Ergodic Theory of Groups: Week 8

Prof. Dr. C. Löh/J. Witzig June 3, 2020

Reading assignment (for the lecture on June 9).

- Read the rest of Chapter 2.3.1 *The space of invariant measures.*
- The argument in Example 2.3.8 should remind you of another proof. Which one?
- Read Chapter 2.3.2 *The ergodic decomposition theorem.*
- Read Chapter 2.3.3 *Sketch proof of ergodic decomposition.*

  If you have never seen conditional expectations before, it could be helpful to have a look at Appendix A.1.8 *Conditional expectations.*
- This is the end of Chapter 2. If you have not done so already, this might be a good time to compile a summary of this chapter.

Reading assignment (for the lecture on June 10).

- Recall the notion of *orbit equivalence.*
- Read Chapter 3.1.1 *Standard equivalence relations.*
- Read Chapter 3.1.2 *Two prototypical arguments.*
- Try to visualise the proofs of the Feldman-Moore theorem and the marker lemma.
- Read Appendix A.3 *A quick introduction to* Isabelle.

  The text is long, but I hope that it will be easy to follow. I don't expect you to understand all details of the implementation of equivalence relations, but that you get a rough idea of how things fit together. We will discuss the implementation of equivalence relations in the lecture.

Implementation (equivalence relations). In the lecture on June 10, we will have a first look at formalising mathematics in Isabelle. In order to keep things simple, we will start with equivalence relations, with the example discussed in Appendix A.3.3 *A toy example.*

Isabelle is a complex system and we will not have the time to get to the bottom of all details. Instead, we take a practical approach and will learn how to formalise mathematics in Isabelle starting from small (but relevant) examples.

Exercises (for the session on June 12). The following exercises (which all are solvable with the material read/discussed in week 7) will be discussed.

Please turn over

Exercise 7.1 (space of invariant measures). Let $\alpha\colon \mathbb{Z}/2020 \curvearrowright S^1$ the rotation action by $1/2020$ and let λ be the Lebesgue measure on S^1. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\mu \in \operatorname{Erg}(\alpha)$, then $1/2020 \cdot \mu + 2019/2020 \cdot \lambda \in \operatorname{Erg}(\alpha)$.
2. We have $\bigl|\operatorname{Erg}(\alpha)\bigr| = 2020$.

Exercise 7.2 (orthogonality of ergodic measures). Let $\Gamma \curvearrowright X$ be an action by a countable group Γ on a standard Borel space X by measurable isomorphisms and let $\mu, \nu \in \operatorname{Erg}(\Gamma \curvearrowright X)$ with $\mu \neq \nu$. Show that there exists a measurable set $A \subset X$ with

$$\mu(A) = 1 \quad \text{and} \quad \nu(A) = 0.$$

Hints. Consider $1/2 \cdot \mu + 1/2 \cdot \nu$.

Exercise 7.3 (finite group actions). Let Γ be a finite group and let $\Gamma \curvearrowright (X, \mu)$ be a free standard probability action without atoms, i.e., $\mu(\{x\}) = 0$ for all $x \in X$.

1. Show that the action admits a measurable fundamental domain.

   Hints. We may assume that $(X, \mu) = ([0,1], \lambda)$ (why?) and $[0,1]$ is ordered in a measurable way.

2. Conclude: The measure μ is *not* ergodic.

Exercise 7.4 (simple normality). Let $d \in \mathbb{N}$. A real number x is *simply normal at base d* if each number in $\{0, \dots, d-1\}$ occurs with frequency $1/d$ in "the" d-adic expansion of x. We consider the periodic decimal number

$$x := 0.012345678901234567890123456789\dots$$

1. Show that x is simply normal at base 10.
2. Show that x is *not* simply normal at every base in $\mathbb{N}_{\geq 2}$.

   Hints. A wise choice of base makes this extremely easy.

Bonus problem (Benford's law).

1. How is Benford's law used in fraud detection and auditing?
2. Give a real-life example in which a comparison with Benford's law caused further investigations that revealed fraudulent data.

Submission before June 10, 8:00, via email to johannes.witzig@ur.de or through git.