23.12.2020

next lecture: 07. 01. 2021

Recap: connections
$\rightsquigarrow$ geodesics, parallel transport

Problem: so far, there are many choices of connections and no relation with a Riem. metric

2.3 THE LEVI-CIVITA CONNECTION

idea: we add two conditions on connections:

- compatibility with the given Riem. metric
- symmetry

$\rightsquigarrow$ ex. & uniqueness (fundamental thm of Riem. geometry)

Preparation: induced connections on tensor bundles
($\rightsquigarrow$ uniform notation)

Remark. Let M be a smooth mfd and let ∇ be a linear connection on M.
$\rightsquigarrow$ get induced connections:

- $T^{0,0}M \cong M \times \mathbb{R}$ ($\rightsquigarrow \Gamma(T^{0,0}M) \cong C^\infty(M)$)

$$\nabla : \Gamma(TM) \times C^\infty(M) \longrightarrow C^\infty(M)$$
$$(X, f) \longmapsto X(f)$$
(indep. of the original ∇)

(this defines a connection on $T^{0,0}M$).

- $T^{2,0}M = (TM^{\otimes 2})^*$ ($\rightsquigarrow \Gamma(T^{2,0}M)$: parametrized bilin. forms "on TM")

$$\nabla : \Gamma(TM) \times \Gamma(T^{2,0}M) \longrightarrow \Gamma(T^{2,0}M)$$

$$(X,g) \longmapsto \text{“}\big((Y,Z) \mapsto X(\langle Y,Z\rangle_g) - \langle \nabla_X Y, Z\rangle_g - \langle Y, \nabla_X Z\rangle_g\big)\text{”}$$

(!)

$C^\infty(M)$-bilin. map:

$$\Gamma(TM) \times \Gamma(TM) \longrightarrow \mathbb{R}$$

$$(Y,Z) \longmapsto X(\underbrace{\langle Y,Z\rangle_g}_{g(Y\otimes Z)}) - \langle \nabla_X Y, Z\rangle_g - \langle Y, \nabla_X Z\rangle_g$$

(this defines a connection on $T^{2,0}M$).

end preparation.

Compatibility:

Definition. (compatible connection). Let (M,g) (!) be a Riem. mfd. Then a linear connection ∇ on M is <u>compatible with g</u> if

$$\forall_{X,Y,Z\in\Gamma(TM)} \quad \nabla_X \underbrace{\langle Y,Z\rangle_g}_{\in C^\infty(M)} = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g .$$

“product rule”

Proposition. (alternative characterisations of compatibility) Let (M,g) be a Riem. mfd and let ∇ be a linear connection on M. Then the following are equivalent:

1. ∇ is compatible with g.

2. g is parallel wrt ∇: $\forall_{X \in \Gamma(TM)}$ $\nabla_X g = 0$ (with $\nabla_X g \in \Gamma(T^{2,0}M)$)

3. For all smooth curves γ on M and all $X, Y \in \Gamma(TM|_\gamma)$, we have

$$\langle X, Y \rangle_g' = \langle D_\gamma X, Y \rangle_g + \langle X, D_\gamma Y \rangle_g.$$

4. For all smooth curves $\gamma : I \to M$ on M and all $s, t \in I$, the parallel transport $P^\gamma_{s,t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is an $\mathbb{R}$-lin isometry (wrt g).

Proof. 1. $\Leftrightarrow$ 2. clear by def of compatibility and of ∇ on $\Gamma(T^{2,0}M)$.

1. $\Leftrightarrow$ 3. this is a local computation; one can use the connection between D_γ and ∇ for extendable vector fields.

3. $\Rightarrow$ 4. We know: $P^\gamma_{s,t}$ is $\mathbb{R}$-linear.

$\rightsquigarrow$ only need to show compatibility with g.

Let $v, w \in T_{\gamma(s)}M$.

$\rightsquigarrow$ extend v, w to parallel vector fields $X, Y \in \Gamma(TM|_\gamma)$ along γ.

$\rightsquigarrow$ $\langle X, Y \rangle_g' \overset{3.}{=} \langle \underbrace{D_\gamma X}_{=0}, Y \rangle_g + \langle X, \underbrace{D_\gamma Y}_{=0} \rangle_g = 0.$

Thus: $\langle X, Y\rangle_g$ is constant, and so

$$\langle P^{\gamma}_{s,t}(v), P^{\gamma}_{s,t}(w)\rangle_g = \langle X(t), Y(t)\rangle_g = \langle X(s), Y(s)\rangle_g = \langle v, w\rangle_g.$$

(def of $P^{\gamma}_{s,t}$)

4. $\Rightarrow$ 3. $\bar{\text{L}}$ (use an orthonormal parallel frame). $\square$

Proposition. (Local description of compatibility).
Let (M,g) be a Riem mfd and let ∇ be a linear connection on M. TFAE:

1. ∇ is compatible with g.
2. For all coordinate frames $(E_j)_{j\in\{1,\dots,n\}}$ ($n = \dim M$) of M on a smooth chart nbhd U, we have

$$\forall_{i,j,k} \quad \nabla_{E_i}\langle E_j, E_k\rangle_g = \langle \nabla_{E_i}E_j, E_k\rangle_g + \langle E_j, \nabla_{E_i}E_k\rangle_g$$

3. For all coordinate frames $(E_j)_{j\in\{1,\dots,n\}}$ of M on a smooth chart nbhd, we have:

$$\forall_{i,j,k} \quad \sum_{l=1}^{n}\left(\Gamma_{ki}^{l}\cdot g_{lj} + \Gamma_{kj}^{l}\cdot g_{il}\right) = E_k(g_{ij})$$

(g_{lj}: coeff. of g wrt this chart)

Proof. straightforward. $\square$

Symmetry:

no Riem. metric!

Definition. (symmetric connection). Let M be a smooth mfd.. A lin. connection ∇ on M is <u>symmetric</u> (or <u>torsion-free</u>) if

$$\forall_{X,Y\in\Gamma(TM)} \quad \nabla_X Y - \nabla_Y X = [X,Y].$$

∇ "refines" the Lie bracket

$[X,Y] \in \Gamma(TM)$ def. by
$C^\infty(M) \longrightarrow \mathbb{R}$
$f \longmapsto X(Y(f)) - Y(X(f))$

Rmk. (torsion tensor). If ∇ is a linear conn. on a smooth mfd M, then

$$\Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(X,Y) \longmapsto \nabla_X Y - \nabla_Y X - [X,Y]$$

is $C^\infty(M)$-bilin. map.

$\rightsquigarrow$ $(2,1)$-tensor field on M, the <u>torsion tensor</u> of ∇.

Proposition. (local description of symmetry). Let M be a smooth mfd and let ∇ be a lin. connection on M. TFAE:

1. ∇ is symmetric.

2. For all coord. frames $(E_j)_{j\in\{1,\dots,n\}}$ on a smooth chart nbhd U of M:

$$\forall_{i,j\in\{1,\dots,n\}} \quad \nabla^U_{E_i} E_j = \nabla^U_{E_j} E_i .$$

3. For all coord. frames $(E_j)_j$ on a smooth chart nbhd: "symmetry"

$$\forall_{i,j,k\in\{1,\dots,n\}} \quad \Gamma^k_{ij} = \Gamma^k_{ji} .$$

conn. coeffs wrt $(E_j)_j$

Proof. 1. $\Longleftrightarrow$ 2. The key point is that

$$\forall_{i,j\in\{1,\dots,n\}} \quad [E_i, E_j] = 0 .$$

↑ coord. frames!

(true for the standard coord. frame on $\mathbb{R}^n$ by Schwarz's thm; then use naturality of the Lie bracket).

2. $\Longleftrightarrow$ 3. Use the def. of the connect. coeffs. □

Examples. The Euclidean lin. connection $\bar\nabla$ on $\mathbb{R}^n$ is compatible with the Euclidean Riem. metric and symmetric.

But: There ex. a lin. connection on $\mathbb{R}^3$ that is compatible with the Euclidean Riem. metric and has the same geodesics as $\bar\nabla$, but that is <u>not</u> symmetric $\bar{u}$

Example. Let $M \subset \mathbb{R}^N$ be a smooth submfd. Then the lin. conn. ∇^T on M induced by $\overline{\nabla}$ is symmetric and compatible with the Riem. metric on M induced by the Euclid. Riem. metric on $\mathbb{R}^N$. (computation!).

Theorem. (fundamental thm. of Riem. geometry).
On every Riem. mfd, there ex. a unique linear connection that is symmetric and compatible with the given Riem. metric.

$\rightsquigarrow$ Levi-Civita connection.