20.01.2021

Recap: Goal: compute sec of the model spaces
Strategy: use: • model spaces are loc. conf. flat
• high degree of symmetry

2.5.2 SYMMETRIES AND CONSTANT CURVATURE

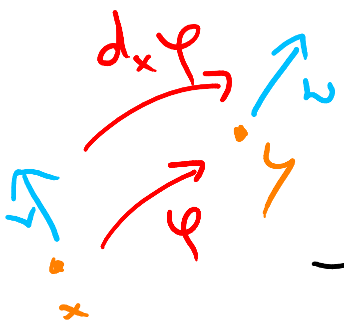
idea: more symmetry $\rightsquigarrow$ more constant curvature

For example: homogeneous spaces have constant scalar curvature (Prop. 2.4.34).

Proposition. Let (M,g) be a Riem. mfd that is homogeneous and isotropic. Then (M,g) has constant Ricci curvature, i.e., there ex. a $c \in \mathbb{R}$ s.t.

$$\forall_{x\in M} \ \forall_{\substack{v\in T_xM \\ \|v\|_g = 1}} \quad Ric_x(v,v) = c.$$

Proof. Let $x,y \in M$, let $v \in T_xM$, $w \in T_yM$ with $\|v\|_g = 1 = \|w\|_g$. Because M is homogeneous and isotropic, there ex. a $\varphi \in \mathrm{Isom}(M,g)$ with



$$\varphi(x) = y \quad \text{and} \quad d_x\varphi(v) = w.$$

Then (because Ric is compatible with isometries):

$$Ric_x(v,v) = Ric_{\underbrace{\varphi(x)}_{=y}}(\underbrace{d_x\varphi(v)}_{=w}, \underbrace{d_x\varphi(v)}_{=w}).$$

$\square$

Definition. (2-isotropic). Let (M,g) be a Riem mfd.

- Let $x \in M$. Then: (M,g) is <u>2-isotropic at x</u> iff: For all orthonormal pairs (v_1, v_2), (v_1', v_2') in T_xM, then ex. $\varphi \in \mathrm{Isom}_x(M,g)$ with
$$d_x\varphi(v_1) = v_1' \quad \text{and} \quad d_x\varphi(v_2) = v_2'.$$
- (M,g) is <u>2-isotropic</u> iff it is 2-isotropic at every pt. in M.

Proposition. Let (M,g) be a Riem. mfd that is homogeneous and 2-isotropic. Then, (M,g) has constant sectional curvature.

Proof. similar to scal and Ric case. $\square$

Remark.
- If a Riem. mfd has constant sectional curvature, then it also has constant Ricci and scalar curvature.
  (averages! Prop. 2.4.28, Prop. 2.4.32)
  The converse, in general, does <u>not</u> hold!
- Riem. quotients of Riem. mfd with constant sectional curvature by isometric proper actions also have constant sectional curvature (and vice versa).

2.5.3 SECTIONAL CURVATURE OF THE MODEL SPACES

Theorem. Let $n \in \mathbb{N}_{\geq 2}$ and $R \in \mathbb{R}_{>0}$. Then:

1. $\mathbb{R}^n$ with the Euclidean Riem. metric has constant sectional curvature 0.
2. The round sphere $\mathbb{S}^n(R)$ has constant sectional curvature $\frac{1}{R^2}$.
3. The hyperbolic space $\mathbb{H}^n(R)$ has constant sectional curvature $-\frac{1}{R^2}$.

Proof. 1. ✓ (Example 2.4.20).

2./3.: We already know:

- $\mathbb{S}^n(R)$ and $\mathbb{H}^n(R)$ are homogeneous and 2-isotropic (§ 1.4)

same proof!

⇝ they have constant sectional curvature

stereographic proj.

(Ü) 4.3

- $\mathbb{S}^n(R)$ and $\mathbb{H}^n(R)$ are locally conformally flat.

Poincaré disk model

locally: on a suitable open nbhd of $0 \in \mathbb{R}^n$

$$g_{\pm} : x \mapsto \frac{4 \cdot R^4}{(R^2 \pm \|x\|_2^2)^2} \cdot g_x$$

Euclidean Riem. metric

+ : sphere
− : hyperbolic

It suffices to compute $\sec_0^{U, g_\pm}(e_1, e_2)$.

Then: (by Thm 2.5.5):

$$\underbrace{R^{\pm}_{1221}} = -e^{2f_\pm} \cdot \Big(\overline{\nabla}_{E_2} \overline{\nabla}_{E_2} f_\pm - \overline{\nabla}_{E_2} f_\pm \cdot \overline{\nabla}_{E_2} f_\pm + \overline{\nabla}_{E_1} \overline{\nabla}_{E_1} f_\pm - \overline{\nabla}_{E_1} f_\pm \cdot \overline{\nabla}_{E_1} f_\pm + \|\operatorname{grad} f_\pm\|_2^2 \Big),$$

where

$$f_\pm : U \to \mathbb{R}$$

$$x \mapsto \frac{1}{2} \cdot \ln \frac{4R^4}{(R^2 \pm \|x\|_2^2)^2}$$

$$= -\ln \frac{1}{4R^4} \cdot (R^2 \pm \|x\|_2^2).$$

$$\rightsquigarrow \left(\overline{\nabla}_{E_1} f_\pm \right)(x) = \mp 2 \cdot \frac{x_1}{R^2 \pm \|x\|^2}$$

$\vdots$

$$\rightsquigarrow R^{\pm}_{1221}(0) = -4 \cdot \left(\mp \frac{2 \cdot R^2}{R^4} - 0 \mp \frac{2 \cdot R^2}{R^4} - 0 + 4 \cdot 0 \right)$$

$$= \pm \frac{16}{R^2}.$$

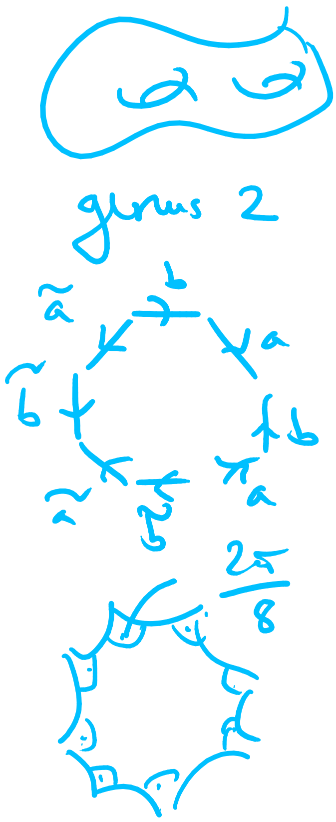
Therefore:

$$\left| \sec_0^{U, g_\pm}(e_1, e_2) \right| = \frac{R^{\pm}_{1221}(0)}{g_{\pm,11}(0) \cdot g_{\pm,22}(0) - 0} = \pm \frac{16}{R^2} \cdot \frac{\cancel{R^4} \cancel{R^4}}{4\cancel{R^4} \cdot 4\cancel{R^4}} = \boxed{\pm \frac{1}{R^2}}. \quad \square$$

Example. • If $n \in \mathbb{N}_{\geq 2}$, then $\mathbb{RP}^n$ admits a Riem. metric of constant sectional curvature 1. (quotient metric of the round metric on $S^n(1)$).

• Oriented closed connected surfaces of genus ≥ 2 admit Riem. metrics of constant sectional curvature -1.

(can obtain such surfaces as Riem quotients of $\mathbb{H}^2(1)$.)



3 RIEMANNIAN GEODESICS

In the following: On Riem mfds, we will always consider the Levi-Civita connection and geodesics with respect to this connection.

Goal: Relate Riem. geodesics to metric geodesics

More precisely, we show:

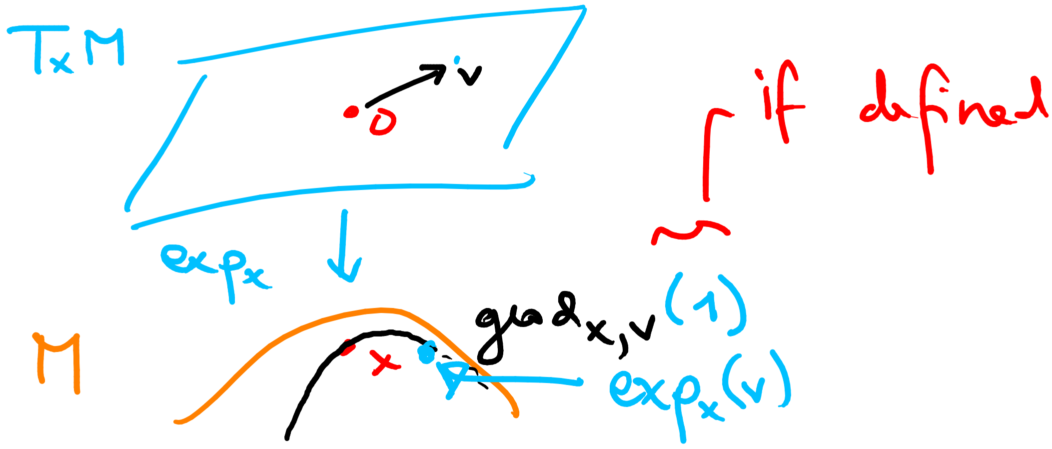
- minimising curves are geodesics
- geodesics are locally length-minimising.

idea: · use nice charts/coordinates (via the exponential map) and radial geodesics

- use variational methods on the length
- next level: Jacobi fields.

3.1 THE EXPONENTIAL MAP

idea:



3.1.1 THE EXPONENTIAL MAP

Definition (exponential map). Let (M,g) be a Riem. mfd.

- Then $\mathrm{Exp} := \bigcup_{x\in M} \mathrm{Exp}_x \subset TM$, where

$$\forall_{x\in M} \quad \mathrm{Exp}_x := \{ v \in T_xM \mid \mathrm{geod}_{x,v} \text{ is defined on } [0,1] \}$$

- The <u>exponential map</u> of (M,g) is defined by

$$\exp : \mathrm{Exp} \longrightarrow M$$

$$T_xM \ni v \longmapsto \exp_x(v) := \mathrm{geod}_{x,v}(1).$$

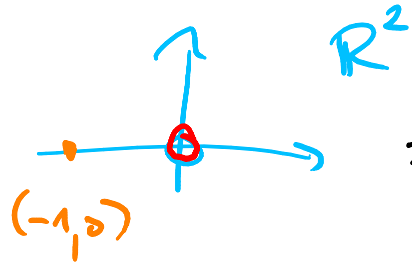
Example (exp. on S^1) The map

$$\mathbb{R} \longrightarrow S^1$$

$$t \longmapsto (\cos t, \sin t) \ \widehat{=}\ \exp(\cancel{2\pi} i t)$$

"is" the exponential map at $(1,0)$ of S^1 (with the round metric).

Example. On $\underbrace{\mathbb{R}^2\setminus\{0\}}_{=:M}$: $\quad \mathrm{Exp}_{(-1,0)} \neq T_{(-1,0)}M.$



Example. The exp. of the round S^2 is azimuthal equidistand projection.

Theorem. (properties of exp). Let (M,g) be a Riem mfd, let $x \in M$.

1. For all $v \in T_xM$, $t \in \mathbb{R}$, we have (whenever either side is defined)

$\bar{\text{L}}$ W $8\frac{1}{2}$

$$\mathrm{geod}_{x,v}(t) = \mathrm{geod}_{x,\, t\cdot v}(1).$$

2. The set $\mathrm{Exp} \subset TM$ is open and $\mathrm{Exp}_x \subset T_xM$ is star-shaped w.r.t. $0 \in T_xM$.

3. The map exp is smooth and

$$d_0 \exp_x \text{ "="} \; \mathrm{id}_{T_xM}.$$

$\leftarrow$ simple comp.

$\Downarrow$ inverse function thm.

$T_0(T_xM) \cong T_xM$

4. In particular: There ex. an open nbhd. $U \subset T_xM$ of 0 s.t. $\exp_x(U) \subset M$ is open and $\exp_x|_U : U \to \exp_x(U)$ is a diffeo.

smooth dependence on the initial conditions in the geodesic eq.

Remark. The exponential map is natural with respect to local isometries.