Group Cohomology

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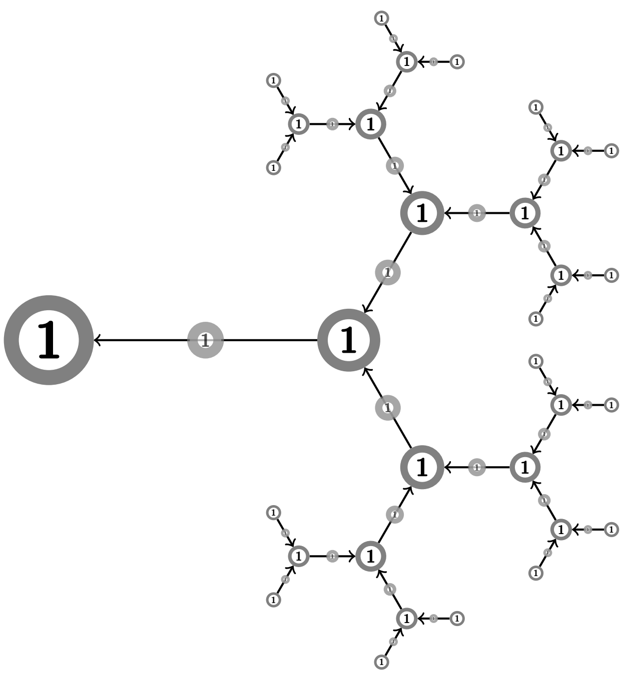

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Contents

Guide to the Literature

This course will not follow a single source and there are many books that cover the standard topics (all with their own advantages and disadvantages). Therefore, you should individually compose your own favourite selection of books.

Group Cohomology and Homological Algebra

- K.S. Brown. *Cohomology of Groups*, Graduate Texts in Mathematics, 82, Springer, 1982.
- S.I. Gelfand, Y.I. Manin. *Methods of Homological Algebra*, Springer Monographs in Mathematics, second edition, Springer, 2003.
- R. Geoghegan. *Topological Methods in Group Theory*, Graduate Texts in Mathematics, 143, Springer, 2008.
- P.J. Hilton, U. Stammbach. *A Course in Homological Algebra*, Graduate Texts in Mathematics, 4, second edition, Springer, 1996.
- C. Weibel. *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, 1995.

Group Theory

- C. Druţu, M. Kapovich. *Geometric Group Theory*, Colloquium Publications, 63, American Mathematical Society, 2018.
- C. Löh. *Geometric Group Theory. An Introduction*, Universitext, Springer, 2018.
  Errata: http://www.mathematik.uni-r.de/loeh/ggt_book/errata.pdf

- R.C. Lyndon, P.E. Schupp. *Combinatorial Group Theory*, Classics in Mathematics, Springer, 2001.
- J.J. Rotman. *An Introduction to the Theory of Groups*, Graduate Texts in Mathematics, 148, third edition, Springer, 1994.

Geometry and Topology

- M. Aguilar, S. Gitler, C. Prieto. *Algebraic Topology from a Homotopical Viewpoint*, Springer, 2002.
- M.W. Davis. *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs, 32, Princeton University Press, 2008.
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- S. Friedl. *Algebraic Topology I, II, III, ...*, lecture notes, Universität Regensburg, 2016–. . . .
- A. Hatcher. *Algebraic Topology*, Cambridge University Press, 2002. http://www.math.cornell.edu/~hatcher/AT/ATpage.html
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- M. Brandenburg. *Einführung in die Kategorientheorie: Mit ausführlichen Erklärungen und zahlreichen Beispielen*, Springer Spektrum, 2015.
- S. MacLane. *Categories for the Working Mathematician*, second edition, Springer, 1998.
- B. Pierce. *Basic Category Theory for Computer Scientists*, Foundations of Computing, MIT University Press, 1991.
- E. Riehl. *Category Theory in Context*, Aurora: Dover Modern Math Originals, 2016.

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Introduction

This course provides an introduction to group cohomology and its applications. More precisely, we will investigate different views on group cohomology and group homology from algebra, geometry, and topology as well as their applications.

What is group cohomology?

Group cohomology and group homology are theories that convert groups and modules over group rings (the so-called "coefficients") into graded groups/graded algebras. In particular, for each group G, each G-module A, and each $n \in \mathbb{N}$, we obtain Abelian groups

- $H_n(G; A)$, the homology of G in degree n with coefficients in A,
- $H^n(G; A)$, the cohomology of G in degree n with coefficients in A.

More precisely, group homology is obtained by deriving the tensor product functor over the group ring; group cohomology is obtained by deriving the Hom-functor over the group ring. While this description is concise and valuable for conceptual (and some computational) reasons, it hides the versatility of these theories and the interaction of group (co)homology with other fields.

Indeed, group (co)homology can be described from different points of view; the most prominent ones are:

- the basic view (via the standard resolutions),
- the derived view (via derived functors),

- the topological view (via classifying spaces),
- the geometric view (via other geometric structures).

Moreover, different choices of coefficient modules exhibit different invariance properties.

Because all these descriptions lead to the same theory, group (co)homology encodes many interesting connections between algebraic and geometric or topological properties of groups.

The art is then to find coefficients and suitable degrees that are well adapted to the target application.

Why group cohomology?

Classical applications of group (co)homology include the following:

Algebra

- generalisations of Hilbert 90 in Galois theory
- classification of group extensions with Abelian kernel; classification of central extensions
- generalisations of the group-theoretic transfer
- generalisations of finiteness properties of groups (such as finiteness, finite generation, finite presentability, ...)
- computations of stable commutator length
- ...

Geometry

- characterisations of amenable groups
- rigidity properties of (non)amenable groups
- study of quasi-morphisms on groups
- differentiation between different dynamical systems of groups
- ...

Topology

- characterisation of finite groups that admit free actions on spheres
- rigidity results in topology and geometry
- ...

Overview of this Course

In this course, we will develop the basics of group (co)homology. We will begin with the basic view (via standard resolutions) and group (co)homology in low degrees. We will then pursue a more geometric direction, with a focus on amenability. Afterwards, we will develop the derived view and discuss some general homological techniques. Finally, we will briefly study the topological view and the description of group (co)homology via classifying spaces and topological group actions. I will try to keep everything rather elementary. Basics in category theory and homological algebra will be recapitulated when necessary.

Moreover, we will discuss some of the applications along the way. The goal is to prepare the participants to access more advanced topics in group (co)homology and related fields and its applications to other fields.

Study note. These lecture notes document the topics covered in the course (as well as some additional optional material). However, these lecture notes are not meant to repace attending the lectures or the exercise classes!

Furthermore, this course will only treat the very beginning of this vast subject. It is therefore recommended to consult other sources (books and research articles!) for further information on this field.

References of the form "Satz I.6.4.11", "Satz II.2.4.33", "Satz III.2.2.25", "Satz IV.2.2.4", or "Corollary AT.1.3.25" point to the corresponding locations in the lecture notes for Linear Algebra I/II, Algebra, Commutative Algebra, Algebraic Topology in previuos semesters:

http://www.mathematik.uni-r.de/loeh/teaching/linalg1_ws1617/lecture_notes.pdf

http://www.mathematik.uni-r.de/loeh/teaching/linalg2_ss17/lecture_notes.pdf

http://www.mathematik.uni-r.de/loeh/teaching/algebra_ws1718/lecture_notes.pdf

http://www.mathematik.uni-r.de/loeh/teaching/calgebra_ss18/lecture_notes.pdf

http://www.mathematik.uni-r.de/loeh/teaching/topologie1_ws1819/lecture_notes.pdf

Literature exercise. Where in the math library can you find books on group cohomology, homological algebra, geometry, algebraic topology, category theory?

Convention. The set $\mathbb{N}$ of natural numbers contains 0. All rings are unital and associative (but very often *not* commutative). We write ${}_R\mathsf{Mod}$ for the category of left R-modules.

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The basic view

The most straightforward definition of group (co)homology is based on the standard resolutions of the trivial module $\mathbb{Z}$ over the group ring. This view on group (co)homology already allows for interesting applications in algebra (e.g., Galois theory and group extensions).

However, this view is not suitable for most concrete calculations. The fundamental insight is that group (co)homology is flexible in the sense that different resolutions will lead to the same theory. We will use this for some simple computations and for a first discussion of the Shapiro lemma and its consequences.

The change of resolution will be exploited systematically in the following chapters.

Overview of this chapter.

Running example. cyclic groups, free groups

1.1 Foundations: The group ring

The basic algebraic object in the context of group (co)homology is the group ring (and its module category).

1.1.1 The group ring

By definition, the group ring of a group G is an extension of the ring $\mathbb{Z}$ with new units coming from the group G.

Definition 1.1.1 (group ring)**.** Let G be a group. The *(integral) group ring* of G is the ring $\mathbb{Z}G$ (sometimes also denoted by $\mathbb{Z}[G]$ to avoid misunderstandings)

- whose underlying Abelian group is the free $\mathbb{Z}$-module $\bigoplus_{g\in G} \mathbb{Z}$, freely generated by G (we denote the basis element corresponding to $g \in G$ simply by g),
- and whose multiplication is the $\mathbb{Z}$-linear extension of composition in G, i.e.:

$$\begin{aligned} \cdot : \mathbb{Z}G \times \mathbb{Z}G &\longrightarrow \mathbb{Z}G \\ \Bigl(\sum_{g\in G} a_g \cdot g, \sum_{g\in G} b_g \cdot g\Bigr) &\longmapsto \sum_{g\in G}\sum_{h\in G} a_h \cdot b_{h^{-1}\cdot g} \cdot g \end{aligned}$$

(where all sums are "finite").

Remark 1.1.2. Let G be a group. A straightforward calculation shows that $\mathbb{Z}G$ is indeed a ring (check!). Moreover, $\mathbb{Z}G$ is unital, with multiplicative unit $1 \cdot e$, where e denotes the neutral element of G (check!).

Example 1.1.3 (group rings)**.**

- The group ring of "the" trivial group 1 is just $\mathbb{Z}[1] \cong_{\mathsf{Ring}} \mathbb{Z}$.
- The group ring $\mathbb{Z}[\mathbb{Z}]$ of the additive group $\mathbb{Z}$ is isomorphic to $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials over $\mathbb{Z}$ (check!).
- Let $n \in \mathbb{N}_{>0}$. Then we have $\mathbb{Z}[\mathbb{Z}/n] \cong_{\mathsf{Ring}} \mathbb{Z}[t]/(t^n - 1)$ (check!). In general, the ring $\mathbb{Z}[\mathbb{Z}/n]$ is *not* isomorphic to the subring $\mathbb{Z}[e^{2\cdot\pi\cdot i/n}]$ of $\mathbb{C}$ (check!).
- In general, group rings are *not* commutative. In fact, a group ring $\mathbb{Z}G$ is commutative if and only if the group G is Abelian (check!). Hence, for example, the group ring $\mathbb{Z}[S_3]$ of the symmetric group S_3 is not commutative.

Caveat 1.1.4 (notation in group rings)**.** When working with elements in group rings, some care is required. For example, the term $4 \cdot 2$ in $\mathbb{Z}[\mathbb{Z}]$ might be interpreted in the following different(!) ways:

- the product of 4 times the ring unit and 2 times the ring unit, or
- 4 times the *group* element 2.

We will circumvent this issue in $\mathbb{Z}[\mathbb{Z}]$, by using the notation "t" for a generator of the additive group $\mathbb{Z}$ and viewing the infinite cyclic group $\mathbb{Z}$ as multiplicative group. Using this convention, the first interpretation would be written as $4 \cdot 2$ (which equals 8) and the second interpretation would be written as $4 \cdot t^2$. Similarly, also in group rings over other groups, we will try to avoid ambiguous notation.

As with every new construction, we should capture the essence of the construction in a universal property:

Proposition 1.1.5 (group ring, universal property)**.** *Let G be a group. Then the group ring $\mathbb{Z}G$, together with the canonical inclusion map $i\colon G \longrightarrow \mathbb{Z}G$ (as standard basis) has the following universal property: For every ring R and every group homomorphism $f\colon G \longrightarrow R^\times$, there exists a unique ring homomorphism $\mathbb{Z}f\colon \mathbb{Z}G \longrightarrow R$ with $\mathbb{Z}f \circ i = f$.*

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "i"'] & R^\times \arrow[r, "\mathrm{incl}"] & R \\
\mathbb{Z}G \arrow[urr, dashed, "\exists!\ \mathbb{Z}f"'] & &
\end{tikzcd}$$

Proof. This is a straightforward calculation (check!). □

Outlook 1.1.6 (Kaplansky conjecture)**.** The ring structure of group rings is not well understood in full generality. For example, the following versions of the Kaplansky conjectures are still open: Let G be a torsion-free group.

- Then the group ring $\mathbb{Z}G$ is a domain (?!).
- The group ring $\mathbb{Z}G$ does not contain non-trivial idempotents (?!). (I.e., if $x \in \mathbb{Z}G$ with $x^2 = x$, then $x = 1$ or $x = 0$).

However, a positive solution is known for many special cases of groups [17, 65, 21][55, Chapter 10] (such proofs often use input from functional analysis or geometry) and no counterexamples are known.

1.1.2 Modules over the group ring

If G is a group, then (left) $\mathbb{Z}G$-modules correspond to Abelian groups that carry a G-representation; more explicitly, if A is a (left) $\mathbb{Z}G$-module, then, in addition to the usual additive structure, we have a map $\cdot\colon G \times A \longrightarrow A$ with

$$\begin{aligned} e \cdot a &= a \\ g \cdot (a + b) &= g \cdot a + g \cdot b \\ (g \cdot h) \cdot a &= g \cdot (h \cdot a) \end{aligned}$$

for all $g, h \in G$ and all $a, b \in A$. By default, $\mathbb{Z}G$-modules mean *left* $\mathbb{Z}G$-modules. Because the group ring is in general non-commutative, we have to be careful when taking tensor products. However, the group ring carries an involution, which simplifies matters:

Remark 1.1.7 (the involution: left vs. right modules)**.** Let G be a group. Then taking inverses is an involution on G. Hence, if A is a left $\mathbb{Z}G$-module, then

$$\begin{aligned} A \times G &\longrightarrow A \\ (a, g) &\longmapsto g^{-1} \cdot a \end{aligned}$$

allows to view A as a right module (check!), and vice versa. We will denote this right $\mathbb{Z}G$-module by $\operatorname{Inv} A$.

Definition 1.1.8 (tensor product and Hom-modules over $\mathbb{Z}G$)**.** Let G be a group and let A, B be left $\mathbb{Z}G$-modules. Then we write

$$\begin{aligned} A \otimes_G B &:= \operatorname{Inv}(A) \otimes_{\mathbb{Z}G} B \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Mod}) \\ \operatorname{Hom}_G(A, B) &:= {}_{\mathbb{Z}G}\operatorname{Hom}(A, B) \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Mod}). \end{aligned}$$

Example 1.1.9 (G-modules)**.** Let G be a group.

- If Z is a $\mathbb{Z}$-module, then the trivial G-action

  $$\begin{aligned} G \times Z &\longrightarrow Z \\ (g, z) &\longmapsto z \end{aligned}$$

  on Z turns Z into a so-called *trivial* (left) $\mathbb{Z}G$-module. Usually, we will regard $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module.

- The group ring $\mathbb{Z}G$ is a $\mathbb{Z}G$-$\mathbb{Z}G$-bimodule with respect to the action of G on the $\mathbb{Z}$-basis G of $\mathbb{Z}G$ by left/right translation.

- Let $\ell^\infty(G, \mathbb{R}) := \{f \colon G \to \mathbb{R} \mid \sup_{g \in G} |f(g)| < \infty\}$. Then $\ell^\infty(G, \mathbb{R})$ is a $\mathbb{Z}$-module with respect to pointwise addition/scalar multiplication. Moreover, $\ell^\infty(G, \mathbb{R})$ is a left $\mathbb{Z}G$-module via translation:

  $$\begin{aligned} G \times \ell^\infty(G, \mathbb{R}) &\longrightarrow \ell^\infty(G, \mathbb{R}) \\ (g, f) &\longmapsto \big(h \mapsto f(h \cdot g)\big). \end{aligned}$$

  Similarly, also $\ell^\infty(G, \mathbb{Z})$, $\ell^2(G, \mathbb{R})$, and $\ell^2(G, \mathbb{C})$ are (left) $\mathbb{Z}G$-modules.

- If X is a topological space equipped with a continuous (left) G-action, then this action turns the singular chain modules $C_n(X)$ into (left) $\mathbb{Z}G$-modules.

Definition 1.1.10 (invariants, coinvariants)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module. Then we introduce the following $\mathbb{Z}$-modules:

- The *invariants* of A are: $A^G := \{a \in A \mid \forall_{g \in G} \quad g \cdot a = a\}$.
- The *coinvariants* of A are: $A_G := A/\operatorname{Span}_{\mathbb{Z}}\{g \cdot a - a \mid g \in G, a \in A\}$.

These definitions are compatible with $\mathbb{Z}G$-linear maps (check!) and thus extend to (covariant) functors

$$\begin{aligned} \cdot^{\,G} \colon {}_{\mathbb{Z}G}\mathsf{Mod} &\longrightarrow {}_{\mathbb{Z}}\mathsf{Mod} \\ \cdot_{\,G} \colon {}_{\mathbb{Z}G}\mathsf{Mod} &\longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}\,. \end{aligned}$$

Study note (categories and functors)**.** For now, we will only need very basic notions from category theory: categories, functors, natural transformations (Chapter IV.1). If you don't know these terms yet, then this is not an issue.

Remark 1.1.11 (an alternative description of (co)invariants)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module. Then (where $\mathbb{Z}$ carries the trivial $\mathbb{Z}G$-module structure)

$$\begin{aligned} A^G &\longrightarrow \operatorname{Hom}_G(\mathbb{Z}, A) \\ a &\longmapsto (n \mapsto n \cdot a) \\ A_G &\longrightarrow A \otimes_G \mathbb{Z} \\ [a] &\longmapsto a \otimes 1 \end{aligned}$$

are $\mathbb{Z}$-isomorphisms (check!). These isomorphisms lead to natural isomorphisms between the invariants functor and $\operatorname{Hom}_G(\mathbb{Z}, \,\cdot\,)$ as well as between the coinvariants functor and $\,\cdot\, \otimes_G \mathbb{Z}$ (check!).

Study note. Which properties do invariants and coinvariants inherit from the description as Hom- and tensor functors (Remark 1.1.11), respectively?

Example 1.1.12. Let G be a group.

- If G is finite, then $(\mathbb{Z}G)^G \cong_{\mathbb{Z}} \mathbb{Z}$, generated by $\sum_{g \in G} g$ (check!).
- If G is infinite, then $(\mathbb{Z}G)^G \cong_{\mathbb{Z}} \{0\}$ and $\ell^2(G, \mathbb{C})^G \cong_{\mathbb{Z}} \{0\}$ (check!).
- Similarly, we have $\ell^\infty(G, \mathbb{Z})^G \cong_{\mathbb{Z}} \mathbb{Z}$ (check!).
- Moreover, $(\mathbb{Z}G)_G \cong_{\mathbb{Z}} \mathbb{Z}G \otimes_G \mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}$.

1.1.3 The domain categories for group (co)homology

As indicated in the introduction, group (co)homology has as arguments a group and a module over this group. In order to work efficiently with this setup, we introduce the categories $\mathsf{GroupMod}$ and $\mathsf{GroupMod}^*$ as domain for group homology and group cohomology, respectively.

Definition 1.1.13 ($\mathsf{GroupMod}$, $\mathsf{GroupMod}^*$)**.** The category $\mathsf{GroupMod}$ consists of the following data:

- objects: The class of all pairs (G, A), where G is a group and A is a (left) $\mathbb{Z}G$-module.
- morphisms: The set of morphisms from (G, A) to (H, B) is the set of pairs (φ, Φ), where
  - $\varphi \colon G \longrightarrow H$ is a group homomorphism, and
  - $\Phi \colon A \longrightarrow \varphi^* B$ is a $\mathbb{Z}G$-module homomorphism; here, $\varphi^* B$ is the $\mathbb{Z}G$-module whose underlying additive group coincides with B and whose $\mathbb{Z}G$-structure is given by

$$\begin{aligned} G \times B &\longrightarrow B \\ (g, b) &\longmapsto \varphi(g) \cdot b. \end{aligned}$$

- compositions: The composition of morphisms is defined by composing both components separately (this is well-defined in the second component; check!).

The category $\mathsf{GroupMod}^*$ consists of the following data:

- objects: We set $\operatorname{Ob}(\mathsf{GroupMod}^*) := \operatorname{Ob}(\mathsf{GroupMod})$.
- morphisms: The set of morphisms from (G, A) to (H, B) is the set of pairs (φ, Φ), where
  - $\varphi \colon G \longrightarrow H$ is a group homomorphism, and
  - $\Phi \colon \varphi^* B \longrightarrow A$ is a $\mathbb{Z}G$-module homomorphism.
- compositions: The composition of morphisms is defined by ordinary composition in the first component and reverse composition in the second component.

The following, simple, example lies at the heart of group (co)homology.

Example 1.1.14 (invariants and coinvariants)**.** Taking coinvariants can be viewed as a functor $\mathsf{GroupMod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$; taking invariants can be viewed as a (contravariant) functor $\mathsf{GroupMod}^* \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ (check!).

1.2 The basic definition of group (co)homology

We will now give a hands-on construction of group (co)homology. The advantage of this definition is that it immediately gives a functorial description of group (co)homology in both arguments. The disadvantage is that it is not immediately clear why this results in a reasonable theory of anything.

1.2.1 The simplicial and the bar resolution

The simplicial resolution of a group G is based on a simple geometric idea: We view the elements of G as the vertices of a (potentially infinite-dimensional) simplex and then take the simplicial chain complex of all (finite ordered) simplices of this huge simplex. Algebraically, this can be concisely formulated as follows (the geometric counterpart is introduced in Definition 4.1.1):

Definition 1.2.1 (the simplicial resolution)**.** Let G be a group. Then the *simplicial resolution* $C_*(G)$ of the group G is the $\mathbb{N}$-indexed $\mathbb{Z}G$-chain complex defined by:

- For $n \in \mathbb{N}$, we consider the chain module

  $$C_n(G) := \bigoplus_{G^{n+1}} \mathbb{Z}.$$

  The $\mathbb{Z}$-basis elements of $C_n(G)$ are denoted as the corresponding tuples in G^{n+1}.

- For $n \in \mathbb{N}$, the G-action on $C_n(G)$ is induced from the diagonal action

  $$\begin{aligned} G \times G^{n+1} &\longrightarrow G^{n+1} \\ \big(g, (g_0, \dots, g_n)\big) &\longmapsto (g \cdot g_0, \dots, g \cdot g_n) \end{aligned}$$

  on the $\mathbb{Z}$-basis G^{n+1}.

- For all $n \in \mathbb{N}_{>0}$, we define the boundary operator ∂_n as the $\mathbb{Z}$-linear map with (which turns out to be $\mathbb{Z}G$-linear)

  $$\begin{aligned} \partial_n \colon C_n(G) &\longrightarrow C_{n-1}(G) \\ G^{n+1} \ni (g_0, \dots, g_n) &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot (g_0, \dots, \widehat{g_j}, \dots, g_n). \end{aligned}$$

  Here, $(g_0, \dots, \widehat{g_j}, \dots, g_n)$ is a shorthand for $(g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$. Moreover, we set $\partial_0 := 0 \colon C_0 \longrightarrow 0$.

The *simplicial augmentation* is defined as the unique $\mathbb{Z}$-linear map with

$$\begin{aligned} \varepsilon\colon C_0(G) = \mathbb{Z}G &\longrightarrow \mathbb{Z} \\ G \ni g &\longmapsto 1. \end{aligned}$$

Remark 1.2.2. Let G be a group. In order to check that $C_*(G)$ is a chain complex, we need to verify that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}_{>0}$. This is a straightforward computation (check!), as in the case of the singular chain complex (Proposition AT.4.1.3) or the chain complex associated with a simplicial set. Moreover, all ∂_n are $\mathbb{Z}G$-homomorphisms and we have $\varepsilon \circ \partial_1 = 0$.

For instance, we prove that $\partial_1 \circ \partial_2 = 0$: It suffices to check this relation on the $\mathbb{Z}$-basis G^3 of $C_2(G)$. For all $(g_0, g_1, g_2) \in G^2$, we have

$$\begin{aligned} \partial_1 \circ \partial_2\bigl((g_0, g_1, g_2)\bigr) &= \partial_1\bigl((g_1, g_2) - (g_0, g_2) + (g_0, g_1)\bigr) \\ &= \partial_1\bigl((g_1, g_2)\bigr) - \partial_1\bigl((g_0, g_2)\bigr) + \partial_1\bigl((g_0, g_1)\bigr) \\ &= g_2 - g_1 - g_2 + g_0 + g_1 - g_0 \\ &= 0, \end{aligned}$$

as claimed.

Study note (homological algebra)**.** For a while, we will only need basic notions from homological algebra (chain complexes, homology of chain complexes, tensor products, chain maps, chain homotopy); this might be a good opportunity to refresh your background on homological algebra, or to acquire it for the first time (Chapter IV.5, Appendix AT.A.6).

While the simplicial resolution is very symmetric and straightforward, it does have the disadvantage that we only have a canonical $\mathbb{Z}$-basis of the chain modules (which are $\mathbb{Z}G$-modules), but we will want to apply $\mathbb{Z}G$-constructions to this complex. Therefore, we will introduce the bar resolution, which comes with a canonical $\mathbb{Z}G$-basis in every degree (while keeping a reasonable description of the boundary operator):

Definition 1.2.3 (the bar resolution)**.** Let G be a group. Then the *bar resolution* $\overline{C}_*(G)$ of G is the $\mathbb{N}$-indexed $\mathbb{Z}G$-chain complex defined by:

- For $n \in \mathbb{N}$, we consider the $\mathbb{Z}G$-module

  $$\overline{C}_n(G) := \bigoplus_{G^n} \mathbb{Z}G.$$

  The $\mathbb{Z}G$-basis element of $\overline{C}_n(G)$ corresponding to $(g_1, \dots, g_n) \in G^n$ is denoted by $[g_1|\dots|g_n]$. The unique element of G^0 is denoted by $[]$.

- For all $n \in \mathbb{N}_{>0}$, we define the boundary operator $\overline{\partial}_n$ as the $\mathbb{Z}G$-linear map with

$$\begin{aligned}
\overline{\partial}_n \colon \overline{C}_n(G) &\longrightarrow \overline{C}_{n-1}(G) \\
[g_1|\dots|g_n] &\longmapsto g_1 \cdot [g_2|\dots|g_n] \\
&\quad + \sum_{j=1}^{n-1} (-1)^j \cdot [g_1|\dots|g_{j-1}|g_j \cdot g_{j+1}|g_{j+2}|\dots|g_n] \\
&\quad + (-1)^n \cdot [g_1|\dots|g_{n-1}].
\end{aligned}$$

Remark 1.2.4 (simplicial vs. bar resolution)**.** Let G be a group. Then the maps

$$\begin{aligned}
C_n(G) &\longrightarrow \overline{C}_n(G) \\
G^{n+1} \ni (g_0, \dots, g_n) &\longmapsto g_0 \cdot [g_0^{-1} \cdot g_1|\dots|g_{n-1}^{-1} \cdot g_n] && (\mathbb{Z}\text{-linear extension}) \\
\overline{C}_n(G) &\longrightarrow C_n(G) \\
[g_1|\dots|g_n] &\longmapsto (e, g_1, g_1 \cdot g_2, \dots, g_1 \cdot g_2 \cdot \dots \cdot g_n) && (\mathbb{Z}G\text{-linear extension})
\end{aligned}$$

for $n \in \mathbb{N}$ form mutually inverse isomorphisms $C(G) \cong_{\mathbb{Z}G} \overline{C}(G)$ of $\mathbb{Z}G$-chain complexes (check!); hence, both resolutions are different views on the same thing. The bar resolution has the advantage that it is easier to read off its coinvariants and related constructions.

A particularly convenient aspect of these constructions is that they are compatible with group homomorphisms (without passing to the homotopy category of the category of chain complexes):

Proposition 1.2.5 (group homomorphisms and the standard resolutions)**.** *Let $\varphi \colon G \longrightarrow H$ be a group homomorphism.*

1. *Then $C_*(\varphi) \colon C_*(G) \longrightarrow \varphi^* C_*(H)$, defined by*

$$\begin{aligned}
C_n(\varphi) \colon C_n(G) &\longrightarrow \varphi^* C_n(H) \\
G^{n+1} \ni (g_0, \dots, g_n) &\longmapsto \bigl(\varphi(g_0), \dots, \varphi(g_n)\bigr)
\end{aligned}$$

   for all $n \in \mathbb{N}$, is a $\mathbb{Z}G$-chain map.

2. *Similarly, $\overline{C}_*(\varphi) \colon \overline{C}_*(G) \longrightarrow \varphi^* \overline{C}_*(H)$, defined by*

$$\begin{aligned}
\overline{C}_n(\varphi) \colon \overline{C}_n(G) &\longrightarrow \varphi^* \overline{C}_n(H) \\
[g_1 \mid \dots \mid g_n] &\longmapsto \bigl[\varphi(g_1) \mid \dots \mid \varphi(g_n)\bigr]
\end{aligned}$$

   for all $n \in \mathbb{N}$, is a $\mathbb{Z}G$-chain map.

Proof. This is a straightforward calculation (check!). □

1.2.2 Group (co)homology

Group cohomology is the cohomology of the invariants of the simplicial resolution; group homology is the homology of the coinvariants of the simplicial resolution. More generally, using Hom-functors and tensor products, we can decorate these constructions with coefficient modules.

Definition 1.2.6 (group homology)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module.

- Then we write $C_*(G;A) := C_*(G) \otimes_G A \in {}_{\mathbb{Z}}\mathsf{Ch}$ for the *standard complex of G with coefficients in A* (Remark 1.2.7).
- For $n \in \mathbb{N}$, we define *group homology of G with coefficients in A* by

$$H_n(G;A) := H_n\bigl(C_*(G;A)\bigr) := \frac{\ker\bigl(\partial_n \otimes_G \mathrm{id}_A \colon C_n(G;A) \to C_{n-1}(G;A)\bigr)}{\mathrm{im}\bigl(\partial_{n+1} \otimes_G \mathrm{id}_A \colon C_{n+1}(G;A) \to C_n(G;A)\bigr)}.$$

Let $(\varphi, \Phi) \colon (G,A) \longrightarrow (H,B)$ be a morphism in $\mathsf{GroupMod}$.

- We write $C_*(\varphi;\Phi) := C_*(\varphi) \otimes \Phi$ for the composition

$$C_*(G) \otimes_G A \xrightarrow{C_*(\varphi)\otimes_G \Phi} \varphi^* C_*(H) \otimes_G \varphi^* B \xrightarrow{\text{can. proj.}} C_*(H) \otimes_H B$$

  of $\mathbb{Z}$-chain maps.
- For $n \in \mathbb{N}$, we then set (which is a $\mathbb{Z}$-linear map)

$$H_n(\varphi;\Phi) := H_n\bigl(C_*(\varphi;\Phi)\bigr) \colon H_n(G;A) \longrightarrow H_n(H;B).$$

We will also write $H_*(G;A)$ for the sequence $(H_n(G;A))_{n\in\mathbb{N}}$ and $H_*(\varphi;\Phi)$ for the sequence $(H_n(\varphi;\Phi))_{n\in\mathbb{N}}$.

Remark 1.2.7 (tensor product of a chain complex with a module)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module. We recall the definition of the functor

$$\cdot \otimes_G A \colon {}_{\mathbb{Z}G}\mathsf{Ch} \longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}\,.$$

Let C_* be a $\mathbb{Z}G$-chain complex. Then $C_* \otimes_G A$ is the $\mathbb{Z}$-chain complex consisting of the chain modules $C_n \otimes_G A$ and the boundary maps

$$\begin{aligned} \partial_n \otimes_G \mathrm{id}_A \colon C_n \otimes_G A &\longrightarrow C_{n-1} \otimes_G A \\ x \otimes a &\longmapsto \partial_n(x) \otimes a. \end{aligned}$$

If $f_* \colon C_* \longrightarrow D_*$ is a $\mathbb{Z}G$-chain map, then the sequence $f_* \otimes_G \mathrm{id}_A := (f_n \otimes_G \mathrm{id}_A)_{n\in\mathbb{N}}$ is a $\mathbb{Z}$-chain map $C_* \otimes_G A \longrightarrow D_* \otimes_G A$ (check!).

Example 1.2.8 (homology of the trivial group)**.** Let 1 denote "the" trivial group. Then $\mathbb{Z}[1] \cong_{\mathsf{Ring}} \mathbb{Z}$, and we will formulate everything in terms of $\mathbb{Z}$-modules (instead of $\mathbb{Z}[1]$-modules). If A is a $\mathbb{Z}$-module, then

$$H_n(1;A) \cong_{\mathbb{Z}} \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

for all $n \in \mathbb{N}$. This follows from the simple observation that the $\mathbb{Z}$-chain complex $C_*(1)$ is isomorphic to the $\mathbb{Z}$-chain complex (check!)

$$\begin{array}{lcccc} \text{degree} & & 2 & 1 & 0 \\ \cdots \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} \xrightarrow[0]{} & \mathbb{Z} \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} \xrightarrow[0]{} & \mathbb{Z} \end{array}$$

Hence, $C_*(1;A) = C_*(1) \otimes_1 A$ is isomorphic to

$$\begin{array}{lcccc} \text{degree} & & 2 & 1 & 0 \\ \cdots \xrightarrow[\mathrm{id}_A]{} & A \xrightarrow[0]{} & A \xrightarrow[\mathrm{id}_A]{} & A \xrightarrow[0]{} & A \end{array}$$

This chain complex has the homology stated above (check!).

Dually, group cohomology is defined in terms of *co*homology of *co*chain complexes. This works in the same way – the only difference between chain complexes and cochain complexes is that the boundary maps *increase* the degree, instead of decreasing it. We denote the categories of cochain complexes by Ch^* instead of Ch and also denote cohomology by an upper index/star instead of a lower index/star.

Definition 1.2.9 (group cohomology)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module.

- Then we write $C^*(G;A) := \operatorname{Hom}_G\bigl(C_*(G), A\bigr) \in {}_{\mathbb{Z}}\mathsf{Ch}^*$ for the *standard cochain complex of G with coefficients in A* (Remark 1.2.10).
- For $n \in \mathbb{N}$, we define *group cohomology of G with coefficients in A* by

$$H^n(G;A) := H^n\bigl(C^*(G;A)\bigr) = \frac{\ker\bigl(\operatorname{Hom}_G(\partial_{n+1}, A)\colon C^n(G;A) \to C^{n+1}(G;A)\bigr)}{\operatorname{im}\bigl(\operatorname{Hom}_G(\partial_n, A)\colon C^{n-1}(G;A) \to C^n(G;A)\bigr)}.$$

Let $(\varphi, \Phi)\colon (G,A) \longrightarrow (H,B)$ be a morphism in $\mathsf{GroupMod}^*$.

- We write $C^*(\varphi;\Phi) := \operatorname{Hom}_G(C_*(\varphi), \Phi)$ for the composition

$$\operatorname{Hom}_H\bigl(C_*(H), B\bigr) \xrightarrow{\text{can. incl.}} \operatorname{Hom}_G\bigl(\varphi^* C_*(H), \varphi^* B\bigr) \xrightarrow{\operatorname{Hom}_G(C_*(\varphi),\Phi)} \operatorname{Hom}_G\bigl(C_*(G), A\bigr)$$

  of $\mathbb{Z}$-cochain maps.
- For $n \in \mathbb{N}$, we then set (which is a $\mathbb{Z}$-linear map)

$$H^n(\varphi;\Phi) := H^n\bigl(C^*(\varphi;\Phi)\bigr)\colon H^n(H;B) \longrightarrow H^n(G;A).$$

We will also write $H^*(G;A)$ for the sequence $(H^n(G;A))_{n\in\mathbb{N}}$ and $H^*(\varphi;\Phi)$ for the sequence $(H^n(\varphi;\Phi))_{n\in\mathbb{N}}$.

Remark 1.2.10 (Hom-cochain complexes)**.** Let G be a group and let A be a (left) $\mathbb{Z}G$-module. We recall the definition of the (contravariant) functor

$$\operatorname{Hom}_G(\,\cdot\,,A)\colon {}_{\mathbb{Z}G}\mathsf{Ch} \longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}^* .$$

Let C_* be a $\mathbb{Z}G$-chain complex. Then $\operatorname{Hom}_G(C_*,A)$ is the $\mathbb{Z}$-cochain complex consisting of the cochain modules $\operatorname{Hom}_G(C_n,A)$ and the coboundary maps

$$\begin{aligned}(-1)^{n+1}\cdot \operatorname{Hom}_G(\partial_{n+1},A)\colon \operatorname{Hom}_G(C_n,A) &\longrightarrow \operatorname{Hom}_G(C_{n+1},A)\\ f &\longmapsto \bigl(x\mapsto (-1)^{n+1}\cdot f(\partial_{n+1}(x))\bigr);\end{aligned}$$

the sign does not affect the corresponding cohomology, but will later result in good sign conventions on (co)chain complexes. If $f_*\colon C_*\longrightarrow D_*$ is a $\mathbb{Z}G$-chain map, then the sequence $\operatorname{Hom}_G(f_*,A) := (\operatorname{Hom}_G(f_n,A))_{n\in\mathbb{N}}$ is a $\mathbb{Z}$-cochain map $\operatorname{Hom}_G(D_*,A)\longrightarrow \operatorname{Hom}_G(C_*,A)$ (check!).

Remark 1.2.11 (functoriality of group (co)homology)**.** The functors (check!)

$$\begin{aligned} C_*(\,\cdot\,;\,\cdot\,)\colon \mathsf{GroupMod} &\longrightarrow {}_{\mathbb{Z}}\mathsf{Ch} &&\text{(covariant)}\\ C^*(\,\cdot\,;\,\cdot\,)\colon \mathsf{GroupMod}^* &\longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}^* &&\text{(contravariant)}\end{aligned}$$

in combination with the (co)homology functors on (co)chain complexes yield for each $n\in\mathbb{N}$ group (co)homology functors of the following types:

$$\begin{aligned} H_n\colon \mathsf{GroupMod} &\longrightarrow {}_{\mathbb{Z}}\mathsf{Mod} &&\text{(covariant)}\\ H^n\colon \mathsf{GroupMod}^* &\longrightarrow {}_{\mathbb{Z}}\mathsf{Mod} &&\text{(contravariant)}\end{aligned}$$

Caveat 1.2.12 (lechts und rinks)**.** There are several, different, conventions for group (co)homology in use. Indeed, instead of taking left modules as coefficients, one could also work with right modules. Because the group ring carries a canonical involution (Remark 1.1.7), we can always canonically translate between these different conventions and all conventions essentially lead to the same theory. However, when working with formulas from the literature, one should always carefully check which conventions are in place in the respective sources.

Example 1.2.13 (cohomology of the trivial group)**.** We continue the considerations from Example 1.2.8: Let 1 denote "the" trivial group; then $\mathbb{Z}[1]\cong_{\mathsf{Ring}}\mathbb{Z}$. If A is a $\mathbb{Z}$-module, then

$$H^n(1;A)\cong_{\mathbb{Z}} \begin{cases} A & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}$$

for all $n \in \mathbb{N}$. Again, using that the $\mathbb{Z}$-chain complex $C_*(1)$ is isomorphic to the $\mathbb{Z}$-chain complex (check!)

$$\begin{array}{ccccccccc} \text{degree} & & & & 2 & & 1 & & 0 \\ \cdots & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} & \xrightarrow[0]{} & \mathbb{Z} & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} & \xrightarrow[0]{} & \mathbb{Z} \end{array}$$

we obtain that $C^*(1;A) = \mathrm{Hom}_1(C_*(1), A)$ is isomorphic to

$$\begin{array}{ccccccccc} \text{degree} & & & & 2 & & 1 & & 0 \\ \cdots & \xleftarrow{\mathrm{id}_A} & A & \xleftarrow{0} & A & \xleftarrow{\mathrm{id}_A} & A & \xleftarrow{0} & A \end{array}$$

This cochain complex has the claimed cohomology (check!).

Study note. If you know basic algebraic topology: What do the computations from Example 1.2.8 and 1.2.13 remind you of? This is part of a deeper connection of group (co)homology with algebraic topology. We will explore this connection in Chapter 4.

Remark 1.2.14 (group (co)homology via bar resolution)**.** Because the simplicial resolution and the bar resolution are (naturally) isomorphic (Remark 1.2.4), we can also use the bar resolution to compute group homology and group cohomology: If G is a group and A is a (left) $\mathbb{Z}G$-module, then the chain isomorphisms from Remark 1.2.4 induce chain isomorphisms

$$C_*(G;A) = C_*(G) \otimes_G A \cong_{\mathbb{Z}\mathsf{Ch}} \overline{C}_*(G) \otimes_G A =: \overline{C}_*(G;A)$$
$$C^*(G;A) = \mathrm{Hom}_G\bigl(C_*(G), A\bigr) \cong_{\mathbb{Z}\mathsf{Ch}^*} \mathrm{Hom}_G\bigl(\overline{C}_*(G), A\bigr) =: \overline{C}^*(G;A).$$

Moreover, these isomorphisms are natural (on $\mathsf{GroupMod}$ and $\mathsf{GroupMod}^*$, respectively).

This is particularly helpful if we work with constant coefficients, because the (co)invariants are easier to read off from the bar resolution than from the simplicial resolution.

In general, computing group (co)homology from the simplicial or the bar resolution is *not* feasible (the chain complexes can be huge even though the homology might be very small). In the following sections, we will see how this basic description of group (co)homology allows to establish connections with more classical invariants/problems. However, for explicit computations, we will usually rely on different descriptions of group (co)homology, which we will develop later.

By definition, (co)homology is the quotient of the (co)cycles by the (co)boundaries. Therefore, we should expect that group (co)homology also carries geometric meaning. We will discuss this in more detail in Chapter 2 and Chapter 4.

1.3 Degree 0: (Co)Invariants

As a warm-up, we compute group (co)homology in degree 0.

Theorem 1.3.1 (group (co)homology in degree 0)**.** *Let G be a group and let A be a $\mathbb{Z}G$-module. Then there are canonical isomorphisms*

$$H_0(G;A) \cong_{\mathbb{Z}} A_G$$
$$H^0(G;A) \cong_{\mathbb{Z}} A^G.$$

More precisely, the functors H_0 and H^0 are canonically naturally isomorphic to the coinvariants functor $\mathsf{GroupMod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ and the invariants functor $\mathsf{GroupMod}^* \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$, respectively.

Proof. We use the bar complex to compute group (co)homology in degree 0 (Remark 1.2.14). The lower part of the bar complex looks as follows:

$$\begin{array}{cccc} \text{degree} & 1 & & 0 \\ & \overline{C}_1(G) & \xrightarrow{\overline{\partial}_1} & \overline{C}_0(G) \xrightarrow{\overline{\partial}_0 = 0} 0 \\ & \| & & \| \\ & \bigoplus_G \mathbb{Z}G & & \mathbb{Z}G \\ & [g] & \longmapsto & g \cdot [\,] - [\,] \end{array}$$

In combination with Remark 1.2.4, we see that the low-degree part of $C_*(G;A) = C_*(G) \otimes_G A$ is canonically naturally isomorphic to

$$\begin{array}{cccc} \text{degree} & 1 & & 0 \\ & \overline{C}_1(G;A) & \xrightarrow{\overline{\partial}_1 \otimes_G \mathrm{id}_A} & \overline{C}_0(G;A) \xrightarrow{0} 0 \\ & \| & & \| \\ & \bigl(\bigoplus_G \mathbb{Z}G\bigr) \otimes_G A & & \mathbb{Z}G \otimes_G A \\ & \wr\|_{\mathbb{Z}} & & \wr\|_{\mathbb{Z}} \\ & \bigoplus_G A & & A \\ & [g] \cdot a & \longmapsto & g^{-1} \cdot a - a. \end{array}$$

Hence,

$$\begin{aligned} H_0(G;A) &\cong_{\mathbb{Z}} \bigl(\overline{C}_0(G;A)\bigr) / \operatorname{im}(\overline{\partial}_1 \otimes_G \mathrm{id}_A) \\ &\cong_{\mathbb{Z}} A / \operatorname{Span}_{\mathbb{Z}}\{g^{-1} \cdot a - a \mid g \in G,\ a \in A\} \\ &= A_G. \end{aligned}$$

Dually, the low-degree part of $C^*(G;A) = \operatorname{Hom}_G(C_*(G),A)$ is canonically naturally isomorphic to

$$
\begin{array}{lccc}
\text{degree} & 1 & & 0 \\
& \overline{C}^1(G;A) & \xleftarrow{-\operatorname{Hom}_G(\overline{\partial}_1,A)} & \overline{C}^0(G;A) \xleftarrow{\;0\;} 0 \\
& \| & & \| \\
& \operatorname{Hom}_G\bigl(\bigoplus_G \mathbb{Z}G, A\bigr) & & \operatorname{Hom}_G(\mathbb{Z}G,A) \\
& \wr\| & & \wr\| \\
& \prod_G A & & A \\
& & & \\
& (a - g\cdot a)_{g\in G} & \longleftarrow\!\!\shortmid & a
\end{array}
$$

Therefore,

$$
\begin{aligned}
H^0(G;A) &\cong_{\mathbb{Z}} \ker\bigl(\operatorname{Hom}_G(\overline{\partial}_1,A)\bigr) \\
&= \{a \in A \mid \forall_{g\in G} \quad a = g\cdot a\} \\
&= A^G.
\end{aligned}
$$

Tracing through the explicit descriptions of these isomorphisms shows that these isomorphisms are natural on GroupMod and GroupMod*, respectively (check!). ☐

Study note. Write down the isomorphisms constructed in the proof of Theorem 1.3.1 in terms of the simplicial resolution (instead of the bar resolution).

Example 1.3.2 (group (co)homology in degree 0)**.** Let G be a group. Plugging in the computations from Example 1.1.12, we obtain the following group (co)homology in degree 0:

- If Z is a trivial G-module, then $H_0(G;Z) \cong_{\mathbb{Z}} Z_G \cong_{\mathbb{Z}} Z$ and $H^0(G;Z) \cong_{\mathbb{Z}} Z^G = Z$.
- We have $H^0(G;\ell^\infty(G,\mathbb{Z})) \cong_{\mathbb{Z}} \ell^\infty(G,\mathbb{Z})^G \cong_{\mathbb{Z}} \mathbb{Z}$ (however, the computation of $H_0(G;\ell^\infty(G,\mathbb{Z}))$ is more delicate; Remark 2.2.18) and we have $H_0(G;\mathbb{Z}G) \cong_{\mathbb{Z}} (\mathbb{Z}G)_G \cong_{\mathbb{Z}} \mathbb{Z}$.
- If G is infinite, then $H^0(G;\mathbb{Z}G) \cong_{\mathbb{Z}} 0$ and $H^0(G;\ell^2(G,\mathbb{C})) \cong_{\mathbb{Z}} 0$.
- If G is finite, then $H^0(G;\mathbb{Z}G) \cong_{\mathbb{Z}} \mathbb{Z}$ and $H^0(G;\ell^2(G,\mathbb{C})) \cong_{\mathbb{Z}} \mathbb{C}$.

In particular: Taking $\mathbb{Z}G$-coefficients or $\ell^2(G;\mathbb{C})$-coefficients allows to characterise finiteness of groups in terms of group cohomology.

In fact, group (co)homology is uniquely determined by its values in degree 0 and the fact that we can apply the dimension shifting trick (Corollary 3.1.13). We will return to this point of view in Chapter 3.

1.4 Degree 1: Abelianisation and homomorphisms

We will now discuss group (co)homology in degree 1. For simplicity, we will first focus on the case of trivial coefficients. Moreover, we will indicate applications to group theory and Galois theory.

1.4.1 Homology in degree 1: Abelianisation

Theorem 1.4.1 (group homology in degree 1). *Let G be a group. Then (where $\mathbb{Z}$ carries the trivial G-action) there is a canonical isomorphism*

$$H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} G_{\mathrm{ab}}.$$

More precisely, the functor $H_1(\,\cdot\,;\mathbb{Z})\colon \mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is canonically naturally isomorphic to the Abelianisation functor.

Remark 1.4.2 (Abelianisation). Recall that for a group G, the *Abelianisation G_{ab}* is the quotient group

$$G_{\mathrm{ab}} := G/[G,G].$$

Here, $[G,G]$ is the *commutator subgroup* of G, i.e., the subgroup of G generated by the set $\{[x,y] \mid x,y \in G\}$ of all commutators

$$[x,y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$$

in G. By construction, G_{ab} is Abelian; more precisely, the Abelianisation G_{ab} is the largest Abelian quotient of G. The construction of the Abelianisation is also compatible with group homomorphisms and thus defines a functor $\mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ (Chapter AT.4.5.1, Chapter III.1.3.4).

Proof of Theorem 1.4.1. Again, we use the bar complex to compute group homology in degree 1 (Remark 1.2.14). The corresponding part of the bar complex looks as follows:

$$\begin{array}{llll}
\text{degree} & 2 & 1 & 0 \\
 & \overline{C}_2(G) \xrightarrow{\quad\overline{\partial}_2\quad} & \overline{C}_1(G) \xrightarrow{\quad\overline{\partial}_1\quad} & \overline{C}_0(G) \\
 & \| & \| & \| \\
 & \bigoplus_{G^2} \mathbb{Z}G & \bigoplus_{G} \mathbb{Z}G & \mathbb{Z}G \\
 & & [g] \longmapsto & g\cdot[\,] - [\,] \\
 & [g_1 \mid g_2] \longmapsto & [g_1] + g_1\cdot[g_2] - [g_1\cdot g_2] &
\end{array}$$

In combination with Remark 1.2.4 (and the fact that G acts trivially on $\mathbb{Z}$), we see that the corresponding part of $C_*(G;\mathbb{Z}) = C_*(G) \otimes_G \mathbb{Z}$ is canonically naturally isomorphic to

$$\begin{array}{llll}
\text{degree} & 2 & 1 & 0 \\
& \overline{C}_2(G;\mathbb{Z}) \xrightarrow{\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}}} & \overline{C}_1(G;\mathbb{Z}) \xrightarrow{\overline{\partial}_1 \otimes_G \mathrm{id}_{\mathbb{Z}}} & \overline{C}_0(G;\mathbb{Z}) \\
& \cong & \cong & \cong \\
& \bigoplus_{G^2} \mathbb{Z} & \bigoplus_G \mathbb{Z} & \mathbb{Z} \\
& & [g] \longmapsto & g \cdot [\,] - [\,] = 0 \\
& [g_1 \mid g_2] \longmapsto [g_1] + [g_2] - [g_1 \cdot g_2] & &
\end{array}$$

Hence, $H_1(G;\mathbb{Z}) = \overline{C}_1(G;\mathbb{Z}) \bigm/ \bigl(\operatorname{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})\bigr)$. A straightforward calculation shows that

$$\begin{aligned}
\overline{C}_1(G;\mathbb{Z}) \bigm/ \bigl(\operatorname{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})\bigr) &\longrightarrow G_{\mathrm{ab}} \\
\overline{\varphi} \colon [g] + \operatorname{im}(\dots) &\longmapsto g \cdot [G,G] \\
[g] + \operatorname{im}(\dots) &\longleftarrow g \cdot [G,G] : \overline{\psi}
\end{aligned}$$

are well-defined mutually inverse isomorphisms:

- The map $\overline{\varphi}$ is a well-defined $\mathbb{Z}$-linear map: We consider the $\mathbb{Z}$-linear map defined by

$$\begin{aligned}
\varphi \colon \overline{C}_1(G;\mathbb{Z}) &\longrightarrow G_{\mathrm{ab}} \\
[g] &\longmapsto g \cdot [G,G];
\end{aligned}$$

  this is possible because G_{ab} is Abelian. In order to show that $\overline{\varphi}$ is well-defined, we only need to show that φ maps all elements of $\operatorname{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})$ to the trivial element of G_{ab}. If $g_1, g_2 \in G$, then we have in G_{ab} that

$$\begin{aligned}
\varphi\bigl(\overline{\partial}_2([g_1 \mid g_2])\bigr) &= \varphi\bigl([g_1] + [g_2] - [g_1 \cdot g_2]\bigr) \\
&= \varphi\bigl([g_1]\bigr) + \varphi\bigl([g_2]\bigr) - \varphi\bigl([g_1 \cdot g_2]\bigr) \\
&= \bigl(g_1 \cdot [G,G]\bigr) \cdot \bigl(g_2 \cdot [G,G]\bigr) \cdot \bigl(g_1 \cdot g_2 \cdot [G,G]\bigr)^{-1} \\
&= e \cdot [G,G].
\end{aligned}$$

- The map $\overline{\psi}$ is a well-defined $\mathbb{Z}$-linear map: The map

$$\begin{aligned}
\psi \colon G &\longrightarrow \overline{C}_1(G;\mathbb{Z}) \bigm/ \bigl(\operatorname{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})\bigr) \\
g &\longmapsto [g] + \operatorname{im}(\dots)
\end{aligned}$$

  is a group homomorphism (by the explicit description of $\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}}$ above). Hence, the universal property of the Abelianisation shows that

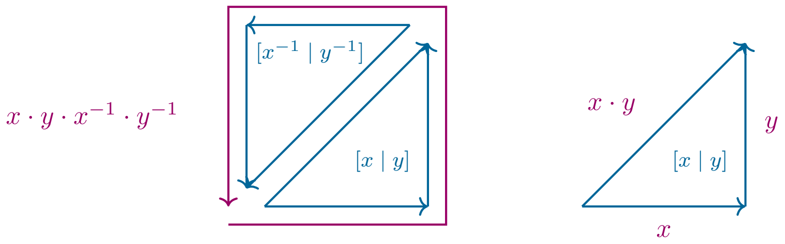


Figure 1.1.: A geometric description of the well-definedness of $\overline{\psi}$; the triangles visualise elements in $\overline{C}_2$, the edges visualise elements in $\overline{C}_1$.

ψ induces a well-defined group homomorphism

$$\begin{aligned} \overline{\psi}\colon G_{\mathrm{ab}} &\longrightarrow \overline{C}_1(G;\mathbb{Z}) \;/\; \big(\mathrm{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})\big) \\ g \cdot [G,G] &\longmapsto [g] + \mathrm{im}(\dots). \end{aligned}$$

- The maps $\overline{\varphi}$ and $\overline{\psi}$ are mutually inverse: This is clear from the construction.

Moreover, φ and ψ are natural on Group (check!). □

Remark 1.4.3 (more on the geometry of commutators)**.** The proof of Theorem 1.4.1 shows that for each commutator $g = [x, y]$ in G with $x, y \in G$ we have that $[g] \in \mathrm{im}(\overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}})$. This can also be seen geometrically as follows: By construction, we have in $\overline{C}_1(G;\mathbb{Z})$ (Figure 1.1)

$$\begin{aligned} [g] &= [x \cdot y \cdot x^{-1} \cdot y^{-1}] \\ &= \overline{\partial}_2 \otimes_G \mathrm{id}_{\mathbb{Z}}\big([x \mid y] + [x^{-1} \mid y^{-1}] - [x \cdot y \mid x^{-1} \cdot y^{-1}] \\ &\quad - [x \mid x^{-1}] - [y \mid y^{-1}] + [e \mid e] + [e \mid e]\big). \end{aligned}$$

This observation can be used to give a more geometric proof of the well-definedness of $\overline{\psi}$ (as we have done in the lecture).

Study note. If you know basic algebraic topology: The statement of Theorem 1.4.1 should look familiar (why?). Again, this is part of a deeper connection of group (co)homology with algebraic topology (Chapter 4).

Example 1.4.4 (group homology in degree 1)**.**

- If G is Abelian, then $H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} G_{\mathrm{ab}} \cong_{\mathbb{Z}} G$.
- Let $n \in \mathbb{N}_{\geq 2}$. Then $(S_n)_{\mathrm{ab}} \cong_{\mathbb{Z}} \mathbb{Z}/2$, and so $H_1(S_n;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2$.

Definition 1.4.5 (perfect group)**.** A group G is *perfect* if $G = [G, G]$.

Corollary 1.4.6 (homological characterisation of perfect groups). *Let G be a group. Then G is perfect if and only if $H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} 0$.*

Proof. Clearly, a group G is perfect if and only if $G_{\mathrm{ab}} \cong_{\mathbb{Z}} 0$. Hence, the claim follows from Theorem 1.4.1. □

Example 1.4.7 (first homology of alternating groups). Let $n \in \mathbb{N}_{\geq 5}$. Then the alternating group A_n is perfect (Satz III.1.3.13). Hence, $H_1(A_n;\mathbb{Z}) \cong_{\mathbb{Z}} 0$.

Outlook 1.4.8 (Thompson's group T). Thompson's group T is an example of a finitely generated infinite simple group [15]. In particular, T is perfect, and so $H_1(T;\mathbb{Z}) \cong_{\mathbb{Z}} 0$.

Example 1.4.9 (first homology of free groups). Let S be a set and let $F(S)$ be the free group, freely generated by S (Appendix A.1). Then the universal properties of free groups and of the Abelianisation show that the homomorphism $F(S)_{\mathrm{ab}} \longrightarrow \bigoplus_S \mathbb{Z}$, that maps S to the standard basis of the free $\mathbb{Z}$-module $\bigoplus_S \mathbb{Z}$ is an isomorphism of $\mathbb{Z}$-modules. In particular, Theorem 1.4.1 shows that

$$H_1\bigl(F(S);\mathbb{Z}\bigr) \cong_{\mathbb{Z}} \bigoplus_S \mathbb{Z}.$$

Moreover, the first group homology also gives a crude lower bound for the rank of a group. Let us first recall the rank of $\mathbb{Z}$-modules: If A is a $\mathbb{Z}$-module, then

$$\operatorname{rk}_{\mathbb{Z}} A = \begin{cases} \text{rank of the free part of } A & \text{if } A \text{ is finitely generated} \\ \infty & \text{if } A \text{ is not finitely generated.} \end{cases}$$

For general groups, we will use the following notion of rank:

Definition 1.4.10 (rank of a group). Let G be a group. The *rank of* G, denoted by $\operatorname{rk} G \in \mathbb{N} \cup \{\infty\}$, is the minimal number of generators needed to generate G.

Corollary 1.4.11 (homological rank estimate). *Let G be a group. Then*

$$\operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) \leq \operatorname{rk} H_1(G;\mathbb{Z}) \leq \operatorname{rk} G.$$

In particular: If $\operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) = \infty$, then G is not *finitely generated.*

Proof. Combining Theorem 1.4.1 with the fact that the rank of a quotient group cannot exceed the rank of the original group (check!), we obtain

$$\operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) \leq \operatorname{rk} H_1(G;\mathbb{Z}) = \operatorname{rk} G_{\mathrm{ab}} = \operatorname{rk}\bigl(G/[G,G]\bigr) \leq \operatorname{rk} G. \qquad □$$

Outlook 1.4.12 (homological rank gradient estimate). In this basic form, the estimate of Corollary 1.4.11 might not seem very impressive and, in general, it is very far from being optimal. Taking $\ell^2(\,\cdot\,;\mathbb{C})$-coefficients instead of $\mathbb{Z}$-coefficients, one can prove a corresponding lower bound for the so-called *rank gradient* of group [64, 29, 48]: If G is a finitely generated residually finite group, then

$$b_1^{(2)}(G) \leq \inf_{H \in \operatorname{FinSub}(G)} \frac{\operatorname{rk} H}{[G:H]},$$

where $\operatorname{FinSub}(G)$ denotes the set of all finite index subgroups of G and the left hand side is a suitable "dimension" of $H_1(G;\ell^2(G,\mathbb{C}))$. There is no example known of a finitely presented residually finite group, for which this inequality is not an equality(!).

1.4.2 Cohomology in degree 1: Homomorphisms

Theorem 1.4.13 (group cohomology in degree 1)**.** *Let G be a group and let Z be a $\mathbb{Z}$-module. Then (where Z carries the trivial G-action) there is a canonical isomorphism*

$$H^1(G;Z) \cong_{\mathbb{Z}} \operatorname{Hom}_{\mathsf{Group}}(G,Z).$$

More precisely, the functor $H^1(\,\cdot\,;Z)\colon \mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is canonically naturally isomorphic to $\operatorname{Hom}_{\mathsf{Group}}(\,\cdot\,,Z)\colon \mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$.

Proof. As in the proof of the corresponding result for homology (Theorem 1.4.1), we use the bar complex to compute $H^1(G;Z)$. In combination with Remark 1.2.14 and the hypothesis that G acts trivially on Z, we obtain that the corresponding part of $C^*(G;Z) = \operatorname{Hom}_G(C^*(G),Z)$ is canonically naturally isomorphic to $\overline{C}^*(G;Z)$, i.e., to

$$\begin{tikzcd}
2 & 1 & 0 \\
\overline{C}^2(G;Z) \arrow[d, equal] & \overline{C}^1(G;Z) \arrow[l, "\operatorname{Hom}_G(\overline{\partial}_2,Z)"'] \arrow[d, equal] & \overline{C}^0(G;Z) \arrow[l, "-\operatorname{Hom}_G(\overline{\partial}_1,Z)"'] \arrow[d, equal] \\
\operatorname{Hom}_G\bigl(\bigoplus_{G^2}\mathbb{Z}G,Z\bigr) \arrow[d, "\cong" sloped] & \operatorname{Hom}_G\bigl(\bigoplus_{G}\mathbb{Z}G,Z\bigr) \arrow[d, "\cong" sloped] & \operatorname{Hom}_G(\mathbb{Z}G,Z) \arrow[d, "\cong" sloped] \\
\operatorname{Hom}_{\mathbb{Z}}\bigl(\bigoplus_{G^2}\mathbb{Z},Z\bigr) \arrow[d, "\cong" sloped] & \operatorname{Hom}_{\mathbb{Z}}\bigl(\bigoplus_{G}\mathbb{Z},Z\bigr) \arrow[d, "\cong" sloped] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},Z) \arrow[d, "\cong" sloped] \\
\bigl([g_1 \mid g_2] \mapsto f([g_1]) + f([g_2]) - f([g_1 \cdot g_2])\bigr) & f \arrow[l, mapsto] & \\
 & \bigl([g] \mapsto f(\overline{\partial}_1[g]) = 0\bigr) & f \arrow[l, mapsto]
\end{tikzcd}$$

Hence, the elements in the kernel of the coboundary map $\operatorname{Hom}_G(\overline{\partial}_2,Z)$ precisely correspond to those maps $G \longrightarrow Z$ that satisfy the homomorphism condition. Because $\operatorname{im}(-\operatorname{Hom}_G(\overline{\partial}_1,Z)) = 0$, it follows that

$$\begin{aligned} H^1(G;Z) \cong_{\mathbb{Z}} \ker\bigl(\operatorname{Hom}_G(\overline{\partial}_2, Z)\bigr) &\longrightarrow \operatorname{Hom}_{\mathsf{Group}}(G, Z) \\ f &\longmapsto \bigl(g \mapsto f([g])\bigr) \\ \bigl([g] \mapsto f(g)\bigr) &\longleftarrow f \end{aligned}$$

are well-defined, mutually inverse, isomorphisms of $\mathbb{Z}$-modules. Moreover, all these isomorphisms are natural on Group (check!). □

In particular, we can apply this to torsion groups. A group G is a *torsion group* if every element of G has finite order. In contrast, a group G is *torsion-free* if *no* non-trivial element of G has finite order.

Example 1.4.14 (torsion groups)**.**

- Every finite group is a torsion group.
- The group $\prod_{\mathbb{N}} \mathbb{Z}/2$ is a torsion group, but infinite.
- There exist finitely generated, infinite, torsion groups [33][53, p. 111ff].
- The additive group $\mathbb{Z}$ is torsion-free.

Corollary 1.4.15. *Let G be a torsion group and let Z be a torsion-free Abelian group (endowed with the trivial G-action). Then $H^1(G;Z) \cong_{\mathbb{Z}} 0$.*

Proof. By Theorem 1.4.13, we have $H^1(G;Z) \cong_{\mathbb{Z}} \operatorname{Hom}_{\mathsf{Group}}(G,Z)$. Because G is a torsion group and Z is torsion-free, all group homomorphisms $G \longrightarrow Z$ are trivial. Hence, the right hand side is the trivial Abelian group. □

Example 1.4.16. By Corollary 1.4.15, we have $H^1(G;\mathbb{Q}) \cong_{\mathbb{Z}} 0$ for every finite group G (we will generalise this in Corollary 1.7.16).

1.4.3 Application: Hilbert 90

In algebraic number theory, number theoretic properties are encoded in field extensions. Problems on Galois extensions can be translated into problems on Galois groups and their cohomology. Therefore, group cohomology contributes to algebraic number theory in various ways. We will briefly discuss a simple example of this type, namely the solution of polynomial equations via radicals. Because the Galois group of a field extension acts on the extension field, we can view the extension field as a module over the Galois group.

Theorem 1.4.17 (Hilbert 90, cohomological version)**.** *Let $L \mid K$ be a finite Galois extension of fields with Galois group G. Then*

$$H^1(G;L^\times) \cong_{\mathbb{Z}} 0,$$

where the G-action on the coefficients $L^\times$ is the Galois action (and we think of the coefficients $L^\times$ as a multiplicative group).

Proof. During this proof, it will be convenient to denote the cohomology group $H^1(G; L^\times)$ multiplicatively (because $L^\times$ should be viewed as multiplicative group). Moreover, in view of Remark 1.2.14, it suffices to prove that $H^1(\overline{C}^*(G; L^\times)) \cong_{\mathbb{Z}} 1$.

We follow the general strategy of averaging: The group G is finite and thus we can sum up (i.e., average) expressions over all elements of G.

Let $f \in \overline{C}^1(G; L^\times) = \operatorname{Hom}_G(\overline{C}_1(G), L^\times)$ be a cocycle. The elements of G yield group homomorphisms $L^\times \longrightarrow L^\times$ and thus can be viewed as characters of L. Such characters are linearly independent over L (Lemma 1.4.18) and thus there exists an $x \in L$ such that

$$\overline{x} := \sum_{\tau \in G} f\bigl(1 \cdot [\tau]\bigr) \cdot \tau(x)$$

is non-zero, whence lies in $L^\times$. We will now prove that the $\mathbb{Z}G$-homomorphism

$$\begin{aligned} b \colon \overline{C}_0(G) &\longrightarrow L^\times \\ \sigma_0 \cdot [] &\longrightarrow \sigma_0(\overline{x}) \end{aligned}$$

witnesses that f is a coboundary: Indeed, for all $\sigma_1 \in G$, we have (where we abbreviate $\overline{\delta}^0 := -\operatorname{Hom}_G(\partial_1, L^\times)$)

$$\begin{aligned}
(\overline{\delta}^0 b)\bigl(1 \cdot [\sigma_1]\bigr) &= b\bigl((-1) \cdot (\sigma_1 \cdot [] - [])\bigr) \\
&= \frac{b\bigl([]\bigr)}{b\bigl(\sigma_1 \cdot []\bigr)} && \text{(multiplicative notation!)} \\
&= \frac{\overline{x}}{\sigma_1(\overline{x})} && \text{(construction of } b\text{)} \\
&= \frac{\overline{x}}{\sum_{\tau \in G} \sigma_1\bigl(f\bigl(1 \cdot [\tau]\bigr)\bigr) \cdot \sigma_1 \circ \tau(x)} && \text{(construction of } \overline{x}\text{)} \\
&= \frac{\overline{x}}{\sum_{\tau \in G} f\bigl(\sigma_1 \cdot [\tau]\bigr) \cdot \sigma_1 \circ \tau(x)} && (f \text{ is } \mathbb{Z}G\text{-linear}) \\
&= \frac{\overline{x}}{\sum_{\tau \in G} \frac{1}{f([\sigma_1])} \cdot f\bigl([\sigma_1 \circ \tau]\bigr) \cdot \sigma_1 \circ \tau(x)} && (f \text{ is a cocycle}) \\
&= \frac{\overline{x} \cdot f\bigl([\sigma_1]\bigr)}{\sum_{\tau \in G} f\bigl([\tau]\bigr) \cdot \tau(x)} && \text{(averaging is invariant)} \\
&= \overline{x} \cdot f\bigl([\sigma_1]\bigr) \cdot \frac{1}{\overline{x}} && \text{(construction of } \overline{x}\text{)} \\
&= f\bigl(1 \cdot [\sigma_1]\bigr).
\end{aligned}$$

Because f and $\overline{\delta}^0(b)$ are $\mathbb{Z}G$-linear, we obtain $\overline{\delta}^0 b = f$. This shows that $H^1(G; L^\times)$ is trivial. $\square$

Proposition 1.4.18 (linear independence of characters)**.** *Let G be a group and let L be a field. If $n \in \mathbb{N}$ and $\chi_1, \dots, \chi_n \colon G \longrightarrow L^\times$ are n distinct group homomorphisms (so-called* characters*), then the family $(\chi_1, \dots, \chi_n)$ in the L-vector space of maps $G \longrightarrow L$ is L-linearly independent.*

Proof. We proceed by induction over n:

- The case $n = 0$: The empty family is always linearly independent.

- The case $n = 1$: Because all values of χ_1 lie in $L^\times$, the function χ_1 is not the zero function and thus linearly independent.

- The induction step: Let $n \in \mathbb{N}_{>1}$ and let the claim already be proved for families of size $n-1$. Then also $(\chi_1, \dots, \chi_n)$ are L-linearly independent: Let $\lambda_1, \dots, \lambda_n \in L$ with

  $$\sum_{j=1}^{n} \lambda_j \cdot \chi_j = 0$$

  (where the 0 on the right hand side is the zero function $G \longrightarrow L$). We will now show that $\lambda_1 = \dots = \lambda_n = 0$:

  Because $\chi_1 \neq \chi_n$, there exists a $g \in G$ with $\chi_1(g) \neq \chi_n(g)$. Applying the equation above to $g \cdot h$ with $h \in G$ shows that

  $$\forall_{h \in H} \quad 0 = \sum_{j=1}^{n} \lambda_j \cdot \chi_j(g \cdot h) = \sum_{j=1}^{n} \lambda_j \cdot \chi_j(g) \cdot \chi_j(h).$$

  On the other hand, multiplying the equation above by $\chi_n(g)$ leads to

  $$\forall_{h \in H} \quad 0 = \chi_n(g) \cdot \sum_{j=1}^{n} \chi_j(h) = \sum_{j=1}^{n} \chi_n(g) \cdot \chi_j(h).$$

  Subtracting the last two equations from each other results in

  $$0 = \sum_{j=1}^{n-1} \lambda_j \cdot \bigl(\chi_j(g) - \chi_n(g)\bigr) \cdot \chi_j.$$

  Because the family $(\chi_1, \dots, \chi_{n-1})$ is L-linearly independent by induction, we obtain that all these coefficients must be zero. In particular,

  $$\lambda_1 \cdot \bigl(\chi_1(g) - \chi_n(g)\bigr) = 0.$$

  By construction, $\chi_1(g) \neq \chi_n(g)$, whence $\lambda_1 = 0$. Inserting this into the original equation shows that also

$$\sum_{j=2}^{n} \lambda_j \cdot \chi_j = 0.$$

Thus, applying the induction hypothesis to $(\chi_2, \dots, \chi_n)$, we also obtain $\lambda_2 = \dots = \lambda_n = 0$. □

We will now apply this cohomological version to cyclic Galois extensions:

Corollary 1.4.19 (Hilbert 90). *Let $L \mid K$ be a finite cyclic Galois extension, let σ be a generator of the Galois group G, and let $x \in L$. Then the following are equivalent:*

1. *The norm $N_{L|K}(x)$ of x equals* 1.
2. *There exists an $a \in L^\times$ satisfying $x = a/\sigma(a)$.*

Before giving the proof, we recall some terminology from Galois theory:

- A Galois extension is *cyclic* if its Galois group is cyclic.
- If $L \mid K$ is a finite Galois extension and $x \in L$, then the *norm* $N_{L|K}(x) \in K$ *of* x is the determinant of the K-linear map $L \longrightarrow L$ given by multiplication with x.
- If $L \mid K$ is a finite Galois extension, then [9, Kapitel 4.7]

  $$N_{L|K}(x) = \prod_{\tau \in \operatorname{Gal}(L,K)} \tau(x)$$

  for all $x \in L$. In particular, $N_{L|K}$ is invariant under the action of the Galois group $\operatorname{Gal}(L,K)$. For our purposes, we could also take this as a definition of the norm.

Proof of Corollary 1.4.19. Ad $2 \Longrightarrow 1$. Let $a \in L^\times$ with $x = a/\sigma(a)$. Then

$$\begin{aligned} N_{L|K}(x) &= \frac{N_{L|K}(a)}{N_{L|K}\big(\sigma(x)\big)} && \text{(multiplicativity of the determinant/norm)} \\ &= \frac{N_{L|K}(a)}{N_{L|K}(a)} && \text{(Galois invariance of the norm)} \\ &= 1. \end{aligned}$$

Ad $1 \Longrightarrow 2$. Conversely, let us suppose that $N_{L|K}(x) = 1$ and that the σ has order n (thus $G \cong_{\mathsf{Group}} \mathbb{Z}/n$). Then the $\mathbb{Z}G$-linear map specified by

$$\begin{aligned} f \colon \overline{C}_1(G) &\longrightarrow L^\times \\ 1 \cdot [\sigma^k] &\longmapsto \prod_{j=0}^{k-1} \sigma^j(x) \end{aligned}$$

is a well-defined cocycle in $\overline{C}^1(G; L^\times)$ (because $\prod_{j=0}^{n-1} \sigma^j(x) = N_{L|K}(x) = 1$; check!). Hence, the cohomological version of Hilbert 90 (Theorem 1.4.17) shows that there exists a $b \in \overline{C}^0(G; L^\times)$ with $\overline{\delta}^0(b) = f$. Then $a := b(1 \cdot [])$ satisfies

$$x = f\bigl(1 \cdot [\sigma]\bigr) = (\overline{\delta}^0 b)\bigl(1 \cdot [\sigma]\bigr) = \frac{b\bigl(1 \cdot []\bigr)}{b\bigl(\sigma \cdot []\bigr)} = \frac{b\bigl(1 \cdot []\bigr)}{\sigma\bigl(b(1 \cdot [])\bigr)} = \frac{a}{\sigma(a)},$$

as desired. □

Remark 1.4.20 (application to number theory)**.** One can now apply Corollary 1.4.19 to roots of unity in the base field: Let $L \mid K$ be a cyclic Galois extension of degree $n \in \mathbb{N}$ and let $\zeta \in K$ be an n-th root of unity. Then $N_{L|K}(\zeta) = \zeta^n = 1$ (we have $\zeta \in K$ and thus the determinant is just $\zeta^{\dim_K L}$). If $\sigma \in \operatorname{Gal}(L, K)$ is a generator of the Galois group of $L \mid K$, then Corollary 1.4.19 provides an element $a \in L$ with $a = \zeta \cdot \sigma(a)$. In particular, $a^n \in L^\sigma = K$ and so $X^n - a^n \in K[X]$.

If ζ is a primitive root of unity (primitive in an algebraic closure of K) and if $\operatorname{char} K \nmid n$, then one can show in this situation that L is the splitting field of $X^n - a^n$ over K.

This observation is the base case of the inductive proof of the characterisation of solvability by radicals in terms of solvability of the corresponding Galois groups [9, Kapitel 6.1] (Chapter III.3.5.2).

Outlook 1.4.21 (generalisations)**.** Clearly, it is possible to prove the classical Hilbert 90 theorem (Corollary 1.4.19) without talking about group cohomology (for instance, we could take the same proof and apply the averaging/character argument directly to the specific cocycle in the proof of Corollary 1.4.19). The cohomological formulation has the advantage that it is easy to guess generalisations:

- There is an additive version of Theorem 1.4.17, involving the trace instead of the norm (which is used to complete the characterisation of cyclic Galois extensions in positive characteristic).
- Further generalisations occur in Kummer theory (i.e., finite Galois extensions with Abelian Galois group).
- There is a version of Theorem 1.4.17 for infinite Galois extensions (in terms of continuous group cohomology of the (profinite) Galois group).

Furthermore, also group cohomology in degree 2 plays an important role in algebraic number theory (as Brauer groups). Therefore, computations in group cohomology are relevant for algebraic number theory.

Literature exercise. Who first proved Corollary 1.4.19? Why is it called "Hilbert 90"? Who first proved Theorem 1.4.17?

1.5 Degree 2: Presentations and extensions

We now move on to group (co)homology in degree 2. In homology, we will see a close relation with presentations of groups; dually, in cohomology, we will be able to classify group extensions with Abelian kernel.

1.5.1 Homology in degree 2: Hopf's formula

Theorem 1.5.1 (Hopf's formula). *Let F be a free group, let $N \subset F$ be a normal subgroup, and let $G := F/N$. Then there is an exact sequence*

$$0 \longrightarrow H_2(G;\mathbb{Z}) \longrightarrow H_1(N;\mathbb{Z})_G \longrightarrow H_1(F;\mathbb{Z}) \longrightarrow H_1(G;\mathbb{Z}) \longrightarrow 0$$

(where the homomorphisms on H_1 are induced by the canonical inclusion and projection, respectively, and G acts on $H_1(N;\mathbb{Z})$ by conjugation of representatives in F). More explicitly,

$$H_2(G;\mathbb{Z}) \cong_{\mathbb{Z}} \frac{N \cap [F,F]}{[F,N]}.$$

Here, $[F,N]$ denotes the subgroup of F generated by the set of commutators $[x,y]$ with $x \in F$ and $y \in N$.

Proof. The Hopf formula admits different proofs, e.g., via spectral sequences or via classifying spaces. We will give a proof later, once we have more tools available (Theorem 3.2.18). A proof that only uses basic homological algebra can be found in the book by Hilton and Stammbach [42, Chapter VI.9]. $\square$

Hopf's formula has many applications in group theory; for example, it can be used to establish an analogue of the rank estimate from Corollary 1.4.11 for the number of relations. In order to formulate this result, we briefly recall presentations of groups in terms of generators and relations:

Remark 1.5.2 (generators and relations). Let S be a set, let $F(S)$ be "the" free group generated by S (Appendix A.1), and let $R \subset F(S)$. Then the *group generated by S with relations R* is defined as

$$\langle S \mid R \rangle := F(S)/N,$$

where $N := \langle R \rangle^{\triangleleft}_{F(S)} \subset F(S)$ is the smallest (with respect to inclusion) normal subgroup of $F(S)$ containing R. It is common to abuse notation and to write $\langle S \mid R \rangle$ also to refer to the presentation, not only the group.

For example, working with the corresponding universal properties shows that [53, Chapter 2.2]

$$\begin{aligned} \langle a \,|\, \rangle &\cong_{\mathsf{Group}} \mathbb{Z} \\ \langle a, b \,|\, aba^{-1}b^{-1} \rangle &\cong_{\mathsf{Group}} \mathbb{Z}^2 \\ \langle a \,|\, a^2 \rangle &\cong_{\mathsf{Group}} \mathbb{Z}/2 \\ \langle s, t \,|\, s^{2019}, t^2, tst^{-1}s \rangle &\cong_{\mathsf{Group}} D_{2019}. \end{aligned}$$

However, it can be proved that there is *no* algorithm that, given a presentation with finitely many generators and finitely many relations, decides whether the corresponding group is trivial or not [70, Chapter 12].

Corollary 1.5.3. *Let G be a finitely generated group. If G admits a finite presentation, then $H_2(G;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.*

In other words: If $H_2(G;\mathbb{Z})$ is not *a finitely generated $\mathbb{Z}$-module, then G does* not *admit a finite presentation.*

Proof. If $\langle S \,|\, R \rangle$ is a finite presentation of G, then

$$G \cong_{\mathsf{Group}} F(S) / \langle R \rangle^{\triangleleft}_{F(S)}$$

and we can apply Hopf's formula (Theorem 1.5.1) to the free group $F := F(S)$ and the normal subgroup $N := \langle R \rangle^{\triangleleft}_{F(S)}$. Hence, $H_2(G;\mathbb{Z})$ is isomorphic to a submodule of $H_1(N;\mathbb{Z})_G$. Because $\operatorname{rk}_{\mathbb{Z}} H_1(N;\mathbb{Z})_G \leq |R|$ (Theorem 1.4.1 and the fact that R *normally* generates N), it follows that $H_2(G;\mathbb{Z})$ is finitely generated. □

We can refine this finiteness property to a quantitative statement:

Corollary 1.5.4. *Let G be a finitely presentable group and let $\langle S \,|\, R \rangle$ be a finite presentation of G. Then*

$$|S| - |R| \leq \operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_2(G;\mathbb{Z}).$$

Proof. Again, we can apply Hopf's formula (Theorem 1.5.1) to the free group $F := F(S)$ and the normal subgroup $N := \langle R \rangle^{\triangleleft}_{F(S)}$. Then the short exact sequence in Hopf's formula shows that

$$\operatorname{rk}_{\mathbb{Z}} H_2(G;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_1(N;\mathbb{Z})_G + \operatorname{rk}_{\mathbb{Z}} H_1(F;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) = 0;$$

this is a generalised version of the dimension formula from linear algebra, which can, e.g., be proved by applying the exact functor $\mathbb{Q} \otimes_{\mathbb{Z}} \,\cdot\,$. Hence,

$$\begin{aligned} |S| - |R| &\leq \operatorname{rk}_{\mathbb{Z}} H_1(F;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_1(N;\mathbb{Z})_G && (\text{Example 1.4.9; } N = \langle R \rangle^{\triangleleft}_{F(S)}) \\ &= \operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_2(G;\mathbb{Z}), && (\text{dimension formula}) \end{aligned}$$

as claimed. □

Outlook 1.5.5 (deficiency)**.** Let G be a finitely generated group and let $P(G)$ denote the "set" of all presentations of G with finite generating set. The *deficiency* of G is defined as

$$\operatorname{def} G := \max\bigl\{|S| - |R| \bigm| \langle S \mid R\rangle \in P(G)\bigr\}.$$

Then Corollary 1.5.4 shows that

$$\operatorname{def} G \leq \operatorname{rk}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_2(G;\mathbb{Z}).$$

In particular, $\operatorname{def} G$ is a well-defined element of $\mathbb{Z} \cup \{-\infty\}$ (and never equal to $+\infty$).

Conversely, we can use group homology in higher degree to introduce further finiteness conditions on groups, going beyond finite presentability. We will return to the point of view in Chapter 4.2.

1.5.2 Cohomology in degree 2: Extensions

One of the most classical applications of group cohomology is the classification of extensions through $H^2(\,\cdot\,;\,\cdot\,)$.

Definition 1.5.6 (extensions)**.** Let Q and A be groups.

- An *extension of* Q *by* A is an exact sequence in Group of the form

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

- Extensions $1 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$ and $1 \longrightarrow A \xrightarrow{i'} G' \xrightarrow{\pi'} Q \longrightarrow 1$ of Q by A are called *equivalent* if there exists a group homomorphism $\varphi\colon G \longrightarrow G'$ fitting into the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & G \arrow[r, "\pi"] \arrow[d, "\varphi"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r, "i'"'] & G' \arrow[r, "\pi'"'] & Q \arrow[r] & 1
\end{tikzcd}$$

(In this case, φ is already an isomorphism; check!)

Remark 1.5.7 (the conjugation action of the quotient on the kernel)**.** Let

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

be an extension of a group Q by an *Abelian* group A. This extension induces a $\mathbb{Z}Q$-module structure on A: The group G acts by conjugation on the normal subgroup $i(A)$, whence on $A \cong_{\mathsf{Group}} i(A)$. Because A is Abelian, the conjugation action of A on itself is trivial, whence the conjugation action by G on A descends to a well-defined action of $Q \cong_{\mathsf{Group}} G/i(A)$ on A:

$$\begin{aligned} Q \times A &\longrightarrow A \\ (q, a) &\longmapsto \text{“}g \cdot a \cdot g^{-1}\text{”} = i^{-1}(g \cdot i(a) \cdot g^{-1}), \text{ where } g \in G \text{ with } \pi(g) = q. \end{aligned}$$

This Q-action on A yields a $\mathbb{Z}Q$-module structure on A.

Equivalent extensions of Q by A lead to the same $\mathbb{Z}Q$-module structure on A (check!).

Example 1.5.8 (extensions and actions)**.**

- Let A be an Abelian group and let Q be a group. Then the action of Q on A induced by the product extension

  $$0 \longrightarrow A \xrightarrow{\text{can. incl.}} A \times Q \xrightarrow{\text{can. proj.}} Q \longrightarrow 1$$

  is the trivial action.

- In the extension

  $$0 \longrightarrow A_3 \xrightarrow{\text{can. incl}} S_3 \xrightarrow{\text{can. proj.}} S_3/A_3 \longrightarrow 1$$

  the non-trivial element of the quotient $S_3/A_3 \cong_{\mathsf{Group}} \mathbb{Z}/2$ acts by taking inverses on $A_3 \cong_{\mathsf{Group}} \mathbb{Z}/3$ (check!).

- The extensions

  $$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{can. incl.}} \mathbb{Z} \times \mathbb{Z}/2 \xrightarrow{\text{can. proj.}} \mathbb{Z}/2 \longrightarrow 1$$

  $$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot\,\cdot} \mathbb{Z} \xrightarrow{\text{can. proj.}} \mathbb{Z}/2 \longrightarrow 1$$

  both lead to the trivial action of $\mathbb{Z}/2$ on $\mathbb{Z}$. But these extensions are *not* equivalent (because the extension groups are *not* isomorphic).

Definition 1.5.9 (extension set)**.** Let Q be a group and let A be a $\mathbb{Z}Q$-module. We then write $E(Q, A)$ for the set(!) of all equivalence classes of extensions of Q by A that induce the given $\mathbb{Z}Q$-module structure on A.

Theorem 1.5.10 (classification of group extensions with Abelian kernel)**.** *Let Q be a group and let A be a $\mathbb{Z}Q$-module. Then the maps*

$$\begin{aligned} H^2(Q; A) &\longrightarrow E(Q, A) \\ [f] &\longmapsto [E_f \colon 0 \to A \to G_f \to Q \to 1] \\ \eta_E &\longleftarrow E \end{aligned}$$

are mutually inverse bijections (the extensions E_f and the cohomology class η_E will be specified in the proof below).

Proof. Again, we will work with the description of $H^2(Q;A)$ in terms of the bar resolution (Remark 1.2.14).

We start with the map from the right hand side to the left hand side; i.e., we explain how an extension defines a 2-cocycle in such a way that equivalent extensions lead to cohomologous cocycles. Let $E \in E(Q,A)$ and let

$$0 \longrightarrow A \xrightarrow{\;i\;} G \xrightarrow{\;\pi\;} Q \longrightarrow 1$$

be an extension of Q by A that induces the given Q-action on A and that represents E. The idea is to measure the failure of $\pi\colon G \longrightarrow Q$ to be a split epimorphism by a 2-cocycle:

- *Choosing a section.* Let $s\colon Q \longrightarrow G$ be a set-theoretic section of the map $\pi\colon G \longrightarrow Q$ (such a section exists by the axiom of choice). Of course, in general, we cannot expect s to be a group homomorphism. Measuring the failure of s being a group homomorphism leads to the map

$$\begin{aligned} F\colon Q \times Q &\longrightarrow A \\ (q_1,q_2) &\longmapsto s(q_1)\cdot s(q_2)\cdot s(q_1\cdot q_2)^{-1}; \end{aligned}$$

  this map F indeed maps to A (because s is a section of π) and will be the key to defining a 2-cocycle.

- *Rewriting the group structure on G.* Using the map F, we can recover the group structure on G from the group structure on Q and the Q-action on A as follows: First of all,

$$\begin{aligned} G &\longrightarrow A \times Q \\ g &\longmapsto \bigl(g\cdot s(\pi(g))^{-1}, \pi(g)\bigr) \\ a\cdot s(q) &\longleftarrow (a,q) \end{aligned}$$

  are mutually inverse bijections (check!). Under these bijections, the composition on G translates into the following composition on $A \times Q$ (where $\bullet$ denotes the Q-action on A induced by the given extension):

$$\begin{aligned} (A\times Q)\times(A\times Q) &\longrightarrow A\times Q \\ \bigl((a,q),(a',q')\bigr) &\longmapsto \bigl(a + q\bullet a' + F(q,q'), q\cdot q'\bigr). \end{aligned}$$

  Indeed, for all $(a,q),(a',q') \in A\times Q$ we have (in G)

$$\begin{aligned} a\cdot s(q)\cdot a'\cdot s(q') &= a\cdot q\bullet a'\cdot s(q)\cdot s(q') \\ &= a\cdot q\bullet a'\cdot F(q,q')\cdot s(q\cdot q'), \end{aligned}$$

because the Q-action on A is given by conjugation in G.

- *Constructing a cocycle.* Because the composition on G is associative, a short calculation shows that

$$F(q_1, q_2) + F(q_1 \cdot q_2, q_3) - q_1 \bullet F(q_2, q_3) - F(q_1, q_2 \cdot q_3) = 0$$

  for all $q_1, q_2, q_3 \in Q$ (check!). Hence,

$$\begin{aligned} f \colon \overline{C}_2(Q) &\longrightarrow A \\ q_0 \cdot [q_1 \mid q_2] &\longmapsto q_0 \bullet F(q_1, q_2) \end{aligned}$$

  is a cocycle in $\overline{C}^*(Q; A)$.

- *Changing the section.* Let $s' \colon Q \longrightarrow G$ be another set-theoretic section of π and let $f' \in C^2(Q; A)$ be the corresponding cocycle. Because s and s' are sections of π, there is a function $B \colon Q \longrightarrow A$ with

$$\forall_{q \in Q} \quad s'(q) = B(q) \cdot s(q).$$

  Then

$$\begin{aligned} b \colon \overline{C}_1(Q) &\longrightarrow A \\ q_0 \cdot [q_1] &\longmapsto q_0 \bullet B(q_1) \end{aligned}$$

  satisfies $\overline{\delta}^1(b) = f' - f$: By construction, for all $q_1, q_2 \in Q$, we have

$$\begin{aligned} f'([q_1 \mid q_2]) &= s'(q_1) \cdot s'(q_2) \cdot s'(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= B(q_1) \cdot s(q_1) \cdot B(q_2) \cdot s(q_2) \cdot s(q_1 \cdot q_2)^{-1} \cdot B(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= B(q_1) \cdot \bigl(q_1 \bullet B(q_2)\bigr) \cdot s(q_1) \cdot s(q_2) \cdot s(q_1 \cdot q_2)^{-1} \cdot B(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= B(q_1) + q_1 \bullet B(q_2) + f\bigl([q_1 \mid q_2]\bigr) - B(q_1 \cdot q_2) && \text{in } A \\ &= (\overline{\delta}^1 b)\bigl([q_1 \mid q_2]\bigr) + f\bigl([q_1 \mid q_2]\bigr). && \text{in } A \end{aligned}$$

- *Changing the extension.* Let

$$0 \longrightarrow A \xrightarrow{i'} G' \xrightarrow{\pi'} Q \longrightarrow 1$$

  be an extension that is equivalent to the previous one. Furthermore, let $\varphi \colon G' \longrightarrow G$ be an isomorphism witnessing that these extensions are equivalent. If $s' \colon Q \longrightarrow G'$ is a section of π', then $\varphi \circ s' \colon Q \longrightarrow G$ is a section of π and the cocycles corresponding to s' and to $\varphi \circ s'$ coincide (check!). Therefore, the previous step shows that the extension G' leads to the same cohomology class as the extension involving G.

Therefore, we obtain a well-defined cohomology class $\eta_E \in H^2(Q; A)$ out of E.

Conversely, let a cohomology class $\eta \in H^2(Q;A)$ be given. We now show how to construct an equivalence class of extensions of Q by A out of this cohomology class. More precisely, we construct extensions out of 2-cocycles in such a way that cohomologous cocycles lead to equivalent extensions:

- *A group structure out of a cocycle.* Let $f \in \overline{C}^2(Q;A)$ be a cocycle and let

$$\begin{aligned} F\colon Q \times Q &\longrightarrow A \\ (q_1, q_2) &\longmapsto f\bigl([q_1 \mid q_2]\bigr). \end{aligned}$$

  Inspired by the first part of the proof, on the set $A \times Q$, we define the composition

$$\begin{aligned} (A \times Q) \times (A \times Q) &\longrightarrow A \times Q \\ \bigl((a,q),(a',q')\bigr) &\longmapsto \bigl(a + q \bullet a' + F(q,q'), q \cdot q'\bigr). \end{aligned}$$

  The same calculation as above shows that f being a cocycle implies that this composition is associative (check!).

  Moreover, using the cocycle property of f once more, we see that (ε, e) is a neutral element for this composition (check!), where

$$\varepsilon := -F(e,e) = -f\bigl([e \mid e]\bigr).$$

  A straightforward computation shows that every element of $A \times Q$ has an inverse element with respect to this composition and the neutral element (ε, e) (check!). So $G_f := A \times Q$ with this composition is a group.

- *An extension out of a cocycle.* The group G_f fits into the extension

$$0 \longrightarrow A \xrightarrow{\ i_f\ } G_f \xrightarrow{\ \pi_f\ } Q \longrightarrow 1,$$

  where the homomorphisms (check!) are given by

$$\begin{aligned} i_f\colon A &\longrightarrow G_f = A \times Q \\ a &\longmapsto (a + \varepsilon, e) \\ \pi_f\colon G_f = A \times Q &\longrightarrow Q \\ (a,q) &\longmapsto q. \end{aligned}$$

- *The induced action on the kernel.* The map

$$\begin{aligned} s_f\colon Q &\longrightarrow G_f = A \times Q \\ q &\longmapsto (0,q) \end{aligned}$$

is a set-theoretic section of π_f. Hence, the Q-action $*$ of the above extension on A is given by

$$\begin{aligned}
q * a &= s_f(q) \cdot (a + \varepsilon, e) \cdot s_f(q)^{-1} \\
&= (0, q) \cdot (a + \varepsilon, e) \cdot (0, q)^{-1} \\
&= \bigl(0 + q \bullet (a + \varepsilon) + F(q, e), q\bigr) \cdot (0', q^{-1}) \\
&= \bigl(q \bullet (a + \varepsilon) + F(q, e) + q \bullet 0' + F(q, q^{-1}), e\bigr) \\
&= \bigl(q \bullet (a + \varepsilon) + F(q, e) + \varepsilon, e\bigr) \qquad ((0', q^{-1}) \text{ is inverse to } (0, q)) \\
&= (q \bullet a + \varepsilon, e) \qquad\qquad (\text{because } f \text{ is a cocycle; for } [q \mid e \mid e]) \\
&= i_f(q \bullet a)
\end{aligned}$$

for all $q \in Q$ and all $a \in A$; here, we write $(0', q^{-1}) := (0, q)^{-1}$. Thus, the extension $E_f := (\ 0 \longrightarrow A \xrightarrow{i_f} G_f \xrightarrow{\pi_f} Q \longrightarrow 1\ )$ induces the given Q-action on A and so represents a class in $E(Q, A)$.

- *Changing the cocycle.* Similarly to the previous arguments, we see that changing the cocycle f by a coboundary leads to an equivalent extension (check!).

Therefore, we obtain a well-defined map $H^2(Q; A) \longrightarrow E(Q, A)$.

These two maps are mutually inverse:

- If $f \in \overline{C}^2(Q; A)$ is a cocycle, then the cocycle associated with the extension E_f and the set-theoretic section $s_f \colon Q \longrightarrow G_f$ is f (check! A is embedded via i_f into G_f). Hence, $\eta_{[E_f]} = [f]$.

- Conversely, if $0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$ is an extension that induces the given Q-action on A, if s is a set-theoretic section of π and $f \in \overline{C}^2(Q; A)$ is the associated cocycle, then the group homomorphism

$$\begin{aligned}
G_f = A \times Q &\longrightarrow G \\
(a, q) &\longmapsto a \cdot s(q)
\end{aligned}$$

shows that E_f is equivalent to the given extension. □

When viewing group cohomology as derived functor of a Hom-functor, we will see that group cohomology can be described as Ext-functor. In fact, the name "Ext" goes back to the above classification of *ext*ensions.

Remark 1.5.11 (trivial extensions)**.** Let Q be a group and let A be a $\mathbb{Z}Q$-module. Then the extension corresponding (under the bijection in Theorem 1.5.10) to the zero class in $H^2(Q; A)$ is the semi-direct product extension

$$0 \longrightarrow A \xrightarrow{\text{can. incl.}} A \rtimes Q \xrightarrow{\text{can. proj.}} Q \longrightarrow 1$$

with respect to the given Q-action on A (as can be seen from the definition of the semi-direct product (Chapter III.1.1.6) and the construction of the extension corresponding to the zero cocycle in the proof of Theorem 1.5.10).

Conversely, non-trivial extensions lead to non-trivial cohomology classes in degree 2. For example, we can use this observation to show that $H^2(\mathbb{Z}/n;\mathbb{Z})$ is non-trivial for all $n \in \mathbb{N}_{\geq 2}$, where $\mathbb{Z}/n$ acts trivially on $\mathbb{Z}$ (Exercise). Similarly, also $H^2(\mathbb{Z}^2;\mathbb{Z})$ and $H^2(\mathrm{Homeo}^+(S^1);\mathbb{Z})$ are non-trivial (Exercise).

Corollary 1.5.12 (cohomology of free groups in degree 2)**.** *Let F be a free group and let A be a $\mathbb{Z}F$-module. Then $H^2(F;A) \cong_{\mathbb{Z}} 0$.*

Proof. In view of the classification theorem (Theorem 1.5.10) and the description of the extensions corresponding to the zero class (Remark 1.5.11), it suffices to show that every extension

$$0 \longrightarrow A \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } F \longrightarrow 1$$

that induces the given $\mathbb{Z}F$-module structure on A is trivial (i.e., equivalent to a semi-direct product extension).

The universal property of the free group F shows that the epimorphism $\pi \colon G \longrightarrow F$ admits a *group*-theoretic section $s \colon F \longrightarrow G$; i.e., s is a group homomorphism with $\pi \circ s = \mathrm{id}_F$.

Then the isomorphism (check! see also the rewriting argument in the proof of Theorem 1.5.10; because s is a group homomorphism, the associated map $F \times F \longrightarrow A$ is the zero map)

$$\begin{aligned} G &\longrightarrow A \rtimes F \\ g &\longmapsto \bigl(g \cdot s(\pi(g))^{-1}, \pi(g)\bigr) \end{aligned}$$

(where the semi-direct product on the right hand side is formed with respect to the given F-action on A) shows that the given extension is trivial. $\square$

Of course, constructing non-trivial cohomology classes out of non-trivial extensions is a "wrong-way" application of the classification theorem; usually, one applies it in the other direction. Indeed, once we have more computational tools available, we will be able to derive group-theoretic results from Theorem 1.5.10.

The classification of extensions with Abelian kernel is functorial in the following sense:

Theorem 1.5.13 (functoriality of the classification of extensions with Abelian kernel)**.** *Let $(\varphi, \Phi) \colon (Q, A) \longrightarrow (Q', A')$ be a morphism in* GroupMod *and let $E \in E(Q,A)$ and $E' \in E(Q',A')$ be represented by the extensions $0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$ and $0 \longrightarrow A' \xrightarrow{i'} G' \xrightarrow{\pi'} Q' \longrightarrow 1$, respectively.*

Then the following are equivalent:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_f"] \arrow[d, "\Phi"'] & G_f \arrow[r, "\pi_f"] \arrow[d, dashed, "\psi"'] & Q \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \varphi^* A' \arrow[r, "i_g"] \arrow[d, equal] & G_g \arrow[r, "\pi_g"] \arrow[d, dashed, "\cong"'] & Q \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \varphi^* A' \arrow[r, "i_g"] \arrow[d, equal] & G_{g'} \arrow[r, "\pi_g"] \arrow[d, dashed, "\psi'"'] & Q \arrow[r] \arrow[d, "\varphi"] & 1 \\
0 \arrow[r] & A' \arrow[r, "i_{f'}"'] & G_{f'} \arrow[r, "\pi_{f'}"'] & Q' \arrow[r] & 1
\end{tikzcd}$$

Figure 1.2.: Comparing extensions

1. *There exists a group homomorphism* $\widetilde{\varphi}\colon G \longrightarrow G'$ *making the following diagram commutative:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\Phi"'] & G \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\varphi}"'] & Q \arrow[r] \arrow[d, "\varphi"] & 1 \\
0 \arrow[r] & A' \arrow[r, "i'"'] & G' \arrow[r, "\pi'"'] & Q' \arrow[r] & 1
\end{tikzcd}$$

2. *In* $H^2\big(Q;\varphi^*(A')\big)$, *we have* $H^2(\mathrm{id}_Q;\Phi)(\eta_E) = H^2(\varphi;\mathrm{id}_{A'})(\eta_{E'})$.

Proof. Ad 1 $\Longrightarrow$ 2. In this situation, we can choose set-theoretic sections $s\colon Q \longrightarrow G$ and $s'\colon Q' \longrightarrow G'$ of π and π', respectively, that satisfy $s' \circ \varphi = \widetilde{\varphi} \circ s$ (check!). Then the explicit construction of the cohomology classes of these extensions in the proof of Theorem 1.5.10 shows that $H^2(\mathrm{id}_Q;\Phi)(\eta_E) = H^2(\varphi;\mathrm{id}_{A'})(\eta_{E'})$ holds already at the level of associated cocycles (check!).

Ad 2 $\Longrightarrow$ 1. Let $H^2(\mathrm{id}_Q;\Phi)(\eta_E) = H^2(\varphi;\mathrm{id}_{A'})(\eta_{E'})$ in $H^2(Q;\varphi^*A')$. We choose cocycles $f \in \overline{C}^2(Q;A)$ and $f' \in \overline{C}^2(Q';A')$ that represent η_E and $\eta_{E'}$, respectively. Without loss of generality, we may assume that the given extensions are the extensions constructed from the cocycles f and f'. Then

$$\begin{aligned}
g\colon \overline{C}_2(Q) &\longrightarrow \varphi^*A' \\
[q_1 \mid q_2] &\longmapsto \Phi\big(f([q_1 \mid q_2])\big) \\
g'\colon \overline{C}_2(Q) &\longrightarrow \varphi^*A' \\
[q_1 \mid q_2] &\longmapsto f'\big([\varphi(q_1) \mid \varphi(q_2)]\big)
\end{aligned}$$

are cocycles representing $H^2(\mathrm{id}_Q; \Phi)(\eta_E)$ and $H^2(\varphi; \mathrm{id}_{A'})(\eta_{E'})$, respectively. Because $[g] = [g']$, we know that the corresponding extensions E_g and $E_{g'}$ of Q by $\varphi^* A'$ are equivalent (Theorem 1.5.10). Moreover, the group homomorphisms (check!)

$$\begin{aligned} \psi \colon G_f = A \times Q &\longrightarrow \varphi^* A' \times Q = G_g \\ (a, q) &\longmapsto \bigl(a, \varphi(q)\bigr) \\ \psi' \colon G_{g'} = \varphi^* A' \times Q &\longrightarrow A' \times Q' = G_{f'} \\ (a', q) &\longmapsto \bigl(\Phi(a'), q\bigr) \end{aligned}$$

fit into the commutative diagram in Figure 1.2 (check!). Therefore, the composition of the middle vertical arrows is a group homomorphism $\widetilde{\varphi} \colon G_f \longrightarrow G_{f'}$ with the desired property. $\Box$

Corollary 1.5.14. *Let Q be a group, let A be a $\mathbb{Z}Q$-module, let $E \in E(Q, A)$, and let* $0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$ *be an extension representing E. Then*

$$H^2(\pi; \mathrm{id}_A)(\eta_E) = 0.$$

Proof. On the one hand, the zero class in $H^2(G; \pi^* A)$ is represented by the semi-direct product extension $0 \longrightarrow \pi^* A \xrightarrow{j} \pi^* A \rtimes G \xrightarrow{q} G \longrightarrow 1$ (Remark 1.5.11). On the other hand, the group homomorphism (check!)

$$\begin{aligned} \widetilde{\pi} \colon \pi^* A \rtimes G &\longrightarrow G \\ (a, g) &\longmapsto a \cdot g \end{aligned}$$

fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^* A \arrow[r, "j"] \arrow[d, "\mathrm{id}_A"'] & \pi^* A \rtimes G \arrow[r, "q"] \arrow[d, dashed, "\widetilde{\pi}"'] & G \arrow[r] \arrow[d, "\pi"] & 1 \\ 0 \arrow[r] & A \arrow[r, "i"'] & G \arrow[r, "\pi"'] & Q \arrow[r] & 1 \end{tikzcd}$$

Therefore, Theorem 1.5.13 and Remark 1.5.11 show that

$$\begin{aligned} H^2(\pi; \mathrm{id}_A)(\eta_E) &= H^2(\mathrm{id}_Q; \mathrm{id}_A)(\eta_{\text{semi-direct product extension}}) \\ &= H^2(\mathrm{id}_Q; \mathrm{id}_A)(0) = 0. \end{aligned}$$ $\Box$

Outlook 1.5.15 (universal central extensions and the Schur multiplier)**.** Let G be a perfect group and let $H := H_2(G; \mathbb{Z})$ (which we consider as trivial G-module). Then $H^2(G; H)$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(H, H)$ and the extension of G by H corresponding to id_H is the *universal central extension*, which is initial among all central extensions of G [87, Chapter 6.9]; an extension is *central* if the kernel is a central subgroup of the extension group.

If G is perfect and finite, then $H_2(G;\mathbb{Z})$ is isomorphic to $H^2(G;\mathbb{C}^\times)$, the *Schur multiplier* of G. Schur used the group $H^2(G;\mathbb{C}^\times)$ to study representations of G. These considerations were one of the precursors of group cohomology.

Outlook 1.5.16 (classification of extensions with non-Abelian kernel)**.** More generally, also extensions of groups by non-Abelian kernels can be classified by means of group cohomology [12, Chapter IV.6]. However, this classification is more delicate; for example, given a quotient group and a kernel group with an action by the quotient, there does not necessarily exist a extension inducing the given action (i.e., there is no non-Abelian analogue of the semi-direct product). This classification can, e.g., be used to give efficient algorithms testing whether finite groups of certain types are isomorphic or not [34].

1.6 Changing the resolution

Using the fundamental theorem of homological algebra, we will obtain that group (co)homology can be computed via many (co)chain complexes. This flexibility is the key to many computations and applications of group (co)homology. We will first briefly review projective resolutions and recall the proof of the fundamental theorem of homological algebra. Then, we will use these results for some concrete computations and later to establish the Shapiro lemma.

1.6.1 Projective resolutions

The basic building blocks in derived homological algebra are projective resolutions. Projective modules are a convenient generalisation of free modules.

Definition 1.6.1 (projective module)**.** Let R be a ring with unit. A left R-module P is *projective* if it has the following lifting property: For every epimorphism $\pi\colon B \longrightarrow C$ in ${}_R\mathsf{Mod}$ and every R-homomorphism $\alpha\colon P \longrightarrow C$, there exists an R-homomorphism $\widetilde{\alpha}\colon P \longrightarrow B$ with $\pi \circ \widetilde{\alpha} = \alpha$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\widetilde{\alpha}"'] \arrow[d, "\alpha"] & \\ B \arrow[r, "\pi"'] & C \arrow[r] & 0 \end{tikzcd}$$

(Analogously, we define projective right modules).

Example 1.6.2 (projective modules)**.**

- All free modules are projective (check!). In fact, for most of our applications free modules will suffice.
- Direct summands of free modules are projective (check!).
- Direct sums of projective modules are projective.
- The $\mathbb{Z}$-module $\mathbb{Z}/2$ is *not* projective: The lifting problem

$$\begin{tikzcd} & \mathbb{Z}/2 \arrow[dl, dashed, "?"'] \arrow[d, "\mathrm{id}_{\mathbb{Z}/2}"] & \\ \mathbb{Z} \arrow[r, "\text{Proj.}"'] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

  in ${}_{\mathbb{Z}}\mathsf{Mod}$ has *no* solution.

- *Not* every projective module is free: For example, the sections of the Möbius strip form a projective module over the ring $C(S^1, \mathbb{R})$ of continuous functions, which is *not* free (Beispiel IV.3.4.12).

Moreover, we have the following equivalent characterisations of projectivity (Proposition IV.5.2.3):

Proposition 1.6.3 (characterisations of projectivity)**.** *Let R be a ring with unit and let P be a left R-module. Then the following are equivalent:*

1. *The R-module P is projective.*
2. *The R-module P is a direct summand of a free R-module.*
3. *Every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ in ${}_R\mathsf{Mod}$ admits a right split.*
4. *The functor ${}_R\mathrm{Hom}(P, \,\cdot\,) \colon {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is exact (i.e., maps exact sequences to exact sequences).*

Definition 1.6.4 (projective resolution)**.** Let R be a ring with unit and let M be a left R-module. A *resoluion of M by R-modules* is a pair (C_*, ε), where

- C_* is an $\mathbb{N}$-indexed chain complex of left R-modules (with boundary operators ∂_*)
- ε is an R-homomorphism $C_0 \longrightarrow M$,
- and the concatenated sequence $P \mathbin{\square} \varepsilon$ is exact:

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

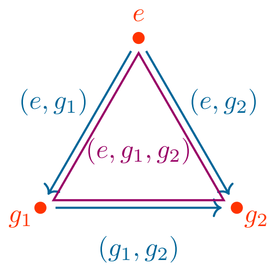


Figure 1.3.: The cone operator in the singular resolution

A resolution (C_*, ε) of M by R-modules is

- *projective* if for each $n \in \mathbb{N}$, the chain module C_n is projective,
- *free* if for reach $n \in \mathbb{N}$, the chain module C_n is free.

Proposition 1.6.5 (simplicial resolution and bar resolution)**.** *Let G be a group and let $\varepsilon \colon \mathbb{Z}G \longrightarrow \mathbb{Z}$ be the augmentation map (Definition 1.2.1). Then $(C_*(G), \varepsilon)$ and $(\overline{C}_*(G), \varepsilon)$ are free (whence projective) resolutions of $\mathbb{Z}$ (with the trivial G-action) over $\mathbb{Z}G$.*

Proof. As the $\mathbb{Z}G$-chain complexes $C_*(G)$ and $\overline{C}_*(G)$ are isomorphic (Remark 1.2.4) and as these isomorphisms are compatible with ε, it suffices to prove that

① $\overline{C}_n(G)$ is for each $n \in \mathbb{N}$ a free $\mathbb{Z}G$-module and that

② $(C_*(G), \varepsilon)$ is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Ad ①. Let $n \in \mathbb{N}$. Then $\overline{C}_n(G) = \bigoplus_{G^n} \mathbb{Z}G$ is a free $\mathbb{Z}G$-module by construction.

Ad ②. Because $C_*(G)$ is a chain complex and $\varepsilon \circ \partial_1 = 0$, the concatenation $C_*(G) \square \varepsilon$ is a $\mathbb{Z}G$-chain complex. In order to prove exactness, we will give an explicit construction of a chain contraction over the ring $\mathbb{Z}$ (geometrically, this corresponds to a cone-operation with cone point e; Figure 1.3): To this end we consider the $\mathbb{Z}$-homomorphisms given by

$$\begin{aligned} s_{-1} \colon \mathbb{Z} &\longrightarrow \mathbb{Z}G = C_0(G) \\ 1 &\longmapsto 1 \end{aligned}$$

as well as, for all $n \in \mathbb{N}$,

$$\begin{aligned} s_n \colon C_n(G) &\longrightarrow C_{n+1}(G) \\ G^{n+1} \ni (g_0, \dots, g_n) &\longmapsto (e, g_0, \dots, g_n). \end{aligned}$$

Then we have

$$\varepsilon \circ s_{-1} = \mathrm{id}_{\mathbb{Z}}$$
$$s_{-1} \circ \varepsilon + \partial_1 \circ s_0 = \mathrm{id}_{C_0(G)}$$

and

$$s_{n-1} \circ \partial_n + \partial_{n+1} \circ s_n = \mathrm{id}_{C_n(G)}$$

for all $n \in \mathbb{N}$ (check! Figure 1.3); i.e., $(s_n)_{n \in \mathbb{N} \cup \{-1\}}$ is a chain contraction of $C_*(G) \mathbin{\square} \varepsilon$ as chain complex of $\mathbb{Z}$-modules (the whole point of the theory is that this, in general, is *not* a chain contraction over the ring $\mathbb{Z}G$). This proves exactness.

More explicitly: By construction, ε is surjective. If $c \in \ker \varepsilon$, then

$$c = (s_{-1} \circ \varepsilon + \partial_1 \circ s_0)(c) = \partial_1\bigl(s_0(c)\bigr) \in \operatorname{im} \partial_1.$$

In the same way, if $n \in \mathbb{N}_{>0}$ and $c \in \ker \partial_n$, then

$$c = (s_{n-1} \circ \partial_n + \partial_{n+1} \circ s_n)(c) = \partial_{n+1}\bigl(s_n(c)\bigr) \in \operatorname{im} \partial_{n+1}. \qquad \square$$

Example 1.6.6 (resolutions from topology)**.** Let X be a path-connected CW-complex with fundamental group G and universal covering $\pi \colon \widetilde{X} \longrightarrow X$. Then the (involution of the) deck transformation action of G on $\widetilde{X}$ turns the singular chain complex $C_*(\widetilde{X};\mathbb{Z})$ of $\widetilde{X}$ into a $\mathbb{Z}G$-chain complex. If $\widetilde{X}$ is contractible, then $C_*(\widetilde{X};\mathbb{Z})$, together with

$$\begin{aligned} C_0(\widetilde{X};\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \operatorname{map}(\Delta^0, \widetilde{X}) \ni \sigma &\longmapsto 1, \end{aligned}$$

is a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over $\mathbb{Z}G$.

If we equip the universal covering $\widetilde{X}$ with the lifted CW-structure from X, then also the cellular chain complex of $\widetilde{X}$ can be viewed as a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We will explore this connection in more detail in Chapter 4.

If G is a group, then the full simplex $\Delta(G)$ on the vertex set G is contractible and its full simplicial chain complex coincides with $C_*(G)$. This gives an alternative proof of the fact that $C_*(G)$ (together with the augmentation map) is a free resolution of $\mathbb{Z} \cong_{\mathbb{Z}} H_0(\bullet;\mathbb{Z})$ over $\mathbb{Z}G$.

1.6.2 The fundamental theorem of group (co)homology

In derived homological algebra, one replaces modules by projective resolutions. The fundamental observation is that every module admits a projective resolution and that projective resolutions are unique up to chain homotopy equivalence. In our situation (i.e., for modules over the group ring), this shows that we can replace the simplicial/bar resolution by other resolutions, which we call the fundamental theorem of group (co)homology.

Theorem 1.6.7 (fundamental theorem of homological algebra)**.** *Let R be a ring with unit, let M and N be left R-modules, and let $f\colon M \longrightarrow N$ be an R-module homomorphism. Let (P_*, ε) be a projective resolution of M over R and let $C_* \square (\gamma\colon C_0 \to N)$ be an exact sequence in ${}_R\mathsf{Mod}$.*

1. *Then $f\colon M \longrightarrow N$ admits an extension $\widetilde{f}_* \square f\colon P_* \square \varepsilon \longrightarrow C_* \square \gamma$ to an R-chain map.*
2. *Moreover, this extension $\widetilde{f}_*$ is unique up to R-chain homotopy.*

Proof. Ad 1. We construct $\widetilde{f}_*$ inductively: The boundary operators of P_* will be denoted by ∂^P_*, the ones of C_* by ∂^C_*. In order to keep notation simple, we will also use the (dangerous) conventions $\partial^P_0 := \varepsilon$, $\partial^C_0 := \gamma$, and $\widetilde{f}_{-1} := f$.

- *The base case.* Because $\gamma\colon C_0 \longrightarrow N$ is surjective and P_0 is projective, there exists an R-homomorphism $\widetilde{f}_0\colon P_0 \longrightarrow C_0$ with $\gamma \circ \widetilde{f}_0 = f \circ \varepsilon$:

$$\begin{tikzcd} P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\widetilde{f}_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ C_0 \arrow[r, "\gamma"'] & N \arrow[r] & 0 \end{tikzcd}$$

- *Induction step.* Let $n \in \mathbb{N}$ and let us suppose that we already constructed an extension $\widetilde{f}\colon P_* \longrightarrow C_*$ up to degree n (as R-chain map). We then obtain an R-homomorphism $\widetilde{f}_{n+1}\colon P_{n+1} \longrightarrow C_{n+1}$ with $\partial^C_{n+1} \circ \widetilde{f}_{n+1} = \widetilde{f}_n \circ \partial^P_{n+1}$ as solution to the following lifting problem:

$$\begin{tikzcd} P_{n+1} \arrow[r, "\partial^P_{n+1}"] \arrow[d, dashed, "\widetilde{f}_{n+1}"'] & \operatorname{im} \partial^P_{n+1} \arrow[d, "\widetilde{f}_n|_{\operatorname{im} \partial^P_{n+1}}"] & \\ C_{n+1} \arrow[r, "\partial^C_{n+1}"'] & \operatorname{im} \partial^C_{n+1} \arrow[r] & 0 \end{tikzcd}$$

  Here, we use that $\widetilde{f}_n(\operatorname{im} \partial^P_{n+1}) \subset \operatorname{im} \partial^C_{n+1} = \ker \partial^C_n$, which follows from $\partial^C_n \circ \widetilde{f}_n \circ \partial^P_{n+1} = \widetilde{f}_{n-1} \circ \partial^P_n \circ \partial^P_{n+1} = 0$.

Ad 2. Uniqueness follows from a similar construction: Let $\widetilde{f}_*$ and $\widetilde{g}_*$ be extensions of f. We then set $h_{-1} := 0\colon M \longrightarrow C_0$ and, inductively, we construct R-homomorphisms $h_n\colon P_n \longrightarrow C_{n+1}$ satisfying the chain homotopy equation

$$\partial^C_{n+1} \circ h_n + h_{n-1} \circ \partial^P_n = \widetilde{g}_n - \widetilde{f}_n.$$

for all $n \in \mathbb{N}$.

- *The base case.* We have $\operatorname{im}(\widetilde{g}_0 - \widetilde{f}_0) \subset \operatorname{im} \partial_1^C$ because

$$\begin{aligned}\partial_0^C \circ (\widetilde{g}_0 - \widetilde{f}_0) &= f \circ \partial_0^P - f \circ \partial_0^P \quad (\widetilde{f}_* \,\square\, f \text{ and } \widetilde{g}_* \,\square\, f \text{ are chain maps})\\ &= 0\end{aligned}$$

  and $C_* \square \gamma$ is exact. Hence, projectivity of P_0 allows to choose $h_0 \colon P_0 \longrightarrow C_1$ as solution to the lifting problem

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "h_0"'] \arrow[d, "\widetilde{g}_0 - \widetilde{f}_0"] & \\ C_1 \arrow[r, "\partial_1^C"'] & \operatorname{im} \partial_1^C \arrow[r] & 0 \end{tikzcd}$$

- *Induction step.* Let $n \in \mathbb{N}$ and let us suppose that we already constructed $h_0, \dots, h_n$ satisfying the chain homotopy equation. The induction hypothesis and exactness of $C_* \square \gamma$ shows that

$$\operatorname{im}(\widetilde{g}_{n+1} - \widetilde{f}_{n+1} - h_n \circ \partial_{n+1}^P) \subset \operatorname{im} \partial_{n+2}^C.$$

  Therefore, projectivity of P_{n+1} guarantees the existence of an R-homomorphism $h_{n+1} \colon P_{n+1} \longrightarrow C_{n+2}$ with

$$\partial_{n+2}^C \circ h_{n+1} = \widetilde{g}_{n+1} - \widetilde{f}_{n+1} - h_n \circ \partial_{n+1}^P. \qquad \square$$

Corollary 1.6.8 (existence and uniqueness of projective resolutions)**.** *Let R be a ring with unit and let M be a left R-module. Then M admits a projective resolution over R, which is unique up to R-chain homotopy equivalence (which is canonical up to R-chain homotopy).*

Proof. Existence. We first establish that the category ${}_R\mathsf{Mod}$ of left R-modules has enough projective objects: If M is a left R-module, then $\bigoplus_M R$ is a free R-module and the map $\mathrm{id}_M \colon M \longrightarrow M$ extends to an R-homomorphism $\varepsilon_M \colon \bigoplus_M R \longrightarrow M$, which is surjective (by construction).

Using this fact, we can construct a projective resolution of M by induction: We set

$$\begin{aligned} P_0 &:= \bigoplus_M R \quad \text{and} \\ P_{n+1} &:= \bigoplus_{\ker \partial_n} R \\ \partial_{n+1} &:= \varepsilon_{\ker \partial_n} \colon P_{n+1} \longrightarrow \ker \partial_n \subset P_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then (P_*, ε_M) is a free resolution of M over R. (Moreover, this construction is functorial on ${}_R\mathsf{Mod}$ in terms of the resolved module).

Uniqueness follows from the usual universal property yoga (which universal property in which category?!): Let (P_*, ε) and (P'_*, ε') be projective resolutions of M over R. By the fundamental theorem (Theorem 1.6.7), the R-homomorphism $\mathrm{id}_M \colon M \longrightarrow M$ extends to R-chain maps

$$\begin{aligned} \widetilde{f}_* \,\square\, \mathrm{id}_M \colon P_* \,\square\, \varepsilon &\longrightarrow P'_* \,\square\, \varepsilon' \\ \widetilde{g}_* \,\square\, \mathrm{id}_M \colon P'_* \,\square\, \varepsilon' &\longrightarrow P_* \,\square\, \varepsilon. \end{aligned}$$

Moreover, the uniqueness part of the fundamental theorem shows that these R-chain maps are unique up to R-chain homotopy and that

$$\begin{aligned} (\widetilde{f}_* \,\square\, \mathrm{id}_M) \circ (\widetilde{g}_* \,\square\, \mathrm{id}_M) &\simeq_R \mathrm{id}_{P'_*} \,\square\, \mathrm{id}_M \\ (\widetilde{g}_* \,\square\, \mathrm{id}_M) \circ (\widetilde{f}_* \,\square\, \mathrm{id}_M) &\simeq_R \mathrm{id}_{P_*} \,\square\, \mathrm{id}_M \,. \end{aligned}$$

This shows the uniqueness claim. □

Corollary 1.6.9 (fundamental theorem of group (co)homology I)**.** *Let G be a group, let A be a $\mathbb{Z}G$-module, and let (P_*, ε) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then, for each $n \in \mathbb{N}$, there exist canonical isomorphisms*

$$\begin{aligned} H_n(G; A) &\cong_{\mathbb{Z}} H_n(P_* \otimes_G A) \\ H^n(G; A) &\cong_{\mathbb{Z}} H^n\bigl(\mathrm{Hom}_G(P_*, A)\bigr). \end{aligned}$$

Proof. By the uniqueness of projective resolutions (Corollary 1.6.8) and because the simplicial resolution is a projective resolution (Proposition 1.6.5), there exists a $\mathbb{Z}G$-chain homotopy equivalence $f \colon C_*(G) \longrightarrow P_*$ extending id_A (which is canonical up to $\mathbb{Z}G$-homotopy).

Then $f \otimes_G \mathrm{id}_A \colon C_*(G; A) \longrightarrow P_* \otimes_G A$ and $\mathrm{Hom}_G(f, A) \colon \mathrm{Hom}_G(P_*, A) \longrightarrow C^*(G; A)$ are $\mathbb{Z}$-(co)chain homotopy equivalences. In particular, these (co)chain homotopy equivalences induce (canonical!) isomorphisms

$$\begin{aligned} H_n(f \otimes_G \mathrm{id}_A) \colon H_n(G; A) &\longrightarrow H_n(P_* \otimes_G A) \\ H^n\bigl(\mathrm{Hom}_R(f, A)\bigr) \colon H^n\bigl(\mathrm{Hom}_G(P_*, A)\bigr) &\longrightarrow H^n(G; A). \end{aligned}$$

of $\mathbb{Z}$-modules for every $n \in \mathbb{N}$. □

The challenge now is to find resolutions over the group ring that are well adapted to the target application.

1.6.3 Example: Finite cyclic groups

As first example, we use the freedom of changing the resolution to compute the (co)homology of finite cyclic groups:

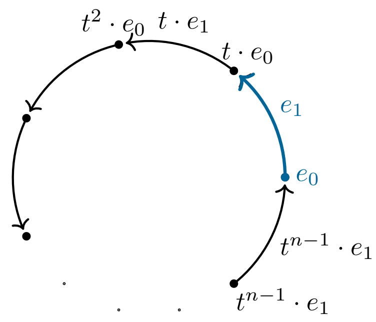


Figure 1.4.: A free $\mathbb{Z}/n$-equivariant CW-structure on S^1, leading to the resolution in Proposition 1.6.11

Definition 1.6.10. Let G be a group and let $a \in \mathbb{Z}G$. Then we denote the associated $\mathbb{Z}G$-homomorphism

$$\begin{aligned} \mathbb{Z}G &\longrightarrow \mathbb{Z}G \\ x &\longmapsto x \cdot a \end{aligned}$$

given by right multiplication with a by M_a.

Study note. Why do we choose right multiplication instead of the more common left multiplication?

Proposition 1.6.11 (a resolution for finite cyclic groups)**.** *Let $n \in \mathbb{N}_{>0}$, let $G := \mathbb{Z}/n$, let $t := [1] \in \mathbb{Z}/n$, and let $N := \sum_{j=0}^{n-1} t^j \in \mathbb{Z}G$. Then*

$$\cdots \xrightarrow{M_N} \mathbb{Z}G \xrightarrow{M_{t-1}} \mathbb{Z}G \xrightarrow{M_N} \mathbb{Z}G \xrightarrow{M_{t-1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ (with trivial G-action) over $\mathbb{Z}G$.

Proof. The chain modules are free $\mathbb{Z}G$-modules, whence projective. Clearly,

$$N \cdot (1 - t) = 0 \in \mathbb{Z}G.$$

This shows that the above chain of $\mathbb{Z}G$-homomorphisms is a chain complex. Exactness follows from concrete computations in finite-dimensional linear algebra (Exercise). □

Remark 1.6.12 (geometric idea)**.** Let $n \in \mathbb{N}_{\geq 2}$. The cyclic group $G := \mathbb{Z}/n$ acts freely on the circle by rotation about $2 \cdot \pi / n$. This leads to a corresponding free G-equivariant CW-structure on S^1 (Figure 1.4). Looking at the cellular homology of this CW-complex results in a short exact sequence

$$0 \longrightarrow H_1(S^1;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{M_{t-1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \cong_{\mathbb{Z}} H_0(S^1;\mathbb{Z}) \longrightarrow 0$$

of $\mathbb{Z}G$-modules, where ε is the usual augmentation and η is multiplication by $N := \sum_{j=0}^{n-1} t^j$. Therefore, we can splice these sequences together to obtain a long exact sequence of the form

$$\cdots \xrightarrow{M_N} \mathbb{Z}G \xrightarrow{M_{t-1}} \mathbb{Z}G \xrightarrow{M_N} \mathbb{Z}G \xrightarrow{M_{t-1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

This is the projective resolution from Proposition 1.6.11.

Corollary 1.6.13 ((co)homology of finite cyclic groups)**.** *Let $n \in \mathbb{N}_{>0}$, let $G := \mathbb{Z}/n$, let $t := [1] \in \mathbb{Z}/n$, let $N := \sum_{j=0}^{n-1} t^j$, and let A be a $\mathbb{Z}G$-module. Then, for all $k \in \mathbb{N}$,*

$$H_k(G;A) \cong_{\mathbb{Z}} \begin{cases} A_G & \text{if } k = 0 \\ A^G/N \cdot A & \text{if } k \text{ is odd} \\ \ker(N\colon A \longrightarrow A)/(t-1)\cdot A & \text{if } k > 0 \text{ is even} \end{cases}$$

$$H^k(G;A) \cong_{\mathbb{Z}} \begin{cases} A^G & \text{if } k = 0 \\ \ker(N\colon A \longrightarrow A)/(t-1)\cdot A & \text{if } k \text{ is odd} \\ A^G/N \cdot A & \text{if } k > 0 \text{ is even.} \end{cases}$$

In particular, for all $k \in \mathbb{N}_{>0}$,

$$H_{k+2}(G;A) \cong_{\mathbb{Z}} H_k(G;A) \quad \text{and} \quad H^{k+2}(G;A) \cong_{\mathbb{Z}} H^k(G;A).$$

Moreover, (where G acts trivially on $\mathbb{Z}$),

$$H_k(G;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/n & \text{if } k \text{ is odd} \\ 0 & \text{if } k > 0 \text{ is even} \end{cases} \quad \text{and} \quad H^k(G;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is odd} \\ \mathbb{Z}/n & \text{if } k > 0 \text{ is even.} \end{cases}$$

Proof. In view of Theorem 1.3.1, we only need to consider (co)homology in positive degrees. We use the fundamental theorem of group (co)homology (Corollary 1.6.9) and the periodic projective resolution (P_*, ε) from Proposition 1.6.11. We only treat the case of homology in detail (the computation in cohomology works in the same way; check!). The $\mathbb{Z}$-chain complex $P_* \otimes_G A$ is isomorphic to

$$\begin{tikzcd}
\text{degree} & 3 & 2 & 1 & 0 \\
\cdots \arrow[r] & \mathbb{Z}G \otimes_G A \arrow[r, "M_{t-1}\otimes_G \mathrm{id}_A"] \arrow[d, "\cong"] & \mathbb{Z}G \otimes_G A \arrow[r, "M_N \otimes_G \mathrm{id}_A"] \arrow[d, "\cong"] & \mathbb{Z}G \otimes_G A \arrow[r, "M_{t-1}\otimes_G \mathrm{id}_A"] \arrow[d, "\cong"] & \mathbb{Z}G \otimes_G A \arrow[d, "\cong"] \\
\cdots \arrow[r] & A \arrow[r, "t^{-1}-1"'] & A \arrow[r, "\sum_{j=0}^{n-1} t^{-j}"'] & A \arrow[r, "t^{-1}-1"'] & A
\end{tikzcd}$$

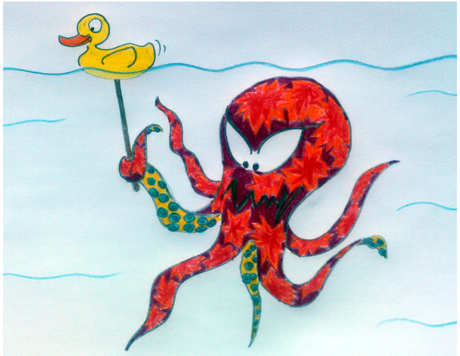


Figure 1.5.: Module (ducky) and a projective resolution (octopus), showing the true nature of the module/ducky.

Moreover, we have $\sum_{j=0}^{n-1} t^{-j} = \sum_{g \in G} g = N$ and $(t^{-1} - 1) \cdot A = (t - 1) \cdot A$. Taking homology proves the claim. $\square$

Study note. It does not make much sense to memorise the formulas in Corollary 1.6.13. Instead, it is much more efficient to memorise the projective resolution in Proposition 1.6.11 and how to compute the (co)homology of finite cyclic groups from this resolution.

Remark 1.6.14. Let $n \in \mathbb{N}_{\geq 2}$ and $G := \mathbb{Z}/n$. Combining the fundamental theorem of group (co)homology (Corollary 1.6.9) and the computation of the cohomology of finite cyclic groups (Corollary 1.6.13 has the following consequence:

> There does *not* exist a projective resolution (P_*, ε) of the $\mathbb{Z}G$-module $\mathbb{Z}$ (with trivial G-action) satisfying $P_k \cong_R 0$ for all large enough k.

In other words, projective resolutions uncover that even though both the $\mathbb{Z}G$-module $\mathbb{Z}$ and the group G look very tame, the module $\mathbb{Z}$ is "complicated" (Figure 1.5).

As sample application of these computations, we consider a purely group-theoretic result, namely the classification of finite p-groups with a unique subgroup of order p (Corollary 1.6.18) and the resulting characterisation of cyclic p-groups (Corollary 1.6.19); for simplicity, we avoid the prime 2.

Theorem 1.6.15 (classification of p-groups with a cyclic subgroup of index p)**.** *Let $p \in \mathbb{N}$ be an odd prime. Every (finite) p-group that contains a cyclic subgroup of index p is isomorphic to exactly one of the groups in the following list:*

A. $\mathbb{Z}/p^n$ for some $n \in \mathbb{N}_{>0}$

B. $\mathbb{Z}/p^n \times \mathbb{Z}/p$ *for some* $n \in \mathbb{N}_{>0}$

C. $\mathbb{Z}/p^n \rtimes \mathbb{Z}/p$ *for some* $n \in \mathbb{N}_{>1}$*, where the generator* $[1]$ *of* $\mathbb{Z}/p$ *acts on* $\mathbb{Z}/p^n$ *by multiplication with* $1 + p^{n-1}$*.*

Before giving the proof, let us observe that

$$(1 + p^{n-1})^p \equiv 1 \mod p^n$$

for all primes $p \in \mathbb{N}$ and all $n \in \mathbb{N}_{>1}$ (divisibility of binomial coefficients ... check!). Hence, the groups of type C in Theorem 1.6.15 indeed exist.

Proof of Theorem 1.6.15. The groups listed in the theorem all fall into different isomorphism classes (we can use the cardinality, the set of orders, and the property of being Abelian as separating invariants).

We now show that the list is complete:

Let G be a finite p-group that contains a cyclic subgroup A of index p. Then A is a normal subgroup of G (Lemma 1.6.16). Hence, we obtain an extension

$$0 \longrightarrow A \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } \mathbb{Z}/p \longrightarrow 1$$

where $i\colon A \longrightarrow G$ is the inclusion. As A is cyclic (and a subgroup of a p-group), we have $A \cong_{\mathsf{Group}} \mathbb{Z}/p^n$ for some $n \in \mathbb{N}$. Without loss of generality, we may assume that $n > 0$ and that $A = \mathbb{Z}/p^n$ (to simplify notation).

The idea is now to use the classification of extensions in terms of group cohomology (Theorem 1.5.10). Therefore, we first need to understand the $\mathbb{Z}/p$-action on A induced by the extension above.

If the $\mathbb{Z}/p$-action on A is trivial, then G is Abelian (because it is then generated by two commuting elements) and thus of type A or B (by the classification of finite Abelian groups; Satz III.1.3.1).

If the $\mathbb{Z}/p$-action on A is non-trivial, then we can argue as follows: In view of the classification of extensions (Theorem 1.5.10) and the characterisation of trivial extensions (Remark 1.5.11), it suffices to prove the following in order to establish that G is of type C:

① There exists a (non-trivial) element in $\mathbb{Z}/p$ that acts by multiplication with $1 + p^{n-1}$ on $A = \mathbb{Z}/p^n$ (which shows that the extension is isomorphic to one, where $[1] \in \mathbb{Z}/p$ acts by multiplication with $1 + p^{n-1}$ on A).

② We have $H^2(\mathbb{Z}/p; A) \cong_{\mathbb{Z}} 0$.

Ad ①. The $\mathbb{Z}/p$-action of $[1] \in \mathbb{Z}/p$ on $A = \mathbb{Z}/p^n$ is given by multiplication by a number $a \in \mathbb{Z}$; because the action is non-trivial, we have $a \not\equiv 1 \mod p^n$. Therefore, by Lemma 1.6.17, there exists a $k \in \{1, \dots, p-1\}$ with $[a^k] = [1 + p^{n-1}]$ in $\mathbb{Z}/p^n$. Then the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & G \arrow[r, "\pi"] \arrow[d, equal] & \mathbb{Z}/p \arrow[r] \arrow[d, "\text{multiplication by } k"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & G \arrow[r, "k\circ\pi"'] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

shows that we can restrict to the case that [1] acts by multiplication by $1 + p^{n-1}$ on A.

Ad ②. By Corollary 1.6.13, we have

$$H^2(\mathbb{Z}/p; A) \cong_{\mathbb{Z}} A^{\mathbb{Z}/p}/\langle N\rangle_A,$$

where $N := \sum_{j=0}^{n-1}[a^j] = \sum_{j=0}^{n-1}[1 + j \cdot p^{n-1}] \in A$. On the one hand, we have

$$\begin{aligned} A^{\mathbb{Z}/p} &= A^{[1+p^{n-1}]} = \bigl\{[x] \in \mathbb{Z}/p^n \bigm| p^{n-1} \cdot x \equiv 0 \mod p^n\bigr\} \\ &= p \cdot \mathbb{Z}/p^n. \end{aligned}$$

On the other hand, we have (in A) that

$$N = \sum_{j=0}^{p-1}[1 + j \cdot p^{n-1}] = [p] + \Bigl[\frac{p \cdot (p-1)}{2} \cdot p^{n-1}\Bigr] = [p],$$

and so $\langle N\rangle_A = p \cdot \mathbb{Z}/p^n$. Therefore, $H^2(\mathbb{Z}/p; A) \cong_{\mathbb{Z}} 0$.

This shows that G is of type A, B, or C. □

Lemma 1.6.16 (subgroups of prime index)**.** *Let $p \in \mathbb{N}$ be prime, let G be a finite p-group, and let $H \subset G$ be a subgroup of index p. Then H is a normal subgroup of G.*

Proof. This statement can be proved in several ways; for instance: The group G acts by left translation on the coset space G/H. This action corresponds to a group homomorphism $\varphi\colon G \longrightarrow \operatorname{Sym}(G/H) \cong_{\mathsf{Group}} S_p$ into the corresponding symmetric group. Let $K := \ker\varphi$. We will now show that $K = H$, which proves that H as a kernel is normal in G.

By construction, $K \subset H$ (as can be seen by evaluation on the coset H). Moreover,

$$[H : K] = \frac{[G : K]}{[G : H]} = \frac{[G : K]}{p} = \frac{|\operatorname{im}\varphi|}{p}.$$

Because G is a p-group, also $|\operatorname{im}\varphi|$ is a power of p. On the other hand, $|\operatorname{im}\varphi|$ divides $|S_p| = p!$. Because p is prime, it follows that $|\operatorname{im}\varphi| = 1$ or $|\operatorname{im}\varphi| = p$. The first case cannot occur (because $[H : K]$ is an integer). Hence, $|\operatorname{im}\varphi| = p$, which implies $[H : K] = p/p = 1$; thus, $H = K$. □

Lemma 1.6.17 ($\mathbb{Z}/p$-actions on $\mathbb{Z}/p^n$)**.** *Let $p \in \mathbb{N}$ be an odd prime, let $n \in \mathbb{N}_{>1}$, and let $a \in \mathbb{Z}$ with $a^p \equiv 1 \mod p^n$.*

1. *Then $a \equiv 1 \mod p^{n-1}$.*
2. *If $a \not\equiv 1 \mod p^n$, then there exists a $k \in \{1, \dots, p-1\}$ with $a^k \equiv 1 + p^{n-1} \mod p^n$.*

Proof. The first part follows from elementary number theory (Exercise; the little Fermat might help). The second part is a consequence of the first part (Exercise). $\square$

Corollary 1.6.18 (classification of p-groups with a unique subgroup of order p)**.** *Let $p \in \mathbb{N}$ be an odd prime and let G be a finite p-group that contains a unique subgroup with p elements. Then G is cyclic.*

Proof. We prove the claim by induction over $|G|$. By assumption G is non-trivial and $|G|$ is a power of p.

- *Base case.* If $|G| = p$, then $G \cong_{\mathsf{Group}} \mathbb{Z}/p$, which is cyclic.
- *Induction step.* Let $|G| > p$ and let us suppose that the claim is already established for all groups with fewer elements. As p-group, G contains a normal subgroup N of index p (Satz III.1.3.33 and Satz III.1.3.27).

  Because $|G| > p$, the normal subgroup N is also a p-group with a unique subgroup with p elements: The subgroup N is a non-trivial p-group and so contains an element of order p; because $N \subset G$, the corresponding subgroup with p elements is unique.

  Hence, by induction, N is cyclic. Therefore, we can apply Theorem 1.6.15 and thus G is of one of the types listed in the theorem. The only one of these types that has a unique subgroup with p elements is type A (which is cyclic). This shows that G is cyclic. $\square$

Corollary 1.6.19 (recognising cyclic p-groups)**.** *Let $p \in \mathbb{N}$ be an odd prime and let G be a finite p-group. Then the following are equivalent:*

1. *The group G is cyclic.*
2. *All Abelian subgroups of G are cyclic.*

Proof. The implication "1 $\Longrightarrow$ 2" follows from the fact that subgroups of cyclic groups are cyclic.

Conversely, let us suppose that all Abelian subgroups of G are cyclic. Without loss of generality, we may assume that G is non-trivial. In view of Corollary 1.6.18, it suffices to show that G contains a unique subgroup of size p.

By a classical result, the centre $Z(G)$ of G is non-trivial (Satz III.1.3.33) and thus contains an element x of order p. Let $H := \langle x \rangle_G \subset G$ be the corresponding subgroup.

Let $K \subset G$ also be a subgroup of size p and let $y \in K$ be of order p. Because x is central in G, the subgroup

$$A := \langle x, y \rangle_G \subset G$$

is Abelian (and a p-group with $|A| \leq p^2$). By assumption, A is cyclic, whence $A = H$ or $A \cong_{\mathbb{Z}} \mathbb{Z}/p^2$. However, $\mathbb{Z}/p^2$ contains only one subgroup of size p and so $K = H$. $\square$

Outlook 1.6.20 (the ~~odd~~ even prime)**.** For the prime 2, in principle, the same arguments apply. However, the situation is a little bit more complicated in the sense that more groups can occur. More precisely [12, Chapter IV.4]:

Every (finite) 2-group that contains a cyclic subgroup of index 2 is isomorphic to exactly one of the groups in the following list:

A. $\mathbb{Z}/2^n$ for some $n \in \mathbb{N}_{>0}$

B. $\mathbb{Z}/2^n \times \mathbb{Z}/2$ for some $n \in \mathbb{N}_{>0}$

C. $\mathbb{Z}/2^n \rtimes \mathbb{Z}/2$ for some $n \in \mathbb{N}_{>1}$, where the generator $[1]$ of $\mathbb{Z}/2$ acts on $\mathbb{Z}/2^n$ by multiplication with $1 + 2^{n-1}$

D. *dihedral* 2*-groups.* $\mathbb{Z}/2^n \rtimes \mathbb{Z}/2$ for some $n \in \mathbb{N}_{>2}$, where the generator $[1]$ of $\mathbb{Z}/2$ acts by multiplication by -1

E. *generalised quaternion* 2*-groups.* Let H be the quaternion algebra and let $n \in \mathbb{N}_{>0}$. Then the *generalised quaternion group* Q_{2^n} is the subgroup of the units of H generated by $e^{\pi \cdot i/2^n}$ and j; the group Q_{2^n} can also be described by the presentation

$$\langle x, y \,|\, y^4 = 1,\ y^2 = x^{2^n},\ y \cdot x \cdot y^{-1} = x^{-1} \rangle.$$

F. $\mathbb{Z}/2^n \rtimes \mathbb{Z}/2$ for some $n \in \mathbb{N}_{>2}$, where the generator $[1]$ of $\mathbb{Z}/2$ acts on $\mathbb{Z}/2^n$ by multiplication by $- + 2^{n-1}$.

Inductively, one then obtains: A finite 2-group that contains a unique subgroup with two elements is cyclic or a generalised quaternion group.

These classification results can, for instance, be used in the investigation of free actions on spheres (Chapter 4.3).

1.6.4 Example: Free groups

As second class of examples, we compute the (co)homology of free groups.

Proposition 1.6.21 (a resolution for free groups)**.** *Let S be a set and let F be the free group, freely generated by S. Then*

$$\cdots \longrightarrow 0 \xrightarrow{\ 0\ } 0 \xrightarrow{\ 0\ } \bigoplus_S \mathbb{Z}F \xrightarrow{\ \partial\ } \mathbb{Z}F \xrightarrow{\ \varepsilon\ } \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ *(with trivial* F*-action) over* $\mathbb{Z}F$*. Here,*

$$\begin{aligned}\partial\colon \bigoplus_S \mathbb{Z}F &\longrightarrow \mathbb{Z}F \\ \sum_{s\in S} x_s \cdot e_s &\longmapsto \sum_{s\in S} x_s \cdot (s-1).\end{aligned}$$

Proof. Clearly, the chain modules are free (whence projective). We have (a fact that does not only hold for free groups)

$$\begin{aligned}\ker\varepsilon &= \operatorname{Span}_{\mathbb{Z}F}\{s-1 \mid s\in S\} && \text{(Exercise)}\\ &= \operatorname{im}\partial. && \text{(by definition of } \partial)\end{aligned}$$

Therefore, it remains to show that ∂ is injective (and this is special for free groups). To this end, we will make use of the following notation: If $y = \sum_{g\in G} y_s \cdot g \in \mathbb{Z}F$, then we define the *support of* y by

$$\operatorname{supp} y := \{g \in F \mid y_g \neq 0\} \subset F.$$

Let $x := \sum_{s\in S} x_s \cdot e_s$ be a non-zero element. Then

$$X := \{y\cdot s \mid s\in S,\ y \in \operatorname{supp} x_s\} \cup \{y \mid s \in S,\ y\in \operatorname{supp} x_s\} \subset F$$

is a finite and non-empty subset of F and $\operatorname{supp}\partial(x) \subset X$. Moreover, every element $g \in X$ of maximal length (when viewing the elements of F as reduced words in $S\cup S^{-1}$) satisfies $g \in \operatorname{supp}\partial x$; geometrically this is related to vertices far away from ε in the tree in Figure 1.6. We can verify this claim algebraically as follows: Let $z \in X$ be an element of maximal length. We only need to show that z can only have one "representation" as an element of X.

- Let $z = y \cdot s$ with $s\in S$ and $y \in \operatorname{supp} x_s$. Let w be the reduced word over $S \cup S^{-1}$ underlying y. Then ws is reduced as well (otherwise y would be longer than $y \cdot s$ and in X) and $z = ws$. We need to consider two cases:
  - We have $z = y' \cdot s'$ with $s' \in S$ and $y' \in \operatorname{supp} x_{s'}$. The same argument as above shows that $w's'$ is also the reduced word representing z. Hence, $w = w'$ (whence $y = y'$) and $s = s'$.
  - We have $z = y'$ with $s' \in S$ and $y' \in \operatorname{supp} x_{s'}$. Then w' ends with s'^{-1} (because otherwise $y' \cdot s'$ would be longer than y' and in X), which cannot happen (because we know that the reduced representative of z is ws, which ends in s and not in s'^{-1}.
- Let $z = y$ with $s \in S$ and $y \in \operatorname{supp} x_s$; let w be the reduced word over $S\cup S^{-1}$ representing y. Then w ends in s^{-1} (because of the maximality). We need to consider two cases:

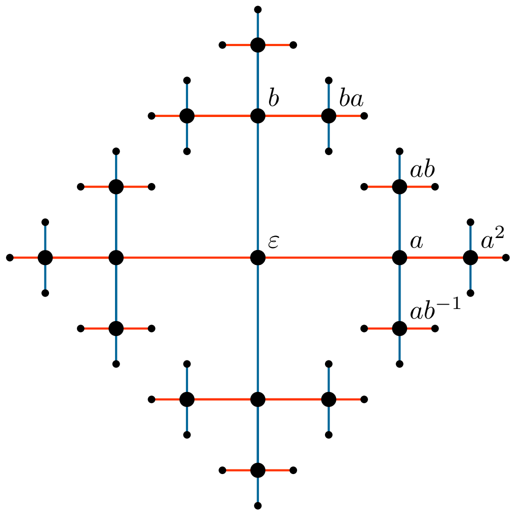


Figure 1.6.: A free F-equivariant CW-structure on the 4-regular tree [53, Figure 3.10]; blue/red edges are the F-orbits of 1-cells.

- We have $z = y' \cdot s'$ with $s' \in S$ and $y' \in \operatorname{supp} x_{s'}$. This cannot happen (see above).
- We have $z = y'$ with $s' \in S$ and $y' \in \operatorname{supp} x_{s'}$. Then w' ends in s'^{-1}, whence $s = s'$.

This shows the uniqueness of the "representation" of z.

Therefore, $\operatorname{supp} \partial(x) \neq \emptyset$ and so $\partial(x) \neq 0$. Thus, ∂ is injective. $\square$

Remark 1.6.22 (geometric idea)**.** Let F be a free group of rank 2, freely generated by a and b. Then $F \cong_{\mathsf{Group}} \pi_1(X, x_0)$, where $(X, x_0) := (S^1, 1) \vee (S^1, 1)$. The universal covering $\widetilde{X}$ of X is homeomorphic to the (geometric realisation of the) 4-regular tree (Figure 1.6) and $\widetilde{X}$ inherits a free F-equivariant CW-structure from the "obvious" CW-structure on X (with one 0-cell and two 1-cells). The corresponding cellular chain complex is isomorphic to

$$\cdots \longrightarrow 0 \xrightarrow{\;0\;} 0 \xrightarrow{\;0\;} \bigoplus_S \mathbb{Z}F \xrightarrow{\;\partial\;} \mathbb{Z}F,$$

where ∂ is defined as in Proposition 1.6.21. Moreover, $\widetilde{X}$ is contractible (as a tree) and so $H_*(X; \mathbb{Z}) \cong_{\mathbb{Z}} H_*(\bullet; \mathbb{Z})$. This shows that the above complex leads to a resolution of $\mathbb{Z}$ over $\mathbb{Z}F$.

Corollary 1.6.23 ((co)homology of free groups)**.** *Let S be a set, let F be the free group freely generated by S, and let A be a $\mathbb{Z}F$-module. Then, for all $k \in \mathbb{N}_{\geq 2}$,*

$$H_k(F;A) \cong_{\mathbb{Z}} 0 \quad \textit{and} \quad H^k(F;A) \cong_{\mathbb{Z}} 0.$$

Moreover, we have (where $\mathbb{Z}$ *carries the trivial action) for all* $k \in \mathbb{N}$

$$H_k(F;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \textit{if } k=0 \\ \bigoplus_S \mathbb{Z} & \textit{if } k=1 \\ 0 & \textit{if } k>0 \end{cases} \quad \textit{and} \quad H^k(F;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \textit{if } k=0 \\ \prod_S \mathbb{Z} & \textit{if } k=1 \\ 0 & \textit{if } k>0. \end{cases}$$

Proof. In view of Theorem 1.3.1, we only need to consider (co)homology in positive degrees. We use the fundamental theorem of group (co)homology (Corollary 1.6.9) and the projective resolution (P_*, ε) from Proposition 1.6.21. We only treat the case of homology in detail (the computation in cohomology works in the same way; check!).

Because the projective resolution is trivial in degree bigger than 1, we know that $H_k(F;A) \cong_Z H_k(P_* \otimes_F A) \cong_{\mathbb{Z}} 0$ for all $k \in \mathbb{N}_{\geq 2}$. Therefore, it remains to compute $H_1(F;\mathbb{Z})$. The corresponding degrees of $P_* \otimes_F \mathbb{Z}$ are isomorphic to (because F acts trivially on $\mathbb{Z}$)

$$\begin{tikzcd} \text{degree} & 2 & 1 & 0 \\ & 0 \otimes_F \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \bigoplus_S \mathbb{Z}F \otimes_F \mathbb{Z} \arrow[r, "\partial \otimes_F \mathrm{id}_{\mathbb{Z}}"] \arrow[d, "\cong"] & \mathbb{Z}F \otimes_F \mathbb{Z} \arrow[d, "\cong"] \\ & 0 \arrow[r] & \bigoplus_S \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \end{tikzcd}$$

Taking homology finishes the proof.

Alternatively, we could compute $H_1(F;\mathbb{Z})$ and $H^1(F;\mathbb{Z})$ also via the methods from Chapter 1.4.1 (Example 1.4.9) and Chapter 1.4.2. The approach via the resolution has the advantage that it, in principle, also works with non-trivial coefficients. □

In particular, this also includes a computation of the (co)homology of the infinite cyclic group $\mathbb{Z}$ (which is freely generated by a single element).

As a sample application we compute the deficiency of free groups:

Corollary 1.6.24 (deficiency of free groups). *Let* S *be a finite set. Then the free group* $F(S)$*, freely generated by* S*, has deficiency*

$$\operatorname{def} F(S) = |S|.$$

Proof. On the one hand, the presentation $\langle S \mid \rangle$ of $F(S)$ shows that

$$\operatorname{def} F(S) \geq |S| - 0 = |S|.$$

On the other hand, we have

$$\begin{aligned}
\operatorname{def} F(S) &\leq \operatorname{rk}_{\mathbb{Z}} H_1(F(S);\mathbb{Z}) - \operatorname{rk}_{\mathbb{Z}} H_2(F(S);\mathbb{Z}) && \text{(Corollary 1.5.4)}\\
&= |S| - 0. && \text{(Example 1.4.9, Corollary 1.6.23)}
\end{aligned}$$

Therefore, we obtain $\operatorname{def} F(S) = |S|$. □

Example 1.6.25. For instance, Corollary 1.6.24 shows that:

- The group described by the presentation $\langle x, y, z \mid x \cdot y^{2019} \cdot z \cdot x \rangle$ is *not* isomorphic to $\mathbb{Z}$ (however, we could also use the Abelianisation to see that).
- The group described by the presentation $\langle a_1, a_2, b_1, b_2 \mid [a_1, b_1] \cdot [b_1, b_2] \rangle$ has deficiency at least 3 and thus is *not* isomorphic to a free group of rank 2 (which we could also derive from the Abelianisation, which is isomorphic to $\mathbb{Z}^4$).

  Moreover, one can show that this group has deficiency equal to 3 (Example 4.1.28), so it is *not* free of rank 4 (and because of the Abelianisation also not free of any other rank).

  This group has geometric meaning: It is (isomorphic to) the fundamental group of an oriented closed connected surface of genus 2.

1.7 (Co)Homology and subgroups

We will now consider the following problem: How does the (co)homology of a subgroup relate to the (co)homology of the ambient group? As a first step, we will first discuss how we can convert modules of the ambient group into modules of the given subgroup and vice versa. We then prove Shapiro's lemma and then discuss the transfer for finite index subgroups.

1.7.1 Restriction and (co)induction

Restricting the action of an ambient group to a subgroup yields the restriction functor.

Definition 1.7.1 (restriction)**.** Let G be a group and let H be a subgroup, and let $i\colon H \longrightarrow G$ be the inclusion. Then we write

$$\operatorname{Res}_H^G := i^* \colon {}_{\mathbb{Z}G}\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}H}\mathsf{Mod}$$

for the *restriction* functor from G to H. In other words, the restriction is given by forgetting the action of the group elements of G that do not lie in H.

Proposition 1.7.2 (projectivity and restriction). *Let G be a group and let H be a subgroup.*

1. *If P is a projective $\mathbb{Z}G$-module, then $\operatorname{Res}_H^G P$ is a projective $\mathbb{Z}H$-module.*
2. *If (P_*, ε) is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then $(\operatorname{Res}_H^G P_*, \operatorname{Res}_H^G \varepsilon)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$.*

Proof. Ad 1. We use the characterisation of projective modules as direct summands of free modules (Proposition 1.6.3. Because the functor Res_H^G is compatible with direct sums (check!), we only need to show that $\operatorname{Res}_H^G \mathbb{Z}G$ is a free $\mathbb{Z}H$-module. By construction,

$$\operatorname{Res}_H^G \mathbb{Z}G \cong_{\mathbb{Z}H} \bigoplus_{G/H} \mathbb{Z}H,$$

which is a free $\mathbb{Z}H$-module.

Ad 2. Because restricting the module structure preserves exactness, the second part follows from the first part. □

Corollary 1.7.3. *Let G be a group that contains an element of non-trivial finite order. Then there is* no *projective resolution of $\mathbb{Z}$ (with trivial G-action) over $\mathbb{Z}G$ of finite length.*

Proof. Let (P_*, ε) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. By hypothesis, there exists a non-trivial finite cyclic subgroup C of G. Then $(\operatorname{Res}_C^G P_*, \operatorname{Res}_C^G \varepsilon)$ is a projective resolution of the trivial $\mathbb{Z}C$-module $\mathbb{Z}$ over $\mathbb{Z}C$ (Proposition 1.7.2). Therefore, $\operatorname{Res}_C^G P_*$ (whence also P_*) does *not* have finite length (Remark 1.6.14). □

Conversely, we can use tensor products to extend module structures over a subgroup to an ambient group. This leads to the induction functor.

Definition 1.7.4 (induction). Let G be a group and let H be a subgroup. Then the *induction* functor from H to G is defined by

$$\operatorname{Ind}_H^G := \mathbb{Z}G \otimes_{\mathbb{Z}H} \ \cdot : {}_{\mathbb{Z}H}\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}G}\mathsf{Mod}\,.$$

Here, we equip $\mathbb{Z}G$ with the $\mathbb{Z}G$-$\mathbb{Z}H$-bimodule structure given by the G-multiplication on G from the left and the H-multiplication on G from the right. More explicitly: If B is a left $\mathbb{Z}H$-module, then the G-action on $\operatorname{Ind}_H^G(B)$ is given by

$$\begin{aligned} G \times (\mathbb{Z}G \otimes_{\mathbb{Z}H} B) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B \\ (g, x \otimes b) &\longmapsto (g \cdot x) \otimes b. \end{aligned}$$

Example 1.7.5 (induction). Let G be a group and let $H \subset G$ be a subgroup.

- There is a canonical isomorphism $\operatorname{Ind}_H^G \mathbb{Z}H = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong_{\mathbb{Z}G} \mathbb{Z}G$.

- We have $\operatorname{Ind}_H^G \mathbb{Z} = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} \cong_{\mathbb{Z}G} \mathbb{Z}[G/H]$, where the G-action on $\mathbb{Z}[G/H] := \bigoplus_{G/H} \mathbb{Z}$ is the one induced by the left translation action of G on the coset space G/H. (If H is a normal subgroup of G, this coincides with the left G-action on the group ring $\mathbb{Z}[G/H]$ of the quotient group G/H; therefore, the ambigous notation is not a problem).

Dually, replacing the tensor product by the corresponding Hom-functor, we can introduce the coinduction:

Definition 1.7.6 (coinduction). Let G be a group and let H be a subgroup. Then the *coinduction* functor from H to G is defined by

$$\operatorname{Coind}_H^G := \operatorname{Hom}_H(\mathbb{Z}G, \,\cdot\,) \colon {}_{\mathbb{Z}H}\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}G}\mathsf{Mod}\,.$$

Here, we equip $\mathbb{Z}G$ with the right $\mathbb{Z}G$-module structure induced by right multiplication of G on G. More explicitly: If B is a left $\mathbb{Z}H$-module, then the G-action on $\operatorname{Coind}_H^G(B)$ is given by

$$\begin{aligned} G \times \operatorname{Hom}_H(\mathbb{Z}G, B) &\longrightarrow \operatorname{Hom}_H(\mathbb{Z}H, B) \\ (g, f) &\longmapsto \bigl(x \mapsto f(x \cdot g)\bigr). \end{aligned}$$

Proposition 1.7.7 ((co)induction for finite index subgroups). *Let G be a group and let $H \subset G$ be a finite index subgroup. Then there is a canonical natural isomorphism*

$$\operatorname{Ind}_H^G(B) \cong_{\mathbb{Z}G} \operatorname{Coind}_H^G(B)$$

for all $\mathbb{Z}H$-modules B.

Proof. A straightforward computation shows that the two homomorphisms

$$\begin{aligned} \varphi \colon \operatorname{Ind}_H^G(B) = \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\longrightarrow \operatorname{Hom}_H(\mathbb{Z}G, B) = \operatorname{Coind}_H^G(B) \\ g \otimes b &\longmapsto \bigl(x \mapsto \chi_H(x \cdot g) \cdot (x \cdot g) \cdot b\bigr) \\ \psi \colon \operatorname{Coind}_H^G B = \operatorname{Hom}_H(\mathbb{Z}G, B) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B) \\ f &\longmapsto \sum_{gH \in G/H} g \otimes f(g^{-1}) \end{aligned}$$

are well-defined and $\mathbb{Z}G$-linear (Exercise). Here, $\chi_H \colon G \longrightarrow \{0, 1\}$ denotes the characteristic function of the subset $H \subset G$; it should be noted that the term following $\chi_H(x \cdot g)$ only makes sense if $x \cdot g \in H$. Moreover, φ and ψ are mutually inverse (Exercise). $\square$

1.7.2 The Shapiro lemma

The Shapiro lemma allows to express the (co)homology of a subgroup in terms of the cohomology of the ambient group – for the price of changing the coefficient module via (co)induction.

Theorem 1.7.8 (Shapiro lemma). *Let G be a group, let H be a subgroup, let B be an H-module, and let $n \in \mathbb{N}$. Then there are canonical isomorphisms*

$$\begin{aligned} H_n(H;B) &\cong_{\mathbb{Z}} H_n(G;\operatorname{Ind}_H^G B) \\ H^n(H;B) &\cong_{\mathbb{Z}} H^n(G;\operatorname{Coind}_H^G B). \end{aligned}$$

More precisely: Let $i\colon H \longrightarrow G$ be the incusion and let

$$\begin{aligned} I\colon B &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B) \\ b &\longmapsto 1 \otimes b \\ C\colon \operatorname{Coind}_H^G(B) = \operatorname{Hom}_H(\mathbb{Z}G, B) &\longrightarrow B \\ f &\longmapsto f(1); \end{aligned}$$

then I and C are $\mathbb{Z}H$-homomorphisms, $(i, I)\colon (H,B) \longrightarrow (G, \operatorname{Ind}_H^G B)$ is a morphism in GroupMod, *and $(i, C)\colon (H,B) \longrightarrow (G, \operatorname{Coind}_H^G B)$ is a morphism in* $\mathsf{GroupMod}^*$. *The induced homomorphisms*

$$\begin{aligned} H_n(i;I)\colon H_n(H;B) &\longrightarrow H_n(G;\operatorname{Ind}_H^G B) \\ H^n(i;C)\colon H^n(G;\operatorname{Coind}_H^G B) &\longrightarrow H^n(H;B) \end{aligned}$$

are $\mathbb{Z}$-isomorphisms.

Proof. The proof for group homology basically consists of the cancellation of $\otimes_G \mathbb{Z}G$ as well as the fundamental theorem of group cohomology (Corollary 1.6.9):

Because the restriction functor turns projective resolutions into projective resolutions (Proposition 1.7.2), $(\operatorname{Res}_H^G C_*(G), \operatorname{Res}_H^G \varepsilon)$ is a projective resolution of the trivial $\mathbb{Z}H$-module $\mathbb{Z}$ over $\mathbb{Z}H$. Hence, there is a canonical isomorphism

$$\begin{aligned} H_n(H;B) &\cong_{\mathbb{Z}} H_n\bigl(\operatorname{Res}_H^G C_*(G) \otimes_H B\bigr) && \text{(Corollary 1.6.9)} \\ &\cong_{\mathbb{Z}} H_n\bigl(C_*(G) \otimes_G (\mathbb{Z}G \otimes_{\mathbb{Z}H} B)\bigr) && \text{(induced by } I\text{)} \\ &= H_n\bigl(C_*(G) \otimes_G \operatorname{Ind}_H^G B\bigr) \\ &= H_n(G;\operatorname{Ind}_H^G B). \end{aligned}$$

More precisely, $C_*(i)\colon C_*(H) \longrightarrow \operatorname{Res}_H^G C_*(G)$ is a chain map that extends the identity of $\mathbb{Z}$. Because both sides are projective resolutions, $C_*(i)$ is already "the" $\mathbb{Z}H$-chain homotopy equivalence. This induces the first isomorphism in the above computation. The second isomorphism is induced by I (and the canonical projection from $\otimes_H$ to $\otimes_G$). Therefore, the whole isomorphism coincides with $H_n(i;I)$.

Similarly, the version for cohomology can be proved using the cancellation of $\operatorname{Hom}_G(\,\cdot\,, \operatorname{Hom}_H(\mathbb{Z}G,\,\cdot\,))$ (check!). □

Corollary 1.7.9 (homological characterisation of the trivial group). *Let G be a group that satisfies $H_n(G;A) \cong_{\mathbb{Z}} H_n(1;A)$ for all $\mathbb{Z}G$-modules A and all $n \in \mathbb{N}$. Then G is trivial.*

Proof. Let C be a cyclic subgroup of G. Then

$$\begin{aligned} C &\cong_{\mathbb{Z}} H_1(C;\mathbb{Z}) && (C \text{ is Abelian; Theorem 1.4.1}) \\ &\cong_{\mathbb{Z}} H_1(G;\operatorname{Ind}_C^G \mathbb{Z}) && (\text{Shapiro lemma; Theorem 1.7.8}) \\ &\cong_{\mathbb{Z}} 0 && (\text{by assumption}). \end{aligned}$$

As every non-trivial element of G would generate a non-trivial cyclic subgroup of G, we obtain that G is the trivial group. □

Corollary 1.7.10. *Let G be a group and let H be a subgroup of finite index. Then, for all $n \in \mathbb{N}$, there is a canonical isomorphism*

$$H^n(G;\mathbb{Z}G) \cong_{\mathbb{Z}} H^n(H;\mathbb{Z}H).$$

Proof. By Shapiro's lemma (Theorem 1.7.8), we have

$$H^n(H;\mathbb{Z}H) \cong_{\mathbb{Z}} H^n(G;\operatorname{Coind}_H^G \mathbb{Z}H).$$

Because the subgroup H has finite index in G, we can convert the coinduction into an induction: $\operatorname{Coind}_H^G \mathbb{Z}H \cong_{\mathbb{Z}} \operatorname{Ind}_H^G \mathbb{Z}H$ (Proposition 1.7.7). Therefore, we obtain

$$\begin{aligned} H^n(H;\mathbb{Z}H) &\cong_{\mathbb{Z}} H^n(G;\operatorname{Ind}_H^G \mathbb{Z}H) \\ &\cong_{\mathbb{Z}} H^n(G;\mathbb{Z}G), && (\text{Example 1.7.5}) \end{aligned}$$

as claimed. □

Study note. Why is the corresponding result for $H_*(G;\mathbb{Z}G)$ not exciting?

Outlook 1.7.11 (geometric meaning of $H^1(G;\mathbb{Z}G)$). Let G be an infinite, finitely generated group. Then $H^1(G;\mathbb{Z}G)$ is related to the number $e(G)$ of ends of G. The *number of ends of G* is the number of path-connected components "at infinity" of (geometric realisations of Cayley graphs of) G [53, Chapter 8.2]. One then has [32, Theorem 13.5.5][80]:

- $e(G) = 1$ if and only if $\operatorname{rk}_{\mathbb{Z}} H^1(G;\mathbb{Z}G) = 0$,
- $e(G) = 2$ if and only if $\operatorname{rk}_{\mathbb{Z}} H^2(G;\mathbb{Z}G) = 1$,
- $e(G) = \infty$ if and only if $\operatorname{rk}_{\mathbb{Z}} H^2(G;\mathbb{Z}G) = \infty$,
- and there are no other cases.

Typical examples of these three cases are $\mathbb{Z}^2$, $\mathbb{Z}$, and the free group of rank 2, respectively (Exercise). The number of ends does *not* change when passing to a finite index subgroup; algebraically, this is reflected in Corollary 1.7.10.

1.7.3 Transfer

For subgroups of finite index, there are also "wrong-way" maps, the transfer maps:

$$\begin{tikzcd} H_*(H; \operatorname{Res}^G_H A) \arrow[d, shift left, "H_n(\mathrm{incl.}; \mathrm{id}_A)"] \\ H_*(G; A) \arrow[u, dashed, shift left, "\text{transfer}"] \end{tikzcd}$$

Transfer maps usually are defined through some averaging process. Firstly, the maps going in the "right" direction are the (co)restriction maps:

Definition 1.7.12 ((co)restriction)**.** Let G be a group, let $H \subset G$ be a subgroup, let $i \colon H \longrightarrow G$ be the inclusion, and let A be a $\mathbb{Z}G$-module. Then we define the

- *restriction map* as $\operatorname{res}^G_H := H^*(i; \mathrm{id}_A) \colon H^*(G; A) \longrightarrow H^*(H; \operatorname{Res}^G_H A)$;
- *corestriction map* as $\operatorname{cor}^G_H := H_*(i; \mathrm{id}_A) \colon H_*(H; \operatorname{Res}^G_H A) \longrightarrow H_*(G; A)$.

For the "wrong-way" maps, we will use the following hands-on description; one should note that even though the maps on the chain level are only well-defined/unique up to (co)chain homotopy, after passage to (co)homology, we obtain well-defined, canonical homomorphisms (because (co)homology is chain homotopy invariant).

Definition 1.7.13 (transfer)**.** Let G be a group, let $H \subset G$ be a finite index subgroup (with inclusion $i \colon H \longrightarrow G$), let A be a $\mathbb{Z}G$-module, and let $n \in \mathbb{N}$.

- *Homological transfer.* The transfer map

  $$\operatorname{tr}^G_H := \operatorname{res}^G_H \colon H_n(G; A) \longrightarrow H_n(H; \operatorname{Res}^G_H A)$$

  is the map on homology induced by composing the averaging map with the "canonical" chain homotopy equivalence:

$$\begin{tikzcd} C_*(G) \otimes_G A \arrow[r] & \operatorname{Res}^G_H C_*(G) \otimes_H \operatorname{Res}^G_H A & C_*(H) \otimes_H \operatorname{Res}^G_H A \arrow[l, "C_*(i) \otimes_H \mathrm{id}_A"', "\simeq_{\mathbb{Z}}"] \\ x \otimes a \arrow[r, mapsto] & \displaystyle\sum_{gH \in G/H} (g^{-1} \cdot x) \otimes (g^{-1} \cdot a) & \end{tikzcd}$$

- *Cohomological transfer.* The transfer map

  $$\operatorname{tr}^G_H := \operatorname{cor}^G_H \colon H^n(H; \operatorname{Res}^G_H A) \longrightarrow H^n(G; A)$$

is the map on cohomology induced by composing the "canonical" cochain homotopy equivalence with the averaging map:

$$\begin{array}{ccccc} \operatorname{Hom}_H(C_*(H), \operatorname{Res}^G_H A) & \xleftarrow[\simeq_{\mathbb{Z}}]{\operatorname{Hom}_H(C_*(i),A)} & \operatorname{Hom}_H(\operatorname{Res}^G_H C_*(G), \operatorname{Res}^G_H A) & \longrightarrow & \operatorname{Hom}_G(C_*(G), A) \\ & & f & \longmapsto & \Bigl(x \mapsto \sum_{gH \in G/H} g \cdot f(g^{-1} \cdot x)\Bigr). \end{array}$$

Remark 1.7.14 (alternative descriptions of the transfer)**.** For every description of group (co)homology, there is a corresponding description of the transfer maps. For example, one can eliminate the concrete choice of projective resolution and hide the explicit averaging with the help of the fundamental theorem (Corollary 1.6.9) and Proposition 1.7.7. Moreover, using covering theory, one can also give a topological description of the transfer (Proposition 4.1.30).

Theorem 1.7.15 (transfer)**.** *Let G be a group, let $H \subset G$ be a subgroup of finite index, let A be a $\mathbb{Z}G$-module, and let $n \in \mathbb{N}$. Then*

$$\operatorname{cor}^G_H \circ \operatorname{res}^G_H(\alpha) = [G : H] \cdot \alpha$$

holds for all $\alpha \in H_n(G; A)$ and all $\alpha \in H^n(G; A)$.

Study note. In Theorem 1.7.15, in the homological case, res^G_H is the transfer map; in the cohomological case, cor^G_H is the transfer map. This slight abuse of notation allows us to state properties of the transfer in a uniform way.

Proof. We prove only the statement in homology; the cohomological case is similar. We can check this on the chain level. On the chain level, the composition $\operatorname{cor}^G_H \circ \operatorname{res}^G_H = \operatorname{cor}^G_H \circ \operatorname{tr}^G_H$ is modelled (up to chain homotopy) by (check!)

$$\begin{aligned} C_*(G) \otimes_G A &\longrightarrow C_*(G) \otimes_G A \\ x \otimes a &\longmapsto \sum_{gH \in G/H} (g^{-1} \cdot x) \otimes (g^{-1} \cdot a) = \sum_{gH \in G/H} x \otimes a, \end{aligned}$$

which is $[G : H]$ times the identity. □

The transfer shows that rationally all finite groups look like the trivial group:

Corollary 1.7.16. *Let G be a finite group and let R be a commutative ring with unit (with trivial G-action) in which $|G|$ is invertible. Then, for all $n \in \mathbb{N}_{>0}$:*

$$H_n(G; R) \cong_{\mathbb{Z}} 0 \quad \text{and} \quad H^n(G; R) \cong_{\mathbb{Z}} 0.$$

In particular, $H_n(G; \mathbb{Q}) \cong_{\mathbb{Z}} 0$ and $H^n(G; \mathbb{Q}) \cong_{\mathbb{Z}} 0$ for all $n \in \mathbb{N}_{>0}$.

Proof. We only consider homology; the cohomological case works in the same way. We apply the transfer (Theorem 1.7.15) to the trivial subgroup $H = 1$: Then $[G : H] \cdot \mathrm{id}_{H_n(G;R)}$ factors over $H_n(1; R) \cong_{\mathbb{Z}} 0$. Because $[G : H]$ is invertible in R, we obtain that also $\mathrm{id}_{H_n(G;R)}$ is the zero map. □

Moreover, we can deduce torsion results of the following type from the transfer formula (Theorem 1.7.15).

Example 1.7.17 (some torsion results).

- The group $\mathrm{SL}_2(\mathbb{Z})$ contains a free group of index 12 [53, Proposition 4.4.2]. Therefore, the transfer (Theorem 1.7.15) and the vanishing of the (co)homology of free groups in higher degrees (Corollary 1.6.23) shows that for all $k \in \mathbb{N}_{\geq 2}$:

$$12 \cdot H_k\bigl(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0 \quad \text{and} \quad 12 \cdot H^k\bigl(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0.$$

- The infinite dihedral group D_∞ (i.e., the isometry group of $\mathbb{Z}$ with respect to the standard metric on $\mathbb{R}$) is isomorphic to a semi-direct product of the form $\mathbb{Z} \rtimes \mathbb{Z}/2$ and thus contains an infinite cyclic group of index 2 (Exercise). Therefore, for all $k \in \mathbb{N}_{\geq 2}$:

$$2 \cdot H_k(D_\infty; \mathbb{Z}) \cong_{\mathbb{Z}} 0 \quad \text{and} \quad 2 \cdot H^k(D_\infty; \mathbb{Z}) \cong_{\mathbb{Z}} 0.$$

Outlook 1.7.18 (classical transfer). The (co)homological transfer is a generalisation of the classical, group-theoretic, transfer: Let G be a group and let $H \subset G$ be a subgroup of finite index. Then the *classical transfer* is the well-defined(!) homomorphism

$$\begin{aligned} G_{\mathrm{ab}} &\longrightarrow H_{\mathrm{ab}} \\ [g] &\longmapsto \Biggl[\prod_{k=1}^{[G:H]} g_k \cdot g \cdot R(g_k \cdot g)^{-1}\Biggr], \end{aligned}$$

where $\{g_1, \dots, g_{[G:H]}\}$ is a set of representatives of $H \backslash G$, and where $R \colon G \longrightarrow \{g_1, \dots, g_{[G:H]}\}$ selects for $g \in G$ the representative of the coset $H \cdot g$.

Under the canonical natural isomorphism between $H_1(\,\cdot\,; \mathbb{Z})$ and the Abelianisation functor (Theorem 1.4.1), the transfer $H_1(G; \mathbb{Z}) \longrightarrow H_1(H; \mathbb{Z})$ then corresponds to the classical transfer (check!).

Hence, the (co)homological transfer can be viewed as a generalisation of the classical transfer.

Outlook 1.7.19 (Legendre symbol). Let $p \in \mathbb{N}$ be an odd prime. Then the transfer map on $H_1(\,\cdot\,; \mathbb{Z})$ for the subgroup $\{-1, +1\}$ of the multiplicative group $(\mathbb{Z}/(p))^\times$ coincides with the Legendre symbol associated with p; this is a consequence of the Gauß lemma on quadratic resdiues (Exercise).

2

The geometric view

We will now turn to a more geometric setting, viewing groups as (geo)metric objects. Therefore, we will first briefly introduce notions from geometric group theory; in particular, we will consider the class of amenable groups.

We will then focus on two aspects:

- On the one hand, we will see how suitable choices of coefficients lead to geometric invariance properties of group homology.
- On the other hand, we will modify the simplicial cochain complex in a metric way.

In both settings, we will investigate the contrasting behaviour of amenable groups vs. free groups.

Overview of this chapter.

Running example. amenable groups, free groups

2.1 Foundations: Geometric group theory

The aim of *geometric group theory* is to investigate the interplay between geometric and algebraic properties of groups [35, 37, 39, 25, 53]. One flavour of this theory starts by viewing groups as geometric objects.

2.1.1 Quasi-isometry

Using word metrics, we can view groups as metric spaces:

Definition 2.1.1 (word metric)**.** Let G be a group and let $S \subset G$ be a generating set. Then the *word metric of G with respect to S* is given by

$$\begin{aligned} d_S \colon G \times G &\longrightarrow \mathbb{N} \subset \mathbb{R}_{\geq 0} \\ (g, h) &\longmapsto \min\{n \in \mathbb{N} \mid \exists_{s_1,\dots,s_n \in S \cup S^{-1}} \quad g^{-1} h = s_1 \cdot \dots \cdot s_n\}. \end{aligned}$$

Study note. A straightforward computation shows that the word metric on a group G with respect to a generating set S of G indeed is a metric (check!). What would happen if the subset $S \subset G$ is *not* a generating set?

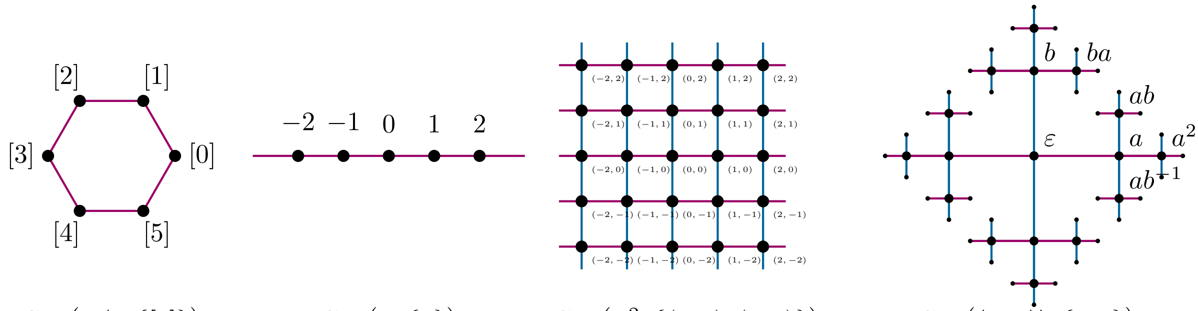
Example 2.1.2 (word metrics)**.** Let $n \in \mathbb{N}$ and let $S \subset \mathbb{Z}^n$ be the standard basis. Then the word metric d_S on $\mathbb{Z}^n$ is the ℓ^1-metric on $\mathbb{Z}^n \subset \mathbb{R}^n$. The word metrics on $\mathbb{Z}$ associated with the generating sets $\{2, 2019\}$ and $\mathbb{Z}$ are very different from that (the latter one even has finite diameter).

Remark 2.1.3 (visualisation of word metrics)**.** A convenient way to visualise word metrics is through Cayley graphs: Let G be a group and let $S \subset G$ be a generating set. Then the *Cayley graph of G with respect to S* is the (undirected) graph $\operatorname{Cay}(G, S)$ whose

- vertex set is G and whose
- edge set is $\bigl\{\{g, g \cdot s\} \bigm| \{g \in G,\ s \in S \cup S^{-1}) \setminus \{e\}\bigr\}$.

Then the word metric d_S on G is the graph metric of $\operatorname{Cay}(G, S)$ on its set of vertices (i.e., all edges have length 1 and the distance between to vertices is the length of a shortest path) (check!). A few prototypical examples are shown in Figure 2.1.

The word metric *does* depend on the chosen generating set. But all word metrics associated with *finite* generating sets on a given group lead to the same large-scale geometry. Large-scale geometry arises by relaxing the notion of isometry by allowing for uniformly bounded error terms.



Cay$\big(\mathbb{Z}/6, \{[1]\}\big)$ Cay$\big(\mathbb{Z}, \{1\}\big)$ Cay$\big(\mathbb{Z}^2, \{(1,0),(0,1)\}\big)$ Cay$\big(\langle a, b \,|\, \rangle, \{a, b\}\big)$

Figure 2.1.: Some Cayley graphs

Definition 2.1.4 (bilipschitz embedding/equivalence)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f \colon X \longrightarrow Y$ be a map.

- The map f is a *bilipschitz embedding* if there exists a $c \in \mathbb{R}_{>0}$ with
$$\forall_{x,x' \in X} \quad \frac{1}{c} \cdot d_X(x, x') \leq d_Y\big(f(x), f(x')\big) \leq c \cdot d_X(x, x').$$
- The map f is a *bilipschitz equivalence* if it is a bilipschitz embedding and if there exists a bilipschitz embedding $g \colon Y \longrightarrow X$ with
$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X .$$
- Metric spaces are *bilipschitz equivalent* if there exists a bilipschitz equivalence between them.

Study note. How can bilipschitz equivalences be viewed as isomorphisms in a suitable category?

Proposition 2.1.5. *Let G be a finitely generated group and let $S, T \subset G$ be finite generating sets of G. Then the identity map $(G, d_S) \longrightarrow (G, d_T)$ is a bilipschitz equivalence.*

Proof. As S is finite, we can consider the (finite) maximum
$$c := \max_{s \in S \cup S^{-1}} d_T(e, s).$$
Let $g, h \in G$ and $n := d_S(g, h)$. Then there exist $s_1, \dots, s_n \in S \cup S^{-1}$ with $g^{-1} \cdot h = s_1 \cdot \dots \cdot s_n$. Using the triangle inequality and the fact that the word metric d_T is left-invariant (check!), we find

$$\begin{aligned} d_T(g,h) &= d_T(g, g\cdot s_1 \cdot \dots \cdot s_n) \\ &\leq d_T(g, g\cdot s_1) + d_T(g\cdot s_1, g\cdot s_1\cdot s_2) \\ &\quad + \dots + d_T(g\cdot s_1\cdot\dots\cdot s_{n-1}, g\cdot s_1\cdot\dots\cdot s_n) && \text{(triangle inequality)} \\ &= d_T(e,s_1) + d_T(e,s_2) + \dots + d_T(e,s_n) && \text{(left-invariance)} \\ &\leq c\cdot n && \text{(construction of } c\text{)} \\ &= c\cdot d_S(g,h). && \text{(definition of } n\text{)} \end{aligned}$$

Similarly, we can estimate d_S in terms of d_T. □

Therefore, we can also safely talk about finitely generated groups being bilipschitz equivalent to a metric space. However, as bilipschitz equivalences are bijections, we will not be able to relate the geometry of finitely generated groups to the geometry of Euclidean or hyperbolic spaces in a meaningful way. Thus, one allows additionally a uniform additive error:

Definition 2.1.6 (quasi-isometric embedding, quasi-isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \longrightarrow Y$ be a map.

- The map f is a *quasi-isometric embedding* if there exist $c \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{\geq 0}$ with
$$\forall_{x,x'\in X} \quad \frac{1}{c}\cdot d_X(x,x') - b \leq d_Y\bigl(f(x), f(x')\bigr) \leq c\cdot d_X(x,x') + b.$$
- A map $f'\colon X \longrightarrow Y$ has *finite distance from f* (is *uniformly close to f*) if there is a $c \in \mathbb{R}_{\geq 0}$ with
$$\forall_{x\in X} \quad d_Y\bigl(f(x), f'(x)\bigr) \leq c.$$
- The map f is a *quasi-isometry* if it is a quasi-isometric embedding and if there exists a quasi-isometric embedding $g\colon Y \longrightarrow X$ such that $f\circ g$ and $g\circ f$ have finite distance from id_Y and id_X, respectively.
- The metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $X \longrightarrow Y$. In this case, we write $X \sim_{\mathrm{QI}} Y$.

Study note. If you know about homotopy categories: How can quasi-isometries be viewed as isomorphisms in a suitable homotopy category?

Sometimes, it is convenient to use the following characterisation of quasi-isometries:

Proposition 2.1.7 (alternative characterisation of quasi-isometry)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A quasi-isometric embedding $f\colon X \longrightarrow Y$ is a quasi-isometry if and only if it has* quasi-dense image, *i.e., if there exists a $c \in \mathbb{R}_{>0}$ such that*

$$\forall_{y\in Y} \quad \exists_{x\in X} \quad d_Y\bigl(f(x), y\bigr) \leq c.$$

Proof. If $f\colon X \longrightarrow Y$ is a quasi-isometry, then there exists a quasi-inverse g of f. Hence, there is a $c \in \mathbb{R}_{>0}$ satisfying

$$\forall_{y \in Y} \quad d_Y\bigl(f \circ g(y), y\bigr) \leq c.$$

Therefore, f has quasi-dense image.

For the converse implication, one applies the axiom of choice to obtain a quasi-inverse [53, Proposition 5.1.10]. □

Corollary 2.1.8. *Let G be a finitely generated group and let $S, T \subset G$ be finite generating sets of G. Then the identity map $(G, d_S) \longrightarrow (G, d_T)$ is a quasi-isometry.*

Proof. Every bilipschitz equivalence is a quasi-isometry. Therefore, we can apply Proposition 2.1.5. □

Therefore, we can also safely talk about finitely generated groups being quasi-isometric to a metric space.

Example 2.1.9.

- Let $n \in \mathbb{N}$. The group $\mathbb{Z}^n$ is quasi-isometric to the space $(\mathbb{R}^n, \ell^1\text{-metric})$, and whence also to the Euclidean space $(\mathbb{R}^n, \text{Euclidean metric})$. In particular: In general, quasi-isometries are neither injective nor surjective.
- All finite groups are quasi-isometric to the trivial group (check!). Moreover, every finitely generated group quasi-isometric to the trivial group is finite (check!).

Caveat 2.1.10 (subgroup distortion)**.** In general, injective group homomorphisms between finitely generated groups are *not* quasi-isometric embeddings (distances in the target group can be significantly shorter than in the domain group [53, Exercise 6.E.6]).

One of the central goals of geometric group theory is to understand finitely generated groups up to quasi-isometry. Therefore, it is useful to develop quasi-isometry invariants. Important examples of quasi-isometry invariants are, for instance [53]:

- number of ends
- growth type (and whence also: containing a nilpotent subgroup of finite index (!) [35])
- hyperbolicity
- amenability (Remark 2.1.18, Corollary 2.2.20)
- ...

In Chapter 2.2, we will see an example of a quasi-isometry invariant extracted from group homology; in particular, this will allow us to separate some quasi-isometry types of finitely generated groups.

2.1.2 Amenability

The notion of amenability is based on (almost) invariance properties and has applications in various fields [73][66][53, Chapter 9]. We will use the classical definition via invariant means:

Definition 2.1.11 (amenable group)**.** Let G be a group. The group G is *amenable* if it admits a left-invariant mean. A *left-invariant mean on* G is an $\mathbb{R}$-linear map $m\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$ with the following properties:

- *Normalisation.* We have $m(1) = 1$.
- *Positivity.* We have $m(f) \geq 0$ for all functions $f \in \ell^\infty(G,\mathbb{R})$ that satisfy $f \geq 0$ (pointwise).
- *Left-invariance.* For all $g \in G$ and all $f \in \ell^\infty(G,\mathbb{R})$, we have

$$m(g \cdot f) = m(f).$$

Example 2.1.12 (finite groups are amenable)**.** Let G be a finite group. Then G is amenable; this is witnessed by the invariant mean

$$\begin{aligned} \ell^\infty(G,\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \frac{1}{|G|} \cdot \sum_{x \in G} f(x). \end{aligned}$$

Example 2.1.13 (Abelian groups are amenable)**.** Every Abelian group is amenable; the existence of an invariant mean can, for example, be shown via the Markov-Kakutani fixed point theorem [53, Proposition 9.1.3].

Moreover, there is the following, slightly more geometric, argument: For simplicity, we will only treat the case $\mathbb{Z}$. For $n \in \mathbb{N}$ let

$$F_n := \{-n, \dots, n\} \subset \mathbb{Z}.$$

The idea is now for $f \in \ell^\infty(\mathbb{Z},\mathbb{R})$ to average the values over F_n via $f_n := 1/|F_n| \cdot \sum_{x \in F_n} f(x)$ and then to take the "limit" $n \to \infty$; the sequence $(f_n)_{n \in \mathbb{N}}$ is bounded (because f is bounded), but in general not convergent. We will therefore need an appropriate notion of limit: Let ω be a non-principal ultrafilter on $\mathbb{N}$ (such things exist in suitably rich set theory). We then consider

$$\begin{aligned} m\colon \ell^\infty(\mathbb{Z},\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \lim_{n \in \omega} \frac{1}{|F_n|} \cdot \sum_{x \in F_n} f(x). \end{aligned}$$

The limit $\lim_{n \in \omega}$ along ω allows to choose "systematically" accumulation points in bounded sequences.

A straightforward computation shows that m is normalised and positive. Left-invariance of m follows from the following almost invariance property of $(F_n)_{n\in\mathbb{N}}$: If $g \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \frac{1}{|F_n|} \cdot |F_n \bigtriangleup g \cdot F_n| = 0.$$

Therefore, m is a left-invariant mean on $\mathbb{Z}$, which shows that $\mathbb{Z}$ is amenable. Similarly, one can argue for all finitely generated Abelian groups.

Outlook 2.1.14 (Følner sequences)**.** Let G be a finitely generated group and let $S \subset G$ be a finite generating set of G. A *Følner sequence* for G is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite non-empty subsets of G with the following property: For all $r \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \cdot |\partial_r^S F_n| = 0;$$

for a finite set $F \subset G$, we define the *r-boundary (with respect to S)* by

$$\partial_r^S F := \bigl\{ x \in G \setminus F \bigm| \exists_{f\in F} \;\; d_S(x,f) \leq r \bigr\}.$$

Then the following are equivalent [66][53, Chapter 9.2.1]:

- The group G is amenable.
- The group G admits a Følner sequence (with respect to S).

Example 2.1.15 (non-amenability of free groups of higher rank)**.** Free groups of rank at least 2 are *not* amenable: For notational simplicity, we only consider the case $F := \langle a, b \,|\, \rangle$ of a free group of rank 2.

Assume for a contradiction that F were amenable, i.e., that there existed a left-invariant mean $m \colon \ell^\infty(F,\mathbb{R}) \longrightarrow \mathbb{R}$ on F. Let $A \subset F$ be the set of reduced words ending with a non-trivial power of a. Then

$$A \cup A \cdot a^{-1} = F.$$

Using the properties of the mean m, we obtain

$$\begin{aligned}
1 &= m(1) \\
&= m(\chi_{A\cup(A\cdot a^{-1})}) \\
&\leq m(\chi_A) + m(\chi_{A\cdot a^{-1}}) \\
&= m(\chi_A) + m(a \cdot \chi_A) \\
&= m(\chi_A) + m(\chi_A) && \text{(left-invariance)} \\
&= 2 \cdot m(\chi_A),
\end{aligned}$$

and so $m(\chi_A) \geq 1/2$. However, the sets A, $A \cdot b$ and $A \cdot b^2$ are pairwise disjoint; therefore,

$$
\begin{aligned}
1 &= m(1) \\
&\geq m(\chi_{A \cup A \cdot b \cup A \cdot b^2}) \\
&= m(\chi_A) + m(\chi_{A \cdot b}) + m(\chi_{A \cdot b^2}) \\
&= 3 \cdot m(\chi_A) && \text{(left-invariance)} \\
&\geq \frac{3}{2},
\end{aligned}
$$

which is impossible. Hence, F is *not* amenable.

Proposition 2.1.16 (inheritance properties of amenable groups)**.**

1. *Subgroups of amenable groups are amenable.*
2. *Homomorphic images of amenable groups are amenable.*
3. *Let*
$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$
be an extension of groups. Then G is amenable if and only if N and Q are amenable.

In particular: Every group that contains a solvable subgroup of finite index is amenable.

Proof. These inheritance properties can, e.g., be shown by transforming the corresponding invariant means [53, Proposition 9.1.6]. □

Outlook 2.1.17 (the von Neumann problem)**.** Our previous discussion shows that: A group that contains a free subgroup of rank 2 is *not* amenable. Conversely, the *von Neumann problem* asked whether the converse also holds.

By now, it is known that the converse does *not* hold; i.e., there exist non-amenable groups that do *not* contain a free subgroup of rank 2 [63]. However, for an early candidate of such an example (namely, Thompson's group F, which is known to not contain any free subgroups of rank 2), it is still an open problem to decide whether this group is amenable or not!

The von Neumann problem had a big influence on the development of geometric group theory and measurable group theory. Moreover, it turns out that converse statements of these type do hold in certain geometric situations [90, 30, 77].

Remark 2.1.18 (quasi-isometry invariance of amenability)**.** Let G and H be finitely generated groups with $G \sim_{\mathrm{QI}} H$. If G is amenable, then also H is amenable.

One can show by direct computation that Følner sequences on G can be converted into Følner sequences on H [53, Theorem 9.3.1]. Alternatively, we can also invoke the homological characterisation of amenability and the quasi-isometry invariance of uniformly finite homology (Corollary 2.2.20).

2.2 Uniformly finite homology

We will now give an example of a quasi-isometry invariant of finitely generated groups of homological origin. In general, group homology with (trivial) $\mathbb{Z}$-coefficients is *not* quasi-isometry invariant:

Example 2.2.1.

- The group $\mathbb{Z}/2$ is quasi-isometric to the trivial group, but $H_1(\mathbb{Z}/2;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2 \not\cong_{\mathbb{Z}} 0 \cong_{\mathbb{Z}} H_1(1;\mathbb{Z})$.
- The group $\mathbb{Z}$ is quasi-isometric to the infinite dihedral group D_∞, but $H_3(\mathbb{Z};\mathbb{Z}) \cong_{\mathbb{Z}} 0$ and $H_3(D_\infty;\mathbb{Z}) \not\cong_{\mathbb{Z}} 0$ (the latter can be seen by using a retraction from D_∞ to $\mathbb{Z}/2$; check!).
- Free groups of rank 2 contain free groups of rank 3 of finite index (index 2; Example AT.2.3.49, Corollary AT.5.3.13); therefore, free groups of rank 2 and 3 are quasi-isometric (check!), but $H_1(\langle a,b\,|\,\rangle;\mathbb{Z}) \not\cong_{\mathbb{Z}} H_1(\langle a,b,c\,|\,\rangle;\mathbb{Z})$.

In contrast, taking different coefficients changes the situation: As we will see, group homology with $\ell^\infty(\,\cdot\,;\mathbb{Z})$-coefficients *is* quasi-isometry invariant (Theorem 2.2.12). In order to prove this fact, we will show that group homology with these coefficients admits a purely geometric description in terms of so-called uniformly finite homology. Moreover, we will characterise amenability in terms of uniformly finite homology.

2.2.1 Uniformly finite homology of spaces

As first step, we introduce a large-scale homology for metric spaces, namely uniformly finite homology. For simplicity, we will restrict to the case of so-called (countable) UDBG spaces.

Definition 2.2.2 (UDBG space)**.** Let (X,d) be a metric space.

- The metric space (X,d) is *uniformly discrete* if there are uniform gaps between all points, i.e., if

$$\inf\bigl\{d(x,x') \bigm| x,x' \in X,\ x \neq x'\bigr\} > 0.$$

- The metric space (X,d) has *bounded geometry* if balls of fixed radius are uniformly bounded, i.e., if

$$\forall_{r\in\mathbb{R}_{>0}} \quad \exists_{K_r\in\mathbb{N}} \quad \forall_{x\in X} \quad \bigl|B_r^{X,d}(x)\bigr| \le K_r.$$

A *UDBG space* is a uniformly discrete metric space with bounded geometry that consists only of countably many points (countability is automatic if the metric does not take the value ∞).

Example 2.2.3 (UDBG spaces).

- If G is a finitely generated group and $S \subset G$ is a finite generating set, then (G, d_S) is a UDBG space (check!).
- The Euclidean space $(\mathbb{R}^2, \text{Euclidean metric})$ is neither uniformly discrete nor of bounded geometry.

Uniformly finite homology is defined via the uniformly finite chain complex; the basic idea is as follows: Tuples in a space can be viewed as "simplices" (described by their vertices). We then form chains of such simplices. As in all other constructions of this type, we need a finiteness condition to make the simplicial boundary operator well-defined. In our situation, we will allow chains with infinitely many simplices and impose the following finiteness conditions:

- The coefficients have to be uniformly bounded.
- The simplices have uniformly bounded size.

More precisely: In the following, a *normed ring* is a commutative ring with unit with a multiplicative norm (e.g., $\mathbb{Z}$ or $\mathbb{R}$).

Proposition and Definition 2.2.4 (uniformly finite chain complex). *Let R be a normed ring and let (X, d) be a UDBG space. Then the* uniformly finite chain complex $C^{\mathrm{uf}}_*(X; R)$ *of (X, d) is the $\mathbb{N}$-indexed R-chain complex defined by:*

- *For $n \in \mathbb{N}$, we write $C^{\mathrm{uf}}_n(X; R)$ for the R-module of all bounded functions $c\colon X^{n+1} \longrightarrow R$ with the following property: There is an $r \in \mathbb{R}_{>0}$ with*
  $$\operatorname{supp} c \subset \{x \in X^{n+1} \mid \operatorname{diam} x \leq r\}.$$
  Here, $\operatorname{supp} c := \{x \in X^{n+1} \mid c(x) \neq 0\}$ *is the* support *of c and* $\operatorname{diam} x := \max_{j,k \in \{0,\dots,n\}} d(x_j, x_k)$ *denotes the* diameter *of tuples in X.*

  The elements of $C^{\mathrm{uf}}_n(X; R)$ are called uniformly finite n-chains in X with coefficients in R; *usually, such functions $c\colon X^{n+1} \longrightarrow R$ are denoted as "sums" of the form $\sum_{x \in X^{n+1}} c(x) \cdot x$.*
- *If $n \in \mathbb{N}_{>0}$, then*
  $$\begin{aligned} \partial_n \colon C^{\mathrm{uf}}_n(X; R) &\longrightarrow C^{\mathrm{uf}}_{n-1}(X; R) \\ \sum_{x \in X^{n+1}} c_x \cdot x &\longmapsto \sum_{x \in X^{n+1}} \sum_{j=0}^{n} (-1)^j \cdot c_x \cdot (x_0, \dots, \widehat{x}_j, \dots, x_n) \end{aligned}$$

describes a well-defined R-linear map and this turns $C^{\mathrm{uf}}_*(X;R)$ into an R-chain complex.

Proof. We first have to show that the boundary operator is well-defined; even though the notation is suggestive, as the "sums" are not necessarily finite, this is not just a "linear extension" of the usual definition on the standard basis! Therefore, we first have to correct the description of ∂_n: Let $c = \sum_{x \in X^{n+1}} c_x \cdot x \in C^{\mathrm{uf}}_n(X;R)$. Then

$$\begin{aligned} X^n &\longrightarrow R \\ y &\longmapsto \sum_{j=0}^{k} (-1)^j \cdot \sum_{x \in \{z \in X^{n+1} \mid (z_0,\dots,\widehat{z}_j,\dots,z_n) = y\}} c_x \end{aligned}$$

is a well-defined chain in $C^{\mathrm{uf}}_{n-1}(X;R)$ (Exercise). This is the chain that is meant by the (more suggestive, but less correct) description in the claim.

That $\partial_n \circ \partial_{n+1} = 0$ holds follows as in the standard argument in the simplicial world (check! see also Remark 1.2.2). □

Definition 2.2.5 (uniformly finite homlogy). Let R be a normed ring, let (X,d) be a UDBG space, and let $n \in \mathbb{N}$. Then, we define the *uniformly finite homology of X with coefficients in R in degree n* by (where $\partial_0 := 0$)

$$H^{\mathrm{uf}}_n(X;R) := \frac{\ker\bigl(\partial_n \colon C^{\mathrm{uf}}_n(X;R) \to C^{\mathrm{uf}}_{n-1}(X;R)\bigr)}{\operatorname{im}\bigl(\partial_{n+1} \colon C^{\mathrm{uf}}_{n+1}(X;R) \to C^{\mathrm{uf}}_n(X;R)\bigr)} \in \operatorname{Ob}({}_R\mathsf{Mod}).$$

Example 2.2.6 (uniformly finite homology of the point). Let R be a normed ring, let $\bullet$ be "the" one-point metric space, and let $n \in \mathbb{N}$. The standard computation shows that

$$H^{\mathrm{uf}}_n(\bullet;R) \cong_R \begin{cases} R & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

(check!). Indeed, this is the same computation as for the homology of the trivial group (Example 1.2.8) or the singular homology of the one-point space.

Proposition and Definition 2.2.7 (functoriality of uniformly finite homology). *Let R be a normed ring and let (X,d_X), (X,d_Y) be UDBG spaces.*

1. *Let $f\colon (X,d_X) \longrightarrow (Y,d_Y)$ be a quasi-isometric embedding. Then, for each $n \in \mathbb{N}$,*

$$\begin{aligned} C^{\mathrm{uf}}_n(f;R) \colon C^{\mathrm{uf}}_n(X;R) &\longrightarrow C^{\mathrm{uf}}_n(Y;R) \\ \sum_{x \in X^{n+1}} c_x \cdot x &\longmapsto \sum_{x \in X^{n+1}} c_x \cdot \bigl(f(x_0),\dots,f(x_n)\bigr) \end{aligned}$$

describes a well-defined R-linear map.

The corresponding sequence $C^{\mathrm{uf}}_*(f;R)$ *is an* R*-chain map* $C^{\mathrm{uf}}_*(X;R) \longrightarrow C^{\mathrm{uf}}_*(Y;R)$ *and we denote the induced map on uniformly finite homology by* $H^{\mathrm{uf}}_*(f;R)\colon H^{\mathrm{uf}}_*(X;R) \longrightarrow H^{\mathrm{uf}}_*(Y;R)$.

2. *This construction is functorial (in the sense that it maps identities to identities and is compatible with composition).*
3. *If* $f, f'\colon (X,d_X) \longrightarrow (Y,d_Y)$ *are uniformly close quasi-isometric embeddings, then* $C^{\mathrm{uf}}_*(f;R) \simeq_R C^{\mathrm{uf}}_*(f';R)$. *In particular,* $H^{\mathrm{uf}}_*(f;R) = H^{\mathrm{uf}}_*(f';R)$.

Proof. Ad 1. Again, we first have to convert the description in the proposition into a proper definition. If $c = \sum_{x\in X^{n+1}} c_x \cdot x \in C^{\mathrm{uf}}_n(X;R)$, then

$$\begin{aligned} Y^{n+1} &\longrightarrow R \\ y &\longmapsto \sum_{x\in\{z\in Z^{n+1} \mid (f(z_0),\dots,f(z_n))=y\}} c_x \end{aligned}$$

is a well-defined chain in $C^{\mathrm{uf}}_n(Y;R)$ (check!). Moreover, the standard computation shows that this construction is compatible with the boundary operators on $C^{\mathrm{uf}}_*(X;R)$ and $C^{\mathrm{uf}}_*(Y;R)$, respectively (check!).

Ad 2. It is clear that $C^{\mathrm{uf}}_*(\mathrm{id}_X;R)$ is the identity map on $C^{\mathrm{uf}}_*(X;R)$ and that this construction is compatible with composition (check!).

Ad 3. As f and f' are uniformly close it is not hard to see that $(h_n)_{n\in\mathbb{N}}$, given by

$$\begin{aligned} h_n\colon C^{\mathrm{uf}}_n(X;R) &\longrightarrow C^{\mathrm{uf}}_{n+1}(Y;R) \\ \sum_{x\in X^{n+1}} c_x \cdot x &\longmapsto \sum_{x\in X^{n+1}} \sum_{j=0}^{n} (-1)^j \cdot c_x \cdot \bigl(f(x_0),\dots,f(x_j),f'(x_j),\dots,f'(x_n)\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$, is an R-chain homotopy between $C^{\mathrm{uf}}_*(f;R)$ and $C^{\mathrm{uf}}_*(f';R)$ (check!). This is a quasi-geometric version of the prism construction in the proof of homotopy invariance of singular homology (Lemma AT.4.2.2). □

Corollary 2.2.8 (QI-invariance of uniformly finite homology)**.** *Let* R *be a normed ring, let* $f\colon (X,d_x) \longrightarrow (Y,d_Y)$ *be a quasi-isometry between UDBG spaces, and let* $n \in \mathbb{N}$. *Then* $H^{\mathrm{uf}}_n(f;R)\colon H^{\mathrm{uf}}_n(X;R) \longrightarrow H^{\mathrm{uf}}_n(Y;R)$ *is an* R*-isomorphism.*

Proof. This is a direct consequence of functoriality of $H^{\mathrm{uf}}_*(\,\cdot\,;R)$ with respect to quasi-isometric embeddings and uniform closeness (Proposition 2.2.7). □

Example 2.2.9 (uniformly finite homology of finite spaces)**.** Let (X,d) be a finite metric space. Then X is a UDBG space and $X \sim_{\mathrm{QI}} \bullet$. Hence, quasi-isometry invariance of uniformly finite homology (Corollary 2.2.8) shows that $H^{\mathrm{uf}}_*(X;R) \cong_R H^{\mathrm{uf}}_*(\bullet;R)$ for every normed ring R. The latter homology has been computed in Example 2.2.6.

2.2.2 Uniformly finite homology of groups

As second step, we relate group homology with $\ell^\infty(\,\cdot\,;\mathbb{R})$-coefficients to uniformly finite homology. The conversion between uniformly finite chains and simplicial chains with function-coefficients is based on viewing the coefficients on an orbit of a simplex as values of a function on the given group.

Remark 2.2.10 (uniformly finite homology of finitely generated groups)**.** Let R be a normed ring, let G be a finitely generated group, and let $S, T \subset G$ be finite generating sets. Because $\mathrm{id}_G \colon (G, d_S) \longrightarrow (G, d_T)$ is a quasi-isometry (Corollary 2.1.8), Proposition 2.2.7 shows that $C^{\mathrm{uf}}_*((G, d_S); R)$ and $C^{\mathrm{uf}}_*((G, d_T); R)$ coincide. We will simply write $C^{\mathrm{uf}}_*(G; R)$ and $H^{\mathrm{uf}}_*(G; R)$ for the corresponding uniformly finite chain complex and the uniformly finite homology of G, respectively.

Theorem 2.2.11 (group homology as uniformly finite homology [10])**.** *Let R be a normed ring and let G be a finitely generated group. Then the maps*

$$\begin{aligned} C_n\bigl(G; \ell^\infty(G, R)\bigr) &\longrightarrow C^{\mathrm{uf}}_n(G; R) \\ (x_0, \dots, x_n) \otimes f &\longmapsto \sum_{y \in G} f(y) \cdot (y \cdot x_0, \dots, y \cdot x_n) \\ C^{\mathrm{uf}}_n(G; R) &\longrightarrow C_n\bigl(G; \ell^\infty(G, R)\bigr) \\ \sum_{x \in G^{n+1}} c_x \cdot x &\longmapsto \sum_{z \in G^n} (e, z_1, \dots, z_n) \otimes \bigl(y \mapsto c_{(y, y \cdot z_1, \dots, y \cdot z_n)}\bigr) \end{aligned}$$

are well-defined for each $n \in \mathbb{N}$ and form mutually inverse R-chain isomorphisms $C^{\mathrm{uf}}_*(G; R) \longleftrightarrow C_*(G; \ell^\infty(G, R))$. In particular, there are canonical isomorphisms $H_n\bigl(G; \ell^\infty(G, R)\bigr) \cong_R H^{\mathrm{uf}}_n(G; R)$.

Proof. This follows from lengthy, but straightforward, computations (check! do you see how the finiteness conditions correspond to each other?). ☐

Corollary 2.2.12 (a homological QI-invariant)**.** *Let R be a normed ring, let G and H be finitely generated groups with $G \sim_{\mathrm{QI}} H$, and let $n \in \mathbb{N}$. Then, every quasi-isometry $G \longrightarrow H$ induces an isomorphism*

$$H_n\bigl(G; \ell^\infty(G, R)\bigr) \cong_R H_n\bigl(H; \ell^\infty(H, R)\bigr).$$

Proof. If $f \colon G \longrightarrow H$ is a quasi-isometry, then we have

$$\begin{aligned} H_n\bigl(G; \ell^\infty(G, R)\bigr) &\cong_R H^{\mathrm{uf}}_n(G; R) && \text{(Theorem 2.2.11)} \\ &\cong_R H^{\mathrm{uf}}_n(H; R) && \text{(Corollary 2.2.8 for } H^{\mathrm{uf}}_n(f; R)\text{)} \\ &\cong_R H_n\bigl(H; \ell^\infty(H, R)\bigr) && \text{(Theorem 2.2.11)} \end{aligned}$$

as claimed. ☐

2.2.3 Application: Ponzi schemes and amenability

As an application, we give a homological characterisation of amenability [8], which has applications in geometric group theory.

Theorem 2.2.13 (homological characterisation of amenability)**.** *Let G be a finitely generated group. Then the following are equivalent:*

1. *The group G is amenable.*
2. $H_0\bigl(G;\ell^\infty(G,\mathbb{R})\bigr) \not\cong_{\mathbb{R}} 0$
3. $H_0\bigl(G;\ell^\infty(G,\mathbb{Z})\bigr) \not\cong_{\mathbb{Z}} 0$

In order to prove this theorem, we use the connection with uniformly finite homology and a closer investigation of uniformly finite homology in degree 0.

Definition 2.2.14 (fundamental class in uniformly finite homology)**.** Let R be a normed ring and let (X,d) be a UDBG space. Then $\sum_{x\in X} 1\cdot x$ is a cycle in $C_0^{\mathrm{uf}}(X;R)$ and the corresponding uniformly finite homology class is the *fundamental class of X with R-coefficients*:

$$[X]_R := \Bigl[\sum_{x\in X} 1\cdot x\Bigr] \in H_0^{\mathrm{uf}}(X;R).$$

Theorem 2.2.15 (fundamental class in uniformly finite homology)**.** *Let (X,d) be a UDBG space. Then the following are equivalent:*

1. *We have $H_0^{\mathrm{uf}}(X;\mathbb{Z}) \cong_{\mathbb{Z}} 0$.*
2. *We have $[X]_{\mathbb{Z}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$.*
3. *We have $[X]_{\mathbb{R}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{R})$.*
4. *We have $H_0^{\mathrm{uf}}(X;\mathbb{R}) \cong_{\mathbb{R}} 0$.*

Proof. The implications $1 \Longrightarrow 2$ and $4 \Longrightarrow 3$ are clear. Moreover, $2 \Longrightarrow 3$ holds (because every integral uniformly finite boundary also is a real uniformly finite bonudary).

Therefore, it suffices to prove the following implications:

- *Ad* $2 \Longrightarrow 1$ *and* $2 \Longrightarrow 4$. Let $[X]_{\mathbb{Z}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. Then, by the tail lemma (Lemma 2.2.16), there exists a family $(t_x)_{x\in X}$ of chains in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$ such that $\partial_1 t_x = 1\cdot x$ in $C_0^{\mathrm{uf}}(X;\mathbb{Z})$ for all $x\in X$ and such that "$\sum_{x\in X} t_x$" is a well-defined chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$.

  Let R be a normed ring (e.g., $\mathbb{R}$ or $\mathbb{Z}$) and $\alpha \in H_0^{\mathrm{uf}}(X;R)$. Then α is represented by some chain $c = \sum_{x\in X} c_x \cdot x \in C_0^{\mathrm{uf}}(X;R)$; hence,

$$b := \sum_{x \in X} c_x \cdot t_x$$

describes a well-defined chain in $C_1^{\mathrm{uf}}(X;R)$ (check!) with $\partial_1 b = c$. Therefore, $\alpha = 0 \in H_0^{\mathrm{uf}}(X;R)$.

- *Ad* 3 $\Longrightarrow$ 2. Let $[X]_{\mathbb{R}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{R})$; let $b \in C_1^{\mathrm{uf}}(X;\mathbb{R})$ with $\partial_1 b = \sum_{x \in X} 1 \cdot x$. As in the proof of Lemma 2.2.16, we may assume that $b \geq 0$. We then consider the floored version

$$\begin{aligned} \widetilde{b} \colon X^2 &\longrightarrow \mathbb{Z} \\ x &\longmapsto \bigl\lfloor (2 \cdot K + 1) \cdot b(x) \bigr\rfloor, \end{aligned}$$

where K is defined as follows: Because b is a uniformly finite chain, there exists an $r \in \mathbb{N}$ with $\operatorname{diam} x \leq r$ for all $x \in \operatorname{supp} b$. As X is a UDBG space, we can define the number

$$K := \max\bigl\{ |B_r^{X,d}(x)| \bigm| x \in X \bigr\}.$$

Clearly, $\widetilde{b}$ is a uniformly finite chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$.

We now set $\widetilde{c} := \partial_1 \widetilde{b} \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ and show that $\operatorname{supp} \widetilde{c} = X$: Let $x \in X$. In $(\partial_1 \widetilde{b})(x)$, at most $2 \cdot K$ coefficients of b contribute (by construction of K; check!). Therefore, we obtain the improved floor estimate

$$\begin{aligned} \widetilde{c}(x) &= (\partial_1 \widetilde{b})(x) \\ &= \sum_{y \in X} \widetilde{b}(y,x) - \sum_{y \in X} \widetilde{b}(x,y) && \text{(definition of } \partial_1 \widetilde{b}) \\ &\geq (2 \cdot K + 1) \cdot \Bigl(\sum_{y \in X} b(y,x) - \sum_{y \in X} b(x,y) \Bigr) - 2 \cdot K \cdot 1 && \text{(improved floor estimate)} \\ &= (2 \cdot K + 1) \cdot (\partial_1 b)(x) - 2 \cdot K && \text{(definition of } \partial_1 b) \\ &= (2 \cdot K + 1) \cdot 1 - 2 \cdot K = 1, && \text{(because } \partial_1 b = \sum_{x \in X} 1 \cdot x) \end{aligned}$$

and thus $x \in \operatorname{supp} \widetilde{c}$.

Applying the tail lemma to $\widetilde{c}$ yields a family $(t_x)_{x \in X}$ of tails in $C_(^{\mathrm{uf}} X;\mathbb{Z})$ for all of $\operatorname{supp} \widetilde{c} = X$ with

$$\partial_1 \Bigl(\sum_{x \in X} t_x \Bigr) = \sum_{x \in X} 1 \cdot x,$$

which shows that $[X]_{\mathbb{Z}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. $\square$

Lemma 2.2.16 (the tail lemma by Block and Weinberger)**.** *Let (X,d) be a UDBG space. If $c \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ is a uniformly finite chain with $c \geq 0$ (i.e., all*

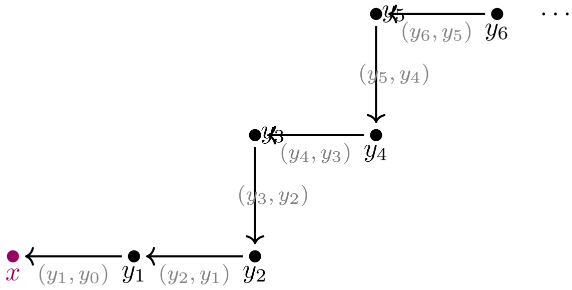


Figure 2.2.: a tail for x

coefficients of c *are non-negative) and* $[c] = 0$ *in* $H_0^{\mathrm{uf}}(X;\mathbb{Z})$*, then there exists a family* $(t_x)_{x\in\operatorname{supp} c}$ *of chains in* $C_1^{\mathrm{uf}}(X;\mathbb{Z})$ *with the following properties:*

- *For all* $x \in \operatorname{supp} c$*, we have* $\partial_1 t_x = 1 \cdot x$ *in* $C_0^{\mathrm{uf}}(X;\mathbb{Z})$
- *and*
$$\sum_{x\in\operatorname{supp} c} t_x = \sum_{y\in X^2} \biggl(\sum_{x\in\operatorname{supp} c} t_x(y)\biggr) \cdot y$$
describes a well-defined chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$*.*

Proof. Let $b = \sum_{x\in X^2} b_x \cdot x \in C_1^{\mathrm{uf}}(X;\mathbb{Z})$ with $\partial_1 b = c$. Using $\partial_1(1\cdot(x_0,x_1)) = -\partial_1(1\cdot(x_1,x_0))$ for all $x \in X^2$, we may assume without loss of generality that $b_x \geq 0$ for all $x \in X^2$ (check!).

We extract the tails t_x out of b by (double) induction. Because X is countable, we can enumerate all points of $\operatorname{supp} c$ and treat one point of $\operatorname{supp} c$ at a time.

Let $x \in \operatorname{supp} c$; we set $y_0 := x$. Because $\partial_1 b = c$ and $b \geq 0$ (and b is integral), there exists a $y_1 \in X \setminus \{y_0\}$ with
$$b_{(y_1,y_0)} \geq 1.$$
Then $y_1 \in \operatorname{supp}(c - 1\cdot y_0 + 1\cdot y_1)$ and $\partial_1(b - b_{(y_1,y_0)}\cdot(y_1,y_0)) = c - 1\cdot y_0 + 1\cdot y_1$. Hence, inductively, we obtain a "tail" $t_x := \sum_{n\in\mathbb{N}} 1\cdot(y_{n+1},y_n)$ (Figure 2.2) with $t_x \leq b$ (pointwise). So t_x indeed is a well-defined uniformly finite chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$. By construction, we have
$$\begin{aligned} \partial_1 t_x = 1\cdot x, \quad c - 1\cdot x \geq 0, \quad \operatorname{supp} c \setminus \{x\} \subset \operatorname{supp}(c - 1\cdot x) \subset \operatorname{supp} c, \\ \partial_1(b - t_x) = c - 1\cdot x, \quad b - t_x \geq 0, \quad \operatorname{supp}(b - t_x) \subset \operatorname{supp} b. \end{aligned}$$

By induction over $\operatorname{supp} c$, we can construct tails for all points in $\operatorname{supp} c$. By construction, the accumulation of these tails is contained in the non-negative

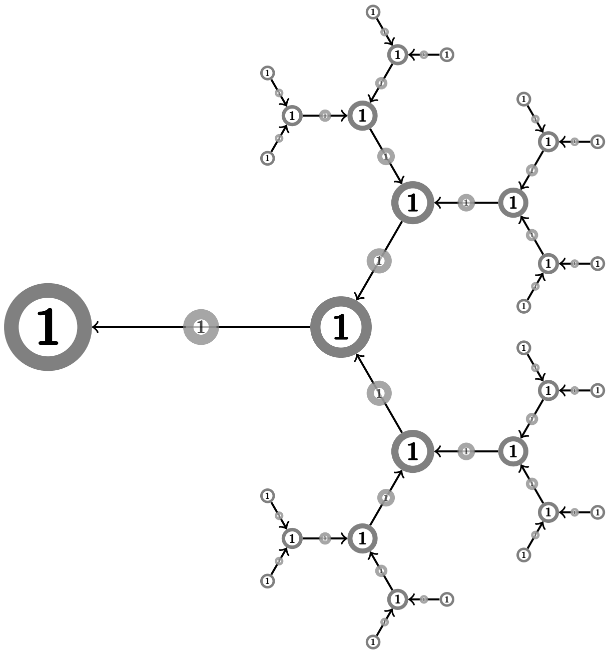


Figure 2.3.: a Ponzi scheme on the rooted 3-regular tree

uniformly finite chain b and whence their "sum" "$\sum_{x\in X} t_x$" is a well-defined uniformly finite chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$ with the desired properties. $\square$

Block and Weinberger proposed the following interpretation of the vanishing of the fundamental class in uniformly finite homology [8]:

Remark 2.2.17 (Ponzi schemes in UDBG spaces)**.** Let X be a UDBG space with $[X]_{\mathbb{R}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{R})$. Then there exists a chain $b \in C_1^{\mathrm{uf}}(X;\mathbb{R})$ with $\partial_1 b = \sum_{x\in X} 1\cdot x$; without loss of generality, we may assume that all coefficients of b are non-negative (by "swapping edges" if necessary).

We now view b as a flow of money: If $(x_0,x_1) \in X$, then x_0 sends the amount $b(x_0,x_1)$ to x_1. The fact that

$$\partial_1 b = \sum_{x\in X} 1\cdot x - 0$$

holds means that afterwards every point in X has something (namely 1) even though they had nothing in the beginning (namely 0). The chain b pushes the problem of generating enough money to infinity (in particular, the total amount of money in this system is infinite ...). Figure 2.3 shows such a flow for the UDBG space given by the vertices of the rooted 3-regular tree (with the path-metric induced by the graph structure).

A famous example of real-world schemes that "worked" in a similar way are the so-called Ponzi schemes (Exercise).

Finally, we can complete the proof of the homological characterisation of amenability:

Proof of Theorem 2.2.13. *Ad* 1 $\Longrightarrow$ 2. Let G be amenable, i.e., G admits a left-invariant mean $m\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$. We now use the mean m for a transfer-like argument:

Let $i\colon \mathbb{R} \longrightarrow \ell^\infty(G,\mathbb{R})$ be the injective $\mathbb{R}$-linear map given by viewing scalars as constant functions on G (this map is $\mathbb{Z}G$-linear!). Because m is normalised, we have $m \circ i = \mathrm{id}_\mathbb{R}$. Moreover, (id_G, i) and (id_G, m) are morphisms in $\mathsf{GroupMod}$ (check!). Therefore, we obtain for every $n \in \mathbb{N}$

$$H_n(\mathrm{id}_G; m) \circ H_n(\mathrm{id}_G; i) = H_n(\mathrm{id}_G; m \circ i) = H_n(\mathrm{id}_G; \mathrm{id}_\mathbb{R}) = \mathrm{id}_{H_n(G;\mathbb{R})}.$$

In particular, $H_0(\mathrm{id}_G; i)\colon H_0(G;\mathbb{R}) \longrightarrow H_0(G;\ell^\infty(G,\mathbb{R}))$ is injective. As $H_0(G;\mathbb{R}) \cong_\mathbb{R} \mathbb{R}$, we obtain $H_0(G;\ell^\infty(G,\mathbb{R})) \not\cong_\mathbb{R} 0$.

Ad 2 $\Longleftrightarrow$ 3. This equivalence follows from the interpretation in terms of uniformly finite homology (Theorem 2.2.11) and Theorem 2.2.15.

Ad 2 $\Longrightarrow$ 1. Let $H_0(G;\ell^\infty(G,\mathbb{R})) \not\cong_\mathbb{R} 0$. By the general computation of group homology in degree 0 (Theorem 1.3.1), this means that $W \neq \ell^\infty(G,\mathbb{R})$, where

$$W := \mathrm{Span}_\mathbb{R}\bigl\{f - g \cdot f \bigm| f \in \ell^\infty(G,\mathbb{R})\bigr\}.$$

In oder to find a left-invariant mean on G, we will first show that $1 \notin \overline{W}$ (where we take the closure with respect to the supremum norm $|\cdot|_\infty$ on the Banach space $\ell^\infty(G,\mathbb{R})$); more precisely, we will show that the distance

$$d := \inf\bigl\{|1 - f|_\infty \bigm| f \in W\bigr\} \in \mathbb{R}_{\geq 0}$$

from 1 to W equals 1: We clearly have $d \leq |1 - 0|_\infty = 1$ (because $0 \in W$). *Assume* for a contradiction that there exists an $f \in W$ with $|1 - f|_\infty < 1$. This means that $\varepsilon := \inf_{x \in G} f(x) > 0$.

Using the translation of Theorem 2.2.11, we therefore obtain a chain $c \in C_0^{\mathrm{uf}}(G;\mathbb{R})$ with $c \geq \varepsilon$ and $[c] = 0 \in H_0^{\mathrm{uf}}(G;\mathbb{R})$. Now the same argument as in the proof of the implication 3 $\Longrightarrow$ 2 of Theorem 2.2.15 (applied to the chain $1/\varepsilon \cdot c$) shows that this already implies $H_0(G;\ell^\infty(G,\mathbb{R})) \cong_\mathbb{R} H_0^{\mathrm{uf}}(G;\mathbb{R}) \cong_\mathbb{R} 0$, which contradicts our hypothesis.

Hence, $d = 1$. Therefore, the Hahn-Banach theorem [72, Theorem 5.16] shows that there exists an $\mathbb{R}$-linear map $m\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$ with the following properties:

- We have $m(1) = 1$ and $m|_W = 0$.
- We have $\bigl|m(f)\bigr| \leq \frac{1}{d} \cdot |f|_\infty = |f|_\infty$ for all $f \in \ell^\infty(G,\mathbb{R})$.

The condition $m|_W = 0$ implies that m is left-invariant. Moreover, one can deduce from these properties that m is positive (Exercise). Hence, m is a left-invariant mean for G and thus G is amenable. □

Remark 2.2.18. Let G be a finitely generated group. Then Theorem 2.2.13 and Theorem 1.3.1 show (and the same arguments also apply to $\ell^\infty(G,\mathbb{R})$):

- If G is amenable, then $\ell^\infty(G,\mathbb{Z})_G \not\cong_{\mathbb{Z}} 0$. In fact, in many cases, this group is huge [7].
- If G is non-amenable, then $\ell^\infty(G,\mathbb{Z})_G \cong_{\mathbb{Z}} 0$.

Remark 2.2.19 (Ponzi schemes on groups)**.** Combining Theorem 2.2.13 and Remark 2.2.17 shows: A finitely generated group admits a Ponzi scheme if and only if it is *not* amenable. This is compatible with the geometric intuition that non-amenable groups have enough branching/space on the way to "infinity".

Corollary 2.2.20 (quasi-isometry invariance of amenability)**.** *Let G and H be finitely generated groups with $G \sim_{\mathrm{QI}} H$. If G is amenable, then also H is amenable.*

Homological proof. This is a direct consequence of the fact that uniformly finite homology characterises amenability (Theorem 2.2.13 and 2.2.11) and that uniformly finite homology is quasi-isometry invariant (Corollary 2.2.8). □

Outlook 2.2.21 (QI-invariance of the Hirsch rank)**.** One can use an amenable transfer argument in uniformly finite homology and the quasi-isometry invariance of uniformly finite homology to show that $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{Z}^m$ if and only if $n = m$ (Exercise).

More generally, one can also use uniformly finite homology to give an alternative proof of the fact that the Hirsch rank is a quasi-isometry invariant of finitely generated (virtually) nilpotent groups [7].

Outlook 2.2.22 (quasi-isometries vs. bilipschitz equivalences)**.** Let $f\colon X \longrightarrow Y$ be a quasi-isometry between UDBG spaces. Then one can show that the following are equivalent [90]:

1. The map f is uniformly close to a bilipschitz equivalence.
2. We have $H_0^{\mathrm{uf}}(f;\mathbb{Z})[X]_{\mathbb{Z}} = [Y]_{\mathbb{Z}}$ in $H_0^{\mathrm{uf}}(Y;\mathbb{Z})$.

In particular, because $H_0^{\mathrm{uf}}(\,\cdot\,;\mathbb{Z})$ is trivial on non-amenable groups, every quasi-isometry between *non*-amenable groups is uniformly close to a bilipschitz equivalence(!).

In contrast, there do exist infinite finitely generated amenable groups that are quasi-isometric but not bilipschitz equivalent [26].

Outlook 2.2.23 (homological invariants in measured group theory)**.** Taking suitable coefficients also leads to homological invariants for different notions of "equivalence", such as orbit/measure equivalence [29, 74, 54].

2.3 Bounded cohomology

Now, in a slightly different direction, we will metrically modify the definition of the simplicial cochain complex (by introducing a boundedness condition). This leads to bounded cohomology, which has various applications in group theory, geometric group theory, and manifold geometry [36, 43, 44, 61, 27, 50].

2.3.1 Bounded cohomology of groups

Bounded cohomology is a functional analytic twin of ordinary group cohomology; in order to describe its construction in terms of the simplicial resolution, we need normed/Banach versions of the ordinary theory.

Definition 2.3.1 (Banach G-module)**.** Let G be a group.

- A *normed G-module* is a (real) normed vector space V together with an isometric (left) G-action on V.
- A *Banach G-module* is a (real) Banach space V together with an isometric (left) G-action on V (i.e., a complete normed G-module).
- Let V and W be normed [Banach] G-modules. A *morphism $V \longrightarrow W$ of normed [Banach] G-modules* is an $\mathbb{R}$-linear map $\varphi\colon V \longrightarrow W$ that
  - is G-equivariant, i.e., $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $g \in G$ and all $x \in V$, and that
  - is bounded, i.e., there exists a $C \in \mathbb{R}_{\geq 0}$ with
  $$\forall_{x \in V} \quad \big\|\varphi(x)\big\| \leq C \cdot \|x\|.$$
  (The infimum of all such bounds C is the *norm* $\|\varphi\|$ of φ).

  We write $\operatorname{BHom}_G(V, W)$ for the space of all morphisms $V \longrightarrow W$ of normed [Banach] G-modules (which is a Banach space with respect to the operator norm as soon as W is Banach).

The category of normed and Banach G-modules is denoted by ${}_G\mathsf{Norm}$ and ${}_G\mathsf{Ban}$, respectively. Moreover, we write $\mathsf{GroupBan}^*$ for the Banach version of $\mathsf{GroupMod}^*$.

Example 2.3.2 (the normed simplicial resolution)**.** Let G be a group. For $n \in \mathbb{N}$, we can equip $C_n^{\mathbb{R}}(G) := \mathbb{R} \otimes_{\mathbb{Z}} C_n(G)$ with the ℓ^1-norm associated with the basis G^{n+1}, i.e.,

$$\begin{aligned}|\cdot|_1\colon C_n^{\mathbb{R}}(G) &\longrightarrow \mathbb{R}_{\geq 0}\\ \sum_{g\in G^{n+1}} a_g\cdot g &\longmapsto \sum_{g\in G^{n+1}} |a_g|.\end{aligned}$$

Then the boundary operator $\mathrm{id}_{\mathbb{R}}\otimes_{\mathbb{Z}}\partial_n\colon C_n^{\mathbb{R}}(G)\longrightarrow C_{n-1}^{\mathbb{R}}(G)$ is a bounded operator (of norm at most $n+1$; check!). Hence, we obtain a chain complex $C_*^{\mathbb{R}}(G)$ in the category of normed G-modules. Moreover, group homomorphisms induce chain maps on $C_*^{\mathbb{R}}$ that consist of bounded operators of norm at most 1.

In this Banach setting, we replace the Hom-functors and dual modules with their bounded versions and then proceed as before:

Example 2.3.3 ((bounded) dual)**.** Let G be a group and let V be a (left) Banach G-module. Then V represents the corresponding functor

$$\mathrm{BHom}_G(\,\cdot\,,V)\colon {}_G\mathsf{Norm}\longrightarrow \mathsf{Ban}$$

to the category Ban of Banach spaces (and bounded linear maps).

Definition 2.3.4 (bounded cohomology)**.** Let G be a group and let V be a (left) Banach G-module.

- Then we write $C_{\mathrm{b}}^*(G;V) := \mathrm{BHom}_G\bigl(C_*^{\mathbb{R}}(G),V\bigr)\in {}_{\mathsf{Ban}}\mathsf{Ch}^*$ for the *bounded cochain complex of G with coefficients in V.*
- For $n\in\mathbb{N}$, we define *bounded cohomology of G with coefficients in V in degree n* by

$$H_{\mathrm{b}}^n(G;V) := H^n\bigl(C_{\mathrm{b}}^*(G;V)\bigr)\in {}_{\mathbb{R}}\mathsf{Mod}\,.$$

Let $(\varphi,\Phi)\colon (G,V)\longrightarrow (H,W)$ be a morphism in $\mathsf{GroupBan}^*$.

- We write $C_{\mathrm{b}}^*(\varphi;\Phi) := \mathrm{BHom}_G(C_*^{\mathbb{R}}(\varphi),\Phi)$ for the composition

$$\mathrm{BHom}_H\bigl(C_*^{\mathbb{R}}(H),W\bigr)\xrightarrow{\text{can. incl.}}\mathrm{BHom}_G\bigl(\varphi^*C_*^{\mathbb{R}}(H),\varphi^*W\bigr)\xrightarrow{\mathrm{BHom}_G(C_*^{\mathbb{R}}(\varphi),\Phi)}\mathrm{BHom}_G\bigl(C_*^{\mathbb{R}}(G),V\bigr)$$

  of (degree-wise) bounded cochain maps.
- For $n\in\mathbb{N}$, we then set (which is $\mathbb{R}$-linear)

$$H_{\mathrm{b}}^n(\varphi;\Phi) := H^n\bigl(C_{\mathrm{b}}^*(\varphi;\Phi)\bigr)\colon H_{\mathrm{b}}^n(H;W)\longrightarrow H_{\mathrm{b}}^n(G;V).$$

Remark 2.3.5 (additional structure on bounded cohomology)**.** By construction, bounded cohomology in each degree is not only an $\mathbb{R}$-vector space, but a semi-normed $\mathbb{R}$-vector space (with respect to the semi-norm induced by the operator norm on the bounded cochain complex). This structure is useful in the context of simplicial volume (Chapter AT.4.4.5) and topological rigidity of volume [36, 50, 27].

Example 2.3.6 (bounded cohomology of the trivial group)**.** Let 1 be "the" trivial group and let V be a Banach space. Then $C^*_b(1;V) \cong_{\mathbb{R}\mathsf{Ch}} C^*(1;V)$ (check!), and therefore, for all $n \in \mathbb{N}$,

$$H^n_b(1;V) \cong_\mathbb{R} \begin{cases} V & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Remark 2.3.7 (bounded cohomology in degree 0)**.** The same argument as in the proof of the computation of group cohomology in degree 0 (Theorem 1.3.1) shows that the functor H^0_b is canonically naturally isomorphic to the invariants functor $\mathsf{GroupBan}^* \longrightarrow \mathsf{Ban}$ (check!).

Remark 2.3.8 (bounded cohomology in degree 1)**.** The same argument as in the computation of $H^1(\,\cdot\,;\mathbb{R})$ (Theorem 1.4.13) shows that for each group G we have (check!):

$$\begin{aligned} H^1_b(G;\mathbb{R}) &\cong_\mathbb{R} \Bigl\{\varphi \in \operatorname{Hom}_{\mathsf{Group}}(G,\mathbb{R}) \Bigm| \sup_{g\in G} |\varphi(g)| < \infty\Bigr\} && \text{(bounded version of Theorem 1.4.13)} \\ &\cong_\mathbb{R} 0 && \text{($\mathbb{R}$ has no non-trivial bounded subgroups)} \end{aligned}$$

In particular, in general, the comparison map $H^*_b(\,\cdot\,;\mathbb{R}) \longrightarrow H^*(\,\cdot\,;\mathbb{R})$ is *not* surjective (for example $H^1(\mathbb{Z};\mathbb{R}) \not\cong_\mathbb{R} 0$).

Bounded cohomology in degree 2 turns out to be interesting: it is related to quasi-morphisms (Theorem 2.3.17).

Remark 2.3.9 (comparison map)**.** The inclusion of bounded G-equivariant bounded linear maps into G-equivariant linear maps induces a natural transformation

$$H^*_b(\,\cdot\,;\,\cdot\,) \Longrightarrow H^*(\,\cdot\,;\,\cdot\,)$$

(check!), the so-called *comparison map*.

In general, the comparison map is neither injective nor surjective (Corollary 2.3.19, Remark 2.3.8).

2.3.2 Application: A characterisation of amenability

Theorem 2.3.10 (cohomological characterisation of amenability)**.** *Let G be a group. Then the following are equivalent:*

1. *The group G is amenable.*
2. *For all right Banach G-modules V and all $n \in \mathbb{N}_{\geq 1}$, we have*
$$H^n_b(G;V^\#) \cong_\mathbb{R} 0.$$
3. *For all right Banach G-modules V, we have $H^1_b(G;V^\#) \cong_\mathbb{R} 0$.*

Notation 2.3.11. For a right Banach G-module V, we write $V^{\#}$ for the Banach G-module consisting of the Banach space of bounded linear functionals $V \longrightarrow \mathbb{R}$ together with the left G-action

$$\begin{aligned} G \times V^{\#} &\longrightarrow V^{\#} \\ (g, f) &\longmapsto \bigl(x \mapsto f(x \cdot g)\bigr). \end{aligned}$$

Proof of Theorem 2.3.10. *Ad* 1 $\Longrightarrow$ 2. Let G be an amenable group and $n \in \mathbb{N}_{\geq 1}$. In order to keep notation simple, we only show that $H^n_{\mathrm{b}}(G;\mathbb{R}) \cong_{\mathbb{R}} 0$, where G acts trivially on $\mathbb{R}$. We argue by transfer:

As G is amenable, G admits a left-invariant mean $m \colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$. We then consider the corresponding transfer $t^* \colon C^*_{\mathrm{b}}(G;\mathbb{R}) \longrightarrow C^* := (C^{\mathbb{R}}_*(G))^{\#}$, given by

$$\begin{aligned} t^k \colon C^k = \bigl(C^{\mathbb{R}}_k(G)\bigr)^{\#} &\longrightarrow = \operatorname{BHom}_G\bigl(C^{\mathbb{R}}_k(G),\mathbb{R}\bigr) C^k_{\mathrm{b}}(G;\mathbb{R}) \\ f &\longmapsto \bigl((g_0,\dots,g_k) \mapsto m\bigl(x \mapsto f(x\cdot g_0,\dots,x\cdot g_k)\bigr)\bigr) \end{aligned}$$

in each degree k; then t^k is a bounded linear map (check!) that is compatible with the simplicial coboundary operators (check!) and that satisfies

$$t^* \circ i^* = \mathrm{id}_{C^*_{\mathrm{b}}(G;\mathbb{R})}$$

(check!), where $i^* \colon C^*_{\mathrm{b}}(G;\mathbb{R}) \longrightarrow C^*$ is the inclusion. In particular, the identity of $H^n_{\mathrm{b}}(G;\mathbb{R})$ factors through $H^n(C^*)$:

$$H^n(t^*) \circ H^n(i^*) = H^n(t^* \circ i^*) = \mathrm{id}_{H^n_{\mathrm{b}}(G;\mathbb{R})}.$$

Therefore, it suffices to show that $H^n(C^*) \cong_{\mathbb{R}} 0$ holds: The standard chain contraction for $C_*(G)$ (given by cones; proof of Proposition 1.6.5) induces a chain contraction for $C^{\mathbb{R}}_*(G)$, which is degreewise bounded with respect to the ℓ^1-norm (check!). Applying the functor $\,\cdot\,^{\#}$ leads to a well-defined cochain contraction for C^* (check!). In particular, $H^n(C^*) \cong_{\mathbb{R}} 0$ (because $n \geq 1$).

For the general case, we instead make use of the transfer map given by

$$\begin{aligned} \operatorname{BHom}\bigl(C^{\mathbb{R}}_k(G), V^{\#}\bigr) &\longrightarrow C^k_{\mathrm{b}}(G;V^{\#}) \\ f &\longmapsto \bigl((g_0,\dots,g_k) \mapsto \bigl(v \mapsto m(x \mapsto f(x\cdot g_0,\dots,x\cdot g_k)(v))\bigr)\bigr) \end{aligned}$$

and show that the cochain complex $\operatorname{BHom}\bigl(C^{\mathbb{R}}_k(G), V^{\#}\bigr)$ is contractible.

Ad 2 $\Longrightarrow$ 3. This is clear.

Ad 3 $\Longrightarrow$ 1. We consider the Banach G-module (check!)

$$V := \ell^\infty(G,\mathbb{R})/C,$$

where $C \subset \ell^\infty(G,\mathbb{R})$ is the subspace of constant functions (which is isomorphic to $\mathbb{R}$ and closed).

The condition $H^1_{\mathrm{b}}(G; V^{\#}) \cong_{\mathbb{R}} 0$ can be used to show that there exists a bounded $\mathbb{R}$-linear functional $\mu\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$ with $\mu(1) = 1$ that is (left) G-invariant (Exercise); unfortunately, it is not a priori clear that the norm of μ is 1 (equivalently, that μ is positive).

One can then apply a decomposition argument from functional analysis to improve this to a bounded linear functional $m\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$ with $m(1) = 1$ that is positive: Similarly to the Hahn decomposition of signed measures of finite total variation [45, Chapter 7.1.2] there is a "minimal" decomposition

$$\mu = \mu_+ - \mu_-$$

into positive functionals $\mu_+, \mu_- \longrightarrow \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$. In addition, this decomposition is unique in a certain sense; this uniqueness is strong enough to show that also μ_+ has to be left-invariant. Because of $\mu(1) = 1$, we know that $\mu_+ \neq 0$. Hence, a suitable normalisation of μ_+ is a left-invariant mean on G.

The left-invariant mean m shows that G is amenable. □

Caveat 2.3.12. Amenable groups cannot be characterised using bounded cohomology only with trivial $\mathbb{R}$-coefficients: There exist many examples of non-example groups G with $H^*_{\mathrm{b}}(G;\mathbb{R}) \cong_{\mathbb{R}} H^*_{\mathrm{b}}(1;\mathbb{R})$ [57, 52].

Study note (transfer)**.** Which transfer arguments in group (co)homology do you know by now? Collect and compare them!

The characterisations of amenability in terms of bounded cohomology and uniformly finite homology complement each other and are actually related [10].

2.3.3 Application: Quasi-morphisms

One classical application of bounded cohomology concerns quasi-morphisms. A quasi-morphism $G \longrightarrow \mathbb{R}$ on a group G is a map that almost satisfies the homomorphism condition:

Definition 2.3.13 (quasi-morphism)**.** Let G be a group.

- A *quasi-morphism on* G is a map $\varphi\colon G \longrightarrow \mathbb{R}$ such that the *defect*

  $$D(\varphi) := \sup_{g,h \in G} \bigl|\varphi(g \cdot h) - \varphi(g) - \varphi(h)\bigr|$$

  is finite.

- A quasi-morphism $\varphi\colon G \longrightarrow \mathbb{R}$ is *trivial* if it is uniformly close to a homomorphism, i.e., if there exists a group homomorphism $\psi\colon G \longrightarrow \mathbb{R}$ with

  $$\sup_{g \in G} \bigl|\varphi(g) - \psi(g)\bigr| < \infty.$$

- A quasi-morphism $\varphi\colon G \longrightarrow \mathbb{R}$ is *homogeneous*, if for each $g \in G$ and each $n \in \mathbb{Z}$ we have $\varphi(g^n) = n \cdot \varphi(g)$.
- We write

$\mathrm{QM}(G)$ for the $\mathbb{R}$-vector space of all quasi-morphisms on G
$\mathrm{QM}_0(G)$ for the $\mathbb{R}$-vector space of all trivial quasi-morphisms on G
$\overline{\mathrm{QM}}(G)$ for the $\mathbb{R}$-vector space of all homogeneous quasi-morphisms on G.

One can now wonder whether every quasi-morphism (i.e., a map that almost satisfies the homomorphism condition) has to be trivial (i.e., uniformly close to a homomorphism). Bounded cohomology answers this question (Theorem 2.3.17). Homogeneous quasi-morphisms allow to express the difference between quasi-morphisms and group homomorphisms in a simplified way:

Proposition 2.3.14 (homogenisation of quasi-morphisms). *Let G be a group and let $\varphi\colon G \longrightarrow \mathbb{R}$ be a quasi-morphism.*

1. *Then the following map is a homogeneous quasi-morphism on G that is uniformly close to φ:*

$$\begin{aligned} \overline{\varphi}\colon G &\longrightarrow \mathbb{R} \\ g &\longmapsto \lim_{n\to\infty} \frac{1}{n} \cdot \varphi(g^n) \end{aligned}$$

2. *Hence, we obtain an isomorphism of $\mathbb{R}$-vector spaces:*

$$\begin{aligned} \mathrm{QM}(G)/\,\mathrm{QM}_0(G) &\longrightarrow \overline{\mathrm{QM}}(G)/\,\mathrm{Hom}_{\mathsf{Group}}(G,\mathbb{R}) \\ [\varphi] &\longmapsto [\overline{\varphi}] \end{aligned}$$

Proof. The first part follows from Lemma 2.3.15 and a straightforward computation (Exercise). The second part follows from the first part and the fact that the homogenisation of a trivial quasi-morphism is a group homomorphism (Exercise). ☐

Lemma 2.3.15 (convergence of normalised sequences). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.*

1. *If $\sup_{n,m\in\mathbb{N}} |a_{n+m} - a_n - a_m|$ is finite, then the sequence $(a_n/n)_{n\in\mathbb{N}_{>0}}$ converges.*
2. *If the sequence $(a_n)_{n\in\mathbb{N}}$ is non-negative and $a_{n+m} \le a_n + a_m$ for all $n, m \in \mathbb{N}$, then the sequence $(a_n/n)_{n\in\mathbb{N}_{>0}}$ converges and*

$$\lim_{n\to\infty} \frac{1}{n} \cdot a_n = \inf_{n\in\mathbb{N}_{>0}} \frac{1}{n} \cdot a_n.$$

Proof. This is elementary analysis (Exercise). ☐

Proposition 2.3.16 (counting quasi-morphism). *Let $F := \langle a, b \,|\, \rangle$ be "the" free group of rank 2 (described by reduced words) and let $w \in F$.*

1. *Then the following map is a quasi-morphism on F, the* counting quasi-morphism associated with w, *where $\#(w, g)$ denotes the number of (possibly overlapping) occurrences of the word w in the reduced word g:*

$$\begin{aligned}\psi_w \colon F &\longrightarrow \mathbb{R}\\ g &\longmapsto \#(w, g) - \#(w^{-1}, g)\end{aligned}$$

2. *If $w = ab$, then $\overline{\psi}_w$ is* not *a group homomorphism (and hence ψ_w is* not *a trivial quasi-morphism).*

Proof. Ad 1. Let $g, h \in F$. By construction, we have

$$\begin{aligned}D(g, h) &:= \psi_w(g \cdot h) - \psi_w(g) - \psi_w(h)\\ &= \#(w, g \cdot h) - \#(w^{-1}, g \cdot h)\\ &\quad - \#(w, g) + \#(w^{-1}, g)\\ &\quad - \#(w, h) + \#(w^{-1}, h).\end{aligned}$$

The word gh, in general, is *not* reduced; therefore, we have to be careful with counting occurrences of w and w^{-1} at the "end" of g and the "beginning" of h. Let r be the tail/initial part of g and h, respectively, that is deleted when forming the product $g \cdot h$; i.e., $g = g'r$ and $h = r^{-1}h'$ are reduced decompositions and $g'h'$ is the reduced word that represents $g \cdot h$.

Then the only terms that can contribute to $D(g, h)$ are occurrences of w or w^{-1} that intersect with r in g or h but are not completely contained in r (Figure 2.4); the number of these occurrences can be bounded in terms of the length $\ell(w)$ of w. More precisely: For all $x, y \in F$ for which xy is reduced, we have

$$\bigl|\#(w, xy) - \#(w, x) - \#(w, y)\bigr| \leq \ell(w).$$

Therefore, we obtain

$$\begin{aligned}\bigl|D(g, h)\bigr| &= \bigl|\#(w, g'h') - \#(w^{-1}, g'h')\\ &\quad - \#(w, g'r) + \#(w^{-1}, g'r)\\ &\quad - \#(w, r^{-1}h') + \#(w^{-1}, r^{-1}h')\bigr|\\ &\leq 6 \cdot \ell(w)\end{aligned}$$

by resolving all six reduced decompositions and the triangle inequality (check!).

Ad 2. It suffices to prove that $\overline{\psi}_w \colon F \longrightarrow \mathbb{R}$ is *not* a group homomorphism (in view of Proposition 2.3.14). By construction,

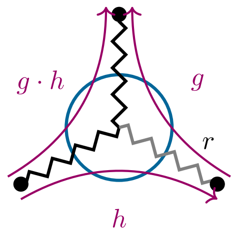


Figure 2.4.: bounding the defect of counting quasi-morphisms

$$\overline{\psi}_w(a) = \lim_{n\to\infty} \frac{1}{n} \cdot \psi_w(a^n) = 0$$
$$\overline{\psi}_w(b) = \lim_{n\to\infty} \frac{1}{n} \cdot \psi_w(b^n) = 0.$$

As $\{a, b\}$ is a generating set of F, if $\overline{\psi}_w$ were a group homomorphism, then $\overline{\psi}_w = 0$. However,

$$\overline{\psi}_w(ab) = \lim_{n\to\infty} \frac{1}{n} \cdot \psi_w\bigl((ab)^n\bigr) = \lim_{n\to\infty} \frac{1}{n} \cdot (n - 0) = 1 \neq 0.$$

Hence, ψ_w is *not* a trivial quasi-morphism. □

Theorem 2.3.17 (quasi-morphisms via bounded cohomology)**.** *Let G be a group. Then the sequence*

$$\operatorname{Hom}_{\mathsf{Group}}(G, \mathbb{R}) \xrightarrow{\text{incl.}} \overline{\mathrm{QM}}(G) \xrightarrow{\overline{\Delta}} H^2_{\mathrm{b}}(G; \mathbb{R}) \xrightarrow[\text{map}]{\text{comparison}} H^2(G; \mathbb{R})$$

of $\mathbb{R}$-vector spaces is exact, where $\overline{\Delta}$ is the map induced by (when describing bounded cohomology via the bar complex)

$$\begin{aligned} \Delta\colon \mathrm{QM}(G) &\longrightarrow \overline{C}^2_{\mathrm{b}}(G; \mathbb{R}) \\ \varphi &\longmapsto \bigl([g_1 \mid g_2] \mapsto \varphi(g_1 \cdot g_2) - \varphi(g_1) - \varphi(g_2)\bigr). \end{aligned}$$

Proof. We briefly comment on the use of the bar complex: As in the situation of ordinary cohomology, we may also describe bounded cohomology in terms of the bar complex (to simplify notation). Let $\overline{C}^{\mathbb{R}}_*(G) := \mathbb{R} \otimes_{\mathbb{Z}} \overline{C}_*(G)$ be the $\mathbb{R}$-version of the bar complex and let

$$\overline{C}^*_{\mathrm{b}}(G; \mathbb{R}) := \operatorname{BHom}_G\bigl(\overline{C}^{\mathbb{R}}_*(G), \mathbb{R}\bigr)$$

be the corresponding bounded dual. Then the isomorphism from Remark 1.2.4 carries over to this bounded setting and shows that we can compute $H^2_{\mathrm{b}}(G; \mathbb{R})$ as $H^2(\overline{C}^*_{\mathrm{b}}(G; \mathbb{R}))$ (check!).

We now proceed in the following steps:

① The maps Δ and $\overline{\Delta}$ are well-defined: Let $\varphi\colon G \longrightarrow \mathbb{R}$ be a quasi-morphism. Then the map

$$\begin{aligned}\Phi\colon \overline{C}_2^{\mathbb{R}}(G) &\longrightarrow \mathbb{R}\\ [g_1 \mid g_2] &\longmapsto \varphi(g_1 \cdot g_2) - \varphi(g_1) - \varphi(g_2)\end{aligned}$$

is bounded (with respect to the norm obtained from the ℓ^1-norm on $C_2^{\mathbb{R}}(G)$) and G-equivariant (by construction); hence, $\Phi \in \overline{C}_{\mathrm{b}}^2(G;\mathbb{R})$. Moreover, Φ is a cocycle in $\overline{C}_{\mathrm{b}}^*(G;\mathbb{R})$, because (in $\overline{C}^*(G;\mathbb{R})$; check!)

$$\Phi = \overline{\delta}^1\bigl([g] \mapsto -\varphi(g)\bigr)$$

and $\overline{\delta}^2 \circ \overline{\delta}^1 = 0$.

② Computation of the kernel of $\overline{\Delta}$: Clearly, $\mathrm{Hom}_{\mathsf{Group}}(G,\mathbb{R}) \subset \ker \overline{\Delta}$. Conversely, let $\varphi \in \ker \overline{\Delta}$, i.e., there exists a bounded cochain $f \in \overline{C}_{\mathrm{b}}^1(G;\mathbb{R})$ with $\overline{\delta}^1 f = \overline{\delta}^1([g] \mapsto -\varphi(g))$. This shows that

$$\begin{aligned}G &\longrightarrow \mathbb{R}\\ g &\longmapsto \varphi(g) + f\bigl([g]\bigr)\end{aligned}$$

is a group homomorphism (check!) that is uniformly close to φ (because f is bounded). Therefore, φ is a trivial homogeneous quasi-morphism, whence a group homomorphism $G \longrightarrow \mathbb{R}$ (Proposition 2.3.14).

③ Computation of the image of $\overline{\Delta}$: The argument in step ① shows that the image of $\overline{\Delta}$ lies in the kernel of the comparison map (check!). Conversely, let $f \in \overline{C}_{\mathrm{b}}^2(G;\mathbb{R})$ be a bounded cocycle whose associated class in ordinary group cohomology $H^2(G;\mathbb{R})$ is zero. Then there exists a chain $b \in \overline{C}^1(G;\mathbb{R})$ with

$$f = \overline{\delta}^1(b) = \bigl([g_1 \mid g_2] \mapsto b([g_1]) - b([g_1 \cdot g_2]) + b([g_2])\bigr).$$

Because f is bounded, the map $\varphi := (g \mapsto -b([g]))\colon G \longrightarrow \mathbb{R}$ is a quasi-morphism and $\Delta(\varphi) = f$. Therefore, $\overline{\Delta}(\overline{\varphi}) = [f]$ in $H_{\mathrm{b}}^2(G;\mathbb{R})$ (check!). □

We will now apply this theorem in two directions: On the one hand, we will use it to show that amenable groups do not admit non-trivial quasi-morphisms. On the other hand, we will use it to exhibit non-trivial bounded cohomology for free groups of higher rank.

Corollary 2.3.18 (quasi-morphisms on amenable groups)**.** *Let G be an amenable group. Then all quasi-morphisms on G are trivial:* $\overline{\mathrm{QM}}(G) = \mathrm{Hom}_{\mathsf{Group}}(G,\mathbb{R})$.

Proof. By Theorem 2.3.17, $\overline{\mathrm{QM}}(G)/\operatorname{Hom}_{\mathsf{Group}}(G,\mathbb{R})$ is isomorphic to the kernel of the comparison map $c_G\colon H^2_{\mathrm{b}}(G;\mathbb{R}) \longrightarrow H^2(G;\mathbb{R})$. Because G is amenable, we know that $H^2_{\mathrm{b}}(G;\mathbb{R}) \cong_{\mathbb{R}} 0$ (Theorem 2.3.10) and so the kernel of c_G is trivial as well. □

Corollary 2.3.19 (bounded cohomology of free groups)**.** *Let F be a free group of rank at least 2. Then $H^2_{\mathrm{b}}(F;\mathbb{R})$ is non-trivial and the comparison map $H^2_{\mathrm{b}}(F;\mathbb{R}) \longrightarrow H^2(F;\mathbb{R})$ is* not *injective.*

Proof. Because F retracts onto a free group of rank 2, we only need to consider the case of $\langle a, b \,|\, \rangle$ (check!).

As the free group $\langle a, b \,|\, \rangle$ admits non-trivial (counting) quasi-morphisms (Proposition 2.3.16), the description of $H^2_{\mathrm{b}}(\,\cdot\,;\mathbb{R})$ in terms of quasi-morphisms (Theorem 2.3.17) shows that $H^2_{\mathrm{b}}(\langle a, b \,|\, \rangle;\mathbb{R}) \not\cong_{\mathbb{R}} 0$ and that the comparison map $H^2_{\mathrm{b}}(\langle a, b \,|\, \rangle;\mathbb{R}) \longrightarrow H^2(\langle a, b \,|\,; \rangle\mathbb{R})$ is *not* injective. □

Caveat 2.3.20. The computation in Corollary 2.3.19 shows that bounded cohomology is, in general, *not* restricted by the length of projective resolutions. The proof of the fundamental theorem of group cohomology (Corollary 1.6.9) does not directly carry over to bounded cohomology: In general, projective resolutions do not carry a norm (and so we cannot talk about bounded cochains) and it is not clear that the canonical chain homotopies would be bounded. In fact, there is a fundamental theorem for bounded cohomology, based on a functional analytic notion of projectivity [36, 43].

Outlook 2.3.21 (higher bounded cohomology of free groups)**.** Let F be a free group of rank at least 2. Constructions from hyperbolic geometry in dimension 3 show that $H^3_{\mathrm{b}}(F;\mathbb{R}) \not\cong_{\mathbb{R}} 0$ [79]. However, it is unknown whether $H^n_{\mathrm{b}}(F;\mathbb{R})$ is non-trivial or not for $n \in \mathbb{N}_{\geq 4}$ (!).

2.3.4 Application: Stable commutator length

We will now move one step further into group theory and briefly discuss stable commutator length.

Definition 2.3.22 ((stable) commutator length)**.** Let G be a group and let $g \in [G, G]$.

- The *commutator length* of g is defined by

  $$\operatorname{cl}_G g := \min\bigl\{n \in \mathbb{N} \bigm| \exists_{a_1,\dots,a_n,b_1,\dots,b_n \in G} \quad [a_1, b_1] \cdot \dots \cdot [a_n, b_n] = g\bigr\} \in \mathbb{N}.$$

- The *stable commutator length* of g is defined by

  $$\operatorname{scl}_G g := \inf_{n \in \mathbb{N}_{>0}} \frac{1}{n} \cdot \operatorname{cl}_G(g^n).$$

Remark 2.3.23 (lim vs. inf)**.** Let G be a group and let $g \in [G, G]$. Then the second part of Lemma 2.3.15 shows that

$$\operatorname{scl}_G g = \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{cl}_G g^n.$$

Of course, the definition of (stable) commutator length only makes sense because the commutator subgroup, in general, does *not* only consist of commutators, but also of elements that can be written as products/inverses of commutators.

Example 2.3.24 (short-cuts)**.** Let G be a group and let $a, b \in G$. Then

$$[a, b]^3 = [a \cdot b \cdot a^{-1},\ b^{-1} \cdot a \cdot b \cdot a^{-2}] \cdot [b^{-1} \cdot a \cdot b,\ b^2]$$

(check! [18]). Therefore, $\operatorname{cl}_G [a, b]^3 \leq 2$ and so

$$\operatorname{scl}_G [a, b] = \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{cl}_G [a, b]^n \leq \frac{2}{3}.$$

Stable commutator length can be expressed in terms of quasi-morphisms (and thus, at least implicitly, in terms of bounded cohomology):

Proposition 2.3.25 (commutator estimate for quasi-morphisms)**.** *Let G be a group, let $\varphi\colon G \longrightarrow \mathbb{R}$ be a homogeneous quasi-morphism, and let $a, b \in G$. Then*

$$\bigl|\varphi\bigl([a, b]\bigr)\bigr| \leq D(\varphi).$$

Proof. As first step, we show that $\varphi(a \cdot b \cdot a^{-1}) = \varphi(b)$: Because φ is homogeneous, we have

$$\begin{aligned}
\bigl|\varphi(a \cdot b \cdot a^{-1}) - \varphi(b)\bigr| &= \frac{1}{n} \cdot \bigl|\varphi((a \cdot b \cdot a^{-1})^n) - \varphi(b^n)\bigr| \\
&= \frac{1}{n} \cdot \bigl|\varphi(a \cdot b^n \cdot a^{-1}) - \varphi(b^n)\bigr| \\
&\leq \frac{1}{n} \cdot \bigl|\varphi(a \cdot b^n \cdot a^{-1}) - \varphi(a \cdot b^n) - \varphi(a^{-1}) + \varphi(a \cdot b^n) + \varphi(a^{-1}) - \varphi(b^n)\bigr| \\
&= \frac{1}{n} \cdot \bigl|\varphi(a \cdot b^n \cdot a^{-1}) - \varphi(a \cdot b^n) - \varphi(a^{-1}) + \varphi(a \cdot b^n) - \varphi(a) - \varphi(b^n)\bigr| \\
&\leq \frac{1}{n} \cdot \bigl|\varphi(a \cdot b^n \cdot a^{-1}) - \varphi(a \cdot b^n) - \varphi(a^{-1})\bigr| + \frac{1}{n} \cdot \bigl|\varphi(a \cdot b^n) - \varphi(a) - \varphi(b^n)\bigr| \\
&\leq \frac{2}{n} \cdot D(\varphi)
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ shows that $|\varphi(a \cdot b \cdot a^{-1} - \varphi(b)| = 0$. Therefore,

$$\begin{aligned}\bigl|\varphi\bigl([a,b]\bigr)\bigr| &= \bigl|\varphi(a\cdot b\cdot a^{-1}\cdot b^{-1}) - \varphi(a\cdot b\cdot a^{-1}) - \varphi(b^{-1}) + \varphi(a\cdot b\cdot a^{-1}) + \varphi(b^{-1})\bigr| \\ &\leq D(\varphi) + \bigl|\varphi(a\cdot b\cdot a^{-1}) + \varphi(b^{-1})\bigr| \\ &= D(\varphi) + \bigl|\varphi(b) - \varphi(b)\bigr| \\ &= D(\varphi),\end{aligned}$$

as claimed. □

Theorem 2.3.26 (Bavard duality)**.** *Let G be a group and let $g \in [G,G]$. Then (where $\sup\emptyset = 0$)*

$$\operatorname{scl}_G g = \frac{1}{2} \cdot \sup_{\varphi\in\overline{\mathrm{QM}}(G)\setminus\operatorname{Hom}_{\mathsf{Group}}(G,\mathbb{R})} \frac{\bigl|\varphi(g)\bigr|}{D(\varphi)}.$$

Proof. We only prove the easy estimate: Let $\varphi \in \overline{\mathrm{QM}}(G)$. Then the commutator estimate (Proposition 2.3.25) and a straightforward induction over the commutator length (check!) show that

$$\forall_{g\in[G,G]} \quad \bigl|\varphi(g)\bigr| \leq 2\cdot D(\varphi)\cdot \operatorname{cl}_G g.$$

Therefore, for all $g \in [G,G]$, we obtain (because φ is homogeneous)

$$\forall_{n\in\mathbb{N}_{>0}} \quad \bigl|\varphi(g)\bigr| = \frac{1}{n}\cdot\bigl|\varphi(g^n)\bigr| \leq \frac{1}{n}\cdot 2\cdot D(\varphi)\cdot\operatorname{cl}_G g^n.$$

Taking the limit shows that $\bigl|\varphi(g)\bigr| \leq 2\cdot D(\varphi)\cdot\operatorname{scl}_G g$.

The proof of the converse estimate is more delicate and relies on topological arguments (involving surfaces) [3, 14]. □

Example 2.3.27 (stable commutator length on amenable groups)**.** If G is an amenable group, then $\overline{\mathrm{QM}}(G) = \operatorname{Hom}_{\mathsf{Group}}(G,\mathbb{R})$ (Corollary 2.3.18). Therefore, Bavard duality (Theorem 2.3.26) implies that

$$\forall_{g\in[G,G]} \quad \operatorname{scl}_G g = 0.$$

This can be viewed as a far-reaching generalisation of the fact that the commutator subgroup of an Abelian group is trivial.

Example 2.3.28 (stable commutator length on free groups)**.** Let $F := \langle a,b \mid \rangle$. Then the homogenisation φ of the counting quasi-morphism associated with the word $aba^{-1}b^{-1}$ satisfies

$$\varphi\bigl([a,b]\bigr) = \lim_{n\to\infty}\frac{1}{n}\cdot\psi_{aba^{-1}b^{-1}}\bigl([a,b]^n\bigr) = \lim_{n\to\infty}\frac{1}{n}\cdot(n-0) = 1.$$

Moreover, φ is *not* a group homomorphism (we can use the same argument as in the proof of Proposition 2.3.16). Therefore, Bavard duality shows that

$$\mathrm{scl}_F\,[a,b] \geq \frac{1}{2} \cdot \frac{1}{D(\varphi)} > 0.$$

Outlook 2.3.29 (values of scl)**.** More precisely, we have in the free group $F := \langle a, b \,|\, \rangle$ the following results on stable commutator length [14]:

- More careful estimates show that $\mathrm{scl}_F\,[a,b] = 1/2$.
- It is known that $\mathrm{scl}_F\, g \in \mathbb{Q}_{\geq 1/2}$ for all $g \in [F,F] \setminus \{e\}$. In particular, scl_F has a gap in $(0, 1/2)$.
- There is a polynomial-time (in the length of the input word) algorithm that computes stable commutator length on F.
- No element $g \in [F,F]$ with $\mathrm{scl}_F\, g \in (1/2, 7/12)$ is known; so, there might be a second scl-gap on F (this is an open problem).

In contrast, it is known that there exist finitely presented groups whose stable commutator length function also assumes transcendental values [91].

Computations of stable commutator length in Thompson-like groups have recently been used to manufacture closed manifolds with arbitrary (non-negative) rational simplicial volume [41]; in particular, this shows that there is *no* simplicial volume gap in dimension at least 4.

3

The derived view

Group homology and group cohomology are derived functors of tensor and Hom-functors, respectively. We will briefly describe this aspect of group (co)homology (leading to an axiomatic description of group homology) and explain how computational tools from homological algebra can be used in group (co)homology (e.g., spectral sequences).

Overview of this chapter.

Running example. tensor and Hom-functors

3.1 Derived functors

Many algebraic constructions only yield semi-exact functors (e.g., tensor products and Hom-functors). Derived functors provide exact extensions of such functors and therefore naturally come up in many computations.

For simplicity, in the following discussion, we will focus on derived functors of right exact functors (and group homology); we will summarise the corresponding results for left exact functors (and group cohomology) in Chapter 3.1.5. Moreover, we will stay in the elementary language of homological algebra of module categories; the refined approach via derived categories will briefly be discussed in Chapter 3.1.6.

3.1.1 Axiomatic description

We first give an axiomatic description of derived functors. A construction will be sketched in Chapter 3.1.2.

Definition 3.1.1 (module category). A *module category* is a category of the form ${}_R\mathsf{Mod}$, where R is a ring. Similarly, a *right module category* is a category of the form Mod_R, where R is a ring.

Study note. In order to avoid introducing the foundations of categories with additional structures (such as additive, Abelian, exact, triangulated, ...) and reproving standard homological facts about them, we will just work in the elementary setup of module categories. If you are familiar with more abstract setups of homological algebra, you should try to remember, which kind of structures are needed in which step. If you are not familiar with any of these setups, it might be a good idea to consult the literature [87, 13].

Definition 3.1.2 (right exact functor). Let $F\colon C \longrightarrow D$ be a (covariant) additive functor, i.e., F is compatible with the additive structure on the Hom-sets (given by pointwise addition). The functor F is *right exact*, if for every short exact sequence $0 \longrightarrow A' \xrightarrow{\;i\;} A \xrightarrow{\;\pi\;} A'' \longrightarrow 0$ in C, the sequence

$$F(A') \xrightarrow{F(i)} F(A) \xrightarrow{F(\pi)} F(A'') \longrightarrow 0$$

is exact in D.

The functor F is *exact* if, in addition, also $0 \longrightarrow F(A') \xrightarrow{F(i)} F(A)$ is exact.

Example 3.1.3. Tensor product functors are (additive and) right exact, as can be seen from the adjunction with the Hom-functor (Korollar IV.1.5.14).

Derived functors of right exact functors are a systematic "optimal" extension of the given functor turning short exact sequences into long exact sequences (in a natural way). We formulate this universal property in terms of homological ∂-functors:

Definition 3.1.4 (homological ∂-functor)**.** Let C and D be module categories. A *homological ∂-functor* $C \longrightarrow D$ consists of the following data:

- a sequence $(T_n : C \longrightarrow D)_{n \in \mathbb{N}}$ of additive functors,
- morphisms $(\partial_n \in \mathrm{Mor}_C(T_n(A''), T_{n-1}(A')))_{n \in \mathbb{N}}$, the *connecting morphisms*, for each short exact sequence $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{\pi} A'' \longrightarrow 0$ in C

such that for every commutative

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "f'"'] & A \arrow[r, "\pi"] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & B' \arrow[r, "j"'] & B \arrow[r, "\varphi"'] & B'' \arrow[r] & 0
\end{tikzcd}$$

diagram in C with exact rows, the corresponding diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_{n+1}(A'') \arrow[r, "\partial_{n+1}"] \arrow[d, "T_{n+1}(f'')"'] & T_n(A') \arrow[r, "T_n(i)"] \arrow[d, "T_n(f')"'] & T_n(A) \arrow[r, "T_n(\pi)"] \arrow[d, "T_n(f)"'] & T_n(A'') \arrow[r, "\partial_n"] \arrow[d, "T_n(f'')"'] & \cdots \\
\cdots \arrow[r] & T_{n+1}(B'') \arrow[r, "\partial_{n+1}"'] & T_n(B') \arrow[r, "T_n(j)"'] & T_n(B) \arrow[r, "T_n(\varphi)"'] & T_n(B'') \arrow[r, "\partial_n"'] & \cdots
\end{tikzcd}$$

in D is commutative and has exact rows. Here, we set $T_{-1} := 0$ and $\partial_0 := 0$.

In order to unclutter notation, we usually just say that $T_* : C \longrightarrow D$ is a homological ∂-functor (even though it consists of more data!) and the short exact sequence is not included in the notation of the connecting morphisms.

Study note. What is a reasonable notion of natural transformations between homological ∂-functors? When are two homological ∂-functors isomorphic? (These notions also have to incorporate the connecting morphisms!)

Definition 3.1.5 (derived functor)**.** Let C and D be module categories and let $F : C \longrightarrow D$ be a right-exact functor. A *left-derived functor of F* is a homological ∂-functor $L_* : C \longrightarrow D$

- together with a natural isomorphism $L_0 \Longrightarrow F$ (i.e., L_* *extends* F),
- and the following *universality property*: If $T_* : C \longrightarrow D$ is a homological ∂-functor and $\tau_0 : T_0 \Longrightarrow F$ is a natural transforation, then there is a unique natural transformation $\tau_* : T_* \Longrightarrow L_*$ of homological ∂-functors that extends τ_0.

Remark 3.1.6 (uniqueness of derived functors)**.** If $F\colon C \longrightarrow D$ is a right-exact functor between module categories, then derived functors of F (if they exist) are unique up to canonical natural isomorphism of homological ∂-functors (check!).

Example 3.1.7 (derived functors of exact functors)**.** Let $F\colon C \longrightarrow D$ be an *exact* functor. Then

$$\begin{aligned} & L_0 := F \\ \forall_{n\in\mathbb{N}_{>0}} \quad & L_n := 0 \\ \forall_{n\in\mathbb{N}} \quad & \partial_n := 0 \end{aligned}$$

defines a ("the") left-derived functor of F (check!).

3.1.2 A construction

In order to construct a left-derived functor of a right-exact functor, we proceed as follows:

- We replace objects by projective resolutions (i.e., by decomposition into homologically simpler objects),
- apply the functor in question to these projective resolutions,
- and measure the failure of exactness via homology.

Theorem 3.1.8 (existence of derived functors)**.** *Let $F\colon C \longrightarrow D$ be a right-exact functor between module categories. Then there exists a left-derived functor of F. (Moreover, by Remark 3.1.6, we have essential uniqueness.)*

Proof. As first step, we construct the functors $(L_n\colon C \longrightarrow D)_{n\in\mathbb{N}}$: For each $A \in \mathrm{Ob}(C)$, we choose a projective resolution (P_*^A, ε^A) of A in C (as module category, C contains enough projectives (Proposition IV.5.2.10). Let $n \in \mathbb{N}$; we then define $L_n\colon C \longrightarrow D$ as follows:

- On objects: For $A \in \mathrm{Ob}(C)$, we set
$$L_n(A) := H_n\bigl(F(P_*^A)\bigr) \in \mathrm{Ob}(D).$$
- On morphisms: Let $A, B \in \mathrm{Ob}(C)$ and let $f \in \mathrm{Mor}_C(A, B)$. By the fundamental theorem (Theorem 1.6.7), there exists a lift $f_* \square f\colon P_*^A \square \varepsilon^A \longrightarrow P_*^B \square \varepsilon^B$ of f (which is unique up to chain homotopy), and we set
$$L_n(f) := H_n\bigl(F(f_*)\bigr) \in \mathrm{Mor}_D\bigl(L_n(A), L_n(B)\bigr).$$
Because f_* is uniquely determined up to chain homotopy and because homology is a chain homotopy invariant, $L_n(f)$ does not depend on the chosen extension f_*.

This construction has the following properties:

① Each L_n is a functor $C \longrightarrow D$:

Clearly, L_n maps identity morphisms to identity morphisms (we can choose the identity as extension). Moreover, the fundamental theorem (Theorem 1.6.7) shows that L_n is compatible with compositions (for the composition, we may choose compositions of extensions as extension).

② If $P \in \mathrm{Ob}(C)$ is projective and $n \in \mathbb{N}_{>0}$, then $L_n(P) \cong_D 0$:

If (Q_*, ε) is any projective resolution of P, then the fundamental theorem (Theorem 1.6.7) shows that there is a canonical isomorphism

$$L_n(P) = H_n\bigl(F(P_*^P)\bigr) \cong_D H_n\bigl(F(Q_*)\bigr).$$

On the other hand, as P is projective, we can choose the trivial projective resolution

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \xrightarrow{\mathrm{id}_P} P \longrightarrow 0$$

of P. Therefore, if $n \in \mathbb{N}_{>0}$, we have $L_n(P) \cong_D 0$.

③ There is a canonical natural isomorphism $L_0 \Longrightarrow F$:

Let $A, B \in \mathrm{Ob}(C)$, let $f \in \mathrm{Mor}_C(A, B)$, and let $f_* \mathbin{\square} f \colon P_*^A \mathbin{\square} \varepsilon^A \longrightarrow P_*^B \mathbin{\square} \varepsilon^B$ be an extension of f. Then, in particular, we have the following commutative diagram in C with exact rows:

$$\begin{tikzcd}
P_1^A \arrow[r, "\partial_1^A"] \arrow[d, "f_1"'] & P_0^A \arrow[r, "\varepsilon^A"] \arrow[d, "f_0"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
P_1^B \arrow[r, "\partial_1^B"'] & P_0^B \arrow[r, "\varepsilon^B"'] & B \arrow[r] & 0
\end{tikzcd}$$

Applying F and homology leads to the following commutative diagram in D (as F is right-exact, we know that $\operatorname{im} F(\partial_1^A) = \ker F(\varepsilon^A)$ and $\operatorname{im} F(\varepsilon^A) = A$, and similarly for B):

$$\begin{tikzcd}
L_0(A) \arrow[r, equal] \arrow[d, "L_0(f)"'] & H_0(F(P_*^A)) \arrow[r, equal] \arrow[d, "\text{“}F(f_1)\text{”}"'] & F(P_0^A)/\operatorname{im} F(\partial_1^A) \arrow[r, equal] \arrow[d, "\text{“}F(f_0)\text{”}"'] & F(P_0^A)/\ker F(\varepsilon^A) \arrow[r, "\text{“}F(\varepsilon_A)\text{”}", "\cong_D"'] \arrow[d, "\text{“}F(f_0)\text{”}"'] & F(A) \arrow[d, "F(f)"'] \\
L_0(B) \arrow[r, equal] & H_0(F(P_*^B)) \arrow[r, equal] & F(P_0^B)/\operatorname{im} F(\partial_1^B) \arrow[r, equal] & F(P_0^B)/\ker F(\varepsilon^B) \arrow[r, "\cong_D", "\text{“}F(\varepsilon_B)\text{”}"'] & F(B)
\end{tikzcd}$$

Hence, $L_0 \cong F$.

As next step, we construct the connecting morphisms, using the horseshoe lemma and the long exact sequence in homology (Appendix A.2):

- Let $0 \longrightarrow A' \xrightarrow{\;i\;} A \xrightarrow{\;\pi\;} A'' \longrightarrow 0$ be a short exact sequence in A. By the horseshoe lemma (Proposition A.2.3), there exists a projective resolution (Q_*, ε) of A and chain maps $i_* \,\square\, i$ as well as $\pi_* \,\square\, \pi$ such that
$$0 \longrightarrow P_*^{A'} \xrightarrow{\;i_*\;} Q_* \xrightarrow{\;\pi_*\;} P_*^{A''} \longrightarrow 0$$
is a degree-wise short exact sequence of chain complexes over C. Because $P_*^{A''}$ consists of projective modules, in each degree, we have a *split exact* sequence. Because F is additive, we hence obtain a degree-wise short exact sequence
$$0 \longrightarrow F(P_*^{A'}) \xrightarrow{F(i_*)} F(Q_*) \xrightarrow{F(\pi_*)} F(P_*^{A''}) \longrightarrow 0$$
of chain complexes over D. Applying homology (and independence of L_* of the chosen projective resolutions), we therefore get a long exact homology sequence (Proposition A.2.2)
$$\cdots \longrightarrow L_{n+1}(A'') \xrightarrow{\partial_{n+1}} L_n(A') \xrightarrow{L_n(i)} L_n(A) \xrightarrow{L_n(\pi)} L_n(A'') \xrightarrow{\partial_n} \cdots$$
We then *define* the connecting morphisms for the given short exact sequence as the connecting morphisms in this long exact sequence.

  What about naturality? The extended horseshoe lemma [87, Theorem 2.4.6] allows to construct compatible projective resolutions for every ladder of short exact sequences. Because the long exact homology sequence is also natural, this naturality carries over to the connecting morphisms (check!).

Hence, L_*, together with these connecting morphisms, defines a homological ∂-functor $C \longrightarrow D$ that extends F.

Therefore, it remains to show universality: Let $T_* \colon C \longrightarrow D$ be a homological ∂-functor and let $\tau_0 \colon T_0 \Longrightarrow F$ be a natural transformation; because $L_0 \cong F$, we can instead also assume that τ_0 is a natural transformation $T_0 \Longrightarrow L_0$ (to simplify notation). We inductively extend τ_0 to a natural transformation of homological ∂-functors from T_* to L_* by dimension shifting (Corollary 3.1.9 below):

Let $n \in \mathbb{N}$ and let us suppose that τ_* is already constructed up to degree n. We now construct τ_{n+1}: Let $A \in \mathrm{Ob}(C)$. Then there exists a short exact sequence
$$0 \longrightarrow K \xrightarrow{\;i\;} P \xrightarrow{\;\pi\;} A \longrightarrow 0$$
in C with a projective P (because C has enough projectives). Because T_* and L_* are homological ∂-functors and because $L_{n+1}(P) \cong_D 0$ (by ②), we obtain the following commutative diagram (the solid arrows)

$$\begin{tikzcd} & T_{n+1}(A) \arrow[r, "\partial^T_{n+1}"] \arrow[d, dashed, "\tau_{n+1}(A)"'] & T_n(K) \arrow[r, "T_n(i)"] \arrow[d, "\tau_n(K)"'] & T_n(P) \arrow[d, "\tau_n(P)"'] \\ 0 \cong_D L_{n+1}(P) \arrow[r] & L_{n+1}(A) \arrow[r, "\partial^L_{n+1}"'] & L_n(K) \arrow[r, "L_n(i)"'] & L_n(P) \end{tikzcd}$$

in D with exact rows. A simple diagram chase shows that there is a unique morphism $\tau_{n+1}(A)$ in D that makes the left-hand square commutative (check!).

One now has to check that $\tau_{n+1}(A)$ indeed is compatible with the connecting morphisms of *all* short exact sequences with A as quotient and that τ_{n+1} is compatible with homomorphisms of modules:

④ The homomoprhism $\tau_{n+1}(A)$ is compatible with the connecting morphisms ∂_{n+1}:

Let $0 \longrightarrow B' \xrightarrow{j} B \xrightarrow{\varphi} A \longrightarrow 0$ be a short exact sequence in C that ends in the given A. As P is projective, a diagram chase reveals that there exist morphisms f and f' in C that fit into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "f'"'] & P \arrow[r, "\pi"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B' \arrow[r, "j"'] & B \arrow[r, "\varphi"'] & A \arrow[r] & 0 \end{tikzcd}$$

in C (check!). We now consider the diagram in Figure 3.1. The large outer square commutes by construction of $\tau_{n+1}(A)$. The small outer four quadrangles commute because T_* and L_* are homological ∂-functors and because τ_n is natural (by induction). Therefore, also the inner square is commutative, as claimed.

⑤ The homomorphisms τ_{n+1} are compatible with morphisms:

Let $f \in \operatorname{Mor}_C(A, B)$ and let

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "\pi"] & A \arrow[r] & 0 \\ 0 \arrow[r] & M \arrow[r, "j"'] & Q \arrow[r, "\varphi"'] & B \arrow[r] & 0 \end{tikzcd}$$

short exact sequences in C with projectives P and Q. Similarly, as above, we can find morphisms p and f' fitting into the commutative diagram

$$\begin{tikzcd}
T_{n+1}(A) \arrow[rrr, "\partial^T_{n+1}"] \arrow[dr, equal] \arrow[ddd, "\tau_{n+1}(A)"'] & & & T_n(K) \arrow[dl, "T_n(f')"'] \arrow[ddd, "\tau_n(K)"] \\
& T_{n+1}(A) \arrow[r, "\partial^T_{n+1}"] \arrow[d, "\tau_{n+1}(A)"'] & T_n(B') \arrow[d, "\tau_n(B')"] & \\
& L_{n+1}(A) \arrow[r, "\partial^L_{n+1}"'] & L_n(B') & \\
L_{n+1}(A) \arrow[rrr, "\partial^L_{n+1}"'] \arrow[ur, equal] & & & L_n(K) \arrow[ul, "L_n(f')"]
\end{tikzcd}$$

Figure 3.1.: Compatibility of $\tau_{n+1}(A)$ with ∂_{n+1}

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "f'"'] & P \arrow[r, "\pi"] \arrow[d, "p"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & M \arrow[r, "j"'] & Q \arrow[r, "\varphi"'] & B \arrow[r] & 0
\end{tikzcd}$$

in C (check!). Considering the diagram in Figure 3.2 and arguing inductively as above, we find that

$$\partial^L_{n+1} \circ \tau_{n+1}(B) \circ T_{n+1}(f) = \partial^L_{n+1} \circ L_{n+1}(f) \circ \tau_{n+1}(A).$$

Because $L_{n+1}(Q) \cong_D 0$, the morphism $\partial^L_{n+1} \colon L_{n+1}(B) \longrightarrow L_n(M)$ is monic, and thus we obtain

$$\tau_{n+1}(B) \circ T_{n+1}(f) = L_{n+1}(f) \circ \tau_{n+1}(A),$$

as desired.

This completes the proof that L_* is a left-derived functor of F. □

During the proof, we also used/established the following property (that allows to express derived functors in higher degree by lower degrees, for the price of changing the objects):

Corollary 3.1.9 (dimension shifting)**.** *Let $F \colon C \longrightarrow D$ be a right-exact functor between module categories and let L_* be a/“the” left-derived functor of F. Let $A \in \mathrm{Ob}(C)$ and let*

$$0 \longrightarrow K \xrightarrow{\;i\;} P \xrightarrow{\;\pi\;} A \longrightarrow 0$$

be a short exact sequence in C with a projective P. Then, for all $n \in \mathbb{N}_{>0}$, the connecting morphism of L_* induces an isomorphism

$$L_{n+1}(A) \cong_D L_n(K).$$

$$\begin{tikzcd}
T_n(K) \arrow[rrr, "T_n(f')"] \arrow[ddd, "\tau_n(K)"'] & & & T_n(M) \arrow[ddd, "\tau_n(M)"] \\
& T_{n+1}(A) \arrow[ul, "\partial^T_{n+1}"'] \arrow[r, "T_{n+1}(f)"] \arrow[d, "\tau_{n+1}(A)"'] & T_{n+1}(B) \arrow[ur, "\partial^T_{n+1}"] \arrow[d, "\tau_{n+1}(B)"] & \\
& L_{n+1}(A) \arrow[dl, "\partial^L_{n+1}"] \arrow[r, "L_{n+1}(f)"'] & L_{n+1}(B) \arrow[dr, "\partial^L_{n+1}"'] & \\
L_n(K) \arrow[rrr, "L_n(f')"'] & & & L_n(M)
\end{tikzcd}$$

Figure 3.2.: Naturality of τ_{n+1}

Moreover,

$$L_1(A) \cong_D \ker\bigl(L_0(i)\colon L_0(K) \to L_0(A)\bigr).$$

Proof. By observation ② of the proof of Theorem 3.1.8, we know that L_n is trivial on projectives, provided that $n \in \mathbb{N}_{>0}$. The claim then follows from the long exact sequence obtained from the given short exact sequence by application of L_*. □

3.1.3 The two sides of Tor

We now apply the previous discussion to the tensor product:

Example 3.1.10 (Tor)**.** derived functor Let R be a ring and let $M \in \mathrm{Ob}(\mathsf{Mod}_R)$. Then "the" derived functor of $M \otimes_R \,\cdot\, : {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is denoted by $\mathrm{Tor}^R_*(M, \,\cdot\,)$. More explicitly: If $A \in \mathrm{Ob}({}_R\mathsf{Mod})$ and (P_*, ε) is a projective R-resolution of A, then, for all $n \in \mathbb{N}$,

$$\mathrm{Tor}^R_n(M, A) \cong_{\mathbb{Z}} H_n(M \otimes_R P_*).$$

Moreover, Tor can also be computed by resolving the first argument:

Theorem 3.1.11 ("symmetry" of Tor)**.** *Let R be a ring and let $M \in \mathrm{Ob}(\mathsf{Mod}_R)$. If (P_*, ε) is a projective R-resolution of M (by right R-modules), then, for all $n \in \mathbb{N}$, there is a natural isomorphism*

$$\mathrm{Tor}^R_n(M, \,\cdot\,) \cong_{\mathbb{Z}} H_n(P_* \otimes_R \,\cdot\,).$$

Proof. We prove this via the universal property of derived functors; therefore, we first introduce an appropriate homological ∂-functor:

- For $n \in \mathbb{N}$, we set $L_n := H_n(P_* \otimes_R \,\cdot\,)\colon {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$.

- If $0 \longrightarrow A' \xrightarrow{\;i\;} A \xrightarrow{\;\pi\;} A'' \longrightarrow 0$ is a short exact sequence of left R-modules, then also the sequence

$$0 \longrightarrow P_* \otimes_R A' \xrightarrow{P_* \otimes_R i} P_* \otimes_R A \xrightarrow{P_* \otimes_R \pi} P_* \otimes_R A'' \longrightarrow 0$$

  in ${}_{\mathbb{Z}}\mathsf{Ch}$ is degree-wise short exact (because all P_n are projective, whence flat over R). Hence, we obtain an associated natural long exact homology sequence

$$\cdots \longrightarrow L_{n+1}(A'') \xrightarrow{\partial_{n+1}} L_n(A') \xrightarrow{L_n(i)} L_n(A) \xrightarrow{L_n(\pi)} L_n(A'') \xrightarrow{\partial_n} \cdots$$

  We take these connecting morphisms as connecting morphisms for our homological ∂-functor.

We now verify that L_* with these connecting morphisms is a left-derived functor of $M \otimes_R \;\cdot\;$:

- Extension of $M \otimes_R \;\cdot\;$: Let $A \in \operatorname{Ob}({}_R\mathsf{Mod})$. Then the right-exactness of $\;\cdot\; \otimes_R A$ shows (as in the proof of Theorem 3.1.8) that

$$\begin{aligned} L_0(A) = H_n(P_* \otimes_R A) &= (P_0 \otimes_R A) \big/ \operatorname{im}(\partial_1^{P_*} \otimes_R A) \\ &= (P_0 \otimes_R A) \big/ \ker(\varepsilon \otimes_R A) \cong_{\mathbb{Z}} (P_0 / \ker \varepsilon) \otimes_R A \\ &\cong_{\mathbb{Z}} M \otimes_R A. \end{aligned}$$

  Clearly, this isomorphism is also natural.

- Universality: We can use the same argument as in the proof of Theorem 3.1.8; indeed, the argument in that proof only used the properties of homological ∂-functors and the vanishing on projectives in higher degree. Therefore, it suffices to establish this vanishing on projectives: If Q is a projective left R-module, then $(P_* \otimes_R Q, \varepsilon \otimes_R Q)$ is exact (because Q is flat over R). In particular, for all $n \in \mathbb{N}_{>0}$,

$$L_n(Q) = H_n(P_* \otimes_R Q) \cong_{\mathbb{Z}} 0.$$

Hence, L_* is a left-derived functor of $M \otimes_R \;\cdot\;$. Therefore, the uniqueness of left-derived functors yields the desired isomorphism. □

3.1.4 Group homology as derived functor

We can interpret group homology as a Tor-functor:

Theorem 3.1.12 (group homology as Tor-functor)**.** *Let G be a group and let $n \in \mathbb{N}$. Then there is a canonical natural isomorphism*

$$H_n(G; \;\cdot\;) \cong_{\mathbb{Z}} \operatorname{Tor}_n^{\mathbb{Z}G}\bigl(\mathbb{Z}, \;\cdot\;\bigr)$$

of functors ${}_{\mathbb{Z}G}\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$. *In particular, for every left* $\mathbb{Z}G$*-module* A, *we have:*

1. *If* (P_*, ε) *is a projective* $\mathbb{Z}G$*-resolution of* A, *then there is a canonical isomorphism*
$$H_n(G;A) \cong_{\mathbb{Z}} H_n(\mathbb{Z} \otimes_G P_*).$$
2. *If* (P_*, ε) *is a projective* $\mathbb{Z}G$*-resolution of* $\mathbb{Z}$, *then there is a canonical isomorphism*
$$H_n(G;A) \cong_{\mathbb{Z}} H_n(P_* \otimes_G A).$$

Proof. We start with the last claim; in fact, we already know this – this is nothing but the fundamental theorem of group homology (Corollary 1.6.9). Then $(\operatorname{Inv} P_*, \operatorname{Inv} \varepsilon)$ is a projective resolution of $\mathbb{Z}$ by right $\mathbb{Z}G$-modules. Hence, Theorem 3.1.11 shows that
$$H_n(G;A) \cong_{\mathbb{Z}} H_n(P_* \otimes_G A) = H_n\bigl((\operatorname{Inv} P_*) \otimes_{\mathbb{Z}G} A\bigr) \cong_{\mathbb{Z}} \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A).$$
Therefore, Example 3.1.10 gives the first description of $H_n(G;A)$. □

Hence, one can characterise group homology concisely as "derived functor of co-invariants". In particular, one could use this description in terms of Tor-functors also to give an axiomatic characterisation of group homology.

Corollary 3.1.13 (derived properties of group homology). *Let* G *be a group.*

1. Vanishing on projectives. *If* A *is a projective (or flat)* $\mathbb{Z}G$*-module and* $n \in \mathbb{N}_{>0}$, *then*
$$H_n(G;A) \cong_{\mathbb{Z}} 0.$$
2. Long exact sequence. *If* $0 \longrightarrow A' \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } A'' \longrightarrow 0$ *is a short exact sequence of* $\mathbb{Z}G$*-modules, then there is an associated (natural) long exact sequence*
$$\cdots \xrightarrow{\partial_{n+1}} H_n(G;A') \xrightarrow{H_n(\mathrm{id}_G;i)} H_n(G;A) \xrightarrow{H_n(\mathrm{id}_G;\pi)} H_n(G;A'') \xrightarrow{\partial_n} H_{n-1}(G;A') \longrightarrow \cdots$$
3. Dimension shifting. *If* $0 \longrightarrow K \xrightarrow{\ i\ } P \xrightarrow{\ \pi\ } A \longrightarrow 0$ *is a short exact sequence of* $\mathbb{Z}G$*-modules and* P *is projective (or flat), then, for each* $n \in \mathbb{N}_{>0}$, *the connecting homomorphism induces an isomorphism*
$$H_{n+1}(G;A) \cong_{\mathbb{Z}} H_n(G;K).$$
Moreover, $H_1(G;A) \cong_{\mathbb{Z}} \ker\bigl(H_0(\mathrm{id}_G;i)\colon H_0(G;K) \to H_0(G;A)\bigr)$.

Proof. By Theorem 3.1.12, $H_*(G;\,\cdot\,) \cong_{\mathbb{Z}} \operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},\,\cdot\,)$. Therefore, the listed properties follow from the corresponding properties of the derived functor $\operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},\,\cdot\,)$ of $\mathbb{Z} \otimes_{\mathbb{Z}G} \,\cdot\,$ (Definition 3.1.5, proof of Theorem 3.1.8, Corollary 3.1.9). □

Example 3.1.14. Let G be a finite group. We consider the Abelian group S^1 as $\mathbb{Z}G$-module with the trivial G-action. Then $S^1 \cong_{\mathsf{Group}} \mathbb{R}/\mathbb{Z}$ and we use the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 0$$

of (trivial) $\mathbb{Z}G$-modules (given by the canonical inclusion and projection). Because G is finite, we know that

$$H_n(G;\mathbb{R}) \cong_{\mathbb{Z}} 0$$

for all $n \in \mathbb{N}_{>0}$ (Corollary 1.7.16). Hence, the long exact sequence in group homology (Corollary 3.1.13) associated with the short exact coefficients sequence above shows that

$$H_n(G;S^1) \cong_{\mathbb{Z}} H_{n-1}(G;\mathbb{Z})$$

for all $n \in \mathbb{N}_{>1}$. In particular, $H_2(G;S^1) \cong_{\mathbb{Z}} H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} G_{\mathrm{ab}}$.

3.1.5 Group cohomology as derived functor

Dually to homological ∂-functors, one can also introduce cohomological δ-functors and define right-derived functors of left-exact functors via injective resolutions.

Example 3.1.15 (Ext)**.** Let R be a ring and let $M \in \mathrm{Ob}({}_R\mathsf{Mod})$. Then "the" right-derived functor of ${}_R\mathrm{Hom}(M,\,\cdot\,)\colon {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is denoted by $\mathrm{Ext}^*_R(M,\,\cdot\,)$. More explicitly: If $A \in \mathrm{Ob}({}_R\mathsf{Mod})$ and (I^*,ε) is an injective R-resolution of A, then, for all $n \in \mathbb{N}$,

$$\mathrm{Ext}^n_R(M,A) \cong_{\mathbb{Z}} H^n\bigl({}_R\mathrm{Hom}(M,I^*)\bigr).$$

Similarly, to the case of Tor, one can also compute Ext by resolving the first variable [87, Chapter 2.7]: If (P_*,ε) is a projective R-resolution of M, then, for all $n \in \mathbb{N}$,

$$\mathrm{Ext}^n_R(M,A) \cong_{\mathbb{Z}} H^n\bigl({}_R\mathrm{Hom}(P_*,A)\bigr).$$

As in the case of group homology, also cohomology can be described in terms of derived functors (check!):

Theorem 3.1.16 (group cohomology as Ext-functor)**.** *Let G be a group and let $n \in \mathbb{N}$. Then there is a canonical natural isomorphism*

$$H^n(G;\,\cdot\,) \cong_{\mathbb{Z}} \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},\,\cdot\,)$$

of functors ${}_{\mathbb{Z}G}\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$. In particular, for every left $\mathbb{Z}G$-module A, we have:

1. *If (I^*, ε) is an injective $\mathbb{Z}G$-resolution of A, then there is a canonical isomorphism*
$$H^n(G; A) \cong_{\mathbb{Z}} H^n\bigl({}_{\mathbb{Z}G}\mathrm{Hom}(\mathbb{Z}, I^*)\bigr).$$
2. *If (P_*, ε) is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then there is a canonical isomorphism*
$$H^n(G; A) \cong_{\mathbb{Z}} H^n\bigl({}_{\mathbb{Z}G}\mathrm{Hom}(P_*, A)\bigr).$$

Thus, group cohomology can be characterised as "derived functor of invariants", which also leads to an axiomatic description of group cohomology. Also, the analogue of Corollary 3.1.13 holds. As mentioned before, this relation between group cohomology and Ext also explains the name Ext: Second group cohomology is related to *ext*ensions (Theorem 1.5.10).

Remark 3.1.17 (duality)**.** The properties of group cohomology are dual to the ones of group homology. One can establish these properties by imitating the homological proof (and making the obvious modifications). Alternatively, one can also pass to the more general setup of homological algebra in Abelian categories and then apply the previous theory to the opposite category.

3.1.6 The derived category

Our definition of derived functors via homological ∂-functors and the construction via resolutions has the advantage of being rather concrete, but also the disadvantage of requiring tedious bookkeeping and working with unruly objects. These issues become even more apparent when one starts working with several derived functors at once or when one wants to transfer the theory of derived functors to other settings.

A remedy is the derived category and the notion of total derived functors. In the following, we will roughly outline this version of homological algebra (for module categories); further details can be found in the literature [87, 46, 69]. The key idea is to postpone/avoid taking (co)homology as long as possible.

Let us recall the computation of the value of the derived functor of a right-exact functor $F\colon C \longrightarrow D$ between module categories on $A \in \mathrm{Ob}(C)$:

- We choose a projective resolution (P_*, ε) of A.
- We apply F to P_*.
- We apply homology.

According to the key idea, we should avoid taking homology in the last step, but work with $F(P_*)$ instead, which leads to the following issues:

- The result $F(P_*)$ will depend more on the choice of P_* than $H_*(F(P_*))$; it is not well-defined up to isomorphism, but only well-defined up to canonical chain homotopy equivalence.

- We might strive for a uniform setup for both objects in C as well as resolutions of objects in C. We can view A as chain complex concentrated in degree 0 and $\varepsilon\colon P_* \longrightarrow A$ as a chain map; however, ε, in general, will *not* be a chain homotopy equivalence, but only a chain map that induces an isomorphism on the level of homology.

Therefore, the derived category of C will be defined as the category constructed out of the category of chain complexes over C by turning all homology isomorphisms into isomorphisms. More precisely, one proceeds as follows:

Definition 3.1.18 (quasi-isomorphism)**.** Let R be a ring and let C_* and D_* be chain complexes of left R-modules. A *quasi-isomorphism* $C_* \longrightarrow D_*$ is an R-chain map $f_*\colon C_* \longrightarrow D_*$ such that for all $n \in \mathbb{N}$, the induced homomorphism

$$H_n(f_*)\colon H_n(C_*) \longrightarrow H_n(D_*).$$

is an R-isomorphism.

Caveat 3.1.19 (quasi-isomorphisms need not be invertible)**.** There exist chain complexes C_* and D_* over the ring $\mathbb{Z}$ such that there exists a quasi-isomorphism $C_* \longrightarrow D_*$ but *no* quasi-isomorphism $D_* \longrightarrow C_*$ (Exercise).

Example 3.1.20 (projective resolutions)**.** Let R be a ring, let A be an R-module and let (P_*, ε) be a projective resolution of A over R. If we view A as an R-chain complex concentrated in degree 0, then ε (in degree 0; together with the 0-morphisms in all other degrees) is a quasi-isomorphism $P_* \longrightarrow A$ (check!):

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
P_1 \arrow[d, "\partial_1"'] & 0 \arrow[d, "0"] \\
P_0 \arrow[r, "\varepsilon"'] & A
\end{tikzcd}$$

For chain complexes of projective modules, quasi-isomorphisms are tame:

Theorem 3.1.21 (quasi-isomorphisms between projective chain complexes)**.** *Let R be a ring, let C_* and D_* be ($\mathbb{N}$-indexed) chain complexes over R that consist of projective modules. Every quasi-isomorphism $C_* \longrightarrow D_*$ is a chain homotopy equivalence.*

Proof. Using mapping cones (Definition 3.1.22, Proposition 3.1.23), one only needs to consider the case that one of the two complexes is trivial (Exercise). In this case, the theorem can be deduced from the fundamental theorem of homological algebra (Theorem 1.6.7) (Exercise). □

Definition 3.1.22 (mapping cone)**.** Let R be a ring and let $f_*\colon C_* \longrightarrow D_*$ be a chain map of R-chain complexes. The *mapping cone of* f_* is the R-chain complex $\operatorname{Cone}_*(f_*)$ consisting of the chain modules

$$\operatorname{Cone}_n(f_*) := C_{n-1} \oplus D_n$$

for all $n \in \mathbb{N}$ (where $C_{-1} := 0$), equipped with the boundary operators

$$\begin{aligned}\partial_n\colon \operatorname{Cone}_n(f_*) &\longrightarrow \operatorname{Cone}_{n-1}(f_*)\\ (x,y) &\longmapsto \bigl(-\partial^C_{n-1}(x), \partial^D_n(y) - f_{n-1}(x)\bigr)\end{aligned}$$

for all $n \in \mathbb{N}_{>0}$.

Proposition 3.1.23 (mapping cone trick)**.** *Let R be a ring, let $f_*\colon C_* \longrightarrow D_*$ be a chain map of R-chain complexes. The the following are equivalent:*

1. *The chain map $f_*\colon C_* \longrightarrow D_*$ is a quasi-isomorphism.*
2. *For all $n \in \mathbb{N}$, we have $H_n\bigl(\operatorname{Cone}(f_*)\bigr) \cong_R 0$.*

Proof. This can be extracted from a suitable long exact homology sequence (Exercise). □

Study note (mapping cones)**.** For the boundary operator on $\operatorname{Cone}(f_*)$, several different sign conventions are in use. Therefore, literature has to be used with care! Of course, the mapping cone of chain maps is an algebraic imitation of the topological mapping cone; however, the algebraic version has slightly "better" properties (Exercise).

The derived category is now defined as the localisation of the category of (bounded below) chain complexes at the class of all quasi-isomorphisms.

We first formalise this localisation via a universal property (similar to the universal property of localisations of (non-commutative) rings); we will then briefly explain why the localisation at the class of quasi-isomorphisms does exist.

Definition 3.1.24 (localisation)**.** Let C be a category and let S be a class of morphisms of C. A *localisation of C at S* is a functor $q\colon C \longrightarrow D$ to a category D with the following universal property:

- For each $f \in S$, the morphism $q(f)$ is an isomorphism in D.
- If E is a category and $F\colon C \longrightarrow E$ is a functor with the property that $F(s)$ is an isomorphism for all $s \in S$, then there exists a unique functor $\widetilde{F}\colon D \longrightarrow E$ that satisfies

$$\widetilde{F} \circ q = F.$$

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "q"'] & E \\ D \arrow[ur, dashed, "\widetilde{F}"'] & \end{tikzcd}$$

(If D and q exist, then they are unique up to natural equivalence, and the category D is usually denoted by $C[S^{-1}]$ or $S^{-1}C$.)

Example 3.1.25 (the homotopy category as localisation)**.** Let R be a ring. Then the homotopy category of ${}_R\mathsf{Ch}$ is the category ${}_R\mathsf{Ch}_\mathsf{h}$ defined as follows:

- objects: We set $\mathrm{Ob}({}_R\mathsf{Ch}_\mathsf{h}) := \mathrm{Ob}({}_R\mathsf{Ch})$.
- morphisms: For all $C_*, D_* \in \mathrm{Ob}({}_R\mathsf{Ch})$, we set
$$\mathrm{Mor}_{{}_R\mathsf{Ch}_\mathsf{h}}(C_*, D_*) := \mathrm{Mor}_{{}_R\mathsf{Ch}}(C_*, D_*) \ /{\simeq_R} \ .$$
- compositions: Induced by ordinary composition of representatives.

Together with the canonical functor ${}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Ch}_\mathsf{h}$, the homotopy category is a localisation of ${}_R\mathsf{Ch}$ at the class of all R-chain homotopy equivalences; this can be proved using mapping cylinders [87, Proposition 10.1.2].

In general, localisation categories do *not* exist and (if they exist) might be hard to compute (for various reasons).

Theorem 3.1.26 (the derived category)**.** *Let R be a ring. Then there exists a localisation $\mathsf{D}(R)$ of ${}_R\mathsf{Ch}$ at the class of all quasi-isomorphisms, the* derived category of ${}_R\mathsf{Mod}$. *More concretely, $\mathsf{D}(R)$ can be constructed as a localisation of ${}_R\mathsf{Ch}_h$ at the class of all quasi-isomorphisms, which in turn can be constructed by the following calculus of fractions:*

- objects: *Let* $\mathrm{Ob}(\mathsf{D}(R)) := \mathrm{Ob}({}_R\mathsf{Ch}_h) = \mathrm{Ob}({}_R\mathsf{Ch})$.
- morphisms $X \longrightarrow Y$: *Equivalence classes of fractions, i.e., of morphisms* $X \xleftarrow{s} X' \longrightarrow Y$ *in* ${}_R\mathsf{Ch}_h$ *with a quasi-isomorphism s. Two such fractions* $X \xleftarrow{s} X' \longrightarrow Y$ *and* $X \xleftarrow{t} X'' \longrightarrow Y$ *are equivalent if there exists a fraction* $X \xleftarrow{u} X''' \longrightarrow Y$ *that fits into a commutative diagram in* ${}_R\mathsf{Ch}_h$ *of the form*

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr] & \\ X & X''' \arrow[l, "u"'] \arrow[u] \arrow[d] \arrow[r] & Y \\ & X'' \arrow[ul, "t"] \arrow[ur] & \end{tikzcd}$$

- compositions: *If* $F\colon X \longrightarrow Y$ *and* $G\colon Y \longrightarrow Z$ *are morphisms represented by fractions* $X \xleftarrow{s} X' \xrightarrow{f} Y$ *and* $Y \xleftarrow{t} Y' \xrightarrow{g} Z$ *, respectively, then the composition* $g \circ f$ *is represented by the fraction*

$$X \xleftarrow{s \circ s'} X'' \xrightarrow{g \circ f'} Z$$

where $s'\colon X'' \longrightarrow X'$ *is a quasi-isomorphism and* $f'\colon X'' \longrightarrow Y'$ *is a chain map that fit into the commutative square (in* ${}_R\mathsf{Ch}_h$*)*

$$\begin{tikzcd} X'' \arrow[r, "f'"] \arrow[d, "s'"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"'] & Y \end{tikzcd}$$

The localisation functor $q_R\colon {}_R\mathsf{Ch} \longrightarrow \mathsf{D}(R)$ *is the identity on objects and turns morphisms* $f\colon X \longrightarrow Y$ *into the corresponding fraction*

$$X \xleftarrow{\mathrm{id}_X} X \xrightarrow{[f]_\simeq} Y \ .$$

Sketch of proof. The main technical problem is the following:

> When naively constructing $\mathsf{D}(R)$ by generators and relations (which enforce the existence of inverses for every quasi-isomorphism) it is not clear a priori that the morphisms between two objects will form a *set* (and not only a class).

This problem can be resolved by a careful examination of the class of quasi-isomorphisms in ${}_R\mathsf{Ch}_\mathsf{h}$ and a crude cardinality argument [87, Chapter 10.4].

Moreover, it should be noted that we need to work in the homotopy category ${}_R\mathsf{Ch}_\mathsf{h}$ for the calculus of fractions to work (e.g., for the composition, which involves a category-theoretic version of the Ore condition); in ${}_R\mathsf{Ch}$, the situation would be more involved.

Then, the proof is mainly a matter of computation [87, Chapter 10.3], analogously to the construction of localisations of non-commutative rings. □

In general, concrete computations in derived categories are hard. However, in our simple case (of derived categories of module categories), one can just compute in the homotopy category of (bounded below) chain complexes of projective modules [87, Theorem 10.4.8]. This again underlines the importance of projectivity.

We will now come to derived functors: The derived category, in general, will not be an Abelian category; therefore, more general terminology is needed to formulate exactness properties, e.g., the language of triangulated categories [87, Chapter 10.2] (we will ignore this here and we will also ignore the technicality that we should work with bounded below chain complexes instead of $\mathbb{N}$-indexed chain complexes).

If $F\colon {}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$ is an exact functor, then it is not difficult to see that there is a corresponding functor $\mathsf{D}(F)\colon \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ (of triangulated categories) that satisfies

$$\mathsf{D}(F) \circ q_R = q_S \circ F,$$

where $q_R\colon {}_R\mathsf{Ch} \longrightarrow \mathsf{D}(R)$, $q_S\colon {}_S\mathsf{Ch} \longrightarrow \mathsf{D}(S)$ are the localisation functors. However, if F is not exact, this will not be possible in general. For right-exact functors, total left-derived functors are "optimal" extensions to the derived categories in the following sense (where the compatibility with the triangulated structure encodes the exactness properties):

Definition 3.1.27 (total derived functor)**.** Let R and S be rings and let $F\colon {}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$ be a right-exact functor. Then a *total left-derived functor of F* is a functor $\mathsf{L}\,F\colon \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ (of triangulated categories) together with a natural transformation $\tau\colon (\mathsf{L}\,F) \circ q_R \Longrightarrow q_S \circ F$ with the following universal property:

If $G\colon \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ is a functor (of triangulated categories) and if $\sigma\colon G \circ q_R \Longrightarrow q_S \circ F$ is a natural transformation, then there exists a unique natural transformation $\widetilde{\sigma}\colon G \Longrightarrow \mathsf{L}\,F$ such that

$$\forall_{A \in \operatorname{Ob}({}_R\mathsf{Ch})} \quad \tau(A) \circ \widetilde{\sigma}\bigl(q_R(A)\bigr) = \sigma(A).$$

$$\begin{tikzcd} \mathsf{D}(R) \arrow[r, "\mathsf{L}\,F"] & \mathsf{D}(S) \\ {}_R\mathsf{Ch} \arrow[u, "q_R"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow\tau"] & {}_S\mathsf{Ch} \arrow[u, "q_S"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D}(R) \arrow[r, "G"] & \mathsf{D}(S) \\ {}_R\mathsf{Ch} \arrow[u, "q_R"] \arrow[r, "F"'] \arrow[ur, phantom, "\sigma"] & {}_S\mathsf{Ch} \arrow[u, "q_S"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D}(R) \arrow[r, "G"] \arrow[dr, phantom, "\widetilde{\sigma}"] & \mathsf{D}(S) \\ {}_R\mathsf{Ch} \arrow[u, "q_R"] \arrow[r, "q_R"'] & \mathsf{D}(R) \arrow[u, "F"'] \end{tikzcd}$$

Here, $q_R\colon {}_R\mathsf{Ch} \longrightarrow \mathsf{D}(R)$ and $q_S\colon {}_S\mathsf{Ch} \longrightarrow \mathsf{D}(S)$ denote the localisation functors.

Study note. Compare this definition with the definition of ordinary left-derived functors! Where did the data/properties shift to in the totally derived setting? Moreover, it might be interesting to look up the notion of a *Kan extension* and to compare it to the definition of total derived functors.

Theorem 3.1.28 (derived functor via total derived functor [87, Corollary 10.5.7, Remark 10.5.8])**.** *Let R and S be rings and let $F\colon {}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$ be a right-exact functor. Then a/"the" total derived functor $\mathsf{L}\,F\colon \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ exists and we have a canonical natural isomorphism (as functors ${}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$)*

$$L_*(F) \cong_S H_*\bigl(\mathsf{L}\,F(q_r(\,\cdot\,))\bigr).$$

(One should note that taking homology H_* is well-defined on $\mathsf{D}(S)$.)

What is the benefit of the total derived functor?

- Composition and combination formulas for several derived functors are simpler on the total derived level than on the level of homology modules [87, Chapter 10.8].

  For example, it is easier to compare $\mathsf{L}(G) \circ \mathsf{L}(F)$ with $\mathsf{L}(G \circ F)$ on the derived categories than to compare $L_*(G) \circ L_*(F)$ (whatever that even means) with $L_*(G \circ F)$ (because in the latter case we would have to unravel the intermediate homology first; Theorem 3.2.11).

- The formalism of derived categories and total derived functors covers a wide range of situations (also in algebraic/arithmetic geometry and homotopy theory).

In order to enjoy the full power of derived categories and derived functors one has to pass to a more general setup (e.g., triangulated categories or model categories) [87, Chapter 10.9][69]. Then, in particular, also the close connection with homotopy theory of topological spaces becomes visible:

Remark 3.1.29 (a derived category in topology)**.** We have the following rough dictionary between homological algebra and homotopy theory of spaces:

algebra	*topology*
module	topological space
chain complex	spectrum
chain homotopy	homotopy of maps of spectra
quasi-isomorphism	weak equivalence
projective resolution	CW-spectrum

The construction of the topological derived category (as localisation at weak equivalences) can then be performed in analogy with the algebraic case [87, Chapter 10.9].

3.2 The Hochschild-Serre spectral sequence

Bookkeeping in homological algebra can quickly get overwhelming when more than a single complex is involved, for example, when taking multi-step filtrations of chain complexes or double complexes. A convenient computational tool is provided by spectral sequences. We will explain the setup of spectral sequences and how to perform computations with them. However, we will refrain from *proving* any convergence results.

Why do we need any of this in group (co)homology? For example, when trying to compute the (co)homology of an extension group in terms of the (co)homology of the kernel and the quotient, our techniques so far are not sufficient to give a satisfactory answer. This problem will be solved by the Hochschild-Serre spectral sequence (Chapter 3.2.3).

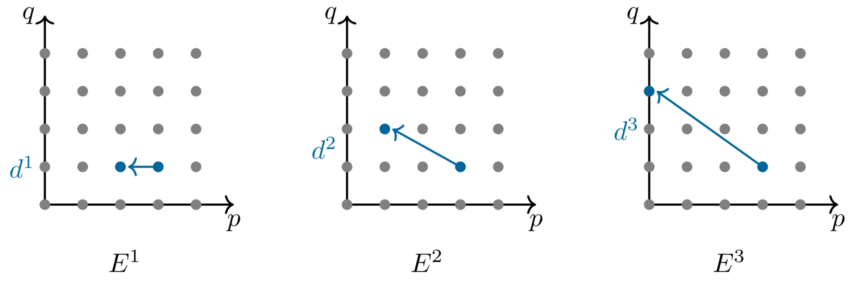


Figure 3.3.: pages of a homological spectral sequence

3.2.1 Terminology for spectral sequences

We will now explain the principle of spectral sequences. In the beginning, spectral sequences and all the notation might seem frightening and very technical; however, with a little bit of practice one will sooner or later appreciate their power and the challenge of tricking spectral sequences into revealing their secrets.

We restrict our discussion to spectral sequences in the first quadrant. Extensive treatments of spectral sequences (including proofs) can for instance be found in the books by Weibel [87], Hatcher [40], McCleary [58], and in the lecture notes of Bauer [2].

A spectral sequence is a sequence of bigraded modules, where the next bigraded module is obtained from the previous one by taking homology (see also Figure 3.3 for an illustration):

Definition 3.2.1 (homological spectral sequence)**.** A *(bigraded, homological) spectral sequence* over a ring R is a sequence $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ (or $r \in \mathbb{N}_{>1}$) of bigraded R-modules (i.e., every E^r is family $(E^r_{pq})_{p,q \in \mathbb{N}}$ of R-modules) and R-homomorphisms $d^r \colon E^r \longrightarrow E^r$ with the following properties:

- For every $r \in \mathbb{N}_{>0}$ the map d^r has degree $(-r, r-1)$, and $d^r \circ d^r = 0$.
- For every $r \in \mathbb{N}_{>0}$ there is an isomorphism
$$E^{r+1} \cong_R H_*(E^r, d^r) = \frac{\ker d^r}{\operatorname{im} d^r},$$
and this isomorphism is also part of the data of the spectral sequence.

The term E^r is also called the *r-th page* of (E^*, d^*) and the isomorphism $E^{r+1} \cong_R H_*(E^r, d^r)$ is the *r-th page-turning isomorphism.*

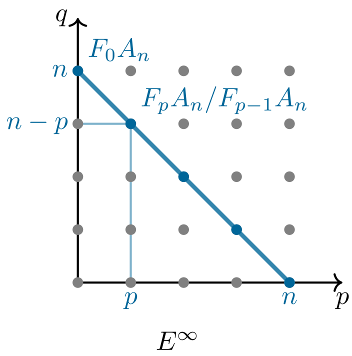


Figure 3.4.: convergence of a homological spectral sequence, schematically

Definition 3.2.2 (∞-page of a spectral sequence)**.** Let $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ be a homological spectral sequence over a ring R. Because all the (E^r, d^r) reside in the first quadrant, for every p, $q \in \mathbb{N}$ there exists an $s \in \mathbb{N}_{>0}$ such that the differentials starting and ending at (p, q) are trivial in all later pages, i.e., such that the page-turning isomorphisms induce isomorphisms

$$E^s_{pq} \cong_R E^{s+1}_{pq} \cong_R E^{s+2}_{pq} \cong_R \cdots ;$$

we then define $E^\infty_{pq} := E^s_{pq}$ (for the minimal such s).

Definition 3.2.3 (degeneration of a spectral sequence)**.** Let $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ be a homological spectral sequence over a ring R and let $s \in \mathbb{N}_{>0}$. We say that this spectral sequence *degenerates at stage* s if, for all $r \in \mathbb{N}_{\geq s}$, we have $d^r = 0$. In particular, the page-turning isomorphisms induce isomorphisms

$$E^s \cong_R E^{s+1} \cong_R E^{s+2} \cong_R \ldots \quad \text{and} \quad E^s \cong_R E^\infty.$$

Until now, nothing happened – we merely introduced some notation. The next definition is crucial for the applications of spectral sequences; it relates a spectral sequence to something we want to compute:

Definition 3.2.4 (convergence of a spectral sequence)**.** Let R be a ring, let A be an $\mathbb{N}$-graded R-module, and let $(F_n A)_{n \in \mathbb{N}}$ be an increasing filtration of A that is compatible with the grading of A. We say that a spectral sequence $(E^r, d^r)_r$ over R *converges to* A if the following conditions are satisfied (see also Figure 3.4):

- For all $p, q \in \mathbb{N}$ we have (with $F_{-1} A := 0$)

$$E^\infty_{pq} \cong_R \frac{F_p A_{p+q}}{F_{p-1} A_{p+q}}.$$

$$0 \longrightarrow F_0 A_n \longrightarrow F_1 A_n \longrightarrow \frac{F_1 A_n}{F_0 A_n} \longrightarrow 0$$
$$0 \longrightarrow F_1 A_n \longrightarrow F_2 A_n \longrightarrow \frac{F_2 A_n}{F_1 A_n} \longrightarrow 0$$
$$\vdots$$
$$0 \longrightarrow F_{n-1} A_n \longrightarrow F_n A_n = A_n \longrightarrow \frac{F_n A_n}{F_{n-1} A_n} \longrightarrow 0$$

Figure 3.5.: convergence of a homological spectral sequence, extensions

- The spectral sequence is exhaustive, i.e., $F_n A_n = A_n$ for all $n \in \mathbb{N}$.

In this case one writes

$$E^2_{pq} \Longrightarrow A_{p+q} \quad \text{or} \quad E^1_{pq} \Longrightarrow A_{p+q}.$$

Remark 3.2.5 (stepping through a spectral sequence)**.** What is the typical "usage" of a spectral sequence? We might be interested in some graded object A (in most cases: homology of something) for which there happens to exist a (homological) spectral sequence $(E^r, d^r)_{r \in \mathbb{N}_{>1}}$ converging to A, where the E^2-term is something accessible:

$$E^2_{pq} \Longrightarrow A_{p+q}.$$

Usually, one then proceeds as follows:

1. Try to compute as many of the modules of the E^2-term as possible; in general, the more zeroes, the better!

2. Try to prove that many of the differentials d^2_{pq} in the E^2-term are zero, e.g., using the degree, torsion phenomena, product structures, ...

3. Using the results of the first two steps, try to compute as much of the E^3-term and the differential d^3 as possible.

   Fortunately, many spectral sequences degenerate (at least to a large extent) at the E^2-stage or the E^3-stage!

4. Carry on like that and try to compute as much of the E^∞-term as possible.

5. Try to solve the extension problems arising when reconstructing A out of E^∞.

Caveat 3.2.6. If a spectral sequence $(E^r, d^r)_{r\in\mathbb{N}_{>0}}$ converges to a graded filtred module A, and if we know this spectral sequence, then this does not necessarily mean that we can actually compute A; we only obtain the quotients $F_*A/F_{*-1}A$ of the associated filtration (as depicted in Figure 3.4)! I.e., we still have to solve a sequence of extension problems (Figure 3.5).

In most cases, one is *not* able to determine the differentials $(d^r)_{r\in\mathbb{N}_{>0}}$ explicitly; however, the degrees of these differentials already tell a lot about the spectral sequence and its long-term development, and additional external input might provide enough information to extract non-trivial conclusions out of a spectral sequence.

In a way, spectral sequences behave more like puzzles than like deterministic processes. We will explain some of the basic techniques for handling spectral sequences below (Chapter 3.2.3).

Dually to the concept of homological spectral sequences there is also a notion of cohomological spectral sequences:

Definition 3.2.7 (cohomological spectral sequence)**.** A *(bigraded) cohomological spectral sequence* over a ring R is a sequence $(E_r, d_r)_{r\in\mathbb{N}_{>0}}$ of bigraded R-modules E_r and R-homomorphisms $d_r\colon E_r \longrightarrow E_r$ with the following properties:

- For every $r \in \mathbb{N}_{>0}$ the map d_r has degree $(r, -r+1)$, and $d_r \circ d_r = 0$.
- For every $r \in \mathbb{N}_{>0}$ there is an isomorphism

$$E_{r+1} \cong_R H_*(E_r, d_r) = \frac{\ker d_r}{\operatorname{im} d_r},$$

  and this isomorphism is also part of the data of the spectral sequence.

Similar to the homological case, the ∞-page and degeneration are defined for cohomological spectral sequences.

Definition 3.2.8 (convergence of a cohomological spectral sequence)**.** Let R be a ring, let A be an $\mathbb{N}$-graded R-module, let A be an $\mathbb{N}$-graded R-module, and let $(F_nA)_{n\in\mathbb{N}}$ be an decreasing filtration of A that is compatible with the grading of A. We say that a cohomological spectral sequence $(E_r, d_r)_{r\in\mathbb{N}_{>0}}$ *converges to* A if the following conditions are satisfied:

- For all $p, q \in \mathbb{N}$ we have

$$E_\infty^{pq} \cong_R \frac{F_pA_{p+q}}{F_{p+1}A_{p+q}}$$

- The filtration F_*A is exhaustive and Hausdorff, i.e., $F_0A = A$ and $F_{n+1}A = 0$.

3.2.2 Classical spectral sequences

Most spectral sequences are based on the following classical prototypes. It all starts with the big brother of the long exact homology sequence associated with a short exact sequence of chain complexes (Proposition A.2.2):

Theorem 3.2.9 (filtration spectral sequence [87, Chapter 5.4]). *Let R be a ring, let C_* be an ($\mathbb{N}$-indexed) R-chain complex, and let $(F_pC)_{p\in\mathbb{N}\cup\{-1\}}$ be a canonically bounded filtration of C_*, i.e.,*

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \cdots \subset C_*$$

is a nested chain of subcomplexes of C_* with $(F_nC)_n = C_n$ for all $n \in \mathbb{N}$. Then there is a natural converging homological spectral sequence $(E^r, d^r)_{r\in\mathbb{N}_{\geq 1}}$ with

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Longrightarrow H_{p+q}(C_*).$$

For many applications it is not relevant to know the corresponding filtration on $H_*(C_*)$ or the construction of the differentials – existence and naturality is often enough!

Applying the filtration spectral sequence to the horizontal/vertical filtration of a double complex leads to the two double complex spectral sequences:

Theorem 3.2.10 (double complex spectral sequences [87, Theorem 5.5.1]). *Let R be a ring and let C_{**} be an ($\mathbb{N}\times\mathbb{N}$-indexed) R-double complex, i.e., a family $(C_{pq})_{p,q\in\mathbb{N}}$ of R-modules together with R-homomorphisms $(\partial^h_{p,q} \colon C_{p,q} \to C_{p-1,q})_{p,q\in\mathbb{N}}$ and $(\partial^v_{p,q} \colon C_{p,q} \to C_{p,q-1})_{p,q\in\mathbb{N}}$ satisfying*

$$\begin{aligned} \partial^v \circ \partial^v &= 0 && \textit{(columns are chain complexes)}\\ \partial^h \circ \partial^h &= 0 && \textit{(rows are chain complexes)}\\ \partial^v \circ \partial^h + \partial^h \circ \partial^v &= 0 && \textit{(compatbility of rows/columns)} \end{aligned}$$

Then there exist two natural converging homological spectral sequences:

$$\begin{aligned} {}^{①}E^1_{pq} &= H_q(C_{p,*}, \partial^v_{p,*}) \Longrightarrow H_{p+q}(\operatorname{Tot} C)\\ {}^{②}E^1_{pq} &= H_q(C_{*,p}, \partial^h_{*,p}) \Longrightarrow H_{p+q}(\operatorname{Tot} C), \end{aligned}$$

where $\operatorname{Tot} C$ is the total chain complex (check!) associated with the double complex C_{**}: For $n \in \mathbb{N}$, we have $(\operatorname{Tot} C)_n := \bigoplus_{j=0}^n C_{j,n-j}$ and the boundary operator

$$\begin{aligned} (\operatorname{Tot} C)_n &\longrightarrow (\operatorname{Tot} C)_{n-1}\\ x &\longmapsto \partial^v(x) + \partial^h(x). \end{aligned}$$

These double complex spectral sequences in turn can be applied to resolutions of chain complexes (leading to double complexes), which results in a spectral sequence for the derived functor of a composition:

Theorem 3.2.11 (Grothendieck spectral sequence [87, Corollary 5.8.4]). *Let $F\colon C \longrightarrow D$ and $G\colon D \longrightarrow E$ be right-exact functors between module categories. Moreover, we assume that F is G-acyclic, i.e., that for every projective $P \in \mathrm{Ob}(C)$ and every $n \in \mathbb{N}_{>0}$, we have $L_n G(F(P)) \cong_E 0$. Then there is a converging homological spectral sequence*

$$E^2_{pq} = (L_p G)\big(L_q F(A)\big) \Longrightarrow L_{p+q}(G \circ F)(A).$$

In the situation of Theorem 3.2.11, the corresponding formula for total derived functors on the derived categories is much simpler, namely [87, Theorem 10.8.2]

$$\mathsf{L}(G \circ F) \cong \mathsf{L}(G) \circ \mathsf{L}(F).$$

3.2.3 The spectral sequence of a group extension

Group (co)homology can be viewed as the derived functor of the (co)invariants functor. If $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ is a short exact sequence of groups, then

$$\begin{aligned} G\text{-invariants functor} &= (Q\text{-invariants functor}) \circ (N\text{-invariants functor})\\ G\text{-coinvariants functor} &= (Q\text{-coinvariants functor}) \circ (N\text{-coinvariants functor}) \end{aligned}$$

The Grothendieck spectral sequence (Theorem 3.2.11) then translates into the following spectral sequence (Figure 3.6):

Theorem 3.2.12 (Hochschild-Serre spectral sequence [87, Theorem 6.8.2]). *Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of groups and let A be a $\mathbb{Z}G$-module.*

1. *Then there is a natural converging homological spectral sequence*

$$E^2_{pq} = H_p\big(Q; H_q(N; \mathrm{Res}^G_N A)\big) \Longrightarrow H_{p+q}(G; A);$$

   here, $Q \cong_{\mathsf{Group}} G/N$ acts on the coefficients $H_*(N; \mathrm{Res}^G_N A)$ as described in Proposition 3.2.13 below.

2. *Similarly, there is a natural converging cohomological spectral sequence*

$$E_2^{pq} = H^p\big(Q; H^q(N; \mathrm{Res}^G_N A)\big) \Longrightarrow H^{p+q}(G; A).$$

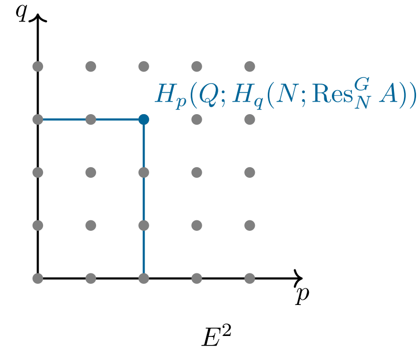


Figure 3.6.: the Hochschild-Serre spectral sequence, schematically

Proposition 3.2.13 (conjugation action on (co)homology)**.** *Let G be a group, let $N \subset G$ be a normal subgroup, let $g \in G$, and let A be a $\mathbb{Z}G$-module. Moreover, let*

$$\begin{aligned} c(g) &:= (x \mapsto g \cdot x \cdot g^{-1},\ x \mapsto g \cdot x) \\ &\in \operatorname{Mor}_{\mathsf{GroupMod}}\bigl((N, \operatorname{Res}_N^G A), (N, \operatorname{Res}_N^G A)\bigr) \\ c^*(g) &:= (x \mapsto g^{-1} \cdot x \cdot g,\ x \mapsto g \cdot x) \\ &\in \operatorname{Mor}_{\mathsf{GroupMod}^*}\bigl((N, \operatorname{Res}_N^G A), (N, \operatorname{Res}_N^G A)\bigr) \end{aligned}$$

be the associated conjugation morphisms in GroupMod *and* GroupMod**, respectively (check!).*

1. *Then, for all $k \in \mathbb{N}$ and all $n \in N$,*

$$\begin{aligned} H_k\bigl(c(n)\bigr) &= \operatorname{id}_{H_k(N; \operatorname{Res}_N^G A)} \\ H^k\bigl(c^*(n)\bigr) &= \operatorname{id}_{H^j(N; \operatorname{Res}_N^G A)} . \end{aligned}$$

2. *In particular: If $Q := G/N$ is the corresponding quotient, then conjugation of G on N induces a well-defined action of Q on $H_*(N; \operatorname{Res}_N^G(A))$ via*

$$\begin{aligned} Q \times H_k(N; \operatorname{Res}_N^G A) &\longrightarrow H_k(N; \operatorname{Res}_N^G A) \\ (g \cdot N, \alpha) &\longmapsto H_k\bigl(c(g)\bigr)(\alpha) \end{aligned}$$

   and similarly on $H^*(N; \operatorname{Res}_N^G(A))$.

Proof. This can, for instance, be proved via the fundamental theorem of group (co)homology (Corollary 1.6.9) (Exercise). $\square$

Clearly, on $H_1(N;\mathbb{Z})$ (with the trivial N-action on $\mathbb{Z}$), this action is just the conjugation action of G on N_{ab}.

Remark 3.2.14 (naturality of the Hochschild-Serre spectral sequence)**.** The Hochschild-Serre spectral sequence is natural in the following sense: For simplicity, let A be a $\mathbb{Z}$-module on which all of the following groups act trivially. If

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & G \arrow[r] \arrow[d, "g"'] & Q \arrow[r] \arrow[d, "h"] & 1 \\ 1 \arrow[r] & N' \arrow[r] & G' \arrow[r] & Q' \arrow[r] & 1 \end{tikzcd}$$

is a commutative diagram of groups with exact rows, then the corresponding induced homomorphisms on homology fit together to form a morphism

$$\begin{tikzcd} E^2_{pq} = H_p\bigl(Q; H_q(N;A)\bigr) \arrow[r, Rightarrow] \arrow[d, "H_p(h;H_q(f;A))"'] & H_{p+q}(G;A) \arrow[d, "H_{p+q}(g;A)"] \\ E^2_{pq} = H_p\bigl(Q'; H_q(N';A)\bigr) \arrow[r, Rightarrow] & H_{p+q}(G';A) \end{tikzcd}$$

of spectral sequences (i.e., homomorphisms between the corresponding pages of the spectral sequences that are compatible with the differentials, and such that the map between the $(r+1)$-st pages is the map induced on homology by the map between the r-th pages).

Similarly, the cohomological Hochschild-Serre spectral sequence is natural with respect to such morphisms.

Outlook 3.2.15 (multiplicativity of the Hochschild-Serre spectral sequence)**.** Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of groups. If A is a $\mathbb{Z}G$-algebra, then the cohomological Hochschild-Serre spectral sequence is multiplicative in the following sense: All pages $(E_r)_{r\in\mathbb{N}_{\geq 2}}$ carry the structure of a differential graded algebra (i.e., they are equipped with a graded commutative product such that the differentials satisfy the Leibniz rule) such that

- the product on the E_2-term coincides with the so-called cup-product on the cohomology $H^*(Q; H^*(N; \operatorname{Res}^G_N A))$ (Outlook 3.2.28), and
- such that the filtration on $H^*(G;A)$ induced by this spectral sequence is also compatible with the cup-product on $H^*(G;A)$.

This additional, multiplicative, information is useful in computations.

We give some simple examples to illustrate basic techniques in spectral sequence computations:

$$
\begin{array}{c|cccccc}
q & & & & & & \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \mathbb{Z}/3 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 \quad p
\end{array}
$$

$$E^2 = E^\infty$$

Figure 3.7.: the Hochschild-Serre spectral sequence for S_3

Degeneration at the E^2-stage, trivial extension problems. We start illustrating the use of the Hochschild-Serre spectral sequence by computing the homology of the symmetric group S_3:

Example 3.2.16 (the symmetric group S_3)**.** The symmetric group S_3 fits into a group extension

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/2$ acts on the kernel $\mathbb{Z}/3$ by taking inverses. The Hochschild-Serre spectral sequence then gives us:

$$E^2_{pq} = H_p\bigl(\mathbb{Z}/2; H_q(\mathbb{Z}/3;\mathbb{Z})\bigr) \Longrightarrow H_{p+q}(S_3;\mathbb{Z}),$$

where $\mathbb{Z}/2$ acts on the coefficients $H_*(\mathbb{Z}/3;\mathbb{Z})$ by the maps induced by taking inverses on $\mathbb{Z}/3$; i.e., for $k \in \mathbb{N}$, the group $\mathbb{Z}/2$ acts by multiplication by $(-1)^{k+1}$ on $H_{2k+1}(\mathbb{Z}/3;\mathbb{Z}) \cong \mathbb{Z}/3$ (Exercise).

1. How does the E^2-term look like? The description of the $\mathbb{Z}/2$-action on the homology of $\mathbb{Z}/3$ gives the vertical axis of the E^2-term of the Hochschild-Serre spectral sequence (recall that zeroth homology is given by taking coinvariants; Theorem 1.3.1).

   Of course, the horizontal axis is nothing but $H_*(\mathbb{Z}/2;\mathbb{Z})$. In view of the torsion results provided by the transfer (Corollary 1.7.16), we obtain

   $$E^2_{pq} = H_p\bigl(\mathbb{Z}/2; H_q(\mathbb{Z}/3;\mathbb{Z})\bigr) \cong_{\mathbb{Z}} 0$$

   for all p, $q \in \mathbb{N}_{>0}$. Therefore, the E^2-term looks as in Figure 3.7.

2. Are there non-trivial differentials? For every $r \in \mathbb{N}_{\geq 2}$, the differential d^r of the Hochschild-Serre spectral sequence has degree $(-r, r-1)$; in particular, the horizontal and the vertical component of the bidegree have different parity. Hence, all differentials $(d^r)_{\geq 2}$ have to be trivial in this example. In other words, the spectral sequence corresponding to the above extension degenerates at the E^2-stage, and therefore $E^\infty \cong_{\mathbb{Z}} E^2$.

3. What about the extension problems? From the E^∞-page of the spectral sequence, for $k \in \mathbb{N}_{>0}$ we obtain short exact sequences of Abelian groups of the following types:

$$\begin{array}{ll}
0 \longrightarrow H_k(S_3;\mathbb{Z}) \longrightarrow 0 & \text{if } k \equiv 0 \bmod 4 \\
0 \longrightarrow H_k(S_3;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0 & \text{if } k \equiv 1 \bmod 4 \\
0 \longrightarrow H_k(S_3;\mathbb{Z}) \longrightarrow 0 & \text{if } k \equiv 2 \bmod 4 \\
0 \longrightarrow \mathbb{Z}/3 \longrightarrow H_k(S_3;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0 & \text{if } k \equiv 3 \bmod 4.
\end{array}$$

The classification of finitely generated Abelian groups tells us that all these extensions have to be trivial. Therefore, we obtain

$$H_k(S_3;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \equiv 0 \bmod 4 \text{ and } k > 0 \\ \mathbb{Z}/2 & \text{if } k \equiv 1 \bmod 4 \\ 0 & \text{if } k \equiv 2 \bmod 4 \\ \mathbb{Z}/6 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Degeneration at the E^2-stage, non-trivial extension problems. We now give an example of a Hochschild-Serre spectral sequence that still degenerates at the E^2-term, but where the resulting extension problems are non-trivial:

Example 3.2.17 (the infinite dihedral group)**.** We consider the infinite dihedral group D_∞, which fits into an extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/2$ acts on the kernel $\mathbb{Z}$ by taking inverses. The Hochschild-Serre spectral sequence then gives us:

$$E^2_{pq} = H_p\big(\mathbb{Z}/2; H_q(\mathbb{Z};\mathbb{Z})\big) \Longrightarrow H_{p+q}(D_\infty;\mathbb{Z}),$$

where $\mathbb{Z}/2$ acts on the coefficients $H_*(\mathbb{Z};\mathbb{Z})$ by the maps induced by taking inverses in $\mathbb{Z}$; i.e., the group $\mathbb{Z}/2$ acts trivially on $H_0(\mathbb{Z};\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$ and by multiplication by -1 on $H_1(\mathbb{Z};\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$ (check!).

1. How does the E^2-term look like? With help of the standard periodic $\mathbb{Z}[\mathbb{Z}/2]$-resolution of $\mathbb{Z}$ (see the proof of Corollary 1.6.13) we see that the E^2-term of this spectral sequence has the shape depicted in Figure 3.8.

$$
\begin{array}{c|cccccc}
q & & & & & & \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 \quad p
\end{array}
$$

$$E^2 = E^\infty$$

Figure 3.8.: the Hochschild-Serre spectral sequence for D_∞

2. Are there non-trivial differentials? Looking at the degrees of the differentials, we see that the differentials all start or end in 0; so, there are no non-trivial differentials. In other words, the spectral sequence degenerates at the E^2-stage, and thus $E^\infty \cong_{\mathbb{Z}} E^2$.

3. What about the extension problems? We obtain $H_0(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$, and $H_k(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} 0$ for all even $k \in \mathbb{N}_{>0}$ from the spectral sequence.

   For all odd $k \in \mathbb{N}$, the E^∞-term of the Hochschild-Serre spectral sequence gives us short exact sequences of the following type:

   $$0 \longrightarrow \mathbb{Z}/2 \longrightarrow H_k(D_\infty;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

   Now the classification of finitely generated Abelian groups shows that $H_k(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/4$ or $H_k(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

   In order to determine which of these alternatives actually happens, we take advantage of the fact that the group extension

   $$1 \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

   we started with splits. In particular, the identity homomorphism $\mathbb{Z}/2 \cong_{\mathbb{Z}} H_k(\mathbb{Z}/2;\mathbb{Z}) \longrightarrow H_k(\mathbb{Z}/2;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2$ factors through $H_k(D_\infty;\mathbb{Z})$. Hence, $H_k(D_\infty;\mathbb{Z})$ cannot be isomorphic to $\mathbb{Z}/4$. Therefore,

   $$H_k(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

   for all odd $k \in \mathbb{N}$. Alternatively, one could also use $H_1(D_\infty;\mathbb{Z}) \cong_{\mathbb{Z}} (D_\infty)_{\mathrm{ab}}$ in degree 1, and in the higher odd degrees a transfer argument (Example 1.7.17).

This completes the computation of $H_*(D_\infty;\mathbb{Z})$ via the Hochschild-Serre spectral sequence. Alternatively, one can also compute the homology of D_∞ using the decomposition $D_\infty \cong_{\mathsf{Group}} \mathbb{Z}/2 * \mathbb{Z}/2$ (Example 4.1.29).

3.2.4 A proof of Hopf's formula

Finally, we also have a suitable tool to give a straightforward proof of Hopf's formula (Theorem 1.5.1):

Theorem 3.2.18 (Hopf's formula)**.** *Let F be a free group, let $N \subset F$ be a normal subgroup, and let $G := F/N$. Then there is an exact sequence (in ${}_{\mathbb{Z}}\mathsf{Mod}$)*

$$0 \longrightarrow H_2(G;\mathbb{Z}) \longrightarrow H_1(N;\mathbb{Z})_G \longrightarrow H_1(F;\mathbb{Z}) \longrightarrow H_1(G;\mathbb{Z}) \longrightarrow 0$$

(where the homomorphisms on H_1 are induced by the canonical inclusion and projection, respectively, and G acts on $H_1(N;\mathbb{Z})$ by conjugation of representatives in F). More explicitly,

$$H_2(G;\mathbb{Z}) \cong_{\mathbb{Z}} \frac{N \cap [F,F]}{[F,N]}.$$

Proof. Let us first prove the explicit description out of the exact sequence: By the exact sequence, $H_2(G;\mathbb{Z})$ is isomorphic to kernel of the homomorphism $H_1(N;\mathbb{Z})_G \longrightarrow H_1(F;\mathbb{Z})$ induced by the inclusion $N \longrightarrow F$. Using the computation of group homology in degree 1 (Theorem 1.4.1), we obtain a canonical isomorphism

$$\begin{aligned} H_1(N;\mathbb{Z})_G &\cong_{\mathbb{Z}} \bigl(N/[N,N]\bigr) \mathbin{/} \operatorname{Span}_{\mathbb{Z}}\{g \cdot [n] - [n] \mid n \in N, g \in G\} \\ &= \bigl(N/[N,N]\bigr) \mathbin{/} \operatorname{Span}_{\mathbb{Z}}\{[f \cdot n \cdot f^{-1}] - [n] \mid n \in N, f \in F\} \\ &= \bigl(N/[N,N]\bigr) \mathbin{/} \operatorname{Span}_{\mathbb{Z}}\{[f \cdot n \cdot f^{-1} \cdot n^{-1}] \mid n \in N, f \in F\} \\ &= N/[F,N]. \end{aligned}$$

Therefore, the kernel of the map $H_1(N;\mathbb{Z})_G \longrightarrow H_1(F;\mathbb{Z})$ induced by the inclusion $N \longrightarrow F$ is isomorphic to $(N \cap [F,F])/[F,N]$, as desired.

It remains to establish the four-term exact sequence: We apply the Hochschild-Serre spectral sequence (Theorem 3.2.12) to the group extension

$$1 \longrightarrow N \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } G \longrightarrow 1$$

and obtain

$$E^2_{pq} = H_p\bigl(G; H_q(N;\mathbb{Z})\bigr) \Longrightarrow H_{p+q}(F;\mathbb{Z}).$$

In contrast with the previous examples, we know the "limit" of this spectral sequence (this is just homology of a free group) and we will try to reverse engineer the spectral sequence in order to compute $H_2(G;\mathbb{Z})$:

1. How does the E^2-term look like? As we do not know $H_2(G;\mathbb{Z})$ yet, we cannot compute all of the E^2-page, but we obtain partial information

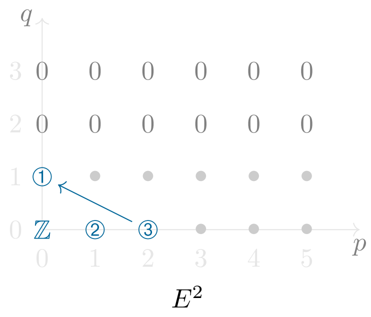


Figure 3.9.: the Hochschild-Serre spectral sequence for the proof of Hopf's formula

from the freeness of F: As subgroup of the free group F, also N is a free group (Theorem AT.2.3.52). Therefore, by Corollary 1.6.23, for all $k \in \mathbb{N}_{\geq 2}$,

$$H_k(F;\mathbb{Z}) \cong_{\mathbb{Z}} 0 \quad \text{and} \quad H_k(N;\mathbb{Z}) \cong_{\mathbb{Z}} 0.$$

In particular, $E^2_{pq} \cong_{\mathbb{Z}} 0$ for all $q \in \mathbb{N}_{\geq 2}$ and so the spectral sequence degenerates at stage 3 (because of the degrees of the differentials; Figure 3.9).

Moreover, we know that (by Theorem 1.3.1)

$$\begin{aligned}
① = E^2_{10} &= H_1\bigl(G; H_0(N;\mathbb{Z})\bigr) \cong_{\mathbb{Z}} H_1(G;\mathbb{Z}) \\
② = E^2_{01} &= H_0\bigl(G; H_1(N;\mathbb{Z})\bigr) \cong_{\mathbb{Z}} H_1(N;\mathbb{Z})_G \\
③ = E^2_{20} &= H_1\bigl(G; H_0(N;\mathbb{Z})\bigr) \cong_{\mathbb{Z}} H_2(G;\mathbb{Z})
\end{aligned}$$

and the only potentially non-trivial differential in this range is

$$d^2_{20} \colon H_2(G;\mathbb{Z}) \cong_{\mathbb{Z}} E^2_{20} \longrightarrow E^2_{01} \cong_{\mathbb{Z}} H_1(N;\mathbb{Z})_G.$$

2. What do we know about the E^∞-term? As the spectral sequence degenerates at stage 3, we have $E^\infty \cong_{\mathbb{Z}} E^3$.
   - Because of $H_2(F;\mathbb{Z}) \cong_{\mathbb{Z}} 0$, we have $E^3_{20} \cong_{\mathbb{Z}} E^\infty_{20} \cong_{\mathbb{Z}} 0$.
   - Convergence of the spectral sequence to $H_*(F;\mathbb{Z})$ implies that there is a short exact sequence

     $$0 \longrightarrow E^\infty_{01} \longrightarrow H_1(F;\mathbb{Z}) \longrightarrow E^\infty_{10} \longrightarrow 0.$$

     Moreover,

$$E^\infty_{01} \cong_\mathbb{Z} E^3_{01} \cong_\mathbb{Z} E^2_{01} / \operatorname{im} d^2_{20} \quad \text{and} \quad E^\infty_{10} \cong_\mathbb{Z} E^3_{10} \cong_\mathbb{Z} E^2_{10}.$$

3. Refining the information on the E^2-term. In view of the previous step and the transition between the pages of a spectral sequence, we have

$$0 \cong_\mathbb{Z} E^3_{20} \cong_\mathbb{Z} \ker d^2_{20} / \operatorname{im} 0,$$

and so d^2_{20} is injective.

We can now put it all together. Thus, we obtain a four-term exact sequence

$$0 \longrightarrow H_2(G;\mathbb{Z}) \xrightarrow{\text{“}d^2_{20}\text{”}} H_1(N;\mathbb{Z})_G \xrightarrow{⑤} H_1(F;\mathbb{Z}) \xrightarrow{④} H_1(G;\mathbb{Z}) \longrightarrow 0.$$

It remains to show that the homomorphisms on H_1 are induced by the group homomorphisms in the original extension (up to sign). Of course, if one knows the inner workings of a concrete construction of the Hochschild-Serre spectral sequence, then one can deduce this fact from the construction. However, as we blackboxed the construction, we will instead rely on the naturality of the Hochschild-Serre spectral sequence.

To this end, we consider the commutative diagram (in Group)

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d, "1"'] & F \arrow[r, "\mathrm{id}_F"] \arrow[d, "\mathrm{id}_F"'] & F \arrow[r] \arrow[d, "\pi"] & 1 \\
1 \arrow[r] & N \arrow[r, "i"'] & F \arrow[r, "\pi"'] & G \arrow[r] & 1
\end{tikzcd}$$

with exact rows. By the naturality of the Hochschild-Serre spectral sequence (Remark 3.2.14), this leads to a corresponding transformation between the associated Hochschild-Serre spectral sequences and thus to the following commutative diagram (which are parts of the four-term exact sequences):

$$\begin{tikzcd}
0 \cong_\mathbb{Z} H_1(1;\mathbb{Z})_F \arrow[r] & H_1(F;\mathbb{Z}) \arrow[r, "\varphi"] \arrow[d, "H_1(\mathrm{id}_F;\mathbb{Z})"'] & H_1(F;\mathbb{Z}) \arrow[r] \arrow[d, "H_1(\pi;\mathbb{Z})"] & 0 \\
& H_1(F;\mathbb{Z}) \arrow[r, "④"'] & H_1(G;\mathbb{Z}) &
\end{tikzcd}$$

Hence, $\varphi\colon H_1(F;\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z})$ is an isomorphism (but we do not know yet, which one). We will now show that $\varphi = \pm \mathrm{id}_{H_1(F;\mathbb{Z})}$, again, by naturality: If $f\colon F \longrightarrow F$ is a group automorphism, then naturality of the Hochschild-Serre spectral sequence and the associated four-term exact sequence shows that

$$\begin{array}{ccc} H_1(F;\mathbb{Z}) & \xrightarrow{\varphi} & H_1(F;\mathbb{Z}) \\ {\scriptstyle H_1(f;\mathbb{Z})}\big\downarrow & & \big\downarrow{\scriptstyle H_1(f;\mathbb{Z})} \\ H_1(F;\mathbb{Z}) & \xrightarrow[\varphi]{} & H_1(F;\mathbb{Z}) \end{array}$$

is a commutative diagram. Because $H_1(F;\mathbb{Z}) \cong_{\mathbb{Z}} F_{\mathrm{ab}}$ is free Abelian (with rank equal to the rank of F), using the right type of automorphisms f of F shows that φ is sufficiently central in the automorphism group of this free Abelian group (check!) and thus equal to $\pm\,\mathrm{id}_{H_1(F;\mathbb{Z})}$ (check!). In combination with the previous commutative diagram, we therefore obtain ④ $= \pm H_1(\pi;\mathbb{Z})$.

Analogously, one can use the extension $1 \longrightarrow F \longrightarrow F \longrightarrow 1 \longrightarrow 1$ given by the identity on F to deduce that ⑤ is induced by $\pm H_1(i;\mathbb{Z})$ (check!). Because sign changes do not affect the exactness properties, we obtain the claimed exact sequence. ☐

3.2.5 Universal coefficients and products

Products of two groups are a special case of extensions. Hence, we could apply the Hochschild-Serre spectral sequence (Theorem 3.2.12) to relate the (co)homology of the product to the (co)homology of the two factors. However, it turns out that we can obtain more refined information if we work with the double complex spectral sequences directly. This strategy leads to the universal coefficient theorem and the Künneth formula. For simplicity, we focus on the case that the base ring is a principal ideal domain (but the theory in principle also allows to treat more general cases):

Theorem 3.2.19 (Künneth spectral sequence). *Let R be a ring, let $P_* \in \mathrm{Ob}(\mathsf{Ch}_R)$ a chain complex consisting of flat R-modules, and let $A \in \mathrm{Ob}({}_R\mathsf{Mod})$. Then there is a natural (both in P_* and A) converging spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^R_p\bigl(H_q(P_*), A\bigr) \Longrightarrow H_{p+q}(P_* \otimes_R A).$$

Proof. We will prove this via the double complex spectral sequences. How do we get a double complex involved? Let (Q_*, ε) be a projective resolution of A over R. Then $P_* \otimes_R Q_*$ (by definition) is the total complex of the double complex C_{**}, which is given by $C_{p,q} := P_p \otimes_R Q_q$ for all $p, q \in \mathbb{N}$ and the horizontal/vertical boundary maps

$$\begin{aligned} \partial^h_{p,q} \colon C_{p,q} &\longrightarrow C_{p-1,q} \\ x \otimes y &\longmapsto \partial^P_p(x) \otimes y \\ \partial^v_{p,q} \colon C_{p,q} &\longrightarrow C_{p,q-1} \\ x \otimes y &\longmapsto (-1)^p \cdot x \otimes \partial^Q_q(y). \end{aligned}$$

Then we have two natural spectral sequences converging to $H_*(P_* \otimes_R Q_*)$ (Theorem 3.2.10):

$$
\begin{aligned}
{}^{①}E^1_{pq} &= H_q(P_p \otimes_R Q_*, \partial^v_*) \Longrightarrow H_{p+q}(\operatorname{Tot} C) = H_{p+q}(P_* \otimes_R Q_*)\\
{}^{②}E^1_{pq} &= H_q(P_* \otimes_R Q_p, \partial^h_*) \Longrightarrow H_{p+q}(\operatorname{Tot} C) = H_{p+q}(P_* \otimes_R Q_*)
\end{aligned}
$$

We will now use the first spectral sequence to identify the limit as $H_*(P_* \otimes_R A)$ and then the second spectral sequence to obtain the short exact sequence.

- Because P_p is flat and (Q_*, ε) is a projective resolution of A, we have

$$
{}^{①}E^1_{pq} \cong_R P_p \otimes_R H_q(Q_*) \cong_R \begin{cases} P_p \otimes_R A & \text{if } q = 0\\ 0 & \text{if } q > 0.\end{cases}
$$

  In particular, the first spectral sequence degenerates at stage 2, the ∞-page is concentrated in the "$q = 0$"-line, and thus $H_n(P_* \otimes_R Q_*) \cong_R {}^{①}E^\infty_{n,0} \cong_R {}^{①}E^2_{n,0} \cong_R H_n(P_* \otimes_R A)$ for all $n \in \mathbb{N}$.

- Therefore, also the second spectral sequence converges to $H_*(P_* \otimes_R A)$. Moreover, because Q_p is projective/flat,

$$
E^1_{pq} = H_q(P_* \otimes_R Q_p) \cong_R H_q(P_*) \otimes_R Q_p
$$

  and so (by construction of $\operatorname{Tor}^R_*(\,\cdot\,, A)$)

$$
{}^{②}E^2_{pq} \cong_R H_p\bigl(H_q(P_*) \otimes_R Q_*\bigr) \cong_R \operatorname{Tor}^R_p\bigl(H_q(P_*), A\bigr).
$$

  Hence, we obtain the desired natural converging spectral sequence

$$
E^2_{pq} = \operatorname{Tor}^R_p\bigl(H_q(P_*), A\bigr) \Longrightarrow H_{p+q}(P_* \otimes_R A). \qquad \square
$$

Corollary 3.2.20 (algebraic universal coefficient theorem)**.** *Let R be a principal ideal domain, let $P_* \in \operatorname{Ob}(\mathsf{Ch}_R)$ be a chain complex consisting of free R-modules, and let $A \in \operatorname{Ob}({}_R\mathsf{Mod})$. Then, for each $n \in \mathbb{N}$, there is a natural short exact sequence*

$$
0 \longrightarrow H_n(P_*) \otimes_R A \xrightarrow{\mu} H_n(P_* \otimes_R A) \longrightarrow \operatorname{Tor}^R_1\bigl(H_{n-1}(P_*), A\bigr) \longrightarrow 0.
$$

of R-modules, where

$$
\begin{aligned}
\mu\colon H_n(P_*) \otimes_R A &\longrightarrow H_n(P_* \otimes_R A)\\
[z] \otimes a &\longmapsto [z \otimes a].
\end{aligned}
$$

The sequence splits (but not naturally); in particular,

$$
H_n(P_* \otimes_R A) \cong_R \bigl(H_n(P_*) \otimes_R A\bigr) \oplus \operatorname{Tor}^R_1\bigl(H_{n-1}(P_*), A\bigr).
$$

Proof. Because R is a principal ideal domain, $\operatorname{Tor}_p^R(\,\cdot\,, A) \cong_R 0$ for all $p \in \mathbb{N}_{\geq 2}$ (every module admits a short projective resolution (Example IV.5.3.13; which also works without finite generation)) and $\operatorname{Tor}_0^R(\,\cdot\,, A) \cong_R \,\cdot\, \otimes_R A$. Therefore, the Künneth spectral sequence is concentrated on the two lines with "$p \in \{0,1\}$", thus degenerates at stage 2, and from convergence we obtain natural short exact sequences of the form

$$0 \longrightarrow H_n(P_*) \otimes_R A \xrightarrow{\mu_{P_*,A}} H_n(P_* \otimes_R A) \longrightarrow \operatorname{Tor}_1^R\bigl(H_{n-1}(P_*), A\bigr) \longrightarrow 0.$$

As next step, we show that $\mu_{P_*,A}$ (up to a unit in R, which we can safely ignore as it does not affect the kernels or images) has the claimed form (via naturality!): We consider the chain complex Q_* that just consists of the module R, concentrated in degree n. Then, in the corresponding short exact sequence for $Q_* \otimes_R R$ we know that $\mu_{Q_*,R} \colon H_n(Q_*) \otimes_R R \longrightarrow H_n(Q_* \otimes_R R)$ is an isomorphism (the right outer term is trivial). Because

$$H_n(Q_*) \otimes_R R \cong_R R \otimes_R R \cong_R R \quad \text{and} \quad H_n(Q_* \otimes_R R) \cong_R H_n(Q_*) \cong_R R,$$

the homomorphism $\mu_{Q_*,R}$ corresponds (under these isomorphisms) to multiplication by a unit $r \in R^\times$. Let $z \in P_n$ be a cycle and let $a \in A$. Then

$$\begin{aligned} R &\longrightarrow P_n \\ 1 &\longmapsto z \end{aligned}$$

defines an R-chain map $f_* \colon Q_* \longrightarrow P_*$ (as z is a cycle). Together with the module homomorphism $g \colon R \longrightarrow A$ with $g(1) = a$ and naturality of the short exact sequence, we obtain the commutative diagram

$$\begin{tikzcd} H_n(Q_*) \otimes_R R \arrow[r, "\mu_{Q_*,R}"] \arrow[d, "H_n(f_*)\otimes_R g"'] & H_n(Q_* \otimes_R R) \arrow[d, "H_n(f_* \otimes_R g)"] \\ H_n(P_*) \otimes_R A \arrow[r, "\mu_{P_*,A}"'] & H_n(P_* \otimes_R A) \end{tikzcd}$$

of R-modules. Therefore, $\mu_{P_*,A}\bigl([z] \otimes a\bigr) = r \cdot [z \otimes a]$ (and every element of $H_n(P_*) \otimes_R A$ is an R-linear combination of such elementary tensors).

Hence, we may assume that μ has the shape in the statement and it remains to show that μ admits a split. Because the chain modules of P_* are free and R is a principal ideal domain, there is an R-homomorphism $p_n \colon P_n \longrightarrow \ker \partial_n$ with $p_n \circ i_n = \mathrm{id}_{\ker \partial_n}$, where $i_n \colon \ker \partial_n \longrightarrow P_n$ is the inclusion (check! $\operatorname{im} \partial_n$ is free as a submodule of P_{n-1} ...). Then a straightforward computation shows that the following map is a well-defined split of μ (check!):

$$\begin{aligned} H_n(P_* \otimes_R A) &\longrightarrow H_n(P_*) \otimes_R A \\ \Bigl[\sum_{j=1}^k z_j \otimes a_j\Bigr] &\longmapsto \sum_{j=1}^k \bigl[p_n(z_j)\bigr] \otimes a_j \end{aligned}$$

□

Alternatively, one can also prove this theorem directly, by manipulating the chain complexes and homology groups by hand [87, Chapter 3.6]. The splitting in Corollary 3.2.20 is *not* natural in P_* [42, proof of Proposition V.2.4]. Moreover, the splitting also exists if the complex P_* consists of flat (instead of free) modules [42, Theorem V.2.1].

Study note. What does the universal coefficient theorem say if the principal ideal domain is a field?! How do these proofs relate to symmetry of Tor ?

Literature exercise (Theorems for free!)**.** Read "Theorems for free!" by Wadler [86]. What does this have to do with the proof of Corollary 3.2.20?

In particular, we can use the algebraic universal coefficient theorem to relate group homology with coefficients with trivial action to group homology with $\mathbb{Z}$-coefficients.

Corollary 3.2.21 (universal coefficients for group homology)**.** *Let G be a group, let A be a $\mathbb{Z}$-module (with trivial G-action), and let $n \in \mathbb{N}$. Then*

$$H_n(G;A) \cong_{\mathbb{Z}} \bigl(H_n(G;\mathbb{Z}) \otimes_{\mathbb{Z}} A\bigr) \oplus \operatorname{Tor}_1^{\mathbb{Z}}\bigl(H_{n-1}(G;\mathbb{Z}), A\bigr).$$

Proof. We have

$$H_n(G;A) = H_n\bigl(C_*(G) \otimes_G A\bigr) \cong_{\mathbb{Z}} H_n\bigl((C_*(G) \otimes_G \mathbb{Z}) \otimes_{\mathbb{Z}} A\bigr).$$

Moreover, $C_*(G) \otimes_G \mathbb{Z}$ consists of free $\mathbb{Z}$-modules (check!) and thus we can apply the universal coefficient theorem (Corollary 3.2.20) to this situation (over the ring $\mathbb{Z}$). □

To understand the homology of product groups, we need to upgrade the universal coefficient theorem to tensor products of two chain complexes:

Theorem 3.2.22 (algebraic Künneth theorem)**.** *Let R be a principal ideal domain, let $P_* \in \operatorname{Ob}(\mathsf{Ch}_R)$ be a chain complex consisting of flat R-modules, and let $Q_* \in \operatorname{Ob}({}_R\mathsf{Ch})$. Then, for each $n \in \mathbb{N}$, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p=0}^{n} H_p(P_*) \otimes_R H_{n-p}(Q_*) \longrightarrow H_n(P_* \otimes_R Q_*) \longrightarrow \bigoplus_{p=0}^{n-1} \operatorname{Tor}_1^R\bigl(H_p(P_*), H_{n-1-p}(Q_*)\bigr) \longrightarrow 0$$

of R-modules. Moreover, the left map is given by $[z] \otimes [w] \mapsto [z \otimes w]$ and the sequence splits (but not naturally).

Sketch of proof. This can be shown similarly to the proof of the universal coefficient theorem (Theorem 3.2.19, Corollary 3.2.20), using so-called Cartan-Eilenberg resolutions of Q_* and using Corollary 3.2.20 to identify the modules in the E^2-term [71, Theorem 10.90].

One should be aware that using the obvious double complex related to $P_* \otimes_R Q_*$ and its two spectral sequences will give the correct limit and good first pages, but it seems hard to figure out how to pass from page to page or what the ∞-page looks like [71, Chapter 10.10]. □

Corollary 3.2.23 (Künneth theorem for group homology). *Let G and H be groups, let R be a principal ideal domain (with trivial G- and H-action), and let $n \in \mathbb{N}$. Then (where the R-module structure is inherited from the coefficients)*

$$\begin{aligned} H_n(G \times H; R) \cong_R & \bigoplus_{p=0}^{n} H_p(G; R) \otimes_R H_{n-p}(H; R) \\ & \oplus \bigoplus_{p=0}^{n-1} \operatorname{Tor}_1^R\bigl(H_p(G; R), H_{n-1-p}(H; R)\bigr). \end{aligned}$$

Proof. With the goal in mind to apply the algebraic Künneth theorem (Theorem 3.2.22), we show that $(C_*(G) \otimes_{\mathbb{Z}} C_*(H), \varepsilon)$ is a projective $\mathbb{Z}[G \times H]$-resolution of $\mathbb{Z}$, where

$$\begin{aligned} \varepsilon \colon C_0(G) \otimes_{\mathbb{Z}} C_0(H) & \longrightarrow \mathbb{Z} \\ x \otimes y & \longmapsto \varepsilon^G(x) \cdot \varepsilon^H(y). \end{aligned}$$

As first step, one checks that

$$\begin{aligned} \mathbb{Z}[G \times H] & \longleftrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}H \\ G \times H \ni (g, h) & \longleftrightarrow g \otimes h \end{aligned}$$

describes well-defined mutually inverse ring isomorphisms (check!).

In particular, $C_*(G) \otimes_{\mathbb{Z}} C_*(H)$ is a $\mathbb{Z}[G \times H]$-chain complex that consists of free $\mathbb{Z}[G \times H]$-modules (because $C_*(G)$ and $C_*(H)$ consist of free modules over $\mathbb{Z}G$ and $\mathbb{Z}H$, respectively, the compatibility of the tensor product with direct sums, and the previous step). Moreover, the algebraic Künneth theorem (Theorem 3.2.22) shows that $(C_*(G) \otimes_{\mathbb{Z}} C_*(H)) \mathbin{\square} \varepsilon$ has trivial homology. Hence, $(C_*(G) \otimes_{\mathbb{Z}} C_*(G), \varepsilon)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times H]$.

Therefore, by the fundamental theorem (Corollary 1.6.9),

$$H_n(G \times H; R) \cong_R H_n\bigl((C_*(G) \otimes_{\mathbb{Z}} C_*(H)) \otimes_{G \times H} R\bigr).$$

Using the mutually inverse chain isomorphisms (R is commutative!)

$$\begin{aligned} \bigl(C_*(G) \otimes_{\mathbb{Z}} C_*(G)\bigr) \otimes_{G \times H} R & \longrightarrow \bigl(C_*(G) \otimes_G R\bigr) \otimes_R \bigl(C_*(H) \otimes_H R\bigr) \\ (x \otimes y) \otimes r & \longmapsto (x \otimes r) \otimes (y \otimes 1), \\ \bigl(C_*(G) \otimes_G R\bigr) \otimes_R \bigl(C_*(H) \otimes_H R\bigr) & \longrightarrow \bigl(C_*(G) \otimes_{\mathbb{Z}} C_*(G)\bigr) \otimes_{G \times H} R \\ (x \otimes r) \otimes (y \otimes s) & \longmapsto (x \otimes y) \otimes (r \cdot s) \end{aligned}$$

we obtain $H_n(G \times H; R) \cong_R H_n\bigl(\bigl(C_*(G) \otimes_G R\bigr) \otimes_R \bigl(C_*(H) \otimes_H R\bigr)\bigr)$. Now the algebraic Künneth theorem (Theorem 3.2.22) finishes the proof. $\square$

Example 3.2.24 (homology of $\mathbb{Z} \times \mathbb{Z}$)**.** Let $G := \mathbb{Z} \times \mathbb{Z}$. Then the Künneth formula (Corollary 3.2.23) allows us to compute $H_*(G; \mathbb{Z})$ (with trivial G-action on $\mathbb{Z}$): For all $n \in \mathbb{N}$, we have

$$\begin{aligned} H_n(G; \mathbb{Z}) \cong_{\mathbb{Z}} & \bigoplus_{p=0}^{n} H_p(\mathbb{Z}; \mathbb{Z}) \otimes_{\mathbb{Z}} H_{n-p}(\mathbb{Z}; \mathbb{Z}) \\ & \oplus \bigoplus_{p=0}^{n-1} \operatorname{Tor}_1^{\mathbb{Z}}\bigl(H_p(\mathbb{Z}; \mathbb{Z}), H_{n-1-p}(\mathbb{Z}; \mathbb{Z})\bigr). \end{aligned}$$

Let us first take care of the Tor-terms: If $k \in \mathbb{N}$, then $H_k(\mathbb{Z}; \mathbb{Z})$ is a free $\mathbb{Z}$-module (Corollary 1.6.23); therefore, all Tor-terms are trivial and we obtain:

- We have

$$\begin{aligned} H_1(G; \mathbb{Z}) & \cong_{\mathbb{Z}} \bigl(H_0(\mathbb{Z}; \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z})\bigr) \oplus \bigl(H_0(\mathbb{Z}; \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z})\bigr) \oplus 0 \\ & \cong_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

- Moreover,

$$\begin{aligned} H_2(G; \mathbb{Z}) & \cong_{\mathbb{Z}} \bigl(H_1(\mathbb{Z}; \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z})\bigr) \oplus 0 \\ & \cong_{\mathbb{Z}} \mathbb{Z}. \end{aligned}$$

- For all $n \in \mathbb{N}_{\geq 2}$, we have (because there is always a contribution by a higher degree homology of $\mathbb{Z}$, which is trivial) $H_n(G; \mathbb{Z}) \cong_{\mathbb{Z}} 0$.

Alternatively, one can also directly figure out a projective resolution for the group $\mathbb{Z} \times \mathbb{Z}$ (Exercise) or use Theorem 1.4.1 for degree 1 and the Hochschild-Serre spectral sequence (Theorem 3.2.12) for degree at least 2 (check!). Moreover, we will see a topological computation (Example 4.1.17).

Example 3.2.25 (homology of $\mathbb{Z}/p \times \mathbb{Z}/p$)**.** Let $p \in \mathbb{N}$ be prime. We consider the group $G := \mathbb{Z}/p \times \mathbb{Z}/p$ and compute the homology $H_*(G; \mathbb{F}_p)$ (with trivial G-action on the field $\mathbb{F}_p$) via the Künneth formula: For each $n \in \mathbb{N}$, we have (Corollary 3.2.23)

$$\begin{aligned} H_n(G; \mathbb{F}_p) \cong_{\mathbb{F}_p} & \bigoplus_{j=0}^{n} H_j(\mathbb{Z}/p; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_{n-j}(\mathbb{Z}/p; \mathbb{F}_p) \\ & \oplus \bigoplus_{j=0}^{n-1} \operatorname{Tor}_1^{\mathbb{F}_p}\bigl(H_j(\mathbb{Z}/p; \mathbb{F}_p), H_{n-1-j}(\mathbb{Z}/p; \mathbb{F}_p)\bigr). \end{aligned}$$

In view of the computation of $H_*(\mathbb{Z}/p; \mathbb{F}_p)$ (Corollary 1.6.13) and the fact that $\mathbb{F}_p$ is a field, we obtain

$$
\begin{aligned}
H_n(G;\mathbb{Z}) &\cong_{\mathbb{F}_p} \bigoplus_{j=0}^{n} H_j(\mathbb{Z}/p;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_{n-j}(\mathbb{Z}/p;\mathbb{F}_p) \oplus 0 \\
&\cong_{\mathbb{F}_p} {\mathbb{F}_p}^n.
\end{aligned}
$$

This behaviour is very different from the group homology of finite cyclic groups or S_3 (which all are "periodic").

With a little more patience, we can also compute $H_*(G;\mathbb{Z})$ (with trivial G-action on $\mathbb{Z}$) via the Künneth theorem.

Outlook 3.2.26 (cohomological universal coefficient theorem and cohomological Künneth theorem)**.** The previous results also have counterparts in cohomology:

- A cohomological universal coefficient theorem (that relates cohomology of the dual cochain complex to the dual of the homology, using Ext for the correction terms) [23, Section VI.4.2].
- A cohomological algebraic Künneth theorem [23, Theorem VI.10.11].
- A Künneth theorem for group cohomology [87, Exercise 6.1.8] (which is also compatible with the product structures) under a mild finiteness assumption.

Outlook 3.2.27 (topological Künneth theorem)**.** The universal coefficient theorem can, of course, also be applied to singular or cellular chain complexes (which consist of free modules!) in algebraic topology. This leads to the universal coefficient theorem for singular/cellular (co)homology [23, Section VI.7.8].

If X and Y are topological spaces, then, by the Eilenberg-Zilber theorem [23, Theorem VI.12.1], there are canonical natural chain homotopy equivalences

$$
C_*(X \times Y) \simeq_{\mathbb{Z}} C_*(X) \otimes_{\mathbb{Z}} C_*(Y)
$$

of the corresponding singular chain complexes. Therefore,

$$
H_*(X \times Y; R) \cong_R H_*\big(C_*(X;R) \otimes_R C_*(Y;R)\big)
$$

for all rings R. If R is a principal ideal domain, we can then apply the algebraic Künneth theorem (Theorem 3.2.22) to obtain the Künneth formula for each $n \in \mathbb{N}$:

$$
\begin{aligned}
H_n(X \times Y; R) \cong_R &\bigoplus_{p=0}^{n} H_p(X;R) \otimes_R H_{n-p}(Y;R) \\
&\oplus \bigoplus_{p=0}^{n} \operatorname{Tor}_1^R\big(H_p(X;R), H_{n-1-p}(Y;R)\big).
\end{aligned}
$$

Outlook 3.2.28 (irresponsible omission: product structures)**.** Let G be a group and let R be a commutative ring. Then the maps

$$\begin{aligned} \cdot \cup \cdot : C^p(G;R) \otimes_R C^q(G;R) &\longrightarrow C^{p+q}(G;R) \\ f \otimes_R g &\longrightarrow \big((g_0,\dots,g_{p+q}) \mapsto (-1)^{p\cdot q} \cdot f(g_0,\dots,g_p) \cdot g(g_p,\dots,g_{p+q})\big) \end{aligned}$$

induce a well-defined product, the so-called *cup-product*,

$$\begin{aligned} H^p(G;R) \otimes_R H^q(G;R) &\longrightarrow H^{p+q}(G;R) \\ [f] \otimes [g] &\longmapsto [f \cup g], \end{aligned}$$

which turns $\bigoplus_{n\in\mathbb{N}} H^n(G;R)$ into a graded ring. This multiplicative structure is functorial with respect to group homomorphisms (check!), carries valuable additional information, and most constructions in group cohomology are compatible with this product structure. For a deeper understanding of group (co)homology this multiplicative structure is essential [12, Chapter V/VI].

4

The topological view

Finally, we arrive at the topological view on group (co)homology. By the fundamental theorem, we can choose our favourite projective resolution to compute group (co)homology; in particular, we can choose projective resolutions of topological origin, namely chain complexes of classifying spaces of the group in question.

We will first briefly survey the translation of group theory into topology via classifying spaces. Nice models of classifying spaces then allow to compute group (co)homology for many groups and lead to additional inheritance properties (e.g., for free products).

We will then discuss higher finiteness properties of groups and sketch the role of group (co)homology for the understanding of free actions of finite groups on spheres.

Overview of this chapter.

Running example. graphs, surfaces

4.1 Classifying spaces

Classifying spaces allow to translate group theory into topology, more precisely, into homotopy theory. In this context, it is convenient to work with topological spaces that admit a nice cellular structure, i.e., with CW-complexes. This will allow us to obtain a topological version of the fundamental theorem (Corollary 1.6.9).

4.1.1 The standard simplicial model

The standard simplicial model of a group is the topological origin of the bar resolution, constructed from simplices spanned by group elements.

Definition 4.1.1 (standard simplicial model)**.** Let G be a group.

- The *standard simplicial model of* G is the topological space $BG := G \setminus EG$, where EG is defined as follows: Let

  $$EG := \Bigl(\coprod_{n \in \mathbb{N}} \Delta^n \times G^{n+1}\Bigr) \,/\, \sim,$$

  where "$\sim$" is the equivalence relation generated by

  $$\bigl(i_j(t), (g_0, \dots, g_n)\bigr) \sim \bigl(t, (g_0, \dots, \widehat{g}_j, \dots, g_n)\bigr)$$

  for all $n \in \mathbb{N}_{>0}$, $j \in \{0, \dots, n\}$, $t \in \Delta^{n+1}$, $g_0, \dots, g_n \in G$ (here, $i_j \colon \Delta^{n-1} \longrightarrow \Delta^n$ is the inclusion of the j-th face; Proposition AT.4.1.3). We equip EG with the quotient topology of the disjoint union topology (where the products G^{n+1} carry the discrete topology). Moreover, we endow EG with the (well-defined and continuous) diagonal G-action:

  $$\begin{aligned} G \times EG &\longrightarrow EG \\ \bigl(g, [t, (g_0, \dots, g_n)]\bigr) &\longmapsto \bigl[t, (g \cdot g_0, \dots, g \cdot g_n)\bigr]. \end{aligned}$$

- If $f \colon G \longrightarrow H$ is a group homomorphism, then we write (which is well-defined and continuous)

  $$\begin{aligned} Ef \colon EG &\longrightarrow EH \\ \bigl[t, (g_0, \dots, g_n)\bigr] &\longmapsto \bigl[t, (f(g_0), \dots, f(g_n))\bigr] \end{aligned}$$

  Moreover, we write $Bf \colon BG \longrightarrow BH$ for the associated (well-defined and continuous; check!) map.

In this way, we obtain a functor $B \colon \mathsf{Group} \longrightarrow \mathsf{Top}$ (check!).

Caveat 4.1.2. Usually, in the literature BG often refers to *any* choice of a model of a classifying space for G (as in Definition 4.1.6). However, it seems to convenient to reserve BG for this concrete model and to make the choice of other models explicit.

Proposition 4.1.3 (properties of EG)**.** *Let G be a group.*

1. *The space EG is contractible and locally path-connected.*
2. *The diagonal action $G \curvearrowright EG$ is properly discontinuous.*

Proof. *Ad 1.* We use a cone construction as in the proof of Proposition 1.6.5. The well-defined and continuous (check!) map

$$\begin{aligned} EG \times [0,1] &\longrightarrow EG \\ \bigl([t,(g_0,\dots,g_n)],s\bigr) &\longmapsto \bigl[s\cdot i_0(t) + (1-s)\cdot e_0, (e,g_0,\dots,g_n)\bigr] \end{aligned}$$

shows that EG is contractible; we use that $[0,1]$ is compact and Hausdorff to ensure that $\,\cdot\, \times [0,1]$ is compatible with quotients [23, Lemma V.2.13].

The "obvious" simplicial structure on EG can be viewed as a CW-structure. Therefore, path-connectedness implies local path-connectedness [28, Theorem 1.3.2] (or one can use the neighbourhoods constructed below).

Ad 2. Let $x \in EG$; we can write $x = [t,(g_0,\dots,g_n)]$ with minimal $n \in \mathbb{N}$. Then t lies in the (relative) interior of Δ^n. Hence, there is an $\varepsilon \in (0,1)$ with

$$t \in U_\varepsilon := \bigl\{(t_0,\dots,t_n) \in (\varepsilon, 1-\varepsilon)^n \bigm| t_0 + \dots + t_n = 1\bigr\} \subset \Delta^n.$$

Then a straightforward calculation (check!) shows that

$$\begin{aligned} U := \bigcup_{m\in\mathbb{N}_{\geq n}} \bigcup_{k_0<\dots<k_n\in\{0,\dots,m\}} \bigl\{[s,(h_0,\dots,h_m)] \bigm| {} & s\in\Delta^m,\ (s_{k_0},\dots,s_{k_n}) \in U_\varepsilon, \\ & \forall_{j\in\{0,\dots,m\}\setminus\{k_0,\dots,k_n\}}\quad s_j \in [0,\varepsilon), \\ & (h_{k_0},\dots,h_{k_n}) = (g_0,\dots,g_n)\bigr\} \end{aligned}$$

is an open neighbourhood of x in EG and with (essentially because G acts freely on $\coprod_{n\in\mathbb{N}} G^{n+1}$)

$$\forall_{g\in G} \quad g\cdot U \cap U \neq \emptyset \Longrightarrow g = e.$$

Therefore, the diagonal action $G \curvearrowright EG$ is properly discontinuous. □

Corollary 4.1.4 (properties of BG)**.** *Let G be a group.*

1. *Then the canonical projection $EG \longrightarrow BG$ is a (contractible) universal covering of BG and there is a canonical isomorphism*

$$\varphi_G \colon \pi_1(BG,x_0) \longrightarrow G$$

of groups through the deck transformation action, where x_0 is the point in BG represented by $e \in G^{0+1}$ of EG.

2. *If* $f\colon G \longrightarrow H$ *is a group homomorphism, then*
$$f = \varphi_H \circ \pi_1(Bf) \circ \varphi_G^{-1}.$$

3. *In particular,* $\pi_1 \circ (B\,\cdot\,, x_0)$ *is canonically naturally isomorphic to the identity functor on* Group.

Proof. This follows from the properties of the action $G \curvearrowright EG$ and covering theory (Corollary AT.2.3.39; check!). □

Using Corollary 4.1.4, we can model Group in Top. In particular, every topological invariant leads to a corresponding invariant for groups.

Outlook 4.1.5 (classifying spaces of categories). Let G be a group. The spaces EG and BG are (fat) geometric realisations of simplicial sets, which in turn are nerves of certain categories [31, Definition II.20] (Exercise).

4.1.2 Changing the classifying space

As in the algebraic case, the standard simplicial model tends to be rather big. Therefore, it is desirable to be able to replace the standard simplicial model by other spaces that are better adapted to the groups/problems under consideration. In the topological setting, we replace

- projective chain complexes by CW-complexes, and
- resolutions by spaces with contractible universal covering

(and whence projective resolutions by classifying spaces). Basic homotopy theory of CW-complexes is collected in Appendix A.3.

Definition 4.1.6 (classifying space). Let G be a group. A *(model of a) classifying space for* G is a pair $((X, x_0), \varphi)$, consisting of

- a path-connected pointed CW-complex (X, x_0) with contractible universal covering,
- and a group isomorphism $\varphi\colon \pi_1(X, x_0) \longrightarrow G$.

In order to minimise notational overhead, one often also simply says that X is classifying space for G (and leaves the rest of the data implicit).

Example 4.1.7 (classifying spaces of the trivial group). The one-point space (together with the only point as base-point and the trivial group homomorphism) clearly is a classifying space for the trivial group.

More generally: If (X, x_0) is a pointed CW-complex that is (pointedly) contractible, then $((X, x_0), \text{trivial homomorphism}\colon \pi_1(X, x_0) \longrightarrow 1)$ also is a classifying space for 1. In particular, classifying spaces are *not* unique up to homeomorphism.

Example 4.1.8 (classifying spaces from group actions)**.** Let G be a group and let $G \curvearrowright Y$ be a continuous group action on a topological space Y with the following properties:

- The action is properly discontinuous.
- The space Y is contractible.
- The space Y is locally path-connected.

Then the orbit projection $p\colon Y \longrightarrow G \setminus Y$ onto quotient space $X := G \setminus Y$ is the universal covering of X and given $x_0 \in X$ and $y_0 \in p^{-1}(x_0)$, there is a canonical group isomorphism $\varphi\colon \pi_1(X, x_0) \longrightarrow G$ (through the deck transformation action; Corollary AT.2.3.39).

In particular: If X admits a CW-structure, then $((X, x_0), \varphi)$ is a classifying space for G.

Example 4.1.9 (standard simplicial models)**.** Let G be a group. Then the standard simplicial model BG can be canonically turned into a classifying space for G: In view of Example 4.1.8 and Corollary 4.1.4, we only need to specify a CW-structure on BG and compatible base-points in BG and EG.

- As CW-structure on BG, we use the CW-structure induced by the simplicial structure on the underlying simplicial complex/set for EG (which descends to BG; check!).
- As base-point x_0, we take the 0-cell that corresponds to $e \in G^{0+1}$ (which we choose as covering base-point in EG).

In particular, every group admits a classifying space (and the standard simplicial construction is functorial).

Theorem 4.1.10 (recognising classifying spaces)**.** *Let G be a group, let (X, x_0) be a path-connected pointed CW-complex, and let $\varphi\colon \pi_1(X, x_0) \longrightarrow G$ be a group isomorphism. Then the following are equivalent:*

1. *The pair $((X, x_0), \varphi)$ is a classifying space for G.*
2. *For all $n \in \mathbb{N}_{\geq 2}$, we have $\pi_n(X, x_0) \cong_{\mathsf{Group}} 1$.*
3. *For all $n \in \mathbb{N}_{\geq 2}$, we have $H_n(\widetilde{X}; \mathbb{Z}) \cong_{\mathbb{Z}} 0$, where $\widetilde{X}$ is "the" universal covering space of X.*

Proof. Let $p\colon \widetilde{X} \longrightarrow X$ be the universal covering map and let $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$.

Ad 1 $\Longrightarrow$ 2. Let $((X, x_0), \varphi)$ be a classifying space for G and let $n \in \mathbb{N}_{\geq 2}$. Then p induces an isomorphism $\pi_n(p)\colon \pi_n(\widetilde{X}, \widetilde{x}_0) \longrightarrow \pi_n(X, x_0)$ (Corollary AT.2.3.25). Because the universal covering $\widetilde{X}$ is contractible, we obtain $\pi_n(X, x_0) \cong_{\mathsf{Group}} \pi_n(\widetilde{X}, \widetilde{x}_0) \cong_{\mathsf{Group}} 1$ (similar to Example AT.2.1.8).

Ad 2 $\Longrightarrow$ 3. This is a consequence of the compatibility of higher homotopy groups with coverings (Corollary AT.2.3.25), the simply connectedness of $\widetilde{X}$, and the Hurewicz theorem (Theorem AT.4.5.6).

Ad 3 $\Longrightarrow$ 1. This follows from the Whitehead theorem (Corollary A.3.6), applied to the constant map from $\widetilde{X}$ to the one-point space. □

Theorem 4.1.11 (uniqueness of classifying spaces). *Let G be a group and let $((X, x_0), \varphi)$ and $((Y, y_0), \psi)$ be classifying spaces of G. Then there exists a (unique up to homotopy) pointed homotopy equivalence $f\colon (X, x_0) \longrightarrow (Y, y_0)$ with*

$$\mathrm{id}_G = \psi \circ \pi_1(f) \circ \varphi^{-1}.$$

Proof. It suffices to consider the case that $((X, x_0), \varphi) = ((BG, x_0), \varphi_G)$ (check!). Moreover, by the Whitehead theorem (Theorem A.3.4), it suffices to construct a *weak* homotopy equivalence $f\colon BG \longrightarrow X$ with $\pi_1(f) = \psi^{-1} \circ \varphi_G$. We will construct such an f inductively, by induction over the dimension of the "simplices" of BG:

- Dimension 0: We take
$$f\colon [\Delta^0 \ni t, g \in G^{0+1}] \longmapsto y_0.$$
- Dimension 1: For $g_0, g_1 \in G$, we map the edge in BG defined by (g_0, g_1) as follows to Y:
$$f\colon \bigl[\Delta^1 \ni t, (g_0, g_1)\bigr] \longmapsto \gamma_{g_0^{-1} \cdot g_1}(t_0),$$
where $\gamma_{g_0^{-1} \cdot g_1}\colon [0,1]/\!\sim \longrightarrow Y$ is a based loop that represents the element
$$\psi^{-1} \circ \varphi_G\bigl(\bigl[s \mapsto [(s, 1-s), (e, g_0^{-1} \cdot g_1)]\bigr]_*\bigr) \in \pi_1(Y, y_0).$$
- Dimension 2: For $g_0, g_1, g_2 \in G$, we map the triangle in BG defined by (g_0, g_1, g_2) to Y by choosing a pointed homotopy in (Y, y_0) between the loops $\gamma_{g_0^{-1} \cdot g_1} * \gamma_{g_1^{-1} \cdot g_2}$ and $\gamma_{g_0^{-1} \cdot g_2}$ (such a homotopy exists!).
- Dimension ≥ 3: For $n \in \mathbb{N}_{\geq 3}$ and $g_0, \dots, g_n \in G$, the already constructed map $\{[t, [g_0, \dots, g_n]] \in BG \mid t \in \partial\Delta^n\} \longrightarrow Y$ represents an element in $\pi_{n-1}(Y, y_0) \cong_{\mathsf{Group}} 1$. Therefore, we can extend our map continuously over the whole n-simplex $\{[t, [g_0, \dots, g_n]] \in BG \mid t \in \Delta^n\}$.

By construction, the resulting map $f\colon BG \longrightarrow X$ induces $\psi \circ \varphi_G$ on π_1 and isomorphisms on the (trivial) higher homotopy groups. Hence, f is a weak equivalence.

Similarly, inductive constructions over $BG \times [0,1]$ show uniqueness up to homotopy (check!). □

Remark 4.1.12 (an inductive construction of classifying spaces). A more hands-on construction of classifying spaces than the standard simplicial model can be performed as follows: Let $\langle S \mid R \rangle$ be a presentation of a group G. Inductively, we construct a CW-complex X:

- 1-skeleton: We take $\bigvee_S (S^1, 1)$.
- 2-skeleton: We attach $|R|$ two-dimensional disks D^2 to $\bigvee_S (S^1, 1)$, according to the relations in R. This two-dimensional CW-complex is the *presentation complex* of $\langle S \mid R \rangle$. However, in general, the universal covering of the presentation complex need *not* be contractible.
- For $n \in \mathbb{N}_{\geq 2}$, from the n-skeleton X_n to the $(n+1)$-skeleton X_{n+1}: Let A_n be a generating set of $\pi_n(X_n, 1)$. Then, for each $a \in A_n$, we choose a map $\gamma\colon S^n \longrightarrow X_n$ representing a and attach an $(n+1)$-cell D^{n+1} via γ on $\partial D^{n+1} = S^n$.

By the Seifert and van Kampen theorem, the fundamental group of the resulting CW-complex X is isomorphic to G. Moreover, the Blakers-Massey theorem (or a careful application of the Hurewicz theorem to the universal covering) allow to prove that the higher homotopy groups of X are trivial. Applying Theorem 4.1.10 shows that X can be extended to a classifying space for G [51, Proposition 3.83].

Theorem 4.1.13 (new classifying spaces out of old). *Let G and H be groups and let $((X, x_0), \varphi)$ and $((Y, y_0), \psi)$ be classifying spaces for G and H, respectively. Then:*

1. Subgroups. *If $K \subset G$ is a subgroup, then "the" pointed path-connected covering $q\colon (Y, y_0) \longrightarrow (X, x_0)$ of X corresponding to the subgroup K of G yields a classifying space $((Y, y_0), \varphi \circ \pi_1(q, y_0))$ of K.*
2. Products. *The product $((X \times Y, (x_0, y_0)), \varphi \times \psi)$ is a classifying space for $G \times H$.*
3. Free products. *The wedge $((X, x_0) \vee (Y, y_0), [x_0], \varphi * \psi)$ is a classifying space for the free product $G * H$.*

Proof. Ad 1. The covering space (Y, y_0) inherits a CW-structure from the CW-structure on (X, x_0). By the classification theorem of coverings (Theorem AT.2.3.43), the homomorphism $\varphi \circ \pi_1(q, y_0)$ is a group isomorphism $\pi_1(Y, y_0) \longrightarrow K$ and the universal covering space of (Y, y_0), which coincides with the one of (X, x_0), is contractible.

Ad 2. Here, $X \times Y$ denotes the product CW-complex (which in general does *not* carry the product topology of X and Y) [28, p. x, Theorem 2.2.2][24]. By Theorem 4.1.10, it suffices to show that all higher homotopy groups of $X \times Y$ are trivial and that $\varphi \times \psi$ is an isomorphism on π_1. This facts on homotopy groups follow from: The canonical projections induce group isomorphisms

$$\pi_n\big(X \times Y, (x_0, y_0)\big) \cong_{\mathsf{Group}} \pi_n(X, x_0) \times \pi_n(Y, y_0)$$

for all $n \in \mathbb{N}_{\geq 1}$ (the argument for the usual product (Proposition AT.2.2.4) also works for this modified product [82]).

Ad 3. We equip the wedge $(Z, z_0) := (X, x_0) \vee (Y, y_0)$ with the combined CW-structure of the summands. By the Seifert and van Kampen theorem (Theorem AT.2.2.6, Example AT.2.2.10), the group homomorphism

$$\varphi * \psi \colon \pi_1\big((X, x_0) \vee (Y, y_0)\big) \longrightarrow G * H$$

is an isomorphism. Therefore, it remains to prove that the total space $\widetilde{Z}$ of the universal covering $p\colon \widetilde{Z} \longrightarrow Z$ of (Z, z_0) is contractible: In view of the Whitehead theorem (Theorem 4.1.10), we only need to show that $H_n(\widetilde{Z}; \mathbb{Z})$ is trivial for all $n \in \mathbb{N}_{\geq 2}$. To this end, we use a Mayer-Vietoris argument:

Let $\overline{X} := p^{-1}(X)$, $\overline{Y} := p^{-1}(Y) \subset \widetilde{Z}$.

- Then $\overline{X} \cup \overline{Y} = \widetilde{Z}$ and $\overline{X} \cap \overline{Y} = p^{-1}([x_0])$, which is a closed discrete subspace of $\widetilde{Z}$.

- The path-connected components of $\overline{X}$ are total spaces of coverings of X (check!). Moreover, each path-connected component $U \subset \overline{X}$ is simply connected: Let $u_0 \in U \cap p^{-1}(x_0)$ (which exists). By construction, we have the following commutative diagram in Group (where the horizontal arrows are induced by the inclusions):

$$\begin{tikzcd}
\pi_1(U, u_0) \arrow[r] \arrow[d, "\pi_1(p|_U)"'] & \pi_1(\widetilde{Z}, u_0) \arrow[d, "\pi_1(p)"] \\
\pi_1(X, x_0) \arrow[r] & \pi_1(Z, z_0)
\end{tikzcd}$$

  The vertical homomorphisms are injective (by covering theory; Corollary AT.2.3.25) and the lower horizontal homomorphism is injective (by the Seifert and van Kampen theorem, see above). Hence, also the upper horizontal homomorphism is injective. Because $\pi_1(\widetilde{Z}, u_0) \cong_{\mathsf{Group}} 1$, we obtain also $\pi_1(U, u_0) \cong_{\mathsf{Group}} 1$.

  Therefore, $p|_U \colon U \longrightarrow X$ is a universal covering of X and thus is contractible; in particular, U has trivial reduced homology. Similarly, we can argue for $\overline{Y}$.

We can now apply the Mayer-Vietoris sequence (Theorem AT.3.3.2) for singular homology $\widetilde{H}_*(\,\cdot\,; \mathbb{Z})$ (strictly speaking, we need a cellular version or we need to thicken up the subspaces suitably). For $n \in \mathbb{N}_{\geq 2}$, we consider the following fragment of the Mayer-Vietoris sequence:

$$H_n(\overline{X}; \mathbb{Z}) \oplus H_n(\overline{Y}; \mathbb{Z}) \longrightarrow H_n(\widetilde{Z}; \mathbb{Z}) \xrightarrow{\Delta_n} H_n(\overline{X} \cap \overline{Y}; \mathbb{Z}).$$

The outer terms all are singular homology groups in non-zero degree of spaces all of whose path-connected components are contractible. Therefore, strong additivity of singular homology (Proposition AT.4.1.14) and homotopy invariance show that these groups are trivial. Therefore, we obtain

$$H_n(\widetilde{Z};\mathbb{Z}) \cong_{\mathbb{Z}} 0$$

If one wants to avoid technical problems in the Seifert and van Kampen argument and the Mayer-Vietoris argument, a reasonable alternative space is the "thick" wedge $\bigl((X,x_0) \sqcup [0,1] \sqcup (Y,y_0)\bigr) \bigm/ (x_0 \sim 0 \wedge 1 \sim y_0)$. □

Outlook 4.1.14 (what do classifying spaces classify?)**.** Let G be a group, equipped with the discrete topology. Then the functor $[\,\cdot\,, BG]\colon \mathsf{Top_h} \longrightarrow \mathsf{Set}$ classifies G-principal bundles. More explicitly: If X is a paracompact topological space and $f\colon X \longrightarrow BG$ is a continuous map, then f^*p (the pullback of the covering $p\colon EG \longrightarrow BG$) is a principal G-bundle over X (as G is discrete, principal G-bundles are just G-covering maps). Then

$$\begin{aligned}[X, BG] &\longrightarrow \text{isomorphism classes of principal } G\text{-bundles over } X\\ [f] &\longmapsto [f^*p]\end{aligned}$$

is a well-defined bijection [85, Chapter 14.4]. In other words, BG (and the universal covering $p\colon EG \longrightarrow BG$) classifies principal G-bundles.

Outlook 4.1.15 (Eilenberg-MacLane spaces)**.** Let G be a group. Every classifying space for G is a $K(G,1)$-Eilenberg-MacLane space: If $n \in \mathbb{N}_{\geq 1}$, then a $K(G,n)$-Eilenberg-MacLane space is a triple $((X,x_0),\varphi)$, consisting of

- a path-connected pointed CW-complex (X,x_0) with
$$\forall_{k\in\mathbb{N}_{>0}\setminus\{n\}} \quad \pi_k(X,x_0) \cong_{\mathsf{Group}} 1,$$
- and a group isomorphism $\varphi\colon \pi_n(X,x_0) \longrightarrow G$.

For each $n \in \mathbb{N}_{\geq 2}$ and each Abelian group G, there exists a $K(G,n)$-Eilenberg-MacLane space.

4.1.3 Examples of classifying spaces

We list some prominent examples of "nice" classifying spaces for groups:

Example 4.1.16 (a classifying space for $\mathbb{Z}$)**.** Clearly, the circle S^1 is a classifying space for $\mathbb{Z}$:

- The circle S^1 admits a CW-structure (e.g., with a single 0-cell and a single 1-cell).

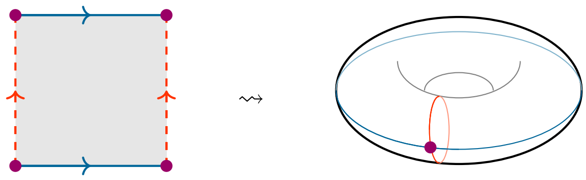


Figure 4.1.: A CW-structure on the 2-torus with one 0-cell, two 1-cells, and one 2-cell

- The fundamental group of S^1 is isomorphic to $\mathbb{Z}$ (Theorem AT.2.3.40).
- The universal covering space of S^1 is $\mathbb{R}$, which is contractible (Example AT.2.3.31).

Example 4.1.17 (a classifying space for $\mathbb{Z} \times \mathbb{Z}$)**.** The 2-torus $S^1 \times S^1$ is a classifying space for the group $T := \mathbb{Z} \times \mathbb{Z}$ (Example 4.1.16, Theorem 4.1.13).

We equip $S^1 \times S^1$ with the CW-structure in Figure 4.1. Then the universal covering space $\mathbb{R}^2$ of $S^1 \times S^1$ inherits the CW-structure in Figure 4.2. The cellular chain complex of this CW-structure on $\mathbb{R}^2$ (with the $\mathbb{Z}^2$-action induced by the deck transformation action) is isomorphic to the following resolution of $\mathbb{Z}$ by $\mathbb{Z}[T]$-modules (check!):

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}T \xrightarrow{\partial_2} \mathbb{Z}T \oplus \mathbb{Z}T \xrightarrow{\partial_1} \mathbb{Z}T \xrightarrow{\varepsilon} \mathbb{Z}$$

Here, we use that $\mathbb{Z}[T] \cong_{\mathsf{Ring}} \mathbb{Z}[a,b]_{\{a^n \cdot b^m | n,m \in \mathbb{N}\}}$ (Exercise) and the maps

$$\begin{aligned} \partial_2 \colon \mathbb{Z}T &\longrightarrow \mathbb{Z}T \oplus \mathbb{Z}T \\ x &\longmapsto \bigl(x \cdot (1-b), x \cdot (a-1)\bigr) \\ \partial_1 \colon \mathbb{Z}T \oplus \mathbb{Z}T &\longrightarrow \mathbb{Z}T \\ (x,y) &\longmapsto x \cdot (a-1) + y \cdot (b-1), \end{aligned}$$

together with the usual augmentation $\varepsilon \colon \mathbb{Z}T \longrightarrow \mathbb{Z}$.

Example 4.1.18 (a classifying space for $\mathbb{Z}/2$)**.** The infinite-dimensional real projective space $\mathbb{R}P^\infty := \operatorname{colim}_{n\to\infty} \mathbb{R}P^n$ is a classifying space for $\mathbb{Z}/2$. This can, for instance, be seen by looking at the antipodal $\mathbb{Z}/2$-action on $S^\infty := \operatorname{colim}_{n\to\infty} S^n$ (which satisfies the hypotheses of Theorem 4.1.10). The corresponding cellular chain complex of S^∞ (with the CW-structure induced from the standard CW-structure on $\mathbb{R}P^\infty$, which in turn has a single cell in each dimension) is $\mathbb{Z}[\mathbb{Z}/2]$-isomorphic to the periodic resolution in Proposition 1.6.11.

Example 4.1.19 (classifying spaces for free groups)**.** Let S be a [finite] set. Then $\bigvee_S (S^1, 1)$ is a classifying space for the free group $F(S)$ (Example 4.1.16,

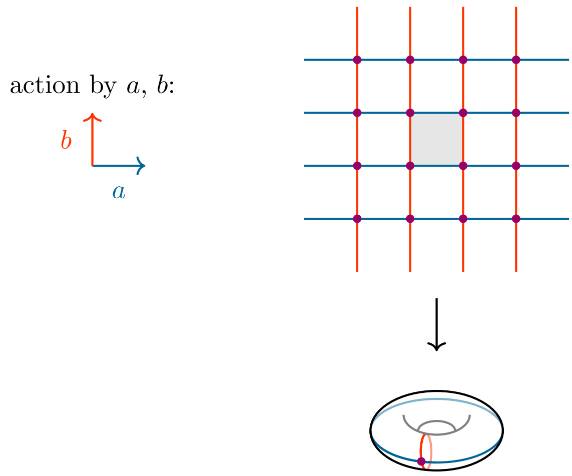


Figure 4.2.: The universal covering of the 2-torus

Theorem 4.1.13). The universal covering of $\bigvee_S(S^1, 1)$ is a $2 \cdot |S|$-regular tree (check!).

Now let $S = \{a, b\}$ with $a \neq b$. We consider the CW-structure on the wedge $\bigvee_{\{a,b\}}(S^1, 1)$ that consists of a single 0-cell and two 1-cells. Then we obtain the corresponding CW-structure on the universal covering (Figure 1.6). The cellular chain complex of this CW-structure on the 4-regular tree then is isomorphic to the resolution from Proposition 1.6.21, as explained in Remark 1.6.22.

Example 4.1.20 (classifying spaces for surface groups)**.** Let $g \in \mathbb{N}_{\geq 2}$ and let

$$\Gamma_g := \bigl\langle a_1, \dots, a_g, b_1, \dots, b_g \bigm| [a_1, b_1] \cdot \dots \cdot [a_g, b_g] \bigr\rangle.$$

Let Σ_g be the topological space that is obtained from a regular $4g$-gon by the identifications indicated in Figure 4.3 (in fact, this is a closed surface with g "holes").

- A straightforward application of the Seifert and van Kampen theorem (Theorem AT.2.2.6) shows that the fundamental group of Σ_g is isomorphic to Γ_g.
- On Σ_g, we can find a CW-structure (e.g., with one 0-cell and $2 \cdot g$ one-cells and a single 2-cell; Exercise).
- Moreover, the universal covering of Σ_g is contractible. This can be proved as follows [4, Proposition B.3.1]: In the hyperbolic plane $\mathbb{H}^2$, there exists a regular geodesic $4g$-gon all of whose angles are $\pi/(2 \cdot g)$.

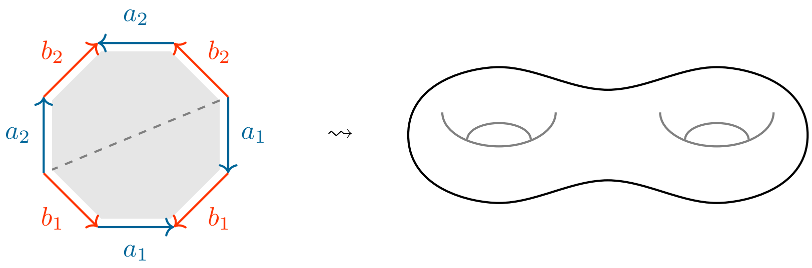


Figure 4.3.: From a $4g$-gon to a surface of genus g (check!)

Then the Riemannian metric of $\mathbb{H}^2$ induces a well-defined Riemannian metric on Σ_g (via this geodesic $4g$-gon). In particular, Σ_g admits a Riemannian metric of constant sectional curvature -1. Hence, the Riemannian universal covering of Σ_g is isometric to $\mathbb{H}^2$. Because $\mathbb{H}^2 \cong_{\mathsf{Top}} \mathbb{R}^2$, the universal covering of Σ_g is contractible.

Alternatively, this can also be proved via a generalisation of Theorem 4.1.13 from free products to amalgamated free products (and the fact that Γ_g is an amalgamated free product of two free groups over $\mathbb{Z}$).

Hence, we can view Σ_g as a classifying space of Γ_g.

The case of surface groups admits the following important generalisations:

Outlook 4.1.21 (manifolds of non-positive sectional curvature)**.** Let (M, g) be a (non-empty) complete Riemannian manifold with $\sec_g \leq 0$ (i.e., of non-positive sectional curvature). By the Cartan-Hadamard theorem [49, Theorem 11.5], the universal covering of M is homeomorphic to $\mathbb{R}^{\dim M}$ and thus contractible. Moreover, M admits a triangulation (as smooth manifold [89]), whence a CW-structure. Therefore, we can view M as classifying space of its fundamental group.

Oriented closed connected surfaces of genus at least 1 fall into this class of examples (Example 4.1.20).

In contrast, by now one knows many examples of groups that admit a closed manifold as a classifying space but which have "exotic" properties [20] (and which thus are far away from being non-positively curved).

Outlook 4.1.22 (classifying spaces for hyperbolic groups)**.** Let G be a hyperbolic group (i.e., a finitely generated group such that G with its word-metrics is a Gromov-hyperbolic metric space [53, Chapter 7][37]; e.g., a surface group or a free group) and let $S \subset G$ be a finite generating set.

The *Rips complex $R_r(G, S)$ of G with respect to S* for the radius $r \in \mathbb{R}_{>0}$ is the geometric realisation of the simplicial complex on the vertex set G with the set

$$\{\sigma \subset G \mid |\sigma| < \infty,\ \forall_{g,h\in\sigma} \quad d_S(g,h) \le r\}$$

of simplices. If G is in addition torsion-free and r is large enough, then the quotient $G \setminus R_r(G,S)$ is a classifying space for G [37, Section 2.2].

Outlook 4.1.23 (classifying spaces for one-relator groups)**.** A *one-relator group* is a group G that admits a presentation $\langle S \,|\, r\rangle$, where $r \in F(S)$ is a single relation. Then the following holds [67, 56, 62]:

- If $r \in F(S)$ is *not* a proper power of an element in $F(S)$, then the presentation complex of $\langle S \,|\, r\rangle$ is a classifying space for $G = \langle S \,|\, r\rangle$.
- If $r \in F(S)$ is a proper power of an element in $F(S)$, then the presentation complex of $\langle S \,|\, r\rangle$ is *not* a classifying space for G (because the group G contains torsion and the presentation complex is finite-dimensional; Corollary 4.2.7). But in this situation, the group $G = \langle S \,|\, r\rangle$ is hyperbolic (and thus we can find a classifying space through the Rips construction; Outlook 4.1.22).

Clearly, the fundamental groups of oriented closed connected surfaces of genus at least 1 are one-relator groups.

4.1.4 Group (co)homology via classifying spaces

Chain complexes of classifying spaces lead to projective resolutions over the fundamental group and thus can be used to compute group (co)homology. In the following, we will use the following notation (on topological spaces/CW-complexes and maps between them):

- C_* for the singular chain complex
- H_* for singular homology
- C_*^{CW} for the cellular chain complex (associated with $H_*(\,\cdot\,;\mathbb{Z})$)
- H_*^{CW} for cellular homology (associated with $H_*(\,\cdot\,;\mathbb{Z})$)

Theorem 4.1.24 (resolutions from classifying spaces)**.** *Let G be a group, let $((X,x_0),\varphi)$ be a classifying space for G, and let $\widetilde{X} \longrightarrow X$ be "the" universal covering of X (with the CW-structure induced from the CW-structure on X).*

1. *Then $(C_*(\widetilde{X};\mathbb{Z}),\varepsilon)$ is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Here, $C_*(\widetilde{X};\mathbb{Z})$ carries the G-action induced from the deck transformation action (and φ) and*

$$\begin{aligned} \varepsilon\colon C_0(\widetilde{X};\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \operatorname{map}(\Delta^0,\widetilde{X}) \ni \sigma &\longmapsto 1. \end{aligned}$$

2. *Similarly, also* $(C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}), \varepsilon^{\mathsf{CW}})$ *is a free resolution of* $\mathbb{Z}$ *by* $\mathbb{Z}G$*-modules. Here,* $C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z})$ *carries the* G*-action induced from the deck transformation action (and* φ*) and*

$$\begin{aligned} \varepsilon^{\mathsf{CW}} \colon C_0^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ 0\text{-cells } \sigma &\longmapsto 1. \end{aligned}$$

Corollary 4.1.25 (group homology via classifying spaces)**.** *Let* G *be a group, let* A *be a* $\mathbb{Z}$*-module (with trivial* G*-action), and let* $n \in \mathbb{N}$*.*

1. *Then there is a canonical isomorphism* $H_n(G;A) \cong_{\mathbb{Z}} H_n(BG;A)$*.*
2. *If* $f \colon G \longrightarrow H$ *is a group homomorphism, then* $H_n(f;A)$ *corresponds under this isomorphism to* $H_n(Bf;A) \colon H_n(BG;A) \longrightarrow H_n(BH;A)$*.*
3. *If* $((X,x_0),\varphi)$ *is a classifying space for* G*, then there is a canonical isomorphism*

$$H_n(G;A) \cong_{\mathbb{Z}} H_n(X;A) \cong_{\mathbb{Z}} H_n^{\mathsf{CW}}(X;A).$$

Proof of Theorem 4.1.24. Because X is a classifying space, its universal covering $\widetilde{X}$ is contractible and so

$$\begin{aligned} H_*\big(C_*(\widetilde{X};\mathbb{Z})\big) &= H_*(\widetilde{X};\mathbb{Z}) \cong_{\mathbb{Z}} H_*(\bullet;\mathbb{Z}) \\ H_*\big(C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z})\big) &= H_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}) \cong_{\mathbb{Z}} H_*(\widetilde{X};\mathbb{Z}) \cong_{\mathbb{Z}} H_*(\bullet;\mathbb{Z}). \end{aligned}$$

Together with the concrete computation of singular/cellular homology in degree 0 (Proposition AT.4.1.15), we see that the homology of $C_*(\widetilde{X};\mathbb{Z}) \square \varepsilon$ and of $C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}) \square \varepsilon^{\mathsf{CW}}$ is trivial in all degrees (check!). Therefore, $(C_*(\widetilde{X};\mathbb{Z}),\varepsilon)$ and $(C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}),\varepsilon^{\mathsf{CW}})$ both are resolutions of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.

Moreover, the chain modules of $C_*(\widetilde{X};\mathbb{Z})$ and $C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z})$ consist of free $\mathbb{Z}$-modules that admit $\mathbb{Z}$-bases on which G-acts freely (in the singular case, the basis of all singular simplices; in the cellular case, the basis of all open cells). Hence the chain modules of $C_*(\widetilde{X};\mathbb{Z})$ and $C_*^{\mathsf{CW}}(\widetilde{X};\mathbb{Z})$ are free $\mathbb{Z}G$-modules. □

Proof of Corollary 4.1.25. Ad 1. By Theorem 4.1.24, $(C_*(EG;\mathbb{Z}),\varepsilon)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. By the fundamental theorem (Corollary 1.6.9), there is thus a canonical isomorphism

$$H_n(G;Z) \cong_{\mathbb{Z}} H_n\big(C_*(EG;\mathbb{Z}) \otimes_G A\big).$$

We now relate the right-hand side to BG: The lifting properties of covering maps show (because simplices are simply connected) that the chain map $C_*(p;\mathbb{Z}) \colon C_*(EG;\mathbb{Z}) \longrightarrow C_*(BG;\mathbb{Z})$ induced by the (covering) projection map p induces a chain isomorphism

$$C_*(EG;\mathbb{Z}) \otimes_G A \cong_{\mathbb{Z}\mathsf{Ch}} C_*(BG;\mathbb{Z}) \otimes_{\mathbb{Z}} A$$

(because G acts trivially on A). This proves the first part.

Ad 2. This can easily be seen by comparing the standard simplicial model construction with the simplicial resolution construction and Corollary 4.1.4 (check!).

Ad 3. This is a direct consequence of the first part and the uniqueness of classifying spaces up to homotopy equivalence (Theorem 4.1.11). Moreover, cellular homology is known to coincide with singular homology (Theorem AT.5.2.13). □

Outlook 4.1.26 (group cohomology via classifying spaces)**.** Analogously, there is also a version of Corollary 4.1.25 for group cohomology and singular/cellular cohomology of classifying spaces. Moreover, the corresponding isomorphisms are also compatible with the respective cup-products [12, Chapter V].

Outlook 4.1.27 (twisted coefficients)**.** The results of Corollary 4.1.25 also extend to twisted coefficients: Let $p\colon \widetilde{X} \longrightarrow X$ be the universal covering of a path-connected CW-complex with fundamental group G and let A be a $\mathbb{Z}G$-module. Then, singular homology of X with twisted coefficients in A is defined as

$$H_*(X;A) := H_*\bigl(C_*(\widetilde{X};\mathbb{Z}) \otimes_G A\bigr),$$

where we equip $C_*(\widetilde{X};\mathbb{Z})$ with "the" deck transformation action of G.

Example 4.1.28 (surface groups)**.** Let $g \in \mathbb{N}_{\geq 2}$ and let Γ_g be the surface group of Example 4.1.20. Then we obtain from the classifying space in Example 4.1.20 and Corollary 4.1.25 (Exercise):

$$H_n(\Gamma_g;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2\cdot g} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n > 2 \end{cases}$$

Together with the above presentation of Γ_g and Corollary 1.5.4, the deficiency of Γ_g can be computed as

$$\operatorname{def} \Gamma_g = 2 \cdot g - 1.$$

This class of examples can also be used to show that, in general, the transfer map is *not* induced by a morphism in $\mathsf{GroupMod}$ or $\mathsf{GroupMod}^*$: There is a double sheeted covering map $p\colon \Sigma_3 \longrightarrow \Sigma_2$ (e.g., given by "rotation around the middle hole of Σ_3 around π"). In particular, we may view Γ_3 as subgroup of Γ_2 of index 2.

We now consider the transfer map $\operatorname{tr}_2\colon H_2(\Sigma_2;\mathbb{Z}) \longrightarrow H_2(\Sigma_3;\mathbb{Z})$ associated with this subgroup. Then

$$H_2(p;\mathbb{Z}) \circ \operatorname{tr}_2 = 2 \cdot \operatorname{id}_{H_2(\Sigma_2;\mathbb{Z})} \neq 0.$$

However, if $g\colon \Sigma_2 \longrightarrow \Sigma_3$ is a group homomorphism, then $H_2(g;\mathbb{Z}) = 0$: *Assume* for a contradiction that $H_2(g;\mathbb{Z}) \neq 0$. For g, there exists a continuous map $f\colon \Sigma_2 \longrightarrow \Sigma_3$ that induces g on the level of π_1 (Theorem 4.1.11, and functoriality of the standard simplicial model). Then a Poincaré duality argument shows that $H_1(f;\mathbb{Z})\colon H_1(\Sigma_2;\mathbb{Z}) \longrightarrow H_1(\Sigma_3;\mathbb{Z})$ is surjective, which contradicts the above computation of $H_1(\Sigma_g;\mathbb{Z})$. Therefore, $H_2(g;\mathbb{Z}) = 0$. Hence, tr_2 is *not* induced by a group homomorphism $\Sigma_2 \longrightarrow \Sigma_3$.

Example 4.1.29 (free products)**.** Let G and H be groups. Then the wedge $X := BG \vee BH$ yields a classifying space for the free product $G * H$ (Theorem 4.1.13). Therefore, the Mayer-Vietoris sequence (Theorem AT.3.3.2) shows that the inclusions $BG \longrightarrow X$ and $BH \longrightarrow X$ induce isomorphisms

$$H_n(G * H;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(G;\mathbb{Z}) \oplus H_n(H;\mathbb{Z})$$

for all $n \in \mathbb{N}_{\geq 1}$.

More generally, in a similar way, also amalgamated free products can be handled [12, Chapter II.7].

Study note. One could also *define* group (co)homology as (co)homology of the corresponding classifying spaces (with twisted coefficients). Try to establish as many properties of group (co)homology as possible via this approach!

Remark 4.1.30 (topological description of transfer)**.** Let G be a group, let $H \subset G$ be a subgroup of finite index, and let $n \in \mathbb{N}$. Then the transfer

$$\operatorname{tr}_H^G\colon H_n(G;\mathbb{Z}) \longrightarrow H_n(H;\mathbb{Z})$$

can be described topologically as follows (Figure 4.4): Let $((X,x_0),\varphi)$ be a classifying space for G and let $p\colon (Y,y_0) \longrightarrow (X,x_0)$ be "the" path-connected covering corresponding to the subgroup $\varphi^{-1}(H) \subset \pi_1(X,x_0)$. Then $((Y,y_0),\varphi \circ \pi_1(p))$ is a classifying space for H (Theorem 4.1.13) and the transfer map tr_H^G corresponds under the canonical isomorphisms from Corollary 4.1.25 to the following homomorphism [12, Chapter III.9]:

$$\begin{aligned} H_n(X;\mathbb{Z}) &\longrightarrow H_n(Y;\mathbb{Z}) \\ \Bigl[\sum_{j=1}^k a_j \cdot \sigma_j\Bigr] &\longmapsto \Bigl[\sum_{j=1}^k a_j \cdot \sum_{\sigma \in p^{-1}(\sigma_j)} \sigma\Bigr], \end{aligned}$$

where $p^{-1}(\sigma_j) := \{\sigma \in \operatorname{map}(\Delta^n, Y) \mid p \circ \sigma = \sigma_j\}$ is the set of all p-lifts of σ_j; by covering theory, $p^{-1}(\sigma_j)$ contains exactly $[G : H]$ elements.

Outlook 4.1.31 (bounded cohomology, topologically)**.** The computation of group (co)homology via classifying spaces (Corollary 4.1.25) also has a counterpart in bounded cohomology: If (X,x_0) is a sufficiently nice path-connected

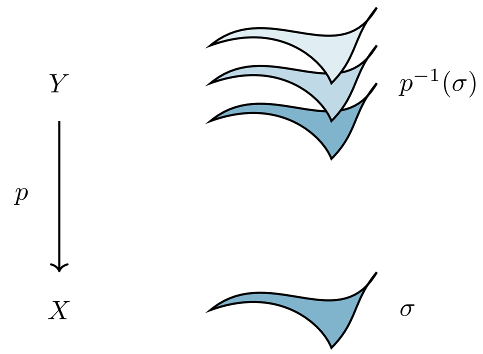


Figure 4.4.: Transfer, topologically: lifting simplices

pointed topological space (e.g., a CW-complex), then there is a canonical isomorphism

$$H^*\big(\mathrm{BHom}(C_*(X;\mathbb{R}),\mathbb{R})\big) \cong_{\mathbb{R}} H_*\big(\pi_1(X,x_0);\mathbb{R}\big),$$

which is isometric with respect to the $\|\cdot\|_\infty$-semi-norm on the respective cohomology spaces [36, 43, 44, 27].

A notable difference between this bounded setting and the ordinary setting is that we do *not* need that the universal covering of X is contractible (!). The, rough, underlying reason is that all higher homotopy groups of X are Abelian (whence amenable) and thus do not contribute to bounded cohomology. This fact has remarkable applications in geometric topology, e.g., in the context of simplicial volume [36, 43, 27, 50, 75].

4.2 Finiteness conditions

When studying groups from the point of view of (co)homology, several natural finiteness conditions/properties emerge: On the one hand, we could try to introduce a "dimension" for groups:

- What is the maximal dimension with non-vanishing cohomology?
- How short can projective resolutions of $\mathbb{Z}$ over the group ring be?
- What is the minimal dimension of a classifying space?

On the other hand, we could also ask for finiteness in certain degrees:

- Does there exist a finite classifying space? Or at least a classifying space that has finitely many cells in many dimensions?
- Does there exist a finite projective resolution? Or at least a projective resolution that is finitely generated in many degrees?

4.2.1 Cohomological dimension

Let us first consider the "dimension" of the group (over $\mathbb{Z}$).

Definition 4.2.1 (cohomological dimension)**.** Let G be a group. The *cohomological dimension of G* is defined as*

$$\operatorname{cd} G := \sup\bigl\{n \in \mathbb{N} \bigm| \exists_{A \in \operatorname{Ob}({}_{\mathbb{Z}G}\mathsf{Mod})} \quad H^n(G; A) \not\cong_{\mathbb{Z}} 0\bigr\} \in \mathbb{N} \cup \{\infty\}.$$

This dimension can also be rephrased in terms of projective resolutions:

Proposition 4.2.2 (cohomological dimension, alternative descriptions)**.** *Let G be a group. If $\operatorname{cd} G < \infty$, then**

$$\begin{aligned} \operatorname{cd} G &= \sup\bigl\{n \in \mathbb{N} \bigm| \exists_{A \in \operatorname{Ob}({}_{\mathbb{Z}G}\mathsf{Mod})} \quad H^n(G; A) \not\cong_{\mathbb{Z}} 0 \text{ and } A \text{ is free}\bigr\} && ① \\ &= \inf\bigl\{n \in \mathbb{N} \bigm| \forall_{A \in \operatorname{Ob}({}_{\mathbb{Z}G}\mathsf{Mod})} \quad \forall_{k \in \mathbb{N}_{>n}} \quad H^k(G; A) \cong_{\mathbb{Z}} 0\bigr\} && ② \\ &= \inf\bigl\{n \in \mathbb{N} \bigm| \forall_{A \in \operatorname{Ob}({}_{\mathbb{Z}G}\mathsf{Mod})} \quad H^{n+1}(G; A) \cong_{\mathbb{Z}} 0\bigr\} && ③ \\ &= \inf\bigl\{n \in \mathbb{N} \bigm| \text{there exists a projective resolution of } \mathbb{Z} \text{ over } \mathbb{Z}G \\ &\qquad \text{of length } n\bigr\}. && ④ \end{aligned}$$

If $\operatorname{cd} G = \infty$, then the descriptions ②, ③, ④ are also valid.

Proof.

- Clearly, $\operatorname{cd} G \geq$ ①. Conversely, let $n := \operatorname{cd} G \in \mathbb{N}$ and let A be a left $\mathbb{Z}G$-module with $H^n(G; A) \not\cong_{\mathbb{Z}} 0$. We then consider a short exact sequence
$$0 \longrightarrow K \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } A \longrightarrow 0$$
of $\mathbb{Z}G$-modules, where F is a free $\mathbb{Z}G$-module (this exists; proof of Corollary 1.6.8). Then we have the following portion of the associated long exact cohomology sequence (for the derived functor Ext; Theorem 3.1.16):
$$H^n(G; F) \xrightarrow{H^n(\operatorname{id}_G; \pi)} H^n(G; A) \xrightarrow{\ \delta^n\ } H^{n+1}(G; K) \cong_{\mathbb{Z}} 0$$
Therefore, $\operatorname{im} H^n(\operatorname{id}_G; \pi) = \ker \delta^n = H^n(G; A) \not\cong_{\mathbb{Z}} 0$. In particular, also $H^n(G; F) \not\cong_{\mathbb{Z}} 0$.
- It is clear that $\operatorname{cd} G =$ ② (with the convention $\inf \emptyset = \infty$ and $\sup \emptyset = 0$).

*There is a small set-theoretic issue here; strictly speaking, the beast on the right-hand side is not a set, and not even a class. However, the proof of the description ④ in Proposition 4.2.2 can be used to show that this beast can be replaced by a set.

- Moreover, ② = ③ follows from dimension shifting (here, we need a cohomological version that is obtained by embedding modules into injective modules [12, Chapters III.4, III.7]).

- Clearly, $\operatorname{cd} G \leq$ ④. For the converse inequality, we proceed as follows: Let $n := \operatorname{cd} G$; without loss of generality, we may assume $n < \infty$. Let (P_*, ε) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. It then suffices to show that $K := \ker \partial_n \subset P_n$ is a projective $\mathbb{Z}G$-module (then we can truncate the original projective resolution by taking K in degree n instead of P_n).

  To this end, we show that K is a direct summand of P_n (which establishes that also K is projective).

$$\begin{tikzcd}
P_{n+2} \arrow[r, "\partial_{n+2}"] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & P_n \arrow[r, "\partial_n"] \arrow[d, dashed, shift left, "p"] & P_{n-1} \\
& & K \arrow[u, shift left] &
\end{tikzcd}$$

  By assumption,

$$\begin{aligned}
0 &\cong_{\mathbb{Z}} H^{n+1}(G; K) && \text{(definition of } n\text{)} \\
&\cong_{\mathbb{Z}} H^{n+1}\bigl(\operatorname{Hom}_G(P_*, K)\bigr) && \text{(Corollary 1.6.9)} \\
&= \frac{\ker \operatorname{Hom}_G(\partial_{n+2}, K)}{\operatorname{im} \operatorname{Hom}_G(\partial_{n+1}, K)}.
\end{aligned}$$

  In particular, we can apply this to the cocycle $\partial_{n+1} \in \operatorname{Hom}_G(P_{n+1}, K)$; thus, there exists a $\mathbb{Z}G$-homomorphism $p \colon P_n \longrightarrow K$ with $\partial_{n+1} = p \circ \partial_{n+1}$. Because, $\operatorname{im} \partial_{n+1} = \ker \partial_n = K$, we have $p|_K = \operatorname{id}_K$, and so K is a direct summand of P_n. □

The cohomological dimension is also related to the geometric dimension, defined in terms of classifying spaces.

Definition 4.2.3 (geometric dimension)**.** Let G be a group. Then the *geometric dimension* of G is (where the dimension of classifying spaces is the dimension in the sense of CW-complexes)

$$\operatorname{gd} G := \inf\bigl\{\dim X \bigm| X \text{ is a classifying space for } G\bigr\} \in \mathbb{N} \cup \{\infty\}.$$

Proposition 4.2.4 (cohomological vs. geometric dimension)**.** *Let G be a group. Then*

$$\operatorname{cd} G \leq \operatorname{gd} G.$$

Proof. We may assume that $n := \operatorname{gd} G$ is finite. Hence, there exists a classifying space X for G that has dimension n (as CW-complex). Then $(C_*^{\mathsf{CW}}(\widetilde{X}; \mathbb{Z}), \varepsilon^{\mathsf{CW}})$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (Theorem 4.1.24); because, $\dim X = n$, we have

$$C_k^{\mathsf{CW}}(\widetilde{X};\mathbb{Z}) \cong_{\mathbb{Z}G} 0$$

for all $k \in \mathbb{N}_{>n}$. Therefore, $\operatorname{cd} G \leq n = \operatorname{gd} G$. □

Proposition 4.2.5 (dimension, inheritance properties)**.** *Let G be a group.*

1. *If $H \subset G$ is a subgroup, then $\operatorname{cd} H \leq \operatorname{cd} G$ and $\operatorname{gd} H \leq \operatorname{gd} G$.*
2. *If $H \subset G$ is a subgroup of finite index and $\operatorname{cd} G < \infty$, then $\operatorname{cd} H = \operatorname{cd} G$.*
3. *If $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ is an extension of groups, then*
$$\operatorname{cd} G \leq \operatorname{cd} N + \operatorname{cd} Q.$$
4. *If H is another group, then*
$$\operatorname{gd}(G * H) = \max\bigl(\operatorname{gd} G, \operatorname{gd} H\bigr).$$

Proof. Ad 1. The claim about cohomological dimension follows, e.g., from the Shapiro lemma (Theorem 1.7.8; Exercise). The claim about geometric dimension follows from covering theory (Theorem 4.1.13).

Ad 2. By the first part, $\operatorname{cd} H \leq \operatorname{cd} G$. For the converse inequality, we use the characterisation of cohomological dimension via free coefficient modules (Proposition 4.2.2): Let $n := \operatorname{cd} G \in \mathbb{N}$. Then there exists a free $\mathbb{Z}G$-module A with $H^n(G;A) \not\cong_{\mathbb{Z}} 0$. Because A is free and $[G:H] < \infty$, there is a free $\mathbb{Z}H$-module B with
$$A \cong_{\mathbb{Z}G} \operatorname{Ind}_H^G B \cong_{\mathbb{Z}G} \operatorname{Coind}_H^G B$$
(the latter isomorphism follows from Proposition 1.7.7). Therefore, by the Shapiro lemma (Theorem 1.7.8), we have
$$H^n(H;B) \cong_{\mathbb{Z}} H^n(G;\operatorname{Coind}_H^G B) \cong_{\mathbb{Z}} H^n(G;A) \not\cong_{\mathbb{Z}} 0,$$
and so $\operatorname{cd} H \geq n = \operatorname{cd} G$.

Ad 3. This is a standard spectral sequence argument (Exercise): We may assume that $n := \operatorname{cd} N$ and $m := \operatorname{cd} Q$ are finite. Let A be a left $\mathbb{Z}G$-module. Then the Hochschild-Serre spectral sequence (Theorem 3.2.12) gives a converging cohomological spectral sequence
$$E_2^{pq} = H^p\bigl(Q; H^q(N;\operatorname{Res}_N^G A)\bigr) \Longrightarrow H^{p+q}(G;A).$$
Let $k \in \mathbb{N}_{>m+n}$ and let $p,q \in \mathbb{N}$ with $p+q=k$. Then $E_2^{pq} \cong_{\mathbb{Z}} 0$. Therefore, also $E_\infty^{pq} \cong_{\mathbb{Z}} 0$; i.e., all terms on the k-"diagonal" are trivial. Convergence thus implies that $H^k(G;A) \cong_{\mathbb{Z}} 0$. Hence, $\operatorname{cd} G \leq n+m$.

Ad 4. This follows from the fact that we can take a wedge of classifying spaces for G and H as classifying space for $G * H$ (Theorem 4.1.13).

We also have $\operatorname{cd}(G*H) = \max\bigl(\operatorname{cd} G, \operatorname{cd} H\bigr)$ [12, Proposition VIII.2.4]; this can, for example, be shown by arguing as in Example 4.1.29 via a Mayer-Vietoris sequence with twisted coefficients. □

Example 4.2.6 (cohomological/geometric dimension).

- If G is a non-trivial finite group, then $\operatorname{cd} G = \operatorname{gd} G = \infty$ (Proposition 4.2.4 and Corollary 1.7.3).

- A group G satisfies $\operatorname{cd} G = 0$ if and only if G is trivial. Clearly, the trivial group has cohomological dimension 0. Conversely, Let $\operatorname{cd} G = 0$ and let C be a cyclic subgrup of G. Then
$$\operatorname{cd} C \leq \operatorname{cd} G = 0$$
and so C is trivial (by the previous example and the computation of $H^1(\mathbb{Z};\mathbb{Z})$).

- If G is a free group, then by Proposition 4.2.4 and using wedges of circles as a classifying space (Example 4.1.19), we obtain
$$\operatorname{cd} G \leq \operatorname{gd} G \leq 1$$
Conversely, a group G satisfies $\operatorname{cd} G \leq 1$ if and only if G is free [84].

- For all $n \in \mathbb{N}$, we have
$$\operatorname{cd} \mathbb{Z}^n = \operatorname{gd} \mathbb{Z}^n = n :$$
On the one hand, $\operatorname{cd} \mathbb{Z}^n \leq \operatorname{gd} \mathbb{Z}^n \leq n$ (by Proposition 4.2.4 and taking the n-torus as classifying space); on the other hand, $\operatorname{cd} \mathbb{Z}^n \geq n$ (by the Künneth theorem; Corollary 3.2.23).

- For all $g \in \mathbb{N}_{\geq 2}$, the surface group Γ_g satisfies
$$\operatorname{cd} \Gamma_g = \operatorname{gd} \Gamma_g = 2 :$$
On the one hand, $\operatorname{cd} \Gamma_g \leq \operatorname{gd} \Gamma_g \leq \dim \Sigma_g = 2$; on the other hand, $\operatorname{cd} \Gamma_g \geq 2$ (by the computation in Example 4.1.28).

  In particular, Γ_g does *not* contain a subgroup that is isomorphic to $\mathbb{Z}^3$; in fact, geometric methods also allow to show that Γ_g does not even contain $\mathbb{Z}^2$ [37][53, Corollary 7.5.15].

- The Heisenberg group $H \subset \mathrm{SL}_3(\mathbb{Z})$ satisfies $\operatorname{cd} H = \operatorname{gd} H = 3$: On the one hand, $\dim H \leq \operatorname{gd} H \leq 3$ (because of the Heisenberg manifold; Exercise); on the other hand, $H_3(H;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z} \not\cong_{\mathbb{Z}} 0$ (Exercise) and so $\operatorname{cd} H \geq 3$.

Corollary 4.2.7. *Let G be a group that contains a non-trivial torsion element. Then*
$$\operatorname{gd} G = \operatorname{cd} G = \infty.$$

Proof. Because G contains a non-trivial torsion element g, there is also a non-trivial finite cyclic subgroup $\langle g \rangle_G$ of G. Hence,

$$\begin{aligned} \infty = \operatorname{cd}\langle g \rangle_G & & \text{(Example 4.2.6)} \\ \leq \operatorname{cd} G & & \text{(Proposition 4.2.5)} \\ \leq \operatorname{gd} G, & & \text{(Proposition 4.2.4)} \end{aligned}$$

as claimed. □

Example 4.2.8. By Corollary 4.2.7, the groups D_∞ and $\mathrm{SL}_2(\mathbb{Z})$ have infinite cohomological/geometric dimension (because they contain torsion elements), but they contain free subgroups of finite index.

Usually, the contraposition is applied: If a group admits a finite-dimensional classifying space, then this group must be torsion-free:

Example 4.2.9 (fundamental groups of manifolds of non-positive curvature)**.** If G is the fundamental group of a closed smooth manifold M that admits a Riemannian of non-positive sectional curvature, then M is a finite-dimensional classifying space for G (Outlook 4.1.21). In particular, G is torsion-free (Corollary 4.2.7).

Caveat 4.2.10 (Eilenberg-Ganea problem)**.** Let G be a group and let $n := \operatorname{cd} G$. If $n \neq 2$, then it is known that also $\operatorname{gd} G = n$ [12, Theorem VIII.7.1]. Moreover, if $\operatorname{cd} G = 2$, then one at least has $\operatorname{gd} G \leq 3$ [12, Theorem VIII.7.1]. However, it is still an open problem if every group of cohomological dimension 2 has geometric dimension 2.

Outlook 4.2.11 (lattices)**.** An important class of finitely generated groups are lattices (i.e., discrete subgroups with finite covolume) of Lie groups. The cohomological and geometric dimension can be determined in terms of the Borel-Serre compactification of the associated symmetric space [12, Chapter VIII.9].

4.2.2 Finite type

We briefly indicate degree-wise finiteness types, both geometrically and algebraically:

Definition 4.2.12 (type F, F_n, F_∞)**.** Let G be a group and let $n \in \mathbb{N}$.

- The group G is of type F if there exists a classifying space for G that consists of only finitely many cells (whence, finite-dimensional).
- The group G is of type F_n if there exists a classifying space for G (possibly of infinite dimension) whose n-skeleton is finite.

- The group G is of type F_∞ if there exists a classifying space for G that, in each dimension, contains only finitely many open cells.

Remark 4.2.13 (relation with classical finiteness conditions)**.** Let G be a group.

- The concrete description of the fundamental group of CW-complexes in terms of their 2-skeleta [83, Chapter 4.1.6f] and the inductive construction of classifying spaces from presentation complexes (Remark 4.1.12) shows that:
  - Every group G is of type F_0.
  - The group G is finitely generated if and only if it is of type F_1.
  - The group G is finitely presented if and only if it is of type F_2.
- If G is of type F, then clearly $\operatorname{cd} G \leq \operatorname{gd} G < \infty$ and G is of type F_∞.
- The converse does *not* hold, in general: If G is finite (and non-trivial), then G is of type F_∞ (e.g., the standard simplicial model works), but *not* of type F (because $\operatorname{cd} G = \infty$; Corollary 4.2.7).

  Moreover, the free group $F(\mathbb{N})$ satisfies $\operatorname{cd} F(\mathbb{N}) = \operatorname{gd} F(\mathbb{N}) = 1 < \infty$, but $F(\mathbb{N})$ is not finitely generated (check!) and thus *not* of type F_1.
- There exist finitely presented groups that are *not* of type F_3 [81].

Definition 4.2.14 (type FP, FP_n, FP_∞ [6])**.** Let G be a group and let $n \in \mathbb{N}$.

- The group G is of type FP if there exists a finite-length projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ that consists of finitely generated $\mathbb{Z}G$-modules.
- The group G is of type FP_n if there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that $P_0, \dots, P_n$ are finitely generated $\mathbb{Z}G$-modules.
- The group G is of type FP_∞ if there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_n is finitely generated for each $n \in \mathbb{N}$.

Remark 4.2.15 (relation between F and FP)**.** Let G be a group and $n \in \mathbb{N}$. In view of Theorem 4.1.24, we have:

- If G has type F, then it has type FP.
- If G has type F_n, then it has type FP_n.
- If G has type F_∞, then it has type FP_∞.

The converse implications are more delicate: There exist groups of type FP that are *not* of type F_2 [5]. However, for finitely presented groups, type FP_n coincides with F_n for all $n \in \mathbb{N}_{\geq 3}$, by work of Eilenberg-Ganea and Wall [12, Chapter VIII.7].

Outlook 4.2.16 (Euler characteristic)**.** If a group G is of type F or FP, then one can, for instance, consider its *Euler characteristic*

$$\chi(G) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \dim_{\mathbb{Q}} H_n(G; \mathbb{Q}).$$

The Euler characteristic can be used to show that certain groups are *not* commensurable (Exercise).

Using the theory of von Neumann dimensions, one sees that the Euler characteristic can also be computed via (co)homology with ℓ^2-coefficients [1][55, Theorem 1.35]. Therefore, the invariance properties of these coefficients also carry over to the Euler characteristic; for example:

- Amenable groups of type F have Euler characteristic 0 [16][55, Corollary 6.75].
- (Fundamental groups of) Oriented closed connected connected hyperbolic manifolds of even dimension have non-zero Euler characteristic [22][55, Theorem 1.62].
- The sign of the Euler characteristic is an orbit equivalence invariant for groups of type F [29].

Moreover, the (virtual) Euler characteristic of (arithmetic) lattices is also related to their arithmetic properties [38, 76][12, Chapter IX].

4.3 Application: Free actions on spheres

As conclusion of this course, we use the theory of group (co)homology to approach a classical problem in geometric topology, the *space form problem*: Which manifolds have a sphere as universal covering? In terms of group actions, one can reformulate a part of this problem as follows:

Questions 4.3.1. Which finite groups can act freely on spheres?

There are some obvious positive examples (but it seems hard to exclude groups by elementary means):

Example 4.3.2 (free actions on spheres)**.**

- Let $n \in \mathbb{N}_{>0}$. Then the finite cyclic group $\mathbb{Z}/n$ acts freely on the circle S^1 (the one-dimensional sphere) by rotation around $2 \cdot \pi/n$.
- The generalised quaternion groups (Outlook 1.6.20) act freely on the unit sphere of the quaternion algebra (which is S^3) by multiplication in the quaternion algebra.

Theorem 4.3.3 (free actions on spheres [78, 59])**.** *Let G be a finite group that admits a free continuous action on a sphere. Then:*

- *If $p \in \mathbb{N}$ is an odd prime, then every p-Sylow subgroup of G is cyclic.*
- *The group G contains at most one element of order 2 and the 2-Sylow subgroups of G are cyclic or isomorphic to a generalised quaternion group.*

The theorem does not mention group homology, but the proof (going back to work by Smith and Milnor) will take advantage of group homology. More precisely, group actions on spheres lead to periodic resolutions (Chapter 4.3.1); we can then use our knowledge on homology of Abelian groups and the recognition of cyclic groups to derive the theorem (Chapter 4.3.2).

Before giving the proof, we illustrate the power of the theorem in simple examples.

Example 4.3.4. The group $\mathbb{Z}/2019 \times \mathbb{Z}/2019$ does *not* admit a free action on a sphere: We have $2019 = 3 \cdot 673$ and the 3-Sylow subgroups of $\mathbb{Z}/2019 \times \mathbb{Z}/2019$ are isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$, which is *not* cyclic. Therefore, Theorem 4.3.3 implies that $\mathbb{Z}/2019 \times \mathbb{Z}/2019$ does *not* admit a free action on a sphere.

Example 4.3.5 (symmetric groups acting on spheres? [59, Corollary 2])**.** Let $n \in \mathbb{N}_{\geq 3}$. Then the symmetric group S_n does *not* admit a free action on a sphere because S_n then contains at least two elements of order 2 (e.g., two different transpositions), which is excluded by Theorem 4.3.3.

Example 4.3.6. Let G be a non-trivial finite group. Then $G \times G$ does *not* admit a free action on a sphere. This can be deduced from the Sylow theorems (Theorem III.1.3.35) and Theorem 4.3.3 (check!).

4.3.1 From sphere actions to periodic resolutions

Theorem 4.3.7 (from sphere actions to periodic resolutions)**.** *Let $n \in \mathbb{N}$ be odd and let $G \curvearrowright S^n$ be a free continuous action of a finite group G on S^n. Then there exists a projective resolution (P_*, ε) of $\mathbb{Z}$ over $\mathbb{Z}G$ that is $(n+1)$-periodic, i.e., for all $k \in \mathbb{N}$, $\ell \in \mathbb{N}_{>0}$, we have*

$$P_{k+n+1} = P_k \quad \textit{and} \quad \partial_{\ell+n+1} = \partial_\ell.$$

The basic idea is to consider a cellular $\mathbb{Z}G$-chain complex of S^n and to use the fact that $H_0(S^n;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z} \cong_{\mathbb{Z}} H_n(S^n;\mathbb{Z})$ to splice together infinitely many copies of the cellular chain complex, which then results in a periodic projective resolution. However, for this to work, we need to know that the G-action on $H_n(S^n;\mathbb{Z})$ is trivial; this problem is solved by the Lefschetz number, an "Euler characteristic" for self-maps (replacing dimensions by traces), and the Lefschetz fixed point theorem:

Definition 4.3.8 (Lefschetz number)**.** Let X be a finite CW-complex and let $f\colon X \longrightarrow X$ be a continuous map. Then the *Lefschetz number of f* is

$$\Lambda(f) := \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr}_{\mathbb{Z}}\bigl(H_n(f;\mathbb{Z})\colon H_n(X;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})\bigr) \in \mathbb{Z}.$$

Here, $\operatorname{tr}_{\mathbb{Z}}$ denotes the trace on the free part of the corresponding finitely generated $\mathbb{Z}$-module.

Example 4.3.9 (Euler characteristic as Lefschetz number)**.** If X is a finite CW-complex, then

$$\begin{aligned}\Lambda(\mathrm{id}_X) &= \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr}_{\mathbb{Z}} H_n(\mathrm{id}_X;\mathbb{Z}) = \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{rk}_{\mathbb{Z}} H_n(X;\mathbb{Z}) \\ &= \chi(X).\end{aligned}$$

Theorem 4.3.10 (Lefschetz fixed point theorem [23, Chapter VII.6])**.** *Let X be a finite CW-complex and let $f\colon X \longrightarrow X$ be a continuous map. If f has no fixed points, then $\Lambda(f) = 0$.*

Corollary 4.3.11 (action on top homology)**.** *Let $n \in \mathbb{N}$ and let $G \curvearrowright S^n$ be a free continuous action of a group G on S^n.*

1. *If n is even, then $G \cong_{\mathsf{Group}} 1$ or $G \cong_{\mathsf{Group}} \mathbb{Z}/2$.*
2. *If n is odd, then the induced G-action on $H_n(S^n;\mathbb{Z})$ is trivial.*

Proof. If $n = 0$, then $|S^n| = 2$ and so $|G| \leq 2$. This implies that G is trivial or isomorphic to $\mathbb{Z}/2$. Therefore, in the following, we only need to consider the case that $n > 0$.

Let $g \in G$ be a non-trivial element and let $f_g\colon S^n \longrightarrow S^n$ be the corresponding homeomorphism; because the action is free, f_g does *not* have a fixed point. As the homology of S^n is concentrated in the (distinct) degrees 0 and n, the Lefschetz fixed point theorem (Theorem 4.3.10) shows that

$$\begin{aligned}0 &= \Lambda(f_g) \\ &= \operatorname{tr}_{\mathbb{Z}} H_0(f_g;\mathbb{Z}) + (-1)^n \cdot \operatorname{tr}_{\mathbb{Z}} H_n(f_g;\mathbb{Z}) \\ &= 1 + (-1)^n \cdot \deg f_g.\end{aligned}$$

We now distinguish two cases:

- If n is odd, then $\deg f_g = 1$. In particular, g acts trivially on $H_n(S^n;\mathbb{Z})$.
- If n is even, then $\deg f_g = -1$. Therefore, *all* non-trivial elements of G act by multiplication by -1 on $H_n(S^n;\mathbb{Z})$. The multiplicativity of the degree shows that, for all $h \in G \setminus \{e\}$, we have

  $$\deg f_{g\cdot h} = \deg(f_g \circ f_h) = \deg f_g \cdot \deg f_h = (-1)\cdot(-1) = 1,$$

  and thus $g \cdot h = e$. In particular, $G \cong_{\mathsf{Group}} 1$ or $G \cong_{\mathsf{Group}} \mathbb{Z}/2$. $\square$

Proof of Theorem 4.3.7. There exists a G-equivariant finite n-dimensional CW-structure on S^n such that G freely permutes the open cells (the quotient space $G \setminus S^n$ is an odd(!)-dimensional closed topological n-manifold, whence homeomorphic to an n-dimensional CW-complex [60][47, p. 107][68]). Let C_* be the associated cellular chain complex. Because G acts freely on the open cells, C_* is a free $\mathbb{Z}G$-chain complex that is concentrated in degrees $0, \dots, n$.

Let

$$\begin{aligned} \eta \colon \mathbb{Z} \cong_{\mathbb{Z}} H_n(S^n;\mathbb{Z}) \cong_{\mathbb{Z}} \ker \partial_n \longrightarrow C_n \\ \varepsilon \colon C_0 \longrightarrow C_0/\operatorname{im} \partial_1 \cong_{\mathbb{Z}} H_0(S^n;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z} \end{aligned}$$

be the inclusion and projection, respectively. Then the doubly augmented sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is exact (the homology of S^n, whence of C_*, in degrees $1, \dots, n-1$ is trivial). The G-action on $H_0(S^n;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$ is trivial (because continuous self-maps of path-connected spaces induce the identity on $H_0(\,\cdot\,;\mathbb{Z})$); moreover, also the G-action on $H_n(S^n;\mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$ is trivial (Corollary 4.3.11). Therefore, the spliced sequence

$$\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\eta\circ\varepsilon} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, which is periodic of period $n+1$. □

4.3.2 From periodic resolutions to Sylow subgroups

We can now complete the proof of Theorem 4.3.3 on finite groups that admit free actions on spheres:

Proof of Theorem 4.3.3. Let $n \in \mathbb{N}$ and let $G \curvearrowright S^n$ be a free continuous action. In view of Corollary 4.3.11, we may assume without loss of generality that n is odd (as the trivial group and $\mathbb{Z}/2$ clearly satisfy the conclusion of the theorem).

Ad 1. Let $p \in \mathbb{N}$ be an odd prime and let $S \subset G$ be a p-Sylow subgroup of G. In order to show that S is cyclic, it suffices to prove that all Abelian subgroups of S are cyclic (Corollary 1.6.19).

Let $H \subset S$ be an Abelian subgroup. *Assume* for a contradiction that H is *not* cyclic. As H is a p-group, this implies that H contains a subgroup $\overline{H}$ that is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$ (check!). As subgroup of G also $\overline{H}$ admits a free action on S^n. Therefore, there exists a periodic projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\overline{H}$ (Theorem 4.3.7); in particular, the homology $H_*(\overline{H};\mathbb{F}_p)$ has to be periodic. However, this contradicts the Künneth computation in Example 3.2.25. Therefore, all Abelian subgroups of the p-Sylow group S are cyclic and so S is cyclic.

Ad 2. Similarly, also the prime 2 can be handled. We first show that G contains at most one element of order 2. As additional ingredient, we need Milnor's generalisation of the Borsuk-Ulam theorem [59, Theorem 1]:

> Let $T\colon S^n \longrightarrow S^n$ be a continuous map without fixed points that satisfies $T \circ T = \mathrm{id}_{S^n}$. Then, for every continuous map $f\colon S^n \longrightarrow S^n$ of odd degree, there exists an $x \in S^n$ with
>
> $$T \circ f(x) = f \circ T(x).$$

Let $f, g\colon S^n \longrightarrow S^n$ be the homeomorphisms corresponding to elements of order 2 in G. Then $f \circ f = \mathrm{id}_{S^n} = g \circ g$ and $\deg f, \deg g \in \{-1, 1\}$; in particular, f and g have odd degree. By the above theorem, $f \circ g$ and $g \circ f$ coincide at one point; because the action is free, we thus have $f \circ g = g \circ f$. Therefore, all elements of order 2 in G commute.

As in the argument for the case of odd primes, we know that G does *not* contain a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Therefore, the elements inducing f and g must be equal, i.e., G contains at most one element of order 2.

This implies that the 2-Sylow subgroups of G are cyclic or a generalised quaternion group (Outlook 1.6.20). $\square$

Further information on the space form problem and on groups with periodic cohomology can be found in the literature [19][12, Chapter VI.9].

More generally, investigating group actions via group (co)homology is a recurring theme in topology, geometry, and the theory of arithmetic groups.

A

Appendix

Overview of this chapter.

A.1 Amalgamated free products

We will now briefly review some concepts from group theory that allow to construct coproducts and pushouts of groups explicitly.

A.1.1 The free group of rank 2

We start with an explicit description of the free group of rank 2, using reduced words [53, Chapter 3.3, Chapter 2.2]. Roughly speaking, this group is the group generated by two different elements with the least possible relations between these elements.

Definition A.1.1 (group of reduced words)**.** Let $a, b, \widehat{a}, \widehat{b}$ four distinct elements. Let W be the set of words (i.e., finite sequences) over $S := \{a, b, \widehat{a}, \widehat{b}\}$.

- Let $n \in \mathbb{N}$ and let $x_1, \dots, x_n \in S$. The word $x_1 \dots x_n \in W$ is *reduced* if
$$x_{j+1} \neq \widehat{x_j} \quad \text{and} \quad \widehat{x_{j+1}} \neq x_j$$
holds for all $j \in \{1, \dots, n-1\}$. In particular, the empty word ε is reduced.
- We write $F(a, b)$ for the set of all reduced words over S.
- On $F(a, b)$, we define a composition by concatenation and reduction:
$$\begin{aligned} \cdot \colon F(a,b) \times F(a,b) &\longrightarrow F(a,b) \\ (x_1 \dots x_n, x_{n+1} \dots x_m) &\longmapsto x_1 \dots x_{n-r} x_{n+1+r} \dots x_{n+m}. \end{aligned}$$
Here,
$$\begin{aligned} r := \max\bigl\{k \in \{0, \dots, \min(n, m-1)\} \bigm| \forall_{j \in \{0, \dots, k-1\}} \quad & x_{n-j} = \widehat{x_{n+1+j}} \\ \vee\ & \widehat{x_{n-j}} = x_{n+1+j} \bigr\}. \end{aligned}$$

Example A.1.2. In the situation of the previous definition, the word $ab\widehat{a}\widehat{b}$ is reduced; the word $ba\widehat{a}b$ is *not* reduced. The elements a and $\widehat{a}$ are inverse to each other with respect to "$\cdot$"; analogously, also b and $\widehat{b}$ are inverse to each other. Hence, one usually writes a^{-1} and b^{-1} instead of $\widehat{a}$ and $\widehat{b}$, respectively.

Proposition A.1.3 (free group of rank 2)**.**

1. *The set $F(a, b)$ is a group with respect to the composition specified in the previous definition.*

2. *The set* $\{a, b\}$ *is a* free generating set *of* $F(a, b)$*, i.e., the following universal property is satisfied:*

   For every group H *and every map* $f\colon \{a, b\} \longrightarrow H$*, there exists a unique group homomorphism* $\overline{f}\colon F(a, b) \longrightarrow H$ *with* $\overline{f}|_{\{a,b\}} = f$*.*

3. *In other words,*

$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & \big\downarrow {\scriptstyle 1\mapsto b} \\ \mathbb{Z} & \xrightarrow[1\mapsto a]{} & F(a, b) \end{array}$$

   is a pushout in Group.

Proof. The first part follows from a straightforward computation (associativity is *not* obvious!) [53, Chapter 3.3].

The second part (and the third part) can be verified directly by hand (check!). □

A.1.2 Free products of groups

More generally, we can also consider the free product of a family of groups. Again, we are looking for a group generated by the given groups with as few relations between them as possible.

Definition A.1.4 (free product of groups)**.** Let $(G_i)_{i\in I}$ be a family of groups; for $g \in \coprod_{i\in I}(G_i \setminus \{1\})$ let $i(g) \in I$ be the unique index with $g \in G_{i(g)}$.

- A finite (possibly empty) sequence $(s_1, \dots, s_n)$ with $n \in \mathbb{N}$ of non-trivial elements of $\coprod_{i\in I} G_i$ is a *reduced word (over the family* $(G_i)_{i\in I}$*)*, if

$$\forall_{j\in\{1,\dots,n-1\}} \quad i(s_j) \neq i(s_{j+1}).$$

- We write $\bigstar_{i\in I} G_i$ for the set of all reduced words over the family $(G_i)_{i\in I}$.
- On $\bigstar_{i\in I} G_i$, we define a composition by concatenation/reduction:

$$\begin{aligned} \cdot\colon \bigstar_{i\in I} G_i \times \bigstar_{i\in I} G_i &\longrightarrow \bigstar_{i\in I} G_i \\ \bigl(s = (s_1, \dots, s_n), t = (t_1, \dots, t_m)\bigr) &\longmapsto \begin{cases} (s_1, \dots, s_{n-k(s,t)}, t_{k(s,t)+1}, \dots, t_m) & ① \\ (s_1, \dots, s_{n-k(s,t)} \cdot t_{k(s,t)+1}, \dots, t_m) & ② \end{cases} \end{aligned}$$

  Here, $k(s, t) \in \{0, \dots, \min(n, m)\}$ is the biggest $k \in \{0, \dots, \min(n, m)\}$ satisfying

$$\forall_{j\in\{1,\dots,k\}} \quad i(s_{n-j+1}) = i(t_j) \wedge s_{n-j+1} = t_j^{-1}.$$

Case ① occurs if $i(s_{n-k(s,t)}) \neq i(t_{k(s,t)+1})$; case ② occurs if $i(s_{n-k(s,t)}) = i(t_{k(s,t)+1})$.

- We call $\bigstar_{i\in I}$, together with this composition, the *free product of* $(G_i)_{i\in I}$.

The free product $G := \bigstar_{i\in I} G_i$ of a family $(G_i)_{i\in I}$ indeed is a group (again, associativity is non-trivial!) and the canonical inclusions $G_i \longrightarrow G$ are group homomorphisms.

Free products are an explicit model of coproducts of groups:

Proposition A.1.5 (coproduct of groups)**.** *Let $(G_i)_{i\in I}$ be a family of groups. Then $\bigstar_{i\in I} G_i$, together with the canonical inclusions $(G_i \longrightarrow \bigstar_{j\in I} G_j)_{i\in I}$, is the coproduct of the family $(G_i)_{i\in I}$ in the category* Group.

Proof. This can be shown by verifying the universal property (check!). □

A.1.3 Amalgamated free products of groups

"Glueing" groups along another group leads to the amalgamated free product:

Definition A.1.6 (amalgamated free product)**.** Let G_0, G_1, and G_2 be groups and let $i_1 \colon G_0 \longrightarrow G_1$ as well as $i_2 \colon G_0 \longrightarrow G_2$ be group homomorphisms. The associated *amalgamated free product of* G_1 *and* G_2 *over* G_0 is defined by

$$G_1 *_{G_0} G_2 := (G_1 * G_2)/N,$$

where $N \subset G_1 * G_2$ is the smallest (with respect to inclusion) normal subgroup of $G_1 * G_2$ that contains the set $\{i_1(g) \cdot i_2(g)^{-1} \mid g \in G_0\}$.

Proposition A.1.7 (pushouts of groups)**.** *Let G_0, G_1, and G_2 be groups and let $i_1 \colon G_0 \longrightarrow G_1$ as well as $i_2 \colon G_0 \longrightarrow G_2$ be group homomorphisms. Let $j_1 \colon G_1 \longrightarrow G_1 *_{G_0} G_2$ and $j_2 \colon G_2 \longrightarrow G_1 *_{G_0} G_2$ be the homomorphisms induced by the canonical inclusions $G_1 \longrightarrow G_1 * G_2$ and $G_2 \longrightarrow G_1 * G_2$, respectively. Then*

$$\begin{tikzcd} G_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"'] & G_1 *_{G_0} G_2 \end{tikzcd}$$

is a pushout in Group.

Proof. This can be shown by the same argument as in the construction of the pushout of topological spaces (Proposition AT.1.1.14), using the universal property of the free product and of quotient groups. □

A.1.4 Free groups

A related generalisation of $F(a,b)$ are general free groups; the universal property of free groups/free generating sets is a group-theoretic version of the universal property of bases of vector spaces.

Definition A.1.8 (free generating set, free group, rank of a free group)**.**

- Let G be a group. A subset $S \subset G$ is a *free generating set of* G if the following universal property is satisfied: The group G is generated by S and for every group H and every map $f\colon S \longrightarrow H$ there exists a unique group homomorphism $\overline{f}\colon G \longrightarrow H$ with $\overline{f}|_S = f$.
- A *free group* is a group that contains a free generating set; the cardinality of such a free generating set is the *rank* of the free group.

Caveat A.1.9. Not every group has a free generating set! For example, the groups $\mathbb{Z}/2$ and $\mathbb{Z}^2$ are *not* free (check!).

Comparing the corresponding universal properties establishes existence of free groups of arbitrary rank:

Proposition A.1.10 (existence of free groups)**.** *Let S be a set, let $G := \bigstar_S \mathbb{Z}$ be the associated free product and for every $s \in S$ let $i_s\colon \mathbb{Z} \longrightarrow G$ be the inclusion of the s-th summand. Then $\{i_s(1) \mid s \in S\}$ is a free generating set of G.*

Proof. We can translate the universal property of coproducts into the universal property of free generating sets (because the building blocks are the groups $\mathbb{Z}$, which are free of rank 1) (check!). $\square$

Proposition A.1.11 (invariance of rank of free groups)**.** *Let G and G' be free groups with free generating sets S and S', respectively. Then G and G' are isomorphic if and only if $|S| = |S'|$.*

Proof. This can be shown, for example, by looking at homomorphisms to $\mathbb{Z}/2$ and a cardinality argument [53, Exercise 2.E.12]. $\square$

A.2 Some homological algebra

In this appendix, we recall two basic results from homological algebra: The long exact homology sequence and the horseshoe lemma. For simplicity, we only consider homological algebra in module categories (instead of general Abelian categories); in view of the Freyd-Mitchell embedding theorem, this is not a substantial limitation.

Setup A.2.1. In the following, R will always be a (not necessarily commutative) ring with unit.

Proposition A.2.2 (algebraic long exact homology sequence). *Let*

$$0 \longrightarrow A_* \xrightarrow{\ i\ } B_* \xrightarrow{\ p\ } C_* \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Ch}$ *(i.e., in every degree, the corresponding sequence in* ${}_R\mathsf{Ch}$ *is exact). Then there is a (natural) long exact sequence*

$$\cdots \xrightarrow{\partial_{k+1}} H_k(A_*) \xrightarrow{H_k(i)} H_k(B_*) \xrightarrow{H_k(p)} H_k(C_*) \xrightarrow{\partial_k} H_{k-1}(A_*) \longrightarrow \cdots$$

This sequence is natural in the following sense: If

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i_*"] \arrow[d, "f_*"'] & B_* \arrow[r, "p_*"] \arrow[d, "g_*"'] & C_* \arrow[r] \arrow[d, "h_*"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i'_*"'] & B'_* \arrow[r, "p'_*"'] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram in ${}_R\mathsf{Ch}$ *with exact rows, then the corresponding ladder*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{k+1}"] & H_k(A_*) \arrow[r, "H_k(i_*)"] \arrow[d, "H_k(f_*)"'] & H_k(B_*) \arrow[r, "H_k(p_*)"] \arrow[d, "H_k(g_*)"'] & H_k(C_*) \arrow[r, "\partial_k"] \arrow[d, "H_k(h_*)"] & H_{k-1}(A_*) \arrow[r] \arrow[d, "H_{k-1}(f_A)"] & \cdots \\
\cdots \arrow[r, "\partial_{k+1}"'] & H_k(A'_*) \arrow[r, "H_k(i'_*)"'] & H_k(B'_*) \arrow[r, "H_k(p'_*)"'] & H_k(C'_*) \arrow[r, "\partial_k"'] & H_{k-1}(A'_*) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative and has exact rows.

Proof. Let $k \in \mathbb{Z}$. We construct the *connecting homomorphism*

$$\partial_k \colon H_k(C_*) \longrightarrow H_{k-1}(A_*)$$

as follows: Let $\gamma \in H_k(C_*)$; let $c \in C_k$ be a cycle representing γ. Because $p_k \colon B_k \longrightarrow C_k$ is surjective, there is a $b \in B_k$ with

$$p_k(b) = c.$$

As p_* is a chain map, we obtain $p_{k-1} \circ \partial_k^B(b) = \partial_k^C \circ p_k(b) = \partial_k^C(c) = 0$; then exactness in degree k shows that there exists an $a \in A_{k-1}$ with

$$i_{k-1}(a) = \partial_k^B(b).$$

In this situation, we call (a, b, c) a *compatible triple for* γ and we define

$$\partial_k(\gamma) := [a] \in H_{k-1}(A_*).$$

Straightforward diagram chases then show (check!):

- If (a, b, c) is a compatible triple for γ, then $a \in A_{k-1}$ is a cycle (and so indeed defines a class in $H_{k-1}(A_*)$).

- If (a, b, c) and (a', b', c') are compatible triples for γ, then $[a] = [a']$ in $H_{k-1}(A_*)$.

These observations show that ∂_k is an R-homomorphism and that ∂_k is natural (check!).

Further diagram chases then show that the resulting long sequence is exact (even more to check ...). $\square$

Proposition A.2.3 (horseshoe lemma). *Let*

$$0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Mod}$ *and let* (P'_*, ε') *and* (P''_*, ε'') *be projective* R*-resolutions of* M' *and* M''*, respectively:*

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P'_0 \arrow[d, "\varepsilon'"'] & ? & P''_0 \arrow[d, "\varepsilon''"'] & \\
0 \arrow[r] & M' \arrow[r, "f'"] \arrow[d] & M \arrow[r, "f''"] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

There there exists a projective R-resolution (P_*, ε) of M and R-chain maps $\widetilde{f}'_* \mathbin{\square} f' \colon P'_* \mathbin{\square} \varepsilon' \longrightarrow P_* \mathbin{\square} \varepsilon$ and $\widetilde{f}''_* \mathbin{\square} f'' \colon P_* \mathbin{\square} \varepsilon \longrightarrow P''_* \mathbin{\square} \varepsilon''$ such that

$$0 \longrightarrow P'_n \xrightarrow{\widetilde{f}'_n} P_n \xrightarrow{\widetilde{f}''_n} P''_n \longrightarrow 0$$

is exact in every degree $n \in \mathbb{N}$.

Proof. On the level of modules, we define (P_*, ε) as direct sum of the outer resolutions and choose $\widetilde{f}'_*$ and $\widetilde{f}''_*$ as the corresponding inclusion and projection, respectively. The boundary operators are constructed as follows:

As P''_0 is projective and $f'' \colon M \longrightarrow M''$ is surjective, there exists an R-module homomorphism $\widetilde{\varepsilon}'' \colon P''_0 \longrightarrow M$ with

$$f'' \circ \widetilde{\varepsilon}'' = \varepsilon''.$$

We set

$$\varepsilon := (f' \circ \varepsilon') \oplus \widetilde{\varepsilon}'' \colon P_0 = P'_0 \oplus P''_0 \longrightarrow M.$$

Inductively, we construct boundary operators $\partial_{n+1} \colon P_{n+1} \longrightarrow P_n$ that are compatible with the boundary operators on P'_* and P''_*:

Let $n \in \mathbb{N}_{>0}$ and let us suppose, by induction, that ∂_n is already constructed. A diagram chase shows that

$$f''_n(\ker \partial_n) = \ker \partial''_n = \operatorname{im} \partial''_{n+1}$$

(check!). Because the module P''_{n+1} is projective, there exists an R-homomorphism $\widetilde{\partial}''_{n+1} \colon P''_{n+1} \longrightarrow \ker \partial_n$ that satisfies

$$f'' \circ \widetilde{\partial}''_{n+1} = \partial''_{n+1}.$$

We then set

$$\partial_{n+1} := (f' \circ \partial'_{n+1}) \oplus \widetilde{\partial}''_{n+1};$$

a diagram chase yields $\partial_n \circ \partial_{n+1} = 0$ (check!).

In this way, we obtain a short exact sequence

$$0 \longrightarrow P' \mathbin{\square} \varepsilon' \xrightarrow{\widetilde{f}' \square f'} P \mathbin{\square} \varepsilon \xrightarrow{\widetilde{f}'' \square f''} P'' \mathbin{\square} \varepsilon'' \longrightarrow 0$$

in ${}_R\mathsf{Ch}$ (indexed over $\mathbb{N} \cup \{-1\}$). Applying the long exact homology sequence (Proposition A.2.2) shows that (P_*, ε) is exact. Hence, (P_*, ε) is a projective resolution of M with the desired properties. $\square$

A.3 Homotopy theory of CW-complexes

Let us review some terminology for CW-complexes and their basic homotopy-theoretic properties. Roughly speaking, a CW-complex is a topological space that is built from inductively attaching disks of increasing dimension to a set of points.

Definition A.3.1 ((relative) CW-complex)**.**

- Let (X, A) be a pair of spaces. A *relative CW-structure on* (X, A) is a sequence
$$A =: X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X$$
of subspaces of X with the following properties:
  - We have $X = \bigcup_{n \in \mathbb{N}} X_n$.
  - The topology on X coincides with the colimit topology of the system $A = X_{-1} \subset X_0 \subset X_1 \subset \dots$; i.e., a subset $U \subset X$ is open if and only if for every $n \in \mathbb{N} \cup \{-1\}$, the intersection $U \cap X_n$ is open in X_n.
  - For every $n \in \mathbb{N}$, the space X_n is obtained from X_{n-1} by attaching n-dimensional cells, i.e., there exists a set I_n and a pushout of the form
$$\begin{tikzcd} \coprod_{I_n} S^{n-1} \arrow[r] \arrow[d, "\text{inclusion}"'] & X_{n-1} \arrow[d, "\text{inclusion}"] \\ \coprod_{I_n} D^n \arrow[r] & X_n \end{tikzcd}$$
in Top; here, we use the convention $S^{-1} := \emptyset$. Then, X_n is the *n-skeleton of X*. The number $|I_n|$ equals the number of path-connected components of $X_n \setminus X_{n-1}$, but the choice of pushouts is *not* part of the data!

- A *relative CW-complex* is a pair (X, A) of spaces together with a relative CW-structure on (X, A). If $A = \emptyset$, then X, together with this CW-structure, is a *CW-complex*. Usually, we will leave the fibration of the CW-structure implicit and say things like "a relative CW-complex (X, A)" if the underlying CW-structure is clear from the context or irrelevant.

- If (X, A) is a relative CW-complex and $n \in \mathbb{N}$, then the path-connected components of $X_n \setminus X_{n-1}$ are homeomorphic to $D^{n\circ}$ (check!) and are called *open n-cells of* (X, A).

- If (X, A) is a relative CW-complex, then the *dimension of* (X, A) is defined as $\dim(X, A) := \min\{n \in \mathbb{N} \mid \forall_{m \in \mathbb{N}_{\geq n}} \quad X_m = X_n\} \in \mathbb{N} \cup \{\infty\}$.
- A (relative) CW-complex is *finite*, if it consists of finitely many open cells. A (relative) CW-complex is *of finite type*, if in each dimension, it has only finitely many open cells.

The strange prefix "CW" refers to the "closure finiteness" condition on cells (which can be derived from the definition above) and the "weak topology" (i.e., the colimit topology).

Example A.3.2 (CW-structures)**.** Examples of CW-structures on the circle, on the sphere S^2, on the torus $S^1 \times S^1$, and on $\mathbb{R}P^2$ are indicated in Figure A.1.

Caveat A.3.3 (products of CW-complexes)**.** Let X and Y be CW-complexes and let $(Z_n)_{n \in \mathbb{N} \cup \{-1\}}$ be given by $Z_{-1} := \emptyset$ and

$$\forall_{n \in \mathbb{N}} \quad Z_n := \bigcup_{k \in \{0, \dots, n\}} X_k \times Y_{n-k}.$$

Then, in general, $(Z_n)_{n \in \mathbb{N} \cup \{-1\}}$ is *no* CW-structure on $X \times Y$ [24] (and the question of when this happens is rather delicate [11]). Therefore, when working with products of (infinite) CW-complexes, it is sometimes convenient to pass to the category of compactly generated spaces [82].

For CW-complexes, homotopy equivalences can be characterised in the following way:

Theorem A.3.4 (Whitehead theorem)**.** *Let X and Y be CW-complexes and let $f\colon X \longrightarrow Y$ be a continuous map. Then the following are equivalent:*

1. *The map $f\colon X \longrightarrow Y$ is a homotopy equivalence (in* Top*).*
2. *The map $f\colon X \longrightarrow Y$ is a* weak equivalence*, i.e., for every $x_0 \in X$ and every $n \in \mathbb{N}$ the induced map $\pi_n(f)\colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is bijective.*
3. *For every CW-complex Z, the map*

$$\begin{aligned} [Z, f]\colon [Z, X] &\longrightarrow [Z, Y] \\ [g] &\longmapsto [f \circ g] \end{aligned}$$

bijective.

Sketch of proof. Ad $1 \Longrightarrow 2$. Let $n \in \mathbb{N}$. Then, by construction, $\pi_n\colon \mathsf{Top}_* \longrightarrow \mathsf{Set}$ is a homotopy invariant functor. Moreover, one can show that π_n also translates unpointed homotopy equivalences to bijections [85, Proposition 6.2.4].

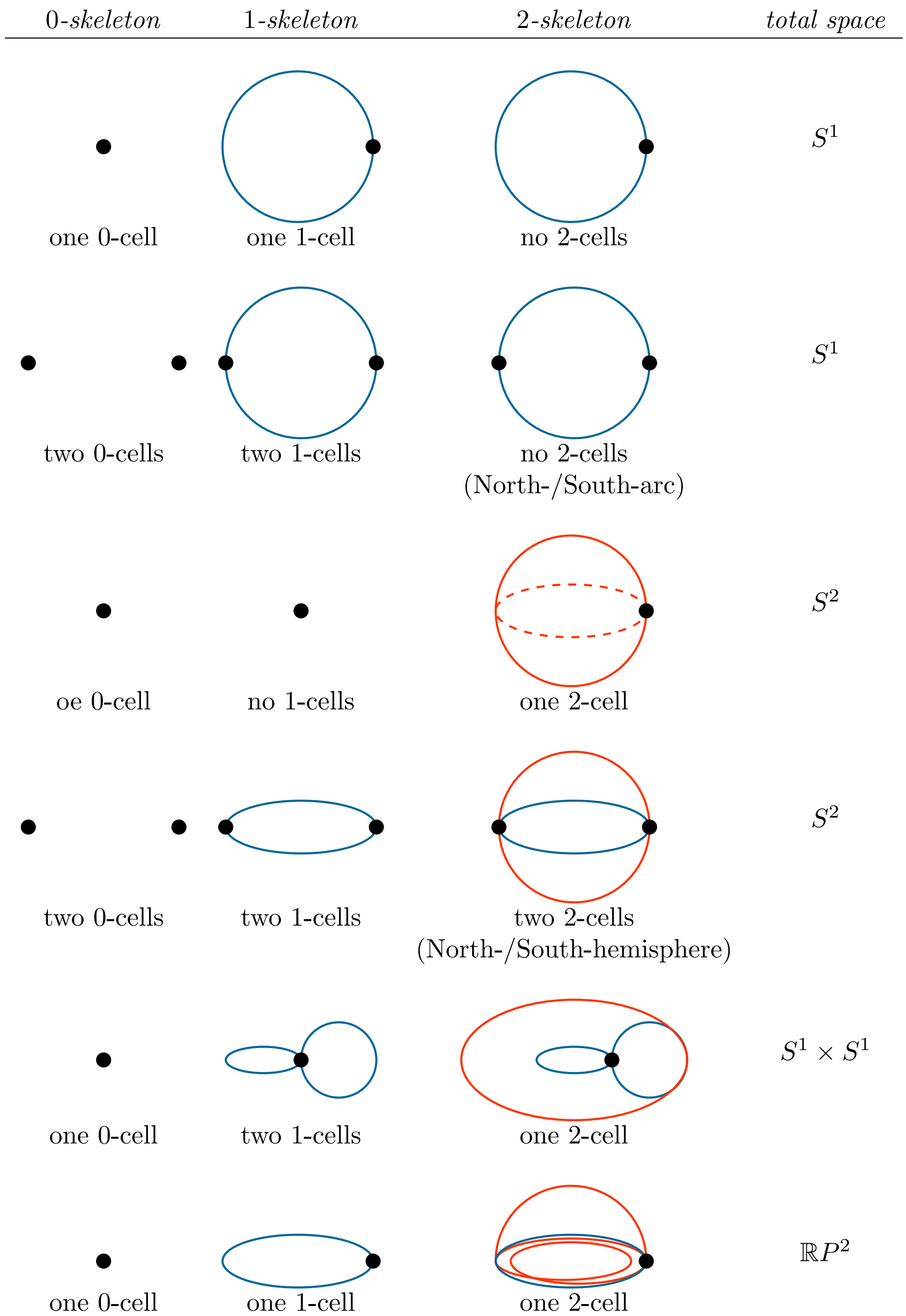


Figure A.1.: Examples of CW-structures

Ad 2 $\Longrightarrow$ 3. Because CW-complexes are built up from cells, one can prove this implication by a careful induction [88, Chapter IV.7, Chapter V.3].

Ad 3 $\Longrightarrow$ 1. If 3. holds, then $[\,\cdot\,, f]$ is a natural isomorphism $[\,\cdot\,, X] \Longrightarrow [\,\cdot\,, Y]$. Then, the Yoneda lemma (Proposition AT.1.2.23) shows $X \simeq Y$. $\square$

Caveat A.3.5.

- The notion of "weak equivalence" is *not* an equivalence relation on the class of topological spaces; in general, symmetry is *not* satisfied [88, p. 221].

  However, the Whitehead theorem shows that on the class of CW-complexes, weak equivalence coincides with homotopy equivalence and thus is an equivalence relation on CW-complexes.

  This is similar to the notion of quasi-isomorphism for chain complexes.

- Abstract isomorphisms between homotopy groups of CW-complexes are *not* sufficient to conclude that the given CW-complexes are homotopy equivalent. It is essential that these isomorphisms are induced by a continuous map.

  For example, the spaces $\mathbb{R}P^2 \times S^3$ and $S^2 \times \mathbb{R}P^3$ (which both admit a CW-structure) have isomorphic homotopy groups (because they have the common covering space $S^2 \times S^3$; Corollary AT.2.3.25), but they are *not* homotopy equivalent (as can be seen from the (cellular) homology in degree 5; check!).

In particular, in the simply connected case, the Whitehead theorem shows that ordinary homology with $\mathbb{Z}$-coefficients is a rather powerful tool.

Corollary A.3.6 (Whitehead theorem, simply connected case)**.** *Let X be a simply connected CW-complex. Then the following are equivalent:*

1. *The space X is contractible (in* Top*).*
2. *For each $x_0 \in X$ and each $n \in \mathbb{N}$, we have $\bigl|\pi_n(X, x_0)\bigr| = 1$.*
3. *For each $n \in \mathbb{N}$, we have $H_n(X; \mathbb{Z}) \cong_{\mathbb{Z}} H_n(\bullet; \mathbb{Z})$.*

Proof. Applying the Whitehead theorem (Theorem A.3.4) to the constant map $X \longrightarrow \bullet$ shows that 1. and 2. are equivalent.

Moreover, because X is simply connected, the equivalence of 2. and 3. is a consequence of the Hurewicz theorem (Corollary AT.4.5.10). $\square$

Remark A.3.7 (homotopy invariants of CW-complexes)**.** Examples of homotopy invariants of CW-complexes are

- homotopy groups (in particular, the fundamental group) and
- cellular/singular homology

B

Exercise Sheets

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 1, April 29, 2019

Exercise 1 (a trace on group rings?). Let G be a group and let

$$\begin{aligned}\tau\colon \mathbb{Z}G &\longrightarrow \mathbb{Z}\\ \sum_{g\in G} a_g \cdot g &\longmapsto a_e.\end{aligned}$$

Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. For all $a, b \in \mathbb{Z}G$, we have $\tau(a \cdot b) = \tau(b \cdot a)$.
2. For all $a \in \mathbb{Z}G$, we have $\tau(a \cdot a^*) \geq 0$, where $(\sum_{g\in G} a_g \cdot g)^* := \sum_{g\in G} a_{g^{-1}} \cdot g$.

Exercise 2 (standard (co)chain complexes in the literature).

1. What is "$C^q(Q, G)$" from the following article in our notation? [I on p. 3/4]

   S. Eilenberg. Topological methods in abstract algebra. Cohomology theory of groups, *Bull. Amer. Math. Soc.*, 55, pp. 3–37, 1949.
2. What is the "bar resolution $C_*(\Gamma)$" from the following article in our notation? [Definition and Lemma 2.1 a)]

   M. Puschnigg. The Kadison-Kaplansky conjecture for word-hyperbolic groups, *Invent. Math.*, 149(1), pp. 153–194, 2002.

Hints. Of course, you do *not* need to read/understand the whole article. It suffices to untangle the terminology and to compare it to our setup. You have to justify your answer in your submission (e.g., by an explicit comparison).

Exercise 3 (the augmentation ideal). Let G be a group and let $I(G) := \ker \varepsilon$ be the *augmentation ideal* (where $\varepsilon\colon C_0(G) = \mathbb{Z}G \longrightarrow \mathbb{Z}$ is the augmentation map).

1. Show that $I(G) = \operatorname{Span}_{\mathbb{Z}}\{g - 1 \mid g \in G\}$.
2. Show that $I(G) = \operatorname{Span}_{\mathbb{Z}G}\{s - 1 \mid s \in S\}$ holds for every generating set $S \subset G$ of G.

Exercise 4 (group rings of cyclic groups). Let $n \in \mathbb{N}_{>0}$, let $G := \mathbb{Z}/n$, let $t := [1] \in G$, and let $N := \sum_{j=0}^{n-1} t^j \in \mathbb{Z}G$. For $a \in \mathbb{Z}G$, we consider the associated $\mathbb{Z}G$-homomorphism $M_a\colon \mathbb{Z}G \longrightarrow \mathbb{Z}G$ given by right multiplication with a.

1. Show that $\operatorname{im} M_N = \ker M_{t-1}$.
2. Show that $\operatorname{im} M_{t-1} = \ker M_N$.

Bonus problem (Kaplansky zero divisor conjecture and unique products).

1. What is the *unique product* property of groups?
2. Give an example of a (non-trivial) group with the unique product property and an example of a group without the unique product property.
3. Show that the group ring $\mathbb{Z}G$ has no non-trivial zero divisors if G is a group with the unique product property.

Submission before May 6, 2019, 10:00, in the mailbox

(Solutions may be submitted in English or German.)

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig — Sheet 2, May 6, 2019

Exercise 1 (induced maps in group homology)**.** Let $\varphi\colon G \longrightarrow H$ be a group homomorphism. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If φ is injective, then $H_1(\varphi;\mathbb{Z})\colon H_1(G;\mathbb{Z}) \longrightarrow H_1(H;\mathbb{Z})$ is injective.
2. If φ is surjective, then $H_1(\varphi;\mathbb{Z})\colon H_1(G;\mathbb{Z}) \longrightarrow H_1(H;\mathbb{Z})$ is surjective.

Exercise 2 (finitary symmetric groups)**.** Let X be a set with $|X| \geq 2$ and let $\operatorname{FSym}(X)$ be the group of all finitary permutations of X; a bijection $f\colon X \longrightarrow X$ is *finitary* if the set $\{x \in X \mid f(x) \neq x\}$ is finite.

1. Compute $H^1\bigl(\operatorname{FSym}(X);\mathbb{Z}\bigr)$.
2. Compute $H_1\bigl(\operatorname{FSym}(X);\mathbb{Z}\bigr)$.

Exercise 3 (certain groups of homeomorphisms in the literature)**.** We consider the following article:

> J. N. Mather. The vanishing of the homology of certain groups of homeomorphisms, *Topology*, 10, pp. 297–298, 1971.

1. What is the main result of this article?
2. Name at least two further published articles whose titles contain the string "certain groups of homeomorphisms".

   Hints. The database https://mathscinet.ams.org/mathscinet might help. Access to Mathscinet requires a subscription; the website can be accessed through the campus network (or using SSH tunnels via UR).
3. *Bonus problem.* Can you make sense of the definition of $C(G)$ in the proof of the lemma on p. 297?
4. *Bonus problem.* What is φ in (a) on p. 298? How do you think that this happened?

Exercise 4 (the integral Heisenberg group)**.** Let

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,z \in \mathbb{Z} \right\} \subset \operatorname{SL}(3,\mathbb{Z})$$

be the *(integral) Heisenberg group.*

1. Compute $H_1(H;\mathbb{Z})$.
2. Show that $\operatorname{rk} H = 2$.

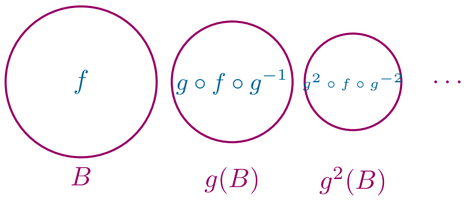
Please turn over

Bonus problem (a perfect homeomorphism group)**.** Let $n \in \mathbb{N}_{>0}$. For a homeomorphism $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we define the *support* by

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq x\}} \subset \mathbb{R}^n.$$

We say that f has *compact support* if $\operatorname{supp} f$ is compact. Let G be the group(!) of all homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ with compact support. Show that G is perfect (whence $H_1(G;\mathbb{Z}) \cong_{\mathbb{Z}} 0$).

Hints. Let $f \in G$, let B be an open ball containing $\operatorname{supp} f$, and let $g \in G$ with $g^k(B) \cap g^m(B) = \emptyset$ for all $k, m \in \mathbb{N}$ with $k \neq m$ (why does such a g exist?). Then consider the following situation:



Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 3, May 13, 2019

Exercise 1 (projectivity)**.** Let G be a group and let $\mathbb{Z}$ be equipped with the trivial G-action. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\mathbb{Z}$ is a a projective $\mathbb{Z}G$-module, then G is finite.
2. If G is finite, then $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module.

Exercise 2 (cohomology of $\mathbb{Z}/3$)**.** Use group extensions to show that $H^2(\mathbb{Z}/3;\mathbb{Z})$ contains at least three elements (where $\mathbb{Z}$ carries the trivial action). More precisely: Provide three extensions of $\mathbb{Z}/3$ by $\mathbb{Z}$ that are pairwise non-equivalent.

Exercise 3 (cohomology of $\mathbb{Z}^2$)**.** Use group extensions to show that $H^2(\mathbb{Z}^2;\mathbb{Z}) \not\cong_{\mathbb{Z}} 0$ (where $\mathbb{Z}$ carries the trivial action).
Hints. The Heisenberg group (Sheet 2, Exercise 4) can serve as a middleman.

Exercise 4 (extensions in the literature)**.** We consider the following article:

> M. Bucher, R. Frigerio, T. Hartnick. A note on semi-conjugacy for circle actions, *L'Enseignement Mathématique (2)*, 62, pp. 317–360, 2016.

1. What is "$e(\xi)$" from this article (paragraph before Lemma 3.1) in our notation? In particular, how does "c_σ" relate to our notation?
2. Give a proof of Lemma 3.1 (in our notation).

Bonus problem (cohomology of homeomorphisms on the circle)**.** We consider the circle $S^1 := \mathbb{R}/\mathbb{Z}$, the group $G := \operatorname{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of S^1, and the subgroup

$$\widetilde{G} := \bigl\{ f \in \operatorname{Homeo}^+(\mathbb{R}) \bigm| \forall_{x\in\mathbb{R}} \quad f(x+1) = f(x)+1 \bigr\}$$

of the orientation-preserving homeomorphisms of $\mathbb{R}$. A homeomorphism $\mathbb{R} \longrightarrow \mathbb{R}$ is *orientation-preserving* if it is monotonically increasing. Moreover, a homeomorphism $S^1 \longrightarrow S^1$ is *orientation-preserving* if it preserves orientations in the sense of linear algebra (which can be defined via suitable determinants). Let $p\colon \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$ be the projection map and let

$$\begin{aligned} \pi\colon \widetilde{G} &\longrightarrow G \\ f &\longmapsto \bigl([x] \mapsto p(f(x))\bigr) \end{aligned}$$

1. Show that π is a well-defined group homomorphism and that there is a central extension of the form
$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{G} \xrightarrow{\ \pi\ } G \longrightarrow 1.$$
Hints. For topologists, surjectivity of π should be easy!
2. Show that this extension is *not* trivial and conclude that $H^2(G;\mathbb{Z}) \not\cong_{\mathbb{Z}} 0$.
Hints. Torsion!

Submission before May 20, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 4, May 20, 2019

Exercise 1 ((co)homology with group ring coefficients)**.** Let $G := \mathbb{Z}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. We have $H_1(G; \mathbb{Z}G) \cong_{\mathbb{Z}} 0$.
2. We have $H^1(G; \mathbb{Z}G) \cong_{\mathbb{Z}} 0$.

Exercise 2 ((co)homology of $\mathbb{Z}^2$)**.** Let $T := \mathbb{Z}^2$.

1. Show that $\mathbb{Z}T \cong_{\mathsf{Ring}} \mathbb{Z}[a,b]_S$, where $S := \{a^n \cdot b^m \mid n, m \in \mathbb{N}\}$.
2. Show that the complex $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}T \xrightarrow{\partial_2} \mathbb{Z}T \oplus \mathbb{Z}T \xrightarrow{\partial_1} \mathbb{Z}T$, where

$$\begin{aligned} \partial_2 \colon \mathbb{Z}T &\longrightarrow \mathbb{Z}T \oplus \mathbb{Z}T \\ x &\longmapsto \bigl(x \cdot (1-b), x \cdot (a-1)\bigr) \\ \partial_1 \colon \mathbb{Z}T \oplus \mathbb{Z}T &\longrightarrow \mathbb{Z}T \\ (x,y) &\longmapsto x \cdot (a-1) + y \cdot (b-1), \end{aligned}$$

   together with the augmentation $\varepsilon \colon \mathbb{Z}T \longrightarrow \mathbb{Z}$ is a projective resolution of the trivial $\mathbb{Z}T$-module $\mathbb{Z}$ over $\mathbb{Z}T$.
3. Compute $H_*(T; \mathbb{Z})$ and $H^*(T; \mathbb{Z})$ (with the trivial T-action on $\mathbb{Z}$).
4. Compute $H^1(T; \mathbb{Z}T)$.
5. *Bonus problem.* What could be the geometric background of the above projective resolution?

Exercise 3 (cohomology of free groups)**.** Let F be a free group of rank 2. Show that $H^1(F; \mathbb{Z}F)$ is *not* a finitely generated $\mathbb{Z}$-module.
Hints. For a free generating set $\{a, b\}$ of F, it might help to consider $(b^n, a^n) \in \mathbb{Z}F \times \mathbb{Z}F$, the reduction to $\mathbb{Z}[F_{\mathrm{ab}}] \times \mathbb{Z}[F_{\mathrm{ab}}]$, and the resolution from Exercise 2.

Exercise 4 (Shapiro lemma in the literature)**.** We consider the following article:

> R.G. Swan. Groups of cohomological dimension one, *Journal of Algebra*, 12, pp. 585–601, 1969.

1. How is $\mathrm{cd}_{\mathbb{Z}}$ defined and what does the Shapiro lemma say about $\mathrm{cd}_{\mathbb{Z}}$?
2. Which result does Swan use to derive Theorem B from Theorem A?

Bonus problem (some congruences for $\mathbb{Z}/p$-actions on $\mathbb{Z}/p^n$)**.** Let $p \in \mathbb{N}$ be an odd prime, let $n \in \mathbb{N}_{>1}$, and let $a \in \mathbb{Z}$ with $a^p \equiv 1 \mod p^n$.

1. Show that $a \equiv 1 \mod p^{n-1}$.
2. Conclude that if $a \not\equiv 1 \mod p^n$, there exists a $k \in \{1, \ldots, p-1\}$ with $a^k \equiv 1 + p^{n-1} \mod p^n$.

Submission before May 27, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 5, May 27, 2019

Exercise 1 (homology of dihedral groups). For $n \in \mathbb{N}_{\geq 3}$, let D_n denote the *dihedral group for* n (i.e., the isometry group of a regular Euclidean n-gon). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $H_{2019}(D_{2021}; \mathbb{Z}/2020) \cong_{\mathbb{Z}} 0$
2. $H_{2019}(D_{2020}; \mathbb{Z}/2021) \cong_{\mathbb{Z}} 0$

Hints. You may use the description $D_n \cong_{\mathsf{Group}} \mathbb{Z}/n \rtimes \mathbb{Z}/2$, where $[1] \in \mathbb{Z}/2$ acts by (additive) inversion on $\mathbb{Z}/n$ (Proposition III.1.1.57).

Exercise 2 (the infinite dihedral group). The *infinite dihedral group* D_∞ is the isometry group of the metric space $\mathbb{Z}$ with respect to the metric inherited from the standard metric on $\mathbb{R}$. Let t denote the reflection at 0, let s denote the translation by 1, and let t' denote the reflection at $1/2 \in \mathbb{R}$. Solve two out of the following four problems:

1. Show that $S := \{s, t\}$ is a generating set of D_∞ and that D_∞ is isomorphic to a suitable semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/2$.
2. Show that the word metric on D_∞ associated with S is isometric to the word metric on $\mathbb{Z} \times \mathbb{Z}/2$ associated with the generating set $\{(1,0), (0,[1])\}$.
3. Show that $T := \{t, t'\}$ is a generating set of D_∞.
4. Show that the word metric on D_∞ associated with T is isometric to the word metric on $\mathbb{Z}$ associated with the generating set $\{1\}$.

Exercise 3 (metric embedding notions in the literature). We consider:

> J. Block, S. Weinberger. Aperiodic tilings, positive scalar curvature, and amenability of spaces, *J. Amer. Math. Soc.*, 5(4), pp. 907–918, 1992.

1. Prove the statement "Also note that a coarse quasi-isometry in the sense of Gromov is an EPL map." (p. 909). More precisely: Show that every quasi-isometric embedding between metric spaces is an effectively proper Lipschitz map (defined on p. 909).
2. Does the converse also hold? Justify your answer!

Exercise 4 ((co)induction of finite index subgroups). Let G be a group and let $H \subset G$ be a subgroup of finite index. We consider

$$\begin{aligned}
\varphi \colon \operatorname{Ind}_H^G(B) = \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\longrightarrow \operatorname{Hom}_H(\mathbb{Z}G, B) = \operatorname{Coind}_H^G(B) \\
g \otimes b &\longmapsto \big(x \mapsto \chi_H(x \cdot g) \cdot (x \cdot g) \cdot b\big) \\
\psi \colon \operatorname{Coind}_H^G B = \operatorname{Hom}_H(\mathbb{Z}G, B) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B) \\
f &\longmapsto \sum_{gH \in G/H} g \otimes f(g^{-1})
\end{aligned}$$

Show that φ and ψ are well-defined $\mathbb{Z}G$-linear maps and that φ and ψ are mutually inverse.

Please turn over

Bonus problem (Legendre symbol and transfer)**.** Let $p \in \mathbb{N}$ be an odd prime.

1. What is the *Legendre symbol* associated with p ?
2. Show that the Legendre symbol "coincides" with the transfer on $H_1(\,\cdot\,;\mathbb{Z})$ of the subgroup $\{-1,+1\}$ of the multiplicative group $\big(\mathbb{Z}/(p)\big)^{\times}$.

Submission before June 3, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 6, June 3, 2019

Exercise 1 (UDBG spaces)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. Every uniformly discrete metric space has bounded geometry.
2. Every metric space with bounded geometry is uniformly discrete.

Exercise 2 (uniformly finite chains)**.** Let R be a normed ring with unit, let (X,d) be a UDBG space, let $n \in \mathbb{N}_{\geq 1}$, let $c = \sum_{x \in X^{n+1}} c_x \cdot x \in C_n^{\mathrm{uf}}(X;R)$, and let $j \in \{0,\dots,n\}$. Show that the following map is a well-defined chain in $C_{n-1}^{\mathrm{uf}}(X;R)$:

$$\begin{aligned} X^n &\longrightarrow R \\ y &\longmapsto \sum_{x \in \{z \in X^{n+1} \mid (z_0,\dots,\widehat{z_j},\dots,z_n) = y\}} c_x. \end{aligned}$$

Exercise 3 (the fundamental class in uniformly finite homology)**.** Let $(X,d) := (\mathbb{Z}, d_{\{1\}})$ and for $A \subset X$ let $[A]_{\mathbb{Z}} \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ be the homology class represented by the uniformly finite cycle $\sum_{x \in A} 1 \cdot x$.

1. Show that for every finite set $A \subset X$, we have $[A]_{\mathbb{Z}} = 0$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$.
2. Show that for each $n \in \mathbb{N}_{>0}$, there exists a class $\alpha_n \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ that satisfies $n \cdot \alpha_n = [X]_{\mathbb{Z}}$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$.

Exercise 4 (means)**.** We consider the following article:

> M. Gromov. Volume and bounded cohomology, *Publ. Math. IHES*, 56, pp. 5–99, 1982.

1. Where and how are amenable groups defined in this article?
2. Show that this notion of amenability is equivalent to ours; more precisely, show that an $\mathbb{R}$-linear map $m\colon \ell^\infty(G,\mathbb{R}) \longrightarrow \mathbb{R}$ is a left-invariant mean on G if and only if all of the following conditions are satisfied:
   - $m(1) = 1$
   - $m(g \cdot f) = m(f)$ for all $f \in \ell^\infty(G,\mathbb{R})$ and all $g \in G$
   - $\bigl|m(f)\bigr| \leq |f|_\infty$ for all $f \in \ell^\infty(G,\mathbb{R})$

Bonus problem (Ponzi schemes)**.**

1. Who was Charles Ponzi?
2. What is a *Ponzi scheme*?

Submission before June 10, 2019, 10:00, in the mailbox

As June 10 is a holiday: extended deadline: June 11, 10:00

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 7, June 10, 2019

Exercise 1 (uniformly finite homology of groups). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $H_0^{\mathrm{uf}}\big(\langle a, b \mid a^{2019}, aba^{-1}b^{-1}\rangle; \mathbb{Z}\big) \cong_{\mathbb{Z}} 0$
2. $H_0^{\mathrm{uf}}\big(\langle a, b, c, d \mid c^{2019}d^{2020}\rangle; \mathbb{Z}\big) \cong_{\mathbb{Z}} 0$

Exercise 2 ($\mathbb{Z} \not\sim_{\mathrm{QI}} \mathbb{Z}^2$). Use uniformly finite homology $H_*^{\mathrm{uf}}(\,\cdot\,;\mathbb{R})$ to prove that $\mathbb{Z}$ and $\mathbb{Z}^2$ are *not* quasi-isometric.
Hints. Amenability calls for transfer! And Sheet 4 might help.

Exercise 3 (quasi-isometry vs. bilipschitz equivalence). We consider the following article:

> K. Whyte. Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture, *Duke Math. J.*, 99(1), pp. 93–112, 1999.

1. Give a proof of the statement “Observe that a quasi-isometry between UDBG spaces is bilipschitz if and only if it is bijective” at the beginning of the proof of Theorem 4.1.
2. How is this fact used in the proof of Theorem 4.1?

Exercise 4 (homology of free groups with ℓ^2-coefficients). Let F be a free group of rank 2. Give an explicit example of a chain $b \in C_1(F; \ell^2(F,\mathbb{R}))$ (with the usual left $\mathbb{Z}F$-module structure on $\ell^2(F,\mathbb{R})$) that satisfies

$$\partial_1 b = e \otimes \chi_{\{e\}} \in C_0\big(F; \ell^2(F,\mathbb{R})\big).$$

Sketch this chain!

Bonus problem (measure equivalence).

1. What is *measure equivalence* of (countable) groups?
2. How does the definition of measure equivalence relate to the dynamical criterion for quasi-isometry of (finitely generated) groups?
3. How does measure equivalence relate to orbit equivalence of group actions?

   Hints. It is enough to cite such a result from the literature; no proof is required.
4. Give an example of (non-trivial) coefficients for group (co)homology that lead to measure equivalence invariants!

   Hints. It is enough to cite such a result from the literature; no proof is required.

Submission before June 17, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 8, June 17, 2019

Exercise 1 (commutator length). Let G be a group. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $g \in [G, G]$ and $h \in G$, then $\mathrm{cl}_G(h \cdot g \cdot h^{-1}) = \mathrm{cl}_G(g)$.
2. If $g, h \in [G, G]$, then $\mathrm{cl}_G(g \cdot h) = \mathrm{cl}_G\, g + \mathrm{cl}_G\, h$.

Exercise 2 (homogenisation). Let G be a group and let $\varphi\colon G \longrightarrow \mathbb{R}$ be a quasi-morphism.

1. Show that the following map is a well-defined homogeneous quasi-morphism on G that is uniformly close to φ (this requires, in particular, a proof of the existence of the limit on the right-hand side):
$$\begin{aligned} \overline{\varphi}\colon G &\longrightarrow \mathbb{R} \\ g &\longmapsto \lim_{n\to\infty} \frac{1}{n} \cdot \varphi(g^n) \end{aligned}$$
2. Conclude that the following map is a well-defined isomorphism of $\mathbb{R}$-vector spaces:
$$\begin{aligned} \mathrm{QM}(G)/\,\mathrm{QM}_0(G) &\longrightarrow \overline{\mathrm{QM}}(G)/\,\mathrm{Hom}_{\mathsf{Group}}(G, \mathbb{R}) \\ [\varphi] &\longmapsto [\overline{\varphi}] \end{aligned}$$

Exercise 3 (rotation number). We consider the following article:

> K. Mann. Rigidity and flexibility of group actions on the circle, *Handbook of group actions*, IV, pp. 705–752, *Adv. Lect. Math.*, 41, International Press, 2018.
> The preprint version is freely available at: https://arxiv.org/abs/1510.00728

Let $\widetilde{G} := \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid \forall_{x\in\mathbb{R}} \quad f(x+1) = f(x) + 1\}$ be the group (with respect to composition) of periodic orientation-preserving (i.e., monotonically increasing) homeomorphisms of $\mathbb{R}$ (see also the bonus exercise on Sheet 3).

1. How is the *rotation number* $\widetilde{\mathrm{rot}}\colon \widetilde{G} \longrightarrow \mathbb{R}$ on $\widetilde{G}$ defined? Why is this definition well-defined? (You should give more details than the article ...)
2. Show that $\widetilde{\mathrm{rot}}$ is a homogeneous quasi-morphism on $\widetilde{G}$.

Exercise 4 (means from vanishing bounded cohomology). Let G be a group, let $V := \ell^\infty(G, \mathbb{R})/C$, where $C \subset \ell^\infty(G, \mathbb{R})$ is the subspace of constant functions, and suppose that $H^1_{\mathrm{b}}(G; V^\#) \cong_{\mathbb{R}} 0$. Show that there exists a bounded $\mathbb{R}$-linear functional $\mu\colon \ell^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ with $\mu(1) = 1$ that is left-invariant.

Bonus problem (acronyms). How do the following acronyms expand in Mathematics? Where are they located?

MFO, MSRI, RIMS, IAS, BIRS, IMPAN,
IML, MPIM, INI, IHÉS, PIMS, KIAS

Submission before June 24, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 9, June 24, 2019

Exercise 1 (second cohomology and stable commutator length)**.** Let G be a group. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $H^2_b(G;\mathbb{R}) \cong_{\mathbb{R}} 0$, then $\mathrm{scl}_G = 0$.
2. If $H^2(G;\mathbb{R}) \cong_{\mathbb{R}} 0$, then $\mathrm{scl}_G = 0$.

Exercise 2 (torsion groups)**.** Let G be a torsion group (i.e., every element in G has finite order).

1. Compute the function $\mathrm{scl}_G \colon [G,G] \longrightarrow \mathbb{R}$ and the space $\overline{\mathrm{QM}}(G)$ (directly, without using Bavard duality).
2. Let G, in addition, have the property that every group extension of the form $0 \longrightarrow \mathbb{R} \longrightarrow ? \longrightarrow G \longrightarrow 1$ splits. Show that $H^2_b(G;\mathbb{R}) \cong_{\mathbb{R}} 0$.

Exercise 3 (quasi-isomorphisms)**.** Let R be a ring. A chain map $f_* \colon C_* \longrightarrow D_*$ in ${}_R\mathsf{Ch}$ is a *quasi-isomorphism* if, for each $n \in \mathbb{N}$, the induced homomorphism $H_n(f_*) \colon H_n(C_*) \longrightarrow H_n(D_*)$ is an isomorphism. Show (via an example over a suitable ring R) that if there exists a quasi-isomorphism $C_* \longrightarrow D_*$, then, in general, there is *no* quasi-isomorphism $D_* \longrightarrow C_*$.

Exercise 4 (exact categories)**.** We consider the article

T. Bühler. Exact categories, *Expo. Math.*, 28, pp. 1–69, 2010.

Before proceeding, you should look up what an *additive category* is.

1. On p. 4, *admissible epics* are defined. Make this definition explicit.
2. What is the *obscure axiom*?
3. Why is it called obscure?
4. How are *exact functors* between exact categories defined?

Bonus problem (duality principle for semi-norms on homology)**.** Let C_* be a chain complex in the category of normed $\mathbb{R}$-vector spaces (and bounded linear operators) and let $D^* := \mathrm{BHom}(C_*, \mathbb{R})$ be the dual cochain complex. Let $n \in \mathbb{N}$ and let $\alpha \in H_n(C_*)$ be represented by the cycle $c \in C_n$.

1. Show that
$$\|\alpha\| = \sup\Big\{ \frac{1}{|f|_\infty} \;\Big|\; f \in D^n,\ \delta^n f = 0,\ f(c) = 1 \Big\}.$$
Here, $\|\cdot\|$ denotes the semi-norm on $H_n(C_*)$ induced by the norm on C_n and δ^* is the coboundary operator of D^*. Moreover, we set $\sup \emptyset := 0$.

   Hints. Hahn-Banach!

2. Does there exist an amenable group G and a class $\alpha \in H_{2019}(G;\mathbb{R})$ with $\|\alpha\|_1 = 2019$? Here, $\|\cdot\|_1$ denotes the semi-norm on $H_*(G;\mathbb{R})$ induced by the ℓ^1-norm on $C^{\mathbb{R}}_*(G)$.

Submission before July 1, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 10, July 1, 2019

Exercise 1 (Tor). In the following, all modules carry the trivial group action. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. $\operatorname{Tor}_2^{\mathbb{Z}[\mathbb{Z}^2]}(\mathbb{Z},\mathbb{Z}) \cong_{\mathbb{Z}} 0$
2. $\operatorname{Tor}_1^{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}^2,\mathbb{Z}^2) \cong_{\mathbb{Z}} 0$

Exercise 2 (algebraic mapping cones). Let R be a ring and let $f_* \colon C_* \longrightarrow D_*$ be a chain map of R-chain complexes. The *mapping cone of* f_* is the R-chain complex $\operatorname{Cone}_*(f_*)$ consisting of the chain modules

$$\operatorname{Cone}_n(f_*) := C_{n-1} \oplus D_n$$

for all $n \in \mathbb{N}$ (where $C_{-1} := 0$), equipped with the boundary operators

$$\begin{aligned} \partial_n \colon \operatorname{Cone}_n(f_*) &\longrightarrow \operatorname{Cone}_{n-1}(f_*) \\ (x,y) &\longmapsto \bigl(-\partial^C_{n-1}(x), \partial^D_n(y) - f_{n-1}(x)\bigr) \end{aligned}$$

for all $n \in \mathbb{N}_{>0}$. Show the mapping cone trick, i.e., that $f_* \colon C_* \longrightarrow D_*$ is a quasi-isomorphism if and only if

$$\forall_{n\in\mathbb{N}} \quad H_n\bigl(\operatorname{Cone}(f_*)\bigr) \cong_R 0.$$

Hints. For the boundary operator on $\operatorname{Cone}(f_*)$, several different sign conventions are in use. Therefore, literature has to be used with care!

Exercise 3 (quasi-isomorphisms of complexes of projectives). Let R be a ring. Prove that if C_* and D_* are ($\mathbb{N}$-indexed) R-chain complexes that consist of projective R-modules, then every quasi-isomorphism $C_* \longrightarrow D_*$ is a chain homotopy equivalence.
Hints. Mapping cone ...

Exercise 4 (injectivity). We consider the article

> N.V. Ivanov. Foundations of the theory of bounded cohomology, *J. Soviet Math.*, 37, pp. 1090–1114, 1987.

1. How are ordinary *injective modules* defined in module categories?
2. How are *relatively injective Banach G-modules* defined in the article?
3. What is the fundamental theorem of homological algebra in this context?
4. Use MathSciNet (https://www.ams.org/mathscinet) to find an article that solves the problem in Remark 3.9.1.

   Hints. Use the "Citations" tool!

Bonus problem (mapping cones in algebra and topology).

1. How can one relate algebraic mapping cones to topological mapping cones?
2. What are differences/similarities between the properties of algebraic mapping cones and topological mapping cones, respectively?

Submission before July 8, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig — Sheet 11, July 8, 2019

Exercise 1 (~~puzzle~~ convergence)**.** Let R be a ring and let $(E^r, d^r)_{r \in \mathbb{N}_{>1}}$ be a homological spectral sequence that converges to a graded R-module A:

$$E^2_{pq} \Longrightarrow A_{p+q}.$$

Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $E^2_{pq} \cong_R 0$ for all $p, q \in \mathbb{N}$ with $q \neq 2019$, then $A_{2020} \cong_R E^2_{1,2019}$.
2. If $E^2_{pq} \cong_R 0$ for all $p, q \in \mathbb{N}$ for which $p + q$ is odd, then $A_{2019} \cong_R 0$.

Exercise 2 (homology of the Heisenberg group)**.** Let $H \subset \mathrm{SL}(3, \mathbb{Z})$ be the integral Heisenberg group (Sheet 2, Exercise 4). Compute $H_n(H; \mathbb{Z})$ for all $n \in \mathbb{N}$ (where H acts trivially on $\mathbb{Z}$) via the Hochschild-Serre spectral sequence.
Hints. You may use the result on $H_1(H; \mathbb{Z})$ from Exercise 4 on Sheet 2.

Exercise 3 (standard spectral arguments)**.** We consider the article

> M.R. Bridson, P.H. Kropholler. Dimension of elementary amenable groups, *J. Reine Angew. Math.*, 699, p. 217–143, 2015.
> Institut Mittag-Leffler, report no. 38, 2011/2012, spring.

In the paragraph after Theorem I.5 it says that "In both cases of course the inequalities $\leq$ follow at once from standard spectral sequence arguments."

1. Which spectral sequence should be applied?
2. Carry out these "standard spectral sequence arguments."

Hints. Cohomological dimension already appeared in Exercise 4 of Sheet 4.

Exercise 4 (the conjugation action on homology)**.**

1. Let G be a group, let A be a $\mathbb{Z}G$-module, and let $g \in G$. Moreover, let
$$c(g) := (x \mapsto g \cdot x \cdot g^{-1},\ x \mapsto g \cdot x) \in \mathrm{Mor}_{\mathsf{GroupMod}}\big((G, A), (G, A)\big).$$
Show that $H_n(c(g)) = \mathrm{id}_{H_n(G;A)}$ for all $n \in \mathbb{N}$.
2. For each $n \in \mathbb{N}$, compute $H_n(\varphi; \mathbb{Z}) \colon H_n(\mathbb{Z}/3; \mathbb{Z}) \longrightarrow H_n(\mathbb{Z}/3; \mathbb{Z})$, where $\varphi \colon \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3$ is the group automorphism $x \mapsto -x$.

Hints. First understand how $H_*(c(g))$ can be described in terms of a projective resolution of $\mathbb{Z}$ and then avoid confusion at all cost.

Bonus problem (centre kills)**.** Let $n, k \in \mathbb{N}_{>0}$ and let K be a field with $\mathrm{char}\, K \neq 2$. Show that $H_k(\mathrm{GL}(n, K); K^n) \cong_K 0$, where $\mathrm{GL}(n, K)$ acts on K^n by matrix multiplication.
Hints. Let elements of the centre act ...

Submission before July 15, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/Dr. D. Fauser/J. P. Quintanilha/J. Witzig Sheet 12, July 15, 2019

Exercise 1 (universal coefficients and Künneth). Let G be a group that satisfies $H_n(G;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(1;\mathbb{Z})$ for all $n \in \mathbb{N}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If H is a group and $n \in \mathbb{N}$, then $H_n(G \times H;\mathbb{Z}) \cong_{\mathbb{Z}} H_n(H;\mathbb{Z})$.
2. If A is a $\mathbb{Z}$-module (with trivial G-action) and $n \in \mathbb{N}$, then $H_n(G;A) \cong_{\mathbb{Z}} H_n(1;A)$.

Exercise 2 (topology of discrete groups). We consider the articles

G. Baumslag, E. Dyer, A. Heller. The topology of discrete groups, *J. Pure Appl. Algebra*, 16(1), pp. 1–47, 1980.

C.R.F Maunder. A short proof of a theorem of Kan and Thurston. *Bull. London Math. Soc.*, 13(4), pp. 325–327, 1981.

1. What is a *mitotic* group?
2. Sketch the proof that mitotic groups have trivial homology.
3. How does the main theorem of the second paper relate to group homology?
4. How/Where is the first paper used in the second paper?

Exercise 3 (a classifying space for the Heisenberg group). Let $H \subset \mathrm{SL}(3,\mathbb{Z})$ be the integral Heisenberg group (Sheet 2, Exercise 4). Show that there exists a classifying space for H that is a compact 3-manifold.
Hints. Start with the real Heisenberg group.

Exercise 4 (surface groups). For $g \in \mathbb{N}_{\geq 2}$, we consider the group

$$\Gamma_g := \bigl\langle a_1,\dots,a_g,b_1,\dots,b_g \bigm| [a_1,b_1]\cdot[a_2,b_2]\cdot\dots\cdot[a_g,b_g]\bigr\rangle.$$

1. Compute $H_*(\Gamma_g;\mathbb{Z})$, using “the” oriented closed connected surface of genus g as classifying space (and a suitable CW-structure on it).
2. Compute the deficiency of the group Γ_g.

Bonus problem (classifying space of a category).

1. How is the classifying space of a (small) category defined?
2. How can one construct classifying spaces for groups out of classifying spaces of a category?

Hints. Rough sketches suffice.

Bonus problem (lecture notes). Find as many typos/mistakes in the lecture notes as you can!

Submission before July 22, 2019, 10:00, in the mailbox

Group Cohomology – Exercises

Prof. Dr. C. Löh/Dr. D. Fauser/J. P. Quintanilha/J. Witzig Sheet 13, July 22, 2019

The ISSSS case. The InterStellar Spectral Sequence Station (ISSSS) has been infiltrated and taken over by a pangalactic group X of con artists. Detective Blorx, an agent of the Cohomological Intelligence Agency, handles the case. He collected the following evidence:

A. The suspects. Only the following groups are sufficiently powerful to be able to infiltrate the ISSSS; here, F_n denotes the free group of rank n and Γ_g denotes the surface group of genus g:

- The Free Group: F_{2018}
- The Freer Group: F_{2019}
- The Bi-Cycle Group: $\mathbb{Z}/4 \times \mathbb{Z}/674 \times F_{2018}$
- GaoS (Group avalaible on Surfaces): Γ_{1010}
- Torus Inc.: $\mathbb{Z}^{2019}$

B. Statements by witnesses.

- The group X was not able to carry invariant means.
- The free products $(F_4 \times F_{674}) * X$ and $(F_4 \times F_{674}) * F_{2018}$ are *not* commensurable.
- Every finite subgroup of X acted freely on some sphere.
- The group X has *no* subgroup that is isomorphic to the fundamental group of an oriented closed connected surface of genus at least 2.

C. Project Euler. Blorx hacked into the servers of the Secret Invariants Service and found the following files on Project Euler:

> Let G be a group of type F, i.e., G admits a classifying space with a finite CW-structure. Then the *Euler characteristic of G* is defined as
>
> $$\chi(G) := \sum_{n\in\mathbb{N}} (-1)^n \cdot \dim_{\mathbb{Q}} H_n(G;\mathbb{Q}).$$

① In this situation, $\chi(G)$ is a well-defined integer and it equals the Euler characteristic of any classifying space for G with finite CW-structure.

② If $H \subset G$ is a subgroup of finite index, then $\chi(H) = [G : H] \cdot \chi(G)$.

③ If H is a group of type F, then $\chi(G \times H) = \chi(G) \cdot \chi(H)$.

④ If H is a group of type F, then $\chi(G * H) = \chi(G) + \chi(H) - 1$.

D. Law of Commensurability. Two groups G and H are *commensurable*, if there exist finite index subgroups $\overline{G} \subset G$ and $\overline{H} \subset H$ with $\overline{G} \cong_{\mathsf{Group}} \overline{H}$.

Please turn over

Exercise ($4 + 8 + 4$ credits)**.** Help Blorx!

1. Establish two of the four claims of Project Euler.

2. Which group infiltrated the ISSSS? Justify your answer!

   Hints. Sphere actions will be discussed in the final lecture.
   How do Project Euler and the Law of Commensurability interact?

3. Help Blorx to get a promotion by working as an informant for him: Pick your favourite field of Mathematics (e.g., algebraic number theory, Riemannian geometry, geometric topology, geometric group theory, operator algebras, functional analysis, graph theory, algebraic geometry, ergodic theory, ...).

   Find (in the literature or on the servers of the Secret Invariants Service) an application of group (co)homology to that field that we did not discuss in the lectures!

Bonus problem (fickle witness)**.** The last witness statement was later withdrawn by the witness and changed to the following:

- The group X is *not* quasi-isometric to F_{2018}.

Does this lead to the same conclusion?
Hints. One can (provided one has access to it) use the secret H2UF-technology.

C

Etudes

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 0, April 25, 2019

Exercise 1 (group rings)**.** Let $G := \mathbb{Z}/8$ and let $t := [1] \in \mathbb{Z}/8$.

1. Is $\mathbb{Z}G$ a commutative ring?
2. Compute $(t-1) \cdot \sum_{j=0}^{7} t^j$ in $\mathbb{Z}G$.
3. Is $\mathbb{Z}G$ isomorphic to $\mathbb{Z}[e^{2\pi i/8}] \subset \mathbb{C}$?
4. Is $t^4 - 1$ a unit of $\mathbb{Z}G$?

Exercise 2 (invariants)**.** Let G be a group. For each of the following $\mathbb{Z}G$-modules, compute the invariants.

1. $\mathbb{R}$ with the trivial G-action
2. $\operatorname{map}(G, \mathbb{R})$, the space of $\mathbb{R}$-valued functions on G, with respect to the G-action by right translation on G
3. $\ell^2(G, \mathbb{C})$, the space of square-summable complex functions on G, with respect to the G-action by right translation on G
4. $A \times B$, where A and B are $\mathbb{Z}G$-modules, with respect to the diagonal G-action on $A \times B$.

Exercise 3 (basic homological algebra)**.**

1. What is the definition of *chain complexes* and *chain maps*?
2. What are typical examples?
3. What is the *homology* of a chain complex?
4. How can homology be computed?
5. How does all this relate to exactness?
6. What is *homotopy* invariance in homological algebra?
7. Why did we introduce chain complexes in Commutative Algebra or Algebraic Topology or Any-Other-Course?

Hints. In case you don't know any homological algebra: Don't panic! Basic notions from homological algebra will also be quickly reviewed in the lectures.

Exercise 4 (basic category theory)**.**

1. What is the definition of a *category*?
2. What is the definition of a *functor* between categories?
3. Give examples of categories and functors between them. (How) Did these arise naturally in previous courses?

Hints. In case you don't know any category theory: Don't panic! Categories and functors will also be quickly reviewed in the lectures.

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no submission!

These problems will be discussed in the exercise classes in the second week.

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 1, May 2, 2019

Exercise 1 (group (co)homology in degree 0)**.** Let $G := \mathbb{Z}/2$. For a $\mathbb{Z}$-module Z, let Z^- be the left $\mathbb{Z}G$-module whose underlying additive group is Z and where the non-trivial element of G acts by multiplication with -1. Compute the following (co)homology groups:

1. $H^0(G; \mathbb{R}^-)$
2. $H_0(G; \mathbb{R}^-)$
3. $H^0(G; \mathbb{Z}^-)$
4. $H_0(G; \mathbb{Z}^-)$
5. $H^0(G; \mathbb{Z}/2^-)$
6. $H_0(G; \mathbb{Z}/2^-)$

Exercise 2 (simplicial chains)**.** Let $G := \mathbb{Z}$ and let $t := 1 \in \mathbb{Z} = G$. Which of the following elements of $C_1(G)$ are cycles (i.e., in the kernel of ∂_1)?

1. $2019 \cdot (t^{2019}, t^{2019})$
2. $2019 \cdot (t, t^{2019})$
3. $1 \cdot (t^0, t) + 1 \cdot (t, t^2) - 1 \cdot (t^0, t^2)$

Exercise 3 (free groups)**.** Let F be the free group of rank 2, freely generated by $\{a, b\}$ (Appendix A.1). Which of the following equalities hold in F ?

1. $(aba^{-1})^{2019} = ab^{2019}a^{-1}$
2. $aba^{-1} \cdot ab^2a^{-1} = b^3$
3. $[a, b]^2 = [a^2, b^2]$
4. $[a^2, a^{-1}b^2] = ab^2a^{-2}b^{-2}a$

Exercise 4 (summary)**.** Write a summary of Chapter 1.1 (Foundations: The group ring) and Chapter 1.2 (The basic definition of group (co)homology), keeping the following questions in mind:

1. How can one work with the group ring? What are basic examples?
2. What are important examples and constructions of $\mathbb{Z}G$-modules?
3. What are the domain categories for group (co)homology?
4. How are the simplicial/bar resolutions constructed? Why?
5. How is group (co)homology defined in terms of the simplicial resolution?
6. Did you check all the little things that we did not discuss in detail in the lectures?

no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 2, May 9, 2019

Exercise 1 (cohomology in degree 1)**.** Compute the following cohomology groups (with respect to the trivial action on the coefficients).

1. $H^1(\mathbb{Z}; \mathbb{Z}/2)$
2. $H^1(\mathbb{Z}/2; \mathbb{Z}/3)$
3. $H^1(S_{2019}; \mathbb{Z}/2019)$
4. $H^1(S_{2020}; \mathbb{Z}/2020)$

Exercise 2 (presentations)**.** Which groups are described by the following presentations? Use universal properties to verify your claims.

1. $\langle a, b \mid a \rangle$
2. $\langle a, b \mid ab \rangle$
3. $\langle a, b \mid ab^2 \rangle$
4. $\langle a, b \mid aba^{-1} \rangle$
5. $\langle a, b \mid a^3, b^2, aba^{-1}b^{-1} \rangle$
6. $\langle a, b \mid a^3, b^2, aba^{-2} \rangle$

Exercise 3 (extensions)**.** Do there exist extensions of the following types?

1. $0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/6 \longrightarrow \mathbb{Z}/3 \longrightarrow 0$
2. $0 \longrightarrow \mathbb{Z}/2 \longrightarrow S_3 \longrightarrow \mathbb{Z}/3 \longrightarrow 0$
3. $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$
4. $0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1$

Exercise 4 (summary)**.** Write a summary of Chapter 1.3 (Degree 0: (Co)Invariants), Chapter 1.4 (Degree 1: Abelianisation and homomorphisms), and Chapter 1.5 (Degree 2: Presentations and extensions) keeping the following questions in mind:

1. How can one compute group (co)homology in low degrees?
2. What are typical examples?
3. What are typical applications?
4. What kind of finiteness obstructions can we get from group homology in low degrees?
5. Did you check all the little things that we did not discuss in detail in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 3, May 16, 2019

Exercise 1 (projectivity). Which of the following modules are projective?

1. $\mathbb{Z}/2019$ over $\mathbb{Z}$?
2. $\mathbb{Q}$ over $\mathbb{Z}$?
3. $\prod_{\mathbb{N}} \mathbb{Q}$ over $\mathbb{Q}$?
4. $\mathbb{Z} \times \{0\} \subset \mathbb{Z} \times \mathbb{Z}$ over $\mathbb{Z} \times \mathbb{Z}$?

Exercise 2 (homology of cyclic groups). Compute the following (co)homology groups (where $\mathbb{Z}/2019$ acts trivially on the coefficients):

1. $H_*(\mathbb{Z}/2019; \mathbb{Z})$
2. $H_*(\mathbb{Z}/2019; \mathbb{Z}/2019)$
3. $H_*(\mathbb{Z}/2019; \mathbb{Q})$
4. $H_*(\mathbb{Z}/3; \mathbb{Z}/2019)$

Exercise 3 (p-groups). Let $p \in \mathbb{N}$ be a prime.

1. How can one prove that p-groups are solvable?
2. Give examples of p-groups that do *not* contain a cyclic subgroup of index p.

Exercise 4 (example list). Start a list of groups and their (co)homology, containing the following data (as far as you know it):

- name of the group
- standard notation of the group
- standard presentations of the group
- geometric relevance of the group
- homology of the group (at least with $\mathbb{Z}$-coefficients) and method of computation
- cohomology of the group (at least with $\mathbb{Z}$-coefficients) and method of computation
- applications of this group (co)homology

Update this list during this course (don't forget to add the examples from the exercises!).

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 4, May 23, 2019

Exercise 1 (tree navigation). Let F be a free group, freely generated by $\{a, b\}$ with $a \neq b$. In "the" 4-regular tree (Figure 1.6), find the vertices corresponding to the following elements of F:

1. $abba$
2. $aba^{-1}b^{-1}$
3. $a^{-2019}b$
4. ab^{2019}

Exercise 2 ((non-)triviality?). Use transfer/functoriality to decide whether the following homology groups are trivial or not (in each of these cases, the action on the coefficients is trivial).

1. $H_{2020}\bigl(\mathbb{Z}/2019 \times \mathbb{Z}/2019; \mathbb{Z}/2018\bigr)$
2. $H_{2019}\bigl(\mathbb{Z}/2018 \times \mathbb{Z}/2018; \mathbb{Z}/2020\bigr)$
3. $H_{2019}(\mathbb{Z} \times \mathbb{Z}/2018; \mathbb{Z})$
4. $H_{2018}(\mathbb{Z} \times \mathbb{Z}/2019; \mathbb{Q})$

Exercise 3 (summary). Write a summary of Chapter 1.6 (Changing the resolution) and Chapter 1.7 ((Co)Homology and subgroups), keeping the following questions in mind:

1. What are projective resolutions?
2. How can projective resolutions be used to compute group (co)homology?
3. What are typical examples?
4. What are applications of this approach?
5. How are the (co)homology of subgroups and ambient groups related?
6. Did you check all the little things that we did not discuss in detail in the lectures?

Exercise 4 (save early, save often). And now for something completely different:

1. What is S.M.A.R.T.?
2. How can S.M.A.R.T. be used to make predictions about disk health?
3. Find a good backup solution for your data!

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 5, May 30, 2019

Exercise 1 (quasi-isometric embeddings)**.** We consider the metric space $\mathbb{R}$ with the standard metric. Which of the following maps $\mathbb{R} \longrightarrow \mathbb{R}$ are quasi-isometric embeddings?

1. $x \mapsto 2019 \cdot x + 2019$
2. $x \mapsto x^{2019}$
3. $x \mapsto \lceil 2019 \cdot x \rceil$
4. $x \mapsto \frac{1}{x^{2019}+1}$

Exercise 2 (Cayley graphs)**.** Sketch the following Cayley graphs!

1. $\mathrm{Cay}\bigl(\mathbb{Z} \times \mathbb{Z}/2, \{(1,0),(0,[1])\}\bigr)$
2. $\mathrm{Cay}\bigl(\mathbb{Z}^2, \{(1,0),(1,1),(0,1)\}\bigr)$
3. $\mathrm{Cay}\bigl(\mathbb{Z}^2, \{(1,2),(2,1)\}\bigr)$
4. $\mathrm{Cay}\bigl(D_{2019}, \{t,t'\}\bigr)$, where t and t' are two different reflections

Exercise 3 (the Hahn-Banach theorem)**.** Recall/look up the following terms and statements from functional analysis:

1. What is a bounded linear map?
2. What is the relation between boundedness and continuity for linear maps?
3. What does the Hahn-Banach theorem say?

Exercise 4 (GAP)**.** Figure out how to use GAP (https://www.gap-system.org/) to compute group (co)homology!

no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 6, June 6, 2019

Exercise 1 (amenability)**.** Which of the following groups are amenable?

1. $\mathbb{Z} \times \mathbb{Z}/2019$
2. D_∞
3. the Heisenberg group
4. the symmetric group $\operatorname{Sym}(\mathbb{N})$ of all bijections $\mathbb{N} \longrightarrow \mathbb{N}$
5. $\operatorname{SL}_2(\mathbb{Z})$
6. $\prod_{\mathbb{N}} \mathbb{R}$

Exercise 2 (uniformly finite chains)**.** Which of the following "sums" describe uniformly finite chains in $C_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$? Draw them! If they describe uniformly finite chains: What is their boundary?

1. $\sum_{n\in\mathbb{Z}} 1 \cdot (n, n+1)$
2. $\sum_{n\in\mathbb{N}} 1 \cdot (n, n+1)$
3. $\sum_{n\in\mathbb{N}} 1 \cdot (1, n)$
4. $\sum_{n\in\mathbb{N}} 1 \cdot (-n, n)$
5. $\sum_{n\in\mathbb{N}} n \cdot (n, n+1)$
6. $\sum_{n\in\mathbb{N}} 2019 \cdot (n, n+2019)$

Exercise 3 (uniformly finite homology of finite groups)**.** Let G be a finite group. Compute $H_*^{\mathrm{uf}}(G;\mathbb{Z})$ in as many ways as you can!

Exercise 4 (summary)**.** Write a summary of Chapter 2.1 (Foundations: Geometric group theory), keeping the following questions in mind:

1. What is the (geo)metric setup in geometric group theory?
2. What are typical examples?
3. What is amenability?
4. Did you check all the little things that we did not discuss in detail in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig — Sheet 7, June 13, 2019

Exercise 1 (group presentations and amenability)**.** Which of the following groups are amenable?

1. $\langle a, b \mid a, b \rangle$
2. $\langle a, b \mid aba^{-1}b^{-1} \rangle$
3. $\langle a, b, c \mid c^2 \rangle$
4. $\langle a, b, c \mid ab \rangle$
5. $\langle a, b \mid a^2, b^2 \rangle$
6. $\langle a \mid a^{2019} \rangle$

Exercise 2 (bounded cochains)**.** Which of the following cochains in $\overline{C}^2(\mathbb{Z}^2; \mathbb{R})$ are bounded? Which of them are cocycles?

1. $[x \mid y] \longmapsto \det(x\ y)$
2. $[x \mid y] \longmapsto \|x\|_2$
3. $[x \mid y] \longmapsto 2019$
4. $[x \mid y] \longmapsto \frac{1}{\|x\|_2^{2020} + \|y\|_2^{2020} + 1}$

Exercise 3 (amenable vs. free)**.** Start a table for amenable groups and free groups, respectively, listing their behaviour with respect to:

1. uniformly finite homology
2. bounded cohomology
3. quasi-morphisms
4. stable commutator length

Exercise 4 (summary)**.** Write a summary of Chapter 2.2 (Uniformly finite homology) keeping the following questions in mind:

1. What is uniformly finite homology of spaces?
2. Which coefficients for group homology lead to uniformly finite homology?
3. What is the fundamental class in uniformly finite homology in degree 0 ?
4. How does all this relate to amenability?
5. Did you check all the little things that we did not discuss in detail in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 8, June 20, 2019

Exercise 1 (counting quasi-morphisms). Which of the following counting quasi-morphisms on $\langle a, b \,|\, \rangle$ are homogeneous? Which are group homomorphisms?

1. ψ_a
2. ψ_{a^2}
3. ψ_{ab}
4. $\psi_{b^{-1}}$
5. $\psi_{aba^{-1}b^{-1}}$

Exercise 2 (functional analysis basics). Recall/loop up the following items from functional analysis:

1. bounded linear operator
2. operator norm
3. Hahn-Banach theorem
4. open mapping theorem

Exercise 3 (Tor). Recall the following terminology:

1. projective module
2. projective resolution
3. construction/properties of Tor

   Hints. In case you don't know anything about Tor: Don't panic! I will quickly review the material in the lectures.

Exercise 4 (summary). Write a summary of Chapter 2.3 (Bounded cohomology) keeping the following questions in mind:

1. What is bounded cohomology of groups?
2. How does bounded cohomology characterise amenability?
3. What are similarities/differences between bounded cohomology and ordinary group cohomology?
4. How does bounded cohomology relate to quasi-morphisms?
5. How do quasi-morphisms relate to stable commutator length?
6. Did you check all the little things that we did not discuss in detail in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig — Sheet 9, June 27, 2019

Exercise 1 (scl)**.** Determine whether the following stable commutator lengths are zero or not:

1. $\mathrm{scl}_{\mathbb{Z}}\bigl([2019, 2018] \cdot [2019, 2018]\bigr)$
2. $\mathrm{scl}_{\langle a,b \,|\, \rangle}\bigl([a, b] \cdot [a, b]\bigr)$
3. $\mathrm{scl}_{\langle a,b \,|\, \rangle}\bigl([a \cdot b, b]\bigr)$
4. $\mathrm{scl}_{\langle a,b \,|\, \rangle}\bigl([a \cdot a, a]\bigr)$
5. $\mathrm{scl}_{\langle a,b \,|\, a^2, b^2 \rangle}\bigl([a, b] \cdot [a, b]\bigr)$

Exercise 2 (Tor)**.** Compute the following Tor-groups!

1. $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/2019, \mathbb{Z})$
2. $\mathrm{Tor}_{2019}^{\mathbb{Z}}(\mathbb{Z}/2019, \mathbb{Z}/2019)$
3. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2019)$
4. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2018, \mathbb{Z}/2019)$
5. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2019, \mathbb{Z}/2019)$
6. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2019, \mathbb{Q})$

Exercise 3 (quasi-isomorphisms)**.** For each choice C_* and D_* of the following chain complexes (over the ring $\mathbb{Z}$), decide whether there exists a quasi-isomorphism $C_* \longrightarrow D_*$ or not.

1. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$
2. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$
3. $\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$
4. $\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2019 \longrightarrow 0$
5. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2019} \mathbb{Z} \longrightarrow 0$

Exercise 4 (homological algebra)**.** Recall/look up the following fundamental results of homological algebra:

1. long exact homology sequence/snake lemma
2. horseshoe lemma

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig   Sheet 10, July 4, 2019

Exercise 1 (spectral sequence). The following diagram is the second page of a homological spectral sequence (of $\mathbb{Z}$-modules; all unlabelled modules are trivial).

1. Which components of the second differential have to be trivial? Which could be non-trivial? What about the third differential? What about all differentials?
2. What does this imply for the ∞-page?

q								
3				$\mathbb{Z}$				
2	$\mathbb{Z}$							
1	$\mathbb{Z}/2$		$\mathbb{Z}$			$\mathbb{Z}$		
0	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}$	
								p

Exercise 2 (five lemma).

1. View the situation of the five lemma as a double complex.
2. Spell out the double complex spectral sequences of this double complex (i.e., what are the first pages? To what do they converge?, ...).
3. Use the double complex spectral sequences to prove the five lemma.

Exercise 3 (long exact homology sequence).

1. Spell out the filtration spectral sequence for the inclusion of a single subcomplex of a chain complex (i.e., what is the first page? To what does it converge?, ...).
2. Use the filtration spectral sequence in this situation to establish the long exact homology sequence associated with a short exact sequence of chain complexes.

Exercise 4 (summary). Write a summary of Chapter 3.1 (Derived functors), keeping the following questions in mind:

1. Why does one need derived functors?
2. What are derived functors?
3. How can derived functors be constructed/computed?
4. How do derived functors relate to group (co)homology?
5. What is the derived category and what are total derived functors?
6. Did you check all the little things that we did not discuss in details in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig Sheet 11, July 11, 2019

Exercise 1 (universal coefficients)**.** Compute the following homology groups (where the groups act trivially on the coefficients):

1. $H_*(S_3; \mathbb{F}_2)$
2. $H_*(S_3; \mathbb{F}_3)$
3. $H_*(D_\infty; \mathbb{F}_2)$
4. $H_*(D_\infty; \mathbb{F}_3)$

Exercise 2 (product groups)**.** Compute the following homology groups (where the groups act trivially on the coefficients):

1. $H_*(\mathbb{Z}/2019 \times \mathbb{Z}/2019; \mathbb{F}_3)$
2. $H_*(\mathbb{Z}/3 \times \mathbb{Z}/2019; \mathbb{F}_3)$
3. $H_*(\mathbb{Z}/3 \times \mathbb{Z}/9; \mathbb{F}_3)$
4. $H_*(\mathbb{Z}/2019 \times \mathbb{Z}/2019; \mathbb{F}_{2017})$ (be lazy!)

Exercise 3 (algebraic topology)**.** Recall the following terminology/facts:

1. contractibility
2. covering map
3. classification of coverings
4. CW-complex
5. singular homology (and its properties)
6. cellular homology (and its properties)

Hints. In case you don't know anything about algebraic topology: Don't panic! I will quickly review some basics in the lectures. However, it might still be helpful to browse literature on algebraic topology to get a first impression.

Exercise 4 (summary)**.** Write a summary of Chapter 3.1 (The Hochschild-Serre spectral sequence), keeping the following questions in mind:

1. What are spectral sequences? How do they work?
2. Which spectral sequences are in your toolbox?
3. Which computational tricks do you know for spectral sequences?
4. How can spectral sequences be used in group (co)homology?
5. Did you check all the little things that we did not discuss in details in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/Dr. D. Fauser/J. P. Quintanilha/J. Witzig Sheet 12, July 18, 2019

Exercise 1 (classifying spaces for $\mathbb{Z}$)**.** Which of the following spaces are classifying spaces of $\mathbb{Z}$ (when equipped with a suitable CW-structure, etc.)?

1. $S^1 \times \mathbb{R}^{2019}$
2. $\mathbb{R}^2 \setminus \{0\}$
3. $\mathbb{R}^2 \setminus \{0, (1,0)\}$
4. $\mathbb{R}^{2019} \setminus \{0\}$
5. $S^1 \vee D^{2019}$
6. $S^1 \times S^{2019}$

Exercise 2 (two-dimensional classifying spaces)**.** Which of the following groups admit a classifying space of dimension 2 ? Here, Γ_g denotes the surface group of genus g and F_n denotes the free group of rank n.

1. $\Gamma_{2018} * \Gamma_{2019} * F_{2019}$
2. $\Gamma_{2018} \times \Gamma_{2019}$
3. $\mathbb{Z}^{2019}$
4. $\mathbb{Z}/2019 * \Gamma_{2019}$
5. $\Gamma_{2018} * (F_{2019} \times F_{2018})$
6. $\mathbb{Z}$

Exercise 3 (presentation complexes)**.** Draw the presentation complexes of the following group presentations. Do you recognise the groups?

1. $\langle x \mid x^2 \rangle$
2. $\langle x, y \mid x \rangle$
3. $\langle x, y \mid [x, y] \rangle$
4. $\langle x, y \mid x^2, y^2 \rangle$

Exercise 4 (summary)**.** Write a summary of Chapter 4.1 (Classifying spaces) keeping the following questions in mind:

1. What is the definition of classifying spaces for groups?
2. What is an example of a functorial construction?
3. What are typical "nice" classifying spaces?
4. Why are classifying spaces useful in group (co)homology?
5. How do classifying spaces compare to projective resolutions?
6. Did you check all the little things that we did not discuss in detail in the lectures?

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no submission!

Group Cohomology – Etudes

Prof. Dr. C. Löh/Dr. D. Fauser/J. P. Quintanilha/J. Witzig Sheet 13, July 25, 2019

Exercise 1 (cohomological dimension). Determine the cohomological dimension of the following groups.

1. $\mathbb{Z}/2019 \times F_{2019}$
2. $\mathrm{GL}_{2019}(\mathbb{F}_{2017})$
3. $\mathrm{GL}_{2019}(\mathbb{Z})$
4. $\Gamma_{2017} * \Gamma_{2018} * \Gamma_{2019}$
5. $\Gamma_{2017} \times \Gamma_{2018} \times \Gamma_{2019}$
6. $\mathbb{Z}^{2019} \times F_{2020}$

Exercise 2 (sphere actions). Which of the following groups admit a free action on some sphere?

$$\mathbb{Z}/2019, \quad \mathbb{Z}/4 \times \mathbb{Z}/9, \quad \mathrm{GL}_{2019}(\mathbb{F}_{2017}), \quad \mathbb{Z}$$

Exercise 3 (summary). Write a summary of Chapter 4.2 (Finiteness conditions) and Chapter 4.3 (Application: Free actions on spheres), keeping the following questions in mind:

1. Which finiteness conditions for groups do you know?
2. How are they related?
3. What are typical examples?
4. How can they be computed?
5. Which necessary conditions do you know for finite groups to admit a free action on a sphere? How does this relate to group (co)homology?
6. How does this work in concrete examples?
7. Did you check all the little things that we did not discuss in detail in the lectures?

Exercise 4 (exam). Due to a tragic malfunction in space-time you end up as examiner in an oral exam on Group Cohomology.

1. Which questions will you ask on basic notions?
2. Which questions will you ask on applications?
3. Which questions will you ask on the different views and methods of computation on group (co)homology?
4. Which examples will you discuss during the exam?

Try out your questions on your fellow students!

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no submission!

D

General Information

Group Cohomology, SS 2019
Organisation

Prof. Dr. C. Löh/D. Fauser/J. P. Quintanilha/J. Witzig April 2019

Homepage. Information and news concerning the lectures, exercise classes, office hours, literature, as well as the exercise sheets can be found on the course homepage and in GRIPS:

http://www.mathematik.uni-regensburg.de/loeh/teaching/grouphom_ss19

https://elearning.uni-regensburg.de

Lectures. The lectures are on Mondays (10:15–12:00; M102) and on Thursdays (10:15–12:00; M104).

Basic lecture notes will be provided, containing an overview of the most important topics of the course. These lecture notes can be found on the course homepage and will be updated after each lecture. Please note that these lectures notes are not meant to replace attending the lectures or the exercise classes!

Exercises. Homework problems will be posted on Mondays (before 10:00) on the course homepage; submission is due one week later (before 10:00, in the mailbox).

Each exercise sheet contains four regular exercises (4 credits each) and more challenging bonus problems (4 credits each).

It is recommended to solve the exercises in small groups; however, solutions need to be written up individually (otherwise, no credits will be awarded). Solutions can be submitted alone or in teams of at most two participants; all participants must be able to present *all* solutions of their team.

The exercise classes start in the *second* week; in this first session, some basics on categories, homological algebra, and free groups will be discussed (as on the sheet Etudes 0).

In addition, we will provide etudes that will help to train elementary techniques and terminology. These problems should ideally be easy enough to be solved within a few minutes. Solutions are not to be submitted and will not be graded.

Registration for the exercise classes. Please register for the exercise classes via GRIPS:

https://elearning.uni-regensburg.de

Please register before Friday, April 26, 2019, 10:00, choosing your preferred time slot. We will try to fill the groups respecting your preferences.

The distribution will be announced at the beginning of the second week via GRIPS.

Credits/Exam. This course can be used as specified in the commented list of courses and in the module catalogue.

- *Studienleistung*: Successful participation in the exercise classes: 50% of the credits (of the regular exercises), presentation of a solution in class, active participation
- *Prüfungsleistung*: Oral exam (25 minutes), by individual appointment at the end of the lecture period/during the break.

You will have to register in FlexNow for the Studienleistung and the Prüfungsleistung (if applicable).

Further information on formalities can be found at:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/studierende-und-studienanfaenger/index.html

Contact.

- If you have questions regarding the organisation of the exercise classes, please contact Daniel Fauser or Johannes Witzig:

  daniel.fauser@ur.de
  johannes.witzig@ur.de

- If you have questions regarding the exercises, please contact your tutor.
- If you have mathematical questions regarding the lectures, please contact your tutor or Clara Löh.
- If you have questions concerning your curriculum or the examination regulations, please contact the student counselling offices or the exam office:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/ansprechpersonen/index.html

In many cases, also the Fachschaft can help:

http://www-cgi.uni-regensburg.de/Studentisches/FS_MathePhysik/cmsms/

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Please note that the bibliography will grow during the semester. Thus, also the numbers of the references will change!

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