Differential Geometry I: Week 11

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In the next weeks, we will cover a lot of material. The underlying ideas are always geometrically simple and the results that we will obtain are definitely worth it! When reading, it is recommended to first focus on general lines of arguments and only then to delve into the technical details.

Reading assignment (for the lecture on January 27)**.** We determine the relation between minimising curves and geodesics, with the help of the first variation formula.

- Recall the *first variation formula*.
- Recall the definition of the *Riemannian distance function*.
- Read Chapter 3.2.3 *Minimising curves are geodesics*.
- Read Chapter 3.2.4 *Geodesics are locally minimising*.

Reading assignment (for the lecture on January 28)**.** The previous results suggest a close relation between metric and Riemannian geometric properties. We clarify this for the notion of "completeness". Moreover, we apply the techniques for geodesics in the case of model spaces and thus to classical geometric problems.

- Read Chapter 3.2.5 *Riemannian isometries*.
- Read Chapter 3.3.1 *Two notions of completeness*.
- Read Chapter 3.3.2 *The Hopf-Rinow theorem*.
- Read Chapter 3.4.1 *Geodesics of the model spaces*.
- (Optional) Read Chapter 3.4.2 *Isometries of the model spaces*.

Next week, as a preparation for the endgame of this course, we will take our handling of geodesics to the next level by studying variations of geodesics through geodesics. This will open the door to local-global results.

Étude (domains of the exponential map)**.** For each of the following manifolds (with the Riemannian Euclidean metric) and each of the given points, determine the domain of the exponential map at this point. Draw pictures!

Manifold	Points
$\mathbb{R}^2 \setminus \{0\}$	$(-1,0),\ (-2,0)$
$\mathbb{R}^2 \setminus \{0,(2,0)\}$	$(-1,0),\ (1,0)$
$\mathbb{R}^2 \setminus (\mathbb{R}_{>0} \times \{0\})$	$(-1,0),\ (1,1)$

Exercises (for the session on February 1/2)**.** The following exercises (which all are solvable with the material read/discussed in week 10) will be discussed.

The exercise series of next week (i.e., Exercises 11.1–11.4) will be the last regular exercise sheet of this course. All later exercises will count as bonus exercises.

Please turn over

Exercise 10.1 (domain of the exponential map)**.** Let (M, g) be a Riemannian manifold and let $x, y \in M$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If $y \in \exp_x(\mathrm{Exp}_x)$, then $x \in \exp_y(\mathrm{Exp}_y)$.
2. If $\exp_x(\mathrm{Exp}_x) = \exp_y(\mathrm{Exp}_y)$, then $x = y$.

Exercise 10.2 (constant sectional curvature)**.** Let (M, g) be a Riemannian manifold that has constant sectional curvature $c \in \mathbb{R}$. Show that then

$$\mathrm{R}(X, Y, Z) = c \cdot \big(\langle Y, Z\rangle_g \cdot X - \langle X, Z\rangle_g \cdot Y\big)$$

holds for all $X, Y, Z \in \Gamma(\mathrm{T}\, M)$.

Exercise 10.3 (negative sectional curvature)**.** Let (M, g) be a Riemannian manifold with $\sec < 0$.

1. If M is compact, show that there exists a $\kappa \in \mathbb{R}_{>0}$ with $\sec \leq -\kappa$.
2. If M is non-compact, show that such a bound does not necessarily exist.
3. *Bonus problem.* What happens in the non-compact, connected case? Justify your answer!

Exercise 10.4 (fixed sets are geodesic; Proposition 3.4.1)**.** Let (M, g) be a Riemannian manifold and let $N \subset M$ be a connected one-dimensional smooth submanifold for which there exists a $\varphi \in \mathrm{Isom}(M, g)$ with

$$N = \big\{x \in M \bigm| \varphi(x) = x\big\}.$$

Moreover, let $x \in N$ and let $v \in \mathrm{T}_x\, N \setminus \{0\}$. Show that

$$\mathrm{im}(\mathrm{geod}_{x,v}) = N.$$

Hints. First show that $\varphi \circ \mathrm{geod}_{x,v} = \mathrm{geod}_{x,v}$ and then apply a connectedness/extension argument.

Bonus problem (UN flag)**.** What does the UN flag have to do with differential geometry? Explain!

Submission before January 28, 2021, 10:00, via email to your tutor.