17.12.2024

Org: • next lectures: 23.12., 07.01.

• (optional) bonus exc sheet ⇆ material will appear in the next days

lin. connection ∇

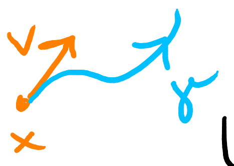
Recap: • geodesic: $D_\gamma \dot{\gamma} = 0$

• geodesic equation: $\forall_k \; \gamma^{k\,\prime\prime} + \sum_i \sum_j \gamma^{i\,\prime} \cdot \gamma^{j\,\prime} \cdot \Gamma^k_{ij} \circ \gamma = 0$

(2nd order ODE)

Theorem. (existence and uniqueness of geodesics). Let M be a smooth mfd, let ∇ be a lin. connection on M, let $x \in M$, let $v \in T_x M$, and let $t \in \mathbb{R}$. Then: There ex. an interval $I \subset \mathbb{R}$ with $t \in I^\circ$ and a geodesic $\gamma : I \to M$ (wrt ∇) with

$$\gamma(t) = x \quad \text{and} \quad \dot{\gamma}(t) = v.$$



Uniqueness: If $\gamma : I \to M$ and $\eta : J \to M$ are geodesics with $t \in I^\circ \cap J^\circ$ and

$$\gamma(t) = x = \eta(t) \quad \text{and} \quad \dot{\gamma}(t) = v = \dot{\eta}(t),$$

then $\gamma|_{I \cap J} = \eta|_{I \cap J}$.

Proof. • This is a local problem.

• Geodesics are characterised by the geodesic equation, a 2nd order ODE; the constraints "$\gamma(t) = x$ and $\dot{\gamma}(t) = v$" give the correct number of initial conditions.

⇝ Apply ODE theory. $\square$

Definition (maximal geodesic). Let M be a smooth mfd with a linear connection ∇. A smooth curve $\gamma : I \to M$ is a <u>maximal geodesic</u> if γ is a geodesic wrt ∇ and if there is no geodesic $\eta : J \to M$ wrt ∇ with $I \subset J$ and

$$\eta|_I = \gamma \quad \text{but} \quad J \neq I.$$

Corollary. (existence and uniqueness of maximal geodesics). Let M be a smooth mfd, let ∇ be a linear connection on M, let $x \in M$, and $v \in T_x M$. Then there ex. a unique maximal geodesic $\gamma : I \to M$ with $0 \in I^\circ$ and

$$\gamma(0) = x \quad \text{and} \quad \dot\gamma(0) = v.$$

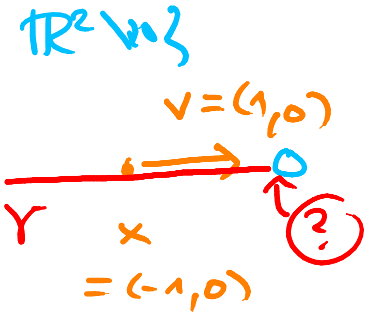
Notation: $\text{geod}_{x,v}$

Proof. previous thm + standard ODE arguments 7.2 $\square$

Examples: (maximal geodesics).

- in $\mathbb{R}^n$ (with the Euclidean lin connection), the maximal geodesics are exactly the (lin. parametrised) affine lines.
  of the form: $$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longmapsto x + t \cdot v \end{aligned}$$
- a maximal geodesic, whose domain is <u>not</u> all of $\mathbb{R}$:

We consider $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean lin. connection.



Let $x := (-1,0)$, $v := (1,0)$. Then

$$\mathrm{geod}^{\mathbb{R}^2\setminus\{0\}}_{x,v} : (-\infty, 1) \longrightarrow \mathbb{R}^2 \setminus \{0\}$$
$$t \longmapsto x + t\cdot v.$$

(compare with $\mathbb{R}^2$ ...)

$\boxed{2.2.5}$ PARALLEL TRANSPORT

<u>idea:</u> vector fields along curves are considered <u>parallel</u> if their covariant derivative is zero.

$\rightsquigarrow$ can use parallel vector fields to move between different tangent spaces.

$\rightsquigarrow$ "geometric" description of connections

Definition (parallel vector field). Let M be a smooth mfd with a lin. connection ∇, and let $X \in \Gamma(TM|_\gamma)$. Then X is <u>parallel along γ (w.r.t. ∇)</u> if

$$D_\gamma X = 0.$$

($\rightsquigarrow$ γ is a geodesic $\iff$ $\dot\gamma$ is parallel along γ)

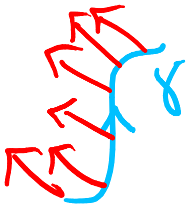
Remark (parallel transport equation). In the situation of the definition, X is parallel along γ

if and only if locally we have

$$X^{k\prime} = - \sum_i \sum_j X^j \cdot \underbrace{\gamma^{i\prime} \cdot \Gamma^k_{ij} \circ \gamma}_{\text{indep of } X}$$

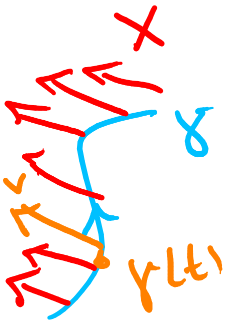
(linear 1st order ODE with smooth coefficients!)

Example. In $\mathbb{R}^n$ with the Euclidean linear connection, vector fields along curves are parallel if and only if they are "constant".



(the connection coeffs wrt standard coord. frame are all 0).

Theorem. (existence and uniqueness of parallel vector fields). Let M be a smooth mfd with a lin. connection ∇, let $\gamma: I \to M$ be a smooth curve, let $t \in I$, and let $v \in T_{\gamma(t)}M$. Then: There ex. a unique vector field $X \in \Gamma(TM|_\gamma)$ with



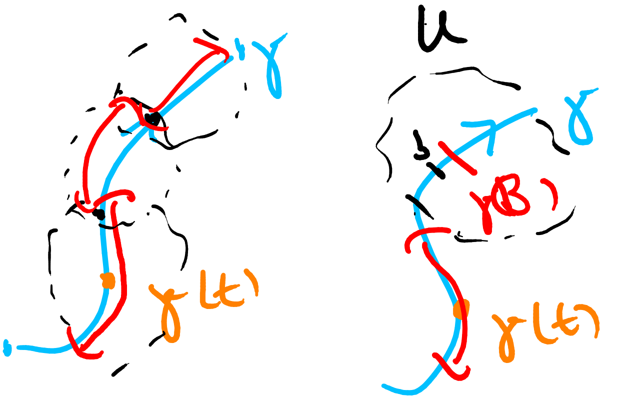
$X(t) = v$ and X is parallel along γ (wrt ∇).

Proof. • Local step: If γ is contained in a single chart domain, then: apply ODE

theory to the parallel transport (eq), which is a linear 1st order ODE with smooth coeff's.
$\rightsquigarrow$ global solution!

• Global step: let

fd (by local step)



$B := \sup\{ b \in I \mid \text{there ex. a solution on } [t,b] \}$

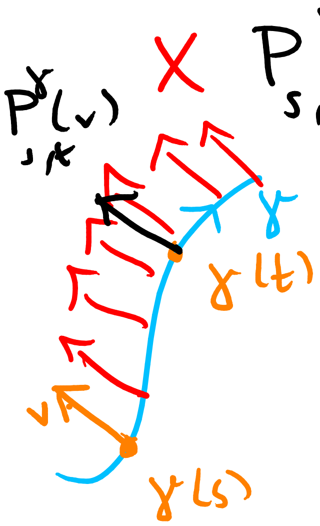
Goal: show that $B = \sup I$.

Assume for a contradiction that $B \neq \sup I$.
Let U be a chart domain around $\gamma(B)$.
$\rightarrow$ there ex. $b \in (t, B)$ and $\varepsilon \in \mathbb{R}_{>0}$ s.t.
$\gamma(\underbrace{(b-\varepsilon, b+\varepsilon)}_{=: J}) \subset U$. and $b+\varepsilon < B$.

Consider the local problem at $b - \frac{1}{2}\cdot\varepsilon$.
Let X be a solution on $[t,b]$.
Use the local step at time $b - \frac{1}{2}\varepsilon$ with direction $X(b-\frac{1}{2}\varepsilon)$.
local step $\rightsquigarrow$ can extend the solution on U.
$\rightsquigarrow$ can extend past B $\lightning$. $\square$

Definition. (parallel transport). Let M be a smooth mfd with lin. connection ∇, let $\gamma : I \to M$

be a smooth curve, and let $s,t \in I$. Then the <u>parallel transport along γ</u> from s to t (wrt ∇) is:



$$P^{\gamma}_{s,t} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$$
$$v \longmapsto X(t), \text{ where } X$$

is the unique parallel vector field along γ with $X(s) = v$.

Proposition. Let M be a smooth mfd with lin. conn. ∇, let $\gamma : I \to M$ be a smooth curve, and let $s,t \in I$. Then

$$P^{\gamma}_{s,t} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$$

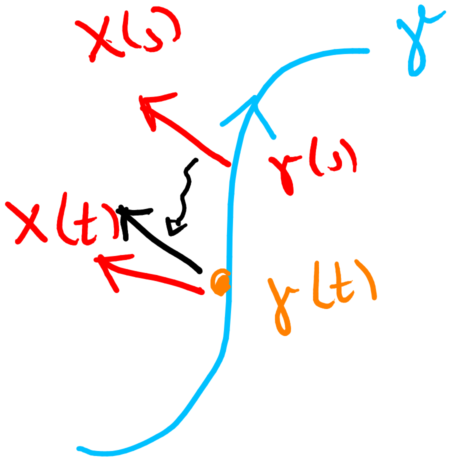
is an $\mathbb{R}$-lin. isomorphism.

Proof.: • $\mathbb{R}$-lin.: $\mathbb{R}$-lin. modification of initial values of a lin. ODE

• iso: $P^{\gamma}_{t,s}$ is the inverse (use the same parallel vector field). $\square$

Theorem. (cov. derivative along curves via parallel transport). Let M be smooth mfd with a lin. conn. ∇, let $\gamma : I \to M$ be a smooth curve, and let $t \in I$. Then

$$\forall_{X \in \Gamma(TM|_\gamma)} \quad (D_\gamma X)(t) = \underbrace{\lim_{s \to t} \frac{P^\gamma_{s,t}(X(s)) - X(t)}{s-t}}_{\text{"differential quotient"}} .$$



Proof. <u>idea:</u> express both sides in terms of a parallel frame.

Let $(E_1, \dots, E_n)$ be a $\underbrace{\text{parallel}}_{E_1,\dots,E_n \text{ parallel along } \gamma}$ frame of TM along γ.

frame ⇝ basis at all pts.

exists: · choose a basis of $T_{\gamma(t)}M$,

· extend it to parallel vector fields

⇝ at each pt a basis, because parallel transport gives lin. isos.

⇝ can write $X = \sum_{j=1}^{n} X^j \cdot E_j$.

$X \in \Gamma(TM|_\gamma)$, $\quad X^j : I \to \mathbb{R}$ smooth

LHS: $\boxed{D_\gamma X} = \sum_{j=1}^{n} D_\gamma (X^j \cdot E_j)$

$= \sum_{j=1}^{n} \big(X^{j\prime} \cdot E_j + X^j \cdot \underbrace{D_\gamma(E_j)}_{=0} \big)$

E_j parallel

$$= \boxed{\sum_{j=1}^{n} X^{j\prime} \cdot E_{j}} .$$

RHS: For all $s \in I$:

$$\boxed{P^{\gamma}_{s,t}(X(s))} = \sum_{j=1}^{n} X^{j}(s) \cdot P^{\gamma}_{s,t}\underbrace{(E_j(s))}_{= E_j(t) \,:\, E_j \text{ parallel}}$$

$$= \boxed{\sum_{j=1}^{n} X^{j}(s) \cdot E_j(t)}$$

⇝ take differential quotient. $\square$

⇝ allows to rewrite ∇ in terms of parallel transport. $\bar{u}$