

Recap: construction of smooth vector bundles from local trivialisations + transition functions (+ wrap. equations)

" $T_{ij} : U_i \cap U_j \xrightarrow{\text{smooth}} GL(V)$ "

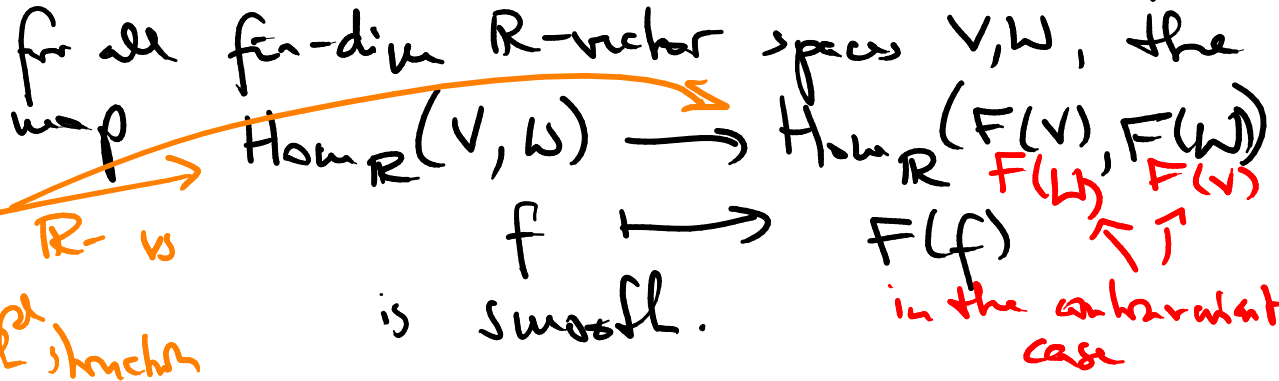
examples: tangent bundle; today: tensor bundles

1.2.4 TENSOR BUNDLES

idea: differential geometry uses multi-variable multi-linear objects on tangent bundles
 → we model these as sections of so-called tensor bundles

apply functors from multilin. algebra to the tangent bundle
 (for the smoothness of the transition functions: need a smoothness condition on the functor)

Definition (smooth functor). let $\text{Vect}_{\mathbb{R}}^{\text{fin}}$ be the category of fin.-dim \mathbb{R} -vector spaces (and \mathbb{R} -lin maps). A functor $F : \text{Vect}_{\mathbb{R}}^{\text{fin}} \rightarrow \text{Vect}_{\mathbb{R}}^{\text{fin}}$ is smooth if



fin.-dim \mathbb{R} -vs
 → standard smooth structure

Remark. Let $F: \text{Vect}_{\mathbb{R}}^F \rightarrow \text{Vect}_{\mathbb{R}}^F$ be a smooth functor and let V be a finite dim \mathbb{R} -v. Then

$$\text{GL}(V) \longrightarrow \text{GL}(F(V))$$

$$f \longmapsto F(f)$$

functors map isos to isos

is a well-def. smooth map.

as a restriction of a smooth map (F is smooth!)

Examples. • taking duals \cdot^* is a contravariant smooth functor

$$\text{Hom}_{\mathbb{R}}(V, W) \longrightarrow \text{Hom}_{\mathbb{R}}(W^*, V^*)$$

$$f \longmapsto f^* = (g \mapsto g \circ f)$$

is \mathbb{R} -linear, hence smooth

• $\otimes_{\mathbb{R}} V: \text{Vect}_{\mathbb{R}}^F \rightarrow \text{Vect}_{\mathbb{R}}^F$ is smooth

(same reason)

• " $V \mapsto V \otimes_{\mathbb{R}} V$ " is smooth

(not linear on the Hom-specs, but smooth \leftarrow uses bases/matrices)

Proposition. (applying smooth functors to smooth vector bundles). Let $F: \text{Vect}_{\mathbb{R}}^F \rightarrow \text{Vect}_{\mathbb{R}}^F$ be a smooth functor. This yields a functor $\text{Vect } B \rightarrow \text{Vect } B$ as follows:

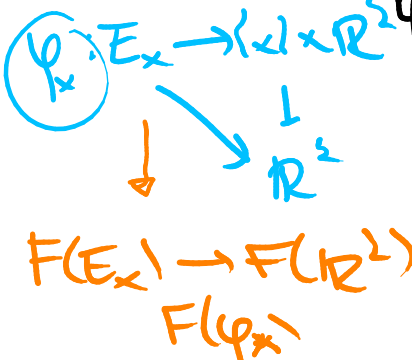
(covariant)

1. On objects: let M be a smooth mfd, let $\pi: E \rightarrow M$ be a smooth vector bundle, let A be the set of all local trivializations of π .

• For $(\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k) \in A$, we set

$$\tilde{\varphi}: \coprod_{x \in U} F(E_x) \rightarrow U \times F(\mathbb{R}^k)$$

$$F(E_x) \ni v \mapsto (x, F(\varphi_x)(v))$$



• For $(\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k), (\psi: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^k) \in A$ we set:

$$\tilde{\tau}_{\varphi, \psi}: U \cap V \rightarrow GL(F(\mathbb{R}^k))$$

$$x \mapsto F(\tilde{\tau}_{\varphi, \psi}(x))$$

$\in GL(\mathbb{R}^k)$
orig. transition func

Then: these maps satisfy the conditions for vector bundle construction (Prop. 1, 2, 3).

2. On morphisms: Applying F fibrewise to smooth bundle maps results in smooth bundle maps.

Proof. Sketch of 1:

• $\tilde{\varphi}$ is fibre-preserving, fibres in linear isos

• $\tilde{\tau}_{\varphi, \psi}$ is smooth \leftarrow see Prop above (F is a smooth functor)

transition function equations hold
 ↑ functionality of F !

We need two extensions: □

- for contravariant smooth functions $c: \text{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^k)$
 (change: for the transition function) need
 to flip back the order: $\in GL(\mathbb{R}^k)$

$$\tilde{T}_{p,q} : x \mapsto F(\tilde{\varphi}_q^{-1} \circ \varphi_p(x))^{-1}$$

→ can form dual vector bundles:

$$\pi \rightsquigarrow \pi^*$$

- for multi-variable smooth functions:

$$\text{Vect}_{\mathbb{R}}^k \times \text{Vect}_{\mathbb{R}}^l \rightarrow \text{Vect}_{\mathbb{R}}^{k+l}$$

→ can form direct sums, tensor products of smooth vector bundles:

$$\pi, \rho \rightsquigarrow \pi \oplus \rho, \pi \otimes_{\mathbb{R}} \rho$$

Definition (tensor bundles). Let M be a smooth manifold with tangent bundle $\pi: TM \rightarrow M$.

- Then $T^*M := \pi^*$ is the cotangent bundle of M .

- let $k, l \in \mathbb{N}$. Then the (k, l) -tensor bundle of M is defined as

$$T^{k,l}(M) := \left(\underbrace{\pi \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi}_k \otimes_{\mathbb{R}} \underbrace{\pi^* \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \pi^*}_l \right)^*$$

The space $\Gamma(T^{k,l}(M))$ of sections is the space of k -covariant l -contravariant tensors on M ((k,l) -tensors on M).

(v_j) , basis of V

$(\lambda \cdot v_j)$, basis of V

coordinates change $\rightarrow \frac{1}{\lambda}$

Remark: important identifications (need canonical isos on the vector space level) does not apply to " $V \cong V^*$ "

We have canonical isos for finite dim vector spaces V :

$T^{k,0}_V \cong (V^{\otimes k})^* \cong (V^*)^{\otimes k}$

$T^{1,0}_V \cong V^*$

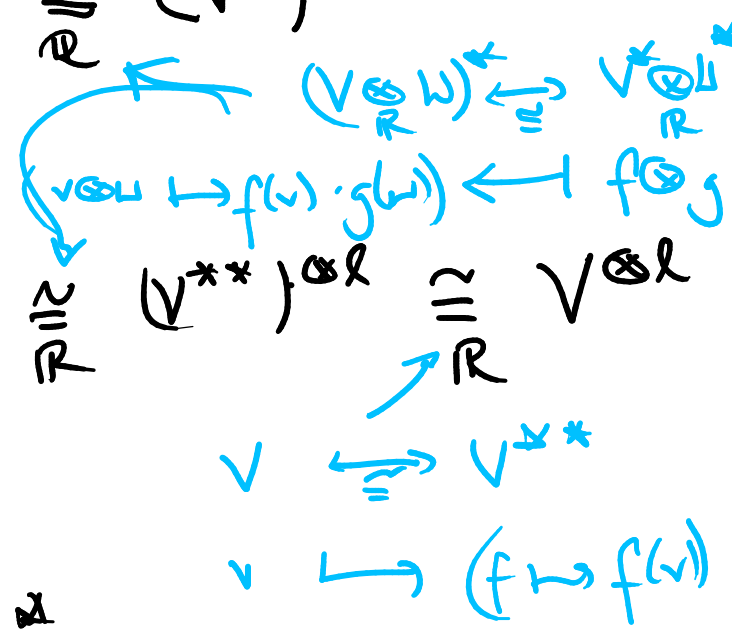
$T^{0,l}_V \cong ((V^*)^{\otimes l})^* \cong (V^{**})^{\otimes l} \cong V^{\otimes l}$

$T^{0,1}_V \cong V$

$T^{0,0}_V \cong (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}^*) \cong \mathbb{R}$

$\Rightarrow \Gamma(T^{0,0}M) \cong C^\infty(M)$

$T^{k,l}_V \cong (V^{\otimes l} \otimes_{\mathbb{R}} V^* \otimes_{\mathbb{R}} \mathbb{R})^k \cong (V^*)^{\otimes k} \otimes_{\mathbb{R}} V^{\otimes l}$



$$T^*M \cong \text{Hom}_{\mathbb{R}}(T_x M, \mathbb{R}) \cong \text{Hom}_{\mathbb{R}}(V, V^*)$$

$$(v \otimes \omega \mapsto \omega(f(v))) \longleftarrow f \quad \text{of dim } n$$

Remark. (tensor fields, explicitly). Let M be a smooth mfd, let $\varphi: U \rightarrow U' \subset \mathbb{R}^n$ be a smooth chart for M with coord. functions $x^1, \dots, x^n: U \rightarrow \mathbb{R}$.

$\Rightarrow (dx^1, \dots, dx^n)$ local frame over U
 $TU \rightarrow T\mathbb{R} \cong \mathbb{R}$ of T^*M

\Rightarrow pointwise dual basis (E_1, \dots, E_n)
 $\in \Gamma(U, TM|_U)$
 is a local frame over U of $(T^*M)^* \cong TM$

(they are called coordinate frames)

\Rightarrow the fibrewise dual of
 $(E_{i_1} \otimes \dots \otimes E_{i_k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_\ell})$
 is a local frame over U
 of $T^{k, \ell} M$. $i_1, \dots, i_k, j_1, \dots, j_\ell \in \{1, \dots, n\}$

$\Rightarrow (dx^{i_1} \otimes \dots \otimes dx^{i_\ell} \otimes E_{j_1} \otimes \dots \otimes E_{j_k})$
 is a local frame over U
 of $T^{k, \ell} M$. $i_1, \dots, i_\ell, j_1, \dots, j_k \in \{1, \dots, n\}$