Recap: sec ≤ 0 : Cartan-Hadamard thm + applications

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4.5 POSITIVE CURVATURE CASE

geometric idea: if the curvature is uniformly positive, then the space has to bend so much that it is "small".

Theorem (Bonnet-Myers thm) Let (M,g) be a complete, connected, non-empty Riem mfd and let $R \in \mathbb{R}_{>0}$ with

$$\forall_{x\in M}\ \forall_{v\in T_xM}\ \boxed{\mathrm{Ric}_x}(v,v) \geq \boxed{\frac{\overset{:=\dim M}{n}-1}{R^2}} \cdot \|v\|_g^2.$$

1. Then the $\underbrace{\text{diameter}}_{\text{wrt } d_g}$ of M is at most $\boxed{\pi \cdot R}$.

2. In particular: M is compact.

[3. Moreover: $\pi_1(M)$ is finite.]

↳ Example: $S^1 \times S^1$ does not admit a Riem. metric of pos. Ricci/sectional curvature. ($\pi_1(S^1\times S^1) \cong \mathbb{Z}^2 \ldots$)

Similarly: $(S^1)^{\times n}$ with $n \geq 2$.

Proof. 1. Assume for a contradiction that the diameter of M is $> \pi \cdot R$.

$\leadsto$ Then there ex. $x, y \in M$ with

$$L := d_g(x,y) > \pi \cdot R.$$

completeness $\leadsto$ there ex. a unit speed minimizing geodesic

Prop. 3.3.5

$$\gamma : [0, L] \longrightarrow M \text{ from } x \text{ to } y.$$

index crit. $\leadsto$ $\forall_{\text{proper normal vector fields } V \text{ along } \gamma}$ $I_\gamma(V,V) \geq 0$.

idea: find V with $I_\gamma(V,V) < 0$.

Let $(E_1, \ldots, E_n)$ be a parallel orthonormal frame along γ with $E_1 = \dot\gamma$. For $j \in \{2, \ldots, n\}$, we consider

$$V_j : [0, L] \longrightarrow TM$$
$$t \longmapsto \sin\left(\frac{\pi \cdot t}{L}\right) \cdot E_j(t).$$

$\leadsto$ By computation:

$$\underbrace{\sum_{j=2}^{n} I_\gamma(V_j, V_j)}_{} \overset{\downarrow}{=} \sum_{j=2}^{n} \int_0^L \sin^2\left(\frac{\pi}{L} \cdot t\right) \cdot \left(\frac{\pi^2}{L^2} - Rm_{\gamma(t)}(E_j(t), \dot\gamma(t), \dot\gamma(t), E_j(t))\right) dt$$

$= sec_{\gamma(t)}(E_j(t), \dot\gamma(t))$

$$= \int_0^L \sin^2\left(\frac{\pi \cdot t}{L}\right) \cdot \left(\frac{(n-1)\cdot\pi^2}{L^2} - \underbrace{\mathrm{Ric}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}_{\geq \frac{n-1}{R^2}\cdot \cancel{\|\dot\gamma(t)\|_g^2}}\right) dt$$

Prop. 2.4.28

$$\leq \int_0^L \sin^2\left(\frac{\pi \cdot t}{L}\right) \cdot \left(\underbrace{\frac{(n-1)\cdot\pi^2}{L^2} - \frac{n-1}{R^2}}_{<0}\right) dt$$

(because $L > \pi \cdot R$)

$\boxed{< 0}$

$\rightsquigarrow$ there ex. a $j \in \{2,..,n\}$ with
$I_\gamma(V_j, V_j) < 0$ ↯.

2. Let $x \in M$. As M is complete: $\mathrm{Exp}_x = T_x M$.

1. + conj-pt thm $\rightsquigarrow$ + Prop. 3.3.5

$$\underbrace{\exp_x}_{\text{cont.}}\Big(\underbrace{\overline{B}^{T_xM}_{\pi\cdot R}(0)}_{\text{compact}}\Big) = M. \Rightarrow \text{compact.}$$

[3. Let $\pi : (\tilde M, \tilde g) \to (M, g)$ be the Riem. universal covering of (M,g) (i.e., $\pi^* g = \tilde g$).

$\rightsquigarrow$ " $\mathrm{Ric}^{\tilde M, \tilde g} \geq \frac{n-1}{R^2}$ " and $(\tilde M, \tilde g)$ is complete

$\overset{2.}{\rightsquigarrow}$ $\tilde M$ is compact

$\rightsquigarrow$ π has only finitely many sheets

$\rightsquigarrow$ $\boxed{|\pi_1(M)|} = \#$ sheets of π $\boxed{< \infty}$. $\square$

Theorem. (constant positive sectional curvature).
Let (M,g) be a complete, simply connected, non-empty Riem. mfd of dim n and let (M,g) have constant sectional curvature $c \in \mathbb{R}_{>0}$. Then (M,g) is isometric to $\underbrace{S^n(R)}_{\text{round sphere of radius } R}$ with $R = \frac{1}{\sqrt{c}}$.

Proof. idea: use normal nbhds (we will need two instead of one!) to construct an isometry $S^n(R) \longrightarrow (M,g)$.

Let $N, S \in S^n(R)$ be the north/south pole. Let $U := S^n(R) \setminus \{S\}$. Then U is a normal nbhd of N.

$\rightsquigarrow$ $\exp_N : B^{T_N S^n(R)}_{\pi \cdot R}(0) \longrightarrow U$ is a diffeo.

$$
\begin{tikzcd}
B^{T_N S^n(R)}_{\pi\cdot R}(0) \arrow[r, dashed, "f"', "\text{isometry}"'] \arrow[d, "\exp_N"', "\text{diffeo}"] & B^{T_xM}_{\pi\cdot R}(0) \arrow[d, "\text{local diffeo}"', "\exp_x"] \\
U \arrow[r, dashed, "F"'] & M
\end{tikzcd}
$$

$(\exp_x \circ f)^* g \qquad \exp_N^* g_R \qquad g_R \qquad g$

Let $x \in M$. Then $\exp_x : B^{T_xM}_{\pi R}(0) \to M$ is a local diffeo (by the conj. pt. comp. thm and Prop. 1.5.14).

Let $f : T_N S^n(R) \to T_xM$ be a linear isometry (w.r.t. $g_{R,N}$ and g_x)

Then: $\exp_N^* g_R$ and $(\exp_x \circ f)^* g$ are Riem. metrics on $B^{T_N S^n(R)}_{\pi \cdot R}(0)$ of const. sect. curvature c.

Moreover: orthonormal coords. on $T_N S^n(R)$ are normal coords for both of these.

Thm 4.2.4 $\Longrightarrow$ $\exp_N^* g_R = (\exp_x \circ f)^* g$.

Thus: $F := \exp_x \circ f \circ \exp_N^{-1} : U \to M$ is a local isometry.

let $P \in S^n(R) \setminus \{N, S\}$, let $V := S^n(R) \setminus \{-P\}$, let $y := F(P) \in M$.

let $\tilde{f} := d_P F : T_P S^n(R) \to T_y M$ (this is a lin. isometry!)

$\leadsto$ $\tilde{F} := \exp_y \circ \tilde{f} \circ \exp_P^{-1} : V \to M$ local isometry.

same construction as before

and $U \cup V = S^n(R)$.

$$F \ \downarrow \ \tilde{F}$$
$$M$$

Why is $F|_{U\cap V} = \tilde{F}|_{U \cap V}$?

Reason: By constr. $F(P) = \tilde{F}(P)$ and $d_P F = d_P \tilde{F}$.

similar to Prop 3.1.15

$\leadsto$ $F|_{U\cap V} = \tilde{F}|_{U\cap V}$.

$(S^n(R), g_R) \to (M, g)$

$\leadsto$ F and $\tilde{F}$ glue to a local isometry φ.

Why is φ a global isometry ?

- The map φ is a covering map (as a local isometry between connected complete Riem mflds).
- Because $S^n(R)$ and M are both simply connected, this covering map is a homeomorphism/diffeomorphism + local isometry → isometry. $n \geq 2$ $\square$

[Example. Let M be a compact mfd with $\pi_1(M) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Then: M does not admit a Riem. metric of constant sectional curvature.

Proof. Assume for a contradiction that there ex. a Riem metric g on M s.t. g has constant sectional curv. $c \in \mathbb{R}$.

M compact $\Rightarrow$ (M, g) complete

- If $c \leq 0$: By Cartan-Hadamard: $|\pi_1(M)| = \infty$ ↯
- If $c > 0$: By the previous thm:

The universal covering of M is a sphere

deck trafo action of $\pi_1(M)$ $\rightsquigarrow$ $\mathbb{Z}/_2 \times \mathbb{Z}/_2$ acts freely on a sphere

$\rightsquigarrow$ $\lightning$

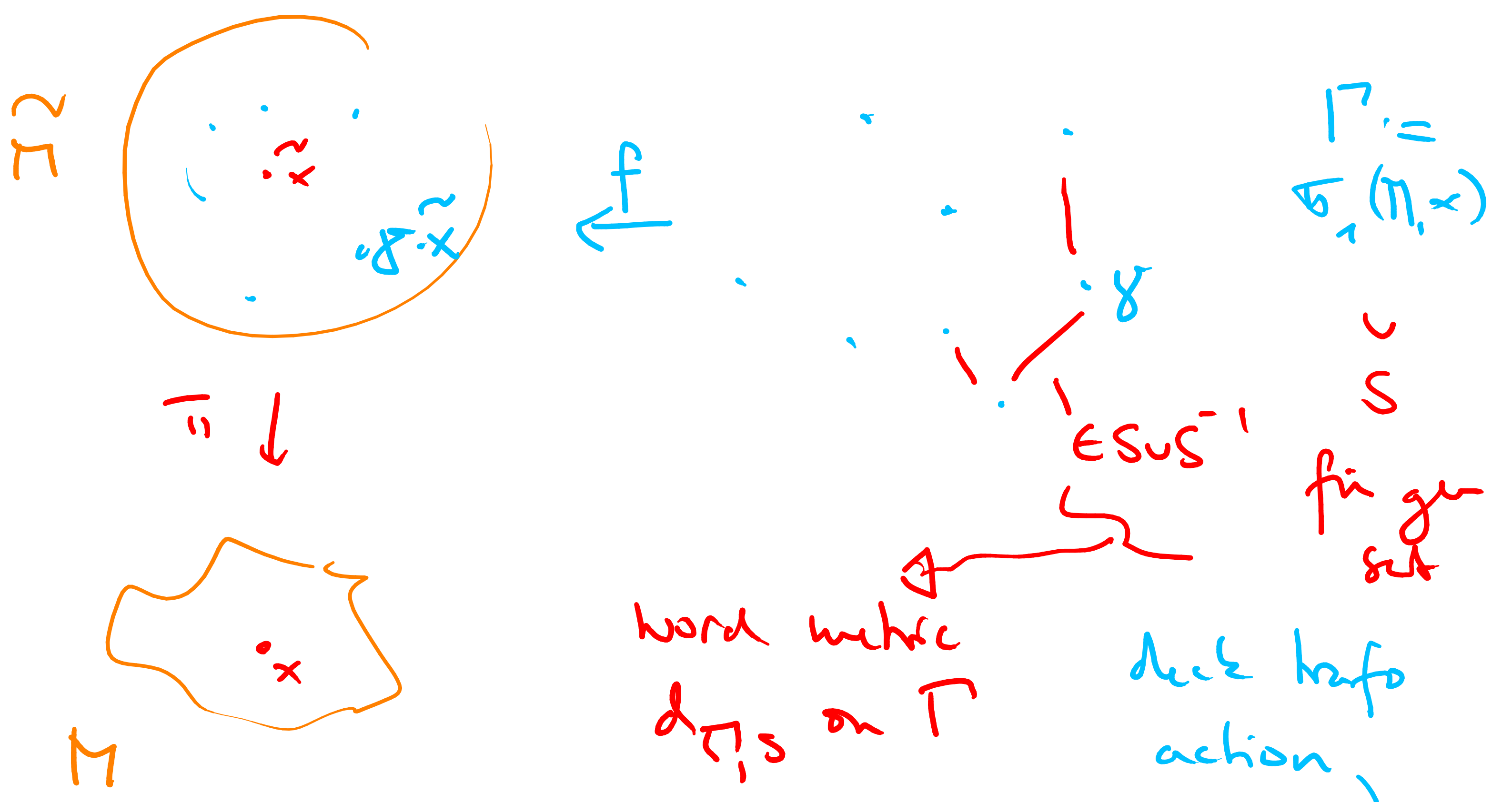
$\mathbb{Z}/_2 \times \mathbb{Z}/_2$ does <u>not</u> have periodic cohomology .]

$\boxed{4.6}$ THE ŠVARC-MILNOR LEMMA

idea: to relate the geometry of Riem. universal coverings of compact Riem. mfds to the "geometry" of the fundamental group.

Theorem (Švarc-Milnor lemma). Let (M,g) be a compact Riem. mfd, let $x \in M$, and let $\pi : (\tilde{M}, \tilde{g}) \to (M,g)$ be the Riem. universal covering of (M,g), let $\tilde{x} \in \pi^{-1}(x)$.

1. Then $\Gamma := \pi_1(M,x)$ is finitely generated.



2. Then the map
$$f : \Gamma \longrightarrow \tilde{M}$$
$$\gamma \longmapsto \gamma \cdot \tilde{x}$$
is a quasi-isometry. (wrt $d_{\Gamma,S}$ and $d_{\tilde{g}}$)

3. Moreover: The growth functions of Γ and of $(\tilde{M}, \tilde{g})$ are quasi-equivalent.

always: at most exponential.

Sample application:

Let (M, g) be a compact Riem. mfd with sec $\boxed{< 0}$. Then $\pi_1(M)$ has exponential growth. In particular: $\pi_1(M) \not\cong \mathbb{Z}^r$.