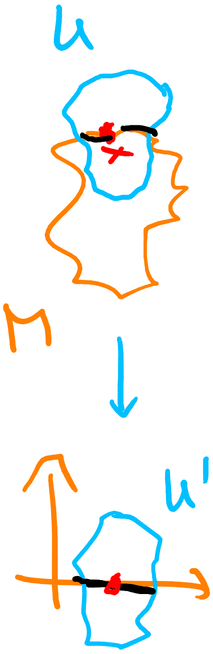
11.11.2020

Recap: smooth mfds := top. mfd + smooth atlas/structure
(clarification: mfds have constant dim)

intrinsic/abstract ⇝ coordinate-free
↕
explicit/concrete via coordinates/submanifolds

$\boxed{1.1.5}$ SUBMANIFOLDS

idea: A submfd of $\mathbb{R}^N$ is a subspace that admits slice/submfd charts



Definition. (submfd chart, submfd). Let $n, N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a subset.

- Let $x \in M$. An n-dim <u>submfd chart</u> of M around x is a diffeo $\varphi : U \to U'$, where
  - $U \subset \mathbb{R}^N$ is an open nbhd of x
  - $U' \subset \mathbb{R}^N$ is an open subset
  - $\varphi(U \cap M) = U' \cap (\mathbb{R}^n \times \{0\})$.
- $M \subset \mathbb{R}^N$ is an n-dim <u>smooth submfd</u> if every point in M admits an n-dim submfd chart around it.

Remark. Let $M \subset \mathbb{R}^N$ be an n-dim smooth submfd. Then

$$\Big\{ \varphi|_{U\cap M} : U \cap M \to \underbrace{\varphi(U\cap M)}_{\subset \mathbb{R}^n \times \cancel{\{0\}}} \;\Big|\; \varphi : \underbrace{U}_{\subset \mathbb{R}^N} \to \underbrace{U'}_{\subset \mathbb{R}^N} \text{ smooth submfd chart for } M \text{ in } \mathbb{R}^N \Big\}$$

is a smooth atlas for M.

Conversely: Every smooth n-mfd is diffeomorphic to a smooth submfd of $\mathbb{R}^{2\cdot n}$.

Remark. (tangent space of a submfd). Let $M \subset \mathbb{R}^N$ be a smooth submfd of dim n and let $x \in M$. Then

viewed as "abstract mfd"

$$T_x M \longrightarrow \{\gamma'(0) \mid \gamma \in C(\mathbb{R}^N; x),\ \operatorname{im}\gamma \subset M\}$$

$$[\alpha] \longmapsto \alpha'(0)$$

is an $\mathbb{R}$-lin. iso.

Theorem. (regular value theorem). Let $N, n \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a subset. Then the following are equivalent: TFAE

1. $M \subset \mathbb{R}^N$ is an n-dim smooth submfd
2. M is locally a set of solutions: For every $x \in M$, there ex. an open nbhd $U \subset \mathbb{R}^N$ of x and a smooth map $f : U \to \mathbb{R}^{N-n}$ s.t. 0 is a regular value of f and $f^{-1}(0) = U \cap M$.

   $\forall_{z \in f^{-1}(0)}\ \operatorname{rk} f'(z) = N - n$
3. M admits local parametrisations: For every $x \in M$, there ex. $\pi \in \Sigma_N$, open subsets $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^{N-n}$, such that $V \times W$ is an open nbhd of x, and an $f \in C^\infty(V, W)$ with $(V \times W) \cap L_\pi(M) = \text{graph of } f$.

   $L_\pi : \mathbb{R}^N \to \mathbb{R}^N$ induced by π

In practice: visualisation / plotting / solution software

Example. (spheres). Let $n \in \mathbb{N}$ and

$$S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \} \subset \mathbb{R}^{n+1}.$$

This is a smooth n-dim submfd of $\mathbb{R}^{n+1}$ via

$$\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^1$$
$$x \longmapsto \underbrace{\|x\|_2^2 - 1}_{\text{smooth, } 0 \text{ is a regular value}}.$$

(standard)

$\rightsquigarrow$ smooth structure on S^n.

Theorem. (smooth submfds via ~~immersions~~). Let $N \in \mathbb{N}$, let W be a smooth mfd and let $f : W \to \mathbb{R}^N$ be an injective smooth ~~map~~ immersion.

Then: TFAE:

1. $f(W) \subset \mathbb{R}^N$ is a smooth submfd
2. $f : W \to f(W)$ ~~is a homeomorphism~~.

i.e.: for all $x \in W$: $d_x f : T_x W \to \mathbb{R}^N$ is injective

1.2 THE TANGENT BUNDLE

idea: smooth mfd M $\rightsquigarrow$ $(T_xM)_{x\in M}$

assemble these into a global object + constructions on these bundles

vector bundles

1.2.1 SMOOTH VECTOR BUNDLES

idea: smooth vector bundle:

$$\begin{array}{ll} E & \leftarrow \text{total space} \\ \downarrow \pi & \text{smooth map} \\ M & \leftarrow \text{base mfd} \end{array}$$

smooth mfd (fibrewise linear)

+ local triviality condition

Definition. (smooth vector bundle). Let M be a smooth mfd and let $k \in \mathbb{N}$. A <u>smooth vector bundle over M of rank k</u> is a smooth map $\pi : E \to M$ with the following data/properties:

data!

- For each $x \in M$, the <u>fibre</u> $E_x := \pi^{-1}(\{x\})$ comes with the structure of an $\mathbb{R}$-vector space.

Faser

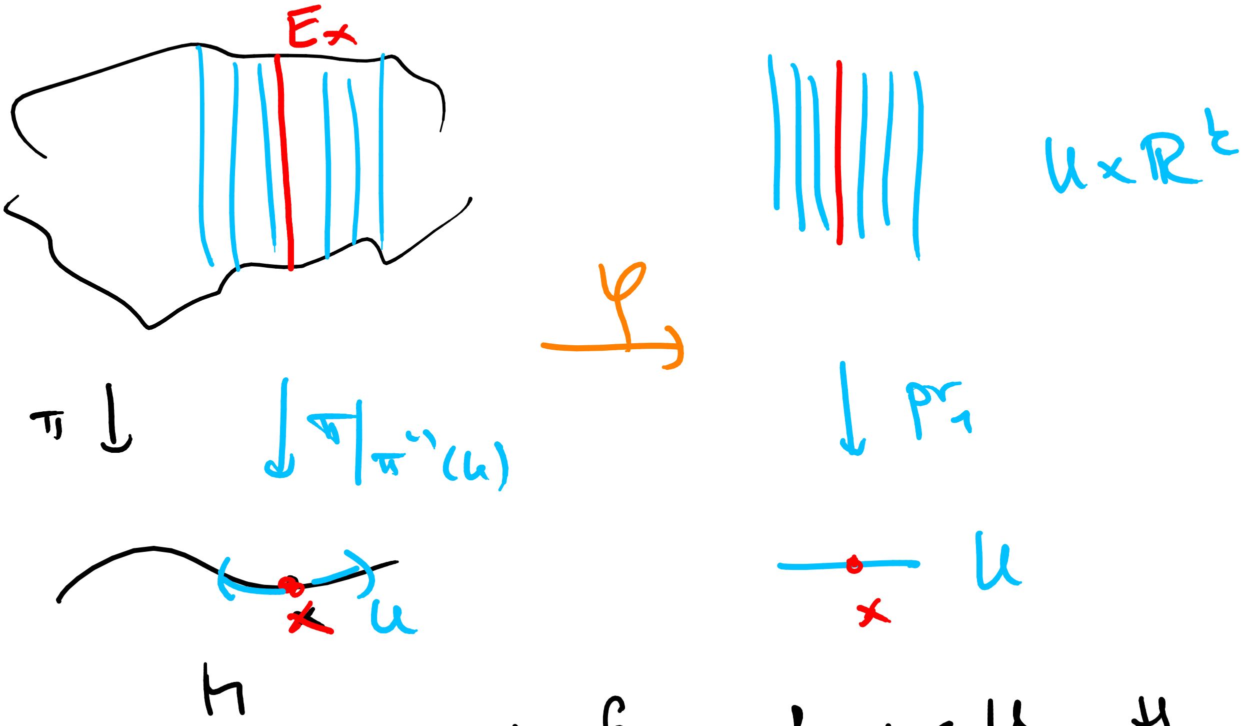
- For each $x \in M$, there ex. an open nbhd $U \subset M$ of x and a diffeo

$$\varphi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

s.t.: .

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^k \\ \pi \downarrow & \circlearrowleft & \downarrow pr_1 \\ U & = & U \end{array}$$

i.e., φ maps fibres to fibres



- for each $y \in U$, the restriction
$$\varphi|_{E_y} : E_y \longrightarrow \{y\} \times \mathbb{R}^k$$
is an $\mathbb{R}$-lin iso.

Example. If M is a smooth mfd and $k \in \mathbb{N}$, then
$$\begin{array}{c} M \times \mathbb{R}^k \\ \downarrow {\scriptstyle pr_1} \\ M \end{array}$$
is a smooth vector bundle of rk k, "the" <u>trivial bundle</u> over M of rank k.

(<u>Not</u> all vector bundles are of this form!)

F is fibre-preserving

Definition. (bundle map). Let $\pi : E \to M$, $\pi' : E' \to M'$ be smooth vector bundles. A <u>bundle map</u> is a pair $(F : E \to E', f : M \to M')$ of smooth maps with $\pi' \circ F = f \circ \pi$ and F is fibre-wise $\mathbb{R}$-linear.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

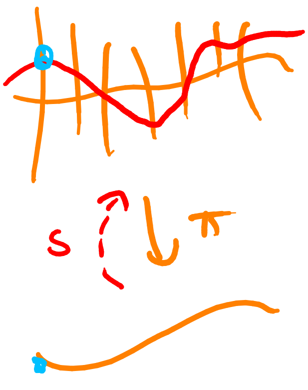
$\rightsquigarrow$ category of smooth vector bundles and bundle maps ← $\mathrm{Vect}\,B$

in particular: a notion of isomorphisms of smooth vector bundles

Definition. (trivial vector bundle) Let M be a smooth mfd and let $k \in \mathbb{N}$. A smooth vector bundle over M of rank k is called <u>trivial</u> if it is isomorphic (in $\mathrm{Vect}\,B$) to the standard trivial vector bundle

$$\begin{array}{c} M \times \mathbb{R}^k \\ \downarrow pr_1 \\ M \end{array}.$$

Definition. (section). Let $\pi : E \to M$ be a smooth vector bundle. A <u>section of π</u> is a smooth map $s : M \to E$ with



$$\pi \circ s = \mathrm{id}_M$$

s maps x to the fibre E_x

We write $\Gamma(\pi)$ (or $\Gamma(E)$) for the set of all sections of π.

- $\mathbb{R}$-vector space (via fibrewise operations)
- $C^\infty(M)$-module

$\boxed{1.2.2}$ CONSTRUCTING VECTOR BUNDLES

idea: · smooth mfds: can be constructed by glueing coordinate patches

· smooth vector bundles: can be constructed by glueing trivial bundles

($\rightsquigarrow$ what are the glueing conditions?)