Differential Geometry I: Week 8

Prof. Dr. C. Löh/AG Ammann December 17, 2020

Reading assignment (for the lecture on December 23). We now let connections and Riemannian metrics interact. In particular, we will formulate the *fundamental theorem of Riemannian geometry* (sounds important!).

- Recall the concept of tensor bundles of the tangent bundle (Chapter 1.2.4)
- Read Chapter 2.3.1 *Connections on tensor bundles.*
- Recall the notion of *Lie brackets* of vector fields.
- Read Chapter 2.3.2 *The Levi-Civita connection* until Theorem 2.3.17 (without proof).

  The technicalities of the proofs of the alternative characterisations of compatibility of a connection with a metric and of symmetry are a good opportunity to practice the various notions and constructions, but they are not that important in the long run. Thus, for now, it is sufficient to have a quick glance at these computations.

I wish you a restful Christmas break and a good start into the year 2021 !

Reading assignment (for the lecture on January 7). Welcome back!

- Recall the terms Riemannian manifolds, connections, geodesics, and parallel transport.
- Recall the defining properties of the Levi-Civita connection.
- Read the rest of Chapter 2.3.2 *The Levi-Civita connection.*
- Read (the beginning of) Chapter 2.4.1 *The Riemannian curvature tensor.*

Next week, we will study first properties of the Riemannian curvature tensor and its siblings.

Étude (decrypting differential geometry). Find the correct bijection between terms (with the "obvious" implicit meaning of all symbols) and descriptions!

$\nabla_{E_i} E_j = \sum_k {\Gamma_{ij}}^k \cdot E_k$	Riemannian metric of the hyperbolic halfspace model
$\sum_\ell ({\Gamma_{ki}}^\ell \cdot g_{\ell j} + {\Gamma_{kj}}^\ell \cdot g_{i\ell}) = E_k(g_{ij})$	parallel transport equation
$\frac{4}{1-\lVert x\rVert_2{}^2} \cdot \sum_i^n (\mathrm{d}\, x^i)^2$	compatibility of a connection with a Riemannian metric
$x^{k\prime\prime} + \sum_i \sum_j x^{i\prime} \cdot x^{j\prime} \cdot {\Gamma_{ij}}^k \circ (x^1, \dots, x^n) = 0$	zero
$X^{k\prime} + \sum_i \sum_j X^i \cdot \gamma^{j\prime} \cdot \Gamma_{ij} \circ \gamma = 0$	defining equation for connection coefficients
$\frac{2020}{2021} \cdot (g_{ij} - g_{ji})$	geodesic equation

Exercises (for the session on January 11/12). The following exercises (which all are solvable with the material read/discussed in week 7) will be discussed.

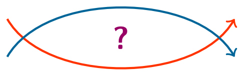
Please turn over

Exercise 7.1 (reparametrising geodesics). Let M be a smooth manifold with a linear connection ∇ and let $\gamma\colon \mathbb{R} \longrightarrow M$ be a smooth curve. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If γ is a geodesic with respect to ∇, then also $t \longmapsto \gamma(-2020 \cdot t)$ is a geodesic with respect to ∇.
2. If γ is a geodesic with respect to ∇, then also $t \longmapsto \gamma(t^{2020})$ is a geodesic with respect to ∇.

Exercise 7.2 (maximal geodesics; Corollary 2.2.24). Let M be a smooth manifold with a linear connection ∇ and let $x \in M$.

1. Show that for each $v \in \mathrm{T}_x M$ there exists a unique maximal geodesic $\gamma\colon I \longrightarrow M$ with $0 \in I^\circ$ and
$$\gamma(0) = x \quad \text{and} \quad \dot\gamma(0) = v.$$
2. Let $y \in M$ with $x \neq y$. Can it happen that there are two different maximal geodesics $\gamma, \eta\colon I \longrightarrow M$ with $\gamma(0) = x = \eta(0)$ and $\gamma(1) = y = \eta(1)$? Justify your answer with a suitable proof or counterexample!



Exercise 7.3 (connections via parallel transport; Corollary 2.2.35). Let M be a smooth manifold with a linear connection ∇, let $X, Y \in \Gamma(\mathrm{T}\,M)$, and let $x \in M$. Moreover, let $\gamma\colon I \longrightarrow M$ is a smooth curve in M with $0 \in I$ and
$$\gamma(0) = x \quad \text{and} \quad \dot\gamma(0) = X(x),$$
Show that
$$(\nabla_X Y)(x) = \lim_{h \to 0} \frac{P^\gamma_{h,0}(Y(\gamma(h)) - Y(x)}{h}.$$

Exercise 7.4 (geodesics and parallel transport). On $\mathbb{R}^3$, we consider the linear connection ∇ that is given by
$$\begin{aligned} &\nabla_X X = 0 && \nabla_Y X = -Z && \nabla_Z X = Y \\ &\nabla_X Y = Z && \nabla_Y Y = 0 && \nabla_Z Y = -X \\ &\nabla_X Z = -Y && \nabla_Y X = X && \nabla_Z Z = 0 \end{aligned}$$
in terms of the standard coordinate frame (X, Y, Z) of $\mathrm{T}\,\mathbb{R}^3$.

1. Show that the maximal geodesics with respect to ∇ are exactly the constant speed affine lines.
2. We consider $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^3, t \longmapsto (t, 0, 0)$. Compute the parallel transport $P^\gamma_{0,1}\colon \mathrm{T}_{\gamma(0)}\,\mathbb{R}^3 \longrightarrow \mathrm{T}_{\gamma(1)}\,\mathbb{R}^3$ and illustrate the result suitably!

Bonus problem (SageMath).

1. Install SageMath (https://www.sagemath.org).
2. Use SageMath to compute the connection coefficients of the Euclidean linear connection on $\mathbb{R}^2$ with respect to polar coordinates and document the individual steps.

Submission before January 7, 2021, 10:00, via email to your tutor.