13.01.2021

Org: · FlexNow registration open! (Studienleistung, Prüfungsleistung)
· exams: oral online exams (25 min)
dates: 25.02., 24.03.
registration for a timeslot: rosina.bonn@uni.de

Recap: (M,g) Riem mfd $\rightsquigarrow$ Levi-Civita connection ∇
$\rightsquigarrow$ Riemannian curvature $(3,1)$-tensor:

$$R: \quad R(X,Y,Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Definition. (Riem. $(4,0)$-tensor). Let (M,g) be a Riem mfd. Then the Riemannian $(4,0)$-tensor of (M,g) is def. as

$$Rm := R^{\flat}, \qquad (\flat: \text{flat}, \quad \rightsquigarrow R = Rm^{\#}, \ \#: \text{sharp})$$

i.e.: $Rm: \Gamma(TM)^{\times 4} \longrightarrow C^\infty(M)$

$$(X,Y,Z,W) \longmapsto \langle R(X,Y,Z), W\rangle_g$$

Rmk. (local descriptions of R and Rm). Let (M,g) be a Riem mfd. Wrt a chart, we can locally write

$$R = \sum_{i,j,k,l} R_{ijk}^{\ \ \ l} \cdot dx^i \otimes dx^j \otimes dx^k \otimes E_l$$

$$= E_i(\Gamma_{jk}^{\ \ l}) - E_j(\Gamma_{ik}^{\ \ l}) + \sum_m \left(\Gamma_{jk}^{\ \ m} \cdot \Gamma_{im}^{\ \ l} - \Gamma_{ik}^{\ \ m} \cdot \Gamma_{jm}^{\ \ l}\right)$$

and

$$Rm = \sum_{i,j,k,l} R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

R_{ijkl}: lowered index ($\rightsquigarrow b$)

$$= \langle R(E_i, E_j, E_k), E_l \rangle_g = \Big\langle \sum_{m=1}^{n} R_{ijk}{}^m \cdot E_m, E_l \Big\rangle_g$$

$$= \sum_{m=1}^{n} R_{ijk}{}^m \cdot \langle E_m, E_l \rangle_g$$

$$= g_{ml} = g_{lm}$$

$$= \sum_{m=1}^{n} g_{lm} \cdot R_{ijk}{}^m .$$

Proposition. (naturality of the Riemannian curvature tensor).
Let (M_1, g_1) and (M_2, g_2) be Riem. mfds (with Riem. curvature tensors R_1, Rm_1, R_2, Rm_2) and let $\varphi : (M_1, g_1) \longrightarrow (M_2, g_2)$ be an isometry. i.e., φ is a diffeo with $\varphi^* g_2 = g_1$

Then:

$$\forall_{X,Y,Z \in \Gamma(TM_1)} \quad \varphi_* R_1(X,Y,Z) = R_2(\varphi_* X, \varphi_* Y, \varphi_* Z)$$

and

$$\forall_{X,Y,Z,W \in \Gamma(TM_1)} \quad Rm_1(X,Y,Z,W) = Rm_2(\varphi_* X, \ldots, \varphi_* W) \circ \varphi .$$

Proof. Key input: use naturality of LC-connection (or: restate everything locally). □

Similarly: Riem. curvature tensors are preserved by local isometries.

$Rm(X,Y,Z,U) = \langle R(X,Y,Z),U\rangle_g$ $\quad R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

Proposition. (Symmetries of the Riem. curvature tensor).

Let (M,g) be a Riem. mfd. Then: for all $X,Y,Z,W \in \Gamma(TM)$, we have:

1. $Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$
2. $Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$
3. $Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$
4. (First) Bianchi identity:

$$Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0.$$

Sketch of proof.

1. clear from the def.
2. idea: use compatibility of ∇ with g.
4. idea: Jacobi identity for the Lie bracket and symmetry of ∇.
3. follows from 1., 2., 4.

"$\square$"

2.4.2 FLAT MANIFOLDS

(first test whether R and Ric model the right thing)

Definition. (flat mfd). A Riem. mfd is <u>flat</u> if its Riemannian curvature tensor is zero.

(3,1)-

(4,0)- ← doesn't matter! ($\langle \cdot, \cdot \rangle_g$ is non-deg).

Example. $\mathbb{R}^n$ with the Euclidean Riem. metric is flat (Example 2.4.3).

Are there all examples?
No! e.g., flat torus $\mathbb{Z}^2 \backslash \mathbb{R}^2$ (with quotient Riem. metric)

Theorem. (flatness). A Riem. mfd is flat iff and only if it is locally isometric to a Euclidean space (wrt the Euclidean Riem. metric).

Proof. "$\Leftarrow$": clear. (Riem. curv. tensor is a local isometry invariant; Euclidean $\mathbb{R}^n$ is flat)

"$\Rightarrow$": Let (M,g) be a flat Riem mfd of dim n and let $x \in M$.

Goal: locally around x, the mfd (M,g) is isometric to an open subset of Euclid. $\mathbb{R}^n$.