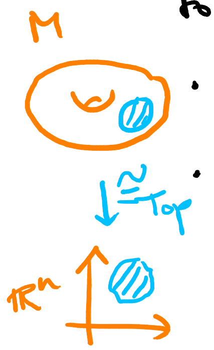
05.11.2020

og: • week2.pdf
• Ü: register!

Recap: Topological mfd of dim n:



top. space M with:
- locally homeomorphic to (open subsets of) $\mathbb{R}^n$
- Hausdorff and second countable

} $\Rightarrow$ para-compact

Remark: For non-empty top. mfds, the dimension is uniquely determined. (use invariants from algebraic topology)

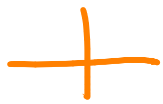
Examples:
- spheres $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$

stereographic proj: $S^n \setminus pt \cong_{Top} \mathbb{R}^n$.

- torus $S^1 \times S^1$;

more generally: cartesian products of fin. many top. mfds
- $\mathbb{R}^n$, all open subsets of $\mathbb{R}^n$
- projective spaces $\mathbb{RP}^n$ ($\rightsquigarrow$ later)
- hyperbolic space $\mathbb{H}^n \cong_{Top} \mathbb{R}^n$

(Ü 1.3?)
- not a top. mfd



1.1.2 SMOOTH MANIFOLDS

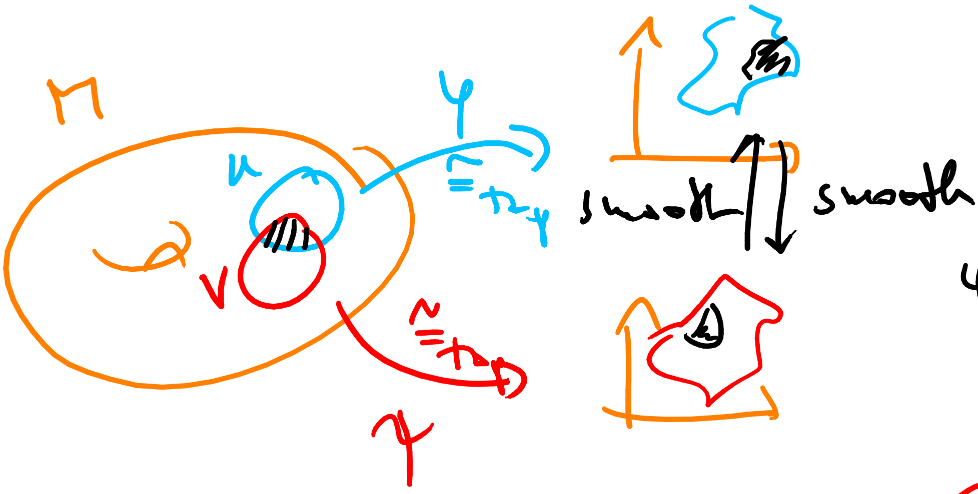
"space" that locally has the same smooth structure as $\mathbb{R}^n$

(idea: use charts)

Definition. (chart) (Karte) Let M be a top. mfd of dim n.
- A chart is a homeomorphism $U \to U'$, where $U \subset M$, $U' \subset \mathbb{R}^n$ are open subsets.

- Two charts $\varphi: U \to U'$, $\psi: V \to V'$ of M are <u>smoothly compatible</u> if

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)} : \varphi(U \cap V) \to \psi(U\cap V)$$



$$\varphi \circ \psi^{-1}|_{\psi(U\cap V)} : \psi(U\cap V) \to \varphi(U \cap V)$$

are smooth maps.

Definition. (atlas). Let M be a top. mfd.

- An <u>atlas</u> of M is a set of charts s.t. the domains cover all of M.
- An atlas A of M is <u>smooth</u> if any two charts in A are smoothly compatible.
- A <u>smooth structure</u> of M is a maximal smooth atlas of M.

Definition. (smooth mfd). A <u>smooth mfd</u> is a pair (M, A), where M is a top. mfd and A is a smooth structure on M.

Examples.
- spheres, tori (⟶ submfds)
- $\mathbb{R}^n$, open subsets U of $\mathbb{R}^n$ (smooth atlas: $\{id_U : U \to U\}$)
- all fin-dim $\mathbb{R}$-vector spaces
- finite products of smooth mfds

1.1.3 THE CATEGORY OF SMOOTH MFDS

- objects: smooth mfds
- morphisms: smooth maps

Definition. (smooth map). A map $f: M \to N$ is <u>smooth</u> if for every $x \in M$ there ex. charts (for all)

$\varphi: U \to U'$ around x and

$\psi: V \to V'$ around $f(x)$

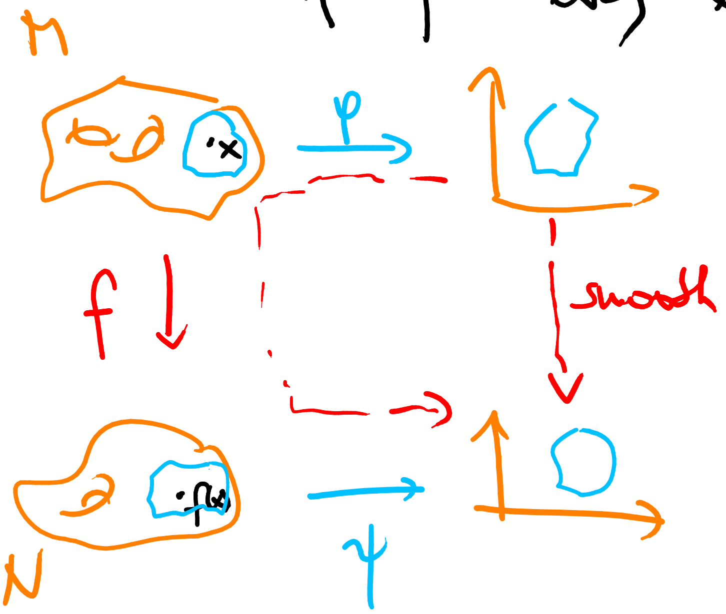
with

$f(U) \subset V$

and

$\psi \circ f \circ \varphi^{-1}: U' \to V'$

is smooth.



Examples:

- on open subsets of Euclidean spaces: the same notion of smoothness as in the classical case.
- "projections $M \times N \to M$" (locally: just a Euclidean projection)
- $id_M : M \to M$ (same smooth structure on both sides!)
- Constant maps are smooth
- composition of smooth maps is smooth

Definition. The category Mfd consists of:

- objects: the class of all smooth mfds
- morphisms: for smooth mfds M, N, we set
$$\mathrm{Mor}_{\mathrm{Mfd}}(M, N) := C^{\infty}(M, N)$$

$C^\infty(M) := C^\infty(M,\mathbb{R})$

$:=$ set of all smooth maps $M \to N$

- compositions: usual composition of maps.

⇝ notion of isomorphism!

Definition. (diffeomorphism). A <u>diffeomorphism</u> $M \to N$ between smooth mfds M, N is an iso in the category Mfd, i.e., a smooth map $f: M \to N$ s.t. there ex. a smooth map $g: N \to M$ with

$$g \circ f = id_M \quad \text{and} \quad f \circ g = id_N .$$

Example. If M is a smooth mfd and $\varphi: U \to U'$ is a smooth chart of M, then φ is a diffeo $U \to U'$

smooth structure induced by M (→ U)

smooth structure induced by $\mathbb{R}^{\dim M}$ (→ U')

1.1.4 TANGENT SPACES

in multivariable analysis: smooth maps (by def) are locally approximable by linear maps

want this also for smooth maps betw. smooth mfds

⇝ need: <u>suitable vector spaces</u> (tangent spaces)

- geometric construction: via curves
- analytic description: via derivations

Definition. (tangent vectors via curves). Let M be a smooth mfd and let $x \in M$.

- Let $C(M;x)$ be the set of all smooth maps $\alpha : I \to M$, where $I \subset \mathbb{R}$ is an open interval with $0 \in I$, with $\alpha(0) = x$.

- $\alpha, \beta \in C(M;x)$ are <u>equivalent</u> if: there ex. a smooth chart $\varphi : U \to U'$ around x s.t.

$$(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0).$$

interval $\to M \xrightarrow{\varphi} U'$
U

- The equivalence classes are called <u>tangent vectors</u> ~~at x~~.

Proposition / def. (tangent space). Let M be a smooth mfd of dim n and let $x \in M$.

1. The above "equivalence" is an equivalence relation on $C(M;x)$. We write

tangent space of M at x $\longrightarrow$ $T_xM := C(M;x)/\sim$ Notation: $\dot{\alpha}(0) := [\alpha]$

2. Let $\varphi : U \to U'$ be a chart around x. Then

$$T_xM \longleftrightarrow \mathbb{R}^n$$
$$[\alpha] \longmapsto (\varphi \circ \alpha)'(0)$$
$$[t \mapsto \varphi^{-1}(t \cdot v + \varphi(x))] \longleftarrow v$$

($\mathbb{R}^n$ has $\mathbb{R}$-linear structure)

are mutually inverse bijections.

3. The $\mathbb{R}$-lin. structure on T_xM inherited from $\mathbb{R}^n$ as in 2. is independent of the chosen chart.

Proof. computations... [Analysis IV, §2.3]. □

Remark. (characterisation via derivations). Let M be a smooth mfd and let $x \in M$.

- Then a <u>derivation</u> at x is an $\mathbb{R}$-lin map $D: C^\infty(M) \to \mathbb{R}$ with

$$\forall_{f,g \in C^\infty(M)} \quad D(f\cdot g) = D(f)\cdot g(x) + f(x)\cdot D(g)$$

- Example: if $v \in T_xM$ ($v = \dot{\alpha}(0)$), then

$$\partial_v : C^\infty(M) \longrightarrow \mathbb{R}$$
$$f \longmapsto (f\circ\alpha)'(0)$$

is a derivation at x.

Then

$$T_xM \longrightarrow \text{set of all derivations at } x$$
$$v \longmapsto \partial_v$$

is an iso of $\mathbb{R}$-vector spaces [AIV, Satz 2.35].

Definition. (differential) Let $f: M \to N$ be a smooth map between smooth mfds, let $x \in M$. Then the

$$d_xf : T_xM \longrightarrow T_{f(x)}N$$
$$\dot{\alpha}(0) \longmapsto (\dot{f\circ\alpha})(0)$$

Examples: • let V be a fin. dim $\mathbb{R}$-vector space, let $x \in V$. Then

$$V \longrightarrow T_x V$$
$$v \longmapsto \dot{\alpha}_v(0), \text{ where } \alpha_v : t \longmapsto t \cdot v + x$$

is an $\mathbb{R}$-lin. iso.

⇝ we will usually use this canonical identification.

If $f : V \to W$ is a linear map between fin-dim $\mathbb{R}$-vector spaces, then

$$d_x f = f$$

• $d_x \, id_M = id_{T_x M}$

• $d_x(g \circ f) = (d_{f(x)} g) \circ (d_x f)$. (chain rule)