb{R})$$

and hence to *cubical singular homology* $H_*^{\square}(X; \mathbb{R})$ (which admits a natural extension to a functor). Dividing out degenerate singular cubes is necessary in order for cubical singular homology of a point to be concentrated in degree 0.

We write $|\cdot|_1^{\square}$ for the norm on $C_*^{\square}(\,\cdot\,; \mathbb{R})$ induced from the ℓ^1-norm on $Q_*(\,\cdot\,; \mathbb{R})$ with respect to the basis consisting of singular cubes; notice that this norm coincides with the ℓ^1-norm given by the basis of all non-degenerate singular cubes.

It is well known that there is a canonical natural (both in spaces and coefficients) isomorphism $H_*^{\square} \longrightarrow H_*$ [10, Theorem V] between cubical singular homology and ordinary singular homology. However, in general this isomorphism is *not* isometric with respect to the corresponding ℓ^1-semi-norms.

If M is an oriented closed manifold, we denote the corresponding cubical $\mathbb{R}$-fundamental class by $[M]_{\mathbb{R}}^{\square} \in H_*^{\square}(M; \mathbb{R})$.

Definition 2.9 (cubical simplicial volume, absolute case)**.** The *cubical simplicial volume* of an oriented closed n-manifold M is defined as

$$\|M\|^{\square} := \inf \left\{ |c|_1^{\square} \;\middle|\; c \in C_n^{\square}(M; \mathbb{R}), \partial c = 0, [c] = [M]_{\mathbb{R}}^{\square} \in H_n^{\square}(M; \mathbb{R}) \right\}.$$

It is not hard to show that for every $n \in \mathbb{N}$ there are $T_n, V_n \in \mathbb{R}_{\geq 0}$ such that for all oriented closed n-manifolds M we have

$$T_n \cdot \|M\| \leq \|M\|^{\square} \leq V_n \cdot \|M\|$$

(see Corollary 4.7 below). However, inheritance results such as (sub-)additivity cannot be derived directly from these estimates.

For the divide and conquer approach to computing cubical simplicial volume of 3-manifolds we will need relative and locally finite versions of cubical simplicial volume and corresponding (sub-)additivity statements. While we could this literally in the same way as in the simplicial case, we prefer to develop these tools in slightly larger generality in the subsequent sections.

3. Normed models of the singular chain complex

We will now develop a framework that allows for a convenient translation from ordinary simplicial volume to other simplicial volumes, such as cubical simplicial volume.

3.1. **Basic terminology for normed models.** Let $\mathsf{Ch}_{\mathbb{R}}^{\mathsf{n}}$ be the category of normed $\mathbb{R}$-chain complexes, i.e., the category of chain complexes whose chain modules are normed $\mathbb{R}$-vector spaces, whose boundary maps are continuous $\mathbb{R}$-linear maps and whose chain maps consist of $\mathbb{R}$-linear maps of norm at most 1. Moreover, we write Top for the category of topological spaces and continuous maps.

Definition 3.1 (normed models of the singular chain complex).
- A *functorial normed chain complex* is a functor $\mathsf{Top} \longrightarrow \mathsf{Ch}_{\mathbb{R}}^{\mathsf{n}}$.
- Let $F, F' \colon \mathsf{Top} \longrightarrow \mathsf{Ch}_{\mathbb{R}}^{\mathsf{n}}$ be functorial normed chain complexes. A *natural continuous chain map* $F \Longrightarrow F'$ is a natural transformation $\varphi \colon F \Longrightarrow F'$ viewed in the category of $\mathbb{R}$-chain complexes, where for every space X and $n \in \mathbb{N}$ the linear map $\varphi_n^X \colon F_n(X) \longrightarrow F_n'(X)$ is continuous (but not necessarily of norm at most 1).
- Let $\varphi, \varphi' \colon F \Longrightarrow F'$ be natural continuous chain maps between functorial normed chain complexes. A *natural normed chain homotopy* $h \colon \varphi \simeq \varphi'$ is a family $(h^X \colon \varphi^X \simeq \varphi'^X)_{X \in \mathrm{Ob}(\mathsf{Top})}$ of $\mathbb{R}$-linear chain homotopies such that for all spaces X and all $n \in \mathbb{N}$ the linear map $h_n^X \colon F_n(X) \longrightarrow F_{n+1}'(X)$ is continuous.
- Natural continuous chain maps $\varphi \colon F \Longrightarrow F'$ and $\psi \colon F' \Longrightarrow F$ are *mutually inverse* if there exist natural normed chain homotopies $\psi \circ \varphi \simeq \mathrm{id}_F$ and $\varphi \circ \psi \simeq \mathrm{id}_{F'}$. In this case, φ and ψ are *natural normed chain homotopy equivalences*.
- A *normed model of the singular chain complex* is a triple (F, φ, ψ), where $F \colon \mathsf{Top} \longrightarrow \mathsf{Ch}_{\mathbb{R}}^{\mathsf{n}}$ is a functorial normed chain complex, and where $\varphi \colon F \Longrightarrow C_*(\,\cdot\,;\mathbb{R})$ and $\psi \colon C_*(\,\cdot\,;\mathbb{R}) \Longrightarrow F$ are mutually inverse natural normed chain homotopy equivalences. Here, $C_*(\,\cdot\,;\mathbb{R})$ is equipped with the ℓ^1-norm.

For example, $(C_*(\,\cdot\,;\mathbb{R}), \mathrm{id}_{C_*(\,\cdot\,;\mathbb{R})}, \mathrm{id}_{C_*(\,\cdot\,;\mathbb{R})})$ is a normed model of the singular chain complex.

Example 3.2 (ignoring degenerate singular simplices). In analogy with the cubical case, we can also look at the singular chain complex modulo degenerate simplices: For a topological space X, let $E_*(X;\mathbb{R}) \subset C_*(X;\mathbb{R})$ be the subcomplex generated by all degenerate singular simplices. Then we define

$$C_*^{\triangle}(\,\cdot\,;\mathbb{R}) := C_*(\,\cdot\,;\mathbb{R})/E_*(\,\cdot\,;\mathbb{R}) \colon \mathsf{Top} \longrightarrow \mathsf{Ch}_{\mathbb{R}}^{\mathsf{n}},$$

where we equip $C_*^{\triangle}(\,\cdot\,;\mathbb{R})$ with the norm $|\cdot|_1^{\triangle}$ induced from the ℓ^1-norm (this coincides with the ℓ^1-norm with respect to the basis of non-degenerated singular simplices).

We will now construct a normed model of $C_*(\,\cdot\,;\mathbb{R})$ on $C_*^{\triangle}(\,\cdot\,;\mathbb{R})$: Let $\psi \colon C_*(\,\cdot\,;\mathbb{R}) \Longrightarrow C_*^{\triangle}(\,\cdot\,;\mathbb{R})$ be the natural continuous chain map given by

the canonical projection. Conversely, we consider the map

$$\varphi_n^X \colon C_n^{\triangle}(X;\mathbb{R}) \longrightarrow C_n(X;\mathbb{R})$$

$$[\sigma \colon \Delta^n \to X] \longmapsto \frac{1}{(n+1)!} \cdot \sum_{\pi \in \mathrm{Isom}(\Delta^n)} (-1)^{\mathrm{sgn}\,\pi} \cdot \sigma \circ \pi$$

where $\mathrm{sgn}\,\pi \in \{0,1\}$ encodes whether π is orientation preserving or not. It is not hard to show that φ_n^X defines a well-defined natural chain map that is continuous in every degree. Thus, we obtain a natural continuous chain map $\varphi \colon C_*^{\triangle}(\,\cdot\,;\mathbb{R}) \Longrightarrow C_*(\,\cdot\,;\mathbb{R})$.

A standard argument (e.g., via acyclic models [10]) shows that φ and ψ are mutually inverse natural normed chain homotopy equivalences. Hence, $(C_*^{\triangle}(\,\cdot\,;\mathbb{R}), \varphi, \psi)$ is a normed model of $C_*(\,\cdot\,;\mathbb{R})$. Notice that, by construction, $\|\varphi_n^X\| \leq 1$ and $\|\psi_n^X\| \leq 1$ for all spaces X and all $n \in \mathbb{N}$.

Cubical normed models of the singular chain complex will be constructed in Section 4.

3.2. **Extending normed models.** Normed models of $C_*(\,\cdot\,;\mathbb{R})$ also lead to corresponding notions for manifolds with boundary and for open manifolds: Let Top^2 be the category of pairs of topological spaces (and maps of pairs) and let Top_p be the category of topological spaces and proper continuous maps. Moreover, let $\mathsf{Ch}_\mathbb{R}^{\mathsf{sn}}$ be the category of semi-normed $\mathbb{R}$-chain complexes and let $\mathsf{Ch}_\mathbb{R}^{\mathsf{sn},\infty}$ be the category of semi-normed $\mathbb{R}$-chain complexes where also the value ∞ is allowed for the norms. The notions of functorial normed chain complexes etc. from Definition 3.1 easily generalise to these variations.

We first extend normed models to the relative case:

Definition 3.3 (normed models, relative case). Let (F, φ, ψ) be a normed model of $C_*(\,\cdot\,;\mathbb{R})$ with norm $|\cdot|^F$. For a pair (X, A) of spaces with inclusion $i \colon A \hookrightarrow X$ we set

$$F(X, A) := F(X) \,/\, F(i)(F(A)),$$

which we endow with the quotient semi-norm $|\cdot|^F$ of $F(X)$. Hence, $F(\cdot, \cdot)$ defines a functor from Top^2 to $\mathsf{Ch}_\mathbb{R}^{\mathsf{sn}}$. Moreover, φ^X and ψ^X induce well-defined natural continuous chain maps $\varphi^{(X,A)} \colon F(X, A) \longrightarrow C_*(X, A;\mathbb{R})$ and $\psi^{(X,A)} \colon C_*(X, A;\mathbb{R}) \longrightarrow F(X, A)$ (which are also naturally mutually inverse chain homotopy equivalences through degree-wise continuous chain homotopies).

Example 3.4. Notice that for the trivial normed model $(C_*(\,\cdot\,;\mathbb{R}), \mathrm{id}, \mathrm{id})$ of $C_*(\,\cdot\,;\mathbb{R})$ this definition yields the same normed structure on the relative singular chain complex as the one used in the definition of relative simplicial volume (Definition 2.2).

Definition 3.5 (locally finite normed models). Let (F, φ, ψ) be a normed model of $C_*(\,\cdot\,;\mathbb{R})$. For a topological space X we let $K(X)$ denote the set of all compact subspaces of X (directed by inclusion). For X we then define

$$F^{\mathrm{lf}}(X) := \varprojlim_{K \in K(X)} F(X, X \setminus K),$$

where the inverse limit is taken with respect to the maps induced by the inclusions. For $c = (c_K)_{K \in K(X)} \in F_n^{\mathrm{lf}}(X)$ we set

$$|c|^F := \inf_{\bar{c} \in A(c)} \lim_{K \in K(X)} |\bar{c}_K|^F \in [0, \infty],$$

where

$$A(c) := \big\{ (\bar{c}_K \in F_n(X))_{K \in K(X)} \mid \bar{c}_K \text{ represents } c_K \text{ in } F_n(X, X \setminus K)$$
$$\text{and } (\bar{c}_K)_{K \in K(X)} \text{ is } |\cdot|^F\text{-Cauchy} \big\}$$

is the (possibly empty) set of all Cauchy families associated with c. This defines a (potentially infinite) semi-norm on $F^{\mathrm{lf}}(X)$. Hence, $F^{\mathrm{lf}}(\cdot)$ defines a functor from $\mathsf{Top}_{\mathsf{p}}$ to $\mathsf{Ch}_{\mathbb{R}}^{\mathrm{sn},\infty}$. Moreover, φ^X and ψ^X induce well-defined natural continuous chain maps $\varphi^{\mathrm{lf},X} : F^{\mathrm{lf}}(X) \longrightarrow C_*^{\mathrm{lf}}(X; \mathbb{R})$ and $\psi^{\mathrm{lf},X} : C_*^{\mathrm{lf}}(X; \mathbb{R}) \longrightarrow F^{\mathrm{lf}}(X)$ (which are also naturally mutually inverse chain homotopy equivalences through degree-wise continuous chain homotopies).

Notice that we have

$$|c|^F \geq \sup_{K \in K(X)} |c_K|^F$$

for all $c \in F^{\mathrm{lf}}(X)$ and that equality holds for the trivial model $C_*(\,\cdot\,; \mathbb{R})$ and all cubical models. It is not clear whether equality holds for all normed models of $C_*(\,\cdot\,; \mathbb{R})$, but the above definition seems to be more suitable for geometric applications.

Example 3.6. For the trivial normed model $(C_*(\,\cdot\,; \mathbb{R}), \mathrm{id}, \mathrm{id})$ of $C_*(\,\cdot\,; \mathbb{R})$ the above definition coincides with the locally finite singular complex and yields the same normed structure on the locally finite singular chain complex as the one used in the definition of simplicial volume of open manifolds (Definition 2.3).

3.3. **Generalised simplicial volume.** Normed models of the singular chain complex induce semi-norms on singular homology:

Definition 3.7 (induced semi-norm on homology). Let (F, φ, ψ) be a normed model of $C_*(\,\cdot\,; \mathbb{R})$. For a pair (X, A) of spaces we define

$$\|\cdot\|^F \colon H_n(X, A; \mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$$
$$\alpha \longmapsto \inf\big\{ |c|^F \mid c \in F_n(X, A), \partial c = 0, [\varphi_n^{(X,A)}(c)] = \alpha \big\}.$$

Similarly, for a topological space X we define

$$\|\cdot\|^F \colon H_n^{\mathrm{lf}}(X; \mathbb{R}) \longrightarrow [0, \infty]$$
$$\alpha \longmapsto \inf\big\{ |c|^F \mid c \in F_n^{\mathrm{lf}}(X), \partial c = 0, [\varphi_n^{\mathrm{lf},X}(c)] = \alpha \big\}.$$

In particular, we obtain corresponding simplicial volumes.

Definition 3.8 (generalised simplicial volume). Let (F, φ, ψ) be a normed model of $C_*(\,\cdot\,; \mathbb{R})$, and let M be an oriented compact n-manifold. Then the *F-simplicial volume of M* is defined as

$$\|M, \partial M\|^F := \big\| H_n(\psi_n^{(M,\partial M)})([M, \partial M]_{\mathbb{R}}) \big\|^F.$$

If $\partial M = \varnothing$, we abbreviate $\|M\|^F := \|M, \partial M\|^F$. If M is an oriented n-manifold without boundary, then we define

$$\|M\|_{\mathrm{lf}}^F := \|H_n(\psi_n^{\mathrm{lf},M})([M]_{\mathbb{R}}^{\mathrm{lf}})\|^F.$$

One should note that the F-simplicial volume, in general, also depends on φ and ψ; however, in the cubical case, we will add a normalisation condition that removes this ambiguity (Remark 4.2).

For simplicity, in the compact case, cycles in $F_n(M, \partial M)$ that represent $H_n(\psi_n^{(M,\partial M)})([M, \partial M]_{\mathbb{R}})$ are also called *(relative) F-fundamental cycles*; in the non-compact case, cycles in $F_n^{\mathrm{lf}}(M)$ that represent $H_n(\psi_n^{\mathrm{lf},M})([M]_{\mathbb{R}}^{\mathrm{lf}})$ are also called *locally finite F-fundamental cycles*.

It is not hard to show that all these simplicial volumes are relative/proper homotopy invariants of the manifolds in question. At this point it is essential that normed models map (proper) continuous maps to chain maps of norm at most 1.

By the very definitions, all generalised norm chain complexes yield equivalent simplicial volumes in the following sense:

Proposition 3.9 (equivalence of generalised simplicial volumes)**.** *Let (F, φ, ψ) be a normed model for $C_*(\,\cdot\,;\mathbb{R})$, and let $n \in \mathbb{N}$.*

(1) *Then for all oriented compact n-manifolds M we have*

$$\|M, \partial M\|^F \leq \|\psi_n^{\Delta^n}\| \cdot \|M, \partial M\|,$$

$$\|M, \partial M\| \leq \|\varphi_n^{(M,\partial M)}\| \cdot \|M, \partial M\|^F.$$

(2) *For all oriented n-manifolds M without boundary we have*

$$\|M\|_{\mathrm{lf}}^F \leq \|\psi_n^{\Delta^n}\| \cdot \|M\|_{\mathrm{lf}},$$

$$\|M\|_{\mathrm{lf}} \leq \|\varphi_n^{\mathrm{lf},M}\| \cdot \|M\|_{\mathrm{lf}}^F.$$

Proof. Ad 1. The estimate $\|M, \partial M\| \leq \|\varphi_n^{(M,\partial M)}\| \cdot \|M, \partial M\|^F$ easily follows from the definitions. Conversely, let $c = \sum_{j=1}^k a_j \cdot \sigma_j \in C_n(M; \mathbb{R})$ be a chain with $\partial c \in C_{n-1}(\partial M; \mathbb{R})$ that represents $[M, \partial M]_{\mathbb{R}}$. Then

$$\left|\psi_n^{(M,\partial M)}[c]\right|^F \leq \sum_{j=1}^k |a_j| \cdot \left|\psi_n^M(\sigma_j \circ \mathrm{id}_{\Delta^n})\right|^F = \sum_{j=1}^k |a_j| \cdot \left|F(\sigma_j)(\psi_n^{\Delta^n}(\mathrm{id}_{\Delta^n}))\right|^F$$

$$\leq \sum_{j=1}^k |a_j| \cdot 1 \cdot \|\psi_n^{\Delta^n}\| \cdot 1$$

$$\leq \|\psi_n^{\Delta^n}\| \cdot |c|_1.$$

Hence, $\|M, \partial M\|^F \leq \|\psi_n^{\Delta^n}\| \cdot \|M, \partial M\|$.

Ad 2. This follows via the same arguments as in the first part, keeping in mind that continuous linear maps map Cauchy families to Cauchy families. $\square$

Example 3.10. For example, the normed model of $C_*(\,\cdot\,;\mathbb{R})$ on $C_*^{\triangle}(\,\cdot\,;\mathbb{R})$ given in Example 3.2 leads to the ordinary simplicial volume because the corresponding chain homotopy equivalences all have norm at most 1.

In particular, vanishing of generalised simplicial volumes is equivalent for all normed models of $C_*(\cdot ; \mathbb{R})$. In contrast, it is not clear that sub-additivity properties of type $\|M\|^F \leq \|N_1\|^F + \|N_2\|^F$ are also inherited in the same way.

The whole point of introducing the framework of normed models of the singular chain complex and the discussion in Section 7 is that we can in fact prove inheritance of sub-additivity in sufficiently benign situations.

3.4. **The uniform boundary condition.**
We now review the uniform boundary condition [26], which will play an important role in our gluing constructions below.

Definition 3.11 (UBC). Let $q \in \mathbb{N}$.

  – A normed chain complex $(C, | \cdot |)$ satisfies the *uniform boundary condition in degree q* (q-UBC, for short) if there exists a $\kappa \in \mathbb{R}_{\geq 0}$ with the following property: For all $z \in \operatorname{im} \partial_{q+1} \subset C_q$ there is a $b \in C_{q+1}$ with
  $$\partial_{q+1}(b) = z \quad \text{and} \quad |b| \leq \kappa \cdot |z|.$$
  – Let $F \colon \mathsf{Top} \longrightarrow \mathsf{Ch}_{\mathbb{R}}^n$ be a functorial normed chain complex. A space X *satisfies q-UBC with respect to F* if the normed chain complex $F(X)$ satisfies q-UBC.

For example, the singular chain complex $C_*(X; \mathbb{R})$ satisfies the uniform boundary condition in all degrees if X is a connected CW-complex with amenable fundamental group [26]. This includes, in particular, all tori and all simply connected spaces.

Proposition 3.12 (UBC inheritance). *Let (F, φ, ψ) be a normed model of the singular chain complex, let $q \in \mathbb{N}$, and let X be a topological space. Then X satisfies q-UBC with respect to the trivial model $C_*(\cdot ; \mathbb{R})$ if and only if X satisfies q-UBC with respect to F.*

Proof. This is a straightforward calculation: Because the situation basically is symmetric, we only show that if $C_*(X; \mathbb{R})$ satisfies q-UBC, then also $F(X)$ satisfies q-UBC.

Let $\kappa \in \mathbb{R}_{\geq 0}$ be a constant witnessing that $C_*(X; \mathbb{R})$ satisfies q-UBC and let $z \in \partial(F_{q+1}(X))$. Then $z' := \varphi_q^X(z)$ lies in $\partial(C_{q+1}(X))$ and in view of the uniform boundary condition there exists a chain $b' \in C_{q+1}(X; \mathbb{R})$ with
$$\partial b' = z' \quad \text{and} \quad |b'|_1 \leq \kappa \cdot |z'|_1.$$

Let $h_* \colon \psi^X \circ \varphi^X \simeq \operatorname{id}_{F(X)}$ be a chain homotopy that is continuous in every degree and let
$$b := \psi_{q+1}^X(b') - h_q(z) \in F_{q+1}(X).$$
Then
$$\partial b = \psi_q^X(z') - \psi_q^X \circ \varphi_q^X(z) + z - h_{q-1}(\partial(z)) = z$$
and
$$|b|^F \leq \left(\|\psi_{q+1}^X\| \cdot \kappa \cdot \|\varphi_q^X\| + \|h_q\| \right) \cdot |z|^F.$$
Hence, $F(X)$ satisfies q-UBC.

Alternatively, one can also use the characterisation of the uniform boundary condition in terms of bounded cohomology [26] to prove this inheritance statement (because normed models are set up in such a way that they yield naturally continuously isomorphic bounded cohomology theories). $\square$

Hence, we can just say that a space satisfies q-UBC without specifying a normed model of the singular chain complex (the UBC constants, however, in general will depend on the chosen model).

4. CUBICAL MODELS OF THE SINGULAR CHAIN COMPLEX

We will now give general bounds between ordinary and cubical simplicial volume, based on natural chain homotopy equivalences with controlled norms. We formulate these results in terms of normed models of the singular chain complex developed in Section 3.

4.1. **Cubical models.** We introduce the following shorthand:

Definition 4.1 (cubical model). A *cubical normed model of* $C_*(\ \cdot\ ;\mathbb{R})$ is a normed model (F, φ, ψ) of the singular chain complex with $F = C_*^{\square}(\ \cdot\ ;\mathbb{R})$ that satisfies the normalisation

$$\varphi_0^{\square^0}(\mathrm{id}_{\square^0}) = (1 \mapsto 0)\colon \Delta^0 \longrightarrow \square^0.$$

This normalisation rules out unwanted scaling:

Remark 4.2. Let (F, φ, ψ) be a cubical normed model of $C_*(\ \cdot\ ;\mathbb{R})$. Then the acyclic models theorem [10], the normalisation condition, and the local characterisation of fundamental classes show that for all oriented manifolds the cubical fundamental class and the ordinary fundamental class are mapped to each other by the given chain homotopy equivalences. Thus, the F-simplicial volume coincides with cubical simplicial volume (also for the straightforward adaptions to the relative and the non-compact case); so, the slightly sloppy term "F-simplicial volume" (without reference to φ or ψ) is not ambiguous in this case.

4.2. **Construction of cubical models.** As next step, we show that cubical models indeed exist.

Proposition 4.3. *There exist cubical normed models of* $C_*(\ \cdot\ ;\mathbb{R})$.

Proof. Via acyclic models, Eilenberg and Mac Lane [10] constructed a natural chain map $C_*^{\square}(\ \cdot\ ;\mathbb{R}) \Longrightarrow C_*^{\triangle}(\ \cdot\ ;\mathbb{R})$ that satisfies the normalisation condition ($C_*^{\triangle}(\ \cdot\ ;\mathbb{R})$ was defined in Example 3.10). In combination with the natural chain homotopy equivalence $C_*^{\triangle}(\ \cdot\ ;\mathbb{R}) \simeq C_*(\ \cdot\ ;\mathbb{R})$, this produces a natural chain map $C_*^{\square}(\ \cdot\ ;\mathbb{R}) \Longrightarrow C_*(\ \cdot\ ;\mathbb{R})$. Applying Lemma 4.4 below then proves the claim. $\square$

Lemma 4.4. *Let* $\varphi\colon C_*^{\square}(\ \cdot\ ;\mathbb{R}) \Longrightarrow C_*(\ \cdot\ ;\mathbb{R})$ *be a natural chain map with*

$$\varphi_0^{\square^0}(\mathrm{id}_{\square^0}) = (1 \mapsto 0).$$

Then φ *is continuous in the sense of Definition 3.1 and there exists a natural continuous chain map* $\psi\colon C_*(\ \cdot\ ;\mathbb{R}) \Longrightarrow C_*^{\square}(\ \cdot\ ;\mathbb{R})$ *such that* $(C_*^{\square}(\ \cdot\ ;\mathbb{R}), \varphi, \psi)$ *is a cubical normed model of* $C_*(\ \cdot\ ;\mathbb{R})$.
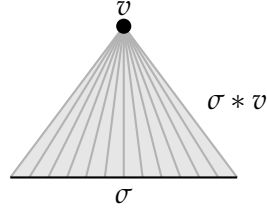
FIGURE 1. Cone $\sigma * v$ of a simplex σ with respect to v

Proof. The acyclic models technique shows that there is a natural chain map $\psi \colon C_*(\, \cdot \, ; \mathbb{R}) \Longrightarrow C_*^\square(\, \cdot \, ; \mathbb{R})$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are naturally chain homotopic to the identity [10].

Because $C_*^\square(\, \cdot \, ; \mathbb{R})$ is basically generated by $(\mathrm{id}_{\square^n})_{n \in \mathbb{N}}$ and naturality, the same arguments as in the proof of Proposition 3.9 show that φ and ψ are continuous in the sense of Definition 3.1 and that also the corresponding natural chain homotopies are continuous. More precisely,

$$\|\varphi_n^X\| \leq \|\varphi_n^{\square^n}\| = |\varphi_n^{\square^n}(\mathrm{id}_{\square^n})|_1 \quad \text{and} \quad \|\psi_n^X\| \leq \|\psi_n^{\Delta^n}\| = |\psi_n^{\Delta^n}(\mathrm{id}_{\Delta^n})|_1^\square,$$

for all spaces X and all $n \in \mathbb{N}$. Hence, $(C_*^\square(\, \cdot \, ; \mathbb{R}), \varphi, \psi)$ is a cubical normed model of $C_*(\, \cdot \, ; \mathbb{R})$. $\square$

We will now give more explicit bounds:

Remark 4.5. The construction of Eilenberg and Mac Lane [10] produces a chain homotopy inverse ψ with $\|\psi_n^{\Delta^n}\| \leq 1$ for all $n \in \mathbb{N}$.

Conversely, an explicit construction of a natural chain map

$$\varphi' \colon C_*^\square(\, \cdot \, ; \mathbb{R}) \Longrightarrow C_*^\triangle(\, \cdot \, ; \mathbb{R})$$

that satisfies the normalisation condition can be obtained by an inductive triangulation procedure via cones:

Definition 4.6 (cones of simplices)**.** Let $K \subset \mathbb{R}^d$ be a convex subset, let $\sigma \colon \Delta^k \longrightarrow K$ be a singular simplex on K and let $v \in K$. Then we define the *cone $\sigma * v$ of σ with respect to v* (Figure 1) as the singular simplex

$$\sigma * v \colon \Delta^{k+1} \longrightarrow K$$

$$(t_0, \ldots, t_{k+1}) \longmapsto t_{k+1} \cdot v + (1 - t_{k+1}) \cdot \sigma\Big(\frac{1}{1 - t_{k+1}} \cdot t_0, \ldots, \frac{1}{1 - t_{k+1}} \cdot t_k\Big).$$

This construction extends linearly to all of $C_k(K; \mathbb{R})$ and passes to $C_k^\triangle(K; \mathbb{R})$; these extensions will also be denoted by $\, \cdot \, * v$.

Roughly speaking, we construct triangulations of cubes by first triangulating the faces and then coning these simplices with respect to the centre of the cube (Figure 2). Using naturality, this can be extended to the whole cubical singular chain complex.

We begin by defining $\varphi_0'^X$ by setting $\varphi_0'^{\square^0}(\mathrm{id}_{\square^0}) := (1 \mapsto 0)$ and extending φ_0' to all of $C_0^\square(\, \cdot \, ; \mathbb{R})$ by naturality.
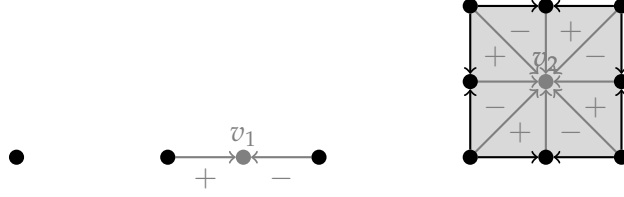
FIGURE 2. Triangulations of cubes via iterated cones, in dimension $0, 1, 2$

Proceeding inductively, for $n \in \mathbb{N}$ we assume that $\varphi_n'^X$ is already constructed for all spaces X and then set

$$\varphi_{n+1}'^X \colon C_{n+1}^{\square}(X; \mathbb{R}) \longrightarrow C_{n+1}^{\triangle}(X; \mathbb{R})$$

$$(\tau \colon \square^{n+1} \to X) \longmapsto C_{n+1}^{\triangle}(\tau; \mathbb{R}) \left(\sum_{j=1}^{n+1} (-1)^j \cdot \varphi_n'^{\square^n} (\square_{(j,0)}^{n+1} - \square_{(j,1)}^{n+1}) * v_{n+1} \right);$$

here, $\square_{(j,0)}^{n+1}, \square_{(j,1)}^{n+1} \colon \square^n \longrightarrow \square^{n+1}$ denote the face maps of $\square^{n+1}$ and

$$v_{n+1} := \left(\frac{1}{2}, \dots, \frac{1}{2} \right) \in \square^{n+1} \subset \mathbb{R}^{n+1}$$

is the centre of the cube $\square^{n+1}$. It is not hard to show that φ'^X indeed is a well-defined chain map.

Because the n-cube has exactly $2 \cdot n$ faces, we inductively obtain

$$\|\varphi_n'^X\| \leq 2^n \cdot n!$$

for all spaces X and all $n \in \mathbb{N}$.

Corollary 4.7. *For all oriented compact n-manifolds M we have*

$$\frac{1}{2^n \cdot n!} \cdot \|M, \partial M\| \leq \|M, \partial M\|^{\square} \leq \|M, \partial M\|$$

and for all oriented n-manifolds M without boundary we have

$$\frac{1}{2^n \cdot n!} \cdot \|M^{\circ}\|_{\mathrm{lf}} \leq \|M^{\circ}\|_{\mathrm{lf}}^{\square} \leq \|M^{\circ}\|_{\mathrm{lf}}.$$

Proof. By Remark 4.5 and the above explicit construction (as well as Example 3.10), there exists a cubical normed model (F, φ, ψ) of $C_*(\,\cdot\,; \mathbb{R})$ that satisfies

$$\|\varphi_n^X\| \leq 2^n \cdot n! \quad \text{and} \quad \|\psi_n^X\| \leq 1$$

for all spaces X and all $n \in \mathbb{N}$. Therefore, Proposition 3.9 gives the desired bounds. $\square$

A more careful analysis of known triangulations of cubes shows that the lower bound in dimension $n \in \mathbb{N}$ can be improved to $2/(2^{n-1} + n!)$ [32]. For simplicity, we will treat such an improvement only in low dimensions.
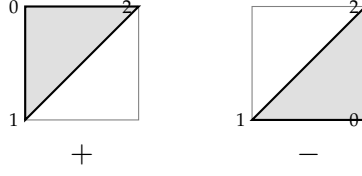
FIGURE 3. Triangulating a square with two triangles

4.3. **Cubical models in low dimensions.** We will now discuss improved lower bounds for cubical simplicial volume in low dimensions.

In the case of surfaces M, it is known that $\|M\|^\square = 1/2 \cdot \|M\|$ holds [22].

Corollary 4.8. *For all oriented compact 3-manifolds M and for all oriented 3-manifolds N without boundary we have*

$$\|M, \partial M\|^\square \geq \frac{1}{5} \cdot \|M, \partial M\| \quad \text{and} \quad \|N\|^\square_{\mathrm{lf}} \geq \frac{1}{5} \cdot \|N\|_{\mathrm{lf}}.$$

Proof. As in the proof of Proposition 4.3 and Corollary 4.7 it suffices to construct a natural chain map $\varphi \colon C^\square_*(\cdot ; \mathbb{R}) \Longrightarrow C^\triangle_*(\cdot ; \mathbb{R})$ that respects the normalisation and satisfies $|\varphi^{\square^3}_3(\mathrm{id}_{\square^3})|^\triangle_1 \leq 5$.

Moreover, in view of the cone construction (or the technique of acyclic models) as used in the previous section, it suffices to construct the first steps $\varphi_0, \ldots, \varphi_3$ compatibly with the normalisation – the higher degrees can then be added through iterated coning as on page 12.

We first define maps $T_j \colon Q_j(\cdot ; \mathbb{R}) \Longrightarrow C_j(\cdot ; \mathbb{R})$ for all $j \in \{0, 1, 2, 3\}$: Here, we use the following notation: For an ascending sequence $n_1, n_2, \ldots n_k$ in $\mathbb{N}_{>0}$, we write $e_{n_1, \ldots, n_k} := \sum_{j=1}^k e_{n_j} \in \mathbb{R}^\infty$ for the corresponding sum of standard basis vectors.

Moreover, for $v_0, \ldots, v_n \in \mathbb{R}^\infty$ we write $[v_0, \ldots, v_n] \colon \Delta^n \longrightarrow \mathbb{R}^\infty$ for the affine singular n-simplex with vertices $v_0, \ldots, v_n$ (in this order).

Dimension 0: We set

$$T^{\square^0}_0(\mathrm{id}_{\square^0}) := [0] \colon \Delta^0 \longrightarrow \square^0$$

and extend T_0 to all of $Q_0(\cdot ; \mathbb{R})$ by naturality. Clearly, this map satisfies the normalisation condition.

Dimension 1: We set

$$T^{\square^1}_1(\mathrm{id}_{\square^1}) := [0, e_1] \colon \Delta^1 \longrightarrow \square^1$$

and extend T_1 to all of $Q_1(\cdot ; \mathbb{R})$ by naturality.

Dimension 2: We set (Figure 3)

$$T^{\square^2}_2(\mathrm{id}_{\square^2}) := [e_2, 0, e_{1,2}] - [e_1, 0, e_{1,2}] \in C_2(\square^2; \mathbb{R})$$

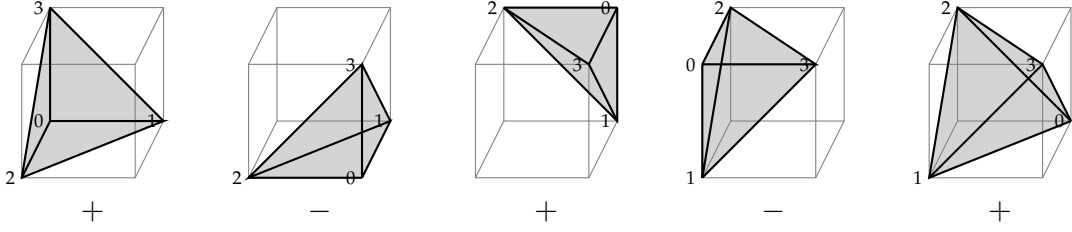and extend T_2 to all of $Q_2(\cdot ; \mathbb{R})$ by naturality.

FIGURE 4. Triangulating a 3-cube with five tetrahedra

Dimension 3: Using the decomposition of the 3-cube into five tetrahedra depicted in Figure 4, we set

$$T_3^{\square^3}(\mathrm{id}_{\square^3}) := [0, e_1, e_2, e_3]$$
$$- [e_{1,2}, e_1, e_2, e_{1,2,3}]$$
$$+ [e_{1,3}, e_1, e_3, e_{1,2,3}]$$
$$- [e_{2,3}, e_2, e_3, e_{1,2,3}]$$
$$+ [e_e, e_2, e_3, e_{1,2,3}] \qquad \in C_3(\square^3; \mathbb{R}),$$

and extend T_3 to all of $Q_3(\,\cdot\,; \mathbb{R})$ by naturality.

These maps are not yet compatible with the cubical and the ordinary singular boundary operators. Therefore, we symmetrise these maps as follows: For $j \in \mathbb{N}$ let

$$\Sigma_j := \frac{1}{|\operatorname{Isom}(\square^j)|} \cdot \sum_{\pi \in \operatorname{Isom}(\square^j)} (-1)^{\operatorname{sgn}\pi} \cdot C_j(\pi; \mathbb{R}) : C_j(\square^j; \mathbb{R}) \longrightarrow C_j(\square^j; \mathbb{R}),$$

where $\operatorname{sgn}\pi \in \{0, 1\}$ encodes whether π is orientation-preserving or not. Now a straightforward calculation shows that our triangulations fit together in the sense that

$$\partial_j \circ \Sigma_j \circ T_j = \Sigma_{j-1} \circ T_{j-1} \circ \partial_j^{\square}$$

holds for all $j \in \{1, 2, 3\}$.

Moreover, it is not hard to see that $\Sigma_j \circ T_j$ induces for all $j \in \{0, \dots, 3\}$ well-defined natural maps

$$\varphi_j : C_j^{\square}(\,\cdot\,; \mathbb{R}) \Longrightarrow C_j^{\triangle}(\,\cdot\,; \mathbb{R})$$

that are compatible with the boundary operators. Indeed, the well-definedness modulo degenerate cubes can be seen as follows: If $\tau : \square^j \longrightarrow X$ is a degenerate singular cube, then the reflection $\pi \in \operatorname{Isom}(\square^j)$ on the middle hyperplane orthogonal to one of the directions of degeneracy of τ ensures that every singular simplex in $\Sigma_j \circ T_j(\tau)$ occurs the same number of times with positive and negative sign.

By construction, we have

$$|\varphi_3^{\square^3}(\mathrm{id}_{\square^3})|_1^{\triangle} \leq \frac{1}{|\operatorname{Isom}(\square^3)|} \cdot |\operatorname{Isom}(\square^3)| \cdot 5 = 5,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. SEIFERT FIBRED PIECES

The goal for this section is to prove that Seifert fibred pieces have trivial cubical simplicial volume. We will first treat a slightly more general situation and then obtain the Seifert fibred case as a special case.

5.1. **Open manifolds with vanishing simplicial volumes.** In the presence of sufficiently nice boundary components, vanishing of the relative simplicial volume implies the vanishing of the locally finite simplicial volume of the interior. We first formulate the boundary condition:

Definition 5.1 (UBC boundary)**.** An oriented compact n-manifold M has *UBC boundary* if every boundary component of M satisfies $(n-1)$-UBC (in the sense of Definition 3.11).

For example, all manifolds with amenable fundamental group satisfy UBC in all degrees [26]. This includes, in particular, tori.

Proposition 5.2. *Let (F, φ, ψ) be a normed model of the singular chain complex and let M be an oriented compact manifold with UBC boundary that satisfies $\|M, \partial M\| = 0$. Then*

$$\|M, \partial M\|^F = 0 = \|M, \partial M\| \quad and \quad \|M^\circ\|_{\mathrm{lf}}^F = 0 = \|M^\circ\|_{\mathrm{lf}}.$$

Proof. In view of the equivalence of vanishing of generalised simplicial volumes (Proposition 3.9), it suffices to prove that $\|M^\circ\|_{\mathrm{lf}} = 0$.

Let $n := \dim M$ and $\kappa \in \mathbb{R}_{>0}$ be an $(n-1)$-UBC constant for ∂M. Let $\varepsilon \in \mathbb{R}_{>0}$. Because of $\|M, \partial M\| = 0$ there is a chain $c \in C_n(M; \mathbb{R})$ with $\partial c \in C_{n-1}(\partial M; \mathbb{R})$ that represents $[M, \partial M]_{\mathbb{R}}$ and satisfies

$$|c|_1 \le \varepsilon.$$

In particular, $|\partial c|_1 \le (n+1) \cdot |c|_1 \le (n+1) \cdot \varepsilon$.

Furthermore, we have $\|\partial M\| \le (n+1) \cdot \|M, \partial M\| = 0$ (Proposition 2.4). Using the uniform boundary condition on ∂M, we hence find fundamental cycles $(z_k)_{k \in \mathbb{N}} \subset C_{n-1}(\partial M; \mathbb{R})$ and chains $(b_k)_{k \in \mathbb{N}} \subset C_n(\partial M; \mathbb{R})$ such that

$$z_0 = \partial c,$$

$$\forall_{k \in \mathbb{N}} \quad |z_k|_1 \le \frac{1}{2^k} \cdot (n+1) \cdot \varepsilon,$$

$$\forall_{k \in \mathbb{N}} \quad \partial b_k = z_{k+1} - z_k,$$

$$\forall_{k \in \mathbb{N}} \quad |b_k|_1 \le \kappa \cdot \frac{2}{2^k} \cdot (n+1) \cdot \varepsilon.$$

Stretching out these chains $(b_k)_{k \in \mathbb{N}}$ along a collar of M and combining the resulting chain with c produces a locally finite fundamental cycle of M° and implies that [20, Proposition 6.4, proof of Theorem 6.1][19, Section 6.2]

$$\|M^\circ\|_{\mathrm{lf}} \le |c|_1 + \sum_{k \in \mathbb{N}} |b_k|_1 + n \cdot \sum_{k \in \mathbb{N}} |z_{k+1}|_1$$

$$\le \varepsilon + 4 \cdot \kappa \cdot (n+1) \cdot \varepsilon + 2 \cdot n \cdot (n+1) \cdot \varepsilon.$$

Taking $\varepsilon \to 0$ shows that $\|M^\circ\|_{\mathrm{lf}} = 0$, as desired. □

Whether Proposition 5.2 holds without the additional UBC condition is an open problem [20, p. 86].

5.2. **Vanishing of simplicial volumes of Seifert fibred pieces.** Let us recall the definition of Seifert fibred 3-manifolds [1].

Definition 5.3 (standard fibred torus)**.** The *standard fibred torus* corresponding to a pair (a, b) of coprime integers with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by $2\pi b / a$, equipped with the natural fibering by circles.

Definition 5.4 (Seifert fibred manifold)**.** A *Seifert fibred* 3-manifold is a compact 3-manifold M with a decomposition of M into disjoint circles, called *fibers*, such that each circle has a tubular neighbourhood in M which is isomorphic to a standard fibred torus.

Notice that in an oriented Seifert fibred manifold the Seifert fibred structure defines a product structure on each boundary component, which implies that each boundary component is an S^1-bundle; thus, each boundary component is a torus.

As a consequence of Proposition 5.2 and the vanishing of classical simplicial volume we obtain:

Corollary 5.5 (Seifert fibred pieces)**.** *Let (F, φ, ψ) be a normed model of the singular chain complex and let M be a Seifert fibred 3-manifold. Then*

$$\|M^\circ\|_{\mathrm{lf}}^F = 0.$$

In particular, $\|M^\circ\|_{\mathrm{lf}}^\square = 0$.

Proof. It is well known that $\|M, \partial M\| = 0$ holds for Seifert fibred 3-manifolds [31, Corollary 6.5.3]. Because the boundary components of M are tori, M has UBC boundary, and thus Proposition 5.2 shows that $\|M^\circ\|_{\mathrm{lf}}^F = 0$. $\square$

6. HYPERBOLIC PIECES

As next step, we discuss the hyperbolic case: we have already recalled in Theorem 2.7 that for a complete oriented hyperbolic n-manifold of finite volume without boundary the ordinary simplicial volume coincides with the ratio between the Riemannian volume and $v_n^\triangle$. The constant $v_n^\triangle$, depending only on the dimension n of the manifold, is the supremum of volumes of geodesic n-simplices in the hyperbolic space $\mathbb{H}^n$, which is equal to the volume of the ideal and regular geodesic n-simplex [16, 27].

In the cubical case, the situation is more involved because the interaction between combinatorics and the geometry of hyperplanes in $\mathbb{H}^n$ is more delicate than in the simplicial case. We review hyperbolic cubes in Section 6.1. In Section 6.2, we prove an upper bound for cubical simplicial volume of hyperbolic manifolds (using a cubical smearing), and in Section 6.3, we prove a lower bound (using a cubical straightening). In dimension 3, we have matching bounds, which gives the exact value (Corollary 6.14). In Section 6.4, we briefly discuss proportionality for cubical simplicial volume of hyperbolic manifolds.

For these arguments, we will use the concrete description of locally finite cubical chains through infinite sums of singular cubes and the corresponding description of locally finite cubical simplicial volume through the ℓ^1-norm of such chains, in analogy with the simplicial case (Example 3.6).

6.1. **Hyperbolic cubes.** We first discuss different types of cubes in $\mathbb{H}^n$. The 2^k vertices of the standard k-cube $\square^k$ will be denoted by $(v_j)_{j \in J^k}$, where $J^k := \{0, 1\}^k$. Taking iterated geodesics produces straight cubes:

Definition 6.1 (straight cube). Let X be a uniquely geodesic metric space (e.g., $\mathbb{H}^n$), let $k \in \mathbb{N}$ and let $x = (x_j)_{j \in J^k}$ be a family of points in X. Then the *straight cube* $[x]^\square$ *with vertices x in X* is defined inductively via

$$[x]^\square : \square^k \longrightarrow X$$
$$(t_1, \ldots, t_k) \longmapsto (1 - t_k) \cdot \left[(x_{j0})_{j \in J^{k-1}}\right]^\square (t_1, \ldots, t_{k-1})$$
$$+ t_k \cdot \left[(x_{j1})_{j \in J^{k-1}}\right]^\square (t_1, \ldots, t_{k-1});$$

if $k = 0$ we just define the straight cube as the corresponding single point.

In other words, straight cubes are defined as iterated geodesic joins of opposite faces. In particular, straight cubes in $\mathbb{H}^n$ are smooth [23, Section 2.1.1].

6.1.1. *Volumes of cubes.* For a smooth cube $c : \square^n \longrightarrow \mathbb{H}^n$ we define the *signed volume* by

$$\mathrm{vol}^{\mathrm{alg}}_{\mathbb{H}^n} c := \int_{\square^n} c^* d\Omega,$$

where Ω is the volume form on $\mathbb{H}^n$, and the *volume* by $\mathrm{vol}_{\mathbb{H}^n} c := |\mathrm{vol}^{\mathrm{alg}}_{\mathbb{H}^n} c|$.

A straight cube is *(non)-degenerate* if its volume is (non)-zero. If all vertices of a non-degenerate cube are different, the cube is *strongly non-degenerate*. If the signed volume is positive (respectively negative) the cube is *positively* (respectively *negatively*) *oriented*.

Definition 6.2 (volume function). Identifying a straight cube in $\mathbb{H}^n$ with its vertices we define the following *signed volume function*

$$\mathrm{vol}^{\mathrm{alg}} : \qquad (\mathbb{H}^n)^{2^n} \longrightarrow \mathbb{R}$$
$$x = (x_j)_{j \in J^n} \longmapsto \mathrm{vol}^{\mathrm{alg}}_{\mathbb{H}^n}([x]^\square).$$

Similarly, we define the volume function $\mathrm{vol} := |\mathrm{vol}^{\mathrm{alg}}|$.

Remark 6.3. Notice that the signed volume function is continuous with respect to the topology on $(\mathbb{H}^n)^{2^n}$ induced by the distance $\times d_{\mathbb{H}^n}$. Indeed, hyperbolic geodesics continuously depend on their endpoints.

By definition of straight cubes, we can express the diameter of $[x]^\square$ in terms of the distances between the elements of x:

(1) $$\mathrm{diam}_{\mathbb{H}^n}[x]^\square = \max_{i,j \in J^k} d_{\mathbb{H}^n}(x_i, x_j).$$

One major difficulty with cubes is that not all straight cubes are geodesic in the following sense:

Definition 6.4 (geodesic cube). A straight cube in $\mathbb{H}^n$ is *geodesic* if each $(n-1)$-face lies in a hyperbolic hyperplane of $\mathbb{H}^n$.

Proposition 6.5 (volume bounds for cubes). *Let $n \in \mathbb{N}$. Then the numbers*

$$v_n^{\square} := \sup\{\mathrm{vol}_{\mathbb{H}^n} c \mid c \text{ is a geodesic } n\text{-cube in } \mathbb{H}^n\}$$

$$w_n^{\square} := \sup\{\mathrm{vol}_{\mathbb{H}^n} c \mid c \text{ is a straight } n\text{-cube in } \mathbb{H}^n\}$$

are finite.

Proof. Let $T(n)$ be the number of n-simplices needed to triangulate convex hulls of 2^n points in $\mathbb{H}^n$. For a straight cube $c : \square^n \to \mathbb{H}^n$, we denote by $\mathrm{Conv}(c(v_j)_{j \in J^n})$ the convex hull of the vertices $(c(v_j))_{j \in J^n}$. By definition of straight cubes, we have $\mathrm{im}(c) \subseteq \mathrm{Conv}(c(v_j)_{j \in J^n})$, and therefore it follows that $v_n^{\square} \leq w_n^{\square} \leq T(n) \cdot v_n^{\triangle} < \infty$. $\qquad\square$

By definition, clearly $v_n^{\square} \leq w_n^{\square}$. However, the exact relation in high dimensions seems to be unknown.

6.1.2. *Ideal geodesic cubes.* Let $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial \mathbb{H}^n$ be the standard compactification of the hyperbolic space. We extend the notion of straight cubes to ideal straight cubes. If $x = (x_j)_{j \in J^k}$ is a family of points in $\partial \mathbb{H}^n$, then the straight cube $[x]^{\square}$ is *ideal*.

Computing the supremum of the volume of geodesic cubes we may include also geodesic cubes with vertices on the boundary by setting

$$\overline{v}_n^{\square} := \sup\{\mathrm{vol}_{\mathbb{H}^n} c \mid c \text{ is a geodesic } n\text{-cube in } \overline{\mathbb{H}^n}\}.$$

Lemma 6.6. *For every $n \in \mathbb{N}_{>0}$, we have that $v_n^{\square} = \overline{v}_n^{\square}$.*

Proof. Using the same construction of Remark 6.8 we deduce that, given a geodesic cube having some vertices on $\partial \mathbb{H}^n$, there exists an ideal one with the same vertices on $\partial \mathbb{H}^n$ containing it. Therefore, it suffices to show that for every positively oriented ideal geodesic n-cube c and all $\varepsilon > 0$ there exists a geodesic n-cube $c_{\varepsilon} : \square^n \to \mathbb{H}^n$ with $\mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_{\varepsilon} \geq \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c - \varepsilon$.

First of all notice that if we pick an ideal cube that is not strongly non-degenerate there exists a strongly non-degenerate one of bigger volume. Therefore, we can restrict ourselves to considering strongly non-degenerate ideal cubes.

Let $B^n \subset \mathbb{R}^n$ be the Poincaré disk model with origin O and boundary S^{n-1}. Let c be a positively oriented strongly non-degenerate ideal geodesic n-cube and let $(c(v_j))_{j \in J^n}$ be its vertices in S^{n-1}. Let $\alpha_j : [0, \infty] \to \mathbb{H}^n$ be the geodesic ray (with constant speed parametrization) that starts at the origin O and is asymptotic to $c(v_j)$. For $L \in \mathbb{R}_{>0}$ we then define the straight cube

$$c_L := \left[(\alpha_j(L))_{j \in J^n}\right]^{\square}$$

in $\mathbb{H}^n$. It is easy to verify that $\lim_{L \to \infty} \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_L = \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c$. Therefore, for every $\varepsilon > 0$ there exists $L \in \mathbb{R}_{>0}$ with $\mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_L \geq \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c - \varepsilon$; in particular, c_L is positively oriented as soon as $\varepsilon < \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c$. $\qquad\square$

Question 6.7. *Is there a geodesic n-cube realizing the maximum of the volume?*

Remark 6.8. One may notice that the maximum of the volume of geodesic n-cubes (possibly with some vertices on $\partial \mathbb{H}^n$), if any, is attained by ideal n-cubes. Indeed, given a geodesic n-cube c, let v be a point in the internal part of $\mathrm{im}(c)$. For each vertex $c(v_j)$ of c consider the geodesic ray starting from v and passing through $c(v_j)$, and pick its endpoint c_∞^j. The ideal n-cube $[(c_\infty^j)_j]^\square$ contains c. Then, for every geodesic n-cube there exists an ideal one with bigger volume.

Question 6.9. *If there exists an ideal n-cube realizing the maximum, is it a regular n-cube?*

Example 6.10. In dimension 3, Coxeter [9] observed that the ideal regular 3-cube can be triangulated by five ideal regular tetrahedra. Then the ideal regular 3-cube realizes the supremum of the volumes of the geodesic 3-cubes and $v_3^\square = 5 \cdot v_3^\triangle$.

Example 6.11. In dimension 4, we have that $15 \cdot v_4^\triangle \leq \overline{v}_4^\square \leq 16 \cdot v_4^\triangle$ [29, 24].

6.1.3. *Approximation of cubes.* We will need that we can approximate large geodesic cubes well enough:

Proposition 6.12 (approximation of cubes). *Let $n \in \mathbb{N}$ and let $\varepsilon \in (0, v_n^\square/2)$. Then there exists a positively oriented geodesic cube c_ε in $\mathbb{H}^n$ and $\delta > 0$ with the following properties:*

(1) *The cube c_ε has large volume, i.e., $\mathrm{vol}_{\mathbb{H}^n} c_\varepsilon \geq v_n^\square - \varepsilon$.*
(2) *Every straight cube close to c_ε has large volume and is positively oriented, i.e., for all straight cubes $c \colon \square^n \longrightarrow \mathbb{H}^n$ with*

$$\max_{j \in J^n} d_{\mathbb{H}^n}\big(c(v_j), c_\varepsilon(v_j)\big) \leq \delta$$

we have $\mathrm{vol}_{\mathbb{H}^n} c \geq \mathrm{vol}_{\mathbb{H}^n} c_\varepsilon - \varepsilon$ and c is positively oriented.

Proof. By definition of $v_n^\square$, there is a geodesic n-cube c_ε in $\mathbb{H}^n$ such that $\mathrm{vol}_{\mathbb{H}^n} c_\varepsilon \geq v_n^\square - \varepsilon$. Without loss of generality we may suppose that c_ε is positively oriented.

Let $\square_n(\mathbb{H}^n)$ be the set of straight n-cubes in $\mathbb{H}^n$. The signed volume function is continuous on $\square_n(\mathbb{H}^n)$ (Remark 6.3), and therefore there exists a $\delta > 0$ such that

$$\forall_{c \in \square_n(\mathbb{H}^n)} \quad \times d_{\mathbb{H}^n}(c, c_\varepsilon) < \delta \implies |\mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_\varepsilon - \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c| < \varepsilon.$$

Notice that $\times d_{\mathbb{H}^n}(c, c_\varepsilon) < \delta$ is equivalent to $\max_{j \in J^n} d_{\mathbb{H}^n}(c(v_j), c_\varepsilon(v_j)) < \delta$; finally, $|\mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c - \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_\varepsilon| < \varepsilon$ implies that

$$\mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c > \mathrm{vol}_{\mathbb{H}^n}^{\mathrm{alg}} c_\varepsilon - \varepsilon \geq v_n^\square - 2 \cdot \varepsilon > 0$$

and hence any such c is positively oriented. $\qquad\square$

6.2. **Hyperbolic pieces – upper bound.** We will now establish an upper bound for cubical simplicial volume of hyperbolic manifolds in terms of geodesic cubes:

Theorem 6.13. *Let M be a complete oriented hyperbolic n-manifold of finite volume. Then*

$$\|M\|_{\mathrm{lf}}^{\square} \leq \frac{\mathrm{vol}(M)}{v_n^{\square}}.$$

Before getting into the details of the proof, we mention that this allows to calculate cubical simplicial volume of hyperbolic 3-manifolds:

Corollary 6.14. *Let M be a complete oriented hyperbolic 3-manifold of finite volume without boundary. Then*

$$\|M\|_{\mathrm{lf}}^{\square} = \frac{1}{5} \cdot \|M\|_{\mathrm{lf}}.$$

Proof. As we have already mentioned it is a well-known fact from hyperbolic geometry that $v_3^{\square} = 5 \cdot v_3^{\triangle}$ holds. Hence, Theorem 2.7, Lemma 6.6, Theorem 6.13, and Corollary 4.8 yield the conclusion. $\square$

We will now prove Theorem 6.13, using a discrete cubical version of the smearing construction by Thurston that already proved useful for ordinary simplicial volume of hyperbolic manifolds [31, 12].

Let us first fix some notation: Let M be a complete oriented hyperbolic n-manifold of finite volume without boundary. Because volume and cubical simplicial volume are additive with respect to connected components, we may assume without loss of generality that M is connected. Let Γ be the fundamental group of M, and let $\pi \colon \mathbb{H}^n \longrightarrow M$ be the universal covering map. Then Γ acts isometrically and properly discontinuously via deck transformations on $\mathbb{H}^n$.

Let G be the isometry group of $\mathbb{H}^n$, endowed with the Haar measure μ_G; we normalise the Haar measure μ_G according to the volume $\mathrm{vol}_{\mathbb{H}^n}$ on $\mathbb{H}^n$ [28, Lemma 11.6.4]; the group G is unimodular [2, Proposition C.4.11], whence μ_G is bi-invariant. Moreover, we let G^+ and G^- be the subset of orientation preserving (or reversing, respectively) isometries.

Roughly speaking, the smearing works as follows: For small ε we pick a geodesic n-cube c_ε as provided by Proposition 6.12, and we would want to consider the chain $\sum_{[g] \in \Gamma \backslash G} \pi \circ (g \cdot c_\varepsilon)$. Because this sum is not locally finite, we discretise the process, using so-called Γ-nets.

Let $R \in \mathbb{R}_{>0}$. A Γ-*net of mesh size at most R in* $\mathbb{H}^n$ is given by a discrete subset $\Lambda \subset \mathbb{H}^n$, called set of *vertices*, and a collection of Borel sets $(B_x)_{x \in \Lambda}$ in $\mathbb{H}^n$, called *cells*, such that the following conditions hold:

(1) The quotient $\pi(\Lambda) \subset M$ is locally finite.
(2) For all $x \in \Lambda$ we have $x \in B_x$. Moreover, the sets $(B_x)_{x \in \Lambda}$ are pairwise disjoint and $\mathbb{H}^n = \bigcup_{x \in \Lambda} B_x$.
(3) For all $\gamma \in \Gamma$, $x \in \Lambda$ we have $\gamma \cdot x \in \Lambda$ and $\gamma \cdot B_x = B_{\gamma \cdot x}$.
(4) For all $x \in \Lambda$ we have $\mathrm{diam}_{\mathbb{H}^n} B_x \leq R$.

Lemma 6.15. *For every $R \in \mathbb{R}_{>0}$ and every finite set $S \subset \mathbb{H}^n$, there exists a Γ-net of mesh size at most R in $\mathbb{H}^n$ such that the points of S are contained in the interior of the cells.*

Proof. Such a net can be obtained by lifting to $\mathbb{H}^n$ and translating with Γ a decomposition of M into small Borel sets [12, Lemma 3.10]. $\square$

We now let $\varepsilon \in (0, 1/2 \cdot v_n^\square)$. By Proposition 6.12, there exists $\delta > 0$ and a geodesic cube $c_\varepsilon \colon \square^n \longrightarrow \mathbb{H}^n$ of large volume with good approximation properties. By the previous lemma, there is a Γ-net $(\Lambda, (B_x)_{x\in\Lambda})$ of mesh size at most δ in $\mathbb{H}^n$ such that the vertices of c_ε are in the interior of their cells. Finally, we can perform the actual smearing construction – smearing the model cube c_ε over all of $\mathbb{H}^n$ and whence M: For $x \in \Lambda^{J^n}$ we consider the associated straight cube

$$\sigma_x := [x]^\square \colon \square^n \longrightarrow \mathbb{H}^n$$

and the Borel set

$$\Omega_{\varepsilon,x}^\pm := \left\{ g \in G^\pm \;\middle|\; \forall_{j\in J^n} \quad g \cdot c_\varepsilon(v_j) \in B_{x_j} \right\} \subset G,$$

and we abbreviate $a_{\varepsilon,x}^\pm := \mu_G(\Omega_{\varepsilon,x}^\pm) \in \mathbb{R}_{\geq 0}$. The group Γ acts diagonally on $\widetilde{X} := \Lambda^{J^n}$, and we write X for the corresponding quotient space.

Lemma 6.16 (smearing). *In this situation, the* smeared chain

$$z_\varepsilon := \sum_{[x]\in X} \left(a_{\varepsilon,x}^+ \cdot \pi \circ \sigma_x - a_{\varepsilon,x}^- \cdot \pi \circ \sigma_x \right)$$

has the following properties:

(1) *The definition of z_ε does not depend on the choice of representatives x of the orbits $[x] \in X$.*
(2) *The chain z_ε indeed is a locally finite chain on M.*
(3) *The chain z_ε is a cycle.*
(4) *The locally finite cubical cycle z_ε represents a non-zero multiple of the cubical (locally finite) fundamental class of M.*

Proof. Ad 1. By construction, we have $\sigma_{\gamma\cdot x} = \gamma \cdot \sigma_x$ for all $x \in \widetilde{X}$, $\gamma \in \Gamma$, and hence $\pi \circ \sigma_{\gamma\cdot x} = \pi \circ \sigma_x$. Furthermore, $a_{\varepsilon,\gamma\cdot x}^\pm = a_{\varepsilon,x}^\pm$ by the equivariance of the net and invariance of the Haar measure μ_G.

Ad 2. This is a size argument: We set $Y := \left\{ x \in \widetilde{X} \mid a_{\varepsilon,x}^+ \neq 0 \text{ or } a_{\varepsilon,x}^- \neq 0 \right\}$. If $x \in Y$, then there is a $g \in G$ with

$$\forall_{j\in J^n} \quad g \cdot c_\varepsilon(v_j) \in B_{x_j}$$

and so $\mathrm{diam}_{\mathbb{H}^n} x \leq 2 \cdot (2 \cdot \delta + \mathrm{diam}_{\mathbb{H}^n} \mathrm{im}\, c_\varepsilon)$. Moreover, the diameter of straight cubes in $\mathbb{H}^n$ is controlled in terms of the diameter of its set of vertices (equation (1)). Hence, there is $L \in \mathbb{R}_{>0}$ such that

$$\forall_{x\in Y} \quad \mathrm{diam}_{\mathbb{H}^n} \mathrm{im}\, \sigma_x \leq L.$$

So, if $K \subset M$ is compact, then

$$Y_K := \left\{ x \in Y \mid \mathrm{im}(\pi \circ \sigma_x) \cap K \neq \varnothing \right\} \subset \left(\Lambda \cap \pi^{-1}(B_{2\cdot L}(K)) \right)^{J^n}.$$

Now the facts that $\pi(\Lambda)$ is locally finite, that the Γ-action on $\mathbb{H}^n$ is properly discontinuous, and that tuples in Y have diameter at most L show that the quotient $\Gamma \setminus Y_K$ is finite. Therefore, the infinite chain z_ε is locally finite.

Ad 3. We use Thurston's reflection trick to show that z_ε is a cycle: At this point it is crucial that every face of the model cube c_ε lies in a hyperplane. More precisely, for all $k \in \{1, \ldots, n\}$ and $i \in \{0, 1\}$ let $\rho_{k,i} \in G$ be the hyperbolic reflection at a hyperplane that contains the (k, i)-face of c_ε.

We can now argue similarly to the simplicial case [2, p. 116]: Clearly, in the expanded expression $\partial_n^\square z_\varepsilon$, only $(n-1)$-cubes of the form $\pi \circ \sigma_y$ with $y \in \Lambda^{J^{n-1}}$ occur. One easily deduces from the construction of z_ε that such a cube has the coefficient

$$b_{\varepsilon,y} := \sum_{k=1}^n \sum_{i=0}^1 (-1)^{k+i} \cdot \Big(\mu_G\{g \in G^+ \mid \forall_{j \in J^{n-1}} \ g \cdot c_\varepsilon(v_{j+_k i}) \in B_{y_j}\}$$
$$- \mu_G\{g \in G^- \mid \forall_{j \in J^{n-1}} \ g \cdot c_\varepsilon(v_{j+_k i}) \in B_{y_j}\}\Big)$$

in $\partial_n^\square z_\varepsilon$; here, $j +_k i$ denotes the n-tuple that results if i is inserted at position k into j, and we used y as the representative of its own Γ-orbit.

Let $k \in \{1, \dots, n\}$ and $i \in \{0, 1\}$. Using $G^- = G^+ \cdot \rho_{k,i}$, the bi-invariance of μ_G, and the fact that $\rho_{k,i}$ fixes the (k,i)-face of c_ε, we obtain

$$\mu_G\{g \in G^- \mid \forall_{j \in J^{n-1}} \ g \cdot c_\varepsilon(v_{j+_k i}) \in B_{y_j}\}$$
$$= \mu_G\{g \in G^+ \mid \forall_{j \in J^{n-1}} \ g \cdot \rho_{k,i} \cdot c_\varepsilon(v_{j+_k i}) \in B_{y_j}\}$$
$$= \mu_G\{g \in G^+ \mid \forall_{j \in J^{n-1}} \ g \cdot c_\varepsilon(v_{j+_k i}) \in B_{y_j}\}$$

Hence, $b_{\varepsilon,y} = 0$. Therefore, z_ε is a cycle.

Ad 4. It suffices to check the claim locally. To this end, let $m \in M$ be chosen in such a way that it π-lifts to the interior of im c_ε. We now show that z_ε represents a non-trivial class in $H_n^\square(M, M \setminus \{m\}; \mathbb{R})$: We choose $\xi \in X$ so that id $\in \Omega_{\varepsilon,\xi}^+$. Then z_ε represents in $H_n^\square(M, M \setminus \{m\}; \mathbb{R})$ the class given by the (finite) sum

$$z_{\varepsilon,m} := \sum_{\substack{[x] \in X, \\ m \in \mathrm{im}\, \pi \circ \sigma_x}} (a_{\varepsilon,x}^+ \cdot \pi \circ \sigma_x - a_{\varepsilon,x}^- \cdot \pi \circ \sigma_x).$$

By choice of m, the cube $\pi \circ \sigma_\xi$ is a relative cycle for $(M, M \setminus \{m\})$ and represents the orientation generator of $H_n^\square(M, M \setminus \{m\}; \mathbb{R})$. Moreover, because the vertices of the cube c_ε lie in the interior of their cells in the chosen Γ-net, it is not hard to see that $a_{\varepsilon,\xi}^+ > 0$.

On the other hand, for all $x \in X$, by construction, $\pi \circ \sigma_x$ is positively oriented if $a_{\varepsilon,x}^+ \neq 0$ and negatively oriented if $a_{\varepsilon,x}^- \neq 0$. Hence, the remaining terms in the defining sum for $z_{\varepsilon,m}$ represent a non-negative multiple of the orientation generator of $H_n(M, M \setminus \{m\}; \mathbb{R})$. Therefore, $z_{\varepsilon,m}$ does not represent the trivial class, and so also z_ε does not represent the trivial class. $\square$

Using Lemma 6.16, it is easy to complete the proof of Theorem 6.13:

Proof of Theorem 6.13. Let $\varepsilon \in (0, 1/2 \cdot v_n^\square)$. The smeared locally finite cycle z_ε of M constructed in Lemma 6.16 has finite ℓ^1-norm because

$$|z_\varepsilon|_1^\square \leq \sum_{[x] \in X} (a_{\varepsilon,x}^+ + a_{\varepsilon,x}^-) \leq 2 \cdot \mathrm{vol}(M).$$

By Lemma 6.16, z_ε represents a non-zero multiple of the locally finite cubical fundamental class of M; let $\alpha \in \mathbb{R} \setminus \{0\}$ be this multiple. Moreover, the straight cubes in z_ε that occur with non-zero coefficients have uniformly bounded diameter (as was shown in the proof of Lemma 6.16) and so can

be seen to be uniformly Lipschitz [23, Section 2.1.1]. Similarly to the simplicial case, we then have

$$\int_{z_\varepsilon} d\Omega_M = \alpha \cdot \mathrm{vol}(M),$$

where Ω_M is the volume form on M. Therefore, using the orientation behaviour of the straight cubes σ_x (as in the proof of Lemma 6.16) and the approximation properties of the model cube c_ε underlying the construction of z_ε, we obtain

$$
\begin{aligned}
\alpha \cdot \mathrm{vol}(M) &= \sum_{[x] \in X} \left(a^+_{\varepsilon,x} \cdot \int \sigma_x^* d\Omega - a^-_{\varepsilon,x} \cdot \int \sigma_x^* d\Omega \right) \\
&= \sum_{[x] \in X} \left(a^+_{\varepsilon,x} \cdot \mathrm{vol}_{\mathbb{H}^n} \sigma_x + a^-_{\varepsilon,x} \cdot \mathrm{vol}_{\mathbb{H}^n} \sigma_x \right) \\
&\geq \sum_{[x] \in X} (a^+_{\varepsilon,x} + a^-_{\varepsilon,x}) \cdot (v_n^\square - 2 \cdot \varepsilon) \\
&= |z_\varepsilon|_1^\square \cdot (v_n^\square - 2 \cdot \varepsilon).
\end{aligned}
$$

This implies

$$\|M\|_{\mathrm{lf}}^\square \leq \frac{1}{\alpha} \cdot |z_\varepsilon|_1^\square \leq \frac{\mathrm{vol}(M)}{v_n^\square - 2 \cdot \varepsilon}.$$

Taking $\varepsilon \to 0$ gives the desired estimate.   □

6.3. **Hyperbolic pieces – lower bound.** Conversely, straight cubes give a lower bound:

Theorem 6.17. *Let M be a complete oriented hyperbolic n-manifold of finite volume. Then*

$$\|M\|_{\mathrm{lf}}^\square \geq \frac{\mathrm{vol}(M)}{w_n^\square}.$$

We will prove this theorem as in the simplicial case via straightening. To every singular k-cube $c : \square^k \to \mathbb{H}^n$ in $\mathbb{H}^n$ we may associate a straight k-cube $[c]^\square := [(c(v_j))_{j \in J^k}]^\square$ where, as before, $(v_j)_{j \in J^k}$ denotes the vertices of the standard k-cube and $J^k := \{0,1\}^k$. Hence, by linearity we define the straightening map

$$[\,\cdot\,]_*^\square : C_*^\square(\mathbb{H}^n; \mathbb{R}) \longrightarrow C_*^\square(\mathbb{H}^n; \mathbb{R}),$$

which is easily seen to be a well-defined chain map.

Let us now extend the straightening operation to locally finite cubical chains: If M is a complete oriented hyperbolic n-manifold of finite volume and $c : \square^k \longrightarrow M$ is a singular k-cube, then we define

$$[c]_k^\square := \pi \circ [\widetilde{c}]_k^\square : \square^n \longrightarrow M,$$

where $\pi : \mathbb{H}^n \longrightarrow M$ is the universal covering map and where $\widetilde{c}$ denotes some π-lift of c. This construction extends to a well-defined chain map

$$[\,\cdot\,]_*^\square : C_*^{\square,\mathrm{lf}}(M; \mathbb{R}) \longrightarrow C_*^{\square,\mathrm{lf}}(M; \mathbb{R}).$$

In order to see that locally finite chains indeed are mapped to locally finite chains, one can proceed as follows: We consider a compact core N of the

complete hyperbolic manifold M, i.e., a subset of M whose complement $M \backslash N$ in M is a disjoint union of finitely many geodesically convex cusps of M. Using such a decomposition it is easy to show that the straightening map indeed maps locally finite chains to locally finite chains (see for instance [17, Lemma 4.3] for the simplicial case).

This straightening map is chain homotopic to the identity map. Indeed, for a singular cube $c : \square^k \to \mathbb{H}^n$, we consider the straight homotopy

$$F_c^k : \quad \square^k \times [0,1] \quad \longrightarrow \quad \mathbb{H}^n$$
$$((t_1, \ldots, t_k), s) \quad \longmapsto \quad (1-s) \cdot c(t_1, \ldots, t_k) + s \cdot [c]^{\square}(t_1, \ldots, t_k).$$

We then define a chain homotopy via

$$h_k : \quad C_k^{\square}(\mathbb{H}^n; \mathbb{R}) \quad \longrightarrow \quad C_{k+1}^{\square}(\mathbb{H}^n; \mathbb{R})$$
$$c \quad \longmapsto \quad F_c^k.$$

It is easy to verify that h_* is a homotopy between the identity and the straightening map on $\mathbb{H}^n$. Moreover, this argument also descends to the locally finite straightening on M.

Proof of Theorem 6.17. Let $c = \sum_i a_i \cdot c_i$ be a locally finite representative of the fundamental class of M and $\widetilde{c} = \sum_i a_i \cdot \widetilde{c}_i$ be a lift to the universal cover. Then, $\widetilde{c}$ is smooth and we have

$$\mathrm{vol}(M) = \int_{[\widetilde{c}]_n^{\square}} d\Omega = \sum_i a_i \cdot \int_{\square^n} [\widetilde{c}_i]_n^{\square *} d\Omega \leq \sum_i |a_i| \, \mathrm{vol}_{\mathbb{H}^n} [\widetilde{c}_i]_n^{\square} \leq |c|_1^{\square} \cdot w_n^{\square},$$

where for the last inequality we are using Proposition 6.5. Passing to the infimum over all the representatives we have

$$\|M\|_{\mathrm{lf}}^{\square} \geq \frac{\mathrm{vol}(M)}{w_n^{\square}}. \qquad \qquad \square$$

Unfortunately, in dimension $n \geq 4$, it is unknown whether the bounds $w_n^{\square}$ and $v_n^{\square}$ match and what the exact relation with the volume of ideal regular cubes is.

6.4. **Proportionality principle for hyperbolic manifolds.** Analogously, to the case of ordinary simplicial volume, also cubical simplicial volume of hyperbolic manifolds satisfies a proportionality principle. For simplicity, we will restrict ourselves to the closed hyperbolic case. Let $n \in \mathbb{N}$ and let M and N be oriented closed connected hyperbolic n-manifolds. Then

$$\frac{\|M\|^{\square}}{\mathrm{vol}(M)} = \frac{\|N\|^{\square}}{\mathrm{vol}(N)}.$$

We sketch how this proportionality can be obtained by a discrete smearing map (similar to ordinary simplicial volume [31, 23]):

Let us first fix some notation (similar to Section 6.2): Let $\Gamma := \pi_1(M)$, let $G := \mathrm{Isom}^+(\mathbb{H}^n)$ and let μ_G be the Haar measure on G normalised by $\mu_G(G/\Gamma) = 1$. Moreover, let $D \subset \mathbb{H}^n$ be a measurable, strict fundamental domain for the deck transformation action of Γ on $\mathbb{H}^n \cong \widetilde{M}$ and let $\pi_M : \mathbb{H}^n \longrightarrow M$, $\pi_N : \mathbb{H}^n \longrightarrow N$ be the universal covering maps. Furthermore, let $\delta \in \mathbb{R}_{>0}$. We then choose a Γ-net $(\Lambda, (B_x)_{x \in \Lambda})$ of mesh size

at most δ, and we let X be the quotient of Λ^{J^n} with respect to the diagonal Γ-action. For $x \in \Lambda^{J^n}$ and a singular cube $c \colon \square^n \longrightarrow \mathbb{H}^n$, we set

$$A_x(c) := \big\{ g \in G \mid \forall_{j \in J^n} \ g \cdot c(v_j) \in B_{x_j} \big\}$$
$$a_x(c) := \mu_G(A_x)$$
$$c_x := [x]^\square \colon \square^n \longrightarrow \mathbb{H}^n.$$

Finally we define the discretised smearing map

$$\mathrm{smear}^\delta_{N,M} \colon C_n^\square(N; \mathbb{R}) \longrightarrow C_n^\square(M; \mathbb{R})$$
$$\mathrm{map}(\square^n, N) \ni c \longmapsto \sum_{[x] \in X} a_x(\widetilde{c}) \cdot \pi_M \circ c_x,$$

where $\widetilde{c}$ is a π_N-lift of c. Straightforward calculations show that $\mathrm{smear}^\delta_{N,M}$ has the following properties:

- The map $\mathrm{smear}^\delta_{N,M}$ is well-defined and induces a well-defined map on cubical singular homology.
- We have $\| \mathrm{smear}^\delta_{N,M} \| \leq 1$ with respect to the cubical ℓ^1-norms.
- If $c \in C_n^\square(N; \mathbb{R})$ is a smooth cubical fundamental cycle of N, then continuity of the volume of straight cubes in $\mathbb{H}^n$ shows that

$$\lim_{\delta \to 0} \int_{\mathrm{smear}^\delta_{N,M}(c)} d\Omega_M = \int_c d\Omega_N = \mathrm{vol}\, N.$$

An inductive smoothing procedure (e.g., through straightening) shows that cubical simplicial volume can be computed via smooth cubical fundamental cycles. Because integration determines the represented class in homology, we obtain for $\delta \to 0$ that

$$\frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot \|M\|^\square \leq \|N\|^\square.$$

By symmetry, this proves proportionality for closed hyperbolic manifolds.

In particular, for every $n \in \mathbb{N}$ there is a constant $C_n \in \mathbb{R}_{>0}$ such that all oriented closed connected hyperbolic n-manifolds M satisfy

$$\|M\|^\square = C_n \cdot \|M\|.$$

In view of Corollary 6.14 we have $C_3 = 1/5$ and from Theorems 6.13 and 6.17 we obtain

$$\frac{w_n^\square}{v_n^\triangle} \geq C_n \geq \frac{v_n^\square}{v_n^\triangle}.$$

Question 6.18. *What are the exact values of the factor C_n for $n \geq 4$? Do we have $\|M\|^\square = C_n \cdot \|M\|$ for* all *oriented closed connected n-manifolds or does the geometry of the manifolds affect the constant?*

7. Gluings

In this section, we will prove (sub-)additivity of generalised simplicial volumes under suitable gluings, provided that the normed model in question is sufficiently geometric.

Theorem 7.1. *Let (F, φ, ψ) be a geometric normed model of the singular chain complex and let N be an oriented compact n-manifold with UBC boundary (e.g., all components are tori). Moreover, let $\partial N = B_+ \sqcup B_-$ be a decomposition of the boundary (into possibly disconnected pieces) and let $f \colon B_- \longrightarrow B_+$ be an orientation reversing homeomorphism. Then the glued n-manifold*

$$M := N \big/ (B_- \cong_f B_+)$$

is closed, inherits an orientation of N, and satisfies

$$\|M\|^F \leq \|N^\circ\|^F_{\mathrm{lf}}.$$

We will explain the notion of geometric normed models in Section 7.1. In Section 7.2 we will prove that there are F-fundamental cycles of N whose boundaries have small F-norm if $\|N^\circ\|^F_{\mathrm{lf}} < \infty$. Using the uniform boundary condition, we will then be able to glue these boundaries with small chains (Section 7.3).

7.1. **Geometric normed models.**

Definition 7.2 (geometric normed model)**.** A functorial normed chain complex $F \colon \mathrm{Top} \longrightarrow \mathrm{Ch}^n_{\mathbb{R}}$ is *geometric* if it satisfies the following conditions for all $n \in \mathbb{N}$ and all spaces X:

- *Compact support.* For all chains $c \in F_n(X)$ there exist $K \in K(X)$ and $z \in F_n(K)$ such that $F_n(K \hookrightarrow X)(z) = c$ and $|z|^F \leq |c|^F$.
- *π_0-Additivity.* If $X = X_1 \sqcup X_2$, then for all chains $c \in F_n(X)$ there exist $z_1 \in F_n(X_1)$ and $z_2 \in F_n(X_2)$ such that

$$c = F_n(X_1 \hookrightarrow X)(z_1) + F_n(X_2 \hookrightarrow X)(z_2)$$
$$|c|^F = |z_1|^F + |z_2|^F.$$

- *Faithfulness.* If $A \subset X$, then $F_n(A \hookrightarrow X) \colon F_n(A) \longrightarrow F_n(X)$ is injective.

A normed model (F, φ, ψ) of $C_*(\, \cdot \, ; \mathbb{R})$ is *geometric* if F is geometric.

In view of faithfulness, for geometric normed models, we will usually omit the explicit notation of homomorphisms induced by inclusions of subspaces. The faithfulness condition is added for convenience; it could be replaced with weaker conditions (which, however, would lead to much more cumbersome notation).

Any faithful normed model on path-connected compact spaces can be extended to a geometric model on all spaces by taking ℓ^1-sums of path-connected components and then colimits over compact subspaces.

Proposition 7.3.
  (1) *The singular chain complex $C_*(\, \cdot \, ; \mathbb{R})$ is geometric.*
  (2) *The cubical singular chain complex $C^\square_*(\, \cdot \, ; \mathbb{R})$ is geometric.*

Proof. This easily follows from the construction of the singular and the cubical singular chain complex as well as the ℓ^1-norm. $\square$

In particular, Theorem 7.1 will hence apply to the locally finite versions of classical simplicial volume and cubical simplicial volume.
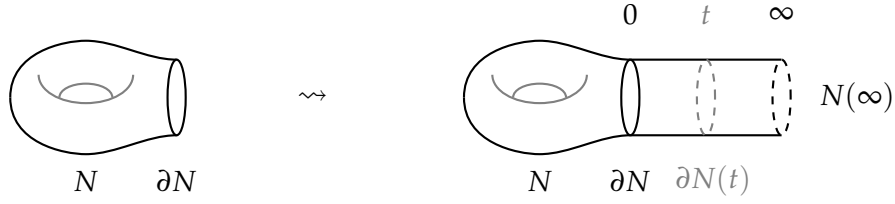
FIGURE 5. Stretching a manifold with boundary

7.2. Small boundaries.

Proposition 7.4. *Let (F, φ, ψ) be a geometric normed model of the singular chain complex and let N be an oriented compact n-manifold with $\|N^\circ\|_{\mathrm{lf}}^F < \infty$. Then for every $\varepsilon \in \mathbb{R}_{>0}$ there is a family $(c_t)_{t\in\mathbb{R}_{>0}}$ in $F_n(N)$ with the following properties:*

(1) *For all $t \in \mathbb{R}_{>0}$ we have $\partial c_t \in F_{n-1}(\partial N)$ and c_t represents a relative F-fundamental cycle of N in $F_n(N, \partial N)$.*

(2) *The family approximates the locally finite F-simplicial volume of N° via*
$$\lim_{t\to\infty} |c_t|^F \leq \|N^\circ\|_{\mathrm{lf}}^F + \varepsilon.$$

(3) *The family has small boundaries in the sense that*
$$\lim_{t\to\infty} |\partial c_t|^F = 0.$$

The proof is a straightforward adaption of the corresponding argument for ordinary simplicial volume [14, 20, p. 17, Chapter 6].

We first introduce some notation: Let N be a compact manifold. Then $N^\circ = N \setminus \partial N$ is homeomorphic [3, 8] to the stretched manifold (Figure 5)
$$N(\infty) := N \cup_{\partial N} (\partial N \times [0, \infty)).$$

For $t \in [0, \infty)$ we write
$$N(t) := N \cup_{\partial N} (\partial N \times [0, t]) \subset N(\infty),$$

which is homeomorphic (relative to the boundary) to N. Furthermore, the homotopy equivalences that collapse the cylinder $\partial N \times [0, \infty)$ to $\partial N \times \{0\}$ are denoted by
$$p_t \colon \bigl(N(\infty), N(\infty) \setminus N(t)\bigr) \longrightarrow (N, \partial N).$$

Clearly, the family $(N(t))_{t\in\mathbb{R}_{>0}}$ is cofinal in the directed set $K(N(\infty))$.

Proof of Proposition 7.4. Because N° is homeomorphic to $N(\infty)$ we also have $\|N(\infty)\|_{\mathrm{lf}}^F < \infty$. Let $c \in F_n^{\mathrm{lf}}(N(\infty))$ be a locally finite F-fundamental cycle of $N(\infty)$ with $|c|^F \leq \|N(\infty)\|_{\mathrm{lf}}^F + 1/2 \cdot \varepsilon < \infty$. In particular, $A(c) \neq \varnothing$ and there exists $\bar c \in A(c)$ with
$$\lim_{t\to\infty} |\bar c_{N(t)}|^F \leq |c|^F + \frac{1}{2} \cdot \varepsilon \leq \|N(\infty)\|_{\mathrm{lf}}^F + \varepsilon.$$

For $t \in \mathbb{R}_{>0}$ we now set
$$c_t := F_n(p_t)(\bar c_{N(t)}) \in F_n(N).$$

By construction, c_t is a chain with $\partial c_t \in F_{n-1}(\partial N)$ that represents a relative F-fundamental cycle of $(N, \partial N)$ in $F_n(N, \partial N)$. Moreover, $|c_t|^F \leq |\bar{c}_{N(t)}|^F$, and so $(c_t)_{t \in \mathbb{R}_{>0}}$ satisfies also the second claim.

We now prove the third claim for this family: By definition of $A(c)$, the family $(\bar{c}_{N(t)})_{t \in \mathbb{R}_{>0}}$ is $|\cdot|^F$-Cauchy. Because F has compact supports, there is a $t' \in \mathbb{R}_{>t+1}$ such that $\bar{c}_{N(t)} \in F_n(N(t'-1))$. By construction,

$$\partial(\bar{c}_{N(t')}) \in F_{n-1}\big(N(\infty) \setminus N(t')\big),$$

$$\partial(\bar{c}_{N(t)}) \in F_{n-1}\big(N(t'-1)\big),$$

$$\partial(\bar{c}_{N(t')} - \bar{c}_{N(t)}) = \partial\bar{c}_{N(t')} - \partial\bar{c}_{N(t)} \in F_{n-1}\big(N(\infty) \setminus N(t') \sqcup N(t'-1)\big).$$

Therefore, faithfulness and π_0-additivity of F show that

$$|\partial c_t|^F \leq |\partial(\bar{c}_{N(t)})|^F \leq |\partial(\bar{c}_{N(t')} - \bar{c}_{N(t)})|^F \leq \|\partial_n^F\| \cdot |\bar{c}_{N(t')} - \bar{c}_{N(t)}|^F.$$

Because of the Cauchy condition, the last term tends to 0 for $t \to \infty$. $\qquad\square$

7.3. **Gluing along UBC boundaries.** We will now glue the small boundaries provided by Proposition 7.4 to obtain efficient fundamental cycles of the glued manifold.

Proof of Theorem 7.1. If $\|N^\circ\|_{\mathrm{lf}}^F = \infty$, then there is nothing to prove. We will hence assume that $\|N^\circ\|_{\mathrm{lf}}^F$ is finite. Let $\varepsilon > 0$ and let $\kappa \in \mathbb{R}_{>0}$ be a common $(n-1)$-UBC constant for all boundary components of N with respect to F. Because F is geometric, we can choose a family $(c_t)_{t \in \mathbb{R}_{>0}}$ as in Proposition 7.4.

Let $t \in \mathbb{R}_{>0}$. Then $\partial c_t \in F_{n-1}(\partial N)$ is an F-fundamental cycle of $\partial N = B_+ \sqcup B_-$. Because F is assumed to be faithful and π_0-additive, we can split

$$\partial c_t = b_{t,+} + b_{t,-}$$

into F-fundamental cycles $b_{t,+} \in F_{n-1}(B_+)$, $b_{t,-} \in F_{n-1}(B_-)$ with

$$|b_{t,+}|^F + |b_{t,-}|^F = |\partial c_t|^F.$$

Moreover, because $f \colon B_- \longrightarrow B_+$ is an orientation reversing homeomorphism, the cycle $w_t := b_{t,+} + F_{n-1}(f)(b_{t,-}) \in F_{n-1}(B_+)$ is a boundary. Therefore, the uniform boundary condition guarantees the existence of a chain $b_t \in F_n(B_+)$ satisfying

$$\partial b_t = w_t \quad \text{and} \quad |b_t|^F \leq \kappa \cdot |w_t|^F \leq \kappa \cdot |\partial c_t|^F.$$

We now consider the gluing projection $\pi \colon N \longrightarrow M$ and the chain

$$z_t := F_n(\pi)(c_t - b_t) \in F_n(M).$$

By construction, $\partial z_t = 0$ and, as can be easily checked locally at points in N°, the cycle z_t is an F-fundamental cycle of M. Moreover,

$$|z_t|^F \leq |c_t|^F + \kappa \cdot |\partial c_t|^F,$$

and thus the properties from Proposition 7.4 imply

$$\|M\|^F \leq \liminf_{t \to \infty} |z_t|^F \leq \|N^\circ\|_{\mathrm{lf}}^F + \varepsilon + \kappa \cdot 0.$$

Taking $\varepsilon \to 0$ gives the desired estimate $\|M\|^F \leq \|N^\circ\|_{\mathrm{lf}}^F$. $\qquad\square$

Alternatively, one could also try to translate the equivalence theorem [14, 6] for weighted semi-norms to the setting of normed models. However, we prefer the argument above because it is more direct and more geometric.

8. Cubical simplicial volume of 3-manifolds

In this section, we complete the proof of Theorem 1.1, using the strategy described in the introduction. We review the decomposition of 3-manifolds in Section 8.1 and then prove Theorem 1.1 in Section 8.2.

8.1. **Decomposition of 3-manifolds.** We recall the Geometrization Theorem describing the decomposition of a 3-manifold in hyperbolic and Seifert pieces.

Definition 8.1 (irreducible 3-manifold). A 3-manifold M is called *irreducible* if every embedded 2-sphere in M bounds an embedded 3-ball in M.

Remark 8.2. It is straightforward that an orientable irreducible 3-manifold cannot be decomposed as a non-trivial connected sum of two manifolds.

Definition 8.3 (incompressible surface). Let S be a connected orientable surface different from a sphere or a disk that is properly embedded into a compact orientable 3-manifold M. Then S is *incompressible* if the map

$$i_* \colon \pi_1(S, x) \to \pi_1(M, x),$$

induced by the inclusion, is injective. Here, *properly* embedded surface means that $\partial S = S \cap \partial M$.

Theorem 8.4 (Geometrization Theorem). *Let M be a compact orientable irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ in M such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is hyperbolic or Seifert fibred. Furthermore, any such collection of tori with a minimal number of components is unique up to isotopy.*

For historical background and detailed references of this statement we refer to the literature [1, Chapter 1.7].

8.2. **Generalised simplicial volume of closed 3-manifolds and proof of Theorem 1.1.** We first formulate and prove a slight generalisation of Theorem 1.1 in the context of normed models. Theorem 1.1 will then be a special case.

Theorem 8.5. *Let (F, φ, ψ) be a geometric normed model of $C_*(\, \cdot \, ; \mathbb{R})$ and let*

$$C_F := \sup\left\{ \frac{v_3^{\triangle} \cdot \|N\|_{\mathrm{lf}}^F}{\mathrm{vol}(N)} \;\middle|\; N \text{ is a complete oriented connected hyperbolic 3-manifold} \right.$$
$$\left. \text{of finite volume} \right\}.$$

Then $C_F \leq \|\psi_3^{\Delta^3}\|$ and for all oriented closed 3-manifolds M we have

$$\|M\|^F \leq C_F \cdot \|M\|.$$

Proof. Because $\|M\|_{\mathrm{lf}} = \mathrm{vol}(M)/v_3^\triangle$ holds for all complete hyperbolic 3-manifolds of finite volume (Theorem 2.7), we obtain $C_F \leq \|\psi_3^{\Delta^3}\|$ from Proposition 3.9.

By the Geometrization Theorem 8.4, there exist $m, n \in \mathbb{N}$ and disjointly embedded incompressible tori $T_1, \ldots, T_m$ in M such that each of the components $N_1, \ldots, N_n$ obtained by cutting M along $T_1 \cup \cdots \cup T_m$ is Seifert fibred or admits a complete finite volume hyperbolic structure on its interior.

Let $S \subset \{1, \ldots, n\}$ be the set of indices belonging to the Seifert fibred pieces (whence the indices in $H := \{1, \ldots, n\} \setminus S$ belong to the hyperbolic pieces). By Corollary 5.5, the definition of C_F, and Theorem 2.7 we obtain

$$\|N_j^\circ\|_{\mathrm{lf}}^F \leq \begin{cases} 0 & \text{if } j \in S \\ C_F \cdot \dfrac{\mathrm{vol}(N_j^\circ)}{v_3^\triangle} & \text{if } j \in H \end{cases}$$

for all $j \in \{1, \ldots, n\}$. All these values are finite. Therefore, we can apply Theorem 7.1 and obtain

$$\|M\|^F \leq \|N_1^\circ \sqcup \cdots \sqcup N_n^\circ\|_{\mathrm{lf}}^F = \sum_{j=1}^n \|N_j^\circ\|_{\mathrm{lf}}^F \leq C_F \cdot \sum_{j \in H} \frac{\mathrm{vol}(N_j^\circ)}{v_3^\triangle}.$$

The last sum is equal to $\|M\|$ by Theorem 2.8. Thus, $\|M\|^F \leq C_F \cdot \|M\|$, as claimed. $\qquad\square$

As a special case, we obtain Theorem 1.1:

Proof of Theorem 1.1. Let M be an oriented closed 3-manifold. In view of the lower bound established in Corollary 4.8, it suffices to prove the upper bound $\|M\|^\square \leq 1/5 \cdot \|M\|$.

By Proposition 7.3, all cubical models (F, φ, ψ) are geometric, and so Theorem 8.5 applies to cubical simplicial volume. On the other hand, Corollary 6.14 tells us that $C_F = 1/5$. Therefore, $\|M\|^\square \leq 1/5 \cdot \|M\|$. $\qquad\square$

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