21.01.2021

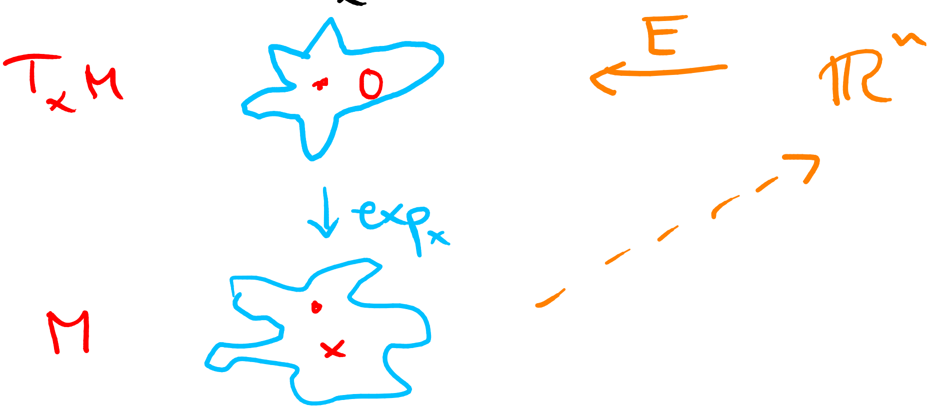
Recap: exponential map:

(M,g) Riem mfd, $x \in M$ $\rightsquigarrow$ $T_xM \supseteq \mathrm{Exp}_x \xrightarrow{\exp_x} M$, $v \longmapsto \mathrm{geod}_{x,v}(1)$

## 3.1.2 Normal Coordinates

**Definition.** (normal nbhd, normal coord's). Let $(M,g)$ be a Riem mfd, let $x \in M$.

- An open nbhd $U \subset M$ of $x$ is <u>normal</u> if it is the diffeomorphic image under $\exp_x$ of an open star-shaped (wrt $0 \in T_xM$) set in $T_xM$.




- Let $U \subset M$ be a normal nbhd of $x$, let $(v_1, \dots, v_n)$ be an orthonormal basis of $T_xM$, and let $E : \mathbb{R}^n \longrightarrow T_xM$ be the corresp. iso.

$$(x^1, \dots, x^n) \longmapsto \sum_{j=1}^n x^j \cdot v_j$$

Then

$$E^{-1} \circ \exp_x^{-1}\big|_U : U \longrightarrow \mathbb{R}^n$$

is a <u>normal coordinate chart around $x$</u>.

**Remark.** If $(M,g)$ is a Riem. mfd and $x \in M$, then there ex. normal nbhds / normal coord's around $x$ (because exp is a local diffeo).

**Definition** (geodesic ball). Let $(M,g)$ be a Riem mfd, let $x \in M$, let $r \in \mathbb{R}_{>0}$ with

$$B_r^{T_xM, g_x}(0) \subset \mathrm{Exp}_x .$$

Then

$$\mathrm{ball}_r^{M,g}(x) := \exp_x\left(B_r^{T_xM, g_x}(0)\right) \subset M$$

is the <u>geodesic ball</u> of radius $r$ at $x$.

Warning: So far, it is not clear how geodesic balls are related to metric balls (w.r.t. Riem. distance function)!

**Proposition** (uniformly normal nbhds). Let $(M,g)$ be a Riem. mfd, let $x \in M$. Then $x$ admits a <u>uniformly normal nbhd</u>, i.e., there ex. an open nbhd $U \subset M$ of $x$ s.t. there ex. an $r \in \mathbb{R}_{>0}$ with: for each $y \in U$, we have that


$\mathrm{ball}_r^{M,g}(y)$

$$U \subset \mathrm{ball}_r^{M,g}(y) \quad \text{and}$$

$\mathrm{ball}_r^{M,g}(y)$ is a normal nbhd of $y$.

*Proof.* This about compactness ... $\square$

Proposition. (initial data for isometries). Let $(M,g)$ be a connected Riem mfd and let $\varphi, \psi \in \operatorname{Isom}(M,g)$.

$\rightsquigarrow$ path-connected

Moreover, let $x \in M$ with

$$\varphi(x) = \psi(x) \quad \text{and} \quad d_x\varphi = d_x\psi.$$

Then: $\varphi = \psi$.

Proof. idea: use uniqueness/naturality of geodesics.

wlog: can consider the case that $\varphi(x) = x$ and $d_x\varphi = \mathrm{id}_{T_xM}$ and show $\varphi = \mathrm{id}_M$.

look at $\varphi \circ \psi^{-1}$

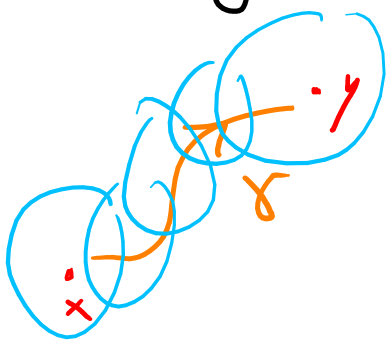
① Special case: $M$ is a uniformly normal nbhd of $x$. Let $y \in M$, say $y = \exp_x(v)$ with $v \in T_xM$. Then

naturality of exp

$$\boxed{\varphi(y)} = \varphi(\exp_x(v)) = \exp_{\underbrace{\varphi(x)}_{=x}}(\underbrace{d_x\varphi(v)}_{=v})$$

$$= \exp_x(v) = \boxed{y}.$$

② general case: Let $y \in M$. Let $\gamma$ be a path from $x$ to $y$ (ex.!)




We can cover $\operatorname{im}\gamma$ by finitely many uniformly normal nbhds, then apply ①. $\square$

successively

# 3.2 RIEMANNIAN GEODESICS

goal: relate geodesics to (locally) minimizing curves

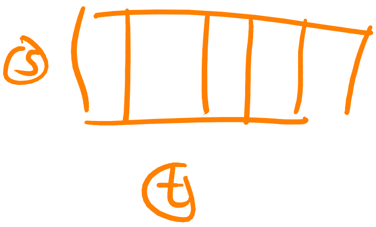
idea: don't work on the space of all curves with given endpoints, but only look at $\underbrace{\text{one-parameter families of curves}}$ ⇝ variations

## 3.2.1 VARIATIONS OF CURVES

**Definition.** (piecewise regular families). Let $M$ be a smooth mfd, let $a, b \in \mathbb{R}$ with $a < b$, let $\varepsilon \in \mathbb{R}_{>0}$.


- A map $G : (-\varepsilon, \varepsilon) \times [a, b] \longrightarrow M$ is <u>piecewise smooth</u> if $G$ is continuous and if there ex. $k \in \mathbb{N}$, $a = a_0 < a_1 < \dots < a_k = b$ s.t. $G|_{(-\varepsilon,\varepsilon)\times[a_j, a_{j+1}]}$ is smooth for each $j \in \{1, \dots, k-1\}$.

$(-\varepsilon, \varepsilon) \times [a, b] \to M$
s t




- A piecewise smooth map $G : (-\varepsilon, \varepsilon) \times [a, b] \to M$ is a <u>piecewise regular family of curves</u> if for each $s \in (-\varepsilon, \varepsilon)$, the map $G(s, \cdot)$ is a piecewise regular curve.

Remark. Let $M$ be a smooth mfd and let $G : (-\varepsilon,\varepsilon)\times[a,b] \to M$ be a piecewise regular family of curves. We then write

in general, not everywhere def'd $\longrightarrow$ $\partial_{②} G(s,t) := \left(G(s,\cdot)\right)'(t)$

def'd everywhere $\longrightarrow$ $\partial_{①} G(s,t) := \left(G(\cdot,t)\right)'(s)$,

wherever they are defined.

Proposition. (symmetry lemma for variations). Let $(M,g)$ be a Riem. mfd, let $G : (-\varepsilon,\varepsilon)\times[a,b]\to M$ be a pw regular family. Then: On every rectangle $Q = (-\varepsilon,\varepsilon)\times[\tilde a,\tilde b] \subset (-\varepsilon,\varepsilon)\times[a,b]$ on which $G$ is smooth, we have

$$\forall_{(s,t)\in Q} \left(D_{G(\cdot,t)} \partial_{②} G\right)(s,t) = \left(D_{G(s,\cdot)} \partial_{①} G\right)(s,t).$$

Proof. This is a consequence of the symmetry of the LC-connection: We prove the prop. in local coord's. In suitable small nbhds in $Q$, we can write

$$\partial_{②} G = \sum_{k=1}^{n} \partial_2 x^k \cdot E_k$$

$$\partial_{①} G = \sum_{k=1}^{n} \partial_1 x^k \cdot E_k.$$

local coords

coord frame

Then:

$$D_{G(\cdot,t)}\partial_{2} G(s,t) = \sum_{k=1}^{n} \underbrace{\partial_1\partial_2 x^k(s,t)} \cdot E_k(G(s,t))$$

$$+ \sum_{i,j,k=1}^{n} \underbrace{\partial_2 x^i(s,t)\cdot \partial_1 x^j(s,t)} \cdot \Gamma_{ji}^{k}(G(s,t))$$

$= (x^k \text{ smooth})$

$$D_{G(s,\cdot)}\partial_{1} G(s,t) = \sum_{k=1}^{n} \underbrace{\partial_2\partial_1 x^k(s,t)} \cdot E_k(G(s,t)) \;=$$

$$+ \sum_{i,j,k=1}^{n} \underbrace{\partial_1 x^{\overset{i}{\cancel{j}}}(s,t)\cdot \partial_2 x^{\overset{j}{\cancel{i}}}(s,t)} \cdot \Gamma_{\underset{ij}{\cancel{ji}}}^{k}(G(s,t))$$

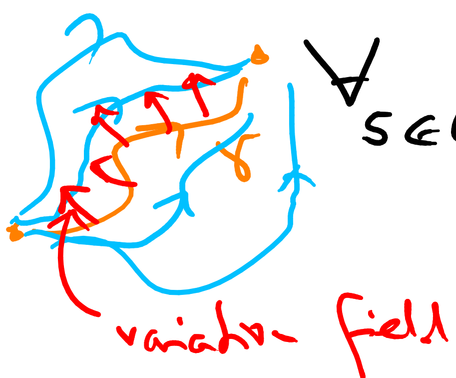
$\square$

LC-conn is symmetric.

**Definition.** (variation of a curve). Let $M$ be a smooth mfd and let $\gamma : [a,b] \to M$ be a piecewise regular curve.

- A <u>variation of $\gamma$</u> is a pw regular family $G : (-\varepsilon,\varepsilon) \times [a,b] \to M$ of curves with
$$G(0,\cdot) = \gamma.$$

- Such a variation is <u>proper</u> if




$$\forall_{s \in (-\varepsilon,\varepsilon)} \quad G(s,a) = \gamma(a) \text{ and } G(s,b) = \gamma(b).$$

## 3.2.2 VARIATION FIELDS AND THE FIRST VARIATION FORMULA

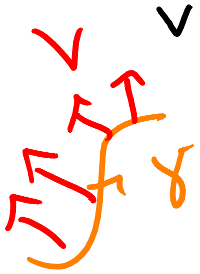
idea: look at the "vertical" change of curves

**Definition.** (variation field). Let $M$ be a smooth mfd, let $G$ be a variation of a pw regular curve $\gamma : [a,b] \to M$ in $M$. The <u>variation field</u> of $G$ is the vector field

$$\begin{aligned} [a,b] &\longrightarrow TM \\ t &\longmapsto \partial_{1} G(0,t) \end{aligned}$$

along $\gamma$.

**Proposition.** (existence of variation fields). Let $(M,g)$ be a Riem mfd, let $\gamma : [a,b] \to M$ be a pw regular curve, and let $V$ be a smooth vector field along $\gamma$. Then $V$ is the variation field of some variation of $\gamma$.




If $V(a) = 0 = V(b)$, then $V$ is the variation field of a proper variation of $\gamma$.

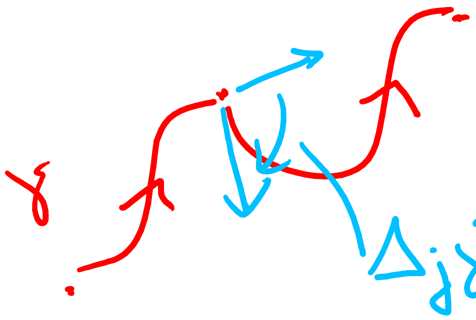
**Proof.** Idea: take

$$G : (s,t) \longmapsto \exp_{\gamma(t)}(s \cdot V(t)).$$

(⚠ need to find a good domain!). □

**Theorem.** (first variation formula). Let $(M,g)$ be a Riem. mfd, let $\gamma:[a,b]\to M$ be a unit speed pw regular curve, let $G:(-\varepsilon,\varepsilon)\times[a,b]\to M$ be a proper variation of $\gamma$, and let $V$ be the variation field of $G$. Let $a=a_0<\dots<a_k=b$ be a partition of $[a,b]$ s.t. each $\gamma|_{[a_{j-1},a_j]}$ is smooth. Then

$$\frac{\partial}{\partial s}\Big|_{s=0} L_g(G(s,\cdot)) = -\int_a^b \langle V(t), D_\gamma \dot\gamma(t)\rangle_g \, dt$$




$$- \sum_{j=1}^{k-1} \langle V(a_j), \Delta_j \dot\gamma\rangle_g .$$

$$\Delta_j\dot\gamma := \dot\gamma(a_j^+) - \dot\gamma(a_j^-)$$

**Proof.** Main input:

- compatibility of the connection with cov. derivatives
- symmetry lemma
- elementary analysis.

read notes $\to \square$