

Algebraic Topology – Exercises

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Exercise 1 (separation theorems?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $f: S^1 \rightarrow S^1 \times S^1$ is continuous and injective, then $(S^1 \times S^1) \setminus f(S^1)$ has exactly two path-connected components.
2. If $f: S^1 \rightarrow \mathbb{R}^{2019}$ is continuous and injective, then $\mathbb{R}^{2019} \setminus f(S^1)$ is path-connected.

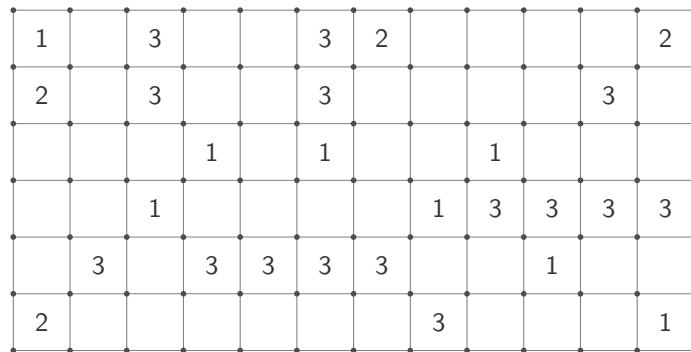
Exercise 2 (Slitherlink). A *Slitherlink puzzle* consists of a square grid; some of the squares have numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.

SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares, the number of edges in the loop is not specified.

SL 3 The loop does not have any self-intersections or branches.

1. Solve the following Slitherlink puzzle:



2. How can the Jordan curve theorem be used to establish global strategies for solving Slitherlink puzzles? Give an example that illustrates this strategy.

Exercise 3 (more on the ℓ^1 -semi-norm). We consider the ℓ^1 -semi-norm on singular homology (Exercise 3 on Sheet 11).

1. Let $f: X \rightarrow Y$ be a homotopy equivalence between topological spaces, let $k \in \mathbb{Z}$, and let $\alpha \in H_k(X; \mathbb{R})$. Show that

$$\|H_k(f; \mathbb{R})(\alpha)\|_1 = \|\alpha\|_1.$$

2. Let $n \in \mathbb{N}_{>0}$ and let $\alpha \in H_n(S^n; \mathbb{R})$. Show that $\|\alpha\|_1 = 0$.

Please turn over

Exercise 4 (non-planarity of the torus).

1. Let $n \in \mathbb{N}$, let M be a compact, non-empty topological manifold of dimension n , let N be a connected topological manifold of dimension n , and let $f: M \rightarrow N$ be continuous and injective. Show that f is surjective.
2. Conclude that there is *no* continuous injective map $S^1 \times S^1 \rightarrow \mathbb{R}^2$.

Bonus problem (the five colour theorem).

1. Choose a book on graph theory from the library that contains a proof of the five colour theorem.
2. Where does the proof use (relatives of) the Jordan curve theorem? Is this made explicit?