DIFFERENTIAL GEOMETRY I

CLARA LÖH

GRIPS + http://www.mathematik.uni-r.de/loeh/teaching/diffgeo_ws2021/org.pdf

LECTURES: WED 8:30–10:00, THU 10:15–12:00

- self-study (lecture notes + reading assignments)
- discussion/question sessions (zoom)
- current notes: http://www.mathematik.uni-r.de/loeh/lecture.pdf
- last lecture: 11.02.2021

EXERCISES: MO/TUE 8:00–10:00
(M. LUDEWIG, J. SEIPEL/R. SCHIEßL)

- weekly exercise sheets (50% + ≥1 presentation)
- digital submission/grading (by email)
- discussion/question sessions (zoom)
- registration/distribution: GRIPS!

STUDIENLEISTUNG: via exercises

PRÜFUNGSLEISTUNG: online oral exam (25 min) in Feb/March 2021

CHAT: invite link: GRIPS

DO NOT RECORD ANY OF THE MEETINGS!

0 INTRODUCTION

parametrized multilinear analysis

DIFFERENTIAL GEOMETRY

= study of geometric objects by analytic means

Riemannian mfds : smooth mfd + Riemannian metric

GOALS:
- define and compare intrinsic notions of curvature
- study global effect of curvature constraints

APPLICATIONS:
- provides a robust quantitative geometric language
- framework for computations
- serves as a blueprint for other fields
- real-world applications

- Geometry / Topology
  - Proof of the Poincaré conjecture (see below)
  - Rigidity phenomena of mfds
  - computational tool for cohomological invariants
  - ⋮
- Groups / Algebra
  - Growth of groups: let Γ be a fin gen. group, let $S \subset \Gamma$ be a fin. gen. set. How does $(|S^n|)_{n\in\mathbb{N}}$ grow?

- Rigidity phenomena of groups
- Approximation results for cohomological invariants of groups
- Representation theory of infinite group

⋮

- Physics
  - Modelling mechanical systems
    fluids
    space-time / general relativity

  ⋮

- Real world:
  - Cartography: Is it possible to draw a plane map of a part of the Earth that is length-preserving? Angle-/Area-pres?
  - Medical imaging
  - Simulating non-Euclidean geometry (e.g. in computer games)

  ⋮

challenge: geometric intuition vs. analytic rigor

OVERVIEW OF THE COURSE:

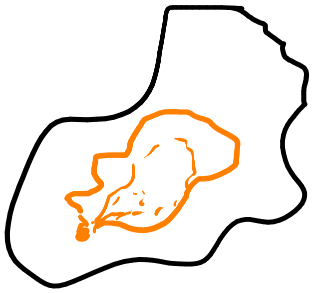
- recall smooth mfds/tangent spaces
- short intro to vector bundles
- introduce Riemannian mfds
- different notions of curvature for connections on vector bundles
- do geometry: - compare different notions of curvature
  - global effect of curvature constraints

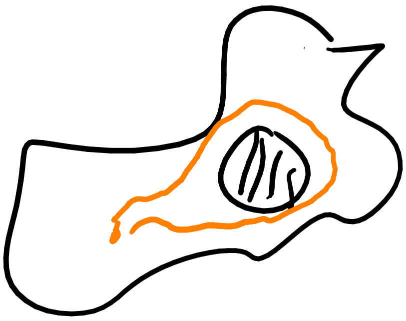
MORE ABOUT THE POINCARÉ CONJECTURE

Theorem (Poincaré Conjecture; Perelman). Every compact 3-mfd (without boundary) that is simply connected is homeomorphic to the 3-sphere S^3.

purely topological statement!

every continuous loop can be continuously deformed into a constant loop





not simply connected

How can differential geometry help?

Let M be compact simply connected 3-mfd.
Then:

- M admits a (n ess. unique) smooth structure
- M admits a Riemannian metric

Let g be a Riemannian metric on M.

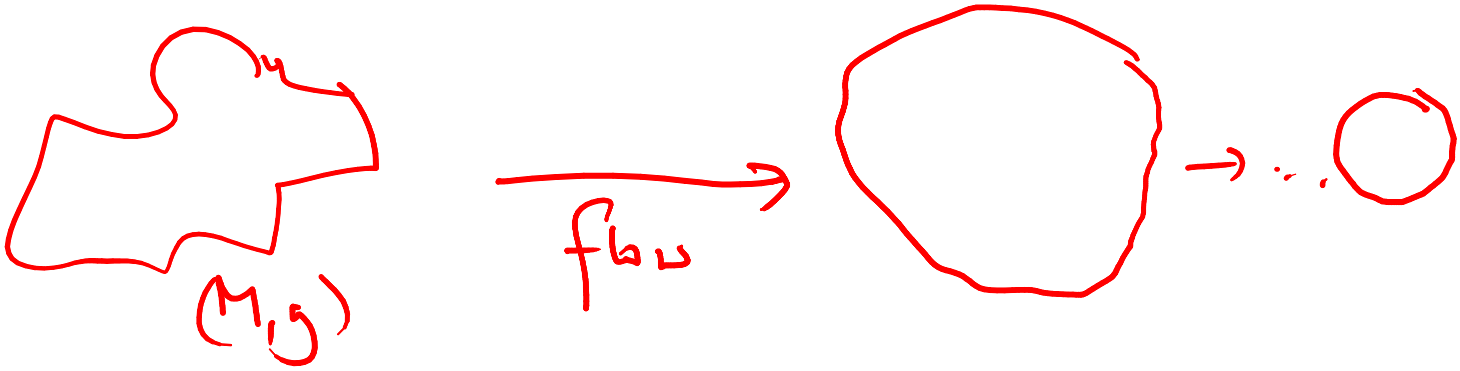
$\rightsquigarrow$ we are now in a geometric situation!

we will prove this

Theorem. Let M be a compact simply connected smooth n-mfd that admits a Riemannian metric of constant sectional curvature. Then M is homeomorphic to S^n.

will define this in the course

Problem: we cannot expect our Riem. metric g to have constant sectional curvature.



Idea [Hamilton] Let the Riem metric g evolve according to Ricci flow:

$$\frac{\partial}{\partial t} g_t = -2 \cdot Ric^{g_t}$$

Ricci curvature will define this

Big technical problem: singularities might occur!

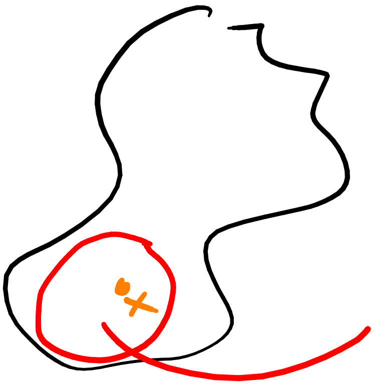
solution by Perelman

1 Riemannian Manifolds

goal: introduce Riemannian mfds and key examples
S^n, $\mathbb{R}^n$, $\mathbb{H}^n$

manifold: a locally Euclidean space

several interpretations
- topologically → topological mfds
- \+ smooth structure → smooth mfds
- \+ inner product → Riemannian mfds



looks Euclidean

introduce vector bundles for this

1.1 Smooth MFDS

1.1.1 Topological MFDS

Definition. (topological mfds). Let $n \in \mathbb{N}$. A <u>topological mfd</u> of dimension n is a topological space M such that:

- For each $x \in M$, there ex. an open nbhd of $x \in M$ that is homeomorphic to an open subset of $\mathbb{R}^n$

technical cond. [• M is Hausdorff and second countable