RESIDUALLY FINITE CATEGORIES

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Abstract. We introduce the notion of residual finiteness for categories. In analogy with the group-theoretic setting, we prove that free categories and finitely generated subcategories of finite-dimensional vector spaces are residually finite. Moreover, finitely generated residually finite categories are Hopfian and finitely presented residually finite categories have solvable word problem.

1. Introduction

Classical mathematics is often concerned with infinite, potentially huge, structures. From a more computational point of view, it is therefore essential to ask which properties can be tested through transformations to finite structures. For example, in group theory, this is captured by the notion of residual finiteness [6]: A group is residually finite, if equality of group elements can be tested via group homomorphisms to finite groups [1, Definition 2.1.1, Proposition 2.1.2].

Definition 1.1 (residually finite group)**.** A group G is *residually finite*, if for all $f,\ g \in G$ with $f \neq g$, there exists a *finite* group D and a group homomorphism $\varphi\colon G \longrightarrow D$ with

$$\varphi(f) \neq \varphi(g).$$

In this note, in analogy with the group-theoretic setting, we introduce the following notion of residual finiteness for categories, based on testing via functors to finite categories:

Definition 1.2 (residually finite category)**.** A category C is *residually finite*, if for all morphisms f and g in C with $f \neq g$, there exists a *finite* category D and a functor $F\colon C \longrightarrow D$ with

$$F(f) \neq F(g).$$

Here, a category C is *finite* if $\mathrm{Ob}(C)$ is finite and for all $X, Y \in \mathrm{Ob}(C)$ also $\mathrm{Mor}_C(X,Y)$ is finite. In Section 2.2, we will say more about the notions of equality of morphisms and finiteness in categories.

For groups (and the canonical interpretations of groups as categories with a single object), the notions of residual finiteness from Definition 1.1 and Definition 1.2 coincide (Proposition 4.1). Moreover, Definition 1.2 also subsumes a notion of residual finiteness for monoids (via categories that only

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contain a single object) [2] and groupoids (categories all of whose morphisms are isomorphisms).

(Non-)Examples. Important examples of residually finite groups are free groups [1, Theorem 2.3.1] and finitely generated linear groups [7]. Analogously, we show that the following categories are residually finite:

- free categories (Proposition 4.7)
- finitely generated subcategories of the category of finite-dimensional vector spaces over a field (Corollary 4.18)

Groups are residually finite if and only if they embed into a product of finite groups [1, Corollary 2.2.6]. Similarly, we prove that a small category is residually finite if and only if it is equivalent to a subcategory of a product of finite categories (Corollary 3.7).

In contrast, the following categories are *not* residually finite (and thus are not amenable to systematic testing via functors to finite categories):

- the category of finite sets (Proposition 4.8)
- the category of finite-dimensional vector spaces over a field (Proposition 4.11)
- the simplex category Δ (Proposition 4.9)

Using residual finiteness. Two classical applications of residual finiteness in group theory are:

- All finitely generated residually finite groups are Hopfian [1, Theorem 2.4.3] (i.e., every surjective endomorphism is already an isomorphism); this shows, for instance, that every self-map of an aspherical oriented closed connected manifold with residually finite group induces an isomorphism on fundamental groups and thus is a homotopy equivalence.

  Similarly, this holds also in other algebraic situations, e.g., for rings and modules [4, 12].
- All finitely presented residually finite groups have solvable word problem [8].

We establish the corresponding versions for residually finite categories: All finitely generated residually finite categories are Hopfian (Theorem 5.1). All finitely presented residually finite categories have solvable word problem (Theorem 5.2); this might be of interest when modelling calculi, deductional systems, or rewriting systems in categories [3].

Organisation of this article. In Section 2, we clarify our setup of category theory. Section 3 contains basic inheritance properties of residual finiteness of categories. Examples of residually finite categories are discussed in Section 4. Finally, in Section 5, we show that finitely generated residually finite categories are Hopfian and that finitely presented residually finite categories have solvable word problem.

2. Setup

2.1. **Categories.** For simplicity and concreteness, we will use classical class-set theory (such as NBG [11]) as ambient theory for category theory; all categories will be *locally small* (i.e., while the objects of a category form a *class*, the morphisms between any two objects form a *set*). A category is *small* if the class of objects is a set. Reference to the axiom of choice will be made explicit. Most of this note can be adapted in a straightforward manner to more synthetic settings or settings with more stages of "sizes" of sets.

2.2. **Equality and finiteness.** Equality of morphisms and finiteness of categories play a central role in the definition of residual finitenes (Definition 1.2). As equality of objects in categories is a delicate subject, we briefly comment on two popular choices:

- Using equality of objects: Let $X, X', Y, Y' \in \operatorname{Ob}(C)$ and let $f \in \operatorname{Mor}_C(X, Y)$, $g \in \operatorname{Mor}_C(X', Y')$. Then the morphisms f and g in C are considered equal if and only if $X = X'$ and $Y = Y'$ and $f = g$ (in the set $\operatorname{Mor}_C(X, Y) = \operatorname{Mor}_C(X', Y')$). In particular, morphisms are supposed to know (at least implicitly) about their domain and target objects.

  In this setting, we can use the naive notion of finiteness of categories: A category C is finite if $\operatorname{Ob}(C)$ is finite and if for all $X, Y \in \operatorname{Ob}(C)$, the set $\operatorname{Mor}_C(X, Y)$ is finite.
- Without using equality of objects: If we want to avoid to speak of equality of objects (in order to obtain equivalence-robust notions), we will only define (in)equality for morphisms in the same morphism set. In this version, it does not make sense to talk about (in)equality of morphisms that have different domains/targets.

  Moreover, in this setting, a category should be considered to be finite if it is weakly finite in the following sense: A category C is *weakly finite* if $\operatorname{Ob}(C)$ contains only finitely many isomorphism classes of objects and if for all $X, Y \in \operatorname{Ob}(C)$, the set $\operatorname{Mor}_C(X, Y)$ is finite.

We will adopt the first, naive, semantics (using equality of objects); in particular, we also will talk about finite generation of categories, etc. in the naive sense. We can then compare the definition using the first semantics and the second semantics. In this case, both interpretations result in the same notion of residual finiteness:

Proposition 2.1. *Let C be a category, let X, X', Y, $Y' \in \operatorname{Ob}(C)$, and let $f \in \operatorname{Mor}_C(X, Y)$, $g \in \operatorname{Mor}_C(X', Y')$ with $X \neq X'$ or $Y \neq Y'$. Then there exists a finite category D and a functor $F\colon C \longrightarrow D$ with $F(f) \neq F(g)$.*

Proof. We consider the complete directed graph on the set $\{X, X', Y, Y'\}$ (with at most four elements) and its associated category D, which is finite (Section 2.3). We then define the following functor $F\colon C \longrightarrow D$:

- on objects: For $Z \in \operatorname{Ob}(C)$, we set
$$F(Z) := \begin{cases} Z & \text{if } Z \in \{X, X', Y, Y'\} \\ X & \text{otherwise.} \end{cases}$$
- on morphisms: For $Z, Z' \in \operatorname{Ob}(C)$ and $h \in \operatorname{Mor}_C(Z, Z')$, we define $F(h)$ as the unique morphism in D from $F(Z)$ to $F(Z')$.

By construction, $F(f) \neq F(g)$ (because the target or the domain objects in D are not equal). □

In particular, this also shows that different objects can always be separated by functors to finite categories.

Proposition 2.2. *Let C be a category, let f and g be morphisms in C with $f \neq g$, let D be a weakly finite category, and let $F\colon C \longrightarrow D$ be a functor with $F(f) \neq F(g)$. Then there exists a finite category D' and a functor $F'\colon C \longrightarrow D'$ with $F'(f) \neq F'(g)$.*

Proof. Every weakly finite category is equivalent to a finite category (one can use the axiom of choice or use a more constructive notion of weak finiteness that includes such an equivalence). Hence, there is a finite category D' and a faithful functor $G\colon D \longrightarrow D'$. We can then take $F' := G \circ F$. □

2.3. **Graphs and quivers.** A *directed graph* is a pair (V, E) consisting of a set V (the *vertices*) and a set $E \subset V \times V$ (the *edges*). If $X := (V, E)$ is a directed graph, then the category C_X associated with X consists of

- objects: We set $\operatorname{Ob}(C_X) := V$.
- morphisms: If u, v are objects in $\operatorname{Ob}(C_X)$, then we set

$$\operatorname{Mor}_{C_X}(u, v) := \begin{cases} \bullet & \text{if there exists a directed path from } u \text{ to } v \text{ in } X \\ \emptyset & \text{otherwise.} \end{cases}$$

- composition of morphisms: The composition of morphism is uniquely determined by the definition of the morphism sets and the fact that concatenation of directed paths in X witnesses that the composition of composable morphisms exists. If $v \in \operatorname{Ob}(C_X)$, then the unique element of $\operatorname{Mor}_{C_X}(v, v)$ is the identity morphism of v.

More generally, a *quiver* is a quadruple (V, E, s, t) consisting of a set V (the *vertices*), a set E (the *edges*), and two maps $s, t\colon E \longrightarrow V$ (the *source* and *target* map, respectively).

3. Basic properties

In order to work efficiently with residually finite categories, we first establish some basic inheritance results.

3.1. Isomorphisms, equivalences, subcategories.

Proposition 3.1. *Let C, C' be isomorphic categories. If C is residually finite, then also C' is residually finite.*

Proof. Composing separating functors with an isomorphism $C' \longrightarrow C$ proves the claim. □

More generally, the same argument shows that residual finiteness is inherited under equivalences of categories:

Proposition 3.2. *Let C and C' be equivalent categories. If C is residually finite, then also C' is residually finite.*

Proof. Let $G\colon C' \longrightarrow C$ be an equivalence of categories and let f and g be morphisms in C' with $f \neq g$. As an equivalence of categories, G is faithful; hence, $G(f) \neq G(g)$ in C. Because C is residually finite, there exists a finite category D and a functor $F\colon C \longrightarrow D$ with $F(G(f)) \neq F(G(g))$. Thus, the functor $F \circ G\colon C' \longrightarrow D$ separates f and g. □

Corollary 3.3. *Let C be a category and let C' be a skeleton of C. Then C is residually finite if and only if C' is residually finite.*

Proof. As a skeleton of C, the category C' is equivalent to C (depending on the setting, we can either use the axiom of choice or a constructive notion of skeleton that requires the existence of such an equivalence). We then only need to use the fact that residual finiteness is inherited under equivalences of categories (Proposition 3.2). □

Proposition 3.4. *Subcategories of residually finite categories are residually finite.*

Proof. This is immediate from the definition (we only need to restrict the corresponding separating functors). □

As in the case of groups, in a residually finite category, we can separate any finite number of morphisms:

Proposition 3.5. *Let C be a residually finite category, let $n \in \mathbb{N}_{\geq 2}$, and let $f_1, \dots, f_n$ be n different morphisms in C. Then there exists a finite category D and a functor $F \colon C \longrightarrow D$ such that the morphisms $F(f_1), \dots, F(f_n)$ are all different.*

Proof. Because C is residually finite, for all $j, k \in \{1, \dots, n\}$ with $j < k$, there exists a finite category $D_{j,k}$ and a functor $F_{j,k} \colon C \longrightarrow D_{j,k}$ with

$$F_{j,k}(f_j) \neq F_{j,k}(f_k).$$

Then also the product category

$$D := \prod_{k=1}^{n} \prod_{j=1}^{k-1} D_{j,k}$$

is finite and the product functor $F := \prod_{k=1}^{n} \prod_{j=1}^{k-1} F_{j,k} \colon C \longrightarrow D$ has the desired property. □

3.2. **Products.**

Proposition 3.6.

(1) *If C and D are residually finite categories, then also $C \times D$ is a residually finite category.*
(2) *If I is a set and $(C_i)_{i \in I}$ is a family of residually finite small categories, then also $\prod_{i \in I} C_i$ is residually finite.*

Proof. We only prove the second part (the first part can be proved in the same way). Let f and g be morphisms in $C := \prod_{i \in I} C_i$ with $f \neq g$. By definition of C, there exist families $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$, where f_i and g_i are morphisms in C_i with $f = (f_i)_{i \in I}$ and $g = (g_i)_{i \in I}$. As $f \neq g$, there is an $i \in I$ with $f_i \neq g_i$. Because C_i is residually finite, there exists a finite category D and a functor $F \colon C_i \longrightarrow D$ with $F(f_i) \neq F(g_i)$. Let $\pi_i \colon C \longrightarrow C_i$ denote the projection functor. Then the composition $\overline{F} := F \circ \pi_i \colon C \longrightarrow D$ satisfies

$$\overline{F}(f) = F(f_i) \neq F(g_i) = \overline{F}(g),$$

as desired. □

Corollary 3.7. *Let C be a small category. Then the following are equivalent:*

(1) *The category C is residually finite.*
(2) *The category C is equivalent to a subcategory of a product (over a set) of finite categories.*

Proof. *Ad* (1) $\Longrightarrow$ (2). Let C be residually finite. We consider the index set

$$I := \bigl\{(f, g) \bigm| f,\, g \text{ morphisms in } C \text{ with } f \neq g\bigr\}$$

(C is small, so this is a set). Because C is residually finite, for each $i = (f, g) \in I$, there exists a finite category D_i and a functor $F_i \colon C \longrightarrow D_i$ with

$$F_i(f) \neq F_i(g).$$

Then, the family $(F_i)_{i\in I}$ defines a functor $F \colon C \longrightarrow \prod_{i\in I} C_i$, which is faithful (by construction). More precisely, the existence of such a functor is guaranteed by the axiom of choice.

Hence, C is equivalent to a subcategory (namely the image category of F) of the product $\prod_{i\in I} C_i$ of finite categories.

Ad (2) $\Longrightarrow$ (1). Products of finite categories are residually finite (Proposition 3.6), subcategories of residualy finite categories are residually finite (Proposition 3.4), and residual finiteness is preserved under equivalences (Proposition 3.2). □

4. Basic examples

4.1. **Groups and groupoids.** If G is a group, then we can consider the associated category C_G, which consists of a single object $\bullet$ and whose morphisms are defined by $\operatorname{Mor}_{C_G}(\bullet, \bullet) := G$ (with the composition given by the composition in G). For groups, the residual finiteness notions in Definition 1.1 and Definition 1.2 coincide:

Proposition 4.1. *Let G be a group. Then G is residually finite if and only if the associated category C_G is residually finite.*

Proof. Let G be residually finite and let f, g be morphisms in C_G with $f \neq g$; in particular, $f, g \in \operatorname{Mor}_{C_G}(\bullet, \bullet) = G$. Because G is residually finite, there is a finite group D and a group homomorphism $\varphi \colon G \longrightarrow D$ with $\varphi(f) \neq \varphi(g)$. The homomorphism φ induces a functor $F \colon C_G \longrightarrow C_D$ mapping the only object $\bullet$ of C_G to the one of C_D and using φ on the morphisms:

$$\begin{aligned} \operatorname{Mor}_{C_G}(\bullet, \bullet) = G &\longrightarrow H = \operatorname{Mor}_{C_D}(\bullet, \bullet) \\ h &\longmapsto \varphi(h). \end{aligned}$$

As D is finite, also the category C_D is finite. Moreover, by construction,

$$F(f) = \varphi(f) \neq \varphi(g) = F(g).$$

Hence, the category C_G is residually finite.

Conversely, let the category C_G be residually finite and let $f, g \in G$ with $f \neq g$. Because C_G is residually finite and $G = \operatorname{Mor}_{C_g}(\bullet, \bullet)$, there exists a finite category D and a functor $F \colon C_G \longrightarrow D$ with $F(f) \neq F(g)$. We then consider the (finite) group

$$H := \operatorname{Aut}_D(X),$$

where $X := F(\bullet)$; the functor F induces a group homomorphism

$$\begin{aligned}\varphi\colon G = \operatorname{Aut}_{C_G}(\bullet) &\longrightarrow \operatorname{Aut}_D(X) = H\\ h &\longmapsto F(h).\end{aligned}$$

Because the category D is finite, also the group H is finite. Moreover, by construction $\varphi(f) = F(f) \neq F(g) = \varphi(g)$. Hence, the group G is residually finite. □

Example 4.2 (finitary symmetric group)**.** Let $\operatorname{FSym}_\infty$ be the group of permutations of $\mathbb{N}$ with finite support. Then $\operatorname{FSym}_\infty$ is *not* residually finite (the subgroup of even permutations in $\operatorname{FSym}_\infty$ is simple and infinite). A similar consideration will show that the category FSet of finite sets is *not* residually finite (Proposition 4.8).

Corollary 4.3. *Let C be a category and let $X \in \operatorname{Ob}(C)$. If C is residually finite, then $\operatorname{Aut}_C(X)$ is a residually finite group.*

Proof. The subcategory C' of C consisting of the object X and the C-automorphisms of X is isomorphic to $C_{\operatorname{Aut}_C(X)}$. If C is residually finite, then also this subcategory C' is residually finite (Proposition 3.4); thus, also $C_{\operatorname{Aut}_C(X)}$ is residually finite (Proposition 3.1). Therefore, the group $\operatorname{Aut}_C(X)$ is residually finie (Proposition 4.1). □

In general, the converse of Corollary 4.3 does *not* hold (Proposition 4.8, Proposition 4.9). However, for groupoids (i.e., small categories all of whose morphisms are isomorphisms), we obtain:

Corollary 4.4. *Let C be a groupoid.*

1. *If C is connected and $X \in \operatorname{Ob}(C)$, then C is residually finite if and only if the group $\operatorname{Aut}_C(X)$ is residually finite.*
2. *The following are equivalent:*
   - (a) *The category C is residually finite.*
   - (b) *For each $X \in \operatorname{Ob}(C)$, the group $\operatorname{Aut}_C(X)$ is residually finite.*

Proof. For the first part, let C' be the full subcategory of C generated by X. Because C is a groupoid, C' is a skeleton of C. Moreover, C' is isomorphic to the category $C_{\operatorname{Aut}_C(X)}$. Applying Corollary 3.3 and Proposition 4.1 finishes the proof of the first part.

For the second part, Corollary 4.3 proves the implication (a) $\Longrightarrow$ (b). For the converse implication, we can argue as follows: Choosing (via the axiom of choice) one object in each connected component of C leads to a skeleton of C. Assuming (b), this skeleton is easily seen to be residually finite (Proposition 4.1 and collapsing all but one components to the one-object category $C_{\{1\}}$). Hence, applying Proposition 3.3 and Proposition 4.1 shows that C is residually finite as well. □

4.2. **Graphs, posets, and free categories.**

Proposition 4.5. *Let X be a directed graph. Then the associated category C_X (Section 2.3) is residually finite.*

Proof. Let $\overline{X}$ be the directed graph

$$\overline{X} := \bigl(V, \{(u,v) \mid (u,v) \in E \vee (v,u) \in E\}\bigr)$$

obtained from X by adding for each edge also the inverse edge. Then the category C_X is a subcategory of $C_{\overline{X}}$, which is a groupoid. Moreover, for each vertex v of $\overline{X}$, the automorphism group $\operatorname{Aut}_{C_{\overline{X}}}(v)$ is trivial, whence residually finite. Therefore, $C_{\overline{X}}$ is residually finite (Corollary 4.4) and so also C_X is residually finite (Proposition 3.4).

Of course, alternatively, we can also invoke the more general statement on free categories (Proposition 4.7). □

Corollary 4.6. *Let I be a poset. Then the poset category of I is residually finite.*

Proof. If I is a poset, then the poset category of I is the same as the category associated to the directed graph

$$\bigl(I, \{(x,y) \mid x,y \in I, x \leq y\}\bigr).$$

Hence, by Proposition 4.5, the poset category of I is residually finite. □

Free groups are residually finite [1, Theorem 2.3.1]; we will now establish the corresponding result for categories.

Proposition 4.7. *Let X be a quiver. Then the free category F_X, freely generated by X, is residually finite.*

Proof. Let f and g be morphisms in F_X with $f \neq g$. We can view f and g as finite (directed) paths in X; because of $f \neq g$, they differ in at least one edge.

Let Y be the quiver obtained from X by identifying all vertices to a single vertex (and keeping distinct edges distinct) and let $\pi\colon X \longrightarrow Y$ be the corresponding quiver projection. Because F_X is the free category, freely generated by X, there exists a functor $P\colon F_X \longrightarrow F_Y$ that induces the quiver morphism π on the underlying quivers. Because f and g are different, also the associated paths in Y are different.

Because Y is a quiver with a single vertex, the category F_Y is the category associated with the free monoid, freely generated by the edges of Y. Hence, we can view F_Y as subcategory of the category C associated with the free group G, freely generated by the edges of Y. Because G is residually finite [1, Theorem 2.3.1], also the category C is residually finite (Proposition 4.1). Hence, there exists a finite category D and a functor $F\colon C \longrightarrow D$ with $F(f) \neq F(g)$. Then the composition

$$F \circ P\colon F_X \longrightarrow D$$

separates f and g. □

4.3. **Sets and simplices.**

Proposition 4.8. *The category* FSet *of finite sets is* not *residually finite.*

Proof. We consider

$$f := \mathrm{id}_{\{1,2,3\}} \in S_3 = \operatorname{Aut}_{\mathsf{FSet}}\bigl(\{1,2,3\}\bigr)$$
$$g := (1\ 2\ 3)_3 \in S_3 = \operatorname{Aut}_{\mathsf{FSet}}\bigl(\{1,2,3\}\bigr)$$

(but in fact any two different maps would work). Let D be a finite category and let $F\colon \mathsf{FSet} \longrightarrow D$ be a functor. We will now show that $F(f) = F(g)$, using a detour via bigger sets.

If $N \in \mathbb{N}$, then the alternating group A_N is a subset of $\operatorname{Aut}_{\mathsf{FSet}}(\{1,\dots,N\})$. Because D is finite, there exists an $N \in \mathbb{N}_{\geq 5}$ with $\bigl|F(A_N)\bigr| < |A_N|$. Because F restricted to A_N is a group homomorphism and because A_N is simple, it follows that

$$F(h) = \mathrm{id}_{F(\{1,\dots,N\})}$$

for all $h \in A_N$. Let $i\colon \{1,2,3\} \longrightarrow \{1,\dots,N\}$ be the inclusion and let $\pi\colon \{1,\dots,N\} \longrightarrow \{1,2,3\}$ be the projection that sends all $j \in \{4,\dots,N\}$ to 3. Then

$$\begin{aligned} f &= \pi \circ i \\ g &= \pi \circ (1\ 2\ 3)_N \circ i. \end{aligned}$$

Because $(1\ 2\ 3)_N$ defines an element of A_N, we obtain

$$\begin{aligned} F(g) &= F(\pi \circ (1\ 2\ 3)_N \circ i) \\ &= F(\pi) \circ F\bigl((1\ 2\ 3)_N\bigr) \circ F(i) \\ &= F(\pi) \circ \mathrm{id}_{F(\{1,\dots,N\})} \circ F(i) \\ &= F(\pi) \circ F(i) \\ &= F(\pi \circ i) \\ &= F(f). \end{aligned}$$

Therefore, f and g cannot be separated by a functor to a finite category. Hence, FSet is *not* residually finite.

Alternatively, one could also argue via the simplex category Δ as subcategory of FSet (Proposition 4.9). □

It should be noted that each object in FSet has a (residually) finite automorphism group. Hence, in the case of FSet the non-residual finiteness originates in the overal interaction of "small" objects with "big" objects.

More drastically, the simplex category Δ is not residually finite (even though all objects have trivial automorphism group). We will use the following version of Δ: Objects are all sets of the form $[n] := \{0,\dots,n\}$ with $n \in \mathbb{N}$ and morphisms are all monotonically increasing functions.

Proposition 4.9. *The simplex category Δ is* not *residually finite.*

Proof. We consider

$$\begin{aligned} f &:= \mathrm{id}_{[1]}\colon [1] \longrightarrow [1] \\ g &:= \mathrm{const}_0\colon [1] \longrightarrow [1]. \end{aligned}$$

Let D be a finite category and let $F\colon \Delta \longrightarrow D$ be a functor. We will now show that $F(f) = f(g)$, using a detour via bigger sets.

Because D is finite, there exists an $N \in \mathbb{N}_{\geq 2}$ such that $N > |\operatorname{Mor}_D(X,X)|$ holds for all $X \in \operatorname{Ob}(D)$. For each $y \in \{1,\dots,N\}$, we set

$$\begin{aligned} \pi_y \colon [N] &\longrightarrow [N] \\ j &\longmapsto \begin{cases} 0 & \text{if } j = 0 \\ y & \text{if } j \geq 1, \end{cases} \end{aligned}$$

which is a morphism in Δ. By the choice of N, there exist $y, z \in \{1,\dots,N\}$ with

$$y < z \quad \text{and} \quad F(\pi_y) = F(\pi_z).$$

Moreover, we look at the following morphisms in Δ:

$$\begin{aligned} i \colon [1] &\longrightarrow [N] \\ x &\longmapsto x \\ r \colon [N] &\longrightarrow [1] \\ j &\longmapsto \begin{cases} 0 & \text{if } j \in \{0,\dots,y\} \\ 1 & \text{if } j > y. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} f &= r \circ \pi_z \circ i \\ g &= r \circ \pi_y \circ i, \end{aligned}$$

and we conclude

$$\begin{aligned} F(f) &= F(r \circ \pi_z \circ i) \\ &= F(r) \circ F(\pi_z) \circ F(i) \\ &= F(r) \circ F(\pi_y) \circ F(i) \\ &= F(r \circ \pi_y \circ i) \\ &= F(g). \end{aligned}$$

Therefore, f and g cannot be separated by a functor to a finite category, which shows that Δ is *not* residually finite. □

4.4. **Module categories.** A key example of residually finite groups are finitely generated linear groups [7, 9]. It is therefore natural to wonder about the residual finiteness of module categories. In the following, a *ring with unit* is a not necessarily commutative ring R that has a multiplicative unit 1 with $1 \neq 0$.

Definition 4.10 (module categories)**.** Let R be a ring with unit and let $n \in \mathbb{N}$. Then we introduce the following categories:

- ${}_R\mathsf{FMod}$: The category of all finitely generated free left R-modules and R-linear maps.
- ${}_R\mathsf{FMod}|_n$: The category of all finitely generated free left R-modules freely generated by at most n elements and R-linear maps.

Proposition 4.11. *Let R be a ring with unit. Then the category ${}_R\mathsf{FMod}$ is* not *residually finite.*

Proof. The category ${}_R\mathsf{FMod}$ contains a subcategory that is isomorphic to the category FSet of finite sets (e.g., taking the objects R^n with $n \in \mathbb{N}$ and all R-linear maps given by right multiplication by matrices that only have entries in $\{0, 1\}$ and all of whose rows contain exactly one 1). Because FSet is *not* residually finite (Proposition 4.8), we obtain that ${}_R\mathsf{FMod}$ is not residually finite (Proposition 3.4). ☐

However, we will see that adding appropriate finiteness conditions on the category, will imply residual finiteness (Corollary 4.18).

Proposition 4.12. *Let R be a ring with unit whose underlying additive Abelian group is* not *residually finite. Then the category ${}_R\mathsf{FMod}\,|_2$ is* not *residually finite.*

Proof. In view of Corollary 4.3, we only need to show that the automorphism group $\operatorname{Aut}_{{}_R\mathsf{FMod}\,|_2}(R^2)$ is *not* residually finite. Because $\operatorname{Aut}_{{}_R\mathsf{FMod}\,|_2}(R^2)$ is isomorphic to the group $\mathrm{GL}_2(R)$, it suffices to show that $\mathrm{GL}_2(R)$ is not residually finite. As

$$\begin{aligned} R &\longrightarrow \mathrm{GL}_2(R) \\ x &\longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is an injective group homomorphism from the additive group R to $\mathrm{GL}_2(R)$ and because the additive Abelian group R is not residually finite, also $\mathrm{GL}_2(R)$ is not residually finite. ☐

Example 4.13. The category ${}_\mathbb{Q}\mathsf{FMod}\,|_2$ is *not* residually finite because the additive group $\mathbb{Q}$ is not residually finite [1, Example 2.1.9].

Proposition 4.14. *Let R be a residually finite ring with unit and let $N \in \mathbb{N}$. Then the category ${}_R\mathsf{FMod}\,|_N$ is residually finite.*

Proof. The full subcategory of ${}_R\mathsf{FMod}\,|_N$ generated by $0, R, \dots, R^N$ is a skeleton of ${}_R\mathsf{FMod}\,|_N$. In view of Corollary 3.3, we can therefore restrict attention to morphisms between these modules.

Let f and g be morphisms in ${}_R\mathsf{Mod}\,|_N$ with $f \neq g$. In view of Proposition 2.1, we may assume that f and g have the same domain R^n and the same target R^m.

Let A and $B \in M_{n\times m}(R)$ be the matrices representing f and g, respectively, with respect to the standard bases (via right multiplication). Because $f \neq g$, there exist $j \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ with $A_{j,k} \neq B_{j,k}$. Because the ring R is residually finite, there exists a finite ring S and a ring epimorphism $\pi\colon R \longrightarrow S$ such that $\pi(A_{j,k}) \neq \pi(B_{j,k})$. In particular,

$$\mathrm{id}_S \otimes_R f \neq \mathrm{id}_S \otimes_R g$$

in ${}_S\mathsf{FMod}\,|_N$ (where the tensor product is taken with respect to π).

Because the ring S is finite, also the category $D := {}_S\mathsf{Mod}\,|_N$ is finite and the reduction functor $S \otimes_R \,\cdot\, : {}_R\mathsf{FMod}\,|_N \longrightarrow D$ separates f and g. ☐

Corollary 4.15. *Let R be a residually finite ring with unit and let C be a finitely generated subcategory of ${}_R\mathsf{FMod}$. Then C is residually finite.*

Proof. Because C is finitely generated, it contains only finitely many objects of ${}_R\mathsf{FMod}$. Hence, there exists an $N \in \mathbb{N}$ such that C is a subcategory of ${}_R\mathsf{FMod}|_N$. We can now apply the previous Proposition 4.14 and Proposition 3.4. □

Example 4.16. For each $N \in \mathbb{N}$, the category ${}_\mathbb{Z}\mathsf{Mod}|_N$ is residually finite: The family of all reductions $\mathbb{Z} \longrightarrow \mathbb{F}_p$ modulo prime numbers p shows that $\mathbb{Z}$ is a residually finite ring. We can therefore apply Proposition 4.14.

Corollary 4.17. *Let R be a finitely generated commutative ring with unit and let C be a finitely generated subcategory of ${}_R\mathsf{FMod}$. Then C is residually finite.*

Proof. Every finitely generated commutative ring is residually finite [10] and thus we can apply Corollary 4.15. □

Finally, we obtain the category-version of Malcev's theorem on linear groups:

Corollary 4.18. *Let k be a field and let C be a finitely generated subcategory of ${}_k\mathsf{FMod}$. Then C is residually finite.*

Proof. Because C is finitely generated, there exists a finitely generated commutative ring R such that all morphisms in C can be represented by matrices over R. Therefore, we can view C as a finitely generated subcategory of ${}_R\mathsf{FMod}$. Therefore, applying Corollary 4.17 proves the claim. □

5. Using residual finiteness

We will now show how residual finiteness of categories can exploited in the presence of finite generation/finite presentation.

5.1. **Hopficity.**

Every finitely generated residually finite category is Hopfian in the following sense:

Theorem 5.1. *Let C be a finitely generated residually finite category and let $E\colon C \longrightarrow C$ be a full functor that is essentially surjective (i.e., for each $X \in \mathrm{Ob}(C)$ there exists a $Y \in \mathrm{Ob}(C)$ with $E(Y) \cong_C X$). Then E is faithful. In particular, in the presence of the axiom of choice, E is an equivalence.*

Proof. We proceed as in the case of the corresponding result for groups: Let f and g be morphisms in C with $f \neq g$; in view of Proposition 2.1, we assume that there are $X, Y \in \mathrm{Ob}(C)$ with $f, g \in \mathrm{Mor}_C(X,Y)$. Because C is residually finite, there exists a finite category D and a functor $F\colon C \longrightarrow D$ with $F(f) \neq F(g)$.

As C is finitely generated and D is a finite category, there exist only finitely many different functors $C \longrightarrow D$. Hence, there are $n, m \in \mathbb{N}$ with $n < m$ and
$$F \circ E^n = F \circ E^m.$$

Inductively, we find sequences $(X_k)_{k\in\mathbb{N}}$, $(Y_k)_{k\in\mathbb{N}}$ in $\mathrm{Ob}(C)$, sequences $(f_k \in \mathrm{Mor}_C(X_k, Y_k))_{k\in\mathbb{N}}$, $(g_k \in \mathrm{Mor}_C(X_k, Y_k))_{k\in\mathbb{N}}$ of morphisms in C, and sequences $(\varphi_k \in \mathrm{Mor}_C(E(X_{k+1}), X_k))_{k\in\mathbb{N}}$, $(\psi_k \in \mathrm{Mor}_C(E(Y_{k+1}), Y_k))_{k\in\mathbb{N}}$ of isomorphisms in C satisfying
$$f_0 = f, \quad g_0 = g, \quad X_0 = X, \quad Y_0 = Y$$

and

$$E(f_{k+1}) = \psi_k^{-1} \circ f_k \circ \varphi_k$$
$$E(g_{k+1}) = \psi_k^{-1} \circ g_k \circ \varphi_k$$

for all $k \in \mathbb{N}$. We now set

$$f' := E^n(f_n) \in \mathrm{Mor}_C(X_n, Y_n)$$
$$g' := E^n(g_n) \in \mathrm{Mor}_C(X_n, Y_n)$$

and show the following:

(1) There exist isomorphisms ψ and φ in C with

$$f' = \psi \circ f \circ \varphi$$
$$g' = \psi \circ g \circ \varphi.$$

(2) We have $F(f') \neq F(g')$.
(3) We have $F \circ E^{m-n-1}(E(f')) \neq F \circ E^{m-n-1}(E(g'))$.
(4) We have $E(f') \neq E(g')$.
(5) We have $E(f) \neq E(g)$ (which proves that E is faithful).

Ad (1). We can take

$$\psi := E^{n-1}(\psi_{n-1}^{-1}) \circ E^{n-2}(\psi_{n-2}^{-1}) \circ \cdots \circ E(\psi_1^{-1}) \circ \psi_0^{-1}$$
$$\varphi := \varphi_0 \circ E(\varphi_1) \circ \cdots \circ E^{n-2}(\varphi_{n-2}) \circ E^{n-1}(\varphi_{n-1}),$$

which clearly are C-isomorphisms with the claimed property.

Ad (2). This follows from (1) and the fact that $F(f) \neq F(g)$.

Ad (3). By construction,

$$\begin{aligned} F \circ E^{m-n-1}\big(E(f')\big) &= F \circ E^{m-n-1} \circ E \circ E^n(f_n) = F \circ E^m(f_n) \\ &= F \circ E^n(f_n) \qquad \text{(because } F \circ E^m = F \circ E^n\text{)} \\ &= F(f') \end{aligned}$$

and, analogously,

$$F \circ E^{m-n-1}\big(E(g')\big) = F(g').$$

Therefore, we can use (2) to prove (3).

Ad (4). This is an immediate consequence of (3).

Ad (5). By (1), we have

$$E(f) = E(\psi^{-1} \circ f' \circ \varphi^{-1}) = E(\psi^{-1}) \circ E(f') \circ E(\varphi^{-1})$$
$$E(g) = E(\psi^{-1} \circ g' \circ \varphi^{-1}) = E(\psi^{-1}) \circ E(g') \circ E(\varphi^{-1}).$$

Moreover, (4) shows that $E(f') \neq E(g')$. Because $E(\psi^{-1})$ and $E(\varphi^{-1})$ are isomorphisms in C, we conclude that $E(f) \neq E(g)$. □

5.2. **Solving the word problem.**

Theorem 5.2. *Let (X, R) be a finite presentation of a residually finite category. Then the word problem is solvable for (X, R) (via an explicit algorithm, specified in the proof).*

Let us first recall the corresponding notions: As in the case of groups, presentations of categories are defined via quotient categories of free categories [5, Chapter II.8].

Definition 5.3 (finite presentation)**.** A *finite presentation* of a category is a pair (X, R) consisting of

- a finite quiver X,
- a finite set R of finite (directed) paths in X.

The category $\langle X \mid R\rangle$ *presented* by such a finite presentation (X, R) is the quotient category of the free category F_X, freely generated by X, modulo the smallest congruence relation on morphisms of F_X containing R.

Definition 5.4 (solvability of the word problem)**.** Let (X, R) be a finite presentation of a category. Then the *word problem for* (X, R) *is solvable* if the following holds: There exists an algorithm that given as input two finite (directed) paths in X (specified as finite lists of directed edges) decides whether the morphisms in the category $\langle X \mid R\rangle$ represented by these paths are equal or not.

Proof of Theorem 5.2. Again, we proceed as in the corresponding result for groups. Let $C := \langle X \mid R\rangle$, let p and q be finite directed paths in X, and let $\overline{p}$ and $\overline{q}$ be the morphisms in C represented by p and q, respectively.

We simultaneously perform the following tasks (e.g., by interleaving):

- We enumerate the congruence relation $\overline{R}$ on the morphisms of the free category F_X generated by R and check whether (p, q) belongs to (these initials of) $\overline{R}$. If $(p, q) \in \overline{R}$, then the answer is *yes* (i.e., $\overline{p} = \overline{q}$).
- We diagonally enumerate all natural numbers n, m and all categories with object set $\{0, \dots, n\}$ and morphism set $\{0, \dots, m\}$. For each such finite category D, we construct the finite set of functors $F\colon \langle X \mid R\rangle \longrightarrow D$ (using the universal property of $\langle X \mid R\rangle$) and its composition $\overline{F}\colon F_X \longrightarrow D$ with the canonical projection functor $F_X \longrightarrow \langle X \mid R\rangle = C$. For each such functor F, we check whether $\overline{F}(p) = \overline{F}(q)$. If $\overline{F}(p) \neq \overline{F}(q)$, then the answer is *no* (i.e., $\overline{p} \neq \overline{q}$).

We briefly explain why this algorithm is correct and terminates after a finite number of steps. To this end, we distinguish the following cases:

- If $\overline{p} = \overline{q}$, then $(p, q) \in \overline{R}$; hence, (p, q) will be found after a finite number of enumeration steps of $\overline{R}$.

  Moreover, because $\overline{p} = \overline{q}$, the morphisms p and q cannot be separated by a functor in the second branch of the algorithm. Hence, the algorithm correctly terminates with *yes*.
- If $\overline{p} \neq \overline{q}$, then we can invoke residual finiteness of C: There exists a finite category D and a functor $F\colon C \longrightarrow D$ with $F(\overline{p}) \neq F(\overline{q})$. Composing F with the canonical projection functor, leads to a functor $\overline{F}\colon F_X \longrightarrow D$ such that

  $$\overline{F}(p) = F(\overline{p}) \neq F(\overline{q}) = \overline{F}(q).$$

  Because every finite category is isomorphic to one of the categories enumerated in the second branch of the algorithm, such a separating functor will be found in a finite number of steps.

Moreover, because $\overline{p} \neq \overline{q}$, we have $(p, q) \notin \overline{R}$ and thus the algorithm will not stop in the first branch. Hence, the algorithm correctly terminates with *no*. □

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