10.12.2020

Recap: goal: derivatives of sections of vector bundles

2.2 CONNECTIONS

idea: "directional derivatives"

2.2.1 CONNECTIONS

Definition (connection). Let $\pi: E \longrightarrow M$ be a smooth vector bundle over a smooth mfd M.

A <u>connection</u> on $\cancel{\pi}$ E is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

with the following properties:

(FL1) $C^\infty(M)$-linearity in the first argument:

$$\forall_{X_1, X_2 \in \Gamma(TM)} \forall_{Y \in \Gamma(E)} \forall_{f_1, f_2 \in C^\infty(M)}$$

$$\nabla_{f_1 \cdot X_1 + f_2 \cdot X_2} Y = f_1 \cdot \nabla_{X_1} Y + f_2 \cdot \nabla_{X_2} Y.$$

(L2) $\mathbb{R}$-linearity in the second argument:

$$\forall_{X \in \Gamma(TM)} \forall_{Y_1, Y_2 \in \Gamma(E)} \forall_{\lambda_1, \lambda_2 \in \mathbb{R}}$$

$$\nabla_X (\lambda_1 \cdot Y_1 + \lambda_2 \cdot Y_2) = \lambda_1 \cdot \nabla_X Y_1 + \lambda_2 \cdot \nabla_X Y_2.$$

(F2) Product rule in the second argument:

$$\forall_{X \in \Gamma(TM)} \forall_{Y \in \Gamma(E)} \forall_{f \in C^\infty(M)}$$

$$\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y.$$

- $\nabla_X Y$: covariant derivative of Y in the direction X
- a connection on TM is a <u>linear connection</u>.

Example. (Euclidean linear connection). Let $n \in \mathbb{N}$. The <u>Euclidean linear connection</u> is defined by

$$\overline{\nabla} : \Gamma(T\mathbb{R}^n) \times \Gamma(T\mathbb{R}^n) \longrightarrow \Gamma(T\mathbb{R}^n)$$

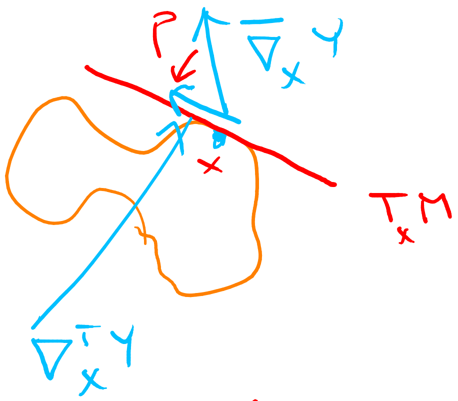
$$(X, Y) \longmapsto \sum_{j=1}^{n} X(Y^j) \cdot E_j$$

$E_1, \dots, E_n$: sections of $T\mathbb{R}^n$ given by the standard basis

$$Y = \sum_{j=1}^{n} Y^j \cdot E_j$$

Example. (induced connection on submfds). Let $N \in \mathbb{N}$, let $M \subset \mathbb{R}^N$ be a smooth submfd. Then the <u>induced connection</u> is def'd by

$$\nabla^T : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM) \quad \tilde{\subset}\, T\mathbb{R}^n$$



$$(X, Y) \longmapsto p\left(\overline{\nabla}_{\tilde{X}} \tilde{Y}\right)$$

$M \to TM$

p: orth. proj. $T\mathbb{R}^N \to TM$

$\tilde{X}, \tilde{Y}$: extensions of X, Y $\in \Gamma(T\mathbb{R}^n)$

(is a well-def. linear connection on M (Ü 6.3))

Proposition (pullback of connections). Let $\varphi : M \to N$ be a diffeomorphism of smooth mfds, let ∇ be linear connection on N. Then

$$\varphi^*\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$(X, Y) \longmapsto d\varphi^{-1} \circ \nabla_{\varphi_* X}\, \varphi_* Y$$

is a linear connection on M.

Proof. Ü 6.3. $\square$

$$N \to TN$$
$$y \mapsto d_{\varphi^{-1}(y)}\varphi\big(\underbrace{X(\varphi^{-1}(y))}_{\in TM}\big)$$

Proposition. ("convex" combinations of connections). Let M be a smooth mfd, let $E \to M$ be a smooth vector bundle, let ∇^1, ∇^2 be connections on E, let $f_1, f_2 \in C^\infty(M)$ with $f_1 + f_2 = 1$. Then

$$\underbrace{f_1 \cdot \nabla^1 + f_2 \cdot \nabla^2}_{=:\nabla} : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$
$$(X, Y) \mapsto f_1 \cdot \nabla^1_X Y + f_2 \cdot \nabla^2_X Y$$

is a connection on E.

Proof. computation; For the product rule: let $X \in \Gamma(TM)$, let $Y \in \Gamma(E)$, let $f \in C^\infty(M)$. Then

$$\nabla_X(f \cdot Y) = f_1 \cdot \nabla^1_X(f \cdot Y) + f_2 \cdot \nabla^2_X(f \cdot Y)$$

(F2) for ∇^1, ∇^2

$$= f_1 \cdot \big(X(f) \cdot Y + f \cdot \nabla^1_X Y\big) + f_2 \cdot \big(X(f) \cdot Y + f \cdot \nabla^2_X Y\big)$$
$$= \underbrace{(f_1 + f_2)}_{=1} \cdot X(f) \cdot Y + f \cdot \nabla_X Y$$
$$= X(f) \cdot Y + f \cdot \nabla_X Y.$$

$\square$

Theorem. (existence of linear connections). Every smooth mfd admits a linear connection.

Proof. • local: pullback along charts of the Euclidean case
• global: glueing via partition of unity. $\square$

2.2.2 LOCAL DESCRIPTIONS OF CONNECTIONS

goal: show that connections are local operators, and find local descriptions

Proposition. (locality of connections). Let M be a smooth mfd, let $E \to M$ be a smooth vector bundle, let ∇ be a connection on E, and let $x \in M$.

Then $\big(\underset{\in \Gamma(TM)}{\nabla_X} \overset{\in \Gamma(E)}{Y}\big)(x)$ only

asymmetric! • depends on Y in a nbhd of x
• depends on $X(x)$.

Proof. uses: bump functions around x. "$\square$"

Proposition. (restrictions of connections). Let M be a smooth mfd, let $\pi : E \to M$ be a smooth vector bundle, let $U \subset M$ be open, and let ∇ be a connection on E (over M).

Then: There ex. a unique connection ∇^U on $\pi| : \pi^{-1}(U) \to U$ with

$$\forall_{X \in \Gamma(TM)} \forall_{Y \in \Gamma(E)} \quad \nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U.$$

Proof. (ü) 6.3. $\square$

Remark. (connection coefficients). (Christoffel symbols)

Let M be a smooth mfd of dim n, let ∇ be a linear connection on M.

Let $\varphi: U \to U' \subset \mathbb{R}^n$ be a smooth chart for M, let $(E_1, \dots, E_n)$ be the corresponding frame for TM over U.

We define the <u>connections coefficients</u> $(\underbrace{\Gamma_{ij}^{k}}_{\in C^\infty(U)})_{i,j,k \in \{1,\dots,n\}}$ via

$$\forall_{\substack{i,j \\ \in\{1,\dots,n\}}} \quad \nabla^U_{E_i} E_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} \cdot E_k .$$

Then we get the following description of ∇^U:

For all $X, Y \in \Gamma(TU)$, we have local reps $X = \sum_{i=1}^{n} \underbrace{X^i}_{\in C^\infty(U)} \cdot E_i$ and $Y = \sum_{i=1}^{n} \underbrace{Y^i}_{\in C^\infty(U)} \cdot E_i$.

$$\leadsto \nabla^U_X Y \overset{(F1)}{=} \sum_{i=1}^{n} X^i \cdot \nabla^U_{E_i} Y$$

$$\overset{(F2)}{=} \sum_{i=1}^{n} X^i \left(\sum_{j=1}^{n} E_i(Y^j) \cdot E_j + Y^j \cdot \nabla^U_{E_i} E_j \right)$$

$$= \sum_{j=1}^{n} X(Y^j) \cdot E_j + \sum_{i,j,k=1}^{n} X^i \cdot Y^j \cdot \Gamma_{ij}^{k} \cdot E_k$$

Example: connection coeff. of $\overline{\nabla}$ in standard coords. are 0