03.12.2020

Recap: lengths of piecewise smooth curves:

$$L_g(\underbrace{\gamma}_{[a,b]\to M}) := \int_a^b \|\dot\gamma(t)\|_{g_{\gamma(t)}}\, dt \in \mathbb{R}_{\geq 0}.$$

1.5.2 THE RIEMANNIAN DISTANCE FUNCTION

<u>idea</u>: distance between two points
:= inf over lengths of curves between these pts.

<u>goal</u>: this gives a metric that induces the same topology

Definition. (Riemannian distance function). Let (M,g) be a Riemannian mfd. Then the <u>distance function</u> on M assoc. with g is def'd by

$$d_g : M \times M \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

$$(x,y) \longmapsto \inf\{ L_g(\gamma) \mid \gamma : [a,b] \to M \text{ piecewise regular curve with } \gamma(a) = x \text{ and } \gamma(b) = y,\ a,b \in \mathbb{R},\ a<b \}.$$

$\inf \emptyset := \infty$

Proposition. (isometry invariance of the Riemannian distance). Let $f : (M,g) \to (M',g')$ be a local isometry and let $x,y \in M$.

1. Then $d_{g'}(f(x), f(y)) \leq d_g(x,y)$.
2. If f is an isometry, then
$$d_{g'}(f(x), f(y)) = d_g(x,y).$$

Proof. • Let $\gamma : [a,b] \to M$ be a piecewise regular curve from x to y.

piecewise regular

Claim: $L_{g'}(f \circ \gamma) = L_g(\gamma)$.

Proof: By definition,

$$\boxed{L_{g'}(f\circ\gamma)} = \int_a^b \| \dot{\overline{f\circ\gamma}}(t) \|_{g'_{f(\gamma(t))}} \, dt$$

chain rule

$$= \int_a^b \| d_{\gamma(t)} f (\dot\gamma(t)) \|_{g'_{f(\gamma(t))}} \, dt$$

f local isometry

$$= \int_a^b \| \dot\gamma(t) \|_{g_{\gamma(t)}} \, dt = \boxed{L_g(\gamma)}.$$

• For all piecewise regular curves γ from x to y, we obtain

$$d_{g'}(f(x), f(y)) \leq L_{g'}(f\circ\gamma) \overset{\text{claim}}{=} L_g(\gamma).$$

inf over all such γ

$$d_{g'}(f(x), f(y)) \leq \inf \ldots = d_g(x,y).$$

• If f is an isometry: apply 1. to f and f^{-1}. $\square$

Proposition. (recovering the Euclidean metric). Let $n \in \mathbb{N}$ and let g be the Euclidean Riem. metric on $\mathbb{R}^n$. Then d_g is the Euclidean metric on $\mathbb{R}^n$.

Proof. Let $x, y \in \mathbb{R}^n$. We split the proof:

• We have $d_g(x,y) \le d(x,y)$: Euclidean distance

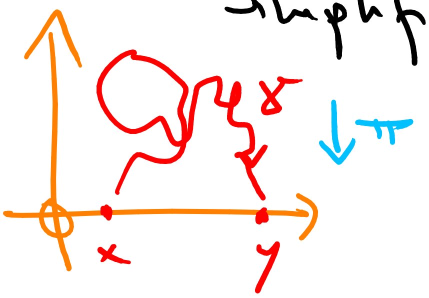
The line segment $\gamma : [0,1] \to \mathbb{R}^n$
$t \mapsto x + t\cdot(y-x)$

shows:

$$\boxed{d_g(x,y)} \le L_g(\gamma) = \|y-x\|_2 = \boxed{d(x,y)}$$

• We have $d_g(x,y) \geq d(x,y)$:

Simplification: Wlog, let x, y be on the first axis.



Let $\gamma : [a,b] \to \mathbb{R}^n$ be a piecewise regular curve from x to y and let $\pi : \mathbb{R}^n \to \mathbb{R}$ be the orthogonal proj. onto the first axis.

Then:

$$\boxed{L_g(\gamma)} = \int_a^b \|\dot\gamma(t)\|_{\cancel{g_{\gamma(t)}}\,2} \, dt$$

π is an orth. projection
$$\geq \int_a^b \|\pi(\dot\gamma(t))\|_2 \, dt$$

π linear + chain rule
$$= \int_a^b |(\pi\circ\gamma)'(t)| \, dt$$

$$\geq \left| \int_a^b (\pi\circ\gamma)'(t) \, dt \right|$$

HDI
$$= |\underbrace{\pi\circ\gamma(b)}_{=y} - \underbrace{\pi\circ\gamma(a)}_{=x}|$$

x and y lie on the first axis
$$= \boxed{\|y-x\|_2}.$$

Taking the inf over all such γ yields $d_g(x,y) \geq \|x-y\|_2$ □

Proposition. (length estimate for other Riem. metrics on Euclidean space). Let $n \in \mathbb{N}$, let $U \subset \mathbb{R}^n$ be an open subset, let g be a Riem. metric on U. Let $K \subset U$ be $\boxed{\text{compact}}$. Then, there ex. $c, C \in \mathbb{R}_{>0}$ with

$$\forall_{x \in K} \ \forall_{v \in T_x U} \quad c \cdot \|v\|_2 \leq \|v\|_{g_x} \leq C \cdot \|v\|_2 .$$

Proof. Compactness argument: let

$$S := \{ v \in T_x U \mid x \in K, \ \|v\|_2 = 1 \}$$
$$\subset TU \cong U \times \mathbb{R}^n$$

$\leadsto$ S is compact.

Moreover, let

$$f \colon S \longrightarrow \mathbb{R}$$
$$T_x U \ni v \longmapsto \|v\|_{g_x} .$$

$\leadsto$ f is continuous, positive everywhere

$\leadsto$ f attains a minimum c and a maximum C on S and $c, C > 0$. $\square$

Now: little computation.

Proposition. (the local Euclidean estimate). Let (M, g) be a Riemannian mfd, let $x \in M$, and let $U \subset M$ be an open nbhd of x. Then: there ex. a smooth chart $\varphi \colon V \to V'$ around x with $V \subset U$ and $C, D \in \mathbb{R}_{>0}$ with:

1. For all $y \in V$, we have

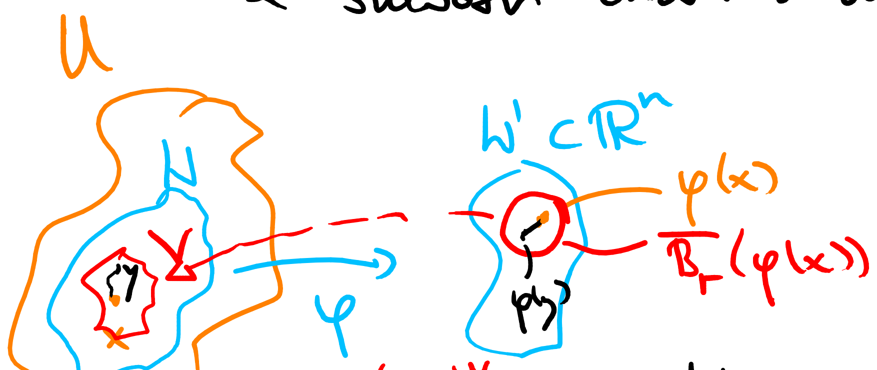
Euclidean distance

$$d_g(x,y) \le C \cdot d(\varphi(x), \varphi(y)).$$

2. For all $y \in M \setminus \overline{V}$, we have

$$d_g(x,y) \ge D.$$

Proof. · Construction of V: Let $\varphi : W \to W'$ be a smooth chart around x with $W \subset U$.



Let $r \in \mathbb{R}_{>0}$ be small enough s.t. $\overline{B}_r(\varphi(x)) \subset W'$.

$(\varphi^{-1})^*(g|_W)$ We set $V := \varphi^{-1}(B_r(\varphi(x)))$.

Then: · $\varphi|_V : V \to V' := B_r(\varphi(x))$ is a smooth chart around x with $V \subset U$.

· $\underbrace{(\varphi^{-1})^*(g|_W)}_{=:\bar{g}}$ is a Riem. metric on W'.

previous prop. $\rightsquigarrow$ Then ex. $c, C \in \mathbb{R}_{>0}$ with

$$\forall_{x \in \overbrace{\overline{B_r(\varphi(x))}}^{\text{compact}}} \ \forall_{v \in T_x W'} \ c \cdot \|v\|_2 \le \|v\|_{\bar{g}_x} \le C \cdot \|v\|_2$$

$\rightsquigarrow$ If γ is a piecewise regular curve in V, then

$$\boxed{*} \quad c \cdot L_{\text{Euclid}}(\varphi \circ \gamma) \le \underbrace{L_g(\gamma)}_{= L_{\bar{g}}(\varphi \circ \gamma)} \le C \cdot L_{\text{Euclid.}}(\varphi \circ \gamma).$$

1. Let $y \in V$. Then

$$d_g(x,y) \le L_g(\gamma)$$

preimage of straight line from $\varphi(x)$ to $\varphi(y)$.

$$\overset{\boxed{*}}{\le} C \cdot L_{Eucl}(\overset{\text{straight line}}{\varphi\circ\gamma})$$

$$= C \cdot d(\varphi(x), \varphi(y)).$$

2. Let $y \in M \setminus \overline{V}$. Let $\gamma : [a,b] \to M$ be a piecewise regular curve from x to y. Let $t_0 := \inf\{t \in [a,b] \mid \gamma(t) \notin V\} \in [a,b]$.

$\rightsquigarrow \varphi\circ\gamma([a,t_0]) \subset \overline{B}_r(\varphi(x))$ and $d(\varphi(x), \varphi(\gamma(t_0))) = r$.

Then:

$$L_g(\gamma) \ge L_g(\gamma|_{[a,t_0]})$$

$$\overset{\boxed{*}}{\ge} c \cdot L_{Eucl}(\varphi\circ\gamma|_{[a,t_0]})$$

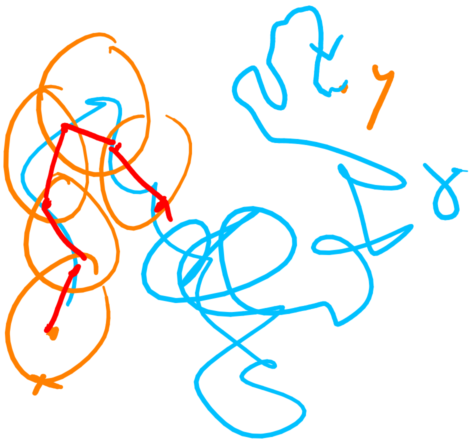
recovering the Euclidean distance

$$\ge c \cdot d(\varphi(x), \varphi(\gamma(t_0))) = \underbrace{c \cdot r}_{=:D}. \quad \square$$

Theorem. (the metric of a Riem. metric). Let (M,g) be a connected Riem. mfd.

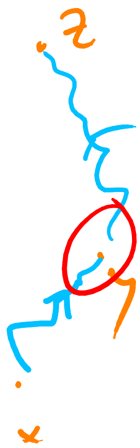
1. Then the Riem. distance function d_g is a metric on M.
2. The topology on M generated by d_g is the original topology on M.

Proof. 1. • For all $x, y \in M$, we have $d_g(x,y) < \infty$:



As M is (path-)connected, there ex. a continuous path γ from x to y.

$\rightsquigarrow$ piecewise regular curve: by gluing pullbacks of line segments in $\mathbb{R}^n$.



• symmetry of d_g: reverse curves

• triangle inequality: concatenate curves (convenient to be in the piecewise setting!)

• positive definiteness: let $x, y \in M$ with $x \neq y$.

Then $d_g(x,y) > 0$ because:

let $U \subset M$ be an open nbhd of x with $y \notin \overline{U}$.

From the previous prop, we obtain an open nbhd $V \subset U$ of x and $D \in \mathbb{R}_{>0}$ with

$$\forall_{z \in M \setminus \overline{V}} \quad d_g(x,z) \geq D.$$

$\rightsquigarrow$ In particular: $d_g(x,y) \geq D > 0$.

$\rightsquigarrow$ d_g is a metric on M.

2. similar arguments. (see lecture notes). $\square$