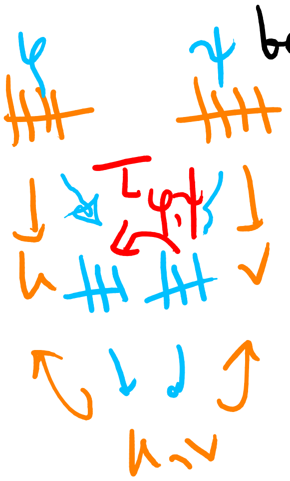
12.11.2020

Recap: vector bundle: locally trivial bundle with linear structure on the fibres

$$\begin{array}{c} E \\ \pi \downarrow \\ M \end{array} \qquad \text{"locally looks like } \begin{array}{c} U \times \mathbb{R}^k \\ \downarrow \text{proj.} \\ U \end{array} \text{"}$$

Proposition (transition functions). Let $\pi : E \to M$ be a smooth vector bundle of rank k, let $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, $\psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ be local trivialisations of π. (with $U \cap V \neq \emptyset$).



Then there ex. a smooth map

$$\tau_{\varphi,\psi} : U \cap V \to GL(\mathbb{R}^k)$$

← transition function

$\subset Gl(\mathbb{R}^k)$

$\subset Hom_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^k)$ open (det!)

with

$$\forall_{x \in U \cap V} \forall_{v \in \mathbb{R}^k} \quad \varphi \circ \psi^{-1}(x, v) = (x, \tau_{\varphi,\psi}(x)(v)).$$

Proof. see lecture notes + (Lu) 1.4. $\square$

Proposition. (constructing vector bundles from patches). Let M be a smooth mfd, let $k \in \mathbb{N}$,

- let $(E_x)_{x \in M}$ be a family of $\mathbb{R}$-vector spaces of dim k, let $E := \coprod_{x \in M} E_x$ and let $\pi : E \to M$, $E_x \ni v \mapsto x$.

Moreover,

- let $(U_i)_{i \in I}$ be an open cover of M,

- let $(\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k)_{i\in I}$
- let $(\tau_{i,j} : U_i \cap U_j \to GL(\mathbb{R}^k))_{i,j\in I}$ be a family of smooth maps

with the following properties:

- for each $i \in I$: φ_i is bijective, fibre-pres., and a fibrewise $\mathbb{R}$-lin. iso
- for all $i,j \in I$, for all $x \in U_i \cap U_j$ and all $v \in E_x$, we have

$$\varphi_i \circ \varphi_j^{-1}(x,v) = (x, \tau_{ij}(x)(v)).$$

Then: There ex. a unique smooth structure on E s.t. $\pi : E \to M$ is a smooth vector bundle with local trivialisations $(\varphi_i)_{i\in I}$ and $(\tau_{i,j})_{i,j\in I}$ as transition functions.

Proof. • Smooth structure on E: let $x \in M$ and let $i \in I$ with $x \in U_i$. let $\psi_x : V_x \to V_x'$ be a smooth chart of M around x with $V_x \subset U_i$.

let

$$\tilde\psi_x := (\psi_x \times id_{\mathbb{R}^k}) \circ \varphi_i| : \pi^{-1}(V_x) \to \underbrace{\underbrace{V_x'}_{\subset \mathbb{R}^n} \times \mathbb{R}^k}_{\subset \mathbb{R}^{n+k}}$$

Then: $(\tilde\psi_x)_{x\in M}$ induces a topology on E and is a smooth atlas on E. (because the τ_{ij} are smooth)

local property

- Smoothness of
  - φ_i: check with $\tilde\psi_x$ on E: locally: id map

- π: check with charts $\tilde{\varphi}_x$ on E
  ψ_x on M

  $\rightsquigarrow$ locally: a projection
- By construction: $(\varphi_i)_i$ are local triv, $(\tau_{i,j})_{i,j}$ transition maps.

Uniqueness: s. lecture notes. $\square$

Corollary (constructing vector bundles from cocycles). Let M be a smooth mfd, let $(U_i)_{i\in I}$ be an open cover of M, let V be a fin-dim $\mathbb{R}$-vector space, and let $(\tau_{i,j}: U_i \cap U_j \to GL(V))_{i,j\in I}$ be smooth maps satisfying the cocycle condition

non-abuse of notation!

$$\forall_{i,j,h\in I}\ \forall_{x\in U_i\cap U_j\cap U_h}\quad \tau_{ij}(x)\circ\tau_{jh}(x) = \tau_{ih}(x).$$

Then: there ex. a "unique" smooth vector bundle over M of rank $\dim_{\mathbb{R}} V$ with trivialisations over the $(U_i)_{i\in I}$ whose transition functions are the $(\tau_{i,j})_{i,j\in I}$.

Sketch of proof: (take $\coprod_{i\in I}(U_i\times V)/\sim$ via $(\tau_{i,j})_{i,j}$)

Reduction to the previous prop:

- set $E_x := V \overset{\text{wlog}}{\cong} \mathbb{R}^k$ $\quad k := \dim_{\mathbb{R}} V$
- for $i \in I$: $\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$

  $E_x \ni v \longmapsto (x, \tau_{i,i_x}(x)(v))$ $\quad x \in U_i$

(Choice of i_x will not matter: cocycle cond.)

Transition function equation is satisfied because: let $x \in U_i \cap U_j$, let $v \in V$.

Then

by def of φ_i, φ_j

$$\boxed{\varphi_i \circ \varphi_j^{-1}(x,v)} = \left(x, \tau_{i,i_x}(x) \circ \tau_{j,i_x}^{-1}(x)\,(v)\right)$$

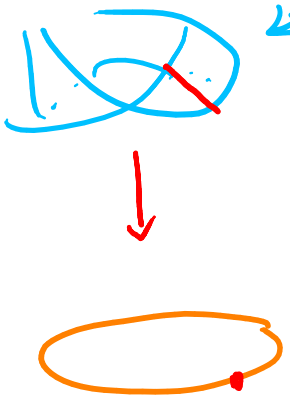
$$= \left(x, \tau_{i,i_x}(x) \circ \tau_{i_x,j}(x)\,(v)\right)$$

$\tau_{ij}^{-1} = \tau_{ji}$

cocycle cond.

$$= \boxed{\left(x, \tau_{ij}(x)\,(v)\right)}. \qquad \square$$

Example. (Möbius strip).



S^1

can be viewed as a line bundle over S^1

DE: Geradenbündel

vector bundle rank 1

e.g., using the cocycle cond. (ü2.3).

This is a non-trivial vector bundle over S^1.

1.2.3 THE TANGENT BUNDLE

idea:
- assemble the tangent spaces into a vector bundle (using the differentials of charts)
- this construction is functorial with respect to smooth maps

Proposition & Definition. (tangent bundle). Let M be a smooth mfd of dim n with smooth structure A.

- For $(\varphi : U \to \underbrace{U'}_{\subset \mathbb{R}^n}) \in A$, we set

$$\tilde{\varphi} : \coprod_{x \in U} T_x M \longrightarrow U \times \mathbb{R}^n$$
$$T_x M \ni v \longmapsto (x, d_x\varphi(v)).$$

- For $(\varphi : U \to U'), (\psi : V \to V') \in A$, we set

$$\tau_{\varphi,\psi} : U \cap V \longrightarrow GL(\mathbb{R}^n)$$
$$x \longmapsto d_{\psi(x)}(\varphi \circ \psi^{-1})$$

Then $(\tilde{\varphi})_{\varphi \in A}$ and $(\tau_{\varphi,\psi})_{\varphi,\psi \in A}$ satisfy the conditions of the prop. on construction of vector bundles.

The resulting vector bundle $TM \to M$ is the <u>tangent bundle</u> of M and has fibres $(T_x M)_{x \in M}$

Proof: • $\tilde{\varphi}$ is fibre-pres and $\underbrace{\text{fibrewise lin iso}}$

by construction (fibre-pres) $\qquad$ φ is diffeo $\rightsquigarrow d_x\varphi$ are lin. isos

• smoothness of $\tau_{\varphi,\psi}$: because $\varphi\circ\psi^{-1}$ is smooth $(\dots)$

• transition function equation: let $(\varphi: U \to U')$, $(\psi: V \to V') \in \mathcal{A}$ and let $x \in U \cap V$, $v \in T_xM$. Then

def of $\tilde{\varphi}, \tilde{\psi}$

$$\boxed{\tilde{\varphi}\circ\tilde{\psi}^{-1}(x,v)} = (x, d_x\varphi \circ (d_x\psi)^{-1}(v))$$

chain rule

$$= (x, d_x\varphi \circ d(\psi^{-1})(v))$$

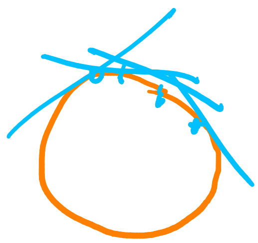
$$= (x, \underbrace{d_{\psi(x)}(\varphi\circ\psi^{-1})}_{=\tau_{\varphi,\psi}(x)}(v)). \qquad \square$$

local coordinates

$\rightsquigarrow$ get explicit charts for TM.

Example. • The tangent bundle of S^1 is trivial!

(idea: choose "good" charts so that all transition functions are constant 1)



S^1

$\Downarrow$

construct a trivialisation from this.

• The tangent bundle of S^2 is non-trivial (needs an argument)

Proposition & Definition. (the differential). Let $f: M \to N$ be a smooth map between smooth mfds.

Then
$$df : TM \longrightarrow TN$$
$$T_xM \ni v \longmapsto \underbrace{d_x f}_{T_xM \to T_{f(x)}N}(v)$$
is a bundle map (over $f: M \to N$).

Proof: • fibre-pres, fibrewise linear: by construction

• smoothness: check it locally (choosing charts of TM and TN coming from charts of M and N, respectively). □

Remark (chain rule). The tangent bundle defines a functor $F: \mathrm{Mfd} \longrightarrow \mathrm{VectB}$:

• on objects: $M \longmapsto F(M) := \begin{array}{c} TM \\ \downarrow \\ M \end{array}$ tangent bundle

• on morphisms: $\underset{\text{smooth}}{f: M \to N} \longmapsto F(f) := df : TM \to TN$

• $\forall_{M \text{ smooth mfd}}$ $F(\mathrm{id}_M) = \mathrm{id}_{TM}$

• $\forall_{\text{composable smooth maps } f, g}$ $\underbrace{F(g \circ f)}_{d(g \circ f)} = \underbrace{F(g)}_{d(g)} \circ \underbrace{F(f)}_{df}$

chain rule (!)