Differential Geometry I: Week 13

Prof. Dr. C. Löh/AG Ammann February 4, 2021

Some of the local-global results require a background in algebraic topology. If you don't have this background, then just view these results as an outlook on further topics (without the obligation to understand all the details, but maybe as a motivation to attend a course on algebraic topology in the future).

Reading assignment (for the lecture on February 10)**.** Using Jacobi fields, we will establish several comparison results; this will also lead to a classification of complete Riemannian manifolds with constant sectional curvature.

- Read Chapter 4.3 *Analytic and geometric comparison theorems.*
- Recall the notion of *covering maps.*
- Read Chapter 4.4.1 *The Cartan-Hadamard theorem.*
- Read Chapter 4.4.2 *Constant non-positive sectional curvature.*

Reading assignment (for the lecture on February 11)**.**

- Read Chapter 4.5.1 *The Bonnet-Myers theorem.*
- Read Chapter 4.5.2 *Constant positive sectional curvature.*
- Read Chapter 4.6 *The Švarc-Milnor lemma.*

This is it!

Étude (theorems)**.** Recall the statements and proof ideas of these theorems:

1. Bonnet-Myers
2. Cartan-Hadamard
3. Gauß lemma
4. Hopf-Rinow

Please turn over

Exercise 12.1 (constant sectional curvature)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. The smooth manifold $\mathbb{R}^{2021}$ admits a Riemannian metric of constant positive sectional curvature.

2. The smooth manifold $\mathbb{S}^1 \times \mathbb{R}^{2021}$ admits a Riemannian metric of constant negative sectional curvature.

Exercise 12.2 (conjugate points vs. minimising)**.** Show that the cylinder $\mathbb{R} \times \mathbb{S}^1$ with the product metric of the Euclidean Riemannian metric and the round metric (of radius 1) has the following property: There exist geodesics without conjugate points that are *not* minimising.

Exercise 12.3 (Jacobi fields along radial geodesics; Proposition 4.2.3)**.** Let (M, g) be a Riemannian manifold of dimension n, let $x \in M$, let $U \subset M$ be a normal neighbourhood of x, and let $(E_j)_{j\in\{1,\dots,n\}}$ be the coordinate frame associated with some normal coordinates on U, centred at x. Let $\gamma\colon [0, b] \longrightarrow U$ be a radial geodesic starting at x and let $v = \sum_{j=1}^n v^j \cdot E_j(x) \in \mathrm{T}_x M$. Show that then the Jacobi field V along γ with

$$V(0) = 0 \quad \text{and} \quad D_\gamma V(0) = v$$

is given by

$$\begin{aligned} V\colon [0, b] &\longrightarrow \mathrm{T}\, M \\ t &\longmapsto t \cdot \sum_{j=1}^n v^j \cdot E_j\bigl(\gamma(t)\bigr). \end{aligned}$$

Exercise 12.4 (volume growth in negative curvature; Corollary 4.4.5)**.** Let (M, g) be a Riemannian manifold that satisfies $\sec \leq c$ for some $c \in \mathbb{R}_{<0}$. Moreover, let $x \in M$.

1. Let $n \in \mathbb{N}$ and let $A, B \in M_{n\times n}(\mathbb{R})$ be symmetric positive definite matrices with
$$\forall_{x\in\mathbb{R}^n} \quad x^\mathsf{T} \cdot A \cdot x \geq x^\mathsf{T} \cdot B \cdot x.$$
Show that $\det A \geq \det B$.

   Hints. Principal axis theorem (Hauptachsentransformation) or Courant minimax principle.

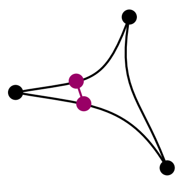
2. Show that (M, g) has (at least) exponential growth at x.

   Hints. What about normal neighbourhoods and radial distances? Apply the metric comparison theorem, the first part, and Exercise 6.2.

Bonus problem (comparison geometry)**.**

1. Look up *Toponogov's theorem* in the literature.

2. Look up the notion of CAT(κ)-*spaces* in the literature.



All these exercises count as bonus exercises. Optional submission!