Differential Geometry I: Week 10

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Reading assignment (for the lecture on January 20). We will now use symmetry to compute the sectional curvature of the model spaces. Moreover, we start with the exploration of Riemannian geodesics.

- Read Chapter 2.5.2 *Symmetries and constant curvature.*
- Recall the construction of the *model spaces* and their basic properties.
- Read Chapter 2.5.3 *Sectional curvature of the model spaces.*
- Recall the notion of *geodesic* (including the geodesic equation) and of *maximal geodesics.*
- Read Chapter 3.1.1 *The exponential map.*

Reading assignment (for the lecture on January 21). Our next goal is to understand the relation between Riemannian and metric geodesics, using a variational approach.

- Read Chapter 3.1.2 *Normal coordinates.*
- Recall the notion of *piecewise regular curves* and *length of curves.*
- Read Chapter 3.2.1 *Variation of curves.*
- Read Chapter 3.2.2 *Variation fields and the first variation formula.*

Étude (using curvature). Which of the following manifolds are isometric? Locally isometric?

1. $\mathbb{S}^1(1)$
2. $\mathbb{S}^1(2021)$
3. $\mathbb{S}^{2021}(1)$
4. $\mathbb{S}^{2021}(2020)$
5. $\mathbb{R}^1$
6. $\mathbb{R}^{2021}$

Exercises (for the session on January 25/26). The following exercises (which all are solvable with the material read/discussed in week 9) will be discussed.

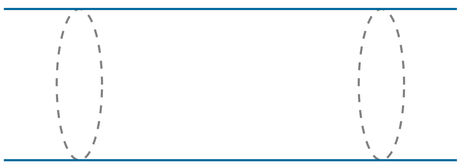
Please turn over

Exercise 9.1 (flat manifolds?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The cylinder $\mathbb{S}^1 \times \mathbb{R}$ (with the product metric of the round metric and the Euclidean Riemannian metric) is flat.

2. The Klein bottle admits a flat Riemannian metric.

3. *Bonus problem (if you know Algebraic Topology).* The torus $\mathbb{S}^1 \times \mathbb{S}^1$ and the Klein bottle admit Riemannian metrics that make them isometric?!



Exercise 9.2 (one-argument Ricci curvature; Remark 2.4.27). Let (M, g) be a Riemannian manifold.

1. Show that Ric is symmetric in its two arguments.

2. Show that Ric can be recovered from only knowing the map

$$\begin{aligned} \mathrm{T}\,M &\longrightarrow \mathbb{R} \\ \mathrm{T}_x\,M \ni v &\longmapsto \mathrm{Ric}_x(v, v). \end{aligned}$$

Exercise 9.3 (sectional curvature determines Riemannian curvature; Proposition 2.4.23). Let M be a smooth manifold and let R_1, R_2 be $(4, 0)$-tensor fields on M that satisfy the symmetries in Proposition 2.4.9. Moreover, for all $x \in M$ and all linearly independent $v, w \in \mathrm{T}_x\,M$ we assume that

$$R_1(v, w, w, v) = R_2(v, w, w, v).$$

Show that then $R_1 = R_2$ follows.
Hints. Look at $R_1 - R_2$.

Exercise 9.4 (Riemannian curvature and conformal changes; Theorem 2.5.5). Prove two of the first three claims of Theorem 2.5.5.

Bonus problem (positive curvature).

1. Give a reasonable definition of "positive Ricci curvature".

2. Give an example of a Riemannian manifold that has positive Ricci curvature but that does *not* have positive sectional curvature.

   Hints. You should not search for such examples in dimensions 1, 2, 3.

Submission before January 21, 2021, 10:00, via email to your tutor.