Differential Geometry I: Week 12

Prof. Dr. C. Löh/AG Ammann January 28, 2021

Reading assignment (for the lecture on February 3)**.** We will now study the second derivative of the length functional of curves and Jacobi fields. Again, the calculations might look discouraging, but the underlying strategies are straightforward.

- Recall the *first variation formula*.
- Read Chapter 3.5.1 *The second variation formula and the index form*.
- Read Chapter 3.5.2 *Jacobi fields*.

Reading assignment (for the lecture on February 4)**.**

- Read Chapter 3.5.3 *Conjugate points*.
- (Optional) Recall the Gauß-Bonnet theorem.
- (Optional) Recall the Euler characteristic and its properties.
- Read Chapter 4.1 *Local-global results*.
- Recall the *Gauß lemma*.
- Read Chapter 4.2 *Constant sectional curvature, locally*.

In the final week, we will put the theory developed in this course to good use: We will establish several local-global results.

Étude (variations)**.** Let (M, g) be a Riemannian manifold, let $x \in M$ and let $v \in \mathrm{T}_x M$. We consider the following expressions (on sufficiently small rectangles in normal neighbourhoods of x). Which of them are variations through geodesics? What can you say about their variation fields? Draw pictures!

1. $(s,t) \longmapsto \exp_x(t \cdot v)$
2. $(s,t) \longmapsto \exp_x(s \cdot v)$
3. $(s,t) \longmapsto \exp_x(s \cdot t \cdot v)$
4. $(s,t) \longmapsto \exp_x((s+t) \cdot v)$
5. $(s,t) \longmapsto \exp_x(s^2 \cdot t \cdot v)$
6. $(s,t) \longmapsto \exp_x(s \cdot t^2 \cdot v)$

Exercises (for the session on February 1/2)**.** The following exercises (which all are solvable with the material read/discussed in week 11) will be discussed.

This is the last regular exercise sheet of this course. All later exercises will count as bonus exercises.

Please turn over

Exercise 11.1 (inheriting completeness?). Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry of Riemannian manifolds. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If (M_1, g_1) is complete, then (M_2, g_2) is complete.
2. If (M_2, g_2) is complete, then (M_1, g_1) is complete.

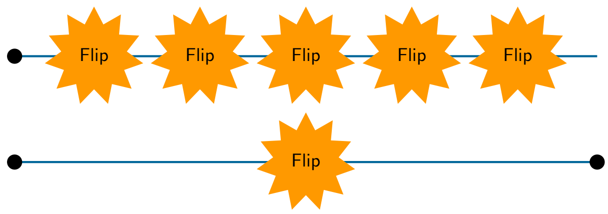
Exercise 11.2 (the unit radial vector field in normal coordinates). Let (M, g) be a Riemannian manifold, let $x \in M$ and let $U \subset M$ be a normal neighbourhood centred at x.

1. Express the radial distance function ϱ and the unit radial vector field $\partial/\partial\varrho$ centred at x in normal coordinates (and supply proofs).
2. Let $y \in U \setminus \{x\}$. Show that
$$\mathrm{d}_y\, \varrho\Bigl(\frac{\partial}{\partial\varrho}(y)\Bigr) = 1.$$

Exercise 11.3 (orthogonal decomposition of vector fields along curves). Let (M, g) be a Riemannian manifold, let γ be a unit speed geodesic on M, and let V be a smooth vector field along γ. Moreover, let
$$\overline{V} := \langle V, \dot\gamma\rangle_g \cdot \dot\gamma \quad \text{and} \quad V^\perp := V - \overline{V}.$$
Show the following compatibility statements (and understand their geometric meaning):
$$\begin{aligned} D_\gamma \overline{V} &= \overline{D_\gamma V} \\ D_\gamma (V^\perp) &= (D_\gamma V)^\perp \\ \|D_\gamma V\|_g^2 &= \langle D_\gamma V, \dot\gamma\rangle_g^2 + \bigl\|(D_\gamma V)^\perp\bigr\|_g^2 \\ \mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) &= \mathrm{Rm}(V^\perp, \dot\gamma, \dot\gamma, V^\perp). \end{aligned}$$

Exercise 11.4 (symmetric spaces).

1. Show that every symmetric space is complete.
2. Show that every symmetric space is homogeneous.



Bonus problem (coverings from geometry). Let $\varphi\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ be a local isometry of Riemannian manifolds and let M_2 be connected.

1. Show that φ is a covering map if (M_1, g_1) is complete.
2. Does this also hold if (M_1, g_1) is not complete? Justify your answer!

Submission before February 4, 2021, 10:00, via email to your tutor.