Applied Algebraic Topology

An introductory course
Wintersemester 2022/23
Universität Regensburg

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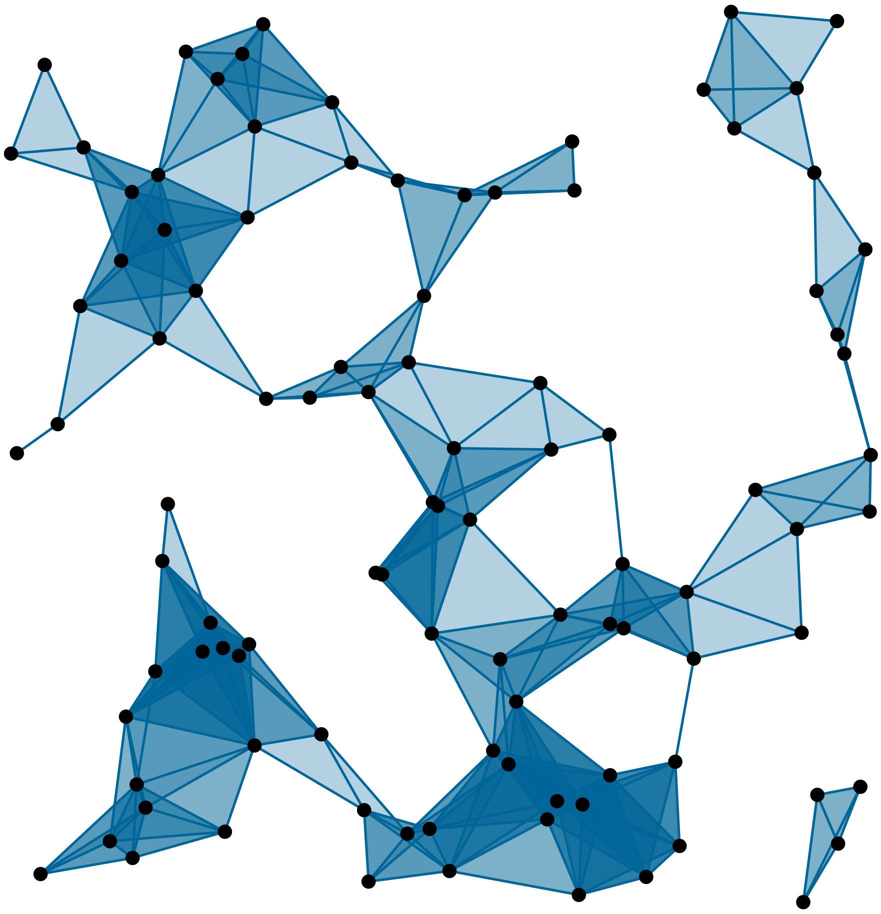

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Contents

Guide to the Literature

This course will not follow a single source and there are many books that cover the standard topics (all with their own advantages and disadvantages). Therefore, you should individually compose your own favourite selection of books.

Algebraic Topology

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Introduction

Applied algebraic topology studies real-world problems using the language, objects, and techniques from algebraic topology.

Algebraic Topology

The basic idea of algebraic topology is to translate topological problems into algebraic problems; topological spaces are translated into algebraic objects (e.g., vector spaces) and continuous maps are translated into homomorphisms (e.g., linear maps).

Topology	$\rightsquigarrow$	**Algebra**
topological spaces		e.g., vector spaces
continuous maps		linear maps
flexible		*rigid*

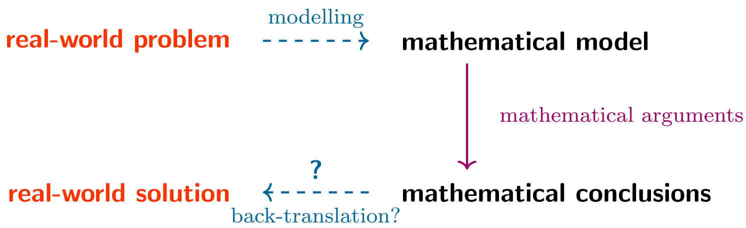
Algebraic topology then is concerned with the classification of topological spaces and continuous maps up to "continuous deformation", i.e., up to so-called homotopy. To this end, one constructs and studies homotopy invariant functors. The main design problem consists of finding functors that

- are fine enough to recover interesting features of topological spaces, but that also
- are coarse enough to be computable in many cases.

A classical example of a homotopy invariant functor is ordinary homology, which is algorithmically computable in the context of simplicial complexes. A first, intuitive, description is that homology measures "which and how many holes" topological spaces have.

Applied Algebraic Topology

Applied mathematics relies on the following principle: A real-world problem is modelled in mathematical terms. Then, mathematical arguments are used to come to a mathematical conclusion or solution. Sometimes (but not always) it is possible to translate the mathematical results back into a solution in the real world.



The model is merely an abstract and simplified approximation of the real world. In most cases, one cannot *prove* that such a model is "correct" because usually there is no formal and provably correct formalisation of the real-world problem. In particular, the conclusions for the original real-world problem cannot be stronger than the mathematical model. Moreover, one should be aware that this setup is suited better to *disprove* claims on the real world rather than to *confirm* them.

There are two types of applications: On the one hand, we can formulate and (dis)prove claims using mathematical language and theory. On the other hand, we can perform actual computations on real-world data.

In applied algebraic topology, we model real-world problems in the language of algebraic topology. A particularly well-suited subfield of algebraic topology in this context is simplicial topology.

Graphs are combinatorial structures that can model connectivity of various kinds, e.g., connections between people in social networks, genetic proximity in biology, or dependencies between software components. Simplicial complexes are a higher-dimensional generalisation of graphs and thus allow for more fine-grained models. The most common uses of simplicial complexes are of the following types:

- to discretely approximate general topological spaces (so-called triangulations);
- to model point clouds in metric spaces (e.g., multi-dimensional big data);
- to model relations between entities (e.g., dependencies/decentralised computations between agents in distributed systems)

The combinatorial flavour of simplicial complexes makes them algorithmically accessible. For example, the homology of (finite) simplicial complexes is algorithmically computable.

In addition to simplicial algebraic topology, also the abstract language of homotopy theory itself has found a different type of applications, namely in the foundations of mathematics and computer science: There are many shades of "equality" and the question of how "equality" persists through various constructions is delicate. In mathematics such questions arise in the guise of "identifications" or "canonical isomorphisms"; in computer science, they appear in the design and implementation of programming languages when different levels of "equality" of entities need to be considered. Homotopy theory provides a setup that makes it possible to describe coherent notions of "equality" in various settings.

Typical areas of applied algebraic topology include:

- configuration spaces for robotics;
- existence of Nash equilibria in game theory;
- impossibility results in social choice;
- lower complexity bounds for distributed algorithms;
- higher statistics and big data;
- de-centralised computations in sensor networks;
- (combinatorial) distribution/colouring problems;
- foundations of computing/homotopy type theory;
- ...

Overview of this Course

The field of applied algebraic topology is rather broad and rapidly growing. Therefore, this course will only consist of a selection of topics.

We will learn the basics of simplicial complexes, simplicial homology, and homotopy invariance. We will explore modelling and real-world applications of these notions and invariants. Whenever feasible, we will also look at implementation matters.

We will develop the topological language far enough to be able to understand and use results from algebraic topology; because the focus will be on the applications, we might not prove all of the underlying results from algebraic topology. Such results are marked as **Black box** and may be used without proof (unless explicitly stated otherwise).

We will start with a brief introduction to the central notion of homotopy and the concept of homotopy invariance (Chapter 1). As a warm-up application, we will consider

- a motion planning problem.

We will then introduce the language of simplicial complexes and how to use them in modelling (Chapter 2). As our main homotopy invariant tool, we will use simplicial homology (Chapter 3). This basic setup already leads to a variety of applications:

- the existence of Nash equilibria in game theory/economics;
- the analysis of (im)possibility results on consensus in distributed systems;
- a proof of Arrow's theorem on social choice;
- sensor network coverage problems.

Topological analysis of big data often is based on persistent homology of filtered simplicial complexes. We introduce the abstract concept of persistent homology along with the corresponding major structure and stability theorems (Chapter 4). We then outline applications of persistent homology to the analysis of multi-dimensional big data:

- analysis of evolution in biology;
- analysis of progression through diseases;

A refined version of simplicial homology is the cup-product structure on simplicial cohomology (Chapter **??**). Using this cup-product, we can tackle

- combinatorial distribution problems;
- refined motion planning problems.

Finally, in Chapter **??**, we will see how the language of modern homotopy theory can be applied to deal with

- coherent notions and implementations of equality in computer science.

Study note. These lecture notes document the topics covered in the course (as well as some additional optional material). However, these lectures notes are not meant to replace attending the lectures or the exercise classes.

Furthermore, this course will only treat a small fraction of applied algebraic topology. It is therefore recommended to consult other sources (books!) for further information on this field.

Literature exercise. Where in the math library (including electronic resources) can you find books on algebraic topology and related fields?

Convention. The set $\mathbb{N}$ of natural numbers contains 0. All rings are unital and associative. Usually, we assume manifolds to be non-empty (but we might not always mention this explicitly).

1

Homotopy

Algebraic topology is concerned with topological spaces "up to continuous deformation" and translates topology up to such deformations into the more rigid world of algebra.

There are several interpretations of "up to continuous deformation". The most rigid one is to consider spaces up to homeomorphism. One of the key insights in algebraic topology is that the weaker notion of homotopy/homotopy equivalence provides additional flexibility and captures a wide range of phenomena. Moreover, homotopy interacts well with algberaic methods.

We briefly introduce the language of homotopies, homotopy equivalences, and the concept of homotopy invariance. As a first application we will see how homotopy equivalences naturally arise in the context of motion planning problems.

In case you are not comfortable with working with general topological spaces, for most of this course, it is sufficient to think of metric spaces and the associated notion of continuity. Basics on topological spaces and categories/functors are collected in Appendix A.1 and Appendix A.2.

Overview of this chapter.

Running example. subsets of Euclidean space, spheres, path spaces

1.1 Homotopy and homotopy equivalence

The notion of homotopy equivalence encodes the basic shape of spaces (Figure 1.1). Two maps are homotopic if there is a continuous deformation between them (Figure 1.2). Spaces are homotopy equivalent if they are related by continuous maps that are inverse to each other up to homotopy.

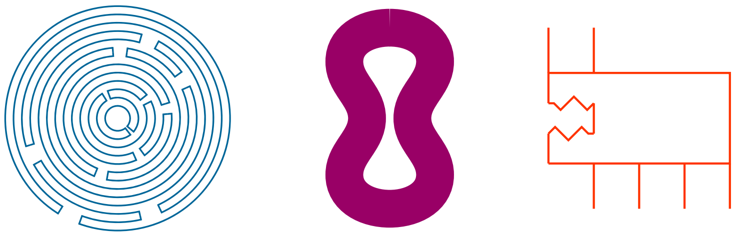


Figure 1.1.: These spaces are *not* homeomorphic (check!), but they all share the same "principal shape", namely a "circle" with a "hole".

Definition 1.1.1 (homotopy, homotopic, homotopy equivalence, null-homotopic, contractible)**.** Let X and Y be topological spaces.

- Let $f, g\colon X \longrightarrow Y$ be continuous maps. Then f is *homotopic* to g, if f can be deformed continuously into g, i.e., if there exists a homotopy from f to g (Figure 1.2).

  A *homotopy* from f to g is a continuous map $h\colon X \times [0,1] \longrightarrow Y$ with
  $$h(\,\cdot\,,0) = f \quad \text{and} \quad h(\,\cdot\,,1) = g.$$
  In this case, we write $f \simeq g$.
- Maps that are homotopic to constant maps are called *null-homotopic*.
- The topological spaces X and Y are *homotopy equivalent*, if there exist continuous maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ satisfying
  $$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y;$$
  such maps are called *homotopy equivalences*. We then write $X \simeq Y$.
- Topological spaces that are homotopy equivalent to one-point spaces are called *contractible*.

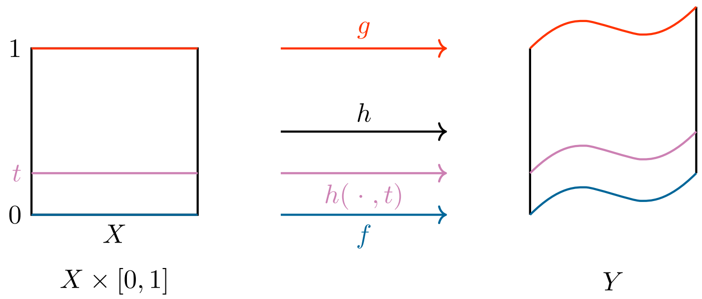


Figure 1.2.: Homotopies are "movies" between continuous maps.

Example 1.1.2 (star-shaped sets)**.** Let $n \in \mathbb{N}$. A subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists a point $x_0 \in X$ such that for every $x \in X$, the segment

$$\{t \cdot x + (1-t) \cdot x_0 \mid t \in [0,1]\}$$

from x_0 to x is also contained in X (Figure 1.3).

For example, every non-empty convex subset in $\mathbb{R}^n$ is star-shaped (the converse does *not* hold in general; check!). The n-ball

$$D^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$$

is star-shaped (for instance, with respect to 0).

Every star-shaped set $X \in \mathbb{R}^n$ is contractible (with respect to the subspace topology): Indeed, let $x_0 \in X$ be a star-point for X. We consider the maps

$$\begin{aligned} f\colon X &\longrightarrow \{x_0\} \\ x &\longmapsto x_0 \\ g\colon \{x_0\} &\longrightarrow X \\ x_0 &\longmapsto x_0. \end{aligned}$$

Then f and g are continuous and $f \circ g = \mathrm{id}_{\{x_0\}}$. Moreover, the homotopy

$$\begin{aligned} X \times [0,1] &\longrightarrow X \\ (x,t) &\longmapsto t \cdot x + (1-t) \cdot x_0 \end{aligned}$$

shows that $g \circ f \simeq \mathrm{id}_X$ (check!). Hence, $X \simeq \{x_0\}$.

Black box 1.1.3 (spheres are not contractible)**.** For every $n \in \mathbb{N}$, the *n-sphere*

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$$

is *not* contractible. (We will see a proof of this fact in Theorem 3.4.8.)

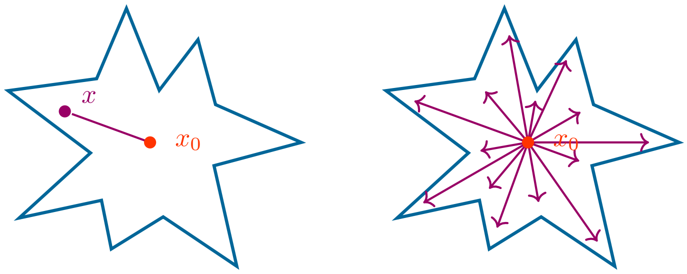


Figure 1.3.: A star-shaped set and a contracting homotopy for a star-shaped set. The illustration of the homotopy does *not* indicate a deformations of the space but of maps.

Example 1.1.4 (thick spheres). Let $n \in \mathbb{N}$. Then $S^n \simeq \mathbb{R}^{n+1} \setminus \{0\}$ (Figure 1.4): We consider the maps

$$\begin{aligned} f\colon S^n &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ x &\longmapsto x \\ g\colon \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow S^n \\ x &\longmapsto \frac{1}{\|x\|_2} \cdot x. \end{aligned}$$

Then $g \circ f = \mathrm{id}_{S^n}$, whence $g \circ f \simeq \mathrm{id}_{S^n}$. Moreover, the (well-defined!) homotopy

$$\begin{aligned} \bigl(\mathbb{R}^{n+1} \setminus \{0\}\bigr) \times [0,1] &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ (x,t) &\longmapsto \frac{t \cdot \|x\|_2 + (1-t)}{\|x\|_2} \cdot x \end{aligned}$$

shows that $f \circ g \simeq \mathrm{id}_{\mathbb{R}^{n+1} \setminus \{0\}}$ (check!). Hence, $S^n \simeq \mathbb{R}^{n+1} \setminus \{0\}$ and we may view the punctured space $\mathbb{R}^{n+1} \setminus \{0\}$ as a "thick sphere".

Caveat 1.1.5. Every homeomorphism is a homotopy equivalence. The converse does *not* hold in general. For example, by Example 1.1.2, $D^1 \simeq \{0\}$, but D^1 and $\{0\}$ do not even have the same cardinality. This example also shows that homotopy equivalences, in general, are neither injective nor surjective.

Remark 1.1.6 (deformation of maps vs. paths of maps). For sufficiently nice topological spaces, the exponential law for mapping spaces shows that homotopies between maps are the same as continuous paths between these maps in mapping spaces.

More precisely: For topological spaces X and Y, we write $\mathrm{map}(X, Y)$ for the set of all continuous maps $X \longrightarrow Y$. The set $\mathrm{map}(X, Y)$ carries the *compact-open topology*, i.e., the topology on $\mathrm{map}(X, Y)$ that is generated by

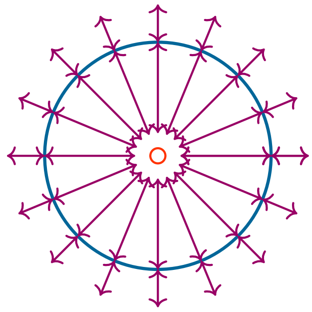


Figure 1.4.: The homotopy equivalence of Example 1.1.4.

all sets of the form

$$M_{K,U} := \{f \in \operatorname{map}(X,Y) \mid f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. If Y is a metric space and X is compact, then this is the same as the topology of uniform convergence (i.e., the topology induced by the sup-metric; Exercise).

Let X be a *locally compact* topological space, i.e., for every $x \in X$ and every open neighbourhood U of x there exists a compact neighbourhood K of x with $K \subset U$. Then, for every topological space Y the currying map

$$\begin{aligned} \operatorname{map}\bigl(X \times [0,1], Y\bigr) &\longmapsto \operatorname{map}\bigl([0,1], \operatorname{map}(X,Y)\bigr) \\ h &\longmapsto \bigl(t \mapsto h(\,\cdot\,,t)\bigr) \end{aligned}$$

is well-defined and bijective [1, Proposition 1.3.1][60, Corollary 3.4]. On the right-hand side, $\operatorname{map}(X,Y)$ carries the compact-open topology.

Proposition 1.1.7 (elementary properties of homotopy)**.**

1. *Let X and Y be topological spaces. Then "$\simeq$" is an equivalence relation on* $\operatorname{map}(X,Y)$.

2. *Let X, Y, Z be topological spaces and let $f, f' \colon X \longrightarrow Y$, $g, g' \colon Y \longrightarrow Z$ be continuous maps with $f \simeq f'$ and $g \simeq g'$. Then*

$$g \circ f \simeq g' \circ f'.$$

3. *If X is a contractible topological space and Y is a topological space, then all continuous maps $X \longrightarrow Y$ and $Y \longrightarrow X$ are null-homotopic.*

Proof. Ad 1. Reflexivity. Let $f \in \operatorname{map}(X,Y)$. Then $f \simeq f$ follows from the constant movie, i.e., via the homotopy:

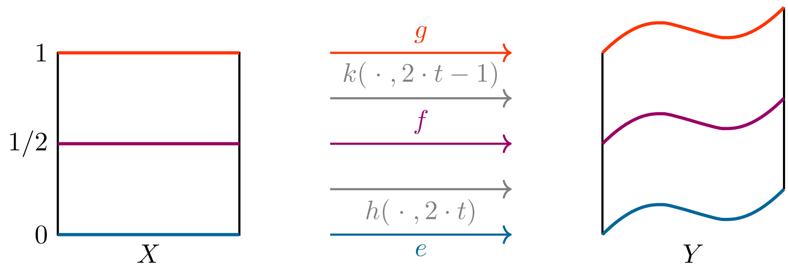


Figure 1.5.: Transitivity of being homotopic, schematically

$$\begin{aligned} X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto f(x) \end{aligned}$$

Symmetry. Let $f, g \in \operatorname{map}(X,Y)$ with $f \simeq g$ and let $h\colon X \times [0,1] \longrightarrow Y$ be such a homotopy. Then $g \simeq f$ follows from the inverse movie, i.e., via the homotopy

$$\begin{aligned} X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto h(x, 1-t). \end{aligned}$$

Transitivity. Let $e, f, g \in \operatorname{map}(X,Y)$ with $e \simeq f$ and $f \simeq g$; let $h, k\colon X \times [0,1] \longrightarrow Y$ be such homotopies. Then $e \simeq g$ follows from the concatenated (and reparametrised) movie (Figure 1.5), i.e., via the homotopy

$$\begin{aligned} X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto \begin{cases} h(x, 2\cdot t) & \text{if } t \in [0,1/2] \\ k(x, 2\cdot t - 1) & \text{if } t \in [1/2,1]; \end{cases} \end{aligned}$$

it should be noted that this map is indeed well-defined and continuous (Proposition A.1.17).

Ad 2. Let $h\colon X \times [0,1] \longrightarrow Y$ and $k\colon Y \times [0,1] \longrightarrow Z$ be homotopies from f to f' and from g to g', respectively. Then

$$\begin{aligned} X \times [0,1] &\longrightarrow Z \\ (x,t) &\longmapsto k\bigl(h(x,t), t\bigr) \end{aligned}$$

is a homotopy showing that $g \circ f \simeq g' \circ f'$ (check!).

Ad 3. Let X be contractible. Then $\mathrm{id}_X \simeq c$, where $c\colon X \longrightarrow X$ is a constant map (check!). If $f\colon X \longrightarrow Y$ is continuous, then the second part shows that

$$f = f \circ \mathrm{id}_X \simeq f \circ c.$$

Moreover, because c is constant, also $f \circ c$ is constant. Hence, f is null-homotopic. Analogously, one can handle the case of maps to X, as well as the pointed case. □

Example 1.1.8 (boring paths)**.** If X is path-connected space, then all continuous maps $[0,1] \longrightarrow X$ are homotopic to each other: By the third part of Proposition 1.1.7, all continuous maps $[0,1] \longrightarrow X$ are null-homotopic (because $[0,1]$ is star-shaped, whence contractible; Example 1.1.2). Moreover, all constant maps to X are homotopic to each other, because X is path-connected.

Corollary 1.1.9 (a characterisation of contractibility)**.** *Let X be a non-empty topological space. Then X is contractible if and only if id_X is null-homotopic.*

Proof. Let X be contractible. Then the map $\operatorname{id}_X : X \longrightarrow X$ is null-homotopic by Proposition 1.1.7.3.

Conversely, let $\operatorname{id}_X : X \longrightarrow X$ be null-homotopic. I.e., there exists an $x_0 \in X$ and a homotopy $h\colon X \times [0,1] \longrightarrow X$ from id_X to the constant map with value x_0. Let $c\colon X \longrightarrow \{x_0\}$ be the constant map. Then $c \circ \operatorname{id}_X = \operatorname{id}_{\{x_0\}}$ and h shows that $\operatorname{id}_X \simeq \operatorname{id}_X \circ c$. Hence, X is contractible. □

Outlook 1.1.10 (homotopy in other fields)**.** More generally, in order to define a notion of homotopy and homotopy equivalences we merely need suitable products and a suitable model of the unit interval. For example, translating this concept into homological algebra leads to the notion of chain homotopy and chain homotopy equivalence for chain complexes (Appendix A.3.3). In geometric group theory, the notion of "having finite distance" can be interpreted as a notion of homotopy between maps. In $\mathbb{A}^1$-homotopy theory (a branch of algebraic geometry inspired by homotopy theory), the affine line $\mathbb{A}^1$ plays a role similar to the unit interval $[0,1]$ in classical homotopy theory.

1.2 Application: Basic motion planning

We consider a basic motion planning problem and explain how homotopy appears in the solution.

Real-world problem 1.2.1 (motion planning)**.** For a factory floor X, given any two positions x, y in X, provide a path $s(x,y)$ subject to the following constraints:

- For all x, y in X, the path $s(x,y)$ is an actual path from x to y within X;
- Stability: For all x, x', y, y' in X, whenever x' is close enough to x and y' is close enough to y, then also $s(x',y')$ is close to $s(x,y)$.

More generally, one may consider all kinds of state sets of systems (e.g., configuration spaces of robot arms) and ask for motion planning to move between given states.

This situation can be modelled as follows [31]:

Model 1.2.2 (motion planning)**.** We model Problem 1.2.1 by:

- State set: A topological space X.

  Explanation. We could start with a plain set. However, to model paths as continuous paths (see below), we need a topology.

- Paths in the state set: Continuous maps $[0,1] \longrightarrow X$.

  We write $\operatorname{P} X$ for the set $\operatorname{map}([0,1], X)$, equipped with the compact-open topology (Remark 1.1.6).

  Explanation. Depending on the abilities of the moving entities (e.g., robots on wheels), it is natural to require that the paths need to be continuous. The compact-open topology on $\operatorname{P} X$ captures closeness of paths.

- Motion planning: A *motion planning* is a map $s \colon X \times X \longrightarrow \operatorname{P} X$ with

  $$\forall_{x,y \in X} \quad s(x,y)(0) = x \quad \text{and} \quad s(x,y)(1) = y.$$

  I.e., a motion planning is a section of the *endpoints map*

  $$\begin{aligned} \pi \colon \operatorname{P} X &\longrightarrow X \times X \\ \gamma &\longmapsto \bigl(\gamma(0), \gamma(1)\bigr). \end{aligned}$$

  Explanation. This constraint reflects that paths have the given initial/terminal points.

- Stability: Continuity of motion planning maps.

  Explanation. In the above setup, stability translates into continuity of the motion planning map.

Thus, Problem 1.2.1 translates into the following problem:

Question 1.2.3. Given a topological space X, find a continuous motion planning $X \times X \longrightarrow \operatorname{P} X$ on X. More fundamentally: Which topological spaces admit a continuous motion planning at all?

The answer involves the notion of homotopy:

Theorem 1.2.4 ([31])**.** *Let X be a non-empty path-connected topological space. Then X admits a continuous motion planning if and only if X is contractible.*

Proof. Let $s\colon X \times X \longrightarrow \mathrm{P}\,X$ be a continuous motion planning for X. As $X \neq \emptyset$, there exists an $x_0 \in X$. Then

$$\begin{aligned} h\colon X \times [0,1] &\longrightarrow X \\ (x,t) &\longmapsto s(x_0,x)(t) \end{aligned}$$

is continuous (Lemma 1.2.5 and composition of continuous maps) and

$$h(\,\cdot\,,0) = s(x_0,\,\cdot\,)(0) = x_0 \quad \text{and} \quad \forall_{x\in X}\quad h(x,1) = s(x_0,x)(0) = x.$$

Hence, the constant map x_0 is homotopic to id_X and so X is contractible (Corollary 1.1.9).

Conversely, let X be contractible. Hence, there exists an $x_0 \in X$ and a homotopy $h\colon X \times [0,1] \longrightarrow X$ from id_X to the constant map at x_0 (Corollary 1.1.9). Then the map

$$\begin{aligned} s\colon X \times X &\longrightarrow \mathrm{P}\,X \\ (x,y) &\longmapsto h(x,\,\cdot\,) * \overline{h(y,\,\cdot\,)} \end{aligned}$$

is continuous (Lemma 1.2.5 and composition of continuous maps). Here, $\overline{\,\cdot\,}$ denotes the reversal of paths and "$*$" denotes the concatenation of paths: For $\gamma,\eta \in \mathrm{P}\,X$ with $\gamma(1) = \eta(0)$, we set (which is well-defined and continuous; check!)

$$\begin{aligned} \gamma * \eta\colon [0,1] &\longrightarrow X \\ t &\longmapsto \begin{cases} \gamma(2\cdot t) & \text{if } t \in [0,1/2] \\ \eta(2\cdot t - 1) & \text{if } t \in [1/2,1]. \end{cases} \end{aligned}$$

By construction, for all $x,y \in X$, we have

$$s(x,y)(0) = h(x,0) = x \quad \text{and} \quad s(x,y)(1) = \overline{h(y,1)} = h(y,0) = y.$$

Therefore, s is a continuous motion planning for X. □

Lemma 1.2.5 (some properties of the compact-open topology on path spaces)**.** *Let X be a topological space.*

1. *The evaluation map is continuous:*

$$\begin{aligned} e\colon \mathrm{P}\,X \times [0,1] &\longrightarrow X \\ (\gamma,t) &\longmapsto \gamma(t) \end{aligned}$$

2. *The reversion and concatenation of paths is continuous (where we write $Y := \bigl\{(\gamma,\eta) \in \mathrm{P}\,X \times \mathrm{P}\,X \bigm| \gamma(1) = \eta(0)\bigr\}$) for the space of concatenable paths, with respect to the subspace topology of the product topology):*

$$\begin{aligned} r\colon \operatorname{P} X &\longrightarrow \operatorname{P} X \\ \gamma &\longmapsto \overline{\gamma} \\ c\colon Y &\longrightarrow \operatorname{P} X \\ (\gamma, \eta) &\longmapsto \gamma * \eta \end{aligned}$$

Proof. Ad 1. Let $U \subset X$ be open and let $(\gamma, t) \in e^{-1}(U)$. We show that $e^{-1}(U)$ contains a neighbourhood of (γ, t):

Because γ is continuous, there is an open neighbourhood $V \subset [0,1]$ of t in $[0,1]$ with $\gamma(V) \subset U$. As $[0,1]$ is locally compact, there exists a compact neighbourhood $K \subset V$ of t. Then $M_{K,U} \times K$ is a neighbourhood of (γ, t) in the product space $\operatorname{P} X \times [0,1]$ and

$$e(M_{K,U} \times K) \subset U.$$

Ad 2. In both cases, it suffices to consider open sets of the form $M_{K,U}$ in $\operatorname{P} X$, where $U \subset X$ is open and $K \subset [0,1]$ is compact.

Concerning the reversion of paths: Let $L := \{1 - t \mid t \in K\} \subset [0,1]$, which is compact. Then

$$r^{-1}(M_{K,U}) = M_{L,U}.$$

Because $M_{L,U}$ is open in $\operatorname{P} X$, we see that r is continuous.

Concerning concatenation: Let $K_1 := K \cap [0, 1/2]$ and $K_2 := K \cap [1/2, 1]$. Then K_1 and K_2 are compact and $M_{K,U} = M_{K_1,U} \cap M_{K_2,U}$. Similarly, to the proof for the reversion of paths, we may consider the compact sets

$$L_1 := \{2 \cdot t \mid t \in K_1\} \quad \text{and} \quad L_2 := \{2 \cdot t - 1 \mid t \in K_2\}$$

in $[0,1]$ and obtain

$$\begin{aligned} c^{-1}(M_{K,U}) &= c^{-1}(M_{K_1,U} \cap M_{K_2,U}) \\ &= c^{-1}(M_{K_1,U}) \cap c^{-1}(M_{K_2,U}) \\ &= \bigl(Y \cap (M_{L_1,U} \times \operatorname{P} X)\bigr) \cap \bigl(Y \cap (\operatorname{P} X \times M_{L_2,U})\bigr), \end{aligned}$$

which is open in Y. □

Remark 1.2.6. Let X be a topological space. Then X is path-connected if and only if there exists a (not necessarily continuous!) section $X \times X \longrightarrow \operatorname{P} X$ of the endpoints map (check!).

We apply Theorem 1.2.4 to concrete situations:

Example 1.2.7 (convex state spaces)**.** Let $n \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be non-empty and convex. Then X admits a continuous motion planning because X is contractible (Example 1.1.2). More concretely, one can easily specify a concrete motion planning in this case (Exercise).

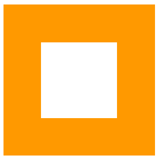


Figure 1.6.: A factory floor

Example 1.2.8 (a circular factory floor)**.** The factory floor depicted in Figure 1.6 does *not* admit a continuous motion planning: The factory floor can be modelled by the subspace $X := \{x \in [-2,2]^2 \mid |x| \geq 1\}$ of $\mathbb{R}^2$, which is homotopy equivalent to S^1 (check!). Hence, X is *not* contractible (Black box 1.1.3 and transitivity of homotopy equivalence) and therefore X does not admit a continuous motion planning.

Example 1.2.9 (motion planning on Earth)**.** There is no continuous motion planning for the surface of planet Earth: This surface should be modelled by S^2, which is *not* contractible (Black box 1.1.3).

If we are more liberal and also allow routes that go through the interior of the planet, then there does exist a continuous motion planning: The solid planet should be modelled by D^3, which is contractible (Example 1.1.2).

Remark 1.2.10 (undecidability of contractibility)**.** In general, even for "nice" spaces (e.g., spaces that are given by a finite triangulation), it is *not* algorithmically decidable whether they are contractible or not (Outlook 2.8.5). Thus, even in the presence of Theorem 1.2.4 it might be difficult to assess whether a continuous motion planning exists in the given situation or not.

Outlook 1.2.11 (topological complexity [31])**.** Topological complexity quantifies the failure of existence of a continuous motion planning: The topological complexity is the minimal number of continuous patches of motion planning that are required to cover the whole state space. More precisely: Let X be a topological space. The *topological complexity* $\operatorname{TC}(X)$ of X is the minimal $n \in \mathbb{N}$ such that there exist open subsets $U_1, \dots, U_n \subset X \times X$ such that

$$X \times X = U_1 \cup \dots \cup U_n$$

and such that for each $j \in \{1, \dots, n\}$, there exists a continuous section $U_j \longrightarrow \operatorname{P} X$ of the endpoints map. If no such n exists, we set $\operatorname{TC}(X) := \infty$.

By definition, $\operatorname{TC}(X) = 1$ if and only if X admits a continuous motion planning. It can be shown that [31] (Example **??**):

- $\operatorname{TC}(S^1) = 2$ and, more generally, that
- $\operatorname{TC}\bigl((S^1)^{\times n}\bigr) = 2 \cdot n$ for all $n \in \mathbb{N}_{>0}$.

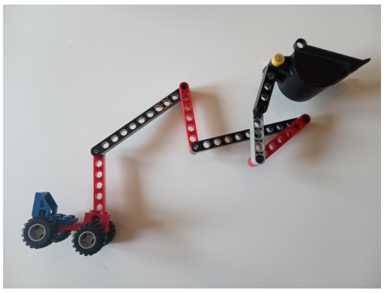


Figure 1.7.: A robot arm

We have already seen a real-world situation that leads to S^1 (Example 1.2.8). It might not be so clear how state spaces like $(S^1)^{\times n}$ occur. A typical situation that leads to high-dimensional models is when systems with many degrees of freedom are considered – for instance, robot arms with many joints:

Let $n \in \mathbb{N}$. We consider a robot arm that is composed as a sequence of $n+1$ linear bars and that is fixed at the beginning of the first bar; every two subsequent bars are connected by a joint that allows for full rotations in a plane; moreover, we assume that all these planes are parallel (Figure 1.7). The configurations of such a robot arm correspond one-to-one to the n angles at the n joints. Thus, we may model this configuration space as $(S^1)^{\times n}$.

1.3 Homotopy invariance

One of the main goals of algebraic topology is to study the homotopy equivalence problem

> Classify topological spaces up to homotopy equivalence!

As in the case of the homeomorphism problem, also this problem is not solvable in full generality (Outlook 2.8.5). However, the problem can be solved for many concrete examples, using suitable functors as invariants.

We will model the translation of topological problems into algebraic problems via the language of categories and functors. More precisely, mathematical theories will be modelled as categories, translations as functors, and the comparison between different translations by natural transformations. Basic terminology from category is reviewed in Appendix A.2.

Homotopy invariant functors are functors that map homotopic maps to the same morphisms. Equivalently, this can be formulated in terms of the homotopy category.

Definition 1.3.1 (homotopy invariant functor)**.** Let T be a category with a notion of homotopy (e.g., Top) and let C be a category. A functor $F\colon T \longrightarrow C$ is *homotopy invariant* if the following holds: For all $X, Y \in \operatorname{Ob}(T)$ and all $f, g \in \operatorname{Mor}_T(X, Y)$ with $f \simeq g$, we have

$$F(f) = F(g).$$

Proposition 1.3.2 (homotopy invariant functors yield homotopy invariants)**.** *Let T be a category with a notion of homotopy (e.g.,* Top*), let C be a category, and let $F\colon T \longrightarrow C$ be a homotopy invariant functor. Let X, $Y \in \operatorname{Ob}(T)$. Then the following hold:*

1. *If $X \simeq Y$, then $F(X) \cong_C F(Y)$.*
2. *If $F(X) \not\cong_C F(Y)$, then $X \not\simeq Y$.*

Proof. It suffices to prove the first part. Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ be mutually homotopy inverse homotopy equivalences. Then the induced morphisms $F(f)\colon F(X) \longrightarrow F(Y)$ and $F(g)\colon F(Y) \longrightarrow F(X)$ are inverse isomorphisms in C, because

$$\begin{aligned} F(g) \circ F(f) &= F(g \circ f) && (F \text{ is a functor}) \\ &= F(\operatorname{id}_X) && (g \circ f \simeq \operatorname{id}_X \text{ and } F \text{ is homotopy invariant}) \\ &= \operatorname{id}_{F(X)} && (F \text{ is a functor}) \end{aligned}$$

and similarly $F(f) \circ F(g) = \operatorname{id}_{F(Y)}$. □

Caveat 1.3.3. In general, in the situation of Proposition 1.3.2, from $F(X) \cong_C F(Y)$ we *cannot* conclude that $X \simeq Y$!

The homotopy category of a category such as Top is the category obtained by identifying maps in Top that are homotopic. Proposition 1.1.7 shows that the corresponding homotopy category indeed is well-defined (check!).

Definition 1.3.4 (homotopy category of topological spaces)**.** The *homotopy category of topological spaces* is the category Top_h consisting of:

- objects: Let $\operatorname{Ob}(\mathsf{Top}_\mathsf{h}) := \operatorname{Ob}(\mathsf{Top})$.
- morphisms: For all topological spaces X, Y, we set

  $$[X, Y] := \operatorname{Mor}_{\mathsf{Top}_\mathsf{h}}(X, Y) := \operatorname{map}(X, Y) \,/\, {\simeq}.$$

  Homotopy classes of maps will be denoted by "$[\,\cdot\,]$".
- compositions: The compositions of morphisms are defined by ordinary composition of representatives.

Similarly, one can define homotopy categories for all categories with a notion of homotopy that satisfies the analogue of Proposition 1.1.7.

Remark 1.3.5 (homotopy invariance via the homotopy category)**.** Topological spaces are homotopy equivalent if and only if they are isomorphic in the category $\mathsf{Top_h}$ (check!). Moreover, if C is a category, then a functor $F\colon \mathsf{Top} \longrightarrow C$ is homotopy invariant if and only if it factors over the *homotopy classes functor* $\mathsf{Top} \longrightarrow \mathsf{Top_h}$, which is the identity on objects and maps each continuous map to its homotopy class (check!). The advantage of this reformulation is that we can now apply the usual results on categories and functors.

Example 1.3.6 (trivial homotopy invariant functors)**.** There are two trivial examples of homotopy invariant functors:

- *Constant functors.* Let C be a category. Then every constant functor $F\colon \mathsf{Top} \longrightarrow C$ (mapping all topological spaces to the same object and all continuous maps to the identity morphism of this object) is homotopy invariant. By construction, we have

  $$\forall_{X,Y \in \operatorname{Ob}(\mathsf{Top})} \quad F(X) \cong_C F(Y).$$

  This functor is easy to compute, but does not contain any information on homotopy theory.

- *Homotopy classes functor.* The homotopy classes functor $H\colon \mathsf{Top} \longrightarrow \mathsf{Top_h}$ is homotopy invariant. By construction, we have

  $$\forall_{X,Y \in \operatorname{Ob}(\mathsf{Top})} \quad H(X) \cong_{\mathsf{Top_h}} H(Y) \Longleftrightarrow X \simeq Y,$$

  i.e., H contains perfect information on the classification of topological spaces up to homotopy equivalence. However, H is not easy to compute (in fact, it is not computable in any reasonable sense; Outlook 2.8.5).

The major design problem of algebraic topology is to find homotopy invariant functors that strike a balance between computability and preservation of homotopy-theoretic information. In Chapter 3, we will construct one such example: simplicial homology.

Literature exercise. Read about the origin of the notion of homotopy and homotopy invariance [24, p. 43].

2

Simplicial complexes

Graphs are combinatorial structures that can be used to model connectivity of various kinds, e.g., connections between people in social networks, genetic proximity in biology, or dependencies between software components. Simplicial complexes are a higher-dimensional generalisation of graphs and thus allow for more fine-grained models, e.g., for discrete approximations of geometric shapes, connectivity of high-dimensional data, decentralised computations in sensor networks, configuration spaces for robots, or dependencies between agents in distributed systems.

We start with basic terminology from graph theory and then introduce the language of simplicial complexes. The connection to actual topology is given by the geometric realisation of simplicial complexes.

In addition, we illustrate these concepts in the three typical modelling situations: Consistency relations between entities, point clouds, approximation of topological spaces.

Overview of this chapter.

Running example. simplicial complexes from relations, nerves of open covers

2.1 Graphs

Graphs are combinatorial structures that model connections ("edges") between entities ("vertices"). We review basic terminology and examples from graph theory [23, 35]. In the following, we will focus on undirected, unlabeled, unweighted, simple graphs without loops.

Definition 2.1.1 (graph)**.** A *graph* is a pair $X = (V, E)$, consisting of a set V and a subset

$$E \subset V^{[2]} := \bigl\{\{v, w\} \bigm| v, w \in V,\ v \neq w\bigr\}$$

with $V \cap E = \emptyset$. The elements of V are called *vertices* of X, the elements of E are called *edges* of X. A graph X is *finite* if V (whence E) is finite.

A *subgraph* of a graph (V, E) is a graph (V', E') with $V' \subset V$ and $E' \subset E$.

Definition 2.1.2 (adjacent, neighbour, degree, incident)**.** Let $X = (V, E)$ be a graph and let $v \in V$.

- A vertex w is a *neighbour of* v *in* X (or *adjacent to* v *in* X) if $\{v, w\} \in E$.
- The number of neighbours in X of v is the *degree* $\deg_X v$ of v.
- An edge in X is *incident to* v *in* X if it contains v.

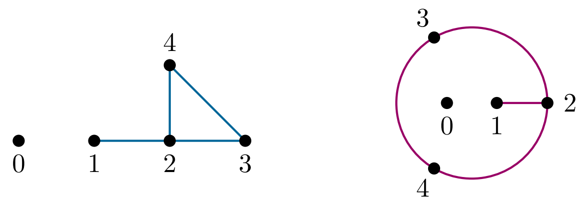


Figure 2.1.: The graph from Example 2.1.3, schematically; both illustrations represent the same graph.

Example 2.1.3 (a small graph)**.** Let

$$\begin{aligned} V &:= \{0, \dots, 4\}, \\ E &:= \bigl\{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\bigr\}. \end{aligned}$$

Then (V, E) is a graph. We can illustrate this graph (V, E) as in Figure 2.1. Both illustrations represent the same graph: Graphs are combinatorial structures and it only matters which vertices exist and which vertices are connected; it is *not* relevant *how* these vertices/connections are drawn.

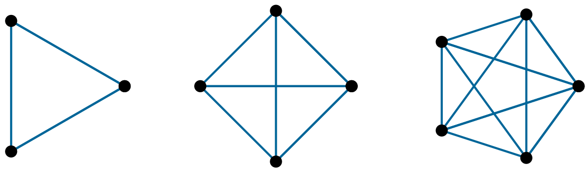


Figure 2.2.: The complete graphs K_3, K_4, K_5, schematically

Example 2.1.4 (complete graph)**.** A graph $X = (V, E)$ is *complete* if $E = V^{[2]}$. For $n \in \mathbb{N}$, we write

$$K_n := \bigl(\{0, \dots, n\}, \{\{j, k\} \mid j, k \in \{0, \dots, n\},\ j \neq k\}\bigr)$$

for the complete graph on $\{0, \dots, n\}$ (Figure 2.2).

Example 2.1.5 (Cayley graphs)**.** Let G be a group and let $S \subset G$ be a generating set. The *Cayley graph of G with respect to S* is defined by

$$\mathrm{Cay}(G, S) := \bigl(G, \{\{g, g \cdot s\} \mid g \in G,\ s \in S \cup S^{-1} \setminus \{e\}\}\bigr).$$

Cayley graphs with respect to finite generating sets are one of the central objects of study in geometric group theory [48]. All Cayley graphs are *regular*, i.e., every vertex has the same degree. Indeed, Cayley graphs lead to interesting examples of constructions of regular graphs.

Proposition 2.1.6 (handshake lemma)**.** *Let $X = (V, E)$ be a finite graph. Then*

$$\sum_{v \in V} \deg_X v = 2 \cdot \#E.$$

Proof. Because every edge connects exactly two vertices, on the left-hand side we count each edge exactly twice. □

Example 2.1.7 (social graphs)**.** Let V be a set of people. The following edge sets define graphs with vertex set V:

$$\begin{aligned} E_1 := \bigl\{\{x, y\} \bigm| {} & x, y \in V,\ x \neq y, \\ & \text{and } x \text{ and } y \text{ have shaken hands}\bigr\}, \\ E_1 := \bigl\{\{x, y\} \bigm| {} & x, y \in V,\ x \neq y, \\ & \text{and } x \text{ and } y \text{ are connected on your-favourite-social-network}\bigr\}, \\ E_2 := \bigl\{\{x, y\} \bigm| {} & x, y \in V,\ x \neq y, \\ & \text{and } x \text{ and } y \text{ have an increased risked of transmitting} \\ & \text{your-favourite-disease between each other}\bigr\}. \end{aligned}$$

From the handshake lemma (Proposition 2.1.6) we can conclude for instance: The number of people that are connected to an odd number of people on your-favourite-social-network is even, because the overall sum of these numbers is even. This probably does not count as the most useful consequence of graph theory.

Definition 2.1.8 (path, cycle, connected). Let $X = (V, E)$ be a graph and let $n \in \mathbb{N}$.

- A *path in* X *of length* n is a sequence $v_0, \dots, v_n$ of vertices $v_0, \dots, v_n \in V$ with the property that $\{v_j, v_{j+1}\} \in E$ for all $j \in \{0, \dots, n-1\}$. We say that this path *connects* v_0 and v_n. A path is *reduced* if the vertices in the sequence are all different.
- A *cycle in* X *of length* n is a path $v_0, \dots, v_n$ with $v_0 = v_n$. Such a cycle is *reduced* if the vertices $v_0, \dots, v_{n-1}$ are all different.
- The graph X is *connected* if every pair of its vertices can be connected by a path in X.

Example 2.1.9 (connected graphs).

- All complete graphs are connected.
- All Cayley graphs are connected (check!).
- The graph from Example 2.1.3 is *not* connected.
- The goal of social distancing measures is to keep graphs of increased risk of transmission as disconnected as possible.
- The goal of network architecture is to keep network graphs as connected as possible even if “few” edges are removed.

For the sake of completeness, we introduce the following (ad hoc) notion of graph isomorphism; a more conceptual discussion of morphisms and isomorphisms will be given in the context of simplicial complexes (Chapter 2.3.3).

Definition 2.1.10 (graph isomorphism). Let $X = (V, E)$ and $X' = (V', E')$ be graphs. The graphs X and X' are *isomorphic* if there is a *graph isomorphism* between X and X', i.e., a bijection $f \colon V \longrightarrow V'$ such that

$$\forall_{v,w \in V} \quad \{v, w\} \in E \iff \bigl\{f(v), f(w)\bigr\} \in E'.$$

2.2 Application: The seven bridges of Königsberg

As a sample application, we consider Euler’s problem on the seven bridges of Königsberg [30].

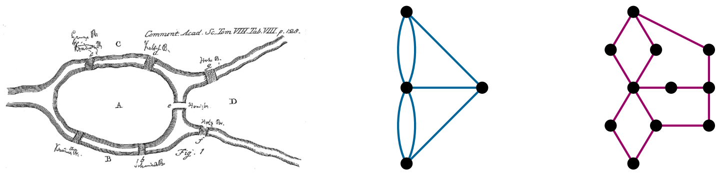


Figure 2.3.: The seven bridges of Königsberg: Euler's map [30]; a corresponding multigraph; a corresponding graph

Real-world problem 2.2.1 (the seven bridges of Königsberg)**.** Is it possible to take a round-trip walk through the city of Königsberg (of ∼1735; see the map in Figure 2.3) that crosses each of the seven bridges exactly once?

Of course, swimming, teleporting, walking around the source of Pregel, etc. are *not* allowed.

This situation can be modelled as follows:

Model 2.2.2 (the seven bridges of Königsberg)**.** We model Problem 2.2.1 by the following graph:

- Vertices: The two river banks of Pregel and the two Pregel islands are vertices of the graph. Moreover, we add vertices for each of the seven bridges.
- Edges: We connect the vertex corresponding to a bank/island by an edge to the vertex corresponding to a bridge if and only if one end of the bridge is on this bank/island. This leads to the graph in Figure 2.3.

Explanation. We model the situation by a graph as the only information that is relevant to this problem is which parts of the city are connected by which bridges; exact locations, distances, etc. are not relevant to this problem. We included vertices for the bridges in order to obtain a simple graph and not a multigraph.

- A valid round-trip walk is now a cycle in this graph that contains every edge exactly once.

  Explanation. The edges correspond to (half)bridges.

Definition 2.2.3 (Eulerian cycle/path)**.** Let $X = (V, E)$ be a graph.

- A *partial Eulerian path* in X is a path in X all of whose edges are different. A partial Eulerian path in X is an *Eulerian path* in X if it contains *all* edges of X (whence exactly once).
- A *partial Eulerian cycle* in X is a cycle in X all of whose edges are different. A partial Eulerian cycle in X is an *Eulerian cycle* in X if it contains *all* edges of X (whence exactly once).

Thus, Problem 2.2.1 translates into the following problem:

Question 2.2.4. Does the graph from Model 2.2.2 admit an Eulerian cycle? Given a finite graph X, (how) can we decide whether X admits an Eulerian cycle or not?

The answer is quite simple in modern terminology. Indeed, Euler's analysis of this bridge problem is the origin of graph theory.

Theorem 2.2.5 (characterisation of existence of Eulerian cycles [30]). *Let $X = (V, E)$ be a finite connected non-empty graph. Then X admits an Eulerian cycle if and only if for each vertex $v \in V$ the degree $\deg_X v$ is even.*

The key observation behind the proof are the following two parity properties:

Lemma 2.2.6 (partial Euler cycles have even degrees). *Let X be a connected graph, let $c = v_0, \dots, v_k$ be a partial Eulerian cycle in X, and let*

$$X[c] := \big(\{v_0, \dots, v_k\}, \{\{v_0, v_1\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v_0\}\}\big)$$

be the subgraph of X generated by c. Then all vertex degrees in $X[c]$ are even.

Proof. Whenever c passes through a vertex v, it needs to enter the vertex v through an edge and leave this vertex v through an edge. As c is a partial Eulerian cycle, all these edges are different. Therefore, c contains an even number of edges incident to v. In other words, $\deg_{X[c]} v$ is even. □

Lemma 2.2.7 (even degrees lead to partial Euler cycles). *Let $X = (V, E)$ be a finite graph all of whose vertices have even degree. Let $v \in V$ with $\deg_X v > 0$. Then there exists a partial Eulerian cycle starting at v of non-zero length.*

Proof. We inductively construct a partial Eulerian cycle c starting at v. Let $v_0 := v$. Because $\deg_X v_0 = \deg_X v > 0$, there exists an edge $\{v_0, v_1\}$ in X. The path v_0, v_1 is a partial Eulerian path.

Inductively, we assume that we have already constructed a partial Eulerian path $v_0, \dots, v_k$ in X with $k \geq 1$. We distinguish two cases:

① If $v_k = v_0$, then $k \neq 0$ and $v_0, \dots, v_{k-1}$ is a partial Eulerian cycle in X of non-zero length.

② If $v_k \neq v_0$, then the number of edges incident to v_k that appears in $v_0, \dots, v_k$ is odd (by the same entering-leaving argument as in the proof of Lemma 2.2.6). As $\deg_X v_k$ is even, there must be an edge $\{v_k, v_{k+1}\}$ incident to v_k that does not appear in $v_0, \dots, v_k$. Therefore, $v_0, \dots, v_{k+1}$ is a partial Eulerian path.

As X is finite, we will reach case ① in a finite number of steps and thus obtain the desired partial Eulerian cycle of non-zero length. □

Moreover, we record the following consequence of connectedness:

Lemma 2.2.8 (subgraphs and connectedness)**.** *Let X be a finite connected graph. Let $X' = (V', E')$ be a subgraph of X with*

$$\forall_{v \in V'} \quad \deg_{X'} v = \deg_X v.$$

Then $V' = \emptyset$ or $X' = X$.

Proof. This statement admits a straightforward proof by contradiction (Exercise). □

Proof of Theorem 2.2.5. If X admits an Eulerian cycle, then all vertices of X have even degree by Lemma 2.2.6.

Conversely, let all vertex degrees of X be even. We show that X admits an Eulerian cycle: Let c be a partial Eulerian cycle in X of maximal length (possibly 0) and let $X[c]$ be the subgraph of X generated by c.

Assume for a contradiction that there exists a vertex v of c with $\deg_{X[c]} v < \deg_X v$. Let X' be the subgraph of X with vertex set V that contains all edges of X that are *not* in $X[c]$. Then all vertices in X' have even degree (as differences of the even degrees in X and $X[c]$; Lemma 2.2.6).

By construction, $\deg_{X'} v > 0$. Hence, X' contains a partial Eulerian cycle c' of non-zero length that starts at v (Lemma 2.2.7). Splicing c' into c produces a partial Eulerian cycle in X that is longer than c. This contradicts the maximality of c.

Hence, this case cannot occur and $\deg_{X[c]} v = \deg_X v$ holds for all vertices v of the subgraph $X[c]$ of X generated by c. Because X is connected and $X[c]$ is non-empty (c contains at least one vertex), we obtain that $X[c] = X$ (Lemma 2.2.8). In other words, c is an Eulerian cycle. □

Example 2.2.9 (the seven bridges of Königsberg)**.** The graph corresponding to the original problem of the seven bridges of Königsberg (Model 2.2.2) contains vertices of odd degree (in fact, there are four vertices of odd degree). Therefore, Theorem 2.2.5 shows that this graph admits no Eulerian cycle. Hence, there is no round-trip walk through the city of Königsberg that crosses each of the seven bridges exactly once.

Caveat 2.2.10 (the travelling salesman problem and Hamiltonian cycles)**.** The "dual" problem of

> Decide whether a given finite graph admits a cycle that visits every vertex exactly once (a so-called *Hamiltonian cycle*) or not!

does not seem to have a similarly simple and algorithmically efficient solution; indeed, this problem, which for obvious reasons is also known as the *travelling salesman problem*, is an NP-complete problem [19].

Outlook 2.2.11 (DNA reconstruction). A typical problem in bioinformatics is to reconstruct DNA strings from partial sequential reads of DNA strings. This problem has various different aspects, including the necessity of error correction. One of the aspects is the assembly of partial reads along common suffixes/prefixes. This problem is usally encoded as a graph-theoretic problem. One approach uses so-called de Bruijn graphs and makes the reconstruction problem an instance of the problem of finding Eulerian paths [62, 51].

2.3 Simplicial complexes and simplicial maps

We turn to the higher-dimensional setting and introduce simplicial complexes, simplicial maps, and basic constructions on them. Simplicial complexes are purely combinatorial and thus easily formalised in proof assistants [49].

2.3.1 Geometric idea

Graphs consist of vertices and edges between vertices, where the edges are modelled as two-element sets of vertices. Simplicial complexes are a higher-dimensional version of graphs (Figure 2.4):

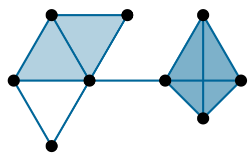


Figure 2.4.: A simplicial complex, schematically

A simplicial complex is a set of simplices, where the simplices are combinatorial versions of vertices, edges, triangles, tetrahedra, ..., i.e., of the geometric standard simplices

$$\Delta^n := \{t \in \mathbb{R}^{n+1}_{\geq 0} \mid t_0 + \cdots + t_n = 1\}.$$

As in the case of graphs, the combinatorics of such a simplicial complex is captured by the information of which vertices span a common simplex. We will thus model simplices by finite sets ("the sets of vertices that span a simplex"). As faces of goemetric simplices are simplices, we require simplicial complexes to be closed under taking subsets.

2.3.2 Simplicial complexes

We give the formal definition of simplicial complexes and some related notions such as finiteness and the dimension. Moreover, we consider basic examples. Application-oriented examples will follow in subsequent sections.

Definition 2.3.1 (simplicial complex)**.** A *simplicial complex* is a set X of finite sets that is closed under taking subsets:

$$\forall_{\sigma \in X} \quad \forall_{\tau \subset \sigma} \quad \tau \in X.$$

The elements of X are called *simplices of* X. The elements of

$$V(X) := \bigcup X = \{x \mid \exists_{\sigma \in X} \quad x \in \sigma\}$$

are called *vertices of* X.

A *subcomplex* of a simplicial complex X is a simplicial complex X' with $X' \subset X$.

Example 2.3.2 (a small simplicial complex)**.** The simplicial complex in Figure 2.4 could be formalised as (check!)

$$\begin{aligned}
&\{\emptyset,\\
&\ \{0\}, \dots, \{8\},\\
&\ \{0,1\}, \{0,2\}, \{0,4\}, \{1,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\},\\
&\quad \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{6,8\}, \{7,8\},\\
&\ \{0,2,4\}, \{2,3,4\}, \{5,6,7\}, \{5,6,8\}, \{5,7,8\}, \{6,7,8\}\\
&\ \{5,6,7,8\}\}.
\end{aligned}$$

Definition 2.3.3 (dimension)**.** Let X be a simplicial complex.

- Let $\sigma \in X$. Then the *dimension of* σ is defined as (where $\#\sigma$ denotes the cardinality of σ)

  $$\dim \sigma := \#\sigma - 1 \in \mathbb{N} \cup \{-1\}.$$

  If $\dim \sigma = n$, then we also say that σ is an *n-simplex of* X.

- For $n \in \mathbb{N}$, we denote the set of n-simplices of X by

  $$X(n) := \{\sigma \in X \mid \dim \sigma = n\}.$$

- The *dimension of* X is defined as (with $\sup \emptyset := -1$)

  $$\dim X := \sup\{\dim \sigma \mid \sigma \in X\} \in \mathbb{N} \cup \{-1, \infty\}.$$

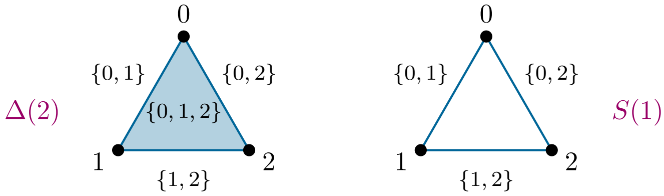


Figure 2.5.: The simplicial standard 2-simplex and the simplicial 1-sphere, respectively

Remark 2.3.4 (vertices and 0-simplices)**.** Let X be a simplicial complex. Then (check!)

$$V(X) = \{x \mid \{x\} \in X(0)\}.$$

Example 2.3.5 (empty)**.** The empty set is a simplicial complex, which has dimension -1.

Example 2.3.6 (standard simplex)**.** If V is a finite set, then the power set $P(V)$ is a simplicial complex, the *simplex spanned by* V. We have $\dim P(V) = \#V - 1$.

For $n \in \mathbb{N}$, we call $\Delta(n) := P(\{0, \dots, n\})$ the *(simplicial) standard n-simplex* (Figure 2.5). By construction, $\dim \Delta(n) = n$. This simplicial complex $\Delta(n)$ can be viewed as a combinatorial model of the affine standard n-simplex $\Delta^n = \{t \in \mathbb{R}^{n+1}_{\geq 0} \mid t_0 + \dots + t_n = 1\}$.

Example 2.3.7 (simplicial sphere)**.** Let V be a finite set with $\#V \geq 2$. Then $S(V) := P(V) \setminus \{V\}$ is a simplicial complex (check!) of dimension $\#V - 2$ (check!).

For $n \in \mathbb{N}$, we call $S(n) := S(\{0, \dots, n+1\})$ the *simplicial n-sphere* (Figure 2.5). In particular, $\dim S(n) = n$.

Example 2.3.8 (the real line)**.** The set

$$\{\{k, k+1\} \mid k \in \mathbb{Z}\} \cup \{\{k\} \mid k \in \mathbb{Z}\} \cup \{\emptyset\}$$

is a simplicial complex (check!) of dimension 1 (check!). It can be viewed as a combinatorial model of the real line. Similarly, for $n \in \mathbb{N}$, we can consider the simplicial complex (check!)

$$[0, n]_\Delta := \{\{k, k+1\} \mid k \in \{0, \dots, n-1\}\} \cup \{\{k\} \mid k \in \{0, \dots, n\}\} \cup \{\emptyset\},$$

which is a combinatorial model of the interval $[0, n]$.

Example 2.3.9 (graphs as one-dimensional simplicial complexes)**.** If (V, E) is a graph, then

$$E \cup V \cup \{\emptyset\}$$

is a simplicial complex of dimension at most 1 (check!), which represents the same connectivity information on V. Conversely, if X is a simplicial complex with $\dim X \leq 1$, then

$$\bigl(V(X), X(1)\bigr)$$

is a graph (check!), which represents the same connectivity information as X on $V(X)$.

Proposition and Definition 2.3.10 (finiteness of simplicial complexes). *Let X be a simplicial complex. Then the following are equivalent:*

1. *The set $V(X)$ is finite.*
2. *The set X is finite.*

In this case, we call X a finite *simplicial complex.*

Proof. Let $V(X)$ be finite. Then also $P_{\text{fin}}(V(X))$ is finite. Moreover, by definition, $X \subset P_{\text{fin}}(V(X))$. Therefore, also X is finite.

Conversely, let X be a finite set. Then $V(X) = \bigcup X$ is also finite. □

Remark 2.3.11 (finiteness and finite-dimensionality). Every finite simplicial complex is finite-dimensional. However, there exist simplicial complexes that are finite-dimensional even though they are not finite (Example 2.3.8).

In this course, we will mainly be concerned with finite simplicial complexes.

We can also describe simplicial complexes by specifying a set of finite sets and then taking the set of all subsets of these sets:

Definition 2.3.12 (generated simplicial complex). Let S be a set of finite sets. We then write

$$\langle S \rangle_\Delta := \bigl\{ \tau \bigm| \exists_{\sigma \in S} \quad \tau \subset \sigma \bigr\}$$

for the *simplicial complex generated by S*.

If S is a set of finite sets, then $\langle S \rangle$ indeed is a simplicial complex (check!). Moreover, if S is finite, then $\langle S \rangle$ is a finite simplicial complex (check!).

Example 2.3.13 (a small simplicial complex, again). The simplicial complex from Example 2.3.2 could alternatively and more efficiently be described as (check!)

$$\bigl\langle \{0,1\}, \{1,2\}, \{2,5\}\{0,2,4\}, \{2,3,4\}, \{5,6,7,8\} \bigr\rangle_\Delta.$$

2.3.3 Simplicial maps

Simplicial maps are structure-preserving maps between simplicial complexes. More precisely, simplicial maps are maps between the sets of vertices that map simplices to simplices (possibly of different dimension).

Definition 2.3.14 (simplicial map)**.** Let X and Y be simplicial complexes. A *simplicial map* $X \longrightarrow Y$ is a map $f\colon V(X) \longrightarrow V(Y)$ with

$$\forall_{\sigma \in X} \quad f(\sigma) \in Y.$$

Here, "$f(\sigma)$" denotes the image $\{f(x) \mid x \in \sigma\}$ of σ under f.

Proposition 2.3.15 (monotonicity of dimension)**.** *Let X and Y be simplicial complexes, let $f\colon X \longrightarrow Y$ be a simplicial map, and let $\sigma \in X$. Then*

$$\dim f(\sigma) \leq \dim(\sigma).$$

Proof. By definition, we have

$$\dim f(\sigma) = \#f(\sigma) - 1 \leq \#\sigma - 1 = \dim \sigma. \qquad \square$$

Example 2.3.16 (identity map)**.** Let X be a simplicial complex. Then

$$\begin{aligned} V(X) &\longrightarrow V(X) \\ x &\longmapsto x \end{aligned}$$

is a simplicial map, which we will denote by id_X. More generally, if X' is a subcomplex of X, then the inclusion $V(X') \hookrightarrow V(X)$ is a simplicial map.

Example 2.3.17 (constant maps)**.** Let X and Y be simplicial complexes and let $y \in V(Y)$. Then the constant map

$$\begin{aligned} V(X) &\longrightarrow V(Y) \\ x &\longmapsto y \end{aligned}$$

is a simplicial map (Exercise).

Example 2.3.18 (a non-simplicial map)**.** The identity map $\{0,1\} \longrightarrow \{0,1\}$ is *not* a simplicial map $\Delta(1) \longrightarrow S(0)$ (the simplex $\{0,1\}$ of $\Delta(1)$ is not mapped to a simplex of $S(0)$). This can be viewed as an instance of a combinatorial version of the intermediate value theorem.

Proposition 2.3.19 (composition of simplicial maps)**.** *Let X, Y, Z be simplicial complexes and let $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ be simplicial maps. Then the composition $h := g \circ f\colon V(X) \longrightarrow V(Z)$ of the underlying maps $f\colon V(X) \longrightarrow V(Y)$ and $g\colon V(Y) \longrightarrow V(Z)$ (by abuse of notation denoted by the same letters) is a simplicial map $X \longrightarrow Z$. This simplicial map will be denoted by $g \circ f$.*

Proof. Let $\sigma \in X$. Then

$$h(\sigma) = (g \circ f)(\sigma) = g\bigl(f(\sigma)\bigr).$$

Because f is simplicial and σ is a simplex of X, we know that $f(\sigma) \in Y$. Because g simplicial, we hence obtain $h(\sigma) = g(f(\sigma)) \in Z$. $\square$

Hence, we can organise simplicial complexes into a category (check!):

Definition 2.3.20 (the category of simplicial complexes)**.** The category SC of simplicial complexes is the category consisting of:

- objects: Let $\mathrm{Ob}(\mathsf{SC})$ be the class(!) of all simplicial complexes.
- morphisms: If X and Y are simplicial complexes, we define $\mathrm{Mor}_{\mathsf{SC}}(X,Y)$ as the set of all simplicial maps $X \longrightarrow Y$.

  We write $\mathrm{map}_\Delta(X,Y) := \mathrm{Mor}_{\mathsf{SC}}(X,Y)$.
- compositions: The compositions of morphisms are defined as in Proposition 2.3.19.

In particular, we obtain a corresponding notion of isomorphism.

Definition 2.3.21 (simplicial isomorphism)**.** Let X and Y be simplicial complexes. Isomorphisms in the category SC are called *simplicial isomorphisms*. More explicitly: A simplicial map $f\colon X \longrightarrow Y$ is a simplicial isomorphism if there exists a simplicial map $g\colon Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

For simplicial complexes of dimension at most 1, this notion of isomorphism is the same as the notion of isomorphism of graphs (under the transition specified in Example 2.3.9; check!).

Using the combinatorial intervals from Example 2.3.8, we obtain a notion of paths in simplicial complexes and thus a notion of connectedness:

Definition 2.3.22 (path, connected)**.** Let X be a simplicial complex.

- A *path in* X is a simplicial map $[0,n]_\Delta \longrightarrow X$ with $n \in \mathbb{N}$.
- The simplicial complex X is *connected* if for all vertices $x, y \in V(X)$, there exists an $n \in \mathbb{N}$ and a path $\gamma\colon [0,n]_\Delta \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(n) = y$.

Example 2.3.23. The simplicial complex $\Delta(1)$ is connected, but $S(0)$ is not.

For simplicial complexes of dimension at most 1, this notion of connectedness is the same as the notion of connectedness of graphs (under the transition specified in Example 2.3.9; check!). However, the notions of paths are not exactly the same.

2.3.4 Basic constructions

We consider basic constructions of simplicial complexes and their categorical properties: unions, intersections, and one version of products.

Definition 2.3.24 (union of simplicial complexes)**.** Let X and Y be simplicial complexes. Then $X \cup Y$ is a simplicial complex (check!), called the *union of X and Y*.

Definition 2.3.25 (intersection of simplicial complexes)**.** Let X and Y be simplicial complexes. Then $X \cap Y$ is a simplicial complex (check!), called the *intersection of X and Y*.

Remark 2.3.26 (unions/intersections lead to pushouts)**.** Let X and Y be simplicial complexes. Then $X \cap Y$ is a subcomplex of X and Y, respectively, and both X and Y are subcomplexes of $X \cup Y$. The inclusions of subcomplexes form a pushout diagram in the category SC (Exercise) (and also a pullback):

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup Y \end{array}$$

The category of simplicial complexes is complete and cocomplete; inverse limits and colimits are recalled in Appendix A.2.4. However, because simplicial complexes are very rigid, already in the case of pushouts and products, the corresponding limiting objects might look different than what one expects from geometry.

Proposition 2.3.27 (inverse limits and colimits in SC)**.** *The category* SC *contains all small colimits and all small inverse limits. More precisely:*

1. Inverse limits. *Let $F\colon I \longrightarrow \mathsf{SC}$ be a small diagram in SC. Let $Y := \varprojlim_{i\in I} V(F(i))$ be the inverse limit in Set of the underlying vertex sets, with corresponding structure maps $(p_i\colon Y \to V(F(i)))_{i\in I}$. Then*
$$X := \bigl\{\sigma \in P_{\text{fin}}(Y) \bigm| \forall_{i\in I} \quad p_i(\sigma) \in F(i)\bigr\}$$
is a simplicial complex with $V(X) = Y$. Moreover, X together with the $(p_i)_{i\in I}$ is an inverse limit of F in SC.

2. Colimits. *Let $F\colon I \longrightarrow \mathsf{SC}$ be a small diagram in SC. Let $Y := \varinjlim_{i\in I} V(F(i))$ be the colimit in Set of the underlying vertex sets, with corresponding structure maps $(j_i\colon V(F(i)) \to Y)$. Then*
$$X := \bigl\{\sigma \in P_{\text{fin}}(Y) \bigm| \exists_{i\in I} \quad \exists_{\tau\in F(i)} \quad j_i(\tau) = \sigma\bigr\}$$
is a simplicial complex with $V(X) = Y$. Moreover, X together with the $(j_i)_{i\in I}$ is a colimit of F in SC.

Proof. We give the proof for the case of inverse limits; the proof for colimits is similar (check!).

Because each $F(i)$ is closed under taking subsets, also X is closed under taking subsets and thus a simplicial complex. Morever, by construction, $V(X) = Y$ (check!).

We show that X together with the $(p_i)_{i\in I}$ has the claimed universal property: First of all, by construction of X, each p_i is a simplicial map $X \longrightarrow F(i)$. Because $(Y,(p_i)_{i\in I})$ is an inverse limit for $V \circ F\colon I \longrightarrow \mathsf{Set}$, we know that $(X,(p_i)_{i\in I})$ is a cone over F in SC.

Let $(Z,(f_i)_{i\in I})$ be a cone over F in SC; in particular, $(V(Z),(f_i)_{i\in I})$ is a cone over $V \circ F$ in Set. By the universal property of the inverse limit in Set, we obtain that there is a unique map $f\colon V(Z) \longrightarrow Y$ with

$$\forall_{i\in I} \quad p_i \circ f = f_i.$$

It thus suffices to show that f is a simplicial map $Z \longrightarrow X$. This is clear by construction: Let $\sigma \in Z$. Then, we have $f_i(\sigma) \in F(i)$ for all $i \in I$, because each f_i is a simplicial map. We hence obtain

$$\forall_{i\in I} \quad p_i\bigl(f(\sigma)\bigr) = f_i(\sigma) \in F(i).$$

By definition of X, this means that $f(\sigma) \in X$. □

In particular, the category SC contains all binary products and all pushouts.

Definition 2.3.28 (simplicial product of simplicial complexes)**.** Let X and Y be simplicial complexes. The simplicial complex (check!)

$$X \boxtimes Y := \bigl\{\sigma \in P_{\text{fin}}(V(X) \times V(Y)) \bigm| p_1(\sigma) \in X,\ p_2(\sigma) \in Y\bigr\}$$

is called the *simplicial product of X and Y*. Here, p_1 and p_2 denote the projections from $V(X) \times V(Y)$ to the first and second factor, respectively.

Caveat 2.3.29 (products of simplicial complexes)**.** By the concrete description of inverse limits in Proposition 2.3.27, the simplicial product of simplicial complexes (together with the canonical projection maps) is "the" categorical product of two simplicial complexes in SC.

Geometrically, we would expect the product of $\Delta(1)$ with $\Delta(1)$ to be a "square", but a straightforward computation shows that (Exercise)

$$\Delta(1) \boxtimes \Delta(1) \cong_{\mathsf{SC}} \Delta(3).$$

Despite of this geometric effect, the simplicial product is a useful construction; for example, we can use products of the form $\,\cdot\, \boxtimes \Delta(1)$ to obtain a notion of homotopy between simplicial maps (Chapter 2.3.5).

Caveat 2.3.30 (pushouts of simplicial complexes)**.** A straightforward calculation shows that the diagram

$$\begin{array}{ccc} S(0) & \xrightarrow{\text{incl}} & \Delta(1) \\ {\scriptstyle\text{const}}\downarrow & & \downarrow{\scriptstyle\text{const}} \\ \langle\{0\}\rangle_\Delta & \xrightarrow[\text{id}]{} & \langle\{0\}\rangle_\Delta \end{array}$$

is a pushout diagram in SC (check!). However, geometrically, we would expect the pushout to describe a "circle" instead of a "point".

Outlook 2.3.31 (simplicial sets)**.** Simplicial sets [32] are a conceptually further evolved version of simplicial structures. The category of simplicial sets and simplicial maps is more flexible and reflects the category of topological spaces better than the rigid category of simplicial complexes. But for many applications simplicial complexes are slicker and can be used more directly.

2.3.5 Simplicial homotopy

Following the standard blueprint of defining "homotopic" via products with intervals and the inclusions at the endpoints of the intervals, we obtain a notion of homotopy in the category of simplicial complexes.

Definition 2.3.32 (simplicial homotopy)**.** Let X and Y be simplicial complexes and let $f, g\colon X \longrightarrow Y$ be simplicial maps. Then f is *simplicially homotopic* to g if there exists a simplicial homotopy from f to g.

A *simplicial homotopy* from f to g is a simplicial map $h\colon X \boxtimes \Delta(1) \longrightarrow Y$ with

$$h \circ i_0 = f \quad \text{and} \quad h \circ i_1 = g,$$

where $i_0, i_1\colon X \longrightarrow X \boxtimes \Delta(1)$ are the simplicial maps (check!) given by the inclusion into the 0- and 1-component, respectively.

If f and g are simplicially homotopic, we write $f \simeq_\Delta g$.

Example 2.3.33 (simplicial maps on $S(1)$)**.** The map $f\colon \{0,1,2\} \longrightarrow \{0,1,2\}$ given by the "projection"

$$0 \longmapsto 1, \quad 1 \longmapsto 1, \quad 2 \longmapsto 2$$

is a simplicial map $S(1) \longrightarrow S(1)$ (Figure 2.6). The maps f and the constant simplicial map $S(1) \longrightarrow S(1)$ with value 1 are simplicially homotopic: Indeed,

$$\begin{aligned} \{0,1,2\} \times \{0,1\} &\longmapsto \{0,1,2\} \\ (x,0) &\longmapsto f(x) \\ (x,1) &\longmapsto 1 \end{aligned}$$

defines a simplicial homotopy $S(1) \boxtimes \Delta(1) \longrightarrow S(1)$ from f to the constant map 1 (check!).

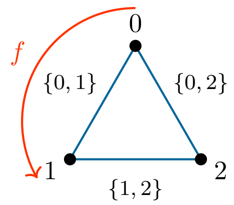


Figure 2.6.: The simplicial map from Example 2.3.33

The simplicial maps $f\colon S(1) \longrightarrow S(1)$ and the simplicial map $\mathrm{id}_{\{0,1,2\}}$ are *not* simplicially homotopic: *Assume* for a contradiction that there exists a simplicial homotopy $h\colon S(1) \boxtimes \Delta(1) \longrightarrow S(1)$ from f to $\mathrm{id}_{\{0,1,2\}}$. In particular, because of $\sigma := \{(0,2)\} \times \{0,1\} \in S(1) \boxtimes \Delta(1)$, we obtain $h(\sigma) \in S(1)$, but

$$\begin{aligned} h(\sigma) &= h\bigl(\{0,2\} \times \{0,1\}\bigr) = \bigl\{h(0,0), h(2,0), h(0,1), h(2,1)\bigr\} \\ &= \bigl\{f(0), g(2), \mathrm{id}_{\{0,1,2\}}(0), \mathrm{id}_{\{0,1,2\}}(2)\bigr\} \\ &= \{1,2,0\}, \end{aligned}$$

which is *not* a simplex of $S(1)$. Thus, $f \not\simeq_\Delta \mathrm{id}_{\{0,1,2\}}$.

In contrast with Top, simplicial homotopy is more rigid: Showing that simplicial maps are not simplicially homotopic can (on finite simplicial complexes) be done via a (finite) exhaustive search. Classically, being simplicially homotopic is expressed in the more concrete terms of contiguity:

Proposition 2.3.34 (contiguous vs. simplicially homotopic)**.** *Let X and Y be simplicial complexes and let $f, g\colon X \longrightarrow Y$ be simplicial maps. Then the following are equivalent:*

1. *We have $f \simeq_\Delta g$.*
2. *The maps f and g are* contiguous*, i.e.,*

$$\forall_{\sigma \in X} \quad f(\sigma) \cup g(\sigma) \in Y.$$

Proof. Ad $1 \Longrightarrow 2$. Let $f \simeq_\Delta g$ and let $h\colon X \boxtimes \Delta(1) \longrightarrow Y$ be a simplicial homotopy from f to g. We show that f and g are contiguous: Let $\sigma \in X$. Then $\sigma \times \{0,1\}$ is a simplex of $X \boxtimes \Delta(1)$ (check!) and

$$\begin{aligned} f(\sigma) \cup g(\sigma) &= h \circ i_0(\sigma) \cup h \circ i_1(\sigma) && (h \text{ is a simplicial homotopy from } f \text{ to } g) \\ &= h\bigl(\sigma \times \{0,1\}\bigr) && \text{(calculation)} \\ &\in Y. && (h \text{ is simplicial}) \end{aligned}$$

Hence, f and g are contiguous.

Ad 2 $\Longrightarrow$ 1. Conversely, let f and g be contiguous. We consider the map

$$\begin{aligned} h\colon V(X) \times \{0,1\} &\longrightarrow V(Y) \\ (x,0) &\longmapsto f(x) \\ (x,1) &\longmapsto g(x) \end{aligned}$$

and show that h is a simplicial map $X \boxtimes \{0,1\} \longrightarrow Y$: Let $\sigma \in X \boxtimes \{0,1\}$; in particular, $p_1(\sigma) \in X$. Hence, we obtain

$$\begin{aligned} h(\sigma) &= f\bigl(\{x \in V(X) \mid (x,0) \in \sigma\}\bigr) \cup g\bigl(\{x \in V(X) \mid (x,1) \in \sigma\}\bigr) && \text{(by definition of } h) \\ &\subset f\bigl(p_1(\sigma)\bigr) \cup g\bigl(p_1(\sigma)\bigr) && \text{(by definition of the projection } p_1) \\ &\in Y. && (p_1(\sigma) \in X \text{ and contiguity}) \end{aligned}$$

Therefore, h is a simplicial map. By construction, $h \circ i_0 = f$ and $h \circ i_1 = g$. So, h is a simplicial homotopy from f to g. □

Caveat 2.3.35 (simplicial homotopy as equivalence relation)**.** Let X and Y be simplicial complexes. The relation "$\simeq_\Delta$" on $\operatorname{map}_\Delta(X,Y)$ is reflexive and symmetric, but in general it is *not* transitive (Exercise). Thus, "$\simeq_\Delta$" in general does not define equivalence relations on the sets of simplicial maps.

We denote the transitive closure of "$\simeq_\Delta$" by "$\simeq_\Delta^*$". By construction, this is the smallest transitive relation that contains "$\simeq_\Delta$"; more concretely, $f,g \in \operatorname{map}_\Delta(X,Y)$ satisfy $f \simeq_\Delta^* g$ if and only if there exists an $n \in \mathbb{N}$ and $f_0,\dots,f_n \in \operatorname{map}_\Delta(X,Y)$ with

$$\forall_{j \in \{0,\dots,n-1\}} \quad f_j \simeq_\Delta f_{j+1}$$

and $f = f_0$ as well as $f_n = g$. Because "$\simeq_\Delta$" is reflexive and symmetric, the relation "$\simeq_\Delta^*$" on $\operatorname{map}_\Delta(X,Y)$ indeed is an equivalence relation.

Moreover, "$\simeq_\Delta$" (whence "$\simeq_\Delta^*$") is compatible with the composition of simplicial maps: For instance, this is immediate in the description via contiguity (check!). Therefore, one can also construct a homotopy category of SC by dividing the morphism set by "$\simeq_\Delta^*$".

2.4 Modelling: Consistency relations

Simplicial complexes can be used to model consistent states of several entities. We illustrate two versions of this technique. Further examples will arise in the analysis of consensus problems in distributed computing (Chapter 3.6). In all these examples it will become apparent how the high-dimensional nature of simplicial complexes captures multilateral consistency conditions more directly than graphs.

Real-world problem 2.4.1 (the incompatible food triad problem [36]). Are there three food ingredients with the following property: Any two of them taste good together, but all three of them together do not taste good?

This problem can be modelled as follows:

Model 2.4.2 (the incompatible food triad problem). We model Problem 2.4.1 by the following simplicial complex X: Let F be the set of all food items.

- Simplices: A (finite) subset of F is a simplex of X if and only if these ingredients together taste good (to a fixed observer).

Explanation. The construction of this simplicial complex makes a (possibly non-trivial?) assumption: If a set of ingredients tastes good together, then the same also holds for every subset of these ingredients.

Problem 2.4.1 then directly translates into the following problem: Does X contain a "hollow triangle", i.e., does X contain three vertices x, y, z such that

$$\{x, y\}, \{y, z\}, \{x, z\} \in X \quad \text{and} \quad \{x, y, z\} \notin X \text{ ?}$$

Even though the problem in this form looks like a feasible mathematical problem, some care is necessary: "taste good together" and "food ingredient" are not easily put into an objective framework and will depend on the chosen observer. Therefore, different observers might find different answers/-solutions. While this model does not help directly with solving the underlying real-world problem, it shows how such information can be represented as a simplicial complex.

Similarly, in music theory, one can consider the simplicial complex of consonant pitches (Exercise).

A more rigorous example comes from the (dual) management of preferences, e.g., used in the analysis of social choice problems (Chapter 3.7):

Real-world problem 2.4.3 (voting preferences). A set S of people wants to figure out a way to rank a set A of alternatives. To this end, every person in S can provide his own preferences concerning A (i.e., which alternatives they consider better than others).

The problem is then to aggregate these preferences into an order of A, usually under additional compatibility constraints (e.g., unanimity, non-dictatorship, independence of irrelevant alternatives).

Model 2.4.4 (voting preferences). We model the relation between the voting preferences in Problem 2.4.3 by the following simplicial complex: Let A be the set of available alternatives and let S be the set of people.

- Voting preferences: If $s \in S$, then we model the voting preferences of s as an order on A.

  Let P be the set of all orders on A that arise in this way from S.

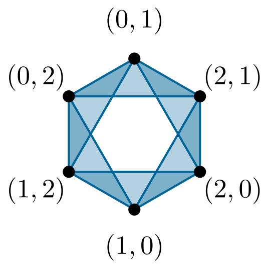


Figure 2.7.: Voting preferences for three alternatives

- Simplicial complex: We set

$$V := \{(x, y) \mid x, y \in A,\ x \neq y\}$$

and

$$X := \{\sigma \in P_{\text{fin}}(V) \mid \exists_{p \in P} \quad \forall_{(x,y) \in \sigma} \quad x >_p y\}.$$

Then X indeed is a simplicial complex (check!).

Explanation. We assume that the preference of each person in S is consistent in the sense that it defines an order on A. The simplices of X correspond to finite sets of binary comparisons that are consistent within P. We will see in Chapter 3.7 how finding aggregation maps of preferences corresponds to finding simplicial maps to X from a suitable other simplicial complex and how the constraints on aggregation maps translate into properties of simplicial maps to X.

Example 2.4.5 (preferences for three alternatives)**.** We spell out the simplicial complex X from Model 2.4.4 in a simple example: Let $A := \{0, 1, 2\}$, i.e., we consider three alternatives and let S be diverse enough to lead to P consisting of all orders on A. Then X is the "ring-shaped" simplicial complex in Figure 2.7. For instance, a partial order that witnesses the existence of the 2-simplex $\{(0, 1), (2, 1), (2, 0)\}$ is $2 > 0 > 1$.

2.5 Modelling: Complexes from point clouds

Topological data analysis applies topological tools to "big data". In this context, we convert "big data" into topological objects. Usually, this is a two-step process:

1. We first convert "big data" into simplicial complexes.

2. We then (convert these simplicial complexes into topological spaces and) apply homotopy invariants to these simplicial complexes/spaces.

In this section, we describe the first step in more detail. The second step is explained in Chapter 2.6 and Chapter 3.

To convert "big data" into simplicial complexes, we first need to agree how to model "big data". In applications, such "big data" usually arises as multi-parameter measurements or observations. It is customary to model such "big data" as finite subsets of $\mathbb{R}^n$ (called *point clouds*) and to endow $\mathbb{R}^n$ with a suitable metric (e.g., the Euclidean metric d_2 or the ℓ_1-metric d_1 or the ℓ_∞-metric d_∞); here, n corresponds to the number of parameters captured in the measurements/retrieval of the data.

Example 2.5.1 (point clouds from MRI images)**.** MRI images (magnetic resonance imaging) represent activation information for locations in the human brain. More precisely, the brain area is subdivided into a cubical grid (so-called *voxels*) and for each voxel the extent of activation is measured. If the scanner has a resolution of n voxels, then MRI images can be viewed as points in $\mathbb{R}^d$. A set of MRI images gives rise to a point cloud in $\mathbb{R}^n$. Such sets arise as sets of scans of multiple subjects or as sets of scans of a single subject at multiple times.

We can build simplicial complexes from such point clouds by "connecting the dots": Two points that have "small" distance should be connected by edges; three points that have "small" distance should span a triangle; ... Such constructions depend on the exact notion of "small" and "small distance". In many cases, there is no good a priori estimate for what "small" should mean. Therefore, one considers the system of simplicial complexes obtained by looking at all scales.

The most prominent examples of simplicial complexes from point clouds are the Čech and Rips complexes. Čech complexes are special cases of nerves of covers. While the Čech complexes are more accurate in a topological sense (Chapter 2.6), the Rips complexes are easier to compute.

Definition 2.5.2 (nerve of a cover)**.** Let X be a set and let $U := (U_i)_{i\in I}$ be a cover of X; i.e., for each $i \in I$, we have $U_i \subset X$ and $\bigcup_{i\in I} U_i = X$. The *nerve of* U is the simplicial complex (check!)

$$N(U) := \Big\{\sigma \in P_{\text{fin}}(I) \;\Big|\; \bigcap_{i\in\sigma} U_i \neq \emptyset\Big\}.$$

Example 2.5.3 (complex of preferences, as nerve)**.** The simplicial complex modelling voting preferences from Model 2.4.4 can also be understood as a nerve of the following cover:

Let A be a set and let P be a set of orders on A. For $x, y \in A$, we set

$$U^+_{x,y} := \{p \in P \mid x >_p y\} \quad \text{and} \quad U^-_{x,y} := \{p \in P \mid x <_p y\}.$$

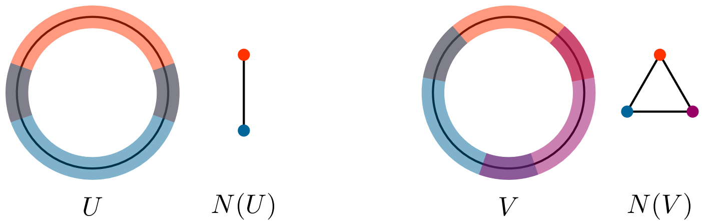


Figure 2.8.: Two covers of S^1 and their nerves

Then $(U^+_{x,y})_{x,y\in A, x\neq y}$ is a cover of P (check!) and the nerve of this cover is the simplicial complex X of consistent voting preferences from Model 2.4.4 (check!).

Example 2.5.4 (nerves of covers of the circle)**.** We consider the covers U and V depicted in Figure 2.8. Then (check!)

$$N(U) \cong_{\mathsf{SC}} \Delta(1) \quad \text{and} \quad N(V) \cong_{\mathsf{SC}} S(1).$$

Definition 2.5.5 (Čech complex)**.** Let (Y,d) be a metric space, let $X \subset Y$, and let $\varepsilon \in \mathbb{R}_{>0}$. The *Čech complex* $\check{C}_\varepsilon(X,Y,d)$ of X in (Y,d) with radius ε is the nerve of the open cover $(U_\varepsilon(x))_{x\in X}$ by all open ε-balls centered at points in X.

Remark 2.5.6 (extremal radii)**.** For finite point clouds in metric spaces, of course, we always obtain:

- If ε is small enough, then the Čech complex with radius ε is discrete (consisting of the points of the point cloud);
- if ε is large enough, then the Čech complex with radius ε is a full simplex (spanned by the point cloud).

Both of these extremal cases do not contain interesting topological information on the point cloud. We thus need techniques that allow us to study Čech complexes at all resolutions simultaneously. For example, one can look at persistent homology (Chapter 4).

Example 2.5.7 (Čech complexes)**.** Let $X := \{0,1\} \times \{0,1\} \subset \mathbb{R}^2$, equipped with the Euclidean metric d_2. For $\varepsilon \in \mathbb{R}_{>0}$, we then obtain (Figure 2.9)

$$\check{C}_\varepsilon(X,\mathbb{R}^2,d_2) = \begin{cases} \bigl\langle \{(0,0)\},\{(1,0)\},\{(0,1)\},\{(1,1)\} \bigr\rangle_\Delta & \text{if } \varepsilon \leq 1/2 \\ \bigl\langle \{(0,0),(1,0)\},\{(1,0),(1,1)\},\{(1,1),(0,1)\},\{(0,1),(0,0)\} \bigr\rangle_\Delta & \text{if } \varepsilon \in (1/2,\sqrt{2}/2] \\ \bigl\langle \{(0,0),(1,0),(0,1),(1,1)\} \bigr\rangle_\Delta & \text{if } \varepsilon > \sqrt{2}/2. \end{cases}$$

Figure 2.9.: Čech complexes for the vertices of a square in the Euclidean plane, schematically

Computing Čech complexes in general is a complex task. Rips complexes are a modified version, which are easier to compute: Instead of checking whether intersections of open balls in the ambient space are inhabited, we only need to compute diameters of finite sets.

Definition 2.5.8 (Rips complex)**.** Let (X, d) be a metric space and let $\varepsilon \in \mathbb{R}_{>0}$. The *Rips complex* (also called *Vietoris–Rips complex*) of (X, d) with radius ε is the simplicial complex (check!)

$$R_\varepsilon(X, d) := \bigl\{\sigma \in P_{\text{fin}}(X) \bigm| \operatorname{diam} \sigma < \varepsilon\bigr\}.$$

Here $\operatorname{diam} \sigma := \sup\{d(x, y) \mid x, y \in \sigma\}$ denotes the *diameter* of σ in (X, d).

Example 2.5.9 (Rips complexes)**.** In the situation of Example 2.5.7, we obtain for all $\varepsilon \in \mathbb{R}_{>0}$ (check!):

$$R_\varepsilon(X, d_2) = \begin{cases} \bigl\langle\{(0,0)\}, \{(1,0)\}, \{(0,1)\}, \{(1,1)\}\bigr\rangle_\Delta & \text{if } \varepsilon \le 1 \\ \bigl\langle\{(0,0),(1,0)\}, \{(1,0),(1,1)\}, \{(1,1),(0,1)\}, \{(0,1),(0,0)\}\bigr\rangle_\Delta & \text{if } \varepsilon \in (1, \sqrt{2}] \\ \bigl\langle\{(0,0),(1,0),(0,1),(1,1)\}\bigr\rangle_\Delta & \text{if } \varepsilon > \sqrt{2}. \end{cases}$$

Example 2.5.10 (“random” Rips complexes)**.** To illustrate the Rips complex construction, we consider Rips complexes of 30 random (uniformly distributed) points on the circle and of 30 random (uniformly distributed) points in the square $[-1, 1]^2$ (Figure 2.10).

Čech and Rips complexes are nested as follows:

Proposition 2.5.11 (Čech and Rips complexes)**.** *Let (Y, d) be a metric space, let $X \subset Y$ be non-empty, and let $\varepsilon, \delta \in \mathbb{R}_{>0}$.*

1. *If $\delta < \varepsilon$, then*

$$\check{C}_\delta(X, Y, d) \subset \check{C}_\varepsilon(X, Y, d) \quad \text{and} \quad R_\delta(X, d) \subset R_\varepsilon(X, d)$$

2. *Moreover, we have*

$$\check{C}_{\varepsilon/2}(X, Y, d) \subset R_\varepsilon(X, d) \subset \check{C}_\varepsilon(X, Y, d).$$

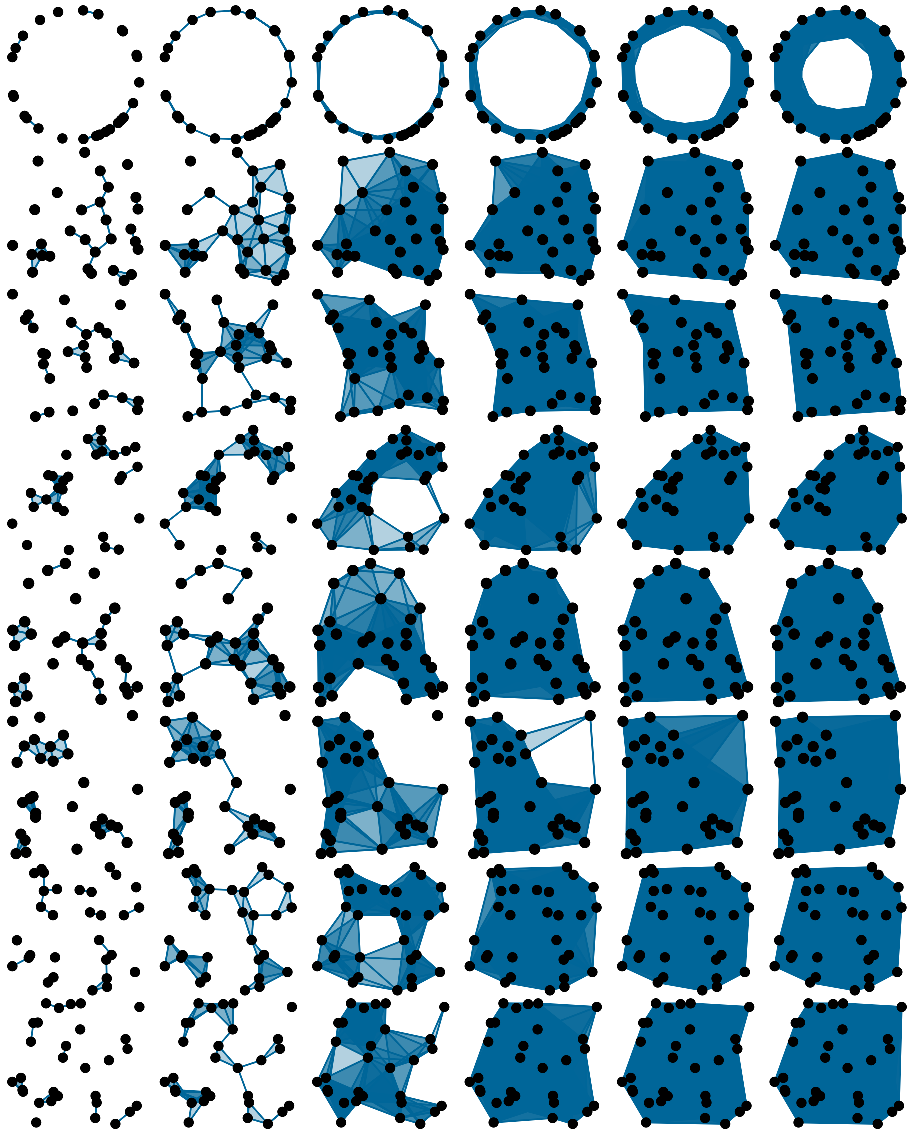


Figure 2.10.: Top: Rips complexes of 30 random (uniformly distributed) points on the unit circle; the radii of the Rips complexes are $0.3, 0.5, 0.8, 1.1, 1.4, 1.7$.
Lower rows: Rips complexes of 30 random (uniformly) distributed) points in the square $[-1, 1]^2$; the radii of the Rips complexes are $0.3, 0.5, 0.8, 1.1, 1.4, 1.7$

Proof. All of these statements follow from straightforward computations (check!).

We only give the details for the inclusion $\check{C}_{\varepsilon/2}(X,Y,d) \subset R_\varepsilon(X,d)$: Let $\sigma \in \check{C}_{\varepsilon/2}(X,Y,d)$; without loss of generality, we may assume that $\sigma \neq \emptyset$. We show that $\sigma \in R_\varepsilon(X,d)$: Let $x_0 \in \bigcap_{x\in\sigma} U_{\varepsilon/2}(x)$. Then the triangle inequality shows that

$$d(x,y) \leq d(x,x_0) + d(x_0,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\operatorname{diam}\sigma < \varepsilon$ and so $\sigma \in R_\varepsilon(X,d)$. □

2.6 Geometric realisation

We defined simplicial complexes are combinatorial objects, based on an informal association with the geometry of affine simplices in Euclidean space. Using the same analogy, we now construct the geometric realisation functor from simplicial complexes to (polyhedral) topological spaces.

Topological spaces are triangulable if they are homeomorphic to the geometric realisation of some simplicial complex. Such simplicial structures also reflect the topology of maps: By the simplicial approximation theorem, continuous maps between triangulable spaces are homotopic to simplicial maps – provided we refine the domain complex by subdivisions.

We first introduce geometric realisation and the corresponding notions of coordinates. We then study the barycentric subdivision of simplicial complexes and use it to prove the simplicial approximation theorem.

2.6.1 Geometric realisation

The geometric realisation functor is defined by replacing combinatorial simplices by actual Euclidean affine simplices (Figure 2.4, Figure 2.11). Points in the goemetric realisation are described in terms of convex coordinates.

Definition 2.6.1 (geometric realisation of simplicial complexes)**.** For a simplicial complex X, we define the *geometric realisation* of X as the subset

$$\begin{aligned} |X| &:= \bigcup_{\sigma\in X} \Bigl\{ \xi \in \bigoplus_{V(X)} \mathbb{R} \Bigm| \forall_{x\in V(X)\setminus\sigma} \quad \xi_x = 0,\ \forall_{x\in V(X)} \quad \xi_x \in [0,1],\ \sum_{x\in\sigma} \xi_x = 1 \Bigr\} \\ &= \bigcup_{\sigma\in X} \operatorname{conv}\{e_x \mid x \in \sigma\} \end{aligned}$$

endowed with the following topology: A subset $U \subset |X|$ is open if and only if for every $\sigma \in X$, the intersection $U \cap |X| \cap |\sigma|$ is open in the subspace topology of $|\sigma|$ induced by the Euclidean topology on $\bigoplus_\sigma \mathbb{R}$. Here, $|\sigma|$ denotes

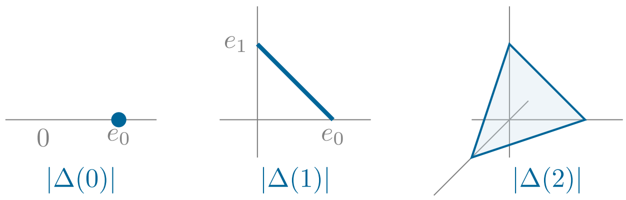


Figure 2.11.: Geometric realisation of simplices

the convex hull

$$|\sigma| := \operatorname{conv}\{e_x \mid x \in \sigma\} \subset |X|.$$

Moreover, for $\xi \in |X|$, we write

$$\operatorname{supp} \xi := \bigl\{x \in V(X) \bigm| \xi_x \neq 0\bigr\} \subset V(X).$$

Simplicial maps induce continuous maps between geometric realisations through affine extension:

Definition 2.6.2 (geometric realisation of simplicial maps)**.** For a simplicial map $f \colon X \longrightarrow Y$, we define the *geometric realisation* of f as the well-defined (check!) map

$$\begin{aligned} |f| \colon |X| &\longrightarrow |Y| \\ \xi &\longmapsto \sum_{x \in V(X)} \xi_x \cdot e_{f(x)}. \end{aligned}$$

Remark 2.6.3 (on the topology of the geometric realisation)**.** By construction, the topology on the geometric realisation has the following property: If X is a simplicial complex and Z is a topological space, then a map $\varphi \colon |X| \longrightarrow Z$ is continuous if and only if for each $\sigma \in X$ the restriction

$$\varphi|_{|\sigma|} \colon |\sigma| \longrightarrow Z$$

is continuous (with respect to the Euclidean topology on $|\sigma|$.

If X is a finite simplicial complex, then the topology on the geometric realisation $|X|$ is the same as the subspace topology induced by the finite-dimensional Euclidean space $\bigoplus_{V(X)} \mathbb{R}$ (check!) and $|X|$ is compact (Exercise).

If X is infinite, then the Euclidean topology on $|X|$ is coarser than the topology from Definition 2.6.1 (check!).

Proposition 2.6.4 (geometric realisation is a functor)**.** *Geometric realisation defines a functor* $|\cdot| \colon \mathsf{SC} \longrightarrow \mathsf{Top}$*. In particular:*

1. *If* $f \colon X \longrightarrow Y$ *is a simplicial map, then* $|f| \colon |X| \longrightarrow |Y|$ *is continuous.*

2. *If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are simplicial maps, then*

$$|g \circ f| = |g| \circ |f|.$$

3. *If X is a simplicial complex, then $|\mathrm{id}_X| = \mathrm{id}_{|X|}$.*

Proof. Ad 1. We use the characterisation of continuity on the geometric realisation of simplices (Remark 2.6.3): Let $\sigma \in X$. By definition, we have that the restriction $|f||_{|\sigma|}$ is given through

$$\begin{aligned} |f|\,\big|_{|\sigma|}\colon |\sigma| &\longrightarrow |Y| \\ \xi &\longmapsto \sum_{x\in\sigma} \xi_x \cdot e_{f(x)}. \end{aligned}$$

This is a finite sum of continuous maps (as multiplication on $\mathbb{R}$ is continuous) whose domains are a finite dimensional real vector space; therefore, $|f||_{|\sigma|}$ is continuous. In view of Remark 2.6.3, we obtain that f is continuous.

Ad 2. This is a straightforward computation: Let $\xi \in |X|$. By construction, we have

$$\begin{aligned} |g|\bigl(|f|(\xi)\bigr) &= |g|\biggl(\sum_{x\in V(X)} \xi_x \cdot e_{f(x)}\biggr) = \sum_{x\in V(X)} \xi_x \cdot e_{g(f(x))} = \sum_{x\in V(X)} \xi_x \cdot e_{g\circ f(x)} \\ &= |g\circ f|(\xi). \end{aligned}$$

Ad 3. This is immediate from the definition. □

Example 2.6.5 (geometric realisations)**.** Let $n \in \mathbb{N}$.

- We have $|\Delta(n)| \cong_{\mathsf{Top}} \Delta^n$ because the map

$$\begin{aligned} \bigl|\Delta(n)\bigr| &\longrightarrow \Delta^n \\ \xi &\longmapsto \sum_{j=0}^{n} \xi_j \cdot e_j \end{aligned}$$

  clearly is a homeomorphism.

- We have $|S(n)| \cong_{\mathsf{Top}} S^n$ because: The idea is to use the central projection map $c\colon \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n, x \longmapsto x/\|x\|_2$. There are several ways of carrying out this idea. One that generalises well to other setups is the following: We pick $n+2$ affinely independent points $v_0, \dots, v_{n+1}$ on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and then consider

$$\begin{aligned} \varphi\colon \bigl|S(n)\bigr| &\longrightarrow S^n \\ \xi &\longmapsto c\biggl(\sum_{x\in\{0,\dots,n+1\}} \xi_x \cdot v_x\biggr). \end{aligned}$$

This map φ is well-defined, bijective, and continuous (check!). Moreover, $|S(n)|$ is compact and S^n is Hausdorff. Therefore, we can apply the compact-Hausdorff trick (Corollary A.1.40) to conclude that φ is a homeomorphism.

- More generally: If P is a convex simplicial polytope in $\mathbb{R}^{n+1}$ with non-empty interior, then the geometric realisation of the simplicial complex associated with the combinatorial structure of the faces of P is homeomorphic to S^n (check!).

The vertices of a simplicial complex have special open neighbourhoods: the open stars. While closed stars could also be defined on the combinatorial level, the concept of open stars is more difficult to describe on the combinatorial side: open stars in general do *not* form subcomplexes.

Definition 2.6.6 (open star, closed star)**.** Let X be a simplicial complex and let $x \in V(X)$.

- The *closed star* of x in X is the subset

$$\operatorname{star}^c_X x := \bigl|\langle\{\sigma \in X \mid x \in \sigma\}\rangle_\Delta\bigr| = \bigcup_{\sigma \in X, x \in \sigma} |\sigma| \subset |X|.$$

- The *open star* of x in X is the subset

$$\operatorname{star}^\circ_X x := \bigcup_{\sigma \in X, x \in \sigma} \bigl\{\xi \in |\sigma| \bigm| \forall_{x \in \sigma} \quad \xi_x > 0\bigr\} \subset |X|.$$

Remark 2.6.7 (open stars are open and stars)**.** Let X be a simplicial ocmplex and let $x \in V(X)$. Then the open star $\operatorname{star}^\circ_X x$ is open in $|X|$; this can be derived directly from the definition of the topology on $|X|$ (check!). Moreover, $\operatorname{star}^\circ_X x$ is star-shaped with star-point x (check!).

Example 2.6.8 (an open star)**.** We consider the simplicial complex

$$X := \bigl\langle \{0,1,2\}, \{0,2,3\}, \{0,4\}, \{0,5\}, \{5,6\} \bigr\rangle_\Delta$$

and the vertex 0. The open star $\operatorname{star}^\circ_X 0$ is depicted in Figure 2.12.

Proposition 2.6.9 (simplices via open stars)**.** *Let X be a simplicial complex and let $\sigma \subset V(X)$ be a finite (non-empty) subset. Then σ is a simplex of X if and only if*

$$\bigcap_{x \in \sigma} \operatorname{star}^\circ_X x \neq \emptyset.$$

Proof. This is straightforward from the definitions (Exercise). □

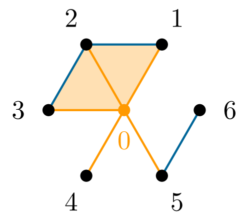


Figure 2.12.: An open star (yellow) in a simplicial complex (Example 2.6.8)

Example 2.6.10 (nerves of open stars)**.** Let X be a simplicial complex and let $U := \{\operatorname{star}^\circ_X x \mid x \in V(X)\}$. Then U is a cover of $|X|$ (check!) and Proposition 2.6.9 shows that the map

$$\begin{aligned} V(X) &\longrightarrow V(X) \\ x &\longmapsto \operatorname{star}^\circ_X x \end{aligned}$$

is a simplicial isomorphism $X \cong_{\mathsf{SC}} N(U)$ (check!).

Conversely, in certain situations, topological spaces can be recovered up to homotopy equivalence from the nerves of suitable open covers (Chapter 2.7).

Topological spaces that admit a combinatorial description in terms of simplicial complexes are called triangulable:

Definition 2.6.11 (triangulation)**.** A topological space is *triangulable* if it admits a triangulation. A *triangulation* of a topological space Z is a pair (X, φ), consisting of

- a simplicial complex X and
- a homeomorphism $\varphi\colon |X| \longrightarrow Z$.

Example 2.6.12 (triangulation)**.** From Example 2.6.5, we obtain triangulations of the standard affine n-simplex Δ^n and of the sphere S^n for all $n \in \mathbb{N}$. In particular, we see that triangulations are far from being unique in general.

Outlook 2.6.13 (triangulable spaces)**.** Every smooth manifold admits a triangulation; every compact smooth manifold admits a finite triangulation [75].

Every triangulation leads to a CW-structure (taking the simplices as cells). Conversely, every [finite] CW-complex is homotopy equivalent to a [finite] simplicial complex [37, Theorem 2C.5].

Caveat 2.6.14 (non-triangulability)**.** Not every topological space admits a triangulation! For example, geometric realisations of simplicial complexes always are Hausdorff (check!); in particular, non-Hausdorff spaces do *not* admit triangulations.

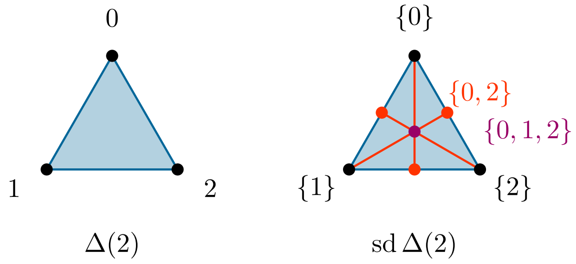


Figure 2.13.: Barycentric subdivision

A less exotic example is the following: The subspace $\{1/n \mid n \in \mathbb{N}_{>0}\} \cup \{0\}$ of $\mathbb{R}$ is *not* triangulable (Exercise).

Our next goal is to prove the simplicial approximation theorem. For this, we will need subdivisions.

2.6.2 Subdivision

The barycentric subdivision refines simplicial complexes by replacing each simplex by its barycentric subdivision; the barycentric subdivision of a simplex geometrically is obtained by inductively adding barycentres of simplices and coning of the previous lower-dimensional steps using the barycentre as cone point (Figure 2.13). On the combinatorial level, this corresponds to replacing a simplex by the simplicial complex given by chains of faces of the original complex; geometrically, in this abstract description, one should view the simplices in the chains as "barycentres" of the corresponding original simplices.

Definition 2.6.15 (barycentric subdivision)**.** Let X be a simplicial complex. The *barycentric subdivision* of X is the simplicial complex

$$\begin{aligned}\operatorname{sd} X := \bigl\{\{\sigma_0,\dots,\sigma_n\} \bigm| n \in \mathbb{N},\ \sigma_0,\dots,\sigma_n \in X \setminus \{\emptyset\},\ \sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_n\bigr\}\\ \cup \{\emptyset\}.\end{aligned}$$

Remark 2.6.16 (functoriality of barycentric subdivision)**.** Let X and Y be simplicial complexes. By construction, $V(\operatorname{sd} X) = X \setminus \{\emptyset\}$ (check!). If $f\colon X \longrightarrow Y$ is a simplicial map, then

$$\begin{aligned}V(\operatorname{sd} X) &\longrightarrow V(\operatorname{sd} Y)\\ \sigma &\longmapsto f(\sigma)\end{aligned}$$

is a well-defined simplicial map (check!), denoted by $\operatorname{sd} f\colon \operatorname{sd} X \longrightarrow \operatorname{sd} Y$. Moreover, this construction is compatible with identity maps and composition; thus, we obtain a functor $\operatorname{sd}\colon \mathsf{SC} \longrightarrow \mathsf{SC}$.

Barycentric subdivision is compatible with geometric realisations:

Proposition 2.6.17 (geometric realisation of the barycentric subdivision)**.** *Let X be a simplicial complex. Then there is a canonical homeomorphism*

$$\beta_X\colon |\operatorname{sd} X| \longrightarrow |X|$$

that is natural with respect to simplicial maps.

Proof. Let $\beta_X\colon |\operatorname{sd} X| \longrightarrow |X|$ be the affine linear extension of the map

$$\begin{aligned} V(\operatorname{sd} X) &\longrightarrow |X| \\ \sigma &\longmapsto \beta(\sigma) \end{aligned}$$

where

$$\beta(\sigma) := \frac{1}{\#\sigma} \cdot \sum_{x\in\sigma} e_x \in |\sigma| \subset |X|$$

denotes the *barycentre* of $\sigma \in X$. This affine linear extension β_X indeed has image in $|X|$ (check!) and is continuous (by definition of the topology on geometric realisations). Moreover, β_X is clearly natural with respect to simplicial maps.

We show that β_X is a homeomorphism. To this end, we establish the following:

① If $\sigma \in X$ and $\sigma \neq \emptyset$, then $\beta_{X_\sigma}\colon |\operatorname{sd} X_\sigma| \longrightarrow |\sigma|$ is a homeomorphism, where $X_\sigma := P(\sigma)$.

② If $\sigma \in X$, then $\beta_X|_{|\sigma|} = \beta_{X_\sigma}$.

From ① and ② we obtain that β_X is a homeomorphism: The compatibility in ② ensures that the maps of type ① glue to give the map β_X. By definition of $|\operatorname{sd} X|$ and $|X|$, therefore claim ① shows that the glued map β_X is surjective (check!), injective (check!?!), and its inverse also is continuous (check!).

Claim ② is immediate from the construction. For claim ①, we argue as follows: Because $|\operatorname{sd} X_\sigma|$ is compact and $|\sigma|$ is Hausdorff, it suffices to show that β_{X_σ} is continuous and bijective.

- Continuity follows from the construction as affine linear extension.
- Surjectivity: We proceed by induction on $\dim \sigma$.

  Base case: If $\dim \sigma = 0$, then $\sigma = \{x\}$ for some $x \in V(X)$. If $\xi \in |\sigma|$, then $\zeta := \xi_x \cdot e_{\{x\}} \in |\operatorname{sd} X_\sigma|$ and by construction we have

$$\beta_{X_\sigma}(\zeta) = \xi_x \cdot \beta\bigl(\{x\}\bigr) = \xi_x \cdot e_x = \xi.$$

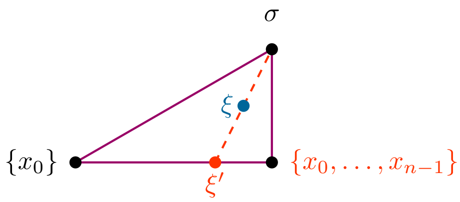


Figure 2.14.: Projection to a lower-dimensional simplex

Induction step: Let $n := \dim \sigma > 0$ and let us assume that surjectivity holds for all simplices of dimension $< n$. Let $\xi \in |\sigma|$; we write $\sigma = \{x_0, \dots, x_n\}$ with $x_0, \dots, x_n \in V(X)$ in such an order that

$$\xi_{x_0} \geq \xi_{x_1} \geq \dots \geq \xi_{x_n}.$$

In particular, $\xi_{x_n} \leq 1/(n+1)$ and so $t := (n+1) \cdot \xi_{x_n}$ lies in $[0, 1]$.

We consider the projected point (Figure 2.14)

$$\xi' := \sum_{j=0}^{n-1} \xi'_{x_j} \cdot e_{x_j},$$

where

$$\forall_{j \in \{0,\dots,n-1\}} \quad \xi'_{x_j} := \frac{1}{1-t} \cdot (\xi_{x_j} - \xi_{x_n}) \in [0, 1].$$

Then $\xi' \in \bigl|\{x_0, \dots, x_{n-1}\}\bigr|$, because

$$\begin{aligned}
\sum_{j=0}^{n-1} \xi'_{x_j} &= \frac{1}{1-t} \cdot \sum_{j=0}^{n-1} \xi_{x_j} - \frac{1}{1-t} \cdot n \cdot \xi_{x_n} && \text{(by definition of } \xi'\text{)} \\
&= \frac{1 - \xi_{x_n}}{1-t} - \frac{n \cdot \xi_{x_n}}{1-t} && \text{(because } \xi \in |\sigma|\text{)} \\
&= \frac{1-t}{1-t} = 1. && \text{(by definition of } t\text{)}
\end{aligned}$$

By the induction hypothesis (applied to the lower-dimensional simplex $\sigma' := \{x_0, \dots, x_{n-1}\}$), there exists a $\zeta' \in |\operatorname{sd} X_{\sigma'}| \subset |\operatorname{sd} X_\sigma|$ with $\beta_{X_{\sigma'}}(\zeta') = \xi'$. Then

$$\zeta := (1-t) \cdot \zeta' + t \cdot e_\sigma$$

is a point of $|\operatorname{sd} X_\sigma|$ and by construction, we have

$$\begin{aligned}
\beta_{X_\sigma}(\zeta) &= (1-t)\cdot\beta_{X_\sigma}(\zeta') + t\cdot\beta(\sigma) = (1-t)\cdot\beta_{X_{\sigma'}}(\zeta') + t\cdot\beta(\sigma)\\
&= (1-t)\cdot\xi' + t\cdot\beta(\sigma)\\
&= (1-t)\cdot\sum_{j=0}^{n-1}\xi'_{x_j}\cdot e_{x_j} + \frac{t}{n+1}\cdot\sum_{j=0}^{n} e_{x_j}\\
&= (1-t)\cdot\sum_{j=0}^{n-1}\frac{1}{1-t}\cdot(\xi_{x_j}-\xi_{x_n})\cdot e_{x_j} + \frac{t}{n+1}\cdot\sum_{j=0}^{n-1} e_{x_j} + \xi_{x_n}\cdot e_{x_n}\\
&= \sum_{j=0}^{n-1}\Bigl(\xi_{x_j} - \xi_{x_n} + \frac{t}{n+1}\Bigr)\cdot e_{x_j} + \xi_{x_n}\cdot e_{x_n}\\
&= \sum_{j=0}^{n-1}\xi_{x_j}\cdot e_{x_j} + \xi_{x_n}\cdot e_{x_n}\\
&= \xi.
\end{aligned}$$

Therefore, β_{X_σ} is surjective, as claimed.

- Injectivity: Let $\xi \in |\sigma|$ and let $\zeta, \zeta' \in |\operatorname{sd} X_\sigma|$ with $\beta_{X_\sigma}(\zeta) = \xi = \beta_{X_\sigma}(\zeta')$. Let $n := \dim\sigma$. We may pick an enumeration $\sigma = \{x_0,\dots,x_n\}$ and a permutation $\pi \in S_{\{0,\dots,n\}}$ such that ζ comes from the simplex $\{\{x_0\},\{x_0,x_1\},\dots\} \in \operatorname{sd} X_\sigma$ and such that ζ' comes from the simplex $\{\{x_{\pi(0)}\},\{x_{\pi(0)},x_{\pi(1)}\},\dots\}$. Rearranging the terms leads to

$$\begin{aligned}
\xi &= \beta(\zeta)\\
&= \sum_{\tau\subset\sigma}\zeta_\tau\cdot\sum_{x\in\tau}\frac{1}{\#\tau}\cdot e_x\\
&= \sum_{j=0}^{n}\Bigl(\sum_{k=j}^{n}\frac{1}{k+1}\cdot\zeta_{\{x_0,\dots,x_k\}}\Bigr)\cdot e_{x_j}.
\end{aligned}$$

In particular, we obtain

$$\forall_{j\in\{0,\dots,n\}}\quad \xi_{x_j} = \sum_{k=j}^{n}\frac{1}{k+1}\cdot\zeta_{\{x_0,\dots,x_k\}}.$$

As the convex coordinates of ζ are all non-negative this shows that

$$\xi_{x_0} \geq \dots \geq \xi_{x_n}$$

and that the indices with strict inequalities encode the set $\operatorname{supp}\zeta$. Similarly, from $\xi = \beta(\zeta')$, we deduce that

$$\forall_{j\in\{0,\dots,n\}}\quad \xi_{x_{\pi(j)}} = \sum_{k=j}^{n}\frac{1}{k+1}\cdot\zeta'_{\{x_{\pi(0)},\dots,x_{\pi(k)}\}}$$

and (where the strict inequalities encode $\operatorname{supp} \zeta'$)

$$\xi_{x_{\pi(0)}} \geq \cdots \geq \xi_{x_{\pi(n)}}.$$

We combine both chains of inequalities; in particular, $\operatorname{supp} \zeta = \operatorname{supp} \zeta'$. Equalities mean that the corresponding additional convex coordinate is zero; thus, we may swap such indices and retain the same points. Therefore, we may assume without loss of generality that π is the identity permutation. Overall, we have

$$\forall_{j \in \{0,\dots,n\}} \quad \sum_{k=j}^{n} \frac{1}{k+1} \cdot \zeta_{\{x_0,\dots,x_k\}} = \xi_{e_j} = \sum_{k=j}^{n} \frac{1}{k+1} \cdot \zeta'_{\{x_0,\dots,x_k\}}.$$

A downward induction over j then shows that the convex coordinates of ζ and ζ' coincide. Hence, $\zeta = \zeta'$, which proves injectivity of β_{X_σ}.

Hence, $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$ is a homeomorphism. □

Iterated barycentric subdivision leads to simplicial complexes with arbitrarily small mesh size inside the geometric realisation of the original simplicial complex:

Proposition 2.6.18 (mesh size of barycentric subdivisions)**.** *Let X be a simplicial complex and let $n \in \mathbb{N}$. We equip $|\operatorname{sd} X|$ with the metric induced by the Euclidean metric on $|X|$ under the homeomorphism $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$. Then, for each n-simplex σ of $\operatorname{sd} X$, we obtain*

$$\operatorname{diam} |\sigma| \leq \frac{n}{n+1} \cdot \sqrt{2}.$$

More generally, for each $N \in \mathbb{N}$ every simplex σ of the N-fold iterated barycentric subdivision $\operatorname{sd}^N X$ satisfies (with iterated pull-back metric)

$$\operatorname{diam} |\sigma| \leq \Bigl(\frac{n}{n+1}\Bigr)^{N} \cdot \sqrt{2}.$$

Proof. This is standard convex/Euclidan geometry [55, Theorem 15.4]. □

2.6.3 Simplicial approximation

Using barycentric subdivision, we prove the simplicial approximation theorem, which states that up to iterated barycentric subdivisions, continuous maps between triangulated topological spaces are “close” to simplicial maps.

Definition 2.6.19 (simplicial approximation)**.** Let X and Y be simplicial complexes and let $\varphi \colon |X| \longrightarrow |Y|$ be a continuous map. A simplicial map $f \colon X \longrightarrow Y$ is a *simplicial approximation of φ* if

$$\forall_{x \in V(X)} \quad \varphi(\operatorname{star}^\circ_X x) \subset \operatorname{star}^\circ_Y \bigl(f(x)\bigr).$$

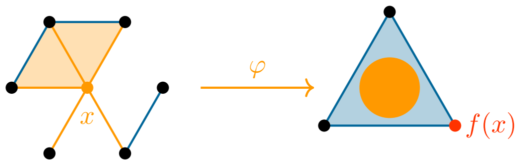


Figure 2.15.: The star condition for simplicial approximation, schematically

The approximation condition on the stars is illustrated in Figure 2.15.

Example 2.6.20 (simplicial approximation)**.**

- Let X be a simplicial complex. Then the identity id_X (Example 2.3.16) is a simplicial approximation of the identity map $\mathrm{id}_{|X|}\colon |X| \longrightarrow |X|$.

  More generally, if $f\colon X \longrightarrow Y$ is a simplicial map, then f is a simplicial approximation of $|f|\colon |X| \longrightarrow |Y|$ (check!).

- Let $X := D(1)$ and $Y := \operatorname{sd} X$. The inverse $\beta_X^{-1}\colon |X| \longrightarrow |Y|$ (Proposition 2.6.17) does *not* admit a simplicial approximation $X \longrightarrow Y$ (check!).

- If X is a simplicial complex, then the barycentric subdivision homeomorphism $\beta_X\colon |\operatorname{sd} X| \longrightarrow |X|$ (Proposition 2.6.17) admits a simplicial approximation $\operatorname{sdm}_X\colon \operatorname{sd} X \longrightarrow X$ (Exercise).

Before getting into the simplicial approximation theorem, we show that simplicial approximations are unique up to homotopy and that the geometric realisation of a simplicial approximation is homotopic to the approximated continuous map:

Proposition 2.6.21 (simplicial approximations are simplicially homotopic)**.** *Let X and Y be simplicial complexes, let $\varphi\colon |X| \longrightarrow |Y|$ be a continuous map, and let $f, g\colon X \longrightarrow Y$ be simplicial approximations of φ. Then $f \simeq_\Delta g$.*

Proof. It suffices to show that f and g are contiguous (Proposition 2.3.34). We use the characterisation of simplices via open stars from Proposition 2.6.9: Let $\sigma \in X$ with $\sigma \neq \emptyset$. Then $U := \bigcap_{x\in\sigma} \operatorname{star}^\circ_X x \neq \emptyset$ (Proposition 2.6.9). Then also the image $\varphi(U)$ is non-empty and thus

$$
\begin{aligned}
\bigcap_{y \in f(\sigma)\cup g(\sigma)} \operatorname{star}^\circ_Y y &= \bigcap_{x\in\sigma} \operatorname{star}^\circ_Y \bigl(f(x)\bigr) \cap \bigcap_{x\in\sigma} \operatorname{star}^\circ_Y \bigl(g(x)\bigr) \\
&\supset \bigcap_{x\in\sigma} \varphi(\operatorname{star}^\circ_X x) \quad (f \text{ and } g \text{ are simplicial approximations of } \varphi) \\
&= \varphi(U)
\end{aligned}
$$

is non-empty. Hence, $f(\sigma) \cup g(\sigma)$ is a simplex of Y (Proposition 2.6.9). □

Proposition 2.6.22 (geometric homotopies for simplicial approximations). *Let X and Y be simplicial complexes.*

1. *If $\varphi\colon |X| \longrightarrow |Y|$ is a continuous map and $f\colon X \longrightarrow Y$ is a simplicial approximation to φ, then $\varphi \simeq |f|$.*
2. *If $f, g\colon X \longrightarrow Y$ are simplicially homotopic, then $|f| \simeq |g|$.*

In particular: If $f, g\colon X \longrightarrow Y$ are simplicial approximations of $\varphi\colon |X| \longrightarrow |Y|$, then $|f| \simeq |g|$.

Proof. Both assertions can be proved using the straight-line homotopy. We only prove the first part in detail: We consider the map

$$\begin{aligned} \eta\colon |X| \times [0,1] &\longrightarrow |Y| \\ (\xi, t) &\longmapsto (1-t) \cdot \varphi(\xi) + t \cdot |f|(\xi). \end{aligned}$$

This map is indeed well-defined: Let $\xi \in |X|$ and let $\sigma := \operatorname{supp} \xi$. Then σ is a simplex of X. We show that $\varphi(\xi)$ and $|f|(\xi)$ both lie in $|f(\sigma)|$. As the latter set is convex in $\bigoplus_{V(Y)} \mathbb{R}$, we then obtain that η maps to $|Y|$.

On the one hand, by construction, $|f|(\xi) \in |f(\operatorname{supp} \xi)| = |f(\sigma)|$. On the other hand, because f is a simplicial approximation to φ, we obtain that

$$\begin{aligned} \varphi(\xi) \in \varphi\Bigl(\bigcap_{x \in \sigma} \operatorname{star}^\circ_X x\Bigr) &\subset \bigcap_{x \in \sigma} \varphi(\operatorname{star}^\circ_X x) \\ &\subset \bigcap_{x \in \sigma} \operatorname{star}^\circ_Y f(x) = \bigcap_{y \in f(\sigma)} \operatorname{star}^\circ_Y y \subset \bigl|f(\sigma)\bigr|. \end{aligned}$$

The map η is continuous: The restriction $\eta|_{|\sigma| \times [0,1]}$ of η is continuous for each $\sigma \in X$. From this one can conclude that η is continuous (if X is finite, this is straightforward; in the general case, one needs an additional technical argument to show that $|X| \times [0,1]$ carries the colimit topology of the $|\sigma| \times [0,1]$ with $\sigma \in X$).

By construction, $\eta(\,\cdot\,, 0) = \varphi$ and $\eta(\,\cdot\,, 1) = |f|$. Hence, the homotopy η witnesses that $\varphi \simeq |f|$. □

As a preparation for the simplicial approximation theorem we show that continuous maps satisfying the star condition admit a simplicial approximation. The general case then follows by applying iterated barycentric subdivisions.

Proposition 2.6.23 (simplicial approximation from star condition). *Let X and Y be simplicial complexes and let $\varphi\colon |X| \longrightarrow |Y|$ be a continuous map that satisfies the* star condition*, i.e.,*

$$\forall_{x \in V(X)} \quad \exists_{y \in V(Y)} \quad \varphi(\operatorname{star}^\circ_X x) \subset \operatorname{star}^\circ_Y y.$$

Then φ admits a simplicial approximation $X \longrightarrow Y$.

Proof. By the star condition (and the axiom of choice), there exists a map $f\colon V(X) \longrightarrow V(Y)$ with

$$\forall_{x\in X} \quad \varphi(\operatorname{star}^\circ_X x) \subset \operatorname{star}^\circ_Y\bigl(f(x)\bigr).$$

It suffices to show that f is a simplicial map $X \longrightarrow Y$: Let $\sigma \in X$; without loss of generality, let $\sigma \neq \emptyset$. We use the characterisation of simplices in terms of open stars (Proposition 2.6.9): Because of $\sigma \in X$, we have that $\bigcap_{x\in\sigma} \operatorname{star}^\circ_X x \neq \emptyset$. Therefore, together with the star-condition, we obtain

$$\bigcap_{y\in f(\sigma)} \operatorname{star}^\circ_Y y = \bigcap_{x\in\sigma} \operatorname{star}^\circ_Y\bigl(f(x)\bigr) \supset \bigcap_{x\in\sigma} \varphi(\operatorname{star}^\circ_X x) \neq \emptyset;$$

in particular, $\bigcap_{y\in f(\sigma)} \operatorname{star}^\circ_Y y \neq \emptyset$ and so $f(\sigma) \in Y$. □

Theorem 2.6.24 (simplicial approximation theorem)**.** *Let X and Y be simplicial complexes, let X be finite, and let $\varphi\colon |X| \longrightarrow |Y|$ be a continuous map. Then there exists an $N \in \mathbb{N}$ such that*

$$\varphi \circ \beta_X^N \colon |\operatorname{sd}^N X| \longrightarrow |Y|$$

admits a simplicial approximation.

Proof. We use a subdivision and Lebesgue lemma argument: Let $V := (\operatorname{star}^\circ_Y y)_{y\in V(Y)}$ be the open cover of $|Y|$ given by the open stars of all vertices in Y. Then $U := (\varphi^{-1}(\operatorname{star}^\circ_Y y))_{y\in V(Y)}$ is an open cover of X. As X is finite, $|X|$ is compact.

We apply the Lebesgue lemma (Lemma 2.6.25) to the open cover U of $|X|$, where we equip $|X|$ with the Euclidean metric on the finite-dimensional space $\bigoplus_{V(X)} \mathbb{R}$. Hence, we obtain an $\varepsilon \in \mathbb{R}_{>0}$ with the following property: For each $x \in |X|$ there exists a $y \in V(Y)$ with

$$U_\varepsilon(x) \subset \varphi^{-1}(\operatorname{star}^\circ_Y y).$$

Then there exists an $N \in \mathbb{N}$ such that the mesh size of $\operatorname{sd}^N X$ is less than $\varepsilon/2$ (Proposition 2.6.18). Hence, the open stars of $|\operatorname{sd}^N X|$ have diameter less than ε when viewed as subsets of $|X|$ via the homeomorphism β_X^N for the N-fold iterated barycentric subdivision (check!). In particular,

$$\forall_{x\in V(\operatorname{sd}^N X)} \quad \exists_{y\in V(Y)} \quad \varphi\circ\beta_X^N(\operatorname{star}^\circ_{\operatorname{sd}^N X} x) \subset \varphi\bigl(\varphi^{-1}(\operatorname{star}^\circ_Y y)\bigr) \subset \operatorname{star}^\circ_Y y.$$

Therefore, $\varphi\circ\beta_X^N$ admits a simplicial approximation (Proposition 2.6.23). □

Lemma 2.6.25 (Lebesgue lemma)**.** *Let (X,d) be a compact metric space and let $(U_i)_{i\in I}$ be an open cover of X. Then there exists an $\varepsilon \in \mathbb{R}_{>0}$ with the following property: For every $x \in X$ there is an $i \in I$ such that the open ball $U(x,\varepsilon)$ of radius ε around x is contained in U_i.*

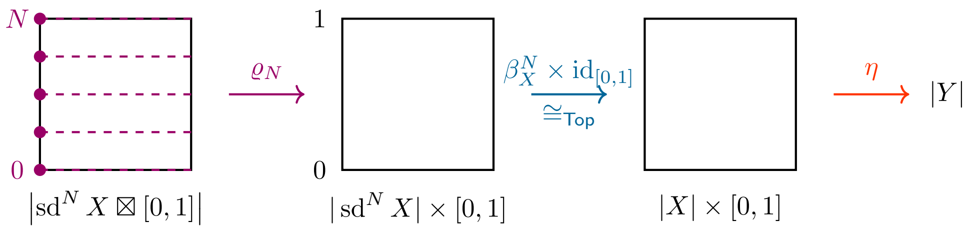


Figure 2.16.: From homotopies to simplicial homotopies, schematically

Any such number ε is called a Lebesgue number *of the cover* $(U_i)_{i\in I}$.

Proof. Because $(U_i)_{i\in I}$ is an open cover of X, for every $x \in X$ there exists an $i_x \in I$ and an $r_x \in \mathbb{R}_{>0}$ with $U(x, r_x) \subset U_{i_x}$. In view of compactness, there is a *finite* set $Y \subset X$ with $\bigcup_{y\in Y} U(y, r_y/2) = X$. Then

$$\varepsilon := \frac{1}{2} \cdot \min_{y\in Y} r_y \in \mathbb{R}_{>0}$$

has the desired property: Let $x \in X$. Hence, there is a $y \in Y$ with $x \in U(y, r_y/2)$. Therefore, we obtain $U(x, \varepsilon) \subset U(y, r_y) \subset U_{i_y}$, as desired. □

There is also an adapted version of the simplicial approximation theorem for infinite domain complexes [56, Theorem 16.5].

The same reasoning as in the proof of the simplicial approximation theorem also leads to the corresponding statement for homotopies:

Theorem 2.6.26 (homotopy and simplicial homotopy)**.** *Let X and Y be simplicial complexes, let X be finite, let $\varphi, \psi \colon |X| \longrightarrow |Y|$ be continuous maps, and let $f, g \colon X \longrightarrow Y$ be simplicial approximations of φ and ψ, respectively. Then $\varphi \simeq \psi$ if and only if there exists an $N \in \mathbb{N}$ such that $f \circ \operatorname{sdm}_X^N \simeq_\Delta^* g \circ \operatorname{sdm}_Y^N$. Here, sdm_X^N denotes an iterated simplicial approximation of β_X^N.*

Proof. First, let $f \circ \operatorname{sdm}_X^N \simeq_\Delta^* g \circ \operatorname{sdm}_Y^N$. Because f and sdm_X^N are simplicial approximations of φ and β_X^N, respectively, we see that $f \circ \operatorname{sdm}_X^N$ is a simplicial approximation of $\varphi \circ \beta_X^N$ (Exercise). Then Proposition 2.6.22 shows that

$$\begin{aligned} \varphi \circ \beta_X^N &\simeq |f \circ \operatorname{sdm}_X^N| && (f \circ \operatorname{sdm}_X^N \text{ is a simplicial approximation of } \varphi \circ \beta_X^N) \\ &\simeq |g \circ \operatorname{sdm}_X^N| && (f \circ \operatorname{sdm}_X^N \simeq_\Delta^* g \circ \operatorname{sdm}_X^N, \text{ which is a finite sequence of “}\simeq_\Delta\text{”}) \\ &\simeq \psi \circ \beta_X^N. && (g \circ \operatorname{sdm}_X^N \text{ is a simplicial approximation of } \psi \circ \beta_X^N) \end{aligned}$$

Pre-composing with the inverse homeomorphism of β_X^N shows that $\varphi \simeq \psi$.

Conversely, let $\varphi \simeq \psi$. Let $\eta \colon |X| \times [0,1] \longrightarrow |Y|$ be a homotopy from φ to ψ. We apply a subdivision and Lebesgue lemma argument to a suitable product (Figure 2.16). To this end, for $N \in \mathbb{N}_{>0}$, we consider the simplicial interval $[0,N]_\Delta$ (Example 2.3.8) and the continuous map $\alpha_N \colon |[0,N]_\Delta| \longrightarrow [0,1]$ that uniformly shrinks the "long" interval to $[0,1]$. Then α_N and the geometric realisations of the projections from $(\operatorname{sd}^N X) \boxtimes [0,N]_\Delta$ to the two factors induce a continuous map

$$\varrho_N \colon \bigl|(\operatorname{sd}^N X) \boxtimes [0,N]_\Delta\bigr| \longrightarrow |\operatorname{sd}^N X| \times \bigl|[0,N]_\Delta\bigr| \longrightarrow |\operatorname{sd}^N X| \times [0,1].$$

By construction, the corresponding inclusions are compatible:

$$\varrho_N \circ |i_0| = \iota_0 \quad \text{and} \quad \varrho_N \circ |i_N| = \iota_N.$$

If N is large enough, then the same kind of argument as in the proof of the simplicial approximation theorem (Theorem 2.6.24) shows that the composition $\eta \circ (\beta_X^N \times \operatorname{id}_{[0,1]}) \circ \varrho_N$ admits a simplicial approximation $h \colon \operatorname{sd}^N X \boxtimes [0,N]_\Delta \longrightarrow Y$, where we write $\beta_X^N \colon |\operatorname{sd}^N X| \longrightarrow |X|$ for the N-fold iterated barycentric subdivision homeomorphism (Proposition 2.6.17). Then:

- $h \circ i_0$ is a simplicial approximation of $\eta \circ (\beta_X^N \times \operatorname{id}_{[0,1]}) \circ \iota_0 = \varphi \circ \beta_X^N$;
- $h \circ i_1$ is a simplicial approximation of $\eta \circ (\beta_X^N \times \operatorname{id}_{[0,1]}) \circ \iota_1 = \psi \circ \beta_X^N$;
- h induces simplicial homotopies

$$h \circ i_0 \simeq_\Delta h \circ i_1, \quad h \circ i_1 \simeq_\Delta h \circ i_2, \quad \dots \quad h \circ i_{N-1} \simeq_\Delta h \circ i_N.$$

Because $f \circ \operatorname{sdm}_X^N$ is also a simplicial approximation of $\varphi \circ \beta_X^N$, we obtain $f \circ \operatorname{sdm}_X^N \simeq_\Delta h \circ i_0$ (Proposition 2.6.21) and analogously $g \circ \operatorname{sdm}_X^N \simeq_\Delta h \circ i_N$. Combining all these simplicial homotopies, we conclude that $f \circ \operatorname{sdm}_X^N \simeq_\Delta^* g \circ \operatorname{sdm}_X^N$, as claimed. □

Corollary 2.6.27 (counting homotopy types)**.**

1. *There exist only countably many homotopy types of topological spaces that admit a finite triangulation.*
2. *If W and Z are topological spaces that admit finite triangulations, then there are only countably many homotopy classes of continuous maps $W \longrightarrow Z$.*

Proof. Ad 1. There exist only countably many isomorphism types of finite simplicial complexes (check!). In particular, there exist only countably many homotopy types of spaces homeomorphic [even: homotopy equivalent] to finite simplicial complexes.

Ad 2. Let X and Y be finite triangulations of W and Z, respectively. From the simplicial approximation theorem (Theorem 2.6.24) and the fact that the geometric realisation of a simplicial approximation is homotopic to the original map (Proposition 2.6.22), we obtain that geometric realisation induces a surjection

$$\bigcup_{N \in \mathbb{N}} \operatorname{map}_\Delta(\operatorname{sd}^N X, Y) \longrightarrow \big[|X|, |Y|\big] \cong_{\mathsf{Set}} [W, Z].$$

The left-hand side is countable (check!). Thus, also $[W, Z]$ is countable. □

In particular, Corollary 2.6.27 can be applied in various ways in the context of compact smooth manifolds (Outlook 2.6.13). Moreover, Theorem 2.6.24 allows to show many basic null-homotopy results (Exercise).

2.7 Modelling: Approximation by simplicial structures

In many applications, simplicial complexes are used as combinatorial representations/approximations of smooth shapes – i.e., one considers triangulations.

Example 2.7.1 (computer graphics)**.** Triangulations allow us to reduce many geometric problems to linear algebra. For example, the intersection of geometric realisations of finite simplicial complexes can be computed via linear algebra. In contrast, the computation of intersections of solutions to polynomial equations or equations involving more general smooth functions would be considerably more difficult.

Intersection problems are ubiquitous in computer graphics, e.g., in raytracing or collision detection in computer games.

Therefore, triangulations (and other polygonal structures) are popular tools in computer aided design (CAD), 3D printing, computer games, ...

Example 2.7.2 (finite element method)**.** The finite element method is an approach to numerically (approximately) solve partial differential equations on subsets of Euclidean spaces. The key idea is to triangulate the subsets and to replace the original problem by piecewise (polynomial) problems. Typically, the triangulation is then called a *mesh* and each of the piecewise problems is called a *finite element*. One first solves the finite elements and then uses a variational approach to minimise a global error function to obtain reasonable global approximate solutions.

We now return to the conversion of point clouds into topological spaces and to the question in which sense the associated Čech complexes are valid approximations of the underlying "true" space.

As one result in this direction, we have a look at the sampling theorem by Niyogi, Smale, and Weinberger [59]:

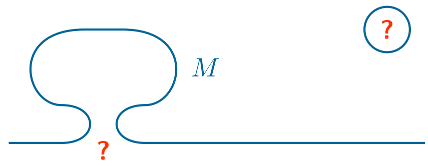


Figure 2.17.: Towards the condition number: Resolution necessary to locally and globally distinguish parts of submanifolds

Theorem 2.7.3 (high confidence simplicial approximation through iid sampling [59, (proof of) Theorem 3.1]). *Let M be a closed k-dimensional smooth submanifold of $\mathbb{R}^N$ with condition number τ. Let $\varepsilon \in (0, \tau/2)$, let $\delta \in (0,1)$, and let*

$$n > \beta_1 \cdot \Bigl(\log \beta_2 + \log \frac{1}{\delta}\Bigr),$$

where ($\operatorname{vol} M$ is the volume of M as Riemannian submanifold of $\mathbb{R}^N$)

$$\beta_1 := \frac{\operatorname{vol} M}{\cos^k(\vartheta_1) \cdot \operatorname{vol}(B^k_{\varepsilon/4})}, \quad \beta_2 := \frac{\operatorname{vol} M}{\cos^k(\vartheta_2) \cdot \operatorname{vol}(B^k_{\varepsilon/8})}$$

$$\vartheta_1 := \arcsin\Bigl(\frac{\varepsilon}{8 \cdot \tau}\Bigr), \quad \vartheta_2 := \arcsin\Bigl(\frac{\varepsilon}{16 \cdot \tau}\Bigr)$$

and where $B^k_{\varepsilon/4}$ is a closed $\varepsilon/4$-ball in $\mathbb{R}^k$. Let $X_1, \dots, X_n$ be a family of independent identically uniformly distributed M-valued random variables. Then

$$\operatorname{Prob}\bigl(\bigl|\check{C}_\varepsilon(\{X_1, \dots, X_n\}, \mathbb{R}^N, d_2)\bigr| \simeq M\bigr) > 1 - \delta.$$

The condition number of a submanifold of $\mathbb{R}^N$ describes the resolution that is necessary to distinguish parts of the submanifold that do not belong together – both locally and globally (Figure 2.17).

Definition 2.7.4 (condition number [59, Section 2]). Let M be a closed submanifold of $\mathbb{R}^N$.

- The *medial axis* of M is the closure $\overline{A}_M$ of the set

  $$A_M := \bigl\{a \in \mathbb{R}^N \bigm| \exists_{x,y \in M} \quad x \neq y \land \|a - x\|_2 = d_2(a, M) = \|a - y\|_2\bigr\} \subset \mathbb{R}^N.$$

- For $x \in M$, then $\sigma_M(x) := d_2(x, \overline{A}_M)$ is the *local feature size* of M at x.

- The *condition number* of M is $\inf_{x \in M} \sigma_M(x)$. (Sometimes, the condition number also refers to the inverse of this number.)

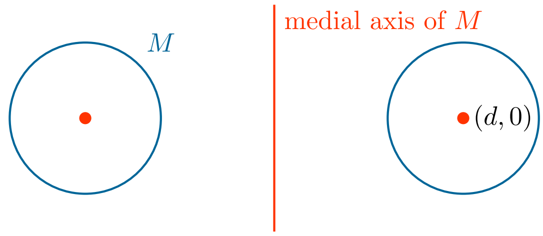


Figure 2.18.: An example of a medial axis (Example 2.7.5)

Alternatively, the condition number can also be described via the open normal bundle of M in $\mathbb{R}^N$ [59, Section 2] or in terms of the second fundamental form of the submanifold M of $\mathbb{R}^N$ [59, Section 6].

Example 2.7.5 (condition number)**.**

- The medial axis of a circle in $\mathbb{R}^2$ at the origin of radius $r \in \mathbb{R}_{>0}$ is the singleton containing the origin. The condition number of this circle in $\mathbb{R}^2$ is thus r.

  If we embed this circle into $\mathbb{R}^3$ via the embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ into the first two coordinates, then the medial axis is $\{(0,0)\} \times \mathbb{R}$; the condition number remains r.

- Let $d \in \mathbb{R}_{>2}$. The medial axis of $M := S^1 \cup (S^1 + (d, 0))$ in $\mathbb{R}^2$ is depicted in Figure 2.18. The condition number is thus the minimum of 1 and $(d-2)/2$.

Sketch of proof of Theorem 2.7.3. The proof consists of the following steps (Figure 2.19):

1. The subset

$$\bigl\{(x_1, \dots, x_n) \in M^n \bigm| |\check{C}_\varepsilon(\{x_1, \dots, x_n\}, \mathbb{R}^N, d_2)| \simeq M\bigr\} \subset M^n$$

   is measurable (with respect to the product σ-algebra; Exercise). In particular, the probability considered in the statement of the theorem is well-defined.

2. With probability at least $1 - \delta$, the sampled points form an $\varepsilon/2$-net in M, i.e., every point in M is $\varepsilon/2$-close to one of the sampled points [59, Proposition 3.2].

3. If $\{x_1, \dots, x_n\} \subset M$ is an $\varepsilon/2$-net, then the thickened up space

$$U := \bigcup_{j=1}^{n} U_\varepsilon(x_j)$$

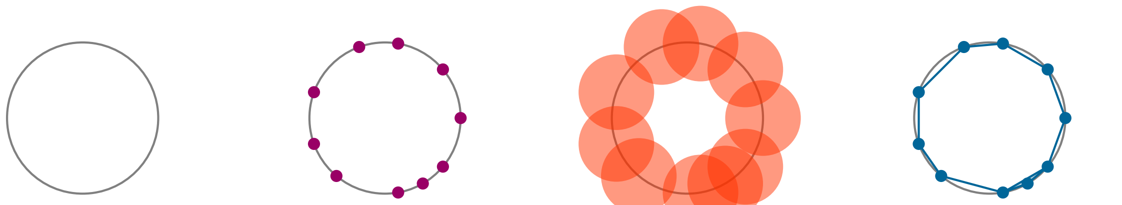


Figure 2.19.: Proof of Theorem 2.7.3, schematically

deformation retracts onto M [59, Proposition 3.1]; here, $U_\varepsilon(x_j)$ denotes the open ε-ball around x_j in $\mathbb{R}^N$. In particular, $M \simeq U$.

4. We apply the nerve lemma (Theorem 2.7.6) to $(U_\varepsilon(x_j))_{j\in\{1,\dots,n\}}$, as an open cover of U, to conclude that

$$M \simeq U \simeq \bigl|\check{C}_\varepsilon(\{x_1,\dots,x_n\},\mathbb{R}^N,d_2)\bigr|. \qquad \square$$

The proof of Theorem 2.7.3 used the nerve lemma. The nerve lemma is a key tool in homotopy theory and approximation through simplicial structures.

Theorem 2.7.6 (nerve lemma)**.** *Let Z be a paracompact space (e.g., a manifold or a CW-complex) and let $U = (U_i)_{i\in I}$ be an open cover of Z with the following property: For all non-empty finite subsets $J \subset I$, the intersection $\bigcap_{i\in J} U_i$ is empty or contractible. Then*

$$\bigl|N(U)\bigr| \simeq X.$$

More concretely: If $\varphi = (\varphi_i)_{i\in I}$ is a partition of unity on Z that is subordinate to U, then the nerve map *defined by*

$$\begin{aligned} Z &\longrightarrow \bigl|N(U)\bigr| \\ \zeta &\longmapsto \sum_{i\in I} \varphi_i(\zeta)\cdot e_i \end{aligned}$$

is a homotopy equivalence. Different partitions of unity on Z subordinate to U lead to homotopic nerve maps.

Sketch of proof [37, Chapter 4.G]. As Z is paracompact, there exists a partition of unity $\varphi = (\varphi_i)_{i\in I}$ subordinate to U. The nerve map $\nu\colon Z \longrightarrow |N(U)|$ induced by φ is well-defined and continuous (Exercise) and other choices of partition of unity lead to homotopic nerve maps (Exercise).

In order to show that ν is a homotopy equivalence, we consider an intermediate stage, namely a homotopy-theoretic thickening of $\bigcup_{i\in I} U_i$: Let Y be the homotopy colimit of the diagram given by all finite non-empty intersections of members of U and their respective chains of inclusion maps; i.e., the diagram is based on $\operatorname{sd} N(U)$, directed by decreasing dimension of the underlying simplices of $N(U)$. This homotopy colimit is constructed by taking iterated mapping cylinders of the corresponding chains of inclusion maps (as a thick replacement of the set-theoretic union).

We obtain the following diagram:

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & \bigl|\operatorname{sd} N(U)\bigr| \\ {\scriptstyle \text{flatten}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \cong_{\mathsf{Top}}}\big\downarrow{\scriptstyle \beta_{N(U)}} \\ Z & \underset{\nu}{\longrightarrow} & \bigl|N(U)\bigr| \end{array}$$

The left vertical map is given by "flattening" the mapping cylinders, which is a homotopy equivalence. The upper horizontal map φ is obtained from the universal property of the homotopy colimit, applied to the constant maps from sets of the form $\bigcap_{i\in J} U_i$ to the one-point space. Tracing through the constructions shows that the diagram is commutative up to homotopy.

The whole point of homotopy colimits is that they are (in contrast with ordinary colimits in Top) compatible with homotopy equivalences. By the standard construction of the homotopy colimits, it turns out that $|\operatorname{sd} N(U)|$ can be viewed as the homotopy colimit space of the diagram over $\operatorname{sd} N(U)$ that consists only of one-point spaces. Moreover, by hypothesis on U, whenever $J \in P_{\text{fin}}(I)$ is non-empty, the intersection $\bigcap_{i\in J} U_i$ is (empty or) contractible; hence, the constant map $\bigcap_{i\in J} U_i \longrightarrow \bullet$ is a homotopy equivalence. Therefore, $\varphi\colon Y \longrightarrow |\operatorname{sd} N(U)|$ is a homotopy equivalence.

Therefore, the diagram shows that also ν is a homotopy equivalence. □

Remark 2.7.7 (the nerve theorem for convex covers)**.** For the proof of Theorem 2.7.3, a weaker version of the nerve lemma would be sufficient: We consider only open covers consisting of Euclidean balls in Euclidean space. Here, the contractibility of the members of the open cover and of their (non-empty) intersections is provided by convexity.

In the case of open covers of subsets of $\mathbb{R}^N$ consisting of convex subsets, the nerve lemma can be approached in a more concrete way [5]. However, also in this case, one of the homotopies requires an inductive homotopy-theoretic construction.

Real-world problem 2.7.8 (shape from data)**.** Let P be a real-world phenomenon and let there be $N \in \mathbb{N}$ observable real-valued parameters of objects involved in P. The goal is to describe P as accurately as possible through a finite number of data points, composed of the N observables, obtained by taking a finite number of measurements on instances in which P occurs.

Model 2.7.9 (shape from data)**.** We model the situation of Problem 2.7.8 as follows:

- We model the observables as a function $f\colon P \longrightarrow \mathbb{R}^N$. We then consider the observable subset
$$M := f(P) \subset \mathbb{R}^N.$$
  The best overall outcome is to be able to reconstruct M (as we can access P only through the N observables).

- Let $n \in \mathbb{N}$. We model n measurements as a family of n independent identically uniformly distributed M-valued random variables.

  It is debatable whether this assumption is realistic.

The sampling theorem (Theorem 2.7.3) then allows us to reconstruct the homotopy type of M through the Čech complexes associated with the sampled points. However, this assumes that

- M indeed is a smooth submanifold of $\mathbb{R}^N$ (which sometimes might be realistic and sometimes not),
- that we choose the radii small enough and that we have enough points to compensate for the (a priori unknown!) condition number of M,
- that there is no noise or imprecision in the measurements,
- that the measurements are independent and uniformly distributed over M,
- that we work with Čech complexes instead of the easier to compute Rips complexes.

Some of these issues can be resolved [59, 43].

2.8 Implementation: Simplicial complexes

We briefly address the problem of how to implement simplicial structures in the context of algorithmic computations.

As always in algorithmic contexts:

- There are no uniformly optimal solutions; the setup needs to be adapted to the problem at hand.
- Algorithms and data structures are symbiotic and need to be considered simultaneously.
- The algorithms/data structures should fit the used programming language/programming paradigm.

Simplicial complexes, by definition, are sets of finite sets. In most applications, we only need to handle finite simplicial complexes; thus, the following discussion will focus on this case.

A priori this set of finite sets is unstructured. In many applications, we will need to be able to access simplices of a given dimension. Therefore, often, it is useful to implement simplicial complexes as

- a "finite collection" X
- of "finite sets" $X(n)$, indexed by dimension n,
- which in turn consist of "finite sets" over some base type a.

Thus, "finite collection" should be a datatype that is indexed by natural numbers and contains "finite sets" of "finite sets"; depending on whether the dimension of the considered simplicial complexes is known beforehand or not, one could take an array-like structure or a hash table or, more generally, a function from natural numbers to "finite sets" of "finite sets".

The "finite sets" should model mathematical finite sets in the sense that they satisfy extensionality (in particular, the order of "elements" is not relevant) and that they are traversable. Depending on the size and the base types, internally such "finite sets" might be represented by binary search trees (if the base type is ordered) or by hash tables.

In the case of point clouds, the sets $X(n)$ have a strong locality property and usually are sparse. Depending on the concrete application, it can make sense to choose a data structure for $X(n)$ that is optimised in that respect.

The vertices will be elements of the chosen base type a. In many applications, a will be a data type that does not only contain the labels of the vertices, but also further data.

As a first concrete example let us consider the problem of computing the connected components of a simplicial complex (i.e., the set of maximal connected subcomplexes). There are various algorithmic solutions. We will use the union-find framework [19, Chapter 22] to compute the connected components [29, Chapter 1]:

Definition 2.8.1 (union-find)**.** Let V be a (finite) set of elements of a datatype a. A *union-find structure* over V consists of

- a data structure S, representing a (dynamic) partition of V into subsets of V,
- and the operations make-sets, union, find

with the following properties:

- *Make-sets.* The function make-sets initialises S as the partition of V consisting of the singletons of the elements of V.

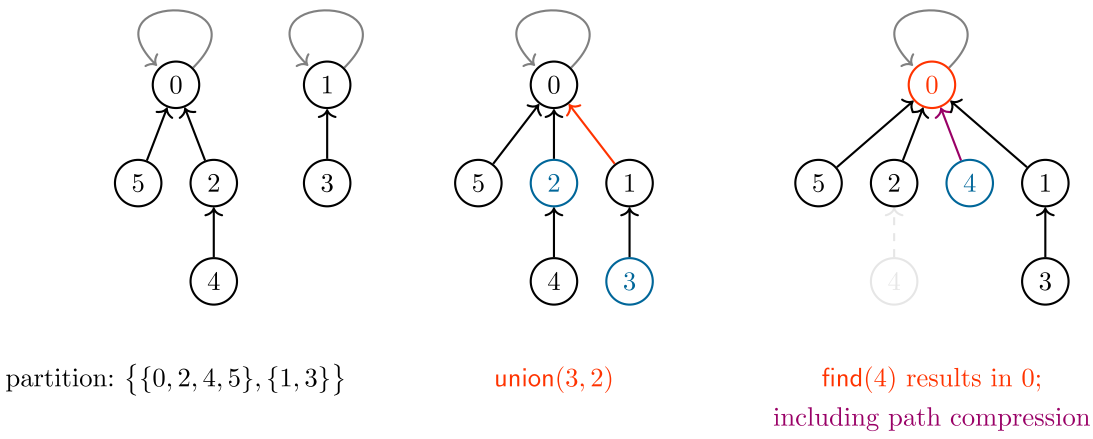


Figure 2.20.: Union-find via rooted forests, schematically

- *Find.* The function find takes an element $x \in V$ as argument and returns an element (a "representative") of the subset of V that contains x. Here, elements of the same partition set result in the same representative.
- *Union.* The function union takes two elements $x, y \in V$ as argument and updates the partition S by merging the subsets in S that contain x and y, respectively.

If also the underlying finite set V is dynamic, then in addition to make-sets, one considers a function make-set that creates the singleton of an element (provided this element is not yet contained in the union over S).

Remark 2.8.2 (union-find, implementation)**.** A standard way to implement a union-find structure over a finite set V is to realise the partition S as a rooted directed forest (Figure 2.20) such that

- the rooted trees of the forest correspond to the sets in the partition,
- in each rooted tree, from each vertex, one can access its parent vertex (which is the vertex itself if the vertex is a root), and
- the roots are the representatives provided by find.

More concretely, one could realise V as an array and represent the forest structure of S through corresponding pointers from each vertex to its parent.

- Then make-sets(V) creates the forest in which every element of V is a root and has no children.

- For $x \in V$, one calculates $\mathsf{find}(x)$ by following the path of parents until one reaches a root.

  In order to improve the aggregated efficiency, one can use *path compression* in every call to find: After finding the associated root, every element lying on the path from the given element to the root, has its parent changed to the root. This does not only return the desired representative but also updates the underlying data structure S!

- For $x, y \in V$, one calculates $\mathsf{union}(x, y)$ as follows:

  If $\mathsf{find}(x) \neq \mathsf{find}(y)$, then the structure S is updated as follows:

  - If the tree for x is at most as big as the one for y, we reset the parent of the root obtained by $\mathsf{find}(x)$ to be the root obtained by $\mathsf{find}(y)$.
  - If the tree for x is larger than the one for y, we reset the root of the parent obtained by $\mathsf{find}(y)$ to be the root obtained by $\mathsf{find}(x)$.

  To make this efficient, the sizes of subtrees also need to be updated accordingly in the union step.

If V contains n elements and one uses a total number m of union and find operations (there are at most $n - 1$ unions possible on n elements), then the complexity with path compression lies in $O(m \cdot \alpha(m, n))$, where α denotes the "inverse" of the Ackermann function [19, Chapter 22]. As the Ackermann function grows extremely fast, one may view α in practice as being constant.

Algorithm 2.8.3 (connected components of simplicial complexes)**.** Given a finite simplicial complex X, do the following:

- Initialise a union-find structure on $V(X)$ via $\mathsf{make\text{-}sets}(V(X))$.
- For each $\sigma \in X(1)$:

  Let x, y be the two vertices of σ.

  If $\mathsf{find}(x) \neq \mathsf{find}(y)$, then $\mathsf{union}(x, y)$.

- Return the resulting union-find structure.

Here, we assume that X is given in a representation as discussed on p. 64. Strictly speaking, we only have $X(0)$ available instead of $V(X)$. Using Remark 2.3.4, we can easily convert between these two sets and we will freely use both versions in the algorithms.

Every algorithm needs a termination, correctness, and runtime analysis:

Proposition 2.8.4. *The algorithm specified in Algorithm 2.8.3*

1. *terminates on every input and*

2. *computes for a given finite simplicial complex X, the vertex sets of the connected components of X (in the form of a union-find partition structure on $V(X)$).*

3. *Runtime analysis: If X is a finite simplicial complex, then the algorithm uses one call to* make-set *on $\#V(X)$ elements and $2 \cdot \#X(1)$ calls to* find*, and at most $\#X(1)$ calls to* union*. Moreover, a traversal over $V(X)$ (or $X(0)$) and over $X(1)$ is performed.*

Proof. Let X be a finite simplicial complex with n vertices and m edges.

Ad 1./3. The first part of the algorithm consists of a call to make-sets on a set of n elements. The second part of the algorithm consists of $2 \cdot m$ calls to find and at most m calls to union.

In particular: As make-sets, find, and union terminate on all inputs, also Algorithm 2.8.3 terminates on every input.

Ad 2. The algorithm and the connected components depend only on the 1-skeleton of X (i.e., the subcomplex of vertices and 1-simplices of X). Let $\sigma_1, \dots, \sigma_m$ be the 1-simplices of X, ordered as in the traversal in the algorithm.

We show inductively: For each $k \in \{0, \dots, m\}$, after handling the 1-simplices $\sigma_1, \dots, \sigma_k$, the union-find structure consists exactly of the connected components of the subcomplex $Y_k := \{\sigma_1, \dots, \sigma_k\} \cup X(0) \cup X(-1)$.

In the base case $k = 0$, we have no edges. This means that every vertex of Y_0 constitutes its own connected component. This clearly is also the result computed by the algorithm (which only consists of the make-sets step).

For the induction step, let $k \in \{1, \dots, m\}$, and let us assume that the claim holds for $k - 1$. We write $\sigma_k = \{x, y\}$ and distinguish two cases:

- The vertices x and y lie in the same connected component of Y_{k-1}. Then they also lie in the same connected component of Y_k and the connected components of Y_k are the same as those of Y_{k-1}.

  On the side of the algorithm: Because x and y lie in the same component of Y_{k-1}, by induction, we obtain $\mathsf{find}(x) = \mathsf{find}(y)$. Hence, the algorithm gives the same partition for Y_k as for Y_{k-1}.

  Therefore, in this case, the claim holds for k.

- The vertices x and y do *not* lie in the same connected component of Y_{k-1}. Then σ_k witnesses that the connected components of Y_k are the same as the ones of Y_k, except for the component in Y_k of x and y, which is the union of the components of x and y in Y_{k-1}.

  On the side of the algorithm: Because x and y do not lie in the same component of Y_{k-1}, by induction, we obtain that $\mathsf{find}(x) \neq \mathsf{find}(y)$. Hence, the algorithm merges the sets for x and y via $\mathsf{union}(x, y)$ and leaves the other partition sets untouched.

  Therefore, also in this case, the claim holds for k. □

The concrete runtime analysis depends on the complexity of traversals and the complexity of the implementation of the union-find structure.

Outlook 2.8.5 (the limitation of algorithms)**.** The problem

- Given a finite simplicial complex X,
- decide whether $|X|$ is contractible or not.

is algorithmically undecidable. More precisely, using the undecidability of the word problem for finitely presented groups [67, Chapter 12], one can prove that there cannot be an algorithm (in the sense of Turing machines) that solves this problem and that terminates on every input [50, 8].

In particular, also the problem

- Given finite simplicial complexes X and Y,
- decide whether $|X| \simeq |Y|$ or not.

is algorithmically undecidable (otherwise, we could solve the previous problem by taking one input to be a simplicial complex consisting of a single vertex).

Similarly, also the problem

- Given finite simplicial complexes X and Y,
- decide whether $|X| \cong_{\mathsf{Top}} |Y|$ or not.

is algorithmically undecidable [50, 8].

3

Simplicial homology

Simplicial homology is an invariant of simplicial complexes. Simplicial homology is the homology of the simplicial chain complex, which is built from the simplices (graded by dimension) and their boundaries.

On the one hand simplicial homology can be computed by the usual divide-and-conquer approach and it can be computed algorithmically.

On the other hand, simplicial homology turns out to be topologically homotopy invariant and thus defines a homotopy invariant on topological spaces that admit a triangulation.

Using simplicial homology, we can prove several classical "abstract" applications of algebraic topology: the existence of Nash equilibria and impossibility results on distributed systems and social choice.

Overview of this chapter.

Running example. simplicial spheres, simplicial complexes from relations

3.1 The construction of simplicial homology

Simplicial homology is an algebraic linearisation of the simplicial structure, graded by dimension; the different dimensions are linked by taking faces of simplices.

Geometrically, simplicial homology detects certain types of “holes” in spaces by engulfing them with linear combinations of simplices.

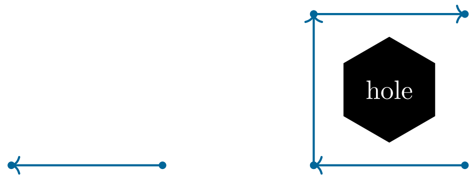


Figure 3.1.: Simplicial chains

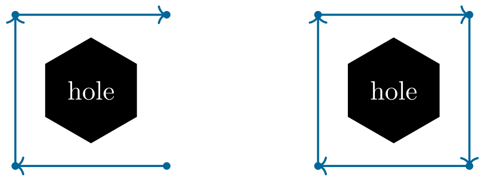


Figure 3.2.: Simplicial cycle; the chain on the left cannot surround a “hole”, the chain on the right does surround a “hole”.

More precisely, we will proceed as follows (Figure 3.1–3.3):

- The simplicial chain complex of a simplicial complex consists of linear combinations of oriented simplices (of a given dimension). These linear combinations are called *simplicial chains*.

- Candidates for chains that detect a “hole” are chains that have “no boundary”, i.e., where the faces of all involved simplices cancel. Such chains are called *cycles*.

- Cycles only detect a “proper hole” if they are not the *boundary* of a higher-dimensional chain.

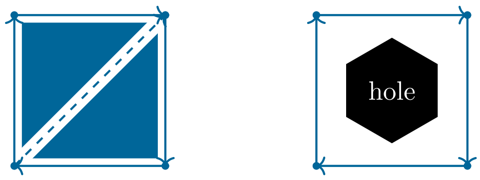


Figure 3.3.: Simplicial boundary; the cycle on the left is the boundary of a simplicial chain and hence cannot detect a "hole"; the cycle on the right is *not* the boundary of a simplicial chain (because the "hole" is in the way).

For a simplicial complex X and $n \in \mathbb{N}$, we will set

$$\text{simplicial homology of } X \text{ in degree } n := \frac{\text{simplicial } n\text{-cycles of } X}{\text{boundaries of simplicial } (n+1)\text{-chains}}.$$

This idea will be formalised in the language of chain complexes and basic homological algebra (Appendix A.3). Historically, the terminology in basic homological algebra geos back to simplicial homology and its variations.

The simplicial chain complex is introduced in Chapter 3.1.1; simplicial homology is defined in Chapter 3.1.2.

3.1.1 The simplicial chain complex

To define the simplicial chain complex as algebraic linearisation of simplicial complexes, we need to take orientations of simplices into account. Orientations of simplices arise from ordering the vertices in a simplex. In non-zero dimension, there are essentially two ways of picking orders. This can be expressed in terms of even/odd permutations. Flipping the orientation will introduce a sign on the chain level.

Definition 3.1.1 (oriented simplex)**.** Let X be a simplicial complex and let $n \in \mathbb{N}$. We write

$$X\langle n\rangle := \bigl\{(x_0, \dots, x_n) \in V(X)^{n+1} \bigm| \{x_0, \dots, x_n\} \in X(n)\bigr\}$$

for the set of all *ordered n-simplices* of X. The symmetric group S_{n+1} acts on $X\langle n\rangle$ through permutation of components. The quotient

$$X[n] := X\langle n\rangle / A_{n+1}$$

by the alternating group $A_{n+1} \subset S_{n+1}$ is the set of *oriented n-simplices* of X. If $(x_0, \dots, x_n) \in X\langle n\rangle$, then we write $[x_0, \dots, x_n]$ for the corresponding oriented simplex in $X[n]$.

Remark 3.1.2 (opposite orientation)**.** Let X be a simplicial complex and let $n \in \mathbb{N}$.

- If $n > 0$, then $[S_{n+1} : A_{n+1}] = 2$ and so $X[n]$ contains two oriented simplices for each simplex in $X(n)$. If $(x_0, \dots, x_n) \in X\langle n\rangle$, then applying an odd permutation (e.g., a transposition) τ leads to the corresponding opposite orientation on $\{x_0, \dots, x_n\}$.

  If $\sigma := [x_0, \dots, x_n] \in X[n]$, then we write $\overline{\sigma} \in X[n]$ for the this oppositely oriented simplex $[x_{\tau(0)}, \dots, x_{\tau(n)}]$.
- In degree 0, we have $A_0 = S_0$ and thus $X[0]$ is canonically (as a set) isomorphic to $X(0)$ and $X\langle 0\rangle$.

Definition 3.1.3 (simplicial chain)**.** Let X be a simplicial complex and $n \in \mathbb{N}$. We define the *simplicial chain group of X in degree n* by

$$C_n(X) := F_n(X) \big/ T_n(X),$$

where

$$F_n(X) := \bigoplus_{\sigma \in X[n]} \mathbb{Z} \cdot \sigma,$$
$$T_n(X) := \begin{cases} \operatorname{Span}_{\mathbb{Z}}\bigl\{\sigma + \overline{\sigma} \bigm| \sigma \in X[n]\bigr\} & \text{if } n > 0 \\ 0 & \text{if } n = 0. \end{cases}$$

The elements of $C_n(X)$ are called *simplicial n-chains* of X.

In order to keep the notation lightweight, we usually denote simplicial chains by representatives:

Example 3.1.4. We consider the simplicial circle $S(1)$ (Example 2.3.7). Then

$$[0,1] + 2 \cdot [2,1] \quad \text{and} \quad [0,1] + [1,2] + [2,0] = [0,1] + [1,2] - [0,2]$$

are simplicial 1-chains of $S(1)$.

Remark 3.1.5 (ordered simplicial complex)**.** Let X be a simplicial complex and let "$<$" be a total order on $V(X)$. For $n \in \mathbb{N}$, the canonical projection $X\langle n\rangle_< \longrightarrow X[n]$ induces an isomorphism (check!)

$$C_n(X) \cong_{\mathbb{Z}} \bigoplus_{X\langle n\rangle_<} \mathbb{Z},$$

where

$$X\langle n\rangle_< := \bigl\{(x_0, \dots, x_n) \in X\langle n\rangle \bigm| x_0 < x_1 < \dots < x_n\bigr\}.$$

In particular, $C_n(X)$ is a free $\mathbb{Z}$-module and this description lets us check whether chains are trivial/equal or not.

The boundary operator of the simplicial chain complex is defined as the alternating sum of the face maps. The signs are chosen in such a way that a coherent orientation emerges on each simplex and its faces.

Proposition and Definition 3.1.6 (simplicial chain complex)**.** *Let X be a simplicial complex and let $n \in \mathbb{N}$. Then*

$$\begin{aligned}\partial_n \colon C_n(X) &\longrightarrow C_{n-1}(X)\\ [x_0,\dots,x_n] &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot [x_0,\dots,\widehat{x_j},\dots,x_n]\end{aligned}$$

gives a well-defined $\mathbb{Z}$-linear map; here, the hat indicates that the corresponding element is omitted. For $n = 0$, we interpret the definition as $\partial_0 = 0$. Moreover,

$$\partial_n \circ \partial_{n+1} = 0.$$

We call the chain complex $C(X) := ((C_n(X))_{n\in\mathbb{N}}, (\partial_n)_{n\in\mathbb{N}})$ the simplicial chain complex of X*.*

Proof. Without loss of generality, we may assume $n > 0$. To show that ∂_n is well-defined, it suffices to show that

$$\partial_n([x_{\tau(0)},\dots,x_{\tau(n)}]) = -\partial_n([x_0,\dots,x_n])$$

holds for all $[x_0,\dots,x_n] \in X[n]$ and for all transpositions $\tau \in S_{n+1}$ (check!). Let $\tau = (r\ s)$ with $r < s$ and let $\overline{x} := \tau \cdot x$. Then we obtain

$$\begin{aligned}
\partial_n[\overline{x}_0,\dots,\overline{x}_n] &= \sum_{j\in\{0,\dots,n\}\setminus\{r,s\}} (-1)^j \cdot [\overline{x}_0,\dots,\widehat{\overline{x}}_j,\dots,\overline{x}_n]\\
&\quad + (-1)^r \cdot [\overline{x}_0,\dots,\widehat{\overline{x}}_r,\dots,\overline{x}_n]\\
&\quad + (-1)^s \cdot [\overline{x}_0,\dots,\widehat{\overline{x}}_s,\dots,\overline{x}_n]\\
&= \sum_{j\in\{0,\dots,n\}\setminus\{r,s\}} (-1)^j \cdot (-1) \cdot [x_0,\dots,\widehat{x}_j,\dots,x_n] \text{(because τ is $(r\ s)$ on $\{0,\dots,n\}\setminus\{j\}$)}\\
&\quad + (-1)^r \cdot [x_0,\dots,x_{r-1},x_{r+1},\dots,x_{s-1},x_r,x_s,\dots,x_n] && \text{(by definition of τ)}\\
&\quad + (-1)^s \cdot [x_0,\dots,x_{r-1},x_s,x_{r+1},\dots,x_{s-1},x_s,\dots,x_n] && \text{(by definition of τ)}\\
&= \sum_{j\in\{0,\dots,n\}\setminus\{r,s\}} (-1)^{j+1} \cdot [x_0,\dots,\widehat{x}_j,\dots,x_n]\\
&\quad + (-1)^r \cdot (-1)^{s-r-1} \cdot [x_0,\dots,\widehat{x}_s,\dots,x_n] && \text{(we need $s-r-1$ flips)}\\
&\quad + (-1)^s \cdot (-1)^{s-r-1} \cdot [x_0,\dots,\widehat{x}_r,\dots,x_n] && \text{(we need $s-r-1$ flips)}\\
&= -\partial_n[x_0,\dots,x_n].
\end{aligned}$$

Moreover, we have $\partial_n \circ \partial_{n+1} = 0$: This is the standard computation for "simplicial" boundary operators. For all $[x_0,\dots,x_{n+1}] \in X[n+1]$, we calculate

$$
\begin{aligned}
\partial_n \circ \partial_{n+1}[x_0, \dots, x_{n+1}] &= \partial_n \biggl(\sum_{j=0}^{n+1} (-1)^j \cdot [x_0, \dots, \widehat{x}_j, \dots, x_{n+1}] \biggr) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k-1} (-1)^{k+j} \cdot [x_0, \dots, \widehat{x}_j, \dots, \widehat{x}_{k+1}, \dots, x_{n+1}] \\
&\quad + \sum_{k=0}^{n} \sum_{j=k+1}^{n+1} (-1)^{k+j} \cdot [x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{n+1}] \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k-1} \bigl((-1)^{k+j} + (-1)^{k+j+1} \bigr) \\
&\qquad \cdot [x_0, \dots, \widehat{x}_j, \dots, \widehat{x}_{k+1}, \dots, x_{n+1}] \qquad \text{(re-indexing the second sum)} \\
&= 0 \qquad \text{(the coefficients cancel)}
\end{aligned}
$$

This completes the proof. □

Definition 3.1.7 (cycle, boundary)**.** Let X be a simplicial complex and let $n \in \mathbb{N}$.

- The elements of $\ker \partial_n \subset C_n(X)$ are the *(simplicial) n-cycles* of X.
- The elements of $\operatorname{im} \partial_{n+1} \subset C_n(X)$ are the *(simplicial) n-boundaries* of X.

Example 3.1.8. We consider the simplicial circle $S(1)$:

- The 1-chain $c := [0,1] + [1,2] + [2,0]$ is a 1-cycle, because
$$\partial_1(c) = [1] - [0] + [2] - [1] + [0] - [2] = 0.$$
However, c is *not* a boundary of $S(1)$, because $S(1)$ has no 2-simplices.
- The 1-chain $c' := [0,1] + 2 \cdot [2,1]$ is *not* a cycle, because
$$\partial_1(c) = [1] - [0] + 2 \cdot [1] - 2 \cdot [2] = -[0] + 3 \cdot [1] - 2 \cdot [2].$$
This chain is non-zero as can be seen from the description through an ordering on the vertices (Remark 3.1.5).

The chain complex $C_*(S(1))$ is isomorphic to (check!)

$$
\begin{array}{ccccc}
0 & & 1 & & 2 \\
\mathbb{Z}^3 & \xleftarrow[\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}]{} & \mathbb{Z}^3 & \xleftarrow[0]{} & 0.
\end{array}
$$

Here, in degree 0, we chose the basis $([0],[1],[2])$; in degree 1, we chose the basis $([0,1],[1,2],[0,2])$.

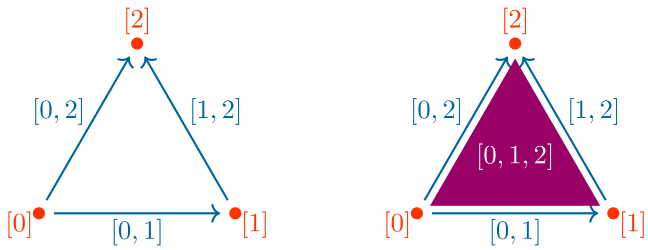


Figure 3.4.: Simplicial chains on $S(1)$ and $\Delta(2)$, respectively.

In $\Delta(2)$, the chain c is a boundary, namely of $[0,1,2]$ (Figure 3.4). The chain complex $C_*(\Delta(2))$ is isomorphic to (check!)

$$\begin{array}{ccccc} 0 & & 1 & & 2 \\ \mathbb{Z}^3 & \xleftarrow[\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}]{} & \mathbb{Z}^3 & \xleftarrow[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}]{} & \mathbb{Z}. \end{array}$$

Here, in degree 0, we chose the basis $([0],[1],[2])$; in degree 1, we chose the basis $([0,1],[1,2],[0,2])$; in degree 2, we chose the basis consisting of the single element $[0,1,2]$.

Proposition 3.1.9 (simplicial chain complex: functoriality)**.** *Let X and Y be simplicial complexes, let $f\colon X \longrightarrow Y$ be a simplicial map, and let $n \in \mathbb{N}$. Then*

$$\begin{aligned} C_n(f)\colon C_n(X) &\longrightarrow C_n(Y) \\ X[n] \ni [x_0,\dots,x_n] &\longmapsto \begin{cases} [f(x_0),\dots,f(x_n)] & \text{if } \#\{f(x_0),\dots,f(x_n)\} = n+1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

gives a well-defined $\mathbb{Z}$-linear map. Moreover, $C(f) := (C_n(f))_{n\in\mathbb{N}}$ is a chain map, i.e.,

$$\forall_{n\in\mathbb{N}} \quad \partial_{n+1} \circ C_{n+1}(f) = C_n(f) \circ \partial_{n+1}.$$

This turns C into a functor $\mathsf{SC} \longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}$.

Proof. It is clear that the definition of ∂_n is compatible with the sign of permutations on $\{0,\dots,n\}$ (the case distinction is permutation invariant).

Moreover, it is clear that C is functorial (check!). Therefore, it remains to prove that $C(f)$ is a chain map: Let $n \in \mathbb{N}$ and let $[x_0,\dots,x_{n+1}] \in X[n+1]$. If $\#\{f(x_0),\dots,f(x_{n+1})\} = n+2$, then it is not difficult to check that the chain map condition is satisfied (check!).

We consider the case that $\#\{f(x_0),\dots,f(x_{n+1})\} \leq n+1$, say there are $r,s \in \{0,\dots,n+1\}$ with $r < s$ and $f(x_r) = f(x_s)$. On the one hand, by definition,

$$\partial_{n+1} \circ C_{n+1}(f)[x_0, \dots, x_{n+1}] = 0.$$

On the other hand, we compute

$$\begin{aligned}
C_n(f) \circ \partial_{n+1}[x_0, \dots, x_{n+1}] &= \sum_{j \in \{0,\dots,n+1\} \setminus \{r,s\}} (-1)^j \cdot C_n(f)[x_0, \dots, \widehat{x}_j, \dots, x_{n+1}] \\
&\quad + (-1)^r \cdot C_n(f)[x_0, \dots, \widehat{x}_r, \dots, x_{n+1}] \\
&\quad + (-1)^r \cdot C_n(f)[x_0, \dots, \widehat{x}_s, \dots, x_{n+1}] \\
&= 0 && \text{(because } f(x_r) = f(x_s)\text{)} \\
&\quad + (-1)^r \cdot C_n(f)[x_0, \dots, \widehat{x}_r, \dots, x_{n+1}] \\
&\quad + (-1)^{r+s-r-1} \cdot C_n(f)[x_0, \dots, \widehat{x}_r, \dots, x_{n+1}] && (f(x_r) = f(x_s);\ s-r-1 \text{ flips}) \\
&= 0.
\end{aligned}$$

Therefore, we have $\partial_{n+1} \circ C_{n+1}(f)[x_0, \dots, x_{n+1}] = C_n(f) \circ \partial_{n+1}[x_0, \dots, x_{n+1}]$ also in this degenerate case. □

Notation 3.1.10 (generalised simplices)**.** In view of the phenomenon in Proposition 3.1.9 and since flipping two equal entries in a tuple does not change the tuple, it is sometimes convenient to introduce the following notation: If X is a simplicial complex, $n \in \mathbb{N}$, and $\{x_0, \dots, x_n\} \in X$ and $\#\{x_0, \dots, x_n\} \leq n$, one could write

$$[x_0, \dots, x_n] := 0 \in C_n(X).$$

This should be handled with care as it might incur additional proof obligations!

3.1.2 Simplicial homology of simplicial complexes

We now define the simplicial homology as the homology of the simplicial chain complex, i.e., as cycles modulo boundaries:

Definition 3.1.11 (simplicial homology)**.** Let $n \in \mathbb{N}$. The *simplicial homology functor in degree n* is defined as

$$H_n := H_n \circ C \colon \mathsf{SC} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod},$$

where the H_n on the right-hand side refers to the algebraic homology functor ${}_{\mathbb{Z}}\mathsf{Ch} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ in degree n. More explicitly:

- If X is a simplicial complex, then

$$H_n(X) = H_n\bigl(C(X)\bigr) = \frac{\ker \partial_n \colon C_n(X) \to C_{n-1}(X)}{\operatorname{im} \partial_{n+1} \colon C_{n+1}(X) \to C_n(X)}.$$

- If $f\colon X \longrightarrow Y$ is a simplicial map, then

$$\begin{aligned} H_n(f) = H_n\bigl(C(f)\bigr)\colon H_n(X) &\longrightarrow H_n(Y) \\ [c] &\longmapsto \bigl[C_n(f)(c)\bigr]. \end{aligned}$$

Remark 3.1.12 (simplicial homology, functoriality)**.** Simplicial homology is indeed well-defined on simplicial maps: Let $f\colon X \longrightarrow Y$ be a simplicial map between simplicial complexes and let $n \in \mathbb{N}$.

- If $c \in C_n(X)$ is a cycle, then $C_n(f)(c)$ indeed is a cycle, because

$$\begin{aligned} \partial_n\bigl(C_n(f)(c)\bigr) &= C_{n-1}(f)\bigl(\partial_n(c)\bigr) && \text{(Proposition 3.1.9)} \\ &= C_{n-1}(f)(0) && \text{(c is a cycle)} \\ &= 0. \end{aligned}$$

- If $c, c' \in C_n(X)$ are cycles that represent the same homology class, then $C_n(f)(c)$ and $C_n(f)(c')$ represent the same homology class: Let $b \in C_{n+1}(X)$ with $\partial_{n+1}(b) = c - c'$. Then $C_{n+1}(f)(b)$ witnesses that $C_n(f)(c)$ and $C_n(f)(c')$ represent the same class, because

$$\begin{aligned} \partial_{n+1}\bigl(C_{n+1}(f)(b)\bigr) &= C_n(f)\bigl(\partial_{n+1}(b)\bigr) && \text{(Proposition 3.1.9)} \\ &= C_n(f)(c - c') && \text{(choice of b)} \\ &= C_n(f)(c) - C_n(f)(c'). \end{aligned}$$

Example 3.1.13 (simplicial homology of the simplicial circle)**.** Using the description of the simplicial chain complex of $S(1)$ from Example 3.1.8, we can compute the simplicial homology of $S(1)$ through linear algebra (over $\mathbb{Z}$):

- Degree 1: Because isomorphisms of chain complexes induce isomorphisms on the level of homology (check!), we can compute (check!)

$$\begin{aligned} H_1\bigl(S(1)\bigr) &\cong_{\mathbb{Z}} \ker \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Big/ 0 \cong_{\mathbb{Z}} \ker \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \operatorname{Span}_{\mathbb{Z}}(1, 1, -1)^T \\ &\cong_{\mathbb{Z}} \mathbb{Z}. \end{aligned}$$

  Translating the chosen basis of $\mathbb{Z}^3$ (in degree 1) back into simplicial chains shows that the simplicial 1-cycle

$$[0,1] + [1,2] + [2,0] = [0,1] + [1,2] - [0,2]$$

  represents a generator of $H_1(S(1)) \cong_{\mathbb{Z}} \mathbb{Z}$. In particular, this coincides with our intuition that this cycle detects the "hole" in $S(1)$.

- Degree 0: Again, using the isomorphism of chain complexes, we compute (check!)

$$\begin{aligned} H_0\bigl(S(1)\bigr) &\cong_{\mathbb{Z}} \mathbb{Z}^3 \Big/ \operatorname{im} \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &\cong_{\mathbb{Z}} \mathbb{Z}; \end{aligned}$$

  the last isomorphism is induced by

$$\begin{aligned} \mathbb{Z}^3 &\longrightarrow \mathbb{Z} \\ (x, y, z)^T &\longmapsto x + y + z \end{aligned}$$

  Translating the chosen basis of $\mathbb{Z}^3$ (in degree 0) back into simplicial chains shows that the simplicial 0-cycles [0], [1], [2] all represent the same homology class, which moreover is a generator of $H_0(S(1)) \cong_{\mathbb{Z}} \mathbb{Z}$.

- Higher degrees: For all $n \in \mathbb{N}_{\geq 2}$, we have $H_n(S(1)) \cong_{\mathbb{Z}} 0$, because $C_n(S(1)) \cong_{\mathbb{Z}} 0$.

Example 3.1.14 (simplicial homology of the standard 2-simplex)**.** We can extend the computation of the simplicial homology of the simplicial circle $S(1)$ from Example 3.1.13 to obtain the computation of the simplicial homology of $\Delta(2)$. Using the description of $C(\Delta(2))$ from Example 3.1.8, we obtain:

- Degree 0: Because $C_0(\Delta(2)) = C_0(S(1))$ and $C_1(\Delta(2)) = C_1(S(1))$ as well as $\partial_1^{\Delta(2)} = \partial_1^{S(1)}$, we have

$$H_0\bigl(\Delta(2)\bigr) \cong_{\mathbb{Z}} H_0\bigl(S(1)\bigr) \cong_{\mathbb{Z}} \mathbb{Z}.$$

- Degree 1: Because $[0,1] + [1,2] + [2,0]$ is a 1-boundary in $\Delta(2)$, because the chain complexes of $\Delta(2)$ and $S(1)$ coincide in degrees 0 and 1, and because $[0,1] + [1,2] + [2,0]$ is a cycle representing a generator of $H_1(S(1)) \cong_{\mathbb{Z}} \mathbb{Z}$, we conclude that

$$H_1\bigl(\Delta(2)\bigr) \cong_{\mathbb{Z}} 0.$$

  Alternatively, of course, we could also compute $H_1(\Delta(2))$ via linear algebra over $\mathbb{Z}$ directly from the description of $C(\Delta(2))$.

- Degree 2: We compute

$$\begin{aligned} H_2\bigl(\Delta(2)\bigr) &\cong_{\mathbb{Z}} \ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Big/ 0 \\ &\cong_{\mathbb{Z}} 0. \end{aligned}$$

- Higher degrees: For all $n \in \mathbb{N}_{\geq 3}$, we have $H_n(\Delta(2)) \cong_{\mathbb{Z}} 0$, because $C_n(\Delta(2)) \cong_{\mathbb{Z}} 0$.

Remark 3.1.15 (simplicial homology in degree 0)**.** If X is a simplicial complex, then $H_0(X)$ is isomorphic to $\bigoplus_S \mathbb{Z}$, where S denotes the set of connected components of X (Exercise).

Example 3.1.16 (a reflection in simplicial homology of the simplicial circle)**.** We consider the simplicial map $f \colon S(1) \longrightarrow S(1)$ given by the transposition $(0\ 1)$ on the vertices. Then $H_1(f) \colon H_1(S(1)) \longrightarrow H_1(S(1))$ is $-\operatorname{id}_{H_1(S(1))}$. This can be seen by applying $C_1(f)$ to the generating 1-cycle $c := [0,1] + [1,2] + [2,0]$ of $H_1(S(1)) \cong_{\mathbb{Z}} \mathbb{Z}$ (Example 3.1.13):

$$\begin{aligned} C_1(f)(c) &= \bigl[f(0), f(1)\bigr] + \bigl[f(1), f(2)\bigr] + \bigl[f(2), f(0)\bigr] \\ &= [1,0] + [0,2] + [2,1] \\ &= -[0,1] - [2,0] - [1,2] \\ &= -c. \end{aligned}$$

Hence, the induced map on homology is an "algebraic reflection".

Remark 3.1.17 (simplicial homology with coefficients)**.** Let R be a commutative ring with unit and let Z be an R-module. Let $n \in \mathbb{N}$. We define the *simplicial chain complex with coefficients in the R-module Z* by

$$C(\,\cdot\,;Z) := Z \otimes_{\mathbb{Z}} C(\,\cdot\,) \in {}_R\mathsf{Ch}$$

and *simplicial homology in degree n with coefficients in the R-module Z* as the composition

$$H_n(\,\cdot\,;Z) := H_n \circ \bigl(C(\,\cdot\,;Z)\bigr) \colon \mathsf{SC} \longrightarrow {}_R\mathsf{Mod}\,.$$

By construction, there is a canonical natural isomorphism $H_n(\,\cdot\,;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(\,\cdot\,) \colon \mathsf{SC} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$.

A convenient choice of coefficients is the $\mathbb{Z}$-module $\mathbb{Z}/2$: Then no orientations/signs need to be considered.

Particularly interesting is the case of $Z = R$ being a field or a principal ideal domain: The size of homology groups can then be measured in terms of dimensions/ranks.

Definition 3.1.18 (Betti number)**.** Let X be a finite simplicial complex, let $n \in \mathbb{N}$, and let R be a principal ideal domain (or another ring with a "reasonable" notion of rank). The *n-th Betti number of X with coefficients in R* is defined as

$$b_n(X;R) := \operatorname{rk}_R H_n(X;R).$$

Proposition 3.1.19 (Betti number estimate)**.** *Let X be a finite simplicial complex, let $n \in \mathbb{N}$, and let R be a principal ideal domain (or another ring with a "reasonable" notion of rank). Then*

$$b_n(X;R) \leq \#X(n).$$

Proof. By definition, $H_n(X;R)$ is isomorphic to a quotient of a submodule $Z_n(X;R)$ of $C_n(X;R)$.

Because $C_n(X)$ is a free $\mathbb{Z}$-module (Remark 3.1.5) of rank $\#X(n)$, the R-module $C_n(X;R) = R \otimes_{\mathbb{Z}} C_n(X)$ is free of rank $\#X(n)$. As R is a principal ideal domain, also $Z_n(X;R)$ is free and of rank at most $\#X(n)$. Therefore, also the quotient $H_n(X;R)$ has rank at most $\#X(n)$. □

Caveat 3.1.20 (the universal coefficient theorem)**.** Let R be a commutative ring with unit and let Z be an R-module. Let $n \in \mathbb{N}$. If Z is flat over R, then basic homological algebra shows that there is a natural isomorphism

$$H_n(\,\cdot\,;Z) \cong Z \otimes_{\mathbb{Z}} H_n(\,\cdot\,;R)\colon\ \mathsf{SC} \longrightarrow {}_R\mathsf{Mod}\,.$$

For example, $\mathbb{Q}$ and $\mathbb{R}$ are flat over $\mathbb{Z}$; finite fields are *not* flat over $\mathbb{Z}$.

For general coefficients that are not necessarily flat over the base ring, the relation can be described in terms of the derived functor of the tensor product. By the *universal coefficient theorem* [73, Chapter 3.6], we have: If R is a principal ideal domain, then there is a natural short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & Z \otimes_R H_n(X;R) & \longrightarrow & H_n(X;Z) & \longrightarrow & \operatorname{Tor}_1^R\bigl(Z, H_{n-1}(X;R)\bigr) & \longrightarrow 0 \\ & z \otimes [c] & \longmapsto & [z \otimes c] & & & \end{array}$$

of R-modules. This short exact sequence splits (but, in general, there is no natural splitting).

3.2 Computations: Divide and conquer

We give three classical instances of the divide and conquer paradigm to compute homology from smaller pieces: The Mayer–Vietoris sequence, the long exact sequence of pairs, and simplicial homotopy invariance. To keep notation simple, we formulate everything in terms of $\mathbb{Z}$-coefficients, but the statements also hold for general coefficients.

3.2.1 The Mayer–Vietoris sequence

The Mayer–Vietoris is a homological version of the inclusion-exclusion principle, relating the homology of a union to the homology of the components and the intersection.

Theorem 3.2.1 (Mayer–Vietoris sequence)**.** *Let X and Y be simplicial complexes. Then there is a natural long exact sequence*

$$\cdots \xrightarrow{\Delta_{n+1}} H_n(X \cap Y) \xrightarrow{(H_n(i_X), -H_n(i_Y))} H_n(X) \oplus H_n(Y) \xrightarrow{H_n(j_X) \oplus H_n(j_Y)} H_n(X \cup Y) \xrightarrow{\Delta_n} H_{n-1}(X \cap Y) \longrightarrow \cdots$$

Here, i_X, i_Y, j_X, j_Y denote the corresponding inclusions.

Proof. A straightforward computation shows that the sequence

$$0 \longrightarrow C(X \cap Y) \xrightarrow{(C(i_X), -C(i_Y))} C(X) \oplus C(Y) \xrightarrow{C(j_X) \oplus C(j_Y)} C(X \cup Y) \longrightarrow 0$$

in ${}_{\mathbb{Z}}\mathsf{Ch}$ is (split) exact in every degree (check!). Moreover, this sequence is natural.

Applying the algebraic long exact sequence (Proposition A.3.20) to this setting proves the theorem. In particular, this also gives an explicit description of Δ_n. □

Example 3.2.2 (simplicial figure eight)**.** We compute the simplicial homology of the complex

$$\bigl\langle \{0,1\}, \{1,2\}, \{0,2\}, \{0,1'\}, \{1',2'\}, \{0,2'\} \bigr\rangle_\Delta$$

We can write this complex as $X \cup Y$, where

$$X := S(1) \quad \text{and} \quad Y := \bigl\langle \{0,1'\}, \{1',2'\}, \{0,2'\} \bigr\rangle_\Delta.$$

Looking at the dimensions of the simplices, we see that we only need to compute $H_0(X \cup Y)$ and $H_1(X \cup Y)$. Because $X \cup Y$ is connected, we obtain (Remark 3.1.15) that

$$H_0(X \cup Y) \cong_{\mathbb{Z}} \mathbb{Z}.$$

For degree 1, we use the Mayer–Vietoris sequence (Theorem 3.2.1): We thus obtain the following exact sequence:

$$H_1(X \cap Y) \xrightarrow{(H_1(i_X), -H_1(i_Y))} H_1(X) \oplus H_1(Y) \xrightarrow{H_1(j_X) \oplus H_1(j_Y)} H_1(X \cup Y) \xrightarrow{\Delta_1} H_0(X \cap Y) \xrightarrow{H_0(i_x), -H_0(i_Y)} H_0(X) \oplus H_0(Y)$$

On the one hand, because $X \cap Y = \{\emptyset, \{0\}\}$ has no 1-simplices, $H_1(X \cap Y) \cong_{\mathbb{Z}} 0$. On the other hand, because $X \cap Y$ and X, Y are connected, the homomorphisms $H_0(i_X)$ and $H_0(i_Y)$ are injective (check!); hence, exactness shows that $\Delta_1 = 0$. The exact sequence therefore reduces to

$$0 \longrightarrow H_1(X) \oplus H_1(Y) \xrightarrow{H_1(j_X) \oplus H_1(j_Y)} H_1(X \cup Y) \xrightarrow{0} 0.$$

Because X and Y are isomorphic to $S(1)$, we obtain with Example 3.1.13 that

$$H_1(X \cup Y) \cong_{\mathbb{Z}} H_1(X) \oplus H_1(Y) \cong_{\mathbb{Z}} H_1\bigl(S(1)\bigr) \oplus H_1\bigl(S(1)\bigr) \cong_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}.$$

Remark 3.2.3 (Mayer–Vietoris sequence with coefficients)**.** There is also a corresponding long exact Mayer–Vietoris sequence for simplicial homology with coefficients. The proof extends to this general case, because the tensor product functor turns *split* exact sequences into exact sequences.

3.2.2 The long exact sequence of pairs

A common strategy to study the difference between spaces in topology is to consider pairs of spaces (instead of quotients, which often lead to pathologies). In our simplicial setting, this amounts to considering pairs, consisting of a simplicial complex and a subcomplex. The simplicial homology of such pairs is defined as the homology of the quotient of the corresponding simplicial chain complexes:

Definition 3.2.4 (relative simplicial chain complex)**.** Let X be a simplicial complex and let $Y \subset X$ be a subcomplex. Then the *relative simplicial chain complex of* (X, Y) is defined as the degree-wise quotient

$$C(X, Y) := C(X)/C(Y)$$

(with the boundary operator induced by the boundary operator on $C(X)$).

This indeed leads to a well-defined chain complex (check!). The construction of the relative simplicial chain complex is functorial with respect to simplicial maps that map the domain subcomplex to the target subcomplex (check!).

Remark 3.2.5. Let X be a simplicial complex (with a total ordering on $V(X)$), let $Y \subset X$ be a subcomplex, and let $n \in \mathbb{N}$. Then $C_n(X, Y) = C_n(X)/C_n(Y)$ is a free $\mathbb{Z}$-module, freely generated by all ordered n-simplices of X that are not completely contained in Y (check!).

Definition 3.2.6 (relative simplicial homology)**.** Let $n \in \mathbb{N}$, let X be a simplicial complex, and let $Y \subset X$ be a subcomplex. Then the *relative simplicial homology of* (X, Y) *in degree* n is defined as

$$H_n(X, Y) := H_n\bigl(C(X, Y)\bigr) \in {}_{\mathbb{Z}}\mathsf{Mod}$$

(where H_n on the right-hand side refers to the algebraic homology of chain complexes).

If X' is a simplicial complex with a subcomplex $Y' \subset X'$ and $f \colon X \longrightarrow X'$ is a simplicial map with $f(\sigma) \in Y'$ for all $\sigma \in Y$, then we define

$$H_n(f) := H_n\bigl(C(f) \colon C(X, Y) \longrightarrow C(X', Y')\bigr).$$

A straightforward computation shows that relative simplicial homology is functorial (check!).

The simplicial homology of a pair and the simplicial homologies of the two involved complexes are related through a long exact sequence:

Theorem 3.2.7 (long exact sequence of pairs)**.** *Let X be a simplicial complex and let $Y \subset X$ be a subcomplex. Then there is a natural long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} H_n(Y) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,Y) \xrightarrow{\partial_n} H_{n-1}(Y) \longrightarrow \cdots$$

Here, i and j denote the corresponding inclusions and ∂_n is induced by the simplicial boundary operator.

Proof. By construction, the sequence

$$0 \longrightarrow C(Y) \xrightarrow{C(i)} C(X) \xrightarrow{C(j)} C(X,Y) \longrightarrow 0$$

in $_{\mathbb{Z}}\mathsf{Ch}$ is exact in every degree (even split exact; check!) and natural.

Applying the algebraic long exact sequence (Proposition A.3.20) to this setting proves the theorem. □

Example 3.2.8 (2-simplex relative to the simplicial circle)**.** We compute the relative homology $H_n(\Delta(2), S(1))$ for all $n \in \mathbb{N}$, using the computations from Example 3.1.13, Example 3.1.14, and the long exact sequence of pairs (Theorem 3.2.7): Because of $\dim \Delta(2) = 2$, we have

$$\forall_{n \in \mathbb{N}_{\geq 3}} \quad H_n\bigl(\Delta(2), S(1)\bigr) \cong_{\mathbb{Z}} 0.$$

In degree 2, we obtain the pair sequence

$$0 \cong_{\mathbb{Z}} H_2\bigl(\Delta(2)\bigr) \longrightarrow H_2\bigl(\Delta(2), S(1)\bigr) \longrightarrow H_1\bigl(S(1)\bigr) \longrightarrow H_1\bigl(\Delta(2)\bigr) \cong_{\mathbb{Z}} 0.$$

In particular, we have $H_2\bigl(\Delta(2), S(1)\bigr) \cong_{\mathbb{Z}} H_1\bigl(S(1)\bigr) \cong_{\mathbb{Z}} \mathbb{Z}$.

Moreover, tracing through the isomorphisms shows that this homology group is generated by the relative cycle $[0, 1, 2]$ (check!).

In degree 1, we have the pair sequence

$$0 \cong_{\mathbb{Z}} H_1\bigl(\Delta(2)\bigr) \longrightarrow H_1\bigl(\Delta(2), S(1)\bigr) \longrightarrow H_0\bigl(S(1)\bigr) \xrightarrow{(*)} H_0\bigl(\Delta(2)\bigr),$$

where $(*)$ is induced by the inclusion $S(1) \subset \Delta(2)$. Because $S(1)$ and $\Delta(2)$ are connected, the homomorphism $(*)$ is an isomorphism. Therefore,

$$H_1\bigl(\Delta(2), S(1)\bigr) \cong_{\mathbb{Z}} 0.$$

Similarly, one can show $H_0(\Delta(2), S(1)) \cong_{\mathbb{Z}} 0$ (check!).

Remark 3.2.9 (long exact sequence of pairs with coefficients)**.** There is also a corresponding definition of relative simplicial homology with coefficients and a long exact sequence of pairs for simplicial homology with coefficients. The proof extends to shi general case, because the tensor product functor turns *split* exact sequences into exact sequences.

3.2.3 Simplicial homotopy invariance

Simplicial homology is invariant under simplicial homotopy. More precisely, simplicially homotopic maps lead to chain homotopic chain maps and thus to the same maps on simplicial homology:

Theorem 3.2.10 (simplicial homotopy invariance)**.** *Let X and Y be simplicial complexes and let $f, g\colon X \longrightarrow Y$ be simplicial maps with $f \simeq_\Delta^* g$.*

1. *Then*
$$C(f) \simeq_{\mathbb{Z}} C(g)\colon C(X) \longrightarrow C(Y),$$
i.e., there exists a chain homotopy $h = (h_n\colon C_n(X) \to C_{n+1}(Y))_{n\in\mathbb{N}}$ from $C(f)$ to $C(g)$, which means that for all $n \in \mathbb{N}$, we have
$$\partial^Y_{n+1} \circ h_n + h_{n-1} \circ \partial^X_n = C_n(g) - C_n(f).$$
2. *In particular, $H_n(f) = H_n(g)$ for all $n \in \mathbb{N}$.*

Proof. The second part is a purely algebraic consequence of the first part (Proposition A.3.32).

For the first part, by induction, it suffices to consider the case that $f \simeq_\Delta g$. We construct a chain homotopy $C(f) \simeq_{\mathbb{Z}} C(g)$ from a simplicial homotopy $k\colon X \boxtimes \Delta(1) \longrightarrow Y$ from f to g:

To this end, we first consider the universal case: Let $i_0, i_1\colon X \longrightarrow X \boxtimes \Delta(1)$ be the inclusions into the 0- and 1-component, respectively. We choose a total ordering on $V(X)$. Then

$$\begin{aligned} h_n\colon C_n(X) &\longrightarrow C_{n+1}\bigl(X \boxtimes \Delta(1)\bigr) \\ X\langle n\rangle \ni (x_0, \dots, x_n) &\longmapsto \sum_{j=0}^n (-1)^j \cdot \bigl[(x_0, 0), \dots, (x_j, 0), (x_j, 1), \dots (x_n, n)\bigr] \end{aligned}$$

defines a chain homotopy $C(i_0) \simeq_{\mathbb{Z}} C(i_1)$. This follows from a straightforward calculation (check!); the underlying geometric idea is that this chain map is based on a suitable triangulation of prisms over the standard simplex (Figure 3.5).

Then $(C_{n+1}(k) \circ h_n)_{n\in\mathbb{N}}$ is a chain homotopy from

$$C(k) \circ C(i_0) = C(k \circ i_0) = C(f)$$

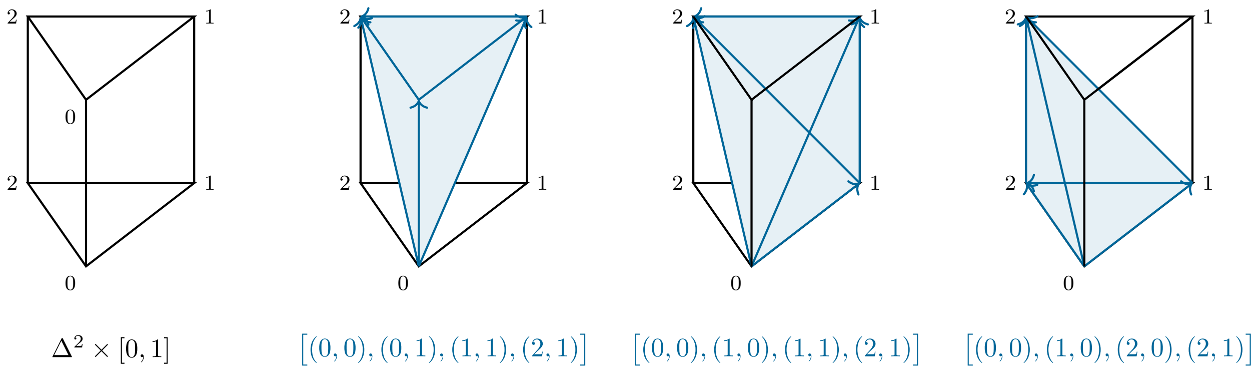


Figure 3.5.: Triangulating the prism $\Delta^2 \times [0,1]$

to

$$C(k) \circ C(i_1) = C(k \circ i_1) = C(g)$$

(check!). $\square$

Example 3.2.11 (simplicial homology of the simplicial standard simplex)**.** Let $d \in \mathbb{N}$. We show that

$$H_n\bigl(\Delta(d)\bigr) \cong_{\mathbb{Z}} H_n\bigl(\Delta(0)\bigr) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

holds for all $n \in \mathbb{N}$: We consider the inclusion map $f \colon \Delta(0) \longrightarrow \Delta(d)$ and the constant map $g \colon \Delta(d) \longrightarrow \Delta(0)$. By construction $g \circ f = \mathrm{id}_{\Delta(0)}$. Conversely, because $\Delta(d)$ contains all simplices on its vertex set, $f \circ g$ is contiguous to $\mathrm{id}_{\Delta(d)}$, whence (Proposition 2.3.34)

$$f \circ g \simeq_\Delta \mathrm{id}_{\Delta(d)}\,.$$

Therefore, $H_n(f)$ and $H_n(g)$ are mutually inverse isomorphisms by simplicial homotopy invariance of simplicial homology (Theorem 3.2.10, Proposition 1.3.2), and so $H_n(\Delta(d)) \cong_{\mathbb{Z}} H_n(\Delta(0))$. Moreover, from the chain complex $C(\Delta(0))$, we can immediately calculate that $H_n(\Delta(0))$ has the claimed isomorphism type (check!).

This result on simplicial homology can be used in combination with the dimension formula to show that (Exercise)

$$\sum_{k=1}^{d+1} (-1)^{k-1} \cdot \binom{d+1}{k} = 1.$$

Example 3.2.12 (homology of simplicial spheres). Let $d \in \mathbb{N}_{>0}$. Using a direct computation of $H_*(\Delta(d+1), S(d))$, the computation of $H_*(\Delta(d+1))$ (Example 3.2.11), and the long exact sequence of the pair $(\Delta(d+1), S(d))$ (Theorem 3.2.7), one can show that

$$H_n\bigl(S(d)\bigr) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } n \in \{0, d\} \\ 0 & \text{otherwise} \end{cases}$$

holds for all $n \in \mathbb{N}$ (Exercise).

Remark 3.2.13 (simplicial homotopy invariance with coefficients). Simplicial homotopy invariance of simplicial homology extends to simplicial homology with coefficients: The tensor product functors turn chain homotopies into chain homotopies and thus chain homotopic chain maps into chain homotopic chain maps.

We will see in Chapter 3.4 that simplicial homotopy is not only invariant under simplicial homotopies, but also under topological homotopies(!). This will have far-reaching consequences.

3.3 Implementation: Simplicial homology

The computation of simplicial homology of finite simplicial complexes reduces to finite-dimensional linear algebra and can be carried out algorithmically over base rings such as $\mathbb{Q}$, $\mathbb{Z}$, or finite fields. There are different levels of detail that we might want to compute over such rings R:

- The Betti numbers with R-coefficients;
- the isomorphism types of the homology modules with R-coefficients;
- the isomorphism types and suitable generating cycles of the homology modules with R-coefficients.

In the following, we briefly outline some simple-minded approaches to these problems. For large-scale applications more refined and more efficient algorithms are necessary.

Our (meta) algorithms are based on the following building blocks:

- An algorithm $\mathsf{rank}(\,\cdot\,, R)$ that computes the rank of the given matrix over R (e.g., by suitable row- and column operations).
- An algorithm $\mathsf{SNF}(\,\cdot\,, R)$ that computes the Smith normal form of the given matrix over R.

- An algorithm extSNF(· , R) that computes the Smith normal form A' of the given matrix A over R as well as invertible R-linear transformations S and T with
$$S \cdot A \cdot T = A'.$$

Remark 3.3.1 (Smith normal form)**.** We briefly recall the Smith normal form: Let R be a field or a principal ideal domain and let $m, n \in \mathbb{N}$. A matrix in $M_{m\times n}(R)$ is in *Smith normal form* if it is of the "diagonal" form

$$\begin{pmatrix} a_1 & 0 & \dots & \\ 0 & a_2 & 0 & \dots \\ \vdots & & \ddots & \end{pmatrix}$$

with $a_1|a_2$, $a_2|a_3$, ...

For every $A \in M_{m\times n}(R)$, there exist invertible matrices $S \in \mathrm{GL}_m(R)$ and $T \in \mathrm{GL}_n(R)$ such that $S \cdot A \cdot T$ is in Smith normal form. The corresponding diagonal entries are unique up to units and called the *elementary divisors of* A [45, Chapter 2.5.2]. If R is Euclidean (or a field), then there exist elimination algorithms to compute "the" Smith normal form via row and column operations [45, Chapter 2.5.2].

We assume that all computations are carried out in exact arithmetic; this is available in many programming languages for integers, rational numbers, and finite fields. Exact arithmetic is usually less efficient than floating point arithmetic, but has the advantage that the results will be exact (provided the implementation is correct) and not only numerical approximations. While the issue of numerical stability need not be considered in this exact setting, similar problems should be analysed, e.g., such as the growth of the numbers involved in the computations (as this can lead to performance problem – both in time and space).

Moreover, in practice, the matrices of simplicial complexes appearing in applications tend to be large, but very sparse. For efficient implementations, this should be exploited.

For simplicity, we will not consider any of these problems or optimisations Further details on improved versions can be found in the literature [27, 54].

Setup 3.3.2 (marked free chain complex)**.** Let R be a field or a principal ideal domain. Let $C = ((C_n)_{n\in\mathbb{N}}, (\partial_n)_{n\in\mathbb{N}})$ be a chain complex over R that consists of finitely generated free R-modules. Moreover, we assume that B_n for each $n \in \mathbb{N}$ is an R-basis of C_n. We write

$$d_n := \mathrm{rk}_R\, C_n = \#B_n.$$

Let $A_n \in M_{d_n \times d_{n+1}}(R)$ be the matrix representing ∂_n with respect to the bases B_{n+1} and B_n.

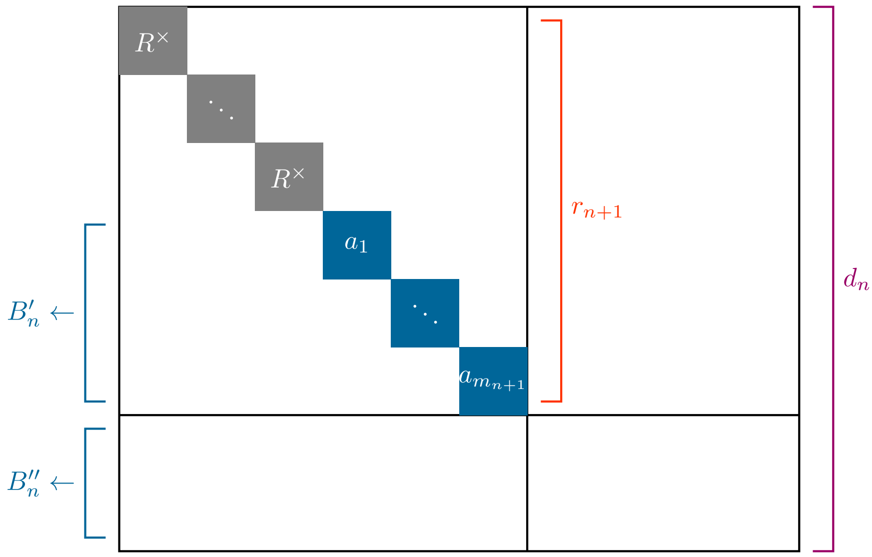


Figure 3.6.: Computing homology and a generating set, schematically; all unmarked entries are zero

Remark 3.3.3 (homology of a marked free chain complex)**.** In the situation of Setup 3.3.2, we collect the following facts from linear algebra: We abbreviate

$$r_n := \operatorname{rk}_R(\operatorname{im} \partial_n) = \operatorname{rk}_R A_n \in \mathbb{N}.$$

Then the dimension formula (over R or its quotient field) shows that

$$\begin{aligned} \operatorname{rk}_R H_n(C) &= \operatorname{rk}_R \ker \partial_n / \operatorname{im} \partial_{n+1} \\ &= \bigl(\operatorname{rk}_R C_n - \operatorname{rk}_R(\operatorname{im} \partial_n)\bigr) - \operatorname{rk}_R(\operatorname{im} \partial_{n+1}) \\ &= d_n - r_n - r_{n+1}. \end{aligned}$$

Let $a_1, \dots, a_{m_{n+1}}$ be the non-unit and non-zero elementary divisors of A_{n+1} (Figure 3.6). Then

$$\begin{aligned} H_n(C) &= \ker \partial_n / \operatorname{im} \partial_{n+1} \\ &\cong_R R^{d_n - r_n - r_{n+1}} \oplus \bigoplus_{j=1}^{m_{n+1}} R/a_j. \end{aligned}$$

Let S, T be invertible matrices such that $S \cdot A_{n+1} \cdot T$ is in Smith normal form. The columns of S define a basis of C_n, viewed as coefficients for vectors in terms of B_n. The first r_{n+1} columns all lie in $\ker \partial_n$ because they form a basis of $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$. Let

- B'_n be the family in C_n corresponding to the columns $r_{n+1} - m_{n+1} + 1, \dots, r_{n+1}$ of S, and
- B''_n be the family in C_n corresponding to the columns $r_{n+1} + 1, \dots, d_n$.

If B'''_n is a basis of $\ker \partial_n \cap \operatorname{Span}_R B''_n = \ker \partial_n|_{\operatorname{Span}_R B'''_n}$, then $B'_n \cup B'''_n$ represents a generating set of $H_n(C)$, which is moreover minimal with respect to inclusion. The torsion type of the elements in $H_n(C)$ represented by B'_n are given by the corresponding elementary divisors; the elements represented by B'''_n are a basis of a free summand of full rank.

We can rephrase these observations in algorithmic terms:

Algorithm 3.3.4 (computation of Betti numbers)**.** Given the situation in Setup 3.3.2 and $n \in \mathbb{N}$, do the following:

- compute r_n via $\mathsf{rank}(A_n, R)$;
- compute r_{n+1} via $\mathsf{rank}(A_{n+1}, R)$;
- return $d_n - r_n - r_{n+1}$.

Algorithm 3.3.5 (computation of the isomorphism type of homology)**.** Given the situation in Setup 3.3.2 and $n \in \mathbb{N}$, do the following:

- compute r_n via $\mathsf{rank}(A_n, R)$;
- compute r_{n+1} via $\mathsf{rank}(A_{n+1}, R)$;
- compute the Smith normal form A' of A_{n+1} via $\mathsf{SNF}(A_{n+1}, R)$;
- extract the non-unit and non-zero elementary divisors $a_1, \dots, a_{m_{n+1}}$ of A_{n+1} from the "diagonal" elements of A';
- return (a suitable representation of) $R^{d_n - r_n - r_{n+1}} \oplus \bigoplus_{j=1}^{m_{n+1}} R/a_j$.

Algorithm 3.3.6 (computation of the isomorphism type of homology and generating cycles)**.** Given the situation in Setup 3.3.2 and $n \in \mathbb{N}$, do the following:

- compute r_n via $\mathsf{rank}(A_n, R)$;
- compute r_{n+1} via $\mathsf{rank}(A_{n+1}, R)$;
- compute the Smith normal form $A' = S \cdot A_{n+1} \cdot T$ of A_{n+1} and corresponding transformations S, T via $\mathsf{extSNF}(A_{n+1}, R)$;
- extract the non-unit and non-zero elementary divisors $a_1, \dots, a_{m_{n+1}}$ of A_{n+1} from the "diagonal" elements of A';
- let B'_n be the vectors given by the columns $r_{n+1} - m_{m+1} + 1, \dots, m_{n+1}$ of S; let B''_n be the columns $m_{n+1} + 1, \dots, d_n$ of S.

- compute the matrix A'_n representing $\partial_n|_{\operatorname{Span}_R B''_n} \colon \operatorname{Span}_R B''_n \longrightarrow C_{n-1}$.
- compute a basis B'''_n of the kernel of A'_n via $\mathsf{extSNF}(A'_n, R)$.
- return (a suitable representation of) $R^{d_n - r_n - r_{n+1}} \oplus \bigoplus_{j=1}^{m_{n+1}} R/a_j$ and the family $B'_n \cup B'''_n$. The torsion type of the elements in $H_n(C)$ represented by B'_n are given by the corresponding elementary divisors; the elements represented by B'''_n are a basis of a free summand of full rank.

Proposition 3.3.7. *In the situation of Setup 3.3.2, we have:*

1. *The algorithm specified in Algorithm 3.3.4 terminates on every input and computes for a given $n \in \mathbb{N}$ the Betti number $\operatorname{rk}_R H_n(C)$.*
2. *The algorithm specified in Algorithm 3.3.5 terminates on every input and computes for a given $n \in \mathbb{N}$ the isomorphism type of $H_n(C)$ as R-module.*
3. *The algorithm specified in Algorithm 3.3.6 terminates on every input and computes for a given $n \in \mathbb{N}$ the isomorphism type of $H_n(C)$ as R-module and a (minimal with respect to inclusion) family of cycles that represent a generating set of $H_n(C)$, including their torsion types in $H_n(C)$.*

Proof. Correctness follows from $H_n(C) \cong_R \ker \partial_n / \operatorname{im} \partial_{n+1}$ and basic considerations in linear algebra (Remark 3.3.3). □

Over $\mathbb{Q}$, $\mathbb{Z}$, and finite fields, with the standard type of row/column algorithms, the time complexity is polynomial in the number of arithmetic operations.

Straightforward modifications of these algorithms also allow us to compute whether a given cycle represents the trivial class in homology, etc..

Example 3.3.8 (computation of simplicial homology)**.** Given a field or a principal ideal domain R, a finite simplicial complex X (with ordered vertices), and $n \in \mathbb{N}$, we can use Algorithm 3.3.4, Algorithm 3.3.5, Algorithm 3.3.6 to compute

- the Betti number $b_n(X; R)$,
- the isomorphism type of $H_n(X; R)$,
- and a family of simplicial n-cycles that represent a generating set of $H_n(X; R)$ (including the torsion types), which is minimal with respect to inclusion,

respectively. Indeed, we apply these algorithms to the chain complex $C(X; R)$, with an R-basis obtained as in Remark 3.1.5.

Example 3.3.9 (computing homology with the Python library simplicial)**.** The Python library simplicial [25] provides an interface to construct and manipulate finite simplicial complexes and to compute their Betti numbers with $\mathbb{F}_2$-coefficients. The underlying computation of the Smith normal form over $\mathbb{F}_2$ follows the algorithm described by Edelsbrunner and Harer [29, Chapter IV.2].

We give some simple examples on how to use this library, starting with the simplicial circle:

```
from typing import *
from simplicial import *

# the simplicial circle
circle = SimplicialComplex()
circle.addSimplexWithBasis(bs = [0,1])
circle.addSimplexWithBasis(bs = [1,2])
circle.addSimplexWithBasis(bs = [0,2])
```

In an interactive Python shell, we can perform queries and computations:

```
>>> circle.numberOfSimplicesOfOrder()
[3, 3]
>>> circle.simplices()
[0, 1, 2, '1d0', '1d1', '1d2']
>>> circle.bettiNumbers()
{0: 1, 1: 1}
```

Similarly, we can model $\Delta(2)$ and the simplicial spheres:

```
# the simplicial disk
disk = SimplicialComplex()
disk.addSimplexWithBasis(bs = [0,1,2])

# the simplicial sphere S(2)
sphere = SimplicialComplex()
sphere_simplices = [[0,1,2], [0,1,3], [0,2,3], [1,2,3]]
for sigma in sphere_simplices :
    sphere.addSimplexWithBasis(bs = sigma)

# simplicial sphere of a given dimension
def sphereOfDim (n: int) -> SimplicialComplex:
    c = SimplicialComplex()
    vertices = list(range(0,n+2))
    c.addSimplexWithBasis(bs = vertices, id = 'max_simplex')
    c.deleteSimplex('max_simplex')
    return c
```

We obtain the expected results for the $\mathbb{F}_2$-Betti numbers:

```
>>> disk.bettiNumbers()
{0: 1, 1: 0, 2: 0}
>>> sphere.bettiNumbers()
```

```
{0: 1, 1: 0, 2: 1}
>>> sphereOfDim(3).bettiNumbers()
{0: 1, 1: 0, 2: 0, 3: 1}
```

Continuing these examples, we define a more convenient generation function and compute the $\mathbb{F}_2$-Betti numbers of triangulations of the 2-torus, of the Klein bottle, and of the real projective plane. To this end, we first construct suitable simplicial complexes [49, Chapter 3.5]:

```
# generating a simplicial complex from a list of simplices (with the given "bases")
def genSimplicialComplex(simplices: List[Any]) -> SimplicialComplex:
    c = SimplicialComplex()
    for sigma in simplices :
        c.addSimplexWithBasis(bs = sigma)
    return c

# torus
torus_simplices = [[0, 1, 4], [0, 3, 4], [1, 2, 5], [1, 4, 5], [2, 0, 3], [2, 5, 3],
                   [3, 4, 7], [3, 6, 7], [4, 5, 8], [4, 7, 8], [5, 3, 6], [5, 8, 6],
                   [6, 7, 1], [6, 0, 1], [7, 8, 2], [7, 1, 2], [8, 6, 0], [8, 2, 0]]
torus = genSimplicialComplex(torus_simplices)

# Klein bottle
kbottle_simplices = [[0, 1, 4], [0, 3, 4], [1, 2, 5], [1, 4, 5], [2, 0, 6], [2, 5, 6],
                     [3, 4, 7], [3, 6, 7], [4, 5, 8], [4, 7, 8], [5, 3, 6], [5, 8, 3],
                     [6, 7, 1], [6, 0, 1], [7, 8, 2], [7, 1, 2], [8, 3, 0], [8, 2, 0]]
kbottle = genSimplicialComplex(kbottle_simplices)

# projective plane
rp2_simplices = [[0, 1, 5], [0, 4, 5], [1, 2, 6], [1, 5, 6], [2, 3, 6], [3, 6, 7],
                 [4, 5, 8], [4, 7, 8], [5, 6, 9], [5, 8, 9], [6, 7, 4], [6, 4, 9],
                 [7, 8, 2], [7, 3, 2], [8, 9, 1], [8, 2, 1], [9, 4, 0], [9, 1, 0]]
rp2 = genSimplicialComplex(rp2_simplices)
```

We can then compute their $\mathbb{F}_2$-Betti numbers:

```
>>> torus.bettiNumbers()
{0: 1, 1: 2, 2: 1}
>>> kbottle.bettiNumbers()
{0: 1, 1: 2, 2: 1}
>>> rp2.bettiNumbers()
{0: 1, 1: 1, 2: 1}
```

However, further experiments show that this library does not scale too well and that handling large simplicial complexes does not seem to be very efficient.

Remark 3.3.10 (incremental computation)**.** Alternatively to the algorithms discussed above, it is sometimes convenient to compute the homology incrementally, adding one simplex at a time [20]. This approach is particularly useful in situations where simplicial complexes are gradually extended. This is a higher-dimensional version of the incremental computation of the connected components of simplicial complexes (Chapter 2.8).

3.4 Homotopy invariance

Our next goal is to prove that simplicial homology is invariant under topological homotopies. To this end, we first show that simplicial homology is compatible with barycentric subdivisions. We can then use the simplicial approximation of topological homotopies and simplicial homotopy invariance.

3.4.1 Simplicial homology and barycentric subdivision

We have the following chain-level version of the inverse of the barycentric subdivision homeomorphism (Proposition 2.6.17).

Proposition 3.4.1 (barycentric subdivision on simplicial chains). *Let X be a simplicial complex. Then the sequence $(B_{X,n})_{n\in\mathbb{N}}$ with*

$$\begin{aligned} B_{X,n} \colon C_n(X) &\longrightarrow C_n(\operatorname{sd} X) \\ [x_0,\dots,x_n] &\longmapsto \sum_{\pi\in S_{n+1}} \operatorname{sgn}(\pi)\cdot \bigl[\{x_{\pi(0)}\},\{x_{\pi(0)},x_{\pi(1)}\},\dots,\{x_{\pi(0)},\dots,x_{\pi(n)}\}\bigr] \end{aligned}$$

defines a natural chain map $B_X \colon C(X) \longrightarrow C(\operatorname{sd} X)$. The chain map B_X is a chain homotopy equivalence; if $s\colon \operatorname{sd} X \longrightarrow X$ is a simplicial approximation of the homeomorphism $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$, then s is a chain homotopy inverse of B_X.

Proof. That B_X is a well-defined and natural chain map follows from a lengthy but straightforward computation (check!).

As all simplicial approximations of $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$ lead to the same chain homotopy class of chain maps (Proposition 2.6.21, Theorem 3.2.10), it suffices to prove the claim for a specific such simplicial approximation (check!). We choose a total ordering of $V(X)$; then

$$\begin{aligned} s\colon V(\operatorname{sd} X) &\longrightarrow V(X) \\ X\setminus\{\emptyset\} \ni \sigma &\mapsto \max\sigma \end{aligned}$$

defines such a simplicial approximation $s\colon \operatorname{sd} X \longrightarrow X$ (Exercise). By construction, we have (check!)

$$C(s)\circ B_X = \operatorname{id}_{C(X)}.$$

It remains to show that $B_X \circ C(s) \simeq_{\mathbb{Z}} \operatorname{id}_{C(\operatorname{sd} X)}$. We prove this claim by inductively constructing a chain homotopy H_X: We set

$$\begin{aligned} H_{X,0} \colon C_0(\operatorname{sd} X) &\longrightarrow C_1(\operatorname{sd} X) \\ (\operatorname{sd} X)[0] \ni [\sigma] &\longmapsto \begin{cases} 0 & \text{if } \dim \sigma = 0 \\ [\{\max \sigma\}, \sigma] & \text{if } \dim \sigma > 0 \end{cases} \end{aligned}$$

If $n \in \mathbb{N}_{>0}$ and $H_{X,n-1}$ is already constructed, we define (inspired by the chain homotopy formula and the corresponding proof in singular homology)

$$\begin{aligned} H_{X,n} \colon C_n(\operatorname{sd} X) &\longrightarrow C_{n+1}(\operatorname{sd} X) \\ (\operatorname{sd} X)[n] \ni \sigma &\longmapsto \left(\bigcup \sigma\right) * \left(B_{X,n}(C(s)(\sigma)) - \sigma - H_{X,n-1}(\partial_n^{\operatorname{sd} X} \sigma)\right) \end{aligned}$$

Here, "$\bigcup \sigma$" should be viewed as an algebraic version of the barycentre and "$*$" denotes the linear extension of following "coning" operation: If $\tau = [\tau_0, \dots, \tau_n] \in (\operatorname{sd} X)[n]$ and $\varrho \in X$ with $\bigcup \tau \subset \varrho$, then

$$\sigma * \tau := \begin{cases} 0 & \text{if } \varrho \text{ occurs in } \tau \\ [\sigma, \tau_0, \dots, \tau_n] & \text{if } \varrho \text{ does not occur in } \tau. \end{cases}$$

A straightforward computation then shows that H_X is a chain homotopy between $B_X \circ C(s)$ and $\mathrm{id}_{C(\operatorname{sd} X)}$:

Indeed, the case of $n = 0$ is easy to check (check!). For the induction step, in the previous notation, we have $\partial(\varrho * \tau) = \tau - \varrho * \partial\tau$ (check!). Using the induction hypothesis, we obtain for all $n \in \mathbb{N}_{>0}$ and all $\sigma \in (\operatorname{sd} X)[n]$ that $H_{X,n}(\sigma)$ is well-defined (check!) and that

$$\begin{aligned} \partial H_{X,n}(\sigma) = {} & B_{X,n}\bigl(C(s)(\sigma)\bigr) - \sigma - H_{X,n-1}(\partial\sigma) \\ & - \left(\bigcup \sigma\right) * \bigl(\partial \circ B_{X,n} \circ C(s)(\sigma) - \partial\sigma - \partial \circ H_{X,n-1}(\partial\sigma)\bigr) \end{aligned}$$

The second part is zero because

$$\begin{aligned} & \partial \circ B_{X,n} \circ C(s)(\sigma) - \partial\sigma - \partial \circ H_{X,n-1}(\partial\sigma) \\ & = B_{X,n-1} \circ C(s)(\partial\sigma) - \partial\sigma - \partial \circ H_{X,n-1}(\partial\sigma) && (B_X \text{ and } C(s) \text{ are chain maps}) \\ & = B_{X,n-1} \circ C(s)(\partial\sigma) - \partial\sigma - \bigl(B_{X,n-1} \circ C(s)(\partial\sigma) - \partial\sigma - H_{X,n-2}(\partial \circ \partial(\sigma))\bigr) && (\text{by induction}) \\ & = 0 - 0 + H_{X,n-2}(0) \\ & = 0. \end{aligned}$$

This completes the induction step. □

Corollary 3.4.2 (barycentric subdivision on simplicial homology)**.** *Let X be a simplicial complex and let $n \in \mathbb{N}$. Then*

$$H_n(B_X) \colon H_n(X) \longrightarrow H_n(\operatorname{sd} X)$$

is an isomorphism. Moreover, if $s \colon \operatorname{sd} X \longrightarrow X$ is a simplicial approximation of $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$, then $H_n(s)$ is the inverse isomorphism of $H_n(B_X)$.

Proof. This follows from applying the algebraic homology functor to the corresponding statement on the chain level (Proposition 3.4.1). ☐

3.4.2 Simplicial homology and simplicial approximation

Using the barycentric subdivision homology isomorphism and simplicial approximation, we extend the simplicial homology functor to continuous maps on geometric realisations. As we proved the simplicial approximation theorem only for finite simplicial complexes, we restrict to this case.

More precisely, if X and Y are simplicial complexes and $\varphi\colon |X| \longrightarrow |Y|$ is a continuous map, then the idea is to define for $n \in \mathbb{N}$ the induced map

$$H_n(\varphi) := H_n(f) \circ H_n(B_X^N)\colon H_n(X) \longrightarrow H_n(Y),$$

where B_X^N denotes the N-fold iteration of the construction from Proposition 3.4.1 and f is a simplicial approximation of $\varphi \circ \beta_X^N\colon |\operatorname{sd}^N X| \longrightarrow |Y|$:

$$\begin{array}{ccc} H_n(X) & \overset{H_n(\varphi)}{\dashrightarrow} & H_n(Y) \\ {\scriptstyle H_n(B_X^N)}\downarrow{\scriptstyle \cong_{\mathbb{Z}}} & & \Vert \\ H_n(\operatorname{sd}^N X) & \underset{H_n(f)}{\longrightarrow} & H_n(Y) \end{array}$$

If X is finite, then such a simplicial approximation indeed exists by the simplicial approximation theorem (Theorem 2.6.24). We now show that this construction is independent of the chosen simplicial approximation and that this construction is functorial:

Proposition and Definition 3.4.3 (simplicial homology and continuous maps). *Let X and Y be simplicial complexes, let $\varphi\colon |X| \longrightarrow |Y|$ be a continuous map, let $N, M \in \mathbb{N}$, let $f\colon \operatorname{sd}^N X \longrightarrow Y$ be a simplicial approximation of $\varphi \circ \beta_X^N\colon |\operatorname{sd}^N X| \longrightarrow |Y|$, and let $g\colon \operatorname{sd}^M X \longrightarrow Y$ be a simplicial approximation of $\varphi \circ \beta_X^M\colon |\operatorname{sd}^M X| \longrightarrow |Y|$. Then*

$$H_n(f) \circ H_n(B_X^N) = H_n(g) \circ H_n(B_X^M).$$

This common map is denoted by $H_n(\varphi)\colon H_n(X) \longrightarrow H_n(Y)$.

Proof. Without loss of generality, we may assume that $M \geq N$.

Let $s\colon \operatorname{sd}^M X \longrightarrow \operatorname{sd}^N X$ be a simplicial approximation of $\beta^{M-N}_{\operatorname{sd}^N X}$. Then the composition $f \circ s\colon \operatorname{sd}^M X \longrightarrow Y$ is also a simplicial approximation of $\varphi \circ \beta_X^N \circ \beta^{M-N}_{\operatorname{sd}^N X} = \varphi \circ \beta_X^M\colon |\operatorname{sd}^M X| \longrightarrow |Y|$; in particular,

$$H_n(f \circ s) = H_n(g),$$

by Proposition 2.6.21 and Theorem 3.2.10. Moreover, the induced homomorphism $H_n(s)$ is inverse to $H_n(B^{M-N}_{\operatorname{sd}^N X})$ (Corollary 3.4.2). Therefore, the diagram

$$\begin{array}{ccc} H_n(\operatorname{sd}^N X) & \xrightarrow{H_n(f)} & H_n(Y) \\ {\scriptstyle H_n(B^{M-N}_{\operatorname{sd}^N X})}\downarrow \cong_{\mathbb{Z}} & & \| \\ H_n(\operatorname{sd}^M X) & \xrightarrow[H_n(g)]{} & H_n(Y) \end{array}$$

commutes and we obtain

$$\begin{aligned} H_n(g) \circ H_n(B^M_X) &= H_n(g) \circ H_n(B^{M-N}_{\operatorname{sd}^N X}) \circ H_n(B^N_X) \\ &= H_n(f \circ s) \circ H_n(B^{M-N}_{\operatorname{sd}^N X}) \circ H_n(B^N_X) \\ &= H_n(f) \circ H_n(s) \circ H_n(B^{M-N}_{\operatorname{sd}^N X}) \circ H_n(B^N_X) \\ &= H_n(f) \circ H_n(B^N_X), \end{aligned}$$

as claimed. □

Proposition 3.4.4 (simplicial homology, functoriality for continuous maps)**.** *Let X, Y, Z be simplicial complexes [where X and Y are finite] and let $\varphi\colon |X| \longrightarrow |Y|$ and $\psi\colon |Y| \longrightarrow |Z|$ be continuous maps. Then, for all $n \in \mathbb{N}$, we have*

$$H_n(\psi \circ \varphi) = H_n(\psi) \circ H_n(\varphi).$$

Proof. Let $N \in \mathbb{N}$ be large enough such that there exists a simplicial approximation $f\colon \operatorname{sd}^N X \longrightarrow Y$ of $\varphi \circ \beta^N_X\colon |\operatorname{sd}^N X| \longrightarrow |Y|$. Let $M \in \mathbb{N}$ be such that there exists a simplicial approximation $g\colon \operatorname{sd}^M Y \longrightarrow Z$ of $\psi \circ \beta^M_Y\colon |\operatorname{sd}^M Y| \longrightarrow |Z|$. Then

$$g \circ \operatorname{sd}^M f\colon \operatorname{sd}^{N+M} X \longrightarrow Z$$

is a simplicial approximation of $\psi \circ \varphi \circ \beta^{N+M}_X\colon |\operatorname{sd}^{N+M} X| \longrightarrow |Z|$. Therefore, we obtain

$$\begin{aligned} H_n(\psi \circ \varphi) &= H_n(g \circ \operatorname{sd}^M f) \circ H_n(B^{N+M}_X) \\ &= H_n(g) \circ H_n(\operatorname{sd}^M f) \circ H_n(B^M_{\operatorname{sd}^N X}) \circ H_n(B^N_X) && \text{(functoriality of } H_n \text{ for simplicial maps)} \\ &= H_n(g) \circ H_n(B^M_Y) \circ H_n(f) \circ H_n(B^N_X) && \text{(naturality of } B) \\ &= H_n(\psi) \circ H_n(\varphi), \end{aligned}$$

which proves functoriality. □

Remark 3.4.5. The construction of induced maps for continuous maps between [finite] simplicial complexes also works for general coefficients. The reason is that the the barycentric subdivision chain homotopy equivalence from Proposition 3.4.1 passes through the tensor product to the simplicial chain complex with coefficients.

3.4.3 Topological homotopy invariance

We finally collected all the ingredients that simplicial homology (which a priori is a "rigid" construction) is flexible enough to be invariant under *topological* homotopy:

Theorem 3.4.6 (topological homotopy invariance). *Let X and Y be simplicial complexes [where X is finite] and let $\varphi, \psi\colon |X| \longrightarrow |Y|$ be continuous maps with $\varphi \simeq \psi$. Then, for all $n \in \mathbb{N}$, we have*

$$H_n(\varphi) = H_n(\psi)\colon H_n(X) \longrightarrow H_n(Y).$$

Proof. By the simplicial approximation theorem (Theorem 2.6.24), there exists an $N \in \mathbb{N}$ such that there exist simplicial approximations

$$\begin{aligned} f\colon \operatorname{sd}^N X &\longrightarrow Y, \\ g\colon \operatorname{sd}^N X &\longrightarrow Y \end{aligned}$$

of $\varphi \circ \beta_X^N$ and $\psi \circ \beta_X^N$, respectively (by passing to the maximum, we may assume that the same number of iterations can be used).

By the simplicial approximation theorem for homotopies (Theorem 2.6.26), there exists an $M \in \mathbb{N}$ such that

$$f \circ \operatorname{sdm}^M_{\operatorname{sd}^N X} \simeq^*_\Delta g \circ \operatorname{sdm}^M_{\operatorname{sd}^N X},$$

where $\operatorname{sdm}^M_{\operatorname{sd}^N X}$ is an iterated simplicial approximation of $\beta^M_{\operatorname{sd}^N X}$.

Using simplicial homotopy invariance of simplicial homology and the barycentric subdivision isomorphism, we therefore obtain

$$\begin{aligned} H_n(\varphi) &= H_n(f) \circ H_n(B_X^N) \\ &= H_n(f) \circ H_n(\operatorname{sdm}^M_{\operatorname{sd}^N X}) \circ H_n(B^M_{\operatorname{sd}^N X}) \circ H_n(B_X^N) && \text{(Corollary 3.4.2)} \\ &= H_n(f \circ \operatorname{sdm}^M_{\operatorname{sd}^N X}) \circ H_n(B^M_{\operatorname{sd}^N X}) \circ H_n(B_X^N) && \text{(functoriality)} \\ &= H_n(g \circ \operatorname{sdm}^M_{\operatorname{sd}^N X}) \circ H_n(B^M_{\operatorname{sd}^N X}) \circ H_n(B_X^N) && \text{(Theorem 3.2.10)} \\ &= H_n(g) \circ H_n(\operatorname{sdm}^M_{\operatorname{sd}^N X}) \circ H_n(B^M_{\operatorname{sd}^N X}) \circ H_n(B_X^N) && \text{(functoriality)} \\ &= H_n(g) \circ H_n(B_X^N) && \text{(Corollary 3.4.2)} \\ &= H_n(\psi), \end{aligned}$$

as desired. □

Corollary 3.4.7 (simplicial homology and homotopy equivalences). *Let X and Y be [finite] simplicial complexes with $|X| \simeq |Y|$. Then, for all $n \in \mathbb{N}$, we have*

$$H_n(X) \cong_{\mathbb{Z}} H_n(Y).$$

Proof. This is immediate from Theorem 3.4.6 and the general homotopy invariance principle (Proposition 1.3.2). □

In particular, we now have tools at hand to resolve Black box 1.1.3:

Theorem 3.4.8. *Let $n \in \mathbb{N}$. Then the n-sphere S^n is* not *contractible.*

Proof. It suffices to prove that $|S(n)|$ is not contractible (Example 2.6.5). *Assume* for a contradiction that $|S(n)|$ were contractible. Then Corollary 3.4.7 shows that

$$H_n\bigl(S(n)\bigr) \cong_{\mathbb{Z}} H_n\bigl(\Delta(0)\bigr) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

However, this contradicts the computation of $H_n(S(n))$ (Example 3.2.12). □

Remark 3.4.9 (topological homotopy invariance with coefficients). Topological homotopy invariance of simplicial homology also holds with general coefficients, because the ingredients work in the same way also with coefficients.

3.4.4 Simplicial homology of triangulable spaces

Using the topological invariance of simplicial homology, we could use simplicial homology to define a homology theory on all topological spaces that admit a [finite] triangulation.

This can be done by choosing for every [finitely] triangulable topological space a reference [finite] triangulation or by a universal construction encompassing all [finite] triangulations. Then the considerations from Chapter 3.4.2 and Theorem 3.4.6 can be used to show that this results in a well-defined sequence of homotopy invariant functors [55].

On [finitely] triangulable spaces, these functors coincide with ordinary homology with $\mathbb{Z}$-coefficients, whence with singular and cellular homology. This can be proved by an inductive Mayer–Vietoris argument – adding simplex by simplex and comparing the resulting Mayer–Vietoris sequences.

While simplicial homology is straightforward to construct and compute on simplicial chain complexes, topological and homotopy invariance are somewhat more cumbersome to establish.

Conversely, the construction of singular homology is based on much larger chain complexes, which makes topological and homotopy invariance easy to prove. However, in the case of singular homology, the proof of the Mayer–Vietoris sequence (or equivalently of excision) is technically more involved than in the case of simplicial homology.

Moreover, comparing the proofs for simplicial and singular homology, we see that they rely on the same basic geometric ideas.

3.5 The Brouwer fixed point theorem

As a first application of simplicial homology, we consider the Brouwer fixed point theorem and related results: The Lefschetz fixed point theorem and Sperner's lemma.

3.5.1 The Lefschetz and Brouwer fixed point theorems

We derive the Brouwer fixed point theorem as special case of the Lefschetz fixed point theorem. The Lefschetz fixed point theorem makes use of the Lefschetz number – an Euler characteristic invariant for maps:

Definition 3.5.1 (Lefschetz number)**.** Let X be a finite simplicial complex, let $\varphi\colon |X| \longrightarrow |X|$ be a continuous map, and let K be a field. Then the *Lefschetz number of* φ *over* K is defined as

$$\Lambda(\varphi; K) := \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(\varphi; \mathbb{Q}).$$

Theorem 3.5.2 (Lefschetz fixed point theorem)**.** *Let X be a finite simplicial complex, let $\varphi\colon |X| \longrightarrow |X|$ be a continuous map, and let K be a field. If φ has no fixed point, then*

$$\Lambda(\varphi; K) = 0.$$

In particular: If $\Lambda(\varphi; K) \neq 0$, then φ has a fixed point, i.e., there is a $\xi \in |X|$ with $\varphi(\xi) = \xi$.

Proof. The idea is to apply simplicial approximation and barycentric subdivisions to replace φ by a chain map that has trivial Lefschetz number for obvious reasons. This is a prototypical argument that benefits from the simplicial language and toolbox.

We first refine the complex X such that φ properly moves simplices away from each other: Let φ have no fixed point. Because $|X|$ is compact, the infimum $\inf_{\xi\in|X|} d_2\bigl(\xi, \varphi(\xi)\bigr)$ is non-zero. Therefore, there exists an $N \in \mathbb{N}$ such that mesh size of $\operatorname{sd}^N X$ is so "small" (measured via β_X^N in $|X|$; Proposition 2.6.18) that (check!)

$$\forall_{x\in V(\operatorname{sd}^N X)} \quad \operatorname{star}^c_{\operatorname{sd}^N}\bigl((\beta_X^N)^{-1} \circ \varphi \circ \beta_X^N(\operatorname{star}^c_{\operatorname{sd}^N} x)\bigr) \cap \operatorname{star}^c_{\operatorname{sd}^N} x = \emptyset;$$

we extended the notation to $\operatorname{star}^c_{\operatorname{sd}^N X} Z := \bigcup\{|\sigma| \mid \sigma \in \operatorname{sd}^N X,\ |\sigma| \cap Z \neq \emptyset\}$ for every subset $Z \subset |\operatorname{sd}^N X|$. Then $\psi := (\beta_X^N)^{-1} \circ \varphi \circ \beta_X^N \colon |\operatorname{sd}^N X| \longrightarrow$

$|\operatorname{sd}^N X|$ has the same number of fixed points as φ (check!) and, by the trace property (check!),

$$\Lambda(\psi; K) = \Lambda(\varphi; K).$$

Therefore, for the rest of the proof, we may assume without loss of generality that $\operatorname{star}^c_X\bigl(\varphi(\operatorname{star}^c_X x)\bigr) \cap \operatorname{star}^c_X x = \emptyset$ holds for all $x \in V(X)$.

We now replace φ by a simplicial approximation: By the simplicial approximation theorem (Theorem 2.6.24), there exists an $M \in \mathbb{N}$ such that $\varphi \circ \beta^M_X \colon |\operatorname{sd}^M X| \longrightarrow |X|$ admits a simplicial approximation $f \colon \operatorname{sd}^M X \longrightarrow X$. By the above avoidance property of φ, we see that f satisfies

$$\forall_{\sigma \in \operatorname{sd}^M X} \quad f(\sigma) \cap (\bigcup \cdots \bigcup \sigma) = \emptyset.$$

This already looks good in terms of traces on the simplicial chain complex; however, we need to fix the issue that f is *not* a self-map anymore. This fix will be performed on the chain level: We set

$$F := (K \otimes_{\mathbb{Z}} B^M_X) \circ C(f; K) \colon C(\operatorname{sd}^M X; K) \longrightarrow C(\operatorname{sd}^M X; K).$$

The avoidance property for f and the construction of B^M_X show: If $n \in \mathbb{N}$ and $\sigma \in (\operatorname{sd}^M X)[n]$, then the coefficient of σ in $F_n(\sigma)$ is zero. In particular, this makes it clear that

$$\operatorname{tr} F_n = \text{“sum of diagonal matrix entries”} = 0.$$

By construction, $F \colon C(\operatorname{sd}^M X) \longrightarrow C(\operatorname{sd}^M X)$ is a chain map. Because the Lefschetz number can be computed on the chain level (Lemma 3.5.3), we obtain

$$\sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(F) = \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr} F_n = 0.$$

Therefore, overall, it follows that

$$\begin{aligned}
\Lambda(\varphi; K) &= \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(\varphi; K) \\
&= \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr}\bigl(H_n(f) \circ H_n(K \otimes_{\mathbb{Z}} B^M_X)\bigr) && \text{(definition of } H_n(\varphi; K)) \\
&= \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr}\bigl(H_n(K \otimes_{\mathbb{Z}} B^M_X)^{-1} \circ H_n(F) \circ H_n(K \otimes_{\mathbb{Z}} B^M_X)\bigr) && \text{(definition of } F\text{; Corollary 3.4.2)} \\
&= \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(F) && \text{(trace property)} \\
&= 0,
\end{aligned}$$

as claimed. □

Lemma 3.5.3 (Lefschetz number of chain maps). *Let K be a field and let C be a chain complex over K that has only finitely many non-zero chain modules and such that each chain module is finite-dimensional. Let $f\colon C \longrightarrow C$ be a chain map. Show that*

$$\sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(f) = \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} f_n.$$

Proof. This can be seen by a suitable choice of bases and careful bookkeeping of the alternating cancellations (Exercise). □

Corollary 3.5.4. *Let X be a finite simplicial complex such that $|X|$ is contractible. Then every continuous map $|X| \longrightarrow |X|$ has a fixed point.*

Proof. Let $\varphi\colon |X| \longrightarrow |X|$ be a continuous map and let $d := \dim X$. Because $|X|$ is contractible, topological homotopy invariance of simplicial homology (Corollary 3.4.7) shows that

$$H_n(X;\mathbb{Q}) \cong_{\mathbb{Q}} H_n\bigl(\Delta(0);\mathbb{Q}\bigr) \cong_{\mathbb{Q}} \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Moreover, $H_0(\varphi;\mathbb{Q}) = \operatorname{id}_{H_0(X;\mathbb{Q})}$, by direct computation. Therefore, we obtain

$$\begin{aligned} \Lambda(\varphi;\mathbb{Q}) &= \sum_{n=0}^{d} (-1)^n \cdot \operatorname{tr} H_n(\varphi;\mathbb{Q}) = \operatorname{tr} H_0(\varphi;\mathbb{Q}) + \sum_{n=1}^{d} (-1)^n \cdot \operatorname{tr} H_n(\varphi;\mathbb{Q}) \\ &= 1 + \sum_{n=1}^{d} (-1)^n \cdot 0 = 1 \\ &\neq 0. \end{aligned}$$

The Lefschetz fixed point theorem (Theorem 3.5.2) therefore implies that φ has a fixed point. □

Corollary 3.5.5 (Brouwer fixed point theorem). *Let $n \in \mathbb{N}$. Then, every continuous map $f\colon D^n \longrightarrow D^n$ has a fixed point, i.e., there exists an $x \in D^n$ with*

$$f(x) = x.$$

The same applies to self-maps of $|\Delta(n)|$.

Proof. Because D^n is contractible (Example 1.1.2) and admits a finite triangulation on $\Delta(n)$ (check!), this is a direct consequence of Corollary 3.5.4. □

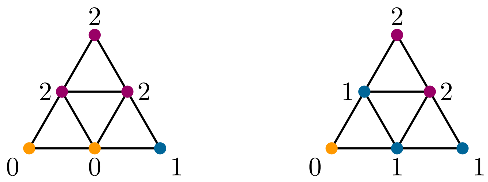


Figure 3.7.: LHS: A Sperner colouring; RHS: *not* a Sperner colouring

3.5.2 Sperner's lemma

A combinatorial relative of the Brouwer fixed point theorem is Sperner's lemma on colourings. We first derive the classical version for subdivisions of simplices. Furthermore, we show how a manifold version of Sperner's lemma can be obtained from simplicial homology.

Definition 3.5.6 (colouring, rainbow simplex)**.** Let $n \in \mathbb{N}$ and let X be a simplicial complex.

- An $[n]$-*colouring of* X is a simplicial map $X \longrightarrow \Delta(n)$.
- Let $c \colon X \longrightarrow \Delta(n)$ be an $[n]$-colouring of X. A simplex $\sigma \in X(n)$ is a *rainbow simplex* of c if $c(\sigma) = \{0, \dots, n\}$.

Definition 3.5.7 (sudivision, Sperner colouring)**.** Let $n \in \mathbb{N}$.

- A *subdivision* of $\Delta(n)$ is a finite triangulation (X, φ) of $|\Delta(n)|$ with the following property: For every $\sigma \in \Delta(n)$, there is a subcomplex $A \subset X$ with $\dim A = \dim \sigma$ and
$$|\sigma| = \varphi\bigl(|A|\bigr).$$
In this situation, we also say that each $\tau \in A$ *lies in* σ.
- Let (X, φ) be a subdivision of $\Delta(n)$. A *Sperner colouring* of (X, φ) is an $[n]$-colouring $c \colon X \longrightarrow \Delta(n)$ with the following property: If $\sigma \in \Delta(n)$ and $\tau \in X$ lies in σ, then $c(\tau) \subset \sigma$.

Example 3.5.8 (Sperner colouring)**.** In Figure 3.7, the left hand side depicts a Sperner colouring (with a rainbow simplex in the lower right corner), but the right hand side does not.

Theorem 3.5.9 (Sperner's lemma)**.** *Let $n \in \mathbb{N}$. Let (X, φ) be a subdivision of $\Delta(n)$ and let $c \colon X \longrightarrow \Delta(n)$ be a Sperner colouring of (X, φ). Then c contains a rainbow simplex.*

Proof. We apply the Brouwer fixed point theorem to a map $|\Delta(n)| \longrightarrow |\Delta(n)|$ obtained from c:

Let $s\colon \Delta(n) \longrightarrow \Delta(n)$ be the simplicial map induced by the cyclic permutation $(0\ 1\ \dots\ n)$. We then consider the continuous map

$$\psi := |s \circ c| \circ \varphi^{-1}\colon |\Delta(n)| \longrightarrow |\Delta(n)|.$$

By the Brouwer fixed point theorem (Corollary 3.5.5), the map ψ has a fixed point $\xi \in |\Delta(n)|$. Let $\sigma \in X$ be the support of $\varphi^{-1}(\xi)$. We show that σ is a rainbow simplex for c:

By construction of c, the simplex σ is a rainbow simplex if and only if $|c|(\varphi^{-1}(\xi)) \in |\Delta(n)| \setminus |S(n-1)|$. Because s is a simplicial isomorphism, this is equivalent to $\psi(\xi) \in |\Delta(n)| \setminus |S(n-1)|$.

Assume for a contradiction that $\xi = \psi(\xi) \in |S(n-1)|$. Thus, $\tau := \operatorname{supp} \xi \in S(n-1)$ and σ lies in τ. As c is a Sperner colouring, we obtain $c(\sigma) \subset \tau$. Because s shifts the indices, this implies $s \circ c(\sigma) \subset s(\tau) \neq \tau$. Therefore, we obtain that

$$\tau = \operatorname{supp} \xi = \operatorname{supp} \psi(\xi) = \operatorname{supp} |s \circ c| \circ \varphi^{-1}(\xi) \subset s(\tau).$$

However, this is impossible, because τ and $s(\tau) \neq \tau$ have the same finite cardinality. This contradiction shows that σ indeed is a rainbow simplex. ☐

Remark 3.5.10 (from Sperner to Brouwer; and cake)**.** Conversely, one can also prove the Brouwer fixed point theorem from Sperner's lemma; indeed, this is the original context in which Sperner formulated and proved it [70].

A related application of Sperner's lemma is the Simmons–Su protocol for envy-free division (Exercise).

Another nice application of Sperner's lemma and 2-adic numbers is *Monsky's theorem*: If a Euclidean square is subdivided into triangles of equal area, then the number of these triangles must be even [53].

Definition 3.5.11 (pseudomanifold)**.** Let $n \in \mathbb{N}$. An *n-pseudomanifold with boundary* is a finite simplicial complex M with the following properties:

- *Purity.* For all $\tau \in M$, there exists a $\sigma \in M(n)$ with $\tau \subset \sigma$.
- *Non-singularity.* For all $\tau \in M(n-1)$, we have $\#\{\sigma \in M(n) \mid \tau \subset \sigma\} \in \{1, 2\}$.
- *Strong connectedness.* The *dual graph*

  $$\bigl(M(n), \{\{\sigma, \tau\} \mid \dim(\sigma \cap \tau) = n - 1\}\bigr)$$

  is connected.

We write $\partial M := \langle \{\tau \in M(n-1) \mid \#\{\sigma \in M(n) \mid \tau \subset \sigma\} = 1\}\rangle_\Delta \subset M$ for the *boundary* of M.

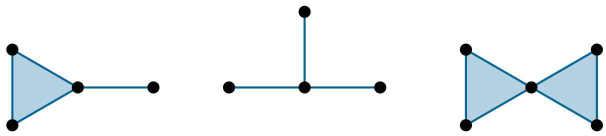


Figure 3.8.: Examples of simplicial complexes that fail to be pseudomanifolds (Example 3.5.12).

Example 3.5.12 (pseudomanifolds)**.**

- Let $n \in \mathbb{N}$. Then $\Delta(n)$ is an n-pseudomanifold with boundary; moreover, $\partial\Delta(n) = S(n-1)$.

  Moreover, $S(n)$ is an n-pseudomanifold with boundary and $\partial S(n) = \emptyset$.

- If M is an n-pseudomanifold with boundary, then also $\operatorname{sd} M$ is an n-pseudomanifold with boundary (check!).

- The simplicial complexes in Figure 3.8 are *no* pseudomanifolds: The leftmost example violates the purity condition, the middle example violates the non-singularity condition, and the right example violates the connectedness condition.

Theorem 3.5.13 (Sperner's lemma for manifolds)**.** *Let $n \in \mathbb{N}_{>0}$, let M be an n-pseudomanifold with boundary, and let $c\colon M \longrightarrow \Delta(n)$ be an $[n]$-colouring of M. Moreover, let*

$$\#\bigl\{\tau \in \partial M \bigm| c(\tau) = \{0,\dots,n-1\}\bigr\}$$

be odd. Then M contains an odd number of rainbow simplices. In particular, M contains at least one rainbow simplex.

Proof. We use simplicial homology with $\mathbb{F}_2$-coefficients. We pick an ordering on $V(M)$ and let

$$z := \sum_{\sigma \in X(n)_<} \sigma \in C_n(M;\mathbb{F}_2).$$

By the non-singularity property of M, we have $\partial_n z \in C_{n-1}(\partial M;\mathbb{F}_2)$. Hence, z represents a class $\alpha \in H_n(M,\partial M;\mathbb{F}_2)$. We first express the rainbow condition in terms of α: The pseudomanifold M contains an odd number of rainbow simplices for c if and only if (where we view z as relative chain)

$$C_n(c;\mathbb{F}_2)(z) = 1 \cdot [0,\dots,n] \in C_n\bigl(\Delta(n), S(n-1);\mathbb{F}_2\bigr).$$

The latter property is (by direct computation) equivalent to α being the(!) generator of $H_n(\Delta(n), S(n-1);\mathbb{F}_2) \cong_{\mathbb{F}_2} \mathbb{F}_2$ (check!).

We now relate this homological property to the boundary, using the long exact sequence of the pair $(M, \partial M)$: Let $A \subset S(n-1)$ be the subcomplex generated by $S(n-1) \setminus \{\{0, \dots, n-1\}\}$. We consider the following commutative (check!) diagram:

$$\begin{array}{ccccc}
H_n(M, \partial M; \mathbb{F}_2) & \xrightarrow{H_n(c;\mathbb{F}_2)} & H_n(\Delta(n), S(n-1); \mathbb{F}_2) & \xrightarrow{\cong_{\mathbb{F}_2}} & \mathbb{F}_2 \\
\Big\downarrow{\scriptstyle \partial_n} & & \Big\downarrow{\scriptstyle \partial_n} & \searrow & \Big|{\scriptstyle \cong_{\mathbb{F}_2}} \\
H_{n-1}(\partial M; \mathbb{F}_2) & \xrightarrow[H_{n-1}(c|_{\partial M};\mathbb{F}_2)]{} & H_{n-1}(S(n-1); \mathbb{F}_2) & \xrightarrow[H_{n-1}(j;\mathbb{F}_2)]{} & H_n(S(n-1), A; \mathbb{F}_2)
\end{array}$$

Here, the left and middle vertical arrows are the connecting homomorphisms of the corresponding long exact sequences of pairs (Theorem 3.2.7) and are thus induced by the simplicial boundary operators. For dimension reasons, c maps ∂M to $S(n-1)$; therefore, the left horizontal arrows are well-defined. The lower right arrow is induced by the inclusion j and the right diagonal arrow is given by the composition.

Looking at the relative chain complexes shows $H_{n-1}(S(n-1), A; \mathbb{F}_2) \cong_{\mathbb{F}_2} \mathbb{F}_2$ and that the right diagonal arrow is an isomorphism (check!).

Because $\#\bigl\{\tau \in \partial M \bigm| c(\tau) = \{0, \dots, n-1\}\bigr\}$ is odd, the class α (which is represented by z) is mapped to the(!) generator of $H_{n-1}(S(n-1), A; \mathbb{F}_2)$ under the composition $H_{n-1}(j; \mathbb{F}_2) \circ H_{n-1}(c|_{\partial M}\ F_2) \circ \partial_n$. By commutativity of the diagram, we therefore obtain that α is mapped via $H_n(c; \mathbb{F}_2)$ to the(!) generator of $H_n(\Delta(n), S(n-1); \mathbb{F}_2)$. As seen above, this means exactly that M contains an odd number of rainbow simplices. □

Remark 3.5.14 (Sperner's lemma: from manifolds to the classical case)**.** The classical case of Sperner's lemma (Theorem 3.5.9) can be derived from Sperner's lemma for manifolds (Theorem 3.5.13) by showing the case of dimension 1 directly by hand and then proceeding by induction (Exercise).

Remark 3.5.15 (Sperner's lemma: from graphs to manifolds)**.** Sperner's lemma for manifolds admits an elementary proof (Exercise): One can consider the dual graph of the given pseudomanifold (plus an "external" vertex) and then argue via the handshake lemma (Proposition 2.1.6).

In combination this gives a proof of Brouwer's theorem from graph-theory and elementary topology (Remark 3.5.10, Remark 3.5.14).

We will apply Sperner's lemma in the context of distributed systems (Chapter 3.6).

3.6 Application: Consensus in distributed systems

Computation and coordination in distributed systems faces various challenges. A recurring problem is that multiple processes need to agree on something – while only having minimal guarantees on aliveness, synchronisation, means of communication, access to shared resources, fault-freeness, and non-maliciousness of other processes.

In the following, we will focus on the case of the computational model of asynchronous, wait-free, shared read/write memory, multi-layer immediate snapshot computations [38, Chapters 8/9].

Real-world problem 3.6.1 (set agreement). Let $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. The *k-set agreement task* is:

- Given a finite set A and given $n+1$ processes, each with an input value from A,
- each process $j \in \{0, \dots, n\}$ should output a value a_j such that the following conditions are satisfied:
  - For each $j \in \{0, \dots, n\}$, the value a_j is one of the given input values;
  - the set $\{a_0, \dots, a_n\}$ contains at most k elements.

The 1-set agreement task is also known as the *consensus task*, because all processes have to agree on a single value.

The key idea is to model such tasks and corresponding protocols in the language of simplicial complexes, simplicial/carrier maps, and colourings. Here,

- the simplices model "consistent states" of inputs, outputs or computations;
- the colours model the process ids, and
- the carrier maps model the allowed ranges of tasks or protocols from the given input/state.

We will refrain from introducing the most general setup and only use a fragment of the theory, sufficient to discuss the set agreement problem.

Definition 3.6.2 (chromatic simplicial complex). Let $n \in \mathbb{N}$ and let A be a finite set. An *$[n]$-chromatic simplicial complex with labels in A* is a simplicial complex X with $V(X) \subset \{0, \dots, n\} \times A$ and the following properties:

- The simplicial complex X is pure of dimension n;
- For every simplex $\sigma \in X$, the first components of σ are all different.

Definition 3.6.3 (chromatic carrier map). Let $n \in \mathbb{N}$, let A, B be finite sets, and let X, Y be $[n]$-chromatic simplicial complexes with labels in A and B, respectively. An *$[n]$-chromatic carrier map from X to Y* is a map T from X to the set of subcomplexes of Y with the following properties:

- For all $\sigma \in X$, the subcomplex $T(\sigma)$ is pure of dimension $\dim \sigma$.

- For all $\sigma, \tau \in X$, we have
$$\sigma \subset \tau \Longrightarrow T(\sigma) \subset T(\tau).$$

- For all $\sigma \in X$, we have (where π_1 denotes the projection onto the first coordinate)
$$\pi_1(\sigma) = \bigl\{\pi_1(y) \bigm| y \in V(T(\sigma))\bigr\}.$$

Definition 3.6.4 (task)**.** Let $n \in \mathbb{N}$ and let A be a finite set. A *task for $n+1$ processes on A* is a triple (I, O, T), consisting of:

- The *input complex* I, an $[n]$-chromatic simplicial complex with labels in A; the labels are called *input values*.

- The *output complex* O, an $[n]$-chromatic simplicial complex with labels in A; the labels are called *output values*.

- The *task map* T, a carrier map from I to O.

Definition 3.6.5 (protocol, solving a task, decision map)**.** Let $n \in \mathbb{N}$ and let A be a finite set. A *protocol* for $n + 1$ processes on A is a triple (I, P, K), consisting of:

- The *input complex*, an $[n]$-chromatic simplicial complex I with labels in A.

- The *protocol complex* P, an $[n]$-chromatic simplicial complex with labels in some finite set B.

- The *protocol map*, a chromatic carrier map K from I to P with $P = \bigcup_{\sigma \in I} K(\sigma)$ that is *strict* in the sense that:
$$\forall_{\sigma,\tau \in I} \quad K(\sigma \cap \tau) = K(\sigma) \cap K(\tau).$$

The protocol (I, P, K) *solves* the task (I, O, T) for $n+1$ processes on A (with the same input complex) with *decision map* δ if $\delta\colon P \longrightarrow O$ is a simplicial map with
$$\forall_{\sigma \in I} \quad \delta\bigl(K(\sigma)\bigr) \subset T(\sigma).$$

In particular, the notion of protocol does not make the exact execution of the protocol explicit, but only summarises the results of all possible runs (taking into account different speeds or failures of all involved processes) for each input configuration. In concrete applications, the labels of the vertices in the protocol complexes usually are views of the corresponding processes, including memory state etc.. The decision map selects for each “possible run” of the protocol a corresponding result.

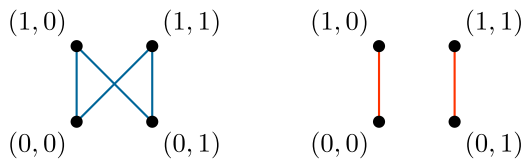


Figure 3.9.: Left: The input complex for consensus for two complexes; right: the 1-simplices in the image of the protocol map (Example 3.6.7).

Model 3.6.6 (set agreement)**.** Let $n \in \mathbb{N}$, let $k \in \{0, \dots, n\}$, and let A be a finite set. In this setting, the set agreement task (Problem 3.6.1 can be modelled by the task (I, O, T), where:

$$\begin{aligned} I &:= \bigl\langle \bigl\{\{(0, a_0), \dots, (n, a_n)\} \bigm| a_0, \dots, a_n \in A\bigr\} \bigr\rangle_\Delta \\ O &:= I \\ T &: \sigma \longmapsto \bigl\{\tau \in O \bigm| \pi_1(\tau) \subset \pi_1(\sigma),\ \pi_2(\tau) \subset \pi_2(\sigma),\ \#\pi_2(\tau) \leq k\bigr\} \end{aligned}$$

The input states are all combinations of process ids and available values; the output states are of the same shape. The task map encodes the conditions in the set agreement task: We are only allowed to use values that occur as one of the input values, and the set of output values may not exceed the cardinality k.

The problem then becomes to find a protocol that solves this task.

Example 3.6.7 (consensus for two processes)**.** The input complex for consensus for two processes on the label set $\{0, 1\}$ as in Model 3.6.6 is depicted in Figure 3.9.

In contrast, the only 1-simplices in the image of the protocol map are depicted in Figure 3.9 on the right. We see that the right-hand side is less connected than the left-hand side. This already indicates that this approach to distributed systems is susceptible to topological methods and homotopy invariants.

In connection with the concrete computation model that we are considering, we record the following fact:

Black box 3.6.8 ([38, Chapter 9.2])**.** Every layered immediate snapshot protocol (I, P, K) is a *manifold protocol*, i.e.:

- For all $\sigma \in I$, the subcomplex $K(\sigma)$ is a pseudomanifold with boundary (of dimension $\dim \sigma$).
- For all $\sigma \in I$, we have

$$\partial\bigl(K(\sigma)\bigr) = K(\partial\sigma).$$

Moreover, the following holds: If I is a pseudomanifold with boundary, then P is a pseudomanifold with boundary and $\partial P = K(\partial I)$ [38, Theorem 9.1.7].

Theorem 3.6.9. *Let $n \in \mathbb{N}$. There is no manifold protocol for n-set agreement for $n+1$ processes on a label set with at least $n+1$ elements.*

Proof. As we are proving an impossibility result, without loss of generality, we may take $A := \{0, \dots, n\}$. Let (I, O, T) be the n-set agreement task for $n+1$ processes with labels in A, as described in Model 3.6.6. *Assume* for a contradiction that (I, P, K) is a manifold protocol for $n+1$ processes solving this n-set agreement task with the decision map $\delta \colon P \longrightarrow O$.

We consider the special simplex

$$\sigma_\Delta := \bigl\{(0,0), \dots, (n,n)\bigr\} \in I$$

in which each process takes its own id as input value; let $M := K(\sigma_\Delta)$. Then M is an n-pseudomanifold with boundary and the decision map δ induces an $[n]$-colouring $c \colon M \longrightarrow \Delta(n)$ of M via

$$\begin{aligned} V(M) &\longrightarrow \{0, \dots, n\} \\ x &\longmapsto \pi_2\bigl(\delta(x)\bigr). \end{aligned}$$

Then c has the following "Sperner" property: For all $\sigma \in \Delta(n)$, the simplex $\overline{\sigma} := \{(j,j) \mid j \in \sigma\}$ satisfies

$$c\bigl(K(\overline{\sigma})\bigr) \subset \sigma.$$

Indeed, by construction, we have

$$\begin{aligned} c\bigl(K(\overline{\sigma})\bigr) &= \pi_2\bigl(\delta(K(\overline{\sigma}))\bigr) && \text{(construction of } c) \\ &\subset \pi_2\bigl(T(\overline{\sigma})\bigr) && \text{(the protocol solves the task)} \\ &\subset \pi_2(\overline{\sigma}) && \text{(by definition of the task map } T) \\ &= \sigma. && \text{(construction of } \overline{\sigma}) \end{aligned}$$

Inductively applying Sperner's lemma for manifolds (Theorem 3.5.13) shows (similarly to Remark 3.5.14) that M contains an odd number of rainbow simplices for c (check!). In particular, M contains at least one rainbow simplex σ. By definition of protocols, there exists a $\sigma' \in I$ with $\sigma \in K(\sigma')$. This means that

$$\begin{aligned} \#\pi_2\bigl(T(\sigma')\bigr) &\geq \#\pi_2\bigl(\delta(K(\sigma'))\bigr) && \text{(because } \delta(K(\sigma')) \subset T(\sigma')) \\ &\geq \#\pi_2\bigl(\delta(\sigma)\bigr) && \text{(by the choice of } \sigma') \\ &\geq \#c(\sigma) && \text{(by construction of } c) \\ &= n+1 && (\sigma \text{ is a rainbow simplex}), \end{aligned}$$

which contradicts the n-set agreement property of the task map. Therefore, such a manifold protocol cannot exist. □

Corollary 3.6.10. *Let $n \in \mathbb{N}$. In the layered immediate snapshot model, there is no protocol for n-set agreement for $n+1$ processes on a label set with at least $n+1$ elements.*

Proof. This is immediate from Theorem 3.6.9 and Black box 3.6.8. □

In particular, in the layered execution model, there is no protocol for consensus for more than one process. The same technique also applies to a wide range of other problems in distributed computing [39, 38].

3.7 Application: Social choice

Following up on the management of voting preferences (Problem 2.4.3), we give a simplicial proof of Arrow's impossibility theorem [2] in social choice [16, 3, 28, 74, 65]. Let us recall the general problem in social choice (Problem 2.4.3): Given a set A of available alternatives and $n \in \mathbb{N}$, the goal is to aggregate each n-tuple of total orders on A (i.e., the preferences of n voters) into a single total order on A (the "result" of the vote). Ideally, this aggregation should adhere to coherence and fairness properties. Arrow's theorem shows that already unanimity and independence of irrelevant alternatives force the existence of a dictator:

Theorem 3.7.1 (Arrow's impossibility theorem)**.** *Let $n \in \mathbb{N}_{\geq 2}$, let A be a finite set with $\#A \geq 3$, and let P be the set of all total orders on A. Let $f \colon P^n \longrightarrow P$ satisfy the following properties:*

- Unanimity. *If all voters agree on the order between two alternatives, then this order appears in the result:*

$$\forall_{x,y \in A} \quad \forall_{p \in P^n} \quad \bigl((\forall_{j \in \{0,\dots,n\}} \quad x <_{p_j} y) \Rightarrow x <_{f(p)} y)\bigr).$$

- Independence of irrelevant alternatives. *Adding further alternatives does not change the resulting aggregated order:*

$$\forall_{x,y \in A} \quad \forall_{p,q \in P^n} \quad \bigl((\forall_{j \in \{0,\dots,n\}} \quad x <_{p_j} y \Leftrightarrow x <_{q_j} y) \Rightarrow (x <_{f(p)} y \Leftrightarrow x <_{f(q)} y)\bigr).$$

Then there exists a dictator*, i.e., there is a $j \in \{1,\dots,n\}$ such that*

$$\forall_{x,y \in A} \quad \forall_{p \in P^n} \quad (x <_{p_j} y \Rightarrow x <_{f(p)} y).$$

The general case can be reduced to the case of $n = 2$ voters and the set $A = \{0,1,2\}$ of alternatives through a combinatorial argument [65]. We

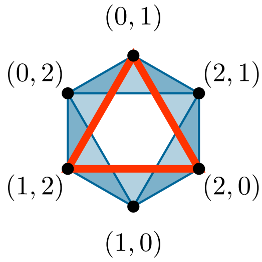


Figure 3.10.: Voting preferences for two voters and three alternatives; a generating cycle for $H_1(X)$ is highlighted in orange.

will therefore only prove the theorem in this special case. In principle, this case could be brute-forced by enumerating all possible functions and checking that the claim is satisfied. Because this gives no understanding of the underlying problem, we will use a different strategy.

As in the case of distributed systems, we use simplices to model consistent choices (Model 2.4.4, Example 2.4.5, Example 2.5.3). More precisely, let

$$V := \bigl\{(x, y) \bigm| x, y \in A,\ x \neq y\bigr\};$$

we consider the simplicial complex

$$X := \bigl\{\sigma \in P_{\text{fin}}(V) \bigm| \exists_{p \in P} \quad \forall_{(x,y) \in \sigma} \quad x >_p y\bigr\}$$

of consistent preferences (Figure 3.10).

The key to proving Arrow's theorem (Theorem 3.7.1) is to define a corresponding complex X' for P^2, to realise that an aggregation map $P^2 \longrightarrow P$ as in the theorem leads to a simplicial map $X' \longrightarrow X$, and to then to apply the simplicial homology functor in degree 1.

Example 3.7.2 (pairs of preferences for three alternatives)**.** Let

$$V' := \bigl\{((x, y), (x', y')) \bigm| (x, y), (x', y') \in V,\ \{x', y'\} = \{x, y\}\bigr\} \subset V \times V.$$

We consider the following subcomplex of $X \boxtimes X$:

$$X' := \bigl\{\sigma \in P_{\text{fin}}(V') \bigm| \pi_1(\sigma) \in X,\ \pi_2(\sigma) \in X\bigr\}.$$

As in the case of the complex X, also this complex can be described as the nerve of a straightforward cover of P^2 (check!).

Lemma 3.7.3 (a simplicial aggregation map)**.** *In the situation of Theorem 3.7.1, the map f induces a well-defined simplicial map $F \colon X' \longrightarrow X$ via*

$$
\begin{aligned}
V' = V(X') &\longrightarrow V(X) = V \\
\bigl((x,y),(x',y')\bigr) &\longmapsto \begin{cases} (x,y) & \text{if there exists } (p,p') \in P^2 \text{ with } x >_p y,\ x' >_{p'} y' \text{ and } x >_{f(p,p')} y \\ (y,x) & \text{if there exists } (p,p') \in P^2 \text{ with } x >_p y,\ x' >_{p'} y' \text{ and } y >_{f(p,p')} x. \end{cases}
\end{aligned}
$$

Proof. By the independence of irrelevant alternatives, the map F is well-defined on the vertices. That F is simplicial is a also consequence of the independence of irrelevant alternatives (Exercise). □

In the following, we will abbreviate elements (x,y) of $V(X)$ by the string xy and elements $((x,y),(x'y'))$ of $V(X')$ by the string $xyx'y'$.

Lemma 3.7.4 (a troublemaker cycle). *The simplicial cycle*

$$c := [01, 12] + [12, 20] + [20, 01] \in C_1(X)$$

represents a generator of $H_1(X) \cong_{\mathbb{Z}} \mathbb{Z}$.

Proof. This follows from a direct computation or by using homotopy invarance of H_1 and the computation of $H_1(S(1))$ (check!). □

In addition to the simplicial aggregation map $F\colon X' \longrightarrow X$, we consider the diagonal map $\Delta\colon X \longrightarrow X'$ (which is simplicial; check!) and the two inclusions $i_1, i_2\colon X \longrightarrow X'$, defined as follows: Let $q := (0 > 1 > 2) \in P$. We then define i_1 and i_2 on $xy \in V(X)$ by

$$
\begin{aligned}
i_1(xy) &:= \begin{cases} (xy, xy) & \text{if } x >_q y \\ (xy, yx) & \text{if } y >_q x \end{cases} \\
i_2(xy) &:= \begin{cases} (xy, xy) & \text{if } x >_q y \\ (yx, xy) & \text{if } y >_q x. \end{cases}
\end{aligned}
$$

Also, i_1 and i_2 are indeed simplicial (check!).

Lemma 3.7.5 (homology of maps to the paired preference complex). *We have*

$$H_1(\Delta) = H_1(i_1) + H_1(i_2)\colon H_1(X) \longrightarrow H_1(X').$$

Proof. This is an instance of a general phenomenon for the homology of products. For the sake of simplicity, we prove this by an explicit computation: In view of Lemma 3.7.4, it suffices to show that

$$H_1(\Delta)([c]) = H_1(i_1)([c]) + H_1(i_2)([c]).$$

Let

$$b := [1212, 2002, 0101] + [1212, 0220, 0101] - [1212, 2020, 0101] \in C_2(X').$$

Then $\partial_2 b = C_1(\Delta)(c) - C_1(i_1)(c) - C_1(i_2)(c)$ (check!), proving the claim. □

Lemma 3.7.6 (simplicial maps on the preference complex)**.** *Let $g\colon X \longrightarrow X$ be a simplicial map.*

1. *Then $H_1(g) = d \cdot \mathrm{id}_{H_1(X)}$ with $d \in \{-1, 0, 1\}$. We write $\deg g := d$.*
2. *If $\deg g = 1$ and $g(01) = (01)$, then $g = \mathrm{id}_X$.*

Proof. Ad 1. Because of $H_1(X) \cong_{\mathbb{Z}} \mathbb{Z}$ (Lemma 3.7.4), the degree $\deg g$ of g is well-defined. The rigidity of the simplicial structure of X forces $\deg g \in \{-1, 0, 1\}$ (Exercise).

Ad 2. We first investigate what happens to c under g. Because of $\deg g = 1$ and because $[c] \neq 0$ in $H_1(X)$, the cycle $C_1(g)(c)$ represents a non-zero class in $H_1(X)$ and thus cannot consist of fewer than three 1-simplices (check!). Therefore, $\{g(01), g(12)\}$, $\{g(12), g(20)\}$, and $\{g(20), g(01)\}$ are 1-simplices.

Using that $g(01) = 01$, there are only four options for $g(12)$, namely the neighbours of 01 in X. Because of

$$\bigl[C_1(g)(c)\bigr] = H_1(g)([c]) = \deg g \cdot [c] = 2 \cdot [c] = [c],$$

we obtain that $C_1(g)(c)$ must not be the boundary of a 2-simplex of X. Therefore, only the following two options remain:

- If $g(12) = 12$ and $g(20) = 20$, then also the values on all other vertices are uniquely determined (because of the 2-simplices) and we obtain $g = \mathrm{id}_X$.
- If $g(12) = 20$ and $g(20) = 12$, then

  $$[c] = \deg g \cdot [c] = H_1(g)([c]) = \bigl[[01, 20] + [12, 12] + [12, 01]\bigr] = -[c],$$

  which is impossible. Hence, this case cannot occur.

This shows that $g = \mathrm{id}_X$. □

Proof of Theorem 3.7.1. By the unanimity property, the simplicial aggregation map F (Lemma 3.7.3, which uses the independence of irrelevant alternatives) satisfies $F \circ \Delta = \mathrm{id}_X$. Applying the functor H_1, we therefore obtain

$$\begin{aligned}
\mathrm{id}_{H_1(X)} &= H_1(F \circ \Delta) = H_1(F) \circ H_1(\Delta) && \text{(functoriality of } H_1) \\
&= H_1(F) \circ \bigl(H_1(i_1) + H_1(i_2)\bigr) && \text{(Lemma 3.7.5)} \\
&= H_1(F \circ i_1) + H_1(F \circ i_2). && \text{(functoriality)}
\end{aligned}$$

Hence, $\deg(F \circ i_1) + \deg(F \circ_2) = 1$. Because these degrees lie in $\{-1, 0, 1\}$ (Lemma 3.7.6), the only possiblity is that there is a $k \in \{1, 2\}$ with $\deg(F \circ i_k) = 1$.

Unanimity also shows that $F \circ i_k(01) = 01$ (check!). Therefore, we obtain $F \circ i_k = \mathrm{id}_X$ (Lemma 3.7.6). However, this just means that entity k is a dictator. □

Alternatively, one can also formulate this proof in the same language as the consensus problems in distributed systems [65] or use other techniques from homotopy theory.

3.8 Application: Nash equilibria

In game theory, there are various notions of equilibria. The study of such equilibria is not only relevant for actual games, but is also part of economic and social theories. Nash introduced a suitable notion of equilibrium in non-cooperative multi-person games and showed that such equilibria always exist [57, 58]. His first approach was based on the Kakutani fixed point theorem; he then improved his argument, using the Brouwer fixed point theorem instead. We will follow this second version.

Real-world problem 3.8.1 (games and strategies)**.** A number of players (people, companies, states, ...) interacts with each other (without cooperation). Each player has a finite list of alternative strategies for this interaction; different players can have different such lists. Moreover, each player receives a payoff (money, energy, reputation, survival, ...), depending on the chosen strategies of all entities.

The problem is to determine how rational players should behave in such a situation if they aim at maximising their payoffs. There are several interpretations of this problem. In general, globally maximising the payoff for each player will not be achievable. One interpretation therefore asks for finding "equilibria", i.e., combinations of strategies of all players that make it unattractive for each player to choose a different strategy.

One way to model such problems is via Nash's notion of games, strategies, and equilibria:

Definition 3.8.2 (game, pure/mixed strategy)**.** Let $n \in \mathbb{N}$. An $(n+1)$-*person game* is a tuple $(S_0, \dots, S_n, p_0, \dots, p_n)$, consisting of finite sets $S_0, \dots, S_n$ and *payoff functions* (or *utility functions*) $p_0, \dots, p_n \colon \prod_{j=0}^n S_j \longrightarrow \mathbb{R}$.

For $j \in \{0, \dots, n\}$, the elements of S_j are called the *pure strategies* of player j. The elements of $\bigl|\Delta(S_j)\bigr|$ are called the *mixed strategies* of player j.

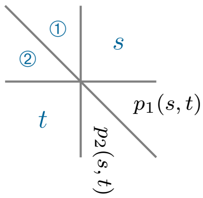
The mixed strategies of a player are just convex combinations of pure strategies and thus can be viewed as probability measures on the set of pure strategies, i.e., as randomised strategies.

A Nash equilibrium is a tuple of mixed strategies so that no player can single-handedly improve his payoff by changing his own strategy. In general, this does *not* mean that the payoff is globally maximised for each player.

Definition 3.8.3 (extended payoff function, Nash equilibrium)**.** Let $n \in \mathbb{N}$ and let $G := (S_0, \dots, S_n, p_0, \dots, p_n)$ be an $(n+1)$-person game.

- We write $S(G) := \prod_{j=0}^{n} |\Delta(S_j)|$. For $j \in \{0, \dots, n\}$, we extend the payoff function $p_j \colon \prod_{j=0}^{n} S_j \longrightarrow \mathbb{R}$ to $S(G) \longrightarrow \mathbb{R}$ in the unique $(n+1)$-affine-linear way. For simplicity, also the extended function will be denoted by p_j.
- If $\xi \in S(G)$, $j \in \{0, \dots, n\}$, and $\alpha \in |\Delta(S_j)|$, then we write
$$\xi[j : \alpha] := (\xi_0, \dots, \xi_{j-1}, \alpha, \xi_{j+1}, \dots, \xi_n) \in S(G).$$
- A point $\xi \in S(G)$ is a *Nash equilibrium* of G if
$$\forall_{j \in \{0,\dots,n\}} \quad p_j(\xi) = \max_{\alpha \in |\Delta(S_j)|} p_j\bigl(\xi[j : \alpha]\bigr).$$

Example 3.8.4 (rock-paper-scissors)**.** We model the game rock-paper-scissors in the terminology of Definition 3.8.3. This game is a two-player game. Each player has three pure strategies: rock, paper, and scissors. The payoff functions on the pure strategies are given by Figure 3.11, where the notation is to be interpreted as in the following scheme:



A case-by-case analysis shows that this game has *no* Nash equilibrium that consists only of pure strategies. For instance, (S, P) is not a Nash equilibrium, because
$$p_1\bigl((\mathrm{S},\mathrm{P})[2 : \mathrm{R}]\bigr) = p_2(\mathrm{S},\mathrm{R}) = 1 > -1 = p_2(\mathrm{S},\mathrm{P}).$$
One can show that (r, r) with $r := 1/3 \cdot \mathrm{R} + 1/3 \cdot \mathrm{S} + 1/3 \cdot \mathrm{P}$ is a Nash equilibrium; this corresponds to both players randomly picking rock, paper, or scissors (with equal probabilities).

Theorem 3.8.5 (existence of Nash equilibria)**.** *Every game (in the sense of Definition 3.8.3) has a Nash equilibrium.*

Proof. We apply the Brouwer fixed point theorem (Corollary 3.5.5) to the normalised "gain functions". Let $n \in \mathbb{N}$ and let $G := (S_0, \dots, S_n, p_0, \dots, p_n)$ be an $(n+1)$-person game. For $j \in \{0, \dots, n\}$ and $\alpha \in |\Delta(S_j)|$, we consider the gain function
$$\begin{aligned} g_{j,\alpha} \colon S(G) &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \max\bigl(0, p_j(\xi[j : \alpha]) - p_j(\xi)\bigr), \end{aligned}$$
which describes the gain of player j by switching his strategy to α. We then set

② \ ①	R	P	S
R	0 / 0	1 / −1	−1 / 1
P	−1 / 1	0 / 0	1 / −1
S	1 / −1	−1 / 1	0 / 0

Figure 3.11.: The game rock-paper-scissors; R stands for rock, P for paper, and S for scissors.

$$\begin{aligned}\varphi\colon S(G) &\longrightarrow S(G)\\ \xi &\longmapsto \Bigl(\frac{\xi_j + \sum_{\alpha\in S_j} g_{j,\alpha}(\xi)\cdot e_\alpha}{1+\sum_{\alpha\in S_j} g_{j,\alpha}(\xi)}\Bigr)_{j\in\{0,\dots,n\}}.\end{aligned}$$

We may view $S(G)$ as subset of a Euclidean space; then $S(G)$ is convex and non-empty, thus contractible. Moreover, $\varphi\colon S(G) \longrightarrow S(G)$ is well-defined (check!) and continuous (check!). Because $S(G)$ admits a finite triangulation (check!), the (generalised) Brouwer fixed point theorem (Corollary 3.5.4) implies that φ has a fixed point $\xi \in S(G)$.

We show that ξ is a Nash equilibrium of G: It suffices to show for all $j \in \{0,\dots,n\}$ and all *pure* strategies $\alpha \in S_j$ that $p_j(\xi) \geq p_j(\xi[j:\alpha])$ (Exercise), or, equivalently, that $g_{j,\alpha}(\xi) = 0$.

Let $j \in \{0,\dots,n\}$ and $\alpha \in S_j$. Let $\beta \in S_j$ be a least profitable pure strategy for player j at ξ, i.e., $p_j(\xi[j:\beta]) = \min_{\gamma\in S_j} p_j(\xi[j:\gamma])$. By convexity, then $p_j(\xi[j:\beta]) \leq p_j(\xi)$ (check!), whence $g_{j,\beta}(\xi) = 0$. Because ξ is a fixed point of φ, the β-contribution of ξ_j is not decreased by φ. As $g_{j,\beta}(\xi) = 0$, this implies that

$$1 + \sum_{\alpha\in S_j} g_{j,\alpha}(\xi) = 1.$$

Hence, $g_{j,\alpha}(\xi) = 0$ for all $\alpha \in S_j$, as desired. □

Literature exercise. Read about the life of John Forbes Nash Jr., including his work (and awards) in economic theory and his work (and awards) in theoretical mathematics.

3.9 Application: Sensor network coverage

If multiple sensors are used in surveillance tasks, one is interested in having a guarantee that the full region of interest is covered by the sensors. For sensors with fixed positions or with known absolute positions, this coverage problem can be solved by means of (computational) geometry.

In situations, where absolute positions are not known or cannot be obtained with reasonable effort (e.g., certain indoor setups do not allow for use of GPS), other methods are needed. Simplicial homology can be used to organise local information into a sufficient coverage criterion [21]:

Real-world problem 3.9.1 (sensor network coverage)**.** We consider a region in the plane that is fenced off by a "cycle" of sensors. Within this region, we consider finitely many sensors. All sensors have unique ids. Each sensor can survey a circular region around it. Moreover, each sensor broadcasts its unique id to all sensors that are close enough; in addition, the sensors can also communicate other data (such as information that they gathered on other sensors) to sensors that are close enough. For simplicity, we assume that the surveillance and communication radii all agree (Figure 3.12).

The problem is to find sufficient conditions that guarantee that the sensor network covers the whole fenced off region; these sufficient conditions should be checkable by the fence sensors of the network.

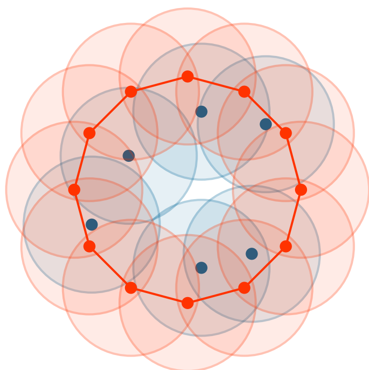


Figure 3.12.: Sensor network coverage; the fence is orange

Model 3.9.2 (sensor network coverage)**.** We model the situation of Problem 3.9.1 as follows:

- Every sensor is modelled as a point in $\mathbb{R}^2$; let $S \subset \mathbb{R}^2$ be the finite set of sensors. All sensors are assumed to have the same communication and sensing radius $\varepsilon \in \mathbb{R}_{>0}$ with respect to the Euclidean metric d_2.

  The region under surveillance by the sensors is thus $\bigcup_{s \in S} B_\varepsilon(x, d_2)$.

- Let $X := R_\varepsilon(S, d_2)$ be the associated Rips complex. The *shadow map* is the map $\varphi\colon |X| \longrightarrow \mathbb{R}^2$ induced by the inclusion $S \hookrightarrow \mathbb{R}^2$ and affine linear extension.

  In particular, $\varphi(|X|) \subset \mathbb{R}_2$ is a subset of the region $\bigcup_{s \in S} B_\varepsilon(x, d_2)$ covered by the sensors.

- The fence is modelled as a one-dimensional subcomplex $C \subset X$ with the following properties:

  1. The restriction $\varphi|_{|C|}$ to $|C|$ is a homeomorphism onto its image
  2. and this image is homeomorphic to S^1.

  These two conditions imply that the 1-simplices of C form a cycle in $C_1(C; \mathbb{F}_2) \subset C_1(X; \mathbb{F}_2)$ (check!).

  By the Jordan curve theorem [47, Chapter 4.4.2], $\varphi(|C|)$ separates $\mathbb{R}^2$ into two connected components, exactly one of which is bounded. This bounded component is the *fenced region* D and $\partial D = \varphi(|C|) \cong_{\mathsf{Top}} S^1$. By the Jordan–Schönflies theorem [52], we furthermore have $D \cong_{\mathsf{Top}} D^2$. the Jordan–Schönflies theorem, we could also add these properties as conditions.)

We call (S, C) a *sensor network in* $\mathbb{R}^2$ *with radius* ε.

The class represented by the "cycle" C in $H_1(C; \mathbb{F}_2)$ will be denoted by $[C]$. The class represented by C in $H_1(X; \mathbb{F}_2)$ will be denoted by α_C.

We will see in Theorem 3.9.3 a sufficient condition in terms of α_C that guarantees that $\varphi(|X|)$ (and whence $\bigcup_{s \in S} B_\varepsilon(x, d_2)$) contains D.

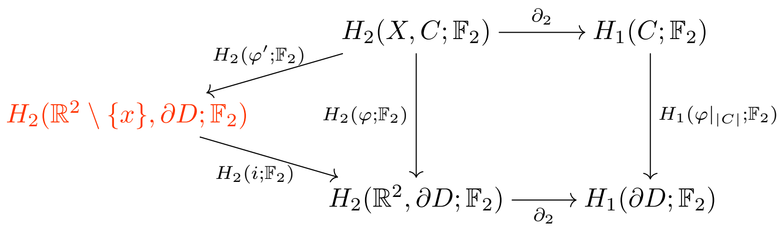
Theorem 3.9.3 (a sufficient condition for sensor network coverage [21])**.** *Let (S, C) be a sensor network in $\mathbb{R}^2$ with radius $\varepsilon \in \mathbb{R}_{>0}$ and let $X := R_\varepsilon(S, d_2)$ be the associated Rips complex. If the fence class $\alpha_C \in H_1(X; \mathbb{F}_2)$ is trivial, then the sensor network covers the whole fenced region D.*

Proof. Let $\alpha_C = 0$ in $H_1(X; \mathbb{F}_2)$. The long exact sequence of the pair (X, C) shows that $\alpha_C = 0 \in H_1(X; \mathbb{F}_2)$ is equivalent to the existence of a relative class $\beta \in H_2(X, C; \mathbb{F}_2)$ with $\partial_2(\beta) = [C]$ (check!).

Assume for a contradiction that there exists a point $x \in D$ that is *not* covered by the sensor network. Then the shadow map $\varphi\colon |X| \longrightarrow \mathbb{R}^2$ factors over $\mathbb{R}^2 \setminus \{x\}$, i.e., we have

$$\varphi = i \circ \varphi',$$

where $i\colon \mathbb{R}^2 \setminus \{x\} \hookrightarrow \mathbb{R}^2$ is the inclusion and $\varphi'\colon |X| \longrightarrow \mathbb{R}^2 \setminus \{x\}$. Applying homology leads to the following commutative diagram:



The maps ∂_2 denote the connecting homomorphisms from the corresponding long exact sequences of pairs.

We slightly abused notation. In fact, we defined homology only for (finitely) triangulable spaces; in particular, for the homology of $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{x\}$ we would need to make replacements by suitable finitely triangulable spaces. This is not difficult to achieve (check!), but in order to keep the notation lightweight, we omit this step. Moreover, a straightforward computation (e.g., using the long exact homology sequence for the pair, homotopy invariance, and the homotopy equivalence $\partial D \hookrightarrow \mathbb{R}^2 \setminus \{x\}$; check!) shows that

$$H_2(\mathbb{R}^2 \setminus \{x\}, \partial D; \mathbb{F}_2) \cong_{\mathbb{F}_2} 0.$$

By construction, $[C] \in H_1(C; \mathbb{F}_2) \cong_{\mathbb{F}_2} \mathbb{F}_2$ is non-zero (check!). By assumption, $\varphi|_{|C|} \colon |C| \longrightarrow \partial D$ is a homeomorphism. Hence, the right vertical arrow is an isomorphism (Theorem 3.4.6). Therefore, we obtain

$$\begin{aligned} 0 &\neq H_1(\varphi|_{|C|}; \mathbb{F}_2)([C]) \\ &= H_1(\varphi|_{|C|}; \mathbb{F}_2)\bigl(\partial_2(\beta)\bigr) && \text{(by the choice of } \beta) \\ &= \partial_2 \circ H_2(i; \mathbb{F}_2) \circ H_2(\varphi'; \mathbb{F}_2)(\beta) && \text{(commutativity of the diagram)} \\ &= \partial_2 \circ H_2(i; \mathbb{F}_2)(0) && (\text{because } H_2(\mathbb{R}^2 \setminus \{x\}, \partial D; \mathbb{F}_2) \cong_{\mathbb{F}_2} 0) \\ &= 0. \end{aligned}$$

This contradiction shows that the whole region D is covered. □

Remark 3.9.4 (reduced networks)**.** In the situation of Theorem 3.9.3, let $\alpha_C = 0$ in $H_1(X; \mathbb{F}_2)$. In the proof, we used a relative class $\beta \in H_2(X, C; \mathbb{F}_2)$ with $\partial_2(\beta) = \alpha_C$. We can then also find a chain $z \in C_2(X; \mathbb{F}_2)$ with $\partial_2(z) = C$. Such explicit chains z can be used to select a smaller network that also covers the whole region, thus allowing for a reduced use of energy etc..

Remark 3.9.5 (computability by the network and implementation)**.** In the situation of Theorem 3.9.3, the local information present in the sensors can be aggregated by the fence sensors into a computation that decides whether $\alpha_C = 0$ in $H_1(X; \mathbb{F}_2)$ or not. Several algorithms are available. Particularly efficient algorithms are based on spanning trees [69].

Outlook 3.9.6 (sensing vs. communicating)**.** For simplicity, we considered the case that the sensing radius and the communication radius of the sensors agree. If these radii are different, a slightly more refined analysis is available [22], which is related to a simple case of persistent homology (Chapter 4).

4

Persistent homology

Topological data analysis constructs simplicial complexes from “big data”, depending on threshold/mesh parameters (Chapter 2.5). Typically, in such applications, there is only limited a priori knowledge on which ranges of parameters are appropriate. Therefore, one considers a sequence of such simplicial complexes, where each member represents a choice of parameters.

The usual Betti numbers of such sequences do not directly provide meaningful information as they are not stable under small perturbations. Instead, one considers so-called persistent Betti numbers. The structure theorem for persistent homology allows to represent this information in terms of their associated barcodes. These barcodes encode “how long homology classes persist” in the sequence of complexes and turn out to satisfy stability under suitable perturbations. Persistent homology and barcodes (over fields) can be computed efficiently.

In addition to the theory, we will explain the persistent homology workflow in practice and consider applications from biology and medicine.

Overview of this chapter.

Running example. persistent homology of Rips filtrations of point clouds

4.1 Persistent homology

We introduce the basic terminology for persistent homology, i.e., for the persistence of homology classes/Betti numbers under sequences of chain maps.

4.1.1 Filtrations

Simplicial complexes from point clouds (Chapter 2.5) depend on threshold/mesh parameters. This leads to filtrations:

Definition 4.1.1 (filtration). Let K be a simplicial complex. A *filtration* of K is a sequence $(K^n)_{n\in\mathbb{N}}$ of subcomplexes of K that are nested via

$$K^0 \subset K^1 \subset \cdots \subset K$$

and that satisfy $K = \bigcup_{n\in\mathbb{N}} K^n$. A filtration is *of finite type* if the sequence stabilises after finitely many steps and if each simplicial complex is finite.

Example 4.1.2 (Rips filtration). Let d be a metric on $\mathbb{R}^N$, let $X \subset \mathbb{R}^N$ be a finite set, and let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{>0}$ be a monotonically increasing sequence with $\lim_{n\to\infty} \varepsilon_n = \infty$. Then $(R_{\varepsilon_n}(X,d))_{n\in\mathbb{N}}$ is a filtration of $\Delta(X)$ of finite type (check!).

Example 4.1.3 (from filtrations to sequences of chain complexes and persistent homology). If R is a commutative ring with unit and $(K^n)_{n\in\mathbb{N}}$ is a filtration of a simplicial complex K, then applying the simplicial chain complex functor to the inclusion maps $(i^n \colon K^n \to K^{n+1})_{n\in\mathbb{N}}$ leads to a sequence

$$C(K^0;R) \xrightarrow{C(i^0;R)} C(K^1;R) \xrightarrow{C(i^1;R)} C(K^2;R) \longrightarrow \cdots \subset C(K;R)$$

in the category ${}_R\mathsf{Ch}$ of R-chain complexes. We can now apply the homology functor to obtain in each degree $k \in \mathbb{N}$ a corresponding sequence

$$H_k(K^0;R) \xrightarrow{H_k(i^0;R)} H_k(K^1;R) \xrightarrow{H_k(i^1;R)} H_k(K^2;R) \longrightarrow \cdots \subset H_k(K;R)$$

of R-modules. The idea behind persistent homology is to ask

- at which stages in this sequence "new" classes are "born" and
- at which stages in this sequence classes "die".

The lifespan of a homology class is its *persistence*. Classes with large persistence should then correspond to genuine topological features and not to irrelevant noise.

4.1.2 Persistence objects

Before proceeding with the ideas from Example 4.1.3, we first introduce a more general abstract framework to handle such filtrations [77]. Persistence chain complexes are persistence objects in the category of chain complexes; persistence modules are persistence objects in the category of modules. More explicitly:

Definition 4.1.4 (persistence chain complex, persistence module)**.** Let R be a ring.

- A *persistence R-chain complex* is a sequence

$$C^0 \xrightarrow{f^0} C^1 \xrightarrow{f^1} C^2 \longrightarrow \cdots$$

  in ${}_R\mathsf{Ch}$ (i.e., each C^n is an R-chain complex and each f^n is an R-chain map).

- A persistence R-chain complex (C^*, f^*) is *of finite type* if the following hold: Each C^n is a finite R-chain complex (i.e., finitely generated in all degrees as well as non-trivial only in finitely many degrees) and there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$, the chain map f^n is an R-chain isomorphism.

- A *persistence R-module* is a sequence

$$M^0 \xrightarrow{f^0} M^1 \xrightarrow{f^1} M^2 \longrightarrow \cdots$$

  in ${}_R\mathsf{Mod}$ (i.e., each M^n is an R-module and each f^n is an R-linear map).

- A persistence R-module (M^*, f^*) is *of finite type* if each M^n is finitely generated and there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$, the map f^n is an R-isomorphism.

Example 4.1.5 (persistence chain complexes lead to persistence modules)**.** Let (C^*, f^*) be a persistence R-chain complex over some ring R and let $k \in \mathbb{N}$. Applying the homology functor H_k in degree k give the persistence R-module

$$H_k(C^0) \xrightarrow{H_k(f^0)} H_k(C^1) \xrightarrow{H_k(f^1)} H_k(C^2) \longrightarrow \cdots$$

Example 4.1.6 (filtrations lead to persistence modules)**.** In Example 4.1.3, the simplicial chain complexes of a filtration of a simplicial complex define a persistence chain complex; for each degree $k \in \mathbb{N}$, simplicial homology leads to a persistence module.

Remark 4.1.7 (spectral sequences)**.** Spectral sequences apply to homological questions for sequences of chain complexes. However, the focus is different: For example, the spectral sequence of a filtration gives a strategy to compute the homology of the filtered simplicial complex (or topological space) from the homology groups of the filtration stages. In our situation, the homology of the filtered simplicial complex often will be trivial (because the filtered simplicial complex is a simplex). Instead, we are interested in a more quantitative evolution of homology classes through the filtration stages [29, Chapter VII.4].

4.1.3 Persistent homology and persistent Betti numbers

Persistent homology of a persistence chain complex keeps track of which homology classes survive to which stages:

Definition 4.1.8 (persistent homology)**.** Let R be a ring and let (C^*, f^*) be a persistence R-chain complex. Let $i, j \in \mathbb{N}$ with $j \geq i$ and let $k \in \mathbb{N}$. Then the *(i, j)-persistent homology of (C^*, f^*) in degree k* is defined as

$$\begin{aligned} H_k^{i,j}(C^*, f^*) &:= \frac{f^{j-1} \circ \cdots \circ f^i(\ker \partial_k^i)}{\operatorname{im} \partial_{k+1}^j \cap f^{j-1} \circ \cdots \circ f^i(\ker \partial_k^i)} \\ &= H_k(f^{j-1} \circ \cdots \circ f^i)\bigl(H_k(C^i)\bigr) \subset H_k(C^j). \end{aligned}$$

Example 4.1.9 (persistent homology of persistence 0)**.** Let R be a ring and let (C^*, f^*) be a persistence R-chain complex. Let $i \in \mathbb{N}$ and $k \in \mathbb{N}$. Then, by definition,

$$H_k^{i,i}(C^*, f^*) = H_k(C^i).$$

Persistent homology is functorial with respect to morphisms of persistence chain complexes; a morphism of persistence objects is nothing but a natural transformation between the corresponding diagrams, i.e., a sequence of stage-preserving morphisms such that the resulting ladder is commutative.

Definition 4.1.10 (persistent Betti number)**.** Let R be a principal ideal domain (or another ring with a "reasonable" notion of rank) and let (C^*, f^*) be a persistent R-chain complex of finite type. Let $i, j \in \mathbb{N}$ with $j \geq i$ and let $k \in \mathbb{N}$. Then the *(i, j)-persistent Betti number of (C^*, f^*) in degree k* is defined as

$$b_k^{i,j}(C^*, f^*) := \operatorname{rk}_R H_k^{i,j}(C^*, f^*) \in \mathbb{N}.$$

Example 4.1.11 (persistent homology/Betti numbers of point clouds)**.** Let $X \subset \mathbb{R}^N$ be a finite set, let d be a metric on $\mathbb{R}^N$, and let $(\varepsilon_n)_{n \in \mathbb{N}}$ be an increasing sequence with $\lim_{n \to \infty} \varepsilon_n = \infty$. If R is a principal ideal domain and $i, j \in \mathbb{N}$ with $i < j$ and $k \in \mathbb{N}$, then

$$H_k^{i,j}(X, d, \varepsilon_*; R) := H_k^{i,j}\bigl(C(R_{\varepsilon_*}(X, d); R), C(\text{inclusions}; R)\bigr)$$

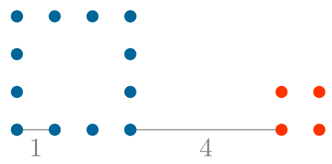


Figure 4.1.: A point cloud in the Euclidean plane

is the (i,j)*-persistent homology of* X *(with respect to* d *and* $(\varepsilon_n)_{n\in\mathbb{N}}$*) with* R*-coefficients in degree* k and

$$b_k^{i,j}(X,d,\varepsilon_*;R) := b_k^{i,j}\bigl(C(R_{\varepsilon_*}(X,d);R), C(\text{inclusions};R)\bigr)$$

is the (i,j)*-persistent Betti number of* X *(with respect to* d *and* $(\varepsilon_n)_{n\in\mathbb{N}}$*) over* R *in degree* k.

Definition 4.1.12 (persistence of a homology class)**.** Let R be a commutative ring and let (C^*, f^*) be a persistent R-chain complex. Let $i,j\in\mathbb{N}$ with $i<j$ and let $k\in\mathbb{N}$.

- A homology class $\alpha \in H_k^{i,i}(C^*,f^*)$ is *born at stage* i if $\alpha \notin H_k^{i-1,i}(C^*,f^*)$.
- A homology class $\alpha \in H_k^{i,i}(C^*,f^*)$ born at stage i *dies at stage* j if $H_k(f^{j-1}\circ\dots\circ f^i)(\alpha) \in H_k^{i-1,j}(C^*,f^*)$ but $H_k(f^{j-2}\circ\dots\circ f^i)(\alpha) \notin H_k^{i-1,j-1}(C^*,f^*)$ (this is the *elder rule*).
- If α is born at stage i and dies at stage j, we call $j-i$ the *(index) persistence of* α.

Example 4.1.13 (persistence of classes in point clouds)**.** We consider the finite subset X of $\mathbb{R}^2$ depicted in Figure 4.1 and the sequence $\varepsilon_* := (0.1, 1.1, 2.1, 3.1, 4.1, 100, 101, \dots)$ in $\mathbb{R}_{>0}$. A straightforward computation shows that (Figure 4.2; check!)

$$\begin{aligned}
b_1^{1,1}(X,d_2,\varepsilon_*;\mathbb{Q}) &= 1+1=2 && \text{(classes are born in both "squares")}\\
b_1^{1,2}(X,d_2,\varepsilon_*;\mathbb{Q}) &= 1 && \text{(the "right cycle" dies)}\\
b_1^{1,3}(X,d_2,\varepsilon_*;\mathbb{Q}) &= 0 && \text{(weird intermediate stage in the left "square")}\\
b_1^{1,4}(X,d_2),\varepsilon_*;\mathbb{Q}) &= 0 && \text{(the left hand side dies; the emerging middle class does not come from stage 1)}\\
b_1^{1,100}(X,d_2,\varepsilon_*;\mathbb{Q}) &= 0. && \text{(all classes are dead)}
\end{aligned}$$

The persistence of the "left cycle" is 2, while the one of the "right cycle" is 1.

Remark 4.1.14 (weighted persistence)**.** In the situation of Example 4.1.11 it is usually better to consider the weighted persistence, which is measured in terms of the underlying radius parameter instead of the sampling indices. We will return to this aspect in Chapter 4.4.3.

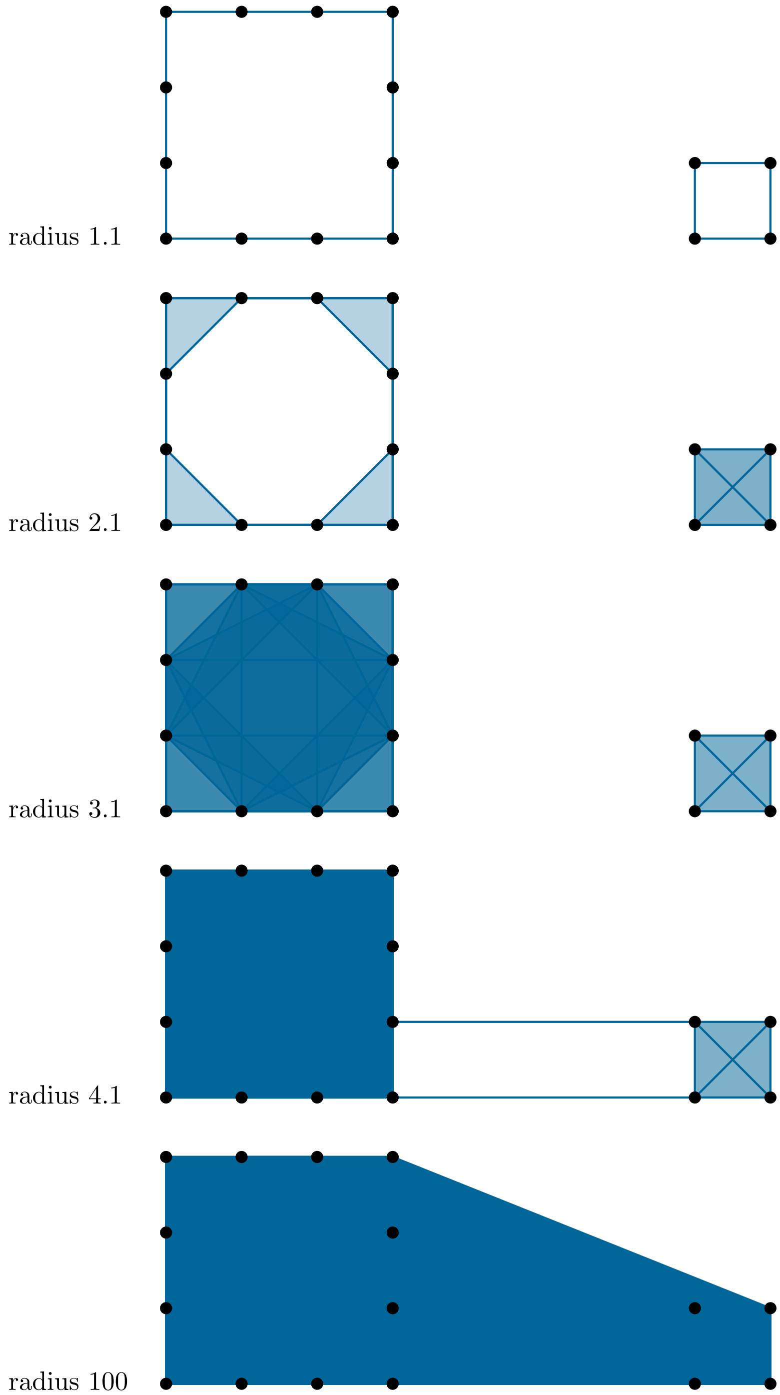


Figure 4.2.: The ($\mathbb{R}^2$-shadows of the) Rips complexes in Example 4.1.13

4.2 The structure theorem for persistent homology

We reorganise the information contained in the persistent Betti numbers in terms of barcodes. One could do this directly by hand. We use an approach based on graded commutative algebra – this more abstract approach has the benefit that it is conceptually transparent and that it is closer to generalisations to other settings, such as zigzag persistence.

The idea is to view persistence modules as *graded* modules over the *graded* polynomial ring of the coefficient ring. If the coefficient ring is a field, this polynomial ring is a principal ideal domain. A graded version of the structure theorem for finitely generated modules over principal ideal domains then gives the structure theorem for persistence modules. This structure can be reinterpreted in terms of "barcodes".

4.2.1 Persistence modules and polynomial rings

We introduce some notions on graded rings and modules and explain the fundamental example of persistence modules as graded modules.

Definition 4.2.1 (graded ring)**.** A *graded ring* is a triple $(R, (R_n)_{n\in\mathbb{N}}, \varphi)$, where R is a ring, the R_n are Abelian groups, and $\varphi\colon \bigoplus_{n\in\mathbb{N}} R_n \longrightarrow (R,+)$ is an isomorphism of Abelian groups with the property that

$$\forall_{n,m\in\mathbb{N}} \quad \varphi(R_n)\cdot\varphi(R_m) \subset \varphi(R_{n+m}).$$

For $n \in \mathbb{N}$, the elements in $\varphi(R_n)$ are called *homogeneous of degree* n. An element of R is *homogenous* if there exists an $n \in \mathbb{N}$ such that the element is homogeneous of degree n. Usually, one leaves φ implicit and omits it from the notation.

Example 4.2.2 (polynomial rings)**.** Let K be a ring. Then the usual degree on monomials in the polynomial ring $K[T]$ turns $K[T]$ into a graded ring via $K[T] \cong_{\mathsf{Ab}} \bigoplus_{n\in\mathbb{N}} K\cdot T^n$ (check!). We will always consider this graded structure on polynomial rings.

Definition 4.2.3 (graded module)**.** Let R be a graded ring. A *graded module over* R is a triple $(M,(M_n)_{n\in\mathbb{N}},\varphi)$, consisting of an R-module M, Abelian groups M_n, and an isomorphism $\varphi\colon \bigoplus_{n\in\mathbb{N}} M_n \longrightarrow (M,+)$ of Abelian groups with

$$\forall_{n,m\in\mathbb{N}} \quad R_n\cdot\varphi(M_m) \subset \varphi(M_{n+m}).$$

Elements of $\varphi(M_m)$ are called *homogeneous of degree m*. Again, usually, one leaves φ implicit and omits it from the notation.

Example 4.2.4 (shifted graded modules)**.** Let R be a graded ring, let M be a graded module over R, and let $n \in \mathbb{N}$. Then $\Sigma^n M$ denotes the graded R-module (check!) given by the shifted decomposition $0 \oplus \cdots \oplus 0 \oplus \bigoplus_{j \in \mathbb{N}_{\geq n}} M_{j-n}$.

Example 4.2.5 (from persistence modules to graded modules)**.** Let K be a domain and let (M^*, f^*) be a persistence K-module. Then $M := \bigoplus_{n \in \mathbb{N}} M^n$ carries a $K[T]$-module structure, given by

$$\forall_{x \in M^n} \quad T \cdot x := f^n(x) \in M^{n+1}.$$

If we view $K[T]$ as a graded ring (Example 4.2.2), then this $K[T]$-module structure and this direct sum decomposition of M turn M into a graded $K[T]$-module (check!).

Remark 4.2.6 (the category of graded modules)**.** Let R be a graded ring. *Homomorphisms* between graded R-modules are R-linear maps that preserve the grading. Graded R-modules and homomorphisms of R-modules form the category ${}_R\mathsf{Mod}^*$ of graded R-modules.

4.2.2 The structure theorem

We now consider the graded version of the structure theorem for graded modules over graded principal ideal domains (similarly to the one of Zomorodian and Carlsson [77, Theorem 2.1]) and apply it to the case of persistence modules.

Theorem 4.2.7 (structure theorem for modules over graded PIDs)**.** *Let R be a graded ring that is a principal ideal domain (as ungraded ring) with $R \neq R_0$. Let M be a graded R-module that is finitely generated as (ungraded) R-module. Then there exist $N \in \mathbb{N}$, homogeneous elements $f_1, \dots, f_N \in R$, and $n_1, \dots, n_N \in \mathbb{N}$ such that*

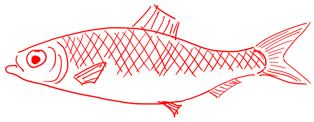
$$M \cong_{{}_R\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j} \bigl(R/(f_j)\bigr)$$

and $f_j \mid f_{j+1}$ for all $j \in \{1, \dots, N-1\}$. Here, the right-hand side carries the canonical grading.

This decomposition is unique in the sense that the multiset consisting of all pairs $(n_j, R^\times \cdot f_j)$ with $j \in \{1, \dots, N\}$ is uniquely determined by M. The elements $f_1, \dots, f_N$ are called elementary divisors of M*.*

As in the ungraded case: The free part of the decompositions corresponds to the elementary divisors that are 0.

The statement of the theorem provides a convenient perspective on the structure of persistence modules. However, one should be aware that it is a red herring:



In fact, the theory of graded rings that are principal ideal domains (as ungraded rings) encompasses only two classes of examples (Proposition 4.2.8):

- Principal ideal domains with the 0-grading. For these rings, there is a prime power version of the graded structure theorem, but in general no graded elementary divisor version (Example 4.2.9).
- Polynomial rings over fields with a multiple of the canonical grading from Example 4.2.2; for these rings, we will complement the ungraded structure theorem with an ad-hoc argument.

Proposition 4.2.8 (graded PIDs). *Let R be a graded ring that is a principal ideal domain (as ungraded ring). Then R is of one of the following types:*

- *We have $R = R_0$, i.e., R is an ordinary principal ideal domain with the 0-grading.*
- *The subring R_0 is a field and R is isomorphic to the graded ring $R_0[T]$, where the grading on $R_0[T]$ is a multiple of the canonical grading from Example 4.2.2.*

Proof. Let $R \neq R_0$ and let $n \in \mathbb{N}_{>0}$ be the minimal degree with $R_n \neq 0$. Then

$$R_{\geq n} := \bigoplus_{j \in \mathbb{N}_{\geq n}} R_j$$

is a homogeneous ideal in R (check!); as R is a principal ideal domain, there exists a $t \in R$ with $R_{\geq n} = (t)$. We show that t is homogeneous of degree n: Let $x \in R_n \setminus \{0\}$. Then t divides x and thus also t is homogeneous (Exercise). The grading implies that t has degree n.

We show that the canonical R_0-algebra homomorphism $\varphi\colon R_0[T] \longrightarrow R$ given by $\varphi(T) := t$ is an isomorphism.

- We first show that φ is injective: Because R is graded and t is homogeneous, it suffices to show that $a \cdot t^k \neq 0$ for all $a \in R_0 \setminus \{0\}$ and all $k \in \mathbb{N}$. However, this is guaranteed by the hypothesis that R is a domain.
- Regarding surjectivity, let $y \in R$. It suffices to consider the case that y is homogeneous of degree $m \geq n$. Because $(t) = R_{\geq n}$, we know that t

divides y, say $y = t \cdot y'$. Then y' is homogeneous and we can iterate the argument for y'. Proceeding inductively, we obtain that m is a multiple of n and that there exists an $a \in R_0$ with $y = a \cdot t^{m/n}$. Hence, φ is surjective.

This establishes that R is isomorphic as a graded ring to $R_0[T]$, where $R_0[T]$ carries the grading of Example 4.2.2 scaled by n.

It remains to show that R_0 is a field. By hypothesis, $R \cong_{\mathsf{Ring}} R_0[T]$ is a principal ideal domain. Therefore, R_0 is a field: The ideal (T) is non-zero and prime (because R is a domain); as R is a principal ideal domain, the ideal (T) is maximal. Therefore, $R_0 \cong_{\mathsf{Ring}} R[T]/(T)$ is a field. □

Example 4.2.9 (decompositions over trivially graded PIDs)**.** We consider the principal ideal domain $\mathbb{Z}$ with the 0-grading and the graded (check!) $\mathbb{Z}$-module $M := M_0 \oplus M_1$ given by $M_0 := \mathbb{Z}/(2)$ and $M_1 := \mathbb{Z}/(3)$. This graded module does *not* admit a *graded* elementary divisor decomposition: Indeed, if there were a graded elementary divisor decomposition of M, then the corresponding elementary divisors would have to coincide with the ungraded elementary divisors. The only ungraded elementary divisor of M is 6. However, M does *not* contain a homogeneous element with annihilator ideal (6). Therefore, M does not admit a graded elementary divisor decomposition.

Thus, Theorem 4.2.7 really is only about graded modules over polynomial rings over fields. We restate and prove the theorem in this form:

Theorem 4.2.10 (structure theorem for graded modules over polynomial rings)**.** *Let K be a field and let M be a graded $K[T]$-module that is finitely generated as (ungraded) $K[T]$-module. Then there exist $N \in \mathbb{N}$, $k_1, \dots, k_N \in \mathbb{N}_{>0} \cup \{\infty\}$, and $n_1, \dots, n_N \in \mathbb{N}$ such that*

$$M \cong_{{}_{K[T]}\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j} \bigl(K[T]/(T^{k_j})\bigr)$$

Here, $T^\infty := 0$ and the right-hand side carries the canonical grading.

This decomposition is unique in the sense that the multiset consisting of all pairs (n_j, k_j) with $j \in \{1, \dots, N\}$ is uniquely determined by M.

Proof. As K is a field, the polynomial ring $R := K[T]$ is a principal ideal domain (as ungraded ring).

Uniqueness. We take the given graded decomposition of M via a graded R-isomorphism $\varphi\colon \bigoplus_{j=1}^{N} \Sigma^{n_j} K[T]/(f_j) \longrightarrow M$ with $f_j = T^{k_j}$ and show how the multiset of all (n_j, k_j) is uniquely determined by M. We proceed by induction, using the following decomposition: Let

$$N' := \bigoplus_{j \in \{1,\dots,N\}, n_j = 0} \Sigma^{n_j} R/(f_j),$$
$$N'' := \bigoplus_{j \in \{1,\dots,N\}, n_j > 0} \Sigma^{n_j} R/(f_j).$$

First, we consider the graded submodule $M' := \operatorname{Span}_R M_0 \subset M$. We claim that $M' = \varphi(N')$. Indeed, as φ is graded, we have $\varphi(N') \subset M'$ and $\varphi(N'') \cap M_0 0$. Therefore,

$$\begin{aligned} M_0 &= \varphi(N' \oplus N'') \cap M_0 && (\varphi \text{ is surjective}) \\ &= \bigl(\varphi(N') \cap M_0\bigr) + \bigl(\varphi(N'') \cap M_0\bigr) && (\varphi \text{ is graded}) \\ &= \varphi(N') \cap M_0. && (\text{because } \varphi(N'') \cap M_0 = 0) \end{aligned}$$

In particular, $M' = \operatorname{Span}_R M_0 = \varphi(N')$. Moroever, M' is finitely generated over R (as R is a principal ideal domain and M is finitely gernerated).

Therefore, the uniqueness statement of the ungraded structure theorem for M' uniquely determines the multiset of all k_j with $n_j = 0$.

For the induction, we pass to the R-module $M'' := M/M'$. As M' is a graded submodule of M, also M'' is a graded R-module. Because M is finitely generated, M'' is finitely generated as well. Moreover, the discussion above shows that M'' is isomorphic to $\varphi(N'')$ as a graded R-module. By construction $M''_0 = 0$. Shifting the degrees on M'' by -1, we can apply the first case and iterate.

Because M is finitely generated, this procedure terminates after finitely many steps (check!) and gives the desired uniqueness.

Existence. In view of Lemma 4.2.11 below, all torsion in M comes from the prime T and its powers. Therefore, the ungraded structure theorem provides us with an ungraded decomposition

$$M \cong_{K[T]} \bigoplus_{j=1}^{N} K[T]/(T^{k_j})$$

for some $N \in \mathbb{N}$ and $k_1, \dots, k_N \in \mathbb{N} \cup \{\infty\}$. All of the remaining work goes into showing that we have a *graded* direct sum decomposition of this type. We give an ad-hoc argument (more generally, one could look at quiver representations over K [61]): Let

$$L := \max\bigl\{k_j \bigm| j \in \{1, \dots, N\}, k_j \neq \infty\bigr\} \in \mathbb{N}.$$

We first reduce to the torsion case: Let $N \subset M$ be the torsion submodule. Then N is a graded submodule of M (because all torsion is T-power torsion) and thus $F := M/N$ is a *graded* R-module without torsion. By Lemma 4.2.12 below, R is a free R-module that has a homogeneous R-basis; because F is finitely generated, this basis is finite. We thus find a *graded* split $F \longrightarrow M$

of the canonical projection $M \longrightarrow M/N = F$. This shows that there exists a graded decomposition (check!)

$$M \cong_{_R\mathsf{Mod}^*} N \oplus F.$$

Moreover, N is finitely generated. Therefore, we may assume without loss of generality that M is a torsion R-module.

Let $L \in \mathbb{N}$ be the maximal T-power that is non-trivial on M. We construct a decomposition by a nested induction. The idea is to proceed by increasing module degree and decreasing power/length of torsion. This corresponds to the "elder rule" in persistent homology and is (of course) similar to the computation of Jordan normal form bases. The construction will produce K-subspaces $M_{n,L+1}, \dots, M_{n,0}$ of M_n and graded R-submodules $A_{-1}, \dots, A_L$ of M.

We start with $A_{-1} := 0$. For the induction step, we let $n \in \mathbb{N}$ and assume that ($M_{m,L+1}, \dots, M_{m,0}$ with $m \in \{0, \dots, n\}$ as well as) $A_{-1}, \dots, A_n$ are already constructed. Let $M_{n+1,L+1} := 0$. For $k \in \{-1, \dots, L-1\}$, let $M_{n+1,L+1}, \dots, M_{n+1,L-k}$ be already constructed. We then set

$$M_{n+1,L-(k+1)} := \operatorname{Span}_K\{x_1, \dots, x_r\},$$

where $(T^{L-(k+1)} \cdot x_1, \dots, T^{L-(k+1)} \cdot x_r)$ is a K-basis of a $\bigoplus_K$-complement of

$$T^{L-k} \cdot (M_{n+1,L} + \dots + M_{n+1,L-k}) + A_0 + \dots + A_n \cap \dots$$

in $T^{L-(k+1)} \cdot (M_{n+1} \cap \ker T^{L-k})$. By construction, the annihilators of elements of the level j-part $M_{n+1,j}$ is (T^{j+1}) and $\operatorname{Span}_R M_{n+1,L-(k+1)}$ is the internal direct sum of $R \cdot x_1, \dots, R \cdot x_r$ (in the notation above). Hence, as a graded R-module, $\operatorname{Span}_R M_{n+1,L-(k+1)}$ is isomorphic to a direct sum of cyclic modules. Finally, we set

$$A_{n+1} := \operatorname{Span}_R(M_{n+1,L} + \dots + M_{n+1,0}) \subset \bigoplus_{j \in \mathbb{N}_{\geq n}} M_j.$$

From the order of the construction, one can inductively prove (check!) that M is (as a graded R-module) the internal direct sum of $A_0, \dots, A_D$, where $D \in \mathbb{N}$ is maximal with $M_D \not\cong 0$ (such an index exists because M is finitely generated and torsion); moreover, each A_n is the internal direct sum of $\operatorname{Span}_R M_{n,L}, \dots, \operatorname{Span}_R M_{n,0}$ (check!). This is a straightforward but rather lengthy calculation (check!). Combining all parts, we obtain a graded direct sum decomposition of M of the claimed shape. We will see an alternative (algorithmic) proof in Chapter 4.3.1. □

Lemma 4.2.11 (torsion in graded modules over graded PIDs). *Let K be a field and let M be a finitely generated graded $K[T]$-module and let $M' \subset M$ be the torsion submodule of M. Then there exists a $k \in \mathbb{N}$ with $T^k \cdot M' = 0$.*

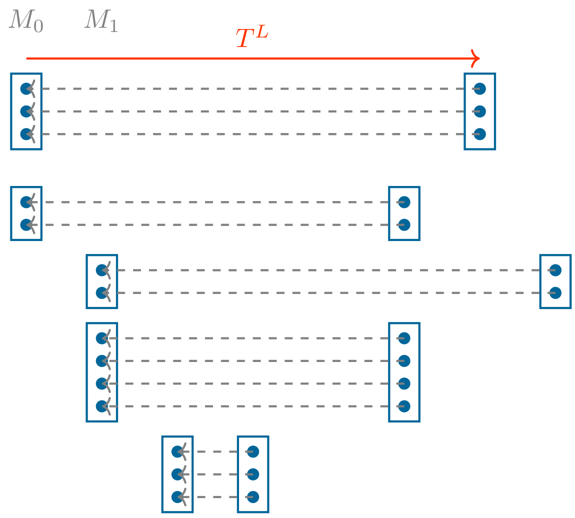


Figure 4.3.: Existence of a graded decomposition

Proof. With the help of the ungraded structure theorem over $K[T]$ and induction, we see that it suffices to show (check!) that M contains a (homogeneous) T-power torsion element.

Let $x \in M \setminus \{0\}$ be a torsion element and let $f \in K[T] \setminus \{0\}$ with $f \cdot x = 0$. We may write $x = x' + x''$ and $f = f' + f''$, where x' and f' are the lowest degree non-trivial homogeneous parts of x and f, respectively, and $x'' := x - x'$, $f'' := f - f'$. Then

$$0 = f \cdot x = f' \cdot x' + f' \cdot x'' + f'' \cdot x' + f'' \cdot x''.$$

The grading shows that that $f' \cdot x' = 0$. Therefore, M contains non-trivial f'-torsion. Because $f' \in K[T]$ is homogeneous and the only homogeneous elements of $K[T]$ are monomials, we obtain that x' is T-power torsion. □

The proof of Lemma 4.2.11 would also work for finitely generated graded modules over "general" graded PIDs to show that all torsion is homogeneous.

Lemma 4.2.12 (graded free modules [72])**.** *Let K be a field and let M be a graded module over $K[T]$ that has no torsion. Then M has a homogeneous R-basis.*

Proof. We write $M_{-1} := 0$. For $n \in \mathbb{N}$, we choose a splitting

$$M_n \cong_K T \cdot M_{n-1} \oplus N_n$$

of K-vector spaces. Let B_n be a K-basis of N_n; in particular, B_n consists of homogeneous elements. Then $B := \bigcup_{n\in\mathbb{N}} B_n$ is a homogeneous $K[T]$-basis of M (check!). □

Corollary 4.2.13 (structure of persistence modules over fields)**.** *Let K be a field and let (M^*, f^*) be a persistence K-module of finite type. Then the associated graded $K[T]$-module M (Example 4.2.5) has a graded decomposition*

$$M \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j}\bigl(K[T]/(T^{k_j})\bigr)$$

for certain $N \in \mathbb{N}$, $k_1, \dots, k_N \in \mathbb{N}_{>0} \cup \{\infty\}$, and $n_1, \dots, n_N \in \mathbb{N}$. The multiset consisting of all pairs (n_j, k_j) with $j \in \{1, \dots, N\}$ is uniquely determined by (M^*, f^*).

Proof. Because the persistence K-module (M^*, f^*) is of finite type, the graded $K[T]$-module M is finitely generated over $K[T]$ (check!). Hence, we can apply the structure theorem (Theorem 4.2.10) to M. □

Caveat 4.2.14. In Theorem 4.2.10, if the base ring K were not a field, the situation is less clear, because the graded polynomial ring $K[T]$ will have a more involved structure theory.

4.2.3 Barcodes

Barcodes of persistence modules over fields can now be defined in terms of the degree shifts and the torsion lengths appearing in the graded decomposition from Corollary 4.2.13; in view of the uniqueness part, this is indeed well-defined.

Definition 4.2.15 (barcode)**.**

- A *barcode* is a finite multiset of pairs (n, k) with $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$.
- Let K be a field and let (M^*, f^*) be a persistence K-module of finite type. The *barcode* of (M^*, f^*) is the multiset consisting of all $(n_j, k_j - 1)$ with $j \in \{1, \dots, N\}$ coming from a graded $K[T]$-decomposition $\bigoplus_{j=1}^{N} \Sigma^{n_j}\bigl(K[T]/(T^{k_j})\bigr)$ of the graded $K[T]$-module associated with the persistence module (M^*, f^*).

In particular, the persistent homology (in a given degree) over fields of filtrations of finite simplicial complexes or of persistence chain complexes of finite type has an associated barcode.

Remark 4.2.16 (persistence of classes)**.** Let K be a field. If a direct summand $\Sigma^n K[T]/(T^k)$ appears in the graded decomposition of persistent homology over K, then the corresponding cyclic generator in degree n is born

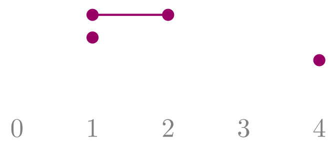


Figure 4.4.: The barcode in degree 1 for Example 4.1.13

at stage n and dies at stage $n + k - 1$. In this sense, the bars in the barcode correspond to the lifetimes of classes in persistent homology.

Remark 4.2.17 (persistent Betti numbers from barcode)**.** Let K be a field. It is not difficult to extract the persistent Betti numbers of a persistent K-chain complex of finite type from the barcodes of its persistent homology (Exercise).

Example 4.2.18. The persistent homology in degree 1 of the point cloud X from Example 4.1.13 with coefficients in $\mathbb{Q}$ and the sequence ε_* from Example 4.1.13 has the barcode (check!)

$$(1, 1), (1, 0), (4, 0)$$

We visualise these abstract barcodes by actual barcodes as in Figure 4.4.

Example 4.2.19. We consider the subset X of $\mathbb{R}^2$ depicted in Figure 4.5. Let $\varepsilon_* := (0.1, 1.1, 2.1, 3.1, 4.1, 100, 101, 102, \dots)$. Then the persistent homology in degree 1 of X with coefficients in $\mathbb{Q}$ and the sequence ε_* has the barcode (check!)

$$(1, 1), (1, 1), (2, 0), (2, 0), (2, 0)$$

which is depicted in Figure 4.6.

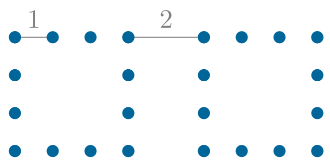


Figure 4.5.: The point cloud in Example 4.2.19.

Further instructive examples can be found in the literature [64, p. 140].

Caveat 4.2.20. In applications, barcodes can give indications of interesting structure in data [64]. However, one should be aware that without additional

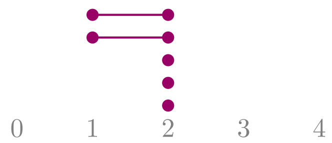


Figure 4.6.: The barcode in degree 1 for Example 4.2.19

a priori information, there is no objective measure for when a bar is long enough to correspond to an "essential" feature.

4.3 Implementation: Computing barcodes

Persistent homology with field coefficients can be algorithmically computed through a matrix reduction algorithm[29, 77, 6].

4.3.1 A homogeneous matrix reduction algorithm

As a preparation for the matrix reduction algorithm for persistent homology, we first discuss a homogeneous matrix reduction algorithm for certain matrices over polynomial rings over fields. This algorithm will also give another proof of the existence part in the structure theorem of graded modules over graded polynomial rings (Theorem 4.2.10).

Definition 4.3.1 (graded matrix)**.** Let K be a field, let $r, s \in \mathbb{N}$, and let $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}$ be monotonically increasing. A matrix $A \in M_{r\times s}(K[T])$ is (n_*, m_*)-*graded* if the following holds: For all $j \in \{1, \dots, r\}, k \in \{1, \dots, s\}$, we have that the entry $A_{jk} \in K[T]$ is a homogeneous polynomial and

- $A_{jk} = 0$ or
- $m_k = n_j + \deg A_{jk}$.

In a graded matrix, the degrees of matrix entries monotonically increase from the left to the right and from the bottom to the top.

Definition 4.3.2 (reduced matrix)**.** Let K be a field, let $r, s \in \mathbb{N}$, and let $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}$ be monotonically increasing, and let $A \in M_{r\times s}(K[T])$ be an (n_*, m_*)-graded matrix.

- For $k \in \{1, \dots, s\}$, we define

$$\operatorname{low}_A(k) := \max\bigl\{j \in \{1, \dots, r\} \bigm| A_{jk} \neq 0\bigr\} \in \mathbb{N}$$

(with $\max \emptyset := 0$). I.e., $\operatorname{low}_A(k)$ is the index of the "lowest" matrix entry in column k that is non-zero.

- The matrix A is *reduced* if all columns have different lowest indices: For all $k, k' \in \mathbb{N}$ with $k \neq k'$, $\operatorname{low}_A(k) \neq 0$, and $\operatorname{low}_A(k') \neq 0$, we have $\operatorname{low}_A(k) \neq \operatorname{low}_A(k')$.

Example 4.3.3 (graded matrices)**.** We consider matrices over $\mathbb{Q}[T]$ and use the grading given by the indices of the rows/columns. Then the left and middle matrices are graded, but the right matrix is not. The left matrix is not reduced, the middle matrix is reduced.

$$\begin{pmatrix} 1 & 3 \cdot T & T^2 & -T^3 \\ 0 & 5 & T & T^2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & T^2 & -T^3 \\ 0 & 0 & T & T^2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \cdot T & T^2 & -T^3 \\ T & 5 & T & T^2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Graded matrices can be transformed into reduced matrices via elementary column operations; these reduced matrices then lead to module decompositions:

Algorithm 4.3.4 (homogeneous matrix reduction [29])**.** Given a field K, numbers $r, s \in \mathbb{N}$, monotonically increasing sequences $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}$, and an (n_*, m_*)-graded matrix $A \in M_{r \times s}(K[t])$, do the following:

- For each k from 1 up to s (in ascending order):

  Let $\ell := \operatorname{low}_A(k)$.

  If $\ell \neq 0$, then:

  - For each j from ℓ down to 1 (in descending order):

    If $A_{jk} \neq 0$ and there exists $k' \in \{1, \dots, k-1\}$ with $\operatorname{low}_A(k') = j$, then: Update the matrix A by subtracting $A_{jk}/A_{jk'}$-times the column k' from column k.

    [Loop invariant observation: Because A is graded, $A_{jk}/A_{jk'}$ indeed is a homogeneous polynomial over K (check!) and the resulting matrix is (n_*, m_*)-graded (check!). This eliminates the entry $A_{jk'}$.]

- Return the resulting matrix A.

Proposition 4.3.5 (homogeneous matrix reduction, properties)**.** *Let K be a field, let $r, s \in \mathbb{N}$, and let $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}$ be monotonically increasing, and let $A \in M_{r \times s}$ be an (n_*, m_*)-graded matrix. Then:*

1. *The homogeneous matrix reduction algorithm (Algorithm 4.3.4) is well-defined and terminates on this input after finitely many steps (relative to arithmetic in K).*

2. *The resulting matrix A' is reduced and there is a graded $s \times s$-matrix B over $K[T]$ that admits a graded inverse and satisfies $A' = A \cdot B$.*

3. *The low-entries of the resulting matrix A' are the elementary divisors of A over $K[T]$.*

4. *We have*

$$F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I} \Sigma^{n_j} K[T]/(T^{m_{k(j)}-n_j}) \oplus \bigoplus_{j\in I'} \Sigma^{n_j} K[T],$$

with $F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T]$ and $I := \{\mathrm{low}_{A'}(k) \mid k \in \{1,\dots,s\}\} \setminus \{0\}$ as well as $I' := \{1,\dots,r\} \setminus I$. For $j \in I$, let $k(j) \in \{1,\dots,s\}$ be the unique (!) index with $\mathrm{low}_{A'}(k(j)) = j$.

Proof. Ad 1. Well-definedness follows from the observation mentioned in the algorithm: As every homogeneous polynomial in $K[T]$ is of the form $\lambda \cdot T^d$ with $\lambda \in K$ and $d \in \mathbb{N}$ and as the matrix is graded, the corresponding division can be performed in $K[T]$ and the gradedness of the matrix is preserved by the elimination operation.

Termination is clear from the algorithm.

Ad 2. As we traverse the columns from left to right, a straightforward induction shows that no two columns can remain that have the same non-zero value of "low". The product decomposition comes from the fact that we only applied elementary homogeneous column operations without swaps.

Ad 3. Because the resulting matrix A' is obtained through elementary column operations from A, the elementary divisors of A' and A coincide. Applying Lemma 4.3.6 to A' proves the claim.

Ad 4. In view of the second part, we have that $F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} A'$. Therefore, the claim is a direct consequence of Lemma 4.3.6. □

Lemma 4.3.6. *Let K be a field, let $r, s \in \mathbb{N}$, let $n_1,\dots,n_r, m_1,\dots,m_s \in \mathbb{N}$ be monotonically increasing, and let $A \in M_{r\times s}(K[T])$ be an (n_*, m_*)-graded matrix that is reduced.*

1. *The* low-*entries of A are the elementary divisors of A over $K[T]$.*

2. *Let $F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T]$ and $I := \{\mathrm{low}_A(k) \mid k \in \{1,\dots,s\}\} \setminus \{0\}$ as well as $I' := \{1,\dots,r\} \setminus I$. Then*

$$F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I} \Sigma^{n_j} K[T]/(T^{m_{k(j)}-n_j}) \oplus \bigoplus_{j\in I'} \Sigma^{n_j} K[T].$$

Proof. Ad 1. Let $k \in \{1,\dots,s\}$ with $\ell := \mathrm{low}_A(k) \neq 0$. Then we can clear out all the entries of A in column k above ℓ by elementary row operations (again, the gradedness of A ensures that this is possible; check!). Swapping zero rows and columns appropriately thus results in a matrix in rectangle "diagonal" form; moreover, as all the "diagonal" entries are monomials, we can swap rows and columns to obtain a matrix A' in Smith normal form that both

- has the same elementary divisors as A and
- whose elementary divisors are precisely the low-entries of A.

In particular, these elementary divisors must coincide.

Ad 2. The claim is clear if A is already in Smith normal form (check!). By construction, there are square matrices B and C that are invertible over $K[T]$ and represent graded $K[T]$-isomorphisms with

$$A' = C \cdot A \cdot B.$$

In particular, $F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} (C \cdot F)/\operatorname{im} A'$. By construction, the values of $\operatorname{low}_{A'}$ and the degrees of A' differ from the ones of A only by compatible index permutations, the claim follows. □

Remark 4.3.7 (rows vs. columns)**.** Symmetrically, one could also perform matrix reduction using only row operations instead of column operations. As we intend to apply the reduction algorithm to matrices coming from boundary operators of simplicial complexes, columns are more convenient: Each column will have only very few non-zero entries. Both for the column and the row version, the worst-case runtime is roughly cubic (with respect to arithmetic operations in the base field) in the number of columns and rows of the input matrix (check!).

Example 4.3.8 (homogeneous matrix reduction)**.** We consider the following graded $\mathbb{Q}[T]$-matrix, where the column grading is given by $1, 2, 3, 3, 4$ and the row grading is given by $1, 2, 2, 2$:

$$A := \begin{pmatrix} 1 & T & 2 \cdot T^2 & 0 & T^3 \\ 0 & 0 & T & 0 & T^2 \\ 0 & 1 & T & T & 0 \\ 0 & 0 & 0 & T & T^2 \end{pmatrix}.$$

Applying homogeneous matrix reduction to A (Algorithm 4.3.4) yields the following matrices:

- After processing column 1:

$$\begin{pmatrix} 1 & T & 2 \cdot T^2 & 0 & T^3 \\ 0 & 0 & T & 0 & T^2 \\ 0 & 1 & T & T & 0 \\ 0 & 0 & 0 & T & T^2 \end{pmatrix}.$$

- After processing column 2:

$$\begin{pmatrix} 1 & 0 & 2 \cdot T^2 & 0 & T^3 \\ 0 & 0 & T & 0 & T^2 \\ 0 & 1 & T & T & 0 \\ 0 & 0 & 0 & T & T^2 \end{pmatrix}.$$

- After processing column 3:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & T^3 \\ 0 & 0 & T & 0 & T^2 \\ 0 & 1 & 0 & T & 0 \\ 0 & 0 & 0 & T & T^2 \end{pmatrix}.$$

- After processing column 4:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & T^3 \\ 0 & 0 & T & 0 & T^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & T^2 \end{pmatrix}.$$

- After processing column 5 (which is the final column):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 \end{pmatrix}.$$

Thus, the result is this final matrix.

Homogeneous matrix reduction also provides a convenient proof of the existence part of the structure theorem for finitely generated graded modules over polynommial rings over fields:

Alternative proof of the existence part of Theorem 4.2.10. We first show that M has a graded finite presentation and then apply homogeneous matrix reduction to this presentation.

Because M is a finitely generated graded $K[T]$-module, there exists a finite *homogeneous* generating set for M (check!). This defines a surjective graded $K[T]$-homomorphism

$$\varphi\colon F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T] \longrightarrow M$$

for suitable $r \in \mathbb{N}$ and monotonically increasing $n_1, \dots, n_r \in \mathbb{N}$. As φ is a graded homomorphism, $\ker \varphi \subset F$ is a graded $K[T]$-submodule and we obtain an isomorphism

$$M \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} \ker \varphi$$

of graded $K[T]$-modules.

Because $K[T]$ is a principal ideal domain, the graded submodule $\ker \varphi \subset F$ is a finitely generated (it is even free as ungraded module and thus there also exists a *homogeneous* $K[T]$-basis for $\ker \varphi$ (Lemma 4.2.12)). Therefore, $\ker \varphi$

has a finite homogeneous generating set; in particular, there exist $s \in \mathbb{N}$, monotonically increasing $m_1, \dots, m_s \in \mathbb{N}$ and a graded $K[T]$-homomorphism

$$\psi\colon E := \bigoplus_{k=1}^{s} \Sigma^{m_k} K[T] \longrightarrow F$$

with $\operatorname{im}\psi = \ker\varphi$. Because ψ is graded and n_*, m_* are monotonically increasing, the $r \times s$-matrix A over $K[T]$ that represents ψ with respect to the canonical graded bases is (n_*, m_*)-graded in the sense of Definition 4.3.1.

Applying the homogeneous matrix reduction algorithm to A shows that

$$M \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} A$$

has the desired decomposition (Proposition 4.3.5). □

4.3.2 Persistent homology via matrix reduction

A straightforward modification of homogeneous matrix reduction allows us to compute persistent homology over fields.

The most direct attempt would be to first compute all persistent homology groups using an algorithm for the computation of marked free chain complexes (Chapter 3.3), then to derive the persistence module structure on these homology groups, and finally to apply homogeneous matrix reduction and the methods from Chapter 4.3.1 to compute the barcodes. However, there is a substantial simplification [77, 29]: We can apply the homogeneous matrix reduction directly to all the boundary operators at once to compute the barcodes without first explicitly computing the persistent homology groups.

Again, we formulate this (slightly modified) version of the standard matrix reduction algorithm for persistent homology [29, Chapter VII] in the graded language. We first give the algorithm for a graded matrices that satisfy the boundary operator condition and then specialise to the case of matrices coming from simplicial fitrations.

Setup 4.3.9 (graded matrices that satisfy the boundary operator condition)**.** Let K be a field, let $r_0, r_1, r_2 \in \mathbb{N}$, and let n_{0*}, n_{1*}, n_{2*} be monotonically increasing sequences in $\mathbb{N}$ of length, r_0, r_1, r_2, respectively. Let $A_2 \in M_{r_1 \times r_2}(K[T])$ be an (n_{1*}, n_{2*})-graded matrix and let $A_1 \in M_{r_0 \times r_1}(K[T])$ be an (n_{0*}, n_{1*})-graded matrix with $A_1 \cdot A_2 = 0$.

Algorithm 4.3.10 (barcode of homology of graded matrices)**.** Given the situation of Setup 4.3.9, do the following:

- Let A_2' be the result of applying the homogeneous matrix reduction algorithm (Algorithm 4.3.4) to A_2.
- Let

$$I_2 := \big\{\mathrm{low}_{A_2'}(k) \bigm| k \in \{1,\dots,r_2\}\big\} \setminus \{0\}$$
$$I_2' := \{1,\dots,r_1\} \setminus I_2.$$

- Let A_1' be the result of applying the homogeneous matrix reduction algorithm (Algorithm 4.3.4) to A_1.
- Let
$$\overline{I}_1 := \big\{k \in \{1,\dots,r_1\} \bigm| \mathrm{low}_{A_1'}(k) \neq 0\big\}.$$
- Let B be the multiset consisting of all pairs $(n_{1j}, n_{2,k(j)} - n_{1j} - 1)$ with $j \in I_2$ and $n_{2,k(j)} - n_{1j} \neq 0$ (where $k(j)$ computed with respect to A_2') and all pairs (n_{1j}, ∞) with $j \in I_2' \setminus \overline{I}_1$.
- Return B.

Remark 4.3.11. In Algorithm 4.3.10, the two matrix reductions could be combined into a single matrix reduction (on a bigger block matrix). For the sake of transparency, we performed them separately.

Proposition 4.3.12 (barcode of homology of graded matrices, properties)**.** *In the situation of Setup 4.3.9, the Algorithm 4.3.10 terminates (relative to arithmetic in K) and computes the graded decomposition of the graded $K[T]$-module* $\ker A_1/\operatorname{im} A_2$*; i.e., if B is the multiset computed by the algorithm, then*
$$\ker A_1/\operatorname{im} A_2 \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{(n,p)\in B} \Sigma^n K[T]/(T^{p+1}).$$

Proof. We use our knowledge on homogeneous matrix reduction and the structure theorem (Theorem 4.2.10). For $\ell \in \{0,1,2\}$, we write $F_\ell := \bigoplus_{j\in\{1,\dots,r_\ell\}} \Sigma^{n_{\ell,j}} K[T]$; moreover, we use the notation from Algorithm 4.3.10.

On the one hand, we already know that (Proposition 4.3.5)

$$\begin{aligned} F_1/\operatorname{im} A_2 &\cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I_2} \Sigma^{n_{1j}} K[T]/(T^{n_{2,k(j)}-n_{1,j}}) \oplus \bigoplus_{j\in I_2'} \Sigma^{n_{1j}} K[T] \\ &\cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I_2,\, n_{2,k(j)}-n_{1,j}\neq 0} \Sigma^{n_{1j}} K[T]/(T^{n_{2,k(j)}-n_{1,j}}) \oplus \bigoplus_{j\in I_2'} \Sigma^{n_{1j}} K[T]. \end{aligned}$$

On the other hand, the column basis associated with the reduced matrix A_1' shows that there exists a finitely generated free graded $K[T]$-submodule $E_1 \subset F_1$ with
$$F_1 \cong_{K[T]\mathsf{Mod}^*} \ker A_1' \oplus E_1 \cong_{K[T]\mathsf{Mod}^*} \ker A_1 \oplus E_1;$$
more precisely, the matrix A_1' shows that $E_1 \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in\overline{I}_1} \Sigma^{n_{1j}} K[T]$.

Moreover, the hypothesis $A_1 \cdot A_2 = 0$ implies that $\operatorname{im} A_2 \subset \ker A_1$ and so
$$F_1/\ker A_1 \cong_{K[T]\mathsf{Mod}^*} (\ker A_1/\operatorname{im} A_2) \oplus E_1.$$

Therefore, the combination (direct sum/disjoint union of multisets) of a graded decomposition of $\ker A_1/\operatorname{im} A_2$ and of E_1, respectively, gives a graded decomposition for $F_1/\ker A_1$. Thus, uniqueness of graded decompositions for $F_1/\ker A_1$ shows that we obtain a graded decomposition for $\ker A_1/\operatorname{im} A_2$ by removing the decomposition data for E_1 from the one for $F_1/\ker A_1$. This is exactly what is reflected in the algorithm. □

Setup 4.3.13 (filtered marked free chain complexes)**.** Let K be a field and $N \in \mathbb{N}$. A *filtered marked free chain complex over K of length N* consists of a nested sequence

$$C^0 \subset C^1 \subset \dots \subset C^N = C^{N+1} = \dots$$

of marked free chain complexes over K and bases $(X^n[k])_{n\in\{0,\dots,N\},k\in\mathbb{N}}$ with the following properties:

$$\begin{aligned}
\forall_{n\in\{0,\dots,N\}} \quad \forall_{k\in\mathbb{N}} \quad & C^n_k = \bigoplus_{X^n[k]} K \\
\forall_{n\in\{0,\dots,N-1\}} \quad \forall_{k\in\mathbb{N}} \quad & X^n[k] \subset X^{n+1}[k] \\
\forall_{n\in\{0,\dots,N-1\}} \quad \forall_{k\in\mathbb{N}} \quad & \partial^{n+1}_k|_{C^n_k} = \partial^n_k.
\end{aligned}$$

Moreover, we assume that the bases are sufficiently disjoint:

$$\forall_{k,k'\in\mathbb{N}} \quad X^N[k] \cap X^N[k'] = \emptyset.$$

For convenience, we set $X^n[-1] := \emptyset$ and $\overline{X}^n[k] := X^n[k] \setminus X^{n-1}[k]$ for all $n \in \{1,\dots,N\}$, $k \in \mathbb{N}$, and we write $f^n\colon C^n \longrightarrow C^{n+1}$ for the canonical inclusions.

Algorithm 4.3.14 (persistent homology)**.** Given the situation in Setup 4.3.13 and $k \in \mathbb{N}$, do the following:

- Let
$$\begin{aligned}
r_2 &:= \#X^N[k+1] \\
r_1 &:= \#X^N[k] \\
r_0 &:= \#X^N[-1].
\end{aligned}$$

- Let n_{2*} be the sequence $\underbrace{0,\dots,0}_{\#\overline{X}^0[k+1]},\dots,\underbrace{N,\dots,N}_{\#\overline{X}^N[k+1]}$.
  Let n_{1*} be the sequence $\underbrace{0,\dots,0}_{\#\overline{X}^0[k]},\dots,\underbrace{N,\dots,N}_{\#\overline{X}^N[k]}$.
  Let n_{0*} be the sequence $\underbrace{0,\dots,0}_{\#\overline{X}^0[k-1]},\dots,\underbrace{N,\dots,N}_{\#\overline{X}^N[k-1]}$.

- For $\ell \in \{0, 1, 2\}$, let
$$F_\ell := \bigoplus_{j=1}^{r_\ell} \Sigma^{n_{\ell j}} K[T].$$

- Let A_2 be the matrix representing ∂_{k+1} viewed as graded homomorphism $F_2 \longrightarrow F_1$ with respect to the canonical graded $K[T]$-bases.

  More specifically, in columns corresponding to an element of $\overline{X}^n[k+1]$, we list the (graded) coefficients of ∂^n_{k+1} on this basis element, expressed in terms of $X^n[k]$.

- Let A_1 be the matrix representing ∂_k viewed as graded homomorphism $F_1 \longrightarrow F_0$ with respect to the canonical graded $K[T]$-bases.

- Let B be the barcode obtained by applying the graded homology algorithm (Algorithm 4.3.10) to the graded (!) matrices A_1 and A_2.

- Return B.

Proposition 4.3.15 (persistent homology algorithm, properties)**.** *Given the situation in Setup 4.3.13 and $k \in \mathbb{N}$, the persistent homology algorithm (Algorithm 4.3.14) terminates (relative to arithmetic of K) and computes the barcode of the persistent homology of the persistence K-chain complex (C^*, f^*) in degree k.*

Proof. The application of Algorithm 4.3.10 is possible because the matrices A_1 and A_2 satisfy the corresponding grading conditions and because $\partial_k \circ \partial_{k+1}$ ensures that $A_1 \cdot A_2 = 0$. Therefore, correctness of Algorithm 4.3.10 (Proposition 4.3.12) allows us to conclude. □

Remark 4.3.16. One can also simultaneously compute the persistent homology in all degrees (up to the maximal dimension) by doing a single matrix reduction on a large combined matrix [29, Chapter 7].

Example 4.3.17 (persistent homology of filtrations)**.** Let X be a finite simplicial complex and let $(X^n)_{n\in\mathbb{N}}$ be a filtration of X. Moreover, let $N \in \mathbb{N}$ be large enough such that $X^N = X$. Let K be a field. Then the simplicial chain complexes of $X^1, \dots, X^N$ of the filtration complexes with coefficients in K form a filtered marked free chain complex as in Setup 4.3.13, taking the oriented simplices as bases (check!).

We can thus apply the matrix reduction algorithm (Algorithm 4.3.14) to compute the barcode for the persistent homology of this filtration. For instance, this can be applied to filtrations coming from point clouds.

In particular, this allows to also compute the persistent Betti numbers of such filtrations (Remark 4.2.17).

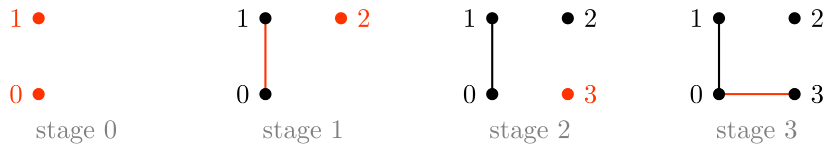


Figure 4.7.: The filtration from Example 4.3.18

Example 4.3.18. We consider the simplicial complex

$$X := \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,3\}\}$$

with the filtration over $\{0,1,2,3\}$ depicted in Figure 4.7.

We compute the barcode for the persistence homology in degree 0 with $\mathbb{Q}$-coefficients via Algorithm 4.3.14: For convenience, we order the vertices according to their labels. As bases of the correspdoning chain modules, we obtain:

n	0	1	2	3
$X^n[1]$		$\{01\}$	$\{01\}$	$\{01, 03\}$
$X^n[0]$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

By construction of the simplicial boundary operator, for all $n \in \mathbb{N}$, we have $\partial_0^n = 0$ (thus nothing needs to be done for the kernel part) and ∂_1^n is represented by the following graded matrix (where the first lines/columns indicate the degrees and basis elements):

$$\begin{array}{cccccc}
 & & 0 & 1 & 2 & 3 \\
 & & & 01 & & 03 \\
0 & 0 & & -T & & -T^3 \\
 & 1 & & T & & 0 \\
1 & 2 & & 0 & & 0 \\
2 & 3 & & 0 & & T \\
3 & & & & &
\end{array}$$

Homogeneous matrix reduction does not change this matrix (check!). Therefore, we can read off the graded decomposition

$$\Sigma^0 K[T]/(T) \oplus \Sigma^2 K[T]/(T) \oplus \Sigma^0 K[T] \oplus \Sigma^1 K[T]$$

of the persistent homology of this filtration in degree 0 with $\mathbb{Q}$-coefficients, which translates into the barcode $(0,0)$, $(2,0)$, $(0,\infty)$, $(1,\infty)$.

Remark 4.3.19 (persistent homology in degree 0)**.** Persistent homology in degree 0 of filtrations of finite simplicial complexes describes the evolution of connected components. From an algorithmic point of view, it is usually more efficient to compute these structure via a suitable version of the union-find algorithm for the computation of connected components (Algorithm 2.8.3).

4.3.3 A sparse implementation

For a practical implementation of the persistent homology algorithm (Algorithm 4.3.14), we carry out several simplifications and optimisations in the underlying homogeneous matrix reduction algorithm:

- The columns in the boundary matrix tend to contain many zeroes (Remark 4.3.7). Therefore, we use a sparse matrix setup.
- We reduce the polynomial coefficients of the graded matrices to field entries – as the degree information is redundant.
- We swap the columns into a convenient order.

This basically leads to the standard matrix reduction algorithm for the computation of persistent homology [29, Chapter VII.2]:

We briefly recall the sparse matrix representation of matrices. The underlying idea is to only record the non-zero entries by filling them column-wise into lists. For matrices with "many" zeroes, this can be significantly more space and time efficient than the standard double array representation. We first recall the differences between arrays and lists:

Remark 4.3.20 (arrays vs. lists)**.** Let a be a datatype. There are two basic linear data structures with entries from a: Arrays and list over a.

- *Arrays.* An array of length n over a provides n slots for values of type a, indexed over $\{0, \dots, n-1\}$ or $\{1, \dots, n\}$. The length n needs to be known at initialisation, the space requirement is linear in this length, and access to/update of the value at a given index is provided in constant time.

  Arrays are usually implemented as contiguous areas in (virtual) memory.

- *Lists.* Lists over a are constructed inductively:
  - The empty list $\emptyset$ has no members; in particular, it has no head member.
  - Given a list x' over a and an element x of type a, we can construct a new list $(x : x')$, starting with x, followed by the (members of the) list x'. In this case, x is the head of the new list.

	1	2	3	4	5	6
1	0	2	0	2	3	0
2	0	0	0	3	0	0
3	0	4	2	0	0	0
4	1	0	0	0	0	0

$\rightsquigarrow$

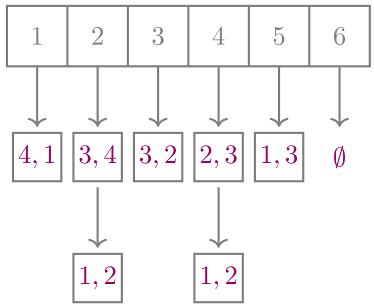


Figure 4.8.: Sparse matrix representation with columns ordered by descending row index

Lists have variable length and they need space proportional to the number of their actual members. Access to/update of the head of the list is provided in constant time; access to/update of other values in the list is in the worst-case linear in the length of the list.

Lists are usually implemented by linked structures in which each member points to the next (sometimes also the previous) member in the list.

Remark 4.3.21 (sparse matrices, by columns)**.** In the sparse matrix representation (by columns), an $r \times s$-matrix A over a ring R is represented as

- an array $\overline{A}$ over the index set $\{1, \dots, s\}$,
- whose entries are lists, containing pairs (j, λ) with $j \in \{1, \dots, r\}$ and $\lambda \in R$

with the following properties (Figure 4.8):

- For each $k \in \{1, \dots, s\}$, the list $\overline{A}[k]$ contains for each $j \in \{1, \dots, r\}$ at most one member with first component j, namely (j, A_{jk}).
- If (j, A_{jk}) is an entry in $\overline{A}[k]$, then $A_{jk} \neq 0$.
- If $A_{jk} \neq 0$, then (j, A_{jk}) is a member of $\overline{A}[k]$.
- All lists in $\overline{A}$ are sorted in ascending or descending row index order.

Sometimes, it is convenient to additionally store further labelling information in the array, together with the column lists.

Setup 4.3.22. Let K be a field, let $r, s \in \mathbb{N}$, let $n_1, \dots, n_r$ and $m_1, \dots, m_s$ be monotonically increasing sequences in $\mathbb{N}$, and let A be a sparse $r \times s$-matrix with columns indexed over $\{1, \dots, s\}$ (each sorted in descending row index order) that represents the field coefficients of a graded (n_*, m_*)-graded matrix over $K[T]$. For $k \in \{1, \dots, s\}$, the entry $A[k]$ is a pair $(A[k].\operatorname{list}, A[k].\deg)$ consisting of

- a list $A[k].\operatorname{list}$ over $\{1, \dots, r\} \times K$ (describing the k-column);
- the natural number $A[k].\deg = m_k$.

Algorithm 4.3.23 (homogeneous matrix reduction, sparse version)**.** Given the situation in Setup 4.3.22, do the following:

- Let A' be a sparse $r \times r$-matrix with row and column degrees n_* that represents the zero matrix.
- For each k from 1 up to s (in ascending order):

  Let L be $A[k].\operatorname{list}$.

  Eliminate along L with index k in A' (see below).
- Let B be the multiset consisting of all pairs $(n_j, p_j - 1)$ with $j \in \{1, \dots r\}$ and $p_j \neq 0$, where

$$p_j := \begin{cases} A'[j].\deg - n_j & \text{if } A'[j].\operatorname{list} \neq \emptyset \\ \infty & \text{if } A'[j].\operatorname{list} = \emptyset. \end{cases}$$

- Return B.

Elimination along a list L with index k in the sparse matrix A':

- If $L \neq \emptyset$, then:
  - Let (i, λ) be the head of L
  - If $A'[i].\operatorname{list} = \emptyset$, then:
    * Update A' by updating $A'[i].\operatorname{list}$ with L
    * Update A' by updating $A'[i].\deg$ with $A[k].\deg$
  - else:
    * Update L by "subtracting"

      $$\frac{\lambda}{\text{second component of the head of } A'[i]}$$

      "times" $A'[i].\operatorname{list}$ from $L[i]$.
    * Eliminate along L with index k in A'.

In the context of Algorithm 4.3.23 "subtracting" and "times" refers to versions of column operations adapted to sparse matrices.

Proposition 4.3.24 (sparse homogeneous matrix reduction, properties). *Given the situation in Setup 4.3.22, Algorithm 4.3.23 terminates. Let B be the barcode computed by applying Algorithm 4.3.23 to the input matrix A. Then, we have*

$$F/\operatorname{im} A \cong_{(n,p)\in B} \Sigma^n K[T]/(T^{p+1}),$$

where $F := \bigoplus_{j=1}^r \Sigma^{n_j} K[T]$ *and where* $\operatorname{im} A$ *denotes the image of the graded* $K[T]$*-homomorphism represented by the sparse matrix A.*

Proof. Let A' be the final state of the helper matrix in the course of the algorithm. Inductively one sees that $A'[k]$. list is empty or it is a column whose low-entry is in row k (check!). Thus A' represents a reduced matrix (but not necessarily graded in our strict sense – because of the order of the column degrees); in this aspect, the algorithm only marginally differs from Algorithm 4.3.4 (Exercise). Moreover, the column operations involved show that $\operatorname{im} A = \operatorname{im} A'$. We can now argue similarly to the proof of Proposition 4.3.5 (check!). □

Therefore, we can adapt Algorithm 4.3.14 by replacing homogeneous matrix reduction with its sparse version (Algorithm 4.3.23) and modifying the barcode computation accordingly (check!). In the literature, descriptions of these algorithms for persistent homology usually treat all (homological) degrees at once in a single matrix.

Remark 4.3.25 (complexity). The worst-case runtime of Algorithm 4.3.23 is cubic in the number of columns/rows relative to the arithmetic of the coefficient field (check!). In practice the algorithm usually performs much better than cubic. However, one should be aware that for the computation of persistent homology of filtrations of simplicial complexes, the number of columns corresponds to the number of simplices in the dimensions under consideration. The total number of simplices is (in the worst case) *exponential* in the number of vertices. Therefore, in practice, for the analysis of big data, one usually only

- computes persistent homology in low homological degrees (mostly 0 and 1)
- and tries to find filtrations with "few" simplices (e.g., via alpha complexes or landmark complexes [29, Chapter III][64, Chapter 2.7]).

Moreover, several optimised versions of the standard reduction algorithms for persistent homology of Rips filtrations are available [4, 6].

Remark 4.3.26 (the benefit of $\mathbb{F}_2$-coefficients). A particularly efficient approach to persistent homology is to use $\mathbb{F}_2$-coefficients. Arithmetic over the field $\mathbb{F}_2$ has a straightforward exact and efficient implementation, there are no size/stability issues for coefficients, and elimination steps consist only of additions of columns.

We give a simple-minded unsophisticated implementation of sparse homogeneous matrix reduction (Algorithm 4.3.23) in Haskell. We begin with basic preparations on the sparse matrix representation and column operations:

```
-- (sparse) homogeneous matrix reduction
module HomogMatrixReduction where

import qualified Data.Vector as V
import Data.List
import Data.Maybe

-----------------------------------------------------------
-- sparse matrix representations
-----------------------------------------------------------

-- representation: by columns,
-- sorted by descending row index
-- SparseMatrix in addition includes the sequence of row degrees

type Column a = [(Int,a)]
type Matrix a = V.Vector (Column a, Int)
type SparseMatrix a = ([Int], Matrix a)

matrix :: SparseMatrix a -> Matrix a
matrix = snd

rowDegs :: SparseMatrix a -> [Int]
rowDegs = fst

-- a zero matrix with the given row degrees
mkZeroMatrixFromDegs :: [Int] -> Matrix a
mkZeroMatrixFromDegs ns = V.fromList [ ([],n) | n <- ns ]

-- generic column operations on columns in sparse matrices:
-- we assume that the lists are ordered by _decreasing_ index component;
-- we assume f 0 0 == 0 and f 0 neq0 /= 0 and f neq0 0 /= 0;
-- we discard zero entries to keep the sparseness condition
zipWithIndex :: (Num a, Eq a) => (a -> a -> a) -> Column a -> Column a -> Column a
zipWithIndex f [] ws = map (\(j,w) -> (j,f 0 w)) ws
zipWithIndex f vs [] = map (\(j,v) -> (j,f v 0)) vs
zipWithIndex f ((i,v):vs) ((j,w):ws)
  = case compare i j of
          EQ -> let z = f v w
                in if z == 0
                   then zipWithIndex f vs ws -- discard zero entries
                   else (i, z):(zipWithIndex f vs ws)
          GT -> (i, f v 0):(zipWithIndex f vs ((j,w):ws))
          LT -> (j, f 0 w):(zipWithIndex f ((i,v):vs) ws)
```

We can now translate the sparse homogeneous matrix reduction into this setting:

```
-----------------------------------------------------------
-- homogeneous matrix reduction
-----------------------------------------------------------
```

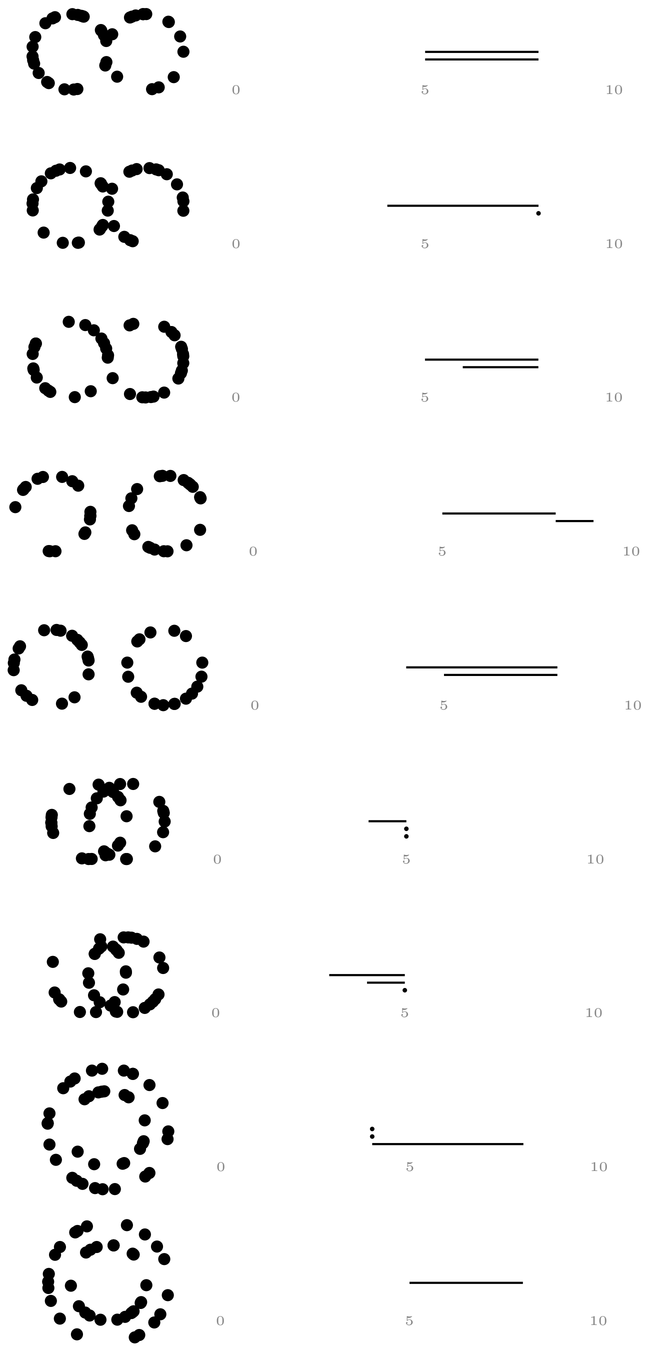
```
reduce :: (Fractional a, Eq a) => SparseMatrix a -> Matrix a
reduce sa = let a  = matrix sa
                a' = mkZeroMatrixFromDegs (rowDegs sa)
              in V.foldl handleColumn a' a
  where -- trigger elimination of the given column
        -- by columns in the given matrix;
        -- update the matrix accordingly with a new pivot column
        handleColumn :: (Fractional a, Eq a) => Matrix a -> (Column a, Int) -> Matrix a
        handleColumn a' (xs, m)
          = let xs' = eliminate (xs, m) a'
            in case xs' of
                    []         -> a'
                    ((j, _):_) -> a' V.// [(j, (xs', m))]

        -- inductive elimination along a column
        eliminate :: (Fractional a, Eq a) => (Column a, Int) -> Matrix a -> Column a
        eliminate (xs, m) a'
          = case xs of
                 []          -> []
                 ((j,ajk):_) -> case a' V.! j of
                                     ([],_) -> xs
                                     (ys,_) -> let xs' = subtractColumn ajk xs ys
                                                in  eliminate (xs',m) a'

        -- subtract ajk times column ys from column xs
        subtractColumn :: (Fractional a, Eq a) => a -> Column a -> Column a -> Column a
        subtractColumn _   xs []        = xs
        subtractColumn ajk xs ys@(y:_) = zipWithIndex (\ v w -> v - ajk/(snd y) * w) xs ys
```

Building on this matrix reduction, we can compute persistent homology and its barcodes. On the one hand, this allows to check computations by hand for correctness. On the other hand, we can perform experiments and actual calculations.

Example 4.3.27 (persistent homology of random points on circles)**.** We consider several constellations of random points on circles and their persistent homology in degree 1 with $\mathbb{Q}$-coefficients (Figure 4.9). In "most" (but not all ...) cases, we obtain the expected barcodes.

Remark 4.3.28 (libraries for persistent homology)**.** Open source libraries/packages to compute persistent homology are available in many programming languages, including highly optimised versions for actual applications [76]. Such packages do not only exist for mainstream languages:

- Julia: https://github.com/Eetion/Eirene.jl
- Haskell: https://hackage.haskell.org/package/Persistence-1.0
- Coq: https://wiki.portal.chalmers.se/cse/pmwiki.php/ForMath/ProofExamples#wp3ex6



In the point clouds, the distance unit is: ———

Figure 4.9.: Point clouds from random points on various circles; the Rips filtration is constructed with respect to $(0, 0.1, 0.2, \ldots, 1.1)$.

4.4 The stability theorem for persistent homology

Before using persistent homology in actual applications, we need to consider the question of stability:

Do small changes in the input lead only to small changes in the output?

For instance, we need to know that persistent homology of Rips filtrations of point clouds satisfy stability with respect to small perturbations of the point cloud.

We first run a basic experiment and then introduce terminology for the stability theorem. Finally, we give a rough sketch of the proof of the stability theorem.

4.4.1 A basic experiment

Example 4.4.1 (persistent homology of random points on circles)**.** We consider several constellations of random points on circles and their persistent homology in degree 1 with $\mathbb{Q}$-coefficients (Figure 4.10). In "most" (but not all ...) cases, we obtain the expected barcodes.

This example indicates that barcodes of point clouds seem to be robust under small perturbations of the point clouds. In the following sections, we will introduce notions that allow for a rigorous treatment of this observation.

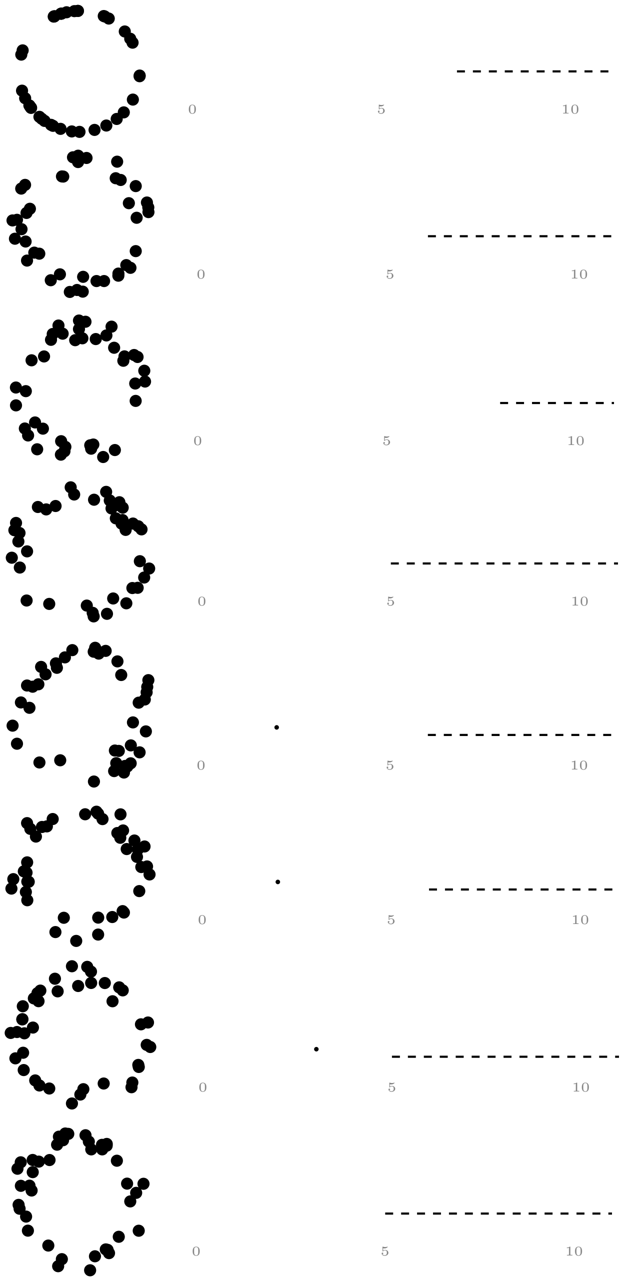
4.4.2 Comparing point clouds: Gromov–Hausdorff distance

As first step, we introduce a metric on the class of finite metric spaces (whence point clouds), the Gromov–Hausdorff distance [34, 12].

The geometric idea of the Gromov–Hausdorff distance is to measure the distance between metric spaces by embedding them into another space and then to compare the images via the Hausdorff distance; the Hausdorff distance measures the distance between (finite) subsets of a metric space.

Definition 4.4.2 (Hausdorff distance)**.** Let (Z, d) be a metric space and let $X, Y \subset Z$ be finite subsets. The *Hausdorff distance* between X and Y is defined as

$$d_{\mathrm{H}}^{(Z,d)}(X,Y) := \inf\bigl\{r \in \mathbb{R}_{>0} \bigm| X \subset U_r(Y) \text{ and } Y \subset U_r(X)\bigr\} \in \mathbb{R}_{\geq 0}.$$



In the point clouds, the distance unit is: ———

Figure 4.10.: Point clouds from random points on a circle without noise and with square noise of order 0.3 added; the Rips filtration is constructed with respect to $(0, 0.1, 0.2, \ldots, 1.1)$.

Remark 4.4.3 (Hausdorff distance, properties)**.** Let (Z,d) be a metric space. Then$d_{\mathrm{H}}^{(Z,d)}$ defines a metric on the set of all finite subsets of Z (check!).

Remark 4.4.4 (the category of finite metric spaces)**.** Let FMet denote the category of all finite metric spaces (i.e., metric spaces on finite sets) and isometric embeddings. An *isometric embedding* is a map $f\colon (X,d) \longrightarrow (X',d')$ between metric spaces that is distance preserving:

$$\forall_{x,y\in X} \quad d'\bigl(f(x),f(y)\bigr) = d(x,y).$$

The category FMet has a small skeleton (check!). In particular, the potential set-theoretic issues in the definition of Gromov–Hausdorff distance (Definition 4.4.5) can easily be resolved.

Definition 4.4.5 (Gromov–Hausdorff distance)**.** Let (X,d_X) and (Y,d_Y) be finite metric spaces. The *Gromov–Hausdorff distance* between (X,d_X) and (Y,d_Y) is defined as

$$\begin{aligned} d_{\mathrm{GH}}\bigl((X,d_X),(Y,d_Y)\bigr) := \inf\bigl\{d_{\mathrm{H}}^{(Z,d)}(f(X),g(Y)) \bigm| \;& (Z,d) \in \mathrm{Ob}(\mathsf{FMet}),\\ & f \in \mathrm{Mor}_{\mathsf{FMet}}((X,d_X),(Z,d)),\\ & g \in \mathrm{Mor}_{\mathsf{FMet}}((Y,d_Y),(Z,d))\bigr\} \in \mathbb{R}_{\geq 0}. \end{aligned}$$

Example 4.4.6 (Gromov–Hausdorff distance)**.**

- Isometric finite metric spaces have Gromov–Hausdorff distance equal to 0 (check!).

  The converse also holds (less obviously so; Exercise).

- We consider the subsets

$$\begin{aligned} X &:= \{(0,0),(1,0)\},\\ Y &:= \{(2023,2023)\},\\ Z &:= \{(0,0),(2,0)\},\\ W &:= \{(1,0),(2,0),(2.1,0)\} \end{aligned}$$

  of $\mathbb{R}^2$. Then (check!)

$$\begin{aligned} d_{\mathrm{GH}}((X,d_2),(Y,d_2)) &= 0.5,\\ d_{\mathrm{GH}}((X,d_2),(Z,d_2)) &= 0.5,\\ d_{\mathrm{GH}}((Y,d_2),(Z,d_2)) &= 1,\\ d_{\mathrm{GH}}((X,d_2),(W,d_2)) &= 0.05. \end{aligned}$$

Remark 4.4.7 (noisy point clouds and Gromov–Hausdorff distance)**.** In the context of topological data analysis, one assumes that noise in the input data leads to point clouds that are "close" with respect to the Gromov–Hausdorff distance.

Proposition 4.4.8 (Gromov–Hausdorff distance, properties). *The Gromov–Hausdorff distance defines a metric on the set (!) of all isometry classes of finite metric spaces: Let* (X, d_X), (Y, d_Y), *and* (Z, d_Z) *be finite metric spaces. Then:*

1. Non-degeneracy. *We have* $d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) = 0$ *if and only if* (X, d_X) *and* (Y, d_Y) *are isometric.*
2. Symmetry. *We have* $d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) = d_{\mathrm{GH}}((Y, d_Y), (X, d_X))$.
3. Triangle inequality. *We have*

$$d_{\mathrm{GH}}\big((X, d_X), (Z, d_Z)\big) \leq d_{\mathrm{GH}}\big((X, d_X), (Y, d_Y)\big) + d_{\mathrm{GH}}\big((Y, d_Y), (Z, d_Z)\big).$$

Proof. Non-degeneracy is mentioned in Example 4.4.6. Symmetry is a direct consequence of the definitions (check!). The triangle inequality can be shown by combining metric spaces/isometric embeddings in a suitable way (Exercise) or via the characterisation in terms of correspondences (Remark 4.4.9). □

Remark 4.4.9 (Gromov–Hausdorff distance via correspondences). The definition of Gromov–Hausdorff distance clearly reflects the underlying geometric idea. However, often it is more convenient to use the description of Gromov–Hausdorff distance in terms of correspondences [12, Theorem 7.3.25]:

Let X and Y be sets. A *correspondence* between X and Y is a subset $C \subset X \times Y$ such that

$$\pi_1(C) = X \quad \text{and} \quad \pi_2(C) = Y,$$

where $\pi_1 \colon X \times Y \longrightarrow X$ and $\pi_1 \colon X \times Y \longrightarrow Y$ denote the canonical projections to the first and second factor, respectively. One can view correspondences as "multi-valued maps" that admit a "multi-valued map" as "inverse".

Let C be a correspondence between finite metric spaces (X, d_X) and (Y, d_Y). The *distortion of* C is defined as

$$\operatorname{dis}(C) := \sup\big\{\big|d_X(x, x') - d_Y(y, y')\big| \bigm| (x, y), (x', y') \in C\big\} \in \mathbb{R}_{\geq 0}.$$

Then one has [12, Theorem 7.3.25]:

$$d_{\mathrm{GH}}\big((X, d_x), (Y, d_Y)\big) = \frac{1}{2} \cdot \inf\big\{\operatorname{dis}(C) \bigm| C \text{ is a correspondence between } X \text{ and } Y\big\}.$$

Outlook 4.4.10 (generalisations and theoretical applications). The notion of Gromov–Hausdorff distance also works for a larger class of metric spaces than only finite metric spaces. For simplicity, we restricted to the finite case because this is the only case relevant for our discussion. The general notion is relevant in geometric group theory [34] and in geometric analysis [12] to study limits of spaces.

4.4.3 Comparing weighted barcodes: bottleneck distance

As second step, we introduce a pseudo-metric on the set of barcodes, the bottleneck distance [18]. Before giving the definition, we make two adjustments to our current treatment to simplify working with barcodes and to simplify the formulation of stability properties:

- We formalise the notion of multiset.
- We switch from index persistence/barcodes to weighted persistence/barcodes.

Remark 4.4.11 (multiset)**.** Let X be a set.

- A *multiset over* X is a map $X \longrightarrow \mathbb{N}$. Multisets over X are equal if and only if they are equal as maps.
- If $M\colon X \longrightarrow \mathbb{N}$ is a multiset and $x \in X$, then we say that x is *an element of* M if $f(x) > 0$. In that case, $f(x)$ is the *multiplicity of* x *in* M.
- A multiset M over X is a *subset* of a multiset N over X if
$$\forall_{x\in M} \quad M(x) \leq N(x).$$
- If M is a multiset over X, we write
$$\overline{M} := \bigcup_{x\in X} \{x\} \times \{1, \dots, M(x)\}.$$
for the associated indexed set. A multiset M is *finite* if $\overline{M}$ is a finite set.
- A *map* from a multiset $M\colon X \longrightarrow \mathbb{N}$ over X to a multiset $N\colon Y \longrightarrow N$ over a set Y is a map $f\colon \overline{M} \longrightarrow \overline{N}$. The *image of* f is defined as the multiset
$$\begin{aligned} f(M)\colon Y &\longrightarrow \mathbb{N} \\ y &\longmapsto \#\bigl(f(\overline{M}) \cap (\{y\} \times \{1, \dots, N(y)\}\bigr). \end{aligned}$$

Unions, intersections, and complements of multisets are defined in a straightforward manner (check!).

Example 4.4.12 (barcodes)**.** Index barcodes are multisets over $\mathbb{N}\times(\mathbb{N}\cup\{\infty\})$.
More generally, one can consider real-valued barcodes (Remark 4.4.13). We write BC for the set of all finite multisets over $\mathbb{R}_{>0} \times [0, \infty]$.

Remark 4.4.13 (weighted barcodes of point clouds). Let (X, d) be a (non-empty) finite metric space. Then we obtain the family $(R_\varepsilon(X, d))_{\varepsilon \in \mathbb{R}_{>0}}$ of associated Rips complexes. This family contains only finitely many different stages, because the family is monotonically increasing in the radius and all complexes are subcomplexes of the finite simplicial complex $\Delta(X)$. Hence, there exists a unique decomposition

$$(0, \varepsilon_0], (\varepsilon_0, \varepsilon_1], \dots (\varepsilon_{N-1}, \varepsilon_N], (\varepsilon_N, \infty)$$

with $N \in \mathbb{N}$ and $\varepsilon_0 < \dots < \varepsilon_N \in \mathbb{R}_{>0}$ of $(0, \infty)$ that has the following properties (with $\varepsilon_{-1} := 0$):

- For all $j \in \{-1, \dots, N-1\}$ we have $R_{\varepsilon_j}(X, d) \neq R_{\varepsilon_{j+1}}(X, d)$ and for all $\varepsilon \in (\varepsilon_j, \varepsilon_{j+1}]$, we have $R_\varepsilon(X, d) = R_{\varepsilon_{j+1}}(X, d)$.
- For all $\varepsilon \in \mathbb{R}_{>\varepsilon_N}$, we have $R_\varepsilon(X, d) = \Delta(X)$.

Let B be the barcode associated with the finite filtration $R_{\varepsilon_0}(X, d), \dots, R_{\varepsilon_N}(X, d), R_{\varepsilon_N+1}(X, d), \dots$ of $\Delta(X)$ over the field K in the degree $k \in \mathbb{N}$.

The *weighted barcode* $\mathrm{BC}_k(X, d; K)$ of (X, d) over K in degree k is the multiset consisting of all $(\varepsilon_n, \varepsilon_{n+p} - \varepsilon_n)$ with $(n, p) \in B$.

Example 4.4.14 (weighted barcode). We return to our good friend from Example 4.1.13. The corresponding sequence of radius thresholds is of the form

$$(0, 1], (1, \sqrt{2}], (\sqrt{2}, \sqrt{5}], \dots, (2, \dots], \dots, (3, \dots], \dots, (4, \dots], \dots, (\sqrt{73}, \infty).$$

Not all of these increments in the Rips complexes lead to changes in the homology. For the weighted barcode $B_1(X, d_2; \mathbb{Q})$, we obtain the multiset consisting of the weighted bars (check!)

$$(1, \sqrt{2} - 1), \ (1, 3 - 1), \ (4, \sqrt{17} - 4).$$

The bottleneck distance between weighted barcodes measures the minimal discrepancy that occurs when matching the barcodes. This notion is defined in such a way that bars of small length can be added or removed for free (Figure 4.11).

Definition 4.4.15 (bottleneck distance). Let $X, Y \in \mathrm{BC}$.

- For $(a, b), (a', b') \in \mathbb{R}_{\geq 0} \times [0, \infty]$, we set

  $$d_\infty\big((a, b), (a', b')\big) := \max\big(|a - a'|, |a + b - (a' + b')|\big) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

- Let $\varepsilon \in \mathbb{R}_{>0}$. An ε-*matching* between X and Y is a bijection $f\colon X' \longrightarrow Y'$ between sub-multisets X' of X and Y' of Y with the following three properties:

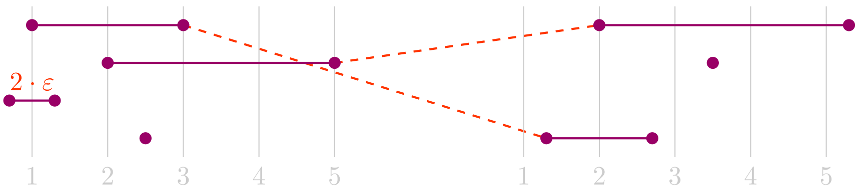


Figure 4.11.: These two weighted barcodes have a bottleneck distance at most ε, as indicated by the dashed matching. Short bars do not need to be matched.

$$\begin{aligned} \forall_{x \in X'} \quad & d_\infty(x, f(x)) \leq \varepsilon \\ \forall_{(a,b) \in X \setminus X'} \quad & |b| \leq 2 \cdot \varepsilon \\ \forall_{(a,b) \in Y \setminus Y'} \quad & |b| \leq 2 \cdot \varepsilon. \end{aligned}$$

- The *bottleneck distance* between X and Y is defined as

$$d_{\mathrm{B}}(X, Y) := \inf\bigl\{\varepsilon \in \mathbb{R}_{>0} \bigm| \text{there exists an } \varepsilon\text{-matching between } X \text{ and } Y\bigr\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Example 4.4.16 (bottleneck distance)**.** The bottleneck distance between the weighted barcode $(0,1), (0,0)$ and the weighted barcode $(0,1)$ is 0 (check!).

Proposition 4.4.17 (bottleneck distance, properties)**.** *The bottleneck distance* d_{B} *is a pseudo-metric on* BC*: Let* $X, Y, Z \in \mathrm{BC}$*. Then:*

1. *We have* $d_{\mathrm{B}}(X, X) = 0$.
2. Symmetry. *We have* $d_{\mathrm{B}}(X, Y) = d_{\mathrm{B}}(Y, X)$.
3. Triangle inequality. *We have* $d_{\mathrm{B}}(X, Z) \leq d_{\mathrm{B}}(X, Y) + d_{\mathrm{B}}(Y, Z)$.

Proof. This is a straightforward computation (Exercise). □

Remark 4.4.18 (persistence diagram)**.** The structure of persistence modules/persistent homology encoded in barcodes can equivalently be described in terms of persistence diagrams. A *persistence diagram* is a finite multiset on $\mathbb{R}_{\geq 0} \times [0, \infty]$. The correspondence between barcodes and persistence diagrams is as follows (Figure 4.12):

To a bar (n, p) corresponds the point $(n, n + p)$ in the associated persistence diagram, and vice versa. Thus, persistence diagrams can be depicted as multisets in the upper triangle of the first quadrant. This also leads to a nice visualisation of the bottleneck distance [29, Chapter VIII.2].

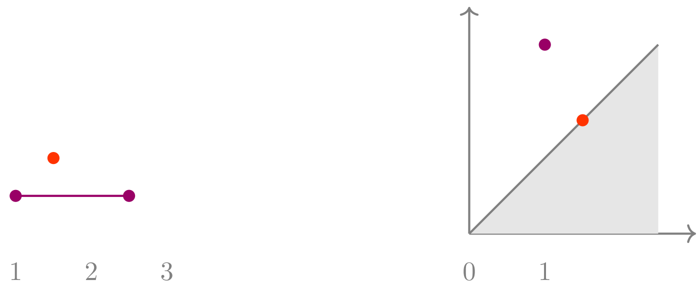


Figure 4.12.: Barcodes and persistence diagrams; "long" bars correspond to points "far away" from the diagonal.

Outlook 4.4.19 (other metrics for barcodes)**.** Other popular pseudo-metrics on the space BC of weighted barcodes include metrics motivated by optimal transport, such as the Wasserstein distance [29, 18].

4.4.4 The stability theorem

Persistent homology is stable under perturbations in the sense that the bottleneck distance between barcodes of persistent homology is controlled by the Gromov–Hausdorff distance of the underlying point clouds or other distance notions on filtrations [18, 14, 10, 11]. For the sake of simplicity, we restrict to Rips filtrations of point clouds.

Theorem 4.4.20 (stability theorem for persistent homology)**.** *Let (X, d_X) and (Y, d_Y) be finite metric spaces, let K be a field, and let $k \in \mathbb{N}$. Then*

$$d_B\bigl(\mathrm{BC}_k(X, d_X; K), \mathrm{BC}_k(Y, d_Y; K)\bigr) \leq 2 \cdot d_{GH}\bigl((X, d_X), (Y, d_Y)\bigr).$$

Sketch of proof [14, 15]. Let $\delta \in \mathbb{R}$ with $\delta > d_{GH}((X, d_X), (Y, d_Y))$. The proof consists of two steps:

① One shows that the persistent homology of the corresponding Rips filtrations (with K-coefficients in degree k) is δ-interleaved.

② One shows that such δ-interleaved persistence modules lead to barcodes whose bottleneck distance is at most $2 \cdot \delta$.

As a preparation, we introduce some notation: For $\varepsilon \in \mathbb{R}_{\geq 0}$, let $M^\varepsilon := H_k(R_\varepsilon(X, d_X); K)$. For $\varepsilon' \in \mathbb{R}_{\geq \varepsilon}$, the inclusion of the Rips complexes induces a K-linear map $f^{\varepsilon,\varepsilon'} \colon M^\varepsilon \longrightarrow M^{\varepsilon'}$. This defines a functor $(M^*, f^{*,*})$ from the partial order category $(\mathbb{R}_{\geq 0}, \leq)$ to the category of finite-dimensional K-vector spaces, a so-called $\mathbb{R}_{\geq 0}$-*graded persistence module.*

Similarly, we obtain a corresponding $\mathbb{R}_{\geq 0}$-persistence module $(N^*, g^{*,*})$ from the homology of the Rips complexes of (Y, d_Y).

Ad ①. Because of $\delta > d_{\mathrm{GH}}((X, d_X), (Y, d_Y))$, we may assume without loss of generality that (X, d_X) and (Y, d_Y) are isometric subspaces of a metric space (Z, d) with $d_{\mathrm{H}}^{(Z,d)}(X, Y) < \delta$. Using this ambient space, we find maps $F \colon X \longrightarrow Y$ and $G \colon Y \longrightarrow X$ with

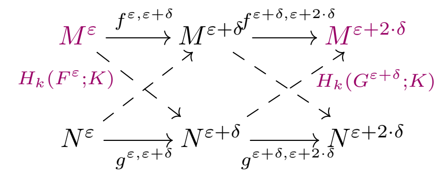
$$\forall_{x \in X} \quad d\bigl(x, F(x)\bigr) < \delta \quad \text{and} \quad \forall_{y \in Y} \quad d\bigl(y, G(y)\bigr) < \delta.$$

In particular, these maps induce for each $\varepsilon \in \mathbb{R}_{\geq 0}$ simplicial maps (check!)

$$F^\varepsilon \colon R_\varepsilon(X, d_X) \longrightarrow R_{\varepsilon + 2 \cdot \delta}(Y, d_Y),$$
$$G^\varepsilon \colon R_\varepsilon(Y, d_Y) \longrightarrow R_{\varepsilon + 2 \cdot \delta}(X, d_X)$$

and the compositions $G^{\varepsilon + 2\cdot\delta} \circ F^\varepsilon$ and $F^{\varepsilon + 2\cdot\delta} \circ G^\varepsilon$ are contiguous to the inclusions (check!). Therefore, applying $H_k(\,\cdot\,; K)$ to these compositions coincides with $f^{\varepsilon, \varepsilon + 4\cdot\delta}$ and $g^{\varepsilon, \varepsilon + 4\cdot\delta}$, respectively (Proposition 2.3.34, Theorem 3.2.10). In the language of persistence modules, this means that $(M^*, f^{*,*})$ and $(N^*, g^{*,*})$ are $2 \cdot \delta$-*interleaved*.

Using correspondences, this argument can be improved to a δ-interleaving [14]:



Ad ②. Interleavings lead to matchings for the corresponding barcodes. This can be seen by a direct calculation [15, Chapter 5]: One first compares the basic building blocks in graded decompositions and then combines these estimates for direct sums. For instance, if $g^{\varepsilon,\varepsilon'} = 0$, then the δ-interleaving shows that $f^{\varepsilon - \delta, \varepsilon' + \delta} = 0$.

In combination with the first step, we therefore obtain

$$d_{\mathrm{B}}\bigl(\mathrm{BC}_k(X, d_X; K), \mathrm{BC}_k(Y, d_Y; K)\bigr) \leq 2 \cdot \delta.$$

Taking the infimum over all $\delta > d_{\mathrm{GH}}((X, d_X), (Y, d_Y))$ proves the claim. □

Caveat 4.4.21 (Betti numbers)**.** The ordinary Betti numbers of Rips complexes associated with point clouds are *not* stable under small changes with respect to the Gromov–Hausdorff distance (Exercise).

Outlook 4.4.22 (sublevel filtrations)**.** There is a large variety of stability theorems for persistent homology, covering different metrics on barcodes and different types/metrics on filtrations. The first stability theorem was formulated for filtrations coming from sublevel sets of maps [18]:

Let X be a finite simplicial complex and let $f\colon X \longrightarrow \mathbb{R}_{\geq 0}$ be a map with the following monotonicity property: For all $\sigma, \tau \in X$ with $\sigma \subset \tau$, we have

$$f(\sigma) \leq f(\tau).$$

Then the sublevel sets $(f^{-1}((-\infty, \varepsilon]))_{\varepsilon \in \mathbb{R}_{\geq 0}}$ form an $\mathbb{R}_{\geq 0}$-filtration of X (check!). Applying simplicial homology to this filtration gives rise to an $\mathbb{R}_{\geq 0}$-persistence module. One can then control the bottleneck distance for the resulting barcodes (or persistence diagrams) in terms of supremum norms of differences of functions $X \longrightarrow \mathbb{R}_{\geq 0}$. Also more general types of spaces and functions can be considered in this setup [18, 29].

4.5 Application: Horizontal evolution

A central question in biology is to understand the evolution and interaction of species.

Real-world problem 4.5.1 (evolution)**.** How can species be classified and how can the evolution of species be described?

In the following, we outline an application of persistent homology to the detection and analysis of horizontal evolution. More detailed information can be found in the excellent book by Rabadán and Blumberg [64], on which this section is based.

We first need to clarify the question in Problem 4.5.1. Organisms have a *genotype* (as given by the information stored in the genome) and a *phenotype* (observable characteristics of an individual, e.g., fur colour). The phenotype arises from a combination of the genotype and environmental factors.

Classically, answers have been sought and formulated in terms of the phenotype (e.g., by Linné or Darwin). With the discovery and better understanding of the genotype, the question is also to be interpreted in terms of the genotype instead of the phenotype. In addition, this leads to the question of how genotypes and phenotypes are related:

Real-world problem 4.5.2 (from genotype to phenotype)**.** How do the genotype and environmental factors translate into the phenotype?

In the following, we will focus on the second part of Problem 4.5.1, i.e., the description of evolutionary processes.

The first attempts of descriptions of classification and evolution are based on tree structures. In the presence of genome sequencing, one can attempt to reconstruct the "evolutionary tree" of genotypes as follows:

Model 4.5.3 (phylogenetic trees from genome sequencing)**.** Genomes (or significant segments of genomes) are modelled as finite sequences of the letters G,

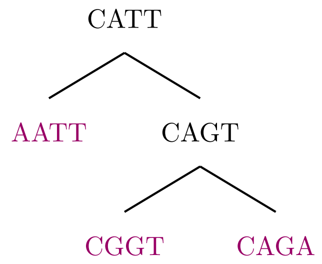


Figure 4.13.: A simple Hamming distance tree (Example 4.5.4)

A, T, C. For simplicity, we will assume that we consider sequences of the same length. Let S be a set of such finite sequences, coming from the set of species under consideration.

The similarity between gene sequences is measured in terms of the Hamming distance: Let $x_*, y_* \in S$. Then the *Hamming distance* between x_* and y_* is given by

$$d_{\mathrm{Ham}}(x_*, y_*) := \#\text{positions in which } x_* \text{ and } y_* \text{ differ.}$$

The Hamming distance defines a metric on S (check!).

The underlying assumption is that the Hamming distance relates to the proximity in evolution. I.e., species that are related through “few steps” of evolution should have genomes with a “small” difference in the Hamming metric.

Assuming that all current species are leaves (or nodes) of the genotype evolution tree (the so-called *phylogenetic tree*), Problem 4.5.1 then translates into the following problem: Find a finite (weighted) tree such that the induced metric on the nodes/leaves corresponds to the Hamming distances on S.

Of course, this is a crudely simplified version of the problem, neglecting many biology intermediate steps; in particular, usually only specific blocks of the genome are considered – namely those blocks that are roughly constant across a species and not as much affected by individual differences.

Example 4.5.4 (a simple tree)**.** We consider the following sequences:

$$\text{AATT}, \quad \text{CGGT}, \quad \text{CAGA}.$$

We have

$$\begin{aligned} d_{\mathrm{Ham}}(\text{AATT}, \text{CGCT}) &= 3, \\ d_{\mathrm{Ham}}(\text{AATT}, \text{CAGA}) &= 3, \\ d_{\mathrm{Ham}}(\text{CGCT}, \text{CAGA}) &= 2. \end{aligned}$$

The tree in Figure 4.13 is a solution to the tree-finding problem: all three sequences could be viewed as an ancestor of CATT (or of CAGT).

However, it turns out that evolution is substantially more complicated than a tree structure. Several discoveries point to that:

- It is observed that different parts of the genome can lead to incompatible phylogenetic trees.
- There exist small-scale mutation patterns that are not compatible with a tree structure.

We illustrate the second observation in a simple example:

Example 4.5.5 (the four gamete test)**.** The *infinite-site hypothesis* postulates that it is extremely unlikely that during the evolution of a single species several mutations occur at the same site. This is considered valid in situations with long genomes and a low mutation rate (e.g., in mammals, but not in bacteria).

This means that, for example, the simultaneous co-existence of all four combinations of

$$\text{AA}, \quad \text{AG}, \quad \text{GA}, \quad \text{GG}.$$

(at the same two sites) in four species is under the infinite-site hypothesis *not* compatible with a tree-shaped evolution: This can be established by a case-by-case analysis (check!) or by studying the Hamming metric directly.

Detecting non-tree shaped evolution in this way is the basis of the *four gamete test*, applied to haploid chromosomes.

Therefore, the tree model of evolution is extended to a model that includes *horizontal* interaction; one distinguishes between clonal events and reticulate evolutionary events.

clonal evolutionary events vertical interaction	reticulate evolutionary events horizontal interaction
⇣	⇣
phylogenetic tree	reticulate network

How can such reticulate events be detected? The four gamete test (Example 4.5.5) can sometimes expose reticulate events; however, this only works in very special situations. As the non-tree property in graphs is witnessed by non-trivial cycles, it is natural to consider (persistent) homology.

Definition 4.5.6 (tree-like metric space)**.** A finite metric space is *tree-like* if it is isometric to the leave space of a weighted finite tree.

Theorem 4.5.7 (persistent homology of tree-like metric spaces [13])**.** *Let (X, d) be a tree-like metric space, let K be a field, and let $k \in \mathbb{N}_{\geq 1}$. Then $\mathrm{BC}_k(X, d; K)$ is empty.*

Sketch of proof. One shows by induction over the number of points of X that all Rips complexes associated with (X, d) are disjoint unions of simplicial complexes whose simplicial homology coincides with that of $\Delta(0)$. The induction step makes use of the Mayer–Vietoris sequence (Theorem 3.2.1) and a combinatorial/metric argument. □

Corollary 4.5.8. *If (X, d) is a finite metric space with $B_1(X, d; \mathbb{F}_2) \neq \emptyset$, then (X, d) is* not *tree-like.*

Proof. This is a special case of the contraposition of Theorem 4.5.7. □

Moreover, in the situation of Corollary 4.5.8, refined algorithmic computations of the persistent homology barcode can give concrete information on how the tree property is violated.

In practice, we need to take one further aspect into account: Even if evolution is tree-shaped, then we cannot expect experimental data to lead to an exact tree structure. However, we can expect that it leads to a metric space that is "close" to a tree-like space with respect to the Gromov–Hausdorff distance. In view of the stability theorem (Theorem 4.4.20), the persistent homology strategy remains valid: "Long" bars in barcodes in degree 1 with coefficient in $\mathbb{F}_2$ point to reticulate evolutionary events.

Example 4.5.9 (reassortment in influenza A [13][64, Chapter 5])**.** There are several types of the influenza A virus, adapted to pigs, birds, or humans; moreover, all these types continue to evolve and cross-infections can occur.

The genome of the influenza virus consists of eight distinguished RNA segments. A careful analysis of the persistent homology of the RNA of influenza has been perforemd:

- Persistent homology in degree 0 recovers the standard hemagglutinin classification (H1–H16) of the main influenza A types.
- Persistent homology in degree 1, separately on each of the eight segments: No significant reticulate evolutionary events are detected.
- Persistent homology in degree 1, on the whole RNA sequence: In some strains, a significant amount of "long bars", whence reticulate evolutionary events, is detected.

In combination, this is compatible with a horizontal evolution by so-called *reassortment* of segments: I.e., whole blocks are swapped between strains.

Strains with such reticulate evolutionary events correlate with influenza pandemics (e.g., H1N1 in 1918). In order to sequence the influenza virus of the 1918 pandemic, genetic material from permafrost graves in Alaska has been used.

This data analysis on the evolution of influenza is compatible with the following biological explanation of when/how reassortment can occur: If a host is infected with different influenza strains (e.g., a human is infected with one adapted to pigs and one adapted to humans) and if during such an infection, single cells are simultaneously infected by the two strains, then inside of such cells, reassortment can occur. Depending on the circumstances and overall fitness of the newly combined virus, this virus can spread – first within this host and, if it has retained the ability to be transmitted host-to-host, also to others.

In particular, this influenza example displays the risks emerging from cross-infection of humans with a zoonotic virus and vice versa.

Example 4.5.10 (SARS-CoV-2 [7])**.** Similar investigations are also carried out on the Sars-Cov-2 virus, in particular, on the Spike protein [7].

Literature exercise. Read about the similarities and dissimilarities between the 1918 influenza pandemic and the SARS-CoV-2 pandemic. For instance, the following is an excerpt of a letter of an army physician from *Camp Devens* [33]:

> "Camp Devens is near Boston, and has about 50 000 men, or did have before this epidemic broke loose. It also has the Base Hospital for the Div. of the N. East. This epidemic started about four weeks ago, and has developed so rapidly that the camp is demoralized and all ordinary work is held up till it has passed.
>
> [. . .]
>
> "These men start with what appears to be an ordinary attack of La-Grippe or Influenza, and when brought to the Hosp. they very rapidly develop the most viscious type of Pneumonia that has ever been seen. Two hours after admission they have the Mahogony spots over the cheek bones, and a few hours later you can begin to see the Cyanosis extending from their ears and spreading all over the face, until it is hard to distinguish the colored men from the white. It is only a matter of a few hours then until death comes, and it is simply a struggle for air until they suffocate. It is horrible. One can stand it to see one, two or twenty men die, but to see these poor devils dropping out like flies sort of gets on your nerves. We have been averaging about 100 deaths per day, and still keeping it up. There is no doubt in my mind that there is a new mixed infection here, but what I don't know.
>
> [. . .]
>
> "We have lost an outrageous number of Nurses and Drs., and the little town of Ayer is a sight. It takes Special trains to carry away the dead. For several days there were no coffins and the bodies piled up something fierce, we used to go down to the morgue (which is just back of imy ward) and look at the boys laid out in long rows. It beats any sight they ever had in France after a battle. An extra long barracks has been vacated for the use of the Morgue, and it would make any man sit up and take notice to walk down the long lines of dead soldiers all dressed

and laid out in double rows. We have no relief here, you get up in the miorninig at 5.30 and work steady till about 9.30 P.M., sleep, then go at it again. Some of the men of course have been here all the time, and they are TIRED.

[...]

"The men here are all good fellows, but I get so damned sick of Pneumonia that when I go to eat I want to find some fellow who will not 'Talk Shop' but there aint none nohow. We eat it live it, sleep it, and dream it, to say nothing of breathing it 16 hours a day.

[...]

"Each man here gets a ward with about 150 beds, (Mine has 168) and has an Asst. Chief to boss him, and you can imagine what the paper work alone is—fierce—and the Govt. demands that all paper work be kept up in good shape."

4.6 Application: Exploring multi-dimensional data

In data analysis, it is also common practice to "explore" data. This means that in contrast to classical experiments and investigations, one does not test a hypothesis formulated before the experiment.

The standard persistent homology pipeline for data exploration is:

- Measure/acquire data;
- Clean up the data (e.g., by removing outliers and normalising the data);
- Reduce the complexity by projecting the data to low-dimensional Euclidean space or clustering (this step usually involves statistical methods; there is a wide range of parameters and choices in this stage);
- Compute suitable filtrations of simplicial complexes (e.g., Rips filtrations);
- Compute weighted barcodes and possibly also cycles that correspond to "long" bars;
- Analyse/interpret the results.

Caveat 4.6.1. This method is not very robust in the sense that the reduction and normalisation steps involve choices of parameters; in many situations, there are no good a priori estimates for these parameters. Moreover, at some stage in the exploration one has to consider whether an actual effect is measured or whether the computed shape is an artefact, resulting of some systematic bias (in the data or in the reduction steps).

Such an analysis can lead to a hypothesis that can then be tested in the classical sense through an appropriate experiment or study.

A

Appendix

Overview of this chapter.

A.1 Point-set topology

We collect basic notions and facts from point-set topology, as taught in introductory courses. Detailed explanations, proofs, and examples can be found in all books on point-set topology [41, 42, 56, 71].

A.1.1 Topological spaces

The category of topological spaces consists of topological spaces and continuous maps between them. The main idea of topological spaces is to express "being close" not by distances but by a system of subsets, the so-called open subsets.

Definition A.1.1 (topological space, topology, open, closed)**.** A *topological space* is a pair (X, T) consisting of a set X and a *topology* T on X, i.e., T is a subset of the power set $P(X)$ of X with the following properties:

- We have $\emptyset \in T$ and $X \in T$.
- If $U \subset T$, then $\bigcup U \in T$ (i.e., T is closed with respect to taking unions).
- If $U \subset T$ is finite, then $\bigcap U \in T$ (i.e., T is closed with respect to taking finite intersections).

The elements of T are called *open sets (with respect to T)*; if $A \subset X$ and $X \setminus A \in T$, then A is *closed (with respect to T)*.

Convention A.1.2. In algebraic topology, whenever the topology T on a set X is clear from the context, we will abuse notation and also speak of the "topological space X" instead of the "topological space (X, T)". This slight imprecision will save us from a lot of notational clutter.

That the axioms for open sets do make sense can be easily seen in the case of topologies induced by a metric:

Proposition A.1.3 (topology induced by a metric)**.** *Let (X, d) be a metric space. Then*

$$T := \bigl\{U \subset X \bigm| \forall_{x \in U} \quad \exists_{\varepsilon \in \mathbb{R}_{>0}} \quad U(x, \varepsilon) \subset U\bigr\}$$

is a topology on X, the metric topology induced by *d. Here, for $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$, we write*

$$U(x, \varepsilon) := \bigl\{y \in X \bigm| d(y, x) < \varepsilon\bigr\}$$

for the open ε-ball around x in (X, d).

Remark A.1.4.

- For $\mathbb{R}^n$, the notion of open sets with respect to the topology induced by the Euclidean metric, coincides with the standard notion of open sets (as considered in the Analysis courses). We will call this topology the *standard topology on* $\mathbb{R}^n$.
- Moreover: If (X, d) is a metric space and $A \subset X$, then A is closed (with respect to the metric topology) if and only if it is sequentially closed.

Caveat A.1.5. Not every topological space is metrisable! (Corollary A.1.31).

Example A.1.6 (extremal topologies)**.** Let X be a set. Then there are two extremal topologies on X:

- The set $P(X)$ is a topology on X, the *discrete topology.*
- The set $\{\emptyset, X\}$ is a topology on X, the *trivial topology* (or *indiscrete topology*).

Remark A.1.7 (exotic topological spaces)**.** In algebraic topology, we will usually only work with "nice" topological spaces (that are built from balls, spheres, simplices, etc.) and only consider situations where the point-set topology is tame. In contrast, topological spaces that arise naturally in algebraic geometry usually are more exotic (e.g., the Zariski topology on $\operatorname{Spec} \mathbb{Z}$).

Moreover, we will use the following generalisations of the corresponding notions for metric spaces:

Definition A.1.8 ((open) neighbourhood)**.** Let (X, T) be a topological space and let $x \in X$.

- A subset $U \subset X$ is an *open neighbourhood of* x, if U is open and $x \in U$.
- A subset $U \subset X$ is a *neighbourhood of* x if there exists an open neighbourhood $V \subset X$ of x with $V \subset U$.

Definition A.1.9 (closure, interior, boundary)**.** Let (X, T) be a topological space and let $Y \subset X$.

- The *interior of* Y is
$$Y^\circ := \bigcup \{U \mid U \in T \text{ and } U \subset Y\},$$
i.e., Y° is the largest (with respect to inclusion) open subset of X that is contained in Y.
- The *closure of* Y is
$$\overline{Y} := \bigcap \{A \mid X \setminus A \in T \text{ and } Y \subset A\},$$
i.e., $\overline{Y}$ is the smallest (with respect to inclusion) closed subset of X that contains Y.

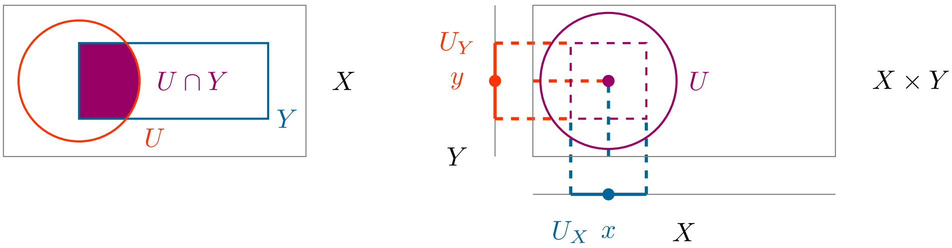


Figure A.1.: The subspace topology/product topology, schematically

- The *boundary of* Y is
$$\partial Y := \overline{Y} \cap \overline{(X \setminus Y)}.$$

Caveat A.1.10 (∂)**.** The symbol ∂ is heavily overloaded in algebraic topology. Most uses relate to some underlying geometric notion of boundary, but one should always make sure to understand what the actual meaning of ∂ is in the given context.

Two elementary constructions of topological spaces are subspaces and products; these constructions are illustrated in Figure A.1:

Remark A.1.11 (subspace topology)**.** Let (X, T) be a topological space and let $Y \subset X$ be a subset. Then
$$\{U \cap Y \mid U \in T\}$$
is a topology on Y, the *subspace topology on* Y. If T on X is induced by a metric d, then the subspace topology on Y is the topology induced by the restriction of the metric d to Y.

Remark A.1.12 (product topology)**.** Let (X, T_X) and (Y, T_Y) be topological spaces. Then
$$\{U \subset X \times Y \mid \forall_{(x,y) \in U} \quad \exists_{U_X \in T_X} \quad \exists_{U_Y \in T_Y} \quad (x, y) \in U_X \times U_Y \subset U\}$$
is a topology on $X \times Y$, the *product topology*. The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ coincides with the product topology of the standard topology on $\mathbb{R}$ (on both factors). Moreover, the product topology satisfies (together with the canonical projections onto the factors) the universal property of the product in the category of topological spaces (Remark **??**).

Remark A.1.13 (general products)**.** Let $(X_i, T_i)_{i \in I}$ be a family of topological spaces and let $X := \prod_{i \in I} X_i$. Then the *product topology on* X is the coarsest topology that makes the canonical projections $(X \longrightarrow X_i)_{i \in I}$ continuous. More explicitly: A subset $U \subset X$ is open if and only if for every $x \in U$ there exists a finite set $J \subset I$ and open subsets $U_j \subset X_j$ for every $j \in J$ with

$$x \in \prod_{j \in J} U_j \times \prod_{i \in I \setminus J} X_i \subset U.$$

This product topology satisfies (together with the canonical projections onto the factors) the universal property of the product in the category of topological spaces.

A.1.2 Continuous maps

Continuous maps are structure preserving maps in the world of topological spaces.

Definition A.1.14 (continuous)**.** Let (X, T_X) and (Y, T_Y) be topological spaces. A map $f\colon X \longrightarrow Y$ is *continuous* (with respect to T_X and T_Y), if

$$\forall_{U \in T_Y} \quad f^{-1}(U) \in T_X,$$

i.e., if preimages of open sets are open.

Remark A.1.15.

- For metric spaces, continuity with respect to the topology induced by the metric coincides with the ε-δ-notion of continuity.
- If X is a set and T, T' are topologies on X, then the identity map $\mathrm{id}_X\colon X \longrightarrow X$ is continuous as a map from (X, T) to (X, T') if and only if $T' \subset T$ (i.e., if T' is coarser than T).
- The maps $+, \cdot, -\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $/\colon \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$ are continuous with respect to the standard topology.
- If (X, T) is a topological space and $Y \subset X$, then the inclusion $Y \hookrightarrow X$ is continuous with respect to the subspace topology on Y.
- Constant maps are continuous.

Proposition A.1.16 (inheritance properties of continuous maps)**.** *Let (X, T_X), (Y, T_Y), and (Z, T_Z) be topological spaces and let $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ be maps.*

1. *If f and g are continuous, then also $g \circ f\colon X \longrightarrow Z$ is continuous.*
2. *If f is continuous and $A \subset X$, then the restriction $f|_A\colon A \longrightarrow Y$ is continuous (with respect to the subspace topology on A).*

3. *The map* $f\colon X \longrightarrow Y$ *is continuous if and only if* $f\colon X \longrightarrow f(X)$ *is continuous (with respect to the subspace topology on* $f(X)$*).*

Proposition A.1.17 (glueing principle for continuous maps)**.** *Let* (X, T_X) *and* (Y, T_Y) *be topological spaces, let* $A, B \subset X$ *be closed subsets with* $A \cup B = X$*, and let* $f\colon A \longrightarrow Y$ *and* $g\colon B \longrightarrow Y$ *be continuous maps (with respect to the subspace topology on* A *and* B*) with* $f|_{A\cap B} = g|_{A\cap B}$*. Then the map*

$$\begin{aligned} f \cup_{A\cap B} g \colon X &\longrightarrow Y \\ x &\longmapsto \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases} \end{aligned}$$

is well-defined and continuous.

Isomorphisms in the category of topological spaces are called *homeomorphisms*:

Definition A.1.18 (homeomorphism)**.** Let (X, T_X) and (T, T_Y) be topological spaces. A continuous map $f\colon X \longrightarrow Y$ is a *homeomorphism* if there exists a continuous map $g\colon Y \longrightarrow X$ such that

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y\,.$$

If there exists a homeomorphism $X \longrightarrow Y$, then X and Y are *homeomorphic*, in symbols: $X \cong_{\mathsf{Top}} Y$.

Caveat A.1.19. In general, *not* every bijective continuous map is a homeomorphism!

Intuitively, topological spaces are homeomorphic if and only if they can be deformed into each other without "tearing" or "glueing".

A.1.3 (Path-)Connectedness

An important property of continuous functions $[0,1] \longrightarrow \mathbb{R}$ is the intermediate value theorem. More generally, in the context of topological spaces, this phenomenon can be described in terms of path-connectedness and connectedness.

Definition A.1.20 (path, path-connected)**.** Let (X, T) be a topological space.

- A *path in* X is a continuous map $\gamma\colon [0,1] \longrightarrow X$ (with respect to the standard topology on $[0,1] \subset \mathbb{R}$). Then $\gamma(0)$ is the *start point* and $\gamma(1)$ is the *end point* of γ. The path γ is *closed* if $\gamma(0) = \gamma(1)$.

- The space X is *path-connected*, if for all $x, y \in X$ there exists a path $\gamma\colon [0,1] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Remark A.1.21.

- The unit interval $[0,1]$ is path-connected.
- For every $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is path-connected (with respect to the standard topology).
- If X is a set with $|X| \geq 2$, then X is *not* path-connected with respect to the discrete topology.

Proposition A.1.22 (continuity preserves path-connectedness)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f: X \longrightarrow Y$ be a continuous map. If X is path-connected, then also $f(X)$ is path-connected (with respect to the subspace topology inherited from Y).*
2. *In particular, path-connectedness is a homeomorphism invariant: If X and Y are homeomorphic, then X is path-connected if and only if Y is path-connected.*

Example A.1.23. Let $n \in \mathbb{N}$. We can use Proposition A.1.22 (and a little trick, involving the removal of a single point) to show that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = 1$.

A meaningful weaker version of path-connectedness is connectedness. A topological space is connected, if the only way to partition X into open sets is the trivial way.

Definition A.1.24 (connected)**.** A topological space (X, T_X) is *connected*, if for all $U, V \in T_X$ with $U \cup V = X$ and $U \cap V = \emptyset$ we have $U = \emptyset$ or $V = \emptyset$.

Remark A.1.25. The unit interval $[0,1]$ is connected. If $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is open, then U is path-connected if and only if U is connected.

Proposition A.1.26 (path-connectedness implies connectedness)**.** *Every path-connected topological space is connected.*

Caveat A.1.27. There exist topological spaces that are connected but *not* path-connected: The standard example is the wild sinus

$$\bigl\{(x, \sin 1/x) \bigm| x \in (0,1]\bigr\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$$

(with the subspace topology of $\mathbb{R}^2$).

The generalisation of the intermediate value theorem then reads as follows:

Proposition A.1.28 (continuity preserves connectedness)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f: X \longrightarrow Y$ be a continuous map. If X is connected, then also $f(X)$ is connected (with respect to the subspace topology inherited from Y).*

2. *In particular, connectedness is a homeomorphism invariant: If X and Y are homeomorphic, then X is connected if and only if Y is connected.*

In algebraic topology, one also studies higher connectedness properties (in the context of higher homotopy groups).

A.1.4 Hausdorff spaces

It is easy to construct weird and unintuitive topological spaces; it is much harder to ensure with simple properties that topological spaces are reasonably well-behaved. A key example is the folllowing separation property:

Definition A.1.29 (Hausdorff)**.** A topological space (X, T_X) is *Hausdorff*, if every two points can be separated by open sets, i.e., if for all $x, y \in X$ with $x \neq y$, there exist open subsets $U, V \subset X$ such that

$$x \in U, \; y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Proposition A.1.30 (metric spaces are Hausdorff)**.** *Let (X, d) be a metric space. Then the metric topology on X is Hausdorff.*

Corollary A.1.31. *If X is a set with $|X| \geq 2$, then the trivial topology on X is* not *induced by a metric on X.*

Proposition A.1.32. *Being Hausdorff is a homeomorphism invariant: If two topological spaces are homeomorphic, then one of them is Hausdorff if and only if they are both Hausdorff.*

There is a zoo of further separation properties of topological spaces [71]. Whenever possible, we will avoid these pitfalls.

A.1.5 Compactness

Roughly speaking, compactness is a finiteness property of topological spaces, defined in terms of open covers.

Definition A.1.33 (compact)**.** A topological space (X, T) is *compact*, if every open cover of X contains a finite subcover. More precisely: The topological space (X, T) is compact, if for every family $(U_i)_{i \in I}$ of open subsets of X with $X = \bigcup_{i \in I} U_i$ there exists a finite subset $J \subset I$ with $X = \bigcup_{i \in J} U_i$.

Caveat A.1.34 (cover/Überdeckung)**.** Sometimes, also the term "covering" is used instead of "cover". We will always use "cover" (German: Überdeckung; family of subsets of a spaces whose union is the given space) in order to distinguish it from the "covering" notion in covering theory (German: Überlagerung; a map with special properties).

Remark A.1.35. Let X be a set.

- Then X is compact with respect to the trivial topology.
- Moreover, X is compact with respect to the discrete topology if and only if X is finite.

The unit interval $[0, 1]$ is compact with respect to the standard topology; this implies that every continuous map $[0, 1] \longrightarrow \mathbb{R}$ has a minimum and a maximum. More generally, we have:

Proposition A.1.36 (generalised maximum principle)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f\colon X \longrightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is compact (with respect to the subspace topology of Y).*
2. *In particular, compactness is a homeomorphism invariant: If X and Y are homeomorphic, then X is compact if and only if Y is compact.*

In Euclidean spaces, we have a simple characterisation of compact sets:

Theorem A.1.37 (Heine-Borel)**.** *Let $n \in \mathbb{N}$ and let $A \subset \mathbb{R}^n$ (endowed with the subspace topology of the standard topology on $\mathbb{R}^n$). Then the following are equivalent:*

1. *The space A is compact.*
2. *The set A is closed and bounded with respect to the Euclidean metric on $\mathbb{R}^n$.*
3. *The set A is sequentially compact with respect to the Euclidean metric on $\mathbb{R}^n$ (i.e., every sequence in A has a subsequence that converges to a limit in A).*

Caveat A.1.38. In fact, every compact subspace of a metric space is closed and bounded. However, in general, the converse is *not* true in general metric spaces! For example, infinite sets are closed and bounded with respect to the discrete metric, but *not* compact.

More generally, we have the following relationship between closedness and compactness (which leads to a highly useful sufficient homeomorphism criterion).

Proposition A.1.39 (closed vs. compact)**.** *Let (X, T) be a topological space and let $Y \subset X$.*

1. *If X is compact and Y is closed in X, then Y is also compact (with respect to the subspace topology).*
2. *If X is Hausdorff and Y is compact (with respect to the subspace topology), then Y is closed in X.*

Corollary A.1.40 (compact-Hausdorff trick). *Let (X, T_X) be a compact topological space, let (Y, T_Y) be a Hausdorff topological space, and let $f\colon X \longrightarrow Y$ be continuous and bijective. Then f is a homeomorphism(!).*

Proof. Because f is bijective, it admits a set-theoretic inverse $g\colon Y \longrightarrow X$. It suffices to show that g is continuous (i.e., that g-preimages of open/closed sets are open/closed). Equivalently, it suffices to show that f-images of closed subsets of X are closed in Y.

Let $A \subset X$ be a closed subset. Because X is compact, also A is compact (Proposition A.1.39). Hence, $f(A)$ is compact by the generalised maximum principle (Proposition A.1.36). As Y is Hausdorff, this implies that $f(A)$ is closed in Y (Proposition A.1.39), as desired. □

Finally, we briefly discuss the preservation of compactness under taking products:

Proposition A.1.41 (product of two compact spaces). *Let (X, T_X) and (Y, T_Y) be compact topological spaces. Then the product $X \times Y$ is compact with respect to the product topology.*

Caveat A.1.42 (the Tychonoff Theorem). The Tychonoff Theorem

> Every product (including infinite products!) of compact spaces is compact.

is equivalent to the Axiom of Choice(!) (whence also to Zorn's Lemma and the Well-Ordering Theorem) [40, Chapter 4.8].

A.2 Categories and functors

We quickly review basic terminology from elementary category theory: categories, functors, and natural transformations.

A.2.1 Categories

Mathematical theories consist of objects (e.g., groups, topological spaces, ...) and structure preserving maps (e.g., group homomorphisms, continuous maps, ...). This can be abstracted to the notion of a category [44, 9, 63, 66].

Definition A.2.1 (category). A *category* C consists of the following data:

- A class $\operatorname{Ob}(C)$; the elements of $\operatorname{Ob}(C)$ are called *objects of* C.
- For all objects $X, Y \in \operatorname{Ob}(C)$ a set $\operatorname{Mor}_C(X,Y)$; the elements of the set $\operatorname{Mor}_C(X,Y)$ are called *morphisms from* X *to* Y *in* C. (Implicitly, we will assume that the morphism sets between different pairs of objects are disjoint and that we can recover the source and target object from a morphism.)
- For all objects $X, Y, Z \in \operatorname{Ob}(C)$ a composition of morphisms:

$$\begin{aligned} \circ\colon \operatorname{Mor}_C(Y,Z) \times \operatorname{Mor}_C(X,Y) &\longrightarrow \operatorname{Mor}_C(X,Z) \\ (g,f) &\longmapsto g \circ f \end{aligned}$$

This data is required to satisfy the following conditions:

- For each object X in C there exists a morphism $\operatorname{id}_X \in \operatorname{Mor}_C(X,X)$ such that: For all $Y \in \operatorname{Ob}(C)$ and all morphisms $f \in \operatorname{Mor}_C(X,Y)$ and $g \in \operatorname{Mor}_C(Y,X)$, we have

$$f \circ \operatorname{id}_X = f \quad \text{and} \quad \operatorname{id}_X \circ g = g.$$

  (The morphism id_X is uniquely determined by this property (check!); it is the *identity morphism of* X *in* C.)
- The composition of morphisms is associative: For all objects W, X, Y, Z in C and all morphisms $f \in \operatorname{Mor}_C(W,X)$, $g \in \operatorname{Mor}_C(X,Y)$, and $h \in \operatorname{Mor}_C(Y,Z)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Remark A.2.2 (classes)**.** Classes are a tool to escape the set-theoretic paradoxon of the "set of all sets" [68]. In case you are not familiar with von Neumann–Bernays–Gödel set theory, you can use the slogan that classes are "potentially large", "generalised" sets.

All concepts and facts in mathematical theories that can be expressed in terms of objects, identity morphisms, and (the composition of) morphisms also admit a category theoretic version. For instance, in this way, we obtain a general notion of isomorphism:

Definition A.2.3 (isomorphism)**.** Let C be a category. Objects $X, Y \in \operatorname{Ob}(C)$ are *isomorphic in* C, if there exist morphisms $f \in \operatorname{Mor}_C(X,Y)$ and $g \in \operatorname{Mor}_C(Y,X)$ with

$$g \circ f = \operatorname{id}_X \quad \text{and} \quad f \circ g = \operatorname{id}_Y .$$

In this case, f and g are *isomorphisms in* C and we write $X \cong_C Y$. If the category is clear from the context, we might also write $X \cong Y$.

Proposition A.2.4 (elementary properties of isomorphisms)**.** *Let C be a category and let $X, Y, Z \in \operatorname{Ob}(C)$.*

1. *Then the identity morphism id_X is an isomorphism in C (from X to X).*
2. *If $f \in \operatorname{Mor}_C(X,Y)$ is an isomorphism in C, then there is a unique morphism $g \in \operatorname{Mor}_C(Y,X)$ that satisfies $g \circ f = \operatorname{id}_X$ and $f \circ g = \operatorname{id}_Y$.*
3. *Compositions of (composable) isomorphisms are isomorphisms.*
4. *If $X \cong_C Y$, then $Y \cong_C X$.*
5. *If $X \cong_C Y$ and $Y \cong_C Z$, then $X \cong_C Z$.*

Proof. All claims follow easily follow from the definitions (check!). □

Moreover, the setup of categories can be used to give a general definition of commutative diagrams [46, Chapter 1.1.4].

We collect some basic examples of categories and introduce some categories that are relevant in algebraic topology.

Example A.2.5 (set theory)**.** The category Set of sets consists of:

- objects: Let $\operatorname{Ob}(\mathsf{Set})$ be the class(!) of all sets.
- morphisms: If X and Y are sets, then we define $\operatorname{Mor}_{\mathsf{Set}}(X,Y)$ as the set of all set-theoretic maps $X \longrightarrow Y$.
- compositions: If X, Y, and Z are sets, then the composition map $\operatorname{Mor}_{\mathsf{Set}}(Y,Z) \times \operatorname{Mor}_{\mathsf{Set}}(X,Y) \longrightarrow \operatorname{Mor}_{\mathsf{Set}}(X,Z)$ is ordinary composition of maps.

Clearly, this composition is associative. If X is a set, then the usual identity map

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

is the identity morphism id_X of X in Set. Objects in Set are isomorphic if and only if there exists a bijection between them, i.e., if they have the same cardinality.

Caveat A.2.6. The concept of morphisms and compositions in the definition of categories is modelled on the example of maps between sets and ordinary composition of maps. In general categories, morphisms are not necessarily maps between sets and the composition of morphisms is not necessarily ordinary composition of maps!

Example A.2.7 (algebra)**.** Let K be a field. The category Vect_K of K-vector spaces consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Vect}_K)$ be the class(!) of all K-vector spaces.
- morphisms: If X, Y are K-vector spaces, then we define $\mathrm{Mor}_{\mathsf{Vect}_K}(X, Y)$ as the set of all K-linear maps $X \longrightarrow Y$. In this case, we also write $\mathrm{Hom}_K(X, Y)$ for the set of morphisms.
- compositions: As composition we take the ordinary composition of maps.

Objects in Vect_K are isomorphic if and only if they are isomorphic in the classical sense from linear algebra.

Analogously, we can define the category Group of groups, the category Ab of Abelian groups, the category ${}_R\mathsf{Mod}$ of left-modules over a ring R, the category Mod_R of right-modules over a ring R, ...

Example A.2.8 (topology)**.** The category Top of topological spaces consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Top})$ be the class(!) of all topological spaces.
- morphisms: If X and Y are topological spaces, then we define

  $$\mathrm{map}(X, Y) := \mathrm{Mor}_{\mathsf{Top}}(X, Y)$$

  to be the set of all continuous maps $X \longrightarrow Y$.
- compositions: As composition we take the ordinary composition of maps.

Objects in Top are isomorphic if and only if they are homeomorphic.

Often, we are only interested in the difference between a topological space and a certain subspace. For example, we can model this situation through quotient spaces. However, in general, the quotient topology tends to have bad properties. Alternatively, we can use the following trick to handle differences between spaces and subspaces:

Example A.2.9 (relative topology, pairs of spaces)**.** The category Top^2 of pairs of spaces consists of:

- objects: Let
$$\mathrm{Ob}(\mathsf{Top}^2) := \bigl\{(X,A) \bigm| X \in \mathrm{Ob}(\mathsf{Top}),\ A \subset X\bigr\}.$$
- morphisms: If (X,A) and (Y,B) are pairs of spaces, then we define
$$\begin{aligned}\mathrm{map}\bigl((X,A),(Y,B)\bigr) &:= \mathrm{Mor}_{\mathsf{Top}^2}\bigl((X,A),(Y,B)\bigr)\\ &:= \bigl\{f \in \mathrm{map}(X,Y) \bigm| f(A) \subset B\bigr\}.\end{aligned}$$
- compositions: As composition we take the ordinary composition of maps (this is well-defined!).

The absolute case corresponds to pairs of spaces with empty subspace. A particularly important special case is the case where the subspace consists of a single point. This leads to the category Top_* of pointed spaces (which is used in homotopy theory; Definition **??**).

Finally, let us introduce a category that (at least implicitly) plays a key role in simplicial topology and the definition of various homology theories:

Definition A.2.10 (the simplex category)**.** The *simplex category* Δ consists of:

- objects: Let $\mathrm{Ob}(\Delta) := \bigl\{\Delta(n) \bigm| n \in \mathbb{N}\bigr\}$. Here, for $n \in \mathbb{N}$, we write
$$\Delta(n) := \{0,\dots,n\}.$$
- morphisms: If $n, m \in \mathbb{N}$, then $\mathrm{Mor}_\Delta\bigl(\Delta(n),\Delta(m)\bigr)$ is defined to be the set of all maps $\{0,\dots,n\} \longrightarrow \{0,\dots,m\}$ that are monotonically increasing.
- compositions: As compositions we take the ordinary composition of maps (this is well-defined!).

In Δ, objects are isomorphic if and only if they are equal.

A.2.2 Functors

As next step, we formalise translations between mathematical theories, using functors. Roughly speaking, functors are "structure preserving maps be-

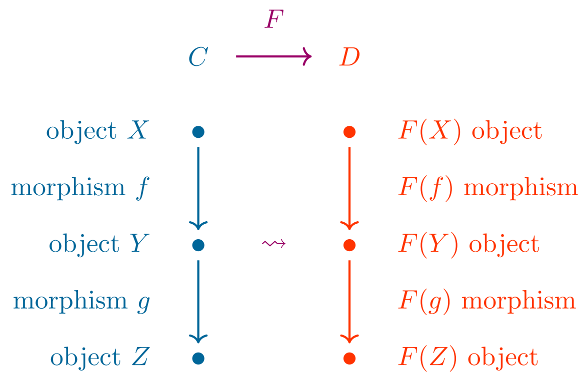


Figure A.2.: Functor, schematically

tween categories" (Figure A.2). In particular, functors preserve isomorphisms (Proposition A.2.18).

Definition A.2.11 (functor)**.** Let C and D be categories. A *(covariant) functor* $F\colon C \longrightarrow D$ consists of the following data:

- A map $F\colon \operatorname{Ob}(C) \longrightarrow \operatorname{Ob}(D)$.
- For all objects $X, Y \in \operatorname{Ob}(C)$ a map

$$F\colon \operatorname{Mor}_C(X,Y) \longrightarrow \operatorname{Mor}_D\bigl(F(X),F(Y)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \operatorname{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all $X, Y, Z \in \operatorname{Ob}(C)$ and all $f \in \operatorname{Mor}_C(X,Y)$, $g \in \operatorname{Mor}_C(Y,Z)$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $F\colon C \longrightarrow D$ consists of the following data:

- A map $F\colon \operatorname{Ob}(C) \longrightarrow \operatorname{Ob}(D)$.
- For all objects $X, Y \in \operatorname{Ob}(C)$ a map

$$F\colon \operatorname{Mor}_C(X,Y) \longrightarrow \operatorname{Mor}_D\bigl(F(Y),F(X)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \operatorname{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

- For all $X, Y, Z \in \mathrm{Ob}(C)$ and all $f \in \mathrm{Mor}_C(X,Y)$, $g \in \mathrm{Mor}_C(Y,Z)$, we have
$$F(g \circ f) = F(f) \circ F(g).$$

In other words, contravariant functors reverse the direction of arrows. More concisely, contravariant functors $C \longrightarrow D$ are the same as covariant functors $C \longrightarrow D^{\mathrm{op}}$, where D^{op} denotes the dual category of D.

Example A.2.12 (identity functor)**.** Let C be a category. Then the *identity functor* $\mathrm{Id}_C \colon C \longrightarrow C$ is defined as follows:

- on objects: We consider the map
$$\begin{aligned} \mathrm{Ob}(C) &\longrightarrow \mathrm{Ob}(C) \\ X &\longmapsto X. \end{aligned}$$
- on morphisms: For objects X, $Y \in \mathrm{Ob}(C)$, we consider the map
$$\begin{aligned} \mathrm{Mor}_C(X,Y) &\longrightarrow \mathrm{Mor}_C(X,Y) \\ f &\longmapsto f. \end{aligned}$$

Clearly, this defines a functor $C \longrightarrow C$.

Example A.2.13 (composition of functors)**.** Let C, D, E be categories and let $F\colon C \longrightarrow D$, $G\colon D \longrightarrow E$ be functors. Then the functor $G \circ F\colon C \longrightarrow E$ is defined as follows:

- on objects: Let
$$\begin{aligned} G \circ F\colon C &\longrightarrow E \\ X &\longmapsto G\bigl(F(X)\bigr). \end{aligned}$$
- on morphisms: For all X, $Y \in \mathrm{Ob}(C)$, we set
$$\begin{aligned} G \circ F\colon\ \mathrm{Mor}_C(X,Y) &\longrightarrow \mathrm{Mor}_E\bigl(G(F(X)), G(F(Y))\bigr) \\ f &\longmapsto G(F(f)). \end{aligned}$$

Clearly, this defines a functor $C \longrightarrow E$. Moreover, composition of functors is associative.

Caveat A.2.14 (the category of categories)**.** In view of the previous examples, it is tempting to introduce the "category of all catgories" (whose objects would be categories and whose morphisms would be functors). However, constructions of this type require set-theoretic precautions [17]. In the following, we will only use basic category theory and hence we will avoid these issues.

Three important, general, sources for functors are forgetful functors (by forgetting structure), free generation functors (by freely generating objects), and represented/representable functors (by viewing a category through the eyes of a given object).

Example A.2.15 (forgetful functor)**.** The *forgetful functor* $\mathsf{Top} \longrightarrow \mathsf{Set}$ is defined as follows:

- on objects: We take the map $\operatorname{Ob}(\mathsf{Top}) \longrightarrow \operatorname{Ob}(\mathsf{Set})$ that maps a topological space to its underlying set.
- on morphisms: For all topological spaces X and Y, we consider the map

$$\begin{aligned} \operatorname{Mor}_{\mathsf{Vect}_\mathbb{R}}(X,Y) = \operatorname{Hom}_\mathbb{R}(X,Y) &\longrightarrow \operatorname{Mor}_{\mathsf{Set}}(X,Y) \\ f &\longmapsto f. \end{aligned}$$

Hence, this functor "forgets" the topological structure and only retains the underlying set-theoretic information. Analogously, we can define forgetful functors $\mathsf{Vect}_\mathbb{R} \longrightarrow \mathsf{Set}$, $\mathsf{Vect}_\mathbb{R} \longrightarrow \mathsf{Ab}$, ...

Example A.2.16 (free generation functor)**.** We can translate set theory to linear algebra via the following functor $F\colon \mathsf{Set} \longrightarrow \mathsf{Vect}_\mathbb{R}$:

- on objects: We define

$$\begin{aligned} F\colon \operatorname{Ob}(\mathsf{Set}) &\longrightarrow \operatorname{Ob}(\mathsf{Vect}_\mathbb{R}) \\ X &\longmapsto \bigoplus_X \mathbb{R}. \end{aligned}$$

- on morphisms: If X and Y are sets and if $f\colon X \longrightarrow Y$ is a map, we define $F(f)\colon \bigoplus_X \mathbb{R} \longrightarrow \bigoplus_Y \mathbb{R}$ as the unique $\mathbb{R}$-linear map that extends f from the basis X to all of $\bigoplus_X \mathbb{R}$.

Example A.2.17 (represented functor)**.** Let C be a category and let $X \in \operatorname{Ob}(C)$. Then the functor $\operatorname{Mor}_C(X,\,\cdot\,)\colon C \longrightarrow \mathsf{Set}$ *represented by* X is defined as follows:

- on objects: Let

$$\begin{aligned} \operatorname{Mor}_C(X,\,\cdot\,)\colon \operatorname{Ob}(C) &\longrightarrow \operatorname{Ob}(\mathsf{Set}) \\ Y &\longmapsto \operatorname{Mor}_C(X,Y). \end{aligned}$$

- on morphisms: Let

$$\begin{aligned} \operatorname{Mor}_C(X,\,\cdot\,)\colon \operatorname{Mor}_C(Y,Z) &\longrightarrow \operatorname{Mor}_{\mathsf{Set}}\bigl(\operatorname{Mor}_C(X,Y), \operatorname{Mor}_C(X,Z)\bigr) \\ g &\longmapsto (f \mapsto g \circ f). \end{aligned}$$

Additional structure on the object X allows us to refine the represented functor $\mathrm{Mor}_C(X, \,\cdot\,)$ to a functor from C to categories with more structure than Set.

Analogously, one can define the contravariant functor $\mathrm{Mor}_C(\,\cdot\,, X)$ represented by X.

Fundamental examples of algebraic functors are tensor product functors (Bemerkung IV.1.5.7).

The key property of functors is that they preserve isomorphisms. In particular, functors provide a good notion of invariants.

Proposition A.2.18 (functors preserve isomorphism)**.** *Let C and D be categories, let $F\colon C \longrightarrow D$ be a functor, and let $X, Y \in \mathrm{Ob}(C)$.*

1. *If $f \in \mathrm{Mor}_C(X,Y)$ is an isomorphism in C, then the translated morphism $F(f) \in \mathrm{Mor}_D(F(X), F(Y))$ is an isomorphism in D.*
2. *In particular: If $X \cong_C Y$, then $F(X) \cong_D F(Y)$. In other words: If $F(X) \not\cong_D F(Y)$, then $X \not\cong_C Y$.*

Proof. The *first part* follows from the defining properties of functors: Because f is an isomorphism, there is a morphism $g \in \mathrm{Mor}_C(Y,X)$ with

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

Hence, we obtain

$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$

and $F(f) \circ F(g) = \mathrm{id}_{F(Y)}$. Thus, $F(f)$ is an isomorphism from $F(X)$ to $F(Y)$ in D.

The *second part* is a direct consequence of the first part. □

Therefore, suitable functors can help to prove that certain objects are *not* isomorphic.

Caveat A.2.19. In general, the converse is *not* true! I.e., objects that are mapped via a functor to isomorphic objects are, in general, *not* isomorphic (check!).

A.2.3 Natural transformations

Functors are compared through natural transformations; roughly speaking, natural transformations are "structure preserving maps between functors".

Definition A.2.20 (natural transformation, natural isomorphism)**.** Let C and D be categories and let $F, G\colon C \longrightarrow D$ be functors.

- A *natural transformation T from F to G*, in short $T\colon F \Longrightarrow G$, is a family $\bigl(T(X) \in \operatorname{Mor}_D(F(X), G(X))\bigr)_{X \in \operatorname{Ob}(C)}$ of morphisms such that for all objects $X, Y \in \operatorname{Ob}(C)$ and all(!) morphisms $f \in \operatorname{Mor}_C(X, Y)$ the equation
$$G(f) \circ T(X) = T(Y) \circ F(f)$$
holds in D. In other words, the following diagrams in D are commutative:
$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T(X)}\big\downarrow & & \big\downarrow{\scriptstyle T(Y)} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

- A *natural isomorphism* is a natural transformation that consists of isomorphisms (equivalently, a natural isomorphism is a natural transformation that admits an object-wise inverse natural transformation; check!).

Study note. The definition of natural transformation can easily be reconstructed: From (linear) algebra we already know examples of "natural isomorphisms". Natural isomorphisms only receive objects as input; hence, it is clear what type of families natural transformations have to be. Moreover, naturality should contain compatibility with morphisms. The only reasonable notion that can be formulated with this amount of data is the one in the commutative diagram above. That's it!

Remark A.2.21 (natural)**.** The attribute "natural" is used in two related ways: On the one hand, it refers to functorial constructions; on the other hand, it refers to things based on natural transformations.

Natural transformations between represented functors can be completely classified; the key trick is to evaluate on identity morphisms:

Example A.2.22 (morphisms lead to natural transformations between represented functors)**.** Let C be a category, let X, $Y \in \operatorname{Ob}(C)$, and let $f \in \operatorname{Mor}_C(X, Y)$. Then
$$T_f := \begin{pmatrix} \operatorname{Mor}_C(Y, Z) \longrightarrow \operatorname{Mor}_C(X, Z) \\ g \longmapsto g \circ f \end{pmatrix}_{Z \in \operatorname{Ob}(C)}$$
defines a natural transformation $\operatorname{Mor}_C(Y, \,\cdot\,) \Longrightarrow \operatorname{Mor}_C(X, \,\cdot\,)$ (check!).

Proposition A.2.23 (Yoneda Lemma)**.** *Let C be a category, let X, $Y \in \operatorname{Ob}(C)$, and let $N(Y, X)$ be the collection (which turns out to be describable as a set) of all natural transformations* $\operatorname{Mor}_C(Y, \,\cdot\,) \Longrightarrow \operatorname{Mor}_C(X, \,\cdot\,)$.

1. *Then*

$$\begin{aligned}\varphi\colon \operatorname{Mor}_C(X,Y) &\longrightarrow N(Y,X)\\ f &\longmapsto T_f\\ \psi\colon N(Y,X) &\longrightarrow \operatorname{Mor}_C(X,Y)\\ T &\longmapsto \bigl(T(Y)\bigr)(\mathrm{id}_Y)\end{aligned}$$

are mutually inverse bijections.

2. *In particular: The functors* $\operatorname{Mor}_C(X,\,\cdot\,), \operatorname{Mor}_C(Y,\,\cdot\,)\colon C \longrightarrow \mathsf{Set}$ *are isomorphic if and only if* X *and* Y *are isomorphic in* C*.*

Proof. The *first part* follows from a straightforward calculation: It should be noted that the map φ is indeed well-defined by Example A.2.22. The maps φ and ψ are mutually inverse:

The composition $\psi \circ \varphi$: On the one hand, by definition, we have

$$\psi \circ \varphi(f) = \psi\bigl(\varphi(f)\bigr) = \bigl(T_f(Y)\bigr)(\mathrm{id}_Y) = \mathrm{id}_Y \circ f = f$$

for all $f \in \operatorname{Mor}_C(X,Y)$.

The composition $\varphi \circ \psi$: On the other hand, let $T \in N(Y,X)$ and let $Z \in \operatorname{Ob}(C)$, $g \in \operatorname{Mor}_C(Y,Z)$. Then we obtain

$$\begin{aligned}\bigl(T(Z)\bigr)(g) &= \bigl(T(Z)\bigr)(g \circ \mathrm{id}_Y)\\ &= T(Z)\bigl(\operatorname{Mor}_C(Y,g)(\mathrm{id}_Y)\bigr) && \text{(by definition of } \operatorname{Mor}_C(Y,\,\cdot\,))\\ &= \operatorname{Mor}_C(X,g)\bigl(T(Y)(\mathrm{id}_Y)\bigr) && \text{(because } T \text{ is a natural transformation)}\\ &= g \circ \psi(T) && \text{(by construction of } \psi)\\ &= \bigl(T_{\psi(T)}(Z)\bigr)(g) && \text{(by construction of } T_{\psi(T)})\\ &= \Bigl(\bigl(\varphi \circ \psi(T)\bigr)(Z)\Bigr)(g) && \text{(by definition of } \varphi).\end{aligned}$$

Hence, $\varphi \circ \psi(T) = T$, as desired.

The *second part* can be derived from the first part: The maps φ and ψ are compatible with identity morphisms/transformations and with the composition of morphisms/natural transformations. Hence, isomorphisms in C correspond under φ and ψ to natural isomorphisms. Alternatively, one can use the same proof strategy as in the first part [46, Proposition 1.3.6]. □

Definition A.2.24 (representable functor)**.** Let C be a category. A functor $F\colon C \longrightarrow \mathsf{Set}$ is *representable* if there exists an object $X \in \operatorname{Ob}(C)$ such that F and the represented functor $\operatorname{Mor}_C(X,\,\cdot\,)\colon C \longrightarrow \mathsf{Set}$ are naturally isomorphic. In this case, X is a *representing object for* F.

(In view of Proposition A.2.23, representing objects of representable functors are unique up to isomorphism.)

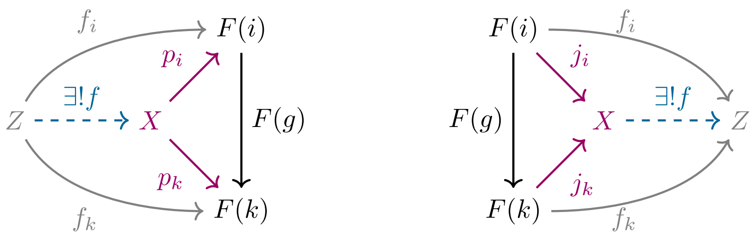


Figure A.3.: Cones/inverse limits and cocones/colimits, schematically

Remark A.2.25 (compatibility with inverse limits)**.** One advantage of representable functors is that we gain compatibility with inverse limits (e.g., with products) for free [46, Bemerkung 1.4.12].

Outlook A.2.26. By now, category theory is a foundational language that is not only used in mathematics, but also in other fields such as computer science [63] or linguistics.

Literature exercise. Read about "The Birth of Categories and Functors" [24, p. 96f].

A.2.4 Limits

Limits and colimits are a fundamental tool that encompass constructions such as products and pushouts; to avoid misunderstandings, we will always speak of "inverse limits" and "colimits". We briefly outline the general terminology and some basic examples.

(Commutative) Diagrams are formalised as functors from small categories.

Definition A.2.27 (small diagram)**.** Let I be a small category (i.e., $\mathrm{Ob}(I)$ is a set) and let C be a category. An *I-shaped diagram in C* is a functor $I \longrightarrow C$.

An inverse limit/colimit of such a diagram is an terminal/initial solution to the corresponding mapping problems (Figure A.3):

Definition A.2.28 (cone, cocone, inverse limit, colimit)**.** Let I be a small category, let C be a category, and let $F\colon I \longrightarrow C$ be an I-shaped diagram in C.

- A *cone* over F in C is a pair $(Z, (f_i)_{i\in\mathrm{Ob}(I)})$, consisting of an object Z in C and morphisms $f_i \in \mathrm{Mor}_C(Z, F(i))$ for all $i \in \mathrm{Ob}(I)$ with the following naturality property:

$$\forall_{i,k\in\mathrm{Ob}(I)} \quad \forall_{g\in\mathrm{Mor}_I(i,k)} \quad F(g) \circ f_i = f_k.$$

- An *inverse limit* over F in C is a cone $(X, (p_i)_{i \in \mathrm{Ob}(I)})$ that is terminal among all cones over F, i.e., that has the following universal property: For every cone $(Z, (f_i)_{i \in \mathrm{Ob}(I)})$ over F, there exists a unique morphism $f \in \mathrm{Mor}_C(Z, X)$ with
$$\forall_{i \in \mathrm{Ob}(I)} \quad p_i \circ f = f_i.$$

- A *cocone* over F in C is a pair $(Z, (f_i)_{i \in \mathrm{Ob}(I)})$, consisting of an object Z in C and morphisms $f_i \in \mathrm{Mor}_C(F(i), Z)$ for all $i \in \mathrm{Ob}(I)$ with the following naturality property:
$$\forall_{i,k \in \mathrm{Ob}(I)} \quad \forall_{g \in \mathrm{Mor}_I(i,k)} \quad f_k \circ F(g) = f_i.$$

- A *colimit* over F in C is a cocone $(X, (j_i)_{i \in \mathrm{Ob}(I)})$ that is initial among all cocones over F, i.e., that has the following universal property: For every cocone $(Z, (f_i)_{i \in \mathrm{Ob}(I)})$ over F, there exists a unique morphism $f \in \mathrm{Mor}_C(X, Z)$ with
$$\forall_{i \in \mathrm{Ob}(I)} \quad f \circ j_i = f_i.$$

Example A.2.29 (special diagrams)**.** In the following examples, we only indicate the non-identity morphisms.

1. If I is a small discrete category (i.e., a category that only contains identity morphisms), then inverse limits of I-diagrams are the same as categorical products.

2. If I is a small discrete category, then colimits of I-diagrams are the same as categorical coproducts.

3. If I is the category
$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$
then colimits of I-diagrams are the same as pushouts.

4. If I is the category
$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$
then inverse limits of I-diagrams are the same as pullbacks.

Proposition A.2.30 (uniqueness of inverse limits/colimits)**.** *Let I be a small category, let C be a category, and let $F \colon I \longrightarrow C$ be an I-shaped diagram in C.*

1. *If* $(X, (p_i)_{i \in \mathrm{Ob}(I)})$ *and* $(X', (p'_i)_{i \in \mathrm{Ob}(I)})$ *are inverse limits of* F *in* C*, then* $X \cong_C X'$*. Moreover, there is a unique such isomorphism in* C *that is compatible with* $(p_i)_{i \in \mathrm{Ob}(I)}$ *and* $(p'_i)_{i \in \mathrm{Ob}(I)}$*.*

2. *If* $(X, (j_i)_{i \in \mathrm{Ob}(I)})$ *and* $(X', (j'_i)_{i \in \mathrm{Ob}(I)})$ *are colimits of* F *in* C*, then* $X \cong_C X'$*. Moreover, there is a unique such isomorphism in* C *that is compatible with* $(j_i)_{i \in \mathrm{Ob}(I)}$ *and* $(j'_i)_{i \in \mathrm{Ob}(I)}$*.*

Proof. This is the usual uniqueness argument for universal properties: We apply the universal properties in both directions to obtain morphisms in both directions. We then use the uniqueness in the universal property to show that both compositions are the respective identity morphisms. □

In view of Proposition A.2.30, we also sometimes use the (somewhat sloppy!) formulation to say that $\varprojlim_{i \in I} F(I)$ is a/the inverse limit of F or that $\varinjlim_{i \in I} F(I)$ is a/the colimit of F.

In the category of sets, all small inverse limits and colimits exist:

Proposition A.2.31 (inverse limits and colimits in Set)**.** *Let* I *be a small category and let* $F\colon I \longrightarrow \mathsf{Set}$ *be an* I*-shaped diagram in* Set*.*

1. *The set*

$$X := \Bigl\{ x \in \prod_{i \in \mathrm{Ob}(I)} F(i) \Bigm| \forall_{i,k \in \mathrm{Ob}(I)} \quad \forall_{g \in \mathrm{Mor}_I(i,k)} \quad F(g)(x_i) = x_k \Bigr\},$$

together with the maps $X \longrightarrow F(i)$ *induced by the canonical projections* $\prod_{k \in \mathrm{Ob}(I)} F(k) \longrightarrow F(i)$*, is an inverse limit of* F *in* Set*.*

2. *The set*

$$X := \Bigl(\bigsqcup_{i \in \mathrm{Ob}(I)} F(i) \Bigr) \Big/ \bigl(\forall_{i,k \in \mathrm{Ob}(I)} \quad \forall_{g \in \mathrm{Mor}_I(i,k)} \quad \forall_{x \in F(i)} \quad j_k \circ F(g)(x) \sim j_i(x) \bigr),$$

together with the maps $F(i) \longrightarrow X$ *induced by the canonical inclusions* $j_i\colon F(i) \longrightarrow \bigsqcup_{k \in \mathrm{Ob}(I)} F(k)$*, is an inverse limit of* F *in* Set*.*

Proof. A straightforward calculation shows that these constructions have the corresponding universal properties in Set (check!). □

A.3 Basic homological algebra

We collect basic notions and facts from homological algebra. Homological algebra is the algebraic theory of [non-]exact sequences and functors that [do not] preserve exactness.

For simplicity, we will only consider homological algebra in module categories (instead of general Abelian categories); in view of the Freyd–Mitchell embedding theorem, this is not a substantial limitation.

Setup A.3.1. In the following, R will always be a (not necessarily commutative) ring with unit.

A.3.1 Exact sequences

We briefly recall exact sequences; we will stick to left modules (but clearly the analogous statements for right modules also hold).

Definition A.3.2 ((short) exact sequence)**.**

- A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms in ${}_R\mathsf{Mod}$ ist *exact (at the middle position B)*, if $\operatorname{im} f = \ker g$.
- A sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
in ${}_R\mathsf{Mod}$ is a *short exact sequence in ${}_R\mathsf{Mod}$*, if the sequence is exact at all positions (i.e., f is injective, g is surjective, and $\operatorname{im} f = \ker g$).
- An $\mathbb{N}$-indexed or $\mathbb{Z}$-indexed sequence
$$\cdots \longrightarrow A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} A_{k-1} \xrightarrow{f_{k-1}} A_{k-2} \longrightarrow \cdots$$
in ${}_R\mathsf{Mod}$ is *exact*, if it is exact at all positions.

Example A.3.3 (exact sequences)**.** The sequences

$$\begin{array}{ccccccccc} & & x & \longmapsto & (x,0) & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & & & (x,y) & \longmapsto & y & & \end{array}$$

and

$$\begin{array}{c} x \longmapsto 2 \cdot x \\ 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\ x \longmapsto [x] \end{array}$$

in ${}_{\mathbb{Z}}\mathsf{Mod}$ are exact; it should be noted that the middle modules are *not* isomorphic even though the outer terms are isomorphic. The sequence

$$\begin{array}{c} x \longmapsto x \\ 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \\ x \longmapsto x \end{array}$$

is *not* exact.

Caveat A.3.4. If S is a ring with unit, then additive functors ${}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$ in general do *not* map exact sequences to exact sequences. For example, tensor product functors, in general, do *not* preserve exactness!

Remark A.3.5 (flatness)**.** A right R-module M is *flat*, if the tensor product functor $M \otimes_R \ \cdot : {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is exact, i.e., it maps exact sequences to exact sequences (Definition IV.3.2.15, Beispiel IV.3.2.16, Beispiel IV.3.2.18, Lemma IV.3.4.7, Korollar IV.5.2.5, Satz IV.3.2.14). For example:

- The R-module R is flat.
- Direct sums of flat modules are flat. Therefore, all free modules are flat. In particular: If R is a field, then every R-module is flat.
- Direct summands of flat modules are flat. Therefore, all projective modules are flat.
- Localisations are flat; e.g., $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
- The $\mathbb{Z}$-module $\mathbb{Z}/2$ is *not* flat.

Particularly well-behaved exact sequences are the split short exact sequences:

Proposition A.3.6 (split exact sequence)**.** *Let*

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Mod}$*. Then the following are equivalent:*

1. *There exists an R-module homomorphism $r\colon C \longrightarrow B$ with $p \circ r = \mathrm{id}_C$.*
2. *There exists an R-module homomorphism $s\colon B \longrightarrow A$ with $s \circ i = \mathrm{id}_A$.*

If these conditions hold, then the sequence above is a split exact sequence *in* ${}_R\mathsf{Mod}$*, and*

$$
\begin{aligned}
A \oplus C &\longrightarrow B \\
(a, c) &\longmapsto i(a) + r(c) \\
B &\longrightarrow A \oplus C \\
b &\longmapsto \bigl(s(b), p(b)\bigr)
\end{aligned}
$$

are isomorphisms in ${}_R\mathsf{Mod}$*.*

Proof. We first show the implication $2 \Longrightarrow 1$: Let $s\colon B \longrightarrow A$ be an R-homomorphism with $s \circ i = \mathrm{id}_A$. We then consider the R-homomorphism

$$
\begin{aligned}
\widetilde{r}\colon B &\longrightarrow B \\
b &\longmapsto b - i \circ s(b).
\end{aligned}
$$

We have $\ker p \subset \ker \widetilde{r}$, because: Let $b \in \ker p$. In view of exactness, there is an $a \in A$ with $i(a) = b$; thus,

$$\widetilde{r}(b) = i(a) - i \circ s(i(a)) = i(a) - i(\mathrm{id}_A(a)) = 0.$$

By the universal property of the quotient module, $\widetilde{r}$ induces an R-homomorphism $r\colon C \cong_R B/\ker p \longrightarrow B$, which, by construction, satisfies $p \circ r = \mathrm{id}_C$.

Similarly, one can show the implication $1 \Longrightarrow 2$.

If the statements 1 and 2 are satisfied, then a straightforward calculation shows that the given R-homomorphisms between B and $A \oplus C$ are bijective (check!), whence isomorphisms. □

When comparing exact sequences, the five lemma is very useful:

Proposition A.3.7 (five lemma)**.** *Let*

$$
\begin{array}{ccccccccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_D} & & \downarrow{\scriptstyle f_E} \\
A' & \xrightarrow[a']{} & B' & \xrightarrow[b']{} & C' & \xrightarrow[c']{} & D' & \xrightarrow[d']{} & E'
\end{array}
$$

be a commutative diagram in ${}_R\mathsf{Mod}$ *with exact rows. Then the following holds:*

1. *If f_B, f_D are injective and f_A is surjective, then f_C is injective.*
2. *If f_B, f_D are surjective and f_E is injective, then f_C is surjective.*
3. *In particular: If f_A, f_B, f_D, f_E are isomorphisms, then f_C is an isomorphism.*

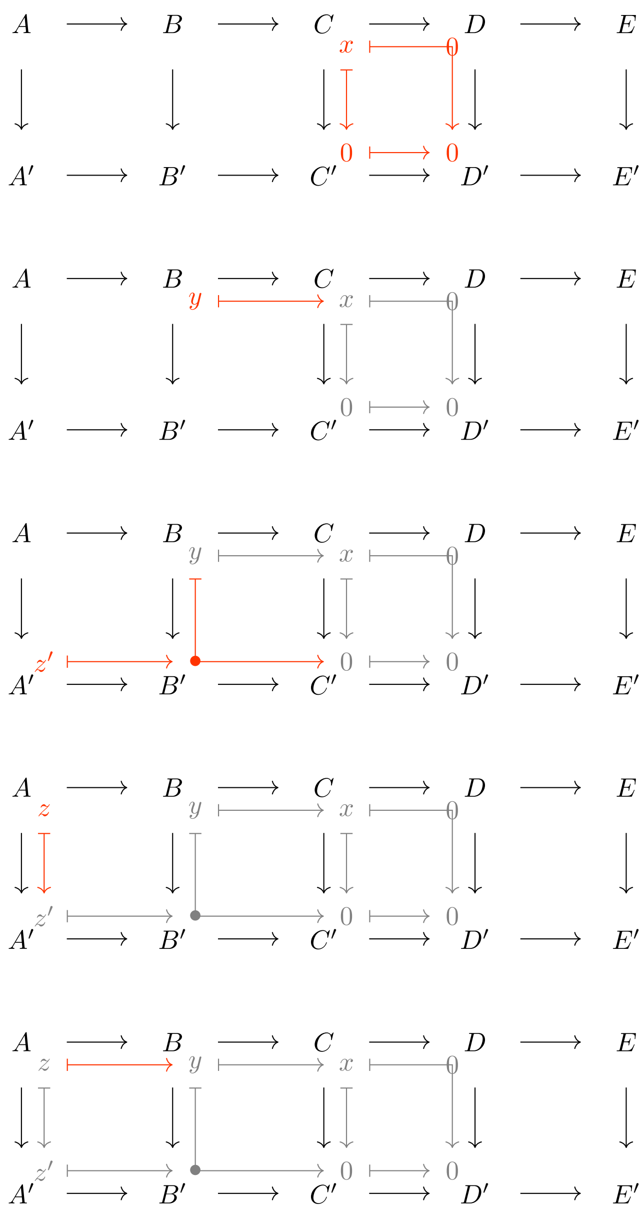


Figure A.4.: The diagram chase in the proof of the five lemma

Proof. We prove the first part via a so-called diagram chase (many statements in homological algebra are established in this way). The second part can be proved in a similar way; the third part is a direct consequence of the first two parts.

Ad 1. Let f_B and f_D be injective and let f_A be surjective. Let $x \in C$ with $f_C(x) = 0$. Then we have $x = 0$, because (Figure A.4):

- Because of $f_D \circ c(x) = c' \circ f_C(x) = c'(0) = 0$ and the injectivity of f_D, we obtain $c(x) = 0$.
- In view of $\operatorname{im} b = \ker c$, there exists a $y \in B$ with $b(y) = x$.
- As $b' \circ f_B(y) = f_C \circ b(y) = f_C(x) = 0$ and $\operatorname{im} a' = \ker b'$, we have: There exists a $z' \in A'$ with $a'(z') = f_B(y)$.
- Because f_A is surjective, there is a $z \in A$ with $f_A(z) = z'$.
- Then $a(z) = y$, because: We have $f_B\bigl(a(z)\bigr) = a' \circ f_A(z) = a'(z') = f_B(y)$ and f_B is injective.
- Thus (because $\operatorname{im} a \subset \ker b$)
$$x = b(y) = b \circ a(z) = 0,$$
as desired. □

For the proof of the Mayer–Vietoris sequence (Theorem 3.2.1) we will need the following construction of long exact sequences:

Proposition A.3.8 (algebraic Mayer–Vietoris sequence)**.** *Let*

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{c_{k+1}} & A_k & \xrightarrow{a_k} & B_k & \xrightarrow{b_k} & C_k & \xrightarrow{c_k} & A_{k-1} & \xrightarrow{a_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{A,k}} & & \downarrow{\scriptstyle f_{B,k}} & & \downarrow{\scriptstyle f_{C,k}} & & \downarrow{\scriptstyle f_{A,k-1}} & & \\
\cdots & \xrightarrow[c'_{k+1}]{} & A'_k & \xrightarrow[a'_k]{} & B'_k & \xrightarrow[b'_k]{} & C'_k & \xrightarrow[c'_k]{} & A'_{k-1} & \xrightarrow[a'_{k-1}]{} & \cdots
\end{array}$$

be a ($\mathbb{Z}$-indexed) commutative ladder in ${}_R\mathsf{Mod}$ with exact rows. Moreover, for every $k \in \mathbb{Z}$, let $f_{C,k}\colon C_k \longrightarrow C'_k$ be an isomorphism and let

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k \colon B'_k \longrightarrow A_{k-1}.$$

Then the following sequence in ${}_R\mathsf{Mod}$ is exact:

$$\cdots \xrightarrow{\Delta_{k+1}} A_k \xrightarrow{(f_{A,k},-a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\Delta_k} A_{k-1} \longrightarrow \cdots$$

Proof. This follows from a diagram chase (Exercise). □

A.3.2 Chain complexes and homology

Chain complexes are a generalisation of exact sequences. The non-exactness of chain complexes is measured in terms of homology.

Definition A.3.9 (chain complex)**.** An R-chain complex is a pair $C = (C_*, \partial_*)$, consisting of

- a sequence $C_* = (C_k)_{k\in\mathbb{Z}}$ of left R-modules (the *chain modules*), and
- a sequence $\partial_* = (\partial_k \colon C_k \longrightarrow C_{k-1})_{k\in\mathbb{Z}}$ of R-homomorphisms (the *boundary operators* or *differentials*) with

$$\forall_{k\in\mathbb{Z}} \quad \partial_k \circ \partial_{k+1} = 0.$$

Let $k \in \mathbb{Z}$.

- The elements of C_k are the *k-chains*,
- the elements of $Z_k C := \ker \partial_k \subset C_k$ are the *k-cycles*,
- the elements of $B_k C := \operatorname{im} \partial_{k+1} \subset C_k$ are the *k-boundaries*.

In the same way, one can also define chain complexes that are indexed over $\mathbb{N}$ instead of $\mathbb{Z}$. In this case, one defines $Z_0 C := C_0$.

Example A.3.10 (chain complexes)**.**

- Every long exact sequence is a chain complex.
- The sequence

$$\cdots \xrightarrow{\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\mathrm{id}_\mathbb{Z}} \cdots$$

  is *no* chain complex of $\mathbb{Z}$-modules, because the composition of successive homomorphisms is not the zero map.

The terms cycle, boundary, chain, … originate from algebraic topology. This can be seen in the construction of singular or simplicial homology (Chapter 3).

Remark A.3.11 (Co)**.** Reversing the direction of arrows in the definition of chain complexes, leads to *cochain complexes* (and cochains, cocycles, coboundaries, coboundary operators, cochain maps, cohomology, …). Usually, one denotes coboundary operators in cochain complexes with δ (instead of ∂) and indices are denoted as superscripts (instead of subscripts).

Such objects naturally arise in the study of smooth manifolds and differential forms: the de Rham cochain complex and de Rham cohomology.

In order to obtain a category of chain complexes, we introduce chain maps as structure-preserving maps between chain complexes:

Definition A.3.12 (chain map)**.** Let $C = (C_*, \partial_*)$ and (C'_*, ∂'_*) be R-chain complexes. An *R-chain map* $C \longrightarrow C'$ is a sequence $(f_k \in {}_R\mathrm{Hom}(C_k, C'_k))_{k\in\mathbb{Z}}$ with

$$\forall_{k\in\mathbb{Z}} \quad f_k \circ \partial_{k+1} = \partial'_{k+1} \circ f_{k+1}.$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_n & \longrightarrow & \cdots \\
 & & {\scriptstyle f_{k+1}}\big\downarrow & & \big\downarrow{\scriptstyle f_k} & & \\
\cdots & \longrightarrow & C'_{k+1} & \xrightarrow[\partial'_{k+1}]{} & C'_k & \longrightarrow & \cdots
\end{array}$$

Definition A.3.13 (category of chain complexes)**.** The category ${}_R\mathsf{Ch}$ of R-chain complexes consists of:

- objects: the class of all R-chain complexes
- morphisms: R-chain maps
- compositions: degree-wise ordinary composition of maps.

Example A.3.14 (tensor product of a module and a chain complex)**.** Let Z be a right(!) R-module and let $C = (C_*, \partial_*) \in \mathrm{Ob}({}_R\mathsf{Ch})$. Then

$$Z \otimes_R C := \big((Z \otimes_R C_k)_{k\in\mathbb{Z}}, (\mathrm{id}_Z \otimes_R \partial_k)_{k\in\mathbb{Z}}\big)$$

is a $\mathbb{Z}$-chain complex (check!). If R is non-commutative, then $Z \otimes_R C$, in general, will *not* be an R-chain complex (for this, we need a bimodule structure on Z). Moreover, it should be noted that, in general, homology is *not* compatible with taking tensor products!

Taking the degree-wise tensor product with id_Z turns this construction into a functor (check!)

$$Z \otimes_R \;\cdot\; : {}_R\mathsf{Ch} \longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}\,.$$

Example A.3.15 (chain complexes of simplicial modules)**.** Let $S \colon \Delta^{\mathrm{op}} \longrightarrow {}_R\mathsf{Mod}$ be a functor (a so-called *simplicial left R-module*); here, Δ^{op} is the dual of the simplex category (obtained from Δ by reversing morphisms). For $k \in \mathbb{Z}$, we define

$$C_k(S) := \begin{cases} S\big(\Delta(k)\big) & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases}$$

$$\partial_k := \begin{cases} \sum_{j=0}^{k} (-1)^j \cdot S(d_j^k) & \text{if } k > 0 \\ 0 & \text{if } k \leq 0; \end{cases}$$

here, $d_j^k \in \operatorname{Mor}_\Delta(\Delta(k-1), \Delta(k))$ is the morphism, whose image is $\{0, \dots, k\} \setminus \{j\}$. We write

$$C(S) := \bigl((C_k(S))_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr).$$

Then $C(S)$ is an R-chain complex (check!). This is one of the key constructions that underlies many homology theories.

This construction can also be extended to a functor

$$C\colon \Delta({}_R\mathsf{Mod}) \longrightarrow {}_R\mathsf{Ch}\,.$$

Here, $\Delta({}_R\mathsf{Mod})$ is the category whose objects are functors $\Delta^{\mathrm{op}} \longrightarrow {}_R\mathsf{Mod}$ and whose morphisms are natural transformations between such functors.

The (non-)exactness of chain complexes is measured in terms of homology:

Definition A.3.16 (homology)**.** Let $C = (C_*, \partial_*)$ be an R-chain complex. For $k \in \mathbb{Z}$, the *k-th homology of C* is defined as

$$H_k(C) := \frac{Z_k(C)}{B_k(C)} = \frac{\ker(\partial_k\colon C_k \to C_{k-1})}{\operatorname{im}(\partial_{k+1}\colon C_{k+1} \to C_k)} \in \operatorname{Ob}({}_R\mathsf{Mod}).$$

Remark A.3.17 (homology and exactness)**.** A chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is an exact sequence if and only if $H_k(C) \cong_R 0$ for all $k \in \mathbb{Z}$.

Remark A.3.18 (computation of homology)**.** Algorithmically, homology of (sufficiently finite) chain complexes can be computed with the tools developed in linear algebra (over fields: Gaussian elimination (Satz I.5.2.8); over Euclidean domains/principal ideal domains: Smith normal form (Satz II.2.5.6)).

Proposition A.3.19 (homology as functor)**.** *Let $k \in \mathbb{Z}$.*

1. *Let $C, C' \in \operatorname{Ob}({}_R\mathsf{Ch})$, let $f\colon C \longrightarrow C'$ be an R-chain map. Then*

$$\begin{aligned} H_k(f)\colon H_k(C) &\longrightarrow H_k(C') \\ [z] &\longmapsto \bigl[f_k(z)\bigr] \end{aligned}$$

   is a well-defined R-homomorphism.

2. *In this way, H_k becomes a functor ${}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Mod}$.*

Proof. Ad 1. The map $H_k(f)$ is well-defined: Because f is a chain map, f_k maps cycles to cycles (check!). Let $z, z' \in Z_k(C)$ with $z - z' \in B_k(C)$; let $b \in C_{k+1}$ such that $\partial_{k+1} b = z - z'$. Then we obtain in $H_k(C')$:

$$\begin{aligned} \bigl[f_k(z)\bigr] - \bigl[f_k(z')\bigr] &= \bigl[f_k(z) - f_k(z')\bigr] \\ &= \bigl[f_k(z - z')\bigr] \\ &= \bigl[f_k(\partial_{k+1} b)\bigr] && \text{(choice of } b) \\ &= \bigl[\partial'_k f_{k+1}(b)\bigr] && (f \text{ is a chain map}) \\ &= 0. && (\text{definition of } H_k(C')) \end{aligned}$$

Hence, $H_k(f)$ is well-defined. By construction, $H_k(f)$ is R-linear (because f_k is R-linear).

Ad 2. This is a straightforward computation (check!). □

When computing homology, inheritance results and computational tricks can save a lot of time and space. One key tool is the long exact homology sequence:

Proposition A.3.20 (algebraic long exact homology sequence)**.** *Let*

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Ch}$ (i.e., in every degree, the corresponding sequence in ${}_R\mathsf{Ch}$ is exact). Then there is a (natural) long exact sequence

$$\cdots \xrightarrow{\partial_{k+1}} H_k(A) \xrightarrow{H_k(i)} H_k(B) \xrightarrow{H_k(p)} H_k(C) \xrightarrow{\partial_k} H_{k-1}(A) \longrightarrow \cdots$$

This sequence is natural in the following sense: If

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ i\ } & B & \xrightarrow{\ p\ } & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \\
0 & \longrightarrow & A' & \xrightarrow[i']{} & B' & \xrightarrow[p']{} & C' & \longrightarrow & 0
\end{array}$$

is a commutative diagram in ${}_R\mathsf{Ch}$ with exact rows, then the corresponding ladder

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_{k+1}} & H_k(A) & \xrightarrow{H_k(i)} & H_k(B) & \xrightarrow{H_k(p)} & H_k(C) & \xrightarrow{\partial_k} & H_{k-1}(A) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle H_k(f_A)} & & \downarrow{\scriptstyle H_k(f_B)} & & \downarrow{\scriptstyle H_k(f_C)} & & \downarrow{\scriptstyle H_{k-1}(f_A)} & \\
\cdots \xrightarrow[\partial_{k+1}]{} & H_k(A') & \xrightarrow[H_k(i')]{} & H_k(B') & \xrightarrow[H_k(p')]{} & H_k(C') & \xrightarrow[\partial_k]{} & H_{k-1}(A') & \longrightarrow \cdots
\end{array}$$

is commutative and has exact rows.

Proof. Let $k \in \mathbb{Z}$. We construct the *connecting homomorphism*

$$\partial_k \colon H_k(C) \longrightarrow H_{k-1}(A)$$

as follows: Let $\gamma \in H_k(C)$; let $c \in C_k$ be a cycle representing γ. Because $p_k \colon B_k \longrightarrow C_k$ is surjective, there is a $b \in B_k$ with

$$p_k(b) = c.$$

As p is a chain map, we obtain $p_{k-1} \circ \partial_k^B(b) = \partial_k^C \circ p_k(b) = \partial_k^C(c) = 0$; then exactness in degree k shows that there exists an $a \in A_{k-1}$ with

$$i_{k-1}(a) = \partial_k^B(b).$$

In this situation, we call (a, b, c) a *compatible triple for* γ and we define

$$\partial_k(\gamma) := [a] \in H_{k-1}(A).$$

Straightforward diagram chases then show (check!):

- If (a, b, c) is a compatible triple for γ, then $a \in A_{k-1}$ is a cycle (and so indeed defines a class in $H_{k-1}(A)$).
- If (a, b, c) and (a', b', c') are compatible triples for γ, then $[a] = [a']$ in $H_{k-1}(A)$.

These observations show that ∂_k is an R-homomorphism and that ∂_k is natural (check!).

Further diagram chases then show that the resulting long sequence is exact (even more to check ...). □

Combining the five lemma and the algebraic long exact sequence gives us:

Example A.3.21 (drie halen, twee betalen)**.** Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\;i\;} & B & \xrightarrow{\;p\;} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \\
0 & \longrightarrow & A' & \xrightarrow[\;i'\;]{} & B' & \xrightarrow[\;p'\;]{} & C' & \longrightarrow & 0
\end{array}$$

be a commutative diagram in ${}_R\mathsf{Ch}$ with exact rows. Then: If two of the three sequences $(H_k(f_A))_{k\in\mathbb{Z}}$, $(H_k(f_B))_{k\in\mathbb{Z}}$, $(H_k(f_C))_{k\in\mathbb{Z}}$ consist of isomorphisms, then so does the third. This can be seen as follows:

The long exact homology sequences of the rows lead to a commutative ladder with exact rows (Proposition A.3.20). We can then apply the five lemma (Proposition A.3.7) to five successive rungs (where we put the mystery homomorphism into the middle).

A.3.3 Homotopy invariance

A key property of homology of chain complexes is homotopy invariance. This algebraic homotopy invariance is the source of homotopy invariance of many functors in geometry and topology; moreover, algebraic homotopy invariance often simplifies the computation of homology.

We briefly explain how topological considerations naturally lead to the notion of chain homotopy (Definition A.3.28):

In Top, homotopy is defined as follows: Continuous maps $f, g\colon X \longrightarrow Y$ are homotopic, if there exists a continuous map $h\colon X \times [0,1] \longrightarrow Y$ with

$$\begin{array}{ccc} \text{degree } 1 & & \text{degree } 0 \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ x & \longmapsto & (-x, x) \end{array}$$

Figure A.5.: an algebraic model of $[0,1]$

$$h \circ i_0 = f \quad \text{and} \quad h \circ i_1 = g;$$

here, $i_0 \colon X \hookrightarrow X \times \{0\} \hookrightarrow X \times [0,1]$ and $i_1 \colon X \hookrightarrow X \times \{1\} \hookrightarrow X \times [0,1]$ denote the canonical inclusions of the bottom and the top into the cylinder $X \times [0,1]$ over X, respectively.

We model this situation in the category ${}_R\mathsf{Ch}$: As first step, we model the unit interval $[0,1]$ by a suitable chain complex (Figure A.5).

Definition A.3.22 (algebraic model $[0,1]$)**.** Let $I \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$ be the chain complex

$$\begin{array}{ccccccccccccccc} \text{degree} & & & & 2 & & 1 & & 0 & & -1 & & & & \\ \dots & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \dots \\ & & & & & & x & \longmapsto & (-x, x) & & & & & & \end{array}$$

As analogy of the product of topological spaces, we consider the tensor product of chain complexes; the basic idea is that chain modules of the product in degree k should contain information on k-dimensional phenomena and thus the degree of the tensor factors should add up to k. Geometrically, one can show that cellular chain complexes of products of finite CW-complexes (with respect to the product cell structure) are isomorphic to the tensor product of the cellular chain complexes of the factors [26, V.3.9]. More generally, the Eilenberg–Zilber theorem shows that the singular chain complex of a product of two spaces is chain homotopy equivalent to the tensor product of the singular chain complexes of the factors [26, Chapter VI.12].

Definition A.3.23 (tensor product of chain complexes)**.** Let $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ and $D \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$. Then we define $C \otimes_{\mathbb{Z}} D \in \operatorname{Ob}({}_R\mathsf{Ch})$ by

$$(C \otimes_R D)_k := \bigoplus_{j \in \mathbb{Z}} C_j \otimes_R D_{k-j}$$

and the boundary operators

$$\begin{aligned} (C \otimes_R D)_k &\longrightarrow (C \otimes_R D)_{k-1} \\ C_j \otimes_R D_{k-j} \ni c \otimes d &\longmapsto \partial_j^C c \otimes d + (-1)^j \cdot c \otimes \partial_{k-j}^D d \end{aligned}$$

for all $k \in \mathbb{Z}$. (This indeed defines a chain complex!)

Study note. This definition generalises the tensor product of a module and a chain complex (Example A.3.14). Do you see why/how?

More generally, if C is an (S,R)-bimodule chain complex and D is a left R-chain complex, then one can also define the left S-chain complex $C \otimes_R D$.

Remark A.3.24 (sign convention)**.** We use the following convention for the choice of signs: If a boundary operator is "moved past" an element, then we introduce the sign

$$(-1)^{\text{degree of that element}}.$$

It should be noted that different authours use different sign conventions. Therefore, for all formulae in the literature concerning products of chain complexes or products on (co)homology, one has to carefully check the sign conventions used in that source.

Remark A.3.25 (functoriality of the tensor product)**.** Let $C, C' \in \operatorname{Ob}({}_R\mathsf{Ch})$, let $D, D' \in \operatorname{Ob}({}_\mathbb{Z}\mathsf{Ch})$, and let $f \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, C')$ und $g \in \operatorname{Mor}_{{}_\mathbb{Z}\mathsf{Ch}}(D, D')$. Then

$$\begin{aligned} f \otimes_R g \colon C \otimes_\mathbb{Z} D &\longrightarrow C' \otimes_\mathbb{Z} D' \\ c \otimes d &\longmapsto f(c) \otimes g(d) \end{aligned}$$

yields a well-defined chain map in ${}_R\mathsf{Ch}$.

As next step, we model the inclusions of the bottom and the top of cylinders in the algebraic setting.

Definition A.3.26 (algebraic model of inclusion of top/bottom of cylinders)**.** Let $C \in \operatorname{Ob}({}_R\mathsf{Ch})$. Then, we define the R-chain maps (check!)

$$\begin{aligned} i_0 \colon C &\longrightarrow C \otimes_\mathbb{Z} I \\ C_k \ni c &\longmapsto (c, 0, 0) \in C_k \oplus C_{k-1} \oplus C_k \cong_R (C \otimes_\mathbb{Z} I)_k \\ i_1 \colon C &\longrightarrow C \otimes_\mathbb{Z} I \\ C_k \ni c &\longmapsto (0, 0, c) \in C_k \oplus C_{k-1} \oplus C_k \cong_R (C \otimes_\mathbb{Z} I)_k. \end{aligned}$$

Under the correspondence indicated in Figure A.5, these chain maps are an algebraic version of the geometric inclusions of bottom and top, respectively.

Remark A.3.27. Let $C, D \in \operatorname{Ob}({}_R\mathsf{Ch})$ and let $f, g \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, D)$. A chain map $h \colon C \otimes_\mathbb{Z} I \longrightarrow D$ in ${}_R\mathsf{Ch}$ with $h \circ i_0 = f$ and $h \circ i_1 = g$ corresponds to a family $(\widetilde{h}_k \in \operatorname{Mor}_{{}_R\mathsf{Mod}}(C_k, D_{k+1}))_{k \in \mathbb{Z}}$ satisfying

$$\partial^D_{k+1} \circ \widetilde{h}_k = \widetilde{h}_{k-1} \circ \partial^C_k + (-1)^k \cdot g_k - (-1)^k \cdot f_k$$

(Figure A.6) for all $k \in \mathbb{Z}$. This last equation can be rewritten as

$$\partial^D_{k+1} \circ (-1)^k \cdot \widetilde{h}_k + (-1)^{k-1} \cdot \widetilde{h}_{k-1} \circ \partial^C_k = g_k - f_k.$$

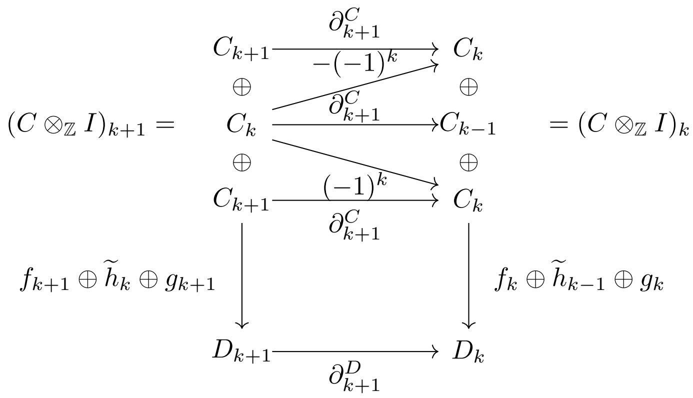


Figure A.6.: discovering the notion of chain homotopy

Therefore, one defines the notion of chain homotopy (and related terms) as follows:

Definition A.3.28 (chain homotopy, null-homotopic, contractible)**.** Let $C, D \in \operatorname{Ob}({}_R\mathsf{Ch})$.

- Chain maps $f, g \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, D)$ are *chain homotopic (in ${}_R\mathsf{Ch}$)*, if there exists a sequence $h = (h_k \in \operatorname{Mor}_{{}_R\mathsf{Mod}}(C_k, D_{k+1}))_{k\in\mathbb{Z}}$ with

$$\partial^D_{k+1} \circ h_k + h_{k-1} \circ \partial^C_k = g_k - f_k$$

  for all $k \in \mathbb{Z}$. In this case, h is a *chain homotopy from f to g (in ${}_R\mathsf{Ch}$)*, and we write $f \simeq_{{}_R\mathsf{Ch}} g$.

- A chain map $f \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, D)$ is a *chain homotopy equivalence (in ${}_R\mathsf{Ch}$)*, if there exists a chain map $g \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(D, C)$ with

$$g \circ f \simeq_{{}_R\mathsf{Ch}} \operatorname{id}_C \quad \text{and} \quad f \circ g \simeq_{{}_R\mathsf{Ch}} \operatorname{id}_D .$$

  We then write $C \simeq_{{}_R\mathsf{Ch}} D$.

- Chain maps that are (in ${}_R\mathsf{Ch}$) chain homotopic to the zero map are *null-homotopic (in ${}_R\mathsf{Ch}$)*.

- The chain complex C ist *contractible (in ${}_R\mathsf{Ch}$)*, if id_C is null-homotopic in ${}_R\mathsf{Ch}$ (equivalently, if C is chain homotopic to the zero chain complex). Homotopies in ${}_R\mathsf{Ch}$ from id_C to the zero map are also called *chain contractions (in ${}_R\mathsf{Ch}$)*.

Example A.3.29. Let $C \in \operatorname{Ob}({}_R\mathsf{Ch})$. Then $i_0 \simeq_{{}_R\mathsf{Ch}} i_1 \colon C \longrightarrow C \otimes_{\mathbb{Z}} I$ (check!). Moreover, we consider the chain map

$$p\colon C \otimes_{\mathbb{Z}} I \longrightarrow C$$
$$C_k \oplus C_{k-1} \oplus C_k \ni (c_0, c, c_1) \longmapsto c_0 + c_1 \in C_k$$

in ${}_R\mathsf{Ch}$. Then $p \circ i_0 = \mathrm{id}_C$ and $i_0 \circ p \simeq_{{}_R\mathsf{Ch}} \mathrm{id}_{C \otimes_{\mathbb{R}} I}$. Hence, $C \simeq_{{}_R\mathsf{Ch}} C \otimes_{\mathbb{Z}} I$, as we would expect from topology.

Proposition A.3.30 (basic properties of chain homotopy)**.**

1. *Let $C, D \in \mathrm{Ob}({}_R\mathsf{Ch})$ and let $f, f', g, g' \in \mathrm{Mor}({}_R\mathsf{Ch})$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_R\mathsf{Ch}} g'$. Then, we have*
$$a \cdot f + b \cdot g \simeq_{{}_R\mathsf{Ch}} a \cdot f' + b \cdot g'$$
for all $a, b \in R$.
2. *Let $C, D \in \mathrm{Ob}({}_R\mathsf{Ch})$. Then "$\simeq_{{}_R\mathsf{Ch}}$" is an equivalence relation on the morphism set $\mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D)$.*
3. *Let $C, D, E \in \mathrm{Ob}({}_R\mathsf{Ch})$, let $f, f' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D)$ and let $g, g' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(D, E)$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_R\mathsf{Ch}} g'$. Then, we have*
$$g \circ f \simeq_{{}_R\mathsf{Ch}} g' \circ f'.$$
4. *Let $C, D \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$, let $Z \in \mathrm{Ob}({}_R\mathsf{Mod})$, and let $f, f' \in \mathrm{Mor}_{{}_{\mathbb{Z}}\mathsf{Ch}}(C, D)$ mit $f \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} f'$. Then,*
$$Z \otimes_{\mathbb{Z}} f \simeq_{{}_R\mathsf{Ch}} Z \otimes_{\mathbb{Z}} f'.$$
5. *Let $C, C' \in \mathrm{Ob}({}_R\mathsf{Ch})$, $D, D' \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$, and let $f, f' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, C')$, $g, g' \in \mathrm{Mor}_{{}_{\mathbb{Z}}\mathsf{Ch}}(D, D')$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} g'$. Then*
$$f \otimes_{\mathbb{Z}} g \simeq_{{}_R\mathsf{Ch}} f' \otimes_{\mathbb{Z}} g'.$$

Proof. All these properties follow via straightforward calculations directly from the definitions (check!). □

In particular, we can pass to the corresponding homotopy category:

Definition A.3.31 (homotopy category of chain complexes)**.** The *homotopy category of left R-chain complexes* is the category ${}_R\mathsf{Ch}_\mathsf{h}$ consisting of:

- objects: Let $\mathrm{Ob}({}_R\mathsf{Ch}_\mathsf{h}) := \mathrm{Ob}({}_R\mathsf{Ch})$.
- morphisms: For all left R-chain complexes C, D, we set
$$[C, D] := \mathrm{Mor}_{{}_R\mathsf{Ch}_\mathsf{h}}(C, D) := \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D) \big/ {\simeq_{{}_R\mathsf{Ch}}} .$$
- compositions: The compositions of morphisms are defined by ordinary (degree-wise) composition of representatives.

As mentioned before, a key property of homology of chain complexes is homotopy invariance in the following sense:

Proposition A.3.32 (homotopy invariance of homology of chain complexes)**.** *Let* $k \in \mathbb{Z}$*. Then the functor* $H_k \colon {}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Mod}$ *factors over* ${}_R\mathsf{Ch}_h$*. More explicitly: Let* $C, C' \in \operatorname{Ob}({}_R\mathsf{Ch})$ *and let* $f, g \colon C \longrightarrow C'$ *be* R*-chain maps with* $f \simeq_{{}_R\mathsf{Ch}} g$*. Then,*

$$H_k(f) = H_k(g).$$

Proof. Let h be a chain homotopy from f to g in ${}_R\mathsf{Ch}$. Moreover, let $z \in Z_k(C)$ be a k-cycle. Then, we obtain in $H_k(C')$:

$$\begin{aligned}
H_k(f)\bigl([z]\bigr) - H_k(g)\bigl([z]\bigr) &= \bigl[f_k(z) - g_k(z)\bigr] \\
&= \bigl[\partial'_{k+1} \circ h_k(z) + h_{k-1} \circ \partial_k(z)\bigr] && (h \text{ is a chain homotopy}) \\
&= \bigl[\partial'_{k+1} \circ h_k(z) + 0\bigr] && (z \text{ is a cycle}) \\
&= [0] && (\text{definition of } H_k(C'))
\end{aligned}$$

Hence, $H_k(f) = H_k(g)$. □

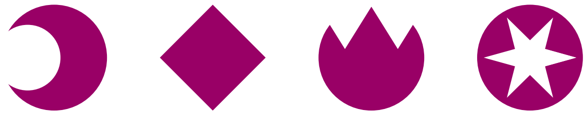
B

Exercise Sheets

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 1, October 21, 2022

Quick check A (contractibility?). Which of the following subspaces of $\mathbb{R}^2$ are convex, star-shaped, contractible?



Quick check B (homotopies). Give explicit homotopies between the following maps; visualise your homotopies via suitable illustrations:

1. $[0,1] \longrightarrow S^1$, $x \longmapsto (\cos x, \sin x)$ and $[0,1] \longrightarrow S^1$, $x \longmapsto (\cos x, -\sin x)$;
2. id_{S^1} and $S^1 \longrightarrow S^1$, $(x_1, x_2) \longmapsto (-x_2, x_1)$;

Quick check C (motion planning on S^1). Sketch a reasonable motion planning for the state space S^1 (of course, this will not be continuous!).

Quick check D (deformations of images).

1. Read/explain the statement on hashing of images in the following tweet:
   https://twitter.com/marcan42/status/1428578906412437507
2. Read/learn more on hashing of images and technical and other issues connected with it:
   https://twitter.com/marcan42/status/1427896137696960513
3. Enjoy the result of the challenge:
   https://twitter.com/marcan42/status/1428758281476927488

Exercise 1 (contractibility? 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$ is non-empty, path-connected, and a finite union of star-shaped sets, then X is contractible.

Exercise 2 (spheres and stars; 3 credits). Let $n \in \mathbb{N}$. Show directly that the sphere S^n is *not* a star-shaped subset of $\mathbb{R}^{n+1}$.
Hints. You might need to resurrect some Euclidean geometry.

Exercise 3 (convex motion planning; 3 credits). Let $n \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be non-empty and convex. Give an explicit continuous motion planning for X (and prove that it has the claimed property).

Exercise 4 (compact-open topology vs. uniform convergence; 3 credits). Let X be a compact space and let Y be a metric space. Show one of the two inclusions of the fact that the compact-open topology on $\mathrm{map}(X, Y)$ coincides with the topology of uniform convergence.

Bonus problem (real-life homotopy; 3 credits). Find three real-life situations that could be modelled by homotopies and explain your model.

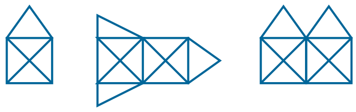
Submission before October 28, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on October 27, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 2, October 28, 2022

Quick check A (Haus/Rakete des Nikolaus). Which of the following doodles can be drawn in a single stroke (without re-drawing lines)? Can you do it in such a way that the path closes up?



Quick check B (Eulerian cycles in infinite graphs?). Let X be an infinite connected graph, all of whose vertices have finite even degree. Does X then necessarily contain a partial Eulerian cycle of non-zero length?

Quick check C (real-life simplicial complexes). Construct (fragments of) the simplicial complexes described as follows:

- Food ingredients that taste well together.
- Pitches that sound consonantly.
- Football players that could play well on the same team.

Exercise 1 (regular polyhedra; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If P is a regular polyhedron, then the graph of vertices/edges of P does *not* admit an Eulerian cycle.

Exercise 2 (subgraphs and connectedness; 3 credits). Let X be a finite connected graph. Let $X' = (V', E')$ be a subgraph of X with $V' \neq \emptyset$ and

$$\forall_{v \in V'} \quad \deg_{X'} v = \deg_X v.$$

Show that $X' = X$. Illustrate your proof with suitable pictures!

Exercise 3 (Eulerian paths; 3 credits). State and prove a characterisation for the existence of Eulerian paths in finite connected graphs! Argue efficiently!

Exercise 4 (independence! 3 credits). Let K be a field and let V be a K-vector space. Then we consider the simplicial complex

$$I(V) := \{\sigma \in P_{\text{fin}}(V) \mid \sigma \text{ is linearly independent over } K\}.$$

In the following, let $X := I(\mathbb{F}_2{}^3)$. Justify your answers!

1. What is $\dim X$?
2. How many edges does X have?
3. Do there exist vertices $x, y, z \in X(0)$ with $\{x, y\}, \{x, z\}, \{y, z\} \in X(1)$ and $\{x, y, z\} \notin X(2)$?

Bonus problem (de Bruijn graphs in DNA reconstruction; 3 credits). Give a formal definition of directed graphs. How are de Bruijn graphs in DNA reconstruction defined? How can Eulerian paths in de Bruijn graphs help in DNA reconstruction? Do not forget to cite your sources!

Submission before November 4, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 3, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 3, November 4, 2022

Quick check A (constant maps). Let X and Y be simplicial complexes and let $y \in V(Y)$. Show that then the constant map $V(X) \longrightarrow V(Y)$ with value y indeed is a simplicial map. What is there to prove anyway?!

Quick check B (pushouts and products). Recall/Learn about the universal property of *pushouts* and *products* in categories.

Quick check C (unions/intersections lead to pushouts). Let X and Y be simplicial complexes. Show that the following diagram of inclusions of simplicial complexes is a pushout diagram in SC:

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup Y \end{array}$$

Quick check D (simplicial maps and connectedness). Let X and Y be simplicial complexes and let $f\colon X \longrightarrow Y$ be a simplicial map. Show that $f(X)$ is a subcomplex of Y and that $f(X)$ is connected if X is connected. Compare this connectedness result/proof with the topological case!

Exercise 1 (simplicial complexes with three vertices; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exist exactly five isomorphism classes of simplicial complexes with exactly three vertices.

Exercise 2 (simplicial products of standard simplices; 3 credits). Let $n, m \in \mathbb{N}$. What is $\Delta(n) \boxtimes \Delta(m)$? Give a concrete description and prove your claim!

Exercise 3 (connectedness from covers; 3 credits). Let X be a simplicial complex and let $(U_i)_{i \in I}$ be a family of connected subcomplexes of X with $\bigcup_{i \in I} V(U_i) = V(X)$. Let N be the graph $(I, \{\{i,j\} \mid i,j \in I,\ i \neq j,\ V(U_i) \cap V(U_j) \neq \emptyset\})$. Show that X is connected if N is connected. Illustrate!

Exercise 4 (transitivity of contiguity? 3 credits). Let X and Y be simplicial complexes. Two maps $f, g\colon X \longrightarrow Y$ are *contiguous* if the following holds:

$$\forall_{\sigma \in X} \quad f(\sigma) \cup g(\sigma) \in Y.$$

Show that contiguity in general is *not* transitive on the set $\operatorname{map}_\Delta(X, Y)$.

Bonus problem (adjoints of the vertex functor; 3 credits). Let $V\colon \mathsf{SC} \longrightarrow \mathsf{Set}$ be the functor mapping simplicial complexes to the underlying set of vertices and mapping simplicial maps to the underlying map between the sets of vertices. Show that this functor V has both a left adjoint and a right adjoint.
Hints. There are two generic ways to convert sets into simplicial complexes with the given set as set of vertices. These ways are the desired left/right adjoints of V.

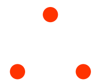
Submission before November 11, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 10, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 4, November 11, 2022

Quick check A (Čech triangle). Let $X \subset \mathbb{R}^2$ be the vertices of a regular Euclidean triangle of side length 1. Compute and illustrate $\check{C}_\varepsilon(X, \mathbb{R}^2, d_2)$ for all $\varepsilon \in \mathbb{R}_{>0}$.



Quick check B (a non-triangulable space). Show that $\{1/n \mid n \in \mathbb{N}_{>0}\} \cup \{0\}$ is *not* triangulable (with respect to the subspace topology of $\mathbb{R}$).

Quick check C (finiteness leads to compactness). Let X be a finite simplicial complex. Show that $|X|$ is compact.

Exercise 1 (open simplices? 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

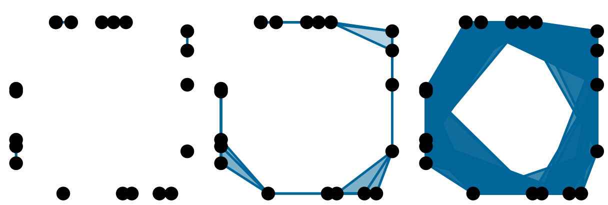
> If X is a simplicial complex and $\sigma \in X$, then $\bigl\{\xi \in |\sigma| \bigm| \forall_{x \in \sigma} \quad \xi_x > 0\bigr\}$ is open in $|X|$.

Exercise 2 (simplices via open stars; 3 credits). Let X be a simplicial complex and let $\sigma \subset V(X)$ be a finite non-empty subset. Show that σ is a simplex of X if and only if $\bigcap_{x \in \sigma} \operatorname{star}^\circ_X x \neq \emptyset$.

Exercise 3 (Rips complexes are flag; 3 credits). Look up the notion of *flag* simplicial complexes and prove that Rips complexes are flag.
Hints. Add the definition of "flag" that you use and add a reference for it.

Exercise 4 (small data; 3 credits). Pick four locations in Regensburg, two north of the Danube and two south of the Danube. Compute the Rips complexes of these four points with respect to the metrics "shortest way by car" and "shortest way on foot" for all radii. Add documentation/maps for your distance calculations.

Bonus problem (random Rips complexes; 3 credits). Write a program that generates pictures of Rips complexes of 20 random (uniformly distributed) points on the boundary of the square $[-1,1] \times [-1,1]$ with respect to the Euclidean metric. Display five samples, each with the radii 0.3, 0.9, 1.8. Explain your approach to the program.



Submission before November 18, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 17, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 5, November 18, 2022

Quick check A (barycentric counting)**.** Let $n \in \mathbb{N}$. How many vertices does the iterated subdivision $\operatorname{sd}^2 \Delta(2)$ have?

Quick check B (simplicial approximation and compositions)**.** Show that compositions of simplicial approximations are simplicial approximations of the composition.

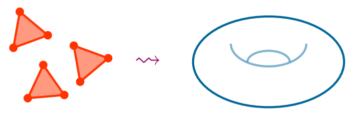
Quick check C (approximating triple wrapping)**.** Give an $N \in \mathbb{N}$ and a simplicial approximation $\operatorname{sd}^N S(1) \longrightarrow S(1)$ of "the" map $|S(1)| \longrightarrow |S(1)|$ wrapping three times around the circle.

Quick check D (real-world triangulations)**.** Give a real-world example, where triangulations are used to approximate geometric objects.

Exercise 1 (counting homotopy classes? 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If X is a countable simplicial complex, then $[|X|, |X|]$ is (at most) countable.

Exercise 2 (triangulating the torus; 3 credits)**.** Give an example of a simplicial complex X such that $|X|$ is homeomorphic to the torus $S^1 \times S^1$.
Hints. Give a precise specification of X. It is not necessary to give a formal proof that $|X|$ is homeomorphic to $S^1 \times S^1$; it is sufficient to give pictures and explanations that make it plausible.



Exercise 3 (simplicial approximation and barycentric subdivision; 3 credits)**.** Let X be a simplicial complex. Show that then the barycentric subdivision homeomorphism $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$ admits a simplicial approximation.
Hints. Choose a total ordering on $V(X)$ (you may use this without proof), then go for the minimum.

Exercise 4 (the nerve map; 3 credits)**.** Let Z be a paracompact topological space, let $U = (U_i)_{i \in I}$ be an open cover of Z, and let $\varphi = (\varphi_i)_{i \in I}$ be a paritition of unity on Z that is subordinate to U. Show that the *nerve map*

$$\begin{aligned} \nu_\varphi \colon Z &\longrightarrow \bigl|N(U)\bigr| \\ \zeta &\longmapsto \sum_{i \in I} \varphi_i(\zeta) \cdot e_i \end{aligned}$$

is well-defined and continous. Moreover, show that if $\nu_{\varphi'}$ is another partition of unity on Z subordinate to U, then $\nu_\varphi \simeq \nu_{\varphi'}$.

Bonus problem (measurability of Čech-realising a given space; 3 credits)**.** Let Z be a topological space, let $n, N \in \mathbb{N}$, and let $\varepsilon \in \mathbb{R}_{>0}$. Show that the subset

$$\bigl\{ x \in (\mathbb{R}^N)^n \bigm| \bigl|\check{C}_\varepsilon(\{x_1, \dots, x_n\}, \mathbb{R}^N, d_2)\bigr| \simeq Z \bigr\}$$

of $(\mathbb{R}^N)^n$ is measurable (with respect to the Borel σ-algebra).

Submission before November 25, 2022, 8:30, via GRIPS (in English or German)

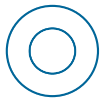
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 24, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold

Sheet 6, November 25, 2022

Quick check A (medial axis, condition number). Compute the medial axis and the condition number of the subset $S^1 \cup 2 \cdot S^1 \subset \mathbb{R}^2$ (with respect to the Euclidean metric).



Quick check B (components). Compute the connected components of the simplicial complex $\{\emptyset, \{0\}, \dots, \{5\}, \{0,2\}, \{0,3\}, \{0,5\}, \{1,4\}, \{2,5\}, \{0,2,5\}\}$ via the algorithm discussed in the lecture. Illustrate!

Quick check C (homological algebra). Refresh your memory on chain complexes, homology, chain maps, and chain homotopy. We will briefly recall these terms in the lectures, but this is an additional opportunity to ask questions.

Exercise 1 (condition numbers; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If M, N are closed smooth submanifolds of $\mathbb{R}^2$ with $M \cap N = \emptyset$, the the condition number of $M \cup N$ is the minimum of the condition numbers of M and N.

Exercise 2 (number of components, algorithmically; 3 credits). Provide an algorithm that, given a finite simplicial complex X, computes the number of connected components of X. Explain your implementation model of simplicial complexes and prove that your algorithm is correct.
Hints. You may base this on algorithmic material from the lecture or start from scratch.

Exercise 3 (degrees of vertices, algorithmically; 3 credits). Provide an algorithm that, given a finite simplicial complex X and a vertex x of X computes the degree of x in the graph determined by the vertices and 1-simplices of X. Explain your implementation model of simplicial complexes and prove that your algorithm is correct.

Exercise 4 (null-homotopic maps of spheres; 3 credits). Let $m, n \in \mathbb{N}$ with $m < n$. Show that every continuous map $S^m \longrightarrow S^n$ is null-homotopic.
Hints. Use ? to show that every continuous map $S^m \longrightarrow S^n$ is homotopic to a map that is not ??. Then apply the ??? to conclude.

Bonus problem (implementation; 3 credits each). Implement these algorithms in your favourite programming language (document your code appropriately!):

1. Computation of connected components of finite simplicial complexes, using the union-find framework.
2. Computation of the number of connected components of finite simplicial complexes as in Exercise 2.
3. Computation of degrees of vertices in finite simplicial complexes as in Exercise 3.

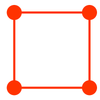
Submission before December 2, 2022, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 1, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 7, December 2, 2022

Quick check A (simplicial homology of simplicial intervals). Compute $H_n([0,N]_\Delta)$ for all $n, N \in \mathbb{N}$.

Quick check B (simplicial homology of the hollow square). Compute the simplicial homology (in all degrees) of the following simplicial complex:



Quick check C (simplicial homology of not so hollow squares). Compute the simplicial homology (in all degrees) of the following simplicial complexes:

Exercise 1 (trivial simplicial homology; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exists an infinite simplicial complex X that satisfies $H_n(X) \cong_{\mathbb{Z}} 0$ for all $n \in \mathbb{N}_{>0}$.

Exercise 2 (simplicial homology of the simplicial 2-sphere; 3 credits). Compute $H_n(S(2))$ for all $n \in \mathbb{N}$.

Exercise 3 (simplicial homology of the simplicial Möbius strip; 3 credits). Give a triangulation of the (closed) Möbius strip and compute the simplicial homology of this simplicial complex. Illustrate!

Exercise 4 (simplicial homology in degree 0; 3 credits). Let X be a finite simplicial complex with m connected components. Show that $H_0(X) \cong_{\mathbb{Z}} \mathbb{Z}^m$.

Bonus problem (custom-made simplicial complex; 3 credits). Give an example of a finite simplicial complex X with

$$H_0(X) \cong_{\mathbb{Z}} \mathbb{Z} \quad \text{and} \quad H_1(X) \cong_{\mathbb{Z}} \mathbb{Z}/2022 \quad \text{and} \quad H_{2022}(X) \cong_{\mathbb{Z}} \mathbb{Z}.$$

Prove that your example does have this property!

Bonus problem (Nikolausaufgabe; 3 credits). The Blorx Building Trust once more has won the bid to construct the *Haus des Nikolaus*. The construction turned out to be cheap, but also somewhat lacking: Blorx only delivered the set

$$\{(0,0),(1,0),(1,1),(0,1),(0.5,2)\} \subset (\mathbb{R}^2, d_2)$$

of vertices, the brand-new Čech-Complex-Constructor™, and the instruction: Setting the radius appropriately will maximise your homological experience!

Which radii lead to the biggest (in terms of rank) simplicial homology of the corresponding Čech complexes in degree 1 ? Justify your answer!

Submission before December 9, 2022, 8:30, via GRIPS (in English or German)

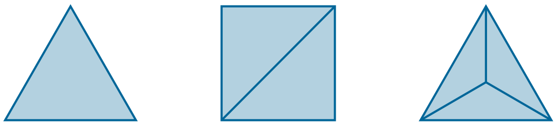
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 8, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 8, December 9, 2022

The Python library simplicial ("Simplicial topology in Python") and its documentation are available at https://simplicial.readthedocs.io/en/latest/index.html.

Quick check A (Mayer–Vietoris sequence). Compute $H_n(S(2))$ for all $n \in \mathbb{N}$ inductively via the Mayer–Vietoris sequence, starting from a 2-simplex and then adding one 2-simplex at a time.



$S(2)$

Quick check B (Smith normal form). Compute the Smith normal form (over $\mathbb{Z}$) of the following matrix:

$$\begin{pmatrix} 2 & 3 & 7 & 1 \\ 3 & 1 & -2 & 1 \\ 1 & 0 & 2 & 3 \end{pmatrix}$$

Quick check C (algorithmic computation of simplicial homology). Compute the $\mathbb{F}_2$-Betti numbers of the simplicial complexes on Sheet 7, using the simplicial library for Python.

Exercise 1 (simplicial homology of unions; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

If X and Y are finite simplicial complexes, then

$$b_2(X \cup Y; \mathbb{Z}) = b_2(X; \mathbb{Z}) + b_2(Y; \mathbb{Z}) - b_2(X \cap Y; \mathbb{Z}).$$

Exercise 2 (homology of simplicial spheres; 3 credits). Let $d \in \mathbb{N}_{>0}$. Compute the homology $H_n(\Delta(d+1), S(d))$ for all $n \in \mathbb{N}$ directly from the definition. Use this result to compute $H_n(S(d))$ for all $n \in \mathbb{N}$.

Exercise 3 (an alternating sum of binomial coefficients via homology; 3 credits). Let $d \in \mathbb{N}$. Use the computation of $H_n(\Delta(d))$ for all $n \in \mathbb{N}$ to show that

$$\sum_{k=1}^{d+1} (-1)^{k-1} \cdot \binom{d+1}{k} = 1,$$

Exercise 4 (barycentric subdivision in the Python library simplicial; 3 credits). What does the method SimplicialComplex.barycentricSubdivide(simplex) from the Python library simplicial do? Give a mathematical definition of this subdivision. Illustrate!

Bonus problem (simplicial products in Python; 3 credits). Write a Python method that computes the simplicial product of two finite simplicial complexes. Use this method to compute the $\mathbb{F}_2$-Betti numbers of $S(1) \boxtimes \Delta(0)$, $S(1) \boxtimes \Delta(1)$, $S(1) \boxtimes S(1)$, and $S(1) \boxtimes S(2)$. Document your code!

Submission before December 16, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 15, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 9, December 16, 2022

Quick check A (simplicial homology of continuous maps)**.** Let X be a finite simplicial complex and let $n \in \mathbb{N}$. Determine $H_n(\beta_X)\colon H_n(\operatorname{sd} X) \longrightarrow H_n(X)$.

Quick check B (self-maps of the circle I)**.** Let $\varphi\colon |S(1)| \longrightarrow |S(1)|$ be the map "wrapping the simplicial circle around itself twice". Give a precise description of φ and compute $H_1(\varphi)\colon H_1(S(1)) \longrightarrow H_1(S(1))$.

Quick check C (self-maps of the circle II)**.** Does every continuous map $S^1 \longrightarrow S^1$ have a fixed point?

Quick check D (ƨtɘdɔƨʇɘ⅃?)**.** Reflect about the question whether there exists a finite simplicial complex X and a continuous map $\varphi\colon |X| \longrightarrow |X|$ that has a fixed point and satisfies $\Lambda(\varphi; K) = 0$ for all fields K.

Exercise 1 (self-maps of spheres; 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

If $f\colon S^{2022} \longrightarrow S^{2022}$ is a continuous map, then f^{2022} has a fixed point.

Exercise 2 (Lefschetz number of chain maps; 3 credits)**.** Let K be a field and let C be a chain complex over K that has only finitely many non-zero chain modules and such that each chain module is finite-dimensional. Let $f\colon C \longrightarrow C$ be a chain map. Show that

$$\sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} H_n(f) = \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr} f_n.$$

Hints. Choose convenient bases.

Exercise 3 (Sperner's lemma; 3 credits)**.** Derive the classical version of Sperner's lemma from the manifold version.

Hints. Show the case of dimension 1 by hand. Then, proceed by induction. You may use without proof: If (X, φ) is a subdivision of $\Delta(n)$, then X is an n-pseudomanifold with boundary.

Exercise 4 (piece of cake; 3 credits)**.** What is the Simmons–Su protocol for envy-free cake division? Explain the terminology, the protocol, and the role of Sperner's lemma.

Hints. Don't forget to cite your sources!

Bonus problem (graph-theoretic proof of Sperner's lemma; 3 credits)**.** Give a homology-free proof of Sperner's lemma for manifolds by considering the dual graph (plus an "external" vertex) and shaking hands.

Submission before December 23, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on December 22, 2022.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 10, December 23, 2022

Quick check A (consensus)**.** Draw the input complex for 3-process consensus on $\{0,1\}$. Which 2-simplices may occur in the image of the task map?

Quick check B (prisoner's dilemma)**.** Look up the *prisoner's dilemma*. Formalise this as a 2-person game. Does this game have pure Nash equilibria? Interpret!

Quick check C (mapping degrees for self-maps of the preference complex)**.** Let X be the preference complex in the proof of Arrow's theorem and let $g\colon X \longrightarrow X$ be a simplicial map. Show that then $g = d \cdot \mathrm{id}_{H_1(X)}$ with $d \in \{-1,0,1\}$.

Exercise 1 (unanimity and dictators; 3 credits)**.** Let $n \in \mathbb{N}_{\geq 2}$, let A be a finite set with $\#A \geq 3$, and let P be the set of total orders on A. Is the following statement true? Justify your answer with a suitable proof or counterexample.

If $f\colon P^n \longrightarrow P$ has a dictator, then f satisfies unanimity.

Exercise 2 (the simplicial aggregation map; 3 credits)**.** In the proof of Arrow's theorem, show that the map $F\colon V' \longrightarrow V$ obtained from the aggregation map $f\colon P^2 \longrightarrow P$ indeed defines a simplicial map $X' \longrightarrow X$.

Exercise 3 ("games"; 3 credits)**.** We consider the two-person games described by the following payoff functions:

② \ ①	L	R
L	1 / 1	−1 / −1
R	−1 / −1	1 / 1

② \ ①	L	R
L	1 / −1	−1 / 1
R	−1 / 1	1 / −1

How can the left one be interpreted as "evasion on a narrow road" and the right one as "penalty kick"? Do these games have pure Nash equilibria? Justify your answer!

Exercise 4 (Nash equilibria; 3 credits)**.** Let $G = (S_0, \dots, S_n, p_0, \dots, p_n)$ be an $(n+1)$-person game and let $\xi \in S(G)$. Show that ξ is a Nash equilibrium for G if and only if

$$\forall_{j \in \{0,\dots,n\}} \quad p_j(\xi) = \max_{\alpha \in S_j} p_j\bigl(\xi[j:\alpha]\bigr).$$

Bonus problem (social choice checking; 3 credits)**.** Write a program that given $n \in \mathbb{N}$, a finite set A, the set P of total orders on A, and a map $f\colon P^n \longrightarrow P$ checks which of the properties "unanimity", "independence of irrelevant alternatives", "existence of a dictator" are satisfied by f. Document your code and apply your program to interesting examples.

Please turn over

Bonus problem (simplicial star; 3 credits). Construct a simplicial complex X that resembles a "nice" "star" shape; the simplicial complex X should have a simplicial automorphism group with more than 20 elements and $H_1(X)$ should be non-trivial. Illustrate your complex!

Bonus problem (Lefschetz fixed point theorem; 3 credits). Write a poem containing the statement and proof of the Lefschetz fixed point theorem.

Bonus problem (approximate agreement; 3 credits). Look up what the *approximate agreement* task is in distributed computing. Formalise this task via a suitable input complex, output complex, and task map.

Bonus problem (homology of preferences; 3 credits). For the simplicial complex X' in the proof of Arrow's theorem, use a computer program to compute $H_n(X'; \mathbb{F}_2)$ for all $n \in \mathbb{N}$. Document your code/solution!

Submission before January 13, 2023, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 12, 2023.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 11, January 13, 2023

Quick check A (simplicial homology: summary). Recall the construction of simplicial homology, different strategies for the computation of simplicial homology, and applications of simplicial homology.

Quick check B (long-term persistence?). Let $X \subset \mathbb{R}^N$ be a finite set, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}_{>0}$ with $\lim_{n\to\infty} \varepsilon_n = \infty$. What can be said about $b_k^{i,j}(X, d_2, \varepsilon_*)$ for "large" j ?

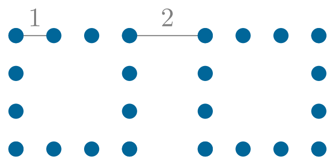
Quick check C (the graded ring $\mathbb{Z}[T]$). Show that not every homogeneous ideal (i.e., generated by homogeneous elements) in $\mathbb{Z}[T]$ is principal. Here, we consider the grading of $\mathbb{Z}[T]$ given by the usual degree of polynomials.

Quick check D (elementary divisors). Determine the elementary divisors of the $\mathbb{Z}$-module $\mathbb{Z}/(2) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(9) \oplus \mathbb{Z}/(81) \oplus \mathbb{Z}/(25) \oplus \mathbb{Z}/(25)$.

Exercise 1 (filtrations from functions; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If X is a finite simplicial complex and $f\colon X \longrightarrow \mathbb{N}$ is a map, then the preimages sequence $(f^{-1}(\{0,\dots,n\}))_{n\in\mathbb{N}}$ are a filtration of X.

Exercise 2 (persistent Betti numbers; 3 credits). Let X be the following subset of $\mathbb{R}^2$:



Let $\varepsilon_* := (0.1, 1.1, 2.1, 100, 101, 102, \dots)$. Compute the persistent Betti numbers $b_1^{i,j}(X, d_2, \varepsilon_*; \mathbb{Q})$ for all $(i,j) \in \{(1,1),(1,2),(1,3)\}$.

Exercise 3 (homogeneous elements; 3 credits). Let R be a graded principal ideal domain. Show one of the following:

1. If $f, g, h \in R$ with $f = g \cdot h$ and $f \neq 0$ is homogeneous, then g and h are homogeneous.
2. If M is a graded R-module and $x \in M$ is homogeneous, then the annihilator $\operatorname{Ann}(x) := \{f \in R \mid f \cdot x = 0\}$ is a principal ideal that is generated by a homogeneous element of R.

Exercise 4 (sensor networks; 3 credits). Give an example of a sensor network that does *not* satisfy the sufficient coverage condition, but such that the network still covers the whole fenced region.

Bonus problem (zigzag persistence; 3 credits). What is zigzag persistence? What is the structure theorem for zigzag persistence and on which theory is it based? As always: Cite all sources!

Submission before January 20, 2023, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 19, 2023.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 12, January 20, 2023

Quick check A (barcodes)**.** Draw a "random" (or not) finite set of points in $\mathbb{R}^2$. What barcodes (over $\mathbb{Q}$) you would expect for a given sequence of radii?

Quick check B (graded modules)**.** Let K be a field and let M be a finitely generated graded torsion $K[T]$-module. What can be said about $(T+1)\cdot M$?

Quick check C (barcodes and disjoint unions)**.** How can the barcode of persistent homology of the disjoint union of two filtered finite simplicial complexes be computed from the individual barcodes?

Exercise 1 (persistent Betti numbers; 3 credits)**.** Let K be a field and let (C^*, f^*) be a persistence K-chain complex of finite type. Let $i, j \in \mathbb{N}$ with $i \leq j$ and let $k \in \mathbb{N}$. Is the following statement true? Justify your answer with a suitable proof or counterexample.

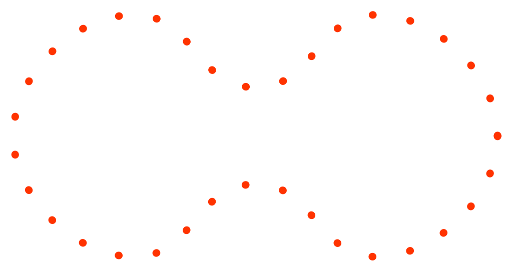
If $b_k^{(i,j)}(C^*, f^*) = 0$, then $b_k^{(i+2022,j)}(C^*, f^*) = 0$.

Exercise 2 (graded structure; 3 credits)**.** Let $\varphi\colon \bigoplus_{j=1}^{6} \Sigma^j \mathbb{F}_2[T] \longrightarrow \bigoplus_{j=1}^{5} \Sigma^j \mathbb{F}_2[T] =: F$ be the homomorphism of graded $\mathbb{F}_2[T]$-modules given by the matrix

$$\begin{pmatrix} 1 & T & T^2 & T^3 & T^4 & 0 \\ 0 & 0 & T & 0 & T^3 & T^4 \\ 0 & 0 & 0 & 0 & T^2 & T^3 \\ 0 & 0 & 0 & 1 & 0 & T^2 \\ 0 & 0 & 0 & 0 & 1 & T \end{pmatrix}.$$

Determine a graded decomposition of $F / \operatorname{im} \varphi$ as in the structure theorem.

Exercise 3 (persistent eight; 3 credits)**.** We consider the following finite subset of $\mathbb{R}^2$ and $\varepsilon_* := (n \cdot \varepsilon')_{n \in \mathbb{N}}$, where ε' is roughly the minimal distance between any of the two points. What do you expect for the barcode of the Rips complexes $R_{\varepsilon_*}(X, d_2)$ (connected by inclusion) with $\mathbb{Q}$-coefficients? Justify your answer!



Exercise 4 (persistent Betti numbers from barcodes; 3 credits)**.** How can the persistent Betti numbers be computed from the barcodes of persistent homology? Formulate a precise statement and provide a proof.

Bonus problem (applications of persistent homology; 3 credits)**.** Find three research papers that describe (different) applications of persistent homology outside of mathematics and appeared after 2017.

Submission before January 27, 2023, 8:30, via GRIPS (in English or German)

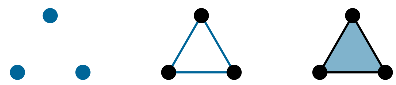
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 26, 2023.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 13, January 27, 2023

Quick check A (homogeneous matrix reduction via rows). Formulate a row-version of the homgeneous matrix reduction. Explain how to read off elementary divisors and graded module decompositions from the result.

Quick check B (persistent homology). Compute the barcode of persistent homology in degree 1 over a field of the following three-step filtration of $\Delta(2)$:



Exercise 1 (empty barcodes; 3 credits). Let K be a field and let (C^*, f^*) be a persistence K-chain complex of finite type with $C_k^n \not\cong 0$ for all $n \in \mathbb{N}$, $k \in \{0, \dots, 2023\}$. Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exists a $k \in \mathbb{N}$ such that the barcode for persistent homology of (C^*, f^*) in degree k is non-empty.

Exercise 2 (persistent homology; 3 credits). Compute the barcode of persistent homology in degree 1 over a field of the following four-step filtration:

Exercise 3 (persistent homology, reverse engineering; 3 credits). Give a filtration of a simplicial complex with four vertices whose persistent homology in degree 1 over $\mathbb{Q}$ has the barcode $(1, 4), (2, 2), (2, 0)$. Justify your answer!

Exercise 4 (a modified homogeneous matrix reduction algorithm; 3 credits). Show that the following algorithm results in a reduced graded matrix. How can one read off a graded decomposition for $\bigoplus_{j=1}^{r} \Sigma^{n_j} K[T]/\operatorname{im} A$ from the resulting matrix ? Justify your answer!
Given a field K, numbers $r, s \in \mathbb{N}$, monotonically increasing sequences $n_1, \dots, n_r$, $m_1, \dots, m_s \in \mathbb{N}$, and an (n_*, m_*)-graded matrix $A \in M_{r \times s}(K[t])$, do:

- For each k from 1 up to s (in ascending order):
  Let $\ell := \operatorname{low}_A(k)$.
  If $\ell \neq 0$, then:
  - For each j from ℓ down to 1 (in descending order):
    If $\operatorname{low}_A(k) = j$ and there exists $k' \in \{1, \dots, k-1\}$ with $\operatorname{low}_A(k') = j$, then: Update the matrix A by subtracting $A_{jk}/A_{jk'}$-times the column k' from column k.
- Return the resulting matrix A.

Bonus problem (persistent homology, implementation; 3 credits). Use a publicly available persistent homology library in a programming language of your choice to solve Exercise 2 over $\mathbb{Q}$ or over $\mathbb{F}_2$. Document your code/solution!

Submission before February 3, 2023, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on February 2, 2023.
This is the last regular exercise sheet.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 14, February 3, 2023

Quick check A (persistent homology: summary). Recall the definition of persistent homology, the definition of barcodes, strategies for the computation, and important properties.

Quick check B (bottleneck distance). Compute the bottleneck distance between the following two weighted barcodes:

$$(0,42),(0,2023),(2023,0) \quad \text{and} \quad (2023,2022)$$

Quick check C (Gromov–Hausdorff distance). Compute the Gromov–Hausdorff distance between the following two subsets of $\mathbb{R}^2$ (with the Euclidean metric):

$$\bigl\{(0,0),(1,0),(1,1),(0,1)\bigr\} \quad \text{and} \quad \bigl\{(0,0),(1.1,0),(1,1),(0,1)\bigr\}$$

Exercise 1 (stability of Betti numbers; 3 credits). Let $X, Y \subset \mathbb{R}^2$ be finite sets (with the Euclidean metric). Is the following statement true? Justify your answer with a suitable proof or counterexample.

$$\text{Then } \bigl|b_1(R_1(X,d_2);\mathbb{Q}) - b_1(R_1(Y,d_2);\mathbb{Q})\bigr| \leq 2 \cdot d_{\mathrm{GH}}\bigl((X,d_2),(Y,d_2)\bigr).$$

Exercise 2 (bottleneck distance, properties; 3 credits). Show that the bottleneck distance defines a pseudo-metric on the set of all weighted barcodes.

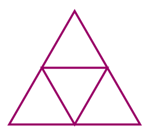
Exercise 3 (Gromov–Hausdorff distance, non-degeneracy; 3 credits). Solve one of the following:

1. Show that the Gromov–Hausdorff distance satisfies the triangle inequality.
2. Let (X,d), (X',d') be finite metric spaces with $d_{\mathrm{GH}}((X,d),(X',d')) = 0$. Show that (X,d) and (X',d') are isometric.

   Hints. This is less obvious than it looks.

Exercise 4 (persistent homology of point clouds; 3 credits). Sketch an implementation plan for the computation of barcodes of persistent homology in degree 1 for point clouds in $\mathbb{R}^2$, taking the following into account: Which input is needed and how could it be represented? How could the output be represented? Which intermediate steps are necessary? How could one solve these intermediate steps?

Bonus problem (persistent triforce; 3 credits). Use a persistent homology library of your choice to generate 42 random points on the lines of



and to compute the barcodes of associated Rips filtrations (with respect to reasonable radii) in degree 1. Document your code and the results! Are the results consistent with your expectations?

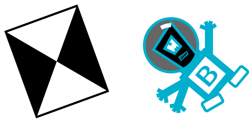
Optional submission before February 10, 2023, 8:30, via GRIPS

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on February 9, 2023.
All credits on this exercise sheet count as bonus credits.

Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 15, February 10, 2023

Commander Blorx currently resides on planet Apalto, recovering from past adventures and structure theorems.



Problem 1 (the hourglass). While reading "Foundation", a note drops from the book into Blorx's hands. The note smells of time travel and clearly says:

Of all the timeless objects,
⧖ is most simplicial and complex.
From the following instructions,
just make homology deductions:

$$\{\emptyset, \{0\}, \dots, \{4\}, \{0,1\} \dots, \{0,4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{0,1,2\}, \{0,3,4\}\}$$

Which of the following homology deductions are valid?

$H_0(\lvert⧖\rvert) \cong_{\mathbb{Z}} \mathbb{Z}^2$	$(1, -1)$
$H_1(\lvert⧖\rvert) \cong_{\mathbb{Z}} \mathbb{Z}^2$	$(3.625, 1.875)$
$H_2(\lvert⧖\rvert) \cong_{\mathbb{Z}} \mathbb{Z}^2$	$(-3, 1.25)$

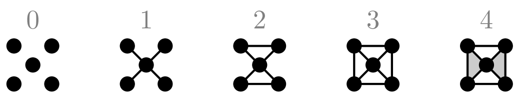
Problem 2 (to move or not to move?). Blorx is tempted to follow up on the note – but also indulges the benefits of laziness. Thus, before taking further steps, he develops a theory of laziness: A simplicial complex X is *lazy* if every continuous map $|X| \longrightarrow |X|$ has a fixed point. Which of the following simplicial complexes are lazy?

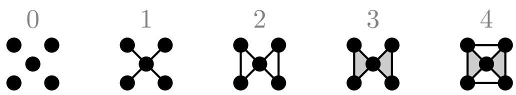
$\Delta(0)$	$(2.3, 0.45)$
$S(0)$	$(4.5, 0)$
$S(1)$	$(-2, 1)$
$\Delta(2023)$	$(6.1.3)$
$\{\emptyset, 0, 1, 2, \{0,1\}, \{0,2\}\}$	$(1.75, 2.25)$

Problem 3 (structure). After completing the theory and practice of laziness, Blorx cannot continue to resist to get hold of the hourglass ⧖. He reads the complete library of Apalto; at least the barcodes. As a service for future readers, he translates the barcodes:

$(2, 0)$	0.4	$\Sigma^0\mathbb{Q}[T]$	3
$(0, \infty)$	3.5	$\Sigma^2\mathbb{Q}[T]/(T^2)$	1
$(2, 1)$	0.6	$\Sigma^2\mathbb{Q}[T]/(T)$	0

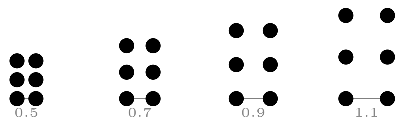
Problem 4 (manufacturing the hourglass). Finally! The book describing how to manufacture the hourglass ⧖ has the barcode $(2, \infty), (2, \infty), (3, 0), (3, 0)$. For physics reasons, only one of the following processes is possible. Which one is it?

$(5.5, 1.5)$ 

$(5, 2.875)$ 

Please turn over

Problem 5 (the Blorx molecule)**.** To power the hourglass, Hallam's electron pump or the Blorx molecule is needed. Since Blorx is fresh out of tungsten, he synthesises the Blorx molecule. For which of the following finite subsets X of $\mathbb{R}^2$ is $|R_1(X, d_2)|$ homotopy equivalent to B ?



$(3.2, 4)$ $(2, 3.33)$ $(6.7, 3)$ $(7, 0)$

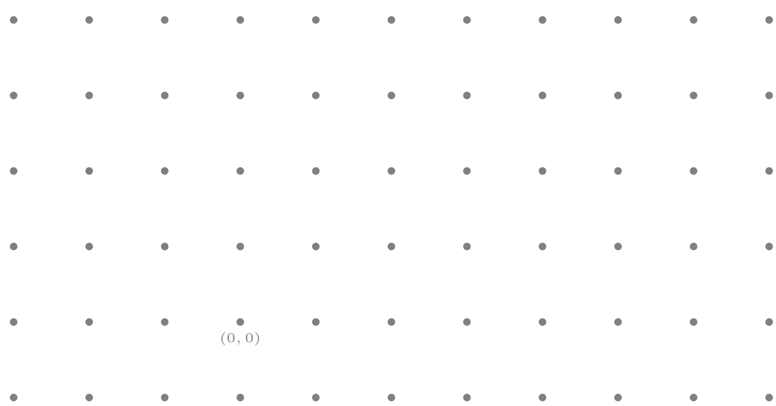
Problem 6 (making the hourglass ready for export)**.** In order to transport the hourglass ⧖, Blorx applies a simplicial map $f\colon ⧖ \longrightarrow ⧖$. What are possible outcomes for $H_1(|f|)$?

$0 \cdot \mathrm{id}_{H_1(⧖)}$ $(-2, 3)$
$1 \cdot \mathrm{id}_{H_1(⧖)}$ $(5, 1)$
$2 \cdot \mathrm{id}_{H_1(⧖)}$ $(4.3, 2.3)$
$-1 \cdot \mathrm{id}_{H_1(⧖)}$ $(-1, 1.5)$

Problem 7 (Blorx evolution)**.** The superior intellectual powers and the remarkable moral compass of Blorx can be explained by his genome BLORX. Which of the following sequences could be common ancestors of BLORX, BLOBR, and ROXOR in tree-only evolution?

BLORB $(-1, -1)$
BLORR $(6.5, -0.2)$
RLORR $(4, 0.8)$
ROORR $(-0.125, 2.625)$
LOLOL $(5, -1.125)$

Problem 8 (escape!)**.** Encouraged by his moral compass, Blorx jumps into the Sea of Tranquility and decides to leave Apalto as soon as possible with the priceless artefact. Connect the dots using brand-new raster image processors of type 2.023 and help Blorx to escape Apalto with a suitable vehicle!



No submission

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General Information

Applied Algebraic Topology: Admin

Prof. Dr. C. Löh/M. Uschold October 2022

Homepage. Information and news concerning the lectures, exercise classes, office hours, literature, as well as the exercise sheets can be found on the course homepage and in GRIPS:

https://loeh.app.ur.de/teaching/aat_ws2223

https://elearning.uni-regensburg.de

Lectures. The lectures are on Tuesdays (8:30–10:00; M104) and on Fridays (8:30–10:00; M104).

Basic lecture notes will be provided, containing an overview of the most important topics of the course. These lecture notes can be found on the course homepage and will be updated after each lecture. Please note that these lectures notes are not meant to replace attending the lectures or the exercise classes!

According to current plans (13.10.2022): This course will be taught on campus in person. On request, this could be turned into a hybrid format (with live zoom streaming). Please note that there will be no recordings of the lectures. The lectures are a precious opportunity for live interaction and I want to keep the atmosphere as casual and unintimidating as possible. For asynchronous self-study, lecture notes will be made available. Please send an email to Clara Löh in case there is a need for the hybrid option!

Exercises. Homework problems will be posted on Fridays (before 8:30) on the course homepage; submission is due one week later (before 8:30, via GRIPS).

Each exercise sheet contains regular exercises (12 credits in total) and more challenging bonus problems (3 credits each).

It is recommended to solve the exercises in small groups; however, solutions need to be written up individually (otherwise, no credits will be awarded). Solutions can be submitted alone or in teams of at most two participants; all participants must be able to present *all* solutions of their team.

The first exercise sheet will appear on Friday, October 21. The exercise classes start in the *second* week.

In addition, the exercise sheets will contain simple problems that will be solved and discussed during the exercise classes. These problems should ideally be easy enough to be solved within a few minutes. Solutions are not to be submitted and will not be graded.

Registration for the exercise classes. Please register for the exercise classes via GRIPS:

https://elearning.uni-regensburg.de

Please register before Wednesday, October 19, 2022, 10:00.

Credits/Exam. This course can be used as specified in the commented list of courses and in the module catalogue.

- *Studienleistung*: Successful participation in the exercise classes: 50% of the credits (of the regular exercises), presentation of solutions in class (twice).
- *Prüfungsleistung*: Oral exam (25 minutes), by individual appointment at the end of the lecture period/during the break.

You will have to register in FlexNow for the Studienleistung and the Prüfungsleistung (if applicable). Registration will open at the end of the lecture period.

Further information on formalities can be found at:

https://www.uni-regensburg.de/mathematik/fakultaet/studium/studierende/index.html

Contact.

- If you have questions regarding the organisation of the exercise classes or the exercises, please contact Matthias Uschold:

  matthias.uschold@ur.de

- If you have mathematical questions regarding the lectures, please contact Matthias Uschold or Clara Löh.
- If you have questions concerning your curriculum or the examination regulations, please contact the student counselling offices or the exam office:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/ansprechpersonen/index.html

- In many cases, also the Fachschaft can help:

https://www-app.uni-regensburg.de/Studentisches/FS_MathePhysik/cmsms/

- Official information of the administration related to the COVID-19 pandemic can be found at:

  https://go.ur.de/corona

Bibliography

Please note that the bibliography will grow during the semester. Thus, also the numbers of the references will change!

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F

Färbung	colouring	102
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Funktor	functor	A.15

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Homotopie	homotopy	6
Homotopieäquivalenz	homotopy equivalence	6
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