25.11.2020

Recap: • Riemannian metric on a smooth mfd M: smooth section of $T^{2,0}M = (TM \otimes_{\mathbb{R}} TM)^* \cong T^*M \otimes_{\mathbb{R}} T^*M$ that is pointwise symmetric and pos. definite

• An isometry $(M,g) \to (M',g')$ is a diffeo f with

$$f^*g' = g.$$

with the Euclidean Riem. metric $\sum_j (dx^j)^2$

Example. ((affine) linear isometries on $\mathbb{R}^n$). Let $n \in \mathbb{N}$ and let $f: \mathbb{R}^n \to \mathbb{R}^n$ be an $\mathbb{R}$-lin. map. Then f is an isometry of Riem mfds if and only if f is rep. (wrt. to standard basis) by a matrix in $\boxed{O(n).}$

Also, all $\boxed{\text{translations}}$ of $\mathbb{R}^n$ are Riem. isometries of $\mathbb{R}^n$.

Then

standard action $O(n) \curvearrowright \mathbb{R}^n$

$$\mathbb{R}^n \rtimes O(n) \longrightarrow \mathrm{Isom}(\mathbb{R}^n, \text{Euclid. Riem. metric})$$

$$(v, A) \longmapsto (x \longmapsto A \cdot x + v)$$

is a well-def. group hom.

semi-direct product:

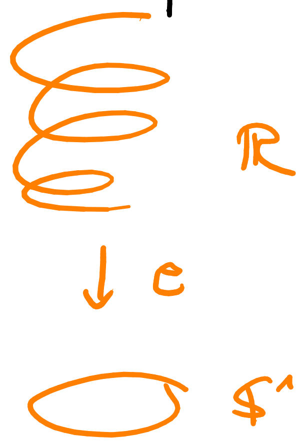
• as a set: $\mathbb{R}^n \times O(n)$

• composition:

$$(v, A) \cdot (w, B) := (v + A \cdot w, \ A \cdot B)$$

Example. We consider the map

$$e : \mathbb{R} \longrightarrow S^1$$
$$t \longmapsto (\cos t, \sin t)$$



and the Euclidean Riem metric g on $\mathbb{R}$ and the round Riem. metric g° on S^1.

Then:

① e is a local isometry

② e is not an isometry

(e is not injective $\rightsquigarrow$ e is no diffeo)

Pf of ①: • e is a local diffeo ✓

• We have $e^* g^\circ = \cancel{e}\, g$: Let $t \in \mathbb{R}$. Then for all $v, w \in T_t\mathbb{R}$, we have:

$$\boxed{(e^* g^\circ)_t (v \otimes w)} = g^\circ_{e(t)}\Big(\underbrace{d_t e(v)}_{= (-\sin t \cdot v,\ \cos t \cdot v) \in \mathbb{R}^2} \otimes \underbrace{d_t e(w)}_{= (-\sin t \cdot w,\ \cos t \cdot w)}\Big)$$

$$= \left\langle \begin{pmatrix} -\sin t \cdot v \\ \cos t \cdot v \end{pmatrix}, \begin{pmatrix} -\sin t \cdot w \\ \cos t \cdot w \end{pmatrix} \right\rangle_2$$

$$= (\sin t)^2 \cdot vw + (\cos t)^2 \cdot vw$$

$$= v \cdot w$$

$$= \boxed{g_t (v \otimes w)}.$$

1.4 MODEL SPACES

We will look at

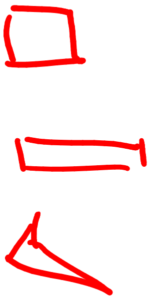
spheres, Euclidean spaces, hyperbolic spaces

model for positive curvature — vanishing curvature — negative curvature

and their symmetries.

1.4.1 HOMOGENEOUS SPACES

idea: · Symmetries are maps that preserve the geometric structure (elts. of $\mathrm{Isom}(M,g)$)



· How to measure the degree of symmetry that an object has?
Using transitivity properties of actions of the isometry group!

Remark. (actions of the isometry group). Let (M,g) be a Riem. mfd.

- We have the action of $\mathrm{Isom}(M,g)$ on M:

$$\mathrm{Isom}(M,g)\times M \longrightarrow M$$
$$(f,x) \longmapsto f(x).$$

- For $x\in M$, set

stabiliser/isotropy gp at x

$$\mathrm{Isom}(M,g)_x := \{ f\in \mathrm{Isom}(M,g) \mid f(x)=x \}.$$

- Let $x \in M$. Then the *isotropy representation of M at x* is the action

$$\mathrm{Isom}(M,g)_x \times T_x M \longrightarrow T_x M$$
$$(f, v) \longmapsto d_x f(v).$$

Definition. (homogeneous mfd). A Riem. mfd (M,g) is *homogeneous* if the action $\mathrm{Isom}(M,g) \curvearrowright M$ is $\underbrace{\text{transitive}}_{\forall_{x,y \in M} \exists_{f \in \mathrm{Isom}(M,g)} f(x)=y}$.

(M looks the same at every pt)

Definition. (isotropic mfd). Let (M,g) be a Riem mfd.

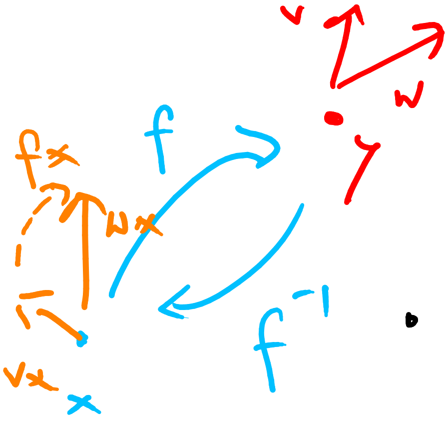
- (M,g) is *isotropic* if it is isotropic at every point.
- (M,g) is *isotropic at $x \in M$* if the isotropy rep. at x is transitive on the unit sphere

$$\{v \in T_x M \mid g_x(v \otimes v) = 1\} \subset T_x M.$$

(at x, the mfd M looks the same in every direction)

Proposition. Let (M,g) be a homogeneous Riem mfd that is isotropic at a single point. Then (M,g) is isotropic.

Proof. Let (M,g) be isotropic at $x \in M$. Let $y \in M$. Then (M,g) is isotropic at y:



Let $v, w \in T_y M$ with $g_y(v \otimes v) = 1$
$= g_y(w \otimes w)$.

- Because (M,g) is homogeneous there ex. an $f \in \mathrm{Isom}(M,g)$ with $f(x) = y$.

$\rightsquigarrow$ $d_y(f^{-1})(v)$, $d_y(f^{-1})(w)$ are g_x-unit vectors in T_xM.

- Because (M,g) is isotropic at x, there ex. an $f_x \in \mathrm{Isom}(M,g)_x$ with $d_x f_x(v_x) = w_x$.

Let us consider

$$\bar{f} := f \circ f_x \circ f^{-1} \in \mathrm{Isom}(M,g).$$

Then

$$\boxed{\bar{f}(y)} = f \circ f_x \circ \underbrace{f^{-1}(y)}_{=x} = f \circ \underbrace{f_x(x)}_{=x}$$

$$= f(x) = \boxed{y}.$$

and $d_y \bar{f}(v) = \overset{\text{chain rule + choice of } f_x}{\cdots} = w$. $\square$

Definition. ((locally) symmetric space). Let (M,g) be a Riem. mfd.

- Let $x \in M$. Then a <u>point reflection</u> of M at x is an isometry $f \in \mathrm{Isom}(M,g)$ with
$$f(x) = x \quad \text{and} \quad d_x f = -\mathrm{id}_{T_xM}.$$
- Then (M,g) is a (Riem) <u>symmetric space</u> if M is connected and M admits a point reflection at every point of M.
- (M,g) is a <u>locally symmetric space</u> if for any point $x \in M$ there ex. an open nbhd $U \subset M$ of x such that $(U, g|_U)$ admits a point reflection at x.

Proposition. Let (M,g) be a homogeneous connected Riem mfd that admits a point reflection at a single point. Then (M,g) is a symmetric space.

Proof. as for "isotropic", via conjugation. $\square$

1.4.2 EUCLIDEAN SPACES

Recall: • $\mathbb{R}^n$ with the Riem metric $g = \sum_{j=1}^{n} (dx^j)^2$

• $\mathbb{R}^n \rtimes O(n) \longrightarrow \mathrm{Isom}(\mathbb{R}^n, g)$

$(v, A) \longmapsto (x \mapsto A \cdot x + v)$

is a group hom.

$\rightsquigarrow$ $(\mathbb{R}^n, g)$ is • homogeneous (translation by the difference)

• isotropic at 0 ($O(n)$ acts transitively on S^{n-1})

• isotropic

• symmetric (at 0: $x \mapsto -x$, use homogeneity).

1.4.3 SPHERES

Definition. (round spheres). Let $n \in \mathbb{N}$ and $R \in \mathbb{R}_{>0}$. We set

sphere of radius R

$$S^n(R) := \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = R\} \subset \mathbb{R}^{n+1}$$

and we equip $S^n(R)$ with the first fundamental form.

Proposition. Let $n \in \mathbb{N}$. Then $O(n+1) \longrightarrow \mathrm{Isom}(S^n(R))$

$A \longmapsto (x \mapsto A \cdot x)$

is a well-def. ✓ group hom. ✓

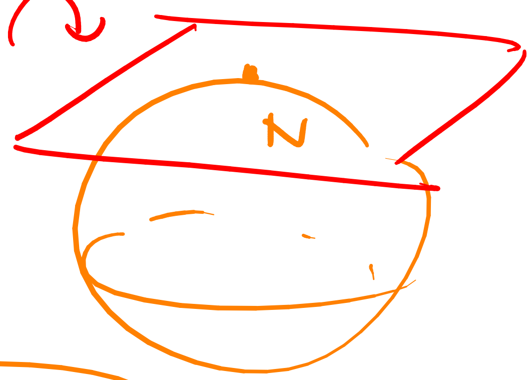
In particular, $\mathbb{S}^n(R)$ (with the round metric) is homogeneous, isotropic, and symmetric (if $n>0$)

Proof. • homogeneous: $O(n+1)$ acts transitively on $\mathbb{S}^n$ (Gram-Schmidt), whence also on $\mathbb{S}^n(R)$.

isotropic

• ~~symmetric~~ isotropic at $N := (0,\dots,0,R)$:

$$\begin{pmatrix} \underset{\in O(n)}{A} & 0 \\ 0 & 1 \end{pmatrix} \curvearrowright T_N \mathbb{S}^n(R)$$ is transitive on the unit sphere



homogeneous +

• point reflection at N:

symmetric

$$\begin{pmatrix} -E_n & 0 \\ 0 & 1 \end{pmatrix} \curvearrowright T_N \mathbb{S}^n(R)$$

$\square$