The spectrum of simplicial volume
of non-compact manifolds

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Abstract

We show that, in dimension at least 4, the set of locally finite simplicial volumes of oriented connected open manifolds is $[0, \infty]$. Moreover, we consider the case of tame open manifolds and some low-dimensional examples.

1   Introduction

Simplicial volumes are invariants of manifolds defined in terms of the ℓ^1-seminorm on singular homology [Gro82].

Definition 1.1 (simplicial volume). Let M be an oriented connected d-manifold without boundary. Then the *simplicial volume of M* is defined by

$$\|M\|^{\mathrm{lf}} := \inf\big\{|c|_1 \;\big|\; c \in C_d^{\mathrm{lf}}(M; \mathbb{R}) \text{ is a fundamental cycle of } M\big\},$$

where C_*^{lf} denotes the locally finite singular chain complex. If M is compact, then we also write $\|M\| := \|M\|^{\mathrm{lf}}$. Using relative fundamental cycles, the notion of simplicial volume can be extended to oriented manifolds with boundary.

Simplicial volumes are related to negative curvature, volume estimates, and amenability [Gro82]. In the present article, we focus on simplicial volumes of *non-compact* manifolds. Only few concrete results are known in this context: There are computations for certain locally symmetric spaces [LS09a, LS09b, BKK14, KK15] as well as the general volume estimates [Gro82], vanishing results [Gro82, FM18], and finiteness results [Gro82, Löh08].

Let $d \in \mathbb{N}$, let $M(d)$ be the class of all oriented closed connected d-manifolds, and let $M^{\mathrm{lf}}(d)$ be the class of all oriented connected manifolds without boundary. Then we set $\mathrm{SV}(d) := \big\{\|M\| \;\big|\; M \in M(d)\big\}$ and

$$\mathrm{SV}^{\mathrm{lf}}(d) := \big\{\|M\|^{\mathrm{lf}} \;\big|\; M \in M^{\mathrm{lf}}(d)\big\}.$$

It is known that $\mathrm{SV}(d)$ is countable and that this set has no gap at 0 if $d \geq 4$:

Theorem 1.2 ([HL20, Theorem A]). *Let $d \in \mathbb{N}_{\geq 4}$. Then $\mathrm{SV}(d)$ is dense in $\mathbb{R}_{\geq 0}$ and $0 \in \mathrm{SV}(d)$.*

In contrast, if we allow non-compact manifolds, we can realise *all* non-negative real numbers:

Theorem A. *Let $d \in \mathbb{N}_{\geq 4}$. Then $\mathrm{SV}^{\mathrm{lf}}(d) = [0, \infty]$.*

The proof uses the no-gap theorem Theorem 1.2 and a suitable connected sum construction.

If we restrict to tame manifolds, then we are in a similar situation as in the closed case:

Theorem B. *Let $d \in \mathbb{N}$. Then the set $\mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(d) \subset [0, \infty]$ is countable. In particular, the set $[0, \infty] \setminus \mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(d)$ is uncountable.*

As an explicit example, we compute $\mathrm{SV}^{\mathrm{lf}}(2)$ and $\mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(2)$ (Proposition 4.2) as well as $\mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(3)$ (Proposition 4.3). The case of non-tame 3-manifolds seems to be fairly tricky.

Question 1.3. What is $\mathrm{SV}^{\mathrm{lf}}(3)$?

As $\mathrm{SV}(4) \subset \mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(4)$, we know that $\mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(4)$ contains arbitrarily small transcendental numbers [HL19].

From a geometric point of view, the so-called Lipschitz simplicial volume is more suitable for Riemannian non-compact manifolds than the locally finite simplicial volume. It is therefore natural to ask the following:

Question 1.4. Do Theorem A and Theorem B also hold for the Lipschitz simplicial volume of oriented connected open Riemannian manifolds?

Organisation of this article

Section 2 contains the proof of Theorem A. The proof of Theorem B is given in Section 3. The low-dimensional case is treated in Section 4.

2   Proof of Theorem A

Let $d \in \mathbb{N}_{\geq 4}$ and let $\alpha \in [0, \infty]$. Because $\mathrm{SV}(d)$ is dense in $\mathbb{R}_{\geq 0}$ (Theorem 1.2), there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $\mathrm{SV}(d)$ with $\sum_{n=0}^{\infty} \alpha_n = \alpha$.
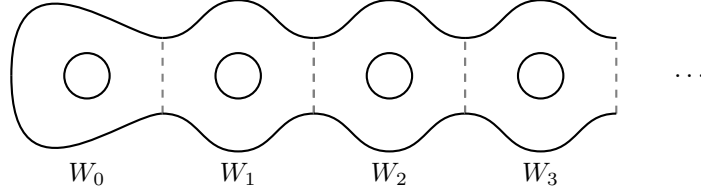
2.1   Construction

We first describe the construction of a corresponding oriented connected open manifold M: For each $n \in \mathbb{N}$, we choose an oriented closed connected d-manifold M_n with $\|M_n\| = \alpha_n$. Moreover, for $n > 0$, we set

$$W_n := M_n \setminus (B^{\circ}_{n,-} \sqcup B^{\circ}_{n,+}),$$

where $B_{n,-} = i_{n,-}(D^d)$ and $B_{n,+} = i_{n,+}(D^d)$ are two disjointly embedded closed d-balls in M_n. Similarly, we set $W_0 := M_0 \setminus B^{\circ}_{0,+}$. Furthermore, we choose an orientation-reversing homeomorphism $f_n \colon S^{d-1} \to S^{d-1}$. We then consider the infinite "linear" connected sum manifold (Figure 1)

$$\begin{aligned} M &:= M_0 \# M_1 \# M_2 \# \dots \\ &= (W_0 \sqcup W_1 \sqcup W_n \sqcup \dots)/\sim, \end{aligned}$$

Figure 1: The construction of M for the proof of Theorem A

where $\sim$ is the equivalence relation generated by

$$i_{n+1,-}(x) \sim i_{n,+}\big(f_n(x)\big)$$

for all $n \in \mathbb{N}$ and all $x \in S^{d-1} \subset D^d$; we denote the induced inclusion $W_n \to M$ by i_n. By construction, M is connected and inherits an orientation from the M_n.

2.2 Computation of the simplicial volume

We will now verify that $\|M\|^{\mathrm{lf}} = \alpha$:

Claim 2.1. *We have $\|M\|^{\mathrm{lf}} \leq \alpha$.*

Proof. The proof is a straightforward adaption of the chain-level proof of sub-additivity of simplicial volume with respect to amenable glueings.

In particular, we will use the uniform boundary condition [MM85] and the equivalence theorem [Gro82, BBF$^+$14]:

UBC The chain complex $C_*(S^{d-1};\mathbb{R})$ satisfies $(d-1)$-UBC, i.e., there is a constant K such that: For each $c \in \operatorname{im}\partial_d \subset C_{d-1}(S^{d-1};\mathbb{R})$, there exists a chain $b \in C_d(S^{d-1};\mathbb{R})$ with

$$\partial_d b = c \quad\text{and}\quad |b|_1 \leq K \cdot |c|_1.$$

EQT Let N be an oriented closed connected d-manifold, let $B_1, \ldots, B_k$ be disjointly embedded d-balls in N, and let $W := N \backslash (B_1^{\circ} \cup \ldots, \cup B_1^{\circ})$. Moreover, let $\epsilon \in \mathbb{R}_{>0}$. Then

$$\|N\| = \inf\big\{|z|_1 \mid z \in Z(W;\mathbb{R}),\ |\partial_d z|_1 \leq \epsilon\big\},$$

where $Z(W;\mathbb{R}) \subset C_d(W;\mathbb{R})$ denotes the set of all relative fundamental cycles of W.

Let $\epsilon \in \mathbb{R}_{>0}$. By EQT, for each $n \in \mathbb{N}$, there exists a relative fundamental cycle $z_n \in Z(W_n;\mathbb{R})$ with

$$|z_n|_1 \leq \alpha_n + \frac{1}{2^n} \cdot \epsilon \quad\text{and}\quad |\partial_d z_n|_1 \leq \frac{1}{2^n} \cdot \epsilon.$$

We now use UBC to construct a locally finite fundamental cycle of M out of these relative cycles: For $n \in \mathbb{N}$, the boundary parts $C_{d-1}(i_n;\mathbb{R})(\partial_d z_n|_{B_{n,+}})$

and $-C_{d-1}(i_{n+1}; \mathbb{R})(\partial_d z_{n+1}|_{B_{n+1,-}})$ are fundamental cycles of the sphere S^{d-1} (embedded via $i_n \circ i_{n,+}$ and $i_{n+1} \circ i_{n+1,-}$ into M, which implicitly uses the orientation-reversing homeomorphism f_n). By UBC, there exists a chain $b_n \in C_d(S^{d-1}; \mathbb{R})$ with

$$\partial_d C_d(i_n \circ i_{n,+}; \mathbb{R})(b_n) = C_{d-1}(i_n; \mathbb{R})(\partial_d z_n|_{B_{n,+}})$$
$$+ C_{d-1}(i_{n+1}; \mathbb{R})(\partial_d z_{n+1}|_{B_{n+1,-}})$$

and

$$|b_n|_1 \le K \cdot \left(\frac{1}{2^n} + \frac{1}{2^{n+1}}\right) \cdot \epsilon \le K \cdot \frac{1}{2^{n-1}} \cdot \epsilon.$$

A straightforward computation shows that

$$c := \sum_{n=0}^{\infty} C_d(i_n; \mathbb{R})\big(z_n - C_d(i_{n,+}; \mathbb{R})(b_n)\big)$$

is a locally finite d-cycle on M. Moreover, the local contribution on W_0 shows that c is a locally finite fundamental cycle of M. By construction,

$$|c|_1 \le \sum_{n=0}^{\infty}\big(|z_n|_1 + |b_n|_1\big)$$
$$\le \sum_{n=0}^{\infty}\left(\alpha_n + \frac{1}{2^n} \cdot \epsilon + K \cdot \frac{1}{2^{n-1}} \cdot \epsilon\right) \le \sum_{n=0}^{\infty} \alpha_n + (2 + 4 \cdot K) \cdot \epsilon$$
$$= \alpha + (2 + 4 \cdot K) \cdot \epsilon.$$

Thus, taking $\epsilon \to 0$, we obtain $\|M\|^{\mathrm{lf}} \le \alpha$. $\square$

Claim 2.2. *We have* $\|M\|^{\mathrm{lf}} \ge \alpha$.

Proof. Without loss of generality we may assume that $\|M\|^{\mathrm{lf}}$ is finite. Let $c \in C_d^{\mathrm{lf}}(M; \mathbb{R})$ be a locally finite fundamental cycle of M with $|c|_1 < \infty$. For $n \in \mathbb{N}$, we consider the subchain $c_n := c|_{W_{(n)}}$ of c, consisting of all simplices whose images touch $W_{(n)} := \bigcup_{k=0}^{n} i_k(W_k) \subset M$. Because c is locally finite, each c_n is a finite singular chain and $(|c_n|_1)_{n \in \mathbb{N}}$ is a monotonically increasing sequence with limit $|c|_1$.

Let $\epsilon \in \mathbb{R}_{>0}$. Then there is an $n \in \mathbb{N}_{>0}$ that satisfies $|c - c_n|_1 \le \epsilon$ and $\alpha - \sum_{k=0}^{n} \alpha_k \le \epsilon$. Let

$$p \colon M \to W_{(n)}/i_n(B_{n,+}) =: W$$

be the map that collapses everything beyond stage $n + 1$ to a single point x. Then $z := C_d(p; \mathbb{R})(c_n) \in C_d(W, \{x\}; \mathbb{R})$ is a relative cycle and

$$|\partial_d z|_1 \le |\partial_d c_n|_1 \le |\partial_d(c - c_n)|_1 \le (d+1) \cdot |c - c_n|_1 \le (d+1) \cdot \epsilon.$$

Because $d > 1$, there exists a chain $b \in C_d(\{x\}; \mathbb{R})$ with

$$\partial_d b = \partial_d z \quad \text{and} \quad |b|_1 \le |\partial_d z| \le (d+1) \cdot \epsilon.$$

Then

$$\overline{z} := z - b \in C_d(W; \mathbb{R})$$

is a cycle on W; because z and $\overline{z}$ have the same local contribution on W_0, the cycle z is a fundamental cycle of the manifold

$$W \cong M_0 \# \cdots \# M_n.$$

As $d > 2$, the construction of our chains and additivity of simplicial volume under connected sums [Gro82, BBF$^+$14] show that

$$|c|_1 \geq |c_n|_1 \geq |z|_1 \geq |\overline{z}|_1 - |b|_1$$
$$\geq \|W\| - (d+1) \cdot \epsilon = \sum_{k=0}^{n} \|M_n\| - (d+1) \cdot \epsilon$$
$$\geq \alpha - (d+2) \cdot \epsilon.$$

Thus, taking $\epsilon \to 0$, we obtain $|c|_1 \geq \alpha$; hence, $\|M\|^{\mathrm{lf}} \geq \alpha$. $\square$

This completes the proof of Theorem A.

Remark 2.3 (adding geometric structures). In fact, this argument can also be performed smoothly: The constructions leading to Theorem 1.2 can be carried out in the smooth setting. Therefore, we can choose the $(M_n)_{n \in \mathbb{N}}$ to be smooth and equip M with a corresponding smooth structure. Moreover, we can endow these smooth pieces with Riemannian metrics. Scaling these Riemannian metrics appropriately shows that we can turn M into a Riemannian manifold of finite volume.

3 Proof of Theorem B

In this section, we prove Theorem B, i.e., that the set of simplicial volumes of tame manifolds is countable.

Definition 3.1. A manifold M without boundary is *tame* if there exists a compact connected manifold W with boundary such that M is homeormorphic to $W^\circ := W \setminus \partial W$.

As in the closed case, our proof is based on a counting argument:

Proposition 3.2. *There are only countably many proper homotopy types of tame manifolds.*

As we could not find a proof of this statement in the literature, we will give a complete proof in Section 3.1 below. Theorem B is a direct consequence of Proposition 3.2:

Proof of Theorem B. The simplicial volume $\|\cdot\|^{\mathrm{lf}}$ is invariant under proper homotopy equivalence (this can be shown as in the compact case). Therefore, the countability of $\mathrm{SV}^{\mathrm{lf}}(d)$ follows from the countability of the set of proper homotopy types of tame d-manifolds (Proposition 3.2). $\square$

Remark 3.3. Let $d \in \mathbb{N}_{\geq 3}$. Then $\infty \in \mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(d)$: Let N be an oriented closed connected hyperbolic $(d-1)$-manifold and let $M := N \times \mathbb{R}$. Then M is tame (as interior of $N \times [0,1]$) and $\|N\| > 0$ [Gro82, Section 0.3][Thu97, Theorem 6.2]. Hence, by the finiteness criterion [Gro82, p. 17][Löh08, Theorem 6.4], we obtain that $\|M\|^{\mathrm{lf}} = \infty$.

3.1   Counting tame manifolds

It remains to prove Proposition 3.2. We use the following observations:

Definition 3.4 (models of tame manifolds)**.**

- A *model* of a tame manifold M is a finite CW-pair (X, A) (i.e., a finite CW-complex X with a finite subcomplex A) that is homotopy equivalent (as pairs of spaces) to $(W, \partial W)$, where W is a compact connected manifold with boundary whose interior is homeomorphic to M.

- Two models of tame manifolds are *equivalent* if they are homotopy equivalent as pairs of spaces.

Lemma 3.5 (existence of models)**.** *Let W be a compact connected manifold. Then there exists a finite CW-pair (X, A) such that $(W, \partial W)$ and (X, A) are homotopy equivalent pairs of spaces.*

In particular: Every tame manifold admits a model.

Proof. It should be noted that we work with topological manifolds; hence, we cannot argue directly via triangulations. Of course, the main ingredient is the fact that every compact manifold is homotopy equivalent to a finite complex [Sie68, KS69].

Hence, there exist finite CW-complexes A and Y with homotopy equivalences $f \colon A \to \partial W$ and $g \colon Y \to W$. Let $j := \overline{g} \circ i \circ f$, where $i \colon \partial W \hookrightarrow W$ is the inclusion and $\overline{g}$ is a homotopy inverse of g. By construction, the upper square in the diagram in Figure 2 is homotopy commutative.

As next step, we replace $j \colon A \to Y$ by a homotopic map $j_c \colon A \to Y$ that is cellular (second square in Figure 2).

The mapping cylinder Z of j_c has a finite CW-structure (as j_c is cellular) and the canonical map $p \colon Z \to Y$ allows to factor j_c into an inclusion J of a subcomplex and the homotopy equivalence p (third square in Figure 2).

We thus obtain a homotopy commutative square

$$
\begin{array}{ccc}
\partial W & \xrightarrow{\ i\ } & W \\
f \uparrow & \ \ \circledcirc_{(h)} & \uparrow F := g \circ p \\
A & \xrightarrow{\ J\ } & Z
\end{array}
$$

where the vertical arrows are homotopy equivalences, the upper horizontal arrow is the inclusion, and the lower horizontal arrow is the inclusion of a subcomplex.

Using a homotopy between $i \circ f$ and $F \circ J$ and adding another cylinder to Z, we can replace Z by a finite CW-complex X (that still contains A as subcomplex) to obtain a *strictly* commutative diagram

$$
\begin{array}{ccc}
\partial W & \xrightarrow{\ i\ } & W \\
f \uparrow \simeq & & \simeq \uparrow \\
A & \xrightarrow{\quad\quad} & X
\end{array}
$$

whose vertical arrows are homotopy equivalences and whose horizontal arrows are inclusions.

$$
\begin{array}{ccc}
\partial W & \xrightarrow{\ i\ } & W \\
{\scriptstyle f}\uparrow & {\scriptstyle (h)\quad g} & \uparrow\ |\ {\scriptstyle \overline{g}} \\
A & \xrightarrow{\ j\ } & Y \\
\| & {\scriptstyle (h)} & \| \\
A & \xrightarrow{\ j_c\ } & Y \\
\| & & \uparrow {\scriptstyle p} \\
A & \xrightarrow{\ J\ } & Z
\end{array}
$$

Figure 2: Finding a model

Because the inclusions $\partial W \hookrightarrow W$ (as inclusion of the boundary of a compact topological manifold) and $A \hookrightarrow X$ (as inclusion of a subcomplex) are cofibrations, this already implies that the vertical arrows form a homotopy equivalence $(X, A) \to (W, \partial W)$ of pairs [May99, Chapter 6.5]. $\qquad\square$

Lemma 3.6 (equivalence of models). *If M and N are tame manifolds with equivalent models, then M and N are properly homotopy equivalent.*

Proof. As M and N admit equivalent models, there exist compact connected manifolds W and V with boundary such that $M \cong W^\circ$ and $N \cong V^\circ$ and such that the pairs $(W, \partial W)$ and $(V, \partial V)$ are homotopy equivalent (by transitivity of homotopy equivalence of pairs of spaces). Let $(f, f_\partial)\colon (W, \partial W) \to (V, \partial V)$ and $(g, g_\partial)\colon (V, \partial V) \to (W, \partial W)$ be mutually homotopy inverse homotopy equivalences of pairs.

By the topological collar theorem [Bro62, Con71], we have homeomorphisms

$$
M \cong W \cup_{\partial W} \big(\partial W \times [0, \infty)\big)
$$
$$
N \cong V \cup_{\partial V} \big(\partial V \times [0, \infty)\big),
$$

where the glueing occurs via the canonical inclusions $\partial W \hookrightarrow \partial W \times [0, \infty)$ and $\partial V \hookrightarrow \partial V \times [0, \infty)$ at parameter 0.

Then the maps f and $f_\partial \times \mathrm{id}_{[0,\infty)}$ glue to a well-defined proper continuous map $F\colon M \to N$ and the maps g and $g_\partial \times \mathrm{id}_{[0,\infty)}$ glue to a well-defined proper continuous map $G\colon N \to M$.

Moreover, the homotopy of pairs between $(f \circ g, f_\partial \circ g_\partial)$ and $(\mathrm{id}_V, \mathrm{id}_{\partial V})$ glues into a proper homotopy between $F \circ G$ and id_M. In the same way, there is a proper homotopy between $G \circ F$ and id_N. Hence, the spaces M and N are properly homotopy equivalent. $\qquad\square$

Lemma 3.7 (countability of models). *There exist only countably many equivalence classes of models.*

Proof. There are only countably many homotopy types of finite CW-complexes (because every finite CW-complex is homotopy equivalent to a finite simplicial

complex). Moreover, every finite CW-complex has only finitely many subcomplexes. Therefore, there are only countably many homotopy types (of pairs of spaces) of finite CW-pairs.   □

Proof of Proposition 3.2. We only need to combine Lemma 3.5, Lemma 3.6, and Lemma 3.7.   □

4   Low dimensions

4.1   Dimension 2

We now compute the set of simplicial volumes of surfaces. We first consider the tame case:

Example 4.1 (tame surfaces)**.** Let W be an oriented compact connected surface with $g \in \mathbb{N}$ handles and $b \in \mathbb{N}$ boundary components. Then the proportionality principle for simplicial volume of hyperbolic manifolds [Gro82, p. 11] (a thorough exposition is given, for instance, by Fujiwara and Manning [FM11, Appendix A]) gives

$$\|W^\circ\|^{\mathrm{lf}} = \begin{cases} 4 \cdot (g-1) + 2 \cdot b & \text{if } g > 0 \\ 2 \cdot b - 4 & \text{if } g = 0 \text{ and } b > 1 \\ 0 & \text{if } g = 0 \text{ and } b \in \{0,1\}. \end{cases}$$

Proposition 4.2. *We have* $\mathrm{SV}^{\mathrm{lf}}(2) = 2 \cdot \mathbb{N} \cup \{\infty\}$ *and* $\mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(2) = 2 \cdot \mathbb{N}$.

Proof. We first prove $2 \cdot \mathbb{N} \subset \mathrm{SV}^{\mathrm{lf}}_{\mathrm{tame}}(2) \subset \mathrm{SV}^{\mathrm{lf}}(2)$ and $\infty \in \mathrm{SV}^{\mathrm{lf}}(2)$, i.e., that all the given values may be realised: In view of Example 4.1, all even numbers occur as simplicial volume of some (possibly open) tame surface.

   Let

$$M := T^2 \# T^2 \# T^2 \# \dots$$

be an infinite "linear" connected sum of tori T^2. Collapsing M to the first $g \in \mathbb{N}$ summands and an argument as in the proof of Claim 2.2 shows that

$$\|M\|^{\mathrm{lf}} \geq \|\Sigma_g\| = 4 \cdot g - 4$$

for all $g \in \mathbb{N}_{\geq 1}$. Hence, $\|M\|^{\mathrm{lf}} = \infty$.

   It remains to show that $\mathrm{SV}^{\mathrm{lf}}(2) \subset 2 \cdot \mathbb{N} \cup \{\infty\}$: Let M be an oriented connected (topological, separable, Hausdorff) 2-manifold without boundary. Then M admits a smooth structure [Moi77] and whence a proper smooth map $p \colon M \to \mathbb{R}$. Using suitable regular values of p, we can thus write M as an ascending union

$$M = \bigcup_{n \in \mathbb{N}} M_n$$

of oriented connected compact submanifolds (possibly with boundary) M_n that are nested via $M_0 \subset M_1 \subset \dots$. Then one of the following cases occurs:

1. There exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ the inclusion $M_n \hookrightarrow M_{n+1}$ is a homotopy equivalence.

2. For each $N \in \mathbb{N}$ there exists an $n \in \mathbb{N}_{\geq N}$ such that the inclusion $M_n \hookrightarrow M_{n+1}$ is *not* a homotopy equivalence.

In the first case, the classification of compact surfaces with boundary shows that M is tame. Hence $\|M\|^{\mathrm{lf}} \in 2 \cdot \mathbb{N}$ (Example 4.1).

In the second case, the manifold M is *not* tame (which can, e.g., be derived from the classification of compact surfaces with boundary). We show that $\|M\|^{\mathrm{lf}} = \infty$. To this end. we distinguish two cases:

a. The sequence $(h(M_n))_{n \in \mathbb{N}}$ is unbounded, where $h(\,\cdot\,)$ denotes the number of handles of the surface.

b. The sequence $(h(M_n))_{n \in \mathbb{N}}$ is bounded.

In the unbounded case, a collapsing argument (similar to the argument for $T^2 \# T^2 \# \ldots$ and Claim 2.2) shows that $\|M\|^{\mathrm{lf}} = \infty$.

We claim that also in the bounded case we have $\|M\|^{\mathrm{lf}} = \infty$: Shifting the sequence in such a way that all handles are collected in M_0, we may assume without loss of generality that the sequence $(h(M_n))_{n \in \mathbb{N}}$ is constant. Thus, for each $n \in \mathbb{N}$, the surface M_{n+1} is obtained from M_n by adding a finite disjoint union of disks and of spheres with finitely many (at least two) disks removed; we can reorganise this sequence in such a way that no disks are added. Hence, we may assume that M_n is a retract of M_{n+1} for each $n \in \mathbb{N}$. Furthermore, because we are in case 2, the classification of compact surfaces shows (with the help of Example 4.1) that

$$\lim_{n \to \infty} \|M_n\| = \infty.$$

Let $c \in C_2^{\mathrm{lf}}(M; \mathbb{R})$ be a locally finite fundamental cycle of M and let $n \in \mathbb{N}$. Because c is locally finite, there is a $k \in \mathbb{N}$ such that $c|_{M_n}$ is supported on M_{n+k}; the restriction $c|_{M_n}$ consists of all summands of c whose supports intersect with M_n. Because M_n is a retract of M_{n+k}, we obtain from $c|_{M_n}$ a relative fundamental cycle c_n of M_n by pushing the chain $c|_{M_n}$ to M_n via a retraction $M_{n+k} \to M_n$. Therefore,

$$|c|_1 \geq |c|_{M_n}|_1 \geq |c_n|_1 \geq \|M_n\|.$$

Taking $n \to \infty$ shows that $|c|_1 = \infty$. Taking the infimum over all locally finite fundamental cycles c of M proves that $\|M\|^{\mathrm{lf}} = \infty$.

Moreover, Example 4.1 shows that $\infty \notin \mathrm{SV}_{\mathrm{tame}}^{\mathrm{lf}}(2)$. □

4.2 Dimension 3

The general case of non-compact 3-manifolds seems to be rather involved (as the structure of non-compact 3-manifolds can get fairly complicated). We can at least deal with the tame case:

Proposition 4.3. *We have* $\mathrm{SV}_{\mathrm{tame}}^{\mathrm{lf}}(3) = \mathrm{SV}(3) \cup \{\infty\}$.

Proof. Clearly, $\mathrm{SV}(3) \subset \mathrm{SV}_{\mathrm{tame}}^{\mathrm{lf}}(3)$ and $\infty \in \mathrm{SV}_{\mathrm{tame}}^{\mathrm{lf}}(3)$ (Remark 3.3).

Conversely, let W be an oriented compact connected 3-manifold and let $M := W^\circ$. We distinguish the following cases:

• If at least one of the boundary components of W has genus at least 2, then the finiteness criterion [Gro82, p. 17][Löh08, Theorem 6.4] shows that $\|M\|^{\mathrm{lf}} = \infty$.

- If the boundary of W consists only of spheres and tori, then we proceed as follows: In a first step, we fill in all spherical boundary components of W by 3-balls and thus obtain an oriented compact connected 3-manifold V all of whose boundary components are tori. In view of considerations on tame manifolds with amenable boundary [KK15] and glueing results for bounded cohomology [Gro82][BBF$^+$14], we obtain that

$$\|M\|^{\text{lf}} = \|W\| = \|V\|.$$

By Kneser's prime decomposition theorem [AFW15, Theorem 1.2.1] and the additivity of (relative) simplicial volume with respect to connected sums [Gro82][BBF$^+$14] in dimension 3, we may assume that V is prime (i.e., admits no non-trivial decomposition as a connected sum). Moreover, because $\|S^1 \times S^2\| = 0$, we may even assume that V is irreducible [AFW15, p. 3].

By geometrisation [AFW15, Theorem 1.7.6], then V admits a decomposition along finitely many incompressible tori into Seifert fibred manifolds (which have trivial simplicial volume [Thu97, Corollary 6.5.3]) and hyperbolic pieces $V_1, \ldots, V_k$. As the tori are incompressible, we can now again apply additivity [Gro82][BBF$^+$14] to conclude that

$$\|V\| = \sum_{j=1}^{k} \|V_j\|.$$

Let $j \in \{1, \ldots, k\}$. Then the boundary components of V_j are π_1-injective tori (as the interior of V_j admits a complete hyperbolic metric of finite volume) [BP92, Proposition D.3.18]. Let S be a Seifert 3-manifold whose boundary is a π_1-injective torus (e.g., the knot complement of a non-trivial torus knot [Mos71, Theorem 2][Lüc02, Lemma 4.4]). Filling each boundary component of V_j with a copy of S results in an oriented closed connected 3-manifold N_j, which satisfies (again, by additivity)

$$\|N_j\| = \|V_j\| + 0 = \|V_j\|.$$

Therefore, the oriented closed connected 3-manifold $N := N_1 \# \cdots \# N_k$ satisfies

$$\|N\| = \sum_{j=1}^{k} \|N_j\| = \sum_{j=1}^{k} \|V_j\| = \|V\|.$$

In particular, $\|M\|^{\text{lf}} = \|V\| = \|N\| \in \text{SV}(3)$.   □

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