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Theorem (fundamental theorem of Riem. geometry). Let (M,g) be a Riem. mfd. Then there ex. a unique lin. connection ∇ on M with:

- ∇ is compatible with g
  $$\nabla_X \langle Y,Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$$
- ∇ is symmetric
  $$\nabla_X Y - \nabla_Y X = [X,Y]$$

We call ∇ the Levi-Civita connection of (M,g).

Proof. ① Koszul formula: let ∇ be a lin. conn. on M that is compatible with g and symm.

Claim: For all $X,Y,Z \in \Gamma(TM)$:

$$\langle \boxed{\nabla_X Y}, Z\rangle_g = \frac{1}{2}\cdot\Big(X(\langle Y,Z\rangle_g) + Y(\langle Z,X\rangle_g) - Z(\langle X,Y\rangle_g)$$
$$\underbrace{- \langle Y,[X,Z]\rangle_g - \langle Z,[Y,X]\rangle_g + \langle X,[Z,Y]\rangle_g\Big)}_{\text{independent of } \nabla}$$

② From ①: we obtain uniqueness! (g is pos. def.)

Proof of the Koszul formula:

$$X(\langle Y,Z\rangle_g) = \nabla_X(\langle Y,Z\rangle_g)$$
$$\overset{\nabla \text{ compatible with } g}{=} \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$$

∇ symmetric $\rightarrow = \boxed{\langle \nabla_X Y, Z\rangle_g} + \langle Y, \nabla_Z X\rangle_g + \langle Y, [X,Z]\rangle_g$

similarly: for YZX and ZXY.

$\rightsquigarrow \boxed{XYZ} + \boxed{YZX} - \boxed{ZXY}$

and solve for $\langle \nabla_X Y, Z\rangle_g$.

③ local coordinate formula: If ∇ is compatible with g and symmetric, then locally we obtain for the conn. coeffs:

$$\forall_{i,j,k} \quad \Gamma_{ij}^k = \frac{1}{2} \cdot \sum_{l=1}^n g^{kl} \cdot \left(E_i(g_{jl}) + E_j(g_{il}) - E_l(g_{ij})\right)$$

Christoffel symbols

Proof: We have

$$\boxed{\sum_{k=1}^n \Gamma_{ij}^k \cdot g_{kl}} = \sum_{k=1}^n \Gamma_{ij}^k \cdot \langle E_k, E_l\rangle_g$$

Def of Γ $\rightarrow$ $= \langle \nabla_{E_i} E_j, E_l\rangle_g$

① $\rightarrow = \frac{1}{2} \cdot \left(E_i(\langle E_j, E_l\rangle_g) + E_j(\langle E_l, E_i\rangle_g) - E_l(\langle E_i, E_j\rangle_g)\right.$

$+$ terms involving $\underbrace{\text{Lie bracket of } E_i \ldots}_{}$

$= 0$ because $(E_i)_i$ are a coord frame

$$= \frac{1}{2} \left(E_i(g_{jl}) + E_j(g_{li}) - E_l(g_{ij})\right).$$

Now: apply the inverse matrix of $(g_{k\ell})_{k,\ell}$.

④ Locally, using ③, we can construct lin. connections that satisfy ③. These local lin. conn. are:

- symmetric
- compatible with g.

(can be checked via the local characterisation Prop. 2.3.12, 2.3.8)

⑤ Proof of existence: The local constructions of ④ are compatible by uniqueness.

$\rightsquigarrow$ can be glued to show global existence.

$\square$

Examples (Levi-Civita connections).

- The Euclidean lin. connection on $\mathbb{R}^n$ (wrt Euclidean Riem. metric on $\mathbb{R}^n$).
- If $M \subset \mathbb{R}^N$ is a smooth submfd, then the submfd connection on M is the LC-conn. of M (wrt. Riem. metric on M induced from the Euclidean Riem. metric on $\mathbb{R}^N$).

$\rightsquigarrow$ works for round spheres

Proposition. (Naturality of the Levi-Civita connection). Let (M_1,g_1), (M_2,g_2) be Riem. mfds. and let $\varphi : (M_1,g_1) \longrightarrow (M_2,g_2)$ be an isometry. Then the LC-connections ∇^1, ∇^2 of M_1, M_2, resp., satisfy:

$$\varphi^* \nabla^2 = \nabla^1.$$

Proof. In view of uniqueness of LC-conn, it suffices to show that $\varphi^* \nabla^2$ is symmetric and compatible with g_1. (straightforward computation). $\square$

2.4 CURVATURE TENSORS

idea: we assign curvatures (ℝ-valued) to 2-dim subthingies of a Riem. mfd.
planes in tangent spaces

geometrically:

$v, w \in T_xM \rightsquigarrow$

- infinitesimal rectangles around x, spanned by v and w
- look at the difference between parallel transport around these rectangles and id_{T_xM}.
- measure this difference

hard to work with

$\rightsquigarrow$ more efficient to organise this via LC-connection into a tensor field.

$\rightsquigarrow$ Riemannian curvature tensor (has many arguments and redundancies)

$\rightsquigarrow$ reductions: · sectional curvature

· Ricci curvature

· scalar curvature

2.4.1 THE RIEMANNIAN CURVATURE TENSOR

Proposition & Definition. (the Riem. curvature (3,1)-tensor).

Let (M,g) be a Riem. manifold with LC-conn ∇. Then

$$R : \Gamma(TM)^{\times 3} \longrightarrow \Gamma(TM)$$

$$(X,Y,Z) \longmapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

depends on g!

is $C^\infty(M)$-multilinear.

The assoc. (3,1)-tensor field on M is the Riem. curvature (3,1)-tensor of (M,g).

Proof. computation. $\square$

Example (Euclidean space). Let $n \in \mathbb{N}$ and let $\overline{\nabla}$ be the Euclid. lin. conn. on $\mathbb{R}^n$ (this is the LC-connection for $\mathbb{R}^n$ wrt. Euclidean Riem. metric).

Let $X, Y, Z \in \Gamma(T\mathbb{R}^n)$ and let $(E_i)_i$ be the standard coord. frame and we can write

$$Z = \sum_{i=1}^{n} Z^i \cdot E_i \quad \text{with } Z^i \in C^\infty(\mathbb{R}^n).$$

Then:

$$R(X,Y,Z) = \overline{\nabla}_X \underbrace{\overline{\nabla}_Y Z}_{= \sum_{i=1}^{n} Y(Z^i) \cdot E_i} - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$$

$$= \overline{\nabla}_X \left(\sum_{i=1}^{n} Y(Z^i) \cdot E_i \right) - \overline{\nabla}_Y \left(\sum_{i=1}^{n} X(Z^i) \cdot E_i \right) - \overline{\nabla}_{[X,Y]} Z$$

$$= \sum_{i=1}^{n} \left(\underbrace{X(Y(Z^i))} \cdot E_i - \underbrace{Y(X(Z^i))} \cdot E_i \right) - \sum_{i=1}^{n} \underbrace{[X,Y](Z^i)} \cdot E_i$$

$$= 0.$$