09.12.2020

Recap: (M,g) Riem mfd.

$\rightsquigarrow$ metric on M: Riemannian distance function

⚠ LECTURE ON DEC 10: START: 10^{45}

1.5.3 VOLUME AND ORIENTATION

idea: (M,g) Riem mfd $\rightsquigarrow$ $\mathrm{vol}(M,g)$:

- locally: use charts and usual multivariable integration (measured via Riem. metric)
- globally: sum up via partition of unity.

Remark. (integration on Riem mfds). Let (M,g) be a Riem mfd, let $(\varphi_i : U_i \to U_i')_{i\in I}$ be a countable smooth atlas for M, let $(\eta_i)_{i\in I}$ be a partition of unity that is subordinate to $(\varphi_i)_{i\in I}$.

For $i \in I$, let $(g^i_{jk})_{j,k\in\{1,\dots,n\}}$, $n = \dim M$, be the local description of g w.r.t. φ_i.

- A map $f: M \to \mathbb{R}$ is measurable if f is measurable wrt the Borel σ-algebra on $\mathbb{R}$ and M.

(equivalently: for each $i \in I$, the map $f \circ \varphi_i^{-1} : \underbrace{U_i'}_{\subset \mathbb{R}^n} \to \mathbb{R}$ is measurable)

- A measurable map $f: M \to \mathbb{R}$ is <u>integrable</u> if

$$\sum_{i\in I} \int_{U_i'} \psi_i(x)\, |f\circ\varphi_i^{-1}(x)| \sqrt{\det\big(\underbrace{g^i\circ\varphi_i^{-1}(x)}_{\in M_{n\times n}(\mathbb{R})}\big)}\, dx$$

is finite.

- For an integrable map $f: M \to \mathbb{R}$, we set

$$\int_M f \, d\mu_{M,g} := \sum_{i\in I} \int_{U_i'} \psi_i \cdot f\circ\varphi_i^{-1} \sqrt{\det g^i \circ\varphi_i^{-1}}\, dx^n$$

integration of char. functions $\rightsquigarrow$ measure $\mu_{M,g}$ on M.

Definition. (volume) Let (M,g) be a Riem. mfd. The <u>volume</u> of (M,g) is

$$\operatorname{vol}(M,g) := \begin{cases} \int_M 1 \, d\mu_{M,g} & \text{if } 1: M\to\mathbb{R} \text{ is integrable} \\ \infty & \text{otherwise.} \end{cases}$$

Example. Let $n\in\mathbb{N}_{\geq 1}$ and $R\in\mathbb{R}_{>0}$. Then

$$\operatorname{vol}(S^n(R), \text{round metric of radius } R) = \frac{2\cdot\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2}+1)}\cdot R^n$$

$$\operatorname{vol}\big(B_R^{\mathbb{R}^n,\text{Euclid.}}(0), \text{Euclid. Riem. metric}\big) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\cdot R^n$$

Questions: Let $R, R_1, R_2 \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_{\geq 1}$.

① When are $\mathbb{S}^n(R_1)$ and $\mathbb{S}^n(R_2)$ isometric? (locally isometric?)

② When are $\mathbb{R}^n$ and $\mathbb{S}^n(R)$ isometric?

③ When are $\mathbb{R}^n$ and $\mathbb{S}^n(R)$ locally isometric? Real world problem: maps!

④ When are $\mathbb{H}^n(R_1)$ and $\mathbb{H}^n(R_2)$ isometric? (locally isometric?)

⑤ When are $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ isometric?

⑥ When are $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ locally isometric?

Some answers:

①: Only if $R_1 = R_2$, because: volume of Riemannian mfds is invariant under Riemannian isometries! (transformation formula)

②: Never: $\mathbb{S}^n(R)$ is compact but $\mathbb{R}^n$ is not. Riem. isometries are homeomorphisms!

⑤,⑥: Never: Problem: $\mathbb{R}^n$ and $\mathbb{H}^n(R)$ have infinite volume!

$\rightsquigarrow$ idea: compare volumes of metric balls.

Definition (volume growth). Let (M,g) be a Riem. mfd and let $x \in M$. The <u>volume growth function</u> of M at x is

$$\varrho_x^{(M,g)} : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$
$$r \longmapsto \mu_{M_g}\left(B_r^{M,d_g}(x)\right).$$

Proposition (isometry invariance of volume growth). Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map and let $x \in M_1$. Then:

1. If f is an isometry, then $\varrho_x^{(M_1,g_1)} = \varrho_{f(x)}^{(M_2,g_2)}$.
2. If f is a local isometry, then there ex. an $\varepsilon \in \mathbb{R}_{>0}$ s.t.
$$\forall_{r \in [0,\varepsilon)} \quad \varrho_x^{(M_1,g_1)}(r) = \varrho_{f(x)}^{(M_2,g_2)}(r).$$

Proof idea:

1. isometries map balls to balls (preserving the radius) and preserve the volume.
2. local isometries map balls of small enough radius to balls (preserving the radius). Then reduce to 1. $\Box$

Back to ⑤/⑥: $\mathbb{R}^n$ vs $\mathbb{H}^n(\mathbb{R})$?!

- For each $x \in \mathbb{R}^n$, the function $\varsigma_x^{\mathbb{R}^n, \text{Euclid.}}$ is polynomial of degree n.
- There ex. $c_{n,R}, a_{n,R} \in \mathbb{R}_{>0}$ s.t.:

$$\forall_{x \in \mathbb{H}^n(\mathbb{R})} \ \forall_{r \in \mathbb{R}_{\geq 0}}$$

(h) 6.2
$$\varsigma_x^{\mathbb{H}^n(\mathbb{R})}(r) \geq c_{n,R} \cdot r^{n-1} \cdot a_{n,R}^{\,r}.$$

⇝ $\mathbb{R}^n$ and $\mathbb{H}^n(\mathbb{R})$ are <u>not</u> (locally) isometric.

Remark. The Riem. volume can also be expressed in terms of differential forms (on orientable mfds).
If (M,g) is ~~a~~ an oriented Riem. mfd, then

$$\operatorname{vol}(M,g) = \int_M \underbrace{\operatorname{Vol}_{M,g}}_{\text{volume form of } (M,g)}.$$

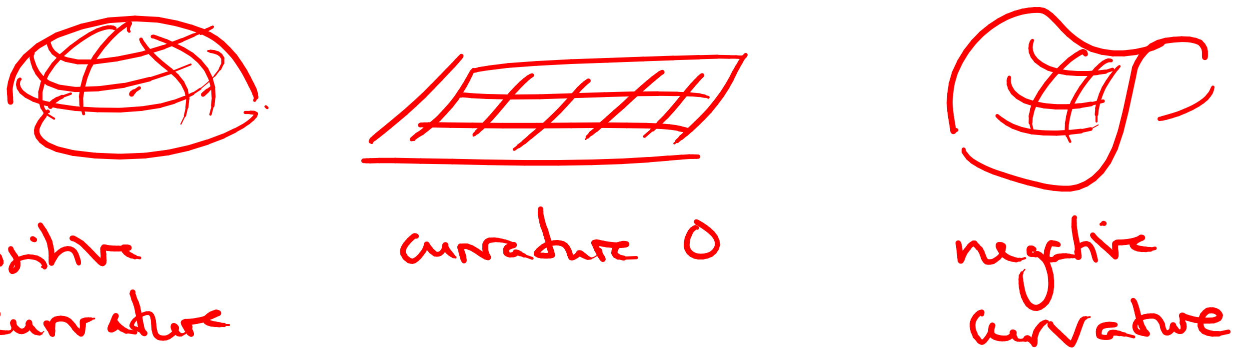
= locally: pull-back of the Euclidean volume form via orientation-preserving charts.

2 CURVATURE: FOUNDATIONS

Goal: give an intrinsic (!) def. of curvature of Riem. mfds.

2.1 THE IDEA OF CURVATURE

Idea: Curvature quantifies how much a geometric object "bends" in comparison with "flat" Euclidean space.
Intuitively, this should result in:



Possible approaches:

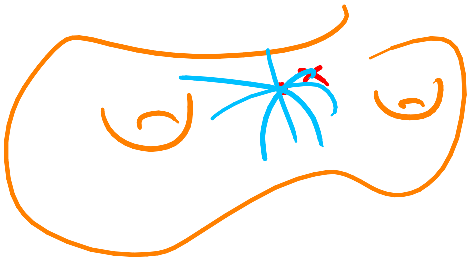
→ ① Via curvature of curves

② Via angle sums in geodesic triangles

③ Via local and global properties of volume growth functions.

We follow ①. Assuming we have a notion of curvature of curves, how to define curvature of high-dim mfds ②?

- curvature of Riem. surfaces (M,g):

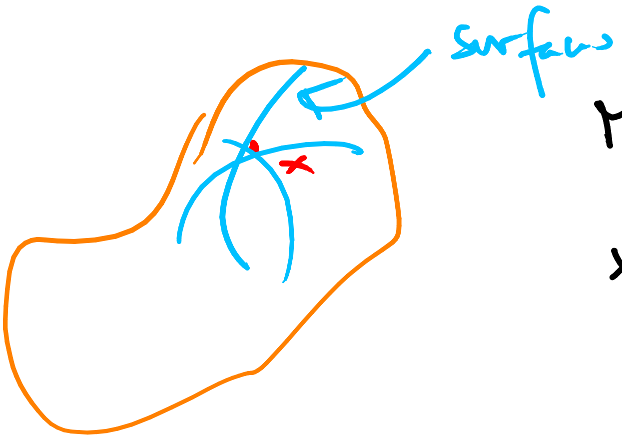


$$M \longrightarrow \mathbb{R}$$

$$x \longmapsto \text{(minimal curvature of a curve through } x\text{)}$$

- (maximal curvature)

- curvature of high-dim Riem. mfd:



$$M \longrightarrow \left(\begin{matrix}\text{tangent} \\ \text{2-plane} \\ \text{at } x\end{matrix} \longrightarrow \mathbb{R}\right)$$

$$x \longmapsto \left(\text{“}\Sigma\text{”} \longmapsto \begin{matrix}\text{curvature} \\ \text{as above} \\ \text{at } x\end{matrix}\right)$$

"Σ" corresp. to the tangent plane

Thus: How can we define "curvature of curves" ③?

Use "acceleration", i.e., second derivatives!

⇝ How to define second derivatives?

Need: a way to differentiate vector fields (sections of vector bundles)

Two naive approaches:

1. Second derivatives in local coordinates

   $\rightsquigarrow$ not independent of the charts! (see example in the lecture notes).

   $\rightsquigarrow$ need correction terms

2. Higher tangent bundles:

   $\gamma : I \to M$ smooth curve

   $\rightsquigarrow d\gamma : TI \longrightarrow TM$ $\qquad \rightsquigarrow \dot{\gamma} : I \to TM$

   $\rightsquigarrow d\dot{\gamma} : TI \longrightarrow \underbrace{T(TM)}$

   but: we want something in TM!

   $\rightsquigarrow$ need to extract the correct part.

In the presence of a Riem. metric, there is a canonical solution, using connections on smooth vector bundles.