19.11.2020

Recap: $T^{k,l}M = \left((TM)^{\otimes k} \otimes_{\mathbb{R}} (T^*M)^{\otimes l}\right)^*$

$T^{2,0}M \cong T^*M \otimes_{\mathbb{R}} T^*M$

$\boxed{1.3}$ RIEMANNIAN MANIFOLDS

idea: there are many approaches to Euclidean geometry (e.g., axiomatically or computationally)

linear algebra: $\mathbb{R}^n$ + Euclidean inner product

covariant 2-tensor on $\mathbb{R}^n$ that is symmetric and pos. definite

- Riemannian manifold: smooth mfd + $\left.\begin{array}{l}\text{"smooth" family of inner products on}\\ \text{the tangent spaces.}\end{array}\right\}$ Riemannian metric

$\boxed{1.3.1}$ RIEMANNIAN METRICS

Definition (Riemannian metric). Let M be a smooth mfd. A Riemannian metric on M is a $g \in \Gamma(T^{2,0}M)$ with:

a section

- symmetry: $\forall_{x \in M} \forall_{v,w \in T_xM}\ g_x(v \otimes w) = g_x(w \otimes v)$
- positive definite: $\forall_{x \in M} \forall_{v \in T_xM \setminus \{0\}}\ g_x(v \otimes v) > 0$.

$\rightsquigarrow$ write $\langle \cdot , \cdot \rangle^g_x : T_xM \times T_xM \to \mathbb{R}$

$(v, w) \longmapsto g_x(v \otimes w)$

(is an inner product on T_xM).

Example. (Euclidean space). Let $n \in \mathbb{N}$. Then

$$\mathbb{R}^n \longrightarrow T^{2,0}\mathbb{R}^n$$
$$x \longmapsto \text{standard Euclidean inner product on } \mathbb{R}^n \underset{\text{canonical iso}}{\cong} T_x\mathbb{R}^n$$

is a Riemannian metric on $\mathbb{R}^n$, the <u>Euclidean Riemannian metric</u>.

(similarly for open subsets of $\mathbb{R}^n$)

goal: every smooth mfd admits a Riemannian metric

strategy:
- locally: use smooth charts to pullback the Euclidean Riemannian metric
- globally: glue these via a partition of unity.

Proposition. (pullback of Riemannian metrics). Let $f: M \to N$ local diffeomorphism (smooth immersion) and let g be a Riemannian metric on N.
Then: f^*g is a Riemannian metric on M, where

$$f^*g : M \longrightarrow T^{2,0}M$$
$$x \longmapsto \Big(\underset{T_xM}{v} \otimes \underset{T_xM}{w} \longmapsto g_{f(x)}\big(\underbrace{d_xf(v) \otimes d_xf(w)}_{\in T_{f(x)}N}\big)\Big)$$

Proof:
- $f^*g \in \Gamma(T^{2,0}M)$, because $g \in \Gamma(T^{2,0}N)$ and f is smooth
- symmetry: because g is symmetric

• positive definiteness: let $x \in M$ and $v \in T_xM \setminus \{0\}$.

Then:

$$\boxed{(f^*g)_x(v \otimes v)} = g_{f(x)}(\underbrace{d_xf(v)}_{\neq 0} \otimes \underbrace{d_xf(v)}) \quad \neq 0 \text{ because } v \neq 0 \text{ and } f \text{ is a local differ}$$

$g_{f(x)}$ is pos. definite

$$\boxed{> 0}$$

□

Proposition. (scaling of Riem. metrics). Let M be a smooth mfd, let $g_1, g_2 \in \mathrm{Riem}(M)$, let $f_1, f_2 \in C^\infty(M, \mathbb{R}_{\geq 0})$ with $f_1 + f_2 > 0$ (pointwise).

Then

$$f_1 \cdot g_1 + f_2 \cdot g_2 : M \longrightarrow T^{2,0}M$$

$$x \longmapsto \Big(v \otimes w \longmapsto \underbrace{f_1(x)}_{\in \mathbb{R}_{\geq 0}} \cdot (g_1)_x(v \otimes w) + f_2(x) \cdot (g_2)_x(v \otimes w)\Big)$$

is a Riemannian metric on M.

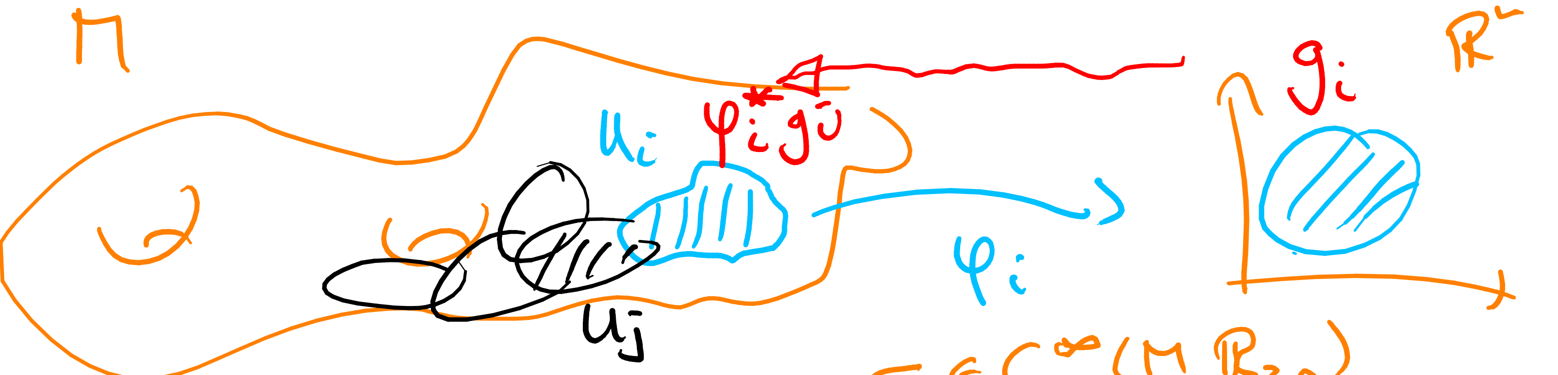
Proof. (ü) 3.2 □

Theorem. (existence of Riem. metrics). Every smooth mfd admits a Riemannian metric.

Proof. (use the strategy above!). Let M be a smooth mfd of dim n. Then there ex. a family $(\varphi_i : U_i \to U_i' \subset \mathbb{R}^n)_{i \in I}$ of smooth charts on M with $\bigcup_{i \in I} U_i = M$.

• local step: let $i \in I$, let g_i be the Euclidean Riem. metric on $U_i' \subset \mathbb{R}^n$.

Then $\varphi_i^* g_i$ is a Riem. metric on U_i
(as a smooth chart $\varphi_i : U_i \to U_i'$ is a diffeo!)



- global step: let $(\psi_i)_{i \in I}$ be a $\underbrace{\text{partition of unity}}$ of M that is subordinate to $(U_i)_{i \in I}$.

  $\psi_i \in C^\infty(M, \mathbb{R}_{\geq 0})$

  - for all $i \in I$: $\operatorname{supp} \psi_i \subset U_i$
  - local finiteness: for all $x \in M$, the set $\{ i \in I \mid \psi_i(x) \neq 0 \}$ is finite
  - $\forall_{x \in M} \sum_{i \in I} \psi_i(x) = 1$.

  Then: $\sum_{i \in I} \psi_i \cdot \varphi_i^* g_i$ is a Riem. metric on M. $\square$

Corollary (non-uniqueness of Riem. metrics). Let M be a non-empty smooth mfd of non-zero dim! Then $\operatorname{Riem}(M)$ is uncountable.

Proof. By the theorem, there ex. a Riem. metric g on M. By scaling, for each $f \in \underbrace{C^\infty(M, \mathbb{R}_{>0})}_{\text{uncountable as } M \neq \emptyset}$, also $f \cdot g$ is a Riem. metric on M.

as $\dim M \neq 0$ $\longrightarrow$ changes with f

$\square$

Remark. (Riemannian metrics in local coordinates).
Let M be a smooth mfd of dim n, let $\varphi : U \to U' \subset \mathbb{R}^n$ be a smooth chart of M, and let $x^1, \dots, x^n : U \to \mathbb{R}$ be the assoc. coord. functions of φ.

$\rightsquigarrow$ $(dx^i \otimes dx^j)_{i,j \in \{1,\dots,n\}}$ is a local frame of $\underbrace{T^{2,0}M}_{\cong (T^*M \otimes_{\mathbb{R}} T^*M)}$ over U.

$\rightsquigarrow$ every Riem. metric g on M can be written as

$$g|_U = \sum_{i,j \in \{1,\dots,n\}} g_{ij} \cdot \underbrace{dx^i \otimes dx^j}_{=: dx^i dx^j}$$

with $(g_{ij} \in C^\infty(U, \mathbb{R}))_{i,j \in \{1,\dots,n\}}$.
For each $x \in U$, the matrix $(g_{ij}(x))_{i,j \in \{1,\dots,n\}} \in M_{n \times n}(\mathbb{R})$ is symmetric and pos. definite.

works also this way

Example. • Euclidean Riemannian metric on $\mathbb{R}^n$ in standard coordinates $x^1, \dots, x^n$:

$$\sum_{i,j \in \{1,\dots,n\}} \delta_{ij} \, dx^i \otimes dx^j = \delta_{ij} \, dx^i dx^j$$
$$= \sum_{i=1}^n dx^i \otimes dx^i$$
$$= \sum_{i=1}^n (dx^i)^2 .$$

gets smaller if x^2 gets larger

- $\frac{1}{(x^2)^2+1} \cdot ((dx^1)^2 + (dx^2)^2)$ is a Riem. metric on $\mathbb{R}^2$ (rescaling of Euclidean Riem metric).

- on $\mathbb{R}^2$: $(dx^1)^2 - (dx^2)^2$ is no Riem metric on $\mathbb{R}^2$ (it's not pos. def.)

1.3.2 RIEMANNIAN MANIFOLDS

Definition. (Riemannian mfd). A Riemannian mfd is a pair (M, g), consisting of a smooth mfd M and a Riem. metric g on M.

Example. ($\mathbb{R}^n$, Euclidean Riem. metric).

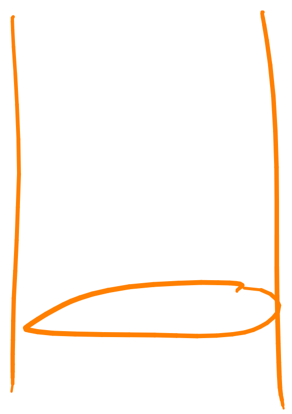
Example. (first fundamental form) Let $N \in \mathbb{N}$ and let $M \subset \mathbb{R}^N$ be a smooth submfd. Then the Euclidean Riem. metric on $\mathbb{R}^N$ induces a Riemannian metric on M (via the geometric spaces), the first fundamental form on M.

$\leadsto$ Riem. metric on $S^n \subset \mathbb{R}^{n+1}$, the round Riem. metric on S^n.

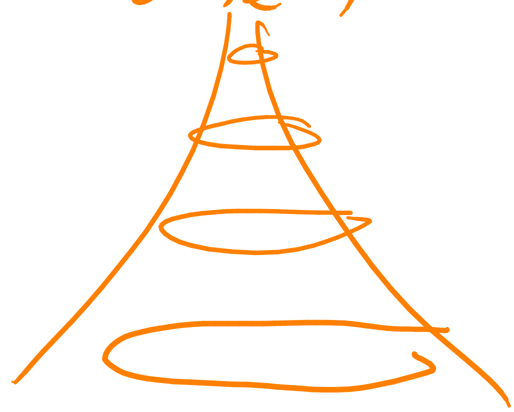
Example. (warped products). ← DE: verzerrtes Produkt

(M_1, g_1), (M_2, g_2) Riem. mfds, $f \in C^\infty(M_1, \mathbb{R}_{>0})$ $\rightsquigarrow$ $\big(M_1 \times M_2,\ (x,y) \mapsto (g_1)_x + f^2(x) \cdot (g_2)_y\big)$ Riem mfd

$S^1 \times \mathbb{R}_{>0}$



$S^1 \times_{(t \mapsto e^{-t})} \mathbb{R}_{>0}$



$(S^1 \times S^1,\ \text{round} \oplus \text{round})$: 2-torus

(geometry is not well reflected in this picture).

Definition. ((local) isometries). Let (M,g), (M',g') be Riem. mfds.

- A <u>local isometry</u> $(M,g) \to (M',g')$ is a local diffeo $f : (M,g) \to (M',g')$ with $f^*g' = g$.

$\mathrm{Isom}(M,g)$

- An <u>isometry</u> $(M,g) \to (M',g')$ is a diffeo $f : (M,g) \to (M',g')$ with $f^*g' = g$.