THE ℓ^∞-SEMI-NORM ON UNIFORMLY FINITE HOMOLOGY

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ABSTRACT. Uniformly finite homology is a coarse homology theory, defined via chains that satisfy a uniform boundedness condition. By construction, uniformly finite homology carries a canonical ℓ^∞-semi-norm. We show that, for uniformly discrete spaces of bounded geometry, this semi-norm on uniformly finite homology in degree 0 with $\mathbb{Z}$-coefficients allows for a new formulation of Whyte's rigidity result. In contrast, we prove that this semi-norm is trivial on uniformly finite homology with $\mathbb{R}$-coefficients in higher degrees.

1. INTRODUCTION

Enriching (co)homology theories with semi-norms allows for a rich interaction between geometry, group theory, and topology; a prominent example of this phenomenon is given by bounded cohomology and simplicial volume [13]. In the present article, we study the ℓ^∞-semi-norm on uniformly finite homology.

Uniformly finite homology is a coarse homology theory, defined via simplicial chains that satisfy a uniform boundedness condition, originally introduced by Block and Weinberger [4]. Uniformly finite homology has various applications to amenability [4], rigidity [21], aperiodic tilings [4], large-scale notions of dimension [10], and positive scalar curvature [4].

By construction, uniformly finite homology carries additional structure, namely a canonical ℓ^∞-semi-norm (Section 2). It is therefore a natural question to study which refined information is encoded in this semi-norm.

In this article, we show that, for uniformly discrete spaces of bounded geometry (UDBG spaces, see Appendix A), this semi-norm is rigid in degree 0 (Section 3): For example, the $\mathbb{R}$-fundamental class of an amenable space has semi-norm 1, and the ℓ^∞-semi-norm with $\mathbb{Z}$-coefficients allows for a new formulation of Whyte's rigidity result:

Theorem 1.1 (rigidity)**.** *Let X and Y be UDBG spaces and let $f\colon X \longrightarrow Y$ be a quasi-isometry. Then the following are equivalent:*

(1) *The map $f\colon X \longrightarrow Y$ is uniformly close to a bilipschitz equivalence.*
(2) *For all $k \in \mathbb{N}$, the map $H_k^{\mathrm{uf}}(f;\mathbb{Z})\colon H_k^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_k^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm.*
(3) *The map $H_0^{\mathrm{uf}}(f;\mathbb{Z})\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm.*

The corresponding equivalence for $\mathbb{R}$-coefficients does *not* hold (Proposition 3.5).

In contrast, in higher degrees, the ℓ^∞-semi-norm on uniformly finite homology (with $\mathbb{R}$-coefficients) is trivial:

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Proposition 1.2 (vanishing: non-amenable case)**.** *Let X be a non-amenable UDBG space and let $k \in \mathbb{N}$. Then the ℓ^∞-semi-norm on $H_k^{\mathrm{uf}}(X;\mathbb{R})$ is trivial.*

Proposition 1.3 (vanishing: amenable case)**.** *Let G be a finitely generated amenable group and let $k \in \mathbb{N}_{>0}$. Then the ℓ^∞-semi-norm on $H_k^{\mathrm{uf}}(G;\mathbb{R})$ is trivial.*

In particular, the corresponding reduced uniformly finite homology with $\mathbb{R}$-coefficients is trivial for all non-amenable UDBG spaces and trivial in higher degrees for finitely generated amenable groups. Moreover, the vanishing in higher degrees shows that the ℓ^∞-semi-norm on uniformly finite homology is not suitable for Engel's approach to the rough Novikov conjecture [12].

Organisation of this article. We briefly review uniformly finite homology for UDBG spaces and its basic properties in Appendix A. The ℓ^∞-semi-norm on uniformly finite homology is introduced in Section 2. The rigidity result Theorem 1.1 is proved in Section 3. The vanishing in higher degrees (Proposition 1.2 and 1.3) is treated in Section 4.

2. The ℓ^∞-semi-norm on uniformly finite homology

We now introduce the ℓ^∞-semi-norm on uniformly finite homology. We start with UDBG spaces in Section 2.1 and then discuss the case of finitely generated groups in Section 2.2.

Let R be a ring with unit endowed with a norm $|\cdot|$ (in the sense of Definition A.5) and let A be an R-module. A *norm on A* is a function

$$\|\cdot\|\colon A \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following conditions:

(1) For all $a \in A$ we have $\|a\| = 0$ if and only if $a = 0$.
(2) For all $a, a' \in A$ we have $\|a + a'\| \leq \|a\| + \|a'\|$.
(3) For all $a \in A$, $r \in R$ we have $\|r \cdot a\| = |r| \cdot \|a\|$.

A function $\|\cdot\|\colon A \longrightarrow \mathbb{R}_{\geq 0}$ is a *semi-norm on A* if it satisfies condition (2) above and the following condition:

(3′) For all $a \in A$, $r \in R$ we have $\|r \cdot a\| \leq |r| \cdot \|a\|$.

2.1. Semi-norm for UDBG spaces.

We define the ℓ^∞-norm on the module of uniformly finite chains with coefficients in a normed ring with unit. We, then, have a corresponding semi-norm on uniformly finite homology.

Definition 2.1. (ℓ^∞-semi-norm) Let R be a normed ring with unit. Let X be a *UDBG* space and let $n \in \mathbb{N}$. The *ℓ^∞-norm* on $C_n^{\mathrm{uf}}(X;R)$ is defined by

$$\begin{aligned} \|\cdot\|_\infty\colon C_n^{\mathrm{uf}}(X;R) &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{x \in X^{n+1}} c_x \cdot x &\longmapsto \sup_{x \in X^{n+1}} |c_x|. \end{aligned}$$

The *ℓ^∞-semi-norm* on $H_n^{\mathrm{uf}}(X;R)$ is the corresponding semi-norm induced on the quotient. More explicitly, for every $\alpha \in H_n^{\mathrm{uf}}(X;R)$, it is defined by

$$\|\alpha\|_\infty := \inf\bigl\{\|c\|_\infty \bigm| c \in C_n^{\mathrm{uf}}(X;R),\ \partial(c) = 0,\ \alpha = [c]\bigr\}.$$

The ℓ^∞-semi-norm on uniformly finite homology with $\mathbb{Z}$-coefficients is *not* homogeneous, in general (Example 3.7). Moreover, the ℓ^∞-semi-norm on uniformly finite homology is *not* a functorial semi-norm in the sense of Gromov [14, 17] (Example 3.8).

Proposition 2.2. *Let R be a normed ring with unit. If $f\colon X \longrightarrow Y$ is a quasi-isometry between UDBG spaces that is uniformly close to a bilipschitz equivalence, then for each $n \in \mathbb{N}$ the induced map $H_n^{\mathrm{uf}}(f;R)\colon H_n^{\mathrm{uf}}(X;R) \longrightarrow H_n^{\mathrm{uf}}(Y;R)$ is an isometric isomorphism with respect to the ℓ^∞-semi-norms.*

Proof. In view of Proposition A.7 it suffices to show that any bilipschitz equivalence induces an isometry in uniformly finite homology. Let $n \in \mathbb{N}$. Because a bilipschitz equivalence $g\colon X \longrightarrow Y$ is a bijection, it induces a bijection

$$\begin{aligned} X^{n+1} &\longrightarrow Y^{n+1} \\ (x_0,\dots,x_n) &\longmapsto \big(g(x_0),\dots,g(x_n)\big). \end{aligned}$$

Thus, for all $c = \sum_{x\in X^{n+1}} c_x \cdot x \in C_n^{\mathrm{uf}}(X;R)$ we have

$$C_n^{\mathrm{uf}}(g;R)(c) = \sum_{y\in Y^{n+1}} c_{g^{-1}(y)} \cdot y.$$

In particular, $\|C_n^{\mathrm{uf}}(g;R)(c)\|_\infty = \|c\|_\infty$. Therefore, $\|H_n^{\mathrm{uf}}(g;R)(\alpha)\|_\infty \le \|\alpha\|_\infty$ for all $\alpha \in H_n^{\mathrm{uf}}(X;R)$. Applying the same argument to the inverse of g shows that $\|H_n^{\mathrm{uf}}(g;R)(\alpha)\|_\infty = \|\alpha\|_\infty$ holds for all $\alpha \in H_n^{\mathrm{uf}}(X;R)$. $\square$

The same definition of ℓ^∞-semi-norm can be considered on uniformly finite homology for general metric spaces [4]. Notice that for metric spaces without isolated points, the ℓ^∞-semi-norm is trivial in any degree in uniformly finite homology [9, Proposition 4.2.3]. In particular, the corresponding reduced uniformly finite homology vanishes in any degree.

2.2. **Semi-norm for groups.** Proposition 2.2 yields a well-defined semi-norm on uniformly finite homology of finitely generated groups:

Corollary 2.3 (ℓ^∞-semi-norm on uniformly finite homology of groups)**.** *Let R be a normed ring with unit. If G is a finitely generated group and $S, T \subset G$ are finite generating sets of G, then the identity map $\mathrm{id}_G\colon (G,d_S) \longrightarrow (G,d_T)$ is a bilipschitz equivalence with respect to the word metrics on G associated with S and T. In particular, the induced map $H_*^{\mathrm{uf}}(G,d_S;R) \longrightarrow H_*^{\mathrm{uf}}(G,d_T;R)$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm. Hence, the ℓ^∞-semi-norm on $H_*^{\mathrm{uf}}(G;R)$ is independent of the chosen generating set.*

Definition 2.4 (ℓ^∞-semi-norm on group homology with twisted coefficients)**.** Let $n \in \mathbb{N}$. Every chain $c \in C_n(G;\ell^\infty(G,R))$ can be written uniquely as a finite sum of the type $\sum_{t\in G^n} (e,t_1,\dots,t_n) \otimes \varphi_t$, where almost all of the $\varphi_t \in \ell^\infty(G,R)$ are zero; we then define

$$\|c\|_\infty := \sup_{t\in G^n} \|\varphi_t\|_\infty,$$

where for all $t \in G^n$ the norm $\|\varphi_t\|_\infty$ is the supremum norm on $\ell^\infty(G,R)$. The ℓ^∞-*semi-norm* on $H_n(G;\ell^\infty(G,R))$ is defined for all $\alpha \in H_n(G;\ell^\infty(G,R))$

by

$$\|\alpha\|_\infty := \inf\bigl\{\|c\|_\infty \bigm| c \in C_n(G;\ell^\infty(G,R)),\ \partial(c)=0,\ \alpha=[c]\bigr\}.$$

Proposition 2.5. *Let G be a group and R be a normed ring with unit. The chain isomorphism $\rho_*\colon C_*^{\mathrm{uf}}(G;R) \longrightarrow C_*(G;\ell^\infty(G,R))$ given in Proposition A.10 induces an isometric isomorphism $H_*(\rho_*)\colon H_*^{\mathrm{uf}}(G;R) \longrightarrow H_*(G;\ell^\infty(G,R))$.*

Proof. We prove that the chain isomorphism ρ_* is isometric in every degree: let $n \in \mathbb{N}$ and let $c = \sum_{g\in G^{n+1}} c_g \cdot g \in C_n^{\mathrm{uf}}(G;R)$. Then with the notation from Proposition A.10 we obtain

$$\begin{aligned}\|\rho_n(c)\|_\infty &= \Bigl\|\sum_{t\in G^n}(e,t_1,\dots,t_n)\otimes\varphi_{c,t}\Bigr\|_\infty\\ &= \sup_{t\in G^n}\|\varphi_{c,t}\|_\infty = \sup_{t\in G^n}\sup_{g\in G}|\varphi_{c,t}(g)| = \sup_{t\in G^n}\sup_{g\in G}|c_{g^{-1}\cdot(e,t_1,\dots,t_n)}|\\ &= \|c\|_\infty.\end{aligned}$$

Because ρ_* is a chain isomorphism, it follows that for all $\alpha \in H_*^{\mathrm{uf}}(G;\mathbb{R})$ we have $\|H_*(\rho_*)(\alpha)\|_\infty = \|\alpha\|_\infty$. $\square$

For finitely generated groups, Definition A.14 gives a notion of amenability via Følner sequences. Alternatively, amenable groups can be characterised through the existence of invariant means [7, Theorem 4.9.2]. Let G be a finitely generated amenable group and let $M(G)$ denote the set of all invariant means $\ell^\infty(G;\mathbb{R}) \longrightarrow \mathbb{R}$ on G. Then every mean $m \in M(G)$ induces a transfer map [1, Proposition 2.15]

$$m_* = H_*(\mathrm{id}_G;m)\colon H_*^{\mathrm{uf}}(G;\mathbb{R}) \cong H_*\bigl(G;\ell^\infty(G;\mathbb{R})\bigr) \longrightarrow H_*(G;\mathbb{R}).$$

In particular, in degree 0, we obtain a map $m_0\colon H_0^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow \mathbb{R}$ mapping $[G]_\mathbb{R}$ to 1 with $\|m_0\| \le 1$.

Invariant means give an alternative description of classes with trivial ℓ^∞-semi-norm in uniformly finite homology in degree 0 [18, Theorem 1][2, Corollary 6.7.5]:

Proposition 2.6 (mean-invisibility and ℓ^∞-semi-norm in degree 0)**.** *Let G be a finitely generated amenable group and let $\alpha \in H_0^{\mathrm{uf}}(G;\mathbb{R})$. Then $\|\alpha\|_\infty = 0$ if and only if α is mean-invisible. Here, a class $\alpha \in H_0^{\mathrm{uf}}(G;\mathbb{R})$ is* mean-invisible [3, Definition 3.5] *if*

$$\forall_{m\in M(G)}\ \ m_0(\alpha) = 0 \in \mathbb{R}.$$

3. Degree 0: Rigidity

We will prove Theorem 1.1 in Section 3.1. The basic idea is to identify the fundamental class as the "largest" class in $H_0^{\mathrm{uf}}(\,\cdot\,;\mathbb{Z})$ of ℓ^∞-semi-norm 1 (Proposition 3.3), and then to apply Whyte's rigidity result Theorem A.12. In contrast, we will show in Section 3.2 that the analogue of Theorem 1.1 for $\mathbb{R}$-coefficients does *not* hold. In Section 3.3, we will translate isometry properties of group homomorphisms into the context of ℓ^∞-semi-norms on uniformly finite homology.

3.1. **Integral coefficients.** As a preparation for the proof of Theorem 1.1, we look at those classes in uniformly finite homology that can be represented via non-negative coefficients:

Definition 3.1 (non-negative classes)**.** Let X be a UDBG-space. We then define

$$C_0^{\mathrm{uf},(+)}(X;\mathbb{Z}) := \left\{ \sum_{x\in X} c_x \cdot x \in C_0^{\mathrm{uf}}(X;\mathbb{Z}) \;\middle|\; \forall_{x\in X}\;\; c_x \geq 0 \right\}$$

and we write $H_0^{\mathrm{uf},(+)}(X;\mathbb{Z}) \subset H_0^{\mathrm{uf}}(X;\mathbb{Z})$ for the subset of classes that admit a representing cycle in $C_0^{\mathrm{uf},(+)}(X;\mathbb{Z})$.

Remark 3.2. Let $f\colon X \longrightarrow Y$ be a quasi-isometry between UDBG-spaces. By definition of non-negative chains and the induced map $C_0^{\mathrm{uf}}(f;\mathbb{Z})$ we obtain $C_0^{\mathrm{uf}}(f;\mathbb{Z})(C_0^{\mathrm{uf},(+)}(X;\mathbb{Z})) \subset C_0^{\mathrm{uf},(+)}(Y;\mathbb{Z})$; in particular, it follows that $H_0^{\mathrm{uf}}(f;\mathbb{Z})(H_0^{\mathrm{uf},(+)}(X;\mathbb{Z})) \subset H_0^{\mathrm{uf},(+)}(Y;\mathbb{Z})$. Looking at a quasi-inverse of f shows that

$$H_0^{\mathrm{uf}}(f;\mathbb{Z})\big(H_0^{\mathrm{uf},(+)}(X;\mathbb{Z})\big) = H_0^{\mathrm{uf},(+)}(Y;\mathbb{Z}).$$

Proposition 3.3 (the fundamental class is large)**.** *Let X be a UDBG space.*

(1) *Let $\alpha \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ with $\|\alpha\|_\infty \leq 1$ and $\alpha \neq [X]_{\mathbb{Z}}$. Then there exists an element $\beta \in H_0^{\mathrm{uf},(+)}(X;\mathbb{Z}) \setminus \{0\}$ with $\|\alpha+\beta\|_\infty \leq 1$.*
(2) *For all $\beta \in H_0^{\mathrm{uf},(+)}(X;\mathbb{Z}) \setminus \{0\}$ we have $\|[X]_{\mathbb{Z}}+\beta\|_\infty > 1$.*

Proof. Ad 1. By assumption, there is a cycle $c = \sum_{x\in X} c_x \cdot x \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ representing α with $|c_x| \leq 1$ for all $x \in X$. We then consider

$$b := \sum_{x\in X} (1-c_x)\cdot x \in C_0^{\mathrm{uf},(+)}(X;\mathbb{Z}).$$

So $\beta := [b] = [X]_{\mathbb{Z}} - \alpha \neq 0 \in H_0^{\mathrm{uf},(+)}(X;\mathbb{Z})$ and $\|\alpha+\beta\|_\infty = \|[X]_{\mathbb{Z}}\|_\infty \leq 1$.

Ad 2. Assume for a contradiction that there is a $\beta \in H_0^{\mathrm{uf},(+)}(X;\mathbb{Z}) \setminus \{0\}$ with $\|[X]_{\mathbb{Z}}+\beta\|_\infty \leq 1$. Let $b = \sum_{x\in X} b_x \cdot x \in C_0^{\mathrm{uf},(+)}(X;\mathbb{Z})$ be a non-negative cycle representing β, and let $c = \sum_{x\in X} c_x \cdot x \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ be a cycle with $[c] = [X]_{\mathbb{Z}} + \beta$ and $|c_x| \leq 1$ for all $x \in X$. Thus,

$$0 = [X]_{\mathbb{Z}} + \beta - \big([X]_{\mathbb{Z}}+\beta\big) = \left[\sum_{x\in X} \big(1+b_x-c_x\big)\cdot x\right].$$

In view of the vanishing criterion (Theorem A.13) there hence exist constants $C, r \in \mathbb{N}$ satisfying

$$\forall_{F\subset X \text{ finite}}\;\; C\cdot|\partial_r F| \geq \left|\sum_{x\in F} 1+b_x-c_x\right| \geq \left|\sum_{x\in F} b_x\right|.$$

Therefore, the vanishing criterion implies that also $\beta = [\sum_{x\in X} b_x \cdot x] = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$, contradicting our assumption on β. $\square$

Proof of Theorem 1.1. The implication "(1) $\Rightarrow$ (2)" is a consequence of Proposition 2.2, and "(2) $\Rightarrow$ (3)" is trivial. We now prove "(3) $\Rightarrow$ (1)":

Suppose that $H_0^{\mathrm{uf}}(f;\mathbb{Z})\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometric isomorphism. In view of Whyte's rigidity result Theorem A.12 it suffices to show that $H_0^{\mathrm{uf}}(f;\mathbb{Z})([X]_\mathbb{Z}) = [Y]_\mathbb{Z}$.

Let $\alpha := H_0^{\mathrm{uf}}(f;\mathbb{Z})([X]_\mathbb{Z}) \in H_0^{\mathrm{uf},(+)}(Y;\mathbb{Z})$. Because $H_0^{\mathrm{uf}}(f;\mathbb{Z})$ is isometric, we have $\|\alpha\|_\infty = \|[X]_\mathbb{Z}\|_\infty \leq 1$. Moreover, $H_0^{\mathrm{uf}}(f;\mathbb{Z})$ induces a bijection between $H_0^{\mathrm{uf},(+)}(X;\mathbb{Z})$ and $H_0^{\mathrm{uf},(+)}(Y;\mathbb{Z})$ (Remark 3.2). Hence, Proposition 3.3(2) and the fact that $H_0^{\mathrm{uf}}(f;\mathbb{Z})$ is an isometric isomorphism show that

$$\|\alpha + \beta\|_\infty > 1$$

holds for all $\beta \in H^{\mathrm{uf},(+)}(Y;\mathbb{Z}) \setminus \{0\}$. Therefore, Proposition 3.3(1) implies that $\alpha = [Y]_\mathbb{Z}$. □

The inclusion $i\colon H \longrightarrow G$ of a subgroup $H < G$ of finite index of a finitely generated group is a quasi-isometry. If G is amenable and H is a proper subgroup, then i is not uniformly close to any bilipschitz equivalence [11, Theorem 3.5] (see also Corollary 3.9). We use this to give an example of a quasi-isometry that is not uniformly close to a bilipschitz equivalence but whose induced maps in uniformly finite homology with integral coefficients are isometric isomorphisms in positive degrees.

Example 3.4. As observed above, the inclusion $i\colon 2\mathbb{Z} \longrightarrow \mathbb{Z}$ is not uniformly close to a bilipschitz equivalence. In particular, by Theorem 1.1 the induced map $H_0^{\mathrm{uf}}(i;\mathbb{Z})\colon H_0^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is not an isometry with respect to the ℓ^∞-semi-norm. On the other hand, we will now show that for all $k \in \mathbb{N}_{>0}$ the map $H_k^{\mathrm{uf}}(i;\mathbb{Z})\colon H_k^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_k^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm.

Using a "fine" (simplicial) version of uniformly finite homology [1, Definition 2.2], one can easily see that for $k \in \mathbb{N}_{>1}$ we have $H_k^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) = 0$. Indeed, $\mathbb{Z}$ is quasi-isometric to a uniformly contractible locally finite simplicial complex of dimension 1 (namely, $\mathbb{R}$ with the standard triangulation). In particular, for all $k \in \mathbb{N}_{>1}$ the map $H_k^{\mathrm{uf}}(i;\mathbb{Z})\colon H_k^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_k^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is trivially an isometric isomorphism with respect to the ℓ^∞-semi-norm. To prove the claim in degree $k = 1$, it suffices to show that every non-trivial class in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ and in $H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z})$ has ℓ^∞-semi-norm equal to 1 and thus the map $H_1^{\mathrm{uf}}(i;\mathbb{Z})\colon H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is necessarily an isometric isomorphism.

As above, using fine uniformly finite homology, one can prove that the group $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is generated by the class $\gamma = \bigl[\sum_{z\in\mathbb{Z}}(z,z+1)\bigr]$. In particular, $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$. Let $\alpha \in H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) \setminus \{0\}$ and let $n \in \mathbb{Z} \setminus \{0\}$ be such that $\alpha = n \cdot \gamma$. From Figure 1 it is easy to see that

$$\begin{aligned}\sum_{z\in\mathbb{Z}}(z,z+1) - \sum_{z\in n\mathbb{Z}}(z,z+n) = \sum_{z\in n\mathbb{Z}} \partial_1\biggl(&\sum_{j=0}^{n-1}(z+j,z+j+1,z+n)\\ &- (z+n,z+n,z+n)\biggr).\end{aligned}$$

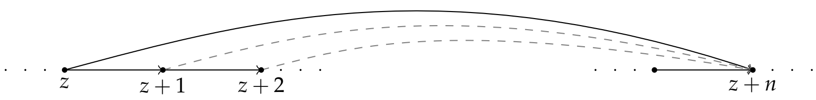

FIGURE 1. homologous chains in $C_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$, schematically

In particular,

$$\gamma = \Bigl[\sum_{z\in n\mathbb{Z}} (z, z+n)\Bigr].$$

Similarly, for all $k \in \{0,\dots,n-1\}$ we consider $c_k := \sum_{z\in n\mathbb{Z}+k}(z,z+n)$. Because translation by k is uniformly close to $\mathrm{id}_{\mathbb{Z}}$, we have $\gamma = [c_k]$ for all $k \in \{0,\dots,n-1\}$. Thus,

$$\alpha = n\cdot\gamma = \Bigl[\sum_{k=0}^{n-1} c_k\Bigr].$$

Clearly, for any $k \in \{0,\dots,n-1\}$ we have $\|c_k\|_\infty = 1$. Moreover, the cycles $c_0,\dots,c_{n-1} \in C_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ are supported on pairwise disjoint subsets of $\mathbb{Z}^2$, and so we have:

$$\|\alpha\|_\infty = \Bigl\|\Bigl[\sum_{k=0}^{n-1} c_k\Bigr]\Bigr\|_\infty = 1.$$

Thus every non-trivial class in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ has ℓ^∞-semi-norm 1. Analogously, one can prove that every non-trivial class in $H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z})$ has ℓ^∞-semi-norm 1. Thus the map $H_1^{\mathrm{uf}}(i;\mathbb{Z})\colon H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is an isometric isomorphism.

3.2. **Real coefficients.** The analogue of Theorem 1.1 for $\mathbb{R}$-coefficients does not hold as the following simple example shows:

Proposition 3.5. *Let $A := \{n^2 \mid n \in \mathbb{N}\} \subset \mathbb{Z}$, and consider $\mathbb{Z}\setminus A$ with the metric induced from $\mathbb{Z}$. Then the inclusion $i\colon \mathbb{Z}\setminus A \longrightarrow \mathbb{Z}$ is a quasi-isometry inducing for all $k \in \mathbb{N}$ an isometric isomorphism*

$$H_k^{\mathrm{uf}}(i;\mathbb{R})\colon H_k^{\mathrm{uf}}(\mathbb{Z}\setminus A;\mathbb{R}) \longrightarrow H_k^{\mathrm{uf}}(\mathbb{Z};\mathbb{R}),$$

but i is not *uniformly close to a bilipschitz equivalence.*

Proof. The inclusion $i\colon \mathbb{Z}\setminus A \longrightarrow \mathbb{Z}$ is an isometric embedding with quasi-dense image, whence a quasi-isometry. Thus, for all $k \in \mathbb{N}$ the induced map $H_k^{\mathrm{uf}}(i;\mathbb{R})\colon H_k^{\mathrm{uf}}(\mathbb{Z}\setminus A;\mathbb{R}) \longrightarrow H_k^{\mathrm{uf}}(\mathbb{Z};\mathbb{R})$ is an isomorphism.

Moreover, $H_k^{\mathrm{uf}}(i;\mathbb{R})$ is isometric: Let $\alpha \in H_k^{\mathrm{uf}}(\mathbb{Z}\setminus A;\mathbb{R})$. Because i is injective, we have $\|H_k^{\mathrm{uf}}(i;\mathbb{R})(\alpha)\|_\infty \le \|\alpha\|_\infty$. Conversely, let $n \in \mathbb{N}_{>0}$. For $j \in \{1,\dots,n\}$ we consider the map

$$\begin{aligned} f_j\colon \mathbb{Z} &\longrightarrow \mathbb{Z}\setminus A \\ x &\longmapsto \begin{cases} x & \text{if } x \in \mathbb{Z}\setminus A \\ x+j & \text{if } x \in A \text{ and } x \ge n^2 \\ -n\cdot x - j & \text{if } x \in A \text{ and } x < n^2. \end{cases} \end{aligned}$$

Clearly, f_j is a quasi-isometric embedding and $f_j \circ i = \mathrm{id}_{\mathbb{Z}\setminus A}$. We then set

$$\varphi_* := \frac{1}{n} \cdot \sum_{j=1}^{n} C_*^{\mathrm{uf}}(f_j;\mathbb{R}) \colon C_*^{\mathrm{uf}}(\mathbb{Z};\mathbb{R}) \longrightarrow C_*^{\mathrm{uf}}(\mathbb{Z}\setminus A;\mathbb{R}).$$

Notice that φ_* is a chain map satisfying $\varphi_* \circ C_*^{\mathrm{uf}}(i;\mathbb{R}) = \mathrm{id}$. The restrictions $f_1|_A, \dots, f_n|_A$ are injective and have pairwise disjoint images. By counting pre-images and types of elements in $(\mathbb{Z}\setminus A)^{d+1}$ we obtain that

$$\|\varphi_k\| \leq 1 + \frac{2^{d+1}-1}{n}$$

holds with respect to the corresponding ℓ^∞-norms. In particular,

$$\|\alpha\|_\infty = \big\|H_0(\varphi_*) \circ H_0^{\mathrm{uf}}(i;\mathbb{R})(\alpha)\big\|_\infty \leq \Big(1 + \frac{2^{d+1}-1}{n}\Big) \cdot \big\|H_0^{\mathrm{uf}}(i;\mathbb{R})(\alpha)\big\|_\infty.$$

Taking the infimum over all $n \in \mathbb{N}_{>0}$ gives the desired estimate.

On the other hand, Whyte's vanishing criterion (Theorem A.13) shows that $[\chi_A] \neq 0$ in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$. Hence, in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ we obtain

$$H_0^{\mathrm{uf}}(i;\mathbb{Z})[\mathbb{Z}\setminus A]_\mathbb{Z} = [\mathbb{Z}]_\mathbb{Z} - [\chi_A] \neq [\mathbb{Z}]_\mathbb{Z},$$

and so i is *not* uniformly close to a bilipschitz equivalence (Theorem A.12). □

As we have seen in Proposition 3.3 in the case of $\mathbb{Z}$-coefficients, for an amenable UDBG space X we have $\|[X]_\mathbb{Z}\|_\infty \leq 1$. More precisely, since a non-trivial class in uniformly finite homology with $\mathbb{Z}$-coefficients cannot have ℓ^∞-semi-norm strictly smaller than 1, we have $\|[X]_\mathbb{Z}\|_\infty = 1$. Similarly, in the amenable case also the fundamental class with $\mathbb{R}$-coefficients is rigid with respect to the ℓ^∞-semi-norm in the following sense:

Proposition 3.6 (semi-norm of the fundamental class)**.** *Let X be an amenable UDBG space and let $[X]_\mathbb{R} \in H_0^{\mathrm{uf}}(X;\mathbb{R})$ be its fundamental class. Then*

$$\|[X]_\mathbb{R}\|_\infty = 1.$$

Proof. By definition of the fundamental class, we have $\|[X]_\mathbb{R}\|_\infty \leq 1$. For the converse estimate, we use an averaging argument: Let $(S_n)_{n\in\mathbb{N}}$ be a Følner sequence for X (Definition A.14), and let ω be a non-principal ultrafilter on $\mathbb{N}$. A straightforward calculation using the Følner condition shows that the map

$$\begin{aligned} C_0^{\mathrm{uf}}(X;\mathbb{R}) &\longrightarrow \mathbb{R} \\ \sum_{x\in X} c_x \cdot x &\longmapsto \lim_{n\in\omega} \frac{1}{|S_n|} \cdot \sum_{x\in S_n} c_x \end{aligned}$$

induces a well-defined map $m \colon H_0^{\mathrm{uf}}(X;\mathbb{R}) \longrightarrow \mathbb{R}$ with $m([X]_\mathbb{R}) = 1$ and

$$\forall_{\alpha\in H_0^{\mathrm{uf}}(X;\mathbb{R})} \quad \big|m(\alpha)\big| \leq \|\alpha\|_\infty.$$

Hence, $\|[X]_\mathbb{R}\|_\infty \geq 1$. □

This semi-norm rigidity of the fundamental class in the amenable case leads to the following examples:

Example 3.7. The ℓ^∞-semi-norm on uniformly finite homology with $\mathbb{Z}$-coefficients is *not* homogeneous, in general. For example: In $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ we have

$$2 \cdot [\chi_{2\mathbb{Z}}] = [\chi_{2\mathbb{Z}}] + [\chi_{2\mathbb{Z}+1}] = [\mathbb{Z}]_{\mathbb{Z}},$$

which is non-trivial because $\mathbb{Z}$ is amenable (Theorem A.16). So, $[\chi_{2\mathbb{Z}}] \neq 0$ in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ and $\big\|[\chi_{2\mathbb{Z}}]\big\|_\infty = 1 = \big\|[\mathbb{Z}]_{\mathbb{Z}}\big\|_\infty$, but

$$\big\|2 \cdot [\chi_{2\mathbb{Z}}]\big\|_\infty = \big\|[\mathbb{Z}]_{\mathbb{Z}}\big\|_\infty = 1 < 2 = 2 \cdot \big\|[\chi_{2\mathbb{Z}}]\big\|_\infty.$$

Example 3.8. The ℓ^∞-semi-norm is *not* a functorial semi-norm on $H_*^{\mathrm{uf}}(\,\cdot\,;\mathbb{Z})$ and $H_*^{\mathrm{uf}}(\,\cdot\,;\mathbb{R})$ in the sense of Gromov [14, 17]: For example, the quasi-isometric embedding

$$\begin{aligned} f \colon \mathbb{Z} &\longrightarrow \mathbb{Z} \\ n &\longmapsto \Big\lfloor \frac{n}{2} \Big\rfloor \end{aligned}$$

does not satisfy $\|H_0^{\mathrm{uf}}(f;\mathbb{Z})\| \leq 1$ or $\|H_0^{\mathrm{uf}}(f;\mathbb{R})\| \leq 1$ because

$$\begin{aligned} \big\|H_0^{\mathrm{uf}}(f;\mathbb{R})[\mathbb{Z}]_{\mathbb{R}}\big\|_\infty &= \big\|2 \cdot [\mathbb{Z}]_{\mathbb{R}}\big\|_\infty = 2 \cdot \big\|[\mathbb{Z}]_{\mathbb{R}}\big\|_\infty = 2 \\ &> 1 = \big\|[\mathbb{Z}]_{\mathbb{R}}\big\|_\infty, \\ \big\|H_0^{\mathrm{uf}}(f;\mathbb{Z})[\mathbb{Z}]_{\mathbb{Z}}\big\|_\infty &= \big\|2 \cdot [\mathbb{Z}]_{\mathbb{Z}}\big\|_\infty \geq \big\|2 \cdot [\mathbb{Z}]_{\mathbb{R}}\big\|_\infty = 2 \\ &> 1 = \big\|[\mathbb{Z}]_{\mathbb{Z}}\big\|_\infty. \end{aligned}$$

3.3. **Group homomorphisms.** We can translate a result by Dymarz [11] into the context of the ℓ^∞-semi-norm on uniformly finite homology in degree 0. In particular, we can characterise when a group homomorphism between finitely generated amenable groups with finite kernel and cokernel is uniformly close to a bilipschitz equivalence using isometric isomorphisms on uniformly finite homology:

Corollary 3.9. *Let G, H be finitely generated amenable groups, let $f \colon G \longrightarrow H$ be a homomorphism with finite kernel and finite cokernel. Then the following are equivalent:*

1. *We have $|\ker(f)| = |\operatorname{coker}(f)|$.*
2. *The map f is uniformly close to a bilipschitz equivalence.*
3. *The induced map $H_0^{\mathrm{uf}}(f;\mathbb{Z}) \colon H_0^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(H;\mathbb{Z})$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm.*
4. *The induced map $H_0^{\mathrm{uf}}(f;\mathbb{R}) \colon H_0^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_0^{\mathrm{uf}}(H;\mathbb{R})$ is an isometric isomorphism with respect to the ℓ^∞-semi-norm.*

Proof. Notice that f is a quasi-isometry because f has finite kernel and cokernel. The equivalence "(1) $\Leftrightarrow$ (2)" is a result of Dymarz [11, Theorem 3.6]. The equivalence "(2) $\Leftrightarrow$ (3)" follows from Theorem 1.1. The implication "(2) $\Rightarrow$ (4)" is a consequence of Proposition 2.2. The implication "(4) $\Rightarrow$ (1)" follows from the fact that the $\mathbb{R}$-fundamental class has ℓ^∞-semi-norm equal to 1 (Proposition 3.6) and that

$$H_0^{\mathrm{uf}}(f;\mathbb{R})([G]_{\mathbb{R}}) = \frac{|\ker(f)|}{|\operatorname{coker}(f)|} \cdot [H]_{\mathbb{R}}. \qquad \square$$

4. Higher degree: Vanishing

In this section we will show that the ℓ^∞-semi-norm on uniformly finite homology with $\mathbb{R}$-coefficients is trivial in higher degrees.

4.1. **The non-amenable case.** We begin with the proof of Proposition 1.2: To this end, we shrink the involved coefficients by spreading the chain over the space; non-amenability allows us to keep control over the ℓ^∞-semi-norms of the classes in question.

Proof of Proposition 1.2. We consider the double $Y := X \times \{0,1\}$ of X with respect to the sum metric

$$\begin{aligned} Y \times Y &\longrightarrow \mathbb{R}_{\geq 0} \\ \big((x,j),(x',j')\big) &\longmapsto d(x,x') + |j - j'|, \end{aligned}$$

where d is the metric on X. Then Y is a UDBG space and

$$\begin{aligned} p\colon Y &\longrightarrow X \\ (x,j) &\longmapsto x \end{aligned}$$

is a quasi-isometry (for example, a quasi-inverse is given by the inclusion into the 0-factor). Because X and hence also Y are non-amenable, p is uniformly close to a bilipschitz equivalence (Corollary A.17). Hence, for all $k \in \mathbb{N}$ the induced map $H_k^{\mathrm{uf}}(p;\mathbb{R})\colon H_k^{\mathrm{uf}}(Y;\mathbb{R}) \longrightarrow H_k^{\mathrm{uf}}(X;\mathbb{R})$ is an isometric isomorphism (Proposition 2.2). Let $\alpha \in H_k^{\mathrm{uf}}(X;\mathbb{R})$. Then

$$\beta := \frac{1}{2} \cdot \alpha \times 0 + \frac{1}{2} \cdot \alpha \times 1 \in H_k^{\mathrm{uf}}(Y;\mathbb{R})$$

satisfies $H_k^{\mathrm{uf}}(p;\mathbb{R})(\beta) = 1/2 \cdot \alpha + 1/2 \cdot \alpha = \alpha$ and

$$\|\alpha\|_\infty = \big\|H_k^{\mathrm{uf}}(p;\mathbb{R})(\beta)\big\|_\infty = \|\beta\|_\infty = \Big\|\frac{1}{2} \cdot \alpha \times 0 + \frac{1}{2} \cdot \alpha \times 1\Big\|_\infty \leq \frac{1}{2} \cdot \|\alpha\|_\infty.$$

Therefore, we obtain $\|\alpha\|_\infty = 0$. □

4.2. **The amenable case.** Finally, we will prove Proposition 1.3: The key is to use the interpretation of uniformly finite homology of groups in terms of group homology with ℓ^∞-coefficients and to apply a vanishing result on ℓ^1-homology.

Proof of Proposition 1.3. We have $H_k^{\mathrm{uf}}(G;\mathbb{R}) \cong H_k(G;\ell^\infty(G;\mathbb{R}))$ and the corresponding isomorphism is isometric with respect to the ℓ^∞-semi-norms (Proposition 2.5). Hence, it suffices to show that the ℓ^∞-semi-norm $\|\cdot\|_\infty$ is trivial on $H_k(G;\ell^\infty(G;\mathbb{R}))$.

Clearly, $\|\cdot\|_\infty \leq \|\cdot\|_1$ on $H_k(G;\ell^\infty(G;\mathbb{R}))$, where $\|\cdot\|_1$ is the ℓ^1-semi-norm induced from the bar complex. However, since $k > 0$ and G is amenable we know that $\|\cdot\|_1 = 0$ on $H_k(G;\ell^\infty(G;\mathbb{R}))$: In fact, the comparison map

$$H_k\big(G;\ell^\infty(G;\mathbb{R})\big) \longrightarrow H_k^{\ell^1}\big(G;\ell^\infty(G;\mathbb{R})\big)$$

is isometric with respect to the ℓ^1-semi-norm and $H_k^{\ell^1}(G;\ell^\infty(G;\mathbb{R}))$ is trivial [16, Proposition 2.4, Corollary 5.5]. □

Appendix A. Review of uniformly finite homology

We start by recalling the category of UDBG spaces (Section A.1) and the definition of uniformly finite homology for UDBG spaces (Section A.2) and for finitely generated groups (Section A.3). In Section A.4, we review the fundamental class in uniformly finite homology and its relation with rigidity and amenability.

A.1. **UDBG spaces.** For simplicity, we will work in the category of UDBG spaces. We briefly recall the definitions:

Definition A.1 (UDBG space)**.** A metric space X is a *UDBG space* if it is uniformly discrete and of bounded geometry, i.e., if

- There exists $\varepsilon > 0$ such that
$$\forall_{x,y \in X} \;\; d(x,y) < \varepsilon \Longleftrightarrow x = y.$$
- For every $r \in \mathbb{R}_{>0}$ there exists $K > 0$ such that
$$\forall_{x \in X} \;\; |B_r(x)| < K.$$

Example A.2. Every finitely generated group equipped with some word metric of a finite generating set is a UDBG space. The space $\{n^2 \mid n \in \mathbb{N}\}$ with the metric induced by the standard metric in $\mathbb{R}$ is a UDBG space. Every manifold with a Riemannian metric of bounded geometry is quasi-isometric to a UDBG space contained in it (namely the space of vertices of a suitable triangulation) [1, Theorem 1.14].

Definition A.3 (quasi-isometric embedding, quasi-isometry, and bilipschitz equivalence)**.** Let (X, d_X) and (Y, d_Y) be UDBG spaces and let $f \colon X \longrightarrow Y$ be a map.

- We say that f is a *quasi-isometric embedding* if there are $C, D \in \mathbb{R}_{>0}$ such that for all $x, x' \in X$ we have
$$\frac{1}{C} \cdot d_X(x,x') - D \leq d_Y\big(f(x), f(x')\big) \leq C \cdot d_X(x,x') + D.$$
- The map f is *uniformly close* to a map $f' \colon X \longrightarrow Y$ if
$$\sup_{x \in X} d_Y\big(f(x), f'(x)\big) < \infty$$
- We say that f is a *quasi-isometry* if it is a quasi-isometric embedding and if it admits a quasi-inverse quasi-isometric embedding, i.e., if there exists a quasi-isometric embedding $g \colon Y \longrightarrow X$ such that $f \circ g$ is uniformly close to id_Y and $g \circ f$ is uniformly close to id_X.
- The map f is a *bilipschitz embedding* if there is a $C \in \mathbb{R}_{>0}$ such that
$$\forall_{x,x' \in X} \;\; \frac{1}{C} \cdot d_X(x,x') \leq d_Y\big(f(x), f(x')\big) \leq C \cdot d_X(x,x').$$
- We say that f is a *bilipschitz equivalence* if it is a bilipschitz embedding and if there is a bilipschitz embedding $g \colon Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. (For UDBG spaces this is equivalent to f being a bijective quasi-isometry.)

Definition A.4. (category of UDBG spaces) We define the category UDBG as follows:

– The objects in UDBG are UDBG spaces.
– The set of morphisms between two objects X and Y in UDBG is given by the set

$$\operatorname{QIE}(X,Y) := \{f \colon X \longrightarrow Y \mid f \text{ quasi-isometric embedding}\}/\sim$$

where $f \sim f'$ if and only if f is uniformly close to f'.
– Composition of morphisms is given by ordinary composition of representatives.

Clearly, quasi-isometries of UDBG spaces correspond to isomorphisms in the category UDBG.

A.2. **Uniformly finite homology of UDBG spaces.** Uniformly finite chains are combinatorial infinite chains on UDBG spaces that satisfy certain geometric finiteness conditions. We consider uniformly finite chains with coefficients in a normed ring with unit.

Definition A.5 (normed ring)**.** Let R be a ring with unit. A *norm on* R is a function $|\cdot| \colon R \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions:
(1) For all $r \in R$ we have $|r| = 0$ if and only if $r = 0$.
(2) For all $r, r' \in R$ we have $|r + r'| \leq |r| + |r'|$.
(3) For all $r, r' \in R$ we have $|r \cdot r'| = |r| \cdot |r'|$.

Definition A.6. (uniformly finite homology) Let R be a normed ring with unit and X be a UDBG space. For each $n \in \mathbb{N}$ the space of *uniformly finite n-chains* is the R-module $C_n^{\mathrm{uf}}(X;R)$ whose elements are functions of type $X^{n+1} \longrightarrow R$, written as formal sums of the form

$$c = \sum_{x \in X^{n+1}} c_x \cdot x,$$

satisfying the following conditions:

– For any $x \in X^{n+1}$, we have $c_x \in R$. Moreover, there exists a constant $K \in \mathbb{R}_{>0}$ such that

$$\forall_{x \in X^{n+1}} \ \ |c_x| < K$$

– There exists a constant $R \in \mathbb{R}_{>0}$ such that:

$$\forall_{x = (x_0, \dots, x_n) \in X^{n+1}} \quad \sup_{i,j \in \{0,\dots,n\}} d(x_i, x_j) > R \Longrightarrow c_x = 0.$$

For each $n \in \mathbb{N}$, we define a boundary operator

$$\partial_n \colon C_n^{\mathrm{uf}}(X;R) \longrightarrow C_{n-1}^{\mathrm{uf}}(X;R)$$

that takes every $x \in X^{n+1}$ to

$$\partial_n(x) = \sum_{j=0}^{n} (-1)^j \cdot (x_0, \dots, \widehat{x}_j, \dots, x_n)$$

and is extended in the obvious way to all of $C_n^{\mathrm{uf}}(X;R)$; this map is indeed well-defined. Moreover, for each $n \in \mathbb{N}$ we have $\partial_n \circ \partial_{n+1} = 0$. In this way we get a well-defined chain complex. The homology of $(C_n^{\mathrm{uf}}(X;R), \partial_n)_{n \in \mathbb{N}}$ is the *uniformly finite homology of* X and it is denoted by $H_*^{\mathrm{uf}}(X;R)$.

Block and Weinberger [4, Proposition 2.1] observed that uniformly finite homology is quasi-isometry invariant:

Proposition A.7 (quasi-isometry invariance)**.** *Let R be a normed ring with unit. Let X, Y be UDBG spaces and let $f\colon X \longrightarrow Y$ be a quasi-isometric embedding. Then f induces a chain map, defined for each $n \in \mathbb{N}$ by*

$$\begin{aligned} C_n^{\mathrm{uf}}(f;R)\colon C_n^{\mathrm{uf}}(X;R) &\longrightarrow C_n^{\mathrm{uf}}(Y;R) \\ \sum_{x\in X^{n+1}} c_x \cdot x &\longmapsto \sum_{x\in X^{n+1}} c_x \cdot \big(f(x_0),\dots,f(x_n)\big). \end{aligned}$$

If f is uniformly close to a quasi-isometric embedding $f'\colon X \longrightarrow Y$, then

$$H_*^{\mathrm{uf}}(f;R) = H_*^{\mathrm{uf}}(f';R)\colon H_n^{\mathrm{uf}}(X;R) \longrightarrow H_n^{\mathrm{uf}}(Y;R).$$

In particular, any quasi-isometry induces an isomorphism in uniformly finite homology.

In view of Proposition A.7, uniformly finite homology with coefficients in a normed ring R is a functor from the category UDBG to the category Mod_*^R of graded R-modules:

$$\begin{array}{rccc} H_*^{\mathrm{uf}}(\,\cdot\,;R)\colon & \mathsf{UDBG} & \longrightarrow & \mathsf{Mod}_*^R \\ \text{on objects}: & X & \longmapsto & H_*^{\mathrm{uf}}(X;R) \\ \text{on morphisms}: & [f]\colon X \to Y & \longmapsto & H_*^{\mathrm{uf}}(f;\mathbb{R})\colon H_*^{\mathrm{uf}}(X;R) \to H_*^{\mathrm{uf}}(Y;R). \end{array}$$

Uniformly finite homology was introduced by Block and Weinberger to study the large scale structure of metric spaces of bounded geometry [4, 19]. One of the main applications provided by Block and Weinberger is a characterisation of amenability for metric spaces of bounded geometry [4, Theorem 3.1]. Whyte used uniformly finite homology to develop a criterion to distinguish between quasi-isometries and bilipschitz equivalences in the case of UDBG spaces. We recall these two applications in Section A.4 (Theorem A.12 and Theorem A.16).

A.3. **Uniformly finite homology of groups.** As a consequence of quasi-isometry invariance (Proposition A.7) we obtain that for finitely generated groups uniformly finite homology is independent from the chosen word metric.

Corollary A.8. *Let G be a finitely generated group and let d_S, d_T be the word metrics on G associated to finite generating sets $S, T \subset G$. Then the identity map $\mathrm{id}_G\colon (G, d_S) \longrightarrow (G, d_T)$ is a bilipschitz equivalence (whence a quasi-isometry). Thus, the induced map $H_*^{\mathrm{uf}}(\mathrm{id}_G;R)\colon H_*^{\mathrm{uf}}(G,d_S;R) \longrightarrow H_*^{\mathrm{uf}}(G,d_T;R)$ is an isomorphism for every normed ring R with unit.*

We now recall homology of groups with twisted coefficients. Proposition A.10 shows that in the case of finitely generated groups uniformly finite homology is isomorphic to group homology with coefficients in the module of bounded functions on the group.

Definition A.9 (group homology)**.** Let G be a group and R be a ring with unit. We consider $\big(C_n(G;R),\partial_n\big)_{n\in\mathbb{N}}$ to be the $R[G]$-chain complex, where for each $n \in \mathbb{N}$ we have:

- The module $C_n(G;R)$ is the free R-module with the basis G^{n+1} (with the $R[G]$-structure induced from the diagonal action on G^{n+1}).
- The operator ∂_n is the standard boundary map given by

$$\begin{aligned} \partial_n\colon C_n(G;R) &\longrightarrow C_{n-1}(G;R) \\ (g_0,\dots,g_n) &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot (g_0,\dots,\widehat{g}_j,\dots,g_n). \end{aligned}$$

Let A be a left $R[G]$-module and let $\overline{C}_*(G;R)$ be the right $R[G]$-module obtained from $C_*(G;R)$ via the canonical involution $G \to G$, $g \mapsto g^{-1}$. The *homology of G with coefficients in A* is the homology of the R-chain complex $C_*(G;A) := \overline{C}_*(G;R) \otimes_{R[G]} A$.

Let R be a ring with unit endowed with a norm $|\cdot|$. The space $\ell^\infty(G,R)$ of functions $\varphi\colon G \to R$ that are bounded with respect to the supremum norm $\|\varphi\|_\infty := \sup_{g\in G} |\varphi(g)|$ has a natural $R[G]$-module structure with respect to the action

$$\begin{aligned} G \times \ell^\infty(G,R) &\longrightarrow \ell^\infty(G,R) \\ (g,\varphi) &\longmapsto \bigl(g\cdot\varphi\colon g' \longmapsto \varphi(g^{-1}\cdot g')\bigr). \end{aligned}$$

Notice that, in the case of uniformly finite homology, the simplices of a given uniformly finite chain are tuples in G^{n+1} of uniformly bounded diameter; therefore they are contained in the G-orbit of finitely many simplices of the form $(e,t_1,\dots,t_n) \in G^{n+1}$. Hence, we have [5][9, Proposition 2.2.4]:

Proposition A.10 (uniformly finite homology as group homology)**.** *Let G be a finitely generated group endowed with the word metric with respect to some finite generating set and let R be a normed ring with unit. For $n \in \mathbb{N}$ consider*

$$\begin{aligned} \rho_n\colon C_n^{\mathrm{uf}}(G;R) &\longrightarrow C_n\bigl(G;\ell^\infty(G,R)\bigr) \\ \sum_{g\in G^{n+1}} c_g\cdot g &\longmapsto \sum_{t=(t_1,\dots,t_n)\in G^n} (e,t_1,\dots,t_n)\otimes\varphi_{c,t} \end{aligned}$$

where for all $t \in G^n$ the map $\varphi_{c,t} \in \ell^\infty(G,R)$ is given by

$$\varphi_{c,t}\colon g \longmapsto c_{g^{-1}\cdot(e,t_1,\dots,t_n)}.$$

Then $\rho_*\colon C_*^{\mathrm{uf}}(G;R) \longrightarrow C_*(G;\ell^\infty(G,R))$ is a chain isomorphism; in particular, ρ_* induces an isomorphism $H_*(\rho_*)\colon H_*^{\mathrm{uf}}(G;R) \longrightarrow H_*(G;\ell^\infty(G,R))$.

A.4. **The fundamental class in uniformly finite homology.** In degree 0 there is a canonical uniformly finite homology class, the fundamental class:

Definition A.11 (fundamental class)**.** Let X be a UDBG space and let R be a normed ring with unit. The *R-fundamental class of X in $H_0^{\mathrm{uf}}(X;R)$* is the class $[X]_R \in H_0^{\mathrm{uf}}(X;R)$ represented by the cycle $\sum_{x\in X} 1\cdot x \in C_0^{\mathrm{uf}}(X;R)$.

We recall now a central application of uniformly finite homology, due to Whyte [21, Theorem 1.1], namely a criterion to distinguish between quasi-isometries and bilipschitz equivalences in the case of UDBG spaces.

Theorem A.12 (bilipschitz equivalence rigidity)**.** *Let X, Y be UDBG spaces and let $f\colon X \longrightarrow Y$ be a quasi-isometry. Then f is uniformly close to a bilipschitz equivalence if and only if $H_0^{\mathrm{uf}}(f;\mathbb{Z})([X]_{\mathbb{Z}}) = [Y]_{\mathbb{Z}}$.*

One step in Whyte's proof is the following characterisation of the trivial class in uniformly finite homology in degree 0 [21, Theorem 7.6]:

Theorem A.13 (vanishing criterion in degree 0)**.** *Let X be a UDBG space, and let $c = \sum_{x \in X} c_x \cdot x$ be a cycle in $C_0^{\mathrm{uf}}(X;\mathbb{Z})$. Then $[c] = 0 \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ if and only if there exist constants $C, r \in \mathbb{N}$ such that for all finite subsets $F \subset X$ we have*

$$C \cdot |\partial_r F| \geq \left| \sum_{x \in F} c_x \right|.$$

These boundary conditions are closely related to amenability; let us recall the definition of amenability for UDBG spaces:

Definition A.14 (amenable UDBG space)**.** A UDBG space X is *amenable* if it admits a *Følner sequence*, i.e., sequence $(S_n)_{n \in \mathbb{N}}$ of non-empty finite subsets $S_n \subset X$ such that

$$\forall_{r \in \mathbb{R}_{>0}} \quad \lim_{n \to \infty} \frac{|\partial_r(S_n)|}{|S_n|} = 0.$$

Remark A.15. We can consider finitely generated groups as UDBG spaces by endowing them with word metrics for finite generating sets. In this case, Definition A.14 is equivalent to the standard definition of amenability using Følner sequences [7, Definition 4.7.2].

Whyte used the vanishing criterion Theorem A.13 to provide a new proof of a characterisation of amenability for UDBG spaces [21, Theorem 7.1]. The original result is due to Block and Weinberger [4, Theorem 3.1] and it is one of the key applications of uniformly finite homology.

Theorem A.16 (characterisation of amenability)**.** *Let X be a UDBG space. The following are equivalent:*

(1) *The UDBG space X is non-amenable.*
(2) *We have $H_0^{\mathrm{uf}}(X;\mathbb{R}) = 0$.*
(3) *We have $H_0^{\mathrm{uf}}(X;\mathbb{Z}) = 0$.*
(4) *We have $[X]_{\mathbb{Z}} = 0$.*

Theorem A.12 and Theorem A.16 imply the following rigidity result for non-amenable spaces:

Corollary A.17. *Any quasi-isometry between non-amenable UDBG spaces is uniformly close to a bilipschitz equivalence.*

This answered a question originally stated by Gromov [15], which was answered by several authors using different tools [6, 20, 8].

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