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Recap: construction of smooth vector bundles from local trivialisations + transition functions (+ corresp. equations)

"$T_{ij} : U_i \cap U_j \xrightarrow{\text{smooth}} GL(V)$"

examples: tangent bundle; today: tensor bundles

1.2.4 TENSOR BUNDLES

idea: differential geometry uses multi-variable multi-linear objects on tangent bundles

⇝ we model these as sections of so-called $\underbrace{\text{tensor bundles}}_{}$

apply functors from multilin. algebra to the tangent bundle

(for the smoothness of the transition functions: need a smoothness condition on the functor)

Definition (smooth functor). Let $\mathrm{Vect}^{\mathrm{fin}}_{\mathbb{R}}$ be the category of fin.-dim. $\mathbb{R}$-vector spaces (and $\mathbb{R}$-lin maps). A functor $F : \mathrm{Vect}^{\mathrm{fin}}_{\mathbb{R}} \to \mathrm{Vect}^{\mathrm{fin}}_{\mathbb{R}}$ is <u>smooth</u> if for all fin-dim $\mathbb{R}$-vector spaces V, W, the map

$$\mathrm{Hom}_{\mathbb{R}}(V,W) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(F(V),F(W))$$

$$f \longmapsto F(f)$$

is smooth.

$F(W), F(V)$ in the contravariant case

fin-dim $\mathbb{R}$-vs ⇝ standard smooth structure

Remark. Let $F: \mathrm{Vect}^{fi}_{\mathbb{R}} \to \mathrm{Vect}^{fi}_{\mathbb{R}}$ be a smooth functor and let V be a fin-dim $\mathbb{R}$-vs. Then

$$\begin{aligned} GL(V) &\longrightarrow GL(F(V)) \\ f &\longmapsto F(f) \end{aligned}$$

functors map isos to isos

is a well-def. smooth map.

as a restriction of a smooth map (F is smooth!)

Examples. • taking duals $\cdot^*$ is a contravariant smooth functor

similarly: $\mathrm{Hom}_{\mathbb{R}}(\cdot, V)$

$\mathrm{Hom}_{\mathbb{R}}(V, \cdot)$

($\mathrm{Hom}_{\mathbb{R}}(V,W) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W^*, V^*)$

$f \longmapsto f^* = (g \mapsto g \circ f)$

is $\mathbb{R}$-linear, whence smooth)

• $\cdot \otimes_{\mathbb{R}} V : \mathrm{Vect}^{fi}_{\mathbb{R}} \longrightarrow \mathrm{Vect}^{fi}_{\mathbb{R}}$ is smooth

(same reason)

• "$V \mapsto V \otimes_{\mathbb{R}} V$" is smooth

(not linear on the Hom-spaces, but smooth $\Leftarrow$ uses bases/matrices)

Proposition. (applying smooth functors to smooth vector bundles). Let $F: \mathrm{Vect}^{fi}_{\mathbb{R}} \to \mathrm{Vect}^{fi}_{\mathbb{R}}$ be a smooth functor. This yields a functor $\mathrm{Vect}\,B \longrightarrow \mathrm{Vect}\,B$ as follows:

(covariant)

1. On objects: let M be a smooth mfd, let $\pi: E \to M$ be a smooth vector bundle, let A be the set of all local trivializations of rk k of π.

$\pi \rightsquigarrow$ "$F(\pi)$"

- For $(\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^k) \in A$, we set

$\varphi_x: E_x \to \{x\} \times \mathbb{R}^k \to \mathbb{R}^k$

$F(E_x) \xrightarrow{F(\varphi_x)} F(\mathbb{R}^k)$

$$\tilde{\varphi}: \coprod_{x \in U} F(E_x) \longrightarrow U \times F(\mathbb{R}^k)$$
$$F(E_x) \ni v \longmapsto (x, F(\varphi_x)(v)).$$

- For $(\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^k), (\psi: \pi^{-1}(V) \to V \times \mathbb{R}^k) \in A$

we set:

$$\tilde{\tau}_{\varphi,\psi}: U \cap V \longrightarrow GL(F(\mathbb{R}^k))$$
$$x \longmapsto F(\underbrace{\tau_{\varphi,\psi}(x)}_{\in GL(\mathbb{R}^k)}).$$

orig. transition function

Then: these maps satisfy the conditions for vector bundle construction (Prop. 1.2.8).

2. On morphisms: Applying F fibrewise to smooth bundle maps results in smooth bundle maps.

Proof. Sketch of 1:

by constr. $\quad$ F is a functor

- $\tilde{\varphi}$ is fibre-preserving, fibrewise linear isos
- $\tilde{\tau}_{\varphi,\psi}$ is smooth $\leftarrow$ see Rmk above (F is a smooth functor)

- transition function equations hold
  ↑ functoriality of F!

∎

We need two extensions:

- for contravariant smooth functors $\subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{k},\mathbb{R}^{k})$
  (change: for the transition function) need to flip back the order:

$$\tilde{\tau}_{\varphi,\psi} : x \longmapsto F(\underbrace{\tau_{\varphi,\psi}(x)}_{\in GL(\mathbb{R}^{k})}{}^{-1})$$

  ⇝ can form dual vector bundles:

$$\pi \rightsquigarrow \pi^{*}$$

- for multi-variable smooth functors:
  $\mathrm{Vect}^{f}_{\mathbb{R}} \times \mathrm{Vect}^{f}_{\mathbb{R}} \longrightarrow \mathrm{Vect}^{fin}_{\mathbb{R}}$.

  ⇝ can form direct sums, tensor products of smooth vector bundles:

$$\pi, \rho \rightsquigarrow \pi \oplus \rho, \ \pi \otimes_{\mathbb{R}} \rho.$$

Definition (tensor bundles). Let M be a smooth manifold with tangent bundle $\pi : TM \to M$.

- Then $T^{*}M := \pi^{*}$ is the <u>cotangent bundle of M</u>.
- Let $k, \ell \in \mathbb{N}$. Then the <u>(k,ℓ)-tensor bundle</u> of M is defined as

$$T^{k,\ell}(M) := \Big(\underbrace{\pi \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi}_{k} \otimes_{\mathbb{R}} \underbrace{\pi^{*} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi^{*}}_{\ell}\Big)^{*}$$

The space $\Gamma(T^{k,l}(M))$ of sections is the space of <u>k-covariant l-contravariant</u> <u>tensors on M</u> (<u>(k,l)-tensors</u> on M).

($(v^j)_j$ basis of V
↓
$(\lambda \cdot v^j)_j$ basis of V
≠ ← coordinates transform
$\hookrightarrow \frac{1}{\lambda}$
)

does not apply to "$V \underset{\mathbb{R}}{\cong} V^{**}$"

Remark: important identifications
(need canonical iso's on the vector space level)

We have canonical isos for fin-dim vector spaces V:

- $T^{k,0}V \underset{\mathbb{R}}{\cong} (V^{\otimes k})^* \underset{\mathbb{R}}{\cong} (V^*)^{\otimes k}$

$(V \underset{\mathbb{R}}{\otimes} W)^* \xleftrightarrow{\cong} V^* \underset{\mathbb{R}}{\otimes} W^*$

$v \otimes w \mapsto f(v) \cdot g(w) \longleftarrow f \otimes g$

$T^{1,0}V \underset{\mathbb{R}}{\cong} V^*$

- $T^{0,l}V \underset{\mathbb{R}}{\cong} ((V^*)^{\otimes l})^* \underset{\mathbb{R}}{\cong} (V^{**})^{\otimes l} \underset{\mathbb{R}}{\cong} V^{\otimes l}$

$V \xleftrightarrow{\cong} V^{**}$

$v \longmapsto (f \mapsto f(v))$

$T^{0,1}V \underset{\mathbb{R}}{\cong} V$

- $T^{0,0}V \underset{\mathbb{R}}{\cong} (\mathbb{R} \underset{\mathbb{R}}{\otimes} \mathbb{R}^*)^* \underset{\mathbb{R}}{\cong} \mathbb{R}$

$\leadsto \Gamma(T^{0,0}M) \underset{\mathbb{R}}{\cong} C^\infty(M)$

- $T^{k,l}V \underset{\mathbb{R}}{\cong} (V^{\otimes k} \underset{\mathbb{R}}{\otimes} V^{*\otimes l})^* \underset{\mathbb{R}}{\cong} (V^*)^{\otimes k} \underset{\mathbb{R}}{\otimes} V^{l}$

- $T^{1,1}V \underset{\mathbb{R}}{\cong} (V \otimes_{\mathbb{R}} V^*)^* \underset{\mathbb{R}}{\cong} \mathrm{Hom}_{\mathbb{R}}(V, V)$

$(v \otimes \omega \mapsto \omega(f(v))) \longleftarrow\!\shortmid f$

Remark. (tensor fields, explicitly). Let M be a smooth mfd, of dim n, let $\varphi : U \to U' \subset \mathbb{R}^n$ be a smooth chart for M with coord. functions $x^1, \dots, x^n : U \to \mathbb{R}$.

$\rightsquigarrow$ $(\underbrace{dx^1}_{TU \to T\mathbb{R} \cong \mathbb{R}}, \dots, dx^n)$ local frame over U of T^*M

$\rightsquigarrow$ pointwise dual basis $(E_1, \dots, E_n)$ $\in \Gamma(U, TM|_U)$ is a local frame over U of $(T^*M)^* \underset{\mathbb{R}}{\cong} TM$

(they are called <u>coordinate frames</u>)

$\rightsquigarrow$ the fibrewise dual of

$$\left(E_{i_1} \otimes \dots \otimes E_{i_k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_\ell}\right)_{\substack{i_1, \dots, i_k, \\ j_1, \dots, j_\ell \\ \in \{1, \dots, n\}}}$$

is a local frame over U of $T^{k,\ell}M$.

$$\rightsquigarrow \left(dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes E_{j_1} \otimes \dots \otimes E_{j_\ell}\right)_{\substack{i_1, \dots, i_k, \\ j_1, \dots, j_\ell \\ \in \{1, \dots, n\}}}$$

is a local frame over U of $T^{k,\ell}M$.