16.12.2020

Recap: (linear) connections on a smooth mfd M

$$\nabla : \Gamma(TM) \times \Gamma(\overset{E}{TM}) \longrightarrow \Gamma(\overset{E}{TM})$$

properties:
$C^\infty(M)$-linearity in first arg, $\mathbb{R}$-lin in the 2nd arg,
product rule in 2nd arg:

$$\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y$$

$\boxed{2.2.3}$ COVARIANT DERIVATIVES ALONG CURVES

<u>idea:</u> connection on M $\rightsquigarrow$ differentiate vector fields along curves in M in the direction of the velocity

"live on the curve, but with values in TM"

Definition (extendable vector field). Let M be a smooth mfd and let $\gamma : I \to M$ be a smooth curve in M.

- We write

vector fields in TM along γ

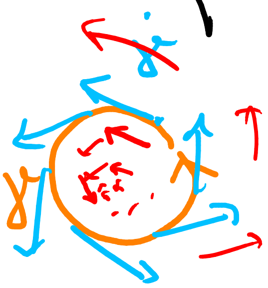
$\Gamma(TM|_\gamma) :=$ set of all smooth maps $X : I \to TM$ with $\forall_{t \in I}$ $X(t) \in T_{\gamma(t)}M$.

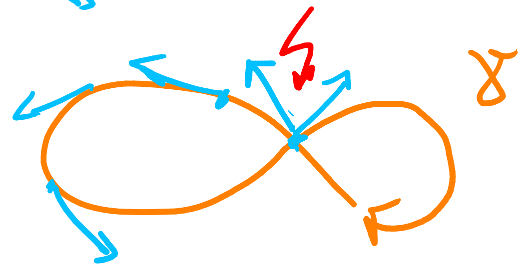
e.g. $\dot\gamma \in \Gamma(TM|_\gamma)$

- An $X \in \Gamma(TM|_\gamma)$ is <u>extendable</u> if there ex. an $\tilde X \in \Gamma(TM)$ with

$$\forall_{t \in I} \quad X(t) = \tilde X(\gamma(t)).$$

Example.

- extendable: • 0 vector field (e.g., by 0)
- 

- <u>not</u> extendable: 

Theorem (covariant derivative along curves). Let M be a smooth mfd, let ∇ be a linear connection on M, and let $\gamma : I \to M$ be a smooth curve in M. Then there ex. a unique map $D_\gamma : \Gamma(TM|_\gamma) \longrightarrow \Gamma(TM|_\gamma)$ covariant derivative along γ with:

$\boxed{\circledast}$

- Compatibility with the connection: If $X \in \Gamma(TM|_\gamma)$ is extendable by $\tilde{X} \in \Gamma(TM)$, then

$$\forall_{t \in I} \quad (D_\gamma X)(t) = \left(\nabla_{\dot\gamma(t)} \tilde{X}\right)(\gamma(t)).$$

$\dot\gamma(t) \in T_{\gamma(t)}M$

- $\mathbb{R}$-linearity:

$$\forall_{\lambda_1,\lambda_2 \in \mathbb{R}} \ \forall_{X_1,X_2 \in \Gamma(TM|_\gamma)} \ D_\gamma(\lambda_1 \cdot X_1 + \lambda_2 \cdot X_2) = \lambda_1 \cdot D_\gamma(X_1) + \lambda_2 \cdot D_\gamma(X_2).$$

- Product rule:

$$\forall_{f \in C^\infty(I)} \ \forall_{X \in \Gamma(TM|_\gamma)} \ D_\gamma(f \cdot X) = f' \cdot X + f \cdot D_\gamma(X).$$

Proof. "strategy": (1) use $\boxed{*}$ to show locality of D_γ

uniqueness ← (2) use $\boxed{*}$ and locality to derive a local formula

existence ← (3) show that everything that is constructed by the formula in (2) satisfies $\boxed{*}$ and can be globalized

(1) as in the proof of locality of connections: use bump functions ...

(2) let $t \in I$ and let $\varphi : U \to U' \subset \mathbb{R}^n$ be a smooth chart around $\gamma(t)$ in M, let $(E_1, \dots, E_n)$ be the corresponding frame for TM over U, and let $X \in \Gamma(TM|_\gamma)$. Then: on a small enough interval $J \subset I$ around t, we can write

$$X|_J = \sum_{i=1}^{n} X^j \cdot E_j \circ \gamma .$$

$X^j : J \to \mathbb{R}$ smooth

Then (if D_γ satisfies $\boxed{*}$)

$$D_\gamma X(t) \overset{(1) + \text{linearity}}{=} \sum_{i=1}^{n} D_\gamma (X^j \cdot E_j \circ \gamma)(t)$$

extendable on U by E_j

$$\overset{\text{product rule}}{=} \sum_{j=1}^{n} \Big(X^{j\prime}(t) \cdot E_j \circ \gamma(t) + X^j(t) \cdot D_\gamma (E_j \circ \gamma)(t) \Big)$$

$$\overset{\text{compatibility}}{=} \sum_{j=1}^{n} \Big(X^{j\prime}(t) \cdot E_j \circ \gamma(t) + X^j(t) \cdot (\nabla_{\dot\gamma(t)} E_j)(\gamma(t)) \Big)$$

$$\overbrace{\text{only contains given data!}} \quad \square \text{②}$$

$$\boxed{**}\left\{ \begin{aligned} &= \sum_{j=1}^{n} X^{j\prime}(t) \cdot E_j \circ \gamma(t) \\ &+ \sum_{k=1}^{k} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{i\prime}(t) \cdot X^j(t) \cdot \Gamma_{ij}^{k}(\gamma(t)) \cdot E_k \circ \gamma(t) \end{aligned}\right.$$

③ . Use $\boxed{**}$ to define D_γ on chart domains

- check that in each chart domain $\boxed{*}$ is satisfied.
- by ① and ②: these definitions are compatible on the intersection of their domains

$\rightsquigarrow$ · can define $D_\gamma : \Gamma(TM|_\gamma) \to \Gamma(TM|_\gamma)$

· check that $\boxed{*}$ is satisfied. $\square$

Example. (covariant derivatives along curves).

- in $\mathbb{R}^n$, with the Euclidean linear connection $\bar{\nabla}$:

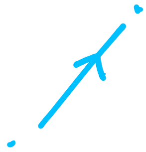
let $\gamma: I \to \mathbb{R}^n$ be a smooth curve. Then

$$\forall_{t \in I} \quad (D_\gamma \dot{\gamma})(t) = \gamma''(t).$$

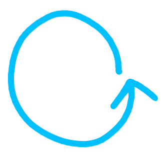
Concrete examples:

- $\gamma : I \to \mathbb{R}^n$ (given $a, b \in \mathbb{R}^n$)

  $t \mapsto a + t \cdot (b-a)$



$$\rightsquigarrow \quad D_\gamma \dot\gamma = 0.$$



- $\gamma : I \to \mathbb{R}^2$

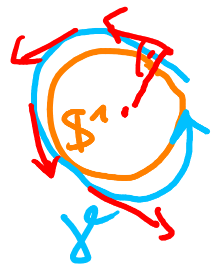
  $t \mapsto (\cos t, \sin t)$

$$\rightsquigarrow \quad \forall_{t \in I} \; (D_\gamma \dot\gamma)(t) = (-\cos t, -\sin t) \neq 0.$$

- $S^1 \subset \mathbb{R}^2$ with the linear connection ∇^T on S^1 induced from $\overline{\nabla}$ on $\mathbb{R}^2$:

  We consider

$$\gamma : \mathbb{R} \to S^1$$
$$t \mapsto (\cos t, \sin t).$$



What is $\underbrace{D_\gamma \dot\gamma}_{=: X \in \Gamma(TS^1|_\gamma)}$?

X is extendable, even to $\mathbb{R}^2$, by:

$$\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (-y, x)$$

Then:

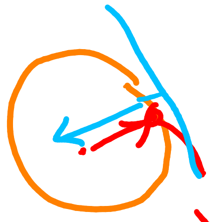
$$\boxed{(D_\gamma \dot\gamma)(t)} \overset{\text{compatibility with the connection}}{=} \left(\nabla^T_X (\tilde{X}|_{S^1})\right)(\gamma(t))$$

$$\overset{\text{orth. projection onto } T_{\gamma(t)} S^1}{=} P_{\gamma(t)}\left(\left(\overline{\nabla}_{\tilde{X}} \tilde{X}\right)(\gamma(t))\right)$$

$$= P_{\gamma(t)}\Big(-\sin t \cdot (\nabla_{E_1}\tilde{X})(\gamma(t)) + \cos t \cdot (\nabla_{E_2}\tilde{X})(\gamma(t))\Big)$$

$T_{\gamma(t)} S^1$

$$= P_{\gamma(t)}\big(-\sin t \cdot (0,1) + \cos t \cdot (-1,0)\big)$$



$$= P_{\gamma(t)}\boxed{(-\cos t, -\sin t)} \quad \boxed{= 0.}$$

$\gamma(t) = (\cos t, \sin t)$

S^1

2.2.4 GEODESICS

<u>idea:</u> given a connection ∇

$\overset{2.2.3}{\rightsquigarrow}$ covariant derivatives along curves

in particular: can define acceleration of curves as $D_\gamma \dot{\gamma}$.

in $\mathbb{R}^n$: straight lines are exactly the curves with 0 acceleration

$\rightsquigarrow$ turn this property into a def!

DE: Geodätische / Geodäte

Definition (geodesic) Let M be a smooth mfd with a linear connection ∇ and let $\gamma: I \to M$ be a smooth curve in M.

Then γ is a <u>geodesic</u> (wrt ∇) if

$$D_\gamma \dot{\gamma} = 0. \quad \rightsquigarrow \text{differential eq}$$

Example. (see previous example)

- in $\mathbb{R}^n$: geodesics (wrt Euclidean linear connection) are exactly the straight lines of the form $t \mapsto a + t \cdot (b-a)$.

  In particular: $\gamma: t \mapsto (\cos t, \sin t)$ is <u>not</u> a geodesic in $\mathbb{R}^2$ (wrt $\bar{\nabla}$).

- on S^1: $\gamma: t \mapsto (\cos t, \sin t)$ is a geodesic (wrt ∇^T).

Remark. (geodesic equation). Let M be a smooth mfd with a linear connection ∇ and let $\gamma: I \to M$ be a smooth curve. Then γ is a <u>geodesic</u> wrt ∇ if and only if γ locally satisfies (follows from $\boxed{**}$)

$$\forall_{k \in \{1,\dots,n\}} \text{ "}\forall_s\text{" } \gamma^{k\,\prime\prime}(s) + \sum_{i=1}^n \sum_{j=1}^n \gamma^{i\,\prime}(s) \cdot \gamma^{j\,\prime}(s) \cdot \Gamma_{ij}^k(\gamma(s)) = 0$$

second order ODE.