VARIATIONS ON THE THEME OF
THE UNIFORM BOUNDARY CONDITION

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ABSTRACT. The uniform boundary condition in a normed chain complex asks for a uniform linear bound on fillings of null-homologous cycles. For the ℓ^1-norm on the singular chain complex, Matsumoto and Morita established a characterisation of the uniform boundary condition in terms of bounded cohomology. In particular, spaces with amenable fundamental group satisfy the uniform boundary condition in every degree. We will give an alternative proof of statements of this type, using geometric Følner arguments on the chain level instead of passing to the dual cochain complex. These geometric methods have the advantage that they also lead to integral refinements. In particular, we obtain applications in the context of integral foliated simplicial volume.

1. INTRODUCTION

The *uniform boundary condition* in a normed chain complex asks for a uniform linear bound on fillings of null-homologous cycles [16] (Definition 3.1). For the ℓ^1-norm on the singular chain complex, Matsumoto and Morita proved a characterisation of the uniform boundary condition in terms of bounded cohomology of the dual cochain complex [16]. In particular, spaces with amenable fundamental group satisfy the uniform boundary condition in every degree. Efficient fillings of this sort are used in glueing formulae for simplicial volume [9, 11] and the calculation of simplicial volume of smooth manifolds with non-trivial smooth S^1-action [22].

In the present article, we will give an alternative proof of the uniform boundary condition in the presence of amenability, using geometric Følner arguments on the chain level instead of passing to the dual cochain complex. These geometric methods lead to integral refinements (Theorems 1.2, 1.3, 1.4). The prototypical result reads as follows (which is a special case of the results by Matsumoto and Morita):

Proposition 1.1 (UBC for the rational ℓ^1-norm)**.** *Let M be an aspherical topological space with amenable fundamental group and let $n \in \mathbb{N}$. Then the chain complex $C_*(M;\mathbb{Q})$ satisfies n-UBC, i.e.: There is a constant $K \in \mathbb{R}_{>0}$ such that: If $c \in C_n(M;\mathbb{Q})$ is a null-homologous cycle, then there exists a filling chain $b \in C_{n+1}(M;\mathbb{Q})$ satisfying*

$$\partial b = c \quad and \quad |b|_1 \leq K \cdot |c|_1.$$

Here, we call a topological space *aspherical* if it is path-connected, locally path-connected and admits a contractible universal covering.

Our method of proof is related to the Følner filling argument for the vanishing of integral foliated simplicial volume of aspherical oriented closed connected manifolds with amenable fundamental group [8, Section 6]. More precisely, the proof consists of three steps:

(1) Lifting appropriate chains to chains on the universal covering, taking translations by Følner sets, and estimating the size of these translates (lifting lemma).
(2) Filling these chains more efficiently (filling lemma; in this step, asphericity is essential).
(3) Projecting the chains to the original chain complex on the base space and dividing by the order of the Følner sets (in this step, special properties of the coefficients are needed).

Taking the limit along a Følner sequence then gives the desired estimates.

Our statements are weaker than the original result of Matsumoto and Morita because the method requires asphericity (or at least a highly connected universal covering space). However, our proof is more constructive and yields refined information in integral contexts:

Theorem 1.2 (UBC for the stable integral ℓ^1-norm). *Let M be an aspherical topological space with countable amenable residually finite fundamental group and let $n \in \mathbb{N}$. Then there is a constant $K \in \mathbb{R}_{>0}$ such that: If $c \in C_n(M; \mathbb{Z})$ is a null-homologous cycle, then there is a sequence $(b_k)_{k \in \mathbb{N}}$ of chains and a sequence $(M_k)_{k \in \mathbb{N}}$ of covering spaces of M with the following properties:*

- *For each $k \in \mathbb{N}$, there is a regular finite-sheeted covering $p_k \colon M_k \longrightarrow M$ of M, and the covering degrees d_k satisfy $\lim_{k \to \infty} d_k = \infty$.*
- *For each $k \in \mathbb{N}$ we have $b_k \in C_{n+1}(M_k; \mathbb{Z})$ and*

$$\partial b_k = c_k \quad and \quad |b_k|_1 \leq d_k \cdot K \cdot |c|_1,$$

where $c_k \in C_n(M_k; \mathbb{Z})$ denotes the full p_k-lift of c (see Equation (3) on p. 11).

Dynamical versions of Følner sequences lead to corresponding results for the parametrised ℓ^1-norm:

Theorem 1.3 (parametrised UBC for tori). *Let $d \in \mathbb{N}_{>0}$ and let $M := (S^1)^d$ be the d-torus, let $\Gamma := \pi_1(M) \cong \mathbb{Z}^d$, and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an (essentially) free standard Γ-space. Then*

$$C_*(M; \alpha) = L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}\Gamma}{\otimes} C_*(\widetilde{M}; \mathbb{Z})$$

satisfies the uniform boundary condition in every degree, i.e.: For every $n \in \mathbb{N}$ there is a constant $K \in \mathbb{R}_{>0}$ such that: For every null-homologous cycle $c \in C_n(M; \alpha)$ there exists a chain $b \in C_{n+1}(M; \alpha)$ with

$$\partial b = c \quad and \quad |b|_1 \leq K \cdot |c|_1.$$

Theorem 1.4 (mixed UBC for the parametrised ℓ^1-norm). *Let M be an aspherical topological space with amenable fundamental group and let $n \in \mathbb{N}$. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an (essentially) free standard Γ-space. Then there is a constant $K \in \mathbb{R}_{>0}$ such that: For every cycle $c \in C_n(M; \mathbb{Z}) \subset C_n(M; \alpha)$ that is*

null-homologous in $C_*(M;\mathbb{Z})$ there exists a parametrised chain $b \in C_{n+1}(M;\alpha)$
with

$$\partial b = c \quad and \quad |b|_1 \leq K \cdot |c|_1.$$

Applications. Integral foliated simplicial volume is the dynamical sibling of simplicial volume, defined as the parametrised ℓ^1-semi-norm of the fundamental class [10, 20] (see Sections 2.3 and 10.1 for definitions). The main interest in integral foliated simplicial volume comes from the fact that this invariant gives an upper bound for L^2-Betti numbers (and whence the Euler characteristic) [20]. It is therefore natural to investigate for which (aspherical) manifolds vanishing of integral foliated simplicial volume is equivalent to vanishing of ordinary simplicial volume.

Integral foliated simplicial volume of oriented closed connected aspherical manifolds with amenable fundamental group is trivial [8] and oriented closed connected Seifert 3-manifolds with infinite fundamental group have trivial integral foliated simplicial volume [13]. Triviality of integral foliated simplicial volume is preserved under taking cartesian products [20], finite coverings [13], and (in the aspherical case) under ergodic bounded measure equivalence [13]. Integral foliated simplicial volume of aspherical oriented closed connected surfaces and hyperbolic 3-manifolds coincides with ordinary simplicial volume [13]. However, for higher-dimensional hyperbolic manifolds, integral foliated simplicial volume is uniformly bigger than ordinary simplicial volume [8].

The parametrised versions of the uniform boundary condition serve as first step for glueing results for integral foliated simplicial volume. We will give a simple example of such a glueing result along tori in Section 10. Moreover, the uniform boundary condition for the parametrised ℓ^1-norm on S^1 is a crucial ingredient in the treatment of S^1-actions for integral foliated simplicial volume [7].

Another application of the Følner filling technique is that one can re-prove the vanishing of ℓ^1-homology of amenable groups – without using bounded cohomology (Section 11).

Organisation of this article. Section 2 contains a brief introduction into normed chain complexes and the ℓ^1-norms on singular chain complexes. In Section 3, we recall the terminology for the uniform boundary condition and we survey the results of Matsumoto and Morita. Section 4 introduces the two key lemmas (lifting and filling lemma) for the Følner filling argument.

The prototypical case of the uniform boundary condition for the rational ℓ^1-norm (Proposition 1.1) is proved in Section 5. The same proof with refined Følner sequences gives Theorem 1.2 (Section 6). The dynamical versions Theorem 1.3 and Theorem 1.4 are proved in Section 7 and Section 8, respectively. The uniform boundary condition on the ordinary integral singular chain complex is briefly discussed in Section 9.

Integral foliated simplicial volume of glueings along tori is considered in Section 10 and ℓ^1-homology of amenable groups is treated in Section 11.

2. Normed chain complexes

We recall basic notions in the context of normed abelian groups and normed chain complexes.

2.1. **Normed chain complexes.**

Definition 2.1 ((semi-)norms on abelian groups)**.**
- A *semi-norm* on an abelian group A is a map $|\cdot|\colon A \longrightarrow \mathbb{R}_{\geq 0}$ with the following properties:
  - We have $|0| = 0$.
  - For all $x \in A$ we have $|-x| = |x|$.
  - For all $x, y \in A$ we have $|x + y| \leq |x| + |y|$.
- A *norm* on an abelian group A is a semi-norm $|\cdot|$ on A with the property that $|x| = 0$ holds only for $x = 0$.
- A *(semi-)normed abelian group* is an abelian group together with a (semi-)norm.
- A homomorphism $\varphi\colon A \longrightarrow B$ between normed abelian groups is *bounded* if there is a constant $C \in \mathbb{R}_{\geq 0}$ satisfying for all $a \in A$ the estimate
$$\big|\varphi(a)\big| \leq C \cdot |a|.$$
  The least such constant is the *norm of φ*, denoted by $\|\varphi\|$.

Definition 2.2 (normed chain complex, induced semi-norm on homology)**.** A *normed* chain complex is a chain complex in the category of normed abelian groups (with bounded homomorphisms as morphisms). Let C_* be a normed chain complex and let $n \in \mathbb{N}$. Then the norm $|\cdot|$ on C_n induces a semi-norm $\|\cdot\|$ on $H_n(C_*)$ via
$$\|\alpha\| := \inf\big\{|c| \bigm| c \in C_n,\ \partial c = 0,\ [c] = \alpha \in H_n(C_*)\big\} \in \mathbb{R}_{\geq 0}$$
for all $\alpha \in H_n(C_*)$.

We will also need the corresponding equivariant versions:

Definition 2.3 (twisted normed modules)**.** Let Γ be a group. A *normed Γ-module* is a normed abelian group together with an isometric Γ-action.

2.2. **The ℓ^1-norm on the singular chain complex.** A geometrically interesting example of a normed chain complex is given by the singular chain complex:

Definition 2.4 (the twisted ℓ^1-norm)**.** Let M be path-connected, locally path-connected topological space that admits a universal covering $\widetilde{M}$ (e.g., a connected CW-complex). Let $\Gamma := \pi_1(M)$, and let A be a normed right $\mathbb{Z}\Gamma$-module. For $n \in \mathbb{N}$, we define the *twisted ℓ^1-norm*
$$|\cdot|_1\colon C_n(M; A) \longrightarrow \mathbb{R}_{\geq 0}$$
$$\sum_{j=1}^{m} f_j \otimes \sigma_j \longmapsto \sum_{j=1}^{m} |f_j|$$
on $C_n(M; A) := A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z})$, where $C_n(\widetilde{M}; \mathbb{Z})$ carries the left $\mathbb{Z}\Gamma$-module structure induced by the deck transformation action of Γ on $\widetilde{M}$.

Here, we assume that $\sum_{j=1}^{m} f_j \otimes \sigma_j$ is in *reduced form*, i.e., that the singular simplices $\sigma_1, \ldots, \sigma_m \in \mathrm{map}(\Delta^n, \widetilde{M})$ all belong to different Γ-orbits.

In the situation of the previous definition, $C_*(M; A)$ is a normed chain complex with respect to the twisted ℓ^1-semi-norm. We denote the induced *twisted ℓ^1-semi-norm* on $H_*(M; A)$ by $\| \cdot \|_{1,A}$.

Using the ℓ^1-semi-norm on singular homology, we can define (twisted) simplicial volumes:

Definition 2.5 (simplicial volume). Let M be an oriented closed connected n-manifold with fundamental group Γ and let A be a normed $\mathbb{Z}\Gamma$-module together with a $\mathbb{Z}\Gamma$-homomorphism $i: \mathbb{Z} \longrightarrow A$ (where we consider $\mathbb{Z}$ as $\mathbb{Z}\Gamma$-module with respect to the trivial Γ-action). Then the *A-simplicial volume of M* is defined by

$$\|M\|_A := \big\|[M]_A\big\|_{1,A} \in \mathbb{R}_{\geq 0},$$

where $[M]_A \in H_n(M; A)$ denotes the push-forward of the integral fundamental class $[M]_{\mathbb{Z}} \in H_n(M; \mathbb{Z})$ along i.

For example, the real numbers $\mathbb{R}$ with the standard norm and the canonical inclusion $\mathbb{Z} \longrightarrow \mathbb{R}$ (of $\mathbb{Z}\Gamma$-modules with trivial Γ-action) lead to the classical simplicial volume by Gromov [9].

2.3. **The parametrised ℓ^1-norm.** We will now focus on twisted ℓ^1-norms where the coefficients are induced from actions of the fundamental group on probability spaces. Twisted ℓ^1-norms of this type lead to integral foliated simplicial volume [10, 20], a notion also studied in Section 10.

Definition 2.6 (standard Γ-space). Let Γ be a countable group. A *standard Γ-space* is a standard Borel probability space (X, μ) together with a measurable probability measure preserving left Γ-action on (X, μ).

Every countable group Γ admits an essentially free ergodic standard Γ-space, for instance the Bernoulli shift [20].

Definition 2.7 (parametrised ℓ^1-norm). Let M be a path-connected, locally path-connected topological space that admits a universal covering space $\widetilde{M}$, let $\Gamma := \pi_1(M)$, and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. Then $L^\infty(X, \mu; \mathbb{Z})$ together with the 1-norm

$$L^\infty(X, \mu; \mathbb{Z}) \longrightarrow \mathbb{R}$$

$$f \longmapsto \int_X |f| \, d\mu$$

and the canonical right Γ-action is a normed $\mathbb{Z}\Gamma$-module, which we also denote by $L^\infty(X; \mathbb{Z})$ or $L^\infty(\alpha; \mathbb{Z})$. The associated twisted ℓ^1-norm on

$$C_*(M; \alpha) := L^\infty(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$$

is the *α-parametrised ℓ^1-norm*.

Let M be an oriented closed connected n-manifold with fundamental group Γ and let α be a standard Γ-space. Then the *α-parametrised simplicial volume of M* is defined as

$$\big|M\big|^{\alpha} := \|M\|_{L^\infty(\alpha; \mathbb{Z})} \in \mathbb{R}_{\geq 0}.$$

Taking the infimum over the set of all isomorphism classes of standard Γ-spaces leads to the *integral foliated simplicial volume* $|M|$ of M. Integral foliated simplicial volume fits into the following chain of inequalities [20, 13]:

$$\|M\| \leq |M| \leq \|M\|_{\mathbb{Z}}^{\infty}.$$

Here,

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf_{(p:\, N \to M) \in F(M)} \frac{\|N\|_{\mathbb{Z}}}{|\deg p|}$$

denotes the *stable integral simplicial volume of M* and $F(M)$ is the set of all isomorphism classes of finite connected coverings of M.

3. THE UNIFORM BOUNDARY CONDITION

The uniform boundary condition in a normed chain complex asks for a uniform linear bound on fillings of null-homologous cycles.

Definition 3.1 (uniform boundary condition; UBC [16])**.** Let C_* be a normed chain complex and let $n \in \mathbb{N}$. We say that C_* *satisfies the uniform boundary condition in degree n*, or in short *n-UBC*, if there exists a constant $K \in \mathbb{R}_{>0}$ such that: For every null-homologous cycle $c \in C_n$ there exists a filling chain $b \in C_{n+1}$ with

$$\partial b = c \quad \text{and} \quad |b| \leq K \cdot |c|.$$

We briefly recall the results of Matsumoto and Morita [16]:

Theorem 3.2 (UBC [16, Theorem 2.8])**.** *Let M be a topological space and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *The normed chain complex $C_*(M;\mathbb{R})$ satisfies n-UBC.*
(2) *The homomorphism $H_b^{n+1}(M;\mathbb{R}) \longrightarrow H^{n+1}(M;\mathbb{R})$ induced by the inclusion $C_b^{n+1}(M;\mathbb{R}) \longrightarrow C^{n+1}(M;\mathbb{R})$ is injective.*

Here, $C_b^*(M;\mathbb{R}) \subset C^*(M;\mathbb{R})$ denotes the subcomplex of the singular cochain complex of M with real coefficients consisting of bounded linear maps and $H_b^*(M;\mathbb{R})$ denotes the cohomology of $C_b^*(M;\mathbb{R})$, the so-called *bounded cohomology of M (with coefficients in $\mathbb{R}$)*.

Since bounded cohomology of topological spaces with amenable fundamental group is trivial [1, 9], Theorem 3.2 implies the following result of which Proposition 1.1 is a special case:

Corollary 3.3 (UBC and amenability)**.** *Let M be a topological space with amenable fundamental group and let $n \in \mathbb{N}$. Then the chain complex $C_*(M;\mathbb{R})$ satisfies n-UBC.*

4. THE FILLING LEMMA AND THE LIFTING LEMMA

The Følner filling strategy starts with a lifting step and a filling step. These steps mainly rely on the following Lemmas 4.2 and 4.1.

4.1. **The filling lemma.** In contractible spaces, boundaries can be filled efficiently.

Lemma 4.1 (filling lemma). *Let M be a contractible topological space, let A be a normed $\mathbb{Z}$-module, and let $n \in \mathbb{N}$. For every $c \in C_n(M; A)$ there exists a chain $c' \in C_n(M; A)$ with*

$$\partial c' = \partial c \quad and \quad |c'|_1 \leq |\partial c|_1.$$

Statements of this form can be proved by adapting the original proof of Frigerio, Löh, Pagliantini, and Sauer [8, Lemma 6.3] from the case of $\mathbb{Z}$-coefficients to general normed coefficients. We will slightly modify the filling construction in order to improve the filling bound to 1.

Proof. We will construct a cone-type chain contraction

$$h \colon C_*(M; A) \longrightarrow C_{*+1}(M; A)$$

of norm ≤ 1, inductively on the dimension of the simplices:

Let $x_0 \in M$. For each 0-simplex σ in M we choose a path $h(\sigma) \colon \Delta^1 \longrightarrow M$ from x_0 to $\sigma(1)$. We proceed inductively as follows: Let $k \in \mathbb{N}$ and let σ be a k-simplex in M. Consider the map $\partial\Delta^{k+1} \longrightarrow M$ that is given by σ on the 0-th face and $h(\sigma \circ i^k_{l-1})$ on the l-th face for all $l \in \{1, \ldots, k+1\}$. Here, i^k_{l-1} denotes the inclusion of the $(l-1)$-face in the standard k-simplex. Since M is contractible, this extends to a map $h(\sigma) \colon \Delta^{k+1} \longrightarrow M$. Finally, for all $k \in \mathbb{N}$ and all $c = \sum_{j=1}^m a_j \otimes \sigma_j \in C_k(M; A)$ we define

$$h(c) := \sum_{j=1}^m a_j \otimes h(\sigma_j) \in C_{k+1}(M; A)$$

By construction, $\|h\| \leq 1$ and for all $k \in \mathbb{N}$ and all k-simplices σ we have

$$\partial h(\sigma) = \sum_{j=0}^{k+1} (-1)^j \cdot h(\sigma) \circ i^{k+1}_j = \sigma + \sum_{j=1}^{k+1} (-1)^j \cdot h(\sigma \circ i^k_{j-1}) = \sigma + h(-\partial\sigma),$$

and therefore, we have $\partial \circ h + h \circ \partial = \mathrm{id}$.

Let $c \in C_n(M; A)$. We set $c' := h(\partial c) \in C_n(M; A)$. Then we have

$$|c'|_1 \leq \|h\| \cdot |\partial c|_1 \leq |\partial c|_1$$

and

$$\partial c' = \partial h(\partial c) = \partial c - h(\partial\partial c) = \partial c + 0. \qquad \square$$

4.2. **The lifting lemma.** Lifting cycles to the universal covering in general will not lead to cycles; however, the size of the boundary of finite translates of such lifts will basically only grow like the boundary of the finite set of group elements.

Lemma 4.2 (lifting lemma). *Let M be a path-connected and locally path-connected space that admits a universal covering, let $\Gamma := \pi_1(M)$, let A be a normed $\mathbb{Z}\Gamma$-module, and let $\widetilde{a} \in A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$ be a lift of $0 \in A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z})$. Then there is a constant $C \in \mathbb{R}_{>0}$ and a finite set $S \subset \Gamma$ such that the following holds: For every finite subset $F \subset \Gamma$ we have*

$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|,$$

where $|F \cdot \widetilde{a}|_1$ is the ℓ^1-norm on $A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$ induced by the norm on A.

Notation 4.3. Here, $\partial_S F = \{\gamma \in F \mid \exists_{s \in S}\ \gamma \cdot s \notin F\}$ denotes the S-boundary of F in Γ and we set $F \cdot \widetilde{a} := \sum_{\gamma \in F} \gamma \cdot \widetilde{a}$, where the Γ-action on $A \underset{\mathbb{Z}}{\otimes} C_n(\widetilde{M}; \mathbb{Z})$ is given by

$$\gamma \cdot (a \otimes \sigma) := a \cdot \gamma^{-1} \otimes \gamma \cdot \sigma$$

for all $\gamma \in \Gamma$, $a \in A$ and $\sigma \in \mathrm{map}(\Delta^n, \widetilde{M})$. The canonical projection

$$A \underset{\mathbb{Z}}{\otimes} C_n(\widetilde{M}; \mathbb{Z}) \longrightarrow A \underset{\mathbb{Z}\Gamma}{\otimes} C_n(\widetilde{M}; \mathbb{Z}).$$

justifies the term "lift" in the lifting lemma.

Proof. In this case, the generalisation from the case of $\mathbb{Z}$-coefficients [8, Lemma 6.2] is slightly more involved (but the proof in spirit is the same):

Let $\widetilde{a} = \sum_{j=1}^{k} f_{\tau_j} \otimes \tau_j$ be in reduced form in $A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$, i.e., $\tau_i \neq \tau_j$ if $i \neq j$. Let

$$K := \big\{ C_n(\pi; \mathbb{Z})(\tau_j) \mid j \in \{1, \dots, k\} \big\},$$

where $\pi : \widetilde{M} \longrightarrow M$ is the universal covering. For every $\tau \in K$ we choose a lift $\widetilde{\tau} \in C_n(\widetilde{M}; \mathbb{Z})$ that occurs in $\widetilde{a}$ and we set

$$S(\tau) := \{ \gamma \in \Gamma \mid \exists_{j \in \{1, \dots, k\}}\ \tau_j = \gamma \cdot \widetilde{\tau} \}.$$

Let $S := \bigcup_{\tau \in K} (S(\tau) \cup S(\tau)^{-1}) \subset \Gamma$. Using S, we can write

$$\widetilde{a} = \sum_{\tau \in K} \sum_{s \in S(\tau)} f_{s \cdot \widetilde{\tau}} \otimes s \cdot \widetilde{\tau} = \sum_{\tau \in K} \sum_{s \in S} f_{s \cdot \widetilde{\tau}} \otimes s \cdot \widetilde{\tau},$$

where we set $f_{s \cdot \widetilde{\tau}} := 0$ for all $s \in S \setminus S(\tau)$. Because $\widetilde{a}$ is a lift of zero, for every $\tau \in K$, we have

(1) $$\sum_{s \in S} f_{s \cdot \widetilde{\tau}} \cdot s = 0.$$

Let $F \subset \Gamma$ be a finite subset. The goal is to estimate the ℓ^1-norm of

(2) $$F \cdot \widetilde{a} = \sum_{\gamma \in F} \gamma \cdot \widetilde{a} = \sum_{\gamma \in F} \sum_{\tau \in K} \sum_{s \in S} f_{s \cdot \widetilde{\tau}} \cdot \gamma^{-1} \otimes \gamma \cdot s \cdot \widetilde{\tau}.$$

To this end, we split $F \cdot \widetilde{a}$ in the following way as a sum $X + Y$: We set

$$X := \sum_{\gamma \in F \setminus \partial_S F} \sum_{\tau \in K} \sum_{s \in S} f_{s \cdot \widetilde{\tau}} \cdot s \cdot \gamma^{-1} \otimes \gamma \cdot \widetilde{\tau} \in A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z}).$$

By definition of the boundary $\partial_S F$, if $s \in S$ and $\gamma \in F \setminus \partial_S F$, then $\gamma \cdot s^{-1} \in F$ (because S is symmetric). Each summand in X occurs as a summand in $F \cdot \widetilde{a}$ (see Equation (2) with γ set to $\gamma \cdot s^{-1}$) and the pairs in

$$\big\{ (s \cdot \gamma^{-1}, \gamma \cdot \widetilde{\tau}) \mid s \in S,\ \gamma \in F \setminus \partial_S F,\ \tau \in K \big\}$$

are pairwise distinct. Thus, we can write $F \cdot \widetilde{a} = X + Y$, where Y is the sum of summands that occur in $F \cdot \widetilde{a}$ but not in X. We will now estimate X and Y separately:

We start with $|Y|_1$: Let $s \in S$ and $\tau \in K$. Then $f_{s \cdot \widetilde{\tau}}$ occurs exactly $|F|$ times in $F \cdot \widetilde{a}$ and exactly $|F \setminus \partial_S F|$ times in X. Therefore, it follows that each $f_{s \cdot \widetilde{\tau}}$ occurs exactly $|\partial_S F|$ times in Y, which implies

$$|Y|_1 \leq |\partial_S F| \cdot |K| \cdot |S| \cdot m,$$

where $m := \max\{ |f_{s \cdot \widetilde{\tau}}|_A \mid s \in S, \tau \in K \}$.

Finally, the sum X is zero, because for each $\tau \in K$ and $\gamma \in F \setminus \partial_S F$ we have

$$\sum_{s \in S} f_{s \cdot \widetilde{\tau}} \cdot s \cdot \gamma^{-1} \otimes \gamma \cdot \widetilde{\tau} = \left(\sum_{s \in S} f_{s \cdot \widetilde{\tau}} \cdot s \right) \cdot \gamma^{-1} \otimes \gamma \cdot \widetilde{\tau} \stackrel{(1)}{=} 0.$$

We conclude $|F \cdot \widetilde{a}|_1 = |Y|_1 \leq C \cdot |\partial_S F|$, where $C := |K| \cdot |S| \cdot m$ depends only on $\widetilde{a}$, but not on F. □

5. RATIONAL UBC FOR AMENABLE GROUPS

In this section, we will prove Proposition 1.1.

5.1. **Amenable groups and Følner sequences.** We briefly recall the definition of amenable groups via left-invariant means and the characterisation of amenable groups via the Følner criterion. For the details we refer to the literature [18].

Definition 5.1 (amenable). A group Γ is called *amenable* if it admits a left-invariant mean, i.e., an $\mathbb{R}$-linear map $m \colon \ell^\infty(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ that is normalised, positive and left-invariant with respect to the Γ-action on $\ell^\infty(\Gamma, \mathbb{R})$ induced by the right-translation of Γ on itself.

Theorem 5.2 (Følner sequences). *Let Γ be a finitely generated group with finite generating set S. Then the following are equivalent:*

(1) *The group Γ is amenable.*
(2) *The group Γ admits a* Følner *sequence (with respect to S), i.e., a sequence $(F_k)_{k \in \mathbb{N}}$ of non-empty finite subsets of Γ satisfying*

$$\lim_{k \to \infty} \frac{|\partial_S F_k|}{|F_k|} = 0.$$

Recall that finite groups and abelian groups are amenable and that the class of amenable groups is closed under taking subgroups, quotient groups and extensions. Groups that contain a free group of rank 2 as subgroup are *not* amenable.

5.2. **Proof of UBC via Følner sets.** Now we are prepared to prove Proposition 1.1 via the strategy outlined in the Introduction. The basic steps are also illustrated in Figure 1.

Proof of Proposition 1.1. Let $c \in C_n(M; \mathbb{Q})$ be a null-homologous cycle, i.e., there exists $b \in C_{n+1}(M; \mathbb{Q})$ with $\partial b = c$. We will use b to find a more efficient filling of c. Let $\pi \colon \widetilde{M} \longrightarrow M$ be the universal covering of M and let $\Gamma := \pi_1(M)$ be the fundamental group of M.

Lifting step: Let $\widetilde{c} \in C_n(\widetilde{M}; \mathbb{Q})$ be a π-lift of c with $|\widetilde{c}|_1 \leq |c|_1$ (e.g., by lifting c simplex by simplex) and let $\widetilde{b} \in C_{n+1}(\widetilde{M}; \mathbb{Q})$ be a π-lift of b. We consider

$$\widetilde{a} := \partial \widetilde{b} - \widetilde{c} \in C_n(\widetilde{M}; \mathbb{Q}).$$

Then $\widetilde{a}$ is a π-lift of $0 \in C_n(M; \mathbb{Q})$. By the lifting lemma (Lemma 4.2) there exist $C \in \mathbb{R}_{>0}$ and a finite subset $S \subset \Gamma$ such that the following holds: for all finite sets $F \subset \Gamma$ we have

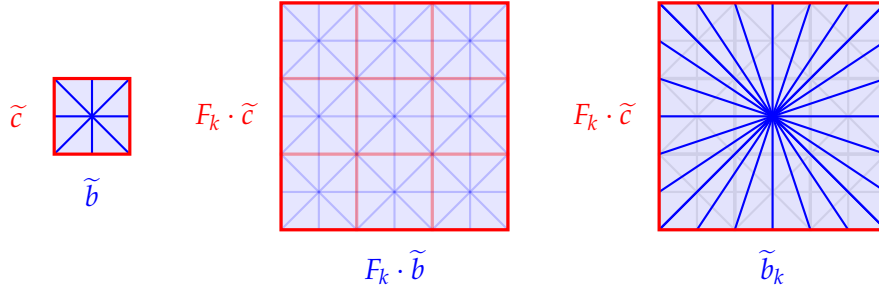$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|.$$

FIGURE 1. Lifting, translating, and filling chains, schematically

The group $\Lambda := \langle S \rangle_\Gamma \subset \Gamma$ is amenable and finitely generated. Let $(F_k)_{k \in \mathbb{N}}$ be a Følner sequence of Λ with respect to S; in particular,

$$\lim_{k \to \infty} \frac{|\partial_S F_k|}{|F_k|} = 0.$$

Then, for all $k \in \mathbb{N}$ we have

$$|\partial(F_k \cdot \widetilde{b})|_1 = |F_k \cdot \widetilde{a} + F_k \cdot \widetilde{c}|_1 \leq C \cdot |\partial_S F_k| + |F_k| \cdot |\widetilde{c}|_1.$$

Filling step: By the filling lemma (Lemma 4.1), for every $k \in \mathbb{N}$ there exists a chain $\widetilde{b}_k \in C_{n+1}(\widetilde{M}; \mathbb{Q})$ with $\partial \widetilde{b}_k = \partial(F_k \cdot \widetilde{b})$ and

$$|\widetilde{b}_k|_1 \leq |\partial(F_k \cdot \widetilde{b})|_1 \leq C \cdot |\partial_S F_k| + |F_k| \cdot |\widetilde{c}|_1.$$

Quotient step: For all $k \in \mathbb{N}$ we define

$$b_k := \frac{1}{|F_k|} \cdot C_{n+1}(\pi; \mathbb{Q})(\widetilde{b}_k) \in C_{n+1}(M; \mathbb{Q}).$$

By construction, we have

$$\partial b_k = \frac{1}{|F_k|} \cdot C_{n+1}(\pi; \mathbb{Q})\big(\partial(F_k \cdot \widetilde{b})\big) = \frac{1}{|F_k|} \cdot \partial(|F_k| \cdot b) = \partial b = c$$

and

$$|b_k|_1 \leq \frac{1}{|F_k|} \cdot |\widetilde{b}_k|_1 \leq C \cdot \frac{|\partial_S F_k|}{|F_k|} + |\widetilde{c}|_1 \leq C \cdot \frac{|\partial_S F_k|}{|F_k|} + |c|_1.$$

Because $(F_k)_{k \in \mathbb{N}}$ is a Følner sequence, for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $k \in \mathbb{N}$ such that $\partial b_k = c$ and

$$|b_k|_1 \leq (1 + \varepsilon) \cdot |c|_1,$$

which is slightly stronger than the statement of Proposition 1.1. $\qquad \square$

6. STABLE INTEGRAL UBC FOR AMENABLE GROUPS

Taking improved Følner sequences allows to prove the uniform boundary condition for the stable integral ℓ^1-norm (Theorem 1.2).

6.1. **Improved Følner sequences.** We use the following result of Deninger and Schmidt [5, Proposition 5.5], which is a reformulation of the Rokhlin Lemma for amenable residually finite groups of Weiss [21, Theorem 1].

Proposition 6.1 (improved Følner sequences). *Let Γ be a countable amenable residually finite group and let $S \subset \Gamma$ be a finite subset. Then there exists a sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of decreasing finite index normal subgroups of Γ with $\bigcap_{k \in \mathbb{N}} \Gamma_k = \{e\}$ and a Følner sequence $(F_k)_{k \in \mathbb{N}}$ of Γ with respect to S such that: For all $k \in \mathbb{N}$ the Følner set F_k is a set of representatives for Γ / Γ_k.*

6.2. **Proof of Theorem 1.2.** Let $c \in C_n(M; \mathbb{Z})$ be a null-homologous cycle, i.e., there exists $b \in C_{n+1}(M; \mathbb{Z})$ with $\partial b = c$. Let $\pi \colon \widetilde{M} \longrightarrow M$ be the universal covering of M and $\Gamma := \pi_1(M)$ the fundamental group of M.

The lifting step is analogous to the lifting step in the proof of Proposition 1.1. By Proposition 6.1 there exists a sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of decreasing finite index normal subgroups of Γ with $\bigcap_{k \in \mathbb{N}} \Gamma_k = \{e\}$ and a Følner sequence $(F_k)_{k \in \mathbb{N}}$ of Γ with respect to S (from the lifting step) such that: For all $k \in \mathbb{N}$ the Følner set F_k is a set of representatives for Γ / Γ_k. We construct $\widetilde{b}_k \in C_{n+1}(\widetilde{M}; \mathbb{Z})$ for all $k \in \mathbb{N}$ as in the filling step in the proof of Proposition 1.1.

Quotient step: Let $k \in \mathbb{N}$. We write $p_k \colon M_k \longrightarrow M$ for the covering of M associated to $\Gamma_k < \Gamma$ (of degree $|F_k|$) and $\pi_k \colon \widetilde{M} \longrightarrow M_k$ for the universal covering of M_k. Then, we define

$$b_k := C_{n+1}(\pi_k; \mathbb{Z})(\widetilde{b}_k) \in C_{n+1}(M_k; \mathbb{Z}).$$

In $C_n(M_k; \mathbb{Z})$ we have

$$\partial b_k = C_{n+1}(\pi_k; \mathbb{Z})\big(\partial(F_k \cdot \widetilde{b})\big) = C_{n+1}(\pi_k; \mathbb{Z})(F_k \cdot \partial\widetilde{b}) =: c_k$$

and by construction c_k is the *full p_k-lift of c*, i.e., if $c = \sum_{j=1}^m a_j \cdot \sigma_j$ then

$$(3) \qquad c_k = \sum_{j=1}^m a_j \cdot \left(\sum_{\substack{\tau \in \mathrm{map}(\Delta^n, M_k) \\ p_k \circ \tau = \sigma_j}} \tau \right).$$

We estimate (where $C \in \mathbb{R}_{>0}$ is the constant found in the lifting step)

$$|b_k|_1 \le |\widetilde{b}_k|_1 \le C \cdot |\partial_S F_k| + |F_k| \cdot |\widetilde{c}|_1 \le |F_k| \cdot \left(C \cdot \frac{|\partial_S F_k|}{|F_k|} + |c|_1 \right).$$

Because $(F_k)_{k \in \mathbb{N}}$ is a Følner sequence, for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $k \in \mathbb{N}$ such that $\partial b_k = c_k$ and

$$|b_k|_1 \le |F_k| \cdot (1 + \varepsilon) \cdot |c|_1,$$

which is slightly stronger than the statement of Theorem 1.2. $\qquad \square$

7. PARAMETRISED UBC FOR TORI

In order to prove the uniform boundary condition for the parametrised ℓ^1-norm, we will perform the division by the order of the Følner sets on the level of the measure space. This is done through suitable versions of the Rokhlin lemma.

7.1. **Rokhlin lemma for free abelian groups.** In the abelian case, we will use the following version of the Roklin lemma [4, Theorem 3.1].

Theorem 7.1 (Rokhlin lemma for free abelian groups). *Let $d \in \mathbb{N}_{>0}$ and let (X, μ) be an (essentially) free standard $\mathbb{Z}^d$-space. Then for every $k \in \mathbb{N}$ and every $\varepsilon \in \mathbb{R}_{>0}$ there exists a measurable subset $A \subset X$ such that the sets*

$$(\gamma \cdot A)_{\gamma \in F_k}$$

with $F_k := \{0, \ldots, k\}^d \subset \mathbb{Z}^d$ are pairwise disjoint and

$$\mu(X \setminus F_k \cdot A) < \varepsilon.$$

7.2. **Proof of Theorem 1.3.** Let $c \in C_n(M; \alpha)$ be a null-homologous cycle, i.e., there exists $b \in C_{n+1}(M; \alpha)$ with $\partial b = c$. We will use b to find a more efficient filling of c. Let

$$p \colon L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_*(\widetilde{M}; \mathbb{Z}) \longrightarrow L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}\Gamma}{\otimes} C_*(\widetilde{M}; \mathbb{Z})$$

be the canonical projection.

Lifting step: Let $\widetilde{c}$ be a p-lift of c with $|\widetilde{c}|_1 \leq |c|_1$ (e.g., by lifting c simplex by simplex) and let $\widetilde{b}$ be a p-lift of b. We now consider

$$\widetilde{a} := \partial\widetilde{b} - \widetilde{c} \in L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_n(\widetilde{M}; \mathbb{Z}).$$

Then $\widetilde{a}$ is a p-lift of $0 \in C_n(M; \alpha)$. By the lifting lemma (Lemma 4.2), there exist $C \in \mathbb{R}_{>0}$ and a finite subset $S \subset \Gamma$ such that the following holds: for all finite sets $F \subset \Gamma$ we have

$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|.$$

For $k \in \mathbb{N}$ we define

$$F_k := \{0, \ldots, k\}^d \subset \Gamma$$

via an isomorphism $\Gamma \cong \mathbb{Z}^d$. Then $(F_k^{-1})_{k \in \mathbb{N}}$ is a Følner sequence for Γ with respect to S in the sense that

$$\lim_{k \to \infty} \frac{|\partial_S(F_k^{-1})|}{|F_k|} = 0.$$

Then, for all $k \in \mathbb{N}$ we have

$$|\partial(F_k^{-1} \cdot \widetilde{b})|_1 = |F_k^{-1} \cdot \widetilde{a} + F_k^{-1} \cdot \widetilde{c}|_1 \leq C \cdot |\partial_S F_k^{-1}| + |F_k| \cdot |\widetilde{c}|_1.$$

Let $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$. By the Rokhlin lemma for free abelian groups (Theorem 7.1), there exists a measurable subset $A_k \subset X$ such that the sets $(\gamma \cdot A_k)_{\gamma \in F_k}$ are pairwise disjoint and the complement

$$B_k := X \setminus F_k \cdot A_k$$

has measure less than ε.

Quotient step: Since $L^\infty(X; \mathbb{Z})$ is a $L^\infty(X; \mathbb{Z})$-$\mathbb{Z}$-bimodule, it follows that

$$L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_n(\widetilde{M}; \mathbb{Z})$$

is a left-$L^\infty(X; \mathbb{Z})$-module. Therefore, we can define

$$\widetilde{b}_{k,\gamma} := \chi_{\gamma \cdot A_k} \cdot (F_k^{-1} \cdot \widetilde{b}) \in L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_n(\widetilde{M}; \mathbb{Z})$$

for all $\gamma \in F_k$. Then we have

$$F_k^{-1} \cdot \widetilde{b} = \chi_X \cdot (F_k^{-1} \cdot \widetilde{b}) = \sum_{\gamma \in F_k} \chi_{\gamma \cdot A_k} \cdot (F_k^{-1} \cdot \widetilde{b}) + \chi_{B_k} \cdot (F_k^{-1} \cdot \widetilde{b})$$

$$= \sum_{\gamma \in F_k} \widetilde{b}_{k,\gamma} + \chi_{B_k} \cdot (F_k^{-1} \cdot \widetilde{b}).$$

Because the chains $(\partial \widetilde{b}_{k,\gamma})_{\gamma \in F_k}$ have pairwise disjoint support, we obtain

$$\left| \sum_{\gamma \in F_k} \partial \widetilde{b}_{k,\gamma} \right|_1 = \sum_{\gamma \in F_k} |\partial \widetilde{b}_{k,\gamma}|_1$$

and by the box principle it follows that there exists $\gamma_0 \in F_k$ with

$$|\partial \widetilde{b}_{k,\gamma_0}|_1 \leq \frac{1}{|F_k|} \cdot \left(|F_k^{-1} \cdot \partial \widetilde{b}|_1 + |\chi_{B_k} \cdot (F_k^{-1} \cdot \partial \widetilde{b})|_1 \right).$$

We write $\widetilde{b} = \sum_{j=1}^m a_j \otimes \sigma_j \in L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_{n+1}(\widetilde{M}; \mathbb{Z})$ in reduced form and set

$$|\widetilde{b}|_{1,\infty} := \sum_{j=1}^m |a_j|_\infty.$$

Then

$$|\partial \widetilde{b}_{k,\gamma_0}|_1 \leq \frac{1}{|F_k|} \cdot \left(|F_k^{-1} \cdot \partial \widetilde{b}|_1 + \mu(B_k) \cdot |F_k| \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty} \right)$$

$$\leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty}.$$

Filling step: By the filling lemma (Lemma 4.1), there exists a parametrised chain $\widetilde{b}'_{k,\gamma_0} \in L^\infty(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} C_{n+1}(\widetilde{M}; \mathbb{Z})$ with $\partial \widetilde{b}'_{k,\gamma_0} = \partial \widetilde{b}_{k,\gamma_0}$ and

$$|\widetilde{b}'_{k,\gamma_0}|_1 \leq |\partial \widetilde{b}_{k,\gamma_0}|_1 \leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty}.$$

Let

$$b_{k,\gamma_0} := p(\widetilde{b}_{k,\gamma_0}) \in C_{n+1}(M; \alpha) \quad \text{and} \quad b'_{k,\gamma_0} := p(\widetilde{b}'_{k,\gamma_0}) \in C_{n+1}(M; \alpha).$$

Because Γ is abelian and $\chi_{\gamma \cdot A_k} = \chi_{A_k} \cdot \gamma^{-1}$ holds for all $\gamma \in \Gamma$, we obtain in $C_n(M; \alpha)$ that

$$\partial b'_{k,\gamma_0} = \partial b_{k,\gamma_0}$$

$$= \sum_{j=1}^m \sum_{\gamma \in F_k} \chi_{\gamma_0 \cdot A_k} \cdot (a_j \cdot \gamma) \otimes \gamma^{-1} \cdot \partial \sigma_j$$

$$= \sum_{j=1}^m \sum_{\gamma \in F_k} \chi_{A_k} \cdot (a_j \cdot \gamma \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \gamma^{-1} \cdot \partial \sigma_j$$

$$= \sum_{j=1}^m \sum_{\gamma \in F_k} \chi_{A_k} \cdot (a_j \cdot \gamma_0 \cdot \gamma) \otimes \gamma^{-1} \cdot \gamma_0^{-1} \cdot \partial \sigma_j$$

$$= \sum_{j=1}^m \sum_{\gamma \in F_k} \chi_{\gamma \cdot A_k} \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \partial \sigma_j,$$

which almost looks like c. We define the correction term

$$r_k := \sum_{j=1}^{m} \chi_{\gamma_0 \cdot B_k} \cdot a_j \otimes \sigma_j = \sum_{j=1}^{m} \chi_{B_k} \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \sigma_j$$

and observe that the following holds in $C_n(M; \alpha)$:

$$\partial(b'_{k,\gamma_0} + r_k) = \sum_{j=1}^{m} \chi_X \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \partial \sigma_j = \sum_{j=1}^{m} a_j \otimes \partial \sigma_j = \partial b = c.$$

Finally, we have

$$\begin{aligned}
|b'_{k,\gamma_0} + r_k|_1 &\leq |b'_{k,\gamma_0}|_1 + |r_k|_1 \\
&\leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty} + \varepsilon \cdot |\widetilde{b}|_{1,\infty} \\
&\leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+3) \cdot |\widetilde{b}|_{1,\infty}.
\end{aligned}$$

Because $(F_k^{-1})_{k \in \mathbb{N}}$ is a Følner sequence for Γ with respect to S, for $k \to \infty$ and $\varepsilon \to 0$ the chains $b'_{k,\gamma_0} + r_k$ are efficient fillings of c. $\qquad\square$

8. MIXED UBC FOR AMENABLE GROUPS

8.1. Ornstein-Weiss.
We need the following modification [19, Theorem 5.2] of the generalized Rokhlin lemma of Ornstein-Weiss [17, Theorem 5]:

Theorem 8.1 (Ornstein-Weiss). *Let Γ be a countable amenable group, let (X, μ) be an (essentially) free standard Γ-space. Then for every finite subset $S \subset \Gamma$ and every $\varepsilon \in \mathbb{R}_{>0}$ there exists an $N \in \mathbb{N}$, finite subsets $F_1, \ldots, F_N \subset \Gamma$, and Borel subsets $A_1, \ldots, A_N \subset X$ such that the following holds:*

- *For every $k \in \{1, \ldots, N\}$ we have*

$$\frac{|\partial_S(F_k^{-1})|}{|F_k|} = \frac{|\{\gamma \in F_k \mid \exists_{s \in S^{-1}} s \cdot \gamma \notin F_k\}|}{|F_k|} < \varepsilon.$$

- *For every $k \in \{1, \ldots, N\}$ the sets $\gamma \cdot A_k$ with $\gamma \in F_k$ are pairwise disjoint.*
- *The sets $F_k \cdot A_k$ with $k \in \{1, \ldots, N\}$ are pairwise disjoint.*
- *The complement $B := X \setminus \bigcup_{k=1}^{N} F_k \cdot A_k$ has measure less than ε.*

8.2. Proof of Theorem 1.4.
Let $c \in C_n(M; \mathbb{Z})$ be a null-homologous cycle, i.e., there exists $b \in C_{n+1}(M; \mathbb{Z})$ with $\partial b = c$. We will use b to find a more efficient filling of c. Let $\pi \colon \widetilde{M} \longrightarrow M$ be the universal covering of M.

Lifting step: Let $\widetilde{c}$ be a π-lift of c with $|\widetilde{c}|_1 \leq |c|_1$ (e.g., by lifting c simplex by simplex) and let $\widetilde{b}$ be a π-lift of b. We now consider

$$\widetilde{a} := \partial \widetilde{b} - \widetilde{c} \in C_n(\widetilde{M}; \mathbb{Z}).$$

Then $\widetilde{a}$ is a π-lift of $0 \in C_n(M; \mathbb{Z})$. By the lifting lemma (Lemma 4.2) there exist $C \in \mathbb{R}_{>0}$ and a finite subset $S \subset \Gamma$ such that the following holds: For all finite sets $F \subset \Gamma$ we have

$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|.$$

The group $\Lambda := \langle S \rangle_\Gamma \subset \Gamma$ is amenable and finitely generated.

Let $\varepsilon \in \mathbb{R}_{>0}$. We apply Theorem 8.1 to the (essentially) free standard Λ-space $\mathrm{res}_\Lambda^\Gamma \alpha = \Lambda \curvearrowright (X, \mu)$: Thus, there exists an $N \in \mathbb{N}$, finite subsets $F_1, \dots, F_N \subset \Lambda$ and Borel subsets $A_1, \dots, A_N, B \subset X$ with the properties listed in Theorem 8.1.

Filling step: By the filling lemma (Lemma 4.1), for all $k \in \{1, \dots, N\}$ there exists $\widetilde{b}_k \in C_{n+1}(\widetilde{M}; \mathbb{Z})$ with $\partial \widetilde{b}_k = \partial(F_k^{-1} \cdot \widetilde{b}) = F_k^{-1} \cdot \partial \widetilde{b}$ and

$$|\widetilde{b}_k|_1 \le |F_k^{-1} \cdot \widetilde{c} + F_k^{-1} \cdot \widetilde{a}|_1 \le |F_k| \cdot |c|_1 + C \cdot |\partial_S(F_k^{-1})| \le |F_k| \cdot |c|_1 + C \cdot \varepsilon \cdot |F_k|.$$

Quotient step: We define

$$b_\varepsilon := \sum_{k=1}^N \chi_{A_k} \otimes \widetilde{b}_k + \chi_B \otimes \widetilde{b} \in C_{n+1}(M; \alpha).$$

Then, in $C_n(M; \alpha)$ the following computation holds

$$\begin{aligned}
\partial b_\varepsilon &= \sum_{k=1}^N \chi_{A_k} \otimes \partial \widetilde{b}_k + \chi_B \otimes \partial \widetilde{b} \\
&= \sum_{k=1}^N \chi_{A_k} \otimes F_k^{-1} \cdot \partial \widetilde{b} + \chi_B \otimes \partial \widetilde{b} \\
&= \sum_{k=1}^N \sum_{\gamma \in F_k} \chi_{\gamma \cdot A_k} \otimes \partial \widetilde{b} + \chi_B \otimes \partial \widetilde{b} \\
&= 1 \otimes \partial \widetilde{b}
\end{aligned}$$

because $X = B \cup \left(\bigcup_{k=1}^N \bigcup_{\gamma \in F_k} \gamma \cdot A_k \right)$ is a disjoint union. Since $\partial \widetilde{b}$ is a π-lift of c, we obtain $\partial \widetilde{b}_\varepsilon = c$, if we view c as a chain in $C_n(M; \mathbb{Z}) \subset C_n(M; \alpha)$ via the inclusion $\mathbb{Z} \hookrightarrow L^\infty(\alpha; \mathbb{Z})$ as constant functions. Finally, we have

$$\begin{aligned}
|b_\varepsilon|_1 &\le \sum_{k=1}^N \mu(A_k) \cdot |\widetilde{b}_k|_1 + \mu(B) \cdot |\widetilde{b}|_1 \\
&\le \sum_{k=1}^N \mu(A_k) \cdot \left(|F_k| \cdot |c|_1 + C \cdot \varepsilon \cdot |F_k| \right) + \mu(B) \cdot |\widetilde{b}|_1 \\
&= \left(|c|_1 + C \cdot \varepsilon \right) \cdot \sum_{k=1}^N \mu(A_k) \cdot |F_k| + \mu(B) \cdot |\widetilde{b}|_1 \\
&= \left(|c|_1 + C \cdot \varepsilon \right) \cdot \mu\left(\bigcup_{k=1}^N F_k \cdot A_k \right) + \mu(B) \cdot |\widetilde{b}|_1 \\
&\le |c|_1 + C \cdot \varepsilon + \varepsilon \cdot |\widetilde{b}|_1.
\end{aligned}$$

Therefore, for $\varepsilon \to 0$ the chains $b_\varepsilon \in C_{n+1}(M; \alpha)$ are efficient fillings of the chain $c \in C_n(M; \mathbb{Z}) \subset C_n(M; \alpha)$. $\square$

9. Integral UBC

We will now briefly discuss the uniform boundary condition for the integral singular chain complex.

9.1. **The integral uniform boundary condition for the circle.** We start with a simple example, namely the circle (and degree 1).

Proposition 9.1 (integral 1-UBC for the circle). *The chain complex $C_*(S^1; \mathbb{Z})$ satisfies 1-UBC. More precisely: If $c \in C_1(S^1; \mathbb{Z})$ is a null-homologous cycle, then there exists a filling chain $c \in C_2(S^1; \mathbb{Z})$ satisfying*

$$\partial b = c \quad and \quad |b|_1 \le 3 \cdot |c|_1.$$

Proof. Let $c \in C_1(S^1; \mathbb{Z})$ be a null-homologous cycle. We use a Hurewicz argument to construct an efficient filling of c. Therefore, it is convenient to normalise c as follows:

- Using the fact that S^1 is path-connected, we can find $b_+ \in C_2(S^1; \mathbb{Z})$ with $|b_+|_1 \le 2 \cdot |c|_1$ such that

$$c_+ := c - \partial b \in C_1(S^1; \mathbb{Z})$$

  satisfies $|c_+|_1 \le |c|_1$ and such that every singular simplex in c_+ maps the boundary of Δ^1 to the basepoint of S^1 [6, Chapter 9.5].
- Splitting the integral coefficients of the chain c_+ into unit steps, we write $c_+ = \sum_{j=1}^m a_j \cdot \sigma_j$ with $a_j \in \{-1, 1\}$ for all $j \in \{1, \dots, m\}$ and $|c_+|_1 = m$.

We now use the connection between the fundamental group and $H_1(S^1; \mathbb{Z})$: Let us consider the based loop

$$f := \sigma_1^{a_j} * \cdots * \sigma_m^{a_m} \colon S^1 \longrightarrow S^1$$

(using the equidistant partition of S^1 into m segments; without loss of generality we may assume $m \ne 0$). Here, if $a_j = -1$, the symbol $\sigma_j^{a_j}$ denotes the reversed loop of σ_j. Because $[c_+] = [c] = 0 \in H_1(S^1; \mathbb{Z})$, we obtain $[f]_* = 0 \in \pi_1(S^1)$ from the Hurewicz theorem. Thus, there exists a continuous map $F \colon D^2 \longrightarrow S^1$ extending f, i.e., $F|_{\partial D^2} = f$.

The filling F of f leads to a filling of c_+: Let $j \in \{1, \dots, m\}$. Then $\tau_j \colon \Delta^2 \longrightarrow S^1$ denotes the restriction of F to the j-th segment of D^2 (Figure 2); if $a_j = 1$ we take the positive orientation, if $a_j = -1$ we take the negative orientation (see Figure 2 for the exact orientation). We then set

$$b_+ := \sum_{j=1}^m a_j \cdot \tau_j \in C_2(S^1; \mathbb{Z}).$$

A straightforward computation shows that

$$\partial b_+ = \sum_{j=1}^m a_j \cdot \partial \tau_j = \sum_{j=1}^m a_j \cdot \sigma_j = c_+$$

(because the "inner" terms cancel) and $|b_+|_1 \le m = |c_+|_1 \le |c|_1$. Combining b and b_+ gives the desired filling of c. $\square$

Remark 9.2. The same Hurewicz argument also can be used to show the following:

(1) Let M be a topological space such that the fundamental group of every path-connected component is abelian. Then $C_*(M; \mathbb{Z})$ satisfies 1-UBC.
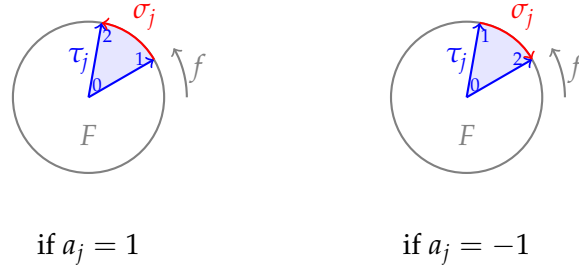
FIGURE 2. Construction of the filling chain

(2) Let $n \in \mathbb{N}_{\geq 2}$ and let M be an $(n-1)$-connected topological space. Then $C_*(M; \mathbb{Z})$ satisfies n-UBC.

9.2. **Discussion of integral UBC for general spaces.** However, for more general spaces and degrees, the situation gets more involved and the general picture is unknown.

Proposition 9.3. *Let* $n \in \mathbb{N}_{\geq 3}$. *Then there exists a simply connected space M such that $C_*(M; \mathbb{Z})$ does* not *satisfy n-UBC.*

Proof. We will construct such an example using the following input: Let N be an oriented closed (connected) n-manifold and let $(M_k)_{k \in \mathbb{N}}$ be a sequence of oriented compact connected $(n+1)$-manifolds such that for every $k \in \mathbb{N}$ we have

$$\partial M_k \cong N \quad \text{and} \quad b_2(M_k; \mathbb{Z}) \geq k.$$

We then set

$$M := \prod_{k \in \mathbb{N}} M_k$$

(if one prefers an example of dimension $n+1$, one can also apply the same arguments to $M := \bigvee_{k \in \mathbb{N}} M_k$).

We now prove that $C_*(M; \mathbb{Z})$ does *not* satisfy n-UBC: Let $c \in C_n(N; \mathbb{Z})$ be a fundamental cycle of N. As a preparation, for $k \in \mathbb{N}$ we consider the fillings of c in $C_*(M_k; \mathbb{Z})$. The Betti number estimate [15, Example 14.28][8, Lemma 4.1] for integral simplicial volume generalises to the relative case and shows that

$$b_j(M_k; \mathbb{Z}) \leq \|M_k, \partial M_k\|_{\mathbb{Z}}$$

holds for all $j \in \mathbb{N}$. In particular,

$$\|M_k, \partial M_k\|_{\mathbb{Z}} \geq b_2(M_k; \mathbb{Z}) \geq k.$$

Let $b \in C_{n+1}(M_k; \mathbb{Z})$ be a chain with $\partial b = c$, where we view $c \in C_n(N; \mathbb{Z})$ as an element of $C_n(M_k; \mathbb{Z})$ via $\partial M_k \cong N$. Then b is a relative fundamental cycle of $(M_k, \partial M_k)$, and so

$$|b|_1 \geq \|M_k, \partial M_k\|_{\mathbb{Z}} \geq k.$$

We now come back to M: For each $k \in \mathbb{N}$, we choose a basepoint in M_k; thus we obtain an inclusion $i_k \colon M_k \longrightarrow M$. If $p_k \colon M \longrightarrow M_k$ denotes the canonical projection, we have $p_k \circ i_k = \mathrm{id}_{M_k}$. For $k \in \mathbb{N}$ we consider

$$c_k := C_n(i_k; \mathbb{Z})(c) \in C_n(M; \mathbb{Z}).$$

Then $|c_k|_1 \leq |c|_1$ and c_k is null-homologous (because it can be filled by any relative fundamental cycle in the factor M_k). However, if $b \in C_{n+1}(M;\mathbb{Z})$ is a filling of c_k, then $b_k := C_{n+1}(p_k;\mathbb{Z})(b) \in C_{n+1}(M_k;\mathbb{Z})$ is a filling of c and thus

$$|b|_1 \geq |b_k|_1 \geq k.$$

Therefore, $C_*(M;\mathbb{Z})$ does *not* satisfy n-UBC.

In order to finish the proof we only need to find the input manifolds N and M_k with the additional property that all M_k are simply connected (because then M will also be simply connected). For example, we can take $N := S^n$ and M_k to be $(S^2 \times S^{n-1})^{\#k}$ minus a small $(n+1)$-ball. $\qquad\square$

However, it is not clear whether one can find aspherical sequences of this type with amenable fundamental group. Therefore, the following problem remains open:

Question 9.4. *Let M be an aspherical space with amenable fundamental group and let $n \in \mathbb{N}$. Does $C_*(M;\mathbb{Z})$ satisfy n-UBC?*

10. APPLICATION: INTEGRAL FOLIATED SIMPLICIAL VOLUME AND GLUEINGS ALONG TORI

As a sample application of the uniform boundary condition for the parametrised ℓ^1-norm, we prove a simple additivity statement for integral foliated simplicial volume and glueings along tori.

10.1. **Integral foliated simplicial volume.** As a first step, we generalise the definition of integral foliated simplicial volume [20] to the case of manifolds with boundary.

Remark 10.1 (parametrised relative fundamental cycles). Let $(M, \partial M)$ be an oriented compact connected n-manifold with boundary; if ∂M is non-empty, we will in addition assume that ∂M is connected and π_1-injective. Let $\pi \colon \widetilde{M} \longrightarrow M$ be the universal covering of M and let $U \subset \pi^{-1}(\partial M)$ be a connected component of $\pi^{-1}(\partial M)$. In particular, $\pi|_U \colon U \longrightarrow \partial M$ is a universal covering for ∂M.

Let $\Gamma := \pi_1(M)$ and let α be a standard Γ-space; then $\pi^{-1}(\partial M)$ is closed under the Γ-action on $\widetilde{M}$ and thus we can consider the subcomplex

$$D_* := L^\infty(\alpha;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*\big(\pi^{-1}(\partial M);\mathbb{Z}\big)$$

of $C_*(M;\alpha)$.

A chain $c \in C_n(M;\alpha)$ *represents* $[M,\partial M]^\alpha$ if there exist $b \in C_{n+1}(M;\alpha)$ and $d \in D_n$ as well as an ordinary relative fundamental cycle $c_{\mathbb{Z}} \in C_n(M;\mathbb{Z})$ (representing $[M,\partial M]_{\mathbb{Z}}$) such that

$$c = c_{\mathbb{Z}} + \partial b + d;$$

here, we view $C_*(M;\mathbb{Z})$ as a subcomplex of $C_*(M;\alpha)$ via the inclusion of constant functions. In particular, in this situation, we have $\partial c \in D_{n-1}$; more precisely, we have the equality

$$\partial c = \partial c_{\mathbb{Z}} + \partial d$$

in D_{n-1}. So, ∂c is a cycle in D_*.

We will now explain how ∂c can be interpreted a parametrised fundamental cycle of ∂M: Let $\Gamma_0 := \pi_1(\partial M) \subset \Gamma$ (the fundamental groups of M and ∂M should be taken with respect to the same basepoint in ∂M). Then the restriction $\alpha_0 := \operatorname{res}^\Gamma_{\Gamma_0} \alpha$ of the Γ-action to the corresponding Γ_0-action is a standard Γ_0-space and we have the canonical (isometric) chain isomorphism

$$
\begin{aligned}
D_* &\cong L^\infty(\alpha; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*\big(\pi^{-1}(\partial M); \mathbb{Z}\big) \\
&\cong L^\infty(\alpha; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma_0} C_*(U; \mathbb{Z}) \\
&\cong L^\infty(\alpha_0; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma_0} C_*(U; \mathbb{Z}) \\
&= C_*(\partial M; \alpha_0).
\end{aligned}
$$

Hence, the equation $\partial c = \partial c_{\mathbb{Z}} + \partial d$ holds in the complex $C_*(\partial M; \alpha_0)$, which shows that ∂c represents $[\partial M]^{\alpha_0}$ (because $\partial c_{\mathbb{Z}}$ is an integral fundamental cycle of ∂M).

Definition 10.2 (integral foliated simplicial volume). Let $(M, \partial M)$ be an oriented compact connected n-manifold (with possibly empty boundary; if the boundary is non-empty, we will assume that ∂M is connected and π_1-injective) and let $\Gamma := \pi_1(M)$.

- If α is a standard Γ-space, then we write

$$
\big|M, \partial M\big|^\alpha := \inf\big\{|c|_1 \;\big|\; c \in C_n(M; \alpha) \text{ represents } [M, \partial M]^\alpha\big\}.
$$

  for the *α-parametrised simplicial volume of $(M, \partial M)$*.
- The *integral foliated simplicial volume of $(M, \partial M)$* is then defined as

$$
\big|M, \partial M\big| := \inf_{\alpha \in P(\Gamma)} \big|M, \partial M\big|^\alpha,
$$

  where $P(\Gamma)$ denotes the set of all (isomorphism classes of) standard Γ-spaces.

10.2. **Glueings along tori.** We can now formulate the glueing result:

Proposition 10.3 (glueings along tori). *Let $n \in \mathbb{N}_{>0}$ and let $(M_+, \partial M_+)$, $(M_-, \partial M_-)$ oriented compact connected $(n+1)$-manifolds whose boundary is π_1-injective and homeomorphic to the n-torus $(S^1)^n$ and let $f \colon \partial M_+ \longrightarrow \partial M_-$ be an orientation-preserving homeomorphism. Let $M := M_+ \cup_f M_-$ be the oriented closed connected $(n+1)$-manifold obtained by glueing M_+ and M_- along the boundary via f and let $\Gamma := \pi_1(M) \cong \pi_1(M_+) *_{\pi_1(f)} \pi_1(M_-)$. Furthermore, let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an essentially free standard Γ-space with*

$$
\big|M_+, \partial M_+\big|^{\operatorname{res}^\Gamma_{\pi_1(M_+)} \alpha} = 0 \quad and \quad \big|M_-, \partial M_-\big|^{\operatorname{res}^\Gamma_{\pi_1(M_-)} \alpha} = 0.
$$

Then $\big|M\big|^\alpha = 0$. In particular, $\big|M\big| = 0$.

In this situation, we equip the glued manifold $M = M_+ \cup_f M_-$ with the orientation inherited from the positive orientation on M_+ and the negative orientation on M_-.

Proof. We prove $\big|M\big|^\alpha = 0$ using the uniform boundary condition for the parametrised ℓ^1-norm on tori: Let $\varepsilon \in \mathbb{R}_{>0}$ and let $\alpha_+ := \operatorname{res}^\Gamma_{\pi_1(M_+)}$, $\alpha_- := \operatorname{res}^\Gamma_{\pi_1(M_-)}$ denote the restricted parameter spaces. By the hypothesis

on the parametrised simplicial volumes of the components M_+ and M_-, there exist chains $c_+ \in C_{n+1}(M_+; \alpha_+)$ and $c_- \in C_{n+1}(M_-; \alpha_-)$ that represent $[M_+, \partial M_+]^{\alpha_+}$ and $[M_-, \partial M_-]^{\alpha_-}$, respectively, and that satisfy

$$|c_+|_1 \leq \varepsilon \quad \text{and} \quad |c_-|_1 \leq \varepsilon.$$

Then

$$c_0 := \partial c_+ - \partial c_-$$

is a null-homologous cycle in $C_n(\partial M_-; \alpha_0)$, where α_0 denotes the restriction of α to the subgroup $\pi_1(\partial M_-)$. We view M_+ and M_- in the canonical way as subspaces of M and use the identification via f to view both ∂c_+ and ∂c_- as chains on ∂M_-. By construction,

$$|c_0|_1 \leq (n+2) \cdot |c_+|_1 + (n+2) \cdot |c_-|_1 \leq 2 \cdot (n+2) \cdot \varepsilon.$$

In view of the uniform boundary condition on the torus ∂M_- (Theorem 1.3), there exists a chain $b \in C_{n+1}(\partial M_-; \alpha_0)$ with

$$\partial b = c_0 \quad \text{and} \quad |b|_1 \leq K \cdot |c_0|_1,$$

where K is a UBC-constant for $C_n(\partial M_-; \alpha_0)$ (which is independent of ε, c_+, and c_-). Then

$$c := c_+ - c_- - b \in C_{n+1}(M; \alpha)$$

is a cycle representing $[M]^\alpha$ and, by construction,

$$\left| M \right|^\alpha \leq |c|_1 \leq |c_+|_1 + |b|_1 + |c_-|_1 \leq 2 \cdot \varepsilon + K \cdot 2 \cdot (n+2) \cdot \varepsilon.$$

Taking the infimum over all $\varepsilon \in \mathbb{R}_{>0}$ shows that $\left| M \right|^\alpha = 0$. $\square$

Corollary 10.4. *Under the hypotheses of Proposition 10.3, if in addition Γ is residually finite and $\alpha = \Gamma \curvearrowright \widehat{\Gamma}$ is the canonical action of Γ on its profinite completion, then*

$$\|M\|_{\mathbb{Z}}^\infty = 0$$

(see p. 6 for the definition of $\|M\|_{\mathbb{Z}}^\infty$).

Proof. Because Γ is residually finite and finitely generated, α is an essentially free standard Γ-space and [8, Theorem 2.6]

$$\|M\|_{\mathbb{Z}}^\infty = \left| M \right|^{\Gamma \curvearrowright \widehat{\Gamma}}.$$

By Proposition 10.3, $\left| M \right|^{\Gamma \curvearrowright \widehat{\Gamma}} = 0$, which proves the corollary. $\square$

Remark 10.5 (growth and gradient invariants). In particular, in these situations, we obtain corresponding vanishing results for homology growth and logarithmic homology torsion growth [8, Theorem 1.6] as well as for the rank gradient [14].

Remark 10.6 (multiple boundary components, self-glueings). In the same way as in Proposition 10.3, one can also treat glueings along disconnected boundaries where each glued component is a torus as well as self-glueings along torus boundary components.

It would be desirable to also obtain additivity formulae for glueings along amenable boundaries in the case of summands with non-zero integral foliated simplicial volume (as in the case of ordinary simplicial volume) [9, 11]. However, one further ingredient for such additivity results is the so-called equivalence theorem [9, 3].

Question 10.7. *Can the Følner filling technique be used to give direct proofs of the equivalence theorem (in the aspherical case)? Can such an argument be refined to lead to an equivalence theorem for integral foliated simplicial volume?*

10.3. **Concrete examples.** We will now give a concrete class of examples for such torus glueings, leading to new vanishing results for integral foliated simplicial volume.

Lemma 10.8. *Let $(N, \partial N)$ be an oriented compact connected manifold with connected boundary (which might be empty), let $k \in \mathbb{N}_{>0}$, and let $M := N \times (S^1)^k$. If α is an essentially free standard $\pi_1(M)$-space, then*

$$\big| M, \partial M \big|^{\alpha} = 0.$$

Proof. This is a relative version of the product inequality by Schmidt [20, Theorem 5.34]. It suffices to consider the case $k = 1$, i.e., $M = N \times S^1$. We write $\Gamma := \pi_1(M)$ and $\Lambda \subset \Gamma$ for the subgroup corresponding to the S^1-factor. Moreover we write $n := \dim M$.

Then $\mathrm{res}_\Lambda^\Gamma \alpha$ is an essentially free standard Λ-space. For every $\varepsilon \in \mathbb{R}_{>0}$ there exists a parametrised fundamental cycle $c \in C_1(S^1; \mathrm{res}_\Lambda^\Gamma \alpha)$ of S^1 with

$$|c|_1 \leq \varepsilon.$$

This follows from an application of the Rokhlin lemma [8, Theorem 1.9][20, Proposition 5.30]. Let $c_N \in C_{n-1}(N; \mathbb{Z})$ be a relative fundamental cycle of N. Then the cross-product

$$c_M := c_N \times c \in C_n(M; \alpha)$$

is a representative of $[M, \partial M]^{\alpha}$ and

$$|c_M|_1 \leq \binom{n}{n-1} \cdot |c_N|_1 \cdot |c|_1 \leq n \cdot |c_N|_1 \cdot \varepsilon.$$

Hence, $\big| M, \partial M \big|^{\alpha} = 0.$ $\square$

Corollary 10.9. *Let $(N_+, \partial N_+)$ and $(N_-, \partial N_-)$ be oriented compact connected manifolds with boundary, let $n_+ := \dim N_+$ and $n_- := \dim N_-$, and suppose that ∂N_+ and ∂N_- are π_1-injective tori of dimension n_+ and n_-, respectively. Furthermore, let $n \in \mathbb{N}_{>\max(n_+, n_-)}$, and let*

$$f \colon \partial N_+ \times (S^1)^{n-n_+} \longrightarrow \partial N_- \times (S^1)^{n-n_-}$$

be an orientation-preserving homeomorphism. Then the glued manifold

$$M := \big(N_+ \times (S^1)^{n-n_+} \big) \cup_f \big(N_- \times (S^1)^{n-n_-} \big).$$

satisfies

$$\big| M \big|^{\alpha} = 0$$

for every essentially free standard $\pi_1(M)$-space α. In particular, $\big| M \big| = 0$.

Proof. We write $M_+ := N_+ \times (S^1)^{n-n_+}$ and $M_- := N_- \times (S^1)^{n-n_-}$ as well as $\Gamma := \pi_1(M)$ for the fundamental group of $M = M_+ \cup_f M_-$.

Let α be an essentially free standard Γ-space. By π_1-injectivity of the boundary tori, the restricted parameter spaces $\mathrm{res}^\Gamma_{\pi_1(M_+)} \alpha$ and $\mathrm{res}^\Gamma_{\pi_1(M_-)} \alpha$ are essentially free as well. Then Lemma 10.8 shows that the hypotheses of Proposition 10.3 are satisfied and hence $|M|^\alpha = 0$ and $|M| = 0$. $\qquad\square$

11. Vanishing of ℓ^1-homology of amenable groups

We will now give an application to ℓ^1-homology of groups. If Γ is a group, we write $C_*(\Gamma)$ for the standard simplicial $\mathbb{R}\Gamma$-resolution of $\mathbb{R}$ [2, p. 18] and $C_*(\Gamma;\mathbb{R}) := \mathbb{R} \otimes_{\mathbb{R}\Gamma} C_*(\Gamma)$ for the associated chain complex. This chain complex is a normed chain complex with respect to the ℓ^1-norm (given by the basis of all $(n+1)$-tuples in Γ whose 0-th vertex is 1). Taking the ℓ^1-completion of $C_*(\Gamma;\mathbb{R})$ leads to the ℓ^1-*chain complex* $C_*^{\ell^1}(\Gamma;\mathbb{R})$ of Γ [16, 12]. Then ℓ^1-*homology* $H_*^{\ell^1}(\Gamma;\mathbb{R})$ is defined as the homology of $C_*^{\ell^1}(\Gamma;\mathbb{R})$. Using the Følner filling technique, we can reprove the following result of Matsumoto and Morita [16] – without using bounded cohomology:

Theorem 11.1. *Let Γ be an amenable group and let $n \in \mathbb{N}_{>0}$. Then*

$$H_n^{\ell^1}(\Gamma;\mathbb{R}) \cong 0.$$

For simplicity, we will only consider the case of trivial coefficients; analogous arguments apply to more general coefficients, including coefficients of a more integral nature (as in the integral foliated case). Moreover, one can also prove the same results for aspherical spaces with amenable fundamental group.

The proof of Theorem 11.1 consists of the following steps: We first prove that the ℓ^1-semi-norm on ordinary group homology of amenable groups is trivial (Proposition 11.2), without using bounded cohomology or multicomplexes. We then show that the image of ordinary group homology in ℓ^1-homology is uniformly trivial (Proposition 11.3). In the final step, we subdivide ℓ^1-cycles into ordinary cycles and apply the previous step.

Proposition 11.2. *Let Γ be an amenable group and let $n \in \mathbb{N}_{>0}$. Then*

$$\|\alpha\|_1 = 0$$

for all $\alpha \in H_n(\Gamma;\mathbb{R})$.

Proof. This can be proved with the Følner filling technique (analogous to the proof of vanishing of integral foliated simplicial volume of aspherical manifolds with amenable fundamental group [8]): Let $n \in \mathbb{N}_{>0}$ and let $c \in C_n(\Gamma;\mathbb{R})$ be a cycle.

Lifting step. We can write $c = \sum_{j=1}^m a_j \cdot [1, \gamma_{j,1}, \ldots, \gamma_{j,n}]$ in reduced form; the chain

$$\widetilde{c} := \sum_{j=1}^m a_j \cdot (1, \gamma_{j,1}, \ldots, \gamma_{j,n}) \in C_n(\Gamma)$$

is a lift of c. Let $S := \{\gamma_{j,1} \mid j \in \{1, \ldots, m\}\}$ and let $(F_k)_{k\in\mathbb{N}}$ be a Følner sequence for the finitely generated amenable group $\Lambda := \langle S \rangle_\Gamma \subset \Gamma$. By construction, the canonical projection of $\widetilde{c}$ to $\mathbb{R} \otimes_{\mathbb{R}\Lambda} C_*(\Gamma)$ is a cycle.

Filling step. Let $k \in \mathbb{N}$. Then $\partial(F_k \cdot \widetilde{c}) \in C_{n-1}(\Gamma)$ is a cycle and the coned off chain

$$\widetilde{b}_k := s\big(\partial(F_k \cdot \widetilde{c})\big) \in C_n(\Gamma),$$

where s is given by $(\eta_1, \ldots, \eta_n) \mapsto (1, \eta_1, \ldots, \eta_n)$, satisfies

$$\partial \widetilde{b}_k = \partial(F_k \cdot \widetilde{c}) \quad \text{and} \quad |\widetilde{b}_k|_1 \leq \big|\partial(F_k \cdot \widetilde{c})\big|_1$$

(because of $\partial \circ s = \text{id} - s \circ \partial$). The same argument as in the lifting lemma (Lemma 4.2) shows that

$$\lim_{k \to \infty} \frac{1}{|F_k|} \cdot |\widetilde{b}_k|_1 = \lim_{k \to \infty} \frac{1}{|F_k|} \cdot \big|\partial(F_k \cdot \widetilde{c})\big|_1 = 0.$$

By construction, $\widetilde{b}_k - F_k \cdot \widetilde{c}_k \in C_n(\Gamma)$ is a cycle, whence null-homologous (the chain complex $C_*(\Gamma)$ is contractible).

Quotient step. Therefore, the chain

$$c_k := \frac{1}{|F_k|} \cdot (\text{canonical projection of } \widetilde{b}_k) \in C_n(\Gamma; \mathbb{R})$$

is a cycle with $[c_k] = 1/|F_k| \cdot |F_k| \cdot [c] = [c]$ and $|c_k|_1 \leq 1/|F_k| \cdot |\widetilde{b}_k|_1$. Hence, $\lim_{k \to \infty} |c_k|_1 = 0$ and so $\|[c]\|_1 = 0$. $\qquad \square$

Proposition 11.3. *Let C_* be a normed $\mathbb{R}$-chain complex, let $\overline{C}_*$ be its completion, and let $n \in \mathbb{N}$. Furthermore, we assume that the induced semi-norm $\|\cdot\|$ on $H_n(C_*)$ is trivial and that C_* satisfies n-UBC. Then the map*

$$H_n(i) \colon H_n(C_*) \longrightarrow H_n(\overline{C}_*)$$

induced by the inclusion $C_* \hookrightarrow \overline{C}_*$ is trivial. More precisely: There exists a constant $K \in \mathbb{R}_{>0}$ with the following property: For every cycle $c \in C_n$ there exists a chain $b \in \overline{C}_{n+1}$ with

$$\partial b = c \quad \text{and} \quad |b|_1 \leq K \cdot |c|_1.$$

Proof. Let $K \in \mathbb{R}_{>0}$ be an n-UBC-constant for C_* and let $c \in C_n$ be a cycle. By hypothesis, $\|[c]\| = 0$. Hence, there is a sequence $(c_k)_{k \in \mathbb{N}}$ of cycles in C_n such that

$$\partial c_k = 0, \qquad |c_k| \leq \frac{1}{2^k} \cdot |c|, \qquad [c_k] = [c] \in H_n(C_*)$$

holds for all $k \in \mathbb{N}$; moreover, we will take $c_0 := 0$.

In view of n-UBC, there exists a sequence $(b_k)_{k \in \mathbb{N}}$ in C_{n+1} such that for all $k \in \mathbb{N}$ we have

$$\partial b_k = c_{k+1} - c_k \quad \text{and} \quad |b_k| \leq K \cdot |c_{k+1} - c_k| \leq K \cdot \frac{1}{2^{k-1}} \cdot |c|.$$

Then $b := \sum_{k=0}^{\infty} b_k$ is a well-defined chain in $\overline{C}_{n+1}$ and one calculates (using continuity of the boundary operator and absolute convergence of all involved series)

$$\partial b = \sum_{k=0}^{\infty} \partial b_k = \sum_{k=0}^{\infty} (c_{k+1} - c_k) = c_0 = c$$

as well as

$$|b| \leq \sum_{k=0}^{\infty} |b_k| \leq K \cdot |c| \cdot \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} \leq 4 \cdot K \cdot |c|.$$

Therefore, the constant $K/4$ has the desired property. $\qquad\square$

Proof of Theorem 11.1. Because Γ is amenable, the chain complex $C_*(\Gamma;\mathbb{R})$ satisfies the uniform boundary condition in each degree (the proof of Proposition 1.1 easily adapts to the group case and $\mathbb{R}$-coefficients). Let $n \in \mathbb{N}_{>0}$ and let $K' \in \mathbb{R}_{>0}$ be an $(n-1)$-UBC-constant for $C_*(\Gamma;\mathbb{R})$.

We now consider a cycle $c \in C_n^{\ell^1}(\Gamma;\mathbb{R})$; the goal is to find an ℓ^1-chain b with $\partial b = c$. As a first step, we show that c can be decomposed into an ℓ^1-sum

$$c = \sum_{k=0}^{\infty} c_k$$

of ordinary cycles $c_k \in C_n(\Gamma;\mathbb{R})$: By definition of the ℓ^1-norm, there is a sequence $(z_k)_{k \in \mathbb{N}}$ in $C_n(\Gamma;\mathbb{R})$ with

$$c = \sum_{k=0}^{\infty} z_k \quad \text{and} \quad |c|_1 \le \sum_{k=0}^{\infty} |z_k|_1.$$

For $k \in \mathbb{N}$ we consider the partial sum $s_k := \sum_{j=0}^{k} z_j \in C_n(\Gamma;\mathbb{R})$. Because c is a cycle, $\lim_{k\to\infty} |\partial s_k|_1 = |\partial c|_1 = 0$. Hence, by regrouping our sequence $(z_k)_{k\in\mathbb{N}}$, we may assume without loss of generality that the sequences $(|\partial s_k|_1)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ both are ℓ^1. By $(n-1)$-UBC, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $C_n(\Gamma;\mathbb{R})$ such that

$$\partial w_k = \partial s_k \quad \text{and} \quad |w_k|_1 \le K' \cdot |\partial s_k|_1$$

holds for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$ we set $c_k := z_k - w_k + w_{k-1}$ (where $w_{-1} := 0$ and $s_{-1} := 0$). Then $\partial c_k = 0$ and $|c_k|_1 \le |z_k|_1 + K' \cdot |\partial s_k|_1 + K' \cdot |\partial s_{k-1}|_1$. Hence, $c = \sum_{k=0}^{\infty} c_k$ is an ℓ^1-sum of ordinary cycles in $C_n(\Gamma;\mathbb{R})$.

We will now apply Proposition 11.3 to $C_*(\Gamma;\mathbb{R})$. In view of Proposition 11.2, the ℓ^1-semi-norm on $H_n(\Gamma;\mathbb{R})$ is trivial. Moreover, $C_*(\Gamma;\mathbb{R})$ satisfies n-UBC (see above). So Proposition 11.3 indeed can be applied. Hence, there is a $K \in \mathbb{R}_{>0}$ such that for each $k \in \mathbb{N}$ there exists a $b_k \in C_{n+1}^{\ell^1}(\Gamma;\mathbb{R})$ with

$$\partial b_k = c_k \quad \text{and} \quad |b_k|_1 \le K \cdot |c_k|_1.$$

Therefore,

$$b := \sum_{k=0}^{\infty} b_k \in C_{n+1}^{\ell^1}(\Gamma;\mathbb{R})$$

is a well-defined ℓ^1-chain that satisfies $\partial b = \sum_{k=0}^{\infty} \partial b_k = \sum_{k=0}^{\infty} c_k = c$. $\qquad\square$

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