23.06.2020

Recap: cost of measured equivalence relations (via graphings)
+ some basic estimates (section, cost only attained by treeings)

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admits a msbl fundamental domain

Proposition (cost of smooth measured eq rels)
Let $(\mathcal{R},\mu)$ be a smooth measured eq rel on a standard Borel space X with $\mu(X)<\infty$ and let $A\subset X$ be a msble fundamental domain for $\mathcal{R}$. Then
$$\operatorname{cost}_\mu \mathcal{R} = \mu(X\setminus A).$$

More precisely:

1. $\mathcal{R}$ admits a treeing
2. If Ψ is a reduced treeing of $\mathcal{R}$, then
$$\operatorname{cost}_\mu \mathcal{R} = \operatorname{cost}_\mu \Psi.$$

Proof. We have
$$\operatorname{cost}_\mu \mathcal{R} \xleftarrow{\text{A section of } \mathcal{R}} = \underbrace{\operatorname{cost}_{\mu|A} \underbrace{\mathcal{R}|_A}_{\Delta_A \text{ (A fund. domain)}}}_{=0} + \mu(X\setminus A)$$
$$= \mu(X\setminus A). \qquad \square$$

Examples:

- If $\Gamma \curvearrowright (X,\mu)$ is a free standard prob. action of a finite group Γ, then $\mathcal{R}_{\Gamma\curvearrowright X}$ is smooth and thus
$$\operatorname{cost}_\mu \mathcal{R}_{\Gamma\curvearrowright X} = 1 - \frac{1}{|\Gamma|}.$$
measure of a fund. domain

- If $(\mathcal{R},\mu)$ is a finite measured eq rel with $\mu(X)<\infty$, then $\mathcal{R}$ is smooth.
each orbit of $\mathcal{R}$ is finite

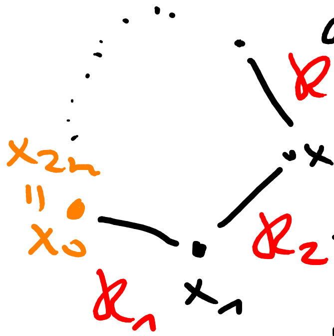
- If $\mathcal{R}'\subset\mathcal{R}$, then $\operatorname{cost}_\mu \mathcal{R}' \leq \operatorname{cost}_\mu \mathcal{R}$.

$\boxed{3.2.4}$ Cost of free products

$R_1 \cap R_2$
$\parallel$
over R_3

Definition (free products of standard eq. rel's). Let X be standard Borel space, let R_1, R_2 be standard eq. rels on X!

- R_1 and R_2 are <u>independent</u> if:
  For all $n \in \mathbb{N}_{\geq 1}$, all $x_0, \dots, x_{2n} \in X$ with $x_{2n} = x_0$ and



$$x_0 \underset{R_1}{\sim} x_1 \underset{R_2}{\sim} x_2 \underset{R_1}{\sim} \cdots \underset{R_2}{\sim} x_{2n} = x_0$$

$x_j \underset{R_3}{\sim} x_{j+1}$

  there ex. a $j \in \{0, \dots, 2n-1\}$ (2) with $x_j = x_{j+1}$ (over R_3).

- If R_1 and R_2 are independent, then we write $R_1 \underset{R_3}{\perp} R_2$ and $R_1 \underset{R_3}{*} R_2 := \underbrace{R_1 \vee R_2}_{\text{eq. rel on } X \text{ gen. by } R_1 \cup R_2}$.

Example. Let $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1, Γ_2 are countable groups and let $\Gamma \curvearrowright (X, \mu)$ be a free standard prob. action. Then

$$R_{\Gamma \curvearrowright X} = R_{\Gamma_1 \curvearrowright X} * R_{\Gamma_2 \curvearrowright X}$$

restricted actions of $\Gamma \curvearrowright X$ via the canonical homs $\Gamma_1, \Gamma_2 \hookrightarrow \Gamma$

Important case: free groups of finite rank

Theorem. (cost of free products). Let $(\mathcal{R},\mu)$ be a measured eq. rel on a standard Borel space with $\mu(X)<\infty$ that splits as a free product $\mathcal{R}_1 * \mathcal{R}_2$ of std eq rels on X of finite cost. Then

$$\operatorname{cost}_\mu \mathcal{R} = \operatorname{cost}_\mu \mathcal{R}_1 + \operatorname{cost}_\mu \mathcal{R}_2 .$$

easy!

Proof. $\leq$ Let Φ_1, Φ_2 be graphings of $\mathcal{R}_1, \mathcal{R}_2$, resp..
Then $\Phi_1 \sqcup \Phi_2$ is a graphing of $\underbrace{\mathcal{R}_1 \vee \mathcal{R}_2}_{=\mathcal{R}}$ and

$$\operatorname{cost}_\mu \mathcal{R} \leq \operatorname{cost}_\mu (\Phi_1 \sqcup \Phi_2)$$
$$= \operatorname{cost}_\mu \Phi_1 + \operatorname{cost}_\mu \Phi_2 .$$

Taking the inf. over all graphings Φ_1, Φ_2 of $\mathcal{R}_1, \mathcal{R}_2$ shows that

$$\operatorname{cost}_\mu \mathcal{R} \leq \operatorname{cost}_\mu \mathcal{R}_1 + \operatorname{cost}_\mu \mathcal{R}_2 .$$

not so easy...

$\geq$ Problem: If Φ is a graphing of $\mathcal{R}$, then it is not so clear how we can decompose Φ into graphings of $\mathcal{R}_1$ and $\mathcal{R}_2$ without increasing the cost too much.

Idea: first consider the decomposable case (but in a slightly more gen. situation)

will later be nec!

Proposition. (lower bound for decomposable graphings).
Let X be a standard Borel space, let $\mathcal{R}_1, \mathcal{R}_2$ be standard eq. rel's on X, let $\mathcal{R} := \mathcal{R}_1 \vee \mathcal{R}_2$, $\mathcal{R}_3 := \mathcal{R}_1 \cap \mathcal{R}_2$.

Let's suppose that $\mathcal{R} = \mathcal{R}_1 \ast_{\mathcal{R}_3} \mathcal{R}_2$ and that $\mathcal{R}_3$ is finite.

Let μ be a finite $\mathcal{R}$-inv. measure on X. and let Φ be a decomposable graphing of $\mathcal{R}$. Then

"Mayer-Vietoris"

$$\operatorname{cost}_\mu \Phi \geq \operatorname{cost}_\mu \mathcal{R}_1 + \operatorname{cost}_\mu \mathcal{R}_2 - \operatorname{cost}_\mu \mathcal{R}_3.$$

Proof. We write $\Phi = \Phi_1 \sqcup \Phi_2$ with $\langle \Phi_1 \rangle \subset \mathcal{R}_1$, $\langle \Phi_2 \rangle \subset \mathcal{R}_2$.

proof: tomorrow

Idea: decomposition lemma: there ex. Ψ_1, Ψ_2 such that:

- $\langle \Phi_1 \sqcup \Psi_1 \rangle = \mathcal{R}_1$
- $\langle \Phi_2 \sqcup \Psi_2 \rangle = \mathcal{R}_2$

efficiency condition!

- $\Psi_1 \sqcup \Psi_2$ is a reduced treeing of a subrel of $\mathcal{R}_3$

finite!

Then:

$$\begin{aligned}\operatorname{cost}_\mu \Phi &= \operatorname{cost}_\mu \Phi_1 + \operatorname{cost}_\mu \Phi_2 \\ &= \underbrace{\operatorname{cost}_\mu(\Phi_1 \sqcup \Psi_1)}_{\geq \operatorname{cost}_\mu \mathcal{R}_1} + \underbrace{\operatorname{cost}_\mu(\Phi_2 \sqcup \Psi_2)}_{\geq \operatorname{cost}_\mu \mathcal{R}_2} - \underbrace{\operatorname{cost}_\mu(\Psi_1 \sqcup \Psi_2)}_{\leq \operatorname{cost}_\mu \mathcal{R}_3} \\ &\geq \operatorname{cost}_\mu \mathcal{R}_1 + \operatorname{cost}_\mu \mathcal{R}_2 - \operatorname{cost}_\mu \mathcal{R}_3. \quad \square\end{aligned}$$

treeing $\subset \mathcal{R}_3$

Back to the main proof: By ($\leq$): $\mathrm{cost}_\mu R < \infty$.

Let $\varepsilon \in \mathbb{R}_{>0}$ and let Φ be a graphing of $\underset{R_1 * R_2}{\overset{\shortparallel}{R}}$ with

$$\mathrm{cost}_\mu \Phi \leq \mathrm{cost}_\mu R + \varepsilon.$$

Problem: Need to decompose Φ!

Let $\Omega = (\omega_i)_{i \in I}$ be a decomposable graphing with $\mathrm{cost}_\mu \Omega < \infty$ of $R = R_1 * R_2$. Idea: Use Ω to decompose Φ.

$(\dots)$

$$\rightsquigarrow \quad \Theta := \underbrace{\Theta_0}_{\boxed{\substack{\text{finite} \\ \subset \Phi}}} \sqcup \underbrace{(\omega_i|_{\underbrace{W(i)_N}_{\mu(\cdot) \leq \frac{\varepsilon}{|J|}}})_{i \in \overset{\text{finite}}{J}} \sqcup \underbrace{(\omega_i)_{i \in I \setminus J}}_{\mathrm{cost}_\mu \dots \leq \varepsilon}}_{\Theta_1 \text{ decomposable } (\Omega \text{ is decomposable!})}$$

graphing of R.

It remains to take care of Θ_0:

Let $\vartheta \in \Theta_0$. Then ~~there~~ ex. a $k(\vartheta) \in \mathbb{N}$ and $\varphi_{\vartheta,1}, \dots, \varphi_{\vartheta,k(\vartheta)}$ s.t.

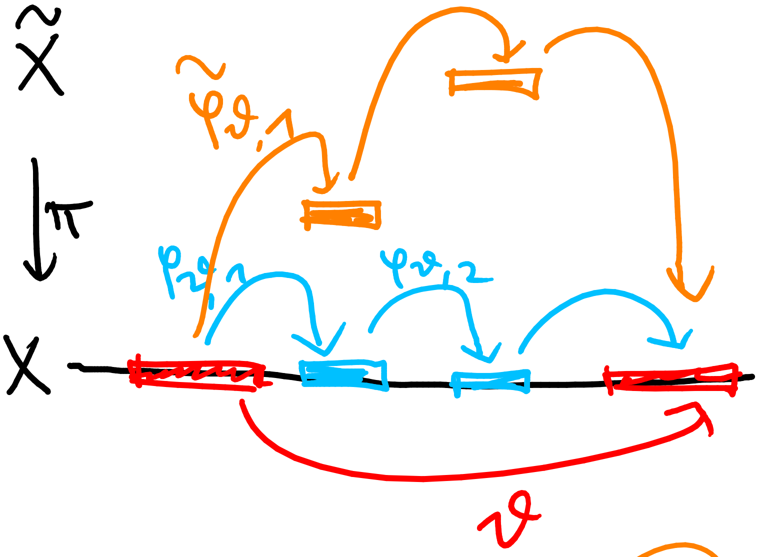
by decomposing domains, we may assume

$$\vartheta = \underbrace{\varphi_{\vartheta,1}}_{\text{restrictions of elements of } \Theta_0 ~~\cup \Theta_0^{-1}~~ \ \Omega \text{ that lie in } R_1 \text{ or } R_2} \circ \dots \circ \varphi_{\vartheta,k(\vartheta)}$$

increases the cost!

$$\rightsquigarrow \quad \overbrace{(\varphi_{\vartheta,j})_{\vartheta \in \Theta_0, \, j \in \{1, \dots, k(\vartheta)\}}} \sqcup \Theta_1 \text{ is a decomposable graphing of } R.$$

Unfolding trick:



$\rightsquigarrow$ $\tilde{\Theta} = \tilde{\Theta}_0 \sqcup \Theta_1$ decomposable graphing of $\tilde{R}$

not $\Delta_{\tilde{X}}$, but a finite equiv. rel.

Let $\tilde{R}, \tilde{R}_1, \tilde{R}_2, \tilde{R}_3$ be the π-pullbacks of $R, R_1, R_2, R_3 = \Delta_X$

Then: • $\tilde{R} = \tilde{R}_1 \ast_{\tilde{R}_3} \tilde{R}_2$

• X is a section of $\tilde{R}, \tilde{R}_1, \tilde{R}_2, \tilde{R}_3$.

• $\tilde{R}|_X = R, \dots$

$\rightsquigarrow$ • $\mathrm{cost}_\mu R - \mu(X) \overset{\text{section}}{=} \dots = \mathrm{cost}_{\tilde{\mu}} \tilde{R} - \tilde{\mu}(\tilde{X})$

(also for R_1, R_2, R_3)

• $\mathrm{cost}_{\tilde{\mu}} \tilde{\Theta} - \tilde{\mu}(\tilde{X}) = \mathrm{cost}_\mu \Theta - \mu(X)$.

Now: apply: lower bound for decomposable graphings to $\tilde{\Theta}$!

Final computation:

$$\boxed{\mathrm{cost}_\mu \mathcal{R}} - \mu(X) + 3\cdot\varepsilon \geq \mathrm{cost}_\mu \Phi - \mu(X) + 2\varepsilon$$

$$\geq \mathrm{cost}_\mu \Theta - \mu(X)$$

$$= \mathrm{cost}_{\tilde\mu} \tilde\Theta - \tilde\mu(\tilde X)$$

$\tilde\Theta$ decomposable

$$\geq \mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_1 + \mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_2 - \mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_3 - \tilde\mu(\tilde X)$$

$$= \left(\mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_1 - \tilde\mu(\tilde X)\right) + \left(\mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_2 - \tilde\mu(\tilde X)\right) - \left(\mathrm{cost}_{\tilde\mu} \tilde{\mathcal{R}}_3 - \tilde\mu(\tilde X)\right)$$

$$= \mathrm{cost}_\mu \mathcal{R}_1 - \mu(X) + \mathrm{cost}_\mu \mathcal{R}_2 - \mu(X) - \left(\mathrm{cost}_\mu \mathcal{R}_3 - \mu(X)\right)$$

$$= \boxed{\mathrm{cost}_\mu \mathcal{R}_1 + \mathrm{cost}_\mu \mathcal{R}_2} \underbrace{- \mathrm{cost}_\mu \underbrace{\mathcal{R}_3}_{= \Delta_X} - \mu(X)}_{= 0}$$

(take $\varepsilon \to 0$). $\square$