Algebraic Topology

An introductory course
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Universität Regensburg

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Contents

Guide to the Literature

This course will not follow a single source and there are many books that cover the standard topics (all with their own advantages and disadvantages). Therefore, you should individually compose your own favourite selection of books.

Algebraic Topology

- M. Aguilar, S. Gitler, C. Prieto. *Algebraic Topology from a Homotopical Viewpoint*, Springer, 2002.
- W.F. Basener. *Topology and Its Applications*, Wiley, 2006.
- J.F. Davis, P. Kirk. *Lecture Notes in Algebraic Topology*, AMS, 2001.
- A. Dold. *Lectures on Algebraic Topology*, Springer, 1980.
- H. Edelsbrunner, J.L. Harer. *Computational Topology: An Introduction*, AMS, 2009.
- S. Friedl. *Algebraic Topology I, II, III, ...*, lecture notes, Universität Regensburg, 2016–. . . .
- A. Hatcher. *Algebraic Topology*, Cambridge University Press, 2002. http://www.math.cornell.edu/~hatcher/AT/ATpage.html
- W. Lück. *Algebraische Topologie: Homologie und Mannigfaltigkeiten*, Vieweg, 2005.
- W.S. Massey. *Algebraic Topology: an Introduction*, seventh edition, Springer, 1989.
- W.S. Massey. *A Basic Course in Algebraic Topology*, third edition, Springer, 1997.
  Hint. This book uses singular (co)homology based on cubes instead of simplices.

- P. May. *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999.
- E.H. Spanier. *Algebraic topology*, corrected reprint of the 1966 original, Springer, 1995.
- J. Strom. *Modern Classical Homotopy Theory*, American Mathematical Society, 2012.
- T. tom Dieck. *Algebraic Topology*, European Mathematical Society, 2008.

Point-Set Topology

- K. Jänich. *Topologie*, eighth edition, Springer, 2008.
- J.L. Kelley. *General Topology*, Springer, 1975.
- A.T. Lundell, S. Weingram. *Topology of CW-complexes.* Van Nostrand, New York, 1969.
- J.R. Munkres. *Topology*, second edition, Pearson, 2003.
- L.A. Steen. *Counterexamples in Topology*, Dover, 1995.

Homological Algebra and Category Theory

- M. Brandenburg. *Einführung in die Kategorientheorie: Mit ausführlichen Erklärungen und zahlreichen Beispielen*, Springer Spektrum, 2015.
- D.-C. Cisinski. *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, 180, Cambridge University Press, 2019.
- S.I. Gelfand, Sergei, Y.I. Manin. *Methods of homological algebra*, second edition, Springer Monographs in Mathematics, Springer, 2003.
- T. Kaczynski, K. Mischaikow, M. Mrozek. *Computational Homology*, Applied Mathematical Sciences, 157, Springer, 2010.
- S. MacLane. *Categories for the Working Mathematician*, second edition, Springer, 1998.
- M. Land. *Introduction to infinity-categories*, Compact Textbooks in Mathematics, Birkhäuser, 2021.
- B. Pierce. *Basic Category Theory for Computer Scientists*, Foundations of Computing, MIT University Press, 1991.
- E. Riehl. *Category Theory in Context*, Aurora: Dover Modern Math Originals, 2016.
- C. Weibel. *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, 1995.

0 Introduction

This course provides an introduction to Algebraic Topology, more precisely, to basic reasoning and constructions in Algebraic Topology and some classical invariants such as the fundamental group, singular homology, and cellular homology.

The basic idea of Algebraic Topology is to translate topological problems into algebraic problems; topological spaces will be translated into algebraic objects (e.g., vector spaces) and continuous maps will be translated into homomorphisms (e.g., linear maps). The right setup for this is the language of categories of functors.

Topology	$\rightsquigarrow$	**Algebra**
topological spaces		e.g., vector spaces
continuous maps		linear maps
flexible		*rigid*

Algebraic Topology then is concerned with the classification of topological spaces and continuous maps up to "continuous deformation", i.e., up to so-called homotopy. To this end, one constructs and studies homotopy invariant functors. The main design problem consists of finding functors that

- are fine enough to recover interesting features of topological spaces, but that also
- are coarse enough to be computable in many cases.

Classical examples of homotopy invariant functors are homotopy groups and (co)homology theories such as singular/cellular homology. A first, intuitive, description of homotopy groups and singular homology is that these functors describe "which and how many holes" topological spaces have.

Why Algebraic Topology?

Algebraic Topology has a large variety of applications, both in theoretical and applied mathematics:

Topology

- Fixed-point theorems
- (Non-)Embeddability theorems
- Study of the geometry and topology of manifolds
- ...

Other fields in theoretical mathematics

- Yet another proof of the fundamental theorem of algebra
- (Non-)Existence of certain division algebras
- Freeness and finiteness properties in group theory
- Blueprint for parts of Algebraic Geometry
- ...

Applied mathematics

- Existence of Nash equilibria in Game Theory
- Configuration spaces for robotics
- Lower complexity bounds for distributed algorithms
- Higher statistics and big data
- Knot theory
- Foundations of computing/Homotopy Type Theory
- ...

Many conclusions in Algebraic Topology are based on arguments by contradiction (if something strange were possible on topological spaces, then one would also obtain a strange situation in algebra, which often can be shown to be impossible). Therefore, Algebraic Topology is particularly good at proving that certain deformations etc. do *not* exist and results in Algebraic Topology are often of a non-constructive flavour.

Overview of this Course

The main goal of this course is to understand basic concepts of Algebraic Topology.

As a preparation, we will first recall some point-set topology and basic constructions of topological spaces as well as basic notions from category theory. We will then introduce the central notion of homotopy and homotopy invariance, and explain how homotopy invariant functors can be used to solve various problems.

The lion share of the course will consist of constructing and analysing examples of homotopy invariant functors:

- First, we will construct the fundamental group functor and investigate its relation with covering theory.
- Second, we will study homology theories, both from the axiomatic point of view, and through concrete constructions and computations.

This course will be complemented with the course *Geometric Group Theory* in the summer semester 2022, where Group Theory, (Algebraic) Topology, and (Metric) Geometry will interact.

Study note. These lecture notes document the topics covered in the course (as well as some additional optional material). However, these lectures notes are not meant to replace attending the lectures or the exercise classes!

Furthermore, this course will only treat a small fraction of Algebraic Topology. It is therefore recommended to consult other sources (books!) for further information on this field.

References of the form "Satz I.6.4.11", "Satz II.2.4.33", "Satz III.2.2.25", or "Satz IV.2.2.4" point to the corresponding locations in the lecture notes for Linear Algebra I/II, Algebra, Commutative Algebra in previuos semesters:

http://www.mathematik.uni-r.de/loeh/teaching/linalg1_ws1617/lecture_notes.pdf
http://www.mathematik.uni-r.de/loeh/teaching/linalg2_ss17/lecture_notes.pdf
http://www.mathematik.uni-r.de/loeh/teaching/algebra_ws1718/lecture_notes.pdf
http://www.mathematik.uni-r.de/loeh/teaching/calgebra_ss18/lecture_notes.pdf

Literature exercise. Where in the math library (including electronic resources) can you find books on Algebraic Topology, Point-Set Topology, Category Theory?

Convention. The set $\mathbb{N}$ of natural numbers contains 0. All rings are unital and associative. Usually, we assume manifolds to be non-empty (but we might not always mention this explicitly).

1

What is Algebraic Topology?

Algebraic Topology translates topology into algebra. We will formalise this translation using categories and functors.

As a preparation, we will first recall some point-set topology and introduce basic constructions of topological spaces; in particular, we will meet important building blocks of topological spaces such as spheres and simplices. Moreover, we will recall basic notions from category theory. We will then introduce the central notion of homotopy and homotopy invariance.

Finally, we will explain how homotopy invariant functors can be used to solve various problems, demonstrating the elegance and power of Algebraic Topology.

Overview of this chapter.

Running example. spheres, balls, standard simplices

1.1 Topological Building Blocks

Topological spaces model a non-quantitative version of geometry, using the system of open sets as a qualitive version of "being close". We will assume familiarity with the notions of topological spaces, continuous maps, compactness, path-connectedness, and the subspace and product topology (Appendix A.1).

In this section, we will discuss basic constructions of topological spaces; these constructions will allow us to generate a large collection of interesting examples.

1.1.1 Construction: Subspaces

Subsets of topological spaces inherit a topology from the ambient space: the subspace topology (Remark A.1.11). Many real-world problems lead to subsets of some $\mathbb{R}^n$. These subsets hence inherit a topology from the metric topology on $\mathbb{R}^n$. In many cases, this topology reflects aspects of the underlying real-world problem in a meaningful way.

The most important building blocks in Algebraic Topology are spheres, balls, and simplices (Figure 1.1).

Definition 1.1.1 (ball, sphere, standard simplex)**.** Let $n \in \mathbb{N}$.

- The *n-dimensional ball* is defined as
$$D^n := \bigl\{x \in \mathbb{R}^n \bigm| \|x\|_2 \leq 1\bigr\} \subset \mathbb{R}^n$$
(endowed with the subspace topology of the standard topology on $\mathbb{R}^n$).
- The *n-dimensional sphere* is defined as
$$S^n := \bigl\{x \in \mathbb{R}^{n+1} \bigm| \|x\|_2 = 1\bigr\} = \partial D^{n+1} \subset \mathbb{R}^{n+1}$$
(endowed with the subspace topology).
- The *n-dimensional standard simplex* is defined as
$$\Delta^n := \Bigl\{x \in \mathbb{R}^{n+1} \Bigm| x_1 \geq 0, \dots, x_{n+1} \geq 0 \text{ and } \sum_{j=1}^{n+1} x_j = 1\Bigr\} \subset \mathbb{R}^{n+1}$$
(endowed with the subspace topology).

In other words: The standard simplex Δ^n is the convex hull of the standard unit vectors $e_1, \dots, e_{n+1} \in \mathbb{R}^{n+1}$. Clearly, we could also define Δ^n as an

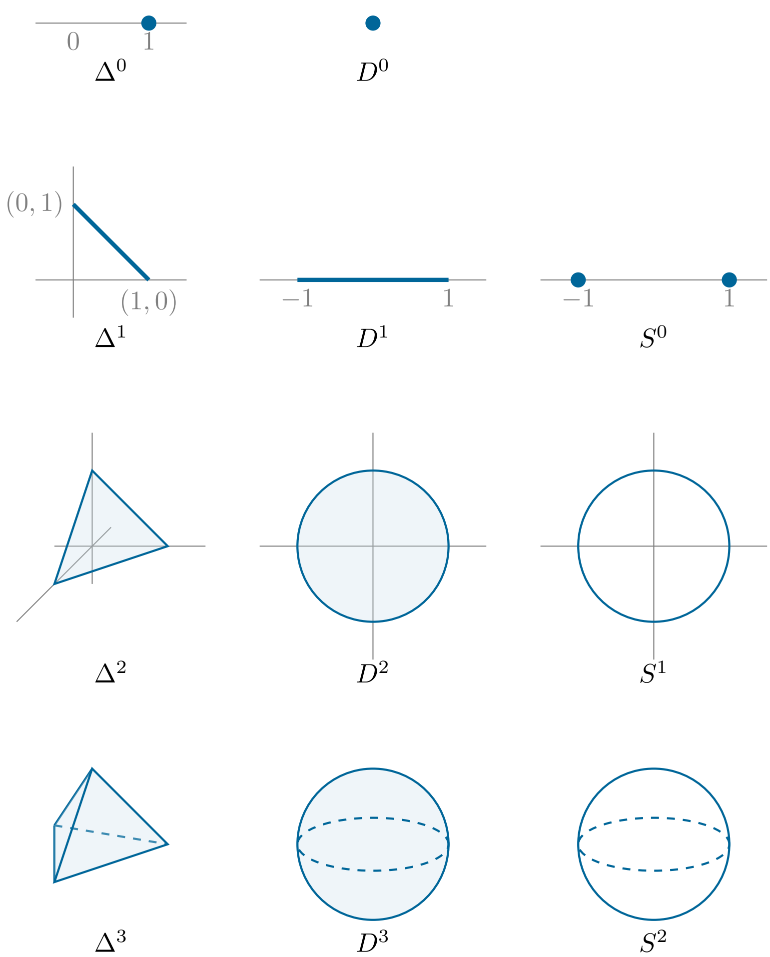


Figure 1.1.: Standard simplices, balls, spheres

appropriate subset of $\mathbb{R}^n$ instead of $\mathbb{R}^{n+1}$; however, passing to $\mathbb{R}^{n+1}$ offers the advantage of a simple symmetric description of this space.

Remark 1.1.2 (balls vs. simplices)**.** Let $n \in \mathbb{N}$. Then D^n and Δ^n are homeomorphic (Exercise). In the context of singular (co)homoloogy, the standard simplex is more convenient because of its combinatorial structure.

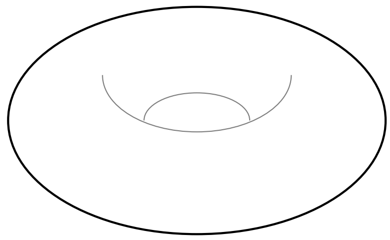


Figure 1.2.: The two-dimensional torus

1.1.2 Construction: Product Spaces

One way to combine two topological spaces into a new space is by taking the product (Remark A.1.12), which satisfies the universal property of products in the category of topological spaces (Remark 1.1.4). For example, taking products of circles leads to tori (Figure 1.2):

Definition 1.1.3 (torus). Let $n \in \mathbb{N}_{>0}$. The *n-dimensional torus* is defined as the n-fold cartesian product $(S^1)^n$, endowed with the product topology.

If X is a topological space, then the product $X \times [0, 1]$ can be used to model what happens to points in X in the "time interval" $[0, 1]$. Therefore, products of the form $X \times [0, 1]$ will be important to define deformation concepts such as homotopy (Chapter 1.3.1).

Let us recall the universal property of products: Roughly speaking, products are characterised by maps *to* the product. Hence, it is easy to construct maps to products; but, in general, it is much harder to construct maps out of products!

Remark 1.1.4 (universal property of product spaces). Let X_1, X_2 be topological spaces, let $X := X_1 \times X_2$ (endowed with the product topology), and let $p_1 : X \longrightarrow X_1$, $p_2 : X \longrightarrow X_2$ be the canonical projections; by definition of the product topology, p_1 and p_2 are continuous. Then X together with p_1 and p_2 satisfies the universal property of the product of X_1 and X_2 in the category of topological spaces (check!):

For every topological space Y and all continuous maps $f_1 : Y \longrightarrow X_1$ and $f_2 : Y \longrightarrow X_2$ there is exactly one continuous map $f : Y \longrightarrow X$ with

$$p_1 \circ f = f_1 \quad \text{and} \quad p_2 \circ f = f_2$$

(namely, $f = (y \mapsto (f_1(y), f_2(y)))$).

More generally, this also holds for products of arbitrary families of topological spaces (check!).

1.1.3 Construction: Quotient Spaces and Glueings

In order to model the glueing of several topological spaces, we proceed as follows:

- We first model glueing parts of a single topological space, using quotient spaces.
- We then model putting topological spaces next to each other, using the disjoint union topology.
- Finally, we model general glueings by combining the first two steps. This leads to so-called pushouts.

Quotient spaces provide a means to model "glueing" parts of a topological space together.

Definition 1.1.5 (quotient topology)**.** Let X be a topological space, let Y be a set, and let $p\colon X \longrightarrow Y$ be a surjective map. The *quotient topology on Y, induced by p*, is defined as

$$\bigl\{U \subset Y \bigm| p^{-1}(U) \text{ is open in } X\bigr\}.$$

This is indeed a topology on Y (check!).

Usually, the surjective map $p\colon X \longrightarrow Y$ arises as the quotient projection of a (geometric) explicit equivalence relation on X (so that Y is the set of equivalence classes).

Caveat 1.1.6. By construction, the quotient map to a quotient space is always continuous. But, in general, the quotient map is *not* open (even though the definition might suggest otherwise).

Example 1.1.7 (Möbius strip)**.** The *Möbius strip* is the quotient space

$$M := \bigl([0,1] \times [0,1]\bigr) \mathbin{/} \sim,$$

where $[0,1] \times [0,1]$ carries the product topology of the standard topology on the unit interval and the equivalence relation "$\sim$" is defined as follows:

> For all $x, y \in [0,1] \times [0,1]$, we have $x \sim y$ if and only if $x = y$ or the condition that $x_1 \in \{0,1\}$ and $y_1 = 1 - x_1$, $y_2 = 1 - x_2$ is satisfied.

We equip M with the quotient topology induced by the canonical projection $[0,1] \times [0,1] \longrightarrow ([0,1] \times [0,1]) \mathbin{/} \sim$. This construction is illustrated in Figure 1.3.

Of course, we could also describe the homeomorphism type of M by an appropriate subset of $\mathbb{R}^3$. However, it is much simpler (and also topologically more illuminating) to use the glueing description of the Möbius strip.

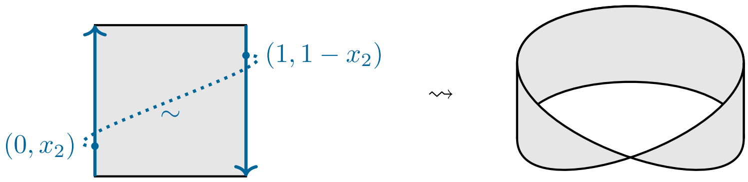


Figure 1.3.: The Möbius strip as quotient space

Example 1.1.8 (real projective spaces). Let $n \in \mathbb{N}$. The *n-dimensional real projective space* is defined as

$$\mathbb{R}P^n := S^n \,/\, \sim,$$

where "$\sim$" is the equivalence relation on S^n generated by

$$\forall_{x \in S^n} \quad x \sim -x.$$

We equip $\mathbb{R}P^n$ with the quotient topology (of the standard topology on the sphere S^n). It should be noted that it is not clear a priori how to describe (the homeomorphism type of) this space as a subspace of some Euclidean space ...

Study note. Where in the building of the Fakultät für Mathematik can you find a model of $\mathbb{R}P^2$?

Example 1.1.9 (collapsing a subspace). Let X be a topological space and let $A \subset X$. We then write X/A for the quotient space $X \,/\, \sim$, where "$\sim$" is the equivalence relation

$$\bigl\{(a, b) \bigm| a, b \in A\bigr\} \cup \bigl\{(x, x) \bigm| x \in X\bigr\} \subset X \times X$$

on X; we equip X/A with the corresponding quotient topology. In other words: In the quotient X/A, all points of A are identified to a single point.

For instance: If $n \in \mathbb{N}_{>0}$, then $D^n/S^{n-1} \cong_{\mathsf{Top}} S^n$ (Exercise). This description is useful, when constructing maps out of spheres (Proposition 1.1.13).

Glueing multiple topological spaces is modelled through so-called pushouts. As intermediate step, we will first model "putting topological spaces next to each other", using the disjoint union topology:

Definition 1.1.10 (disjoint union topology). Let I be a set and let $(X_i)_{i \in I}$ be a family of topological spaces. Then the *disjoint union topology* on the disjoint union $\coprod_{i \in I} X_i$ is given by

$$\left\{ U \subset \bigsqcup_{i \in I} X_i \;\middle|\; \forall_{j \in I} \quad i_j^{-1}(U) \text{ is open in } X_j \right\},$$

where $(i_j \colon X_j \longrightarrow \bigsqcup_{i \in I} X_i)_{j \in I}$ is the corresponding family of canonical inclusion maps. In the following, we will often view the spaces X_i as subsets of $\bigsqcup_{i \in I} X_i$, in order to reduce notational overhead.

Combining the disjoint union topology and the quotient topology leads to the pushout of topological spaces:

Definition 1.1.11 (pushout of topological spaces)**.** Let X_0, X_1, X_2 be topological spaces and let $i_1 \colon X_0 \longrightarrow X_1$ and $i_2 \colon X_0 \longrightarrow X_2$ be continuous maps. The *pushout* of the diagram

$$\begin{tikzcd} X_0 \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \\ X_1 & \end{tikzcd}$$

is the topological space (endowed with the quotient topology of the disjoint union topology)

$$X_1 \cup_{X_0} X_2 := (X_1 \sqcup X_2) \,/ \sim,$$

where "$\sim$" denotes the equivalence relation on $X_1 \sqcup X_2$ that is generated by

$$\forall_{x \in X_0} \quad i_1(x) \sim i_2(x).$$

Example 1.1.12 (some pushouts)**.** Some concrete examples of pushouts of topological spaces are depicted in Figure 1.4.

How can we work rigorously (and not only by handwaving and vague intuition) with quotient spaces, disjoint unions, and pushouts? All these constructions can be viewed as colimits in the category of topological spaces (Chapter IV.1.4); hence, they possess a corresponding universal property that characterises continuous maps *out of* these spaces:

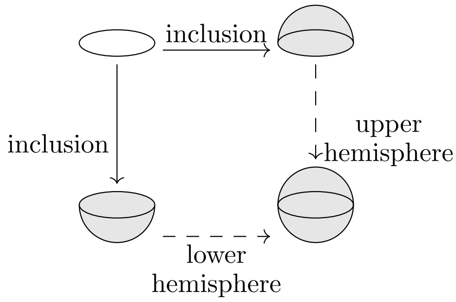
Proposition 1.1.13 (universal property of quotient spaces)**.** *Let X be a topological space, let Y be a set, and let $p \colon X \longrightarrow Y$ be a surjective map. The quotient topology on Y induced by p has the following universal property: For every topological space Z and every map $g \colon Y \longrightarrow Z$, the map g is continuous (with respect to the quotient topology on Y) if and only if $g \circ p \colon X \longrightarrow Z$ is continuous.*

Proof. Let Z be a topological space and let $g \colon Y \longrightarrow Z$ be a map.

If g is continuous, then also the composition $g \circ p$ with the continuous map p is continuous.

Conversely, let $g \circ p$ be continuous. We show that also g is continuous: Let $U \subset Z$ be an open subset. Then $g^{-1}(U) \subset Y$ is open with respect to the

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{inclusion}"] \arrow[d, "\text{inclusion}"'] & D^n \arrow[d, dashed, "\text{upper hemisphere}"] \\ D^n \arrow[r, dashed, "\text{lower hemisphere}"'] & {\color{purple}S^n} \end{tikzcd}$$



$$\begin{tikzcd} \{0,1\} \times [0,1] \arrow[r, "\text{inclusion}"] \arrow[d, "\begin{array}{l}(0,t) \mapsto (0,t) \\ (1,t) \mapsto (1,1-t)\end{array}"'] & {[0,1] \times [0,1]} \arrow[d, dashed] \\ {[0,1] \times [0,1]} \arrow[r, dashed] & {\color{purple}\text{Möbius strip}} \end{tikzcd}$$

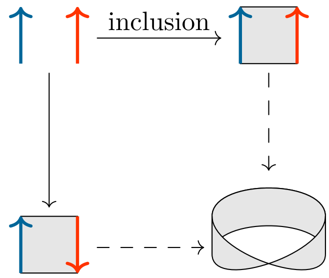


Figure 1.4.: Pushouts in Top

quotient topology on Y because

$$p^{-1}\big(g^{-1}(U)\big) = (g \circ p)^{-1}(U)$$

is open in view of the continuity of $g \circ p$. Hence, g is continuous. $\square$

The disjoint union topology turns the disjoint union of topological spaces into the coproduct of these spaces (in the category of topological spaces, with respect to the canonical inclusions).

Proposition 1.1.14 (universal property of pushouts)**.** *Let X_0, X_1, X_2 be topological spaces and let $i_1 \colon X_0 \longrightarrow X_1$ and $i_2 \colon X_0 \longrightarrow X_2$ be continuous maps. A topological space X together with continuous maps $j_1 \colon X_1 \longrightarrow X$, $j_2 \colon X_2 \longrightarrow X$ satisfies the* universal property of this pushout, *if the following holds: We have $j_1 \circ i_1 = j_2 \circ i_2$ and for every topological space Z and all continuous maps $f_1 \colon X_1 \longrightarrow Z$, $f_2 \colon X_2 \longrightarrow Z$ with $f_1 \circ i_1 = f_2 \circ i_2$ there exists a unique continuous map $f \colon X \longrightarrow Z$ with (Figure 1.5)*

$$f \circ j_1 = f_1 \quad \textit{and} \quad f \circ j_2 = f_2.$$

In this case, we also say that

$$\begin{tikzcd}
X_0 \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \arrow[d, "j_2"] \arrow[ddr, bend left, "f_2"] & \\
X_1 \arrow[r, "j_1"'] \arrow[drr, bend right, "f_1"'] & X \arrow[dr, dashed, "\exists! f"] & \\
 & & Z
\end{tikzcd}$$

Figure 1.5.: The universal property of pushouts

$$\begin{tikzcd}
X_0 \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \arrow[d, "j_2"] \\
X_1 \arrow[r, "j_1"'] & X
\end{tikzcd}$$

is a pushout diagram of topological spaces.

1. *The pushout $X := X_1 \cup_{X_0} X_2$ with respect to i_1 and i_2, together with the continuous maps $j_1 \colon X_1 \longrightarrow X$, $j_2 \colon X_2 \longrightarrow X$ induced by the inclusions into $X_1 \sqcup X_2$, satisfies the universal property of this pushout.*

2. *If X', together with $j_1' \colon X_1 \longrightarrow X'$ and $j_2' \colon X_2 \longrightarrow X'$, satisfies the universal property of this pushout, then there is a unique homeomorphism $f \colon X \longrightarrow X'$ with $f \circ j_1 = j_1'$ and $f \circ j_2 = j_2'$.*

Proof. Ad 1. This is an easy consquence of the universal property of quotient spaces and disjoint union spaces: By construction, $j_1 \circ i_1 = j_2 \circ i_2$. Let Z be a topological space and let $f_1 \colon X_1 \longrightarrow Z$ and $f_2 \colon X_2 \longrightarrow Z$ be continuous maps with $f_1 \circ i_1 = f_2 \circ i_2$.

- *Existence of a glued map:* By definition of the disjoint union topology, the maps f_1 and f_2 yield a continuous map $\widetilde{f} \colon X_1 \sqcup X_2 \longrightarrow Z$ with

  $$\widetilde{f}|_{X_1} = f_1 \quad \text{and} \quad \widetilde{f}|_{X_2} = f_2.$$

  Because of $f_1 \circ j_1 = f_2 \circ j_2$, this map induces a well-defined map

  $$\begin{aligned} f \colon X = X_1 \cup_{X_0} X_2 &\longrightarrow Z \\ [x] &\longmapsto \widetilde{f}(x). \end{aligned}$$

  In view of the universal property of quotient spaces, the map f is also continuous. By construction, it satisfies $f \circ j_1 = f_1$ and $f \circ j_2 = f_2$.

- *Uniqueness of the glued map:* As $j_1(X_1) \cup j_2(X_2) = X$, the glued map is uniquely determined by the composition with j_1 and j_2, whence by f_1 and f_2.

Thus, the pushout construction indeed has the universal property of the pushout.

Ad 2. This follows directly from the standard uniqueness argument for universal properties (Proposition IV.1.4.6). □

In the next section, we will demonstrate in a concrete example how these techniques can be used.

1.1.4 The Homeomorphism Problem

In the previous sections, we have seen how to *construct* interesting examples of topological spaces out of simple building blocks. However, in general, *comparing* different topological spaces is rather difficult. In particular, the obvious classification problem

> Classify topological spaces up to homeomorphism!

is unsolvable in a precise mathematical sense (Outlook 2.2.19). However, there are some strategies that allow us to decide whether certain special topological spaces are homeomorphic or not.

How to prove that two topological spaces are homeomorphic. Let X and Y be topological spaces and let us suppose that it is plausible that X and Y are homeomorphic. How can we prove this rigorously?

One option is to follow the definition of being homeomorphic, by constructing continuous maps $X \longrightarrow Y$ and $Y \longrightarrow X$ that are mutually inverse.

However, in practice, X or Y will often be the result of a specific construction that satisfies a universal property. This universal property will usually grant a continuous map in one of the two directions. In order to conclude that such a map is a homeomorphism, additional information is needed. At this point, the compact-Hausdorff trick (Corollary A.1.40) can be helpful:

Proposition 1.1.15 (compact-Hausdorff trick). *Let X be a compact topological space, let Y be a Hausdorff topological space, and let $f\colon X \longrightarrow Y$ be continuous and bijective. Then f is a homeomorphism(!).*

Example 1.1.16 (the circle). Let

$$C := [0,1]/(0 \sim 1)$$

(endowed with the quotient topology of the standard topology on $[0,1]$); i.e., C is obtained from the unit interval by glueing the two end points. We prove that $C \cong_{\mathsf{Top}} S^1$:

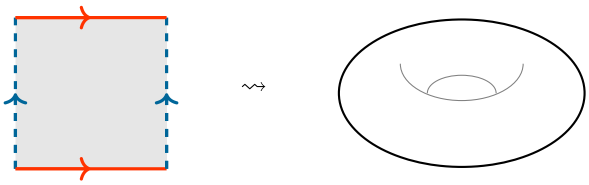


Figure 1.6.: The torus via glueing

- We first observe that C is constructed as a quotient space; hence, we can apply the universal property of quotient spaces (Proposition 1.1.13) to construct continuous maps out of C: Let

  $$\begin{aligned} f \colon [0,1] &\longrightarrow S^1 \\ t &\longmapsto \bigl(\cos(2 \cdot \pi \cdot t), \sin(2 \cdot \pi \cdot t)\bigr). \end{aligned}$$

  Then f is a well-defined continuous map (because cos and sin are continuous and the fact that S^1 carries the subspace topology of the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$).

  By definition, $f(0) = (1,0) = f(1)$. Hence, f induces a well-defined map $\overline{f} \colon C \longrightarrow S^1$. Because f is continuous, the universal property of quotient spaces (Proposition 1.1.13) lets us deduce that $\overline{f}$ is continuous.

- The constructed map $\overline{f} \colon C \longrightarrow S^1$ is bijective (this follows from elementary analysis).

- The space C is compact (as continuous image of the compact space $[0,1]$) and S^1 is Hausdorff (as subspace of the Hausdorff space $\mathbb{R}^2$).

- Applying the compact-Hausdorff trick (Proposition 1.1.15) to $\overline{f}$, we obtain that $\overline{f}$ is a homeomorphism.

In the same way, one can prove that the glueing of the unit square as indicated in Figure 1.6 results in a topological space homeomorphic to the torus $S^1 \times S^1$ (check!).

This shows, for example, that the classical Asteroids [2] game by Atari takes place on a torus.

How to prove that two topological spaces are not homeomorphic. Let X and Y be topological spaces and let us suppose that it is plausible that X and Y are *not* homeomorphic. How can we prove this rigorously?

Determining all continuous maps $X \longrightarrow Y$ and $Y \longrightarrow X$ and checking whether there is a mutually inverse pair among them is certainly not feasible in general.

Therefore, we are interested in finding homeomorphism invariants that have different values for X and Y. We then could argue by contradiction to prove that X and Y are *not* homeomorphic.

Some simple examples of homeomorphism invariant properties of topological spaces are:

- Being path-connected (Proposition A.1.22).
- Being connected (Proposition A.1.28).
- Being Hausdorff (Proposition A.1.32).
- Being compact (Proposition A.1.36).

Moreover, cardinality, the number of path-connected components, and the number of connected components are preserved by homeomorphisms. Also properties such as

- Every continuous self-map has at least one fixed point.

are preserved by homeomorphisms.

The following list of simple-minded questions shows that we will need more sophisticated invariants to understand topological spaces:

- For which $n, m \in \mathbb{N}$ are $\mathbb{R}^n$ and $\mathbb{R}^m$ homeomorphic? For which $n, m \in \mathbb{N}$ do there exist non-empty open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, that are homeomorphic?
- For which $n, m \in \mathbb{N}$ are S^n and S^m homeomorphic?
- For which n is $(S^1)^n$ homeomorphic to S^n ?
- Let $n \in \mathbb{N}$ and let $S \subset \mathbb{R}^{n+1}$ be homeomorphic to S^n. What can be said about the complement $\mathbb{R}^{n+1} \setminus S$?
- Is there a subspace of $\mathbb{R}^3$ that is homeomorphic to $\mathbb{R}P^2$?
- Are there continuous maps $D^n \longrightarrow D^n$ that do *not* have a fixed point?
- Can hedgehogs be combed? For which $n \in \mathbb{N}$ does there exist a nowhere vanishing vector field on S^n ?
- ...

Example 1.1.17 (the dimension problem: 0)**.** Let $m \in \mathbb{N}$. If $\mathbb{R}^0$ is homeomorphic to $\mathbb{R}^m$, then $m = 0$ (as can be seen by comparing cardinalities).

Example 1.1.18 (the dimension problem: 1)**.** Let $m \in \mathbb{N}$ such that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^m$. We show that then $m = 1$, using the *point-removal trick*: Let $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ be a homeomorphism. Then the restriction

$$f|_{\mathbb{R}\setminus\{0\}} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{f(0)\}$$

is also a homeomorphism. Therefore, as $\mathbb{R} \setminus \{0\}$ is not path-connected, also $\mathbb{R}^m \setminus \{f(0)\}$ is not path-connected. This implies $m = 1$ (check!).

Experiment 1.1.19 (the dimension problem: 2 ?!)**.** We try to generalise the argument of Example 1.1.18 to prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are *not* homeomorphic. *Assume* for a contradiction that there exists a homeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. Let $L := \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ be a line. Then the restriction

$$f|_{\mathbb{R}^2\setminus L} : \mathbb{R}^2 \setminus L \longrightarrow \mathbb{R}^m \setminus f(L)$$

is also a homeomorphism and $\mathbb{R}^2 \setminus L$ is not path-connected. However, at this point we are stuck: We do not yet have any tools available to make predictions on $f(L)$ or $\mathbb{R}^3 \setminus f(L)$ (the homeomorphism could do wild stuff ...). We will see later how to overcome this problem with a generalised version of the point-removal trick (Corollary 1.3.24).

In this course, we will see how suitable invariants from Algebraic Topology help to solve problems of this type. To this end, we will first introduce a general notion of invariant, via the language of categories and functors.

1.2 Categories and Functors

We will model the translation of topological problems into algebraic problems via the language of categories and functors. More precisely, mathematical theories will be modelled as categories, translations as functors, and the comparison between different translations by natural transformations. Therefore, we will first quickly review basic terminology from category theory.

1.2.1 Categories

Mathematical theories consist of objects (e.g., groups, topological spaces, ...) and structure preserving maps (e.g., group homomorphisms, continuous maps, ...). This can be abstracted to the notion of a category [33, 5, 58, 60].

Definition 1.2.1 (category)**.** A *category* C consists of the following data:

- A class $\operatorname{Ob}(C)$; the elements of $\operatorname{Ob}(C)$ are called *objects of* C.
- For all objects $X, Y \in \operatorname{Ob}(C)$ a set $\operatorname{Mor}_C(X, Y)$; the elements of the set $\operatorname{Mor}_C(X, Y)$ are called *morphisms from* X *to* Y *in* C. (Implicitly, we will assume that the morphism sets between different pairs of objects are disjoint and that we can recover the source and target object from a morphism.)

- For all objects $X, Y, Z \in \mathrm{Ob}(C)$ a composition of morphisms:

$$\begin{aligned} \circ\colon \mathrm{Mor}_C(Y,Z) \times \mathrm{Mor}_C(X,Y) &\longrightarrow \mathrm{Mor}_C(X,Z) \\ (g,f) &\longmapsto g \circ f \end{aligned}$$

This data is required to satisfy the following conditions:

- For each object X in C there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_C(X,X)$ such that: For all $Y \in \mathrm{Ob}(C)$ and all morphisms $f \in \mathrm{Mor}_C(X,Y)$ and $g \in \mathrm{Mor}_C(Y,X)$, we have

  $$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g.$$

  (The morphism id_X is uniquely determined by this property (check!); it is the *identity morphism of X in C*.)

- The composition of morphisms is associative: For all objects W, X, Y, Z in C and all morphisms $f \in \mathrm{Mor}_C(W,X)$, $g \in \mathrm{Mor}_C(X,Y)$, and $h \in \mathrm{Mor}_C(Y,Z)$ we have

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$

Remark 1.2.2 (classes). Classes are a tool to escape the set-theoretic paradoxon of the "set of all sets" (Chapter I.1.3.3) [21]. In case you are not familiar with von Neumann–Bernays–Gödel set theory, you can use the slogan that classes are "potentially large", "generalised" sets.

All concepts and facts in mathematical theories that can be expressed in terms of objects, identity morphisms, and (the composition of) morphisms also admit a category theoretic version. For instance, in this way, we obtain a general notion of isomorphism:

Definition 1.2.3 (isomorphism). Let C be a category. Objects $X, Y \in \mathrm{Ob}(C)$ are *isomorphic in C*, if there exist morphisms $f \in \mathrm{Mor}_C(X,Y)$ and $g \in \mathrm{Mor}_C(Y,X)$ with

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y\,.$$

In this case, f and g are *isomorphisms in C* and we write $X \cong_C Y$. If the category is clear from the context, we might also write $X \cong Y$.

Proposition 1.2.4 (elementary properties of isomorphisms). *Let C be a category and let $X, Y, Z \in \mathrm{Ob}(C)$.*

1. *Then the identity morphism id_X is an isomorphism in C (from X to X).*
2. *If $f \in \mathrm{Mor}_C(X,Y)$ is an isomorphism in C, then there is a unique morphism $g \in \mathrm{Mor}_C(Y,X)$ that satisfies $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.*

3. *Compositions of (composable) isomorphisms are isomorphisms.*
4. *If $X \cong_C Y$, then $Y \cong_C X$.*
5. *If $X \cong_C Y$ and $Y \cong_C Z$, then $X \cong_C Z$.*

Proof. All claims follow easily follow from the definitions (check!). □

Moreover, the setup of categories can be used to give a general definition of commutative diagrams (Chapter IV.1.1.4).

We collect some basic examples of categories and introduce some categories that are relevant in Algebraic Topology.

Example 1.2.5 (set theory)**.** The category Set of sets consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Set})$ be the class(!) of all sets.
- morphisms: If X and Y are sets, then we define $\mathrm{Mor}_{\mathsf{Set}}(X, Y)$ as the set of all set-theoretic maps $X \longrightarrow Y$.
- compositions: If X, Y, and Z are sets, then the composition map $\mathrm{Mor}_{\mathsf{Set}}(Y, Z) \times \mathrm{Mor}_{\mathsf{Set}}(X, Y) \longrightarrow \mathrm{Mor}_{\mathsf{Set}}(X, Z)$ is ordinary composition of maps.

Clearly, this composition is associative. If X is a set, then the usual identity map

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

is the identity morphism id_X of X in Set. Objects in Set are isomorphic if and only if there exists a bijection between them, i.e., if they have the same cardinality.

Caveat 1.2.6. The concept of morphisms and compositions in the definition of categories is modelled on the example of maps between sets and ordinary composition of maps. In general categories, morphisms are not necessarily maps between sets and the composition of morphisms is not necessarily ordinary composition of maps!

Example 1.2.7 (algebra)**.** Let K be a field. The category Vect_K of K-vector spaces consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Vect}_K)$ be the class(!) of all K-vector spaces.
- morphisms: If X, Y are K-vector spaces, then we define $\mathrm{Mor}_{\mathsf{Vect}_K}(X, Y)$ as the set of all K-linear maps $X \longrightarrow Y$. In this case, we also write $\mathrm{Hom}_K(X, Y)$ for the set of morphisms.
- compositions: As composition we take the ordinary composition of maps.

Objects in Vect_K are isomorphic if and only if they are isomorphic in the classical sense from Linear Algebra.

Analogously, we can define the category Group of groups, the category Ab of Abelian groups, the category ${}_R\mathsf{Mod}$ of left-modules over a ring R, the category Mod_R of right-modules over a ring R, ...

Example 1.2.8 (absolute topology)**.** The category Top of topological spaces consists of:

- objects: Let $\mathrm{Ob}(\mathsf{Top})$ be the class(!) of all topological spaces.
- morphisms: If X and Y are topological spaces, then we define

  $$\mathrm{map}(X, Y) := \mathrm{Mor}_{\mathsf{Top}}(X, Y)$$

  to be the set of all continuous maps $X \longrightarrow Y$.
- compositions: As composition we take the ordinary composition of maps.

Objects in Top are isomorphic if and only if they are homeomorphic.

Often, we are only interested in the difference between a topological space and a certain subspace. For example, we can model this situation through quotient spaces. However, in general, the quotient topology tends to have bad properties. Alternatively, we can use the following trick to handle differences between spaces and subspaces:

Example 1.2.9 (relative topology, pairs of spaces)**.** The category Top^2 of pairs of spaces consists of:

- objects: Let

  $$\mathrm{Ob}(\mathsf{Top}^2) := \bigl\{(X, A) \bigm| X \in \mathrm{Ob}(\mathsf{Top}),\ A \subset X\bigr\}.$$

- morphisms: If (X, A) and (Y, B) are pairs of spaces, then we define

  $$\begin{aligned}\mathrm{map}\bigl((X, A), (Y, B)\bigr) &:= \mathrm{Mor}_{\mathsf{Top}^2}\bigl((X, A), (Y, B)\bigr)\\ &:= \bigl\{f \in \mathrm{map}(X, Y) \bigm| f(A) \subset B\bigr\}.\end{aligned}$$

- compositions: As composition we take the ordinary composition of maps (this is well-defined!).

The absolute case corresponds to pairs of spaces with empty subspace. A particularly important special case is the case where the subspace consists of a single point. This leads to the category Top_* of pointed spaces (which is used in homotopy theory; Definition 1.3.8).

Finally, let us introduce a category that (at least implicitly) plays a key role in the definition of singular homology:

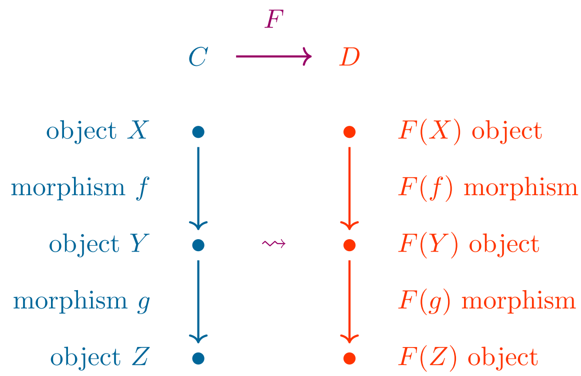


Figure 1.7.: Functor, schematically

Definition 1.2.10 (the simplex category)**.** The *simplex category* Δ consists of:

- objects: Let $\mathrm{Ob}(\Delta) := \{\Delta(n) \mid n \in \mathbb{N}\}$. Here, for $n \in \mathbb{N}$, we write
$$\Delta(n) := \{0, \dots, n\}.$$
- morphisms: If $n, m \in \mathbb{N}$, then $\mathrm{Mor}_\Delta\bigl(\Delta(n), \Delta(m)\bigr)$ is defined to be the set of all maps $\{0, \dots, n\} \longrightarrow \{0, \dots, m\}$ that are monotonically increasing.
- compositions: As compositions we take the ordinary composition of maps (this is well-defined!).

In Δ, objects are isomorphic if and only if they are equal.

1.2.2 Functors

As next step, we will formalise translations between mathematical theories, using functors. Roughly speaking, functors are “structure preserving maps between categories” (Figure 1.7). In particular, functors preserve isomorphisms (Proposition 1.2.18).

Definition 1.2.11 (functor)**.** Let C and D be categories. A *(covariant) functor* $F : C \longrightarrow D$ consists of the following data:

- A map $F\colon\ \mathrm{Ob}(C) \longrightarrow \mathrm{Ob}(D)$.
- For all objects $X, Y \in \mathrm{Ob}(C)$ a map
$$F\colon\ \mathrm{Mor}_C(X, Y) \longrightarrow \mathrm{Mor}_D\bigl(F(X), F(Y)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \mathrm{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all $X, Y, Z \in \mathrm{Ob}(C)$ and all $f \in \mathrm{Mor}_C(X, Y)$, $g \in \mathrm{Mor}_C(Y, Z)$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $F: C \longrightarrow D$ consists of the following data:

- A map $F\colon \mathrm{Ob}(C) \longrightarrow \mathrm{Ob}(D)$.
- For all objects $X, Y \in \mathrm{Ob}(C)$ a map
$$F\colon \mathrm{Mor}_C(X, Y) \longrightarrow \mathrm{Mor}_D\bigl(F(Y), F(X)\bigr).$$

This data is required to satisfy the following conditions:

- For all $X \in \mathrm{Ob}(C)$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all $X, Y, Z \in \mathrm{Ob}(C)$ and all $f \in \mathrm{Mor}_C(X, Y)$, $g \in \mathrm{Mor}_C(Y, Z)$, we have
$$F(g \circ f) = F(f) \circ F(g).$$

In other words, contravariant functors reverse the direction of arrows. More concisely, contravariant functors $C \longrightarrow D$ are the same as covariant functors $C \longrightarrow D^{\mathrm{op}}$, where D^{op} denotes the dual category of D.

Example 1.2.12 (identity functor)**.** Let C be a category. Then the *identity functor* $\mathrm{Id}_C : C \longrightarrow C$ is defined as follows:

- on objects: We consider the map
$$\begin{aligned} \mathrm{Ob}(C) &\longrightarrow \mathrm{Ob}(C) \\ X &\longmapsto X. \end{aligned}$$
- on morphisms: For objects X, $Y \in \mathrm{Ob}(C)$, we consider the map
$$\begin{aligned} \mathrm{Mor}_C(X, Y) &\longrightarrow \mathrm{Mor}_C(X, Y) \\ f &\longmapsto f. \end{aligned}$$

Clearly, this defines a functor $C \longrightarrow C$.

Example 1.2.13 (composition of functors)**.** Let C, D, E be categories and let $F: C \longrightarrow D$, $G: D \longrightarrow E$ be functors. Then the functor $G \circ F : C \longrightarrow E$ is defined as follows:

- on objects: Let

$$\begin{aligned} G \circ F \colon C &\longrightarrow E \\ X &\longmapsto G\bigl(F(X)\bigr). \end{aligned}$$

- on morphisms: For all X, $Y \in \mathrm{Ob}(C)$, we set

$$\begin{aligned} G \circ F \colon \operatorname{Mor}_C(X,Y) &\longrightarrow \operatorname{Mor}_E\bigl(G(F(X)), G(F(Y))\bigr) \\ f &\longmapsto G(F(f)). \end{aligned}$$

Clearly, this defines a functor $C \longrightarrow E$. Moreover, composition of functors is associative.

Caveat 1.2.14 (the category of categories)**.** In view of the previous examples, it is tempting to introduce the "category of all catgories" (whose objects would be categories and whose morphisms would be functors). However, constructions of this type require set-theoretic precautions [10]. In the following, we will only use basic category theory and hence we will avoid these issues.

Three important, general, sources for functors are forgetful functors (by forgetting structure), free generation functors (by freely generating objects), and represented/representable functors (by viewing a category through the eyes of a given object).

Example 1.2.15 (forgetful functor)**.** The *forgetful functor* $\mathsf{Top} \longrightarrow \mathsf{Set}$ is defined as follows:

- on objects: We take the map $\mathrm{Ob}(\mathsf{Top}) \longrightarrow \mathrm{Ob}(\mathsf{Set})$ that maps a topological space to its underlying set.
- on morphisms: For all topological spaces X and Y, we consider the map

$$\begin{aligned} \operatorname{Mor}_{\mathsf{Vect}_\mathbb{R}}(X,Y) = \operatorname{Hom}_\mathbb{R}(X,Y) &\longrightarrow \operatorname{Mor}_{\mathsf{Set}}(X,Y) \\ f &\longmapsto f. \end{aligned}$$

Hence, this functor "forgets" the topological structure and only retains the underlying set-theoretic information. Analogously, we can define forgetful functors $\mathsf{Vect}_\mathbb{R} \longrightarrow \mathsf{Set}$, $\mathsf{Vect}_\mathbb{R} \longrightarrow \mathsf{Ab}$, ...

Example 1.2.16 (free generation functor)**.** We can translate set theory to Linear Algebra via the following functor $F \colon \mathsf{Set} \longrightarrow \mathsf{Vect}_\mathbb{R}$:

- on objects: We define

$$\begin{aligned} F \colon \mathrm{Ob}(\mathsf{Set}) &\longrightarrow \mathrm{Ob}(\mathsf{Vect}_\mathbb{R}) \\ X &\longmapsto \bigoplus_X \mathbb{R}. \end{aligned}$$

- on morphisms: If X and Y are sets and if $f\colon X \longrightarrow Y$ is a map, we define $F(f)\colon \bigoplus_X \mathbb{R} \longrightarrow \bigoplus_Y \mathbb{R}$ as the unique $\mathbb{R}$-linear map that extends f from the basis X to all of $\bigoplus_X \mathbb{R}$.

Example 1.2.17 (represented functor)**.** Let C be a category and let $X \in \operatorname{Ob}(C)$. Then the functor $\operatorname{Mor}_C(X, \,\cdot\,)\colon C \longrightarrow \mathsf{Set}$ *represented by* X is defined as follows:

- on objects: Let

$$\begin{aligned} \operatorname{Mor}_C(X, \,\cdot\,)\colon \operatorname{Ob}(C) &\longrightarrow \operatorname{Ob}(\mathsf{Set}) \\ Y &\longmapsto \operatorname{Mor}_C(X,Y). \end{aligned}$$

- on morphisms: Let

$$\begin{aligned} \operatorname{Mor}_C(X, \,\cdot\,)\colon \operatorname{Mor}_C(Y,Z) &\longrightarrow \operatorname{Mor}_{\mathsf{Set}}\bigl(\operatorname{Mor}_C(X,Y), \operatorname{Mor}_C(X,Z)\bigr) \\ g &\longmapsto (f \mapsto g \circ f). \end{aligned}$$

As we will see, additional structure on the object X will allow us to refine the represented functor $\operatorname{Mor}_C(X, \,\cdot\,)$ to a functor from C to categories with more structure than Set (Outlook 2.1.5).

Analogously, one can define the contravariant functor $\operatorname{Mor}_C(\,\cdot\,, X)$ represented by X.

A fundamental geometric example of a functor is the suspension functor (Exercise; Chapter 3.2.1).

Fundamental examples of algebraic functors are tensor product functors (Bemerkung IV.1.5.7).

The key property of functors is that they preserve isomorphisms. In particular, functors provide a good notion of invariants.

Proposition 1.2.18 (functors preserve isomorphism)**.** *Let C and D be categories, let $F\colon C \longrightarrow D$ be a functor, and let $X, Y \in \operatorname{Ob}(C)$.*

1. *If $f \in \operatorname{Mor}_C(X,Y)$ is an isomorphism in C, then the translated morphism $F(f) \in \operatorname{Mor}_D(F(X), F(Y))$ is an isomorphism in D.*
2. *In particular: If $X \cong_C Y$, then $F(X) \cong_D F(Y)$. In other words: If $F(X) \not\cong_D F(Y)$, then $X \not\cong_C Y$.*

Proof. The *first part* follows from the defining properties of functors: Because f is an isomorphism, there is a morphism $g \in \operatorname{Mor}_C(Y,X)$ with

$$g \circ f = \operatorname{id}_X \quad \text{and} \quad f \circ g = \operatorname{id}_Y .$$

Hence, we obtain

$$F(g) \circ F(f) = F(g \circ f) = F(\operatorname{id}_X) = \operatorname{id}_{F(X)}$$

and $F(f) \circ F(g) = \mathrm{id}_{F(Y)}$. Thus, $F(f)$ is an isomorphism from $F(X)$ to $F(Y)$ in D.

The *second part* is a direct consequence of the first part. □

Therefore, suitable functors can help to prove that certain objects are *not* isomorphic.

Caveat 1.2.19. In general, the converse is *not* true! I.e., objects that are mapped via a functor to isomorphic objects are, in general, *not* isomorphic (check!).

1.2.3 Natural Transformations

Functors are compared through natural transformations; roughly speaking, natural transformations are "structure preserving maps between functors".

Definition 1.2.20 (natural transformation, natural isomorphism)**.** Let C and D be categories and let $F, G \colon C \longrightarrow D$ be functors.

- A *natural transformation T from F to G*, in short $T \colon F \Longrightarrow G$, is a family $\bigl(T(X) \in \mathrm{Mor}_D(F(X), G(X))\bigr)_{X \in \mathrm{Ob}(C)}$ of morphisms such that for all objects $X, Y \in \mathrm{Ob}(C)$ and all(!) morphisms $f \in \mathrm{Mor}_C(X, Y)$ the equation
$$G(f) \circ T(X) = T(Y) \circ F(f)$$
holds in D. In other words, the following diagrams in D are commutative:
$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T(X)"'] & F(Y) \arrow[d, "T(Y)"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

- A *natural isomorphism* is a natural transformation that consists of isomorphisms (equivalently, a natural isomorphism is a natural transformation that admits an object-wise inverse natural transformation; check!).

Study note. The definition of natural transformation can easily be reconstructed: From (Linear) Algebra we already know examples of "natural isomorphisms". Natural isomorphisms only receive objects as input; hence, it is clear what type of families natural transformations have to be. Moreover, naturality should contain compatibility with morphisms. The only reasonable notion that can be formulated with this amount of data is the one in the commutative diagram above. That's it!

Remark 1.2.21 (natural)**.** The attribute "natural" is used in two related ways: On the one hand, it refers to functorial constructions; on the other hand, it refers to things based on natural transformations.

Natural transformations between represented functors can be completely classified; the key trick is to evaluate on identity morphisms:

Example 1.2.22 (morphisms lead to natural transformations between represented functors)**.** Let C be a category, let $X, Y \in \operatorname{Ob}(C)$, and let $f \in \operatorname{Mor}_C(X, Y)$. Then

$$T_f := \left(\begin{aligned} \operatorname{Mor}_C(Y, Z) &\longrightarrow \operatorname{Mor}_C(X, Z) \\ g &\longmapsto g \circ f \end{aligned} \right)_{Z \in \operatorname{Ob}(C)}$$

defines a natural transformation $\operatorname{Mor}_C(Y, \,\cdot\,) \Longrightarrow \operatorname{Mor}_C(X, \,\cdot\,)$ (check!).

Proposition 1.2.23 (Yoneda Lemma)**.** *Let C be a category, let $X, Y \in \operatorname{Ob}(C)$, and let $N(Y, X)$ be the collection (which turns out to be describable as a set) of all natural transformations* $\operatorname{Mor}_C(Y, \,\cdot\,) \Longrightarrow \operatorname{Mor}_C(X, \,\cdot\,)$.

1. *Then*

$$\begin{aligned} \varphi \colon \operatorname{Mor}_C(X, Y) &\longrightarrow N(Y, X) \\ f &\longmapsto T_f \\ \psi \colon N(Y, X) &\longrightarrow \operatorname{Mor}_C(X, Y) \\ T &\longmapsto \big(T(Y)\big)(\operatorname{id}_Y) \end{aligned}$$

   are mutually inverse bijections.

2. *In particular: The functors* $\operatorname{Mor}_C(X, \,\cdot\,), \operatorname{Mor}_C(Y, \,\cdot\,) \colon C \longrightarrow \mathsf{Set}$ *are isomorphic if and only if X and Y are isomorphic in C.*

Proof. The *first part* follows from a straightforward calculation: It should be noted that the map φ is indeed well-defined by Example 1.2.22. The maps φ and ψ are mutually inverse:

The composition $\psi \circ \varphi$: On the one hand, by definition, we have

$$\psi \circ \varphi(f) = \psi\big(\varphi(f)\big) = \big(T_f(Y)\big)(\operatorname{id}_Y) = \operatorname{id}_Y \circ f = f$$

for all $f \in \operatorname{Mor}_C(X, Y)$.

The composition $\varphi \circ \psi$: On the other hand, let $T \in N(Y, X)$ and let $Z \in \operatorname{Ob}(C)$, $g \in \operatorname{Mor}_C(Y, Z)$. Then we obtain

$$\begin{aligned}
\big(T(Z)\big)(g) &= \big(T(Z)\big)(g \circ \mathrm{id}_Y) \\
&= T(Z)\big(\mathrm{Mor}_C(Y,g)(\mathrm{id}_Y)\big) && \text{(by definition of } \mathrm{Mor}_C(Y,\,\cdot\,)) \\
&= \mathrm{Mor}_C(X,g)\big(T(Y)(\mathrm{id}_Y)\big) && \text{(because } T \text{ is a natural transformation)} \\
&= g \circ \psi(T) && \text{(by construction of } \psi) \\
&= \big(T_{\psi(T)}(Z)\big)(g) && \text{(by construction of } T_{\psi(T)}) \\
&= \Big(\big(\varphi \circ \psi(T)\big)(Z)\Big)(g) && \text{(by definition of } \varphi).
\end{aligned}$$

Hence, $\varphi \circ \psi(T) = T$, as desired.

The *second part* can be derived from the first part: The maps φ and ψ are compatible with identity morphisms/transformations and with the composition of morphisms/natural transformations. Hence, isomorphisms in C correspond under φ and ψ to natural isomorphisms. Alternatively, one can use the same proof strategy as in the first part (Proposition IV.1.3.6). □

Definition 1.2.24 (representable functor)**.** Let C be a category. A functor $F\colon C \longrightarrow \mathsf{Set}$ is *representable* if there exists an object $X \in \mathrm{Ob}(C)$ such that F and the represented functor $\mathrm{Mor}_C(X,\,\cdot\,)\colon C \longrightarrow \mathsf{Set}$ are naturally isomorphic. In this case, X is a *representing object for F*.

(In view of Proposition 1.2.23, representing objects of representable functors are unique up to isomorphism.)

Remark 1.2.25 (compatibility with inverse limits)**.** One advantage of representable functors is that we gain compatibility with inverse limits (e.g., with products) for free (Bemerkung IV.1.4.12).

Outlook 1.2.26. By now, category theory is a foundational language that is not only used in mathematics, but also in other fields such as computer science [58] or linguistics.

Literature exercise. Read about “The Birth of Categories and Functors” [12, p. 96f].

After this excursion to category theory, we return to the geometric setting.

1.3 Homotopy and Homotopy Invariance

The notion of homeomorphism is too rigid for many problems and techniques in Algebraic Topology (Figure 1.8). Therefore, we will introduce a weaker notion of isomorphism, namely homotopy equivalence. To this end, we will identify continuous maps that only differ by continuous deformations (Figure 1.9). We will then define the notion of homotopy invariant functors and we will show how the existence of certain homotopy invariant functors allows us to solve concrete problems in topology and geometry (Chapter 1.3.3).

Figure 1.8.: These spaces are *not* homeomorphic (check!), but they all share the same "principal shape", namely a "circle" with a "hole".

1.3.1 Homotopy

Definition 1.3.1 (homotopy, homotopic, homotopy equivalence, null-homotopic, contractible)**.** Let X and Y be topological spaces.

- Let $f, g\colon X \longrightarrow Y$ be continuous maps. Then f is *homotopic* to g, if f can be deformed continuously into g, i.e., if there exists a homotopy from f to g (Figure 1.9).

  A *homotopy* from f to g is a continuous map $h\colon X \times [0,1] \longrightarrow Y$ with

  $$h(\,\cdot\,,0) = f \quad \text{and} \quad h(\,\cdot\,,1) = g.$$

  In this case, we write $f \simeq g$.

- Maps that are homotopic to constant maps are called *null-homotopic*.

- The topological spaces X and Y are *homotopy equivalent*, if there exist continuous maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ satisfying

  $$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y;$$

  such maps are called *homotopy equivalences*. We then write $X \simeq Y$.

- Topological spaces that are homotopy equivalent to one-point spaces are called *contractible*.

Study note (etymology)**.** This might be a good time to briefly investigate the etymology of the different terms:

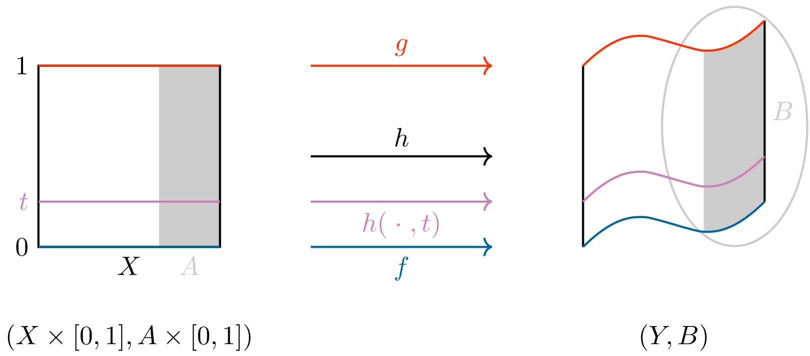


Figure 1.9.: Homotopies are "movies" between continuous maps [of pairs of spaces]; such movies can also be illustrated via flipbooks (Appendix A.2).

Ancient Greek	English
ὅμοιος	similar
μορφή	shape, figure
ὁμός	equal, similar
τόπος	location

Remark 1.3.2 (deformation of maps vs. paths of maps)**.** For sufficiently nice topological spaces, the exponential law for mapping spaces shows that homotopies between maps are the same as continuous paths between these maps in mapping spaces. More precisely: Let X be a *locally compact* topological space, i.e., for every $x \in X$ and every open neighbourhood U of x there exists a compact neighbourhood K of x with $K \subset U$. Then, for every topological space Y the currying map

$$\begin{aligned} \operatorname{map}\bigl(X \times [0,1], Y\bigr) &\longmapsto \operatorname{map}\bigl([0,1], \operatorname{map}(X,Y)\bigr) \\ h &\longmapsto \bigl(t \mapsto h(\,\cdot\,,t)\bigr) \end{aligned}$$

is well-defined and bijective. Here, $\operatorname{map}(X,Y)$ carries the *compact-open topology*, i.e., the topology on $\operatorname{map}(X,Y)$ that is generated by sets of the form

$$\bigl\{ f \in \operatorname{map}(X,Y) \bigm| f(K) \subset U \bigr\}$$

where $K \subset X$ is compact and $U \subset Y$ is open.

Example 1.3.3 (balls are contractible)**.** Let $n \in \mathbb{N}$. Then the ball D^n is contractible (Figure 1.10): We consider the continuous maps

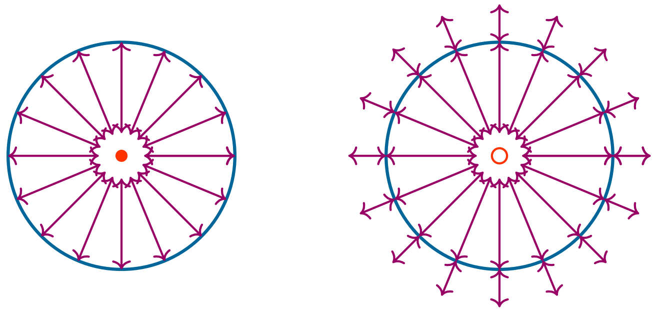


Figure 1.10.: The homotopy equivalences of Example 1.3.3 and Example 1.3.4, schematically; these illustration do *not* indicate an actual deformation of the *spaces*, but of *maps*.

$$\begin{aligned} f\colon D^n &\longrightarrow \{0\} \\ x &\longmapsto 0 \\ g\colon \{0\} &\longrightarrow D^n \\ 0 &\longmapsto 0. \end{aligned}$$

Then $f \circ g = \mathrm{id}_{\{0\}}$, and so $f \circ g \simeq \mathrm{id}_{\{0\}}$. Moreover, the homotopy

$$\begin{aligned} D^n \times [0,1] &\longrightarrow D^n \\ (x,t) &\longmapsto t \cdot x \end{aligned}$$

shows that $g \circ f \simeq \mathrm{id}_{D^n}$ (check!). Hence, $D^n \simeq \{0\}$. In the same way, one can prove that every star-shaped non-empty subspace of $\mathbb{R}^n$ is contractible; in particular, also $\mathbb{R}^n$ is contractible.

We will later develop tools that allow us to prove that the sphere S^n is *not* contractible (Corollary 4.4.2).

Example 1.3.4 (thick spheres)**.** Let $n \in \mathbb{N}$. Then $S^n \simeq \mathbb{R}^{n+1} \setminus \{0\}$ (Figure 1.10): We consider the maps

$$\begin{aligned} f\colon S^n &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ x &\longmapsto x \\ g\colon \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow S^n \\ x &\longmapsto \frac{1}{\|x\|_2} \cdot x. \end{aligned}$$

Then $g \circ f = \mathrm{id}_{S^n}$, whence $g \circ f \simeq \mathrm{id}_{S^n}$. Moreover, the (well-defined!) homotopy

$$\begin{aligned}
\bigl(\mathbb{R}^{n+1} \setminus \{0\}\bigr) \times [0,1] &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\
(x,t) &\longmapsto \frac{t \cdot \|x\|_2 + (1-t)}{\|x\|_2} \cdot x
\end{aligned}$$

shows that $f \circ g \simeq \mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ (check!). Hence, $S^n \simeq \mathbb{R}^{n+1} \setminus \{0\}$ and we may view the punctured space $\mathbb{R}^{n+1} \setminus \{0\}$ as a "thick sphere".

Caveat 1.3.5. Every homeomorphism is a homotopy equivalence. The converse does *not* hold in general. For example, by Example 1.3.3, $D^1 \simeq \{0\}$ and D^1 and $\{0\}$ do not even have the same cardinality. This example also shows that homotopy equivalences, in general, are neither injective nor surjective.

In the same way as in Top, we can also introduce homotopies etc. for pairs of spaces and for pointed spaces:

Definition 1.3.6 (homotopy/homotopy equivalence in Top^2)**.** Let (X, A) and (Y, B) be pairs of spaces.

- We abbreviate

$$(X, A) \times [0,1] := \bigl(X \times [0,1], A \times [0,1]\bigr) \in \mathrm{Ob}(\mathsf{Top}^2).$$

- Let $f, g \colon (X, A) \longrightarrow (Y, B)$ be continuous maps of pairs (i.e., morphisms in Top^2). Then f is *homotopic to g with respect to the subspaces A and B*, if there is a homotopy of pairs from f to g. A *homotopy of pairs* from f to g is a continuous map $h \colon (X, A) \times [0,1] \longrightarrow (Y, B)$ satisfying

$$h(\,\cdot\,, 0) = f \quad \text{and} \quad h(\,\cdot\,, 1) = g.$$

  In this case, we write $f \simeq_{A,B} g$.

  (In particular, in this situation, we have $h(x,t) \in B$ for all $x \in A$ and all $t \in [0,1]$).

- The pairs of spaces (X, A) and (Y, B) are *homotopy equivalent* if there exist continuous maps $f \colon (X, A) \longrightarrow (Y, B)$ and $g \colon (Y, B) \longrightarrow (X, A)$ of pairs such that

$$g \circ f \simeq_{A,A} \mathrm{id}_{(X,A)} \quad \text{and} \quad f \circ g \simeq_{B,B} \mathrm{id}_{(Y,B)};$$

  such maps are called *homotopy equivalences of pairs*. In this case, we write $(X, A) \simeq (Y, B)$.

Caveat 1.3.7 (relative homotopy)**.** For pairs of spaces, there is also a notion of *relative homotopy* (where the homotopy is required to fix each point in the subspace). This is useful when inductively constructing/modifying continuous maps; we will need this concept only at a later stage.

In some situations (e.g., for the construction of homotopy groups), it is convenient to have a distinguished basepoint available. This leads to the category of pointed spaces:

Definition 1.3.8 (pointed spaces)**.** The category Top_* of pointed spaces is the full subcategory of Top^2 whose objects are of the form $(X, \{x_0\})$ with $x_0 \in X$. In order to simplify notation, we will usually write (X, x_0) instead of $(X, \{x_0\})$ and call x_0 the *basepoint of* (X, x_0). More explicitly: The category Top_* consists of:

- objects: We let
$$\operatorname{Ob}(\mathsf{Top}_*) := \bigl\{(X, x_0) \bigm| X \in \operatorname{Ob}(\mathsf{Top}),\ x_0 \in X\bigr\}.$$
- morphisms: For pointed spaces (X, x_0) and (Y, y_0) we set
$$\begin{aligned} \operatorname{map}_*\bigl((X, x_0), (Y, y_0)\bigr) &:= \operatorname{Mor}_{\mathsf{Top}_*}\bigl((X, x_0), (Y, y_0)\bigr) \\ &:= \bigl\{f \in \operatorname{map}(X, Y) \bigm| f(x_0) = y_0\bigr\}. \end{aligned}$$
- compositions: As composition we take the ordinary composition of maps.

Definition 1.3.9 (homotopy/homotopy equivalence in Top_*)**.** Let (X, x_0), (Y, y_0) be pointed spaces.

- We abbreviate
$$(X, x_0) \times [0, 1] := \bigl(X \times [0, 1], \{x_0\} \times [0, 1]\bigr) \in \operatorname{Ob}(\mathsf{Top}^2).$$
- Let $f, g \colon (X, x_0) \longrightarrow (Y, y_0)$ be continuous pointed maps. Then f is *pointedly homotopic to* g if there exists a pointed homotopy from f to g. A *pointed homotopy* from f to g is a continuous map $h \colon (X, x_0) \times [0, 1] \longrightarrow (Y, y_0)$ with
$$h(\,\cdot\,, 0) = f \quad \text{and} \quad h(\,\cdot\,, 1) = g.$$
Let us point out that this includes the condition that h does not move the basepoint:
$$\forall_{t \in [0,1]} \quad h(x_0, t) = y_0.$$
We then write $f \simeq_* g$.
- Pointed maps that are pointedly homotopic to the constant pointed map are called *pointedly null-homotopic.*
- The pointed spaces (X, x_0) and (Y, y_0) are *pointedly homotopy equivalent*, if there exist pointed continuous maps $f \colon (X, x_0) \longrightarrow (Y, y_0)$ and $g \colon (Y, y_0) \longrightarrow (X, x_0)$ such that

$$g \circ f \simeq_* \mathrm{id}_{(X,x_0)} \quad \text{and} \quad f \circ g \simeq_* \mathrm{id}_{(Y,y_0)};$$

such maps are called *pointed homotopy equivalences*. In this case, we write $(X, x_0) \simeq_* (Y, y_0)$.

- Pointed spaces that are pointedly homotopy equivalent to a pointed one-point space are *pointedly contractible*.

Example 1.3.10 (balls are pointedly contractible)**.** The same arguments as in Example 1.3.3 show that $(D^n, 0)$ and $(\mathbb{R}^n, 0)$ are pointedly contractible for every $n \in \mathbb{N}$ (check!).

Caveat 1.3.11 (different notions of homotopy equivalence)**.** In general, absolute homotopy equivalences will not always lead to corresponding relative or pointed homotopy equivalences:

- Even though $D^1 \simeq \mathbb{R}$ and $S^0 \simeq \mathbb{R} \setminus \{0\}$, the pairs (D^1, S^0) and $(\mathbb{R}, \mathbb{R} \setminus \{0\})$ are *not* homotopy equivalent as pairs of spaces: The problem is that $\overline{\mathbb{R} \setminus \{0\}} = \mathbb{R}$ and that S^0 is disconnected. More precisely, we can argue as follows: *Assume* for a contradiction that there exist mutually inverse homotopy equivalences $f\colon (\mathbb{R}, \mathbb{R}\setminus\{0\}) \longrightarrow (D^1, S^0)$ and $g\colon (D^1, S^0) \longrightarrow (\mathbb{R}, \mathbb{R}\setminus\{0\})$ of pairs. In particular, there exists a homotopy $h\colon (D^1, S^0) \times [0,1] \longrightarrow (D^1, S^0)$ with

  $$h(\,\cdot\,, 0) = f \circ g \quad \text{and} \quad h(\,\cdot\,, 1) = \mathrm{id}_{(D^1,S^0)}\,.$$

  Because of

  $$f(\mathbb{R}) = f\bigl(\overline{\mathbb{R} \setminus \{0\}}\bigr) \subset \overline{f(\mathbb{R} \setminus \{0\})} \subset \overline{S^0} = S^0,$$

  we obtain $h(U) \subset S^0$, where

  $$U := \bigl(\{-1\} \times [0,1]\bigr) \cup \bigl(D^1 \times \{0\}\bigr) \cup \bigl(\{1\} \times [0,1]\bigr) \subset D^1 \times [0,1].$$

  However, the U-shaped subspace U is path-connected; thus, the image $h(U) \subset S^0$ is path-connected (Proposition A.1.22), and so $h|_U$ is constant. In particular,

  $$-1 = h(-1,1) = h(1,1) = 1,$$

  which is impossible. Hence, such a homotopy h of pairs cannot exist.

- If (X, x_0) is a pointed space and X is contractible (in Top), then, in general, (X, x_0) is *not* pointedly contractible: We consider the bushy sea urchin (with the subspace topology of $\mathbb{R}^2$)

  $$X := \bigl\{ s \cdot (\cos t, \sin t) \bigm| s \in [0,1],\ t \in \mathbb{Q} \cap [0, 2 \cdot \pi] \bigr\}$$

  with the basepoint $(1,0)$ (Figure 1.11). Then X is contractible (because it is star-shaped, Example 1.3.3), but *not* pointedly contractible (one

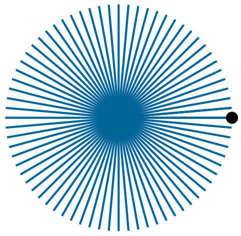


Figure 1.11.: The bushy sea urchin

can argue in a similar way as in the previous example, looking at the spikes "close" to the basepoint).

With a little bit more effort, one can extend this example to show that there are contractible spaces that for *every* choice of basepoint are *not* pointedly contractible.

More generally, in order to define a notion of homotopy and homotopy equivalences we need suitable products and a suitable model of the unit interval. For example, translating this concept into homological algebra leads to the notion of chain homotopy and chain homotopy equivalence for chain complexes (Appendix A.6.3).

Outlook 1.3.12 ($\mathbb{A}^1$-homotopy)**.** In $\mathbb{A}^1$-homotopy theory (a branch of Algebraic Geometry inspired by homotopy theory), the affine line $\mathbb{A}^1$ plays a role similar to the unit interval $[0, 1]$ in classical homotopy theory.

Of course, we can recover the case of homotopies etc. in Top or Top_* from Top^2 (by taking empty subspaces or one-point subspaces, respectively). So, when stating and proving properties of homotopies, it suffices to deal with the case of pairs of spaces.

Proposition 1.3.13 (elementary properties of homotopy)**.**

1. *Let (X, A) and (Y, B) be pairs of spaces. Then "$\simeq_{A,B}$" is an equivalence relation on* $\operatorname{map}\bigl((X, A), (Y, B)\bigr)$.

2. *Let (X, A), (Y, B), and (Z, C) be pairs of spaces and let $f, f' \colon (X, A) \longrightarrow (Y, B)$, $g, g' \colon (Y, B) \longrightarrow (Z, C)$ be continuous maps of pairs with $f \simeq_{A,B} f'$ and $g \simeq_{B,C} g'$. Then*
$$g \circ f \simeq_{A,C} g' \circ f'.$$

3. *If X is a contractible topological space and Y is a topological space, then all continuous maps $X \longrightarrow Y$ and $Y \longrightarrow X$ are null-homotopic. The analogous statement also holds in the pointed setting.*

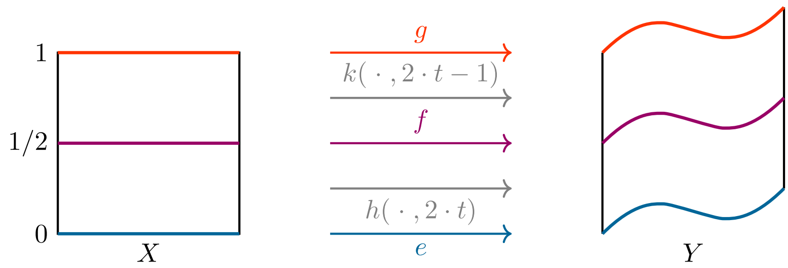


Figure 1.12.: Transitivity of being homotopic, schematically

Proof. Ad 1. Reflexivity. Let $f \in \operatorname{map}\bigl((X,A),(Y,B)\bigr)$. Then $f \simeq_{A,B} f$ follows from the constant movie, i.e., via the homotopy of pairs:

$$\begin{aligned}(X,A) \times [0,1] &\longrightarrow (Y,B)\\ (x,t) &\longmapsto f(x)\end{aligned}$$

Symmetry. Let $f, g \in \operatorname{map}\bigl((X,A),(Y,B)\bigr)$ with $f \simeq_{A,B} g$; moreover, let $h\colon (X,A) \times [0,1] \longrightarrow (Y,B)$ be such a homotopy of pairs of spaces. Then $g \simeq_{A,B} f$ follows from the inverse movie, i.e., via the homotopy

$$\begin{aligned}(X,A) \times [0,1] &\longrightarrow (Y,B)\\ (x,t) &\longmapsto h(x, 1-t)\end{aligned}$$

of pairs.

Transitivity. Let $e, f, g \in \operatorname{map}\bigl((X,A),(Y,B)\bigr)$ with $e \simeq_{A,B} f$ and $f \simeq_{A,B} g$; let $h, k\colon (X,A) \times [0,1] \longrightarrow (Y,B)$ be such homotopies of pairs of spaces. Then $e \simeq_{A,B} g$ follows from the concatenated (and reparametrised) movie (Figure 1.12), i.e., via the homotopy

$$\begin{aligned}(X,A) \times [0,1] &\longrightarrow (Y,B)\\ (x,t) &\longmapsto \begin{cases} h(x, 2\cdot t) & \text{if } t \in [0,1/2]\\ k(x, 2\cdot t-1) & \text{if } t \in [1/2,1]\end{cases}\end{aligned}$$

of pairs; it should be noted that this map is indeed well-defined and continuous (Proposition A.1.17).

Ad 2. Let $h\colon (X,A) \times [0,1] \longrightarrow (Y,B)$ and $k\colon (Y,B) \times [0,1] \longrightarrow (Z,C)$ be homotopies of pairs from f to f' and from g to g', respectively. Then

$$\begin{aligned}(X,A) \times [0,1] &\longrightarrow (Z,C)\\ (x,t) &\longmapsto k\bigl(h(x,t),t\bigr)\end{aligned}$$

is a homotopy of pairs showing that $g \circ f \simeq_{A,C} g' \circ f'$ (check!).

Ad 3. Let X be contractible. Then $\mathrm{id}_X \simeq c$, where $c\colon X \longrightarrow X$ is a constant map (check!). If $f\colon X \longrightarrow Y$ is continuous, then the second part shows that

$$f = f \circ \mathrm{id}_X \simeq f \circ c.$$

Moreover, because c is constant, also $f \circ c$ is constant. Hence, f is null-homotopic. Analogously, one can handle the case of maps to X, as well as the pointed case. □

Example 1.3.14 (boring paths)**.** If X is path-connected space, then all continuous maps $[0,1] \longrightarrow X$ are homotopic to each other (Exercise).

1.3.2 Homotopy Invariance

One of the main goals of Algebraic Topology is to study the homotopy equivalence problem

> Classify topological spaces up to homotopy equivalence!

As in the case of the homeomorphism problem, also this problem is not solvable in full generality (Outlook 2.2.19). However, the problem can be solved for many concrete examples, using suitable functors as invariants.

In order to define the notion of homotopy invariance, we first introduce appropriate categories for the homotopy equivalence problem. The basic idea is to construct such categories by identifying maps in Top, Top^2, Top_*, ... that are homotopic. In view of Proposition 1.3.13, these categories are well-defined (check!).

Definition 1.3.15 (homotopy category of topological spaces)**.** The *homotopy category of topological spaces* is the category Top_h consisting of:

- objects: Let $\mathrm{Ob}(\mathsf{Top}_\mathsf{h}) := \mathrm{Ob}(\mathsf{Top})$.
- morphisms: For all topological spaces X, Y, we set

  $$[X,Y] := \mathrm{Mor}_{\mathsf{Top}_\mathsf{h}}(X,Y) := \mathrm{map}(X,Y) \,/\, \simeq.$$

  Homotopy classes of maps will be denoted by "$[\,\cdot\,]$".
- compositions: The compositions of morphisms are defined by ordinary composition of representatives.

Definition 1.3.16 (homotopy category of pairs of spaces)**.** The *homotopy category of pairs of spaces* is the category ${\mathsf{Top}^2}_\mathsf{h}$ consisting of:

- objects: Let $\mathrm{Ob}({\mathsf{Top}^2}_\mathsf{h}) := \mathrm{Ob}(\mathsf{Top}^2)$.

- morphisms: For all pairs of spaces (X, A), (Y, B), we set

$$\begin{aligned}\big[(X,A),(Y,B)\big] &:= \operatorname{Mor}_{\mathsf{Top^2}_\mathsf{h}}\big((X,A),(Y,B)\big)\\ &:= \operatorname{map}\big((X,A),(Y,B)\big) \,/\simeq_{A,B}.\end{aligned}$$

  Homotopy classes of maps of such pairs will be denoted by "$[\,\cdot\,]_{A,B}$".

- compositions: The compositions of morphisms are defined by ordinary composition of representatives.

Definition 1.3.17 (homotopy category of pointed spaces)**.** The *homotopy category of pointed spaces* is the category $\mathsf{Top}_{*\mathsf{h}}$ consisting of:

- objects: Let $\operatorname{Ob}(\mathsf{Top}_{*\mathsf{h}}) := \operatorname{Ob}(\mathsf{Top}_*)$.

- morphisms: For all pointed spaces (X, x_0), (Y, y_0), we set

$$\begin{aligned}\big[(X,x_0),(Y,y_0)\big]_* &:= \operatorname{Mor}_{\mathsf{Top}_{*\mathsf{h}}}\big((X,x_0),(Y,y_0)\big)\\ &:= \operatorname{map}\big((X,x_0),(Y,y_0)\big) \,/\simeq_*.\end{aligned}$$

  Pointed homotopy classes of pointed maps will be denoted by "$[\,\cdot\,]_*$".

- compositions: The compositions of morphisms are defined by ordinary composition of representatives.

Remark 1.3.18. In this way, we obtain a category theoretic formulation of homotopy equivalence: Topological spaces are homotopy equivalent if and only if they are isomorphic in the category $\mathsf{Top_h}$ (and similarly for the case of Top^2 and Top_*) (check!).

Hence, in order to study spaces up to homotopy equivalence, it is reasonable to study functors mapping out of $\mathsf{Top_h}$, $\mathsf{Top^2}_\mathsf{h}$, $\mathsf{Top}_{*\mathsf{h}}$, ...

Definition 1.3.19 (homotopy invariant functor)**.** Let T be one of the categories Top, Top^2, Top_*, ... (this list will grow during this course) and let C be a category. A functor $F\colon T \longrightarrow C$ is *homotopy invariant* if the following holds: For all $X, Y \in \operatorname{Ob}(T)$ and all $f, g \in \operatorname{Mor}_T(X, Y)$ with $f \simeq_T g$, we have

$$F(f) = F(g).$$

Remark 1.3.20 (homotopy invariant functors and homotopy categories)**.** Let T be one of the categories Top, Top^2, Top_*, ..., let C be a category, and let $F\colon T \longrightarrow C$ be a functor. Then F is homotopy invariant in the sense of the previous definition if and only if it factors over the *homotopy classes functor* $H\colon T \longrightarrow T_\mathsf{h}$ (which is the identity on objects and maps each map to its homotopy class), i.e., if and only if there exists a functor $F_\mathsf{h}\colon T_\mathsf{h} \longrightarrow C$ with $F_\mathsf{h} \circ H = F$:

$$\begin{tikzcd} T \arrow[r, "F"] \arrow[d, "H"'] & C \\ T_{\mathsf{h}} \arrow[ur, dashed, "F_{\mathsf{h}}"'] & \end{tikzcd}$$

Proposition 1.3.21 (homotopy invariant functors yield homotopy invariants). *Let T be one of the categories* Top, Top^2, Top_*, *…, let C be a category, and let $F\colon T \longrightarrow C$ be a homotopy invariant functor. Let $X, Y \in \operatorname{Ob}(T)$. Then the following hold:*

1. *If $X \simeq_T Y$, then $F(X) \cong_C F(Y)$.*
2. *If $F(X) \not\cong_C F(Y)$, then $X \not\simeq_T Y$.*

Proof. It suffices to prove the first part. Because F is a homotopy invariant functor, there is a factorisation F_{h} of F over the homotopy classes functor $H\colon T \longrightarrow T_{\mathsf{h}}$. Because F_{h} preserves isomorphisms (Proposition 1.2.18) and because $X \simeq_T Y$ is equivalent to $H(X) \cong_{T_{\mathsf{h}}} H(Y)$, the claim follows.

Of course, alternatively, one can also go through the definition of homotopy invariant functors and prove the claim "by hand". □

In this language, the aim of this course is to find "good" examples of homotopy invariant functors.

1.3.3 Using Homotopy Invariant Functors

During this course, we will establish the following theorem (Corollary 4.4.2):

Theorem 1.3.22 (existence of "interesting" homotopy invariant functors). *There exists a sequence $(F_n)_{n\in\mathbb{N}}$ of homotopy invariant functors* $\mathsf{Top} \longrightarrow \mathsf{Ab}$ *with the following properties:*

1. *For all $n, m \in \mathbb{N}$, we have*
$$F_m(S^n) \cong_{\mathsf{Ab}} \begin{cases} \{0\} & \text{if } n \neq m \\ \mathbb{Z} & \text{if } n = m. \end{cases}$$
2. *For all $n \in \mathbb{N}$ and all $j \in \{1, \dots, n+1\}$ we have*
$$F_n(r_j^{(n)}) \neq \operatorname{id}_{F_n(S^n)},$$
where
$$\begin{aligned} r_j^{(n)}\colon S^n &\longrightarrow S^n \\ x &\longmapsto (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_{n+1}) \end{aligned}$$
is the reflection at the j-th coordinate hyperplane in $\mathbb{R}^{n+1}$.

Notation 1.3.23. We will now demonstrate how this result can be used in practice; in the following, we mark results by a star if we give a proof that depends on Theorem 1.3.22:

Corollary* 1.3.24 (invariance of dimension I)**.**

1. *For all $n \in \mathbb{N}$, the sphere S^n is* not *contractible.*
2. *For all $n,\ m \in \mathbb{N}$, we have $S^n \simeq S^m$ if and only if $n = m$.*
3. *For all $n,\ m \in \mathbb{N}$, we have $S^n \cong_{\mathsf{Top}} S^m$ if and only if $n = m$.*
4. *For all $n,\ m \in \mathbb{N}$, we have $\mathbb{R}^n \cong_{\mathsf{Top}} \mathbb{R}^m$ if and only if $n = m$.*

Proof. Ad 1. If $n = 0$, then S^0 is not path-connected, whence not homotopy equivalent to a point (Exercise).

Let us consider the case $n > 0$. *Assume* for a contradiction that S^n were contractible. Then all continuous maps $S^n \longrightarrow S^n$ would be null-homotopic; because S^n is path-connected, this would mean that all continuous maps $S^n \longrightarrow S^n$ would be homotopic. In particular, we would obtain

$$\mathrm{id}_{F_n(S^n)} = F_n(\mathrm{id}_{S^n}) = F_n(r_1^{(n)}) \neq \mathrm{id}_{F_n(S^n)}$$

for the functor $F_n \colon \mathsf{Top} \longrightarrow \mathsf{Ab}$ from Theorem 1.3.22, which is impossible. Hence, S^n is *not* contractible.

Ad 2. This follows directly from the first part of Theorem 1.3.22 and homotopy invariance (Proposition 1.3.21).

Ad 3. This follows from the second part (because every homeomorphism is a homotopy equivalence).

Ad 4. We apply the *point-removal trick*: Let $n,\ m \in \mathbb{N}$ and let $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a homeomorphism; without loss of generality, we may assume that $n \neq 0$ and $m \neq 0$ (because cardinality is a homeomorphism invariant). Then f induces a homeomorphism $\mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{f(0)\}$. In particular, using Example 1.3.4, we obtain that

$$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^m \setminus \{f(0)\} \simeq S^{m-1}.$$

Therefore, the second part shows that $n - 1 = m - 1$, whence $n = m$. □

The last part of Corollary 1.3.24 is essential in the study of topological manifolds, in order to obtain a well-defined notion of dimension of (non-empty) topological manifolds. We will return to this problem in Corollary 4.4.2.

Corollary* 1.3.25 (Brouwer fixed point theorem)**.** *Let $n \in \mathbb{N}$. Then every continuous map $D^n \longrightarrow D^n$ has at least one fixed point.*

Proof. Without loss of generality, we may assume that $n \geq 1$ (because D^0 consists of a single point).

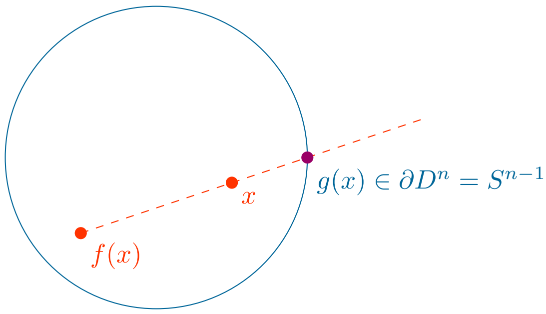


Figure 1.13.: Construction of g, in the proof of the Brouwer fixed point theorem

Assume for a contradiction that there is a continuous map $f\colon D^n \longrightarrow D^n$ with $f(x) \neq x$ for all $x \in D^n$. We will derive a contradiction as follows:

① Using f, we will construct a continuous map $g\colon D^n \longrightarrow S^{n-1}$ satisfying $g \circ i = \mathrm{id}_{S^{n-1}}$, where $i\colon S^{n-1} \longrightarrow D^n$ denotes the inclusion.

② Using ①, we will conclude that S^{n-1} is contractible (which contradicts Corollary 1.3.24).

Ad ①. We consider the map

$$\begin{aligned} g\colon D^n &\longrightarrow S^{n-1} \\ x &\longmapsto x + \frac{t(x)}{\|x - f(x)\|_2{}^2} \cdot \bigl(x - f(x)\bigr), \end{aligned}$$

where

$$\begin{aligned} t(x) := &- \bigl\langle x, x - f(x)\bigr\rangle \\ &+ \sqrt{\bigl\langle x, x - f(x)\bigr\rangle^2 - \bigl\|x - f(x)\bigr\|_2{}^2 \cdot \bigl(\|x\|_2{}^2 - 1\bigr)}. \end{aligned}$$

In other words: Given $x \in D^n$, the point $g(x)$ is the intersection of the unique(!) ray from $f(x)$ through x with the boundary $\partial D^n = S^{n-1}$ (Figure 1.13; check!).

A straightforward calculation shows that g is well-defined and continuous (check!). Moreover, we have (check!)

$$\forall_{x \in S^{n-1}} \quad g(x) = x,$$

and so $g \circ i = \mathrm{id}_{S^{n-1}}$.

Ad ②. By ①, we have $g \circ i = \mathrm{id}_{S^{n-1}} \simeq \mathrm{id}_{S^{n-1}}$. Conversely, we also have that $i \circ g \simeq \mathrm{id}_{D^n}$ (because D^n is contractible; check!). Thus,

$$S^{n-1} \simeq D^n,$$

which means that S^{n-1} is contractible (Example 1.3.3). However, this contradicts Corollary 1.3.24. $\square$

Study note. Where in the proof of Corollary 1.3.25 did we use the assumption that the map under consideration has no fixed points?

Example 1.3.26 (maps)**.** Because filled rectangles are homeomorphic to D^2 and the Brouwer fixed point theorem (Corollary 1.3.25) also applies to spaces homeomorphic to balls (check!), we obtain: If a rectangular rubber map of Regensburg is crumpled/stretched and put on the floor in Regensburg (in the area that is represented on the map), then there exists a point in Regensburg, that lies beneath its corresponding point on the map.

Remark 1.3.27 (on the non-constructive nature)**.** While this proof of the Brouwer fixed point theorem is elegant, it does have the disadvantage that it is highly non-constructive. Also Brouwer's original proof suffers from being non-constructive and Brouwer quite strongly felt that this is a serious issue.

In fact, it can be shown that in dimension bigger than 1, even computable continuous functions need not have computable fixed points [57, 3, 59]. However, as Brouwer already observed, there is a constructive approximate version of this theorem [8].

Study note (fixed point theorems)**.** Compare the Brouwer fixed point theorem with the Banach fixed point theorem, keeping the following questions in mind:

- How do the hypotheses on the spaces differ?
- How do the hypotheses on the self-map differ?
- Which one is more constructive?

Outlook 1.3.28 (Nash equilibria)**.** One of the most famous applications of the Brouwer fixed point theorem is Nash's (second) proof of the existence of Nash equilibria in game theory [56]. Nash was awarded the Nobel Prize (in Economic Sciences, 1994) for his work in game theory. Moreover, Nash made many visionary contributions to different aspects of geometry (e.g., the so-called Nash embedding theorem).

Moreover, Theorem 1.3.22 allows us to solve the hedgehog-combing problem. As a preparation, we will first consider the following variation of the mapping degree principle:

Proposition 1.3.29 (mapping degrees of self-maps)**.** *Let C be a category, let $F\colon C \longrightarrow \mathsf{Ab}$ be a functor, and let $X \in \operatorname{Ob}(C)$ be an object with $F(X) \cong_{\mathsf{Ab}} \mathbb{Z}$. The* degree map *on X with respect to F is given by*

$$\begin{aligned} \deg_F \colon \operatorname{Mor}_C(X, X) &\longrightarrow \mathbb{Z} \\ f &\longmapsto d \in \mathbb{Z} \text{ with } F(f) = d \cdot \mathrm{id}_{F(X)}. \end{aligned}$$

Then the following hold:

1. *The map* $\deg_F \colon \operatorname{Mor}_C(X,X) \longrightarrow \mathbb{Z}$ *is well-defined.*
2. *For all* $f, g \in \operatorname{Mor}_C(X,X)$*, we have* $\deg_F(g \circ f) = \deg_F g \cdot \deg_F f$*.*
3. *We have* $\deg_F(\mathrm{id}_X) = 1$ *and all* C*-isomorphisms* $f \in \operatorname{Mor}_C(X,X)$ *satisfy*
$$\deg_F f \in \{-1, 1\}.$$

Proof. Ad 1. We only need to apply the following basic observation to $F(X)$: If Z is an Abelian group with $Z \cong_{\mathsf{Ab}} \mathbb{Z}$, then every group homomorphism $Z \longrightarrow Z$ is of the form $d \cdot \mathrm{id}_Z$ with $d \in \mathbb{Z}$; moreover, the number d is determined uniquely by the homomorphism.

Ad 2. Because F is a functor, we obtain

$$\begin{aligned}
\deg_F(g \circ f) \cdot \mathrm{id}_{F(X)} &= F(g \circ f) && (\text{definition of } \deg_F)\\
&= F(g) \circ F(f) && (F \text{ is a functor})\\
&= \bigl(\deg_F g \cdot \mathrm{id}_{F(X)}\bigr) \circ \bigl(\deg_F f \cdot \mathrm{id}_{F(X)}\bigr) && (\text{definition of } \deg_F)\\
&= \deg_F g \cdot \deg_F f \cdot \mathrm{id}_{F(X)}\,.
\end{aligned}$$

Therefore, $\deg_F(g \circ f) = \deg_F g \cdot \deg_F f$.

Ad 3. Because F is a functor, we have

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} = 1 \cdot \mathrm{id}_{F(X)};$$

thus, $\deg_F(\mathrm{id}_X) = 1$. The statement about automorphisms of X is hence a consequence of the second part and the fact that -1 and 1 are the only multiplicative units of $\mathbb{Z}$. □

Theorem* 1.3.30 (hedgehog theorem)**.** *Let* $n \in \mathbb{N}_{>0}$*. Then there exists a continuous nowhere vanishing vector field on* S^n *if and only if* n *is odd.*

Remark 1.3.31 (vector fields)**.** Let us recall the following terminology: Let M be a differentiable manifold and let $p \colon TM \longrightarrow M$ be its tangent bundle.

- A *continuous vector field* on M is a continuous map $v \colon M \longrightarrow TM$ with $p \circ v = \mathrm{id}_M$; i.e., v continuously picks a vector in every tangent space.
- A continuous vector field $v \colon M \longrightarrow TM$ is *nowhere vanishing*, if
$$\forall_{x \in M} \quad v(x) \neq 0 \in T_x M.$$
- Let $n \in \mathbb{N}_{>0}$. Then continuous [nowhere vanishing] vector fields on S^n correspond to continuous maps $v \colon S^n \longrightarrow \mathbb{R}^{n+1}$ with
$$\forall_{x \in S^n} \quad \bigl\langle x, v(x) \bigr\rangle = 0 \quad \bigl[\text{and } v(x) \neq 0\bigr].$$

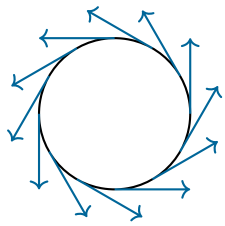


Figure 1.14.: Continuous combing of a one-dimensional hedgehog

In particular, Theorem 1.3.30 shows that the hedgehog (biologically almost correctly modelled by S^2) cannot be combed continuously.

Proof of Theorem 1.3.30. Let $n \in \mathbb{N}$ be odd, say $n = 2 \cdot k + 1$ with $k \in \mathbb{N}$. Then

$$\begin{aligned} S^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto (x_2, -x_1, x_4, -x_3, \dots, x_{2\cdot k+2}, -x_{2\cdot k+1}) \end{aligned}$$

is a continuous nowhere vanishing vector field on S^n (Figure 1.14).

Conversely, let $n \in \mathbb{N}_{>0}$ and let $v\colon S^n \longrightarrow \mathbb{R}^{n+1}$ be a nowhere vanishing vector field on S^n. Then

$$\begin{aligned} h\colon S^n \times [0,1] &\longrightarrow S^n \\ (x,t) &\longmapsto \cos(\pi \cdot t) \cdot x + \sin(\pi \cdot t) \cdot \frac{1}{\|v(x)\|_2} \cdot v(x) \end{aligned}$$

is well-defined (check!) and continuous; by construction, h is a homotopy from id_{S^n} to the antipodal map $-\mathrm{id}_{S^n} = (x \mapsto -x)$.

We will apply the mapping degree principle (Proposition 1.3.29), using a functor F_n as provided by Theorem 1.3.22:

For all $j \in \{1, \dots, n+1\}$, the reflection $r_j^{(n)}\colon S^n \longrightarrow S^n$ is a homeomorphism (being its own inverse); moreover, we have $F_n(r_j^{(n)}) \neq \mathrm{id}_{F_n(S^n)}$ and so $\deg_{F_n}(r_j^{(n)}) = -1$. Because F_n is homotopy invariant, we obtain

$$\begin{aligned} 1 &= \deg_{F_n}(\mathrm{id}_{S^n}) && \text{(Proposition 1.3.29)} \\ &= \deg_{F_n}(-\mathrm{id}_{S^n}) && \text{(F_n is homotopy invariant and $\mathrm{id}_{S^n} \simeq -\mathrm{id}_{S^n}$)} \\ &= \deg_{F_n}(r_{n+1}^{(n)} \circ \dots \circ r_1^{(n)}) && \text{(elementary calculation/geometry)} \\ &= \deg_{F_n}(r_{n+1}^{(n)}) \cdot \dots \cdot \deg_{F_n}(r_1^{(n)}) && \text{(Proposition 1.3.29)} \\ &= (-1)^{n+1}. \end{aligned}$$

In particular, n is odd. $\square$

Literature exercise. Read about the origin of the notion of homotopy and homotopy invariance [12, p. 43].

2

Fundamental Group and Covering Theory

We investigate a first class of homotopy invariant functors, so-called homotopy groups. Geometrically speaking, homotopy groups "count spherical holes" in (pointed) spaces. In terms of category theory, this means to look at the functors

$$\pi_n := \big[(S^n, e_1),\ \cdot\ \big]_*\colon\ \mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Set}$$

represented by the (pointed) spheres.

We focus on the case of the one-dimensional sphere S^1, leading to the fundamental group. In particular, we will study the following questions:

- How can we define a group structure on π_1 ?
- How does π_1 behave with respect to (de)compositions of spaces?
- Can we interpret π_1 as an automorphism group?
- Which applications does π_1 have?

Overview of this chapter.

Running example. the circle and its relatives

2.1 The Fundamental Group

We first give an explicit description of the functor

$$\pi_1 := \bigl[(S^1, e_1),\ \cdot\ \bigr]_* \colon \mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Set}$$

and its group structure. We then briefly also discuss a more conceptual way of thinking about this group structure and the role of the basepoint.

2.1.1 The Group Structure on the Fundamental Group

Remark 2.1.1 (π_1, intuitively)**.** Before going into the technical details, let us first look at π_1 from a geometric point of view: Let (X, x_0) be a pointed space. Then

$$\pi_1(X, x_0) = \bigl[(S^1, e_1), (X, x_0)\bigr]_* = \operatorname{map}_*\bigl((S^1, e_1), (X, x_0)\bigr) \;/\; \simeq_*$$

is the set of pointed homotopy classes of pointed loops in (X, x_0) (Figure 2.1). The pointed loop ① in Figure 2.1 is pointedly null-homotopic. If the pointed loop ② is not pointedly null-homotopic (which we will establish in Theorem 2.3.40), then this loop ② detects the "hole" of this space.

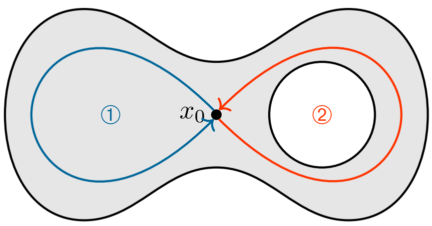


Figure 2.1.: π_1, intuitively

Notation 2.1.2 (parametrisation of S^1)**.** In the following, we use the homeomorphism

$$\begin{aligned} [0,1]/(0 \sim 1) &\longrightarrow S^1 \\ [t] &\longmapsto e^{2\cdot\pi\cdot i\cdot t} \end{aligned}$$

from Example 1.1.16; hence, we parametrise points on S^1 by elements in the unit interval $[0, 1]$.

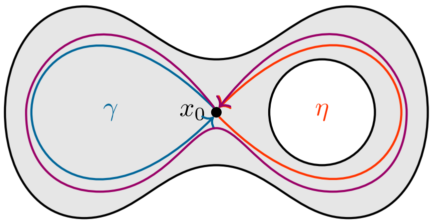


Figure 2.2.: The group structure on π_1: The purple loop is $\gamma * \eta$ and thus represents $[\gamma]_* \cdot [\eta]_*$.

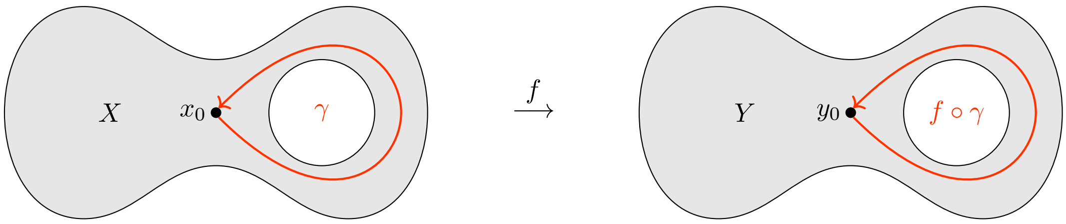


Figure 2.3.: The induced group homomorphism on π_1

Concatenating (and reparametrising) loops at the same basepoint leads to a new loop; this defines a group structure on π_1:

Proposition and Definition 2.1.3 (fundamental group)**.** *Let (X, x_0) be a pointed space.*

1. *Then the map*

$$\begin{aligned} \cdot \colon \pi_1(X,x_0) \times \pi_1(X,x_0) &\longrightarrow \pi_1(X,x_0) \\ \big([\gamma]_*, [\eta]_*\big) &\longmapsto \left[\gamma * \eta := \left([t] \mapsto \begin{cases} \gamma([2\cdot t]) & \text{if } t \in [0,1/2] \\ \eta([2 \cdot t - 1]) & \text{if } t \in [1/2, 1] \end{cases}\right)\right]_* \end{aligned}$$

   is well-defined and $\pi_1(X,x_0)$ is a group with respect to this composition. We call $\pi_1(X,x_0)$ the fundamental group *of (X,x_0).*

2. *Let (Y,y_0) be a pointed space and let $f \in \operatorname{map}_*\big((X,x_0),(Y,y_0)\big)$. Then*

$$\begin{aligned} \pi_1(f) = \operatorname{Mor}_{\mathsf{Top}_{*_h}}\big((S^1,1),f\big) \colon \pi_1(X,x_0) &\longrightarrow \pi_1(Y,y_0) \\ [\gamma]_* &\longmapsto [f \circ \gamma]_* \end{aligned}$$

   is a group homomorphism (with respect to the group structure introduced in the previous item).

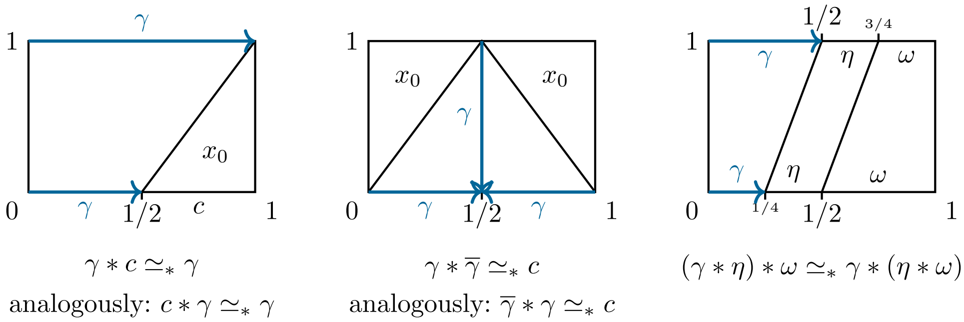


Figure 2.4.: Verifying the group axioms for the composition defined in Proposition 2.1.3; here, $\gamma, \eta, \omega\colon (S^1, e_1) \longrightarrow (X, x_0)$ are pointed continuous maps and $c\colon (S^1, e_1) \longrightarrow (X, x_0)$ denotes the constant loop at x_0.

Proof. Ad 1. The composition $\cdot$ on $\pi_1(X, x_0)$ is *well-defined*:

- If $\gamma, \eta \in \operatorname{map}_*((S^1, e_1), (X, x_0))$ are pointed continuous maps, then $\gamma * \eta$ is a well-defined continuous (Proposition A.1.17) loop, based at x_0; here, the basepoint plays an important role!
- Moreover, if $\gamma \simeq_* \gamma'$ (via h) and $\eta \simeq_* \eta'$ (via k), then the (in S^1-direction concatenated) pointed homotopy (check!)

$$\begin{aligned}(S^1, e_1) \times [0,1] &\longrightarrow (X, x_0) \\ \big([s], t\big) &\longmapsto \begin{cases} h\big([2 \cdot s], t\big) & \text{if } s \in [0, 1/2] \\ k\big([2 \cdot s - 1], t\big) & \text{if } s \in [1/2, 1] \end{cases}\end{aligned}$$

  shows that $\gamma * \eta \simeq_* \gamma' * \eta'$; here, it is important that we use *pointed* homotopies!

Moreover, the composition $\cdot$ satisfies the group axioms; here, it is essential that we pass to (pointed) homotopy classes of loops – so that we can enjoy the freedom of reparametrisation:

- *Existence of a neutral element:* Let $c\colon (S^1, e_1) \longrightarrow (X, x_0)$ be the constant loop at x_0. Then $[c]_* \in \pi_1(X, x_0)$ is neutral with respect to $\,\cdot\,$: For every $\gamma \in \operatorname{map}_*((S^1, e_1), (X, x_0))$ the pointed homotopy (check!)

$$\begin{aligned}(S^1, e_1) \times [0,1] &\longrightarrow (X, x_0) \\ \big([s], t\big) &\longmapsto \begin{cases} \gamma\big([\frac{2}{1+t} \cdot s]\big) & \text{if } s \in [0, 1/2 \cdot (1+t)] \\ x_0 & \text{if } s \in [1/2 \cdot (1+t), 1] \end{cases}\end{aligned}$$

shows that $c * \gamma \simeq_* \gamma$ (Figure 2.4). Similarly, we obtain $\gamma * c \simeq_* \gamma$.

- *Existence of inverses:* Let $\gamma \in \operatorname{map}_*\big((S^1, e_1), (X, x_0)\big)$. We then consider (walking the loop γ backwards)

$$\begin{aligned} \overline{\gamma}\colon (S^1, e_1) &\longrightarrow (X, x_0) \\ [s] &\longmapsto \gamma\big([1-s]\big). \end{aligned}$$

Then $\gamma * \overline{\gamma} \simeq_* c$ through the pointed homotopy (check! Figure 2.4; see also Appendix A.2)

$$\begin{aligned} (S^1, e_1) \times [0,1] &\longrightarrow (X, x_0) \\ \big([s], t\big) &\longmapsto \begin{cases} x_0 & \text{if } s \in [0, 1/2 \cdot t] \cup [1 - 1/2 \cdot t, 1] \\ \gamma\big([2 \cdot s - t]\big) & \text{if } s \in [1/2 \cdot t, 1/2] \\ \overline{\gamma}\big([2 \cdot s - 1 + t]\big) & \text{if } s \in [1/2, 1 - 1/2 \cdot t] \end{cases} \end{aligned}$$

(one should note that $\overline{\gamma}([2 \cdot 1/2 - 1 + t]) = \gamma([2 \cdot 1/2 - t])$ holds for all $t \in [0,1]$). Similarly, we obtain $\overline{\gamma} * \gamma \simeq_* c$. Hence $[\overline{\gamma}]_*$ is an/the inverse of $[\gamma]_*$.

- *Associativity:* Let $\gamma, \eta, \omega \in \operatorname{map}_*\big((S^1, e_1), (X, x_0)\big)$. Then the canonical homotopy associated with the rightmost diagram in Figure 2.4 shows that

$$(\gamma * \eta) * \omega \simeq_* \gamma * (\eta * \omega).$$

Therefore, $\cdot$ is associative.

Hence, $\pi_1(X, x_0)$ indeed forms a group with respect to $\,\cdot\,$.

Ad 2. Let $\gamma, \eta \in \operatorname{map}_*\big((S^1, e_1), (X, x_0)\big)$. Then the definition of the concatenation $*$ shows that

$$f \circ (\gamma * \eta) = (f \circ \gamma) * (f \circ \eta).$$

In particular, $\pi_1(f)\big([\gamma]_* \cdot [\eta]_*\big) = \pi_1(f)\big([\gamma]_*\big) \cdot \pi_1(f)\big([\eta]_*\big)$, as desired. □

Remark 2.1.4 (π_1 as group-valued functor)**.** The group structure on π_1 constructed in Proposition 2.1.3 is compatible with composition of pointed maps/group homomorphisms and identity maps (check!). Hence, we found the desired factorisation of π_1 over the forgetful functor $\mathsf{Group} \longrightarrow \mathsf{Set}$:

$$\begin{tikzcd} & \mathsf{Group} \arrow[d, "\text{forget}"] \\ \mathsf{Top}_{*\mathsf{h}} \arrow[ur, dashed, "?"] \arrow[r, "\pi_1"'] & \mathsf{Set} \end{tikzcd}$$

In order to keep notation simple, we will also denote the resulting functors $\mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Group}$ and $\mathsf{Top}_* \longrightarrow \mathsf{Group}$ by π_1.

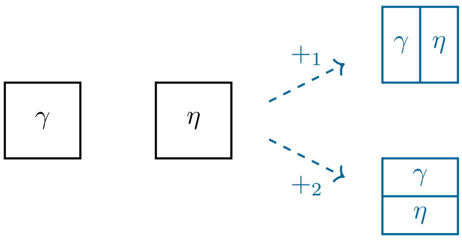


Figure 2.5.: Composition on π_n, schematically

Outlook 2.1.5 (π_1 via cogroup objects)**.** The proof that concatenation of representing loops induces a group structure on π_1 (Proposition 2.1.3) only uses the corresponding properties of (S^1, e_1). More precisely, (S^1, e_1) is a cogroup object in $\mathsf{Top}_{*\mathsf{h}}$ and this cogroup object structure corresponds to a factorisation of π_1 over Group (Appendix A.3). Moreover, one can show that the cogroup object structure on (S^1, e_1) is essentially unique [1, Theorem 7.3].

Because S^1 does *not* admit a cogroup object structure in Top_h (or Top or Top_*), the functor $[S^1, \,\cdot\,]\colon \mathsf{Top}_\mathsf{h} \longrightarrow \mathsf{Set}$ does *not* factor over Group. This shows that the use of basepoints and pointed homotopies is not an artefact of our construction; it is essential in order to obtain a group-valued functor in this way.

Alternative functors that do not require the use of basepoints are the fundamental groupoid (Outlook 2.1.10) as well as the first (singular) homology $H_1(\,\cdot\,;\mathbb{Z})$ (Chapter 4).

Outlook 2.1.6 (higher homotopy groups)**.** If $n \in \mathbb{N}_{\geq 2}$, then similarly to the group structure on π_1, one can also introduce a group structure on π_n (or a cogroup object structure on (S^n, e_1) in $\mathsf{Top}_{*\mathsf{h}}$), resulting in the *$n$-th homotopy group* π_n; more precisely: Let $\square^n := [0,1]^n$ be the unit n-cube. Similarly to Example 1.1.9 one can prove that there is a homeomorphism $\varphi_n\colon \square^n/\partial\square^n \longrightarrow S^n$ with $\varphi_n(\partial\square^n) = e_1$; in the following, we will parametrise points in S^n by $\square^n$ using this homeomorphism φ_n. For pointed spaces (X, x_0) and $j \in \{1, \dots, n\}$ we define the composition (Figure 2.5)

$$
\begin{aligned}
+_j\colon \pi_n(X,x_0) \times \pi_n(X,x_0) &\longrightarrow \pi_n(X,x_0)\\
\bigl([\gamma]_*, [\eta]_*\bigr) &\longmapsto \left[\square^n \ni t \mapsto \begin{cases} \gamma([t_1, \dots, 2\cdot t_j, \dots, t_n]) & \text{if } t_j \in [0,1/2]\\ \eta([t_1, \dots, 2\cdot t_j - 1, \dots, t_n]) & \text{if } t_j \in [1/2,1]\end{cases}\right]_*
\end{aligned}
$$

Similar arguments as in the case of $\pi_1(X, x_0)$ show that $+_j$ indeed yields a well-defined functorial group structure on $\pi_n(X, x_0)$. Moreover, one can prove that $+_j = +_1$ for all $j \in \{1, \dots, n\}$ and that $\pi_n(X, x_0)$ is an *Abelian* group with respect to $+_1$ if $n \geq 2$ (Eckmann–Hilton trick; Exercise).

Remark 2.1.7 (π_0 and path-connected components)**.** If (X, x_0) is a pointed space, then $\pi_0(X, x_0) = \bigl[(S^0, 1), (X, x_0)\bigr]_*$ corresponds to the set of path-connected components of X (Exercise).

As $(S^0, 1)$ does *not* admit the structure of a comonoid object or even cogroup object in $\mathsf{Top}_{*\mathsf{h}}$, the functor π_0 does *not* admit a monoid/group structure. Geometrically, this is plausible as there is no geometric reason why path-connected components should admit a "multiplication" that is compatible with all (pointed) continuous maps.

Example 2.1.8 (trivial fundamental groups)**.**

- If (X, x_0) is pointedly contractible, then $\pi_1(X, x_0)$ is the trivial group (because every pointed loop is pointedly homotopic to the constant pointed loop; Proposition 1.3.13). Moreover, also contractible spaces have trivial fundamental group (Exercise).
- If $n \in \mathbb{N}_{\geq 2}$, then all pointed loops in (S^n, e_1) are pointedly null-homotopic. Hence, also $\pi_1(S^n, e_1)$ is a trivial group. One can prove this by a direct geometric argument or via the theorem of Seifert and van Kampen (Example 2.2.11).

2.1.2 Changing the Basepoint

Before delving into further examples and calculations of π_1, we discuss the effect of changing basepoints on π_1. A first, simple, observation is that $\pi_1(X, x_0)$ can only see the path-connected component of the basepoint x_0 in X. Basepoints in the same path-connected component lead to isomorphic fundamental groups; but, in general, there is *no* canonical isomorphism between these fundamental groups:

Proposition 2.1.9 (fundamental group and change of basepoint)**.** *Let X be a topological space, let $x, y \in X$, and let $\eta, \eta' \colon [0,1] \longrightarrow X$ be paths in X from x to y.*

1. *Then*

$$\begin{aligned} \varphi_\eta \colon \pi_1(X, y) &\longrightarrow \pi_1(X, x) \\ [\gamma]_* &\longmapsto \bigl[(\eta * \gamma) * \overline{\eta}\bigr]_* \end{aligned}$$

is a well-defined map, which is a group isomorphism (Figure 2.6).

2. *We have $c_h \circ \varphi_\eta = \varphi_{\eta'}$, where*

$$\begin{aligned} c_h \colon \pi_1(X, x) &\longrightarrow \pi_1(X, x) \\ g &\longmapsto h \cdot g \cdot h^{-1} \end{aligned}$$

denotes the conjugation by the element $h := [\eta' * \overline{\eta}]_* \in \pi_1(X, x)$.

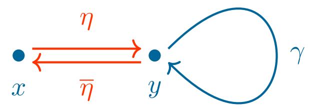


Figure 2.6.: Changing the basepoint in π_1,

Proof. In the formulation of this proposition, we extended the concatenation "$*$" and the inverse parametrisation "$\overline{\cdot}$" in the obvious way from loops to paths $[0,1] \longrightarrow X$. Similarly to the proof of Proposition 2.1.3, this concatenation is also associative with respect to homotopies that fix the endpoints of paths, etc. (check!). In particular, the pointed loops $\overline{\eta} * \eta$ and $\eta * \overline{\eta}$ are pointedly null-homotopic.

Ad 1. A straightforward calculation shows that φ_η indeed is a well-defined map that is compatible with the group structures on $\pi_1(X,y)$ and $\pi_1(X,x)$ (which are also given by concatenation of representatives) and that $\varphi_{\overline{\eta}}$ is the inverse of φ_η (check!). Hence, φ_η is a group isomorphism.

Ad 2. By construction, $\eta' * \overline{\eta}$ is a loop, based at x. Hence, $h := [\eta' * \overline{\eta}]_*$ is an element of $\pi_1(X,x)$. Moreover, for all $\gamma \in \operatorname{map}_*((S^1,e_1),(X,y))$, we obtain

$$\begin{aligned}
\varphi_{\eta'}\bigl([\gamma]_*\bigr) &= [\eta' * \gamma * \overline{\eta'}]_* \\
&= [\eta' * \overline{\eta} * \eta * \gamma * \overline{\eta} * \eta * \overline{\eta'}]_* \\
&= h \cdot [\eta * \gamma * \overline{\eta}]_* \cdot h^{-1} \\
&= c_h\bigl(\varphi_\eta([\gamma]_*)\bigr)
\end{aligned}$$

(we omit various parentheses when concatenating paths, because after taking pointed homotopy classes, concatenation is associative). □

Outlook 2.1.10 (the fundamental groupoid)**.** Sometimes, fixing a basepoint can be inconvenient. In such cases, the fundamental groupoid can serve as a good replacement of the fundamental group: A *groupoid* is a (small) category in which all morphisms are isomorphisms. In this language, groups correspond to groupoids with a single object (and then the group elements correspond to the automorphisms of this unique object). The *fundamental groupoid* of a topological space X consists of:

- objects: the set (of points of) X.
- morphisms: for $x, y \in X$, the set of morphisms from x to y is the set of homotopy classes of paths $[0,1] \longrightarrow X$ from x to y with respect to homotopies that fix the start and end points.

- compositions: concatenation/reparametrisation of representatives.

While the fundamental groupoid allows for more elegant formulations of many results on π_1, it does have the disadvantage of the additional overhead of dealing with groupoids instead of groups (and one hence, would need to translate these results back into the language of groups.)

Definition 2.1.11 (simply connected)**.** A non-empty topological space X is *simply connected* if it is path-connected and if for one (hence every; Proposition 2.1.9) basepoint $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ is trivial.

Example 2.1.12. In this language, pointedly contractible spaces are simply connected; more generally, all contractible spaces are simply connected (Example 2.1.8).

Moreover, spheres of dimension at least 2 are simply connected (Example 2.1.8). In contrast, we will see that the circle S^1 is *not* simply connected (Theorem 2.3.40).

Study note. Where in Analysis did you already come across the notion of simple connectedness?

More generally, one can introduce higher connectedness properties:

Definition 2.1.13 (higher connectedness)**.** Let $n \in \mathbb{N}$. A non-empty topological space X is *n-connected* if (the condition on π_0 ensures that such spaces are path-connected!)

$$\forall_{j \in \{0,\dots,n\}} \quad \forall_{x_0 \in X} \quad \bigl|\pi_n(X, x_0)\bigr| = 1.$$

2.2 Divide and Conquer

We investigate how the fundamental group behaves with respect to (de)compositions of spaces. By construction, the fundamental group is a (covariant) represented functor. Therefore, we may expect that the compatibility of π_1 with respect to inverse limits (such as products) should be easy to obtain. However, the compatibility with colimits (such as glueings) should be more involved. In the following, we study

- products,
- glueings,
- ascending unions

of pointed spaces.

2.2.0 The Fundamental Example

The fundamental group of the circle will serve as basic input for the inductive calculation of fundamental group out of simple building blocks. As one might expect, wrapping loops around the circle multiple times defines an isomorphism between $\mathbb{Z}$ and $\pi_1(S^1, e_1)$. In particular, π_1 indeed detects the "hole" in the circle.

Theorem 2.2.1 (fundamental group of the circle). *The map*

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

is a group isomorphism.

It would be possible to prove this theorem now "by hand". However, we prefer to wait with the proof until Chapter 2.3.4, when we will have the "right" tools to access this problem.

Remark 2.2.2 (complex analysis). The homotopy invariance of complex integration shows that

$$\begin{aligned} \pi_1(S^1, e_1) &\longrightarrow \mathbb{Z} \text{ (!)} \\ [\gamma]_* &\longmapsto \frac{1}{2 \cdot \pi \cdot i} \cdot \int_\gamma \frac{1}{z} \, dz \end{aligned}$$

is well-defined. Moreover, basic computations in complex analysis show that this map is surjective and compatible with the group structures. However, at this point, it is not so clear why this map should also be injective.

2.2.1 Products

Products are special cases of inverse limits. Hence, the represented functor π_1 is compatible with products. More precisely:

Lemma 2.2.3 (products in $\mathsf{Top}_{*\mathsf{h}}$). *Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. Then $\bigl(\prod_{i \in I} X_i, (x_i)_{i \in I}\bigr)$, together with the pointed homotopy classes of the canonical projections onto the factors, satisfies the universal property of the product of $(X_i, x_i)_{i \in I}$ in $\mathsf{Top}_{*\mathsf{h}}$.*

Proof. This is a straightforward computation (Exercise). $\square$

Proposition 2.2.4 (π_n of products). *Let $n \in \mathbb{N}$ and let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. Then*

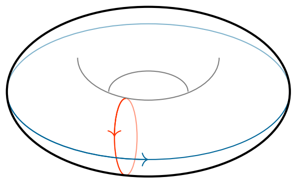


Figure 2.7.: Generators of the fundamental group of the torus

$$\begin{aligned} \pi_n\Bigl(\prod_{i\in I} X_i, (x_i)_{i\in I}\Bigr) &\longrightarrow \prod_{i\in I} \pi_n(X_i, x_i) \\ g &\longmapsto \bigl(\pi_n(p_i)(g)\bigr)_{i\in I} \end{aligned}$$

is a bijection; here, for $i \in I$, we write $p_i \colon \prod_{j\in I} X_j \longrightarrow X_i$ for the projection onto the i-th factor. If $n \geq 1$, then this map is a group isomorphism.

Proof. Because $(\prod_{i\in I} X_i, (x_i)_{i\in I})$, together with $([p_i]_*)_{i\in I}$ is the category theoretic product of $(X_i, x_i)_{i\in I}$ in $\mathsf{Top}_{*\mathsf{h}}$ (Lemma 2.2.3), bijectivity of the map above follows from the universal property of the product, applied to the test space $(S^n, e_1))$ and the definition $\pi_n = \mathrm{Mor}_{\mathsf{Top}_{*\mathsf{h}}}((S^n, e_1), \,\cdot\,)$.

For $n \geq 1$, the maps $\pi_n(p_i)$ are group homomorphisms (by functoriality of π_n as a functor to Group). Therefore, the definition of the group structure on $\prod_{i\in I} \pi_n(X_i, x_i)$ shows that the map above is a group homomorphism (whence an isomorphism). □

Example 2.2.5 (fundamental group of the torus)**.** Combining Theorem 2.2.1 and Proposition 2.2.4, we obtain that

$$\pi_1\bigl(T^2, (e_1, e_1)\bigr) \cong_{\mathsf{Group}} \pi_1(S^1, e_1) \times \pi_1(S^1, e_1) \cong_{\mathsf{Group}} \mathbb{Z} \times \mathbb{Z}.$$

Corresponding generators (constructed from a generator of the fundamental group of the circle) are depicted in Figure 2.7. It should be noted that this is an instance of a perfect computation of a fundamental group: The resulting group is completely understood algebraically; moreover, we can exhibit explicit, geometric, generators.

In particular:

- The torus and the circle are *not* pointedly homotopy equivalent.

- The torus and the two-dimensional sphere are *not* pointedly homotopy equivalent. Hence, the two-dimensional sphere and the torus are *not* homeomorphic. This is a part of the classification of compact surfaces (where one can use the fundamental group to distinguish between different surfaces [51]).

2.2.2 Glueings

We will now show that π_1 is compatible with certain glueings, i.e., special types of pushouts. Pushouts in Group will be discussed in more detail in Chapter 2.2.3 and Appendix A.4.

Theorem 2.2.6 (Seifert and van Kampen). *Let (X, x_0) be a pointed topological space and $X_1, X_2 \subset X$ be path-connected subspaces with the following properties:*

- *We have $X_1^\circ \cup X_2^\circ = X$,*
- *the intersection $X_0 := X_1 \cap X_2$ is path-connected,*
- *and x_0 lies in X_0.*

Then

$$\begin{tikzcd}
\pi_1(X_0, x_0) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1, x_0) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_2, x_0) \arrow[r, "\pi_1(j_2)"'] & \pi_1(X, x_0)
\end{tikzcd}$$

is a pushout diagram in the catgory Group *of groups. Here, we equip X_0, X_1, and X_2 with the subspace topology of X and $i_1, : X_0 \longrightarrow X_1$, $i_2 \colon X_0 \longrightarrow X_2$, $j_1 \colon X_1 \longrightarrow X$, $j_2 \colon X_2 \longrightarrow X$ denote the inclusion maps.*

The basic idea of the proof – as in many glueing or approximation results in Algebraic Topology – is to subdivide loops/homotopies into small pieces (Figure 2.8). In order to ensure that there exist fine enough subdivisions, we will use the Lebesgue lemma:

Lemma 2.2.7 (Lebesgue lemma). *Let (X, d) be a compact metric space and let $(U_i)_{i \in I}$ be an open cover of X. Then there exists an $\varepsilon \in \mathbb{R}_{>0}$ with the following property: For every $x \in X$ there is an $i \in I$ such that the open ball $U(x, \varepsilon)$ of radius ε around x is contained in U_i.*

Any such number ε is called a Lebesgue number *of the cover $(U_i)_{i \in I}$.*

Proof. Because $(U_i)_{i \in I}$ is an open cover of X, for every $x \in X$ there exists an $i_x \in I$ and an $r_x \in \mathbb{R}_{>0}$ with $U(x, r_x) \subset U_{i_x}$. In view of compactness, there is a *finite* set $Y \subset X$ with $\bigcup_{y \in Y} U(y, r_y/2) = X$. Then

$$\varepsilon := \frac{1}{2} \cdot \min_{y \in Y} r_y \in \mathbb{R}_{>0}$$

has the desired property: Let $x \in X$. Hence, there is a $y \in Y$ with $x \in U(y, r_y/2)$. Therefore, we obtain $U(x, \varepsilon) \subset U(y, r_y) \subset U_{i_y}$, as desired. □

Proof of Theorem 2.2.6. We prove that $\pi_1(X, x_0)$, together with the homomorphisms $\pi_1(j_1)$ and $\pi_1(j_2)$ satisfies the universal property of the pushout of

$$\begin{tikzcd} \pi_1(X_0, x_0) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1, x_0) \\ \pi_1(X_2, x_0) & \end{tikzcd}$$

in the category Group. Clearly, $j_1 \circ i_1 = j_2 \circ i_2$ and so

$$\pi_1(j_1) \circ \pi_1(i_1) = \pi_1(j_1 \circ i_1) = \pi_1(j_2 \circ i_2) = \pi_1(j_2) \circ \pi_1(i_2).$$

Let H be a group and let $\varphi_1 \colon \pi_1(X_1, x_0) \longrightarrow H$ and $\varphi_2 \colon \pi_1(X_2, x_0) \longrightarrow H$ be group homomorphisms with $\varphi_1 \circ \pi_1(i_1) = \varphi_2 \circ \pi_1(i_2)$. We have to show that there exists a unique group homomorphism $\varphi \colon \pi_1(X, x_0) \longrightarrow H$ satisfying

$$\varphi \circ \pi_1(j_1) = \varphi_1 \quad \text{and} \quad \varphi \circ \pi_1(j_2) = \varphi_2.$$

The basic, geometric, idea is to subdivide pointed loops and pointed homotopies in (X, x_0) with the help of the Lebesgue lemma (Lemma 2.2.7) into small pieces that lie in the building blocks X_1 or X_2, respectively (Figure 2.8).

As a preparation, we introduce the following notation:

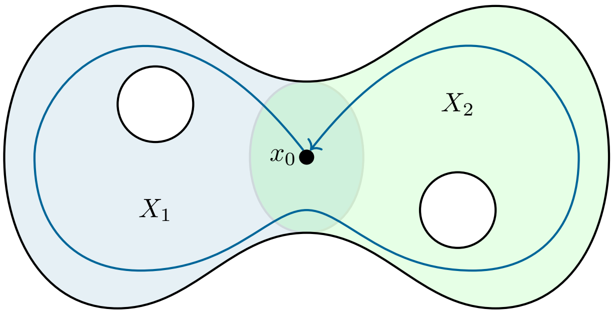
- A continuous path $\gamma \colon [a, b] \longrightarrow X$ (with $a, b \in \mathbb{R}$ and $a \leq b$) is *small* if there exists a $j \in \{1, 2\}$ such that
$$\gamma\bigl([a, b]\bigr) \subset X_j.$$

- Let $\gamma \in \operatorname{map}_*((S^1, e_1), (X, x_0))$. A finite sequence $(\gamma_1, \dots, \gamma_n)$ of paths in X is a *decomposition of* γ if there exists a partition $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ with the following properties:
  - For all $j \in \{0, \dots, n-1\}$, we have $\gamma_j = \gamma|_{[t_j, t_{j+1}]}$
  - and for every $j \in \{0, \dots, n-1\}$ the path γ_j is small.

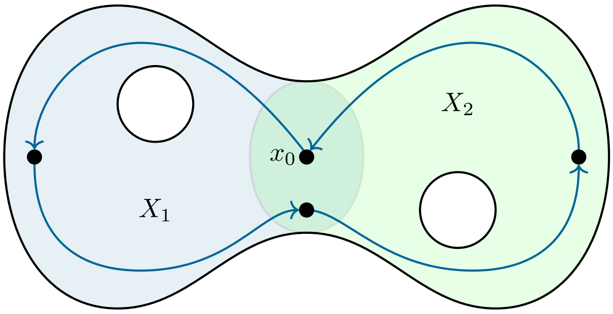
- For every $j \in \{1, 2\}$ and every $x \in X_j$ we choose a continuous path $w_x \colon [0, 1] \longrightarrow X_j$ from x_0 to x. If $x \in X_0$, then we choose w_x in such a way that $w_x([0, 1]) \subset X_0$. For $x = x_0$, we choose the constant path. Such paths do exist in view of the hypothesis that X_0, X_1, and X_2 are path-connected.

- For a small path $\gamma \colon [a, b] \longrightarrow X$, we define the associated loop
$$c(\gamma) := \bigl((w_{\gamma(a)} * R(\gamma)\bigr) * \overline{w_{\gamma(b)}} \colon (S^1, e_1) \longrightarrow (X, x_0),$$

The loop ...



... is subdivided into "small" paths ...



... that are extended to loops based at x_0 by adding paths to the basepoint:

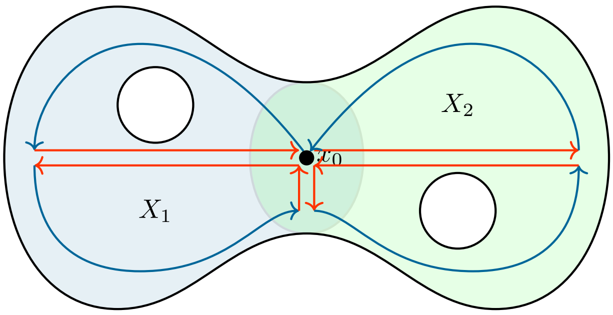


Figure 2.8.: Subdivided loop in the proof of the theorem of Seifert and van Kampen

where $R(\gamma)\colon [0,1] \longrightarrow X$ is the affine linear reparametrisation of γ (which is defined on the interval $[a,b]$). By construction, if $\gamma([a,b]) \subset X_j$, then also $c(\gamma)(S^1) \subset X_j$.

We first prove *uniqueness of φ:* In order to prove uniqueness of φ, it suffices to show that $\pi_1(X,x_0)$ is generated by $\operatorname{im}\pi_1(j_1) \cup \operatorname{im}\pi_1(j_2)$, i.e., that every element in $\pi_1(X,x_0)$ is a product of (pointed homotopy classes of) finitely many small loops. Let $g \in \pi_1(X,x_0)$, say $g = [\gamma]_*$ with $\gamma \in \operatorname{map}_*((S^1,e_1),(X,x_0))$.

① Then γ admits a decomposition, because: In view of $X_1^\circ \cup X_2^\circ = X$, the family $(\gamma^{-1}(X_1^\circ), \gamma^{-1}(X_2^\circ))$ is an open cover of S^1. As $S^1 \cong_{\mathsf{Top}} [0,1]/(0 \sim 1)$ is a compact metrisable space, we obtain from the Lebesgue lemma an $n \in \mathbb{N}$ and $t_0,\dots,t_n \in [0,1]$ with $0 = t_0 < t_1 < \dots < t_n = 1$ such that $\gamma_j := \gamma|_{[t_j,t_{j+1}]}$ is small for every $j \in \{0,\dots,n-1\}$. Then $(\gamma_0,\dots,\gamma_n)$ is a decomposition of γ.

② By construction (with, implicit, leftmost binding priority of "$*$"),

$$\begin{aligned}
\gamma &\simeq_* R(\gamma_0) * \dots * R(\gamma_n) \\
&\simeq_* w_{x_0} * R(\gamma_0) * \overline{w_{\gamma(t_1)}} * w_{\gamma(t_1)} * R(\gamma_1) * \overline{w_{\gamma(t_1)}} * \cdots \\
&\quad * w_{\gamma(t_n)} * R(\gamma_n) * \overline{w_{x_0}} \\
&\simeq_* c(\gamma_0) * \dots * c(\gamma_n),
\end{aligned}$$

and so

$$g = [\gamma]_* = \bigl[c(\gamma_0)\bigr]_*^X \cdot \dots \cdot \bigl[c(\gamma_n)\bigr]_*^X.$$

By construction, each loop $c(\gamma_j)$ lies in X_1 or X_2; hence, $[c(\gamma_j)]_*^X$ lies in $\operatorname{im}\pi_1(j_1) \cup \operatorname{im}\pi_1(j_2)$. Therefore, g is in the subgroup generated by $\operatorname{im}\pi_1(j_1) \cup \operatorname{im}\pi_1(j_2)$.

It remains to show *existence of φ:* By the uniqueness part of the proof, we already know how we have to define φ. The main difficulty is to show that this is actually well-defined.

③ *Definition on small paths.* Let $P \subset \bigcup_{a,b\in\mathbb{R},a\leq b} \operatorname{map}([a,b],X)$ be the set of all small paths in X and let

$$\begin{aligned}
\widetilde{\varphi}\colon P &\longrightarrow H \\
\gamma &\longmapsto \begin{cases} \varphi_1\bigl([c(\gamma)]_*^{X_1}\bigr) & \text{if } \operatorname{im}\gamma \subset X_1 \\ \varphi_2\bigl([c(\gamma)]_*^{X_2}\bigr) & \text{if } \operatorname{im}\gamma \subset X_2. \end{cases}
\end{aligned}$$

This map is well-defined: Let $\gamma \in P$ with $\operatorname{im}\gamma \subset X_1 \cap X_2 = X_0$. Then $\operatorname{im} c(\gamma) \subset X_0$, and so

$$\begin{aligned}
\varphi_1\bigl([c(\gamma)]_*^{X_1}\bigr) &= \varphi_1\bigl([i_1 \circ c(\gamma)]_*^{X_1}\bigr) \\
&= \varphi_1 \circ \pi_1(i_1)\bigl([c(\gamma)]_*^{X_0}\bigr) \\
&= \varphi_2 \circ \pi_1(i_2)\bigl([c(\gamma)]_*^{X_0}\bigr) \quad (\text{because } \varphi_1 \circ \pi_1(i_1) = \varphi_2 \circ \pi_1(i_2)) \\
&= \varphi_2\bigl([i_2 \circ c(\gamma)]_*^{X_2}\bigr) \\
&= \varphi_2\bigl([c(\gamma)]_*^{X_2}\bigr).
\end{aligned}$$

④ *Compatibility with decompositions.* Let $(\gamma\colon [a,b] \longrightarrow X) \in P$, let $t \in [a,b]$, and let $\gamma_1 := \gamma|_{[a,t]}$, $\gamma_2 := \gamma|_{[t,b]}$ (in particular, γ_1 and γ_2 are small). Then we have (if $\operatorname{im}\gamma \subset X_j$)

$$\begin{aligned}
\widetilde{\varphi}(\gamma) &= \varphi_j\bigl([c(\gamma)]_*^{X_j}\bigr) \\
&= \varphi_j\bigl([w_{\gamma(a)} * R(\gamma) * \overline{w_{\gamma(b)}}]_*^{X_j}\bigr) \\
&= \varphi_j\bigl([w_{\gamma(a)} * R(\gamma_1) * \overline{w_{\gamma(t)}} * w_{\gamma(t)} * R(\gamma_2) * \overline{w_{\gamma(b)}}]_*^{X_j}\bigr) \\
&= \varphi_j\bigl([c(\gamma_1)]_*^{X_j} \cdot [c(\gamma_2)]_*^{X_j}\bigr) \\
&= \varphi_j\bigl([c(\gamma_1)]_*^{X_j}\bigr) \cdot \varphi_j\bigl([c(\gamma_2)]_*^{X_j}\bigr) \\
&= \widetilde{\varphi}(\gamma_1) \cdot \widetilde{\varphi}(\gamma_2).
\end{aligned}$$

⑤ *Definition on loops.* We now define

$$\begin{aligned}
\varphi^\circ\colon \operatorname{map}_*\bigl((S^1, e_1), (X, x_0)\bigr) &\longrightarrow H \\
\gamma &\longmapsto \widetilde{\varphi}(\gamma_1) \cdot \dots \cdot \widetilde{\varphi}(\gamma_n) \\
&\qquad \text{if } (\gamma_1, \dots, \gamma_n) \text{ is a decomposition of } \gamma.
\end{aligned}$$

This is indeed well-defined:

- Every loop in (X, x_0) admits a decomposition (by ①),
- and any two decompositions of the same loop have a common refinement and so lead to the same value (by ④).

⑥ *Definition on $\pi_1(X, x_0)$.* As last step, we set

$$\begin{aligned}
\varphi\colon \pi_1(X, x_0) &\longrightarrow H \\
[\gamma]_*^X &\longmapsto \varphi^\circ(\gamma).
\end{aligned}$$

If φ turns out to be well-defined, then it is in fact a group homomorphism (by construction; check!). We prove that φ is well-defined:

Let $\gamma, \eta \in \operatorname{map}_*((S^1, e_1), (X, x_0))$ with $[\gamma]_*^X = [\eta]_*^X$ and let $h\colon (S^1, e_1) \times [0,1] \longrightarrow (X, x_0)$ be a pointed homotopy from γ to η. We apply the subdivision principle to the homotopy h: The sets $h^{-1}(X_1^\circ)$ and $h^{-1}(X_2^\circ)$ form an open cover of the compact metrisable space $S^1 \times [0,1]$. Applying the Lebesgue lemma (Lemma 2.2.7), we obtain an $n \in \mathbb{N}$ such that: For all $j, k \in \{0, \dots, n-1\}$ there exists an $i_{jk} \in \{1, 2\}$ with

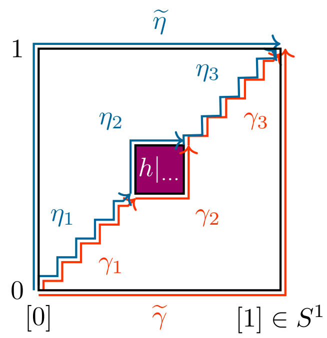


Figure 2.9.: Changing $\widetilde{\gamma}$ "square by square" to $\widetilde{\eta}$ via h

$$h\Bigl(\Bigl[\frac{j}{n}, \frac{j+1}{n}\Bigr] \times \Bigl[\frac{k}{n}, \frac{k+1}{n}\Bigr]\Bigr) \subset X_{i_{jk}}.$$

Proceeding square by square, we prove inductively that $\varphi^\circ(\gamma) = \varphi^\circ(\eta)$: To set up this induction, let

$$\widetilde{\gamma} := \gamma * c \quad \text{and} \quad \widetilde{\eta} := c * \eta,$$

where c denotes the constant loop at x_0. By construction of φ°, we have

$$\begin{aligned} \varphi^\circ(\widetilde{\gamma}) &= \varphi^\circ(\gamma) \cdot \varphi^\circ(c) \\ &= \widetilde{\varphi}(\gamma) \cdot \varphi^\circ(c) \\ &= \varphi^\circ(\gamma), \\ \varphi^\circ(\widetilde{\eta}) &= \varphi^\circ(\eta). \end{aligned}$$

Therefore, it suffices to prove $\varphi^\circ(\widetilde{\gamma}) = \varphi^\circ(\widetilde{\eta})$. Changing $\widetilde{\gamma}$ "square by square" to $\widetilde{\eta}$, inductively, we can restrict attention to the situation in Figure 2.9, where $\eta_1 = \gamma_1$ and $\eta_3 = \gamma_3$, and the paths η_2 and γ_2 are small and related via h as in the sketch.

Out of h, we hence obtain a pointed homotopy

$$c(\gamma_2) \simeq_* c(\eta_2)$$

in X_1 or X_2 (check!). Therefore, $\widetilde{\varphi}(\gamma_2) = \widetilde{\varphi}(\eta_2)$, and so

$$\begin{aligned} \varphi^\circ(\widetilde{\gamma}) &= (\text{contribution of } \gamma_1) \cdot \widetilde{\varphi}(\gamma_2) \cdot (\text{contribution of } \gamma_3) \\ &= (\text{contribution of } \eta_1) \cdot \widetilde{\varphi}(\eta_2) \cdot (\text{contribution of } \eta_3) \\ &= \varphi^\circ(\widetilde{\eta}), \end{aligned}$$

as claimed.

⑦ *Solving the mapping problem.* By construction, we have for all $\gamma \in \operatorname{map}_*((S^1, e_1), (X_1, x_0))$ that

$$\begin{aligned} \varphi \circ \pi_1(j_1)\big([\gamma]_*^{X_1}\big) &= \varphi\big([\gamma]_*^X\big) \\ &= \varphi_1\big([c(\gamma)]_*^{X_1}\big) && \text{(because } \gamma \text{ is } X_1\text{-small)} \\ &= \varphi_1\big([\gamma]_*^{X_1}\big) && \text{(because } \gamma \text{ is a loop at } x_0\text{);} \end{aligned}$$

thus, $\varphi \circ \pi_1(j_1) = \varphi_1$. In the same way, we also obtain $\varphi \circ \pi_1(j_2) = \varphi_2$.

In summary, $\pi_1(X, x_0)$ (together with $\pi_1(j_1)$ and $\pi_1(j_2)$) has the universal property of the pushout specified in the theorem. □

Caveat 2.2.8. The topological situation in the Seifert and van Kampen theorem (Theorem 2.2.6) can be viewed as a pushout in Top_*. Moreover, in the proof of Theorem 2.2.6, it is tempting to work in $\mathsf{Top}_{*\mathsf{h}}$ instead of in Top_*. However, one should note that, in general, the homotopy classes functor $\mathsf{Top}_* \longrightarrow \mathsf{Top}_{*\mathsf{h}}$ does *not* turn pushouts into pushouts! Indeed, the homotopy theory of pushouts (or more general colimits) involves several subtle points [65, Chapter 6].

Outlook 2.2.9 (Blakers-Massey theorem)**.** The analogue of the theorem of Seifert and van Kampen (Theorem 2.2.6) for higher homotopy groups does *not* hold in general: The higher homotopy groups of glueings depend in a more subtle way on the higher homotopy groups of the building blocks [68, Chapter 6.4, Chapter 6.10]; this is the *Blakers–Massey theorem*.

2.2.3 Computations

In order to compute fundamental groups via the theorem of Seifert and van Kampen, we need a better understanding of pushouts of groups. Here, we focus on the most important facts; more details can be found in Appendix A.4.

Example 2.2.10 (pushouts of groups)**.**

- Pushouts of groups are a special type of colimits in the category Group. Hence, all general observations on colimits also apply to these pushouts (e.g., pushout groups are determined up to canonical isomorphism by the input diagram and isomorphic diagrams lead to isomorphic pushout groups).
- All pushouts in Group exist; a concrete construction of the pushout is the amalgamated free product (Appendix A.4). Alternatively, one can also use the Seifert and van Kampen theorem to establish the existence of all pushouts in Group by translating pushout situations in group theory into topology.

- If A is a group, then
$$\begin{tikzcd} A \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$
(where all group homomorphisms are trivial and 1 denotes "the" trivial group) is a pushout in Group (check!).

- If $\varphi\colon A \longrightarrow G$ is a group homomorphism, then
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & G \arrow[d, "p"] \\ 1 \arrow[r] & G/N \end{tikzcd}$$
is a pushout in Group (check!); here, $N \subset G$ denotes the smallest (with respect to inclusion) *normal* subgroup of G that contains $\varphi(A)$ and $p\colon G \longrightarrow G/N$ is the canonical projection.

- Let
$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & F \end{tikzcd}$$
be a pushout in Group. Then F is *not* Abelian (Exercise).

  More precisely, we have the following: Let F_2 be "the" free group of rank 2. A concrete model of F_2 is given by looking at all reduced words in two different elements a and b and taking concatenation (and reduction) as composition (Appendix A.4); then F_2 is freely generated by $\{a, b\}$, i.e., it satisfies the universal property of freeness in Group (Appendix A.4). Then
$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "1\mapsto a"] \\ \mathbb{Z} \arrow[r, "1\mapsto b"'] & F_2 \end{tikzcd}$$
is a pushout in Group.

Example 2.2.11 (fundamental group of higher-dimensional spheres)**.** Let $n \in \mathbb{N}_{\geq 2}$. We write $N := e_{n+1}$ and $S := -e_{n+1}$ for the North and South pole, respectively. We consider the open cover of S^n given by $X_1 := S^n \setminus \{N\}$ and $X_2 := S^n \setminus \{S\}$ (Figure 2.10). Because $X_0 := X_1 \cap X_2$ is path-connected as

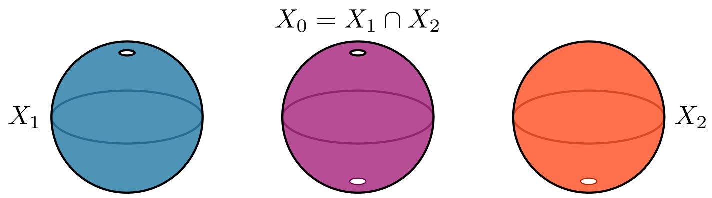


Figure 2.10.: Computing the fundamental group of higher-dimensional spheres via the theorem of Seifert and van Kampen

well, the Seifert and van Kampen theorem (Theorem 2.2.6) applies to this cover. Hence, we obtain a pushout of the form

$$\begin{tikzcd}
\pi_1(X_0, e_1) \arrow[r] \arrow[d] & \pi_1(X_1, e_1) \cong 1 \arrow[d] \\
1 \cong \pi_1(X_2, e_1) \arrow[r] & \pi_1(S^n, e_1)
\end{tikzcd}$$

in Group. As X_1 and X_2 are homeomorphic to $\mathbb{R}^n$ (via the stereographic projection; Exercise) and thus pointedly contractible, we obtain

$$\pi_1(X_1, e_1) \cong_{\mathsf{Group}} 1 \cong_{\mathsf{Group}} \pi_1(X_2, e_1).$$

Therefore, also $\pi_1(S^n, e_1)$ is trivial (Example 2.2.10).

The same argument shows the following: If (X, x_0) is a path-connected pointed topological space, then the fundamental group $\pi_1(\Sigma X, [x_0, 0])$ of the suspension ΣX is trivial.

Study note. Why doesn't the same argument as in Example 2.2.11 also show that $\pi_1(S^1, e_1)$ is trivial?!

Example 2.2.12 (fundamental group of $\mathbb{R}P^2$)**.** Similarly, using the calculation of $\pi_1(S^1, e_1)$ and the theorem of Seifert and van Kampen (Theorem 2.2.6) shows that

$$\pi_1\bigl(\mathbb{R}P^2, [e_1]\bigr) \cong_{\mathsf{Group}} \mathbb{Z}/2$$

and we can exhibit a generator geometrically (Exercise).

Glueing pointed spaces along their basepoints leads to the wedge of spaces (Figure 2.11):

Definition 2.2.13 (wedge)**.** For pointed spaces (X, x_0) and (Y, y_0), we define the *wedge*

$$(X, x_0) \vee (Y, y_0) := \bigl((X \sqcup Y) \bigm/ (x_0 \sim y_0), [x_0] = [y_0]\bigr).$$

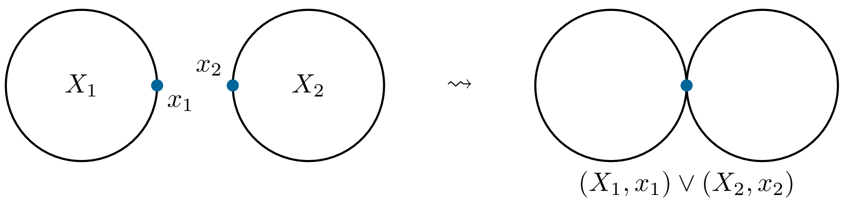


Figure 2.11.: The wedge of pointed spaces

More generally: Let $(X_i, x_i)_{i \in I}$ be a non-empty family of pointed spaces. The *wedge* of this family is defined as

$$\bigvee_{i \in I} (X_i, x_i) := \Bigl(\bigsqcup_{i \in I} X_i / {\sim}, z_0 \Bigr)$$

where $\bigsqcup_{i \in I} X_i / \sim$ carries the quotient topology of the disjoint union topology, where "$\sim$" is the equivalence relation on the disjoint union generated by

$$\forall_{i,j \in I} \quad x_i \sim x_j,$$

and where z_0 is the point represented by all the basepoints x_i with $i \in I$.

The wedge of pointed spaces (together with the pointed homotopy classes of the canonical inclusion maps of the summands) is the coproduct in the category Top_{*h} (check!).

Example 2.2.14 (fundamental group of wedges of finitely many circles)**.** We compute the fundamental group of the wedge $(X, x_0) := (S^1, e_1) \vee (S^1, e_1)$ of two circles, using the theorem of Seifert and van Kampen: Taking the two circles as subspaces would not directly allow to apply the theorem of Seifert and van Kampen (because the union of their interiors would miss the basepoint of the wedge). Therefore, we consider the decomposition indicated in Figure 2.12. The corresponding subspaces X_1, X_2, and $X_0 = X_1 \cap X_2$ satisfy the hypotheses of the Seifert and van Kampen theorem (Theorem 2.2.6).

Hence, we obtain the following pushout in Group (where all homomorphisms are induced by the inclusions of the respective subspaces):

$$\begin{tikzcd}
\pi_1(X_0, x_0) \arrow[r] \arrow[d] & \pi_1(X_2, x_0) \arrow[d] \\
\pi_1(X_1, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

The space (X_0, x_0) is pointedly contractible (check!) and hence has trivial fundamental group. The spaces (X_1, x_0) and (X_2, x_0) are pointedly homotopy

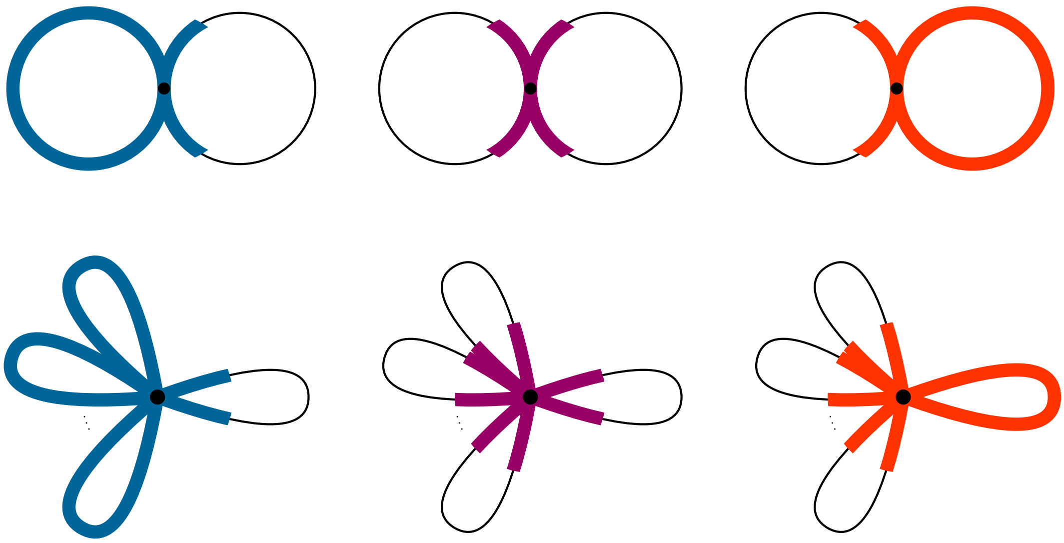


Figure 2.12.: Computing the fundamental group of a wedge of two circles and inductively of a finite wedge of circles

equivalent to (S^1, e_1) (check!) and hence have fundamental group isomorphic to $\mathbb{Z}$, generated by the loops wrapping around the corresponding circle once. Therefore, we obtain a pushout in Group of the following form:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

Using Example 2.2.10, we see that $\pi_1(X, x_0)$ is isomorphic the free group F_2 of rank 2, freely generated by two loops wrapping once around the left/right circle, respectively. In particular, the group $\pi_1(X, x_0)$ is non-abelian (and the loops corresponding to the two circles do *not* commute).

In combination with Example 2.2.5, we obtain: The torus is *not* pointedly homotopy equivalent to $(S^1, e_1) \vee (S^1, e_1)$.

Inductively, in this way we can compute the fundamental group of finite wedges of circles: For $n \in \mathbb{N}_{>0}$, let $(B_n, b_n) := \bigvee_{\{1,\dots,n\}}(S^1, e_1)$ be the wedge of n circles. Then $\pi_1(B_n, b_n)$ is a free group of rank n, freely generated by the circles in the wedge.

Caveat 2.2.15 (fundamental group of wedges)**.** Let (X, x_0) and (Y, y_0) be pointed spaces. In general, $\pi_1((X, x_0) \vee (Y, y_0))$ is *not* isomorphic to the free product of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$. The reason is that, in general, we cannot expect to be able to find a "nice" neighbourhood of the basepoint in order to extend the subspaces (X, x_0) and (Y, y_0) appropriately.

We will now briefly address some questions on the computability of fundamental groups. In order to understand these results, we first need to introduce some terms from group theory:

Outlook 2.2.16 (presentations of groups by generators and relations)**.** Let S be a set, let $F(S) := \bigstar_S \mathbb{Z}$ be "the" free group generated by S (Appendix A.4), and let $R \subset F(S)$. Then the *group generated by S with relations R* is defined as

$$\langle S \mid R \rangle := F(S)/N,$$

where $N \subset F(S)$ is the smallest (with respect to inclusion) normal subgroup of $F(S)$ containing R.

For example, working with the corresponding universal properties shows that [39, Chapter 2.2]

$$\begin{aligned}
\langle \, | \, \rangle &\cong_{\mathsf{Group}} 1 \\
\langle a \, | \, \rangle &\cong_{\mathsf{Group}} \mathbb{Z} \\
\langle a, b \, | \, \rangle &\cong_{\mathsf{Group}} F_2 \\
\langle a, b \mid aba^{-1}b^{-1} \rangle &\cong_{\mathsf{Group}} \mathbb{Z}^2 \\
\langle a \mid a^{2021} \rangle &\cong_{\mathsf{Group}} \mathbb{Z}/2021 \\
\langle s, t \mid s^{2021}, t^2, tst^{-1}s \rangle &\cong_{\mathsf{Group}} D_{2021}.
\end{aligned}$$

However, it can be proved that there is *no* algorithm that, given a presentation with finitely many generators and finitely many relations, decides whether the correspoding group is trivial or not [61, Chapter 12]. Hence, also the analogous general isomorphism problem is *not* algorithmically solvable(!).

Outlook 2.2.17 (from complexes to π_1)**.** Let X be the geometric realisation of a (finite) simplicial complex K, i.e., a space that is constructed from finitely many simplices [54, Chapter 1]. Then, using the theorem of Seifert and van Kampen (Theorem 2.2.6), we can read off a finite presentation of the fundamental group of X from the combinatorics of K [19, Theorem 2.3.1]; roughly speaking, the edeges lead to generators, the triangles lead to relations, and the higher-dimensional simplices do not contribute to the fundamental group.

Outlook 2.2.18 (from presentations to complexes)**.** Conversely, let (S, R) be a finite presentation of a group. Modelling this situation in topology, we can construct the *presentation complex* $P(S, R)$ of (S, R) as the pushout

$$\Bigl(\bigvee_R (D^2, e_1)\Bigr) \cup_\varphi \Bigl(\bigvee_S (S^1, e_1)\Bigr);$$

here, $\varphi\colon \bigvee_R (S^1, e_1) \longrightarrow \bigvee_S (S^1, e_1)$ is a continuous map such that for each $r \in R$, the restriction of φ to the r-th summand corresponds to the element $r \in F(S) \cong_{\mathsf{Group}} \pi_1(\bigvee_S (S^1, e_1))$ under the isomorphism from Example 2.2.14. The theorem of Seifert and van Kampen (Theorem 2.2.6) yields a canonical isomorphism

$$\pi_1\bigl(P(S,R)\bigr) \cong_{\mathsf{Group}} \langle S \mid R \rangle.$$

For example, applying this construction to the presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$ results in a space homeomorphic to the two-dimensional torus.

With careful bookkeeping, this construction can be refined to turn finite presentations (S, R) algorithmically into finite (pointed) simplicial complexes $X(S, R)$ with a canonical isomorphism $\pi_1(X(S, R)) \cong_{\mathsf{Group}} \langle S \mid R \rangle$. Moreover, one can modify this construction in such a way, that

$$X(S,R) \simeq X(\emptyset, \emptyset) \iff \langle S \mid R \rangle \cong_{\mathsf{Group}} 1$$

holds for all finite presentations (S, R) [50].

Outlook 2.2.19 (unsolvability of the homeomorphism/homotopy equivalence problem)**.** Using the observations from Outlook 2.2.16–2.2.18 (and similar, refined constructions), one can prove the following [50]:

- There is *no* algorithm that, given a finite simplicial complex, decides whether it is simply connected or not.
- There is *no* algorithm that, given two finite simplicial complexes, decides whether they are homotopy equivalent or not.
- There is *no* algorithm that, given two finite simplicial complexes, decides whether they are homeomorphic or not.
- There is *no* algorithm that, given two finite triangulations of closed 4-manifolds, decides whether they are homotopy equivalent or not.
- There is *no* algorithm that, given two finite triangulations of closed 4-manifolds, decides whether they are homeomorphic or not.

In particular, the homeomorphism problem and the homotopy equivalence problem are *not* algorithmically solvable.

2.2.4 Ascending Unions

Finally, we consider the behaviour of the fundamental group with respect to ascending unions:

Proposition 2.2.20 (π_n **and ascending unions).** *Let $n \in \mathbb{N}$, let (X, x_0) be a pointed space, let $(I, \leq)$ be a directed set, and let $(X_i)_{i \in I}$ be a directed system of subspaces of X (with respect to inclusion) satisfying*

$$X = \bigcup_{i \in I} X_i^\circ \qquad \text{and} \qquad x_0 \in \bigcap_{i \in I} X_i.$$

Then

1. *We have $\pi_n(X, x_0) = \bigcup_{i \in I} \pi_n(X_i \hookrightarrow X)\bigl(\pi_n(X_i, x_0)\bigr)$.*
2. *If $i, k \in I$ and $g \in \pi_n(X_i, x_0)$, $h \in \pi_n(X_k, x_0)$ with $\pi_n(X_i \hookrightarrow X)(g) = \pi_n(X_k \hookrightarrow X)(h)$, then there is a $j \in I$ with $i \leq j$ and $k \leq j$ satisfying*

$$\pi_n(X_i \hookrightarrow X_j)(g) = \pi_n(X_k \hookrightarrow X_j)(h).$$

In particular, $\pi_n(X, x_0)$ (together with the maps $(\pi_n(X_i \hookrightarrow X))_{i \in I}$) is the colimit of the system $((\pi_n(X_i, x_0))_{i \in I}, (\pi_n(X_i \hookrightarrow X_j))_{i,j \in I, i \leq j})$ in Set *(or* Group*, if $n \geq 1$).*

Recall that a partially ordered set $(I, \leq)$ is *directed* if the following holds: For all $i, k \in I$, there exists a $j \in I$ with $i \leq j$ and $k \leq j$.

Proof. The proof is based on a compactness argument: both S^n and $S^n \times [0,1]$ are compact. Therefore, every pointed map representing an element of $\pi_n(X, x_0)$ and every pointed homotopy between such maps already lives in one of the subspaces X_i. More precisely:

Ad 1. Let $g \in \pi_n(X, x_0)$, say $g = [\gamma]_*$ with $\gamma \in \operatorname{map}_*((S^n, e_1), (X, x_0))$. Because S^n is compact, also $\gamma(S^n)$ is compact. As $(X_i^\circ)_{i \in I}$ is an open cover of X, there exists a finite set $J \subset I$ with

$$\gamma(S^n) \subset \bigcup_{j \in J} X_j.$$

Because $(I, \leq)$ is directed and J is finite, there exists an $i \in I$ with with

$$\forall_{j \in J} \quad j \leq i.$$

Hence, $\gamma(S^n) \subset \bigcup_{j \in J} X_j \subset X_i$. Therefore,

$$g = [\gamma]_* \in \pi_n(X_i \hookrightarrow X)\bigl(\pi_n(X_i, x_0)\bigr).$$

Ad 2. Let $i, k \in I$ and let $\gamma \in \operatorname{map}_*((S^n, e_1), (X_i, x_0))$ as well as $\eta \in \operatorname{map}_*((S^n, e_1), (X_k, x_0))$ with $[\gamma]_*^X = [\eta]_*^X$. Let $h \colon (S^n, e_1) \times [0,1] \longrightarrow (X, x_0)$ be a pointed homotopy from γ to η (in X). Because $S^n \times [0,1]$ is compact, we can apply the same argument as in the first part to conclude that there is a $j \in I$ with $h(S^n \times [0,1]) \subset X_j$ and $i \leq j$, $k \leq j$. Hence, $[\gamma]_*^{X_j} = [\eta]_*^{X_j}$. $\square$

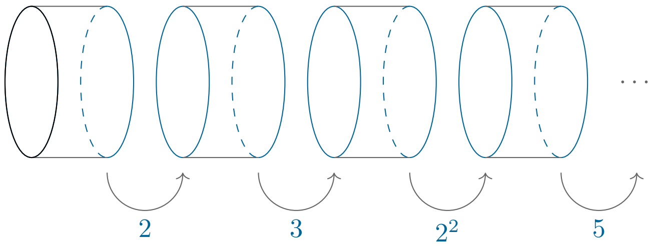


Figure 2.13.: A "telescope"

Example 2.2.21 (fundamental group of general wedges of circles)**.** Let I be a non-empty set. Then the computation of the fundamental group of finite wedges of circles (Example 2.2.14) and Proposition 2.2.20 (applied to a suitable system of subspaces of thickened finite wedges of circles) shows that $\pi_1(\bigvee_I(S^1,1))$ is free of rank $|I|$, where the circles correspond to generators in a free generating set.

Combining Proposition 2.2.20 and the ideas from Outlook 2.2.17 and 2.2.18 allows us to describe fundamental groups of general complexes in terms of generators and relations and to construct complexes with a given fundamental group.

Example 2.2.22 (telescope)**.** We consider the space in Figure 2.13, consisting of a sequence of cylinders $[0,1]\times S^1$ that are glued via the maps wrapping the circle around the circle the given number of times; here, all prime powers occur as such numbers.

We can then use the theorem of Seifert and van Kampen (Theorem 2.2.6) to compute the fundamental groups of the initial pieces (given by finitely many of the first glued cylinders; some care is needed to handle the basepoints!). Viewing these groups as subgroups of $\mathbb{Q}$, we obtain the sequence

$$\mathbb{Z} \subset \frac{1}{2}\cdot\mathbb{Z} \subset \frac{1}{6}\cdot\mathbb{Z} \subset \frac{1}{24}\cdot\mathbb{Z} \subset \frac{1}{120}\cdot\mathbb{Z} \subset \dots$$

of groups. Applying Proposition 2.2.20 then shows that the fundamental group of the whole telescope is isomorphic to $\mathbb{Q}$.

This example can be generalised to the theory of rationalisations and localisations of topological spaces [18].

2.3 Covering Theory

Our next goal is to give an interpretation of fundamental groups as automorphism groups of geometric objects. This will allow us to

- classify certain geometric objects via fundamental groups,
- and to compute the fundamental group geometrically in some cases.

In particular, we will compute the fundamental group of the circle with this method (Theorem 2.3.40).

The key notion in this context are coverings; coverings form a special class of locally trivial bundles. We will first introduce the basic terminology and investigate lifting properties of such maps. We will then study the universal covering and prove the classification theorem. Finally, we sketch how these techniques can be used to give a topological proof of the Nielsen–Schreier theorem in group theory.

2.3.1 Coverings

Coverings are locally trivial bundles with discrete fibre:

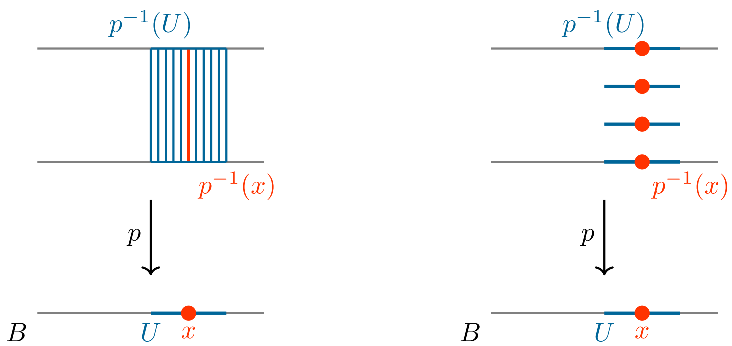


Figure 2.14.: A locally trivial bundle and a covering, schematically

Definition 2.3.1 ((locally) trivial bundle, covering)**.** Let F, E, B be topological spaces.

- A continuous map $p\colon E \longrightarrow B$ is a *trivial bundle over B with fibre F* if there exists a homeomorphism $f\colon E \longrightarrow B \times F$ with

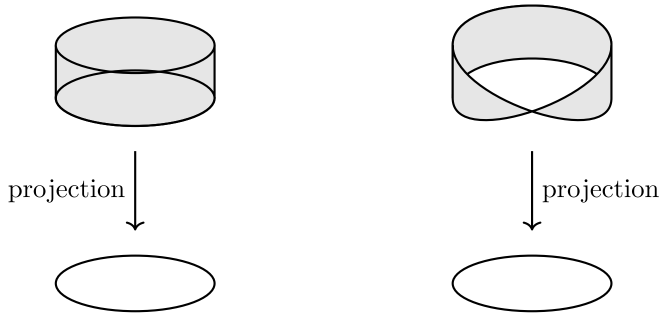


Figure 2.15.: Locally trivial bundles over S^1 with fibre $[0,1]$

$$p_B \circ f = p,$$

where $p_B \colon B \times F \longrightarrow B$ denotes the projection onto the first factor.

- A continuous map $p \colon E \longrightarrow B$ is a *locally trivial bundle over B with fibre F* if: For every $x \in B$ there exists an open neighbourhood $U \subset B$ of X such that the restriction $p|_{p^{-1}(U)} \colon p^{-1}(U) \longrightarrow U$ is a trivial bundle over U with fibre F.

  In this situation, E is the *total space* and B is the *base space*. For $x \in B$, the preimage $p^{-1}(x) \subset E$ is the *fibre of p over x* (Figure 2.14).

- Locally trivial bundles with discrete fibre F are called *coverings*. We then call $|F|$ the *number of sheets* of the covering (Figure 2.14).

Caveat 2.3.2 (covering vs. cover)**.** In this course, we will always use the term "covering" in the sense above and the term "cover" in the sense of Definition A.1.33. In the literature, sometimes these words are used in different ways.

Example 2.3.3 (locally trivial bundles)**.**

- Vector bundles (e.g., tangent bundles of smooth manifolds) are locally trivial bundles; the fibres are (real or complex) vector spaces. For a vector bundle one requires that there exists an atlas of local trivialisations that is compatible with the linear structure on the fibres.

- The continuous maps in Figure 2.15 are locally trivial bundles with fibre $[0,1]$ over S^1. The bundle on the left-hand side is trivial, while the bundle on the right-hand side is *not* trivial (Exercise). Because $[0,1]$ does not carry the discrete topology, these maps are *not* covering maps.

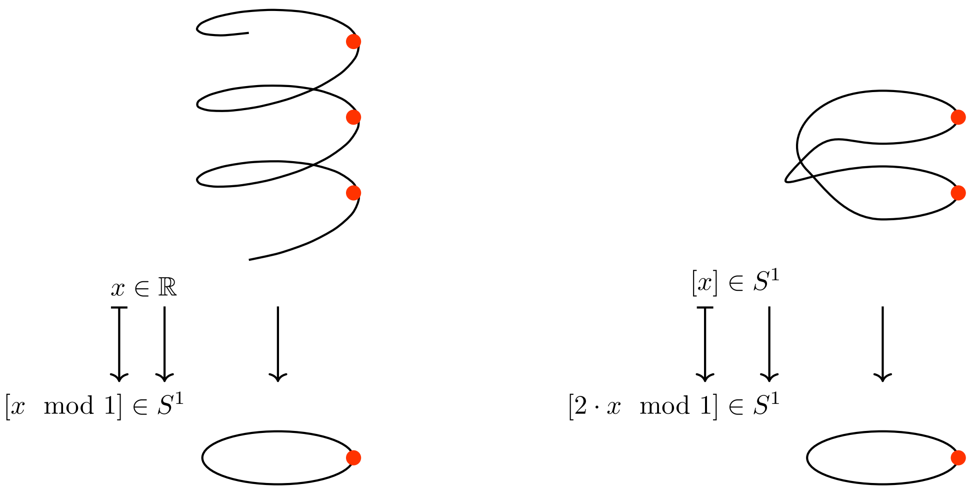


Figure 2.16.: Coverings of S^1

- Viewing S^3 as the unit sphere in $\mathbb{C}^2$, the scalar multiplication with elements of $S^1 \subset \mathbb{C}$ defines a continuous group action $S^1 \curvearrowright S^3$. One can show that the associated quotient map $p\colon S^3 \longrightarrow S^1 \setminus S^3$ is a locally trivial bundle with fibre S^1, the so-called *Hopf fibration*. The base space is homeomorphic to S^2 [26, Example 4.45].

  One of the foundational results of homotopy theory is that the class in $\pi_3(S^2, e_1)$ represented by p is *not* trivial(!) [26, Example 4.51].

- If I is a set and X is a topological space, then $\coprod_I X \xrightarrow{\coprod_I \mathrm{id}_X} X$ is a (trivial) $|I|$-sheeted covering.

- Figure 2.16 shows two coverings of S^1. This can be checked directly by hand (check!) or, alternatively, one can use suitable group actions (Proposition 2.3.7). The total spaces of these coverings are connected and have more than one sheet; hence, these coverings are non-trivial.

Remark 2.3.4. By definition, every covering map is a local homeomorphism. Thus, covering maps are *open* maps, i.e., they map open sets to open sets (check!).

An important source of coverings are group actions. Basic terminology on group actions is reviewed in Appendix A.5.

Definition 2.3.5 (properly discontinuous action)**.** Let $G \curvearrowright X$ be a group action in Top of a group G on a topological space X. This action is *properly* *discontinuous*, if the following holds: For every $x \in X$ there is an open neighbourhood $U \subset X$ of x satisfying

$$\forall_{g \in G \setminus \{e\}} \quad g \cdot U \cap U = \emptyset.$$

Caveat 2.3.6. From the definition it is clear that every properly discontinuous action is free (check!). However, in general, *not* every free action is properly discontinuous: For example, the action of $\mathbb{Z}$ on S^1 given by rotation around an irrational multiple of π is free, but *not* properly discontinuous (check!). Another instructive example is a non-trivial group G acting on itself by left translation, where G is equipped with the trivial topology (check!).

Proposition 2.3.7 (group actions yield coverings)**.** *Let $G \curvearrowright X$ be a properly discontinuous action in* Top*. Then the canonical projection*

$$\begin{aligned} p \colon X &\longrightarrow G \setminus X \\ x &\longmapsto G \cdot x \end{aligned}$$

is a covering map with fibre G. Here, $G \setminus X$ is endowed with the quotient topology induced by p.

Proof. As a first step, we show that the projection $p \colon X \longrightarrow G \setminus X$ is an open map: Let $U \subset X$ be open. Then $V := p(U) \subset G \setminus X$ is open, because: The preimage

$$p^{-1}(V) = \bigcup_{g \in G} g \cdot U$$

is open in X (as a union of homeomorphic images of open sets). By definition of the quotient topology, V is open in $G \setminus X$.

We now prove that p is a covering map with fibre G: Let $z \in G \setminus X$, i.e., there is an $x \in X$ with $z = G \cdot x$. Because the action $G \curvearrowright X$ is properly discontinuous, there exists an open neighbourhood $U \subset X$ of x with

$$\forall_{g \in G \setminus \{e\}} \quad U \cap g \cdot U = \emptyset.$$

As p is an open map, the image $V := p(U)$ is an open neighbourhood of z in $G \setminus X$. Moreover, the map

$$\begin{aligned} p^{-1}(V) = \bigcup_{g \in G} g \cdot U &\longrightarrow V \times G \\ g \cdot u &\longmapsto \bigl(p(u), g\bigr) \end{aligned}$$

(where $V \times G$ carries the product topology of the subspace topology on V and the discrete topology on G) is well-defined (check!), continuous (check!), bijective (check!), and open (because p is open; check!); hence, this map is a homeomorphism. Because this homeomorphism is compatible with p, the restriction $p|_{p^{-1}(V)}$ is a trivial bundle with discrete fibre G. $\square$

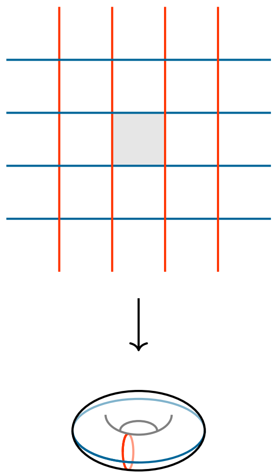


Figure 2.17.: A covering of the 2-torus by the Euclidean plane

Example 2.3.8 (tori)**.** Let $n \in \mathbb{N}$. Then the action $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ given by translation via

$$\begin{aligned} \mathbb{Z}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (z, x) &\longmapsto z + x \end{aligned}$$

is properly discontinuous (check!). Therefore, the projection $\mathbb{R}^n \longrightarrow \mathbb{Z}^n \setminus \mathbb{R}^n$ is a $|\mathbb{Z}^n|$-sheeted covering map of $\mathbb{Z}^n \setminus \mathbb{R}^n$. Moreover, the compact-Hausdorff trick shows (similarly to Example 1.1.16) that

$$\begin{aligned} (S^1)^n &\longrightarrow \mathbb{Z}^n \setminus \mathbb{R}^n \\ \bigl([x_1], \dots, [x_n]\bigr) &\longmapsto \mathbb{Z}^n + (x_1, \dots, x_n) \end{aligned}$$

is a homeomorphism (check!). Hence, if $n > 0$, this action yields an infinite-sheeted covering of the n-dimensional torus by $\mathbb{R}^n$ (Figure 2.17).

Example 2.3.9 (real projective spaces)**.** Let $n \in \mathbb{N}$. Then the action $\mathbb{Z}/2 \curvearrowright S^n$ given by

$$\begin{aligned} \mathbb{Z}/2 \times S^n &\longrightarrow S^n \\ ([0], x) &\longmapsto x \\ ([1], x) &\longmapsto -x \end{aligned}$$

is properly discontinuous (check!). Hence, the projection

$$S^n \longrightarrow \mathbb{Z}/2 \setminus S^n = \mathbb{R}P^n$$

is a 2-sheeted covering of $\mathbb{R}P^n$.

These considerations lead to the following questions:

- Is every covering induced by a properly discontinuous group action?
- (How) Is this related to the fundamental group?
- (How) Can we classify coverings of a given base space?

The link between coverings and fundamental groups is provided by the deck transformation group. The deck transformation group of a covering is its automorphism group in a suitable category of coverings. Therefore, we first introduce categories of coverings.

Definition 2.3.10 (lift, morphism of coverings)**.** Let $p\colon Y \longrightarrow X$ be a covering map.

- Let $f\colon Z \longrightarrow X$ be a continuous map. A *p-lift of f* is a continuous map $\widetilde{f}\colon Z \longrightarrow Y$ with $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

- Let $p'\colon Y' \longrightarrow X'$ be a covering map. A *morphism from p' to p* is a pair $(\widetilde{f}, f)$, where $f \in \operatorname{map}(X', X)$ and $\widetilde{f}\colon Y' \longrightarrow Y$ is a p-lift of $f \circ p'$.

$$\begin{tikzcd} Y' \arrow[r, dashed, "\widetilde{f}"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ X' \arrow[r, dashed, "f"'] & X \end{tikzcd}$$

Straightforward calculations show that coverings and morphisms of coverings form a category:

Definition 2.3.11 (categories of coverings)**.** The category Cov of all covering maps consists of:

- objects: the class of all covering maps with non-empty total space.
- morphisms: morphisms between coverings (as in Definition 2.3.10).
- compositions: ordinary composition of maps of the two components.

Analogously, we define the category Cov_* of pointed coverings:

- objects: the class of all pointed covering maps between pointed spaces with non-empty total space.
- morphisms: morphisms between pointed coverings, where both components are pointed maps.
- compositions: ordinary composition of maps of the two components.

If X is a topological space, the category Cov_X of coverings of X consists of:

- objects: the class of all coverings of X with non-empty total space.
- morphisms: If $p\colon Y \longrightarrow X$ and $p'\colon Y' \longrightarrow X$ are coverings of X, then we set
$$\mathrm{Mor}_{\mathsf{Cov}_X}(p', p) := \bigl\{ f \in \mathrm{map}(Y', Y) \bigm| (f, \mathrm{id}_X) \in \mathrm{Mor}_{\mathsf{Cov}}(p', p) \bigr\}.$$
- compositions: ordinary composition of maps.

Analogously, if (X, x_0) is a pointed space, we define the category $\mathsf{Cov}_{(X,x_0)}$ of all pointed coverings of (X, x_0).

Definition 2.3.12 (deck transformation group)**.** Let $p\colon Y \longrightarrow X$ be a covering map. Then the automorphism group

$$\begin{aligned}\mathrm{Deck}(p) &:= \mathrm{Aut}_{\mathsf{Cov}_X}(p)\\ &= \bigl\{ f\colon Y \longrightarrow Y \bigm| f \text{ is a homeomorphism with } \ p \circ f = p \bigr\}\end{aligned}$$

is the *deck transformation group of* p.

Example 2.3.13 (symmetric groups as deck transformation groups)**.** If I is a set (endowed with the discrete topology), then the trivial covering $I \longrightarrow \{1\}$ (i.e., the constant map) has the deck transformation group $\mathrm{Sym}(I)$ of all bijections $I \longrightarrow I$ (with respect to composition).

The goal is now to understand the category $\mathsf{Cov}_{(X,x_0)}$ for pointed topological spaces (X, x_0) as well as its relation with $\pi_1(X, x_0)$.

Study note. Try to find similarities between the definition of the deck transformation group of a covering map and the definition of the Galois group of a field extension!

2.3.2 Lifting Properties

In order to understand morphisms between coverings and, in particular, the deck transformation group, we study lifting properties of coverings. This leads to an action of the fundamental group on covering spaces and to the π_1-lifting criterion. The key observation is the following lifting of paths (Figure 2.18), which will be the origin of all other lifting results:

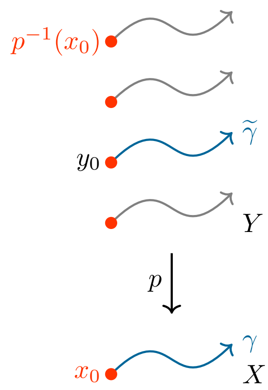


Figure 2.18.: Lifts of paths in coverings

Proposition 2.3.14 (lifts of (homotopies) of paths)**.** *Let $p\colon Y \longrightarrow X$ be a covering, let $x_0 \in X$, and let $y_0 \in p^{-1}(x_0)$.*

1. *If $\gamma\colon [0,1] \longrightarrow X$ is a continuous path with $\gamma(0) = x_0$, then there exists a unique p-lift $\widetilde{\gamma}\colon [0,1] \longrightarrow Y$ of γ with $\widetilde{\gamma}(0) = y_0$.*
2. *If $h\colon [0,1] \times [0,1] \longrightarrow X$ is a homotopy with $h(0,0) = x_0$, then there exists a unique p-lift $\widetilde{h}\colon [0,1] \times [0,1] \longrightarrow Y$ of h with $\widetilde{h}(0,0) = y_0$.*

   In particular: If
   $$\forall_{t\in[0,1]} \quad h(0,t) = x_0,$$
   then also
   $$\forall_{t\in[0,1]} \quad \widetilde{h}(0,t) = y_0.$$

Proof. Ad 1. As first step, we consider the case that the covering is trivial; i.e., there is a trivialisation homeomorphism $f\colon Y \longrightarrow X \times F$ of p, where F is a discrete topological space. Let $z_0 \in F$ with $f(y_0) = (x_0, z_0)$.

- Existence of a p-lift: The map
  $$\begin{aligned} \widetilde{\gamma}\colon [0,1] &\longrightarrow Y \\ t &\longmapsto f^{-1}\bigl(\gamma(t), z_0\bigr) \end{aligned}$$
  is continuous (check!), satisfies $p \circ \widetilde{\gamma} = \gamma$, and
  $$\widetilde{\gamma}(0) = f^{-1}\bigl(\gamma(0), z_0\bigr) = f^{-1}(x_0, z_0) = y_0.$$
- Uniqueness of p-lifts: Let $\eta\colon [0,1] \longrightarrow Y$ also be a p-lift of γ with $\eta(0) = y_0$. Then (where $q\colon X \times F \longrightarrow F$ denotes the projection onto the second

factor)

$$\forall_{t\in[0,1]} \quad q\circ f\big(\eta(t)\big) = q\circ f\big(\eta(0)\big) = q\circ f(y_0) = z_0,$$

because $q\circ f\circ\eta([0,1])$ is path-connected and F is discrete.

Because the map

$$\begin{aligned} X\times\{z_0\} &\longrightarrow X \\ (x,z_0) &\longmapsto x = p\circ f^{-1}(x,z_0) \end{aligned}$$

is injective and

$$p\circ f^{-1}\circ(f\circ\eta) = p\circ\eta = \gamma = p\circ\widetilde{\gamma} = p\circ f^{-1}\circ(f\circ\widetilde{\gamma}),$$

we obtain $f\circ\eta = f\circ\widetilde{\gamma}$, whence $\eta = \widetilde{\gamma}$.

For the general case, we apply the Lebesgue lemma (Lemma 2.2.7) to find $n\in\mathbb{N}$ and a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ such that p is trivial over the pieces $\gamma([t_0,t_1]),\dots,\gamma([t_{n-1},t_n])$.

- Existence of a p-lift: Using the case of trivial coverings, we inductively construct p-lifts $\widetilde{\gamma}_0\colon [t_0,t_1]\longrightarrow Y,\dots,\widetilde{\gamma}_{n-1}\colon [t_{n-1},t_n]\longrightarrow Y$ of $\gamma|_{[t_0,t_1]},\dots,\gamma|_{[t_{n-1},t_n]}$ with

$$\widetilde{\gamma}_0(0) = y_0 \quad\text{and}\quad \forall_{j\in\{1,\dots,n-1\}} \quad \widetilde{\gamma}_j(t_j) = \widetilde{\gamma}_{j-1}(t_j).$$

  Glueing these lifts leads to a p-lift $\widetilde{\gamma}\colon [0,1]\longrightarrow Y$ of γ with $\widetilde{\gamma}(0) = y_0$.

- Uniqueness of p-lifts: If $\eta\colon [0,1]\longrightarrow Y$ also is a p-lift of γ with $\eta(0) = y_0$, then the uniqueness part for trivial coverings shows inductively that

$$\forall_{j\in\{0,\dots,n-1\}} \quad \forall_{t\in[t_j,t_{j+1}]} \quad \eta(t) = \widetilde{\gamma}_j(t) = \widetilde{\gamma}(t).$$

  Hence, $\eta = \widetilde{\gamma}$.

Ad 2. The second claim follows from the first one and the first part (applied to the constant path $h(0,\dots)$. Therefore, it suffices to prove the first part.

There are several ways to prove this statement for homotopies. The most straightforward way is to proceed as in the proof of the first part: One first considers the case of trivial coverings (where one can argue as in the case of paths; check!). For the general case, using the Lebesgue lemma, one subdivides the square $[0,1]\times[0,1]$ into sufficiently small squares and inductively applies the case of trivial coverings (check!); for the compatibility on the glueing intervals, one makes use of the uniqueness properties of part 1. (check!).

Alternatively, one can prove uniqueness using the uniqueness of paths (see also Corollary 2.3.18) or one can use the exponential law to view homotopies as paths in mapping spaces (Remark 1.3.2) and then apply the first part. $\square$

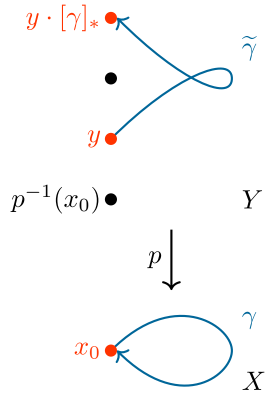


Figure 2.19.: The action of the fundamental group on the fibres

For example, we can use lifts of paths to define a (right) action of the fundamental group of the base space on the fibres of coverings (Figure 2.19):

Corollary 2.3.15 (action of the fundamental group on the fibres)**.** *Let $p\colon Y \longrightarrow X$ be a covering and let $x_0 \in X$.*

1. *Then*

$$\begin{aligned} p^{-1}(x_0) \times \pi_1(X, x_0) &\longrightarrow p^{-1}(x_0) \\ (y, [\gamma]_*) &\longmapsto \widetilde{\gamma}(1) \\ & \text{where } \widetilde{\gamma}\colon [0,1] \longrightarrow Y \text{ is the } p\text{-lift} \\ & \text{of } [0,1] \longrightarrow X, t \mapsto \gamma([t]) \text{ with } \widetilde{\gamma}(0) = y \end{aligned}$$

   is a well-defined right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

2. *If $y \in p^{-1}(x_0)$, then $\pi_1(p)(\pi_1(Y, y))$ is the stabiliser of y with respect to this action.*

3. *This action is transitive if and only if the fibre $p^{-1}(x_0)$ is contained in a single path-connected component of Y, i.e., if all points in $p^{-1}(x_0)$ can be connected by continuous paths.*

Proof. Ad 1. That the action map is well-defined follows from Proposition 2.3.14: First, given a representing pointed loop γ, existence and uniqueness of an appropriate p-lift $\widetilde{\gamma}$ of the path associated with the loop γ is guaranteed by Proposition 2.3.14.

Independence of the chosen representatives: Let $y \in p^{-1}(x_0)$ and let γ and η be pointed loops with $[\gamma]_* = [\eta]_* \in \pi_1(X, x_0)$, say via a pointed homotopy h. By Proposition 2.3.14, there exists a p-lift $\widetilde{h}\colon [0,1] \times [0,1] \longrightarrow Y$ of

$$\begin{aligned}[0,1]\times[0,1] &\longrightarrow X\\ (s,t) &\longmapsto h\bigl([s],t\bigr)\end{aligned}$$

with $\widetilde{h}(0,t) = y$ for all $t \in [0,1]$. Hence, the uniqueness of p-lifts of paths (Proposition 2.3.14) shows that $\widetilde{\gamma} := \widetilde{h}(\,\cdot\,,0)$ is a/the p-lift of the path associated with the loop $\widetilde{\gamma}$ starting at y, that $\widetilde{\eta} := \widetilde{h}(\,\cdot\,,1)$ is a/the p-lift of the path associated with $\widetilde{\eta}$ starting at y, and that $\widetilde{h}(1,\,\cdot\,)$ is constant. Therefore, we obtain

$$\widetilde{\gamma}(1) = \widetilde{h}(1,0) = \widetilde{h}(1,1) = \widetilde{\eta}(1),$$

as desired.

Action of the neutral element. The neutral element can be represented by the constant loop and the constant loop acts trivially on $p^{-1}(x_0)$ (because the lifts of constant paths are constant).

Compatibility with the group structure. We go step by step through the definition and use the lifting properties: Let $y \in p^{-1}(x_0)$, and let $\gamma, \eta \in \operatorname{map}_*((S^1,e_1),(X,x_0))$. If $\widetilde{\gamma}\colon [0,1] \longrightarrow Y$ is a p-lift of $t \longmapsto \gamma([t])$ with $\widetilde{\gamma}(0) = y$ and $\widetilde{\eta}\colon [0,1] \longrightarrow Y$ is a p-lift of $t \longmapsto \eta([t])$ with $\widetilde{\eta}(0) = \widetilde{\gamma}(1)$, then $\widetilde{\gamma} * \widetilde{\eta}$ is a p-lift of $t \mapsto (\gamma * \eta)([t])$ with $\widetilde{\gamma} * \widetilde{\eta}(0) = y$. Hence,

$$\begin{aligned}
y \cdot \bigl([\gamma]_* \cdot [\eta]_*\bigr) &= y \cdot [\gamma * \eta]_* && \text{(group structure on } \pi_1\text{)}\\
&= (\widetilde{\gamma} * \widetilde{\eta})(1) && \text{(definition of the } \pi_1\text{-action)}\\
&= \widetilde{\eta}(1) && \text{(definition of concatenation)}\\
&= \bigl(\widetilde{\gamma}(1)\bigr) \cdot [\eta]_* && \text{(definition of the } \pi_1\text{-action)}\\
&= \bigl(y \cdot [\gamma]_*\bigr) \cdot [\eta]_*. && \text{(definition of the } \pi_1\text{-action)}
\end{aligned}$$

Ad 2. By construction, a pointed loop in (X,x_0) acts trivially on y if and only if it lifts to a *closed* path at y; this is equivalent to lying in the image of $\pi_1(Y,y)$ under $\pi_1(p)$.

Ad 3. By construction, all $\pi_1(X,x_0)$-translates of a point $y \in p^{-1}(X,x_0)$ can be connected by a path to y.

Conversely, if $y' \in p^{-1}(x_0)$ can be connected by a path $\gamma\colon [0,1] \longrightarrow Y$ to y with $\gamma(0) = y$ and $\gamma(1) = y'$, then (by construction)

$$y \cdot [p \circ \gamma]_* = y'$$

(where we view $p \circ \gamma$ as a pointed loop in (X,x_0)). □

Caveat 2.3.16. This action of the fundamental group on the fibres is only interesting, because, in general, lifts of closed paths are *not* closed paths!

Example 2.3.17 (the π_1-action of the fundamental example)**.** We consider the covering map (Example 2.3.3)

$$\begin{aligned}\mathbb{R} &\longrightarrow S^1\\ x &\longmapsto [x \mod 1].\end{aligned}$$

For $d \in \mathbb{Z}$ let $\gamma_d \colon (S^1, e_1) \longrightarrow (S^1, e_1)$ be the pointed loop given by $[t] \longmapsto [d \cdot t \mod 1]$ (wrapping around the circle d times). Then

$$\begin{aligned}\widetilde{\gamma}_d \colon [0,1] &\longrightarrow \mathbb{R}\\ t &\longmapsto d \cdot t\end{aligned}$$

is a p-lift of the path $t \longmapsto \gamma_d([t])$ with $\widetilde{\gamma}_d(0) = 0$ and $\widetilde{\gamma}_d(1) = d$. Hence, we obtain

$$0 \cdot [\gamma_d]_* = \widetilde{\gamma}_d(1) = d$$

for the action from Corollary 2.3.15. In particular, this already shows that the elements $([\gamma_d]_*)_{d\in\mathbb{Z}}$ in $\pi_1(S^1, e_1)$ are pairwise different.

We will now focus on lifts of maps from more general domain spaces. For example, lifts of maps from path-connected domains are unique in the following sense:

Corollary 2.3.18 (uniqueness of lifts)**.** *Let $p \colon Y \longrightarrow X$ be a covering, let Z be a path-connected space, let $f \colon Z \longrightarrow X$ be continuous, and let $z_0 \in Z$. If $\widetilde{f}, g \colon Z \longrightarrow Y$ are p-lifts of f with $\widetilde{f}(z_0) = g(z_0)$, then $\widetilde{f} = g$.*

Proof. We reduce the claim to uniqueness of lifts of paths (Proposition 2.3.14): Let $z \in Z$. Because Z is path-connected, there is a continuous path $\gamma \colon [0,1] \longrightarrow Z$ from z_0 to z. Then $\widetilde{f} \circ \gamma$ and $g \circ \gamma$ are p-lifts of γ with

$$\widetilde{f} \circ \gamma(0) = \widetilde{f}(z_0) = g(z_0) = g \circ \gamma(0).$$

By the uniqueness of p-lifts (Proposition 2.3.14), we obtain $\widetilde{f} \circ \gamma = g \circ \gamma$. In particular, we have

$$\widetilde{f}(z) = \widetilde{f} \circ \gamma(1) = g \circ \gamma(1) = g(z).$$

Hence, $\widetilde{f} = g$. □

In particular, we see that deck transformations on path-connected covering spaces are determined uniquely by their value on a single point!

Remark 2.3.19. The result of Corollary 2.3.18 can be improved in such a way that it also works for connected (instead of path-connected) domains; however, the proof is slightly different in that case [51, Lemma V.3.2].

Furthermore, we can derive the main lifting criterion; in combination with mapping degrees, this criterion is a valuable tool in Geometric Topology.

Theorem 2.3.20 (lifting criterion for coverings via π_1)**.** *Let $p \colon (Y, y_0) \longrightarrow (X, x_0)$ be a pointed covering and let $f \colon (Z, z_0) \longrightarrow (X, x_0)$ be a pointed*

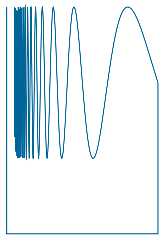


Figure 2.20.: The Warsaw circle

continuous map, where Z is path-connected and locally path-connected. Then f admits a p-lift $(Z, z_0) \longrightarrow (Y, y_0)$ if and only if

$$\pi_1(f)\bigl(\pi_1(Z, z_0)\bigr) \subset \pi_1(p)\bigl(\pi_1(Y, y_0)\bigr).$$

Before proving this theorem, we recall local path-connectedness.

Definition 2.3.21 (locally path-connected)**.** A topological space Z is *locally path-connected* if: For every $z \in Z$ and every open neighbourhood $U \subset Z$ of z there exists a path-connected open neighbourhood $V \subset Z$ of z with $V \subset U$.

Example 2.3.22. Every manifold is locally path-connected.

Caveat 2.3.23 (path-connected vs. locally path-connected)**.** Clearly, not every locally path-connected space is path-connected (as can be seen by considering a discrete space with at least two points). Conversely, also path-connected spaces, in general, are *not* locally path-connected (as can be seen from the Warsaw circle; Figure 2.20).

Study note. It helps to sketch the situations in the following proof!

Proof of Theorem 2.3.20. If $\widetilde{f}\colon (Z, z_0) \longrightarrow (Y, y_0)$ is a p-lift of f, then functoriality of π_1 shows that

$$\begin{aligned}
\pi_1(f)\bigl(\pi_1(Z, z_0)\bigr) &= \pi_1(p \circ \widetilde{f})\bigl(\pi_1(Z, z_0)\bigr) \\
&= \pi_1(p) \circ \pi_1(\widetilde{f})\bigl(\pi_1(Z, z_0)\bigr) \\
&\subset \pi_1(p)\bigl(\pi_1(Y, y_0)\bigr).
\end{aligned}$$

Conversely, let us suppose that $\pi_1(f)\bigl(\pi_1(Z, z_0)\bigr) \subset \pi_1(p)\bigl(\pi_1(Y, y_0)\bigr)$ is satisfied. We construct the desired p-lift $\widetilde{f}\colon (Z, z_0) \longrightarrow (Y, y_0)$ of f using lifts of paths: Let $z \in Z$; because Z is path-connected, there exists a continuous path $\gamma\colon [0, 1] \longrightarrow Z$ from z_0 to z. Let $\widetilde{\gamma}\colon [0, 1] \longrightarrow Y$ be the p-lift of $f \circ \gamma$ with $\widetilde{\gamma}(0) = y_0$ (Proposition 2.3.14). We then set

$$\widetilde{f}(z) := \widetilde{\gamma}(1).$$

We show step by step that this map $\widetilde{f}$ is well-defined and has the desired properties:

- By construction, $p \circ \widetilde{f}(z) = p \circ \widetilde{\gamma}(1) = f \circ \gamma(1) = f(z)$.

- The construction of $\widetilde{f}$ is independent of the chosen path: Let $\eta\colon [0,1] \longrightarrow Z$ be a continuous path in Z from z_0 to z. Then $w := \gamma * \overline{\eta}\colon [0,1] \longrightarrow Z$ is a (closed) continuous path in Z with $w(0) = z_0 = w(1)$. Let $w^\circ\colon (S^1, e_1) \longrightarrow (Z, z_0)$ be the corresponding loop.

  Because $\operatorname{im} \pi_1(f) \subset \operatorname{im} \pi_1(p)$ and because $\pi_1(p)(\pi_1(Y, y_0))$ acts trivially on y_0 (Corollary 2.3.15), we have

  $$y_0 \cdot [f \circ w^\circ]_* = y_0 \cdot \bigl(\pi_1(f)[w^\circ]_*\bigr) = y_0.$$

  In other words, the p-lift of $f \circ w$ that starts at y_0 also ends at y_0. We apply this to a specific construction of the lift of $f \circ w$:

  If $\widetilde{\gamma}\colon [0,1] \longrightarrow Y$ is a p-lift of $f \circ \gamma$ with $\widetilde{\gamma}(0) = y_0$ and if $\widetilde{\overline{\eta}}\colon [0,1] \longrightarrow Y$ is a p-lift of $f \circ \overline{\eta}$ with $\widetilde{\overline{\eta}}(0) = \widetilde{\gamma}(1)$, then $\widetilde{w} := \widetilde{\gamma} * \widetilde{\overline{\eta}}\colon [0,1] \longrightarrow Y$ is a p-lift of $f \circ (\gamma * \overline{\eta}) = f \circ w$ with $\widetilde{w}(0) = \widetilde{\gamma}(0) = y_0$. Hence,

  $$y_0 = y_0 \cdot [f \circ w^\circ]_* = \widetilde{w}(1) = \widetilde{\overline{\eta}}(1).$$

  But this means that $\overline{\widetilde{\overline{\eta}}}\colon [0,1] \longrightarrow Y$ is the p-lift of $f \circ \eta = \overline{f \circ \overline{\eta}}$ that starts at y_0. Because of

  $$\overline{\widetilde{\overline{\eta}}}(1) = \widetilde{\overline{\eta}}(0) = \widetilde{\gamma}(1),$$

  we thus obtain that the value of $\widetilde{f}(z)$ is independent of the choice of γ.

- Considering the constant path at z_0 (and its constant lift) shows that $\widetilde{f}(z_0) = y_0$.

- It remains to show that $\widetilde{f}\colon Z \longrightarrow Y$ is continuous; here, we will use that Z is locally path-connected: Let $z \in Z$ and let $\widetilde{U} \subset Y$ be an open neighbourhood of $\widetilde{f}(z)$. It suffices to show that z has an open neighbourhood that is contained in $\widetilde{f}^{-1}(\widetilde{U})$. Shrinking $\widetilde{U}$ if necessary, we may assume that the restriction $p|_{\widetilde{U}}\colon \widetilde{U} \longrightarrow p(\widetilde{U}) =: U$ is a homeomorphism and that U is open in X (this is possible because p is a covering map).

  Let $V := f^{-1}(U)$, which is an open neighbourhood of z in Z. Because Z is locally path-connected, there exists an open neighbourhood $W \subset V \subset Z$ of z that is path-connected.

  Let us establish that $\widetilde{f}(W) \subset \widetilde{U}$: To this end, let $w \in W$ and $\gamma_w\colon [0,1] \longrightarrow W$ be a continuous path from z to w. Then $f \circ \gamma_w$ is a continuous path from $f(z)$ to $f(w)$ that is contained in $f(W) \subset f(V) \subset U$.

Let $\gamma\colon [0,1] \longrightarrow X$ be a continuous path from z_0 to z and let $\widetilde{\gamma}$ be the p-lift of $f \circ \gamma$ with $\widetilde{\gamma}(0) = y_0$. Then $\eta := \gamma * \gamma_w \colon [0,1] \longrightarrow Z$ is a continuous path in Z from z_0 to w and

$$\widetilde{\eta} := \widetilde{\gamma} * (p|_{\widetilde{U}}^{-1} \circ f \circ \gamma_w) \colon [0,1] \longrightarrow Y$$

is the p-lift of η with

$$\widetilde{\eta}(0) = \widetilde{\gamma}(0) = y_0.$$

Therefore,

$$\widetilde{f}(w) = \widetilde{\eta}(1) = p|_{\widetilde{U}}^{-1} \circ f \circ \gamma_w(1) \in \widetilde{U},$$

as claimed. □

Corollary 2.3.24 (lifts from simply connected domains)**.** *Let $p\colon (Y, y_0) \longrightarrow (X, x_0)$ be a pointed covering map, let Z be locally path-connected and simply connected, let $z_0 \in Z$, and let $f\colon (Z, z_0) \longrightarrow (X, x_0)$ be a continuous map. Then f has exactly one p-lift $(Z, z_0) \longrightarrow (Y, y_0)$.*

Proof. This is a direct consequence of the lifting criterion (Theorem 2.3.20) and the uniqueness of lifts (Corollary 2.3.18). □

Higher-dimensional spheres are simply connected. Hence, we can use the previous corollary to compute higher homotopy groups of covering spaces:

Corollary 2.3.25 (π_*(covering maps))**.** *Let $p\colon (Y, y_0) \longrightarrow (X, x_0)$ be a pointed covering map. Then the following holds:*

1. *The induced homomorphism $\pi_1(p)\colon \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$ is injective.*

2. *For every $n \in \mathbb{N}_{\geq 2}$, the map $\pi_n(p)\colon \pi_n(Y, y_0) \longrightarrow \pi_n(X, x_0)$ is a group isomorphism.*

Proof. Before starting with the actual proof, let us recall that the n-dimensional sphere S^n is

- locally path-connected for all $n \in \mathbb{N}$,
- path-connected for all $n \in \mathbb{N}_{\geq 1}$,
- simply connected for all $n \in \mathbb{N}_{\geq 2}$ (Example 2.2.11).

Injectivity. Let $n \in \mathbb{N}_{\geq 1}$ and let $\gamma, \eta \in \operatorname{map}_*((S^n, e_1), (Y, y_0))$ with $\pi_n(p)[\gamma]_* = \pi_n(p)[\eta]_*$. Let $h\colon (S^n, e_1) \times [0,1] \longrightarrow (X, x_0)$ be a pointed homotopy from $p \circ \gamma$ to $p \circ \eta$.

We lift h to Y via the lifting criterion (Theorem 2.3.20): Because the inclusion $(S^n \times \{0\}, (e_1, 0)) \hookrightarrow (S^n \times [0,1], (e_1, 0))$ is a pointed homotopy equivalence (check!), we obtain

$$\begin{aligned}
&\pi_1(h)\big(\pi_1(S^n \times [0,1], (e_1,0))\big) \\
&= \pi_1(h)\big(\operatorname{im} \pi_1\big((S^n \times \{0\}, (e_1,0)) \hookrightarrow (S^n \times [0,1], (e_1,0))\big)\big) \\
&= \pi_1(p \circ \gamma)\big(\pi_1(S^n, e_1)\big) \\
&\subset \pi_1(p)\big(\pi_1(Y, y_0)\big).
\end{aligned}$$

Also, $h(e_1, \,\cdot\,) = x_0$. Therefore, the lifting criterion is applicable and implies the existence of a p-lift $\widetilde{h}\colon S^n \times [0,1] \longrightarrow Y$ of h with $\widetilde{h}(e_1, 0) = y_0$. Moreover, the uniqueness of lifts shows that

$$\widetilde{h}(\,\cdot\,, 0) = \gamma, \qquad \widetilde{h}(e_1, \,\cdot\,) = y_0, \qquad \widetilde{h}(\,\cdot\,, 1) = \eta.$$

Therefore, $\widetilde{h}$ is a pointed homotopy from γ to η, which shows $[\gamma]_* = [\eta]_*$ in $\pi_n(Y, y_0)$. Thus, $\pi_n(p)$ is injective.

Surjectivity. For $n \in \mathbb{N}_{\geq 2}$, the sphere S^n is simply connected. Hence, every continuous map $(S^n, e_1) \longrightarrow (X, x_0)$ has a p-lift $(S^n, e_1) \longrightarrow (Y, y_0)$ (Corollary 2.3.24). So $\pi_n(p)\colon \pi_n(Y, y_0) \longrightarrow \pi_n(X, x_0)$ is surjective. ☐

Example 2.3.26 (higher homotopy groups of the circle)**.** The covering $\mathbb{R} \longrightarrow S^1$ from Example 2.3.3 and the fact that $(\mathbb{R}, 0)$ is pointedly contractible (Example 1.3.10) show via Corollary 2.3.25 that the group

$$\pi_n(S^1, e_1) \cong_{\mathsf{Ab}} \pi_n(\mathbb{R}, 0)$$

is trivial for all $n \in \mathbb{N}_{\geq 2}$. The same argument also shows that all higher homotopy groups of higher-dimensional tori are trivial.

Caveat 2.3.27. The corresponding result for higher-dimensional spheres is wrong! While it is true that

$$\pi_n(S^n, e_1) \cong \mathbb{Z} \qquad \text{und} \qquad \forall_{k \in \{0,\dots,n-1\}} \;\; \pi_k(S^n, e_1) \cong \{0\}$$

holds for all $n \in \mathbb{N}_{\geq 2}$ (Example 4.5.11), the higher homotopy groups of spheres are non-trivial in general (and rather mysterious); for example $\pi_3(S^2, e_1) \cong_{\mathsf{Ab}} \mathbb{Z}$ [26, Example 4.51].

Outlook 2.3.28 (fibrations)**.** A homotopy-theoretic generalisation of coverings and locally trivial fibre bundles are so-called fibrations; these are defined in terms of suitable homotopy lifting properties. Fibrations allow to decompose spaces along maps. In contrast with glueings of spaces, fibrations are compatible with higher homotopy groups. More precisely, if $(F, f_0) \xrightarrow{i} (E, e_0) \xrightarrow{p} (B, b_0)$ is a fibration, then there is an associated long exact sequence (the *Puppe sequence*)

$$\cdots \longrightarrow \pi_n(F, f_0) \xrightarrow{\pi_n(i)} \pi_n(E, e_0) \xrightarrow{\pi_n(p)} \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, f_0) \longrightarrow \cdots$$

(where on π_0 we only have exactness in pointed sets and on the higher terms we have exactness in Group) [68, Chapter 4.7].

Such sequences can be used to compute the fundamental group of SO(3) as $\pi_1(\mathrm{SO}(3), I_3) \cong_{\mathsf{Group}} \mathbb{Z}/2$, which is related to the so-called belt trick [62].

2.3.3 The Universal Covering

A first step towards understanding the category $\mathsf{Cov}_{(X,x_0)}$ is to think about initial objects in this category. For reasonable topological spaces, being initial can be expressed in terms of the fundamental group; it is customary to use this π_1-description as the defining notion, leading to universal coverings. We show:

- Universal coverings do exist, provided that mild conditions on the topological space are satisfied;
- the deck transformation group of the universal covering is isomorphic to the fundamental groups.

The classification of coverings will be completed in Chapter 2.3.5.

Definition 2.3.29 (universal covering)**.** A covering $\widetilde{X} \longrightarrow X$ of a topological space X is a *universal covering of* X if $\widetilde{X}$ is (non-empty and) simply connected.

Proposition 2.3.30 (universal coverings are initial)**.** *Let X be a locally path-connected topological space that admits a universal covering $p\colon \widetilde{X} \longrightarrow X$. Let $x_0 \in X$ and let $\widetilde{x}_0 \in p^{-1}(x_0)$.*

1. *Then the pointed covering $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ is an initial object in the category $\mathsf{Cov}_{(X,x_0)}$, i.e., for every pointed covering $q\colon (Y, y_0) \longrightarrow (X, x_0)$, there exists exactly one morphism from p to q in $\mathsf{Cov}_{(X,x_0)}$.*
2. *In particular: All pointed universal coverings of (X, x_0) are canonically isomorphic in $\mathsf{Cov}_{(X,x_0)}$.*

Proof. Ad 1. Because p (as a covering map) is a local homeomorphism and being locally path-connected is a local property, also $\widetilde{X}$ is locally path-connected. Moreover, $\widetilde{X}$ is simply connected. Therefore, the lifting criterion (Corollary 2.3.24) shows that there exists exactly one pointed q-lift $(\widetilde{X}, \widetilde{x}_0) \longrightarrow (Y, y_0)$ of $p = p \circ \mathrm{id}_X$.

Ad 2. This follows directly from the standard uniqueness argument for universal properties (Proposition IV.1.4.6). $\square$

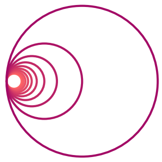


Figure 2.21.: The Hawaiian earring

Example 2.3.31 (universal coverings of tori/projective spaces)**.** Let $n \in \mathbb{N}_{>0}$.

- The projection
$$\begin{aligned} \mathbb{R}^n &\longrightarrow (S^1)^n \\ x &\longmapsto \bigl([x_1 \mod 1], \dots, [x_n \mod 1]\bigr) \end{aligned}$$
is a covering of the n-dimensional torus (Example 2.3.8). Because $\mathbb{R}^n$ is simply connected, this is a/"the" universal covering of the n-dimensional torus.

- The projection
$$\begin{aligned} S^n &\longrightarrow \mathbb{R}P^n \\ x &\longmapsto \{x, -x\} \end{aligned}$$
is a covering of the n-dimensional projective space $\mathbb{R}P^n$ (Example 2.3.9). If $n \geq 2$, then S^n is simply connected; hence, for $n \geq 2$ this is a/"the" universal covering of $\mathbb{R}P^n$.

Caveat 2.3.32 (non-existence of universal coverings)**.** There exist non-empty, path-connected, locally path-connected topological spaces that do *not* admit a universal covering. An example of such a space is the *Hawaiian earring* (Figure 2.21, Exercise).

Theorem 2.3.33 (existence of universal coverings)**.** *Every path-connected, locally path-connected, semi-locally simply connected, non-empty topological space admits a universal covering.*

Definition 2.3.34 (semi-locally simply connected)**.** A topological space X is *semi-locally simply connected* if the following holds: For every point $x \in X$, there exists an open neighbourhood $U \subset X$ of x with the property that the homomorphism
$$\pi_1(U, x) \longrightarrow \pi_1(X, x)$$
induced by the inclusion is trivial (i.e., every pointed loop in (U, x) is pointedly null-homotopic in (X, x)).

Example 2.3.35 (semi-locally simply connected spaces)**.**

- Every simply connected space is semi-locally simply connected.
- Manifolds are semi-locally simply connected (because every point has an open neighbourhood that is homeomorphic to an open ball in a Euclidean space).
- The Hawaiian earring (Figure 2.21) is *not* semi-locally simply connected (check!).

In order to prove Theorem 2.3.33, we reduce everything to our knowledge on (lifts of) paths in coverings. More specifically: In Galois theory, when constructing algebraic/separable closures of fields, we adjoin enough solutions of polynomial equations. Analogously, to construct the universal covering, we will "adjoin" all possible, essentially different, lifts of paths. This is implemented by taking a suitable quotient of the space of all paths with a given start point.

Proof of Theorem 2.3.33. Let X be a path-connected, locally path-connected, semi-locally simply connected topological space and let $x_0 \in X$. We consider the following construction: Let

$$\widetilde{X} := \operatorname{map}_*\bigl(([0,1],0),(X,x_0)\bigr) \;/\sim,$$

where two pointed paths $\gamma, \eta \in \operatorname{map}_*(([0,1],0),(X,x_0))$ satisfy $\gamma \sim \eta$ if and only if $\gamma(1) = \eta(1)$ and the (loop associated with the) closed path $\gamma * \overline{\eta}$ represents the neutral element in $\pi_1(X,x_0)$. We equip $\widetilde{X}$ with the quotient topology of the subspace topology of the compact-open topology on $\operatorname{map}([0,1],X)$ (Remark 1.3.2).

We proceed in the following steps:

① The space $\widetilde{X}$ is path-connected.

② The evaluation map

$$\begin{aligned} p\colon \widetilde{X} &\longrightarrow X \\ [\gamma]_\sim &\longmapsto \gamma(1) \end{aligned}$$

is a covering.

③ The space $\widetilde{X}$ is simply connected.

Ad ①. Let $\gamma, \eta \in \operatorname{map}_*(([0,1],0),(X,x_0))$. Then the curried "homotopy"

$$\begin{aligned} [0,1] &\longrightarrow \widetilde{X} \\ t &\longmapsto \begin{cases} \bigl[s \mapsto \gamma((1-2\cdot t)\cdot s)\bigr]_\sim & \text{if } t \in [0,1/2] \\ \bigl[s \mapsto \eta((2\cdot t-1)\cdot s)\bigr]_\sim & \text{if } t \in [1/2,1] \end{cases} \end{aligned}$$

(that first shrinks γ to the constant path and then extends the constant path to η) is a continuous path in $\widetilde{X}$ from $[\gamma]_\sim$ to $[\eta]_\sim$ (check!). Hence, $\widetilde{X}$ is path-connected.

Ad ②. One can prove this by hand or one can make use of a suitable group action: We consider the well-defined (check!) continuous (check! here, one can use that X is locally path-connected) group action (check!)

$$\begin{aligned} \pi_1(X,x_0)\times\widetilde{X} &\longrightarrow \widetilde{X} \\ \bigl([\gamma]_*,[\eta]_\sim\bigr) &\longmapsto \bigl[(t\mapsto\gamma([t]))*\eta\bigr]_\sim \end{aligned}$$

of $\pi_1(X,x_0)$ on $\widetilde{X}$. Moreover, one can show that this action is properly discontinuous (check! here, one can use that X is semi-locally simply connected) and that p induces a homeomorphism (check! in order to construct an inverse, one needs that X is path-connected)

$$\begin{aligned} \pi_1(X,x_0)\setminus\widetilde{X} &\longrightarrow X \\ \pi_1(X,x_0)\cdot[\gamma]_\sim &\longmapsto \gamma(1). \end{aligned}$$

Hence, Proposition 2.3.7 shows that p is a covering map.

Ad ③. One can prove this by hand or one can make use of the action of $\pi_1(X,x_0)$ on the fibre $p^{-1}(x_0)$: Let $\widetilde{x}_0\in\widetilde{X}$ be the point represented by the constant path at x_0; then $\widetilde{x}_0\in p^{-1}(x_0)$. In view of ①, it suffices to prove that the group $\pi_1(\widetilde{X},\widetilde{x}_0)$ is trivial. Because the map $\pi_1(p)\colon\pi_1(\widetilde{X},\widetilde{x}_0)\longrightarrow\pi_1(X,x_0)$ is injective (Corollary 2.3.25) and the image $\pi_1(p)(\pi_1(\widetilde{X},\widetilde{x}_0))$ coincides with the stabiliser of $\widetilde{x}_0$ under the action of $\pi_1(X,x_0)$ (Corollary 2.3.15), we only need to show that every non-trivial element of $\pi_1(X,x_0)$ acts non-trivially on $\widetilde{x}_0$.

Let $[\gamma]_*\in\pi_1(X,x_0)$ be a non-trivial element. In order to compute $\widetilde{x}_0\cdot[\gamma]_*$, we first have to lift the path associated with γ to $\widetilde{X}$. Such a lift is

$$\begin{aligned} \widetilde{\gamma}\colon [0,1] &\longrightarrow \widetilde{X} \\ t &\longmapsto \bigl[s\mapsto\gamma([t\cdot s])\bigr]_\sim \end{aligned}$$

(check!). Therefore, we obtain

$$\begin{aligned} \widetilde{x}_0\cdot[\gamma]_* &= \widetilde{\gamma}(1) \\ &= \bigl[s\mapsto\gamma([s])\bigr]_\sim. \end{aligned}$$

Because $[\gamma]_*$ is non-trivial in $\pi_1(X,x_0)$, the definition of the relation "$\sim$" shows that $\widetilde{x}_0\not\sim(s\mapsto\gamma([s]))$. Hence, $\widetilde{x}_0\cdot[\gamma]_*\neq\widetilde{x}_0$. This completes the proof that $\widetilde{X}$ is simply connected. □

Caveat 2.3.36. The assumptions on the space in Theorem 2.3.33 are sufficient, but not necessary: For example, the Warsaw circle is *not* locally path-connected, but it is simply connected (Exercise) and thus its own universal covering space.

Study note (the path-connectedness zoo). Complete the following table: In covering theory, the condition of being

path-connected	is used for	...
locally path-connected	is used for	...
semi-locally simply connected	is used for	...

The universal covering allows us to give an interpretation of the fundamental group as an automorphism group:

Theorem 2.3.37 (fundamental group as automorphism group). *Let X be a path-connected, locally path-connected topological space that admits a universal covering $p\colon \widetilde{X} \longrightarrow X$. Let $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. For $g \in \pi_1(X, x_0)$, we write $f_g\colon \widetilde{X} \longrightarrow \widetilde{X}$ for the unique p-lift of p with $f_g(\widetilde{x}_0) = \widetilde{x}_0 \cdot g$.*

1. *Then*

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \operatorname{Deck}(p) \\ g &\longmapsto f_g \end{aligned}$$

is a well-defined group isomorphism.

2. *In particular, the fundamental group $\pi_1(X, x_0)$ acts properly discontinuously on $\widetilde{X}$ (from the left) and $p\colon \widetilde{X} \longrightarrow X$ induces a homeomorphism*

$$\pi_1(X, x_0) \setminus \widetilde{X} \cong_{\mathsf{Top}} X.$$

Caveat 2.3.38 (action on the fibre vs. deck transformation action). In the situation of Theorem 2.3.37, the left action of $\pi_1(X, x_0)$ on $\widetilde{X}$ via deck transformations given by

$$\begin{aligned} \pi_1(X, x_0) \times \widetilde{X} &\longrightarrow \widetilde{X} \\ (g, x) &\longmapsto f_g(x), \end{aligned}$$

in general, coincides only on $\widetilde{x}_0$ with the action of $\pi_1(X, x_0)$ on the fibres of p from Corollary 2.3.15.

Proof of Theorem 2.3.37. Ad 1. Because $\widetilde{X}$ is simply connected and because X (whence $\widetilde{X}$) is locally path-connected, the lifting criterion shows that indeed for every $g \in \pi_1(X, x_0)$ there exists a unique p-lift $f_g\colon \widetilde{X} \longrightarrow \widetilde{X}$ of p with $f_g(\widetilde{x}_0) = \widetilde{x}_0 \cdot g$ (Corollary 2.3.24). Thus, we obtain a *well-defined map*

$$\begin{aligned} \varphi\colon \pi_1(X, x_0) &\longrightarrow \operatorname{Mor}_{\mathsf{Cov}_X}(p, p) \\ g &\longmapsto f_g. \end{aligned}$$

As next step, we show that φ is *compatible with composition*: For all $g, h \in \pi_1(X, x_0)$ we have

$$\varphi(h \cdot g) = \varphi(h) \circ \varphi(g)$$

because: Let $g, h \in \pi_1(X, x_0)$. In view of the uniqueness of lifts of maps with path-connected domain spaces (Corollary 2.3.18), it suffices to show that $f_{h \cdot g}(\widetilde{x}_0) = f_h \circ f_g(\widetilde{x}_0)$.

Let $\gamma \colon (S^1, e_1) \longrightarrow (X, x_0)$ represent g and let $\widetilde{\gamma} \colon [0,1] \longrightarrow \widetilde{X}$ be the unique p-lift of $\gamma' := (t \mapsto \gamma([t]))$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $f_h \circ \widetilde{\gamma}$ is the p-lift of γ' starting in

$$f_h \circ \widetilde{\gamma}(0) = f_h(\widetilde{x}_0) = \widetilde{x}_0 \cdot h.$$

Therefore, we obtain

$$\begin{aligned}
f_{h \cdot g}(\widetilde{x}_0) &= \widetilde{x}_0 \cdot (h \cdot g) && \text{(definition of } f_{h \cdot g}\text{)} \\
&= (\widetilde{x}_0 \cdot h) \cdot g && \text{(right } \pi_1\text{-action on the fibre)} \\
&= f_h \circ \widetilde{\gamma}(1) && \text{(definition of the action on the fibre)} \\
&= f_h(\widetilde{x}_0 \cdot g) && \text{(definition of the action on the fibre)} \\
&= f_h \circ f_g(\widetilde{x}_0), && \text{(definition of } f_g\text{)}
\end{aligned}$$

as desired.

In particular, for all $g \in \pi_1(X, x_0)$, we have

$$\varphi(g) \circ \varphi(g^{-1}) = \varphi(e) = \mathrm{id}_{\widetilde{X}} \quad \text{and} \quad \varphi(g^{-1}) \circ \varphi(g) = \mathrm{id}_{\widetilde{X}},$$

and so $\varphi(g) \in \operatorname{Deck}(p)$. The compatibility with the composition maps thus shows that we can view φ as a *group homomorphism* $\pi_1(X, x_0) \longrightarrow \operatorname{Deck}(p)$.

Moreover, φ is *bijective*, because: It suffices to show that both the $\pi_1(X, x_0)$-action and the $\operatorname{Deck}(p)$-action on $p^{-1}(x_0)$ are free and transitive (check!).

- The $\pi_1(X, x_0)$-action on $p^{-1}(x_0)$ is free and transitive, because $\widetilde{X}$ is simply connected (Corollary 2.3.15).
- The $\operatorname{Deck}(p)$-action on $p^{-1}(x_0)$ is free and transitive by the lifting properties for simply connected domain spaces (Corollary 2.3.24).

Hence, φ is bijective (and so a group isomorphism).

Ad 2. In view of the first part, it suffices to prove the corresponding claim for $\operatorname{Deck}(p)$. The lifting properties show that the deck transformation action

$$\begin{aligned}
\operatorname{Deck}(p) \times \widetilde{X} &\longrightarrow \widetilde{X} \\
(f, x) &\longmapsto f(x)
\end{aligned}$$

is properly discontinuous (check!). In particular, the projection map $q \colon \widetilde{X} \longrightarrow \operatorname{Deck}(p) \setminus \widetilde{X}$ is a covering map (Proposition 2.3.7).

Because deck transformations of p are compatible with p, we obtain a well-defined continuous map

$$\begin{aligned} \overline{p}\colon \operatorname{Deck}(p) \setminus \widetilde{X} &\longrightarrow X \\ [x] &\longmapsto p(x). \end{aligned}$$

Then the map $\overline{p}$ is

- open, because $\overline{p} \circ q = p$ and the covering map p is open;
- surjective, because p is surjective (as $\widetilde{X}$ is non-empty);
- injective, because the deck transformation action is transitive on the fibres of p.

Therefore, $\overline{p}$ is a homeomorphism. $\square$

Study note. In the proofs of Theorem 2.3.33 and 2.3.37, many details are not spelled out. It is highly recommended to practice your covering theory skills by filling in these details!

Conversely, group actions allow us to describe the deck transformation group:

Corollary 2.3.39 (deck transformations via quotients)**.** *Let $G \curvearrowright Y$ be a properly discontinuous action of the group G on a path-connected and locally path-connected non-empty topological space Y, and let $p\colon Y \longrightarrow G \setminus Y$ be the associated covering (Proposition 2.3.7).*

1. *Then*

$$\begin{aligned} G &\longrightarrow \operatorname{Deck}(p) \\ g &\longmapsto (x \mapsto g \cdot x) \end{aligned}$$

is a group isomorphism.

2. *If Y is simply connected, and $x_0 \in G \setminus Y$, then*

$$G \cong_{\mathsf{Group}} \operatorname{Deck}(p) \cong_{\mathsf{Group}} \pi_1(G \setminus Y, x_0).$$

More explicitly: If $y_0 \in p^{-1}(x_0)$, then

$$\begin{aligned} G &\longrightarrow \pi_1(G \setminus Y, x_0) \\ g &\longmapsto \big[[t] \mapsto p \circ \gamma(t)\big]_* \\ &\qquad \textit{where } \gamma\colon [0,1] \to Y \textit{ is a path from } y_0 \textit{ to } g \cdot y_0 \end{aligned}$$

is a well-defined group isomorphism.

Proof. Ad 1. Clearly, the given map is a well-defined group homomorphism. The uniqueness of lifts (Corollary 2.3.18, Theorem 2.3.20) and the definition of the orbit space show that the map is bijective (check!), whence a group isomorphism.

Ad 2. This follows from the first part and Theorem 2.3.37. $\square$

2.3.4 The Fundamental Example

Finally, we have the tools available to comfortably compute the fundamental group of the circle and of real projective spaces.

Theorem 2.3.40 (fundamental group of the circle). *The following map is a group isomorphism:*

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[\gamma_d := ([t] \mapsto [d \cdot t \mod 1])\bigr]_* \end{aligned}$$

Proof. Applying Corollary 2.3.39 to the translation action of $\mathbb{Z}$ on $\mathbb{R}$ (Example 2.3.8) shows that $\mathbb{Z} \cong_{\mathsf{Group}} \pi_1(S^1, e_1)$.

That the map in the theorem is an isomorphism can be shown via the explicit version of this isomorphism in Corollary 2.3.39 and the lifts of the γ_d from Example 2.3.17. □

Example 2.3.41 (fundamental group of real projective spaces). Let $n \in \mathbb{N}_{\geq 2}$. Using the antipodal action of $\mathbb{Z}/2$ on the simply connected sphere S^n, we obtain from Corollary 2.3.39 and Example 2.3.9 that

$$\pi_1\bigl(\mathbb{R}P^n, [e_1]\bigr) \cong_{\mathsf{Group}} \mathbb{Z}/2.$$

2.3.5 The Classification Theorem

We complete the classification of coverings: This is done in terms of the fundamental group of the base space; using the interpretation of the fundamental group as deck transformation group, the classification theorem is similar to the fundamental theorem of Galois theory (Satz III.3.4.23). More precisely, coverings of (X, x_0) will be classified by subgroups of $\pi_1(X, x_0)$.

In order to formulate the theorem, we introduce some notation:

Notation 2.3.42.

- If (X, x_0) is a pointed toplogical space, then $\mathsf{Cov}^\circ_{(X,x_0)}$ denotes the full subcategory of $\mathsf{Cov}_{(X,x_0)}$, whose objects are pointed coverings with *path-connected* total space. More explicitly, the category $\mathsf{Cov}^\circ_{(X,x_0)}$ consists of:
  - objects: the class of all pointed coverings of (X, x_0) with path-connected total space.
  - morphisms: If $p\colon (Y, y_0) \longrightarrow (X, x_0)$ and $p'\colon (Y', y_0') \longrightarrow (X, x_0)$ are pointed coverings of (X, x_0) with path-connected total space, then we set

$$\mathrm{Mor}_{\mathsf{Cov}^\circ_{(X,x_0)}}(p',p) := \bigl\{f \in \mathrm{map}_*((Y',y_0'),(Y,y_0)) \bigm| (f,\mathrm{id}_X) \in \mathrm{Mor}_{\mathsf{Cov}}(p',p)\bigr\}.$$

– compositions: composition is the ordinary composition of maps.

- Let G be a group. Then $\mathsf{Subgroup}_G$ denotes the category of all subgroups of G, where the morphisms are the inclusions.
- A functor $F\colon C \longrightarrow D$ between categories C and D is a *natural equivalence* if there exists a functor $G\colon D \longrightarrow C$ such that $G \circ F$ is naturally isomorphic to id_C and $F \circ G$ is naturally isomorphic to id_D (see Definition 1.2.20 for the notion of natural isomorphism of functors).
- Let G be a group and let $H \subset G$ be a subgroup. Then the *normaliser of H in G* is the subgroup

  $$N_G(H) := \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

  of G. In other words, $N_G(H)$ is the (with respect to inclusion) largest subgroup of G in which H is normal. The quotient $N_G(H)/H$ is the *Weyl group of H in G*.
- A covering is *regular* (or *normal*, or *Galois*) if the deck transformation group acts transitively on the fibres.

Study note. Where was the normaliser used in the Algebra course? How does this definition of regular coverings relate to the notion of Galois field extensions?

Theorem 2.3.43 (classification of coverings)**.** *Let X be a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space, and let $p\colon \widetilde{X} \longrightarrow X$ be the universal covering of X. Let $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$.*

In the following, we consider the deck transformation action of $\pi_1(X,x_0)$ on $\widetilde{X}$ with respect to this basepoint $\widetilde{x}_0$ (Theorem 2.3.37).

1. *Then*

$$\begin{aligned}
\varphi\colon\ \mathsf{Cov}^\circ_{(X,x_0)} &\longleftrightarrow \mathsf{Subgroup}_{\pi_1(X,x_0)} : \psi\\
\bigl(p\colon (Y,y_0) \to (X,x_0)\bigr) &\longmapsto \pi_1(p)\bigl(\pi_1(Y,y_0)\bigr)\\
\mathrm{Mor}_{\mathsf{Cov}^\circ_{(X,x_0)}}(q',q) \ni f &\longmapsto \bigl(\mathrm{im}(\pi_1(q')) \subset \mathrm{im}(\pi_1(q))\bigr)\\
\bigl(q_H\colon (H \setminus \widetilde{X}, H \cdot \widetilde{x}_0) \to (X,x_0)\bigr) &\longleftarrow H\\
(\textit{projection}\colon H' \setminus \widetilde{X} \to H \setminus \widetilde{X}) &\longleftarrow (H' \subset H)
\end{aligned}$$

are mutually "inverse" natural equivalences of categories.

2. *Moreover: If $q\colon (Y,y_0) \longrightarrow (X,x_0)$ is a covering in $\mathsf{Cov}^\circ_{(X,x_0)}$ and $H := \pi_1(q)(\pi_1(Y,y_0))$, then:*

a) The covering q is $[\pi_1(X, x_0) : H]$-sheeted.

b) We have $\operatorname{Deck}(q) \cong_{\mathsf{Group}} N_{\pi_1(X,x_0)}(H)/H$.

c) The covering q is regular if and only if H is normal in $\pi_1(X, x_0)$.

Proof. In the following, we will abbreviate $G := \pi_1(X, x_0)$. If $H \subset G$, then we define

$$\begin{aligned} q_H \colon H \setminus \widetilde{X} &\longrightarrow X \\ H \cdot x &\longmapsto p(x). \end{aligned}$$

Because $p \colon \widetilde{X} \longrightarrow X$ and the projection $p_H \colon \widetilde{X} \longrightarrow H \setminus \widetilde{X}$ are coverings and $q_H \circ p_H = p$, it follows that also q_H is a covering map (check!). Because $\widetilde{X}$ is path-connected, also the total space $H \setminus \widetilde{X}$ is path-connected; moreover, $q_H(H \cdot \widetilde{x}_0) = x_0$. Hence, q_H is an object in $\mathsf{Cov}^\circ_{(X,x_0)}$.

Ad 1.

① Clearly, φ is a functor (because π_1 is a functor).

② Also, ψ is a functor, as can be seen from a straightforward computation (check!).

③ We have $\varphi \circ \psi = \operatorname{Id}_{\mathsf{Subgroup}_G}$, because: Let $H \subset G$ be a subgroup. Then $\varphi \circ \psi(H) = \pi_1(q_H)(\pi_1(H \setminus \widetilde{X}, H \cdot \widetilde{x}_0))$ is the stabiliser of $H \cdot \widetilde{x}_0$ with respect to the $\pi_1(X, x_0)$-action on the fibre $q_H^{-1}(x_0)$ (Corollary 2.3.15).

We connect this to the $\pi_1(X, x_0)$-action on the fibre $p^{-1}(x_0)$: Let $\gamma \in \operatorname{map}_*((S^1, e_1), (X, x_0))$ and let $\widetilde{\gamma} \colon [0, 1] \longrightarrow \widetilde{X}$ be the p-lift of $t \mapsto \gamma([t])$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $p_H \circ \widetilde{\gamma}$ is the q_H-lift that starts at $H \cdot \widetilde{x}_0$ and so $[\gamma]_*$ lies in the stabiliser group $\pi_1(q_H)(\pi_1(H \setminus \widetilde{X}, H \cdot \widetilde{x}_0))$ if and only if

$$p_H\bigl(\widetilde{\gamma}(0)\bigr) = p_H\bigl(\widetilde{\gamma}(1)\bigr).$$

In view of the construction of the deck transformation action of $\pi_1(X, x_0)$ on $\widetilde{X}$ (starting from the fibre $p^{-1}(x_0)$), this is equivalent to $[\gamma]_* \in H$.

Moreover, $\varphi \circ \psi$ maps inclusions of subgroups to the same inclusions. This shows that $\varphi \circ \psi(H) = H$.

④ Conversely, we have $\psi \circ \varphi \cong \operatorname{Id}_{\mathsf{Cov}^\circ_{(X,x_0)}}$, because: Let $q \colon (Y, y_0) \longrightarrow (X, x_0)$ be an object of $\mathsf{Cov}^\circ_{(X,x_0)}$ and let

$$H := \varphi(q) = \pi_1(q)\bigl(\pi_1(Y, y_0)\bigr) \subset G.$$

Then $q_H = \psi \circ \varphi(q)$ and there is exactly one isomorphism $q \cong_{\mathsf{Cov}^\circ_{(X,x_0)}} q_H$ because: By the lifting criterion (Theorem 2.3.20) and ③ there exists

- exactly one $f \in \operatorname{map}_*((Y, y_0), (H \setminus \widetilde{X}, H \cdot \widetilde{x}_0))$ with $q_H \circ f = q$, and

– exactly one $g \in \operatorname{map}_*((H \setminus \widetilde{X}, H \cdot \widetilde{x}_0), (Y, y_0))$ with $q \circ g = q_H$.

Moreover, uniqueness of lifts (Corollary 2.3.18; applied to $f \circ g$ and $g \circ f$) shows that

$$f \circ g = \mathrm{id}_{H \setminus \widetilde{X}} \quad \text{and} \quad g \circ f = \mathrm{id}_Y .$$

This implies that $\psi \circ \varphi$ is naturally isomorphic to the identity (check!).

Ad 2. In view of the first part, it suffices to consider the standard covering $q_H \colon H \setminus \widetilde{X} \longrightarrow X$ (check!).

① The covering q_H has exactly $[G \colon H]$ sheets, because, by construction, we have

$$\begin{aligned} \left|q_H^{-1}(x_0)\right| &= \left|\{H \cdot g \cdot \widetilde{x}_0 \mid g \in G\}\right| && \text{(by definition of } q_H\text{)} \\ &= \left|\{H \cdot g \mid g \in G\}\right| && \text{(because } G \text{ acts freely on } \widetilde{X}\text{)} \\ &= [G : H] && \text{(by definition of the index).} \end{aligned}$$

② We consider the map

$$\begin{aligned} f \colon N_G(H) &\longrightarrow \operatorname{Deck}(q_H) \\ g &\longmapsto f_g, \end{aligned}$$

where, for $g \in N_G(H)$, we define

$$\begin{aligned} f_g \colon H \setminus \widetilde{X} &\longrightarrow H \setminus \widetilde{X} \\ H \cdot x &\longmapsto H \cdot g \cdot x = g \cdot H \cdot x; \end{aligned}$$

the two descriptions of f_g show that f_g is a well-defined deck transformation (check!) and that f is a group homomorphism (check!). We compute the kernel and the image of f:

– We have $\ker(f) = H$, because: By construction, $H \subset \ker f$. Conversely, let $g \in \ker f$. Then

$$H \cdot g \cdot \widetilde{x}_0 = f_g(H \cdot \widetilde{x}_0) = \mathrm{id}_{H \setminus \widetilde{X}}(H \cdot \widetilde{x}_0) = H \cdot \widetilde{x}_0,$$

and so $H \cdot g = H$ (because G acts freely on $\widetilde{X}$). Therefore, $g \in H$.

– Moreover, f is surjective: Let $d \in \operatorname{Deck}(q_H)$. Because the G-action on the fibre $p^{-1}(x_0)$ is transitive, there exists a $g \in G$ with

$$d(H \cdot \widetilde{x}_0) = H \cdot g \cdot \widetilde{x}_0.$$

We show that $g \in N_G(H)$, i.e., that $H = g^{-1} \cdot H \cdot g$: We have

$$
\begin{aligned}
H &= \pi_1(q_H)\bigl(\pi_1(H \setminus \widetilde{X}, H \cdot \widetilde{x}_0))\bigr) && \text{(proof of the first part)}\\
&= \pi_1(q_H) \circ \pi_1(d)\bigl(\pi_1(H \setminus \widetilde{X}, H \cdot \widetilde{x}_0)\bigr) && \text{(because } q_H = q_H \circ d\text{)}\\
&= \pi_1(q_H)\bigl(\pi_1(H \setminus \widetilde{X}, H \cdot g \cdot \widetilde{x}_0)\bigr) && (d \text{ is a homeomorphism; choice of } g)\\
&= \pi_1(q_{g^{-1}\cdot H\cdot g})\bigl(\pi_1(g^{-1} \cdot H \cdot g \setminus \widetilde{X}, g^{-1} \cdot H \cdot g \cdot \widetilde{x}_0)\bigr) && \text{(induced by } x \mapsto g^{-1} \cdot x\text{)}\\
&= g^{-1} \cdot H \cdot g && \text{(proof of the first part).}
\end{aligned}
$$

Uniqueness of lifts proves that $f_g = d$.

Therefore, f induces an isomorphism $N_G(H)/H \cong_{\mathsf{Group}} \operatorname{Deck}(q_H)$ of groups, as claimed.

③ The description of $\operatorname{Deck}(q_H)$ in ② also shows that $\operatorname{Deck}(q_H)$ acts transitively on $q_H^{-1}(x_0)$ if and only if $N_G(H) = G$. This is equivalent to H being normal in G. □

In summary, path-connected coverings correspond to subgroups of the fundamental group.

Remark 2.3.44 (comparison with Galois theory)**.** Using the dictionary below, we can translate between the classification theorem of coverings (Theorem 2.3.43) and the fundamental theorem of Galois theory (Satz III.3.4.23):

covering theory	**Galois theory**
covering	(separable) field extension
	(or the associated morphism of affine schemes)
universal covering	separable closure (as extension)
number of sheets	degree of the extension
regular covering	Galois extension
deck transformation group	Galois group
fundamental group	absolute Galois group
	(or étale fundamental group)
quotient covering	fixed field

Outlook 2.3.45 (generalisations of the classification of coverings)**.** In the situation of Theorem 2.3.43, one can consider the following generalisations [68, Chapter 3.3–3.6]:

- Generalisation to $\mathsf{Cov}_{(X,x_0)}$: The classification theorem can be extended to pointed coverings of (X, x_0) with not necessarily path-connected total space by replacing the category of subgroups of $\pi_1(X, x_0)$ with the category of $\pi_1(X, x_0)$-actions on sets; then subgroups of $\pi_1(X, x_0)$ will correspond to the translation action of $\pi_1(X, x_0)$ on the coset space of this subgroup.

- Generalisation to Cov_X: Moreover, by paying attention to conjugations, one can adapt the classification theorem to the unpointed case.

All these generalisations can be proved with the tools that we already introduced. However, for the sake of simplicity, we focus on the case in Theorem 2.3.43.

Example 2.3.46 (coverings of simply connected spaces)**.** Simply connected, locally path-connected spaces X admit no non-trivial coverings:

By the classification theorem (Theorem 2.3.43), the (path-connected components of pointed) coverings of such a space are determined up to isomorphism by their subgroup of the fundamental group of the base space X. If $x_0 \in X$, then $\mathrm{id}_X \colon (X, x_0) \longrightarrow (X, x_0)$ is the covering that corresponds to $\pi_1(X, x_0)$. Because $\pi_1(X, x_0)$ is trivial, there is no other subgroup of $\pi_1(X, x_0)$. Hence, every (path-connected component of) a covering of X is isomorphic to a trivial covering of X, whence also itself trivial.

In particular, all simply connected manifolds are orientable (because the orientation double covering is trivial).

Caveat 2.3.47 (simply connected wild spaces with non-trivial coverings)**.** However, there exist simply connected spaces that *do* have non-trivial coverings; for example, the Warsaw helix is a non-trivial covering of the Warsaw circle, which is simply connected (Exercise).

Example 2.3.48 (coverings of the circle)**.** The classification theorem (Theorem 2.3.43) lets us easily classify all (path-connected) coverings of (S^1, e_1): We know that $\pi_1(S^1, e_1) \cong_{\mathsf{Group}} \mathbb{Z}$ (Theorem 2.3.40) and that every subgroup of $\mathbb{Z}$ is of the form $d \cdot \mathbb{Z}$ for some $d \in \mathbb{N}$. Then, under this isomorphism $\pi_1(S^1, e_1) \cong_{\mathsf{Group}} \mathbb{Z}$, (Figure 2.16)

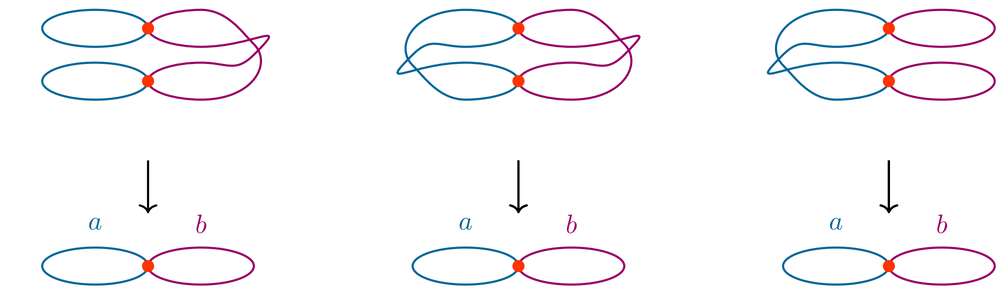
the universal covering $(\mathbb{R}, 0) \longrightarrow (S^1, e_1)$	corresponds to the subgroup $0 \cdot \mathbb{Z} \subset \mathbb{Z}$
the trivial covering $\mathrm{id}_{S^1} \colon (S^1, e_1) \longrightarrow (S^1, e_1)$	corresponds to the subgroup $1 \cdot \mathbb{Z} \subset \mathbb{Z}$
for $d \in \mathbb{N}_{>0}$, the covering $S^1 \longrightarrow S^1, [t] \longmapsto [d \cdot t \mod 1]$	corresponds to the subgroup $d \cdot \mathbb{Z} \subset \mathbb{Z}$

In this example, we can also see that we have to be careful about how subgroups sit in the ambient group: Of course, $1 \cdot \mathbb{Z} \cong_{\mathsf{Group}} 2 \cdot \mathbb{Z}$; however, $1 \cdot \mathbb{Z} \not\cong_{\mathsf{Subgroup}_{\mathbb{Z}}} 2 \cdot \mathbb{Z}$ (and thus these subgroups of $\mathbb{Z}$ lead to different coverings).

Example 2.3.49 (double coverings)**.** Subgroups of index 2 are always normal. Hence, the classification theorem (Theorem 2.3.43) tells us that all (path-connected) double coverings of (sufficiently nice) topological spaces are regular.

Subgroups of index 2 in the free group $F_2 = F(a, b)$ thus correspond to kernels of epimorphisms $F_2 \longrightarrow \mathbb{Z}/2$. Using the universal property of the free generating set $\{a, b\}$ of $F(a, b) = F_2$, we see that there exist precisely the following three group epimorphisms $F_2 \longrightarrow \mathbb{Z}/2$:

$$\begin{aligned} B \colon F_2 &\longrightarrow \mathbb{Z}/2 \\ a &\longmapsto [0] \\ b &\longmapsto [1] \end{aligned}$$



ker B generated by a, bab^{-1}, b^2 | ker AB generated by a^2, ab, b^2 | ker A generated by a^2, aba^{-1}, b

Figure 2.22.: Two-sheeted coverings of $(S^1, e_1) \vee (S^1, e_1)$

$$\begin{aligned} AB\colon F_2 &\longrightarrow \mathbb{Z}/2 \\ a &\longmapsto [1] \\ b &\longmapsto [1] \end{aligned}$$

$$\begin{aligned} A\colon F_2 &\longrightarrow \mathbb{Z}/2 \\ a &\longmapsto [1] \\ b &\longmapsto [0] \end{aligned}$$

We can use this to investigate coverings of $(X, x_0) := (S^1, e_1) \vee (S^1, e_1)$: Because $\pi_1(X, x_0) \cong_{\mathsf{Group}} F(a, b)$ (where the obvious loops in (X, x_0) correspond to a and b; Example 2.2.14) and because X is path-connected, locally path-connected and semi-locally simply connected, we can apply the classification theorem (Theorem 2.3.43) to this situation. We hence obtain that (X, x_0) has (up to isomorphism in $\mathsf{Cov}^\circ_{(X,x_0)}$) exactly three two-sheeted coverings, which are depicted in Figure 2.22. A straightforward calculation on the generators allows us to distinguish which covering must correspond to which kernel (check!).

Moreover, we can use topology to determine generating sets of the kernels of these epimorphisms: As we know generating sets of the fundamental group of the covering spaces, we can use these to compute generating sets of the corresponding images in the fundamental group of (X, x_0). Because we already know from the classification which covering corresponds to which kernel, this gives the desired generating sets (Figure 2.22; check!).

Example 2.3.50 (many coverings)**.** It is a fact from group theory that the free group F_2 of rank 2 has uncountably many different normal subgroups (!) [39, Theorem 2.2.28]. Hence, by the classification theorem (Theorem 2.3.43), there are uncountably many different isomorphism classes of (pointed) regular coverings of $(S^1, e_1) \vee (S^1, e_1)$ with path-connected total space.

Outlook 2.3.51 (gradient invariants of groups and spaces)**.** The correspondence between finite index subgroups and finite coverings leads to a rich interplay between group theory and topology:

- If I is a numerical (isomorphism) invariant of groups, then one can consider the associated gradient invariant $\widehat{I}$, which is defined as follows: If G is a group, then we set
$$\widehat{I}(G) := \inf_{H \in \operatorname{Fin}(G)} \frac{I(H)}{[G : H]},$$
where $\operatorname{Fin}(G)$ denotes the set of all finite index subgroups of G.
- If H is a numerical (homotopy) invariant of spaces, then one can consider the associated gradient invariant $\widehat{H}$, which is defined as follows: If X is a space, then we set
$$\widehat{H}(X) := \inf_{[p\colon Y \to X] \in \operatorname{Fin}(X)} \frac{H(Y)}{[p : X]},$$
where $\operatorname{Fin}(X)$ denotes the set of isomorphism classes of finite coverings of X and $[p : X]$ is the number of sheets of such a covering p.

One can then wonder how gradient invariants of spaces are related to gradient invariants of their fundamental groups and whether such gradient invariants admit an independent global description. Questions of this type are the subject of active research in geometric topology and group theory [44].

2.3.6 Application: The Nielsen–Schreier Theorem

As an application of covering theory, one can prove the Nielsen–Schreier theorem (which is a purely group-theoretic statement):

Theorem 2.3.52 (Nielsen–Schreier theorem)**.** *Subgroups of free groups are free.*

Proof. The basic idea is to apply the classification theorem of coverings (Theorem 2.3.43) the other way around: Starting from groups and subgroups, we construct suitable topological spaces and coverings; we then exploit topological properties to transport properties of the base space to the total space, which finally will tell us something about the subgroup in question.

Let G be a free group, say freely generated by $S \subset G$ and let $H \subset G$ be a subgroup.

In order to translate G into topology, we consider the space

$$(X, x_0) := \bigvee_S (S^1, e_1).$$

Then we have (Example 2.2.21)

$$\pi_1(X, x_0) \cong_{\mathsf{Group}} G;$$

let $f\colon \pi_1(X, x_0) \longrightarrow G$ be such a group isomorphism.

By the classification theorem of coverings (Theorem 2.3.43), there exists a covering $q\colon (Y, y_0) \longrightarrow (X, x_0)$ with

$$\pi_1(q)\bigl(\pi_1(Y, y_0)\bigr) = f^{-1}(H).$$

We now exploit topology: We can view X as a one-dimensional complex (i.e., as a space obtained by glueing points and unit intervals; Exercise). This property is inherited under coverings (Exercise); hence, also Y carries the structure of a one-dimensional complex. Furthermore, fundamental groups of one-dimensional complexes are free (this follows from Theorem 2.2.6 and Proposition 2.2.20; Exercise; or via spanning trees: Example A.7.12). Therefore, $\pi_1(Y, y_0)$ is a free group.

Because $\pi_1(q)\colon \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$ is injective (Corollary 2.3.25), we thus obtain that the group

$$H = f \circ \pi_1(q)\bigl(\pi_1(Y, y_0)\bigr) \cong_{\mathsf{Group}} \pi_1(q)\bigl(\pi_1(Y, y_0)\bigr) \cong_{\mathsf{Group}} \pi_1(Y, y_0)$$

is free. □

A more careful analysis of such coverings allows us to determine the rank of subgroups of free groups in terms of the rank of the ambient free group and the index (Corollary 5.3.13, Example 2.3.49).

Study note. In case you find this proof of the Nielsen–Schreier theorem silly: Try to prove the Nielsen–Schreier theorem by hand, using only elementary group theory ...

Outlook 2.3.53. The above proof of the Nielsen–Schreier theorem is a prototypical example of a topological dimension argument in group theory. Arguments of this type are studied systematically in the context of group (co)homology and classifying spaces [9, 43].

Study note (subobjects of free objects)**.** For which algebraic structures do you already know that sub-thingies of free stuff are free? For which algebraic structures does this *not* hold? What does this have to do with the Möbius strip?

2.4 Applications

Let us collect a list of basic applications of the fundamental group and covering theory:

- Proof of the fundamental theorem of algebra (Exercise).
- Proof of the Brouwer fixed point theorem in dimension 2 (this can be done as in the proof of Corollary 1.3.25; we will use homology to establish the Brouwer theorem in full generality).
- Proof of the Jordan curve theorem (we will use homology to establish the Jordan curve theorem in all dimensions; Theorem 4.4.5)
- Proof of the Nielsen–Schreier theorem (Theorem 2.3.52).
- Proof of the Borsuk–Ulam theorem in dimension 2 (Exercise; higher-dimensional versions can be proved via cohomology rings).
- Topological and geometric classification of compact surfaces [51].
- Translation between geometric and algebraic properties of groups in geometric group theory [39].
- Undecidability results in topology (Outlook 2.2.19).
- ...

Final remark 2.4.1. In this chapter, we met an interesting sequence of Group-valued homotopy invariant functors on Top_*, namely the homotopy groups. In principle, it would be possible to prove a statement similar to the existence of "interesting" homotopy invariant functors (Theorem 1.3.22) via these functors. However, we will now shift focus to homological invariants (which, in many cases, are easier to calculate) and then provide a proof of Theorem 1.3.22 via homology.

3

Axiomatic Homology Theory

We introduce the Eilenberg–Steenrod axioms for homology theories and learn how to compute and work with these axioms. Roughly speaking, homology theories are homotopy invariant functors from pairs of spaces to graded modules that are compatible with glueings of spaces. Hence, we may expect that homology theories are amenable to computations.

While the axioms do not refer to chain complexes or homology of chain complexes, they have a flavour of homological algebra (e.g., they mention exact sequences). Moreover, many constructions of theories satisfying the Eilenberg–Steenrod axioms do involve chain complexes (or related concepts such as spectra).

In Chapter 4 and Chapter 5, we will construct examples of such homology theories.

Overview of this chapter.

Running example. again: spheres, and suspensions

3.1 The Eilenberg–Steenrod Axioms

The Eilenberg–Steenrod axioms concisely list all common properties of homology theories in topology. The advantage of this axiomatic approach is that we learn from the beginning to distinguish between generic properties shared by all homology theories and special properties of specific examples of homology theories.

3.1.1 The Axioms

Definition 3.1.1 (Eilenberg–Steenrod axioms for homology theories)**.** Let R be a ring with unit. A *homology theory on* Top^2 *with values in* ${}_R\mathsf{Mod}$ is a pair $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$, consisting of

- a sequence $(h_k)_{k\in\mathbb{Z}}$ of functors $\mathsf{Top}^2 \longrightarrow {}_R\mathsf{Mod}$ and
- a sequence $(\partial_k)_{k\in\mathbb{Z}}$ of natural transformations $\partial_k \colon h_k \Longrightarrow h_{k-1} \circ U$, the *connecting homomorphisms* (or *boundary operators*), where $U\colon \mathsf{Top}^2 \longrightarrow \mathsf{Top}^2$ is the functor that maps pairs (X, A) to $(A, \emptyset)$ and maps of pairs to the corresponding restrictions,

with the following properties:

- *Homotopy invariance.* For every $k \in \mathbb{Z}$ the functor $h_k\colon \mathsf{Top}^2 \longrightarrow {}_R\mathsf{Mod}$ is homotopy invariant in the sense of Definition 1.3.19.
- *Long exact sequences of pairs.* For every pair (X, A) of spaces, the sequence

$$\cdots \xrightarrow{\partial_{k+1}} h_k(A, \emptyset) \xrightarrow{h_k(i)} h_k(X, \emptyset) \xrightarrow{h_k(j)} h_k(X, A) \xrightarrow{\partial_k} h_{k-1}(A, \emptyset) \xrightarrow{h_{k-1}(i)} \cdots$$

  is exact (Definition A.6.2), where $i\colon (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j\colon (X, \emptyset) \longrightarrow (X, A)$ are the inclusion maps.
- *Excision.* For every pair (X, A) of spaces and all $B \subset A$ with $\overline{B} \subset A^\circ$, the homomorphisms

$$h_k(X \setminus B, A \setminus B) \longrightarrow h_k(X, A)$$

  induced by the inclusion $(X \setminus B, A \setminus B) \longrightarrow (X, A)$ are isomorphisms for every $k \in \mathbb{Z}$ (Figure 3.1).

Let $\bullet := \{\emptyset\}$ be "the" one-point space. One says that $\big(h_k(\bullet, \emptyset)\big)_{k\in\mathbb{Z}}$ are the *coefficients* of the homology theory.

Such a homology theory $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ is an *ordinary homology theory*, if the dimension axiom is satisfied:

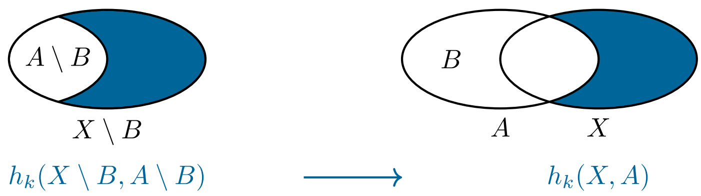


Figure 3.1.: Excision, schematically

- *Dimension axiom.* The one-point space $\bullet$ satisfies for all $k \in \mathbb{Z} \setminus \{0\}$:

$$h_k(\bullet, \emptyset) \cong_R 0.$$

A homology theory $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ is *additive*, if the additivity axiom is satisfied:

- *Additivity.* For all sets I and all families $(X_i)_{i\in I}$ of topological spaces, the canonical inclusions $(X_i \longrightarrow \coprod_{j\in I} X_j)_{i\in I}$ induce for every $k \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{i\in I} h_k(X_i, \emptyset) \longrightarrow h_k\Bigl(\coprod_{i\in I} X_i, \emptyset\Bigr).$$

Remark 3.1.2 (digesting the axioms).

- In the situation of Definition 3.1.1, $h_k(X, A)$, intuitively, measures the "difference" between X and A with respect to h_k.

  As quotient spaces tend to be ill-behaved, it is customary to consider pairs of spaces to measure the "difference" between a space and a subspace.

  The long exact sequence of pairs describes the connection between homology of a space, the given subspace, and the pair formed by these spaces.

- Excision allows to compute homology of (pairs of) spaces by subdivision into smaller parts (divide and conquer!). In contrast with the excision theorem of Blakers–Massey for homotopy groups, excision in homology holds for all degrees (and almost unconditionally). This is the main reason why homology tends to be easier to compute in concrete, geometric, examples than homotopy groups.

- If X is a topological space and $k \in \mathbb{Z}$, then we also abbreviate (in the setting of the Eilenberg–Steenrod axioms)

$$h_k(X) := h_k(X, \emptyset).$$

This is the *absolute homology* of X. Homology of pairs of spaces is also called *relative homology*.

- The Eilenberg–Steenrod axioms also admit versions in other categories of spaces.
- In Chapter 5.2.3, we will see to which extent homology theories are uniquely determined by the Eilenberg–Steenrod axioms and their coefficients.

3.1.2 First Steps

Before turning to glueing results and topological applications of homology theories, we first practice working with the Eilenberg–Steenrod axioms in some simple cases. As the Eilenberg–Steenrod axioms are formulated in terms of exact sequences, we will often need basic facts on exact sequences (Chapter A.6.1).

Setup 3.1.3. In this section, R is a ring with unit and $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Proposition 3.1.4 (more on homotopy invariance)**.** *Let $k \in \mathbb{Z}$.*

1. *If (X, A) is a pair of spaces and if the inclusion $i\colon A \longrightarrow X$ is a homotopy equivalence, then*
$$h_k(X, A) \cong_R 0$$
In particular, $h_k(X, X) \cong_R 0$.
2. *If $f\colon (X, A) \longrightarrow (Y, B)$ is a continuous map of spaces and if both $f\colon X \longrightarrow Y$ and $f|_A\colon A \longrightarrow B$ are homotopy equivalences in* Top*, then*
$$h_k(f)\colon h_k(X, A) \longrightarrow h_k(Y, B)$$
is an isomorphism of R-modules.

Proof. Ad 1. Let $j\colon (X, \emptyset) \longrightarrow (X, A)$ be the inclusion. By the long exact sequence of the pair (X, A), the sequence

$$h_k(A) \xrightarrow{h_k(i)} h_k(X) \xrightarrow{h_k(j)} h_k(X, A) \xrightarrow{\partial_k} h_{k-1}(A) \xrightarrow{h_{k-1}(i)} h_{k-1}(X)$$

in ${}_R\mathsf{Mod}$ is exact. Because h_k and h_{k-1} are homotopy invariant and i is a homotopy equivalence, $h_k(i)$ and $h_{k-1}(i)$ are isomorphisms. We now only need to exploit exactness:

- On the one hand, $\operatorname{im} \partial_k = \ker h_{k-1}(i) = 0$, and so $\partial_k = 0$.

- On the other hand, $\ker h_k(j) = \operatorname{im} h_k(i) = h_k(X)$, and so we also have $h_k(j) = 0$.

Therefore,

$$h_k(X, A) = \ker \partial_k = \operatorname{im} h_k(j) = 0.$$

Ad 2. The second part follows from the long exact sequence of pairs and the five lemma (Proposition A.6.7): We consider the following ladder in ${}_R\mathsf{Mod}$:

$$\begin{tikzcd}
h_k(A) \arrow[r, "h_k(\mathrm{incl})"] \arrow[d, "h_k(f|_A)"'] & h_k(X) \arrow[r, "h_k(\mathrm{incl})"] \arrow[d, "h_k(f)"'] & h_k(X, A) \arrow[r, "\partial_k"] \arrow[d, "h_k(f)"] & h_{k-1}(A) \arrow[r, "h_{k-1}(\mathrm{incl})"] \arrow[d, "h_{k-1}(f|_A)"] & h_{k-1}(X) \arrow[d, "h_{k-1}(f)"] \\
h_k(B) \arrow[r, "h_k(\mathrm{incl})"'] & h_k(Y) \arrow[r, "h_k(\mathrm{incl})"'] & h_k(Y, B) \arrow[r, "\partial_k"'] & h_{k-1}(B) \arrow[r, "h_{k-1}(\mathrm{incl})"'] & h_{k-1}(Y)
\end{tikzcd}$$

The rows are exact (long exact sequences of pairs). The two left squares and the rightmost square are commutative (by functoriality of h_k and h_{k-1}, respectively). The remaining square is commutative, because ∂_k is a natural transformation.

The outer four vertical R-homomorphisms are R-isomorphisms (by homotopy invariance).

Therefore, the five lemma (Proposition A.6.7) shows that also the middle homomorphism is an isomorphism of R-modules. □

Example 3.1.5 (relative homology of thick spheres)**.** Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then the inclusion $(D^{n+1}, S^n) \longrightarrow (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$ induces an isomorphism

$$h_k(D^{n+1}, S^n) \cong_R h_k\bigl(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}\bigr).$$

As next step, we study how we can split off the coefficients from the homology of (non-empty) spaces:

Proposition 3.1.6 (splitting off the coefficients)**.**

1. *Let (X, x_0) be a pointed space. Then the inclusion $(X, \emptyset) \hookrightarrow (X, \{x_0\})$ and the constant map $X \longrightarrow \{x_0\}$ induce for every $k \in \mathbb{Z}$ an isomorphism*
$$h_k(X) \cong_R h_k\bigl(\{x_0\}\bigr) \oplus h_k\bigl(X, \{x_0\}\bigr).$$
2. *These isomorphisms are natural in the following sense: If $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a pointed continuous map and $k \in \mathbb{Z}$, then the diagram*

$$\begin{tikzcd}
h_k(X) \arrow[r, "\cong_R"] \arrow[d, "h_k(f)"'] & h_k\bigl(\{x_0\}\bigr) \oplus h_k\bigl(X, \{x_0\}\bigr) \arrow[d, "h_k(f|_{\{x_0\}}) \oplus h_k(f)"] \\
h_k(Y) \arrow[r, "\cong_R"'] & h_k\bigl(\{f(x_0)\}\bigr) \oplus h_k\bigl(Y, \{f(x_0)\}\bigr)
\end{tikzcd}$$

is commutative, where the horizontal isomorphism are the isomorphisms of the first part.

The second part just says that the isomorphisms from the first part form a natural transformation

$$h_k \circ F \Longrightarrow h_k \circ U \oplus h_k \colon \mathsf{Top}_* \longrightarrow {}_R\mathsf{Mod},$$

where $F\colon \mathsf{Top}_* \longrightarrow \mathsf{Top}^2$ is the functor that replaces the singleton subspace by the empty subspace and U is the subspace functor from Definition 3.1.1.

Proof. Ad 1. Again, this can be shown by investigating the long exact sequence of the pair $(X, \{x_0\})$. Let $i\colon \{x_0\} \longrightarrow X$ and $j\colon (X, \emptyset) \longrightarrow (X, \{x_0\})$ be the inclusions and let $p\colon X \longrightarrow \{x_0\}$ be the constant map. The sequence

$$h_k\bigl(\{x_0\}\bigr) \xrightarrow{h_k(i)} h_k(X) \xrightarrow{h_k(j)} h_k\bigl(X, \{x_0\}\bigr) \xrightarrow{\partial_k} h_{k-1}\bigl(\{x_0\}\bigr) \xrightarrow{h_{k-1}(i)} h_{k-1}(X)$$

of the pair $(X, \{x_0\})$ is exact. Because of $p \circ i = \mathrm{id}_{\{x_0\}}$, we obtain

$$h_k(p) \circ h_k(i) = \mathrm{id}_{h_k(\{x_0\})} \quad \text{and} \quad h_{k-1}(p) \circ h_{k-1}(i) = \mathrm{id}_{h_{k-1}(\{x_0\})}\,.$$

In particular, $h_{k-1}(i)$ is injective, and thus $\partial_k = 0$; and the injective map $h_k(i)$ has a section (namely $h_k(p)$). Therefore, the exact sequence above yields the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & h_k\bigl(\{x_0\}\bigr) \arrow[r, "h_k(i)"] & h_k(X) \arrow[r, "h_k(j)"] \arrow[l, bend left, dashed, "h_k(p)"] & h_k\bigl(X, \{x_0\}\bigr) \arrow[r] & 0 \end{tikzcd}$$

Now basic properties of split exact sequences (Proposition A.6.6) prove the claim.

Ad 2. The naturality follows from the fact that the inclusion of the basepoints and the constant maps are compatible with f and that this property is preserved after applying the functor h_k. $\square$

This splitting can also be expressed in terms of reduced homology:

Remark 3.1.7 (reduced homology)**.** If X is a topological space and $k \in \mathbb{Z}$, we define the *k-th reduced homology of X with respect to* $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ by

$$\widetilde{h}_k(X) := \ker\bigl(h_k(c_X)\colon h_k(X) \longrightarrow h_k(\bullet)\bigr) \subset h_k(X),$$

where $c_X\colon X \longrightarrow \bullet$ is the constant map. Then for every $x_0 \in X$ and every $k \in \mathbb{Z}$, the composition

$$\widetilde{h}_k(X) \longrightarrow h_k(X) \longrightarrow h_k\bigl(X, \{x_0\}\bigr)$$

of the inclusion and the homomorphism induced by the inclusion is an R-isomorphism (Exercise). If $f\colon X \longrightarrow Y$ is a continuous map, then

$$\widetilde{h}_k(f) := h_k(f)|_{\widetilde{h}_k(X)} \colon \widetilde{h}_k(X) \longrightarrow \widetilde{h}_k(Y)$$

is well-defined (Exercise). In this way, we obtain a homotopy invariant functor $\widetilde{h}_k\colon \mathsf{Top} \longrightarrow {}_R\mathsf{Mod}$ with $\widetilde{h}_k(\bullet) \cong 0$.

Therefore, whenever convenient, we can remove the homology of the basepoint from our computations.

In the context of excision arguments, one often considers triples of spaces (i.e., a space together with a subspace and a subspace of this subspace). In this situation, the following sequence is helpful:

Proposition 3.1.8 (long exact sequence of triples). *Let X be a topological space and let $B \subset A \subset X$ be subspaces. Then the sequence*

$$\cdots \xrightarrow{\partial_{k+1}^{(X,A,B)}} h_k(A,B) \longrightarrow h_k(X,B) \longrightarrow h_k(X,A) \xrightarrow{\partial_k^{(X,A,B)}} h_{k-1}(A,B) \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact, where for $k \in \mathbb{Z}$, the connecting homomorphism $\partial_k^{(X,A,B)}$ in the triple sequence is defined as the composition

$$\begin{tikzcd} h_k(X,A) \arrow[rr, dashed, "\partial_k^{(X,A,B)}"] \arrow[dr, "\partial_k^{(X,A)}"'] & & h_{k-1}(A,B) \\ & h_{k-1}(A) \arrow[ur] & \end{tikzcd}$$

(and the unmarked homomorphisms are the R-homomorphisms induced by the respective inclusions of (pairs of) spaces).

Proof. Let $k \in \mathbb{Z}$. We consider the braid diagram in Figure 3.2, consisting of the desired triple sequence and the interwoven long exact sequences of the pairs (A,B), (X,B), and (X,A).

This diagram is commutative by definition of the connecting homomorphism of the sequence of the triple, by functoriality of the $(h_m)_{m\in\mathbb{Z}}$, and the naturality of the connecting homomorphisms of the homology theory.

In the triple sequence, the composition of consecutive homomorphisms is zero:

- At $h_k(X,B)$: The relevant composition factors over $h_k(A,A)$ (which is trivial by Proposition 3.1.4) and thus is trivial.
- At $h_k(X,A)$: The relevant composition factors over the portion

$$h_{k-1}(B) \longrightarrow h_{k-1}(A) \longrightarrow h_{k-1}(A,B)$$

$$
\begin{tikzcd}
 & h_{k+1}(X,A) \arrow[rr, bend left, "\partial_{k+1}^{(X,A,B)}"] \arrow[dr, "\partial_k^{(X,A)}"'] & & h_k(A,B) \arrow[rr, bend left, "\partial_k^{(A,B)}"] \arrow[dr] & & h_{k-1}(B) \arrow[rr, bend left] \arrow[dr] & & h_{k-1}(X) & \\
\cdots \arrow[ur] & & h_k(A) \arrow[ur] \arrow[dr] & & h_k(X,B) \arrow[ur, "\partial_k^{(X,B)}"'] \arrow[dr] & & h_{k-1}(A) \arrow[ur] \arrow[dr] & & \cdots \\
 & h_k(B) \arrow[ur] \arrow[rr, bend right] & & h_k(X) \arrow[ur] \arrow[rr, bend right] & & h_k(X,A) \arrow[ur, "\partial_k^{(X,A)}"] \arrow[rr, bend right, "\partial_k^{(X,A,B)}"'] & & h_{k-1}(A,B) \arrow[ur] &
\end{tikzcd}
$$

Figure 3.2.: The braid diagram of the long exact sequence of a triple; all unmarked R-homomorphisms are induced by the respective inclusions.

of the long exact sequence of the pair (A, B) and hence is trivial.

- At $h_k(A, B)$: In this case, we use the long exact sequence of the pair (X, A).

The exactness of the triple sequence follows now from a diagram chase, using the exactness of the three long exact sequences of pairs (check!). $\square$

Outlook 3.1.9 (spectral sequences)**.** For longer, descending, chains of subspaces, exact sequences are not powerful enough; the right tool in such cases are spectral sequences [69, 9, 43].

3.2 Homology of Spheres and Suspensions

We will now calculate homology of spheres. As spheres can be constructed inductively from lower-dimensional spheres via suspension, we will first compute homology of suspensions.

Setup 3.2.1. In the following, let R be a ring with unit, let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

3.2.1 Suspensions

The suspension of a space is constructed by attaching cones on top and at the bottom (Figure 3.3):

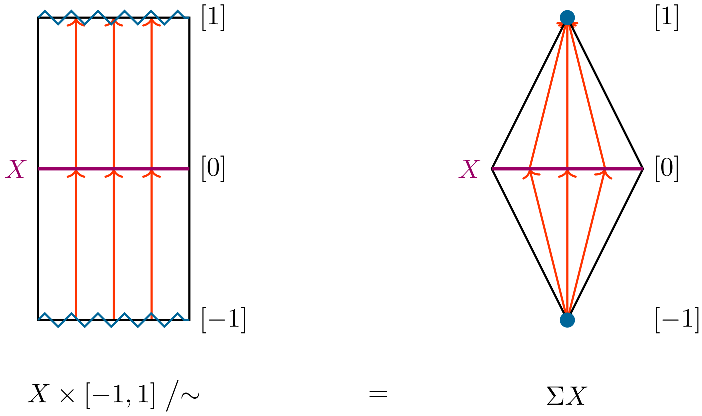


Figure 3.3.: The suspension, schematically

Definition 3.2.2 (suspension)**.**

- The (unreduced) *suspension* of a topological space X is defined as

$$\Sigma X := X \times [-1,1] \,/{\sim}$$

(endowed with the quotient topology of the product topology), where "$\sim$" is the equivalence relation generated by

$$\begin{aligned} \forall_{x,x' \in X} \quad (x,1) &\sim (x',1) \\ \forall_{x,x' \in X} \quad (x,-1) &\sim (x',-1). \end{aligned}$$

- If $f \colon X \longrightarrow Y$ is a continuous map of topological spaces, then we write

$$\begin{aligned} \Sigma f \colon \Sigma X &\longrightarrow \Sigma Y \\ [x,t] &\longmapsto \bigl[f(x),t\bigr] \end{aligned}$$

(which is well-defined and continuous; Exercise).

In this way, we obtain the *suspension functor* $\Sigma \colon \mathsf{Top} \longrightarrow \mathsf{Top}$.

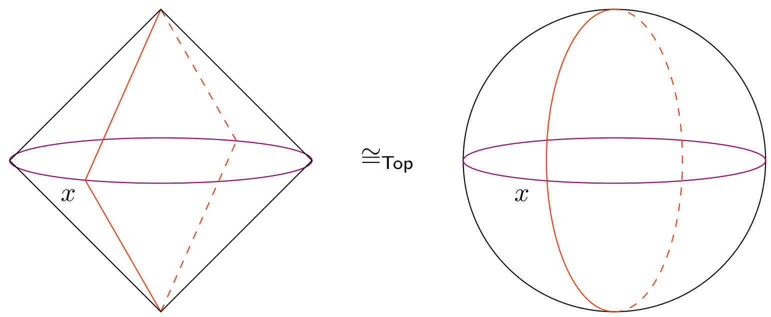


Figure 3.4.: Suspensions of spheres are spheres

Outlook 3.2.3 (join, smash)**.** More conceptually, the unreduced suspension can also be viewed as a join of S^0 with the given space or map. The reduced suspension of a pointed space or map is obtained by taking the so-called smash product with (S^1, e_1).

Study note (suspension in architecture)**.** There are many examples of (usually only one-sided) suspension constructions in architecture (e.g., many famous bridges and the Yoyogi Park National Stadium in Tokyo). Do you know an example close-by?

Example 3.2.4 (suspensions of spheres are spheres)**.** Let $n \in \mathbb{N}$. Then

$$\begin{aligned} \Sigma S^n &\longrightarrow S^{n+1} \\ [x,t] &\longmapsto \bigl(\cos(\pi/2 \cdot t) \cdot x, \sin(\pi/2 \cdot t)\bigr) \end{aligned}$$

is a well-defined homeomorphism (Exercise; Figure 3.4).

Theorem 3.2.5 (homology of suspensions)**.** *Let (X, x_0) be a pointed space and let $k \in \mathbb{Z}$. Then there is a natural (on Top_*) isomorphism, the* suspension isomorphism*,*

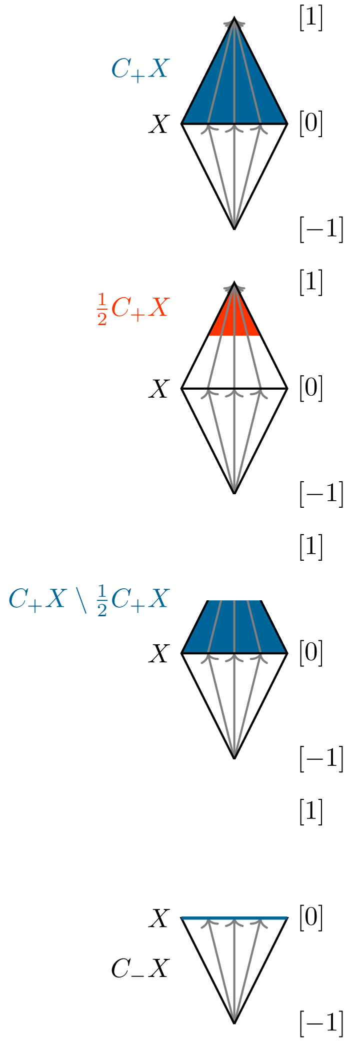
$$\sigma_k(X, x_0) \colon h_k\bigl(X, \{x_0\}\bigr) \longrightarrow h_{k+1}\bigl(\Sigma X, \{[(x_0, 0)]\}\bigr).$$

Proof. In the situation of the theorem, we obtain a chain of natural isomorphisms as in Figure 3.5.

The lowest vertical homomorphism is the connecting homomorphism of the long exact triple sequence (Proposition 3.1.8) of the triple

$$\{x_0\} \subset X \subset C_- X.$$

This connecting homomorphism is an R-isomorphism because: The cone $C_- X$ is contractible; hence, Proposition 3.1.4 implies that $h_m(C_- X, \{x_0\}) \cong_R 0$



$$\begin{tikzcd}
h_{k+1}\big(\Sigma X, \{[x_0, 0]\}\big) \arrow[d, "\cong_R"', "\text{generalised homotopy invariance (Proposition 3.1.4)}"] \\
h_{k+1}(\Sigma X, C_+X) \\
h_{k+1}\big(\Sigma X \setminus \tfrac{1}{2}C_+X, C_+X \setminus \tfrac{1}{2}C_+X\big) \arrow[u, "\cong_R", "\text{excision}"'] \\
h_{k+1}\big(C_-X, X \times \{0\}\big) \arrow[u, "\cong_R", "\text{homotopy invariance}"'] \arrow[d, "\cong_R"', "\text{connecting homomorphism of the triple sequence}"] \\
h_k\big(X, \{x_0\}\big)
\end{tikzcd}$$

Figure 3.5.: Computing the homology of suspensions

holds for all $m \in \mathbb{Z}$. Then the portion

$$h_{k+1}(C_-X, \{x_0\}) \longrightarrow h_{k+1}(C_-X, X) \xrightarrow{\partial_{k+1}^{(C_-X, X, \{x_0\})}} h_k(X, \{x_0\}) \longrightarrow h_k(C_-X, \{x_0\})$$

of the long exact sequence of the triple shows that the connecting homomorphism must be an R-isomorphism. $\square$

Study note. Make the naturality claim in Theorem 3.2.5 explicit! Which diagram do you need to draw? How could you formulate the suspension isomorphism in terms of reduced homology?

Remark 3.2.6 (how to apply excision?)**.** Let X be a topological space. As in the proof of Theorem 3.2.5, we can often compute the homology $(h_k(X))_{k\in\mathbb{Z}}$ via excision as follows:

First, we try to find subspaces $B \subset A \subset X$ with $\overline{B} \subset A^\circ$ such that

- $(h_k(A))_{k\in\mathbb{Z}}$ is accessible, and
- $(h_k(X \setminus B, A \setminus B))_{k\in\mathbb{Z}}$ is accessible (for instance, via the triple sequence (Proposition 3.1.8) or the pair sequence).

We then combine excision with the long exact pair sequence of the pair (X, A). For example, this strategy can be used to compute the homology of the two-dimensional torus (Exercise).

In Chapter 3.3, we will meet a reformulation of excision that allows for a systematic divide and conquer computation of homology.

Outlook 3.2.7 (stable homotopy theory)**.** It turns out that *not* every space is the suspension of another space, not even up to homotopy equivalence (can you find such an example?!). Forcing the suspension functor to be a self-equivalence of a suitable category leads to so-called stable homotopy theory [49, 16]. Stable homotopy theory, among other things, explains how (co)homology theories can be constructed from homotopy groups and allows one to compute examples of generalised homology theories in geometrically relevant examples. In this course, we will focus on a different, even more classical, approach via standard homological algebra (which will lead to a different type of geometric applications).

3.2.2 Homology of Spheres

The computation of the homology of suspensions and the fact that spheres are iterated suspensions of S^0 allow us to compute homology of spheres:

Corollary 3.2.8 (homology of spheres)**.**

1. *For all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$, there is a natural R-isomorphism*

$$h_k\bigl(S^n, \{e_1\}\bigr) \cong_R h_{k+1}\bigl(\Sigma S^n, \{[e_1, 0]\}\bigr) \cong_R h_{k+1}\bigl(S^{n+1}, \{e_1\}\bigr).$$

2. *Inductively, we obtain for all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$:*

$$\begin{aligned} h_k(S^n) &\cong_R h_k\bigl(S^n, \{e_1\}\bigr) \oplus h_k\bigl(\{e_1\}\bigr) && \text{(Proposition 3.1.6)} \\ &\cong_R h_{k-n}\bigl(S^0, \{e_1\}\bigr) \oplus h_k\bigl(\{e_1\}\bigr) && \text{(first part)} \\ &\cong_R h_{k-n}(\bullet) \oplus h_k(\bullet). && \text{(excision on } S^0\text{)} \end{aligned}$$

3. *If* $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ *is an ordinary homology theory,* $n \in \mathbb{N}_{>0}$*, and* $k \in \mathbb{Z}$*, then:*

$$h_k(S^0) \cong_R \begin{cases} h_0(\bullet) \oplus h_0(\bullet) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

$$h_k(S^n) \cong_R \begin{cases} h_0(\bullet) & \text{if } k \in \{0, n\} \\ 0 & \text{if } k \in \mathbb{Z} \setminus \{0, n\}. \end{cases}$$

Proof. The first part follows by combining the computation of homology of suspensions (Theorem 3.2.5) and the fact that suspensions of spheres are spheres (Example 3.2.4). A straightforward induction then shows the second part. The third part is a special case of the second part. □

Caveat 3.2.9 (how *not* to apply excision!)**.** In general, when applying excision, we have to make sure that the closure of the excised subspace lies in the interior of the larger subspace! For example: For all $n \in \mathbb{N}_{>0}$, we have (where $S := -e_{n+1} \in S^n$ and $N := e_{n+1} \in S^n$)

$$\begin{aligned} h_k\big(S^n, \{S\}\big) &\cong_R h_k\big(S^n, S^n \setminus \{N\}\big) && \text{(Proposition 3.1.4)} \\ &\text{??! } h_k\big(S^n \setminus (S^n \setminus \{N\}), (S^n \setminus \{N\}) \setminus (S^n \setminus \{N\})\big) && \text{(excision is } \textit{not} \text{ applicable!)} \\ &\cong_R h_k\big(\{N\}, \emptyset\big) \\ &\cong_R h_k(\bullet). \end{aligned}$$

Corollary 3.2.8 shows that ordinary homology theories with non-trivial coefficients discover the n-dimensional spherical "hole" in the n-sphere. In contrast to homotopy groups, ordinary homology theories cannot detect the "exotic" higher-dimensional spherical holes in spheres (such as the non-trivial elements of $\pi_3(S^2, e_1)$), but homology can also detect non-spherical holes (as can be seen from the homology of the torus; Exercise). We will come back to this point of view in the construction of singular homology.

A first application of the computation in Corollary 3.2.8 is a refined version of invariance of dimension (under the hypothesis that an ordinary homology theory does exist):

Corollary 3.2.10 (invariance of dimension, II)**.** *If there exists an ordinary homology theory* $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ *on* Top^2 *with values in* ${}_{\mathbb{Z}}\mathsf{Mod}$ *and coefficients (isomorphic to)* $\mathbb{Z}$*, then:*

If $n, m \in \mathbb{N}$ *and* $U \subset \mathbb{R}^n$*,* $V \subset \mathbb{R}^m$ *are open and non-empty with* $U \cong_{\mathsf{Top}} V$*, then* $n = m$*.*

Proof. Using excision and homotopy invariance, we can reduce the claim to the computation of the homology of spheres:

Without loss of generality, we may assume that $n, m > 0$. Let $x \in U$, let $f\colon U \longrightarrow V$ be a homeomorphism, and let $y := f(x)$. Because U is open, there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that the open ε-ball $U_\varepsilon(x)$ around x is contained in U. Removing everything outside of this standard neighbourhood via excision, we obtain for all $k \in \mathbb{N}_{>0}$ that

$$\begin{aligned}
h_k\bigl(U, U \setminus \{x\}\bigr) &\cong_{\mathbb{Z}} h_k\bigl(U \setminus (U \setminus U_\varepsilon(x)), (U \setminus \{x\}) \setminus (U \setminus U_\varepsilon(x))\bigr) && \text{(excision; applicable!)}\\
&= h_k\bigl(U_\varepsilon(x), U_\varepsilon(x) \setminus \{x\}\bigr)\\
&\cong_{\mathbb{Z}} h_k\bigl(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\bigr) && \text{(homeomorphism invariance)}\\
&\cong_{\mathbb{Z}} h_{k-1}\bigl(\mathbb{R}^n \setminus \{0\}, \{e_1\}\bigr) && \text{(long exact triple sequence, contractibility of } \mathbb{R}^n,\ n, k > 0)\\
&\cong_{\mathbb{Z}} h_{k-1}\bigl(S^{n-1}, \{e_1\}\bigr); && \text{(homotopy invariance)}
\end{aligned}$$

and, analogously, $h_k\bigl(V, V \setminus \{y\}\bigr) \cong_{\mathbb{Z}} h_{k-1}\bigl(S^{m-1}, \{e_1\}\bigr)$. Because f is a homeomorphism, we have $(U, U \setminus \{x\}) \cong_{\mathsf{Top}^2} (V, V \setminus \{y\})$. Therefore,

$$\begin{aligned}
h_{n-1}(S^{n-1}) &\cong_{\mathbb{Z}} h_{n-1}\bigl(S^{n-1}, \{e_1\}\bigr) \oplus h_{n-1}(\bullet) && \text{(Proposition 3.1.6)}\\
&\cong_{\mathbb{Z}} h_n\bigl(U, U \setminus \{x\}\bigr) \oplus h_{n-1}(\bullet) && \text{(calculation above)}\\
&\cong_{\mathbb{Z}} h_n\bigl(V, V \setminus \{y\}\bigr) \oplus h_{n-1}(\bullet) && \text{(} f \text{ is a homeomorphism)}\\
&\cong_{\mathbb{Z}} h_{n-1}\bigl(S^{m-1}, \{e_1\}\bigr) \oplus h_{n-1}(\bullet) && \text{(calculation above)}\\
&\cong_{\mathbb{Z}} h_{n-1}(S^{m-1}). && \text{(Proposition 3.1.6)}
\end{aligned}$$

Applying Corollary 3.2.8, we thus obtain $n - 1 = m - 1$ (check!). □

In particular, this version of invariance of dimension nicely demonstrates the effect of the dimension axiom (which only involves the homology of a single point!) on the homology of higher-dimensional objects.

3.2.3 Mapping Degrees of Self-Maps of Spheres

As next step, we determine mapping degrees of self-maps of spheres with respect to ordinary homology theories with $\mathbb{Z}$-coefficients. These computations will also play an important role when describing the homology of cell complexes.

Corollary 3.2.11 (mapping degrees on spheres). *Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory. Then:*

1. *If $n \in \mathbb{N}_{>0}$ and $j \in \{1, \dots, n+1\}$, then*

$$h_n(r_j^{(n)}) = -\operatorname{id}_{h_n(S^n)},$$

where $r_j^{(n)}$ is the reflection defined in Theorem 1.3.22.

2. *For $d \in \mathbb{Z}$, we write*

$$\begin{aligned} f_d \colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [d \cdot t \mod 1]. \end{aligned}$$

Then $h_1(f_d) = d \cdot \mathrm{id}_{h_1(S^1)}$.

3. *For all* $n \in \mathbb{N}_{>0}$ *and all* $d \in \mathbb{Z}$*, we have (under the canonical homeomorphism* $\Sigma^{n-1} S^1 \cong_{\mathsf{Top}} S^n$*; Example 3.2.4)*

$$h_n(\Sigma^{n-1} f_d) = d \cdot \mathrm{id}_{h_n(S^n)}.$$

In particular: If $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ *is an ordinary homology theory with values in* ${}_{\mathbb{Z}}\mathsf{Mod}$ *and coefficients (isomorphic to)* $\mathbb{Z}$*, then we obtain for the mapping degree (Proposition 1.3.29):*

- *For all* $n \in \mathbb{N}_{>0}$ *and all* $j \in \{1, \dots, n\}$*, we have*

$$\deg_{h_n} r_j^{(n)} = -1.$$

- *For all* $n \in \mathbb{N}_{>0}$ *and all* $d \in \mathbb{Z}$*, we have*

$$\deg_{h_n} \Sigma^{n-1} f_d = d.$$

Thus, the map $\deg_{h_n} \colon [S^n, S^n] \longrightarrow \mathbb{Z}$ *is surjective.*

The proof is based on the following observation on the addition of maps on spheres, which is related to the cogroup object structure of spheres in $\mathsf{Top}_{*\mathsf{h}}$. This lemma explains how geometric addition of maps translates into purely algebraic addition:

Lemma 3.2.12 (homology of "addition" of maps defined on spheres)**.** *Let* $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ *be an ordinary homology theory and let* $n \in \mathbb{N}_{>0}$*. For* $d \in \mathbb{N}$*, we sloppily write* $\bigvee^d S^n$ *for the topological space underlying the d-fold wedge* $\bigvee^d (S^n, e_1)$*; moreover, let* $(i_j \colon S^n \longrightarrow \bigvee^d S^n)_{j\in\{1,\dots,d\}}$ *be the canonical inclusions of the wedge summands and let* $(p_j \colon \bigvee^d S^n \longrightarrow S^n)_{j\in\{1,\dots,n\}}$ *be the canonical collapse maps that are the identity on one summand and the constant map on all other summands. Then:*

1. *The inclusions* $(i_j)_{j\in\{1,\dots,d\}}$ *and the collapse maps* $(p_j)_{j\in\{1,\dots,d\}}$ *induce for every* $k \in \mathbb{Z} \setminus \{0\}$ *an isomorphism*

$$h_k\Big(\bigvee^d S^n\Big) \cong_R \bigoplus^d h_k(S^n).$$

2. *Let*

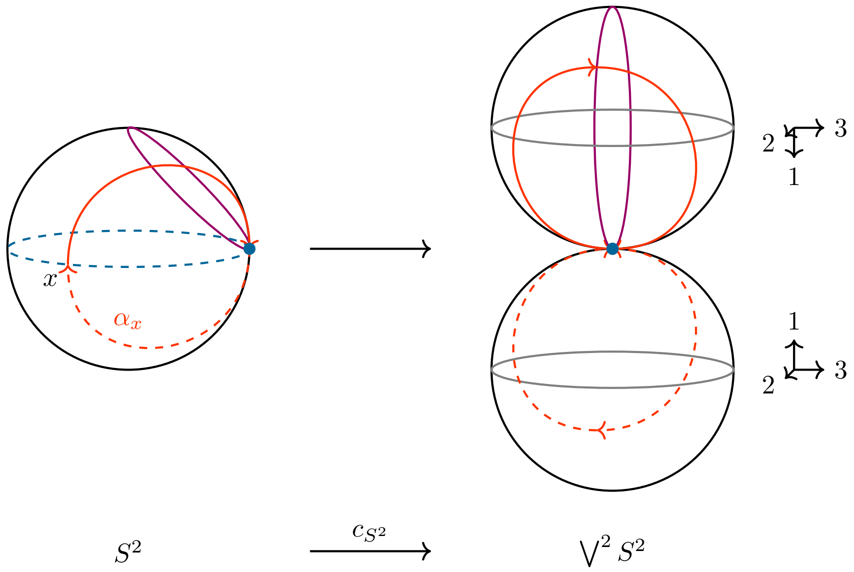


Figure 3.6.: The pinching map $c_{S^n} \colon S^n \longrightarrow \bigvee^2 S^n$.

$$\begin{aligned} c_{S^n} \colon S^n &\longrightarrow \bigvee^2 S^n \\ \alpha_x(t) &\longmapsto \begin{cases} i_1\bigl(\alpha_x(2 \cdot t)\bigr) & \text{if } t \in [0, 1/2] \\ i_2\bigl(\alpha_x(2 \cdot t - 1)\bigr) & \text{if } t \in [1/2, 1] \end{cases} \end{aligned}$$

be the pinching map *(Figure 3.6). Here, for* $x \in S^{n-1} \times \{0\} \setminus \{e_1\}$, *we denote by* $\alpha_x \colon [0,1] \longrightarrow S^n$ *the uniquely determined circle (with constant speed, starting in* e_1 *in direction of* e_{n+1} *that contains* x *and* e_1 *as diametral points and that lies in the plane through* x, e_1, *and* $e_1 + e_{n+1}$. *Then* c_{S^n} *is well-defined and continuous (check!).*

Then: For every topological space X *and all maps* $f, g \in \operatorname{map}(S^n, X)$ *with* $f(e_1) = g(e_1)$, *we have*

$$h_n\bigl((f \vee g) \circ c_{S^n}\bigr) = h_n(f) + h_n(g),$$

where $f \vee g \colon \bigvee^2 S^n \longrightarrow X$ *is the unique continuous map with the property that* $(f \vee g) \circ i_1 = f$ *and* $(f \vee g) \circ i_2 = g$.

Proof of Lemma 3.2.12. The first part follows inductively via excision (where suitable subsets can be constructed as in Example 2.2.14; check!); a convenient framework for such computations will be introduced in Chapter 3.3.

Because $p_1 \circ c_{S^n} \simeq \mathrm{id}_{S^n} \simeq p_2 \circ c_{S^n}$ (check!) and $i_1 \circ p_2$ and $i_2 \circ p_1$ are constant maps, we obtain from homotopy invariance and the first part that

$$\begin{aligned}
h_n\bigl((f \vee g) \circ c_{S^n}\bigr) &= h_n(f \vee g) \circ h_n(i_1) \circ h_n(p_1) \circ h_n(c_{S^n}) \\
&\quad + h_n(f \vee g) \circ h_n(i_2) \circ h_n(p_2) \circ h_n(c_{S^n}) \\
&\quad + h_n(f \vee g) \circ h_n(i_1) \circ h_n(p_2) \circ h_n(c_{S^n}) \\
&\quad + h_n(f \vee g) \circ h_n(i_2) \circ h_n(p_1) \circ h_n(c_{S^n}) && \text{(first part)} \\
&= h_n\bigl((f \vee g) \circ i_1\bigr) \circ h_n(p_1 \circ c_{S^n}) \\
&\quad + h_n\bigl((f \vee g) \circ i_2\bigr) \circ h_n(p_2 \circ c_{S^n}) \\
&\quad + h_n(f \vee g) \circ h_n(i_1 \circ p_2) \circ h_n(c_{S^n}) \\
&\quad + h_n(f \vee g) \circ h_n(i_2 \circ p_1) \circ h_n(c_{S^n}) \\
&= h_n\bigl((f \vee g) \circ i_1\bigr) \circ \mathrm{id}_{h_n(S^n)} \\
&\quad + h_n\bigl((f \vee g) \circ i_2\bigr) \circ \mathrm{id}_{h_n(S^n)} && \text{(because } p_j \circ c_{S^n} \simeq \mathrm{id}_{S^n}\text{)} \\
&\quad + 0 && (h_n(i_1 \circ p_2) \text{ and } h_n(i_2 \circ p_1) \text{ factor over } h_n(\bullet) \cong 0) \\
&= h_n(f) + h_n(g), && \text{(definition of } f \vee g\text{)}
\end{aligned}$$

as desired. □

Remark 3.2.13 (Hurewicz homomorphism)**.** Let $\bigl((h_k)_{k \in \mathbb{Z}}, (\partial_k)_{k \in \mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and coefficients (isomorphic to) $\mathbb{Z}$. Let $n \in \mathbb{N}_{>0}$ and let $[S^n] \in h_n(S^n) \cong_{\mathbb{Z}} \mathbb{Z}$ be a generator. Then, for every pointed space (X, x_0), we obtain a well-defined homomorphism (check! this follows as in the previous lemma)

$$\begin{aligned}
\pi_n(X, x_0) &\longrightarrow h_n(X) \\
[f]_* &\longmapsto h_n(f)\bigl([S^n]\bigr)
\end{aligned}$$

from the n-th homotopy group into homology in degree n, the *Hurewicz homomorphism in degree n*. This leads to a natural transformation $\pi_n \Longrightarrow h_n \circ F$, where $F \colon \mathsf{Top}_* \longrightarrow \mathsf{Top}^2$ is the functor that replaces the basepoint with the empty subspace.

Proof of Corollary 3.2.11. Ad 1. We proceed in the following steps:

① *Reduction to $r_2^{(n)}$.* The reflection $r_j^{(n)}$ is a conjugation of $r_2^{(n)}$: Let

$$\begin{aligned}
g_j \colon S^n &\longrightarrow S^n \\
x &\longmapsto (x_1, x_j, x_3, \dots, x_{j-1}, x_2, x_{j+1}, \dots, x_{n+1})
\end{aligned}$$

be the homeomorphism swapping the second and the j-th coordinate. Then $r_j^{(n)} = g_j \circ r_2^{(n)} \circ g_j^{-1}$. Therefore, if $h_n(r_2^{(n)}) = -\mathrm{id}_{h_n(S^n)}$, then also $h_n(r_k^{(n)}) = -\mathrm{id}_{h_n(S^n)}$ (check!).

② *Computation for $r_2^{(1)}$.* The composition $(\mathrm{id}_{S^1} \vee r_2^{(1)}) \circ c_{S^1}$ is null-homotopic (check!) and thus factors over $\bullet$. Therefore, $h_1(\mathrm{id}_{S^1} \vee r_2^{(1)} \circ c_{S^1})$ factors over $h_1(\bullet) \cong_R 0$. Hence, by Lemma 3.2.12,

$$\begin{aligned} 0 &= h_1\bigl((\mathrm{id}_{S^1} \vee r_2^{(1)}) \circ c_{S^1}\bigr) \\ &= h_1(\mathrm{id}_{S^1}) + h_1(r_2^{(1)}), && \text{(Lemma 3.2.12)} \end{aligned}$$

which implies $h_1(r_2^{(1)}) = -h_1(\mathrm{id}_{S^1}) = -\mathrm{id}_{h_1(S^1)}$.

③ *Computation for $r_2^{(n)}$ in dimensions $n > 1$.* By induction, it suffices to compute $h_n(r_2^{(n)})$ from $h_{n-1}(r_2^{(n-1)})$. To this end, we apply suspension: By construction, under the canonical suspension homeomorphism, $\Sigma r_2^{(n-1)}$ corresponds to $r_2^{(n)}$ and $r_2^{(n)}(e_1) = e_1$. Hence, the commutative diagram

$$\begin{tikzcd} h_{n-1}(S^{n-1}) \arrow[r, two heads] \arrow[d, "h_{n-1}(r_2^{(n-1)})"'] & h_{n-1}(S^{n-1}, e_1) \arrow[r, "\Sigma", "\cong_R"'] \arrow[d, "h_{n-1}(r_2^{(n-1)})"'] & h_n(S^n, e_1) \arrow[d, "h_n(r_2^{(n)})"] & h_n(S^n) \arrow[l, "\cong_R"'] \arrow[d, "h_n(r_2^{(n)})"] \\ h_{n-1}(S^{n-1}) \arrow[r, two heads] & h_{n-1}(S^{n-1}, e_1) \arrow[r, "\cong_R", "\Sigma"'] & h_n(S^n, e_1) & h_n(S^n) \arrow[l, "\cong_R"] \end{tikzcd}$$

shows that $h_n(r_2^{(n)}) = -\mathrm{id}_{h_n(S^n)}$. The left surjections and the right horizontal R-isomorphisms stem from Proposition 3.1.6; the middle isomorphisms are the suspension isomorphisms (Theorem 3.2.5). The three squares are commutative because these isomorphisms are natural.

Ad 2. We distinguish the following three cases:

- If $d = 0$, then $f_d\colon S^1 \longrightarrow S^1$ is constant; hence, $h_1(f_0)$ factors over $h_1(\bullet) \cong_R 0$. Therefore, $h_1(f_0) = 0$.
- If $d \in \mathbb{Z}_{<0}$, then $f_d = r_2^{(1)} \circ f_{|d|}$. In view of the first part, we then obtain $h_1(f_d) = h_1(r_2^{(1)}) \circ h_1(f_{|d|}) = -h_1(f_{|d|})$.
- Thus, it suffices to consider the case $d \in \mathbb{N}_{>0}$: We have (check!)

  $$f_d \simeq (f_{d-1} \vee \mathrm{id}_{S^1}) \circ c_{S^1}.$$

  Therefore, applying Lemma 3.2.12 yields

  $$h_1(f_d) = h_1(f_{d-1}) + h_1(\mathrm{id}_{S^1}) = h_1(f_{d-1}) + \mathrm{id}_{h_1(S^1)};$$

  inductively, we obtain $h_1(f_d) = d \cdot \mathrm{id}_{h_1(S^1)}$.

Ad 3. This follows inductively from the second part, using the naturality of the suspension isomorphism (Theorem 3.2.5) and Example 3.2.4. □

Using mapping degrees, one can also prove the fundamental theorem of algebra (similar to the proof via the fundamental group).

Moreover, homology theories provide a means to prove Theorem 1.3.22:

Corollary 3.2.14 (existence of „interesting“ homotopy invariant functors). *If* $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ *is an ordinary homology theory on* Top^2 *with values in* ${}_{\mathbb{Z}}\mathsf{Mod}$ *and coefficients (isomorphic to)* $\mathbb{Z}$*, then the corresponding reduced homology functors (Remark 3.1.7)*

$$\widetilde{h}_0, \widetilde{h}_1, \dots \colon \mathsf{Top} \longrightarrow \mathsf{Ab}$$

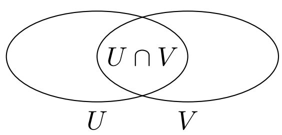
are functors with the properties in Theorem 1.3.22.

Proof. This follows from our previous calculations (Proposition 3.1.6, Corollary 3.2.11) and a straightforward calculation in degree 0 (check!). ☐

Therefore, it is one of the main goals of this course to construct such homology theories. We will do this in Chapter 4 and Chapter 5.

3.3 Glueings: The Mayer–Vietoris Sequence

We derive a version of excision that allows us to compute homology of glueings in a convenient way. More precisely, we will see how to express the homology of a space $X = U \cup V$ in terms of the homology of U, V, and $U \cap V$ (Figure 3.7) via the Mayer–Vietoris sequence (Vietoris was an Austrian mathematician; 1891–2002(!)). This long exact sequence encodes a homological inclusion/exclusion principle.



$$\begin{tikzcd}
 & h_k(U \cap V) \arrow[dl] \arrow[dr] & \\
h_k(U) \arrow[dr] & & h_k(V) \arrow[dl] \\
 & h_k(X) &
\end{tikzcd}$$

Figure 3.7.: The setup for the Mayer–Vietoris sequence

In particular, we will apply the Mayer–Vietoris sequence to mapping cones. On the one hand, this will give access to realisation results for homology groups; on the other hand, this will enable us to interpret relative homology as absolute homology (of mapping cones).

Setup 3.3.1. In the following, let R be a ring with unit, let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Theorem 3.3.2 (Mayer–Vietoris sequence)**.** *Let X be a topological space and let $U, V \subset X$ subspaces of X with the property that the closure of $U \setminus V$ (in $U \cup V$) lies in the interior of U (in $U \cup V$). Moreover, let $A \subset X$ with $A \subset U \cap V$.*

1. *If $U \cup V = X$, then the sequence*

$$\cdots \xrightarrow{\Delta_{k+1}} h_k(U\cap V, A) \xrightarrow{(h_k(i_U), -h_k(i_V))} h_k(U, A) \oplus h_k(V, A) \xrightarrow{h_k(j_U)\oplus h_k(j_V)} h_k(X, A) \xrightarrow{\Delta_k} h_{k-1}(U\cap V, A) \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact. Here, $i_U\colon (U\cap V, A) \longrightarrow (U, A)$, $i_V\colon (U\cap V, A) \longrightarrow (V, A)$, $j_U\colon (U, A) \longrightarrow (X, A)$, and $j_V\colon (V, A) \longrightarrow (X, A)$ are the inclusions. For $k \in \mathbb{Z}$ we write Δ_k for the composition (where unmarked arrows are induced by inclusions)

$$\begin{tikzcd} h_k(X,A) \arrow[r] \arrow[rrr, bend left, "\Delta_k"] & h_k(X,U) & h_k(V, U\cap V) \arrow[l, "\cong"', "\text{excision}"] \arrow[r, "\partial_k^{(V,U\cap V,A)}"'] & h_{k-1}(U\cap V, A). \end{tikzcd}$$

2. *Moreover, the sequence*

$$\cdots \xrightarrow{\Delta_{k+1}} h_k(X, U\cap V) \xrightarrow{(h_k(i_U), -h_k(i_V))} h_k(X, U) \oplus h_k(X, V) \xrightarrow{h_k(j_U)\oplus h_k(j_V)} h_k(X, U\cup V) \xrightarrow{\Delta_k} h_{k-1}(X, U\cap V) \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact. Here, $i_U\colon (X, U\cap V) \longrightarrow (X, U)$, $i_V\colon (X, U\cap V) \longrightarrow (X, V)$, $j_U\colon (X, U) \longrightarrow (X, U\cup V)$, and $j_V\colon (X, V) \longrightarrow (X, U\cup V)$ are the inclusions. For $k \in \mathbb{Z}$ we write Δ_k for the composition (where unmarked arrows are induced by inclusions)

$$\begin{tikzcd} h_k(X, U\cup V) \arrow[r, "\partial_k^{(X,U\cup V,U)}"'] \arrow[rrr, bend left, "\Delta_k"] & h_{k-1}(U\cup V, U) & h_{k-1}(V, U\cap V) \arrow[l, "\cong"', "\text{excision}"] \arrow[r] & h_{k-1}(X, U\cap V). \end{tikzcd}$$

Study note. Instead of memorising the connecting homomorphisms of the Mayer–Vietoris sequences as in the statement of the theorem, it is much more efficient (and much more useful) to remember how to prove the existence of such sequences; the proof, in particular, also shows how to define these connecting homomorphisms.

Proof. The proof is based on the algebraic Mayer–Vietoris sequence (Proposition A.6.8):

Ad 1. We consider the following ladder in ${}_R\mathsf{Mod}$ (where all unmarked arrows are induced by inclusions):

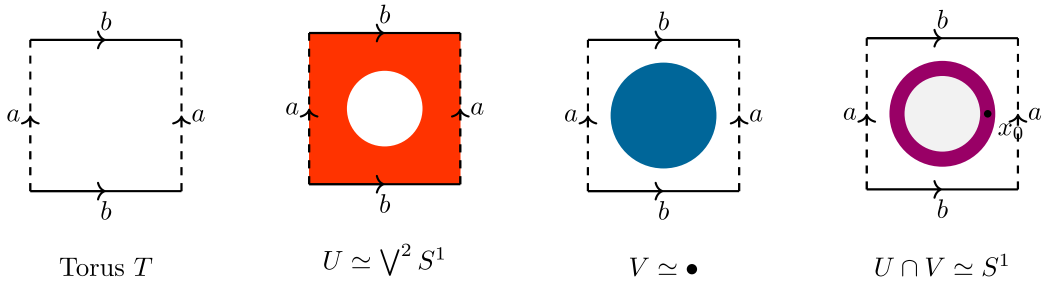


Figure 3.8.: Decomposing a torus

$$\begin{tikzcd}
\cdots \arrow[r] & h_k(U \cap V, A) \arrow[r] \arrow[d] & h_k(V, A) \arrow[r] \arrow[d] & h_k(V, U \cap V) \arrow[r, "\partial_k^{(V, U\cap V, A)}"] \arrow[d, "\cong"] & h_{k-1}(U \cap V, A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & h_k(U, A) \arrow[r] & h_k(X, A) \arrow[r] & h_k(X, U) \arrow[r, "\partial_k^{(X, U, A)}"'] & h_{k-1}(U, A) \arrow[r] & \cdots
\end{tikzcd}$$

The rows are the long exact triple sequences of the triples $(V, U \cap V, A)$ and (X, U, A), respectively; the squares are commutative (by functoriality and naturality of the connecting homomorphisms; check!). Moreover, the middle vertical homomorphism is an isomorphism (by excision; which is applicable!).

Hence, we can apply the algebraic Mayer–Vietoris sequence (Proposition A.6.8) to conclude that the sequence from the theorem is exact.

Ad 2. Analogously, we apply Proposition A.6.8 to the commutative ladder whose rows are the long exact triple sequences of the triples $(X, V, U \cap V)$ and $(X, U \cup V, U)$, respectively. □

Example 3.3.3 (homology of the torus)**.** Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and coefficients (isomorphic to) $\mathbb{Z}$. We compute the homology of the torus $T := S^1 \times S^1$ with help of the Mayer–Vietoris sequence (Theorem 3.3.2). To this end, we consider the decomposition in Figure 3.8 (the vertical and horizontal edges are identified as specified); then the Mayer–Vietoris sequence is applicable (check!) and gives (for the subspace $A = \{x_0\}$) the following long exact sequence (in ${}_{\mathbb{Z}}\mathsf{Mod}$):

$$\cdots \xrightarrow{\Delta_{k+1}} h_k(U \cap V, A) \longrightarrow h_k(U, A) \oplus h_k(V, A) \longrightarrow h_k(T, A) \xrightarrow{\Delta_k} h_{k-1}(U \cap V, A) \longrightarrow \cdots$$

Using the homotopy equivalence $U \cap V \simeq S^1$, the homotopy equivalence $U \simeq \bigvee^2 S^1$, and the fact that V is contractible (check!), we obtain the following commutative diagram in ${}_{\mathbb{Z}}\mathsf{Mod}$, whose top row is exact.

$$\begin{tikzcd}
\cdots \arrow[r, "\Delta_{k+1}"] & h_k(U\cap V, A) \arrow[r] & h_k(U,A)\oplus h_k(V,A) \arrow[r] & h_k(T,A) \arrow[r, "\Delta_k"] & h_{k-1}(U\cap V, A) \arrow[r] & \cdots \\
& h_k(S^1,\{e_1\}) \arrow[u, "\cong"] \arrow[r, dashed, "0"'] & h_k(\bigvee^2 S^1,\{e_1\})\oplus 0 \arrow[u, "\cong"'] & & h_{k-1}(S^1,\{e_1\}) \arrow[u, "\cong"] \arrow[r, dashed, "0"'] & \cdots
\end{tikzcd}$$

Why is the dashed homomorphism trivial? In view of the calculation of homology of spheres, it suffices to consider the case of h_1 (in all other degrees, the modules are already trivial). We use the following geometric input to compute this map in degree 1: The inclusion $U \cap V \hookrightarrow U$ corresponds to the following map $S^1 \longrightarrow \bigvee^2 S^1$:

- walk through the first circle,
- then walk through the second circle in reverse direction,
- then walk through the first circle in reverse direction,
- then walk through the second circle.

The computation of mapping degrees of self-maps of spheres (Corollary 3.2.11) and Lemma 3.2.12 show that this map induces the trivial homomorphism in h_1 (check! it is a good exercise to *write* this down on your own in explicit formulae; there is no point in *reading* such formulae).

Therefore, the above long exact sequence leads to short exact sequences in ${}_{\mathbb{Z}}\mathsf{Mod}$ of the form

$$0 \longrightarrow h_k(\textstyle\bigvee^2 S^1,\{e_1\}) \longrightarrow h_k(T,\{x_0\}) \longrightarrow h_{k-1}(S^1,\{e_1\}) \longrightarrow 0$$

and we obtain for all $k \in \mathbb{Z}$

$$h_k\bigl(T,\{x_0\}\bigr) \cong_{\mathbb{Z}} \begin{cases} h_1\bigl(S^1,\{e_1\}\bigr) \cong_{\mathbb{Z}} \mathbb{Z} & \text{if } k=2 \\ h_1\bigl(\bigvee^2 S^1,\{e_1\}\bigr) \cong_{\mathbb{Z}} \mathbb{Z}\oplus\mathbb{Z} & \text{if } k=1 \\ 0 & \text{if } k \in \mathbb{Z}\setminus\{1,2\} \end{cases}$$

and (Proposition 3.1.6)

$$h_k(T) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } k=1 \\ \mathbb{Z} & \text{if } k=2 \\ 0 & \text{if } k\in\mathbb{Z}\setminus\{0,1,2\}. \end{cases}$$

It should be noted that this computation also shows that the homology of the torus in degree 1 is inherited from $S^1 \times \{e_1\} \cup \{e_1\} \times S^1 \cong_{\mathsf{Top}} \bigvee^2 S^1$.

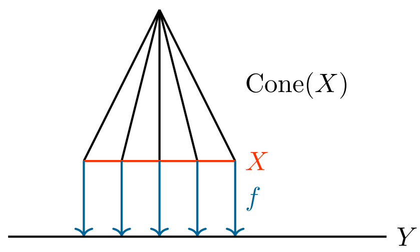


Figure 3.9.: Mapping cone, schematically

Analogously, one can compute the ordinary homology of $\mathbb{R}P^2$, of the Klein bottle, and all other compact surfaces (Exercise).

Another application of the Mayer–Vietoris sequence is the mapping cone trick, which translates questions about maps into questions about spaces. Mapping cones are constructed from continuous maps by attaching a cone over the domain (via the given map) to the target space (Figure 3.9).

Definition 3.3.4 (mapping cone)**.** Let X be a topological space.

- The *cone over* X is defined as

$$\operatorname{Cone}(X) := X \times [0,1] \,/\, X \times \{0\}$$

  if $X \neq \emptyset$. (It is sometimes convenient to set $\operatorname{Cone}(\emptyset) := \bullet$; whenever possible, we will try to avoid considering this case.)

- If $f \colon X \longrightarrow Y$ is a continuous map, then the *mapping cone of* f is the topological space $\operatorname{Cone}(f)$ defined by the pushout in Top, where the left map is the inclusion $X \hookrightarrow X \times \{1\} \hookrightarrow \operatorname{Cone}(X)$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \operatorname{Cone}(X) \arrow[r] & \operatorname{Cone}(f) \end{tikzcd}$$

Example 3.3.5 (mapping cones)**.**

- If X is a (non-empty) topological space, then $\operatorname{Cone}(X)$ is contractible.
- If X is a topological space, then $\operatorname{Cone}(\mathrm{id}_X) \cong_{\mathsf{Top}} \operatorname{Cone}(X)$.
- If X is a (non-empty) topological space and $f \colon X \longrightarrow \bullet$ is the constant map, then $\operatorname{Cone}(f) \cong_{\mathsf{Top}} \Sigma X$.

- If $f\colon X \longrightarrow Y$ is a homotopy equivalence between non-empty spaces, then $\operatorname{Cone}(f)$ is contractible (Exercise). The converse does *not* hold in general (but examples are not that easy to construct).
- We have
$$\operatorname{Cone}(f_2) \cong_{\mathsf{Top}} \mathbb{R}P^2,$$
where $f_2\colon S^1 \longrightarrow S^1$ wraps twice around S^1 (as in Corollary 3.2.11).

Via the Mayer–Vietoris sequence, we obtain the following long exact sequence for mapping cones:

Theorem 3.3.6 (long exact sequence of mapping cones)**.** *Let $f\colon X \longrightarrow Y$ be a continuous map with $X \neq \emptyset$. Then there are (natural) long exact sequences*

$$\cdots \longrightarrow \widetilde{h}_k(X) \xrightarrow{\widetilde{h}_k(f)} \widetilde{h}_k(Y) \xrightarrow{\widetilde{h}_k(Y \hookrightarrow \operatorname{Cone}(f))} \widetilde{h}_k\bigl(\operatorname{Cone}(f)\bigr) \longrightarrow \widetilde{h}_{k-1}(X) \xrightarrow{\widetilde{h}_{k-1}(f)} \cdots$$

and

$$\cdots \longrightarrow h_k(X) \xrightarrow{h_k(f)} h_k(Y) \xrightarrow{h_k(Y \hookrightarrow \operatorname{Cone}(f))} \widetilde{h}_k\bigl(\operatorname{Cone}(f)\bigr) \longrightarrow h_{k-1}(X) \xrightarrow{h_{k-1}(f)} \cdots$$

in ${}_R\mathsf{Mod}$.

Proof. Because X is non-empty, we can choose a point $x_0 \in X$. Then we consider the decomposition of the mapping cone of f as depicted in Figure 3.10. Therefore, we obtain the corresponding Mayer–Vietoris sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & h_k(U\cap V, A) \arrow[r] & h_k(U,A)\oplus h_k(V,A) \arrow[r] \arrow[d, "\cong_R"'] & h_k\bigl(\operatorname{Cone}(f), A\bigr) \arrow[r] \arrow[d, "\cong_R"] & h_{k-1}(U\cap V, A) \arrow[r] & \cdots \\
& h_k(X,\{x_0\}) \arrow[u, "\cong_R"] \arrow[r, dashed, "h_k(f)"'] & h_k(Y,\{f(x_0)\})\oplus 0 \arrow[r, dashed, "h_k(\mathrm{incl})"'] & h_k(\operatorname{Cone}(f),\{f(x_0)\}) & h_{k-1}(X,\{x_0\}) \arrow[u, "\cong_R"] \arrow[r, dashed] & \cdots \\
& \widetilde{h}_k(X) \arrow[u, "\cong_R"] \arrow[r, dashed, "\widetilde{h}_k(f)"'] & \widetilde{h}_k(Y) \arrow[u, "\cong_R"] \arrow[r, dashed, "\widetilde{h}_k(\mathrm{incl})"'] & \widetilde{h}_k(\operatorname{Cone}(f)) \arrow[u, "\cong_R"] & \widetilde{h}_{k-1}(X) \arrow[u, "\cong_R"] \arrow[r, dashed] & \cdots
\end{tikzcd}$$

Here, $A := \{[x_0, 1/2]\} \subset U \cap V$. The lower vertical homomorphisms are the R-isomorphisms from Remark 3.1.7. The upper outer vertical homomorphism is induced by the inclusion (to the layer at $1/2$), the left middle vertical homomorphism is induced by the flattening map $U \longrightarrow Y$, and the right middle homomorphism is given by flattening the lower half of $\operatorname{Cone}(X)$; these are all R-isomorphisms (by Proposition 3.1.4). Moreover, the rectangles all are commutative (check!).

The lowest row proves the existence of the first (natural) long exact sequence in the theorem.

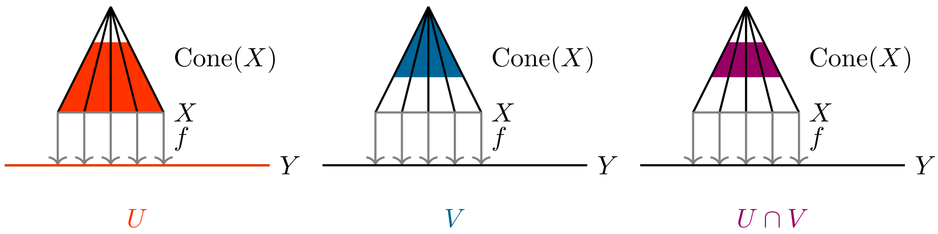


Figure 3.10.: Decomposing mapping cones

For the second sequence, we argue as follows: Adding (via $\oplus$) the long exact sequence

$$\cdots \longrightarrow h_k\bigl(\{x_0\}\bigr) \xrightarrow[\cong_R]{h_k(f|_{\{x_0\}})} h_k\bigl(\{f(x_0)\}\bigr) \longrightarrow 0 \longrightarrow h_{k-1}\bigl(\{x_0\}\bigr) \xrightarrow[\cong_R]{h_k(f|_{\{x_0\}})} \cdots$$

to the middle row, leads (with help of Proposition 3.1.6) to the second (natural) long exact sequence in the theorem. $\square$

In particular, mapping cones give us the following characterisation of homology isomorphisms:

Corollary 3.3.7 (mapping cone trick)**.** *Let $f\colon X \longrightarrow Y$ be a continuous map, where X is non-empty. Then the following are equivalent:*

1. *For all $k \in \mathbb{Z}$, the induced map $h_k(f)\colon h_k(X) \longrightarrow h_k(Y)$ is an R-isomorphism.*
2. *For all $k \in \mathbb{Z}$, we have $\widetilde{h}_k(\operatorname{Cone}(f)) \cong_R 0$.*
3. *For all $k \in \mathbb{Z}$, the induced map $\widetilde{h}_k(f)\colon \widetilde{h}_k(X) \longrightarrow \widetilde{h}_k(Y)$ is an R-isomorphism.*

Proof. This follows through elementary calculus of exact sequences from Theorem 3.3.6 (check!). $\square$

Outlook 3.3.8 (realisation of modules as homology of spaces)**.** Mapping cones can also be used to prove realisation results for homology theories (Exercise); this leads to so-called *Moore spaces*. For example: If $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ is an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and coefficients (isomorphic to) $\mathbb{Z}$, then for every finitely generated Abelian group A and every $k \in \mathbb{N}_{>0}$, there exists a topological space X with

$$h_k(X) \cong_{\mathbb{Z}} A \quad \text{and} \quad \forall_{\ell\in\mathbb{N}_{>0}\setminus\{k\}} \quad h_\ell(X) \cong_{\mathbb{Z}} 0.$$

If carried out carefully, this can be turned into a functor to $\mathsf{Top}_{\mathsf{h}}$.

$$\underbrace{\text{excision} \qquad \begin{array}{c}\text{long exact sequence of pairs}\\ \Downarrow \\ \text{long exact sequence of triples}\end{array}}$$

$$\Downarrow$$

$$\text{(reduced) Mayer–Vietoris sequence}$$

Figure 3.11.: Properties of sequences of functors on $\mathsf{Top}^2{}_{\mathsf{h}}$

Moreover, we can use mapping cones to replace relative homology by absolute (reduced) homology; under this replacement, the long exact homology sequence of the pair then corresponds to the long exact sequence for the mapping cone.

Proposition 3.3.9 (relative homology via mapping cones)**.** *Let (X, A) be a pair of spaces with $A \neq \emptyset$ and let $i \colon A \longrightarrow X$ be the inclusion. Then the inclusions*

$$(X, A) \longrightarrow \big(\operatorname{Cone}(i), \operatorname{Cone}(A)\big) \longleftarrow \big(\operatorname{Cone}(i), \text{cone tip}\big)$$

induce for every $k \in \mathbb{Z}$ a (natural) R-isomorphism

$$h_k(X, A) \cong_R \widetilde{h}_k\big(\operatorname{Cone}(i)\big).$$

Proof. Let $k \in \mathbb{Z}$. Excision (cutting out "$1/2 \cdot \operatorname{Cone}(A)$") and homotopy invariance show that the inclusion $(X, A) \longrightarrow (\operatorname{Cone}(i), \operatorname{Cone}(A))$ induces a natural R-isomorphism

$$h_k(X, A) \cong_R h_k\big(\operatorname{Cone}(i), \operatorname{Cone}(A)\big).$$

Moreover, Proposition 3.1.4 (and $\operatorname{Cone}(A) \simeq \bullet$) and the expression of reduced homology as homology relative to a point (Remark 3.1.7) show that the inclusion $(\operatorname{Cone}(i), \text{cone tip}) \longrightarrow (\operatorname{Cone}(i), \operatorname{Cone}(A))$ induces a natural R-isomorphism

$$h_k\big(\operatorname{Cone}(i), \operatorname{Cone}(A)\big) \cong_R \widetilde{h}_k\big(\operatorname{Cone}(i)\big).$$

Combining both isomorphisms proves the claim.

Alternatively, one could compare the long exact sequence of the pair (X, A) with the mapping cone sequence for the mapping cone of i. $\square$

(reduced) Mayer–Vietoris sequence

$\swarrow \qquad \searrow$

long exact sequence for mapping cones — excision for the extension to $\mathsf{Top}^2{}_\mathsf{h}$

$\Downarrow$

long exact sequence of pairs for the extension to $\mathsf{Top}^2{}_\mathsf{h}$

Figure 3.12.: Properties of sequences of functors on Top_h

Remark 3.3.10 (Mayer–Vietoris vs. excision)**.** In this way, we obtain the dependency graph for properties of $\mathbb{Z}$-indexed sequences of functors on $\mathsf{Top}^2{}_\mathsf{h}$ in Figure 3.11. Conversely, for $\mathbb{Z}$-indexed functors on Top_h, we have the dependencies in Figure 3.12. These observations show that one can formulate the Eilenberg–Steenrod axioms also for homology theories on Top_h instead of $\mathsf{Top}^2{}_\mathsf{h}$.

3.4 Classification of Homology Theories

Before starting with the construction of concrete examples of homology theories, we briefly survey the overall situation:

Existence of homology theories:

- Examples of ordinary homology theories:
  - singular homology on Top^2 (Chapter 4)
  - cellular homology on the category of relative CW-complexes (Chapter 5)
  - simplicial homology on the category of simplicial complexes (or the category of triangulated/triangulable topological spaces) [54]
  - measure homology on Top^2 [25, 71, 37]
  - ...
- Examples of homology theories that are *not* ordinary:
  - bordism [68, Chapter 21]
  - K-homology (a homological version of topological K-theory [66, Chapter 11])

  – ...

Uniqueness of homology theories:

- Uniqueness of ordinary homology theories on CW-complexes (Corollary 5.2.18)
- Comparison theorem for homology theories on manifolds
- Also the Atiyah-Hirzebruch spectral sequence leads to uniqueness properties of homology theories [66, Theorem 15.7].

Classification of homology theories: Stable homotopy theory allows to classify all homology theories via so-called spectra [66, Chapters 8, 9].

4

Singular Homology

We construct ordinary homology theories with given coefficients, namely so-called singular homology. Geometrically, the idea of singular homology is to detect holes in spaces by engulfing them in arrangements of simplices. Algebraically, the construction relies on chain complexes and their homology.

After describing the construction, we prove that singular homology indeed satisfies the Eilenberg–Steenrod axioms.

Finally, we give applications of singular homology: For example, we prove the Jordan curve/separation theorem and discuss additional geometric structure that is visible on singular homology.

In Chapter 5, we will construct cellular homology, which often leads to smaller chain complexes and easier computations.

Overview of this chapter.

Running example. again: simplices, spheres, tori

4.1 Construction

Singular homology attempts to understand topological space in terms of simple building blocks, the singular simplices; these simplices are "singular" in the sense that we only ask them to be continuous maps, without requiring any analytic or geometric regularity.

Definition 4.1.1 (singular simplex)**.** Let X be a topological space, let $k \in \mathbb{N}$. A *singular k-simplex* is a continuous map $\Delta^k \longrightarrow X$, where Δ^k denotes the k-dimensional standard simplex (Definition 1.1.1). We write

$$S_k(X) := \operatorname{map}(\Delta^k, X)$$

for the set of all singular k-simplices in X.

4.1.1 Geometric Idea

The key idea is to detect "holes" in topological spaces by engulfing them with singular simplices. More precisely (the rigorous definition will be given in Chapter 4.1.2):

- Because single singular simplices, in general, will not be sufficient to catch "holes", we will form so-called *chains* of singular simplices (Figure 4.1).
- Candidates for chains that detect a "hole" are chains that have "no boundary", so-called *cycles* (Figure 4.2).
- Cycles only detect a "proper hole" if they are not the *boundary* of a higher-dimensional chain (Figure 4.3).

Hence, if X is a topological space and $k \in \mathbb{N}$, we will set

$$\text{singular homology of } X \text{ in degree } k := \frac{\text{singular } k\text{-cycles of } X}{\text{boundaries of singular } (k+1)\text{-chains}}.$$

In order to make this construction precise, we use the language of chain complexes and basic homological algebra (Appendix A.6.2). Historically, the terminology in basic homological algebra goes back to topological constructions of this type (such as simplicial homology).

It should be noted that (non-pathological) spaces admit uncountably many singular simplices in each dimension; therefore, the singular chain complex tends to be huge. While this might be bad for concrete calculations, this chain complex is flexible enough to allow for straightforward functoriality with respect to all continuous maps and for proving homotopy invariance.

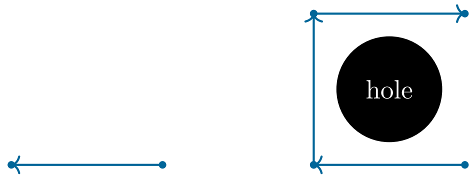


Figure 4.1.: Singular chains

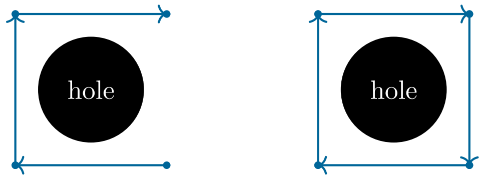


Figure 4.2.: Singular cycle; the chain on the left cannot surround a “hole”, the chain on the right does surround a “hole”.

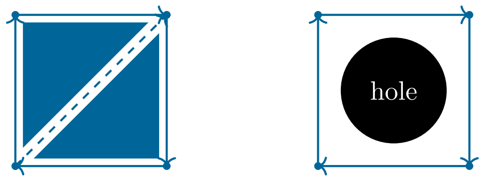


Figure 4.3.: Singular boundary; the cycle on the left is the boundary of a singular chain and hence cannot detect a “hole”; the cycle on the right is *not* the boundary of a singular chain (because the “hole” is in the way).

4.1.2 Singular Homology

Setup 4.1.2. Let R be a ring with unit and let Z be a left R-module.

Singular chains are modelled algebraically as R-linear combinations of singular simplices (which we will describe through the free generation functor); their boundaries are given by the following alternating sums of their facets:

Proposition and Definition 4.1.3 (singular chain complex)**.**

1. *Let X be a topological space and let $k \in \mathbb{Z}$. Then we write*

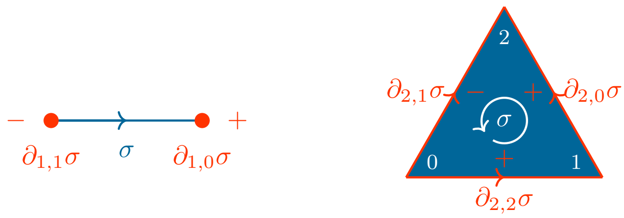


Figure 4.4.: The singular boundary operator in low degrees

$$C_k(X) := \begin{cases} \bigoplus_{\mathrm{map}(\Delta^k, X)} \mathbb{Z} = F\bigl(\mathrm{map}(\Delta^k, X)\bigr) & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases}$$

where $F\colon \mathsf{Set} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ *is the free generation functor (Example 1.2.16). Moreover, we define*

$$\partial_k := \begin{cases} \sum_{j=0}^{k} (-1)^j \cdot \partial_{k,j} & \text{if } k > 0 \\ 0 & \text{if } k \leq 0 \end{cases} : C_k(X) \longrightarrow C_{k-1}(X),$$

where for $k \in \mathbb{N}_{>0}$ *and* $j \in \{0, \dots, k\}$

$$\begin{aligned} i_{k,j}\colon \Delta^{k-1} &\longrightarrow \Delta^k \\ (t_0, \dots, t_{k-1}) &\longmapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{k-1}) \end{aligned}$$

denotes the inclusion of the j*-th face of* Δ^k *and where (Figure 4.4)*

$$\begin{aligned} \partial_{k,j} := F\bigl(\mathrm{map}(i_{k,j}, X)\bigr)\colon C_k(X) &\longrightarrow C_{k-1}(X) \\ S_k(X) \ni \sigma &\longmapsto \sigma \circ i_{k,j}. \end{aligned}$$

Then $C(X) := \bigl(C_*(X), \partial_*\bigr)$ *is a chain complex of* $\mathbb{Z}$*-modules (Definition A.6.9), the* singular chain complex of X*.*

2. *Let* $f\colon X \longrightarrow Y$ *be a continuous map between topological spaces. For* $k \in \mathbb{N}$*, we set*

$$\begin{aligned} C_k(f) := F\bigl(\mathrm{map}(\Delta^k, f)\bigr)\colon C_k(X) &\longrightarrow C_k(Y) \\ S_k(X) \ni \sigma &\longmapsto f \circ \sigma \end{aligned}$$

and for $k \in \mathbb{Z}_{<0}$*, we set* $C_k(f) := 0$*. Then* $C(f) := \bigl(C_k(f)\bigr)_{k \in \mathbb{Z}}$ *is a chain map* $C(X) \longrightarrow C(Y)$ *(Definition A.6.13).*

3. *This defines a functor* $\mathsf{Top} \longrightarrow {}_{\mathbb{Z}}\mathsf{Ch}$*.*

Proof. Ad 1. Let $k \in \mathbb{Z}$. We need to show that $\partial_k \circ \partial_{k+1} = 0$. Without loss of generatliy, we may assume that $k > 0$. Moreover, as $S_{k+1}(X)$ is a $\mathbb{Z}$-basis of $C_{k+1}(X)$, it suffices to show that $\partial_k \circ \partial_{k+1}(\sigma) = 0$ holds for all $\sigma \in \operatorname{map}(\Delta^{k+1}, X)$. Using the fact that

$$i_{k+1,j} \circ i_{k,r} = i_{k+1,r} \circ i_{k,j-1}.$$

holds for all $j \in \{0, \dots, k+1\}$, $r \in \{0, \dots, j-1\}$ (check!), we obtain

$$\begin{aligned}
\partial_k \circ \partial_{k+1}(\sigma) &= \partial_k\biggl(\sum_{j=0}^{k+1} (-1)^j \cdot \sigma \circ i_{k+1,j}\biggr) \\
&= \sum_{j=0}^{k+1} \sum_{r=0}^{k} (-1)^{j+r} \cdot \sigma \circ i_{k+1,j} \circ i_{k,r} \\
&= \sum_{j=0}^{k+1} \sum_{r=0}^{j-1} (-1)^{j+r} \cdot \sigma \circ i_{k+1,j} \circ i_{k,r} && (\text{“}r < j\text{”}) \\
&+ \sum_{j=0}^{k+1} \sum_{r=j}^{k} (-1)^{j+r} \cdot \sigma \circ i_{k+1,j} \circ i_{k,r} && (\text{“}r \geq j\text{”}) \\
&= \sum_{j=0}^{k+1} \sum_{r=0}^{j-1} (-1)^{j+r} \cdot \sigma \circ i_{k+1,r} \circ i_{k,j-1} && (\text{identity above}) \\
&+ \sum_{j=0}^{k+1} \sum_{r=j}^{k} (-1)^{j+r} \cdot \sigma \circ i_{k+1,j} \circ i_{k,r} \\
&= \sum_{r=0}^{k+1} \sum_{j=r+1}^{k+1} (-1)^{j+r} \cdot \sigma \circ i_{k+1,r} \circ i_{k,j-1} && (\text{reorder}) \\
&+ \sum_{j=0}^{k+1} \sum_{r=j}^{k} (-1)^{j+r} \cdot \sigma \circ i_{k+1,j} \circ i_{k,r};
\end{aligned}$$

as both summands only differ by their sign, this implies $\partial_k \circ \partial_{k+1}(\sigma) = 0$.

Ad 2. Again, we only need to consider positive degrees and we only need to show the compatibilitiy of the boundary operator with $C(f)$ on the basis of singular simplices. Let $k \in \mathbb{N}_{>0}$ and let $\sigma \in \operatorname{map}(\Delta^k, X)$. Then, by definition,

$$\begin{aligned}
C_{k-1}(f)\bigl(\partial_k(\sigma)\bigr) &= C_{k-1}(f)\biggl(\sum_{j=0}^{k} (-1)^j \cdot \sigma \circ i_{k,j}\biggr) \\
&= \sum_{j=0}^{k} (-1)^j \cdot f \circ \sigma \circ i_{k,j} = \partial_k(f \circ \sigma) = \partial_k\bigl(C_k(f)(\sigma)\bigr).
\end{aligned}$$

Ad 3. This is a straightforward calculation (check!). □

Outlook 4.1.4 (singular chain complex via simplicial sets)**.** The construction of the singular chain complex in Proposition 4.1.3 coincides with the following construction: Let Δ be the simplex category (Definition 1.2.10). We consider the functor

$$\Delta^{\mathrm{op}} \times \mathsf{Top} \xrightarrow{\Delta_{\mathsf{Top}}^{\mathrm{op}} \times \mathrm{id}_{\mathsf{Top}}} \mathsf{Top}^{\mathrm{op}} \times \mathsf{Top} \xrightarrow{\mathrm{map}(\,\cdot\,,\,\cdot\,)} \mathsf{Set} \xrightarrow{\quad F \quad} {}_{\mathbb{Z}}\mathsf{Mod},$$

where $\Delta_{\mathsf{Top}} \colon \Delta \longrightarrow \mathsf{Top}$ translates abstract simplices and monotonic maps into standard simplices and their corresponding affine linear maps. The composition above then induces a functor $\mathsf{Top} \longrightarrow \Delta({}_{\mathbb{Z}}\mathsf{Mod})$ to simplicial $\mathbb{Z}$-modules and thus a functor

$$\mathsf{Top} \longrightarrow \Delta({}_{\mathbb{Z}}\mathsf{Mod}) \xrightarrow{\quad C \quad} {}_{\mathbb{Z}}\mathsf{Ch}$$

(Example A.6.17). This is the same as the singular chain complex functor from Proposition 4.1.3.

Outlook 4.1.5 (simplices vs. cubes)**.** In principle, similar constructions can also be carried out with other combinatorial models of balls than the standard simplex. For example, one could use singular cubes (thus obtaining the cubical singular chain complex and cubical singular homology) [52].

- Simplices have the advantage that the number of vertices and facets is linear in the dimension; this allows for efficient notation. However, products of simplices in general are not simplices again. Therefore, in the context of products, additional combinatorial overhead occurs.

- Cubes have more vertices/facets than simplices. But products of cubes are cubes again. Therefore, for instance, proving homotopy invariance of cubical singular homology is slightly easier than proving homotopy invariance of classical singular homology.

We could now define singular homology as algebraic homology of this chain complex. However, we prefer to first generalise the whole setting to pairs of spaces and general (constant) coefficients, using tensor products of modules with chain complexes (Example A.6.16).

Definition 4.1.6 (singular chain complex with (constant) coefficients)**.**

- If (X, A) is a pair of spaces, we define

  $$C(X, A; Z) := Z \otimes_{\mathbb{Z}} C(X) \bigm/ \mathrm{im}(Z \otimes_{\mathbb{Z}} C(A \hookrightarrow X)) \in \mathrm{Ob}({}_{R}\mathsf{Ch}),$$

  where the boundary operator is the one induced from the boundary operator on $C(X)$ (this is well-defined; check!). This is the *singular chain complex of* (X, A) *with* Z*-coefficients*. We abbreviate $C(X; Z) := C(X, \emptyset; Z)$.

- If $f\colon (X,A) \longrightarrow (Y,B)$ is a continuous map of pairs, then we define $C(f;Z) = (C_k(f;Z))_{k\in\mathbb{Z}}$ via

$$\begin{aligned} C_k(f;Z)\colon C_k(X,A;Z) &\longrightarrow C_k(Y,B;Z) \\ [c] &\longmapsto \bigl[(Z \otimes_{\mathbb{Z}} C_k(f))(c)\bigr] \end{aligned}$$

  for all $k \in \mathbb{Z}$ (this is a well-defined chain map; check!).

This defines a functor $C(\,\cdot\,,\,\cdot\,;Z)\colon \mathsf{Top}^2 \longrightarrow {}_R\mathsf{Ch}$ (check!).

In more hands-on terms: Chains in $C(X,A;Z)$ can be viewed as formal linear combinations of singular simplices in X with coefficients in Z, where we ignore all terms that live on the subspace A. Elements in the image of the boundary operator are called *singular boundaries* and elements in the kernel of the boundary operator are called *singular cycles*.

Example 4.1.7 (singular chains with coefficients)**.**

- We consider S^1: Let

$$\begin{aligned} \sigma\colon \Delta^1 &\longrightarrow S^1 \\ (1-t,t) &\longmapsto [t] \end{aligned}$$

  be the singular simplex wrapping once around S^1. Then the singular chain $1 \otimes \sigma \in C_1(S^1;\mathbb{Z})$ is a cycle, because

$$\begin{aligned} \partial_1 \sigma &= \sigma \circ i_{1,0} - \sigma \circ i_{1,1} \\ &= \sigma \circ \operatorname{const}_{(0,1)} - \sigma \circ \operatorname{const}_{(1,0)} \\ &= 0 \text{ in } C_0(S^1). \end{aligned}$$

  In this case, we also briefly write $1 \cdot \sigma$ or σ instead of $1 \otimes \sigma$ in order to unclutter the notation. By the same argument, also $1/2 \cdot \sigma \in C_1(S^1;\mathbb{Q})$ is a cycle.

- We consider (D^1, S^0): Let

$$\begin{aligned} \sigma\colon \Delta^1 &\longrightarrow D^1 \\ (1-t,t) &\longmapsto 2\cdot t - 1 \end{aligned}$$

  be the singular simplex dissecting D^2 "horizontally". Then $1 \otimes \sigma \in C_1(D^1,\emptyset;\mathbb{Z})$ is *not* a cycle (check!), but $1 \otimes \sigma$ represents a cycle in $C_1(D^1,S^0;\mathbb{Z})$, because

$$\begin{aligned} \partial_1 \sigma &= \sigma \circ i_{1,0} - \sigma \circ i_{1,1} \\ &= \operatorname{const}_1 - \operatorname{const}_{-1} \\ &\in \operatorname{im}\bigl(\mathbb{Z} \otimes_{\mathbb{Z}} C_0(S_0 \hookrightarrow D^1)\bigr). \end{aligned}$$

Definition 4.1.8 (singular homology with (constant) coefficients)**.** For $k \in \mathbb{Z}$, we define the *k-th singular homology with coefficients in Z* as the composition

$$H_k(\,\cdot\,,\,\cdot\,;Z) := H_k \circ \bigl(C(\,\cdot\,,\,\cdot\,;Z)\bigr)\colon \mathsf{Top}^2 \longrightarrow {}_R\mathsf{Mod}$$

of functors. Here, H_k on the right hand side denotes the homology functor ${}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Mod}$ of R-chain complexes (Definition A.6.18 and Proposition A.6.21). More explicitly, if (X,A) is a pair of spaces, then

$$H_k(X,A;Z) = \frac{\ker\bigl(\partial_k\colon C_k(X,A;Z) \to C_{k-1}(X,A;Z)\bigr)}{\operatorname{im}\bigl(\partial_{k+1}\colon C_{k+1}(X,A;Z) \to C_k(X,A;Z)\bigr)}.$$

If X is a topological space, then we also abbreviate $H_*(X;Z) := H_*(X,\emptyset;Z) = \bigl(H_k(X,\emptyset;Z)\bigr)_{k\in\mathbb{Z}}$.

The construction of singular homology is not only functorial in pairs of spaces but also in the coefficient module.

4.1.3 First Steps

Setup 4.1.9. Let R be a ring with unit and let Z be a left R-module.

Example 4.1.10 (singular homology of the empty set)**.** By definition, $C_k(\emptyset;Z) \cong_R 0$ for all $k \in \mathbb{Z}$. Therefore, for all $k \in \mathbb{Z}$, we obtain

$$H_k(\emptyset;Z) \cong_R 0.$$

Remark 4.1.11 (singular homology in negative degrees)**.** Let (X,A) be a pair of spaces. Then, by construction, we have $C_k(X,A;Z) \cong_R 0$ for all $k \in \mathbb{Z}_{<0}$. Therefore,

$$H_k(X,A;Z) \cong_R 0.$$

for all $k \in \mathbb{Z}_{<0}$.

Example 4.1.12 (singular homology of the point)**.** The singular chain complex $C(\bullet;Z)$ of the point $\bullet$ with Z-coeffcients looks as follows (check!):

$$\begin{array}{lccccccccccccc}
\text{degree} & & & 2 & & 1 & & 0 & & -1 & & & & \\
\cdots & \xrightarrow[\mathrm{id}_Z]{} & Z & \xrightarrow[0]{} & Z & \xrightarrow[\mathrm{id}_Z]{} & Z & \xrightarrow[0]{} & Z & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} \cdots
\end{array}$$

Therefore, we obtain

$$\forall_{k\in\mathbb{Z}}\quad H_k(\bullet;Z) \cong_R \begin{cases} Z & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{Z}\setminus 0. \end{cases}$$

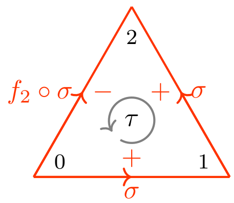


Figure 4.5.: The 2-simplex τ from Example 4.1.13, schematically

Example 4.1.13 (algebraic vs. geometric multiples)**.** Let

$$\begin{aligned} \sigma \colon \Delta^1 &\longrightarrow S^1 \\ (1-t, t) &\longmapsto [t] \end{aligned}$$

(Example 4.1.7). Then $\sigma \in C_1(S^1; \mathbb{Z})$ is a cycle and the chain $1 \cdot \tau \in C_2(S^1; \mathbb{Z})$ (Figure 4.5) with

$$\begin{aligned} \tau \colon \Delta^2 &\longrightarrow S^1 \\ (t_0, t_1, t_2) &\longmapsto [t_2 - t_0 \mod 1] \end{aligned}$$

satisfies

$$\begin{aligned} \partial_2 \tau &= \tau \circ i_{2,0} - \tau \circ i_{2,1} + \tau \circ i_{2,2} \\ &= \bigl((t_0, t_1) \mapsto [t_1]\bigr) - \bigl((t_0, t_1) \mapsto [t_1 - t_0 \mod 1]\bigr) + \bigl((t_0, t_1) \mapsto [-t_0 \mod 1]\bigr) \\ &= \bigl((t_0, t_1) \mapsto [t_1]\bigr) \\ &\quad - \bigl((t_0, t_1) \mapsto [t_1 - (1 - t_1) \mod 1]\bigr) && \text{(convex coordinates!)} \\ &\quad + \bigl((t_0, t_1) \mapsto [-(1 - t_1) \mod 1]\bigr) && \text{(convex coordinates!)} \\ &= \sigma - \bigl((t_0, t_1) \mapsto [2 \cdot t_1 \mod 1]\bigr) + \sigma \\ &= 2 \cdot \sigma - f_2 \circ \sigma. \end{aligned}$$

Therefore,

$$2 \cdot [\sigma] = \bigl[f_2 \circ \sigma\bigr] = H_1(f_2; \mathbb{Z})\bigl([\sigma]\bigr)$$

in $H_1(S^1; \mathbb{Z})$; here, $f_2 \colon S^1 \longrightarrow S^1$ is defined as in Corollary 3.2.11. Similarly, we also obtain $d \cdot [\sigma] = [f_d \circ \sigma]$ for all $d \in \mathbb{Z}$.

Singular homology satisfies the following strong version of additivity; here, "strong" refers to the fact that, in general, path-connected components are neither closed nor open (and hence, in general, topological spaces do *not* carry the disjoint union topology of their path-connected components).

Proposition 4.1.14 (strong additivity of singular homology)**.** *Let X be a topological space and let $(X_i)_{i \in I}$ be the family of the path-connected components* *of X. Then the inclusions $(X_i \hookrightarrow X)_{i\in I}$ induce for every $k \in \mathbb{Z}$ an R-isomorphism*

$$\bigoplus_{i\in I} H_k(X_i; Z) \longrightarrow H_k(X; Z).$$

Proof. We only need to consider the case $k \in \mathbb{N}$. Clearly, Δ^k is path-connected; hence, the inclusions $(X_i \hookrightarrow X)_{i\in I}$ induce a bijection

$$\coprod_{i\in I} \operatorname{map}(\Delta^k, X_i) \longrightarrow \operatorname{map}(\Delta^k, X).$$

Thus, these inclusions $(X_i \hookrightarrow X)_{i\in I}$ induce an isomorphism

$$\bigoplus_{i\in I} C(X_i; Z) \longrightarrow C(X; Z)$$

in ${}_R\mathsf{Ch}$ (the direct sum of chain complexes is defined as degree-wise direct sum of the chain modules and boundary operators). Because homology of chain complexes is compatible with direct sums (check!), the claim follows. □

Theorem 4.1.15 (singular homology in degree 0)**.**

1. *If X is a path-connected, non-empty topological space, then the constant map $c\colon X \longrightarrow \bullet$ induces an R-isomorphism*

$$H_0(c; Z)\colon H_0(X; Z) \longrightarrow H_0(\bullet; Z).$$

2. *Therefore, $H_0(\,\cdot\,; Z)\colon \mathsf{Top} \longrightarrow {}_R\mathsf{Mod}$ is naturally isomorphic to*

$$Z \otimes_{\mathbb{Z}} (F \circ \pi_0^+)\colon \mathsf{Top} \longrightarrow {}_R\mathsf{Mod},$$

where $F\colon \mathsf{Set} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is the free generation functor (Example 1.2.16) and

$$\pi_0^+ := [\bullet, \,\cdot\,]\colon \mathsf{Top} \longrightarrow \mathsf{Set}$$

is the path-component functor.

Proof. The second part immediately follows from the first part and strong additivity of singular homology (Proposition 4.1.14).

Therefore, it suffices to prove the first part: Let $x_0 \in X$ and let $i\colon \bullet \longrightarrow \{x_0\} \hookrightarrow X$ be the inclusion. Because of $c \circ i = \mathrm{id}_\bullet$, functoriality of $H_0(\,\cdot\,; Z)$ shows that

- $H_0(c; Z)$ is surjective and that
- $H_0(c; Z)$ is injective provided that $H_0(i; Z)$ is surjective.

We prove that $H_0(i; Z)$ is surjective: Let $z = \sum_{j=1}^m a_j \cdot x_j \in C_0(X; Z)$ be a 0-cycle with $a_1, \dots, a_m \in Z$ and $x_1, \dots, x_m \in X$; here, we denote singular

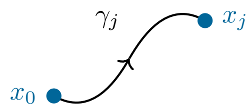


Figure 4.6.: Connecting singular 0-simplices in a path-connected space by a singular 1-simplex

0-simplices just by the corresponding points in the space. Because X is path-connected, for each $j \in \{1, \dots, m\}$, there is a path $\gamma_j \colon \Delta^1 \longrightarrow X$ with

$$\partial_{1,0}(\gamma_j) = x_j \quad \text{and} \quad \partial_{1,1}(\gamma_j) = x_0$$

(Figure 4.6). Then, we obtain in $H_1(X; Z)$:

$$\begin{aligned} [z] &= \biggl[z - \partial_1 \Bigl(\sum_{j=1}^m a_j \cdot \gamma_j\Bigr)\biggr] \\ &= \biggl[\sum_{j=1}^m a_j \cdot x_0\biggr] \in \operatorname{im} H_0(i; Z). \end{aligned}$$

Therefore, $H_0(i; Z)$ is surjective, as claimed. $\square$

In particular, singular homology (with suitable coefficients) is able to detect the number of path-connected components of topological spaces. For example, we will use this when proving the Jordan curve theorem (Theorem 4.4.5).

4.1.4 The Long Exact Sequence of Pairs

Theorem 4.1.16 (long exact sequence of pairs in singular homology)**.** *Let R be a ring with unit and let Z be a left R-module. Let (X, A) be a pair of spaces and let $i \colon A \hookrightarrow X$ and $j \colon (X, \emptyset) \hookrightarrow (X, A)$ be the inclusions. Then*

$$\cdots \xrightarrow{\partial_{k+1}} H_k(A; Z) \xrightarrow{H_k(i;Z)} H_k(X; Z) \xrightarrow{H_k(j;Z)} H_k(X, A; Z) \xrightarrow{\partial_k} H_{k-1}(A; Z) \longrightarrow \cdots$$

is a natural long exact sequence in ${}_R\mathsf{Mod}$, where for $k \in \mathbb{N}_{>0}$ the connecting homomorphism has the explicit description

$$\begin{aligned} \partial_k \colon H_k(X, A; Z) &\longrightarrow H_{k-1}(A; Z) \\ \bigl[(c \in C_k(X; Z)) + Z \otimes_{\mathbb{Z}} \operatorname{im} C_k(i)\bigr] &\longmapsto \bigl[\partial_k(c)\bigr]. \end{aligned}$$

The naturality refers to both naturality in pairs of spaces and in the coefficients.

In particular, for every $k \in \mathbb{Z}$, we have natural transformations

$$\partial_k \colon H_k(\,\cdot\,,\,\cdot\,;Z) \Longrightarrow H_{k-1}(\,\cdot\,,\,\cdot\,;Z) \circ U,$$

where $U\colon \mathsf{Top}^2 \longrightarrow \mathsf{Top}^2$ is the subspace functor (Definition 3.1.1).

Proof. We will derive the claim from the algebraic long exact homology sequence associated with degree-wise short exact sequences of chain complexes (Proposition A.6.23). To this end, we consider the (natural) sequence

$$0 \longrightarrow C(A;Z) \xrightarrow{C(i;Z)} C(X;Z) \xrightarrow{\text{projection}} C(X,A;Z) \longrightarrow 0$$

in ${}_R\mathsf{Ch}$.

We explain why this sequence is degree-wise exact: Exactness at $C(X,A;Z)$ and $C(X;Z)$ is immediate from the construction. What about exactness at $C(A;Z)$? The injection $C(i)\colon C(A) \longrightarrow C(X)$ is *split* injective (because it is induced on the standard bases by the injection mapping singular simplices of A to singular simplices of X via the inclusion $A \longrightarrow X$); therefore, also $Z \otimes_{\mathbb{Z}} C(i)$ is split injective, and so the sequence above is also exact at $C(A;Z)$.

Applying the algebraic long exact homology sequence (Proposition A.6.23, including the construction of the connecting homomorphism) finishes the proof. $\Box$

Our next goal is to prove that

$$\bigl(\bigl(H_k(\,\cdot\,,\,\cdot\,;Z)\bigr)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$$

is an additive ordinary homology theory on Top^2 with values in ${}_R\mathsf{Mod}$ and coefficients (isomorphic to) Z. As we have already shown that

- we have a long exact sequence of pairs (Theorem 4.1.16), that
- singular homology of the point $\bullet$ is concentrated in degree 0 and isomorphic to Z (Theorem 4.1.15), and that
- singular homology ist strongly additive (Proposition 4.1.14),

it remains to prove homotopy invariance and exicision.

4.2 Homotopy Invariance

We prove that singular homology is homotopy invariant; the key idea is to realise that the singular chain maps of homotopic continuous maps satisfy

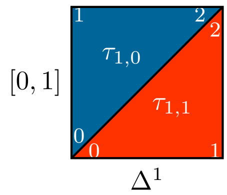


Figure 4.7.: Triangulating the prism $\Delta^1 \times [0, 1]$

an algebraic version of being homotopic, namely being chain homotopic (Appendix A.6.3). Then a simple algebraic argument shows that the induced maps on homology coincide.

Theorem 4.2.1 (homotopy invariance of singular homology)**.** *Let R be a be a ring with unit and let $Z \in \operatorname{Ob}({}_R\mathsf{Mod})$. Let (X, A), (Y, B) be pairs of spaces and let $f, g\colon (X, A) \longrightarrow (Y, B)$ be continuous maps of pairs with $f \simeq_{A,B} g$.*

1. *Then*
$$C(f; Z) \simeq_{{}_R\mathsf{Ch}} C(g; Z)\colon C(X, A; Z) \longrightarrow C(Y, B; Z).$$

2. *In particular: For all $k \in \mathbb{Z}$, we have*
$$H_k(f; Z) = H_k(g; Z)\colon H_k(X, A; Z) \longrightarrow H_k(Y, B; Z).$$
I.e., $H_k(\,\cdot\,, \,\cdot\,; Z)\colon \mathsf{Top}^2 \longrightarrow {}_R\mathsf{Mod}$ is a homotopy invariant functor.

4.2.1 Geometric Idea

In order to prove the first part of Theorem 4.2.1, we first establish the claim in the model case, i.e., for the inclusions $X \longrightarrow X \times [0, 1]$ of bottom and top into the cylinder $X \times [0, 1]$. Using functoriality and a homotopy between the given maps f and g, we can then easily derive the general case from this special case.

The main, technical, difficulty is that, for $k \in \mathbb{N}_{>0}$, the prism $\Delta^k \times [0, 1]$ with its combinatorial structure as a polyhedron is *not* a simplex. Therefore, we decompose such prisms systematically into $(k + 1)$-simplices.

4.2.2 Decomposition of Prisms

Lemma 4.2.2 (decomposition of prisms)**.** *Let X be a topological space. Then the sequence $(h_{X,k})_{k \in \mathbb{Z}}$ with*

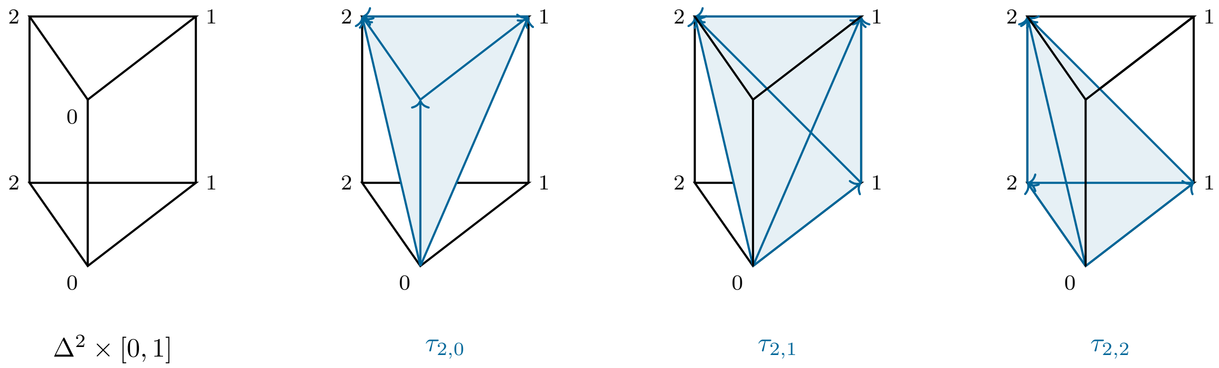


Figure 4.8.: Triangulating the prism $\Delta^2 \times [0,1]$

$$
\begin{aligned}
h_{X,k} \colon C_k(X) &\longrightarrow C_{k+1}\bigl(X \times [0,1]\bigr)\\
\operatorname{map}(\Delta^k, X) \ni \sigma &\longmapsto \sum_{j=0}^{k} (-1)^j \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j}
\end{aligned}
$$

for all $k \in \mathbb{N}$ and $h_{X,k} := 0$ for $k \in \mathbb{Z}_{<0}$ is a chain homotopy in ${}_{\mathbb{Z}}\mathsf{Ch}$ from $C(i_0)$ to $C(i_1)$. Here, $i_0, i_1 \colon X \hookrightarrow X \times [0,1]$ denote the inclusions of the bottom and the top of the cylinder $X \times [0,1]$, respectively, and for all $k \in \mathbb{N}$ and $j \in \{0, \dots, k\}$, we set (Figure 4.7, Figure 4.8)

$$
\begin{aligned}
\tau_{k,j} \colon \Delta^{k+1} &\longrightarrow \Delta^k \times [0,1]\\
(t_0, \dots, t_{k+1}) &\longmapsto \bigl((t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_{k+1}), t_{j+1} + \dots + t_{k+1}\bigr).
\end{aligned}
$$

This chain homotopy is natural in the following sense: For all continuous maps $f \colon X \longrightarrow Y$ and all $k \in \mathbb{Z}$, we have

$$
C_{k+1}(f \times \mathrm{id}_{[0,1]}) \circ h_{X,k} = h_{Y,k} \circ C_k(f).
$$

Proof. Let $k \in \mathbb{N}$ and $\sigma \in \operatorname{map}(\Delta^k, X)$. Then, by construction,

$$
\begin{aligned}
\partial_{k+1} \circ h_{X,k}(\sigma) &= \sum_{r=0}^{k+1} \sum_{j=0}^{k} (-1)^{j+r} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j} \circ i_{k+1,r}\\
&= (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,0} \circ i_{k+1,0} && \text{(index: } (0,0))\\
&\quad + \sum \text{“inner” faces} && \text{(index: } (j+1,j),\ (j+1,j+1))\\
&\quad + \sum \text{“outer” faces} && \text{(index: } (r,j) \text{ with } j \notin \{r, r-1\})\\
&\quad + (-1)^{k+k+1} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,k} \circ i_{k+1,k+1}. && \text{(index: } (k+1,k))
\end{aligned}
$$

We show that the sum of the "inner" faces is 0 and that the sum of the "outer" faces equals $-h_{X,k-1} \circ \partial_k(\sigma)$:

- The sum of the "inner" faces is

$$\begin{aligned}\sum \text{"inner" faces} &= \sum_{j=0}^{k-1} (-1)^{j+j+1} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j} \circ i_{k+1,j+1} \\ &\quad + \sum_{j=0}^{k-1} (-1)^{j+1+j+1} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j+1} \circ i_{k+1,j+1}.\end{aligned}$$

  By construction, $\tau_{k,j} \circ i_{k+1,j+1} = \tau_{k,j+1} \circ i_{k+1,j+1}$ for all $j \in \{0,\dots,k-1\}$. Hence, every "inner" face occurs exactly twice and with opposite signs. In particular, $\sum$ "inner" faces $= 0$.

- The sum of the "outer" faces is

$$\begin{aligned}\sum \text{"outer" faces} &= \sum_{r=0}^{k+1} \sum_{j=0}^{r-2} (-1)^{j+r} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j} \circ i_{k+1,r} \\ &\quad + \sum_{r=0}^{k+1} \sum_{j=r+1}^{k} (-1)^{j+r} \cdot (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,j} \circ i_{k+1,r} \\ &= \sum_{r=0}^{k+1} \sum_{j=0}^{r-2} (-1)^{j+r} \cdot \big((\sigma \circ i_{k,r-1}) \times \mathrm{id}_{[0,1]}\big) \circ \tau_{k-1,j} \\ &\quad + \sum_{r=0}^{k+1} \sum_{j=r+1}^{k} (-1)^{j+r} \cdot \big((\sigma \circ i_{k,r}) \times \mathrm{id}_{[0,1]}\big) \circ \tau_{k-1,j-1} \\ &= \sum_{r=0}^{k-1} \sum_{j=0}^{k} (-1)^{j+r+1} \cdot \big((\sigma \circ i_{k,j}) \times \mathrm{id}_{[0,1]}\big) \circ \tau_{k-1,r} \\ &= -h_{X,k-1}\big(\partial_k(\sigma)\big).\end{aligned}$$

Moreover,

$$\begin{aligned}(\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,0} \circ i_{k+1,0} &= i_1 \circ \sigma && (\text{"top" of the prism}) \\ (\sigma \times \mathrm{id}_{[0,1]}) \circ \tau_{k,k} \circ i_{k+1,k+1} &= i_0 \circ \sigma. && (\text{"bottom" of the prism})\end{aligned}$$

Therefore, we obtain

$$\partial_{k+1} \circ h_{X,k} = C_k(i_1) - h_{X,k-1} \circ \partial_k - C_k(i_0),$$

as desired.

Naturality follows from the construction: This chain homotopy is constructed via composition from the right and the chain maps induced from continuous maps are constructed via composition from the left. □

Study note. How can one use the prism decomposition in dimension 2 (Figure 4.8) to compute the volume of pyramids with triangular base (assuming Cavalieri's principle)?

Caveat 4.2.3. Let X be a topological space and let $k \in \mathbb{N}$. If $c = \sum_{j=1}^{m} a_j \cdot \sigma_j$ and $c' = \sum_{j=1}^{m} a_j \cdot \sigma'_j \in C_k(X)$ are singular cycles with

$$\forall_{j \in \{1,\dots,m\}} \quad \sigma_j \simeq_{\mathsf{Top}} \sigma'_j,$$

then, in general, we *cannot* conclude that $[c] = [c'] \in H_k(X;\mathbb{Z})$. The corresponding homotopies of the singular simplices are, in general, on $\partial\Delta^k$ *not* compatible; therefore, the prism decomposition does *not* provide a chain $b \in C_{k+1}(X)$ with $\partial_{k+1}(b) = c - c'$.

This fact is essential for the construction of singular homology: Because Δ^k is contractible, *all* continuous maps $\Delta^k \longrightarrow X$ are homotopic (if X is path-connected) ....

If in the situation above, there exist homotopies between the singular simplices that are "compatible on the boundary", then one can indeed construct such a boundary $\partial_{k+1}(b)$ (Exercise). This is, for example, useful

- when comparing smooth singular chains in smooth manifolds with ordinary singular chains [36, proof of Theorem 16.6],
- when proving weak homotopy invariance of singular homology [68, Chapter 9.5], or
- when proving the Hurewicz theorem (Theorem 4.5.6).

4.2.3 Proving Homotopy Invariance

Proof of Theorem 4.2.1. In view of homotopy invariance of homology of chain complexes (which is a purely algebraic fact; Proposition A.6.35), it suffices to prove the first part.

Let $h\colon (X,A) \times [0,1] \longrightarrow (Y,B)$ be a homotopy in Top^2 from f to g. In particular, $h \circ i_0 = f$ and $h \circ i_1 = g$, where $i_0, i_1\colon X \longrightarrow X \times [0,1]$ are the inclusion of the bottom and the top, respectively; thus, we can apply the model case of Lemma 4.2.2: Using Lemma 4.2.2 (and a little calculation; check!), it follows that

$$\bigl(C_{k+1}(h) \circ h_{X,k}\colon C_k(X) \longrightarrow C_{k+1}(Y)\bigr)_{k\in\mathbb{Z}}$$

is a chain homotopy in ${}_{\mathbb{Z}}\mathsf{Ch}$ from

$$C(h) \circ C(i_0) = C(h \circ i_0) = C(f)$$

to

$$C(h) \circ C(i_1) = C(h \circ i_1) = C(g).$$

Furthermore, the naturality of the construction in Lemma 4.2.2 shows that

$$\operatorname{im}\bigl(C_{k+1}(h) \circ h_{X,k} \circ C_k(A \hookrightarrow X)\bigr) \subset \operatorname{im}\bigl(C_{k+1}(B \hookrightarrow Y)\bigr)$$

for all $k \in \mathbb{Z}$. Therefore, taking the tensor product $Z \otimes_{\mathbb{Z}} \cdot$ and the quotient by the singular chains living on the subspaces, leads to a well-defined chain homotopy

$$C(f; Z) \simeq_{{}_R\mathsf{Ch}} C(g; Z) \colon C(X, A; Z) \longrightarrow C(Y, B; Z)$$

in ${}_R\mathsf{Ch}$. □

4.3 Excision

It remains to prove that singular homology satisfies excision:

Theorem 4.3.1 (excision in singular homology)**.** *Let R be a ring with unit, let Z be a left R-module, and let $k \in \mathbb{Z}$. Let (X, A) be a pair of spaces and let $B \subset X$ with $\overline{B} \subset A^\circ$. Then the inclusion $(X \setminus B, A \setminus B) \longrightarrow (X, A)$ induces an R-isomorphism*

$$H_k(X \setminus B, A \setminus B; Z) \longrightarrow H_k(X, A; Z).$$

4.3.1 Geometric Idea

The main, technical, difficulty in the proof of Theorem 4.3.1 is that singular simplices in X need not lie in $X \setminus B$ or A (Figure 4.9). Therefore, as in the proof of the Seifert and van Kampen theorem (Theorem 2.2.6), we will subdivide singular simplices into "small" simplices, which lie in $X \setminus B$ or A.

This subdivision will be achieved via iterated barycentric subdivision (Figure 4.10), a systematic way of subdividing simplices.

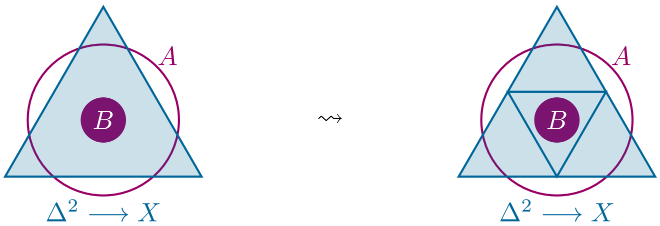


Figure 4.9.: A singular simplex in X that lies neither in $X \setminus B$ nor in A; and a subdivision into "small" simplices

4.3.2 Barycentric Subdivision

The barycentric subdivision of a simplex is obtained by inductively coning off lower dimensional simplices with the barycentre as new cone point (Figure 4.10). We start by subdividing simplices in convex spaces.

Definition 4.3.2 (barycentric subdivision)**.** Let $\mathbb{R}^\infty := \bigoplus_{\mathbb{N}} \mathbb{R}$ (with the topology induced by the Euclidean metric, i.e., the ℓ^2-distance).

- The *barycentre* of a singular simplex $\sigma\colon \Delta^k \longrightarrow \mathbb{R}^\infty$ is given by

$$\beta(\sigma) := \frac{1}{k+1} \cdot \sum_{j=0}^{k} \sigma(e_{j+1}) \in \mathbb{R}^\infty.$$

- Let $v \in \mathbb{R}^\infty$. Then the *cone operator for v* is defined by

$$\begin{aligned} v * \cdot \,\colon C_k(\mathbb{R}^\infty) &\longrightarrow C_{k+1}(\mathbb{R}^\infty) \\ \operatorname{map}(\Delta^k, \mathbb{R}^\infty) \ni \sigma &\longmapsto \begin{pmatrix} \Delta^{k+1} & \longrightarrow & \mathbb{R}^\infty \\ (t_0, \dots, t_{k+1}) & \longmapsto & t_0 \cdot v + (1 - t_0) \cdot \sigma\bigl(\frac{t_1}{1-t_0}, \dots, \frac{t_{k+1}}{1-t_0}\bigr) \end{pmatrix} \end{aligned}$$

  for all $k \in \mathbb{N}$ (and by $v * \cdot := 0$ for $k \in \mathbb{Z}_{<0}$).

- The *barycentric subdivision* $B\colon C(\mathbb{R}^\infty) \longrightarrow C(\mathbb{R}^\infty)$ is defined inductively as follows:
  - For all $k \in \mathbb{Z}_{<0}$, let $B_k := 0$.
  - Let $B_0 := \mathrm{id}_{C_0(\mathbb{R}^\infty)}$.
  - For all $k \in \mathbb{N}_{>0}$,

$$\begin{aligned} B_k\colon C_k(\mathbb{R}^\infty) &\longrightarrow C_k(\mathbb{R}^\infty) \\ \operatorname{map}(\Delta^k, \mathbb{R}^\infty) \ni \sigma &\longmapsto \beta(\sigma) * \bigl(B_{k-1}(\partial_k \sigma)\bigr). \end{aligned}$$

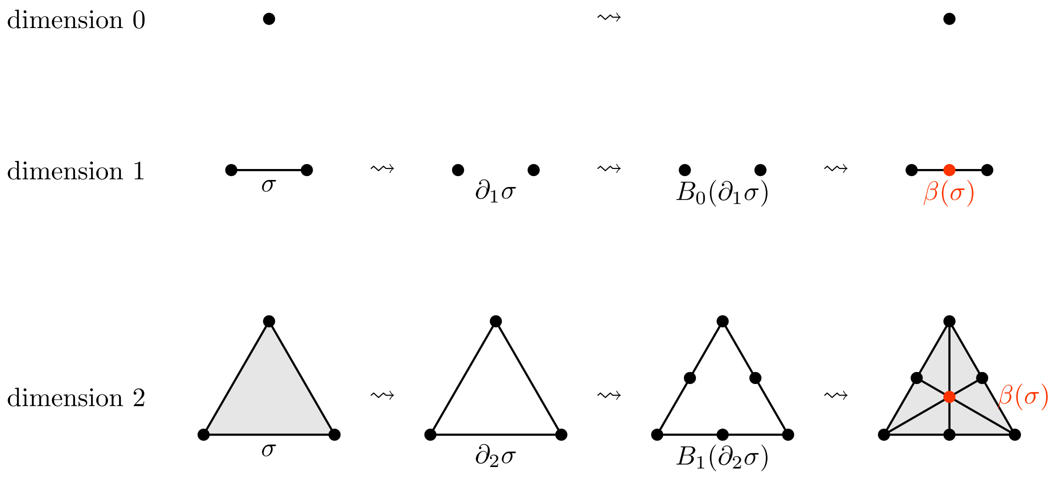


Figure 4.10.: barycentric subdivision

- Moreover, we define $(H_k \colon C_k(\mathbb{R}^\infty) \longrightarrow C_{k+1}(\mathbb{R}^\infty))_{k\in\mathbb{Z}}$ inductively by:
  - For all $k \in \mathbb{Z}_{\leq 0}$, let $H_k := 0$.
  - For all $k \in \mathbb{N}_{>0}$, let

$$\begin{aligned} H_k \colon C_k(\mathbb{R}^\infty) &\longrightarrow C_{k+1}(\mathbb{R}^\infty) \\ \operatorname{map}(\Delta^k, \mathbb{R}^\infty) \ni \sigma &\longmapsto \beta(\sigma) * \bigl(B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma)\bigr) \end{aligned}$$

Remark 4.3.3. Let $\Delta \subset \mathbb{R}^\infty$ be a convex subset and let $k \in \mathbb{N}$. If $\sigma \in \operatorname{map}(\Delta^k, \Delta)$, then $B_k(\sigma) \in C_k(\Delta)$ and $H_k(\sigma) \in C_{k+1}(\Delta)$. In particular, looking at the convex subset Δ^k of $\mathbb{R}^\infty$, we obtain

$$B_k(\mathrm{id}_{\Delta^k}) \in C_k(\Delta^k) \quad \text{and} \quad H_k(\mathrm{id}_{\Delta^k}) \in C_{k+1}(\Delta^k).$$

As next step, we lift this model case to all spaces, by requiring naturality:

Definition 4.3.4 (barycentric subdivision in topological spaces)**.** Let X be a topological space.

- We define $B_X \colon C(X) \longrightarrow C(X)$ as follows: For all $k \in \mathbb{Z}_{<0}$, we set $B_{X,k} := 0$; for all $k \in \mathbb{N}$, we set (by Remark 4.3.3, we have $B_k(\mathrm{id}_{\Delta^k}) \in C_k(\Delta^k)$!)

$$\begin{aligned} B_{X,k} \colon C_k(X) &\longrightarrow C_k(X) \\ \operatorname{map}(\Delta^k, X) \ni \sigma &\longmapsto C_k(\sigma) \circ B_k(\mathrm{id}_{\Delta^k}). \end{aligned}$$

- For $k \in \mathbb{Z}_{<0}$, we set $H_{X,k} := 0 \colon C_k(X) \longrightarrow C_{k+1}(X)$. For all $k \in \mathbb{N}$, we set (by Remark 4.3.3, we have $H_k(\mathrm{id}_{\Delta^k}) \in C_{k+1}(\Delta^k)$!)

$$\begin{aligned} H_{X,k} \colon C_k(X) &\longrightarrow C_{k+1}(X) \\ \operatorname{map}(\Delta^k, X) \ni \sigma &\longmapsto C_{k+1}(\sigma) \circ H_k(\mathrm{id}_{\Delta^k}). \end{aligned}$$

Proposition 4.3.5 (algebraic properties of barycentric subdivision)**.** *Let X be a topological space.*

1. *Then $B_X \colon C(X) \longrightarrow C(X)$ is a chain map in ${}_{\mathbb{Z}}\mathsf{Ch}$.*
2. *The family $(H_{X,k})_{k\in\mathbb{Z}}$ is a chain homotopy $B_X \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} \mathrm{id}_{C(X)}$.*
3. *If $f \colon X \longrightarrow Y$ is continuous, then*

$$\begin{aligned} C(f) \circ B_X &= B_Y \circ C(f), \ \textit{and} \\ C_{k+1}(f) \circ H_{X,k} &= H_{Y,k} \circ C_k(f) \end{aligned}$$

for all $k \in \mathbb{Z}$.

The proof is based on the corresponding properties in the convex case:

Lemma 4.3.6 (algebraic properties of barycentric subdivision, convex case)**.**

0. *For all $v \in \mathbb{R}^\infty$, all $k \in \mathbb{N}$, and all $\sigma \in \operatorname{map}(\Delta^k, \mathbb{R}^\infty)$, we have*

$$\partial_{k+1}(v * \sigma) = \begin{cases} \sigma - \mathrm{const}_v & \textit{if } k = 0 \\ \sigma - v * \partial_k(\sigma) & \textit{if } k > 0. \end{cases}$$

If $f \colon \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ is affine linear, then we also have

$$f \circ (v * \sigma) = f(v) * (f \circ \sigma).$$

1. *The map $B \colon C(\mathbb{R}^\infty) \longrightarrow C(\mathbb{R}^\infty)$ is a chain map in ${}_{\mathbb{Z}}\mathsf{Ch}$.*
2. *The family $(H_k)_{k\in\mathbb{Z}}$ is a chain homotopy $B \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} \mathrm{id}_{C(\mathbb{R}^\infty)}$.*
3. *For all $k \in \mathbb{N}$ and all affine linear maps $\sigma \colon \Delta^k \longrightarrow \mathbb{R}^\infty$, we have*

$$\begin{aligned} B_k(\sigma) &= C_k(\sigma) \circ B_k(\mathrm{id}_{\Delta^k}), \ \textit{and} \\ H_k(\sigma) &= C_{k+1}(\sigma) \circ H_k(\mathrm{id}_{\Delta^k}). \end{aligned}$$

Proof of Lemma 4.3.6. Ad 0. This is a straightforward calculation (check!).

Ad 1. This follows inductively from part 0. Let $k \in \mathbb{N}_{>0}$ and let $\sigma \in \operatorname{map}(\Delta^k, \mathbb{R}^\infty)$. Then

$$
\begin{aligned}
\partial_k\bigl(B_k(\sigma)\bigr) &= \partial_k\bigl(\beta(\sigma) * B_{k-1}(\partial_k(\sigma))\bigr) \\
&= B_{k-1}\bigl(\partial_k(\sigma)\bigr) - \beta(\sigma) * \partial_{k-1} B_{k-1}(\partial_k \sigma) && \text{(by part 0)} \\
&= B_{k-1}\bigl(\partial_k(\sigma)\bigr) - \beta(\sigma) * B_{k-2} \circ \partial_{k-1} \circ \partial_k(\sigma) && \text{(by induction)} \\
&= B_{k-1}\bigl(\partial_k(\sigma)\bigr). && \text{(because } \partial_{k-1} \circ \partial_k = 0\text{)}
\end{aligned}
$$

Ad 2. This follows inductively from part 0 and part 1; indeed, the definition of $(H_k)_{k \in \mathbb{Z}}$ is motivated by trying to inductively (over the dimension) construct a chain homotopy between barycentric subdivision and the identity. More precisely: By construction, we have $\partial_{k+1} \circ H_k - H_{k-1} \circ \partial_k = 0 = B_k - \mathrm{id}_{C_k(\mathbb{R}^\infty)})$ for all $k \in \mathbb{N}_{\leq 0}$ (check!). Let $k \in \mathbb{N}_{>0}$ and let $\sigma \in \mathrm{map}(\Delta^k, \mathbb{R}^\infty)$. Then

$$
\begin{aligned}
\partial_{k+1}\bigl(H_k(\sigma)\bigr) &= \partial_{k+1}\bigl(\beta(\sigma) * (B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma))\bigr) \\
&= B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma) \\
&\quad - \beta(\sigma) * \bigl(\partial_k B_k(\sigma) - \partial_k \sigma - \partial_k H_{k-1}(\partial_k \sigma)\bigr). \qquad \text{(by part 0)}
\end{aligned}
$$

Assuming inductively that the chain homotopy equation is satisfied, we obtain

$$
\begin{aligned}
\partial_{k+1}\bigl(H_k(\sigma)\bigr) &= B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma) \\
&\quad - \beta(\sigma) * \bigl(B_{k-1}(\partial_k \sigma) - \partial_k \sigma \\
&\qquad + H_{k-2}(\partial_{k-1}\partial_k \sigma) - B_{k-1}(\partial_k \sigma) + \partial_k(\sigma)\bigr) && \text{(by part 1 and induction)} \\
&= B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma) \\
&\quad - \beta(\sigma) * \bigl(0 + H_{k-1}(0)\bigr) && \text{(because } \partial_{k-1} \circ \partial_k = 0\text{)} \\
&= B_k(\sigma) - \sigma - H_{k-1}(\partial_k \sigma).
\end{aligned}
$$

Ad 3. This follows inductively from part 0 (check!), using the fact that

$$\beta(\sigma) = \sigma\bigl(\beta(\mathrm{id}_{\Delta^k})\bigr)$$

holds for all *affine linear* maps $\sigma \colon \Delta^k \longrightarrow \mathbb{R}^\infty$. $\square$

Proof of Proposition 4.3.5. Ad 1. Let $k \in \mathbb{N}$ and $\sigma \in \mathrm{map}(\Delta^k, X)$. Because $C(\sigma)$ and B are chain maps, we obtain

$$
\begin{aligned}
\partial_k \circ B_{X,k}(\sigma) &= \partial_k \circ C_k(\sigma) \circ B_k(\mathrm{id}_{\Delta^k}) && \text{(definition of } B_X\text{)} \\
&= C_{k-1}(\sigma) \circ B_{k-1}(\partial_k \,\mathrm{id}_{\Delta^k}) && \text{(}C(\sigma)\text{ is a chain map)} \\
&= \sum_{j=0}^{k} (-1)^j \cdot C_{k-1}(\sigma) \circ B_{k-1}(i_{k,j}) && \text{(linearity of } B_X\text{, definition of } \partial_k\text{).}
\end{aligned}
$$

As $i_{k,j} \colon \Delta^{k-1} \longrightarrow \Delta^k$ is affine linear for every $j \in \{0, \dots, k\}$, the third part of Lemma 4.3.6 can be applied, showing that

$$\begin{aligned}\partial_k \circ B_{X,k}(\sigma) &= \sum_{j=0}^{k} (-1)^j \cdot C_{k-1}(\sigma) \circ C_{k-1}(i_{k,j}) \circ B_{k-1}(\mathrm{id}_{\Delta^{k-1}}) \\ &= \sum_{j=0}^{k} (-1)^j \cdot C_{k-1}(\sigma \circ i_{k,j}) \circ B_{k-1}(\mathrm{id}_{\Delta^{k-1}}) \\ &= B_{X,k-1} \circ \partial_k(\sigma).\end{aligned}$$

Ad 2. This can be shown in a similar fashion.
Ad 3. This is clear from the construction. □

4.3.3 Proving Excision

Using barycentric subdivision, we first show that singular homology can be computed in terms of "small" simplices. We then derive the excision theorem from this fact.

Setup 4.3.7. In the following, let R be a ring with unit and let Z be a left R-module.

Definition 4.3.8 (small simplices)**.** Let X be a topological space and let $U = (U_i)_{i \in I}$ be a *strong cover of* X; i.e., for every $i \in I$, we have $U_i \subset X$, and $\bigcup_{i \in I} U_i^\circ = X$.

- The *chain complex* $C^U(X)$ *of* U*-small singular simplices in* X is defined by

  $$C_k^U(X) := \begin{cases} 0 & \text{falls } k < 0 \\ F(\bigcup_{i \in I} \operatorname{map}(\Delta^k, U_i)) = \bigoplus_{\bigcup_{i \in I} \operatorname{map}(\Delta^k, U_i)} \mathbb{Z} & \text{falls } k \geq 0, \end{cases}$$

  together with the restriction of the boundary operator of $C(X)$ (which is well-defined!).

- If $A \subset X$, then we set

  $$C^U(X, A; Z) := Z \otimes_R C^U(X) \,/\, \operatorname{im}(Z \otimes_R (C^{U \cap A}(A) \hookrightarrow C^U(X))) \in {}_R\mathsf{Ch}$$

  (where $U \cap A := (U_i \cap A)_{i \in I}$) and

  $$H_k^U(X, A; Z) := H_k\bigl(C^U(X, A; Z)\bigr),$$

  as well as $C^U(X; Z) := C^U(X, \emptyset; Z)$ and $H_k^U(X; Z) := H_k^U(X, \emptyset; Z)$.

Theorem 4.3.9 (small simplices compute singular homology)**.** *Let X be a topological space, let U be a strong cover of X, and let $j \colon C^U(X) \longrightarrow C(X)$ be the inclusion. Then, for every $k \in \mathbb{Z}$, the induced homomorphism*

$$H_k(Z \otimes_{\mathbb{Z}} j) \colon H_k^U(X; Z) \longrightarrow H_k(X; Z)$$

is an R-isomorphism.

Remark 4.3.10. The map $Z \otimes_{\mathbb{Z}} j$ is even a chain homotopy equivalence with an explicit chain homotopy inverse. However, the proof of this fact is technically a little bit more demanding (and we will not need it for the proof of excision) [13, Corollary III.7.4][26, p. 123f].

As in the case of the Seifert and van Kampen theorem, we argue via the Lebesgue lemma. In order to apply the Lebesgue lemma successfully, we show that successive barycentric subdivision effectively shrinks simplices:

Proposition 4.3.11 (barycentric subdivision shrinks simplices)**.** *Let $k \in \mathbb{N}$.*

1. *If $\sigma \colon \Delta^k \longrightarrow \mathbb{R}^\infty$ is affine linear, then every summand $\tau \in \operatorname{map}(\Delta^k, \mathbb{R}^\infty)$ in the definition of $B_k(\sigma) \in C_k(\mathbb{R}^\infty)$ (Definition 4.3.2), satisfies (with respect to the Euclidean metric on $\mathbb{R}^\infty$)*

$$\operatorname{diam}\bigl(\tau(\Delta^k)\bigr) \leq \frac{k}{k+1} \cdot \operatorname{diam}\bigl(\sigma(\Delta^k)\bigr).$$

2. *In particular: For every $\varepsilon \in \mathbb{R}_{>0}$, there exists an $n \in \mathbb{N}$ such that for all summands τ in the n-fold barycentric subdivision of id_{Δ^k}, we have*

$$\operatorname{diam}\bigl(\tau(\Delta^k)\bigr) < \varepsilon.$$

Proof. The first part is a straightforward inductive calculation (Exercise). The second part follows from the first part (check!). □

Corollary 4.3.12. *Let X be a topological space, let U be a strong cover of X, let $k \in \mathbb{N}$, and let $\sigma \in \operatorname{map}(\Delta^k, X)$. Then there exists an $n \in \mathbb{N}$ (which might depend on σ!) with*

$$(B_{X,k})^n(\sigma) \in C_k^U(X).$$

Proof. The pulled back family $(\sigma^{-1}(V))_{V \in U}$ is a strong cover of Δ^k. Because Δ^k is a compact metric space (with respect to the Euclidean metric), we can apply the Lebesgue lemma (Lemma 2.2.7) to the corresponding cover U^σ of Δ^k by the interiors.

Then Proposition 4.3.11 provides us with an $n \in \mathbb{N}$ such that the n-fold barycentric subdivision of id_{Δ^k} is U^σ-small. Therefore, the naturality of the barycentric subdivision (Proposition 4.3.5) shows that the n-fold barycentric subdivision of σ is U-small (and thus lies in $C_k^U(X)$). □

Using barycentric subdivision, we prove that small singular simplices compute singular homology:

Proof of Theorem 4.3.9. As in the construction of the long exact sequence of pairs for singular homology (proof of Theorem 4.1.16) it follows that

$$Z \otimes_{\mathbb{Z}} j \colon C^U(X;Z) \longrightarrow C(X;Z)$$

is degree-wise injective. In this way, we can view $C^U(X;Z)$ as a subcomplex of $C(X;Z)$.

Let $k \in \mathbb{Z}$, without loss of generality, $k \geq 0$.

- *The map $H_k(Z \otimes_{\mathbb{Z}} j)$ is surjective:* Let $c \in C_k(X;Z)$ be a cycle, say of the form $c = \sum_{j=1}^m a_j \cdot \sigma_j$ with $a_1, \dots, a_m \in Z$ and $\sigma_1, \dots, \sigma_m \in \operatorname{map}(\Delta^k, X)$. Because this sum is finite, Corollary 4.3.12 shows that there exists an $n \in \mathbb{N}$ with

  $$\forall_{j \in \{1,\dots,m\}} \quad B^n_{X,k}(\sigma_j) \in C^U_k(X),$$

  and so

  $$(Z \otimes_{\mathbb{Z}} (B_{X,k})^n)(c) \in C^U_k(X;Z).$$

  Moreover, we have $Z \otimes_{\mathbb{Z}} (B_X)^n \simeq_{{}_R\mathsf{Ch}} \operatorname{id}_{C(X;Z)}$ (Proposition 4.3.5 and Proposition A.6.33). Therefore, in $H_k(X;Z)$, we obtain that

  $$[c] = \bigl[(Z \otimes_{\mathbb{Z}} (B_{X,k})^n)(c)\bigr] \in \operatorname{im} H_k(Z \otimes_{\mathbb{Z}} j).$$

- *Injectivity of $H_k(Z \otimes_{\mathbb{Z}} j)$* can be proved in the same way as surjectivity: We barycentrically subdivide $(k+1)$-chains in $C(X;Z)$ sufficiently often and use that $Z \otimes B_X|_{C^U(X;Z)}$ is R-chain homotopic to $\operatorname{id}_{C^U(X;Z)}$ (because barycentric subdivision and the corresponding chain homotopies are natural by Proposition 4.3.5). $\square$

Corollary 4.3.13 (small simplices compute singular homology, relative case)**.** *Let (X,A) be a pair of spaces, let U be a strong cover of X, and let $k \in \mathbb{Z}$. Then the canonical chain map $C^U(X,A;Z) \longrightarrow C(X,A;Z)$ induces an R-isomorphism*

$$H^U_k(X,A;Z) \longrightarrow H_k(X,A;Z).$$

Proof. This is a classical case of generalising a statement on absolute homology to relative homology via long exact sequences and the five lemma: We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{U\cap A}(A;Z) \arrow[r] \arrow[d] & C^U(X;Z) \arrow[r] \arrow[d] & C^U(X,A;Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(A;Z) \arrow[r] & C(X;Z) \arrow[r] & C(X,A;Z) \arrow[r] & 0
\end{tikzcd}$$

in ${}_R\mathsf{Ch}$ (where all chain maps are the canonical chain maps). The lower row is degree-wise exact (Proof of Theorem 4.1.16); in the same way, one can show that also the upper row is degree-wise exact.

Applying the algebraic long exact homology sequence (Proposition A.6.23) and the five lemma (Proposition A.6.7), shows that the claim for the relative terms follows from the absolute case (Theorem 4.3.9). $\square$

We can now prove that singular homology satisfies excision by applying the previous corollary to a suitable strong cover:

Proof of Theorem 4.3.1. Let $U := (X \setminus B, A)$; then U is a strong cover of X (because $\overline{B} \subset A^\circ$). We then have an ${}_R\mathsf{Ch}$-isomorphism

$$\begin{aligned}
C^U(X, A; Z) &= \frac{Z \otimes_{\mathbb{Z}} C^U(X)}{Z \otimes_{\mathbb{Z}} \operatorname{im}(C^{U \cap A}(A) \hookrightarrow C^U(X))} \\
&= \frac{Z \otimes_{\mathbb{Z}} \bigl(C(X \setminus B) + C(A)\bigr)}{Z \otimes_{\mathbb{Z}} \operatorname{im}(C(A \setminus B) + C(A) \hookrightarrow C(X \setminus B) + C(A))} \\
&\cong_{{}_R\mathsf{Ch}} \frac{Z \otimes_{\mathbb{Z}} C(X \setminus B)}{Z \otimes_{\mathbb{Z}} \operatorname{im}(C(A \setminus B) \hookrightarrow C(X \setminus B))} \\
&= C(X \setminus B, A \setminus B; Z),
\end{aligned}$$

where the inverse of this isomorphism in ${}_R\mathsf{Ch}$ is induced by the inclusion; moreover, we used that $C(A) \cap C(X \setminus B) = C(A \setminus B)$ (check!).

As U-small simplices compute singular homology (Corollary 4.3.13), we obtain that the maps

$$H_k(X \setminus B, A \setminus B; Z) \longrightarrow H_k^U(X, A; Z) \longrightarrow H_k(X, A; Z)$$

induced by the inclusions are R-isomorphisms. $\square$

4.4 Applications

Finally, we have established that singular homology indeed is an ordinary homology theory (Theorem 4.4.1). In particular, we thus completed the proof of all the applications that depended on the theorem on existence of "interesting" homotopy invariant functors or on the existence of (ordinary) homology theories. Moreover, we will discuss further applications of singular homology such as the Jordan curve theorem and its relatives.

4.4.1 Singular Homology as Homology Theory

We can summarise the results of the previous sections as follows:

Theorem 4.4.1 (singular homology as ordinary homology theory)**.** *Let R be a ring and let $Z \in \operatorname{Ob}({}_R\mathsf{Mod})$. Then, singular homology*

$$\big(H_k(\,\cdot\,,\,\cdot\,;Z)\colon \mathsf{Top}^2 \longrightarrow {}_R\mathsf{Mod}\big)_{k\in\mathbb{Z}},$$

together with the connecting homomorphisms

$$\big(\partial_k\colon H_k(\,\cdot\,,\,\cdot\,;Z) \Longrightarrow H_{k-1}(\,\cdot\,,\,\cdot\,;Z)\circ U\big)_{k\in\mathbb{Z}}$$

from Theorem 4.1.16 (where U is the subspace functor of Definition 3.1.1), is an additive ordinary homology theory on Top^2 *with values in* ${}_R\mathsf{Mod}$ *and whose coefficients are (isomorphic to) Z.*

Proof. We have already proved all the necessary steps:

- Functoriality: (Proposition and) Definition 4.1.3, 4.1.6, 4.1.8
- Construction and naturality of the connecting homomorphisms: Theorem 4.1.16
- Homotopy invariance: Theorem 4.2.1
- Long exact sequence of pairs: Theorem 4.1.16
- Excision: Theorem 4.3.1
- Dimension axiom and computation of the coefficients: Example 4.1.12
- Additivity: Proposition 4.1.14. □

Corollary 4.4.2 (summary of loose ends). *In particular, we completed the proofs of the following results:*

- *For each choice of ring and coefficients, there exists an additive ordinary homology theory with these coefficients.*
- *Computation of singular homology of spheres and mapping degrees for self-maps of spheres (Corollary 3.2.8 and 3.2.11); we will return to this in Example 4.4.3.*
- *In particular, spheres are* not *contractible.*
- *Existence of "interesting" homotopy invariant functors (Theorem 1.3.22, Corollary 3.2.14)*
- *Invariance of dimension I and II (Corollary 1.3.24, Corollary 3.2.10)*
- *In particular: The dimension of non-empty (connected) topological manifolds is well-defined and can be checked locally.*
- *Hedgehog theorem (Theorem 1.3.30)*
- *Fundamental theorem of algebra (Exercise)*

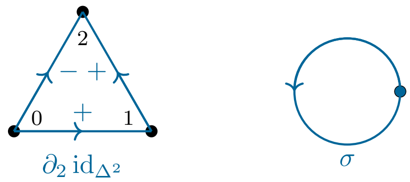


Figure 4.11.: The 1-cycle $\partial_2 \operatorname{id}_{\Delta^2}$ detects the hole in $\partial\Delta^2$; the 1-cycle σ detects the hole in $S^1 \cong_{\mathsf{Top}} \partial\Delta^2$.

- *Brouwer fixed point theorem (Corollary 1.3.25)*
- *Existence of Nash equilibria (Exercise)*
- *All consequences of the Eilenberg–Steenrod axioms apply to singular homology (e.g., the Mayer–Vietoris sequence, ...) (Chapter 3).* □

Moreover, we can confirm now that our original idea for the construction of singular homology (Chapter 4.1.1) was carried out correctly:

Example 4.4.3 (singular homology of spheres, explicitly)**.** Let R be a ring with unit.

- A careful inductive analysis and the computations in (the proof of) Corollary 3.2.8 show (Exercise):
  - Let $n \in \mathbb{N}_{>0}$. The relative homology class of $(\Delta^n, \partial\Delta^n) \cong_{\mathsf{Top}^2} (D^n, S^{n-1})$ represented by the singular chain $1 \cdot \operatorname{id}_{\Delta^n} \in C_n(\Delta^n; R)$ generates the R-module
  $$H_n(\Delta^n, \partial\Delta^n; R) \cong_R H_n(D^n, S^{n-1}; R) \cong_R R.$$
  - Let $n \in \mathbb{N}_{>1}$. The homology class in $H_{n-1}(\partial\Delta^n; R)$ represented by the singular cycle $1 \cdot \partial_n \operatorname{id}_{\Delta^n} \in C_{n-1}(\partial\Delta^n; R)$ (Figure 4.11) generates the R-module
  $$H_{n-1}(\partial\Delta^n; R) \cong_R H_{n-1}(S^{n-1}; R) \cong_R R.$$
- In particular, this leads to explicit generators of $H_{n-1}(S^{n-1}; R)$ and $H_n(D^n, S^{n-1}; R)$ (Exercise).
- Let
$$\begin{aligned} \sigma \colon \Delta^1 &\longrightarrow S^1 \\ (1-t, t) &\longmapsto [t]. \end{aligned}$$

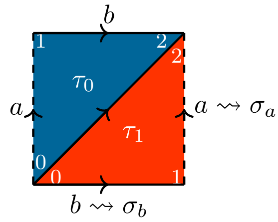


Figure 4.12.: singular homology of the torus, explicitly (using the standard identification of the edges of the square; Figure 1.6)

Then $[1 \cdot \sigma]$ generates the homology $H_1(S^1; R) \cong_R R$, because a straightforward construction provides a singular 2-chain $b \in C_2(S^1; R)$ with $\partial_2 b = \sigma -$ "$\partial_2 \operatorname{id}_{\Delta^2}$" (check!).

Example 4.4.4 (singular homology of the torus, explictly)**.** Going through the calculation of ordinary homology of the torus $S^1 \times S^1$ with $\mathbb{Z}$-coefficients (Example 3.3.3) shows in combination with the previous Example 4.4.3 (using the notation from Figure 4.12):

- The classes $[\sigma_a]$ and $[\sigma_b]$ in $H_1(S^1 \times S^1; \mathbb{Z})$ form a $\mathbb{Z}$-basis of $H_1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- The singular chain $\tau_0 - \tau_1 \in C_2(S^1 \times S^1)$ is a cycle whose homology class $[\tau_0 - \tau_1]$ generates the $\mathbb{Z}$-module $H_2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$.

4.4.2 The Jordan Curve Theorem

Our next goal is to derive the following version of the Jordan curve theorem:

Theorem 4.4.5 (Jordan curve theorem)**.** *Let $n \in \mathbb{N}_{>1}$.*

1. *If $f\colon D^n \longrightarrow \mathbb{R}^n$ is continuous and injective, then $\mathbb{R}^n \setminus f(D^n)$ is path-connected.*

1'. *If $D \subset \mathbb{R}^n$ is homeomorphic to D^n, then $\mathbb{R}^n \setminus D$ is path-connected.*

2. *If $f\colon S^{n-1} \longrightarrow \mathbb{R}^n$ is continuous and injective, then $\mathbb{R}^n \setminus f(S^{n-1})$ has exactly two path-connected components.*

2'. *If $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}, then $\mathbb{R}^n \setminus S$ has exactly two path-connected components.*

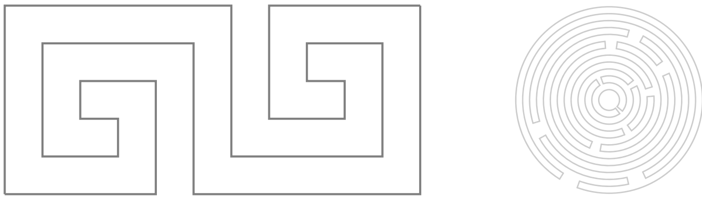
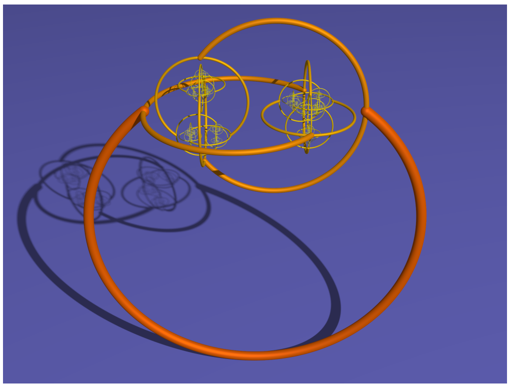


Figure 4.13.: Subspaces of $\mathbb{R}^2$ that are homeomorphic to S^1



http://en.wikipedia.org/wiki/File:Alexander_horned_sphere.png

Figure 4.14.: Alexander horned sphere

The case of continuous injective maps $S^1 \longrightarrow \mathbb{R}^2$ is the classical Jordan *curve* theorem. At first, the theorem might look "obvious"; however, one should keep in mind that subsets of $\mathbb{R}^2$ that are homeomorphic to S^1 can be rather "wild" (Figure 4.13).

Remark 4.4.6. With slightly more effort, one can improve the Jordan curve theorem to the following version [52, Proposition VIII.6.5]: If $n \in \mathbb{N}_{>1}$ and if $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}, then one of the path-connected components of $\mathbb{R}^n \setminus S$ is bounded, one of them is unbounded, and S equals the boundary of both components.

Caveat 4.4.7 (Alexander horned sphere)**.** If $n \in \mathbb{N}_{>2}$ and $D \subset \mathbb{R}^n$ is homeomorphic to D^n, then $\mathbb{R}^n \setminus D$, in general, is *not* homeomorphic to $\mathbb{R}^n \setminus D^n$; an example of this in $\mathbb{R}^3$ is the *Alexander horned sphere* (Figure 4.14) [26, Example 2.B.2]. However, the corresponding statement in dimension 2 *does* hold by the Jordan–Schönflies theorem [35].

In order to prove the Jordan curve theorem, we make use of the fact that $H_0(\,\cdot\,;\mathbb{Z})$ (and the reduced version $\widetilde{H}_0(\,\cdot\,;\mathbb{Z})$; Remark 3.1.7) determines the number of path-connected components (Theorem 4.1.15). Therefore, we first carry out the following computation in singular homology (for the sake of simplicity, we only state it for $\mathbb{Z}$-coefficients).

Lemma 4.4.8. *Let $n, m \in \mathbb{N}$.*

1. *If $f \colon D^m \longrightarrow S^n$ is continuous and injective, then, for all $k \in \mathbb{Z}$,*
$$\widetilde{H}_k\bigl(S^n \setminus f(D^m); \mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0.$$
2. *If $m < n$ and if $f \colon S^m \longrightarrow S^n$ is continuous and injective, then, for all $k \in \mathbb{Z}$,*
$$\widetilde{H}_k\bigl(S^n \setminus f(S^m); \mathbb{Z}\bigr) \cong_{\mathbb{Z}} \widetilde{H}_k(S^{n-m-1}; \mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k = n - m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 4.4.8. Ad 1. We proceed by induction on $m \in \mathbb{N}$. Moreover, in order to keep notation simple, we will sometimes use the cube $I^m := [0,1]^m \cong_{\mathsf{Top}} D^m$ instead of the round ball D^m.

- The case $m = 0$: Because D^m is just a single point, we have (using the stereographic projection)
$$S^n \setminus f(D^0) \cong_{\mathsf{Top}} \mathbb{R}^n \simeq_{\mathsf{Top}} \bullet.$$
Hence, $\widetilde{H}_k(S^n \setminus f(D^m); \mathbb{Z}) \cong_{\mathbb{Z}} \widetilde{H}_k(\bullet; \mathbb{Z}) \cong_{\mathbb{Z}} 0$.
- *Induction step.* Let $m \in \mathbb{N}_{>0}$ and let us suppose that the claim holds for $m - 1$. Then the claim also holds for m:

  Let $f \colon I^m \longrightarrow S^n$ be continuous and injective and let $k \in \mathbb{Z}$. *Assume* for a contradiction that there exists an $\alpha \in \widetilde{H}_k(S^n \setminus f(I^m); \mathbb{Z}) \setminus \{0\}$; in particular, $k \geq 0$. The idea is to use a Mayer–Vietoris argument to bring the induction hypothesis into play. To this end, we consider the complements
$$\begin{aligned} U &:= S^n \setminus f\bigl(I^{m-1} \times [0, 1/2]\bigr) \\ V &:= S^n \setminus f\bigl(I^{m-1} \times [1/2, 1]\bigr). \end{aligned}$$
  of the lower and the upper half of the embedded cube, respectively.

  Then $U \cap V = S^n \setminus f(I^m)$ and $U \cup V = S^n \setminus f(I^{m-1} \times \{1/2\})$ and the hypotheses for the Mayer–Vietoris sequences (Theorem 3.3.2) are satisfied. Because f is injective and $I^m \not\cong_{\mathsf{Top}} S^n$, the intersection $U \cap V$ is non-empty (by the compact-Hausdorff trick); thus, we obtain a corresponding Mayer–Vietoris sequence for reduced singular homology:

$$\widetilde{H}_{k+1}(U \cup V; \mathbb{Z}) \xrightarrow{\Delta_{k+1}} \widetilde{H}_k(U \cap V; \mathbb{Z}) \longrightarrow \widetilde{H}_k(U; \mathbb{Z}) \oplus \widetilde{H}_k(V; \mathbb{Z}) \longrightarrow \widetilde{H}_k(U \cup V; \mathbb{Z})$$

By the induction hypothesis, the outer terms are the trivial module 0. Therefore, by exactness,

- the image of α in $\widetilde{H}_k(U;\mathbb{Z})$ is non-zero, or
- the image of α in $\widetilde{H}_k(V;\mathbb{Z})$ is non-zero.

In this way, inductively, we obtain a sequence $[0,1] = I_0 \supset I_1 \supset I_2 \supset \cdots$ of nested intervals whose intersection consists of a single point t and such that for all $j \in \mathbb{N}$, the image of α in $\widetilde{H}_k(S^n \setminus f(I^{m-1} \times I_j);\mathbb{Z})$ (under the homomorphism induced by the inclusion) is non-zero. However,

$$\bigcup_{j\in\mathbb{N}} S^n \setminus f(I^{m-1} \times I_j) = S^n \setminus f\bigl(I^{m-1} \times \{t\}\bigr)$$

and, by induction,

$$\widetilde{H}_k(S^n \setminus f(I^{m-1} \times \{t\});\mathbb{Z}) \cong_{\mathbb{Z}} 0.$$

This contradicts the compatibility of singular homology with ascending unions (Proposition 4.4.9). Therefore, we have $\widetilde{H}_k(S^n \setminus f(I^m);\mathbb{Z}) \cong_{\mathbb{Z}} 0$.

Ad 2: Again, we proceed by induction on $m \in \{0,\dots,n-1\}$, using a Mayer–Vietoris argument and the first part:

- The case $m = 0 < n$. Again, using stereographic projection, we obtain

  $$S^n \setminus f(S^0) \cong_{\mathsf{Top}} \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$$

  (and thus the desired result on the level of singular homology).

- *Induction step.* Let $m \in \{1,\dots,n-1\}$ and let us suppose that the claim holds for $m-1$. Then the claim also holds for m:

  We consider the decomposition

  $$S^m = D^m_+ \cup D^m_-$$

  into the upper and lower hemisphere, respectively; it should be noted that $D^m_+ \cong_{\mathsf{Top}} D^m \cong_{\mathsf{Top}} D^m_-$ holds (via the "vertical" projection) and that $D^m_+ \cap D^m_- = S^{m-1}$. Let

  $$U := S^n \setminus f(D^m_+)$$
  $$V := S^n \setminus f(D^m_-).$$

  Then $U \cap V = S^n \setminus f(S^m)$ and $U \cup V = S^n \setminus f(S^{m-1})$; because f is injective, $m < n$, and $S^m \not\cong_{\mathsf{Top}} S^n$, we have $U \cap V \neq \emptyset$. The Mayer–Vietoris sequence (Theorem 3.3.2) for reduced singular homology and the first part (applied to U and V) shows that

$$\begin{aligned}\widetilde{H}_k\bigl(S^n \setminus f(S^m);\mathbb{Z}\bigr) &= \widetilde{H}_k(U\cap V;\mathbb{Z})\\ &\cong_{\mathbb{Z}} \widetilde{H}_{k+1}(U\cup V;\mathbb{Z})\\ &= \widetilde{H}_{k+1}\bigl(S^n\setminus f(S^{m-1});\mathbb{Z}\bigr)\end{aligned}$$

holds for all $k \in \mathbb{Z}$. We can now apply the induction hypothesis to obtain the desired result. □

During this proof we used that (reduced) singular homology is compatible with ascending unions:

Proposition 4.4.9 (singular homology and ascending unions)**.** *Let R be a ring with unit, let $Z \in \operatorname{Ob}({}_R\mathsf{Mod})$, and let $k \in \mathbb{Z}$. Let X be a topological space and let $(X_n)_{n\in\mathbb{N}}$ be an ascending sequence of subspaces of X with $\bigcup_{n\in\mathbb{N}} X_n^\circ = X$. Then the inclusions $(X_n \hookrightarrow X)_{n\in\mathbb{N}}$ induce an R-isomorphism*

$$\operatorname*{colim}_{n\in\mathbb{N}} H_k(X_n;Z) \longrightarrow H_k(X;Z).$$

Here,

$$\operatorname*{colim}_{n\in\mathbb{N}} H_k(X_n;Z) := \Bigl(\bigoplus_{n\in\mathbb{N}} H_k(X_n;Z)\Bigr)\Big/\!\sim,$$

where "$\sim$" denotes the equivalence relation generated by

$$\forall_{n\in\mathbb{N}} \quad \forall_{m\in\mathbb{N}_{\geq n}} \quad \forall_{\alpha\in H_k(X_n;Z)} \quad \alpha \sim H_k(i_{n,m})(\alpha) \in H_k(X_m;Z)$$

($i_{n,m}\colon X_n \longrightarrow X_m$ is the corresponding inclusion).

Proof. Because the standard simplices Δ^k and Δ^{k+1} are compact and singular chains are finite sums, we can prove this in the same way as the corresponding statement for homotopy groups (Proposition 2.2.20) (check!). □

Proof of the Jordan curve theorem (Theorem 4.4.5). Clearly, parts 1' and 2' are just reformulations of parts 1 and 2, respectively. Therefore, we will only prove 1 and 2.

Ad 1. Let $f\colon D^n \longrightarrow \mathbb{R}^n$ be continuous and injective. Then also the composition

$$\overline{f} := i\circ f\colon D^n \longrightarrow S^n$$

of f with the inverse $i\colon \mathbb{R}^n \longrightarrow S^n\setminus\{e_{n+1}\}$ of the stereographic projection is continuous and injective.

Because $f(D^n)\subset \mathbb{R}^n$ is compact, the complements $\mathbb{R}^n\setminus f(D^n)$ and $S^n\setminus \overline{f}(D^n)$ have the same number of path-connected components (check!)

Using the fact that singular homology in degree 0 detects the number of path-connected components (Theorem 4.1.15) and the homological computation from Lemma 4.4.8, we see that $\mathbb{R}^n\setminus f(D^n)$ has exactly

$$\operatorname{rk}_{\mathbb{Z}} H_0\bigl(S^n\setminus\overline{f}(D^n);\mathbb{Z}\bigr) = \operatorname{rk}_{\mathbb{Z}} \widetilde{H}_0\bigl(S^n\setminus\overline{f}(D^n);\mathbb{Z}\bigr)+1 = 0+1 = 1$$

path-connected components (by construction, $S^n \setminus \overline{f}(D^n)$ is non-empty; therefore, Remark 3.1.7 and Proposition 3.1.6 can be applied).

Ad 2. We can use the same type of argument and the fact that (by Lemma 4.4.8)

$$\operatorname{rk}_{\mathbb{Z}} \widetilde{H}_0\bigl(S^n \setminus \overline{f}(S^{n-1}); \mathbb{Z}\bigr) = 1$$

holds for all continuous injective maps $\overline{f}\colon S^{n-1} \longrightarrow S^n$. □

Outlook 4.4.10 (graph theory)**.** The Jordan curve theorem in the plane (and its strengthengings) is used often (at least implicitly!) in graph theory; for example, proving non-planarity results for graphs and colouring results for planar graphs usually requires this topological input (Exercise).

Outlook 4.4.11 (Slitherlink)**.** Slitherlink is a combinatorial puzzle developed by the Japanese publisher nikoli. A *Slitherlink puzzle* consists of a square grid; some of the squares have numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.

SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares, the number of edges in the loop is not specified.

SL 3 The loop does not have any self-intersections or branches.

The Jordan curve theorem can be used to prove global strategies for solving Slitherlink puzzles (Exercise) [45, Kapitel 7].

4.4.3 Invariance of Domain and Non-Embeddability

For example, we can use the Jordan curve theorem (Theorem 4.4.5) to prove various non-embeddability results.

Corollary 4.4.12 (non-embeddability I)**.** *Let $n \in \mathbb{N}$.*

1. *There is* no *continuous injective map $S^n \longrightarrow \mathbb{R}^n$.*
2. *If $m \in \mathbb{N}_{>n}$, then there is* no *continuous and injective map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.*

Proof. The second part follows from the first part, using that $S^n \subset \mathbb{R}^{n+1}$.

We now prove the first part: Without loss of generality, we may assume that $n > 0$. *Assume* for a contradiction that there exists a continuous injective map $f\colon S^n \longrightarrow \mathbb{R}^n$. Then also the composition

$$\begin{aligned} \overline{f}\colon S^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto \bigl(f(x), 0\bigr) \end{aligned}$$

is continuous and injective. By the Jordan curve theorem (Theorem 4.4.5), the complement $\mathbb{R}^{n+1} \setminus \overline{f}(S^n)$ has exactly two path-connected components.

However, by construction, the complement $\mathbb{R}^{n+1} \setminus \overline{f}(S^n)$ is path-connected (use the extra dimension!). This contradiction shows that such a map cannot exist. □

Corollary 4.4.13 (invariance of domain). *Let $n \in \mathbb{N}$, let $U \subset \mathbb{R}^n$ be open, and let $f \colon U \longrightarrow \mathbb{R}^n$ be continuous and injective. Then $f(U) \subset \mathbb{R}^n$ is open.*

Proof. Let us first treat the pathological cases: Clearly, the statement holds if $n = 0$ or if $n = 1$ (by the intermediate value theorem; check!). Therefore, in the following, we assume that $n \geq 2$.

We apply the Jordan curve theorem to a suitable local situation: Let $x \in U$. It suffices to show that $f(U)$ contains an open neighbourhood of $f(x)$.

Because U is open, there exists a closed ball $D \subset U$ around x (with non-zero radius). We show that $f(D \setminus \partial D)$ is an open neighbourhood of $f(x)$: By the Jordan curve theorem (Theorem 4.4.5), the complement $\mathbb{R}^n \setminus f(\partial D)$ has exactly two path-connected components. Because

$$\mathbb{R}^n \setminus f(\partial D) = f(D \setminus \partial D) \cup \bigl(\mathbb{R}^n \setminus f(D)\bigr)$$

is a disjoint union and both sets are path-connected (the first as continuous image of a path-connected set, the second by the Jordan curve theorem), we obtain that $f(D \setminus \partial D)$ and $\mathbb{R}^n \setminus f(D)$ are those two path-connected components.

We first show that $f(D \setminus \partial D)$ is open in $\mathbb{R}^n \setminus f(\partial D)$: The set $\mathbb{R}^n \setminus f(\partial D)$ is open in $\mathbb{R}^n$, as complement of a compact (whence closed) set. Because $\mathbb{R}^n$ is locally path-connected, also the open subspace $\mathbb{R}^n \setminus f(\partial D)$ is locally path-connected. Therefore, the path-connected component $f(D \setminus \partial D)$ of $\mathbb{R}^n \setminus f(\partial D)$ is open in $\mathbb{R}^n \setminus f(\partial D)$.

Because $\mathbb{R}^n \setminus f(\partial D)$ is open in $\mathbb{R}^n$, we also obtain that $f(D \setminus \partial D)$ is open in $\mathbb{R}^n$. □

Corollary 4.4.14 (non-embeddability II). *Let $n \in \mathbb{N}$.*

1. *Let $n \in \mathbb{N}$, let M be a compact, non-empty (topological) manifold of dimension n, let N be connected (topological) manifold of dimension n, and let $f \colon M \longrightarrow N$ be continuous and injective. Then f is surjective.*
2. *If $n \in \mathbb{N}_{\geq 2}$, then there is* no *continuous injective map $\mathbb{R}P^n \longrightarrow S^n$.*

Proof. We first briefly recall the notion of topological manifold: A topological space M is a *topological manifold of dimension n* if M is a second countable Hausdorff space such that every point in M has an open neighbourhood that is homeomorphic to $\mathbb{R}^n$.

Ad 1. Using compactness, one can show that f has closed image. Invariance of domain (Corollary 4.4.13) shows that the image of f is open in N (Exercise). By connectedness of N, we have $f(M) = N$ (Exercise).

Ad 2. *Assume* for a contradiction that there exists a continuous injective map $f: \mathbb{R}P^n \longrightarrow S^n$. By the first part, f is surjective. Therefore, the compact-Hausdorff trick (Proposition 1.1.15) shows that f is a homeomorphism.

Then, for $x_0 \in \mathbb{R}P^n$, the homeomorphism f induces an isomorphism

$$\pi_1(\mathbb{R}P^n, x_0) \cong_{\mathsf{Group}} \pi_1(S^n, x_0).$$

However, because of $n \geq 2$, the group on the left hand side is non-trivial (Example 2.3.41) and the group on the right hand side is trivial (Example 2.2.11), which is a contradiction. Therefore, such a continuous injective map f cannot exist. □

Study note. Again, compactness plays the role of a finiteness condition. What are analogues of the first part of Corollary 4.4.14 in set theory or (linear) algebra?

4.4.4 Commutative Division Algebras

The topological applications of the previous section also allow to prove algebraic results of the following type:

Theorem 4.4.15 (Hopf theorem for division algebras)**.** *Let A be a finite-dimensional (not necessarily associative) commutative $\mathbb{R}$-algebra that is a division algebra. Then A is (as $\mathbb{R}$-algebra) isomorphic to $\mathbb{R}$ or to $\mathbb{C}$.*

Recall that a commutative (not necessarily associative) $\mathbb{R}$-algebra A is a *division algebra*, if $1 \neq 0$ and if for all $x, a \in A$ with $x \neq 0$, there exists a unique $y \in A$ with $x \cdot y = a$. In particular, A has no zero-divisors (check!).

Proof. Let $n := \dim_{\mathbb{R}} A$; hence, as $\mathbb{R}$-vector space, $A \cong_{\mathbb{R}} \mathbb{R}^n$. Without loss of generality, we may assume that the underlying vector space of A equals $\mathbb{R}^n$. Let $\cdot: A \times A \longrightarrow A$ be the algebra multiplication on A. Because A is an $\mathbb{R}$-algebra, this map is $\mathbb{R}$-bilinear and thus (in view of finite-dimensionality) continuous with respect to the standard topology on $\mathbb{R}^n$.

Assume for a contradiction that $n > 2$. We then consider the map

$$\begin{aligned} f: S^{n-1} &\longrightarrow S^{n-1} \\ x &\longmapsto \frac{1}{\|x \cdot x\|_2} \cdot x \cdot x; \end{aligned}$$

as multiplication with $\mathbb{R}$-scalars is associative (by $\mathbb{R}$-bilinearity), we do not need additional parentheses. We proceed in the following steps:

- The map f is well-defined and continuous: Let $x \in S^{n-1}$. Then $x \neq 0$. Because A is a division algebra, we have $x \cdot x \neq 0$. Hence, $f(x)$ is a well-defined point in S^{n-1}. Because multiplication on A is continuous, also f is continuous.

- The map f induces a well-defined continuous map

$$\begin{aligned} \overline{f} \colon \mathbb{R}P^{n-1} &\longrightarrow S^{n-1} \\ [x] &\longmapsto f(x), \end{aligned}$$

  because: Let $x \in S^{n-1}$. Then

$$(-x) \cdot (-x) = (-1)^2 \cdot x \cdot x = x \cdot x,$$

  and so $f(-x) = f(x)$. Hence, $\overline{f}$ is well-defined. Moreover, because $\mathbb{R}P^{n-1}$ carries the quotient topology of the antipodal action of $\mathbb{Z}/2$ on S^{n-1}, the map $\overline{f}$ is continuous.

- The map $\overline{f}$ is injective: Let $x, y \in S^{n-1}$ with $f(x) = f(y)$. We write

$$\alpha := \frac{1}{\sqrt{\|x \cdot x\|_2}} \quad \text{and} \quad \beta := \frac{1}{\sqrt{\|y \cdot y\|_2}}.$$

  Then

$$\begin{aligned} (\alpha \cdot x + \beta \cdot y) \cdot (\alpha \cdot x - \beta \cdot y) &= \alpha^2 \cdot x \cdot x - \beta^2 \cdot y \cdot y \quad \text{(Binomi III)} \\ &= f(x) - f(y) \\ &= 0. \end{aligned}$$

  Because A is a division algebra, A has no zero-divisors. Hence, $\alpha \cdot x = -\beta \cdot y$ or $\alpha \cdot x = \beta \cdot y$. Therefore, x and y are points in S^{n-1} that lie on the same line through the origin. Thus, $y = x$ or $y = -x$. This shows that $\overline{f}$ is injective.

However, by Corollary 4.4.14, we know that there is no continuous injective map $\mathbb{R}P^{n-1} \longrightarrow S^{n-1}$. This contradiction shows that $n \leq 2$.

We thus are left with the following cases:

- Because the unit of A is non-zero, we have $n \neq 0$.

- If $n = 1$, then clearly A is isomorphic to the $\mathbb{R}$-algebra $\mathbb{R}$ (check!).

- If $n = 2$, then the following, purely algebraic, argument shows that the $\mathbb{R}$-algebra A is isomorphic to $\mathbb{C}$: Let us first try to find a root of -1 in A: Let $j \in A \setminus \mathbb{R} \cdot 1$. Then $(1, j)$ is a basis of A. In particular, there exist $a, b \in \mathbb{R}$ with

$$j^2 = a + b \cdot j.$$

  Then the element $J := j - b/2$ satisfies

$$J^2 = \Big(j - \frac{b}{2}\Big)^2 = j^2 - j \cdot b + \frac{b^2}{4} = a + \frac{b^2}{4} \in \mathbb{R} \cdot 1.$$

Because j (whence J) is *not* in $\mathbb{R} \cdot 1$ and because A is a division algebra, J^2 is a negative multiple of 1. Let

$$I := \frac{1}{\sqrt{-a - b^2/4}} \cdot J.$$

Then $I^2 = -1$ and

$$\begin{aligned} \mathbb{C} &\longrightarrow A \\ z &\longmapsto \operatorname{Re} z + \operatorname{Im} z \cdot I \end{aligned}$$

is an isomorphism of $\mathbb{R}$-algebras (check!). $\square$

Study note. Where in the previous proof that the dimension of A is at most 2 did we use that A is commutative?!

Outlook 4.4.16 (more division algebras)**.** Hopf's theorem Theorem 4.4.15 is concerned with not necessarily associated commutative $\mathbb{R}$-algebras. If one adds the hypothesis that the algebra has to be associative, then the corresponding result (the *Frobenius theorem*) can be proved by elementary means [15, Satz 8.2.2].

The general case of finite-dimensional division algebras over $\mathbb{R}$ can also be handled by (more advanced) topological means [15, Chapter 11]: Every finite-dimensional (not necessarily associative) division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), or $\mathbb{O}$ (the octonions).

4.4.5 Rigidity in Geometry

We briefly discuss an additional structure on singular homology that has applications in geometry: The singular chain complex is equipped with a canonical basis; hence, we can consider the ℓ^1-norm associated with this basis, which describes "how many" simplices are needed to represent a given homology class.

Definition 4.4.17 (ℓ^1-semi-norm on singular homology)**.** Let X be a topological space and let $k \in \mathbb{N}$. Let $|\cdot|_1$ be the ℓ^1-norm on $C_k(X;\mathbb{R})$ with respect to the $\mathbb{R}$-basis of $C_k(X;\mathbb{R})$ that consists of all singular k-simplices of X. We then define the *ℓ^1-semi-norm* $\|\cdot\|_1 \colon H_k(X;\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\|\alpha\|_1 := \inf\big\{|c|_1 \bigm| c \in C_k(X;\mathbb{R}),\ \partial_k c = 0,\ [c] = \alpha \in H_k(X;\mathbb{R})\big\}$$

for all $\alpha \in H_k(X;\mathbb{R})$.

Proposition 4.4.18 (functoriality of the ℓ^1-semi-norm)**.** *Let X be a topological space and let $k \in \mathbb{N}$.*

1. *Then $\|\cdot\|_1 \colon H_k(X;\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$ is a semi-norm on $H_k(X;\mathbb{R})$.*
2. *If $f \colon X \longrightarrow Y$ is a continuous map, then*
$$\forall_{\alpha \in H_k(X;\mathbb{R})} \quad \big\|H_k(f;\mathbb{R})(\alpha)\big\|_1 \leq \|\alpha\|_1.$$
3. *If $f \colon X \longrightarrow Y$ is a homotopy equivalence, then*
$$\forall_{\alpha \in H_k(X;\mathbb{R})} \quad \big\|H_k(f;\mathbb{R})(\alpha)\big\|_1 = \|\alpha\|_1.$$

Proof. The first two parts follow from straightforward calculations (Exercise). The second part is a direct consequence of the first part (Exercise). □

Example 4.4.19. Let $n \in \mathbb{N}_{>0}$ and let $\alpha \in H_n(S^n;\mathbb{R})$. Then $\|\alpha\|_1 = 0$, as can be seen from explicit, geometric, constructions or by using functoriality (Exercise). In other words, α is a rather "small" homology class.

What is the benefit of this semi-norm on singular homology? On the one hand, the ℓ^1-semi-norm on singular homology is homotopy invariant (Proposition 4.4.18). On the other hand, the ℓ^1-semi-norm is sometimes related to geometric invariants. We outline one instance of this phenomenon:

If M is an oriented closed connected manifold of dimension n, then there exists a canonical(!) homology class $[M]_{\mathbb{R}} \in H_n(M;\mathbb{R})$, the $\mathbb{R}$-*fundamental class* of M [68, Chapters 16.3, 16.4]. Roughly speaking, the fundamental class can be viewed as a singular generalisation of triangulations: If M admits a triangulation, then an appropriate parametrisation of the simplices in such a triangulation is a singular cycle that represents the $\mathbb{R}$-fundamental cycle of M [47, Remark 3.6].

Definition 4.4.20 (simplicial volume)**.** The *simplicial volume* of an oriented closed connected n-manifold M is defined by
$$\|M\| := \big\|[M]_{\mathbb{R}}\big\|_1 \in \mathbb{R}_{\geq 0}.$$

In other words, the simplicial volume "counts the minimal, $\mathbb{R}$-weighted number of singular simplices needed to reconstruct the manifold".

Example 4.4.21 (simplicial volume of spheres and tori)**.** Let $n \in \mathbb{N}_{>0}$. Then $\|S^n\| = 0$ (Example 4.4.19). Similar arguments show that $\|(S^1)^n\| = 0$.

One can use Proposition 4.4.18 to show that the simplicial volume of oriented closed connected manifolds indeed is a homotopy invariant [24, 38].

We now come to the geometric side: A Riemannian manifold is *hyperbolic* if its Riemannian universal covering is isometric to hyperbolic space (of the same dimension); this is equivalent to the existence of a Riemannian metric of constant sectional curvature -1. For hyperbolic manifolds, simplicial volume coincides (up to a factor that only depends on the dimension) with the Riemannian volume [24][67, Chapter 6][4, Theorem C.4.2]:

Theorem 4.4.22 (simplicial volume of hyperbolic manifolds). *Let $n \in \mathbb{N}$ and let M be an oriented closed connected hyperbolic manifold of dimension n. Then*

$$\|M\| = \frac{\operatorname{vol} M}{v_n},$$

where v_n is the volume of the ideal regular geodesic n-simplex in the n-dimensional hyperbolic space.

As a consequence, we obtain the following rigidity result: The Riemannian volume of oriented closed connected hyperbolic manifolds is a homotopy invariant (!). For example, this is a key step in Gromov's alternative proof of Mostow rigidity [53].

Further information on simplicial volume and related invariants can be found in the literature [24, 38, 22]. One example of a current theme in research on simplicial volume is the relationship between simplicial volumes and gradient invariants of groups and spaces (Outlook 2.3.51) [20, 47, 23, 17, 6, 40, 41, 42, 46].

While there are many constructions of ordinary homology theories and while many of them are considered to be more "modern" (using the full power of modern homotopy theory), this additional semi-normed structure is most transparent in singular homology. This explains my personal preference for this approach to homology.

4.5 Singular Homology and Homotopy Groups

So far, we introduced two different types of homotopy invariants: Homotopy groups and homology. We indicate how these invariants are related. The key ingredient is the Hurewicz homomorphism (Remark 3.2.13). We first give a concrete description of the Hurewicz homomorphism for singular homology:

Remark 4.5.1 (Hurewicz homomorphism for singular homology). Let $n \in \mathbb{N}_{>0}$ and let $[S^n] \in H_n(S^n;\mathbb{Z})$ be the generator that corresponds to $\partial_{n+1}(\mathrm{id}_{\Delta^{n+1}})$ under the canonical homeomorphism $\partial\Delta^{n+1} \cong_{\mathsf{Top}} S^n$ (Example 4.4.3). Let (X, x_0) be a pointed space. Then the Hurewicz homomorphism in degree n is given by

$$\begin{aligned} h_{(X,x_0),n} \colon \pi_n(X,x_0) &\longrightarrow H_n(X;\mathbb{Z}) \\ [f]_* &\longmapsto H_n(f;\mathbb{Z})\bigl([S^n]\bigr) = \bigl[C_n(f;\mathbb{Z})(\partial_{n+1}\,\mathrm{id}_{\Delta^{n+1}})\bigr]. \end{aligned}$$

The obvious question is now: To which extent is the Hurewicz homomorphism an isomorphism?

Example 4.5.2 (Hurewicz homomorphism in degree 1)**.** Clearly, the Hurewicz homomorphism in degree 1, in general, cannot be an isomorphism: fundamental groups can be non-Abelian, while homology groups are always Abelian groups. A concrete example of this type is the wedge of two circles: the fundamental group is free of rank 2 (in particular, it is non-Abelian) and the first homology (with $\mathbb{Z}$-coefficients) is isomorphic to $\mathbb{Z}^2$ (Example 2.2.14, Lemma 3.2.12).

Example 4.5.3 (Hurewicz homomorphism in degree 2)**.** Also in degree 2, the Hurewicz homomorphism, in general, cannot be an isomorphism: We know already that $\pi_2(S^1 \times S^1, (e_1, e_1)) \cong_{\mathbb{Z}} 0$, but $H_2(S^1 \times S^2; \mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$ (Example 2.3.26 and Example 3.3.3).

The problem in degree 1 will be solved by passing to the abelianisation. The problem in higher degrees will be circumvented by looking at highly connected spaces and homotopy/homology in low enough degree.

4.5.1 Abelianisation of Groups

In order to understand the Hurewicz isomorphism in degree 1, we first review the abelianisation of groups:

Proposition and Definition 4.5.4 (abelianisation)**.** *The* abelianisation functor $\cdot_{\mathrm{ab}} \colon \mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ *is defined as follows:*

- *on objects: If G is a group, then we write*

$$G_{\mathrm{ab}} := G/[G,G]$$

for the abelianisation of G*. Here,* $[G,G]$ *denotes the* commutator subgroup of G*, i.e., the subgroup of G* generated *by the set of alle commutators in G.*

- *on morphisms: If $f \colon G \longrightarrow H$ is a group homomorphism, then we define*

$$\begin{aligned} f_{\mathrm{ab}} \colon G_{\mathrm{ab}} &\longrightarrow H_{\mathrm{ab}} \\ [g] &\longmapsto \bigl[f(g)\bigr]. \end{aligned}$$

This construction has the following properties:

1. *Then $\cdot_{\mathrm{ab}} \colon \mathsf{Group} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ indeed is a well-defined functor.*
2. *Let G be a group. Then the canonical projection $\pi \colon G \longrightarrow G_{\mathrm{ab}}$ has the following universal property: If A is an Abelian group and $f \colon G \longrightarrow A$ is a group homomorphism, then there exists exactly one group homomorphism $\overline{f} \colon G_{\mathrm{ab}} \longrightarrow A$ with $\overline{f} \circ \pi = f$.*

Proof. *Ad 1.* Let G be a group. A straightforward computation shows that $[G, G]$ is a normal subgroup of G and that G_{ab} is an Abelian group (whence can be viewed in a canonical way as a $\mathbb{Z}$-module) (Proposition III.1.3.20).

Let $f\colon G \longrightarrow H$ be a group homomorphism. As group homomorphisms map commutators to commutators, f_{ab} is a well-defined map; this map is clearly a homomorphism. Moreover, a simple calculation shows that this construction is functorial (check!).

Ad 2. This follows from the universal property of quotient groups and the definition of the commutator subgroup (check!). ☐

Example 4.5.5 (Abelianisation)**.**

- If G is an Abelian group, then the canonical projection $G \longrightarrow G_{\mathrm{ab}}$ is an isomorphism (because the commutator subgroup $[G, G]$ is trivial).
- If F is a free group of rank $r \in \mathbb{N}$, then $F_{\mathrm{ab}} \cong_{\mathbb{Z}} \mathbb{Z}^r$: For example, this can be shown by comparing the the universal properties of "abelianisations of free groups" and of "free Abelian groups" (check!).
- If $n \in \mathbb{N}_{\geq 2}$, then $(S_n)_{\mathrm{ab}} \cong_{\mathbb{Z}} \mathbb{Z}/2$, where S_n denotes the symmetric group on n elements (check! Satz III.1.3.13, proof of Satz III.1.3.24).

Study note. Why did we introduce the commutator subgroup in the algebra course?

4.5.2 The Hurewicz Theorem

Theorem 4.5.6 (Hurewicz theorem)**.** *Let $n \in \mathbb{N}_{>0}$, let X be an $(n-1)$-connected space, and let $x_0 \in X$.*

1. *If $n = 1$, then the Hurewicz homomorphism $h_{(X,x_0),1}\colon \pi_1(X, x_0) \longrightarrow H_1(X; \mathbb{Z})$ induces a (natural) isomorphism*
$$\pi_1(X, x_0)_{\mathrm{ab}} \cong_{\mathbb{Z}} H_1(X; \mathbb{Z}).$$
2. *If $n > 1$, then*
$$\forall_{k \in \{1,\dots,n-1\}} \quad H_k(X; \mathbb{Z}) \cong_{\mathbb{Z}} 0$$
and $h_{(X,x_0),n}\colon \pi_n(X, x_0) \longrightarrow H_n(X; \mathbb{Z})$ is a (natural) isomorphism.

There are many ways to prove the Hurewicz theorem in different levels of generality, both regarding the category of spaces as well as the homology theories. We will sketch a proof that is based on the manipulation of singular chains. This proof has the disadvantage that it does not generalise to other homology theories; however, it does have the advantage that it is based on a technique that is useful in the context of simplicial volume and related invariants (Chapter 4.4.5).

The geometric idea is rather simple: If $\sigma\colon \Delta^n \longrightarrow X$ is a singular simplex with $\sigma(t) = x_0$ for all $t \in \partial\Delta^n$, then σ can be viewed as an element of $\pi_n(X, x_0)$ via the canonical homeomorphism $\Delta^n/\partial\Delta^n \cong_{\mathsf{Top}} S^n$. Using the $(n-1)$-connectedness of X, we can indeed pretend that all singular n-simplices of X are of this type ...

Sketch of proof. As a first step, we replace the singular chain complex $C(X)$ by a subcomplex $C^{x_0,n}(X)$ generated by special singular simplices: For $k \in \mathbb{N}$, we set

$$\begin{aligned} S_k := \bigl\{\sigma \in \operatorname{map}(\Delta^k, X) \bigm| & \text{ for all faces } \Delta \subset \Delta^k \text{ of dimension } \leq n-1 \\ & \text{ we have } \sigma|_\Delta = \operatorname{const}_{x_0}\bigr\}. \end{aligned}$$

Let $C^{x_0,n}(X) \subset C(X)$ be the subcomplex that in degree $k \in \mathbb{N}$ is generated by S_k (instead of $\operatorname{map}(\Delta^k, X)$). This indeed defines a subcomplex as the sequence $(S_k)_{k\in\mathbb{N}}$ is closed under taking faces.

The proof then consists of the following two parts:

① We show that the inclusion $C^{x_0,n}(X) \longrightarrow C(X)$ is a chain homotopy equivalence (this uses the connectedness condition on X).

② We show that the theorem holds for the homology of $C^{x_0,n}(X)$.

In order to deal with ①, we will make use of an instance of the inductive/compatible homotopy principle (Lemma 4.5.7 below): Because X is $(n-1)$-connected, an inductive argument shows that the sequence $(S_k)_{k\in\mathbb{N}}$ defined above satisfies the hypotheses of Lemma 4.5.7 [68, Theorem 9.5.1]; when carrying out this proof in detail, it is useful to know that the inclusion $\partial\Delta^k \longrightarrow \Delta^k$ of the boundary of the standard simplex is a so-called cofibration [68, Chapter 5.1] (Chapter A.7.3). Hence, the inclusion $C^{x_0,n}(X) \longrightarrow C(X)$ is a chain homotopy equivalence. In particular, the induced maps

$$H_k\bigl(C^{x_0,n}(X)\bigr) \longrightarrow H_k(X;\mathbb{Z})$$

in homology are isomorphisms for every $k \in \mathbb{Z}$.

Therefore, it remains to take care of ②: By construction, $C^{x_0,n}(X)$ coincides with $C(\{x_0\})$ up to degree $n-1$.

- In particular, we obtain: For every $k \in \{1, \dots, n-1\}$, we have (check!)

$$H_k(X;\mathbb{Z}) \cong_{\mathbb{Z}} H_k\bigl(C^{x_0,n}(X)\bigr) = H_k\bigl(\{x_0\};\mathbb{Z}\bigr) \cong_{\mathbb{Z}} 0.$$

- Thus, it suffices to show that

$$h_{(X,x_0),n,\mathrm{ab}}\colon \pi_n(X,x_0)_{\mathrm{ab}} \longrightarrow H_n(X;\mathbb{Z})$$

is an isomorphism (here, we implicitly use that the higher homotopy groups are Abelian, which enables us to encode all cases in a single argument): More precisely, we will construct an inverse homomorphism

$$\overline{\varphi}\colon H_n\bigl(C^{x_0,n}(X)\bigr) \longrightarrow \pi_n(X,x_0)_{\mathrm{ab}}$$

as follows: We start with

$$\begin{aligned} \varphi\colon S_n &\longrightarrow \pi_n(X,x_0) \\ \sigma &\longmapsto [f_\sigma]_*, \end{aligned}$$

where $f_\sigma\colon (S^n,e_1) \longrightarrow (X,x_0)$ corresponds under the canonical homeomorphism $S^n \cong_{\mathsf{Top}} \partial\Delta^{n+1}$ to the continuous map that is σ on the 0-face and constantly x_0 on all other faces; this indeed gives a well-defined continuous map (by the glueing principle; Proposition A.1.17), because the singular simplex $\sigma \in S_n$ is constantly x_0 on $\partial\Delta^n$.

In view of the universal property of free generation and the fact that the target $\pi_n(X,x_0)_{\mathrm{ab}}$ is Abelian, the map φ induces a homomorphism

$$\widetilde{\varphi}\colon C_n^{x_0,n}(X) \longrightarrow \pi_n(X,x_0)_{\mathrm{ab}}.$$

In order to pass to homology, we need to verify that this map behaves well with respect to the boundary operator on $C^{x_0,n}(X)$: For all $\sigma \in S_{n+1}$, we have

$$\sum_{j=0}^{n+1}(-1)^j \cdot [f_{\sigma\circ i_{n+1,j}}]_{*,\mathrm{ab}} = 0 \text{ in } \pi_n(X,x_0)_{\mathrm{ab}};$$

this can be shown by hand by a tedious computation or using Corollary 3.2.11 and Lemma 3.2.12 (check!). Therefore, $\widetilde{\varphi}$ induces a well-defined homomorphism

$$\overline{\varphi}\colon H_n\bigl(C^{x_0,n}(X)\bigr) \longrightarrow \pi_n(X,x_0)_{\mathrm{ab}}.$$

Now straightforward steps show that $h_{(X,x_0),n}$ indeed is an isomorphism:

- By construction, for all $\sigma \in S_n$, we have (check!)

  $$h_{(X,x_0),n}\bigl([f_\sigma]_*\bigr) = \bigl[\sigma + (1+(-1)^n)\cdot \mathrm{const}_{x_0}\bigr].$$

  If $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n^{x_0,n}(X)$ is a cycle, then $\sum_{j=1}^m a_j \cdot \mathrm{const}_{x_0}$ is a null-homologous cycle (as can be seen from the constant map $X \longrightarrow \{x_0\}$ and functoriality on the chain level). Therefore, we obtain

$$h_{(X,x_0),n} \circ \overline{\varphi}\bigl([c]\bigr) = \Bigl[\sum_{j=1}^{m} a_j \cdot \sigma_j\Bigr] + \bigl(1 + (-1)^n\bigr) \cdot \Bigl[\sum_{j=1}^{m} a_j \cdot \mathrm{const}_{x_0}\Bigr]$$
$$= [c] + 0.$$

– Conversely, we start on $\pi_n(X, x_0)$. Let $\sigma \in \mathrm{map}_*((S^n, e_1), (X, x_0))$ and let $\tau\colon (\Delta^n, \partial\Delta^n) \longrightarrow (X, x_0)$ be the composition of the canonical wrap-around-map $\Delta^n \longrightarrow S^n$ and σ. Then, $[\sigma]_* = [\sigma_\Delta]_* \in \pi_n(X, x_0)$ and, by construction, we obtain in $\pi_n(X, x_0)_{\mathrm{ab}}$:

$$\begin{aligned}\overline{\varphi} \circ h_{(X,x_0),n}\bigl([\sigma]_*\bigr) &= \overline{\varphi} \circ h_{(X,x_0),n}\bigl([\sigma_\Delta]_*\bigr)\\ &= \sum_{j=1}^{n+1} (-1)^j \cdot [f_{\sigma_\Delta \circ i_{n+1,j}}]_*\\ &= [\sigma_\Delta]_* + 0\\ &= [\sigma]_*\end{aligned}$$ □

Lemma 4.5.7 (compatible homotopies)**.** *Let X be a topological space, let $(S_k \subset \mathrm{map}(\Delta^k, X))_{k\in\mathbb{N}}$ be a family of simplices, and let $(h_\sigma)_{k\in\mathbb{N},\sigma\in\mathrm{map}(\Delta^k,X)}$ be a family of homotopies with the following properties:*

1. *For each $k \in \mathbb{N}$ and each $\sigma \in \mathrm{map}(\Delta^k, X)$, the map $h_\sigma\colon \Delta^k \times [0,1] \longrightarrow X$ is a homotopy from σ to an element of S_k.*
2. *For all $k \in \mathbb{N}$, all $\sigma \in \mathrm{map}(\Delta^k, X)$, and all $j \in \{0, \dots, k\}$, we have*
$$h_{\sigma \circ i_{k,j}} = h_\sigma \circ (i_{k,j} \times \mathrm{id}_{[0,1]}).$$
3. *For all $k \in \mathbb{N}$ and all $\sigma \in S_k$, the homotopy h_σ satisfies*
$$\forall_{x\in\Delta^k} \quad \forall_{t\in[0,1]} \quad h_\sigma(x,t) = \sigma(x).$$

Let $C^S(X) \subset C(X)$ be the subcomplex of the singular chain complex generated in each degree $k \in \mathbb{N}$ by S_k instead of $\mathrm{map}(\Delta^k, X)$.

Then the inclusion $C^S(X) \longrightarrow C(X)$ is a chain homotopy equivalence in $_{\mathbb{Z}}\mathsf{Ch}$ (and thus induces an isomorphism in homology).

Proof. This can be shown by an inductive procedure, using prism decompositions and careful bookkeeping, similar to the proof of Theorem 4.2.1 (Exercise). □

Remark 4.5.8 (Hurewicz and the ℓ^1-semi-norm)**.** Let (X, x_0) be a pointed space and let $n \in \mathbb{N}_{>0}$. Every class $\alpha \in H_n(X; \mathbb{R})$ that is in the image of the composition
$$\pi_n(X, x_0) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{R})$$

of the Hurewicz homomorphism and the natural transformation $H_n(\,\cdot\,;\mathbb{Z}) \Longrightarrow H_n(\,\cdot\,;\mathbb{R})$ induced by the inclusion $\mathbb{Z} \longrightarrow \mathbb{R}$ satisfies $\|\alpha\|_1 = 0$ (by Proposition 4.4.18 and Example 4.4.19).

Example 4.5.9 (existence of finite coverings)**.** Let X be a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space such that $H_1(X;\mathbb{Z})$ is finitely generated with $\operatorname{rk}_{\mathbb{Z}} H_1(X;\mathbb{Z}) \geq 1$. Combining the Hurewicz theorem (Theorem 4.5.6) with the classification of coverings (Theorem 2.3.43) shows that: Then, for every $d \in \mathbb{N}_{>0}$, there exists a connected d-sheeted covering of X.

4.5.3 Some Homotopy Groups

Using the Hurewicz theorem, we can compute the lower-dimensional homotopy groups of spheres and wedges of spheres:

Corollary 4.5.10 (homotopy groups from homology)**.** *Let (X, x_0) be a simply connected pointed space and let $n \in \mathbb{N}_{>1}$ with*

$$\forall_{k\in\{1,\dots,n-1\}} \quad H_k(X;\mathbb{Z}) \cong 0.$$

Then X is $(n-1)$-connected and the Hurewicz homomorphism

$$h_{(X,x_0),n} \colon \pi_n(X, x_0) \longrightarrow H_n(X;\mathbb{Z})$$

is an isomorphism.

Proof. Let

$$N := \min\bigl\{k \in \mathbb{N} \bigm| \pi_k(X, x_0) \not\cong 0\bigr\} \in \mathbb{N} \cup \{\infty\}$$

(alternatively, in order to avoid basepoint issues: $N := 1 + \max\{k \in \mathbb{N} \mid \forall_{x_0 \in X} \quad \pi_k(X, x_0) \cong 0\}$). As X is simply connected, we have $N > 1$. Applying the Hurewicz theorem (Theorem 4.5.6) in degree N proves the claim. $\square$

For example, we can use these results to compute the lower homotopy groups of spheres via homology:

Example 4.5.11 (lower homotopy groups of (wedges of) spheres)**.** Let $n \in \mathbb{N}_{>1}$.

- Because S^n is simply connected (Example 2.2.11), the computation of the homology $H_*(S^n;\mathbb{Z})$ and Corollary 4.5.10 show that

$$\pi_k(S^n, e_1) \cong_{\mathbb{Z}} \begin{cases} 0 & \text{if } k \in \{1, \dots, n-1\} \\ \mathbb{Z} & \text{if } k = n. \end{cases}$$

- More generally: If I is a set, then the same argument shows that

$$\pi_k\Bigl(\bigvee_I (S^n, e_1)\Bigr) \cong_{\mathbb{Z}} \begin{cases} 0 & \text{if } k \in \{1, \dots, n-1\} \\ \bigoplus_I \mathbb{Z} & \text{if } k = n. \end{cases}$$

Caveat 4.5.12. In Corollary 4.5.10, simple connectedness is important: For example, the *Poincaré homology sphere* P is an oriented closed connected 3-manifold with [30]

$$\forall_{k\in\mathbb{Z}} \quad H_k(P;\mathbb{Z}) \cong_{\mathbb{Z}} H_k(S^3;\mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & \text{if } k \in \{0,3\} \\ 0 & \text{otherwise.} \end{cases}$$

However, P is *not* simply connected (indeed its fundamental group is the so-called binary icosahedral group, a group of 120 elements, which is related to A_5). In view of the Hurewicz theorem (Theorem 4.5.6), the fundamental group of P is a perfect group, i.e., a group with trivial abelianisation.

In fact, the Poincaré homology sphere plays an interesting role in manifold topology and in the history of the Poincaré conjecture [12, p. 34f].

5

Cellular Homology

In this final chapter, we will construct cellular homology from singular homology. Usually, singular chain complexes are huge and thus they are not suited for concrete calculations. We therefore want to find "smaller" chain complexes that compute singular/ordinary homology. Classical examples are

- simplicial homology for simplicial complexes (which is very rigid, but easy to implement on computers),
- cellular homology for CW-complexes (which is more flexible and thus better suited for computations by hand).

We first introduce CW-complexes, we then construct cellular homology and compare it to ordinary homology. Moreover, we give sample calculations and applications to the Euler characteristic.

Overview of this chapter.

Running example. again: spheres, tori, projective spaces

5.1 The Category of CW-Complexes

As first step we introduce a suitable category of topological spaces, built inductively from simple, flexible, cells:

Definition 5.1.1 ((relative) CW-complex)**.**

- Let (X, A) be a pair of spaces. A *relative CW-structure on* (X, A) is a sequence
$$A =: X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X$$
of subspaces of X with the following properties:
  - We have $X = \bigcup_{n \in \mathbb{N}} X_n$.
  - The topology on X coincides with the colimit topology of the system $A = X_{-1} \subset X_0 \subset X_1 \subset \dots$; i.e., a subset $U \subset X$ is open if and only if for every $n \in \mathbb{N} \cup \{-1\}$, the intersection $U \cap X_n$ is open in X_n.
  - For every $n \in \mathbb{N}$, the space X_n is obtained from X_{n-1} by attaching n-dimensional cells, i.e., there exists a set I_n and a pushout of the form
$$\begin{tikzcd} \coprod_{I_n} S^{n-1} \arrow[r] \arrow[d, "\text{inclusion}"'] & X_{n-1} \arrow[d, "\text{inclusion}"] \\ \coprod_{I_n} D^n \arrow[r] & X_n \end{tikzcd}$$
in Top; here, we use the convention $S^{-1} := \emptyset$. Then, X_n is the *n-skeleton of* X. The number $|I_n|$ equals the number of path-connected components of $X_n \setminus X_{n-1}$, but the choice of pushouts is *not* part of the data!

- A *relative CW-complex* is a pair (X, A) of spaces together with a relative CW-structure on (X, A). If $A = \emptyset$, then X, together with this CW-structure, is a *CW-complex*. Usually, we will leave the filtration of the CW-structure implicit and say things like "a relative CW-complex (X, A)" if the underlying CW-structure is clear from the context or irrelevant (Caveat 5.1.2).

- If (X, A) is a relative CW-complex and $n \in \mathbb{N}$, then the path-connected components of $X_n \setminus X_{n-1}$ are homeomorphic to $D^{n\circ}$ (check!) and are called *open n-cells of* (X, A).

- If (X, A) is a relative CW-complex, then the *dimension of* (X, A) is defined as $\dim(X, A) := \min\bigl\{n \in \mathbb{N} \bigm| \forall_{m \in \mathbb{N}_{\geq n}} \quad X_m = X_n\bigr\} \in \mathbb{N} \cup \{\infty\}$.

- A (relative) CW-complex is *finite*, if it consists of finitely many open cells. A (relative) CW-complex is *of finite type*, if in each dimension, it has only finitely many open cells.

The prefix "CW" refers to the "closure finiteness" condition on cells (which can be derived from the definition above) and the "weak topology" (i.e., the colimit topology).

Caveat 5.1.2 (existence and uniqueness of CW-structures?!).

- *Not* every topological space is homotopy equivalent to a CW-complex. For example, the Warsaw circle (Figure 2.20) is *not* homotopy equivalent to a CW-complex (proving this fact requires more homotopy theory than we have currently available; Example A.7.4).

- Every closed topological manifold is homotopy equivalent to a finite CW-complex; this is far from simple: By a result of Milnor [34], every closed topological manifold M is homotopy equivalent to a countable CW-complex. One can then, for example, use the work of Kirby and Siebenmann [31, 32] to improve this to the homotopy type of a *finite* CW-complex.

- In the notation, we usually do not mention the chosen CW-structure explicitly. This is just laziness and is not justified by any kind of uniqueness. However, the fact that CW-structures, in general, are not unique is not a bug but a feature: One should always try to find CW-structures that are well adapted to the problem at hand. For example, depending on the problem it might be good to choose a CW-structure with few cells or a very symmetric CW-structure. Whenever the choice of CW-structure is relevant, we will mention it explicitly.

Example 5.1.3 (a relative CW-structure on (D^n, S^{n-1})). We start with the model case of CW-complexes, namely the standard cell: Let $n \in \mathbb{N}$. Then

$$\begin{aligned} X_{n-1} := X_{n-2} := \cdots := X_{-1} := S^{n-1} \\ \forall_{k \in \mathbb{N}_{\geq n}} \quad X_k := D^n \end{aligned}$$

is a relative CW-structure on (D^n, S^{n-1}). A pushout for the passage from the $(n-1)$-skeleton to the n-skeleton is, for example, (where all vertical maps are the inclusions)

$$\begin{tikzcd} S^{n-1} \arrow[r, "\mathrm{id}_{S^{n-1}}"] \arrow[d] & X_{n-1} = S^{n-1} \arrow[d] \\ D^n \arrow[r, "\mathrm{id}_{D^n}"'] & X_n = D^n. \end{tikzcd}$$

Example 5.1.4 (CW-structures). Examples of CW-structures on the circle, on the sphere S^2, on the torus $S^1 \times S^1$, and on $\mathbb{R}P^2$ are indicated in Figure 5.1.

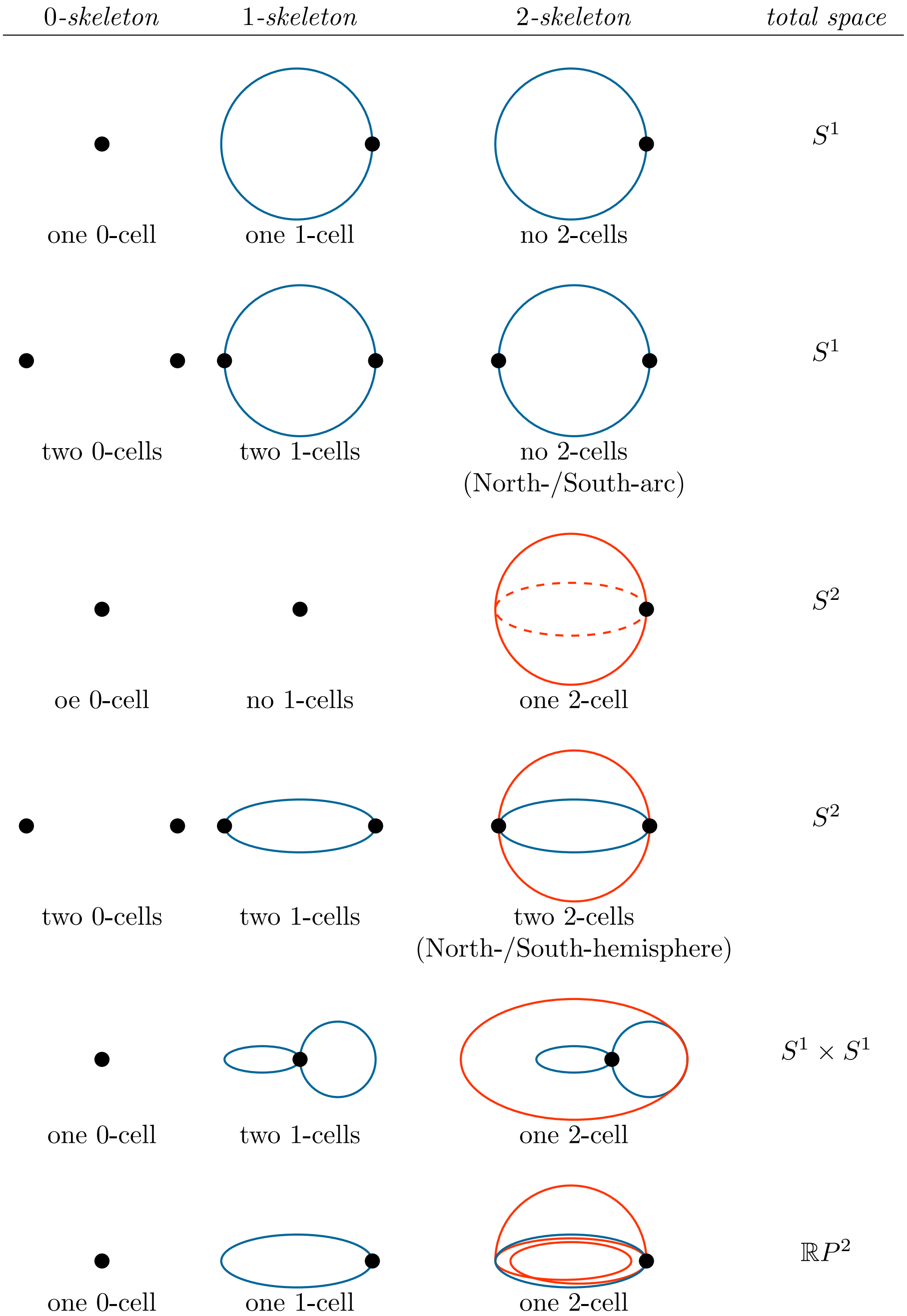


Figure 5.1.: Examples of CW-structures

Example 5.1.5 (one-dimensional complexes). The one-dimensional complexes alluded to in the proof of the Nielsen–Schreier theorem (Theorem 2.3.52) are CW-complexes of dimension 1. We will return to this point of view in Corollary 5.3.13.

Remark 5.1.6 (the point-set topology of CW-complexes). It turns out that the point-set topology of CW-complexes is rather tame. More information about these properties can, for example, be found in the book by Lundell and Weingram [48].

As next step, we introduce a suitable notion of structure preserving morphisms for CW-complexes:

Definition 5.1.7 (cellular map). Let (X, A) and (Y, B) be relative CW-complexes. A *cellular map* $(X, A) \longrightarrow (Y, B)$ is a continuous map $f \colon X \longrightarrow Y$ satisfying

$$\forall_{n \in \mathbb{N} \cup \{-1\}} \quad f(X_n) \subset Y_n$$

(where $(X_n)_{n \in \mathbb{N} \cup \{-1\}}$ and $(Y_n)_{n \in \mathbb{N} \cup \{-1\}}$ denote the relative CW-structures on (X, A) and (Y, B), respectively).

Example 5.1.8 (cellular maps on the circle). We consier S^1 with the first two CW-structures in Figure 5.1.

- The reflection $S^1 \longrightarrow S^1$ at the horizontal axis is a cellular map if we take the first CW-structure on domain and target or if we take the second CW-structure on domain and target.
- The reflection $S^1 \longrightarrow S^1$ at the vertical axes is *not* cellular if we take the first CW-structure on domain and target but it is cellular with respect to the second one.

Example 5.1.9 (generic cellular maps).

- If (X, A) is a relative CW-complex, then $\mathrm{id}_X \colon X \longrightarrow X$ is a cellular map from this relative CW-complex to itself (we take the same CW-structure on domain and target because the CW-structure is part of the data of a relative CW-complex!).
- The composition of cellular maps is cellular (check!).

Definition 5.1.10 (categories of CW-complexes).

- The category CW of CW-complexes consists of:
  - objects: The class of objects is the class of all CW-complexes.
  - morphisms: The set of morphisms between two CW-complexes is the set of cellular maps.
  - compositions: The compositions are given by ordinary composition of maps.

Figure 5.2.: a CW-structure on $[0,1]$

- The category CW^2 of relative CW-complexes consists of:
  - objects: The class of objects is the class of all relative CW-complexes.
  - morphisms: The set of morphisms between two relative CW-complexes is the set of cellular maps.
  - compositions: The compositions are given by ordinary composition of maps.

Moreover, we introduce a cellular version of homotopy. To this end, we first fix cellular models of intervals and cylinders:

Example 5.1.11 (a CW-structure on the unit interval)**.** In the following, we will use the CW-structure

$$\emptyset \subset \{0,1\} \subset [0,1] \subset [0,1] \subset \cdots \subset [0,1]$$

on $[0,1]$ (Figure 5.2, Example 5.1.3).

Proposition 5.1.12 (a CW-structure on cylinders)**.** *Let (X,A) be a relative CW-complex with CW-structure $(X_n)_{n\in\mathbb{N}\cup\{-1\}}$. Then $(Z_n)_{n\in\mathbb{N}\cup\{-1\}}$, given by*

$$\begin{aligned} Z_{-1} &:= A \times [0,1] \\ \forall_{n\in\mathbb{N}} \quad Z_n &:= \bigl(X_n \times \{0,1\}\bigr) \cup \bigl(X_{n-1} \times [0,1]\bigr), \end{aligned}$$

is a relative CW-structure on $(X,A) \times [0,1]$.

Proof. Clearly, $\bigcup_{n\in\mathbb{N}\cup\{-1\}} Z_n = X \times [0,1]$. Moreover, $X \times [0,1]$ carries the colimit topology with respect to this filtration (check!). It remains to take care of the pushout condition: For each $n \in \mathbb{N}$, we choose a pushout

$$\begin{tikzcd} \coprod_{I_n} S^{n-1} \arrow[r, "\varphi_n"] \arrow[d, "\text{inclusion}"'] & X_{n-1} \arrow[d, "\text{inclusion}"] \\ \coprod_{I_n} D^n \arrow[r, "\Phi_n"'] & X_n \end{tikzcd}$$

in Top. Using the fact that products of balls with $[0,1]$ are balls in the sense that

$$\bigl(D^{n-1} \times [0,1], S^{n-2} \times [0,1]) \cup (D^{n-1} \times \{0,1\})\bigr) \cong_{\mathsf{Top}^2} (D^n, S^{n-1})$$

holds for all $n \in \mathbb{N}$ (check!), we can use the pushouts above to construct pushouts of the form

$$\begin{tikzcd}
\coprod_{I_{n-1}} S^{n-1} \sqcup \coprod_{I_n} \bigl(S^{n-1} \times \{0,1\}\bigr) \arrow[r, "\text{``}\Phi_{n-1} \times \mathrm{id}_{[0,1]}|\text{''} \sqcup \varphi_n \times \mathrm{id}_{\{0,1\}}"] \arrow[d, "\text{inclusion}"'] & Z_{n-1} \arrow[d, "\text{inclusion}"] \\
\coprod_{I_{n-1}} D^n \sqcup \coprod_{I_n} \bigl(D^n \times \{0,1\}\bigr) \arrow[r, "\text{``}\Phi_{n-1} \times \mathrm{id}_{[0,1]}\text{''} \sqcup \Phi_n \times \mathrm{id}_{\{0,1\}}"'] & Z_n
\end{tikzcd}$$

Pulling $[0,1]$ through the universal property of the pushout works because $[0,1]$ is compact [13, Lemma V.2.13]. □

Caveat 5.1.13 (products of CW-complexes)**.** Let X and Y be CW-complexes and let $(Z_n)_{n\in\mathbb{N}\cup\{-1\}}$ be given by $Z_{-1} := \emptyset$ and

$$\forall_{n\in\mathbb{N}} \quad Z_n := \bigcup_{k\in\{0,\dots,n\}} X_k \times Y_{n-k}.$$

Then, in general, $(Z_n)_{n\in\mathbb{N}\cup\{-1\}}$ is *no* CW-structure on $X \times Y$ [14] (and the question of when this happens is rather delicate [7]). Therefore, when working with products of (infinite) CW-complexes, it is sometimes convenient to pass to the category of compactly generated spaces [64].

The CW-structure from Proposition 5.1.12 allows us to define cellular homotopy and the cellular homotopy categories:

Definition 5.1.14 (cellular homotopy)**.** Let (X,A) and (Y,B) be relative CW-complexes and let $f,g\colon (X,A) \longrightarrow (Y,B)$ be cellular maps. Then f and g are *cellularly homotopic*, if there exists a cellular map $h\colon (X,A)\times[0,1] \longrightarrow (Y,B)$ with

$$h \circ i_0 = f \quad \text{and} \quad h \circ i_1 = g;$$

here, the cylinder $(X,A) \times [0,1]$ carries the CW-structure from Proposition 5.1.12 and $i_0, i_1\colon (X,A) \longrightarrow (X,A) \times [0,1]$ are the inclusions of bottom and top into the cylinder (which are cellular!).

This notion of homotopy satisfies the usual inheritance properties (as in Proposition 1.3.13; check!). Hence, the following definition makes sense:

Definition 5.1.15 (homotopy categories of CW-complexes)**.**

- The category $\mathsf{CW_h}$ consists of:
  - objects: We let $\mathrm{Ob}(\mathsf{CW_h}) := \mathrm{Ob}(\mathsf{CW})$ be the class of all CW-complexes.
  - morphisms: The set of morphisms between CW-complexes is the set of cellular homotopy classes of cellular maps between these CW-complexes.

  - compositions: The compositions are given by ordinary composition of representatives.

- The category $\mathsf{CW}^2{}_\mathsf{h}$ consists of:
  - objects: We let $\mathrm{Ob}(\mathsf{CW}^2{}_\mathsf{h}) := \mathrm{Ob}(\mathsf{CW}^2)$ be the class of all relative CW-complexes.
  - morphisms: The set of morphisms between relative CW-complexes is the set of cellular homotopy classes of cellular maps between these relative CW-complexes.
  - compositions: The compositions are given by ordinary composition of representatives.

5.2 Cellular Homology

We want to use CW-structures to compute (ordinary) homology. We will therefore introduce cellular homology and compare cellular homology with (ordinary) homology:

5.2.1 Geometric Idea

As first step, we will explain the geometric idea behind the construction of cellular homology; for simplicity, we restrict to the base ring $\mathbb{Z}$: Let X be a CW-complex. In each dimension n, we choose a pushout

$$\begin{tikzcd} \coprod_{I_n} S^{n-1} \arrow[r, "\varphi_n"] \arrow[d, "\text{inclusion}"'] & X_{n-1} \arrow[d, "\text{inclusion}"] \\ \coprod_{I_n} D^n \arrow[r, "\Phi_n"'] & X_n \end{tikzcd}$$

in Top. Then we define

$$C_n := \bigoplus_{I_n} \mathbb{Z}$$

(i.e., every n-cell corresponds to a standard basis element in C_n). In order to define the boundary operator $\partial_n \colon C_n \longrightarrow C_{n-1}$, for each $i \in I_n$ and $j \in I_{n-1}$, we determine the "degree" of φ_n restricted to the i-th sphere S^{n-1} on the j-th cell D^{n-1} in X_{n-1}. Finally, cellular homology of X is defined as the homology of this chain complex (C_*, ∂_*).

- If a CW-structure and corresponding pushouts are known well enough, then computing cellular homology is easy; if the CW-complex is finite, the cellular chain complex is finite and computing its homology is algorithmically solvable (Remark A.6.20).

- However, the construction above suffers from the ambiguity in the choice of the pushouts. Therefore, it is desirable to find a more intrinsic description of the cellular chain complex. The key observation in this context is that

$$H_n(X_n, X_{n-1}; \mathbb{Z}) \cong_{\mathbb{Z}} H_n\Bigl(\coprod_{n \in \mathbb{N}} (D^n, S^{n-1}); \mathbb{Z}\Bigr) \cong_{\mathbb{Z}} \bigoplus_{I_n} \mathbb{Z}$$

  and it turns out that the cellular chain complex and cellular homology can be constructed in a streamlined fashion using such relative homology groups.

5.2.2 The Construction

We now give the precise definition of the cellular chain complex/homology associated with a given ordinary homology theory. Moreover, we explain why this construction coincides with the geometric description above and compute some examples.

Setup 5.2.1. In the following, let R be a ring with unit and let $h := \bigl((h_k)_{k \in \mathbb{Z}}, (\partial_k)_{k \in \mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 with values in ${}_R\mathsf{Mod}$, and $Z := h_0(\bullet)$.

Proposition and Definition 5.2.2 (cellular chain complex)**.** *The* cellular chain complex *functor* $C^h \colon \mathsf{CW}^2 \longrightarrow {}_R\mathsf{Ch}$ *is defined as follows:*

1. *On objects: Let (X, A) be a relative CW-complex with relative CW-structure $(X_n)_{n \in \mathbb{N} \cup \{-1\}}$. For $n \in \mathbb{N}$, we set*

$$C^h_n(X, A) := h_n(X_n, X_{n-1}),$$

   and for all $n \in \mathbb{Z}_{<0}$, we set $C^h_n(X, A) := 0$. Moreover, for $n \in \mathbb{N}_{>0}$, we define $\partial^{h,(X,A)}_n$ by

$$\begin{tikzcd}
C^h_n(X,A) \arrow[r, equal] \arrow[dd, dashed, "\partial^{h,(X,A)}_n"'] & h_n(X_n, X_{n-1}) \arrow[dd, "\partial^{(X_n, X_{n-1}, X_{n-2})}_n"'] \arrow[dr, "\partial^{(X_n, X_{n-1})}_n"] & \\
& & h_{n-1}(X_{n-1}) \arrow[dl, "h_{n-1}(\text{inclusion})"] \\
C^h_{n-1}(X,A) \arrow[r, equal] & h_{n-1}(X_{n-1}, X_{n-2}) &
\end{tikzcd}$$

   Then $C^h(X, A) := \bigl((C^h_n(X, A))_{n \in \mathbb{Z}}, (\partial^{h,(X,A)}_n)_{n \in \mathbb{Z}}\bigr)$ is a chain complex in ${}_R\mathsf{Ch}$, the cellular chain complex of (X, A) with respect to h.

2. *On morphisms: If $f\colon (X,A) \longrightarrow (Y,B)$ is a cellular map of relative CW-complexes, then we define*

$$C_n^h(f) := h_n(f|_{X_n})\colon C_n^h(X,A) = h_n(X_n,X_{n-1}) \longrightarrow h_n(Y_n,Y_{n-1}) = C_n^h(Y,B)$$

for all $n \in \mathbb{N}$; moreover, for $n \in \mathbb{Z}_{<0}$, we set $C_n^h(f) := 0$. Then, $C^h(f) := \bigl(C_n^h(f)\bigr)_{n\in\mathbb{Z}}$ is a chain map $C^h(X,A) \longrightarrow C^h(Y,B)$.

Proof. Ad 1. If $n \in \mathbb{N}_{>0}$, then

$$\partial_n^{h,(X,A)} \circ \partial_{n+1}^{h,(X,A)} = 0,$$

because this composition contains (by construction of the connecting homomorphisms for triples) two subsequent terms of the long exact sequence of the pair (X_n, X_{n-1}).

Ad 2. Because f is cellular, we can view $f|_{X_n}$ as a map $(X_n,X_{n-1}) \longrightarrow (Y_n,Y_{n-1})$. Moreover, the connecting homomorphisms of triple sequences are natural (Proposition 3.1.8). Hence, $C^h(f)$ is a chain map.

By construction, $C^h\colon \mathsf{CW}^2 \longrightarrow {}_R\mathsf{Ch}$ is a functor (check!). □

Taking homology of the cellular chain complex results in cellular homology:

Definition 5.2.3 (cellular homology)**.** Let $n \in \mathbb{Z}$. Then *cellular homology with respect to h, in degree n,* is defined as the composition

$$H_n^h := H_n \circ C^h\colon \mathsf{CW}^2 \longrightarrow {}_R\mathsf{Mod}\,.$$

As indicated in Chapter 5.2.1, the cellular chain complex has a more explicit description, once we choose pushouts. As first step, have a closer look at the effect on homology of attaching cells:

Remark 5.2.4 (homology of attached cells)**.** Let $n \in \mathbb{N}$ and let (Y,X) be a pair of spaces that admits a pushout of the form

$$\begin{tikzcd}
\coprod_I S^{n-1} \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\
\coprod_I D^n \arrow[r, "\Phi"'] & Y
\end{tikzcd}$$

in Top (where the vertical maps are the inclusions). If the homology theory h is not additive, then we add the hypothesis that I is finite.

1. a) If $n \in \mathbb{N}_{>0}$ and $k \in \mathbb{Z}$, then the following composition is an isomorphism in ${}_R\mathsf{Mod}$:

$$h_k(Y,X) \xleftarrow[h_k(\Phi)]{\cong_R} h_k\Bigl(\coprod_I D^n, \coprod_I S^{n-1}\Bigr) \xleftarrow[\bigoplus_I h_k(\mathrm{incl})]{\cong_R} \bigoplus_I h_k(D^n,S^{n-1}) \xrightarrow{\cong_R} \bigoplus_I h_{k-1}\bigl(S^{n-1},\{e_1\}\bigr) \xleftarrow{\cong_R} \bigoplus_I h_{k-n}(\bullet)$$

The first isomorphism is a consequence of excision (after thickening up X), the second isomorphism follows from excision and additivity, the third isomorphism is obtained from the long exact sequence of the triple $(D^n, S^{n-1}, \{e_1\})$, and the last isomorphism is an iterated suspension isomorphism (Corollary 3.2.8).

b) If $n = 0$ and $k \in \mathbb{Z}$, then the analogous composition is an isomorphism in ${}_R\mathsf{Mod}$:

$$h_k(Y,X) \xleftarrow[h_k(\Phi)]{\cong_R} h_k\Bigl(\bigsqcup_I D^n, \bigsqcup_I S^{n-1}\Bigr) \xleftarrow[\bigoplus_I h_k(\mathrm{incl})]{\cong_R} \bigoplus_I h_k(D^n, S^{n-1}) \xrightarrow{\cong_R} \bigoplus_I h_k\bigl(\bullet, \emptyset\bigr) \xleftarrow{\cong_R} \bigoplus_I h_{k-n}(\bullet)$$

2. a) Let $n \in \mathbb{N}_{>0}$ and let $X \neq \emptyset$. Then, the canonical homeomorphism $D^n/S^{n-1} \cong_{\mathsf{Top}} S^n$ induces (together with the pushout above) a homeomorphism (check!)

$$(Y/X, X/X) \cong_{\mathsf{Top}_*} \bigvee_I (S^n, e_1).$$

Then, for each $k \in \mathbb{Z}$, the corresponding composition

$$h_k(Y,X) \xrightarrow[h_k(\mathrm{proj})]{} h_k(Y/X, X/X) \xrightarrow[h_k(\text{see above})]{\cong_R} h_k\Bigl(\bigvee^I S^n, \{e_1\}\Bigr) \xrightarrow{\cong_R} \bigoplus_I h_k\bigl(S^n, \{e_1\}\bigr) \xleftarrow{\cong_R} \bigoplus_I h_{k-n}(\bullet)$$

coincides with the isomorphism in 1a) (check!).

b) If $n = 0$, then the maps $D^0 \longrightarrow \{-1\} \subset S^0$ and $X \longrightarrow \{1\} \subset S^0$ induce a continuous map $Y/X \longrightarrow \bigvee^I S^0$. If $X \neq \emptyset$, then this map is a homeomorphism.

Proposition 5.2.5 (cellular chain complex, explicitly)**.** *Let (X,A) be a relative CW-complex with CW-structure $(X_n)_{n \in \mathbb{N} \cup \{-1\}}$; if h is not additive, then we assume additionally that the CW-complex (X,A) is of finite type. For every $n \in \mathbb{N}$, we choose a pushout*

$$\begin{tikzcd}
\bigsqcup_{I_n} S^{n-1} \arrow[r, "\varphi^{(n)} = \bigsqcup_{i \in I_n} \varphi_i^{(n)}"] \arrow[d] & X_{n-1} \arrow[d] \\
\bigsqcup_{I_n} D^n \arrow[r, "\Phi^{(n)} = \bigsqcup_{i \in I_n} \Phi_i^{(n)}"'] & X_n
\end{tikzcd}$$

in Top *(where the vertical maps are the inclusions).*

1. *The chain modules: For each $n \in \mathbb{N}$, the maps from Remark 5.2.4 yield an isomorphism*

$$C_n^h(X,A) \cong_R \bigoplus_{I_n} Z;$$

moreover, for each $k \in \mathbb{Z} \setminus \{n\}$, we have $h_k(X_n, X_{n-1}) \cong_R 0$.

2. *The boundary operator: Let $n \in \mathbb{N}_{>0}$. For $i \in I_n$ and $j \in I_{n-1}$, we write $f_{i,j}^{(n)}$ for the composition*

$$S^{n-1} \xrightarrow{\varphi_i^{(n)}} X_{n-1} \xrightarrow{\text{proj}} X_{n-1}/X_{n-2} \longrightarrow \bigvee^{I_{n-1}} S^{n-1} \longrightarrow S^{n-1}$$

where the penultimate map is defined as in Remark 5.2.4, and where the last map is the projection onto the j-th summand.

   a) *Let $n \in \mathbb{N}_{>1}$. Under the isomorphism from the first part, the map $\partial_n^{h,(X,A)} \colon C_n^h(X,A) \longrightarrow C_{n-1}^h(X,A)$ corresponds to the "matrix"*

$$\begin{aligned} F_n &:= \big((\sigma^{n-1})^{-1} \circ h_{n-1}(f_{i,j}^{(n)}) \circ \sigma^{n-1}\big)_{j \in I_{n-1}, i \in I_n} \\ &\in M_{I_{n-1} \times I_n}\big(\operatorname{Hom}_R(Z,Z)\big), \end{aligned}$$

   where $\sigma^{n-1} \colon Z = h_0(\bullet) \longrightarrow h_{n-1}(S^{n-1}, \{e_1\}) \cong h_{n-1}(S^{n-1})$ denotes the iterated suspension isomorphism (Corollary 3.2.8).

   In particular, if $Z \cong_{{}_R\mathsf{Mod}} R$, then we can view F_n as a matrix in $M_{I_{n-1} \times I_n}(R)$; the entries of this matrix are called incidence numbers.

   b) *Analogously, the boundary operator $\partial_1^{h,(X,A)}$ corresponds to the matrix*

$$F_1 := \big(d_{i,j}^{(1)} \cdot \mathrm{id}_Z\big)_{j \in I_0, i \in I_1}$$

   where, for all $i \in I_n$, $j \in I_{n-1}$:

$$d_{i,j}^{(1)} := \begin{cases} 0 & \text{if } f_{i,j}^{(1)}(1) = f_{i,j}^{(1)}(-1) \\ 1 & \text{if } f_{i,j}^{(1)}(1) = -1 \neq f_{i,j}^{(1)}(-1) \\ -1 & \text{if } f_{i,j}^{(1)}(1) = 1 \neq f_{i,j}^{(1)}(-1) \end{cases}$$

Proof. The first part is an immediate consequence of the computation of the effect of attaching cells in homology (Remark 5.2.4).

We will now prove the second part in the case $n \in \mathbb{N}_{>1}$ (the description for $\partial_1^{h,(X,A)}$ can be obtained in a similar fashion). To this end, we consider the diagram in Figure 5.3. This diagram is commutative (check!) and the lower/upper rows coincide with the isomorphisms in the first part (Remark 5.2.4). This gives the desired description of $\partial_n^{h,(X,A)}$. $\square$

Recipe for the computation of cellular homology. In view of Proposition 5.2.5, we can compute cellular chain complexes (whence, cellular homology) as follows:

- Choose pushouts in each dimension,
- determine the number of cells in each dimension, and
- study how the attaching maps glue to the lower-dimensional cells.

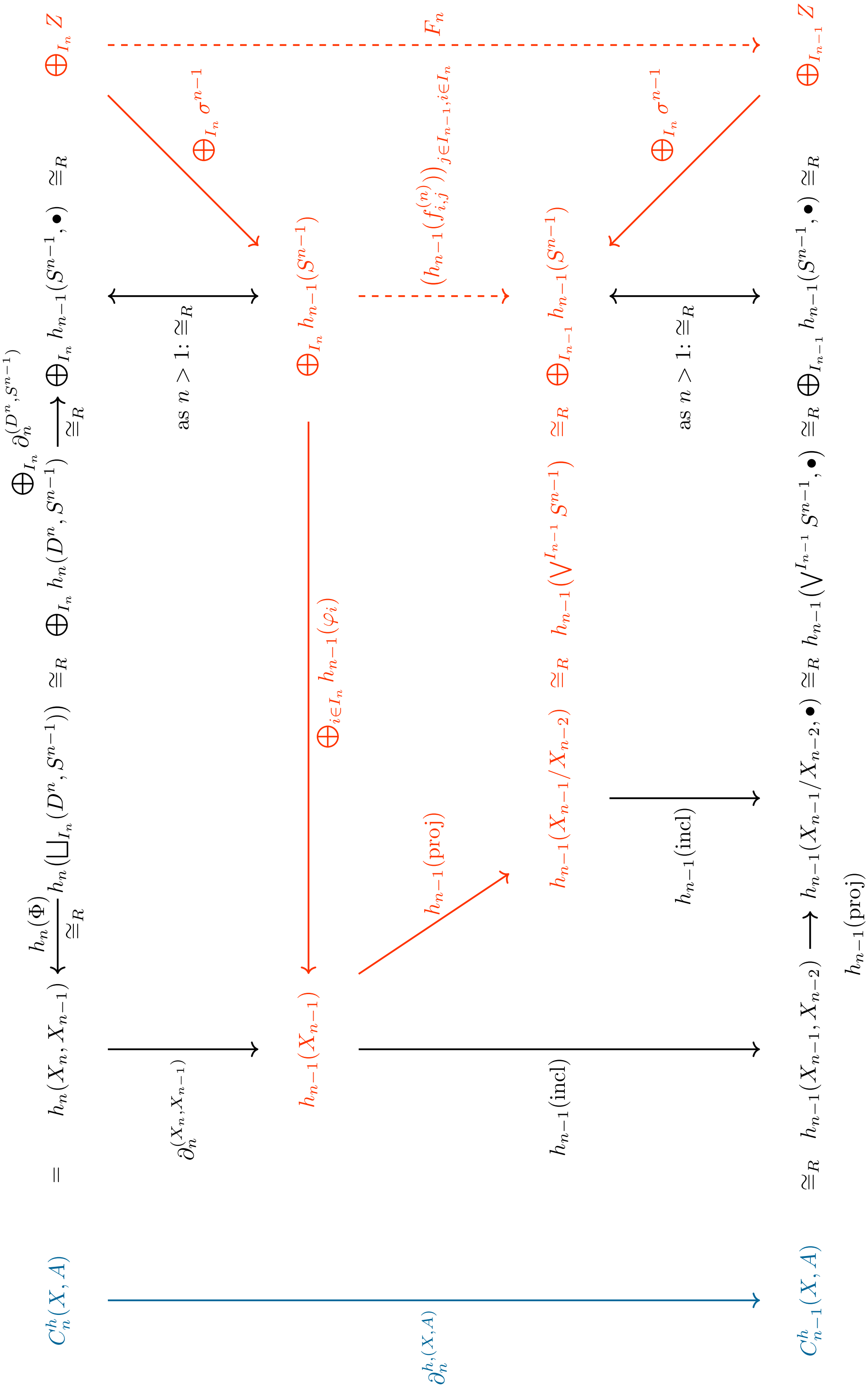


Figure 5.3.: the boundary operator of the cellular chain complex, in dimension bigger than 1

Example 5.2.6 (cellular homology of spheres)**.** Let $n \in \mathbb{N}$; for simplicity, we assume $n \geq 2$. On S^n, we consider the CW-structure

$$\emptyset \subset \{e_1\} = \{e_1\} = \cdots = \{e_1\} \subset S^n = S^n = \cdots,$$

which consists of a single 0-cell and a single n-cell (in analogy with the third example in Figure 5.1). Applying Proposition 5.2.5, we obtain that $C^h(S^n)$ is isomorphic (in ${}_R\mathsf{Ch}$) to the following chain complex:

$$\begin{array}{lcccccccc} \text{degree} & & n+1 & n & n-1 & \cdots & 1 & 0 & -1 \\ & \cdots \xrightarrow[0]{} & 0 \longrightarrow & Z \longrightarrow & 0 \longrightarrow & \cdots \longrightarrow & 0 \longrightarrow & Z \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

Because of $n \geq 2$, all the boundary operators of $C^h(S^n)$ are trivial (because the domain or target is a trivial module). Therefore, we obtain, for all $k \in \mathbb{Z}$,

$$H_k^h(S^n) \cong_R \begin{cases} Z & \text{if } k \in \{0, n\} \\ 0 & \text{if } k \in \mathbb{Z} \setminus \{0, n\} \end{cases} \cong_R h_k(S^n).$$

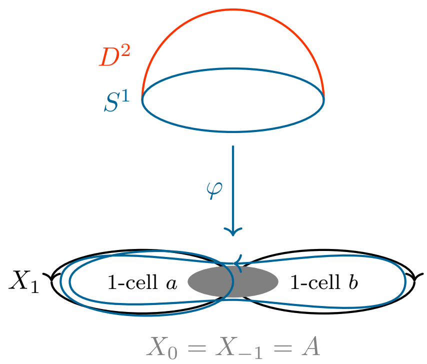
Example 5.2.7 (cellular homology of the unit interval)**.** Let $Z = R = \mathbb{Z}$. On the unit interval $[0,1]$, we consider the CW-structure $\emptyset \subset \{0,1\} \subset [0,1]$. (Example 5.1.11). This CW-structure consists of two 0-cells and one 1-cell. Applying Proposition 5.2.5 shows that the cellular chain complex $C^h([0,1])$ is (in ${}_\mathbb{Z}\mathsf{Ch}$) isomorphic to the chain complex I of Definition A.6.25 (check!):

$$\begin{array}{lccccccc} \text{degree} & & 2 & 1 & 0 & -1 & & \\ & \cdots \xrightarrow[0]{} 0 \xrightarrow[0]{} & 0 \xrightarrow[0]{} & \mathbb{Z} \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[0]{} & 0 \xrightarrow[0]{} & 0 \xrightarrow[0]{} \cdots & \\ & & & x \longmapsto & (-x, x) & & & \end{array}$$

Therefore, for all $k \in \mathbb{Z}$, we obtain

$$H_k^h\bigl([0,1]\bigr) \cong_\mathbb{Z} \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{Z} & \text{if } k = 0 \end{cases} \cong_\mathbb{Z} H_k\bigl([0,1]; \mathbb{Z}\bigr).$$

Remark 5.2.8 (cellular homotopy invariance)**.** Cellular homology is homotopy invariant in the sense that it factors over the homotopy classes functor $\mathsf{CW}^2 \longrightarrow \mathsf{CW}^2{}_\mathsf{h}$. More precisely, cellularly homotopic cellular maps induce chain homotopic chain maps between the cellular chain complexes: In view of functoriality with respect to cellular maps, it suffices to consider the model case of the inclusions i_0, i_1 of top and bottom into the cylinder $X \times [0,1]$ (with the CW-structure of Proposition 5.1.12) of a CW-complex X. Using Example 5.2.7 and a straightforward computation shows that $C^h(X \times [0,1]) \cong_{{}_R\mathsf{Ch}} C^h(X) \otimes_\mathbb{Z} I$ and the discussion in Remark A.6.30 yields $C^h(i_0) \simeq_R C^h(i_1)$ (check!).



$$\begin{tikzcd}
C_2^h(X,A) \arrow[d, "\partial_2^{h,(X,A)}"'] & \cong & \mathbb{Z} \arrow[d, "1 \mapsto (2,-1)"] \\
C_1^h(X,A) & \cong & \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}}
\end{tikzcd}$$

Figure 5.4.: a cellular chain complex

Example 5.2.9 (a more complicated cell attachment)**.** An example for the computation of cellular homology (with respect to an ordinary homology theory h with $\mathbb{Z}$-coefficients) of a CW-complex with a more interesting cell attachment is illustrated in Figure 5.4.

Example 5.2.10 (cellular homology of real projective spaces)**.** Let $n \in \mathbb{N}_{>1}$. A straightforward induction shows that

$$\emptyset \subset \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n = \mathbb{R}P^n = \cdots$$

is a CW-structure on $\mathbb{R}P^n$; the key step is to verify that

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & \mathbb{R}P^{n-1} \arrow[d] \\
D^n \arrow[r] & \mathbb{R}P^n
\end{tikzcd}$$

is a pushout in Top (where the vertical maps are the inclusions, the upper horizontal map is the canonical projection, and the lower horizontal map is the inclusion $D^n \subset S^n$ as Northern hemisphere, followed by the canonical projection $S^n \longrightarrow \mathbb{R}P^n$; Exercise). In particular, this CW-structure has in each of the dimensions $0, \dots, n$ exactly one cell (and no higher-dimensional cells).

Let $Z \cong_R R$. Applying Proposition 5.2.5 shows that the cellular chain complex $C^h(\mathbb{R}P^n)$ is isomorphic (in ${}_R\mathsf{Ch}$) to the following chain complex:

$$\begin{tikzcd}
\text{degree} & & n & n-1 & & 2 & 1 & 0 & -1 & \\
\cdots \arrow[r] & 0 \arrow[r] & R \arrow[r, "\partial_n"] & R \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r] & R \arrow[r, "\partial_2"] & R \arrow[r, "\partial_1"] & R \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
S^{k-1} \arrow[r, "\text{proj}"] \arrow[d, equal] & \mathbb{R}P^{k-1} \arrow[r, "\text{proj}"] & \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \arrow[r, "\text{canoncial homeo}"] & S^{k-1} \arrow[d, equal] \\
S^{k-1} \arrow[rr, "\text{induced by } c_{S^{k-1}}"'] & & \left(\bigvee^2 S^{k-1}\right) / \left(\forall_{x \in S^{k-1}} \quad i_1(x) \sim i_2(x)\right) \arrow[r, "\text{induced by id} \vee -\text{id}"'] & S^{k-1}
\end{tikzcd}$$

Figure 5.5.: determining the incidence numbers for $\mathbb{R}P^n$

We determine the boundary operators $\partial_1, \dots, \partial_n$ via the corresponding incidence numbers:

Because the CW-structure has only a single 0-cell, we have $\partial_1 = 0$. Let $k \in \{2, \dots, n\}$. Then the diagram in Figure 5.5 is commutative up to homotopy; this can be proved by tracing points on the Northern and the Southern hemisphere of S^{k-1} (check!). After applying h_{k-1}, this map induces the homomorphism (Lemma 3.2.12 and Corollary 3.2.11)

$$\left(1 + (-1)^k\right) \cdot \mathrm{id}_{h_{k-1}(S^{k-1})}.$$

Therefore, $C^h(\mathbb{R}P^n, \emptyset)$ is isomorphic (in ${}_R\mathsf{Ch}$) to the chain complex

$$\begin{array}{ccccccccccccccc}
\text{degre} & & & n & & n-1 & & & & 2 & & 1 & & 0 & & -1 \\
\cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{1+(-1)^n} & R & \longrightarrow & \cdots & \xrightarrow{0} & R & \xrightarrow{2} & R & \xrightarrow{0} & R & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Hence, for all $k \in \mathbb{Z}$, we obtain

$$H_k^h(\mathbb{R}P^n) \cong_R \begin{cases} R & \text{if } k = 0 \\ R/(2) & \text{if } k \in \{1, \dots, n-1\} \text{ is odd} \\ \{x \in R \mid 2 \cdot x = 0\} & \text{if } k \in \{2, \dots, n\} \text{ is even} \\ R & \text{if } k = n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

For example, this results in the following table:

R	$H_k^h(\mathbb{R}P^n)$ $k \in \{1, \dots, n-1\}$ odd	$k \in \{2, \dots, n\}$ even	if $k = n$ odd
$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Q}$	0	0	$\mathbb{Q}$

The relation between the results for different coefficients is explained by the universal coefficient theorems [13, VI.7.8] (that involve the relevant Tor-terms).

Remark 5.2.11 (incidence numbers)**.** Let $n \in \mathbb{N}_{>1}$. One can prove that

$$\begin{aligned} [S^{n-1}, S^{n-1}] &\longrightarrow \mathbb{Z} \\ [f] &\longmapsto \deg_{H_{n-1}(\,\cdot\,;\mathbb{Z})} f \end{aligned}$$

is a bijection: By Example 4.5.11, the pointed version of this statement holds. As every continuous map $S^{n-1} \longrightarrow S^{n-1}$ is homotopic to a pointed continuous map $(S^{n-1}, e_1) \longrightarrow (S^n, e_1)$, we obtain a commutative diagram (in Set) of the form

$$\begin{tikzcd} {[S^{n-1}, S^{n-1}]} \arrow[r, "\deg_{H_{n-1}(\,\cdot\,;\mathbb{Z})}"] & \mathbb{Z} \\ \pi_n(S^{n-1}, e_1) \arrow[u, "\mathrm{proj}"] \arrow[r, "\deg_{H_{n-1}(\,\cdot\,;\mathbb{Z})}"'] & \mathbb{Z} \arrow[u, equal] \end{tikzcd}$$

whose left vertical arrow is surjective and whose lower horizontal arrow is bijective. Therefore, also the upper horizontal arrow is bijective.

Moreover, Corollary 3.2.8 and Corollary 3.2.11 give us explicit representatives for each class in $[S^{n-1}, S^{n-1}]$, which are independent of the chosen ordinary homology theory. Thus, under the canonical ring homomorphism $\mathbb{Z} \longrightarrow R$, the incidence numbers can be viewed as integers and can be computed uniformly via $H_{n-1}(\,\cdot\,;\mathbb{Z})$.

These considerations show that cellular homology should be independent of the input (ordinary) homology theory.

5.2.3 Comparison of Homology Theories

We will now compare homology theories on CW-complexes. As first step, we show that cellular homology indeed computes (ordinary) homology:

Setup 5.2.12. Let R be a ring with unit and let $h := \bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

In the following, we will write $\mathsf{CW^2}_{\mathsf{fin}}$ for the category of finite relative CW-complexes (and cellular maps).

Theorem 5.2.13 (cellular homology vs. ordinary homology)**.** *Let h be an* ordinary *homology theory and let $n \in \mathbb{Z}$. Then the functors*

$$\begin{aligned} H_n^h\colon \mathsf{CW^2}_{\mathsf{fin}} &\longrightarrow {}_R\mathsf{Mod} \quad \textit{and} \\ h_n\colon \mathsf{CW^2}_{\mathsf{fin}} &\longrightarrow {}_R\mathsf{Mod} \end{aligned}$$

are naturally isomorphic. More precisely: Let (X, A) be a finite relative CW-complex.

1. *If $n < 0$, then $H_n^h(X,A) \cong_R 0 \cong_R h_n(X,A)$.*
2. *If $n \geq 0$, then the homomorphisms*

$$C_n^h(X,A) = h_n(X_n, X_{n-1}) \longleftarrow h_n(X_n, A) \longrightarrow h_n(X,A)$$

induced by the inclusions induce an R-isomorphism $H_n^h(X,A) \cong_R h_n(X,A)$.

Remark 5.2.14. If h is additive, then the analogous statement holds for the category CW^2 of all relative CW-complexes. In the case of singular homology this follows from a compactness argument (similar to Proposition 4.4.9); for the general case, a more general (homotopy) colimit argument is needed.

Example 5.2.15. In particular, singular homology of finite CW-complexes can be computed via the corresponding cellular homology. Therefore, Example 5.2.10 computes singular homology of the real projective spaces $\mathbb{R}P^n$.

Proof of Theorem 5.2.13. Let (X,A) be a finite relative CW-complex with relative CW-structure $(X_n)_{n\in\mathbb{N}\cup\{-1\}}$. We start with some preparations concerning the relative homology groups that will appear in the proof, the key observation being (A) (everything else is just bookkeeping):

(A). By Remark 5.2.4, we have (because h is an *ordinary* homology theory)

$$\forall_{n\in\mathbb{N}} \quad \forall_{k\in\mathbb{Z}\setminus\{n\}} \quad h_k(X_n, X_{n-1}) \cong_R 0.$$

(B). Inductively, we obtain from (A) and the long exact sequence of the triple (X_{n+1}, X_n, A) that

$$\forall_{n\in\mathbb{N}} \quad \forall_{k\in\mathbb{Z}\setminus\{0,\dots,n\}} \quad h_k(X_n, A) \cong_R 0$$

(C). and (using the long exact triple sequence of (X_N, X_n, X_{n-1}))

$$\forall_{n\in\mathbb{N}\cup\{-1\}} \quad \forall_{N\in\mathbb{N}_{\geq n}} \quad \forall_{k\in\mathbb{Z}_{\leq n}} \quad h_k(X_N, X_n) \cong_R 0.$$

Because (X,A) is a finite relative CW-complex, there is an $N \in \mathbb{N}$ with

$$X_N = X.$$

We will now distinguish two cases:

1. Let $n < 0$. From (B) and the definition of cellular homology, we obtain that
$$h_n(X,A) = h_n(X_N, A) \cong_R 0 \cong_R H_n^h(X,A).$$
2. Let $n \in \mathbb{N}$. We then consider the following diagram in ${}_R\mathsf{Mod}$ with exact rows (the unmarked arrows are induced by the inclusions):

$$\begin{tikzcd}
C^h_{n+1}(X,A) \arrow[r, "\partial^{h,(X,A)}_{n+1}"] \arrow[d, equal] & \ker \partial^{h,(X,A)}_n \arrow[r, "\mathrm{proj}"] \arrow[d, dashed, leftrightarrow, "①"'] & H^h_n(X,A) \arrow[r, "0"] \arrow[d, dashed, "③"'] & 0 \arrow[d, "\cong_R"] \\
h_{n+1}(X_{n+1},X_n) \arrow[r, "\partial^{(X_{n+1},X_n,A)}_{n+1}"'] & h_n(X_n,A) \arrow[r] \arrow[d] & h_n(X_{n+1},A) \arrow[r] \arrow[dl, "②"] & h_n(X_{n+1},X_n) \cong_R 0 \\
& h_n(X,A) & &
\end{tikzcd}$$

The homomorphism ② is an R-isomorphism (by (C) and the long exact sequence of the triple $(X = X_N, X_{n+1}, A)$, where we increase N, if necessary).

How is the homomorphism ① constructed? The long exact sequence of the triple (X_n, X_{n-1}, A) and (B) show that the inclusion induces an R-isomorphism

$$h_n(X_n, A) \cong_R \ker \partial_n^{(X_n,X_{n-1},A)}.$$

Moreover, from the long exact sequence of the triple (X_{n-1}, X_{n-2}, A) and (B), we can derive that

$$\ker \partial_n^{(X_n,X_{n-1},A)} = \ker \partial_n^{h,(X,A)}$$

This yields the R-isomorphism ①.

Because the left square in the diagram above is commutative, the isomorphism ① induces a well-defined R-homomorphism ③. Moreover, by the five lemma (Proposition A.6.7; adding a zero column on the right), the homomorphism ③ is an R-isomorphism.

Therefore, the R-isomorphisms ② and ③ lead to the desired (natural) R-isomorphism $H^h_n(X, A) \cong h_n(X, A)$. □

Corollary 5.2.16 (finiteness/vanishing results for orindary homology theories)**.** *Let h be an ordinary homology theory and let (X, A) be a finite relative CW-complex of dimension N.*

1. *Then, for all $k \in \mathbb{Z} \setminus \{0, \dots, N\}$, we have*

$$h_k(X, A) \cong_R H^h_k(X, A) \cong_R 0.$$

2. *If R is noetherian (e.g, $\mathbb{Z}$ or a field) and if $h_0(\bullet) \cong_R R$, then: For all $k \in \{0, \dots, N\}$ the R-module $h_k(X, A) \cong_R H^h_k(X, A)$ is finitely generated.*

Proof. From Theorem 5.2.13, we obtain $h_k(X, A) \cong_R H^h_k(X, A)$ for all $k \in \mathbb{Z}$. Therefore, it suffices to prove the corresponding statements for cellular homology.

The explicit description of the cellular chain complex shows that $C^h(X,A)$ is concentrated in the degrees $0,\dots,N$ and that all chain modules are finitely generated (Proposition 5.2.5).

Therefore, the first part directly follows from the definition of cellular homology.

For the second part, we only need to recall that $H^h_k(X,A) = H_k(C^h(X,A))$ is a quotient of a submodule of the finitely generated R-module $C^h_k(X,A)$. Because R is noetherian, this module is also finitely generated (Proposition IV.4.1.9). $\square$

Example 5.2.17 (ordinary homology of manifolds)**.** Let M be a closed topological manifold. Then M is homotopy equivalent to a finite CW-complex (Caveat 5.1.2), say of dimension N. Therefore, for every commutative ring R with unit,

- we have $H_k(M;R) \cong_R 0$ for all $k \in \mathbb{Z} \setminus \{0,\dots,N\}$,
- and $H_k(M;R)$ is finitely generated for each $k \in \mathbb{Z}$ (if R is noetherian).

In fact, Poincaré duality allows to prove the first part to vanishing above the topological dimension of M [68, Chapter 16.3, Chapter 18.3].

Corollary 5.2.18 (uniqueness of ordinary homology theories on $\mathsf{CW}^2{}_{\mathsf{fin}}$)**.** *Let h be an ordinary homology theory and let $n \in \mathbb{Z}$.*

1. *If k is an ordinary homology theory on Top^2 with values in ${}_R\mathsf{Mod}$ and if $k_0(\bullet) \cong_R h_0(\bullet)$, then h_n and k_n are naturally isomorphic as functors $\mathsf{CW}^2{}_{\mathsf{fin}} \longrightarrow {}_R\mathsf{Mod}$.*
2. *In particular, h_n und $H_n(\,\cdot\,; h_0(\bullet))$ are naturally isomorphic as functors $\mathsf{CW}^2{}_{\mathsf{fin}} \longrightarrow {}_R\mathsf{Mod}$.*

(If the homology theories in question are additive, then the corresponding statements also hold for CW^2 instead of $\mathsf{CW}^2{}_{\mathsf{fin}}$.)

Proof. The second part is a consequence of the first part and the fact that singular homology is an ordinary homology theory (Theorem 4.4.1).

We will now prove the first part: The explicit description of the cellular chain complex in terms of incidence numbers (Remark 5.2.11) shows that the cellular chain complexes C^h and C^k are naturally isomorphic as functors $\mathsf{CW}^2{}_{\mathsf{fin}} \longrightarrow {}_R\mathsf{Ch}$. Therefore, also

$$H^h_n = H_n \circ C^h \colon\ \mathsf{CW}^2{}_{\mathsf{fin}} \longrightarrow {}_R\mathsf{Mod}$$
$$H^k_n = H_n \circ C^k \colon\ \mathsf{CW}^2{}_{\mathsf{fin}} \longrightarrow {}_R\mathsf{Mod}$$

are naturally isomorphic functors. We then apply Theorem 5.2.13. $\square$

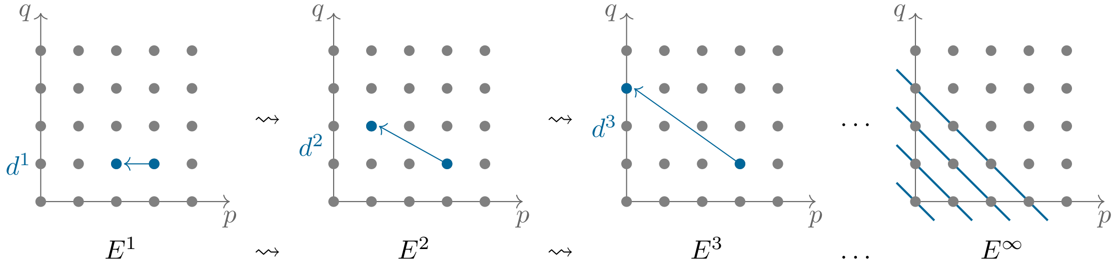


Figure 5.6.: pages of a spectral sequence

Caveat 5.2.19. There exist ordinary homology theories h on Top^2 with values in ${}_{\mathbb{R}}\mathsf{Mod}$ and coefficients (isomorphic to) $\mathbb{R}$ such that there exist (wild) topological spaces X and $n \in \mathbb{N}$ with

$$h_n(X) \not\cong_{\mathbb{R}} H_n(X; \mathbb{R});$$

an example is measure homology [71].

Caveat 5.2.20. If h is a *non-ordinary* homology theory on Top^2, then the cellular homology assoociated with h is *not* isomorphic to h (as can be seen from the one-point space).

By construction, cellular homology of a CW-structure $(X_n)_{n\in\mathbb{N}\cup\{-1\}}$ only keeps the information of the form $h_n(X_n, X_{n-1})$ (and assembles this data in a smart way). For general homology theories, the homology of (X_n, X_{n-1}) will, in general, not be concentrated in degree n. Taking *all* homology modules of (X_n, X_{n-1}) into account and organising this data into a so-called spectral sequence, allows to reconstruct the homology of the original space:

Theorem 5.2.21 (Atiyah-Hirzebruch spectral sequence [70, Chapter XIII.6])**.** *Let R be a ring with unit and let h be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$. If (X, A) is a finite relative CW-complex, then there is a (in (X, A) and h) natural, converging, spectral sequence*

$$\begin{aligned} E^1_{pq} &= h_{p+q}(X_p, X_{p-1}) \Longrightarrow h_{p+q}(X, A) \\ E^2_{pq} &= H_p\bigl(X, A; h_q(\bullet)\bigr) \Longrightarrow h_{p+q}(X, A). \end{aligned}$$

Outlook 5.2.22 (What are spectral sequences?)**.** Spectral sequences are a generalisation of long exact sequences and consist of a sequence $(E^r, d^r)_{r\in\mathbb{N}_{\geq 1}}$ of bigraded chain complexes (the *pages* of the spectral sequence), where the boundary operator d^r has the bidegree $(-r, r-1)$ and E^{r+1} is obtained from E^r via homology with respect to d^r (Figure 5.6).

For example, in the Atiyah-Hirzebruch spectral sequence, the row of E^1 that belongs to "$q = 0$" is nothing but the cellular chain complex $C^h(X, A)$ with respect to h.

In good cases, this process stabilises and yields the so-called ∞-page E^∞ of the spectral sequence. If convergence

$$E^1_{pq} \Longrightarrow B_{p+q}$$

holds, then this means that E^∞ gives an approximation of B; more precisely, the sequence $(E^\infty_{p,n-p})_{p\in\mathbb{Z}}$ is an approximation of B_n. In general, this "limit" B cannot be computed directly from E^∞, but it can only be determined up to so-called extension problems.

Moreover, the boundary operators in the higher pages of spectral sequences, usually cannot be computed explicitly. Nevertheless, spectral sequences and the associated toolbox allow to deduce meaningful results about the limit [69, Chapter 5].

In particular, homology theories on CW-complexes that coincide on the point and that are comparable via a natural transformation (that is an isomorphism on the point) are isomorphic. This result admits a more elementary direct formulation and proof:

Definition 5.2.23 (subcomplex, CW-pair)**.** Let X be a CW-complex with CW-structure $(X_n)_{n\in\mathbb{N}\cup\{-1\}}$ and let $A \subset X$.

- Then, A is a *subcomplex of* X, if for every open cell $e \subset X$ with $e \cap A \neq \emptyset$, we have $\overline{e} \subset A$ (in this case, $(A \cap X_n)_{n\in\mathbb{N}\cup\{-1\}}$ is a CW-structure on A and (X, A) is a relative CW-complex).
- If A is a subcomplex of X, then (X, A) is a *CW-pair*.

Example 5.2.24 (subcomplexes)**.**

- Skeleta of CW-complexes are subcomplexes.
- The inclusion of a subcomplex of a CW-complex is a cellular map.
- We consider the CW-structure $\emptyset \subset \{0, 1\} \subset [0, 1]$ on $[0, 1]$. Then $[0, 1)$ and $[0, 1/2]$ are *no* subcomplexes (check!), but $\{0\}$ and $\{0, 1\}$ are subcomplexes.

Definition 5.2.25 (category of CW-pairs)**.** The category $\mathsf{CW}^{②}$ of CW-pairs consists of:

- objects: The class of objects consists of all CW-pairs.
- morphisms: If (X, A) and (Y, B) are CW-pairs, then we set

$$\operatorname{Mor}_{\mathsf{CW}^{②}}\bigl((X, A), (Y, B)\bigr) := \bigl\{ f \in \operatorname{Mor}_{\mathsf{CW}}(X, Y) \bigm| f(A) \subset B \bigr\}.$$

- compositions: The compositions are given by ordinary composition of maps.

Definition 5.2.26 (homotopy category of CW-pairs)**.**

- Let (X, A) and (Y, B) be CW-pairs and let $f, g\colon (X, A) \longrightarrow (Y, B)$ be morphisms of CW-pairs. Then f and g are *cellularly homotopic as maps of CW-pairs* (abbreviated as $f \simeq_{\mathsf{CW}^{②}} g$, if there exists an $h \in \operatorname{Mor}_{\mathsf{CW}^{②}}((X, A) \times [0, 1], (Y, B))$ with
$$h(\,\cdot\,, 0) = f \quad \text{and} \quad h(\,\cdot\,, 1) = g.$$
Here, $X \times [0, 1]$ carries the CW-structure of Proposition 5.1.12.
- The homotopy category $\mathsf{CW}^{②}_{\mathsf{h}}$ of CW-pairs consists of:
  - objects: Let $\operatorname{Ob}(\mathsf{CW}^{②}_{\mathsf{h}}) := \operatorname{Ob}(\mathsf{CW}^{②})$.
  - morphisms: If (X, A) and (Y, B) are CW-pairs, then
  $$\operatorname{Mor}_{\mathsf{CW}^{②}_{\mathsf{h}}}\bigl((X, A), (Y, B)\bigr) := \operatorname{Mor}_{\mathsf{CW}^{②}}\bigl((X, A), (Y, B)\bigr) \bigm/ \simeq_{\mathsf{CW}^{②}} .$$
  - compositions: The compositions are given by ordinary composition of representatives.

Using CW-pairs, we can define a notion of homology theories on CW-pairs. There are two points that need some attention:

- A straightforward formulation of the long exact sequence for pairs requires that we can apply the homology theory to the subspace in question. Therefore, it is convenient to work with CW-pairs instead of relative CW-complexes. (Alternatively, one could avoid this by involving mapping cones).
- Complements of subcomplexes, in general, do not carry a canonical CW-structure. Therefore, the excision axiom needs to be reformulated.

Definition 5.2.27 (homology theories on CW-pairs)**.** Let R be a ring with unit. A *homology theory on* $\mathsf{CW}^{②}$ *with values in* ${}_R\mathsf{Mod}$ is a pair $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$, consisting of

- a sequence $(h_k)_{k\in\mathbb{Z}}$ of functors $\mathsf{CW}^{②} \longrightarrow {}_R\mathsf{Mod}$ and
- a sequence $(\partial_k)_{k\in\mathbb{Z}}$ of natural transformations
$$\partial_k\colon h_k \Longrightarrow h_{k-1} \circ U_{\mathsf{CW}},$$
where $U_{\mathsf{CW}}\colon \mathsf{CW}^{②} \longrightarrow \mathsf{CW}^{②}$ is the subcomplex functor,

with the following properties:

- *Homotopy invariance.* For every $k \in \mathbb{Z}$, the functor $h_k \colon \mathsf{CW}^{②} \longrightarrow {}_R\mathsf{Mod}$ factors over the homotopy classes functor $\mathsf{CW}^{②} \longrightarrow \mathsf{CW}^{②}_{\mathsf{h}}$.
- *Long exact sequences of pairs.* For every CW-pair (X, A), the sequence

$$\dots \xrightarrow{\partial_{k+1}} h_k(A, \emptyset) \xrightarrow{h_k(i)} h_k(X, \emptyset) \xrightarrow{h_k(j)} h_k(X, A) \xrightarrow{\partial_k} h_{k-1}(A, \emptyset) \xrightarrow{h_{k-1}(i)} \cdots$$

  is exact, where $i \colon (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j \colon (X, \emptyset) \longrightarrow (X, A)$ are the inclusion maps.
- *Excision.* For every CW-pair (X, A) and every subcomplex $C \subset X$, the homomorphisms
$$h_k(C, C \cap A) \longrightarrow h_k(X, A)$$
  induced by the inclusion $(C, C \cap A) \longrightarrow (X, A)$ are R-isomoprhisms for every $k \in \mathbb{Z}$.

Example 5.2.28 (homology theories on CW-pairs)**.** For example, the restriction of a homology theory on Top^2 to $\mathsf{CW}^{②}$ is a homology theory on $\mathsf{CW}^{②}$. On the other hand, also cellular homology (associated with some ordinary homology theory on Top^2) defines a homology theory on $\mathsf{CW}^{②}$ (check!).

As last ingredient, we need the ability to compare homology theories:

Definition 5.2.29 (natural transformations of homology theories)**.** Let R be a ring with unit and let $h := \bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ and $h' := \bigl((h'_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be homology theories on $\mathsf{CW}^{②}$ with values in ${}_R\mathsf{Mod}$. A *natural transformation $h \Longrightarrow h'$ of homology theories on* $\mathsf{CW}^{②}_{\mathsf{h}}$ is a sequence $(T_k \colon h_k \Longrightarrow h'_k)_{k\in\mathbb{Z}}$ of natural transformations that satisfy

$$T_{k-1}(A, \emptyset) \circ \partial_k^{(X,A)} = \partial_k'^{(X,A)} \circ T_k(X, A)$$

for all CW-pairs (X, A).

Finally, we are able to formulate (and prove) the comparison for homology theories on (finite) CW-complexes:

Theorem 5.2.30 (comparison theorem for homology theories)**.** *Let R be a ring with unit, let h and h' be homology theories on $\mathsf{CW}^{②}$ with values in ${}_R\mathsf{Mod}$, and let $(T_k)_{k\in\mathbb{Z}}$ be a natural transformation $h \longrightarrow h'$ of homology theories on $\mathsf{CW}^{②}$ with the property that, for each $k \in \mathbb{Z}$, the homomorphism*

$$T_k(\bullet) \colon h_k(\bullet) \longrightarrow h'_k(\bullet)$$

is an R-isomorphism. Then, for every finite CW-pair (X, A), and every $k \in \mathbb{Z}$, the homomorphism

$$T_k(X, A) \colon h_k(X, A) \longrightarrow h'_k(X, A)$$

is an isomorphism in ${}_R\mathsf{Mod}$.

Proof. This follows from a straightforward induction over the skeleta (check!), using the axiomatic computation of homology of spheres from the homology of the point (Corollary 3.2.8). □

Caveat 5.2.31. Let R be a ring with unit and let h and h' be homology theories on $\mathsf{CW}^{②}$ with values in ${}_R\mathsf{Mod}$ that satisfy

$$h_k(\bullet) \cong_R h'_k(\bullet)$$

for all $k \in \mathbb{Z}$. Then, in general, we *cannot* conclude that h and h' are isomorphic on all finite CW-pairs. In the comparison theorem (Theorem 5.2.30), it is essential that the given isomorphism on the homology of the point is induced by a natural transformation between the homology theories. Concrete examples can be constructed by comparing bordism with a concoction of ordinary homology theories that have the same coefficients as bordism.

Moreover, homology of CW-complexes plays an important role in the homotopy theory of CW-complexes (Appendix A.7; Corollary A.7.3).

5.3 The Euler Characteristic

We conclude this course with a short treatment of one of the oldest and most intriguing topological invariants: the Euler characteristic. The Euler characteristic is a homotopy invariant that can be directly computed from a CW-structure; however, it is just a number and thus has no functoriality properties.

After giving a geometric definition of the Euler characteristic, we will explain a homological interpretation (which also establishes homotopy invariance). Finally, we will give a sample application of the Euler characteristic in group theory.

5.3.1 Geometric Definition of the Euler Characteristik

The Euler characteristic of a finite CW-complex is nothing but the alternating sum of its numbers of cells in each dimension:

Definition 5.3.1 (Euler characteristic of a finite CW-complex)**.** Let X be a finite CW-complex. For each $n \in \mathbb{N}$, we denote by $c_n(X)$ the number of open n-cells of X. The *Euler characteristic of* X is defined as the (finite!) sum

$$\chi(X) := \sum_{n \in \mathbb{N}} (-1)^n \cdot c_n(X) \in \mathbb{Z}.$$

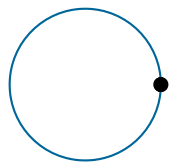
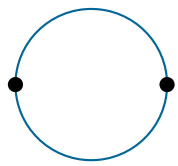
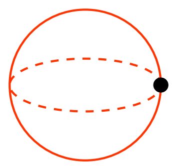
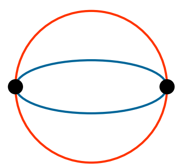
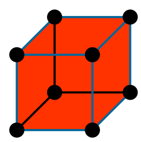
space	*CW-structure*	*Euler characteristic*
$\bullet$		1
$[0,1]$		$2-1=1$
S^1		$1-1=0$
S^1		$2-2=0$
S^2		$1-0+1=2$
S^2		$2-2+2=2$
S^2		$8-12+6=2$

Figure 5.7.: first examples of Euler characteristic calculations

Example 5.3.2 (Euler characteristic)**.** Some basic examples of Euler characteristic computations are collected in Figure 5.7.

The Euler characteristic was first introduced by Euler (Figure 5.8). Clearly, to Euler the terminology of CW-complexes (and homology, homotopy invariance) was not yet available. Originally, Euler introduced the Euler characteristic for (convex) polyhedra (Example 5.3.7, Corollary 5.3.8). At the dawn of

Figure 5.8.: topological characters (Euler (picture by Emanuel Handmann), Poincaré (picture by Eugéne Pirou), Blorx)

topology, Poincaré established parts of the terminology of classical Algebraic Topology and and proved homotopy invariance of the Euler characteristic.

5.3.2 A Homological Description

The little topological miracle that the Euler characteristic is independent of the chosen CW-structure was discovered by Poincaré (Theorem 5.3.5). In order to formulate this result, we introduce the classical notion of Betti numbers:

Definition 5.3.3 (Betti numbers)**.** Let X be a finite CW-complex and let R be a noetherian ring with unit that admits a suitable notion rk_R of rank for finitely generated R-modules (i.e, $\operatorname{rk}_R(R^n) = n$ for all $n \in \mathbb{N}$ and rk_R is additive for all short exact sequences of finitely generated R-modules; such a notion of rank exists for principal ideal domains/fields). For $n \in \mathbb{N}$, the *n-th Betti number of X with R-coefficients* is defined as

$$b_n(X; R) := \operatorname{rk}_R H_n(X; R) \in \mathbb{N}.$$

We also abbreviate the *n-th Betti number of X* by

$$b_n(X) := b_n(X; \mathbb{Z}).$$

Example 5.3.4 (Betti numbers of real projective spaces)**.** For instance, we have (Example 5.2.10, Theorem 5.2.13)

$$b_2(\mathbb{R}P^2; \mathbb{Z}) = 0 \neq 1 = b_2(\mathbb{R}P^2; \mathbb{Z}/2).$$

Even though Betti numbers for different coefficients can be different, their alternating sum always leads to the Euler characteristic:

Theorem 5.3.5 (homotopy invariance of the Euler characteristic)**.** *Let X be a finite CW-complex and let R be a noetherian ring with unit that admits a suitable notion rk_R of rank (Definition 5.3.3). Let X be a finite CW-complex.*

1. *Then, (and this sum is finite in view of Corollary 5.2.16)*
$$\chi(X) = \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n(X;R).$$

2. *In particular: If Y is a finite CW-complex with $X \simeq_{\mathsf{Top}} Y$, then*
$$\chi(X) = \chi(Y).$$
Therefore, the Euler characteristic is independent of the chosen (finite) CW-structure(!).

Proof. The second part follows from the first part (because singular homology and thus also Betti numbers are homotopy invariant).

The first part is an exercise in linear algebra: We have (where we abbreviate singular homology with R-coefficients by h)

$$\begin{aligned}
\chi(X) &= \sum_{n\in\mathbb{N}} (-1)^n \cdot c_n(X) \\
&= \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{rk}_R C_n^h(X) && \text{(Proposition 5.2.5)} \\
&= \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{rk}_R H_n\bigl(C_n^h(X)\bigr) && \text{(dimension formula/additivity of } \operatorname{rk}_R) \\
&= \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{rk}_R H_n(X;R) && \text{(Theorem 5.2.13)} \\
&= \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n(X;R),
\end{aligned}$$

as claimed. □

Example 5.3.6 (Euler characteristic of spheres and projective spaces)**.** Spheres and real projective spaces admit finite CW-structures (Example 5.1.4, Example 5.2.10). We can compute their Euler characteristic, for instance, via the integral Betti numbers. Hence, for all $n \in \mathbb{N}$, we have

$$\chi(S^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even,} \end{cases}$$
$$\chi(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

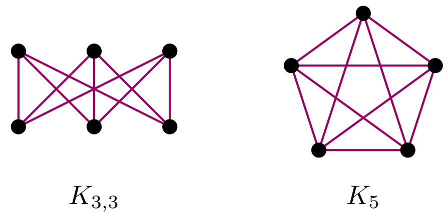


Figure 5.9.: the (non-planar) graphs $K_{3,3}$ and K_5

Example 5.3.7 (Euler's formula for CW-structures). By Theorem 5.3.5, for each (finite, 2-dimensional) CW-structure on S^2 with v vertices (0-cells), e edges (1-cells), and f faces (2-cells), we have

$$v - e + f = \chi(S^2) = 2.$$

Classical examples of applications of this formula (which is also the origin of the more general Euler characteristic) are:

- The classification of Platonic solids (Corollary 5.3.8).
- Non-planarity results for certain graphs, e.g., for K_5 and $K_{3,3}$ (Figure 5.9) [11, Chapter 4.2].
- Colouring theorems for planar graphs [11, Chapter 5.1].

Corollary 5.3.8 (combinatorics of regular polyhedra). *Let $P \subset \mathbb{R}^3$ be a convex, regular, 3-dimensional polyhedron with v vertices, e edges, and f facets. Then (v, e, f) is one of the following triples:*

v	e	f	*has the combinatorics of*
4	6	4	*tetrahedron*
6	12	8	*octahedron*
8	12	6	*hexahedron (cube!)*
20	30	12	*dodecahedron*
12	30	20	*icosahedron*

Proof. Elementary geometry shows that $\partial P \cong_{\mathsf{Top}} S^2$ and that the structure of vertices, edges, and facets defines a CW-structure on S^2 (check!). Therefore, by Example 5.3.7,

$$v - e + f = \chi(S^2) = 2.$$

Because P is regular, there exist $m, n \in \mathbb{N}_{\geq 3}$ with the following properties:

- Each facet of P has exactly m vertices/edges.
- At each vertex of P exactly n edges/facets meet.

As every edge is the edge of exactly two facets and as every edge connects exactly two vertices, we obtain

$$\frac{m \cdot f}{2} = k \quad \text{and} \quad \frac{n \cdot e}{2} = k.$$

Thus, $2 = e - k + f = 2 \cdot k/n - k + 2 \cdot k/m$, and so

$$\frac{1}{2} < \frac{1}{k} + \frac{1}{2} = \frac{1}{n} + \frac{1}{m}.$$

In particular, (m, n) has to be one of the following pairs:

$$(3,3), \quad (3,4), \quad (4,3), \quad (3,5), \quad (5,3).$$

These choices result in the listed possibilities for (e, k, f). $\square$

Moreover, Euclidean geometry shows that each of these combinatorial types is realised only by the regular polyhedron listed above.

Outlook 5.3.9 (classification of surfaces)**.** The Euler characteristic can be used to distinguish the different homotopy types/homeomorphism types/diffeomorphism types of oriented closed connected surfaces [51, Chapter I.8].

Outlook 5.3.10 ($1, 2, 3, \dots, \infty$-category)**.** The Euler characteristic offers the opportunity to explain a general abstraction scheme:

- Level 0: A numerical invariant (Euler characteristic).
- Level 1: An algebraic description of this numerical invariant (Euler characteristic via Betti numbers).
- Level 2: A description in terms of a homotopy invariant functor (Euler characteristic via singular homology).
- Level 3: A description in terms of a functor with refined homotopy invariance properties (Euler characteristic via a chain complex- or spectra-valued homology theory).

  $\vdots$

- Level ω: A description in terms of a homotopy invariant functor in the realm of ∞-categories (which organise the overall bookkeeping of homotopies between homotopies between homotopies $\dots$).

Outlook 5.3.11 (alternating sums)**.** The fact that alternating sums (as in the definition of the Euler characteristic) tend to have better topological properties than the corresponding unsigned sums is a general principle in topology. For example, this also occurs in the Lefschetz number and index theorems.

5.3.3 Divide and Conquer

When computing Euler characteristics, the following inheritance properties turn out to be useful.

Proposition 5.3.12 (Euler characteristic, inheritance properties)**.**

1. *If X and Y are finite CW-complexes, then*
$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

2. *Let*
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$
be a pushout in Top*, where A, B, and X are finite CW-complexes, where $i\colon A \longrightarrow X$ is the inclusion as a subcomplex and $f\colon A \longrightarrow B$ is a cellular map. Then,*
$$\chi(Y) = \chi(X) + \chi(B) - \chi(A)$$

3. *Let X be a finite CW-complex and $p\colon Y \longrightarrow X$ be a finite covering with $d \in \mathbb{N}$ sheets. Then,*
$$\chi(Y) = d \cdot \chi(X).$$

Proof. In each case, we first have to establish that the spaces in question admit a (finite) CW-structure. Then, carefully counting the open cells in these CW-structures and simple computations finish the proof.

Ad 1. Because X and Y are finite CW-complexes, the filtration
$$\Bigl(\bigcup_{k=0}^{n} X_k \times Y_{n-k}\Bigr)_{n\in\mathbb{N}}$$
is a finite CW-structure on $X \times Y$ (check!). Therefore,
$$c_n(X \times Y) = \sum_{k=0}^{n} c_k(X) \cdot c_{n-k}(Y)$$
for all $n \in \mathbb{N}$ (check!), and so a straightforward calculation shows that
$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Ad 2. The filtration $\bigl(g(X_n) \cup j(B_n)\bigr)_{n\in\mathbb{N}}$ is a CW-structure on Y (check!). Therefore, we obtain $c_n(Y) = c_n(X) + c_n(B) - c_n(A)$ for all $n \in \mathbb{N}$ (check!). In particular, $\chi(Y) = \chi(X) + \chi(B) - \chi(A)$.

Ad 3. In this case, the filtration $\bigl(p^{-1}(X_n)\bigr)_{n\in\mathbb{N}}$ is a finite CW-structure on Y (Exercise) and $c_n(Y) = d \cdot c_n(X)$ for all $n \in \mathbb{N}$ (check!). Hence, $\chi(Y) = d \cdot \chi(X)$. □

5.3.4 Application: Nielsen–Schreier, Quantitatively

We conclude with a refined version of the Nielsen–Schreier theorem (Theorem 2.3.52):

Corollary 5.3.13 (Nielsen–Schreier theorem, quantitative version)**.** *Let F be a free group of finite rank $n \in \mathbb{N}$ and let $G \subset F$ be a subgroup of finite index. Then G is free of rank*
$$[F : G] \cdot (n-1) + 1.$$

Proof. We proceed as in the topological proof of the Nielsen–Schreier theorem (Theorem 2.3.52) and relate the ranks of the free groups in question to the Euler characteristic of one-dimensional complexes.

Let $S \subset F$ be a free generating set of F. We consider
$$(X, x_0) := \bigvee_S (S^1, 1)$$
with the obvious CW-structure (having a single 0-cell and $|S|$ open 1-cells). Then $\pi_1(X, x_0) \cong_{\mathsf{Group}} F$ (Example 2.2.14).

By the classification theorem of coverings (Theorem 2.3.43), there exists a connected covering $p\colon (Y, y_0) \longrightarrow (X, x_0)$ associated with the subgroup of $\pi_1(X, x_0)$ that corresponds to $G \subset F$. In particular, we have $\pi_1(Y, y_0) \cong_{\mathsf{Group}} G$ and the covering p is $[F : G]$-sheeted.

We now relate the ranks to the Euler characteristic: Because $|S| = \operatorname{rk} F = n$ is finite, X is a finite CW-complex and
$$\chi(X) = 1 - n = 1 - \operatorname{rk} F.$$
Then also Y is a finite CW-complex (Proposition 5.3.12) and
$$\chi(Y) = [F : G] \cdot \chi(X) = [F : G] \cdot (1 - n).$$
Moreover, $\chi(Y) = 1 - m$, where m is the rank of G (this can be seen by contracting a spanning tree of Y (Example A.7.12) or by inductively computing the Euler charcteristic and the fundamental group of (Y, y_0)). Therefore,
$$(1 - m) = [F : G] \cdot (1 - n)$$
(which is the formula to remember!) and so $m = [F : G] \cdot (n - 1) + 1$. □

The End

The Beginning of the End

A

Appendix

Overview of this chapter.

A.1 Point-Set Topology

We collect basic notions and facts from point-set topology, as taught in introductory courses. Detailed explanations, proofs, and examples can be found in all books on point-set topology [28, 29, 55, 63].

A.1.1 Topological Spaces

The category of topological spaces consists of topological spaces and continuous maps between them. The main idea of topological spaces is to express "being close" not by distances but by a system of subsets, the so-called open subsets.

Definition A.1.1 (topological space, topology, open, closed)**.** A *topological space* is a pair (X, T) consisting of a set X and a *topology* T on X, i.e., T is a subset of the power set $P(X)$ of X with the following properties:

- We have $\emptyset \in T$ and $X \in T$.
- If $U \subset T$, then $\bigcup U \in T$ (i.e., T is closed with respect to taking unions).
- If $U \subset T$ is finite, then $\bigcap U \in T$ (i.e., T is closed with respect to taking finite intersections).

The elements of T are called *open sets (with respect to T)*; if $A \subset X$ and $X \setminus A \in T$, then A is *closed (with respect to T)*.

Convention A.1.2. In Algebraic Topology, whenever the topology T on a set X is clear from the context, we will abuse notation and also speak of the "topological space X" instead of the "topological space (X, T)". This slight imprecision will save us from a lot of notational clutter.

That the axioms for open sets do make sense can be easily seen in the case of topologies induced by a metric:

Proposition A.1.3 (topology induced by a metric)**.** *Let (X, d) be a metric space. Then*

$$T := \bigl\{U \subset X \bigm| \forall_{x \in U} \quad \exists_{\varepsilon \in \mathbb{R}_{>0}} \quad U(x, \varepsilon) \subset U\bigr\}$$

is a topology on X, the metric topology induced by d*. Here, for $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$, we write*

$$U(x, \varepsilon) := \bigl\{y \in X \bigm| d(y, x) < \varepsilon\bigr\}$$

for the open ε-ball around x in (X, d).

Remark A.1.4.

- For $\mathbb{R}^n$, the notion of open sets with respect to the topology induced by the Euclidean metric, coincides with the standard notion of open sets (as considered in the Analysis courses). We will call this topology the *standard topology on* $\mathbb{R}^n$.
- Moreover: If (X, d) is a metric space and $A \subset X$, then A is closed (with respect to the metric topology) if and only if it is sequentially closed.

Caveat A.1.5. Not every topological space is metrisable! (Corollary A.1.31).

Example A.1.6 (extremal topologies)**.** Let X be a set. Then there are two extremal topologies on X:

- The set $P(X)$ is a topology on X, the *discrete topology.*
- The set $\{\emptyset, X\}$ is a topology on X, the *trivial topology* (or *indiscrete topology*).

Remark A.1.7 (exotic topological spaces)**.** In Algebraic Topology, we will usually only work with "nice" topological spaces (that are built from balls, spheres, etc.) and only consider situations where the point-set topology is tame. In contrast, topological spaces that arise naturally in Algebraic Geometry usually are more exotic (e.g., the Zariski topology on $\operatorname{Spec} \mathbb{Z}$).

Moreover, we will use the following generalisations of the corresponding notions for metric spaces:

Definition A.1.8 ((open) neighbourhood)**.** Let (X, T) be a topological space and let $x \in X$.

- A subset $U \subset X$ is an *open neighbourhood of* x, if U is open and $x \in U$.
- A subset $U \subset X$ is a *neighbourhood of* x if there exists an open neighbourhood $V \subset X$ of x with $V \subset U$.

Definition A.1.9 (closure, interior, boundary)**.** Let (X, T) be a topological space and let $Y \subset X$.

- The *interior of* Y is
$$Y^\circ := \bigcup \{U \mid U \in T \text{ and } U \subset Y\},$$
i.e., Y° is the largest (with respect to inclusion) open subset of X that is contained in Y.
- The *closure of* Y is
$$\overline{Y} := \bigcap \{A \mid X \setminus A \in T \text{ and } Y \subset A\},$$
i.e., $\overline{Y}$ is the smallest (with respect to inclusion) closed subset of X that contains Y.

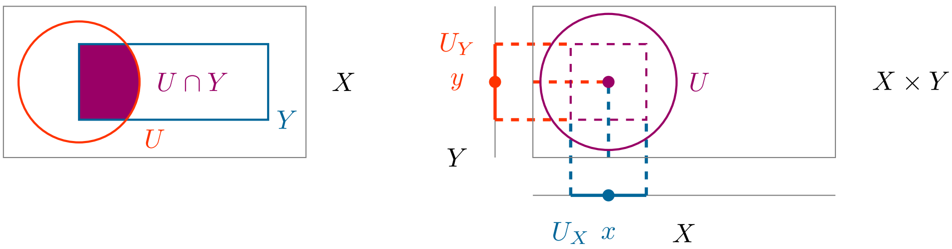


Figure A.1.: The subspace topology/product topology, schematically

- The *boundary of Y* is

$$\partial Y := \overline{Y} \cap \overline{(X \setminus Y)}.$$

Caveat A.1.10 (∂)**.** The symbol ∂ is heavily overloaded in Algebraic Topology. Most uses relate to some underlying geometric notion of boundary, but one should always make sure to understand what the actual meaning of ∂ is in the given context.

Two elementary constructions of topological spaces are subspaces and products; these constructions are illustrated in Figure A.1:

Remark A.1.11 (subspace topology)**.** Let (X, T) be a topological space and let $Y \subset X$ be a subset. Then

$$\{U \cap Y \mid U \in T\}$$

is a topology on Y, the *subspace topology on Y*. If T on X is induced by a metric d, then the subspace topology on Y is the topology induced by the restriction of the metric d to Y.

Remark A.1.12 (product topology)**.** Let (X, T_X) and (Y, T_Y) be topological spaces. Then

$$\{U \subset X \times Y \mid \forall_{(x,y) \in U} \quad \exists_{U_X \in T_X} \quad \exists_{U_Y \in T_Y} \quad (x, y) \in U_X \times U_Y \subset U\}$$

is a topology on $X \times Y$, the *product topology*. The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ coincides with the product topology of the standard topology on $\mathbb{R}$ (on both factors). Moreover, the product topology satisfies (together with the canonical projections onto the factors) the universal property of the product in the category of topological spaces (Remark 1.1.4).

Remark A.1.13 (general products)**.** Let $(X_i, T_i)_{i \in I}$ be a family of topological spaces and let $X := \prod_{i \in I} X_i$. Then the *product topology on X* is the coarsest topology that makes the canonical projections $(X \longrightarrow X_i)_{i\in I}$ continuous. More explicitly: A subset $U \subset X$ is open if and only if for every $x \in U$ there exists a finite set $J \subset I$ and open subsets $U_j \subset X_j$ for every $j \in J$ with

$$x \in \prod_{j\in J} U_j \times \prod_{i\in I\setminus J} X_i \subset U.$$

This product topology satisfies (together with the canonical projections onto the factors) the universal property of the product in the category of topological spaces.

A.1.2 Continuous Maps

Continuous maps are structure preserving maps in the world of topological spaces.

Definition A.1.14 (continuous)**.** Let (X, T_X) and (Y, T_Y) be topological spaces. A map $f\colon X \longrightarrow Y$ is *continuous* (with respect to T_X and T_Y), if

$$\forall_{U\in T_Y} \quad f^{-1}(U) \in T_X,$$

i.e., if preimages of open sets are open.

Remark A.1.15.

- For metric spaces, continuity with respect to the topology induced by the metric coincides with the ε-δ-notion of continuity.
- If X is a set and T, T' are topologies on X, then the identity map $\mathrm{id}_X\colon X \longrightarrow X$ is continuous as a map from (X, T) to (X, T') if and only if $T' \subset T$ (i.e., if T' is coarser than T).
- The maps $+, \cdot, -\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $/\colon \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$ are continuous with respect to the standard topology.
- If (X, T) is a topological space and $Y \subset X$, then the inclusion $Y \hookrightarrow X$ is continuous with respect to the subspace topology on Y.
- Constant maps are continuous.

Proposition A.1.16 (inheritance properties of continuous maps)**.** *Let (X, T_X), (Y, T_Y), and (Z, T_Z) be topological spaces and let $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ be maps.*

1. *If f and g are continuous, then also $g \circ f\colon X \longrightarrow Z$ is continuous.*
2. *If f is continuous and $A \subset X$, then the restriction $f|_A\colon A \longrightarrow Y$ is continuous (with respect to the subspace topology on A).*

3. *The map* $f\colon X \longrightarrow Y$ *is continuous if and only if* $f\colon X \longrightarrow f(X)$ *is continuous (with respect to the subspace topology on* $f(X)$*).*

Proposition A.1.17 (glueing principle for continuous maps)**.** *Let* (X, T_X) *and* (Y, T_Y) *be topological spaces, let* $A, B \subset X$ *be closed subsets with* $A \cup B = X$*, and let* $f\colon A \longrightarrow Y$ *and* $g\colon B \longrightarrow Y$ *be continuous maps (with respect to the subspace topology on* A *and* B*) with* $f|_{A\cap B} = g|_{A\cap B}$*. Then the map*

$$\begin{aligned} f \cup_{A\cap B} g\colon X &\longrightarrow Y \\ x &\longmapsto \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases} \end{aligned}$$

is well-defined and continuous.

Isomorphisms in the category of topological spaces are called *homeomorphisms*:

Definition A.1.18 (homeomorphism)**.** Let (X, T_X) and (T, T_Y) be topological spaces. A continuous map $f\colon X \longrightarrow Y$ is a *homeomorphism* if there exists a continuous map $g\colon Y \longrightarrow X$ such that

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

If there exists a homeomorphism $X \longrightarrow Y$, then X and Y are *homeomorphic*, in symbols: $X \cong_{\mathsf{Top}} Y$.

Caveat A.1.19. In general, *not* every bijective continuous map is a homeomorphism!

Intuitively, topological spaces are homeomorphic if and only if they can be deformed into each other without "tearing" or "glueing".

A.1.3 (Path-)Connectedness

An important property of continuous functions $[0,1] \longrightarrow \mathbb{R}$ is the intermediate value theorem. More generally, in the context of topological spaces, this phenomenon can be described in terms of path-connectedness and connectedness.

Definition A.1.20 (path, path-connected)**.** Let (X, T) be a topological space.

- A *path in* X is a continuous map $\gamma\colon [0,1] \longrightarrow X$ (with respect to the standard topology on $[0,1] \subset \mathbb{R}$). Then $\gamma(0)$ is the *start point* and $\gamma(1)$ is the *end point* of γ. The path γ is *closed* if $\gamma(0) = \gamma(1)$.
- The space X is *path-connected*, if for all $x, y \in X$ there exists a path $\gamma\colon [0,1] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Remark A.1.21.

- The unit interval $[0, 1]$ is path-connected.
- For every $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is path-connected (with respect to the standard topology).
- If X is a set with $|X| \geq 2$, then X is *not* path-connected with respect to the discrete topology.

Proposition A.1.22 (continuity preserves path-connectedness)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f\colon X \longrightarrow Y$ be a continuous map. If X is path-connected, then also $f(X)$ is path-connected (with respect to the subspace topology inherited from Y).*
2. *In particular, path-connectedness is a homeomorphism invariant: If X and Y are homeomorphic, then X is path-connected if and only if Y is path-connected.*

Example A.1.23. Let $n \in \mathbb{N}$. We can use Proposition A.1.22 (and a little trick, involving the removal of a single point) to show that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = 1$ (Example 1.1.18).

A meaningful weaker version of path-connectedness is connectedness. A topological space is connected, if the only way to partition X into open sets is the trivial way.

Definition A.1.24 (connected)**.** A topological space (X, T_X) is *connected*, if for all $U, V \in T_X$ with $U \cup V = X$ and $U \cap V = \emptyset$ we have $U = \emptyset$ or $V = \emptyset$.

Remark A.1.25. The unit interval $[0, 1]$ is connected. If $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is open, then U is path-connected if and only if U is connected.

Proposition A.1.26 (path-connectedness implies connectedness)**.** *Every path-connected topological space is connected.*

Caveat A.1.27. There exist topological spaces that are connected but *not* path-connected: The standard example is the wild sinus

$$\bigl\{(x, \sin 1/x) \bigm| x \in (0, 1]\bigr\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$$

(with the subspace topology of $\mathbb{R}^2$).

The generalisation of the intermediate value theorem then reads as follows:

Proposition A.1.28 (continuity preserves connectedness)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f\colon X \longrightarrow Y$ be a continuous map. If X is connected, then also $f(X)$ is connected (with respect to the subspace topology inherited from Y).*

2. *In particular, connectedness is a homeomorphism invariant: If X and Y are homeomorphic, then X is connected if and only if Y is connected.*

In Algebraic Topology, one also studies higher connectedness properties (in the context of higher homotopy groups).

A.1.4 Hausdorff Spaces

It is easy to construct weird and unintuitive topological spaces; it is much harder to ensure with simple properties that topological spaces are reasonably well-behaved. A key example is the folllowing separation property:

Definition A.1.29 (Hausdorff)**.** A topological space (X, T_X) is *Hausdorff*, if every two points can be separated by open sets, i.e., if for all $x, y \in X$ with $x \neq y$, there exist open subsets $U, V \subset X$ such that

$$x \in U, \ y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Proposition A.1.30 (metric spaces are Hausdorff)**.** *Let (X, d) be a metric space. Then the metric topology on X is Hausdorff.*

Corollary A.1.31. *If X is a set with $|X| \geq 2$, then the trivial topology on X is* not *induced by a metric on X.*

Proposition A.1.32. *Being Hausdorff is a homeomorphism invariant: If two topological spaces are homeomorphic, then one of them is Hausdorff if and only if they are both Hausdorff.*

There is a zoo of further separation properties of topological spaces [63]. Whenever possible, we will avoid these pitfalls.

A.1.5 Compactness

Roughly speaking, compactness is a finiteness property of topological spaces, defined in terms of open covers.

Definition A.1.33 (compact)**.** A topological space (X, T) is *compact*, if every open cover of X contains a finite subcover. More precisely: The topological space (X, T) is compact, if for every family $(U_i)_{i \in I}$ of open subsets of X with $X = \bigcup_{i \in I} U_i$ there exists a finite subset $J \subset I$ with $X = \bigcup_{i \in J} U_i$.

Caveat A.1.34 (cover/Überdeckung)**.** Sometimes, also the term "covering" is used instead of "cover". We will always use "cover" (German: Überdeckung; family of subsets of a spaces whose union is the given space) in order to distinguish it from the "covering" notion in covering theory (German: Überlagerung; a map with special properties).

Remark A.1.35. Let X be a set.

- Then X is compact with respect to the trivial topology.
- Moreover, X is compact with respect to the discrete topology if and only if X is finite.

The unit interval $[0, 1]$ is compact with respect to the standard topology; this implies that every continuous map $[0, 1] \longrightarrow \mathbb{R}$ has a minimum and a maximum. More generally, we have:

Proposition A.1.36 (generalised maximum principle)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces.*

1. *Let $f\colon X \longrightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is compact (with respect to the subspace topology of Y).*
2. *In particular, compactness is a homeomorphism invariant: If X and Y are homeomorphic, then X is compact if and only if Y is compact.*

In Euclidean spaces, we have a simple characterisation of compact sets:

Theorem A.1.37 (Heine-Borel)**.** *Let $n \in \mathbb{N}$ and let $A \subset \mathbb{R}^n$ (endowed with the subspace topology of the standard topology on $\mathbb{R}^n$). Then the following are equivalent:*

1. *The space A is compact.*
2. *The set A is closed and bounded with respect to the Euclidean metric on $\mathbb{R}^n$.*
3. *The set A is sequentially compact with respect to the Euclidean metric on $\mathbb{R}^n$ (i.e., every sequence in A has a subsequence that converges to a limit in A).*

Caveat A.1.38. In fact, every compact subspace of a metric space is closed and bounded. However, in general, the converse is *not* true in general metric spaces! For example, infinite sets are closed and bounded with respect to the discrete metric, but *not* compact.

More generally, we have the following relationship between closedness and compactness (which leads to a highly useful sufficient homeomorphism criterion).

Proposition A.1.39 (closed vs. compact)**.** *Let (X, T) be a topological space and let $Y \subset X$.*

1. *If X is compact and Y is closed in X, then Y is also compact (with respect to the subspace topology).*
2. *If X is Hausdorff and Y is compact (with respect to the subspace topology), then Y is closed in X.*

Corollary A.1.40 (compact-Hausdorff trick). *Let (X, T_X) be a compact topological space, let (Y, T_Y) be a Hausdorff topological space, and let $f : X \longrightarrow Y$ be continuous and bijective. Then f is a homeomorphism(!).*

Proof. Because f is bijective, it admits a set-theoretic inverse $g : Y \longrightarrow X$. It suffices to show that g is continuous (i.e., that g-preimages of open/closed sets are open/closed). Equivalently, it suffices to show that f-images of closed subsets of X are closed in Y.

Let $A \subset X$ be a closed subset. Because X is compact, also A is compact (Proposition A.1.39). Hence, $f(A)$ is compact by the generalised maximum principle (Proposition A.1.36). As Y is Hausdorff, this implies that $f(A)$ is closed in Y (Proposition A.1.39), as desired. □

Finally, we briefly discuss the preservation of compactness under taking products:

Proposition A.1.41 (product of two compact spaces). *Let (X, T_X) and (Y, T_Y) be compact topological spaces. Then the product $X \times Y$ is compact with respect to the product topology.*

Caveat A.1.42 (the Tychonoff Theorem). The Tychonoff Theorem

> Every product (including infinite products!) of compact spaces is compact.

is equivalent to the Axiom of Choice(!) (whence also to Zorn's Lemma and the Well-Ordering Theorem) [27, Chapter 4.8].

A.2 Homotopy Flipbooks

We can illustrate homotopies using flipbooks. The following two pages contain two such examples.

The unloopy loop

A homotopy

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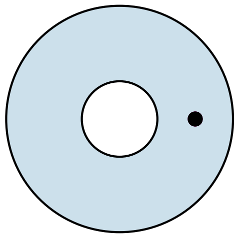
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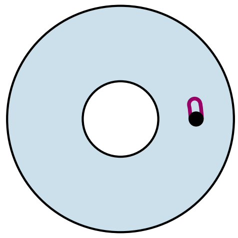
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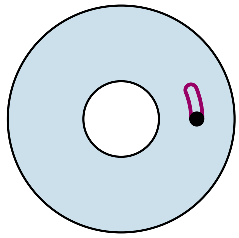
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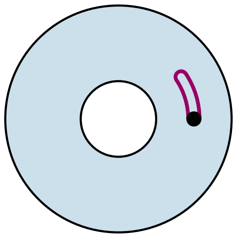
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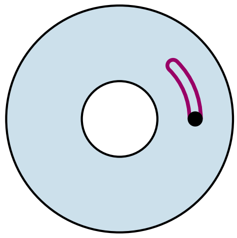
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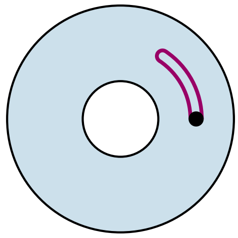
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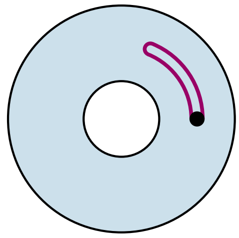
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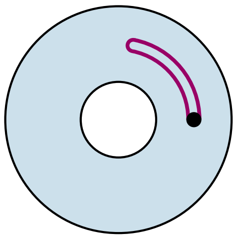
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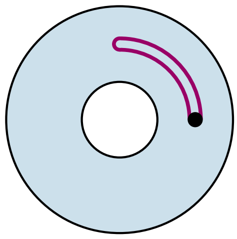
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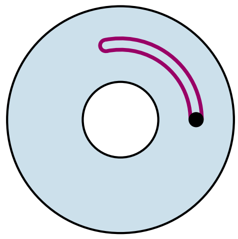
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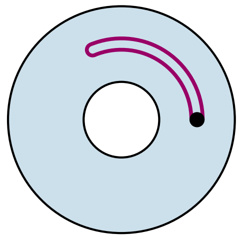
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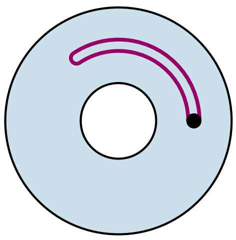
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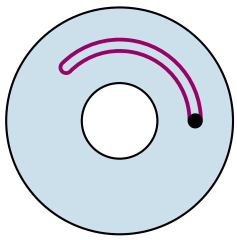
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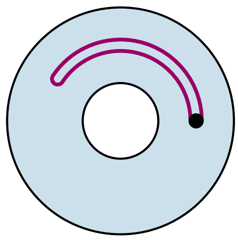
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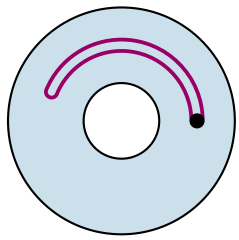
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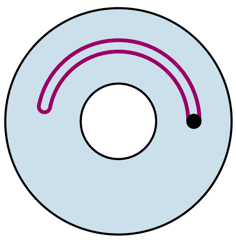
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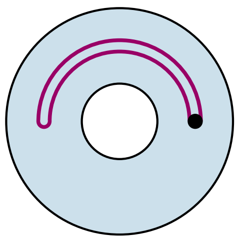
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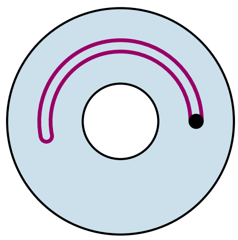
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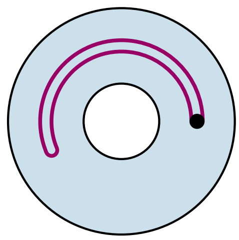
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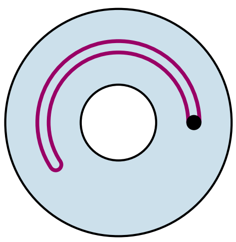
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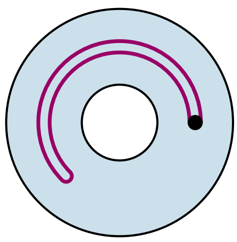
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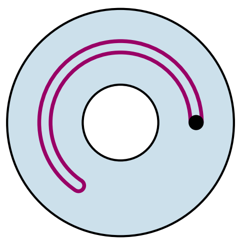
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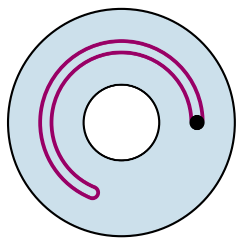
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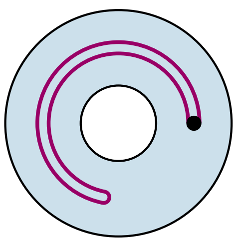
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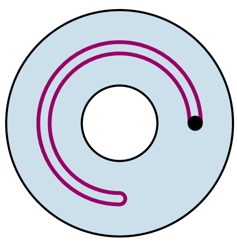
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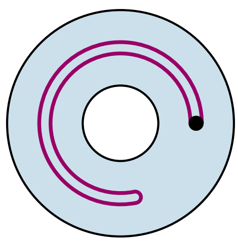
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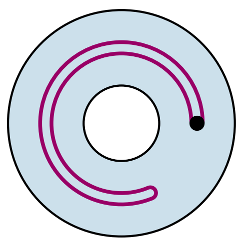
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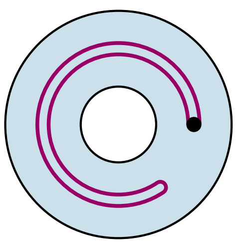
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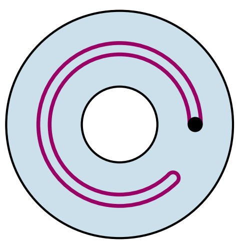
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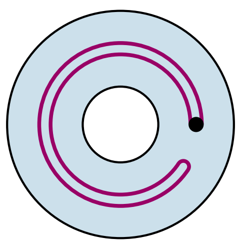
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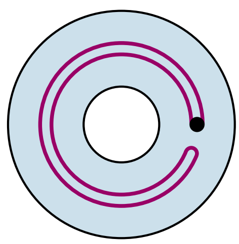

The unstraight line

A homotopy

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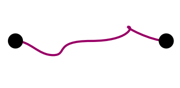

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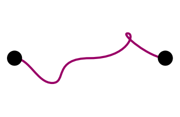

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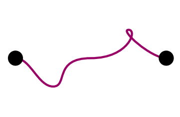

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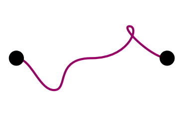

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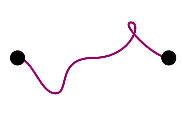

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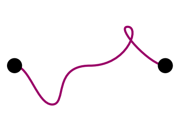

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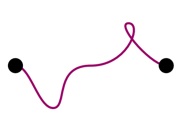

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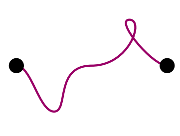

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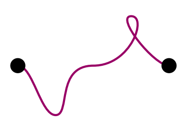

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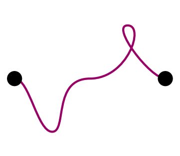

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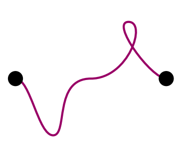

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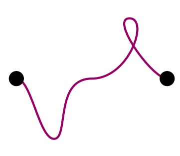

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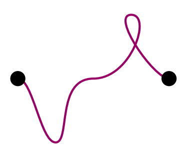

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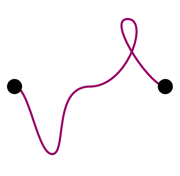

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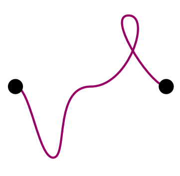

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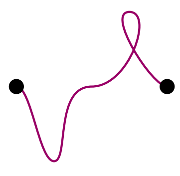

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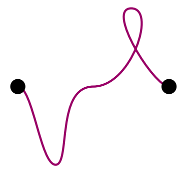

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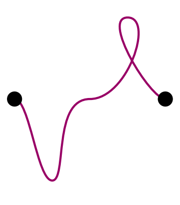

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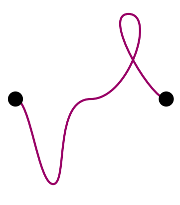

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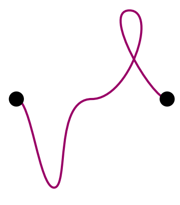

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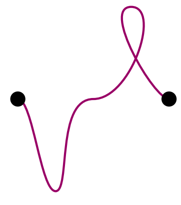

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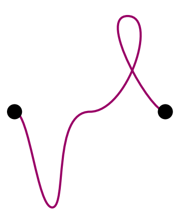

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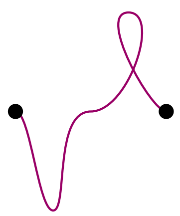

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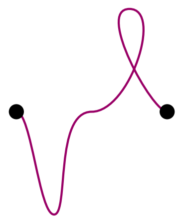

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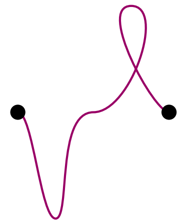


A.3 Cogroup Objects and Group Structures

In the following, we briefly sketch a more conceptual approach to the group structure on π_1, using the category theoretic concept of a cogroup object. The key observation is that the proof that concatenation of representing loops induces a group structure on π_1 (Proposition 2.1.3) only uses the corresponding properties of (S^1, e_1). More precisely, (S^1, e_1) is a cogroup object in $\mathsf{Top}_{*\mathsf{h}}$. We explain these terms in more detail:

In the language of category theory, a "group structure on π_1" is a factorisation of the functor $\pi_1 \colon \mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Set}$ over the category Group and the forgetful functor $\mathsf{Group} \longrightarrow \mathsf{Set}$:

$$\begin{tikzcd} & \mathsf{Group} \arrow[d, "\text{forget}"] \\ \mathsf{Top}_{*\mathsf{h}} \arrow[ur, dashed, "?"] \arrow[r, "\pi_1"'] & \mathsf{Set} \end{tikzcd}$$

Such a factorisation corresponds to a cogroup object structure on (S^1, e_1). We obtain the notion of a cogroup object by first formulating the definition of the notion of a group only through morphisms (instead of individual elements) and then dualising everything (Figure A.2).

Definition A.3.3 (cogroup object). Let C be a category that contains an object $*$ that is both initial and terminal. A *cogroup object in* C is a triple (G, c, i) consisting of an object in G (with the property that the coproducts $G \sqcup G$ and $G \sqcup G \sqcup G$ exist in C) and morphisms $c \in \operatorname{Mor}_C(G, G \sqcup G)$ (the *comultiplication*) and $i \in \operatorname{Mor}_C(G, G)$ satisfying the following properties (Figure A.3):

- We have $(\mathrm{id}_G \sqcup e) \circ c = \mathrm{id}_G = (e \sqcup \mathrm{id}_G) \circ c$, where $e \in \operatorname{Mor}_C(G, G)$ denotes the unique morphism that factors through $*$.
- We have $(\mathrm{id}_G \sqcup i) \circ c = e = (i \sqcup \mathrm{id}_G) \circ c$.
- *Coassociativity.* We have $(c \sqcup \mathrm{id}_G) \circ c = (\mathrm{id}_G \sqcup c) \circ c$.

(The morphisms involving "$\sqcup$" are the ones given by the universal properties of the corresponding coproducts.)

Cogroup objects correspond in the following sense to group structures on the corresponding represented functors:

Theorem A.3.4 (cogroup objects and functorial group structures). *Let C be a category that contains an object that is both intial and terminal and let G be an object in C for which the coproducts $G \sqcup G$ and $G \sqcup G \sqcup G$ exist in C. Then the functor* $\operatorname{Mor}_C(G, \,\cdot\,) \colon C \longrightarrow \mathsf{Set}$ *factors over the forgetful*

Definition A.3.1 (group)**.** A *group* is a pair (G, n), where G is a set and $m \colon G \times G \longrightarrow G$ is a map with the following properties:

- There exists an $e \in G$ with
$$\forall_{g \in G} \quad m(g, e) = g = m(e, g).$$
(This property uniquely determines e.)
- For every $g \in G$ there exists an $i(g) \in G$ with
$$m\bigl(g, i(g)\bigr) = e = m\bigl(i(g), g\bigr).$$
(In combination with associativity, we obtain tha $i(g)$ is determined uniquely by this property.)
- (*Associativity*). For all $g, h, k \in G$, we have
$$m\bigl(m(g, h), k\bigr) = m\bigl(g, m(h, k)\bigr).$$

In other words, we have the following equations:

$$\begin{gathered} m \circ (\mathrm{id}_G, e) = \mathrm{id}_G = m \circ (e, \mathrm{id}_G) \\ m \circ (\mathrm{id}_G, i) = e = m \circ (i, \mathrm{id}_G) \\ m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m). \end{gathered}$$

Definition A.3.2 (group object)**.** Let C be a category that contains an object $*$ that is both initial and terminal (i.e., for every $X \in \mathrm{Ob}(C)$, the sets $\mathrm{Mor}_C(X, *)$ and $\mathrm{Mor}_C(*, X)$ both contain only a single element). A *group object in* C is a triple (G, m, i) consisting of an object G in C (with the property that the products $G \times G$ and $G \times G \times G$ exist in C) and morphisms $m \in \mathrm{Mor}_C(G \times G, G)$ and $i \in \mathrm{Mor}_C(G, G)$ satisfying the following properties:

- We have $m \circ (\mathrm{id}_G, e) = \mathrm{id}_G = m \circ (e, \mathrm{id}_G)$, where $e \in \mathrm{Mor}_C(G, G)$ denotes the unique morphism that factors through $*$.
- We have $m \circ (\mathrm{id}_G, i) = e = m \circ (i, \mathrm{id}_G)$.
- *Associativity.* We have $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$.

(The morphisms involving "$(\,\cdot\,,\,\cdot\,)$" or "$\times$" are the ones given by the universal properties of the corresponding products.) I.e., the following diagrams are commutative:

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "{(\mathrm{id}_G, e)}"] \arrow[rr, "\mathrm{id}_G"'] \arrow[dr, "{(e, \mathrm{id}_G)}"'] & & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "{(\mathrm{id}_G, i)}"] \arrow[rr, "e"'] \arrow[dr, "{(i, \mathrm{id}_G)}"'] & & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \times G \times G \arrow[ur, "m \times \mathrm{id}_G"] \arrow[dr, "\mathrm{id}_G \times m"'] & & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Figure A.2.: Groups and group objects

$$\begin{tikzcd}
& G \sqcup G \arrow[dr, "\mathrm{id}_G \sqcup e"] & \\
G \arrow[ur, "c"] \arrow[rr, "\mathrm{id}_G"'] \arrow[dr, "c"'] & & G \\
& G \sqcup G \arrow[ur, "e \sqcup \mathrm{id}_G"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& G \sqcup G \arrow[dr, "\mathrm{id}_G \sqcup i"] & \\
G \arrow[ur, "c"] \arrow[rr, "e"'] \arrow[dr, "c"'] & & G \\
& G \sqcup G \arrow[ur, "i \sqcup \mathrm{id}_G"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& G \sqcup G \arrow[dr, "\mathrm{id}_G \sqcup c"] & \\
G \arrow[ur, "c"] \arrow[dr, "c"'] & & G \sqcup G \sqcup G \\
& G \sqcup G \arrow[ur, "c \sqcup \mathrm{id}_G"'] &
\end{tikzcd}$$

Figure A.3.: Commutative diagrams encoding the axioms for cogroup objects

functor Group $\longrightarrow$ Set *if and only if G admits a cogroup object structure in C.*

Sketch of proof. If $\operatorname{Mor}_C(G, \,\cdot\,)$ factors over Group, then the morphisms

$$m_{G\sqcup G}(i_1, i_2) \in \operatorname{Mor}_C(G, G \sqcup G)$$
$$i_G \in \operatorname{Mor}_C(G, G)$$

turn G into a cogroup object in C; here, $i_1, i_2 \colon G \longrightarrow G \sqcup G$ denote the structure morphisms of the coproduct $G \sqcup G$, the map $m_{G\sqcup G}$ is the group multiplication on $\operatorname{Mor}_C(G, G \sqcup G)$ given by the factorisation over Group, and i_G is the inversion map on $\operatorname{Mor}_C(G, G \sqcup G)$ given by the factorisation over Group.

Conversely, if (G, c, i) is a cogroup object in C, then for all $X \in \operatorname{Ob}(C)$ the set $\operatorname{Mor}_C(G, X)$ is a group with respect to the multiplication

$$\begin{aligned}
\operatorname{Mor}_C(G, X) \times \operatorname{Mor}_C(G, X) &\longrightarrow \operatorname{Mor}_C(G, X) \\
(g, h) &\longmapsto (g \sqcup h) \circ c
\end{aligned}$$

(and the morphisms in C induce group homomorphisms). This results in the desired factorisation of $\operatorname{Mor}_C(G, \,\cdot\,)$ over the forgetful functor Group $\longrightarrow$ Set. ☐

We apply these general concepts to π_1:

Proposition A.3.5 ((S^1, e_1) as cogroup object in $\mathsf{Top}_{*\mathsf{h}}$)**.** *Let $i_1, i_2 \colon (S^1, e_1) \longrightarrow (S^1, e_1) \vee (S^1, e_1)$ be the canonical inclusions into the one-point union, let*

$$\begin{aligned}
c \colon (S^1, e_1) &\longrightarrow (S^1, e_1) \vee (S^1, e_1) \\
[t] &\longmapsto \begin{cases} i_1([2 \cdot t]) & \text{if } t \in [0, 1/2] \\ i_2([2 \cdot t - 1]) & \text{if } t \in [1/2, 1], \end{cases}
\end{aligned}$$

and let

$$\begin{aligned} i\colon (S^1, e_1) &\longrightarrow (S^1, e_1) \\ [t] &\longmapsto [1-t]. \end{aligned}$$

Then c and i are well-defined pointed continuous maps and $(S^1, [c]_*, [i]_*)$ is a cogroup object in $\mathsf{Top}_{*\mathsf{h}}$.

Sketch of proof. The coproduct in the category $\mathsf{Top}_{*\mathsf{h}}$ is given by the one-point union (and the pointed homotopy classes of the canonical inclusions). In particular, all coproducts exist in $\mathsf{Top}_{*\mathsf{h}}$.

Let $e\colon (S^1, e_1) \longrightarrow (S^1, e_1)$ be the constant pointed map. Then

$$\begin{gathered} (\mathrm{id}_{(S^1,e_1)} \vee e) \circ c \simeq_* \mathrm{id}_{(S^1,e_1)} \simeq_* (e \vee \mathrm{id}_{(S^1,e_1)}) \circ c, \\ (\mathrm{id}_{(S^1,e_1)} \vee i) \circ c \simeq_* e \simeq_* (i \vee \mathrm{id}_{(S^1,e_1)}) \circ c, \\ (\mathrm{id}_{(S^1,e_1)} \vee c) \circ c \simeq_* (c \vee \mathrm{id}_{(S^1,e_1)}) \circ c; \end{gathered}$$

corresponding homotopies can be written down explicitly. $\square$

Corollary A.3.6 (fundamental group). *Hence, the functor*

$$\pi_1 = [(S^1, e_1),\ \cdot\ ]_*\colon \mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Set}$$

factors over the forgetful functor $\mathsf{Group} \longrightarrow \mathsf{Set}$.

The resulting functor $\mathsf{Top}_{*\mathsf{h}} \longrightarrow \mathsf{Group}$ is also denoted by π_1. The group structures from Proposition 2.1.3 and Proposition A.3.5 coincide.

Translating the cogroup object structure on (S^1, e_1) in $\mathsf{Top}_{*\mathsf{h}}$ via π_1 into the language of group theory shows that the cogroup object structure on (S^1, e_1) is essentially unique [1, Theorem 7.3]. Similar arguments also show that S^1 does *not* admit a cogroup object structure in $\mathsf{Top}_{\mathsf{h}}$.

A.4 Amalgamated Free Products

We briefly review some concepts from group theory that allow us to construct coproducts and pushouts of groups explicitly.

A.4.1 The Free Group of Rank 2

We start with an explicit description of the free group of rank 2, using reduced words [39, Chapter 3.3, Chapter 2.2]. Roughly speaking, this group is the group generated by two different elements with the least possible relations between these elements.

Definition A.4.1 (group of reduced words)**.** Let $a, b, \widehat{a}, \widehat{b}$ four distinct elements. Let W be the set of words (i.e., finite sequences) over $S := \{a, b, \widehat{a}, \widehat{b}\}$.

- Let $n \in \mathbb{N}$ and let $x_1, \dots, x_n \in S$. The word $x_1 \dots x_n \in W$ is *reduced* if
$$x_{j+1} \neq \widehat{x_j} \quad \text{and} \quad \widehat{x_{j+1}} \neq x_j$$
holds for all $j \in \{1, \dots, n-1\}$. In particular, the empty word ε is reduced.
- We write $F(a, b)$ for the set of all reduced words over S.
- On $F(a, b)$, we define a composition by concatenation and reduction:
$$\begin{aligned} \cdot \colon F(a, b) \times F(a, b) &\longrightarrow F(a, b) \\ (x_1 \dots x_n, x_{n+1} \dots x_m) &\longmapsto x_1 \dots x_{n-r} x_{n+1+r} \dots x_{n+m}. \end{aligned}$$
Here,
$$\begin{aligned} r := \max\bigl\{k \in \{0, \dots, \min(n, m-1)\} \bigm| \forall_{j \in \{0, \dots, k-1\}} \quad & x_{n-j} = \widehat{x_{n+1+j}} \\ & \vee \widehat{x_{n-j}} = x_{n+1+j}\bigr\}. \end{aligned}$$

Example A.4.2. In the situation of the previous definition, the word $ab\widehat{a}\widehat{b}$ is reduced; the word $ba\widehat{a}b$ is *not* reduced. The elements a and $\widehat{a}$ are inverse to each other with respect to "$\cdot$"; analogously, also b and $\widehat{b}$ are inverse to each other. Hence, one usually writes a^{-1} and b^{-1} instead of $\widehat{a}$ and $\widehat{b}$, respectively.

Proposition A.4.3 (free group of rank 2)**.**

1. *The set $F(a, b)$ is a group with respect to the composition specified in the previous definition.*

2. *The set* $\{a, b\}$ *is a* free generating set *of* $F(a, b)$*, i.e., the following universal property is satisfied:*

   For every group H *and every map* $f : \{a, b\} \longrightarrow H$*, there exists a unique group homomorphism* $\overline{f} : F(a, b) \longrightarrow H$ *with* $\overline{f}|_{\{a,b\}} = f$*.*

$$\begin{tikzcd}
\{a,b\} \arrow[r, "f", "\text{map}"'] \arrow[d, "\text{incl}"'] & H \\
F(a,b) \arrow[ur, dashed, "\exists!\overline{f}\ \text{hom}"'] &
\end{tikzcd}$$

3. *In other words,*

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "1\mapsto b"] \\
\mathbb{Z} \arrow[r, "1\mapsto a"'] & F(a,b)
\end{tikzcd}$$

   is a pushout in Group*.*

Proof. The first part follows from a straightforward computation (associativity is *not* obvious!) [39, Chapter 3.3].

The second part (and the third part) can be verified directly by hand (check!). ☐

A.4.2 Free Products of Groups

More generally, we can consider the free product of a family of groups. Again, we are looking for a group generated by the given groups with as few relations between them as possible.

Definition A.4.4 (free product of groups)**.** Let $(G_i)_{i\in I}$ be a family of groups; for $g \in \bigsqcup_{i\in I}(G_i \setminus \{1\})$ let $i(g) \in I$ be the unique index with $g \in G_{i(g)}$.

- A finite (possibly empty) sequence $(s_1, \dots, s_n)$ with $n \in \mathbb{N}$ of non-trivial elements of $\bigsqcup_{i\in I} G_i$ is a *reduced word (over the family* $(G_i)_{i\in I}$*)*, if

$$\forall_{j\in\{1,\dots,n-1\}} \quad i(s_j) \neq i(s_{j+1}).$$

- We write $\bigstar_{i\in I} G_i$ for the set of all reduced words over the family $(G_i)_{i\in I}$.

- On $\bigstar_{i\in I} G_i$, we define a composition by concatenation/reduction:

$$\begin{aligned}
\cdot \colon \bigstar_{i\in I} G_i \times \bigstar_{i\in I} G_i &\longrightarrow \bigstar_{i\in I} G_i \\
\bigl(s = (s_1, \dots, s_n), t = (t_1, \dots, t_m)\bigr) &\longmapsto \begin{cases} (s_1, \dots, s_{n-k(s,t)}, t_{k(s,t)+1}, \dots, t_m) & ① \\ (s_1, \dots, s_{n-k(s,t)} \cdot t_{k(s,t)+1}, \dots, t_m) & ② \end{cases}
\end{aligned}$$

Here, $k(s,t) \in \{0, \dots, \min(n,m)\}$ is the biggest $k \in \{0, \dots, \min(n,m)\}$ satisfying

$$\forall_{j \in \{1,\dots,k\}} \quad i(s_{n-j+1}) = i(t_j) \land s_{n-j+1} = t_j^{-1}.$$

Case ① occurs if $i(s_{n-k(s,t)}) \neq i(t_{k(s,t)+1})$; case ② occurs if $i(s_{n-k(s,t)}) = i(t_{k(s,t)+1})$.

- We call $\bigstar_{i \in I} G_i$, together with this composition, the *free product of the family* $(G_i)_{i \in I}$.

The free product $G := \bigstar_{i \in I} G_i$ of a family $(G_i)_{i \in I}$ indeed is a group (again, associativity is non-trivial!) and the canonical inclusions $G_i \longrightarrow G$ are group homomorphisms.

Free products are an explicit model of coproducts of groups:

Proposition A.4.5 (coproduct of groups). *Let $(G_i)_{i \in I}$ be a family of groups. Then $\bigstar_{i \in I} G_i$, together with the canonical inclusions $(G_i \longrightarrow \bigstar_{j \in I} G_j)_{i \in I}$, is the coproduct of the family $(G_i)_{i \in I}$ in the category* Group.

Proof. This can be shown by verifying the universal property (check!). □

A.4.3 Amalgamated Free Products of Groups

"Glueing" groups along another group leads to the amalgamated free product:

Definition A.4.6 (amalgamated free product). Let G_0, G_1, and G_2 be groups and let $i_1 \colon G_0 \longrightarrow G_1$ as well as $i_2 \colon G_0 \longrightarrow G_2$ be group homomorphisms. The associated *amalgamated free product of G_1 and G_2 over G_0* is defined by

$$G_1 *_{G_0} G_2 := (G_1 * G_2)/N,$$

where $N \subset G_1 * G_2$ is the smallest (with respect to inclusion) normal subgroup of $G_1 * G_2$ that contains the set $\{i_1(g) \cdot i_2(g)^{-1} \mid g \in G_0\}$.

Caveat A.4.7. In group theory, usually only the case where the homomorphisms i_1 and i_2 both are injective is given the name "amalgamated free product". This case is special, because it admits a nice structure and normal form theory [61].

Proposition A.4.8 (pushouts of groups). *Let G_0, G_1, and G_2 be groups and let $i_1 \colon G_0 \longrightarrow G_1$ as well as $i_2 \colon G_0 \longrightarrow G_2$ be group homomorphisms. Let $j_1 \colon G_1 \longrightarrow G_1 *_{G_0} G_2$ and $j_2 \colon G_2 \longrightarrow G_1 *_{G_0} G_2$ be the homomorphisms induced by the canonical inclusions $G_1 \longrightarrow G_1 * G_2$ and $G_2 \longrightarrow G_1 * G_2$, respectively. Then*

$$\begin{tikzcd} G_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"'] & G_1 *_{G_0} G_2 \end{tikzcd}$$

is a pushout in Group.

Proof. This can be shown by the same argument as in the construction of the pushout of topological spaces (Proposition 1.1.14), using the universal property of the free product and of quotient groups. □

A.4.4 Free Groups

A related generalisation of $F(a,b)$ are general free groups; the universal property of free groups/free generating sets is a group-theoretic version of the universal property of bases of vector spaces.

Definition A.4.9 (free generating set, free group, rank of a free group)**.**

- Let G be a group. A subset $S \subset G$ is a *free generating set of* G if the following universal property is satisfied: The group G is generated by S and for every group H and every map $f \colon S \longrightarrow H$ there exists a unique group homomorphism $\overline{f} \colon G \longrightarrow H$ with $\overline{f}|_S = f$.
- A *free group* is a group that contains a free generating set; the cardinality of such a free generating set is the *rank* of the free group.

Caveat A.4.10. Not every group has a free generating set! For example, the groups $\mathbb{Z}/2$ and $\mathbb{Z}^2$ are *not* free (check!).

Comparing the corresponding universal properties establishes existence of free groups of arbitrary rank:

Proposition A.4.11 (existence of free groups)**.** *Let S be a set, let $G := \bigstar_S \mathbb{Z}$ be the associated free product and for every $s \in S$ let $i_s \colon \mathbb{Z} \longrightarrow G$ be the inclusion of the s-th summand. Then $\{i_s(1) \mid s \in S\}$ is a free generating set of G.*

Proof. We can translate the universal property of coproducts into the universal property of free generating sets (because the building blocks are the groups $\mathbb{Z}$, which are free of rank 1) (check!). □

Proposition A.4.12 (invariance of rank of free groups)**.** *Let G and G' be free groups with free generating sets S and S', respectively. Then G and G' are isomorphic if and only if $|S| = |S'|$.*

Proof. This can be shown, for example, by looking at homomorphisms to $\mathbb{Z}/2$ and a cardinality argument [39, Exercise 2.E.12]. □

A.5 Group Actions

We briefly recall basic terminology of group actions (of discrete groups). Group actions are a generalisation of the notion of symmetry:

Definition A.5.1 (group action). Let C be a category, let X be an object in C, and let G be a group.

- The *automorphism group of* X *in* C is the group(!) $\operatorname{Aut}_C(X)$ (with respect to composition in C) of all isomorphisms from X to X in C.
- A *group action of* G *on* X *in* C is a group homomorphism
$$G \longrightarrow \operatorname{Aut}_C(X).$$
We also (sloppily) denote such an action by $G \curvearrowright X$.
- A *right action of* G *on* X *in* C is a group anti-homomorphism
$$\varphi\colon G \longrightarrow \operatorname{Aut}_C(X),$$
i.e., for all $g, h \in G$ we have $\varphi(g \cdot h) = \varphi(h) \circ \varphi(g)$. We also denote this by $X \curvearrowleft G$.

Remark A.5.2 (group actions in Set and Top). Let G be a group and let X be a set [a topological space]. A map $\varphi\colon G \curvearrowright X$ is a group action of G on X in Set [in Top] if and only if the map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x := \big(\varphi(g)\big)(x) \end{aligned}$$

has the following properties:

- For every $g \in G$, the map $g \cdot \ \cdot : X \longrightarrow X$ is a map of sets [a continuous map]
- For all $x \in X$, we have $e \cdot x = x$.
- For all $x \in X$ and all $g, h \in G$, we have
$$(g \cdot h) \cdot x = g \cdot (h \cdot x).$$

[Group actions in Top are also called *continuous actions*.]

Elementary examples of group actions have been covered in the Algebra course (Chapter III.1.2).

Definition A.5.3 (free action, stabiliser, orbit, orbit space, transitive)**.** Let $G \curvearrowright X$ be a group action of G on X in Set [in Top].

- The action is *free*, if the following holds: For all $x \in X$ and all $g \in G \setminus \{e\}$, we have $g \cdot x \neq x$.
- Let $x \in X$. The *stabiliser* of the action at x is the subgroup

$$G_x := \{g \in G \mid g \cdot x = x\} \subset G.$$

- For $x \in X$, we write

$$G \cdot x := \{g \cdot x \mid g \in G\} \subset X$$

  for the *orbit of* x.
- The quotient space

$$G \setminus X := \{G \cdot x \mid x \in X\}$$

  is the *orbit space* of this action. . [For actions in Top, we endow the orbit space $G \setminus X$ with the quotient topology induced by the canonical projection $X \longrightarrow G \setminus X$.]
- The action is *transitive*, if $|G \setminus X| = 1$, i.e., if all points in X lie in the same orbit.

Analogously, we introduce the corresponding terms for right actions. The orbit space of a right action $X \curvearrowleft G$ is denoted by X/G.

Caveat A.5.4. The quotient topology on the orbit space of a continuous action can be terrible even if the group and the space acted upon are “nice”. In order to have a “nice” quotient space, the *action* needs to have good properties.

A.6 Basic Homological Algebra

We collect basic notions and facts from homological algebra. Homological algebra is the algebraic theory of [non-]exact sequences and functors that [do not] preserve exactness.

For simplicity, we will only consider homological algebra in module categories (instead of general Abelian categories); in view of the Freyd–Mitchell embedding theorem, this is not a substantial limitation.

Setup A.6.1. In the following, R will always be a (not necessarily commutative) ring with unit.

A.6.1 Exact Sequences

We briefly recall exact sequences; we will stick to left modules (but clearly the analogous statements for right modules also hold).

Definition A.6.2 ((short) exact sequence)**.**

- A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms in ${}_R\mathsf{Mod}$ ist *exact (at the middle position B)*, if $\operatorname{im} f = \ker g$.
- A sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
in ${}_R\mathsf{Mod}$ is a *short exact sequence in* ${}_R\mathsf{Mod}$, if the sequence is exact at all positions (i.e., f is injective, g is surjective, and $\operatorname{im} f = \ker g$).
- An $\mathbb{N}$-indexed or $\mathbb{Z}$-indexed sequence
$$\cdots \longrightarrow A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} A_{k-1} \xrightarrow{f_{k-1}} A_{k-2} \longrightarrow \cdots$$
in ${}_R\mathsf{Mod}$ is *exact*, if it is exact at all positions.

Example A.6.3 (exact sequences)**.** The sequences

$$\begin{array}{ccccccccc}
 & & x & \longmapsto & (x,0) & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & & & (x,y) & \longmapsto & y & &
\end{array}$$

and

$$\begin{array}{c} x \longmapsto 2 \cdot x \\ 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\ x \longmapsto [x] \end{array}$$

in ${}_{\mathbb{Z}}\mathsf{Mod}$ are exact; it should be noted that the middle modules are *not* isomorphic even though the outer terms are isomorphic. The sequence

$$\begin{array}{c} x \longmapsto x \\ 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \\ x \longmapsto x \end{array}$$

is *not* exact.

Caveat A.6.4. If S is a ring with unit, then additive functors ${}_R\mathsf{Mod} \longrightarrow {}_S\mathsf{Mod}$ in general do *not* map exact sequences to exact sequences. For example, tensor product functors, in general, do *not* preserve exactness!

Remark A.6.5 (flatness)**.** A right R-module M is *flat*, if the tensor product functor $M \otimes_R \cdot : {}_R\mathsf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is exact, i.e., it maps exact sequences to exact sequences (Definition IV.3.2.15, Beispiel IV.3.2.16, Beispiel IV.3.2.18, Lemma IV.3.4.7, Korollar IV.5.2.5, Satz IV.3.2.14). For example:

- The R-module R is flat.
- Direct sums of flat modules are flat. Therefore, all free modules are flat. In particular: If R is a field, then every R-module is flat.
- Direct summands of flat modules are flat. Therefore, all projective modules are flat.
- Localisations are flat; e.g., $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
- The $\mathbb{Z}$-module $\mathbb{Z}/2$ is *not* flat.

Particularly well-behaved exact sequences are the split short exact sequences:

Proposition A.6.6 (split exact sequence)**.** *Let*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Mod}$*. Then the following are equivalent:*

1. *There exists an R-module homomorphism $r \colon C \longrightarrow B$ with $p \circ r = \mathrm{id}_C$.*
2. *There exists an R-module homomorphism $s \colon B \longrightarrow A$ with $s \circ i = \mathrm{id}_A$.*

If these conditions hold, then the sequence above is a split exact sequence *in* ${}_R\mathsf{Mod}$*, and*

$$\begin{aligned} A \oplus C &\longrightarrow B \\ (a, c) &\longmapsto i(a) + r(c) \\ B &\longrightarrow A \oplus C \\ b &\longmapsto \bigl(s(b), p(b)\bigr) \end{aligned}$$

are isomorphisms in ${}_R\mathsf{Mod}$*.*

Proof. We first show the implication $2 \Longrightarrow 1$: Let $s\colon B \longrightarrow A$ be an R-homomorphism with $s \circ i = \mathrm{id}_A$. We then consider the R-homomorphism

$$\begin{aligned} \widetilde{r}\colon B &\longrightarrow B \\ b &\longmapsto b - i \circ s(b). \end{aligned}$$

We have $\ker p \subset \ker \widetilde{r}$, because: Let $b \in \ker p$. In view of exactness, there is an $a \in A$ with $i(a) = b$; thus,

$$\widetilde{r}(b) = i(a) - i \circ s(i(a)) = i(a) - i(\mathrm{id}_A(a)) = 0.$$

By the universal property of the quotient module, $\widetilde{r}$ induces an R-homomorphism $r\colon C \cong_R B/\ker p \longrightarrow B$, which, by construction, satisfies $p \circ r = \mathrm{id}_C$.

Similarly, one can show the implication $1 \Longrightarrow 2$.

If the statements 1 and 2 are satisfied, then a straightforward calculation shows that the given R-homomorphisms between B and $A \oplus C$ are bijective (check!), whence isomorphisms. $\square$

When comparing exact sequences, the five lemma is very useful:

Proposition A.6.7 (five lemma)**.** *Let*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f_A"'] & B \arrow[r, "b"] \arrow[d, "f_B"'] & C \arrow[r, "c"] \arrow[d, "f_C"'] & D \arrow[r, "d"] \arrow[d, "f_D"] & E \arrow[d, "f_E"] \\ A' \arrow[r, "a'"'] & B' \arrow[r, "b'"'] & C' \arrow[r, "c'"'] & D' \arrow[r, "d'"'] & E' \end{tikzcd}$$

be a commutative diagram in ${}_R\mathsf{Mod}$ *with exact rows. Then the following holds:*

1. *If f_B, f_D are injective and f_A is surjective, then f_C is injective.*
2. *If f_B, f_D are surjective and f_E is injective, then f_C is surjective.*
3. *In particular: If f_A, f_B, f_D, f_E are isomorphisms, then f_C is an isomorphism.*

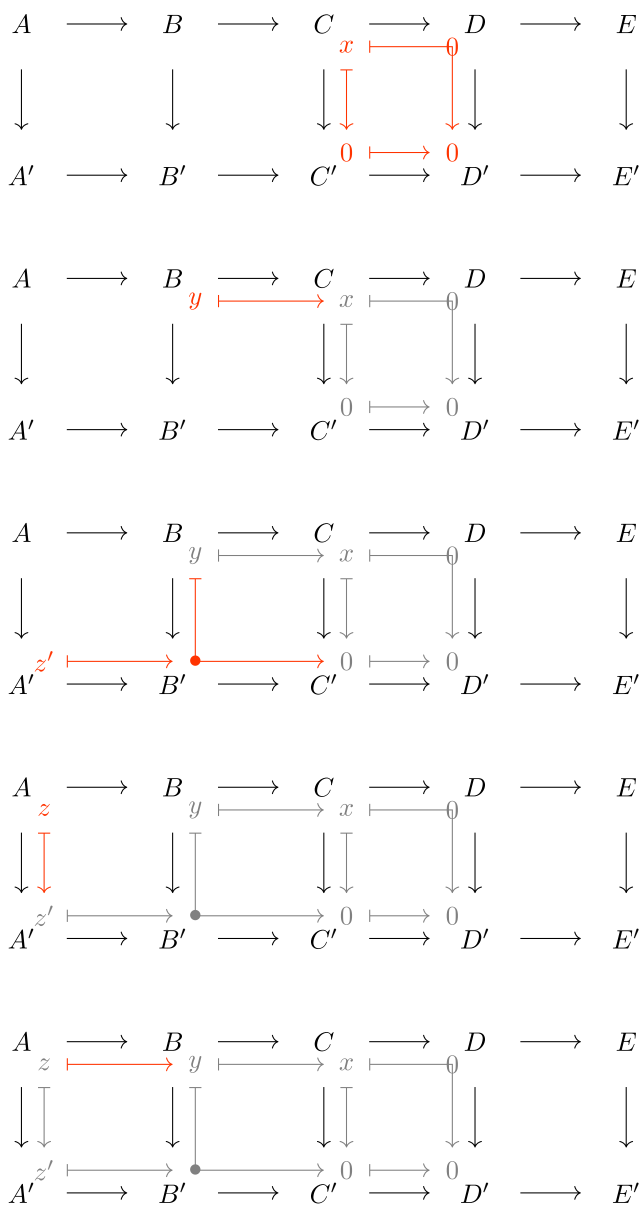


Figure A.4.: The diagram chase in the proof of the five lemma

Proof. We prove the first part via a so-called diagram chase (many statements in homological algebra are established in this way). The second part can be proved in a similar way; the third part is a direct consequence of the first two parts.

Ad 1. Let f_B and f_D be injective and let f_A be surjective. Let $x \in C$ with $f_C(x) = 0$. Then we have $x = 0$, because (Figure A.4):

- Because of $f_D \circ c(x) = c' \circ f_C(x) = c'(0) = 0$ and the injectivity of f_D, we obtain $c(x) = 0$.
- In view of $\operatorname{im} b = \ker c$, there exists a $y \in B$ with $b(y) = x$.
- As $b' \circ f_B(y) = f_C \circ b(y) = f_C(x) = 0$ and $\operatorname{im} a' = \ker b'$, we have: There exists a $z' \in A'$ with $a'(z') = f_B(y)$.
- Because f_A is surjective, there is a $z \in A$ with $f_A(z) = z'$.
- Then $a(z) = y$, because: We have $f_B\bigl(a(z)\bigr) = a' \circ f_A(z) = a'(z') = f_B(y)$ and f_B is injective.
- Thus (because $\operatorname{im} a \subset \ker b$)
  $$x = b(y) = b \circ a(z) = 0,$$
  as desired. □

For the proof of the Mayer–Vietoris sequence (Theorem 3.3.2) we will need the following construction of long exact sequences:

Proposition A.6.8 (algebraic Mayer–Vietoris sequence)**.** *Let*

$$\begin{tikzcd}
\cdots \arrow[r, "c_{k+1}"] & A_k \arrow[r, "a_k"] \arrow[d, "f_{A,k}"'] & B_k \arrow[r, "b_k"] \arrow[d, "f_{B,k}"'] & C_k \arrow[r, "c_k"] \arrow[d, "f_{C,k}"] & A_{k-1} \arrow[r, "a_{k-1}"] \arrow[d, "f_{A,k-1}"] & \cdots \\
\cdots \arrow[r, "c'_{k+1}"'] & A'_k \arrow[r, "a'_k"'] & B'_k \arrow[r, "b'_k"'] & C'_k \arrow[r, "c'_k"'] & A'_{k-1} \arrow[r, "a'_{k-1}"'] & \cdots
\end{tikzcd}$$

be a ($\mathbb{Z}$-indexed) commutative ladder in ${}_R\mathsf{Mod}$ with exact rows. Moreover, for every $k \in \mathbb{Z}$, let $f_{C,k} \colon C_k \longrightarrow C'_k$ be an isomorphism and let

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k \colon B'_k \longrightarrow A_{k-1}.$$

Then the following sequence in ${}_R\mathsf{Mod}$ is exact:

$$\cdots \xrightarrow{\Delta_{k+1}} A_k \xrightarrow{(f_{A,k}, -a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\Delta_k} A_{k-1} \longrightarrow \cdots$$

Proof. This follows from a diagram chase (Exercise). □

A.6.2 Chain Complexes and Homology

Chain complexes are a generalisation of exact sequences. The non-exactness of chain complexes is measured in terms of homology.

Definition A.6.9 (chain complex)**.** An R-chain complex is a pair $C = (C_*, \partial_*)$, consisting of

- a sequence $C_* = (C_k)_{k\in\mathbb{Z}}$ of left R-modules (the *chain modules*), and
- a sequence $\partial_* = (\partial_k\colon C_k \longrightarrow C_{k-1})_{k\in\mathbb{Z}}$ of R-homomorphisms (the *boundary operators* or *differentials*) with
$$\forall_{k\in\mathbb{Z}} \quad \partial_k \circ \partial_{k+1} = 0.$$

Let $k \in \mathbb{Z}$.

- The elements of C_k are the k-*chains*,
- the elements of $Z_k C := \ker \partial_k \subset C_k$ are the k-*cycles*,
- the elements of $B_k C := \operatorname{im} \partial_{k+1} \subset C_k$ are the k-*boundaries*.

In the same way, one can also define chain complexes that are indexed over $\mathbb{N}$ instead of $\mathbb{Z}$. In this case, one defines $Z_0 C := C_0$.

Example A.6.10 (chain complexes)**.**

- Every long exact sequence is a chain complex.
- The sequence
$$\cdots \xrightarrow{\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\mathrm{id}_\mathbb{Z}} \cdots$$
is *no* chain complex of $\mathbb{Z}$-modules, because the composition of successive homomorphisms is not the zero map.

Example A.6.11 (a geometric example)**.** The terms cycle, boundary, chain, ... originate from Algebraic Topology. This can be seen in the construction of singular homology (Chapter 4); moreover, we illustrate this at a related, but slightly simpler, geometric, example (Figure A.5):

We consider the following chain complexes $C = (C_*, \partial_*)$ and $C' = (C'_*, \partial'_*)$ of $\mathbb{Z}$-modules: We set

$$C_0 := \mathbb{Z}^3, \quad C_1 := \mathbb{Z}^3, \quad \forall_{k\in\mathbb{Z}\setminus\{0,1\}} \quad C_k := 0$$

and we let the non-trivial boundary operator be defined by

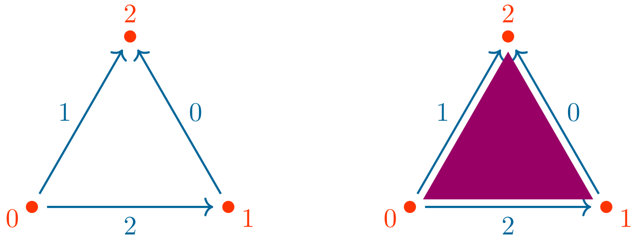


Figure A.5.: The boundary of a triangle and a triangle; degree 0 is red, degree 1 is blue, degree 2 is purple.

$$\begin{aligned} \partial_1 \colon C_1 = \mathbb{Z}^3 &\longrightarrow \mathbb{Z}^3 = C_0 \\ e_0 &\longmapsto e_2 - e_1 \\ e_1 &\longmapsto e_2 - e_0 \\ e_2 &\longmapsto e_1 - e_0; \end{aligned}$$

this clearly defines a chain complex. Moreover, let

$$C'_0 := \mathbb{Z}^3, \quad C'_1 := \mathbb{Z}^3, \quad C'_2 := \mathbb{Z}, \quad \forall_{k \in \mathbb{Z} \setminus \{0,1,2\}} \quad C'_k := 0,$$

and

$$\begin{aligned} \partial'_1 &:= \partial_1 \\ \partial'_2 \colon C'_2 &\longrightarrow C'_1 \\ 1 &\longmapsto e_0 - e_1 + e_2; \end{aligned}$$

then $\partial'_1 \circ \partial'_2 = 0$ (check!); thus, also C' is a chain complex.

These chain complexes can be viewed as simple (simplicial!) algebraic models of the boundary of the 2-simplex and the 2-simplex: The vertices correspond to the standard basis of $C_0 = C'_0$, the edges to the standard basis of $C_1 = C'_1$ and the filled triangle to 1 in C'_2. Moreover, the boundary operators resemble the geometric boundaries.

The 1-chain $z := e_0 - e_1 + e_2$ is a 1-cycle of C (and C'), because

$$\partial_1(z) = \partial_1(e_0 - e_1 + e_2) = 0.$$

This also corresponds to the geometric intuition behind cycles. The complex C is *not* exact: We have $z \in \ker \partial_1$, but $z \notin \{0\} = \operatorname{im} \partial_2$. Therefore, z is *not* a boundary in C (which fits with the geometry).

Howevere, z is a boundary in C', because $\partial'_2(1) = z$ (as expected from the geometric version).

Remark A.6.12 (Co). Reversing the direction of arrows in the definition of chain complexes, leads to *cochain complexes* (and cochains, cocycles, coboundaries, coboundary operators, cochain maps, cohomology, ...). Usually, one denotes coboundary operators in cochain complexes with δ (instead of ∂) and indices are denoted as superscripts (instead of subscripts).

Such objects naturally arise in the study of smooth manifolds and differential forms: the de Rham cochain complex and de Rham cohomology.

In order to obtain a category of chain complexes, we introduce chain maps as structure-preserving maps between chain complexes:

Definition A.6.13 (chain map). Let $C = (C_*, \partial_*)$ and (C'_*, ∂'_*) be R-chain complexes. An *R-chain map* $C \longrightarrow C'$ is a sequence $(f_k \in {}_R\mathrm{Hom}(C_k, C'_k))_{k\in\mathbb{Z}}$ with

$$\forall_{k\in\mathbb{Z}} \quad f_k \circ \partial_{k+1} = \partial'_{k+1} \circ f_{k+1}.$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, "f_{k+1}"'] & C_n \arrow[r] \arrow[d, "f_k"] & \cdots \\
\cdots \arrow[r] & C'_{k+1} \arrow[r, "\partial'_{k+1}"'] & C'_k \arrow[r] & \cdots
\end{tikzcd}$$

Example A.6.14 (a chain map). Let C be the chain complex from Example A.6.11. Then the homomorphisms

$$\begin{aligned}
C_0 &\longmapsto C_0 \\
x &\longmapsto (x_1, x_0, x_2) \\
C_1 &\longmapsto C_1 \\
x &\longmapsto (x_1, x_0, -x_2)
\end{aligned}$$

(together with the zero maps) form a chain map $C \longrightarrow C$ (check!). Geometrically, this map is an algebraic model of reflection of the triangle from Figure A.5 at the vertical axis.

Definition A.6.15 (category of chain complexes). The category ${}_R\mathsf{Ch}$ of R-chain complexes consists of:

- objects: the class of all R-chain complexes
- morphisms: R-chain maps
- compositions: degree-wise ordinary composition of maps.

Example A.6.16 (tensor product of a module and a chain complex). Let Z be a right(!) R-module and let $C = (C_*, \partial_*) \in \mathrm{Ob}({}_R\mathsf{Ch})$. Then

$$Z \otimes_R C := \big((Z \otimes_R C_k)_{k\in\mathbb{Z}}, (\mathrm{id}_Z \otimes_R \partial_k)_{k\in\mathbb{Z}}\big)$$

is a $\mathbb{Z}$-chain complex (check!). If R is non-commutative, then $Z \otimes_R C$, in general, will *not* be an R-chain complex (for this, we need a bimodule structure on Z). Moreover, it should be noted that, in general, homology is *not* compatible with taking tensor products!

Taking the degree-wise tensor product with id_Z turns this construction into a functor (check!)

$$Z \otimes_R \; \cdot \; \colon {}_R\mathsf{Ch} \longrightarrow {}_\mathbb{Z}\mathsf{Ch}\,.$$

Example A.6.17 (chain complexes of simplicial modules)**.** Let $S\colon \Delta^{\mathrm{op}} \longrightarrow {}_R\mathsf{Mod}$ be a functor (a so-called *simplicial left R-module*); here, Δ^{op} is the dual of the simplex category (obtained from Δ by reversing morphisms). For $k \in \mathbb{Z}$, we define

$$C_k(S) := \begin{cases} S\bigl(\Delta(k)\bigr) & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases}$$

$$\partial_k := \begin{cases} \sum_{j=0}^{k} (-1)^j \cdot S(d_j^k) & \text{if } k > 0 \\ 0 & \text{if } k \leq 0; \end{cases}$$

here, $d_j^k \in \mathrm{Mor}_\Delta(\Delta(k-1), \Delta(k))$ is the morphism, whose image is $\{0, \dots, k\} \setminus \{j\}$. We write

$$C(S) := \bigl((C_k(S))_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr).$$

Then $C(S)$ is an R-chain complex (check!). This is one of the key constructions that underlies many homology theories.

This construction can also be extended to a functor

$$C\colon \Delta({}_R\mathsf{Mod}) \longrightarrow {}_R\mathsf{Ch}\,.$$

Here, $\Delta({}_R\mathsf{Mod})$ is the category whose objects are functors $\Delta^{\mathrm{op}} \longrightarrow {}_R\mathsf{Mod}$ and whose morphisms are natural transformations between such functors.

The (non-)exactness of chain complexes is measured in terms of homology:

Definition A.6.18 (homology)**.** Let $C = (C_*, \partial_*)$ be an R-chain complex. For $k \in \mathbb{Z}$, the *k-th homology of C* is defined as

$$H_k(C) := \frac{Z_k(C)}{B_k(C)} = \frac{\ker(\partial_k\colon C_k \to C_{k-1})}{\mathrm{im}(\partial_{k+1}\colon C_{k+1} \to C_k)} \in \mathrm{Ob}({}_R\mathsf{Mod}).$$

Example A.6.19 (homology)**.**

- A chain complex $C \in \mathrm{Ob}({}_R\mathsf{Ch})$ is an exact sequence if and only if $H_k(C) \cong_R 0$ for all $k \in \mathbb{Z}$.
- For the chain complexes C and C' from Example A.6.11, we obtain (check!)

$$
\begin{aligned}
H_1(C) &\cong_{\mathbb{Z}} \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 / \{0\} \cong_{\mathbb{Z}} \mathbb{Z}, && \text{(generated by } [z]) \\
H_1(C') &\cong_{\mathbb{Z}} \ker \partial'_1 / \operatorname{im} \partial'_2 \cong_{\mathbb{Z}} 0. && (z \text{ is a boundary in } C')
\end{aligned}
$$

The homology in degree 1 hence detects the "hole". This basic observation is also the foundation for singular homology (Chapter 4).

Remark A.6.20 (computation of homology)**.** Algorithmically, homology of (sufficiently finite) chain complexes can be computed with the tools developed in Linear Algebra (over fields: Gaussian elimination (Satz I.5.2.8); over Euclidean domains/principal ideal domains: Smith normal form (Satz II.2.5.6)).

Proposition A.6.21 (homology as functor)**.** *Let $k \in \mathbb{Z}$.*

1. *Let $C, C' \in \operatorname{Ob}({}_R\mathsf{Ch})$, let $f \colon C \longrightarrow C'$ be an R-chain map. Then*

$$
\begin{aligned}
H_k(f) \colon H_k(C) &\longrightarrow H_k(C') \\
[z] &\longmapsto \bigl[f_k(z)\bigr]
\end{aligned}
$$

   is a well-defined R-homomorphism.

2. *In this way, H_k becomes a functor ${}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Mod}$.*

Proof. Ad 1. The map $H_k(f)$ is well-defined: Because f is a chain map, f_k maps cycles to cycles (check!). Let $z, z' \in Z_k(C)$ with $z - z' \in B_k(C)$; let $b \in C_{k+1}$ such that $\partial_{k+1} b = z - z'$. Then we obtain in $H_k(C')$:

$$
\begin{aligned}
\bigl[f_k(z)\bigr] - \bigl[f_k(z')\bigr] &= \bigl[f_k(z) - f_k(z')\bigr] \\
&= \bigl[f_k(z - z')\bigr] \\
&= \bigl[f_k(\partial_{k+1} b)\bigr] && \text{(choice of } b) \\
&= \bigl[\partial'_k f_{k+1}(b)\bigr] && (f \text{ is a chain map}) \\
&= 0. && \text{(definition of } H_k(C'))
\end{aligned}
$$

Hence, $H_k(f)$ is well-defined. By construction, $H_k(f)$ is R-linear (because f_k is R-linear).

Ad 2. This is a straightforward computation (check!). □

Example A.6.22 (an induced map in homology)**.** Let $f \colon C \longrightarrow C$ be the chain map from Example A.6.14. As $[z]$ forms a basis of $H_1(C)$ (Example A.6.19), it suffices to determine $H_1(f)([z])$. By definition of f, we have

$$
H_1(f)\bigl([z]\bigr) = \bigl[f_1(z)\bigr] = \bigl[f_1(e_0 - e_1 + e_2)\bigr] = [e_1 - e_0 - e_2] = -[z],
$$

and so $H_1(f) = -\operatorname{id}_{H_1(C)}$.

When computing homology, inheritance results and computational tricks can save a lot of time and space. One key tool is the long exact homology sequence:

Proposition A.6.23 (algebraic long exact homology sequence). *Let*

$$0 \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;p\;} C \longrightarrow 0$$

be a short exact sequence in ${}_R\mathsf{Ch}$ *(i.e., in every degree, the corresponding sequence in* ${}_R\mathsf{Ch}$ *is exact). Then there is a (natural) long exact sequence*

$$\cdots \xrightarrow{\partial_{k+1}} H_k(A) \xrightarrow{H_k(i)} H_k(B) \xrightarrow{H_k(p)} H_k(C) \xrightarrow{\partial_k} H_{k-1}(A) \longrightarrow \cdots$$

This sequence is natural in the following sense: If

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f_A"'] & B \arrow[r, "p"] \arrow[d, "f_B"'] & C \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "p'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in ${}_R\mathsf{Ch}$ *with exact rows, then the corresponding ladder*

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{k+1}"] & H_k(A) \arrow[r, "H_k(i)"] \arrow[d, "H_k(f_A)"'] & H_k(B) \arrow[r, "H_k(p)"] \arrow[d, "H_k(f_B)"'] & H_k(C) \arrow[r, "\partial_k"] \arrow[d, "H_k(f_C)"] & H_{k-1}(A) \arrow[r] \arrow[d, "H_{k-1}(f_A)"] & \cdots \\ \cdots \arrow[r, "\partial_{k+1}"'] & H_k(A') \arrow[r, "H_k(i')"'] & H_k(B') \arrow[r, "H_k(p')"'] & H_k(C') \arrow[r, "\partial_k"'] & H_{k-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

is commutative and has exact rows.

Proof. Let $k \in \mathbb{Z}$. We construct the *connecting homomorphism*

$$\partial_k \colon H_k(C) \longrightarrow H_{k-1}(A)$$

as follows: Let $\gamma \in H_k(C)$; let $c \in C_k$ be a cycle representing γ. Because $p_k \colon B_k \longrightarrow C_k$ is surjective, there is a $b \in B_k$ with

$$p_k(b) = c.$$

As p is a chain map, we obtain $p_{k-1} \circ \partial_k^B(b) = \partial_k^C \circ p_k(b) = \partial_k^C(c) = 0$; then exactness in degree k shows that there exists an $a \in A_{k-1}$ with

$$i_{k-1}(a) = \partial_k^B(b).$$

In this situation, we call (a, b, c) a *compatible triple for* γ and we define

$$\partial_k(\gamma) := [a] \in H_{k-1}(A).$$

Straightforward diagram chases then show (check!):

- If (a, b, c) is a compatible triple for γ, then $a \in A_{k-1}$ is a cycle (and so indeed defines a class in $H_{k-1}(A)$).
- If (a, b, c) and (a', b', c') are compatible triples for γ, then $[a] = [a']$ in $H_{k-1}(A)$.

These observations show that ∂_k is an R-homomorphism and that ∂_k is natural (check!).

Further diagram chases then show that the resulting long sequence is exact (even more to check …). □

Combining the five lemma and the algebraic long exact sequence gives us:

Example A.6.24 (drie halen, twee betalen)**.** Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f_A"'] & B \arrow[r, "p"] \arrow[d, "f_B"'] & C \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "p'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram in ${}_R\mathsf{Ch}$ with exact rows. Then: If two of the three sequences $(H_k(f_A))_{k\in\mathbb{Z}}$, $(H_k(f_B))_{k\in\mathbb{Z}}$, $(H_k(f_C))_{k\in\mathbb{Z}}$ consist of isomorphisms, then so does the third. This can be seen as follows:

The long exact homology sequences of the rows lead to a commutative ladder with exact rows (Proposition A.6.23). We can then apply the five lemma (Proposition A.6.7) to five successive rungs (where we put the mystery homomorphism into the middle).

A.6.3 Homotopy Invariance

A key property of homology of chain complexes is homotopy invariance. This algebraic homotopy invariance is the source of homotopy invariance of many functors in geometry and topology; moreover, algebraic homotopy invariance often simplifies the computation of homology.

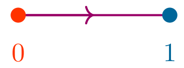
We briefly explain how topological considerations naturally lead to the notion of chain homotopy (Definition A.6.31):

In Top, homotopy is defined as follows: Continuous maps $f, g \colon X \longrightarrow Y$ are homotopic, if there exists a continuous map $h \colon X \times [0,1] \longrightarrow Y$ with

$$h \circ i_0 = f \quad \text{and} \quad h \circ i_1 = g;$$

here, $i_0 \colon X \hookrightarrow X \times \{0\} \hookrightarrow X \times [0,1]$ and $i_1 \colon X \hookrightarrow X \times \{1\} \hookrightarrow X \times [0,1]$ denote the canonical inclusions of the bottom and the top into the cylinder $X \times [0,1]$ over X, respectively.

We model this situation in the category ${}_R\mathsf{Ch}$: As first step, we model the unit interval $[0,1]$ by a suitable chain complex (Figure A.6).



$$\begin{array}{ccc} \text{degree } 1 & & \text{degree } 0 \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ x & \longmapsto & (-x, x) \end{array}$$

Figure A.6.: an algebraic model of $[0,1]$

Definition A.6.25 (algebraic model $[0,1]$)**.** Let $I \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$ be the chain complex

$$\begin{array}{lccccccccc} \text{degree} & & & 2 & 1 & 0 & & -1 & & \\ \dots \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} \mathbb{Z} \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} 0 \xrightarrow[0]{} & \dots \\ & & & & x \longmapsto & (-x, x) & & & & \end{array}$$

As analogy of the product of topological spaces, we consider the tensor product of chain complexes; the basic idea is that chain modules of the product in degree k should contain information on k-dimensional phenomena and thus the degree of the tensor factors should add up to k. Geometrically, one can show that cellular chain complexes of products of finite CW-complexes (with respect to the product cell structure) are isomorphic to the tensor product of the cellular chain complexes of the factors [13, V.3.9]. More generally, the Eilenberg–Zilber theorem shows that the singular chain complex of a product of two spaces is chain homotopy equivalent to the tensor product of the singular chain complexes of the factors [13, Chapter VI.12].

Definition A.6.26 (tensor product of chain complexes)**.** Let $C \in \mathrm{Ob}({}_R\mathsf{Ch})$ and $D \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$. Then we define $C \otimes_{\mathbb{Z}} D \in \mathrm{Ob}({}_R\mathsf{Ch})$ by

$$(C \otimes_R D)_k := \bigoplus_{j \in \mathbb{Z}} C_j \otimes_R D_{k-j}$$

and the boundary operators

$$\begin{aligned} (C \otimes_R D)_k &\longrightarrow (C \otimes_R D)_{k-1} \\ C_j \otimes_R D_{k-j} \ni c \otimes d &\longmapsto \partial_j^C c \otimes d + (-1)^j \cdot c \otimes \partial_{k-j}^D d \end{aligned}$$

for all $k \in \mathbb{Z}$. (This indeed defines a chain complex!)

Study note. This definition generalises the tensor product of a module and a chain complex (Example A.6.16). Do you see why/how?

More generally, if C is an (S,R)-bimodule chain complex and D is a left R-chain complex, then one can also define the left S-chain complex $C \otimes_R D$.

Remark A.6.27 (sign convention)**.** We use the following convention for the choice of signs: If a boundary operator is "moved past" an element, then we introduce the sign

$$(-1)^{\text{degree of that element}}.$$

It should be noted that different authours use different sign conventions. Therefore, for all formulae in the literature concerning products of chain complexes or products on (co)homology, one has to carefully check the sign conventions used in that source.

Remark A.6.28 (functoriality of the tensor product)**.** Let $C, C' \in \operatorname{Ob}({}_R\mathsf{Ch})$, let $D, D' \in \operatorname{Ob}({}_\mathbb{Z}\mathsf{Ch})$, and let $f \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, C')$ und $g \in \operatorname{Mor}_{{}_\mathbb{Z}\mathsf{Ch}}(D, D')$. Then

$$\begin{aligned} f \otimes_R g \colon C \otimes_\mathbb{Z} D &\longrightarrow C' \otimes_\mathbb{Z} D' \\ c \otimes d &\longmapsto f(c) \otimes g(d) \end{aligned}$$

yields a well-defined chain map in ${}_R\mathsf{Ch}$.

As next step, we model the inclusions of the bottom and the top of cylinders in the algebraic setting.

Definition A.6.29 (algebraic model of inclusion of top/bottom of cylinders)**.** Let $C \in \operatorname{Ob}({}_R\mathsf{Ch})$. Then, we define the R-chain maps (check!)

$$\begin{aligned} i_0 \colon C &\longrightarrow C \otimes_\mathbb{Z} I \\ C_k \ni c &\longmapsto (c, 0, 0) \in C_k \oplus C_{k-1} \oplus C_k \cong_R (C \otimes_\mathbb{Z} I)_k \\ i_1 \colon C &\longrightarrow C \otimes_\mathbb{Z} I \\ C_k \ni c &\longmapsto (0, 0, c) \in C_k \oplus C_{k-1} \oplus C_k \cong_R (C \otimes_\mathbb{Z} I)_k. \end{aligned}$$

Under the correspondence indicated in Figure A.6, these chain maps are an algebraic version of the geometric inclusions of bottom and top, respectively.

Remark A.6.30. Let $C, D \in \operatorname{Ob}({}_R\mathsf{Ch})$ and let $f, g \in \operatorname{Mor}_{{}_R\mathsf{Ch}}(C, D)$. A chain map $h \colon C \otimes_\mathbb{Z} I \longrightarrow D$ in ${}_R\mathsf{Ch}$ with $h \circ i_0 = f$ and $h \circ i_1 = g$ corresponds to a family $(\widetilde{h}_k \in \operatorname{Mor}_{{}_R\mathsf{Mod}}(C_k, D_{k+1}))_{k \in \mathbb{Z}}$ satisfying

$$\partial^D_{k+1} \circ \widetilde{h}_k = \widetilde{h}_{k-1} \circ \partial^C_k + (-1)^k \cdot g_k - (-1)^k \cdot f_k$$

(Figure A.7) for all $k \in \mathbb{Z}$. This last equation can be rewritten as

$$\partial^D_{k+1} \circ (-1)^k \cdot \widetilde{h}_k + (-1)^{k-1} \cdot \widetilde{h}_{k-1} \circ \partial^C_k = g_k - f_k.$$

Therefore, one defines the notion of chain homotopy (and related terms) as follows:

Definition A.6.31 (chain homotopy, null-homotopic, contractible)**.** Let $C, D \in \operatorname{Ob}({}_R\mathsf{Ch})$.

$$\begin{tikzcd}
 & C_{k+1} \arrow[r, "\partial^C_{k+1}"] & C_k & \\
 & \oplus & \oplus & \\
(C \otimes_{\mathbb{Z}} I)_{k+1} = & C_k \arrow[r, "\partial^C_{k+1}"] \arrow[ur, "-(-1)^k"] \arrow[dr, "(-1)^k"] & C_{k-1} & = (C \otimes_{\mathbb{Z}} I)_k \\
 & \oplus & \oplus & \\
 & C_{k+1} \arrow[r, "\partial^C_{k+1}"'] \arrow[d, "f_{k+1} \oplus \widetilde{h}_k \oplus g_{k+1}"'] & C_k \arrow[d, "f_k \oplus \widetilde{h}_{k-1} \oplus g_k"] & \\
 & D_{k+1} \arrow[r, "\partial^D_{k+1}"'] & D_k &
\end{tikzcd}$$

Figure A.7.: discovering the notion of chain homotopy

- Chain maps $f, g \in \operatorname{Mor}_{_R\mathsf{Ch}}(C, D)$ are *chain homotopic (in $_R\mathsf{Ch}$)*, if there exists a sequence $h = (h_k \in \operatorname{Mor}_{_R\mathsf{Mod}}(C_k, D_{k+1}))_{k \in \mathbb{Z}}$ with

$$\partial^D_{k+1} \circ h_k + h_{k-1} \circ \partial^C_k = g_k - f_k$$

  for all $k \in \mathbb{Z}$. In this case, h is a *chain homotopy from f to g (in $_R\mathsf{Ch}$)*, and we write $f \simeq_{_R\mathsf{Ch}} g$.

- A chain map $f \in \operatorname{Mor}_{_R\mathsf{Ch}}(C, D)$ is a *chain homotopy equivalence (in $_R\mathsf{Ch}$)*, if there exists a chain map $g \in \operatorname{Mor}_{_R\mathsf{Ch}}(D, C)$ with

$$g \circ f \simeq_{_R\mathsf{Ch}} \mathrm{id}_C \quad \text{and} \quad f \circ g \simeq_{_R\mathsf{Ch}} \mathrm{id}_D .$$

  We then write $C \simeq_{_R\mathsf{Ch}} D$.

- Chain maps that are (in $_R\mathsf{Ch}$) chain homotopic to the zero map are *null-homotopic (in $_R\mathsf{Ch}$)*.

- The chain complex C ist *contractible (in $_R\mathsf{Ch}$)*, if id_C is null-homotopic in $_R\mathsf{Ch}$ (equivalently, if C is chain homotopic to the zero chain complex). Homotopies in $_R\mathsf{Ch}$ from id_C to the zero map are also called *chain contractions (in $_R\mathsf{Ch}$)*.

Example A.6.32. Let $C \in \operatorname{Ob}(_R\mathsf{Ch})$. Then $i_0 \simeq_{_R\mathsf{Ch}} i_1 \colon C \longrightarrow C \otimes_{\mathbb{Z}} I$ (check!). Moreover, we consider the chain map

$$\begin{aligned} p \colon C \otimes_{\mathbb{Z}} I &\longrightarrow C \\ C_k \oplus C_{k-1} \oplus C_k \ni (c_0, c, c_1) &\longmapsto c_0 + c_1 \in C_k \end{aligned}$$

in ${}_R\mathsf{Ch}$. Then $p \circ i_0 = \mathrm{id}_C$ and $i_0 \circ p \simeq_{{}_R\mathsf{Ch}} \mathrm{id}_{C \otimes_{\mathbb{R}} I}$. Hence, $C \simeq_{{}_R\mathsf{Ch}} C \otimes_{\mathbb{Z}} I$, as we would expect from topology.

Proposition A.6.33 (basic properties of chain homotopy)**.**

1. *Let $C, D \in \mathrm{Ob}({}_R\mathsf{Ch})$ and let $f, f', g, g' \in \mathrm{Mor}({}_R\mathsf{Ch})$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_R\mathsf{Ch}} g'$. Then, we have*
$$a \cdot f + b \cdot g \simeq_{{}_R\mathsf{Ch}} a \cdot f' + b \cdot g'$$
for all $a, b \in R$.
2. *Let $C, D \in \mathrm{Ob}({}_R\mathsf{Ch})$. Then "$\simeq_{{}_R\mathsf{Ch}}$" is an equivalence relation on the morphism set $\mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D)$.*
3. *Let $C, D, E \in \mathrm{Ob}({}_R\mathsf{Ch})$, let $f, f' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D)$ and let $g, g' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(D, E)$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_R\mathsf{Ch}} g'$. Then, we have*
$$g \circ f \simeq_{{}_R\mathsf{Ch}} g' \circ f'.$$
4. *Let $C, D \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$, let $Z \in \mathrm{Ob}({}_R\mathsf{Mod})$, and let $f, f' \in \mathrm{Mor}_{{}_{\mathbb{Z}}\mathsf{Ch}}(C, D)$ mit $f \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} f'$. Then,*
$$Z \otimes_{\mathbb{Z}} f \simeq_{{}_R\mathsf{Ch}} Z \otimes_{\mathbb{Z}} f'.$$
5. *Let $C, C' \in \mathrm{Ob}({}_R\mathsf{Ch})$, $D, D' \in \mathrm{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$, and let $f, f' \in \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, C')$, $g, g' \in \mathrm{Mor}_{{}_{\mathbb{Z}}\mathsf{Ch}}(D, D')$ with $f \simeq_{{}_R\mathsf{Ch}} f'$ and $g \simeq_{{}_{\mathbb{Z}}\mathsf{Ch}} g'$. Then*
$$f \otimes_{\mathbb{Z}} g \simeq_{{}_R\mathsf{Ch}} f' \otimes_{\mathbb{Z}} g'.$$

Proof. All these properties follow via straightforward calculations directly from the definitions (check!). □

In particular, we can pass to the corresponding homotopy category:

Definition A.6.34 (homotopy category of chain complexes)**.** The *homotopy category of left R-chain complexes* is the category ${}_R\mathsf{Ch}_\mathsf{h}$ consisting of:

- objects: Let $\mathrm{Ob}({}_R\mathsf{Ch}_\mathsf{h}) := \mathrm{Ob}({}_R\mathsf{Ch})$.
- morphisms: For all left R-chain complexes C, D, we set
$$[C, D] := \mathrm{Mor}_{{}_R\mathsf{Ch}_\mathsf{h}}(C, D) := \mathrm{Mor}_{{}_R\mathsf{Ch}}(C, D) \big/ {\simeq_{{}_R\mathsf{Ch}}} .$$
- compositions: The compositions of morphisms are defined by ordinary (degree-wise) composition of representatives.

As mentioned before, a key property of homology of chain complexes is homotopy invariance in the following sense:

Proposition A.6.35 (homotopy invariance of homology of chain complexes)**.** *Let $k \in \mathbb{Z}$. Then the functor $H_k \colon {}_R\mathsf{Ch} \longrightarrow {}_R\mathsf{Mod}$ factors over ${}_R\mathsf{Ch}_h$. More explicitly: Let $C, C' \in \mathrm{Ob}({}_R\mathsf{Ch})$ and let $f, g \colon C \longrightarrow C'$ be R-chain maps with $f \simeq_{{}_R\mathsf{Ch}} g$. Then,*

$$H_k(f) = H_k(g).$$

Proof. Let h be a chain homotopy from f to g in ${}_R\mathsf{Ch}$. Moreover, let $z \in Z_k(C)$ be a k-cycle. Then, we obtain in $H_k(C')$:

$$\begin{aligned}
H_k(f)\bigl([z]\bigr) - H_k(g)\bigl([z]\bigr) &= \bigl[f_k(z) - g_k(z)\bigr] \\
&= \bigl[\partial'_{k+1} \circ h_k(z) + h_{k-1} \circ \partial_k(z)\bigr] && (h \text{ is a chain homotopy}) \\
&= \bigl[\partial'_{k+1} \circ h_k(z) + 0\bigr] && (z \text{ is a cycle}) \\
&= [0] && (\text{definition of } H_k(C'))
\end{aligned}$$

Hence, $H_k(f) = H_k(g)$. $\square$

A.7 Homotopy Theory of CW-Complexes

For the sake of completeness, we collect some important features of the homotopy theory of CW-complexes: the Whitehead theorem, cellullar approximation, and the cofibration property.

A.7.1 Whitehead Theorem

For CW-complexes, homotopy equivalences can be characterised in the following way:

Theorem A.7.1 (Whitehead theorem)**.** *Let X and Y be CW-complexes and let $f\colon X \longrightarrow Y$ be a continuous map. Then the following are equivalent:*

1. *The map $f\colon X \longrightarrow Y$ is a homotopy equivalence (in* Top*).*
2. *The map $f\colon X \longrightarrow Y$ is a* weak equivalence*, i.e., for every $x_0 \in X$ and every $n \in \mathbb{N}$ the induced map $\pi_n(f)\colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is bijective.*
3. *For every CW-complex Z, the map*

$$\begin{aligned}[][Z, f]\colon [Z, X] &\longrightarrow [Z, Y]\\ [g] &\longmapsto [f \circ g]\end{aligned}$$

bijective.

Sketch of proof. *Ad* $1 \Longrightarrow 2$. Let $n \in \mathbb{N}$. Then, by construction, $\pi_n\colon \mathsf{Top}_* \longrightarrow \mathsf{Set}$ is a homotopy invariant functor. Moreover, one can show that π_n also translates unpointed homotopy equivalences to bijections [68, Proposition 6.2.4].

Ad $2 \Longrightarrow 3$. Because CW-complexes are built up from cells, one can prove this implication by a careful induction [70, Chapter IV.7, Chapter V.3].

Ad $1 \Longrightarrow 2$. If 2. holds, then $[\,\cdot\,, f]$ is a natural isomorphism $[\,\cdot\,, X] \Longrightarrow [\,\cdot\,, Y]$. Then, the Yoneda lemma (Proposition 1.2.23) shows that $X \simeq Y$. □

Caveat A.7.2.

- The notion of "weak equivalence" is *not* an equivalence relation on the class of topological spaces; in general, symmetry is *not* satisfied [70, p. 221].

  However, the Whitehead theorem shows that on the class of CW-complexes, weak equivalence coincides with homotopy equivalence and thus is an equivalence relation on CW-complexes.

- Abstract isomorphisms between homotopy groups of CW-complexes are *not* sufficient to conclude that the given CW-complexes are homotopy equivalent. It is essential that these isomorphisms are induced by a continuous map.

  For example, the spaces $\mathbb{R}P^2 \times S^3$ and $S^2 \times \mathbb{R}P^3$ (which both admit a CW-structure) have isomorphic homotopy groups (because they have the common covering space $S^2 \times S^3$; Corollary 2.3.25), but they are *not* homotopy equivalent (as can be seen from the (cellular) homology in degree 5; check!).

In particular, in the simply connected case, the Whitehead theorem shows that ordinary homology with $\mathbb{Z}$-coefficients is a rather powerful tool.

Corollary A.7.3 (Whitehead theorem, simply connected case)**.** *Let X be a simply connected CW-complex. Then the following are equivalent:*

1. *The space X is contractible (in* Top*).*
2. *For each $x_0 \in X$ and each $n \in \mathbb{N}$, we have $\bigl|\pi_n(X, x_0)\bigr| = 1$.*
3. *For each $n \in \mathbb{N}$, we have $H_n(X; \mathbb{Z}) \cong_{\mathbb{Z}} H_n(\bullet; \mathbb{Z})$.*

Proof. Applying the Whitehead theorem (Theorem A.7.1) to the constant map $X \longrightarrow \bullet$ shows that 1. and 2. are equivalent.

Moreover, because X is simply connected, the equivalence of 2. and 3. is a consequence of the Hurewicz theorem (Corollary 4.5.10). $\square$

Example A.7.4. The Warsaw circle is *not* homotopy equivalent to a CW-complex: On the one hand, all homotopy groups of the Warsaw circle are trivial (this can be shown as in the case of the fundamental group; Exercise); in particular, the Warsaw circle is weakly equivalent to a contractible CW-complex. On the other hand, the Warsaw circle is *not* contractible [70, p. 220]. In combination with Corollary A.7.3, we obtain that the Warsaw circle is not homotopy equivalent to a CW-complex.

A.7.2 Cellular Approximation

While not every topological space has the homotopy type of a CW-complex (Example A.7.4), it at least has the *weak* homotopy type of CW-complex (as can be seen from suitable inductive constructions):

Theorem A.7.5 (cellular approximation of maps [26, Chapter 8.5])**.** *Let X and Y be CW-complexes.*

1. *If $f\colon X \longrightarrow Y$ is a continuous map, then there exists a cellular map $f'\colon X \longrightarrow Y$ with*
$$f \simeq_{\mathsf{Top}} f'.$$

2. *If $f, g\colon X \longrightarrow Y$ are cellular maps with $f \simeq_{\mathsf{Top}} g$, then also $f \simeq_{\mathsf{CW}} g$.*

Theorem A.7.6 (cellular approximation of spaces [26, Chapter 8.6]). *Let X, X' be topological spaces.*

1. *Then there exists a CW-complex Y and a weak homotopy equivalence $f\colon Y \longrightarrow X$. We then call (Y, f) a* cellular approximation *of X.*

2. *If $g\colon X \longrightarrow X'$ is a continuous map and if (Y, f) and (Y', f') are cellular approximations of X and X', respectively, then there exists a cellular map $\widetilde{g}\colon Y \longrightarrow Y'$ (which is unique up to cellular homotopy) with*
$$g \circ f \simeq_{\mathsf{Top}} f' \circ \widetilde{g}.$$

Remark A.7.7 (singular homology and cellular approximation). Singular homology is invariant under weak equivalences (this can be shown by an argument similar to the proof of the Hurewicz theorem [68, Theorem 9.5.3]). Therefore, by cellular approximation, from the point of view of singular homology, there is not much of a difference between $\mathsf{Top}_{\mathsf{h}}$ and CW_{h}.

A.7.3 Subcomplexes and Cofibrations

The inclusions of subcomplexes of CW-complexes into the ambient complex have the following convenient property:

Definition A.7.8 (cofibration). A continuous map $i\colon A \longrightarrow X$ is a *cofibration* if it has the *homotopy extension property*, i.e., if for every topological space Y and every map $f\colon X \longrightarrow Y$ and every continuous map $h\colon A \times [0,1] \longrightarrow Y$ with $h(\,\cdot\,, 0) = f \circ i$, there exists a continuous map $H\colon X \times [0,1] \longrightarrow Y$ with
$$H(\,\cdot\,, 0) = f \quad \text{and} \quad H \circ (i \times \mathrm{id}_{[0,1]}) = h.$$

$$\begin{tikzcd}
A \times [0,1] \arrow[r, "h"] \arrow[d, "i\times \mathrm{id}_{[0,1]}"'] & Y \\
X \times [0,1] \arrow[ur, dashed, "H"] & X \arrow[l, "i_0"] \arrow[u, "f"']
\end{tikzcd}$$

Inclusions of (closed) subspaces that are cofibrations are the “right” kind of subspace inclusions in the sense of homotopy theory:

Proposition A.7.9 (quotients of cofibrations). *Let $i\colon A \longrightarrow X$ be an injective continuous map that is a cofibration. If A is contractible, then the canonical projection $X \longrightarrow X/A$ is a homotopy equivalence.*

Proof. Using the homotopy extension property, one can construct a homotopy inverse of the canonical projection from a contracting homotopy of the identity on A [26, Proposition 0.17]. □

Example A.7.10 (cofibrations).

- Not every injective continuous map is a cofibration. For example, the inclusion $[0, 1) \longrightarrow [0, 1]$ is *not* a cofibration (check!).
- Let $n \in \mathbb{N}_{>0}$. Then the inclusion $S^{n-1} \longrightarrow D^n$ is a cofibration (check!). In the same way, also the inclusion $\partial\Delta^n \longrightarrow \Delta^n$ is a cofibration.
- If $f: X \longrightarrow Y$ is a continuous map and $M(f) := \bigl((X \times [0, 1]) \sqcup Y\bigr) \,/\, \bigl((x, 1) \sim f(x)\bigr)$ is the *mapping cylinder of* f, then the inclusion

$$\begin{aligned} X &\longrightarrow M(f) \\ x &\longmapsto [x, 0] \end{aligned}$$

is a cofibration (check!). Moreover, the canonical continuous map $Y \longrightarrow M(f)$ is a homotopy equivalence (check!). In this way, in homotopy theory, every continuous map can be "replaced" by a cofibration.

Proposition A.7.11 (subcomplexes yield cofibrations [26, Proposition 0.16]). *Let (X, A) be a relative CW-complex. Then the inclusion $A \longrightarrow X$ is a cofibration.*

Sketch of proof. Roughly speaking, the proof relies on the following facts:

- If $n \in \mathbb{N}_{>0}$, then the inclusion $S^{n-1} \longrightarrow D^n$ is a cofibration (Example A.7.10).
- Cofibrations are stable under pushouts.

As (relative) CW-complexes are constructed inductively by attaching cells, it follows that $A \longrightarrow X$ is a cofibration. $\square$

Example A.7.12 (one-dimensional CW-complexes). Let X be a connected one-dimensional CW-complex and let $A \subset X$ be a spanning tree (i.e., a connected subcomplex of X whose 0-cells and 1-cells have the combinatorics of a tree); via Zorn's lemma one can show that such a spanning tree indeed exists. Then A is contractible (check!) and the inclusion $A \longrightarrow X$ is a cofibration (because (X, A) is easily seen to be a relative CW-complex). Because A is a *spanning tree*, we have $X/A \simeq \bigvee^I S^1$, where I is the set of open 1-cells of X that are *not* contained in X. Therefore, Proposition A.7.9 shows that

$$X \simeq X/A \simeq \bigvee^I S^1.$$

Moreover, the inheritance properties of cofibrations and exact sequences associated with cofibrations are also used in other fields in order to mimic parts of homotopy theory in different contexts.

B

Exercise Sheets

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 1, October 19, 2021

Exercise 1 (product topology)**.** Let X and Y be topological spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $B \subset X$ and $C \subset Y$ are closed subsets, then $B \times C \subset X \times Y$ is closed.
2. If $A \subset X \times Y$ is closed, then there are closed sets $B \subset X$ and $C \subset Y$ with $A = B \times C$.

Exercise 2 (TOPOLOGY)**.** Classify the following six subspaces of $\mathbb{R}^2$ up to homeomorphism and prove this classification result.

Hints. Some of the homeomorphisms might be hard to write down explicitly; in these cases, it is sufficient to give an outline on how to construct them and to indicate clearly that a proper formal argument would require more details.

Exercise 3 (stereographic projection)**.** Let $n \in \mathbb{N}_{>0}$ and $N := (0, \dots, 0, 1) \in S^n$; i.e., N is the North Pole of S^n. The map

$$\begin{aligned} s_n \colon S^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) &\longmapsto \frac{1}{1 - x_{n+1}} \cdot (x_1, \dots, x_n) \end{aligned}$$

is called *stereographic projection*. Give a geometric interpretation of this map and prove that it is a homeomorphism. Illustrate your arguments graphically!

Exercise 4 (balls, spheres, simplices)**.** Let $n \in \mathbb{N}_{>0}$. Solve one of the following problems:

1. Prove that Δ^n is homeomorphic to D^n and that $\partial\Delta^n$ is homeomorphic to S^{n-1}.

   Hints. Find the centre and inflate!

2. Prove that D^n/S^{n-1} is homeomorphic to S^n.

   Hints. Quotient spaces will be introduced in the second lecture.

Illustrate your arguments graphically!
Hints. The compact-Hausdorff trick might be useful.

Bonus problem (Peano curves)**.** Show that there exist surjective continuous maps $[0, 1] \longrightarrow [0, 1] \times [0, 1]$. Can such a map be injective?

Submission before October 26, 2021, 8:30, via GRIPS

(Solutions may be submitted in English or German.)

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 2, October 26, 2021

Exercise 1 (relative topology)**.** Let (X, A) and (Y, B) be pairs of topological spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

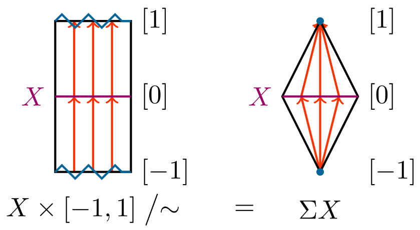
1. If $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$, then $A \cong_{\mathsf{Top}} B$ and $X \cong_{\mathsf{Top}} Y$.
2. If $X/A \cong_{\mathsf{Top}} Y/B$, then $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$.

Exercise 2 (suspension/Einhängung)**.** For a topological space X, we define the *suspension of* X as

$$\Sigma X := \bigl(X \times [-1, 1]\bigr) /{\sim}$$

(endowed with the quotient topology of the product toplogy), where "$\sim$" is the equivalence relation generated by

$$\forall_{x,x' \in X} \quad (x, 1) \sim (x', 1)$$
$$\forall_{x,x' \in X} \quad (x, -1) \sim (x', -1).$$



1. Let $f \colon X \longrightarrow Y$ be a continuous map. Show that the map Σf is well-defined and continuous:
$$\begin{aligned} \Sigma f \colon \Sigma X &\longrightarrow \Sigma Y \\ [x, t] &\longmapsto \bigl[f(x), t\bigr] \end{aligned}$$

2. Let $n \in \mathbb{N}$. Show that the following map is a well-defined homeomorphism:
$$\begin{aligned} \Sigma S^n &\longrightarrow S^{n+1} \\ [x, t] &\longmapsto \bigl(\cos(\pi/2 \cdot t) \cdot x, \sin(\pi/2 \cdot t)\bigr) \end{aligned}$$

Exercise 3 (projective plane via glueings)**.** Prove that there are pushout diagrams of topological spaces of the following type:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \mathbb{R}P^0 \arrow[d] \\ D^1 \arrow[r] & \mathbb{R}P^1 \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & \mathbb{R}P^1 \arrow[d] \\ D^2 \arrow[r] & \mathbb{R}P^2 \end{tikzcd}$$

In particular, describe all of the maps in these diagrams explicitly and illustrate your arguments graphically.

Please turn over

Exercise 4 (morphisms in the simplex category). For $n \in \mathbb{N}_{>0}$ and $j \in \{0, \dots, n\}$ we define

$$\begin{aligned} d_j^n \colon \Delta(n-1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k < j \\ k+1 & \text{if } k \geq j; \end{cases} \end{aligned}$$

for $n \in \mathbb{N}$ and $j \in \{0, \dots, n\}$ we define

$$\begin{aligned} s_j^n \colon \Delta(n+1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j. \end{cases} \end{aligned}$$

Clearly, all of these maps are morphisms in the simplex category Δ.

1. Prove that every morphism in Δ is a composition of finitely many of the morphisms above.

2. Let $n \in \mathbb{N}$. Prove that for all $j, k \in \{0, \dots, n+1\}$ with $j < k$ we have

$$d_k^{n+1} \circ d_j^n = d_j^{n+1} \circ d_{k-1}^n.$$

Bonus problem (Asteroids).

1. Explain how one could design a version of the computer game classic Asteroids on $\mathbb{R}P^2$ (instead of the 2-torus). Of course, screens are still assumed to be of rectangular shape; thus, it is necessary to first explain how to glue a rectangle into $\mathbb{R}P^2$. Moreover, special attention should be paid to the specification of controls (there is a subtlety that one needs to overcome).

2. Implement $\mathbb{R}P^2$-Asteroids. If you deploy it as a web application, then all participants of the course can enjoy it ...

Submission before November 2, 2021, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 3, November 2, 2021

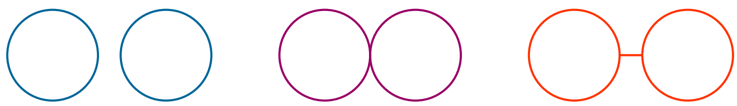
Exercise 1 (point-removal trick for homotopy equivalence?). Let X and Y be topological spaces and let $x \in X$, $y \in Y$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $X \simeq Y$, then $X \setminus \{x\} \simeq Y \setminus \{y\}$.
2. If $X \setminus \{x\} \simeq Y \setminus \{y\}$, then $X \simeq Y$.

Exercise 2 (homotopy equivalence and path-connectedness).

1. Show that a space X is path-connected if and only if all continuous maps $[0,1] \longrightarrow X$ are homotopic to each other.
2. Conclude that path-connectedness is preserved under homotopy equivalences.

Exercise 3 (OOOOOO). We consider the folowing three subspaces of $\mathbb{R}^2$. State and prove the classification of these spaces up to homotopy equivalence.



Exercise 4 (mapping cones/Abbildungskegel). Let X and Y be topological spaces and let $f \in \operatorname{map}(X,Y)$. The *mapping cone of f* is defined by the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ \operatorname{Cone}(X) \arrow[r] & \operatorname{Cone}(f) \end{tikzcd}$$

where $\operatorname{Cone}(X) := (X \times [0,1])/(X \times \{0\})$ denotes the *cone over* X and $i \colon X \longrightarrow \operatorname{Cone}(X)$, $x \longmapsto [x,1]$ is the canonical map to the base of the cone. Prove the following statements (and illustrate your arguments graphically), provided that X is non-empty:

1. The cone $\operatorname{Cone}(X)$ is contractible.
2. If $f \colon X \longrightarrow Y$ is a homotopy equivalence, then the mapping cone $\operatorname{Cone}(f)$ is contractible.

Bonus problem (four-point-circle). Let $X := \{1,2,3,4\}$. We equip X with the topology generated by $\bigl\{\{1\},\{3\},\{1,2,3\},\{1,3,4\}\bigr\}$. In this exercise, you may use that S^1 is *not* contractible (which we will prove later in this course).

1. Show that there exists a surjective continuous map $S^1 \longrightarrow X$.
2. Show that S^1 and X are *not* homotopy equivalent.
3. *Bonus bonus problem.* Show that X is *not* contractible.

Submission before November 9, 2021, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 4, November 9, 2021

Exercise 1 (spheres and π_n)**.** Let $n \in \mathbb{N}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\pi_n(S^n, e_1)$ consists of a single element, then for every pointed topological space (X, x_0) also $\pi_n(X, x_0)$ consists of a single element.

2. If $\pi_n(S^n, e_1)$ is infinite, then for every pointed space (X, x_0) also $\pi_n(S^n \times X, (e_1, x_0))$ is infinite.

Exercise 2 (product in $\mathsf{Top_h}$)**.** Let $(X_i)_{i \in I}$ be a family of topological spaces. Show that $\prod_{i \in I} X_i$ together with the homotopy classes of the canonical projections onto the factors satisfies the universal property of the product of $(X_i)_{i \in I}$ in the homotopy category $\mathsf{Top_h}$.

Exercise 3 (invariance of the boundary)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds.

Let $n \in \mathbb{N}_{>0}$ and let $H^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the upper half-space (with respect to the subspace topology of $\mathbb{R}^n$). Prove *invariance of the boundary*, i.e., show that there is no open neighbourhood of 0 in H^n that is homeomorphic to the open unit ball $(D^n)^\circ$ in $\mathbb{R}^n$.

Hints. As first step, prove the following: If $U \subset H^n$ is an open neighbourhood of 0 in H^n, then $U \setminus \{0\} \simeq U$.

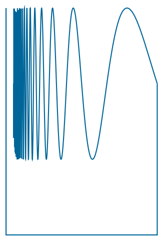
Exercise 4 (the Möbius strip is not boring)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds.

1. Show that the Möbius strip is not homeomorphic to $S^1 \times [0, 1]$.

   Hints. Use Exercise 3.

2. Is the Möbius strip homotopy equivalent to $S^1 \times [0, 1]$? Justify!

Bonus problem (Warsaw circle/Warschauer Kreis)**.** The topological space



$$W := \big\{(x, \sin(2 \cdot \pi/x)) \bigm| x \in (0, 1]\big\} \cup \big(\{1\} \times [-2, 0]\big) \cup \big([0, 1] \times \{-2\}\big) \cup \big(\{0\} \times [-2, 1]\big)$$

(endowed with the subspace topology of $\mathbb{R}^2$) is called *Warsaw circle*. Prove that for every basepoint $w_0 \in W$ the fundamental group $\pi_1(W, w_0)$ is trivial.

Hints. Show first that no loop in W can cross the "gap."

Bonus problem (Nash equilibria)**.** Look up the notion of Nash equilibria in Game Theory. Prove the existence of Nash equilibria using the Brouwer fixed point theorem.

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Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 5, November 16, 2021

Hints. When writing up your solutions, for each problem, first briefly explain the underlying idea and then carry out the arguments in detail.

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (injectivity/surjectivity and π_1)**.** Let $f\colon (X, x_0) \longrightarrow (Y, y_0)$ be a pointed continuous map between pointed spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If f is injective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is injective.
2. If f is surjective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is surjective.

Exercise 2 (π_0 and path-connected components)**.** Look up the definition of path-connected components of topological spaces. Let (X, x_0) be a pointed space and let $\mathrm{PC}(X)$ be the set of path-connected components of X. Prove that the following map is a well-defined bijection:

$$\begin{aligned} \pi_0(X, x_0) &\longrightarrow \mathrm{PC}(X) \\ [\gamma]_* &\longmapsto [\gamma(-1)] \end{aligned}$$

Exercise 3 (π_1 and contractibility)**.** Let X be a topological space.

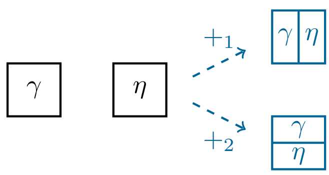
1. Let $\gamma\colon S^1 \longrightarrow X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that then $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$. Illustrate your argument in a suitable way!
2. Conclude: If X is contractible (but not necessarily pointedly contractible!) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Exercise 4 (fundamental theorem of algebra)**.** Use $\pi_1(S^1, e_1)$ to prove that every non-constant polynomial in $\mathbb{C}[X]$ has at least one root in $\mathbb{C}$.
Hints. Show that a non-constant polynomial $p \in \mathbb{C}[X]$ without roots in $\mathbb{C}$ would yield a map $S^1 \longrightarrow S^1$ that is both null-homotopic and homotopic to the map $S^1 \longrightarrow S^1$, $[t] \mapsto [\deg p \cdot t \mod 1]$, ...

Bonus problem (group structure on π_n)**.** Let $n \in \mathbb{N}_{\geq 2}$ and let the composition maps $+_1, \dots, +_n\colon \pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$ be defined as in Outlook 2.1.6.

1. Prove that $+_j = +_1$ for all $j \in \{1, \dots, n\}$.
2. Prove that $\pi_n(X, x_0)$ is an Abelian group with respect to $+_1$.

Hints. Eckmann and Hilton might help!



Submission before November 23, 2021, 8:30, via GRIPS

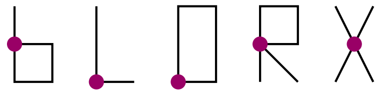
Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 6, November 23, 2021

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (bLORX!)**.** We consider the following five subspaces of $\mathbb{R}^2$, with the indicated basepoints:



Which of the following statements are true? Justify your answer with a suitable proof or counterexample (also, explain what your maps do).

1. These spaces consist of exactly four homeomorphism types.
2. These spaces consist of exactly four pointed homotopy types.

Exercise 2 (pushouts of groups)**.** We consider pushout diagrams in Group of the following type:

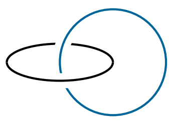
$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"', "\cong"] & B \arrow[d] \\ C \arrow[r] & G \end{tikzcd}$$

(where A, B, C, F, G are groups, $f\colon A \longrightarrow B$ is a group homomorphism, and $g\colon A \longrightarrow C$ is a group isomorphism). Prove the following statements via the universal property of pushouts:

1. The group F is *not* Abelian.
2. The group G is isomorphic to B.

Exercise 3 (fundamental group of the real projective plane)**.** Use the theorem of Seifert and van Kampen to prove that $\pi_1(\mathbb{R}P^2, [e_1]) \cong \mathbb{Z}/2$ and describe a generating loop explicitly.

Exercise 4 (Houdini?)**.** A rubber ring and a steel ring are entangled in $\mathbb{R}^3$ as illustrated below. Moreover, you may assume that the rubber ring is fixed at one point at all times.



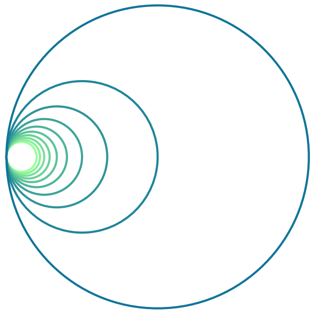
Can these two rings be unlinked in $\mathbb{R}^3$ by deforming the rubber ring (without cutting it)?

1. Model the situation above, using appropriate topological terms.
2. Answer the question above (with a full proof) with respect to your model.
3. *Bonus.* What changes if we look at the same rings in $\mathbb{R}^4$ instead of $\mathbb{R}^3$?

Illustrate your arguments in a suitable way!

Please turn over

Bonus problem (Hawaiian earring)**.** We consider the following subspace H of $\mathbb{R}^2$ with the subspace topology, the so-called *Hawaiian earring*:



$$H := \bigcup_{n \in \mathbb{N}_{>0}} \big\{ x \in \mathbb{R}^2 \bigm| d\big(x, (1/n, 0)\big) = 1/n \big\}$$

Prove that $\pi_1(H, 0)$ is uncountable; illustrate your arguments in a suitable way.

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 7, November 30, 2021

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (coverings)**.** Let X, Y, Z be topological spaces and let $p \colon X \longrightarrow Y$, $q \colon Y \longrightarrow Z$ be continuous maps. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $q \circ p$ is a covering map, then also q is a covering map.
2. If $q \circ p$ is a covering map, then also p is a covering map.

Exercise 2 (Klein bottle/Kleinsche Flasche)**.** Let K be the *Klein bottle*, i.e., the quotient space $K = ([0,1] \times [0,1])/\sim$ defined by the following glueing relation:

1. Show that the fundamental group of K (at the basepoint $([0],[0])$) is non-Abelian.
2. Show that there exists a 2-sheeted covering $S^1 \times S^1 \longrightarrow K$. Draw it!

Exercise 3 (pretzel coverings)**.** Let $(B, b) := (S^1, e_1) \vee (S^1, e_1)$. Solve one of the following:

1. Construct (and draw) two connected 2-sheeted coverings of (B, b) that are *not* isomorphic in $\mathsf{Cov}_{(B,b)}$ (and prove that they are not isomorphic).
2. Construct (and draw) a connected 3-sheeted covering of (B, b) whose deck transformation group does *not* act transitively on the fibres (and prove this fact).

Exercise 4 (coverings of wild spaces)**.** Solve one of the following:

1. Show that the Warsaw circle (Bonus problem of Sheet 4) admits a non-trivial covering. Draw it!

   Hints. Warsaw helix!

2. Show that the Hawaiian earring (Bonus problem of Sheet 6) admits *no* covering with simply connected (non-empty) total space.

Please turn over

Bonus problem (one-dimensional complexes I). A *one-dimensional complex* is a pair (X, X_0), consisting of a topological space X and a discrete subspace X_0 with the following property: There exists a set I and a pushout (in Top) of the form

$$\begin{tikzcd} \coprod_I S^0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \coprod_I D^1 \arrow[r] & X \end{tikzcd}$$

where the left vertical arrow is the canonical inclusion and the right vertical arrow is the inclusion of X_0 into X. I.e., one-dimensional complexes can be obtained by glueing intervals at their end-points.

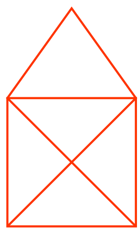
Such a one-dimensional complex is *finite* if both X_0 and I are finite.

1. Let (X, X_0) be a one-dimensional complex and let $x_0 \in X_0$. What does the spread of infectious deseases have to do with $\pi_0(X, x_0)$?

   Hints. Take X_0 as a set of people, take $\coprod_I D^1$ and the glueing maps as ...

2. Find an algorithm that given a finite one-dimensional complex (X, X_0) determines whether X is path-connected or not. In particular, explain how you represent (X, X_0) by finite data, why your algorithm terminates and why your algorithm produces the correct result.

Nikolaus problem (Haus des Nikolaus, all covered). The traditional *Haus des Nikolaus* is the following subspace of $\mathbb{R}^2$:



The new edition of this house will be built by the Blorx Building Trust (which won this contract in a transparent, corruption-free procedure).

Given $n \in \mathbb{N}$, the Blorx Building Trust will construct a path-connected n-sheeted covering space of the Haus des Nikolaus. Write a LaTeX macro \nikolaus with one argument such that \nikolaus{n} draws a beautiful path-connected n-sheeted covering of the Haus des Nikolaus. Execute \nikolaus{8}.

Hints. If you have never seen the Haus des Nikolaus:
http://www.mathematische-basteleien.de/house.html

Submission before December 7, 2021, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 8, December 7, 2021

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \big[[t] \mapsto [d \cdot t \mod 1]\big]_* \end{aligned}$$

Exercise 1 (coverings)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. There exists a covering map $\mathbb{R}P^2 \longrightarrow S^1 \times S^1$.
2. There exists a covering map $\mathbb{C} \setminus \{0, 1\} \longrightarrow \mathbb{C} \setminus \{0\}$.

Exercise 2 (counting sheets)**.** Let $X := S^{2020} \times \mathbb{R}P^{2021} \times \mathbb{R}P^{2022}$. Use the π_1-action on the fibers of covering maps to show that there are no path-connected coverings of X that have

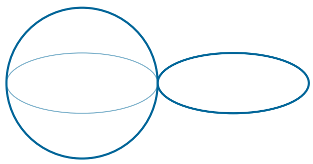
1. infinitely many sheets; or
2. exactly three sheets.

Exercise 3 (The Borsuk-Ulam theorem in dimension 2)**.** A map $f\colon S^2 \longrightarrow S^1$ is *antipodal* if $f(-x) = -f(x)$ holds for all $x \in S^2$.

1. Show that there is no continuous antipodal map $S^2 \longrightarrow S^1$.

   Hints. Try to use $\mathbb{R}P^2$ and consider a path in S^2 from e_1 to $-e_1$.
2. Prove the *Borsuk-Ulam theorem in dimension* 2: If $f\colon S^2 \longrightarrow \mathbb{R}^2$ is continuous, then there exists $x \in S^2$ with $f(x) = f(-x)$.

Exercise 4 (large homotopy groups)**.** Let $(X, x_0) := (S^2, e_1) \vee (S^1, e_1)$.



1. Give a simple geometric description of the universal covering of X and prove that this indeed is a universal covering of X.
2. Let $k \in \mathbb{N}_{\geq 2}$ with the property that $\pi_k(S^2, e_1^2)$ is non-trivial. Prove that then $\pi_k(X, x_0)$ is *not* finitely generated (as Abelian group).

Bonus problem (one-dimensional complexes II)**.** We consider one-dimensional complexes as in the Bonus problem on Sheet 7.

1. Let (X, X_0) be a one-dimensional complex and let $x_0 \in X_0$. Show that $\pi_1(X, x_0)$ is a free group.

   Hints. You may restrict to the finite case.
2. Let (X, X_0) be a one-dimensional complex and let $p\colon Y \longrightarrow X$ be a covering map. Show that then also $(Y, p^{-1}(X_0))$ is a one-dimensional complex.

Submission before December 14, 2021, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 9, December 14, 2021

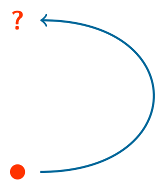
Exercise 1 (regular coverings?)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. All path-connected coverings of $\mathbb{R}P^{2021} \times \mathbb{R}P^{2022}$ are regular.
2. All path-connected coverings of $(\mathbb{R}P^{2021}, [e_1]) \vee (\mathbb{R}P^{2022}, [e_1])$ are regular.

Exercise 2 (no finite coverings)**.** Show that there exists a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space that is *not* simply connected and does *not* admit a finite-sheeted non-trivial covering.
Hints. Don't be irrational!

Exercise 3 (residually finite groups/residuell endliche Gruppen)**.** A group G is *residually finite* if the following holds: For every $g \in G \setminus \{e\}$ there exists a finite group F and a group homomorphism $\varphi \colon G \longrightarrow F$ with $\varphi(g) \neq e$.

1. Let (X, x_0) be a pointed space that admits a universal covering. Give an equivalent characterisation of residual finiteness of $\pi_1(X, x_0)$ in terms of coverings of (X, x_0) and (lifts of) paths.
2. Illustrate these concepts for the circle $(S^1, 1)$.



Exercise 4 (the Heisenberg manifold)**.** We consider the *Heisenberg group*

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset H_{\mathbb{R}} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3(\mathbb{R})$$

and the *Heisenberg manifold* $M := H \setminus H_{\mathbb{R}}$, where H acts on $H_{\mathbb{R}}$ by matrix multiplication. Solve two of the following questions and give detailed proofs for your answers:

1. What is the fundamental group of M ?
2. Is there a covering of the 3-torus by M ?
3. Is there a covering of M by the 3-torus ?
4. Does M admit a path-connected regular covering whose deck transformation group is isomorphic to $\mathbb{Z}^2$?

Please turn over

Bonus problem (classifying spaces of torsion-free groups). Let G be a group and let $\mathrm{P}_{\mathrm{fin}}(G)$ denote the set of all finite subsets of G. For $S \in \mathrm{P}_{\mathrm{fin}}(G)$ we write

$$\Delta_G(S) := \Bigl\{ f \colon G \longrightarrow [0,1] \Bigm| f|_{G \setminus S} = 0 \text{ and } \sum_{s \in S} f(s) = 1 \Bigr\}.$$

We write $\Delta(G) := \bigcup_{S \in \mathrm{P}_{\mathrm{fin}}(G)} \Delta_G(S)$ with the corresponding colimit topology (i.e., a subset of $\Delta(G)$ is closed if and only if the intersection with $\Delta_G(S)$ is closed for all $S \in \mathrm{P}_{\mathrm{fin}}(G)$, where $\Delta_G(S)$ carries the Euclidean topology). Furthermore, we consider the continuous left action

$$\begin{aligned} G \times \Delta(G) &\longrightarrow \Delta(G) \\ (g, f) &\longmapsto \bigl(h \mapsto f(g^{-1} \cdot h)\bigr) \end{aligned}$$

of G on $\Delta(G)$. Solve two of the following:

1. Show that $\Delta(G)$ is pointedly contractible with respect to every basepoint.
2. Determine the fundamental group of the quotient $G \setminus \Delta(G)$ in the case that the group G is torsion-free.
3. Extend this construction to an interesting functor $\mathsf{Group} \longrightarrow \mathsf{Top}$.

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 10, December 21, 2021

Hints. Let $\bullet = \{\emptyset\}$ denote "the" one-point space, let R be a ring with unit and let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Exercise 1 (more homology theories?)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample (a brief explanation is enough!).

1. Then $((h_{k+2022})_{k\in\mathbb{Z}}, (\partial_{k+2022})_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.
2. If R is an integral domain and Q is the quotient field of R, then also the pair $((Q \otimes_R h_k)_{k\in\mathbb{Z}}, (\mathrm{id}_Q \otimes_R \partial_k)_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_Q\mathsf{Mod}$.

Exercise 2 (reduced homology)**.** If X is a topological space and $k \in \mathbb{Z}$, then we define the *k-th reduced homology of X with respect to* $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ by

$$\widetilde{h}_k(X) := \ker\big(h_X(c_X)\colon h_k(X) \to h_k(\bullet)\big) \subset h_k(X),$$

where $c_X\colon X \longrightarrow \bullet$ is the constant map.

1. Show that for all topological spaces X, for all $x_0 \in X$, and for all $k \in \mathbb{Z}$ the composition

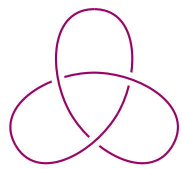
   $$\widetilde{h}_k(X) \longrightarrow h_k(X) \longrightarrow h_k(X, \{x_0\})$$

   of the inclusion and the homomorphism induced by the inclusion is an R-isomorphism.
2. Compute the reduced homology of contractible spaces.

Exercise 3 (knots)**.** An *embedded knot* is a smooth embedding $S^1 \longrightarrow \mathbb{R}^3$. The *knot complement* of an embedded knot $K\colon S^1 \longrightarrow \mathbb{R}^3$ is $\mathbb{R}^3 \setminus K(S^1)$.

1. Does it make sense to study embedded knots up to homotopy of maps?
2. Does it make sense to study embedded knots by considering ordinary homology of knot complements?

   Hints. You may use the following *tubular neighbourhood theorem*: If K is an embedded knot, then there exists a compact subset $N \subset \mathbb{R}^3$ and a homeomorphism $f\colon S^1 \times D^2 \longrightarrow N$ with $f(S^1 \times \{0\}) = K(S^1)$.



Please turn over

Exercise 4 (homology of the torus). Calculate the homology of the two-dimensional torus $T := S^1 \times S^1$ via the following strategy and illustrate your arguments in suitable way! Let $U \subset S^1$ be an appropriate open neighbourhood of $e_1 \in S^1$ and let $S := S^1 \times U \subset T$. Solve two of the following:

1. Use the long exact sequence of the pair and a topological argument to prove the following: For all $k \in \mathbb{Z}$ the inclusions $(T, \emptyset) \hookrightarrow (T, S)$ and $S \hookrightarrow T$ induce an R-isomorphism $h_k(T) \cong h_k(S) \oplus h_k(T, S)$.

2. Use excision to show that for all $k \in \mathbb{Z}$ we have R-isomorphisms

$$h_k(T, S) \cong h_k\big(S^1 \times [0,1], S^1 \times \{0,1\}\big).$$

3. Use the long exact triple sequence and excision to express the homology $h_k\big(S^1 \times [0,1], S^1 \times \{0,1\}\big)$ for all $k \in \mathbb{Z}$ in terms of the homology of S^1.

Bonus problem (tweet). Write a tweet (no more than 140 characters) stating the Seifert and van Kampen theorem, without using mathematical symbols.

Bonus problem (poster). Design a poster that advertises the classification theorem for coverings. Use colours!

Bonus problem (Klein tic-tac-toe). Write instructions on how to play tic-tac-toe on a Klein bottle. Illustrate your instructions with pictures.

Bonus problem (poetry).

And for the classically oriented mind:
don't leave the Brouwer fixed point theorem behind!
Theorem and proof need a formulation,
using rhymes as decoration.

Submission before January 11, 2022, 8:30, via GRIPS
Merry Christmas and a Happy New Year!

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 11, January 11, 2022

Hints. In the following, let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$.

Exercise 1 (surjectivity on homology)**.** Which of the following statements are true for all continuous maps $f \colon X \longrightarrow Y$ of spaces? Justify your answer with a suitable proof or counterexample.

1. If f is surjective, then also $h_{2022}(f) \colon h_{2022}(X) \longrightarrow h_{2022}(Y)$ is surjective.
2. If $h_{2022}(f) \colon h_{2022}(X) \longrightarrow h_{2022}(Y)$ is surjective, then f is surjective.

Exercise 2 (homology of the real projective plane/the Klein bottle)**.** Let K be the Klein bottle (Sheet 7, Exercise 2).

1. Compute $(h_k(\mathbb{R}P^2))_{k\in\mathbb{Z}}$ or $(h_k(K))_{k\in\mathbb{Z}}$.
2. What changes if the coefficients of $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ are isomorphic to $\mathbb{Z}/2$ instead of $\mathbb{Z}$?

Illustrate your arguments in a suitable way!

Exercise 3 (homology vs. homotopy equivalence)**.** Give examples of topological spaces X and Y such that $h_k(X) \cong_{\mathbb{Z}} h_k(Y)$ holds for all $k \in \mathbb{Z}$, but $X \not\simeq Y$.
Hints. Dimension 2 is sufficient.

Exercise 4 (algebraic Mayer-Vietoris sequence)**.** Let R be a ring with unit and let

$$\begin{tikzcd}
\cdots \arrow[r, "c_{k+1}"] & A_k \arrow[r, "a_k"] \arrow[d, "f_{A,k}"'] & B_k \arrow[r, "b_k"] \arrow[d, "f_{B,k}"'] & C_k \arrow[r, "c_k"] \arrow[d, "f_{C,k}"] & A_{k-1} \arrow[r, "a_{k-1}"] \arrow[d, "f_{A,k-1}"] & \cdots \\
\cdots \arrow[r, "c'_{k+1}"'] & A'_k \arrow[r, "a'_k"'] & B'_k \arrow[r, "b'_k"'] & C'_k \arrow[r, "c'_k"'] & A'_{k-1} \arrow[r, "a'_{k-1}"'] & \cdots
\end{tikzcd}$$

be a ($\mathbb{Z}$-indexed) commutative ladder in ${}_R\mathsf{Mod}$ with exact rows. Moreover, for every $k \in \mathbb{Z}$, let $f_{C,k} \colon C_k \longrightarrow C'_k$ be an isomorphism and let

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k \colon B'_k \longrightarrow A_{k-1}.$$

Show (via a diagram chase) that then the sequence

$$\cdots \xrightarrow{\ \Delta_{k+1}\ } A_k \xrightarrow{(f_{A,k},-a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\ \Delta_k\ } A_{k-1} \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact.

Bonus problem (Abelian categories)**.**

1. Look up the term *Abelian category* in the literature.
2. Give an example for an additive category that is *not* Abelian.
3. How can exact sequences be defined in Abelian categories?
4. What does the Freyd–Mitchell embedding theorem say?

Submission before January 18, 2022, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 12, January 18, 2022

Exercise 1 (small singular cycles)**.** Let X be a topological space, let $k \in \mathbb{N}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If $\sigma \in \operatorname{map}(\Delta^k, X) \subset C_k(X)$ is a singular cycle of X, then k is odd.
2. If $\sigma, \tau \in \operatorname{map}(\Delta^k, X)$ and $\sigma + \tau$ is a singular cycle of X, then k is odd.

Exercise 2 (algebraic Euler characteristic)**.** Let R be a ring with unit that admits a nice notion rk_R of rank for finitely generated R-modules (e.g., fields, principal ideal rings, ...). A chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is *finite* if for every $k \in \mathbb{Z}$ the R-module C_k is finitely generated and $\{k \in \mathbb{Z} \mid C_k \not\cong_R 0\}$ is finite. The *Euler characteristic* of a finite chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is defined by

$$\chi(C) := \sum_{k \in \mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R C_k.$$

Show that $\chi(C) = \sum_{k\in\mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R\bigl(H_k(C)\bigr)$ and explain which properties of rk_R you used in your arguments.

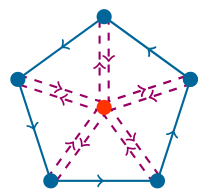
Exercise 3 (the ℓ^1-semi-norm on singular homology)**.** Let X be a topological space and let $k \in \mathbb{N}$. Let $|\cdot|_1$ be the ℓ^1-norm on $C_k(X;\mathbb{R})$ with respect to the $\mathbb{R}$-basis of $C_k(X;\mathbb{R})$ that consists of all singular k-simplices of X. We then define the *ℓ^1-semi-norm* $\|\cdot\|_1 \colon H_k(X;\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_k(X;\mathbb{R}),\ \partial_k c = 0,\ [c] = \alpha \in H_k(X;\mathbb{R})\bigr\}$$

for all $\alpha \in H_k(X;\mathbb{R})$.

1. Show that $\|\cdot\|_1$ is a semi-norm on $H_k(X;\mathbb{R})$.
2. Let $f \colon X \longrightarrow Y$ be a continuous map. Show that $\bigl\|H_k(f;\mathbb{R})(\alpha)\bigr\|_1 \leq \|\alpha\|_1$ holds for all $\alpha \in H_k(X;\mathbb{R})$.

Exercise 4 (singular homology in degree 1)**.** Let X be a path-connected, non-empty topological space. Let $\alpha \in H_1(X;\mathbb{Z})$. Show that there exists a continuous map $f \colon S^1 \longrightarrow X$ with $\alpha \in \operatorname{im} H_1(f;\mathbb{Z})$. Illustrate!



Bonus problem (realisation of homology groups)**.** Let $k \in \mathbb{N}_{>0}$. Construct a functor

$$R_k \colon {}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}} \longrightarrow \mathsf{Top_h}$$

with $h_k \circ R_k \cong \operatorname{Id}_{{}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}}$ and $h_\ell \circ R_k \cong 0$ for all $\ell \in \mathbb{N}_{>0} \setminus \{k\}$. Here, ${}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}$ denotes the category of all finitely generated $\mathbb{Z}$-modules.

Hints. Use spheres and mapping cones!

Submission before January 25, 2022, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 13, January 25, 2022

Exercise 1 (chain homotopy equivalence)**.** Let R be a principal ideal domain and let $C, D \in \mathrm{Ob}({}_R\mathsf{Ch})$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If C is finite and $C \simeq_{{}_R\mathsf{Ch}} D$, then also D is finite.
2. If C and D are finite with $C \simeq_{{}_R\mathsf{Ch}} D$, then $\chi(C) = \chi(D)$.

Hints. Finite chain complexes and the algebraic Euler characteristic were introduced in Exercise 2 on Sheet 12.

Exercise 2 (diameter of affine simplices)**.** Let $k \in \mathbb{N}$ and let $\sigma \colon \Delta^k \longrightarrow \mathbb{R}^\infty$ be an affine linear simplex. Show that every summand τ in the definition of the barycentric subdivision $B_k(\sigma)$ satisfies

$$\operatorname{diam}\bigl(\tau(\Delta^k)\bigr) \leq \frac{k}{k+1} \cdot \operatorname{diam}\bigl(\sigma(\Delta^k)\bigr).$$

Hints. For $A \subset \mathbb{R}^\infty$, we write $\operatorname{diam} A := \sup_{x,y \in A} \|x - y\|_2$. How can the diameter of affine linear simplices be expressed in terms of the vertices?

Exercise 3 (concrete cycles)**.** Give an example of a singular cycle in $C_2(S^2; \mathbb{Z})$ that represents a non-trivial class in $H_2(S^2; \mathbb{Z})$ and prove that this cycle indeed has this property. Illustrate!

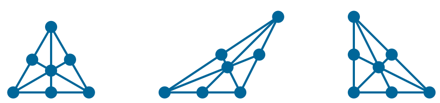
Exercise 4 (compatible homotopies)**.** Let X be a topological space, let $(S_k \subset \operatorname{map}(\Delta^k, X))_{k \in \mathbb{N}}$ be a family of simplices, and let $(h_\sigma)_{k \in \mathbb{N}, \sigma \in \operatorname{map}(\Delta^k, X)}$ be a family of homotopies with the following properties:

1. For each $k \in \mathbb{N}$ and each $\sigma \in \operatorname{map}(\Delta^k, X)$, the map $h_\sigma \colon \Delta^k \times [0,1] \longrightarrow X$ is a homotopy from σ to an element of S_k.
2. For all $k \in \mathbb{N}$, all $\sigma \in \operatorname{map}(\Delta^k, X)$, and all $j \in \{0, \dots, k\}$, we have
$$h_{\sigma \circ i_{k,j}} = h_\sigma \circ (i_{k,j} \times \mathrm{id}_{[0,1]}).$$
3. For all $k \in \mathbb{N}$ and all $\sigma \in S_k$, the homotopy h_σ satisfies
$$\forall_{x \in \Delta^k} \quad \forall_{t \in [0,1]} \quad h_\sigma(x,t) = \sigma(x).$$

Let $C^S(X) \subset C(X)$ be the subcomplex of the singular chain complex generated in each degree $k \in \mathbb{N}$ by S_k instead of $\operatorname{map}(\Delta^k, X)$.

Show that the inclusion $C^S(X) \longrightarrow C(X)$ is a chain homotopy equivalence in ${}_\mathbb{Z}\mathsf{Ch}$ (and thus induces an isomorphism in homology).

Bonus problem (barycentric subdivision)**.** Write a LaTeX-macro that draws the barycentric subdivision of affine 2-simplices (specified by their vertices):



Submission before February 1, 2022, 8:30, via GRIPS

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 14, February 1, 2022

Exercise 1 (separation theorems?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $f \colon S^1 \longrightarrow S^1 \times S^1$ is continuous and injective, then $(S^1 \times S^1) \setminus f(S^1)$ has exactly two path-connected components.
2. If $f \colon S^1 \longrightarrow \mathbb{R}^{2022}$ is continuous and injective, then $\mathbb{R}^{2022} \setminus f(S^1)$ is path-connected.

Exercise 2 (Slitherlink). A *Slitherlink puzzle* consists of a square grid; some of the squares have numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

- SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.
- SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares, the number of edges in the loop is not specified.
- SL 3 The loop does not have any self-intersections or branches.

Solve the following Slitherlink puzzle:

1		3			3	2					2
2		3			3					3	
			1		1			1			
		1					1	3	3	3	3
	3		3	3	3	3			1		
2							3				1

Exercise 3 (Slitherlink and JCT). How can the Jordan curve theorem be used to establish global strategies for solving Slitherlink puzzles (see Exercise 2)? Give an example that illustrates this strategy.

Exercise 4 (non-planarity of the torus).

1. Let $n \in \mathbb{N}$, let M be a compact, non-empty topological manifold of dimension n, let N be a connected topological manifold of dimension n, and let $f \colon M \longrightarrow N$ be continuous and injective. Show that f is surjective.
2. Conclude that there is *no* continuous injective map $S^1 \times S^1 \longrightarrow \mathbb{R}^2$.

Bonus problem (the five colour theorem).

1. Choose a book on graph theory from the library that contains a proof of the five colour theorem.
2. Where does the proof use (relatives of) the Jordan curve theorem? Is this made explicit?

Submission before February 8, 8:30, via GRIPS
This is the last regular exercise sheet. Sheet 15 will consist of bonus problems.

Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 15, February 8, 2022

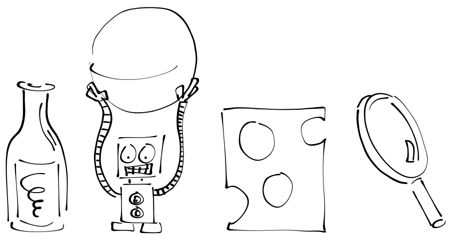
The case of Lord Polygoto. Detective Blorx is called to investigate the murder of Lord Polygoto at the Polygoto Mansion. Blorx quickly figures out that there are only four possible suspects (in alphabetic order):

- the butler
- the cook
- the gardener
- the librarian

Thorough questioning of the suspects revealed that, at the time of the murder, each of the suspects handled exactly one of the four suspicious items (A).

A. The four suspicious items.

- A Klein bottle.
- An ugly trophy made of steel (as depicted; the sphere on top is hollow, the statue is massive).
- A poisoned slice of Swiss cheese (as depicted).
- A magnifying lens (see schematics (D)).



B. Statements by witnesses.

- Drinking from the Klein bottle leads to a severe loss of orientation.
- The librarian or the gardener used the suspicious item with the maximal Euler characteristic.
- The gardener admires commutativity, especially in π_1. He would never touch anything non-commutative.
- The cook experimented with Betti spices and therefore handled an item such that $H_2(\,\cdot\,;\mathbb{Q})$ and $H_3(\,\cdot\,;\mathbb{Q})$ have different $\mathbb{Q}$-dimension.
- The butler never loses his way.

Please turn over

C. The coroner's report.

- The murder weapon admits a connected two-sheeted covering.
- The murder weapon has cyclic $H_1(\cdot ; \mathbb{Z})$.

D. Construction schematics of the magnifying lens.

- Let $G \subset \mathbb{C}^\times$ be the subgroup of the multiplicative group $\mathbb{C}^\times$ consisting of all 2021-th roots of unity.
- View S^3 as unit sphere in $\mathbb{C}^2$.
- Let $\mathbb{C}^\times$ act on S^3 by scalar multiplication in $\mathbb{C}^2$.
- The lens is then the quotient space $G \setminus S^3$.
- 80/SK 340 H361 contains further information on the assembly of lenses.

Hints. This construction is best performed during cellular Yoga while transcending projective space.

Exercise (16 credits)**.** Using the information collected in (A)–(D), help Blorx to answer the following questions:

1. Which item(s) is (are) the murder weapon(s)?
2. Who did it? More generally: Which suspect used which item?

Hints. The *Euler characteristic* of a finite CW-complex X is defined as the alternating sum

$$\chi(X) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \text{number of open } n\text{-cells of } X.$$

You may use that the Euler characteristic is homotopy invariant (among finite CW-complexes).

Bonus problem (pasta!)**.** During the investigation, Blorx discovered that the cook and the librarian secretly worked on a joint book project *Topologhetti*: A dictionary between pasta and (algebraic) topology. Can you imagine how such a dictionary could look like?
Hints. Spaghetti, bucatini, fusilli, tortiglioni, penne rigate, tortellini, ravioli, lasagne, farfalle, ...

Optional submission before February 15, 8:30, via GRIPS

C
Études

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 0, October 19, 2021

Exercise 1 (basic point-set topology).

1. What is the definition of a *topological space*?
2. Which inheritance properties do closed sets in a topological space have?
3. How does a metric induce a topology?
4. What is the definition of a *continuous map* between topological spaces?
5. Why is the composition of two (composable) continuous maps continuous?

Exercise 2 (T as in Topology). We consider

$$T := \bigl\{(t,0) \bigm| t \in [-1,1]\bigr\} \cup \bigl\{(0,t) \bigm| t \in [-1,0]\bigr\} \subset \mathbb{R}^2,$$

endowed with the subspace topology of $\mathbb{R}^2$.

1. Sketch T !
2. What is the definition of *compactness*?
3. Is the topological space T compact? Is $T \setminus \{(0,0)\}$ compact?
4. What is the definition of *path-connectedness*?
5. Is the topological space T path-connected? Is $T \setminus \{(0,0)\}$ path-connected?
6. Is T homeomorphic to $\mathbb{R}$? Or to S^1 ?

Exercise 3 (subspaces of $\mathbb{R}^3$). Give explicit subspaces (as sets, in coordinates) of $\mathbb{R}^3$ that are homeomorphic to the following spaces:

1. the standard 3-simplex Δ^3
2. the torus $S^1 \times S^1$
3. the Möbius strip

Check your formulas with a visualisation tool!

Exercise 4 (Library). Select five books on category theory and five books on point-set topology. For both topics, compare these books:

1. Which books contain many examples?
2. Which books focus on theory?
3. In which books can you find terms/theorems quickly?
4. Which books contain useful exercises?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 1, October 22, 2021

Exercise 1 (collapsing subspaces)**.** Sketch the following quotient spaces!

1. $[0,1]/\{0\}$
2. $[0,1]/[0,1]$
3. $[0,1]/\{0,1\}$
4. $[0,1]/\{0,1/2\}$
5. D^2/S^1
6. S^2/S^1

Exercise 2 (an exotic quotient?)**.** We consider $X := [0,1]/(\mathbb{Q}\cap[0,1])$, endowed with the quotient topology of the standard topology on $[0,1]$.

1. Is X compact?
2. Is X Hausdorff?
3. Is X path-connected?
4. Is X homeomorphic to S^1 ?

Exercise 3 (pushouts)**.** Determine each of the following pushout spaces and draw corresponding pictures!

$$\begin{array}{ccc}\{0,1\} & \xrightarrow{f} & [0,1] \\ {\scriptstyle f}\downarrow & & \downarrow \\ [0,1] & \longrightarrow & ?\end{array} \qquad \begin{array}{ccc}\{0,1\} & \xrightarrow{f} & [0,1] \\ {\scriptstyle g}\downarrow & & \downarrow \\ [0,1] & \longrightarrow & ?\end{array} \qquad \begin{array}{ccc}\{0,1\} & \xrightarrow{g} & [0,1] \\ {\scriptstyle g}\downarrow & & \downarrow \\ [0,1] & \longrightarrow & ?\end{array}$$

Here, f and g are defined as follows:

$$\begin{aligned} f\colon \{0,1\} &\longrightarrow [0,1] \\ x &\longmapsto x \\ g\colon \{0,1\} &\longrightarrow [0,1] \\ x &\longmapsto 0. \end{aligned}$$

Exercise 4 (summary)**.** Write a summary of Chapter 1.1 (Topological building blocks), keeping the following questions in mind:

1. Which construction principles for the construction of topological spaces do you know?
2. (How) Can one construct continuous maps to/from such constructions? Which universal properties do they have?
3. How can one model classical examples of geometric objects with these constructions?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 2, October 29, 2021

Exercise 1 (functors on pairs of spaces)**.** Give reasonable definitions of functors $\mathsf{Top}^2 \longrightarrow \mathsf{Top}$ that deserve the following names:

1. subspace functor
2. ambient space functor
3. quotient space functor

Exercise 2 (morphisms in the simplex category)**.** In the simplex category Δ, describe the following sets of morphisms explicitly and draw schematic pictures of these morphisms:

1. $\mathrm{Mor}_\Delta\bigl(\Delta(1), \Delta(2)\bigr)$
2. $\mathrm{Mor}_\Delta\bigl(\Delta(0), \Delta(2)\bigr)$
3. $\mathrm{Mor}_\Delta\bigl(\Delta(2), \Delta(1)\bigr)$
4. $\mathrm{Mor}_\Delta\bigl(\Delta(2), \Delta(2)\bigr)$

Exercise 3 (homotopy)**.** Describe for the following types of maps all possible homotopy classes:

1. $S^0 \longrightarrow [0,1]$
2. $S^0 \longrightarrow S^0$
3. $[0,1] \longrightarrow S^0$
4. $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Exercise 4 (summary)**.** Write a summary of Chapter 1.2 (Categories and Functors), keeping the following questions in mind:

1. Which examples of categories/functors did you encounter in other courses?
2. Why are categories/functors important in Algebraic Topology?
3. What properties do representable functors have?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 3, November 5, 2021

Exercise 1 (homotopy)**.** Let X be a topological space and let $f \in \operatorname{map}(X, X)$. Prove or disprove:

1. If $f \simeq \mathrm{id}_X$, then f is bijective.
2. If $f \simeq \mathrm{id}_X$, then $f \circ f \simeq f$.
3. If $f \circ f \simeq f$, then $f \simeq \mathrm{id}_X$.
4. If $f^{2021} \simeq \mathrm{id}_X$, then f is a homotopy equivalence.

Exercise 2 (homotopy invariant functors)**.** Let $F\colon \mathsf{Top} \longrightarrow \mathsf{Ab}$ be a homotopy invariant functor. Prove or disprove:

1. $F(D^{2021}) \cong_{\mathsf{Ab}} F(\mathbb{R}^{2022})$.
2. $F(S^{2021}) \not\cong_{\mathsf{Ab}} \mathbb{Z}^{2021}$.
3. $F(\mathbb{R}P^{2021}) \cong_{\mathsf{Ab}} F(\mathbb{R} \times \mathbb{R}P^{2021})$.
4. If X is contractible, then $F(X) \cong_{\mathsf{Ab}} \{0\}$.
5. If $F(\mathbb{R}P^{2021}) \cong_{\mathsf{Ab}} \mathbb{Z}$, then $F(S^{2021}) \not\cong_{\mathsf{Ab}} \mathbb{Z}$.

Hints. Recall that all of these problems are easy!

Exercise 3 (classification problem)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds. Classify the following spaces up to homeomorphism/homotopy equivalence.

1. $\mathbb{R}^{2021}$
2. $\mathbb{R}^{2022}$
3. S^{2021}
4. D^{2022}
5. $S^0 \times S^{2021}$
6. $S^0 \times S^{2022}$

Exercise 4 (summary)**.** Write a summary of Chapter 1.3 (Homotopy and Homotopy Invariance), keeping the following questions in mind:

1. What is homotopy/homotopy equivalence?
2. What are basic examples?
3. What is homotopy invariance?
4. How can homotopy invariance be used?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 4, November 12, 2021

Exercise 1 (concatenation of loops). Let $\gamma := \mathrm{id}_{S^1} \colon (S^1, e_1) \longrightarrow (S^1, e_1)$. Draw pictures of the following loops:

$$\overline{\gamma}, \qquad \gamma * \gamma, \qquad \gamma * (\gamma * \gamma), \qquad \gamma * \overline{\gamma}, \qquad (\gamma * \overline{\gamma}) * \gamma$$

Exercise 2 (loops in spheres).

1. Let $\gamma \colon (S^1, e_1) \longrightarrow (S^2, e_1)$ be the inclusion as equator. Show that $[\gamma]_*$ is trivial in $\pi_1(S^2, e_1)$.
2. Let

$$\begin{aligned} \gamma \colon (S^1, e_1) &\longrightarrow \bigl(S^1 \times S^1, (e_1, e_1)\bigr) \\ [t] &\longmapsto \bigl([t], [2 \cdot t]\bigr), \end{aligned}$$

and let $p_1, p_2 \colon S^1 \times S^1 \longrightarrow S^1$ be the canonical projections. Compute $\pi_1(p_1)([\gamma]_*)$ and $\pi_1(p_2)([\gamma]_*)$ and draw the corresponding pictures!

Exercise 3 (pushouts).

1. Let X be a topological space, let X_1, X_2 be subspaces of X with $X = X_1^\circ \cup X_2^\circ$, and let $X_0 := X_1 \cap X_2$. Show that X (together with the inclusions) is a pushout in Top of the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\text{inclusion}"] \arrow[d, "\text{inclusion}"'] & X_2 \\ X_1 & \end{tikzcd}$$

2. Let $X_1 := S^2 \setminus \{e_2\}$, $X_2 := S^2 \setminus \{-e_2\}$ and $X_0 := X_1 \cap X_2$. Is (S^2, e_1) a pushout of

$$\begin{tikzcd} (X_0, e_1) \arrow[r, "{[\text{inclusion}]_*}"] \arrow[d, "{[\text{inclusion}]_*}"'] & (X_2, e_1) \\ (X_1, e_1) & \end{tikzcd}$$

in $\mathsf{Top}_{*\mathsf{h}}$ (provided that (S^2, e_1) is *not* pointedly contractible)?!

Exercise 4 (summary). Write a summary of Chapter 2.1 (The Fundamental Group), keeping the following questions in mind:

1. How are the functors π_n defined?
2. What is the geometric idea behind π_n ?
3. How is the group structure on π_1 defined?
4. Why/How is the basepoint relevant?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 5, November 19, 2021

Exercise 1 (homotopyc)**.** Let (X, x_0) and (Y, y_0) be pointed spaces and let $f, g \in \operatorname{map}_*((X, x_0), (Y, y_0))$. Find the correct bijection!

$X \simeq Y$	... and ... are pointedly homotopic
$(X, x_0) \simeq_* (Y, y_0)$	... and ... are homotopic
$f \simeq g$	... and ... are pointedly homotopy equivalent
$f \in [g]_*$	... and ... are homotopy equivalent

Exercise 2 (induced homomorphisms on π_1)**.** Illustrate the effect on π_1 of the homomorphisms induced by the following maps $(S^1, e_1) \longrightarrow (S^1, e_1)$:

1. $[t] \longmapsto [t]$
2. $[t] \longmapsto [2 \cdot t \mod 1]$
3. $[t] \longmapsto [-t \mod 1]$
4. $[t] \longmapsto [t^2 \mod 1]$

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 3 (Seifert and van Kampen)**.** For which of the following topological spaces X and subspaces X_1, X_2 are the hypotheses of the theorem of Seifert and van Kampen satisfied?

X	X_1	X_2
$\mathbb{R}$	$\mathbb{R}$	$\{2021\}$
$\mathbb{R}$	$(-\infty, 0)$	$(0, \infty)$
$\mathbb{R}$	$(-\infty, 0]$	$[0, \infty)$
$\mathbb{R}$	$\mathbb{R} \setminus \{0\}$	$(-1, 1)$
$\mathbb{R}$	$(-\infty, 1]$	$[-1, \infty)$
$\mathbb{R}^2$	$\mathbb{R}^2 \setminus \{0\}$	D^2
$\mathbb{R}^2$	$\mathbb{R}^2$	S^1

Exercise 4 (pushouts of groups)**.** Do there exist pushout diagrams in Group of the following shapes? If yes, determine one way to define the appropriate maps.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\ \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[d] \\ 1 \arrow[r] & \mathbb{Z}/2 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & \mathbb{Z} \end{tikzcd} \quad \begin{tikzcd} \mathbb{Z} \arrow[r, "5"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/5 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[d] & S_3 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 6, November 26, 2021

Exercise 1 (coverings of the circle). Find three pairwise non-isomorphic 3-sheeted coverings of S^1 and illustrate these coverings in a suitable way!

Exercise 2 (covering maps?). Illustrate the following maps in a suitable way! Which of them are covering maps and how many sheets do they have?

1. $\mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}, \quad x \longmapsto x^2$
2. $\mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}_{>0}, \quad x \longmapsto x^2$
3. $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}, \quad z \longmapsto z^{2021}$
4. $S^1 \times \mathbb{R} \longrightarrow S^1 \times S^1, \quad ([x], y) \longmapsto ([x], [y])$

Exercise 3 (covering maps from group actions?). Which of the following group actions are properly discontinuous? Determine the corresponding quotient spaces!

1. the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$ by matrix multiplication
2. the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane by Möbius transformations
3. the action of $\mathbb{Z}/2021$ on S^1, where $[1] \in \mathbb{Z}/2021$ acts via
$$\begin{aligned} S^1 &\longrightarrow S^1 \\ [x] &\longmapsto [x + 1/2021 \mod 1] \end{aligned}$$
4. the action of $\mathbb{Z}/2$ on $S^1 \times S^1$, where $[1] \in \mathbb{Z}/2$ acts via
$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ ([x], [y]) &\longmapsto ([y], [x]) \end{aligned}$$
5. the action of $\mathbb{Z}/2$ on $S^1 \times S^1$, where $[1] \in \mathbb{Z}/2$ acts via
$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ ([x], [y]) &\longmapsto ([x + 1/2 \mod 1], [y]) \end{aligned}$$
6. the action of $\mathbb{Z}/2$ on $S^1 \times S^1$, where $[1] \in \mathbb{Z}/2$ acts via
$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ ([x], [y]) &\longmapsto ([1 - x \mod 1], [y]) \end{aligned}$$

Exercise 4 (summary). Write a summary of Chapter 2.2 (Divide and Conquer), keeping the following questions in mind:

1. Which types of constructions of spaces are compatible with π_1 ?
2. Which of the results carry over easily to higher homotopy groups?
3. What are the main ideas of the corresponding proofs?
4. What are the main examples?
5. What are the limits of computability of fundamental groups?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 7, December 3, 2021

Exercise 1 (lifts of loops). We consider the following covering maps

$$\begin{aligned} p\colon \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto [t \mod 1] \\ q\colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [2 \cdot t \mod 1] \\ r\colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [3 \cdot t \mod 1] \end{aligned}$$

and the (pointed) loop

$$\begin{aligned} \gamma\colon (S^1, e_1) &\longrightarrow (S^1, e_1) \\ [t] &\longmapsto [t]. \end{aligned}$$

Which of the following *loops* admit a p-lift, a q-lift, or an r-lift?

$$\gamma, \quad \gamma * \overline{\gamma}, \quad \gamma * \gamma, \quad \gamma^{*2020}, \quad \gamma^{*2021}, \quad \gamma^{*2022}$$

Exercise 2 (deck transformations). Give examples of non-trivial deck transformations of the following covering maps!

1. $\mathbb{R} \longrightarrow S^1$, $t \longmapsto [t \mod 1]$
2. $S^1 \longrightarrow S^1$, $[t] \longmapsto [2 \cdot t \mod 1]$
3. $S^2 \longrightarrow \mathbb{R}P^2$, $x \longmapsto \{x, -x\}$

Exercise 3 (non-coverings). Why are there no coverings of the following types?

1. $S^1 \longrightarrow \mathbb{R}$
2. $\mathbb{R}P^2 \longrightarrow S^2$
3. $S^2 \longrightarrow S^1$
4. $\mathbb{R}^2 \setminus \{-e_1, e_1\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$

Exercise 4 (SchnüffelTron3000).

1. The Blorxian Space Agency launched the satellite SchnüffelTron3000 that constantly monitors the location of all cars on the surface of Earth (i.e., their longitude and latitude, but no height information). Which information on the actual location of a car on a circular car park ramp can be derived from such information? How does this relate to covering theory?

2. How does this relate to the branches of the logarithm function in complex analysis?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 8, December 10, 2021

Exercise 1 (classification of coverings). Apply the classification theorem of coverings to the following spaces and give geometric description of each class of coverings:

1. $\mathbb{R}P^{2021}$
2. $S^1 \times \mathbb{R}P^{2021}$

Exercise 2 (non-trivial coverings). Which of the following spaces admit non-trivial coverings?

1. $\mathbb{R}^2 \setminus \{0\}$
2. $\mathbb{R}^{2021} \setminus \{0\}$
3. $S^{2021} \setminus \{e_1, -e_1\}$
4. $\mathbb{R}^2 \setminus \{e_1, -e_1\}$

Exercise 3 ("random" coverings). Let $(T, t_0) := (S^1, e_1) \times (S^1, e_1)$.

1. Roll four dice; let a, b, c, d be the results and let
$$H := \operatorname{Span}_{\mathbb{Z}} \left\{ \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}, \begin{pmatrix} c-1 \\ d-1 \end{pmatrix} \right\} \subset \mathbb{Z}^2.$$
2. Choose an isomorphism $\pi_1(T, t_0) \cong_{\mathsf{Group}} \mathbb{Z}^2$ and consider the subgroup H' corresponding to H under this isomorphism.
3. Draw the path-connected, pointed, covering of (T, t_0) associated with H'.
4. Iterate!
5. What is the probability that the resulting total space is homeomorphic to $\mathbb{R}^2$?

Exercise 4 (exact sequences). Refresh your memory of the following algebraic terms (Appendix A.6.1):

1. (short) exact sequence
2. split exact sequence
3. five lemma
4. flat module

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 9, December 17, 2021

Exercise 1 (exact sequences)**.** Which of the following sequences of $\mathbb{Z}$-modules are exact?

1. $\mathbb{Z} \xrightarrow{2021} \mathbb{Z} \xrightarrow{2021} \mathbb{Z}$
2. $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$
3. $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2021} \mathbb{Z}$
4. $\mathbb{Z} \xrightarrow{2021} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

Exercise 2 (long exact sequences)**.** Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 and let (X, A) be a pair of spaces.

1. Write down the long exact sequence of this pair with respect to the homology theory $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$. What can you conclude from this sequence if X is contractible? What if A is contractible?
2. Apply this to $(\{0\}, \{0\})$.
3. Apply this to $(\mathbb{R}^2, \mathbb{R}^2 \setminus S^1)$.
4. Apply this to $(S^1, \{e_1\})$.

Exercise 3 (excision)**.** Which of the following pairs of spaces are related by excision (as in the excision axiom)?

1. $(\mathbb{R}^{2021}, \{0\})$ and $(\mathbb{R}^{2021} \setminus \{0\}, \emptyset)$
2. $(\mathbb{R}^2, S^1)$ and $(\mathbb{R}^2 \setminus \{0\}, \emptyset)$
3. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$ and $([0,1] \times [0,1], ([0,1] \times [0,1]) \setminus \{0\})$
4. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{2 \cdot e_1\})$ and $(\mathbb{R}^2 \setminus D^2, \mathbb{R}^2 \setminus (D^2 \cup \{2 \cdot e_1\}))$
5. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{e_1\})$ and $(\mathbb{R}^2 \setminus D^2, \mathbb{R}^2 \setminus D^2)$
6. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0, 2 \cdot e_1\})$ and $(D^2, D^2 \setminus \{0\})$

Exercise 4 (summary)**.** Write a summary of Chapter 2.3 (Covering Theory) and Chapter 2.4 (Applications), keeping the following questions in mind:

1. What are important examples of (non-trivial) coverings?
2. Which lifting properties do coverings have? Why?
3. Why are coverings compatible with homotopy groups?
4. How can coverings be classified?
5. How can covering theory be used to compute fundamental groups?
6. Which applications do fundamental groups and covering theory have?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 10, January 7, 2022

Exercise. Before you start: Recall the Eilenberg–Steenrod axioms for homology theories!

Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$ [alternatively: $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}/2$ or $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Q}$].

Exercise 1 (suspension)**.** Draw the suspensions of the following spaces and compute their (ordinary) homology:

1. $\{0, 1, 2\} \subset \mathbb{R}$
2. D^2
3. $S^1 \sqcup S^1$
4. $(S^1, e_1) \vee (S^1, e_1)$

Exercise 2 (chain complexes and their homology)**.** Recall the notion of chain complexes, chain maps, and their homology (Appendix A.6.2):

1. What is the definition of chain complexes and chain maps?
2. What are typical examples?
3. What is the homology of a chain complex?
4. How can homology be computed?
5. How does all this relate to exactness?
6. Why did we introduce chain complexes in Commutative Algebra?

Exercise 3 (summary)**.** Write a summary of Chapter 3.1 (The Eilenberg–Steenrod Axioms), keeping the following questions in mind:

1. What do the axioms mean geometrically?
2. How can the axioms be used in computations?

Exercise 4 (summary)**.** Write a summary of Chapter 3.2 (Homology of Spheres and Suspensions) keeping the following questions in mind:

1. How can the homology of spheres/suspensions be computed?
2. What can you say about mapping degrees for self-maps of spheres (in ordinary homology)?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 11, January 14, 2022

Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$ [alternatively: $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}/2$ or $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Q}$].

Exercise 1 (Mayer–Vietoris). Let $k \in \mathbb{Z}$.

1. Compute $h_k(S^2)$ via a suitable Mayer–Vietoris sequence from $h_k(S^1)$.
2. Compute $h_k(S^1 \times S^1)$ via a Mayer–Vietoris sequence associated with the decomposition of the torus $S^1 \times S^1$ into two cylinders.

Exercise 2 (mapping cones). Let

$$\begin{aligned} f_3 \colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [3 \cdot t \mod 1] \end{aligned}$$

and let $k \in \mathbb{Z}$.

1. Compute $h_k(\operatorname{Cone}(f_3))$.
2. Compute $h_k(\Sigma \operatorname{Cone}(f_3))$.

Exercise 3 (chain complexes and their homology). Recall the notion of chain complexes, chain maps, and their homology (Appendix A.6.2):

1. What is the definition of chain complexes and chain maps?
2. What are typical examples?
3. What is the homology of a chain complex?
4. How can homology be computed?
5. How does all this relate to exactness?
6. Why did we introduce chain complexes in Commutative Algebra?

Exercise 4 (summary). Write a summary of Chapter 3.3 (Glueings: The Mayer–Vietoris Sequence), keeping the following questions in mind:

1. How can the Mayer-Vietoris sequence be used to compute homology of glueings?
2. What is the mapping cone trick?
3. How can relative homology be viewed as absolute homology?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 12, January 21, 2022

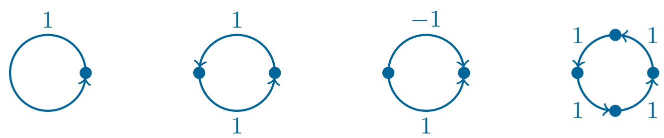
Exercise 1 (singular chains on the torus). We consider the usual description of the torus T as quotient of the unit square $[0,1]\times[0,1]$ (Figure 1.6). In the unit square, we use the following notation for singular 2-simplices: If $v_0, v_1, v_2 \in [0,1]\times[0,1]$, then we consider the associated affine linear 2-simplex

$$\begin{aligned}[v_0, v_1, v_2] \colon \Delta^2 &\longrightarrow [0,1]\times[0,1] \\ (t_0, t_1, t_2) &\longmapsto t_0 \cdot v_0 + t_1 \cdot v_1 + t_2 \cdot v_2.\end{aligned}$$

Which of the following singular chains are cycles in $C_2([0,1]\times[0,1];\mathbb{Z})$? Which of them describe cycles in $C_2(T;\mathbb{Z})$? In $C_2(T;\mathbb{Z}/2)$? Illustrate!

1. $1 \cdot [(0,0),(0,1),(1,1)]$
2. $1 \cdot [(0,0),(0,1),(1,1)] + 1 \cdot [(0,0),(1,0),(1,1)]$
3. $1 \cdot [(0,0),(0,1),(1,1)] - 1 \cdot [(0,0),(1,0),(1,1)]$
4. $1 \cdot [(0,0),(0,1),(1,1)] + 1 \cdot [(0,0),(1,1),(1,0)]$

Exercise 2 (singular homology classes on the circle). Show that the following singular 1-cycles on S^1 all represent the same class in $H_1(S^1;\mathbb{Z})$ (where all the paths are parametrised at constant speed).



Exercise 3 (homology of chain complexes). Compute the homology of the following chain complexes of $\mathbb{Z}$-modules:

1. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z}^2 \xrightarrow{\;0\;} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$
2. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$
3. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\;2\;} \mathbb{Q} \xrightarrow{\;0\;} \mathbb{Q} \longrightarrow 0 \longrightarrow \cdots$
4. $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/2022 \xrightarrow{\;3\;} \mathbb{Z}/2022 \xrightarrow{\;674\;} \mathbb{Z}/2022 \longrightarrow 0 \longrightarrow \cdots$

Exercise 4 (summary). Write a summary of Chapter 4.1 (Construction), keeping the following questions in mind:

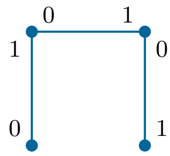
1. What is the geometric idea behind singular homology?
2. Which algebraic objects are used to implement this geometric idea?
3. How can one manipulate singular chains/cycles/boundaries?

No submission!

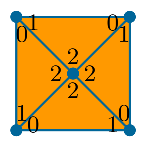
Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 13, January 28, 2022

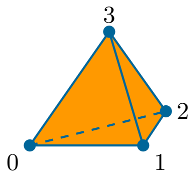
Exercise 1 (barycentric subdivision). Draw the barycentric subdivision of the following singular chains (assuming that everything is parametrised affinely and that every singular simplex has coefficient 1)!



1-chain



2-chain



3-chain

Exercise 2 (Jordan curve theorem, low dimensions).

1. Does the Jordan curve theorem hold in dimension 0 ?
2. Does the Jordan curve theorem hold in dimension 1 ?
3. Is there a Jordan curve theorem for continuous injective maps $S^1 \longrightarrow D^2$?

Exercise 3 (Jordan curve theorem, crayon version). Which of the following subspaces of $\mathbb{R}^2$ are homeomorphic to S^1 ? Why?

Exercise 4 (summary). Write a summary of Chapter 4.2 (Homotopy Invariance) and Chapter 4.3 (Excision), keeping the following questions in mind:

1. What are the geometric ideas behind these proofs?
2. How are these geometric ideas translated into algebra?
3. How do these proofs compare to the proofs of the corresponding results for π_1 ?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 14, February 4, 2022

Exercise 1 (topological embeddings). For which of the following types do there exist continuous and injective maps?

1. $\mathbb{R}^{2022} \longrightarrow S^{2022}$
2. $\mathbb{R}^{2022} \longrightarrow S^{2021}$
3. $S^{2022} \longrightarrow \mathbb{R}P^{2022}$
4. $S^{2022} \times S^{2022} \longrightarrow S^{2022}$

Exercise 2 (homotopy vs. homology). Does there exist a path-connected pointed space (X, x_0) with the following properties?

1. $\pi_1(X, x_0) \cong_{\mathsf{Group}} \mathbb{Z}/2, \quad H_1(X; \mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2$
2. $\pi_1(X, x_0) \cong_{\mathsf{Group}} \mathbb{Z}/2, \quad H_1(X; \mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}$
3. $\pi_1(X, x_0) \cong_{\mathsf{Group}} S_3, \quad H_1(X; \mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/3$
4. $\pi_1(X, x_0) \cong_{\mathsf{Group}} 1, \quad \pi_2(X, x_0) \cong_{\mathsf{Group}} \mathbb{Z}/2, \quad H_1(X; \mathbb{Z}) \cong_{\mathbb{Z}} \mathbb{Z}/2$

Exercise 3 (exam). You are abducted by aliens, transported to planet Blorxifold, and end up as examiner in an oral exam on Algebraic Topology.

1. Which questions will you ask on basic notions?
2. Which questions will you ask on homotopy groups?
3. Which questions will you ask on homology?
4. Which questions will you ask on applications?
5. Which questions will you ask on recurring themes/techniques?
6. Which examples will you discuss during the exam?

Try out your questions on ~~Commander Blorx~~ your fellow students!

Exercise 4 (summary). Write a summary of Chapter 4.4 (Applications) and Chapter 4.5 (Singular Homology and Homotopy Groups), keeping the following questions in mind:

1. What did we achieve by constructing singular homology?
2. What is the key idea of the proof of the Jordan curve theorem?
3. How can the Jordan curve theorem be used to obtain applications in geometry and topology?
4. How can our knowledge on homotopy groups and homology be combined?

No submission!

Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 15, February 11, 2022

Exercise 1 (CW-complexes). Are the following filtrations CW-structures on the unit interval $[0,1]$? If so, compute the cellular chain complex (with respect to $H_*(\,\cdot\,;\mathbb{Z})$), the cellular homology, and the Euler characteristic.

1. $\emptyset \subset \{0\} \subset [0,1]$
2. $\emptyset \subset \{0,1/2,1\} \subset [0,1]$
3. $\emptyset \subset [0,1/2] \subset [0,1]$
4. $\emptyset \subset [0,1) \subset [0,1]$
5. $\emptyset \subset [0,1] \setminus \{1/2\} \subset [0,1]$
6. $\emptyset \subset \{1/n \mid n \in \mathbb{N}_{>0}\} \subset [0,1]$

Exercise 2 (cellular homology). Choose two different CW-structures on $S^1 \times S^1$. In the following, we will consider cellular chain complexes and cellular homology with respect to singular homology with $\mathbb{Z}$-coefficients.

1. Compute the corresponding cellular chain complexes explicitly.
2. Compute the corresponding cellular homology.

Exercise 3 (Yeti vs. Jedi). We consider the following two subspaces of $\mathbb{R}^2$:

1. Are these spaces homeomorphic? Which connected components of YeTI are homeomorphic to which connected components of JEdI?
2. Are YeTI and JEdI homotopy equivalent?
3. Compute all homotopy groups of all connected components.
4. Compute $H_n(\,\cdot\,;\mathbb{Z})$ of these spaces for all $n \in \mathbb{Z}$.
5. Compute the Euler characteristic of these spaces.
6. Which connected components admit a 2022-sheeted connected covering?

Exercise 4 (summary). Write a summary of Chapter 5 (Cellular Homology), keeping the following questions in mind:

1. What are typical examples of CW-complexes and cellular maps?
2. What is the geometric idea of cellular homology? What is the definition?
3. How can cellular homology be computed?
4. How can homology theories on CW-complexes be compared?
5. What consequences does this have for practical computations?
6. What is the Euler characteristic?
7. How can the Euler characteristic be computed?
8. What are typical applications of the Euler characteristic?

No submission!

D

General Information

Algebraic Topology: Admin

Prof. Dr. C. Löh/M. Uschold/J. Witzig October 2021

Homepage. Information and news concerning the lectures, exercise classes, office hours, literature, as well as the exercise sheets can be found on the course homepage and in GRIPS:

http://www.mathematik.uni-regensburg.de/loeh/teaching/topologie1_ws2122
https://elearning.uni-regensburg.de

Lectures. The lectures are on Tuesdays (8:30–10:00; M101) and on Fridays (8:30–10:00; M101). The exact start time will be discussed in the first lecture.

Basic lecture notes will be provided, containing an overview of the most important topics of the course. These lecture notes can be found on the course homepage and will be updated after each lecture. Please note that these lectures notes are not meant to replace attending the lectures or the exercise classes!

According to current plans (13.10.2021): This course will be taught on campus in person. On request, this could be turned into a hybrid format (with live zoom streaming). Please note that there will be no recordings of the lectures. The lectures are a precious opportunity for live interaction and I want to keep the atmosphere as casual and unintimidating as possible. For asynchronous self-study, lecture notes will be made available. Please send an email to Clara Löh in case there is a need for the hybrid option!

Exercises. Homework problems will be posted on Tuesdays (before 8:30) on the course homepage; submission is due one week later (before 8:30, via GRIPS).

Each exercise sheet contains four regular exercises (4 credits each) and more challenging bonus problems (4 credits each).

It is recommended to solve the exercises in small groups; however, solutions need to be written up individually (otherwise, no credits will be awarded). Solutions can be submitted alone or in teams of at most two participants; all participants must be able to present *all* solutions of their team.

The first exercise sheet will appear on Tuesday, October 19. The exercise classes start in the *second* week.

In addition, we will provide études that will help to train elementary techniques and terminology. These problems should ideally be easy enough to be solved within a few minutes. Solutions are not to be submitted and will not be graded.

Registration for the exercise classes. Please register for the exercise classes via GRIPS:

https://elearning.uni-regensburg.de

Please register before Wednesday, October 20, 2021, 10:00, choosing your preferred option. We will try to fill the groups respecting your preferences.

The distribution will be announced at the end of the first week via GRIPS.

Credits/Exam. This course can be used as specified in the commented list of courses and in the module catalogue.

- *Studienleistung*: Successful participation in the exercise classes: 50% of the credits (of the regular exercise), presentation of a solution in class.
- *Prüfungsleistung*: Oral exam (25 minutes), by individual appointment at the end of the lecture period/during the break.

You will have to register in FlexNow for the Studienleistung and the Prüfungsleistung (if applicable). Registration will open at the end of the lecture period.

Further information on formalities can be found at:

https://www.uni-regensburg.de/mathematik/fakultaet/studium/studierende/index.html

Contact.

- On GRIPS, you can find the link to a gather.town meeting space for this course. Please feel free to use this to virtually meet other participants for discussions.
- If you have questions regarding the organisation of the exercise classes, please contact Matthias Uschold or Johannes Witzig:

  matthias.uschold@ur.de
  johannes.witzig@ur.de

- If you have questions regarding the exercises, please contact your tutor.
- If you have mathematical questions regarding the lectures, please contact your tutor or Clara Löh.
- If you have questions concerning your curriculum or the examination regulations, please contact the student counselling offices or the exam office:

http://www.uni-regensburg.de/mathematik/fakultaet/studium/ansprechpersonen/index.html

- In many cases, also the Fachschaft can help:

https://www-app.uni-regensburg.de/Studentisches/FS_MathePhysik/cmsms/

- Official information of the administration related to the COVID-19 pandemic can be found at:

  https://go.ur.de/corona

Bibliography

Please note that the bibliography will grow during the semester. Thus, also the numbers of the references will change!

[1] M. Arkowitz, M. Gutierrez. Comultiplications on free groups and wedges of circles, *Trans. Amer. Math. Soc.*, 350(4), pp. 1663–1680, 1998. Cited on page: 50, A.17

[2] Atari. *Asteroids*, 1979. Cited on page: 15

[3] G. Baigger. Die Nichtkonstruktivit¨at des Brouwerschen Fixpunktsatzes, *Arch. Math. Logik Grundlag.* ,25(3–4), pp. 183–188, 1985. Cited on page: 41

[4] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry.* Universitext, Springer, 1992. Cited on page: 171

[5] M. Brandenburg. *Einführung in die Kategorientheorie: Mit ausführlichen Erklärungen und zahlreichen Beispielen*, Springer Spektrum, 2015. Cited on page: 17

[6] S. Braun. *Simplicial Volume and Macroscopic Scalar Curvature.* PhD thesis, Karlsruhe Institute of Technology, 2018. Cited on page: 171

[7] A. Brooke-Taylor. Products of CW complexes, preprint, available at arXiv: 1710.05296 [math.GN], 2017. Cited on page: 185

[8] L.E.J. Brouwer. Nonconstructible fixed points indicated by classical theorems, *Nederl. Akad. Wetensch. Proc. Ser. A.*, 55; *Indagationes Math.*, 14, pp. 443–445, 1952. Cited on page: 41

[9] K.S. Brown. *Cohomology of Groups*, Graduate Texts in Mathematics, 82, Springer, 1982. Cited on page: 102, 112

[10] D.-C. Cisinski. *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, 180, Cambridge University Press, 2019. Cited on page: 23

[11] R. Diestel. *Graph Theory*, third edition, Graduate Texts in Mathematics, 173, 2006. Cited on page: 207

[12] J. Dieudonné. *A History of Algebraic and Differential Topology, 1900–1960*, reprint of the 1989 edition, Modern Birkhäuser Classics, Birkhäuser, 2009. Cited on page: 27, 44, 178

[13] A. Dold. *Lectures on Algebraic Topology*, Springer, 1980. Cited on page: 155, 185, 194, A.37

[14] C.H. Dowker. Topology of metric complexes. *American Journal of Mathematics*, 74(3), pp. 557–577, 1952. Cited on page: 185

[15] H.-D. Ebbinghaus et. al.. *Zahlen*, Springer, 1992. Cited on page: 169

[16] A.D. Elmendorf, I. Kříž, M.A. Mandell, J.P. May. Modern foundations for stable homotopy theory. *Handbook of algebraic topology*, pp. 213–253, North-Holland, 1995. Cited on page: 116

[17] D. Fauser, S. Friedl, C. Löh. Integral approximation of simplicial volume of graph manifolds, *Bull. Lond. Math. Soc.*, 51(4), pp. 715–731, 2019. Cited on page: 171

[18] Y. Félix, S. Halperin, J.-C. Thomas. *Rational Homotopy Theory*, Graduate Texts in Mathematics, 205, Springer, 2001. Cited on page: 70

[19] B. Fine, A. Gaglione, A. Myasnikov, G. Rosenberger, D. Spellman. *The Elementary Theory of Groups: A Guide through the Proofs of the Tarski Conjectures*, de Gruyter, 2014. Cited on page: 67

[20] S. Francaviglia, R. Frigerio, B. Martelli. Stable complexity and simplicial volume of manifolds, *J. Topol.*, 5(4), pp. 977–1010, 2012. Cited on page: 171

[21] R.M. Smullyan, M. Fitting. *Set theory and the continuum problem*, revised edition, Dover, 2010. Cited on page: 18

[22] R. Frigerio. *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs, 227, AMS, 2017. Cited on page: 171

[23] R. Frigerio, C. Löh, C. Pagliantini, R. Sauer. Integral foliated simplicial volume of aspherical manifolds, *Israel J. Math.*, 216(2), 707–751, 2016. Cited on page: 171

[24] M. Gromov. Volume and bounded cohomology. *Publ. Math. IHES*, 56, pp. 5–99, 1982. Cited on page: 170, 171

[25] S.K. Hansen. Measure homology, *Mathematica Scandinavica*, 83(2), pp. 205–219, 1998. Cited on page: 131

[26] A. Hatcher. *Algebraic Topology*, Cambridge University Press, 2002. http://www.math.cornell.edu/~hatcher/AT/ATpage.html Cited on page: 73, 86, 155, 161, A.44, A.45, A.46

[27] H. Herrlich. *Axiom of Choice*, Springer Lecture Notes in Mathematics, 1876, Springer, 2006. Cited on page: A.10

[28] K. Jänich. *Topologie*, eighth edition, Springer, 2008. Cited on page: A.2

[29] J.L. Kelley. *General Topology*, Springer, 1975. Cited on page: A.2

[30] R.C. Kirby, M.G. Scharlemann. Eight faces of the Poincaré homology 3-sphere. *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, pp. 113–146, Academic Press, 1979. Cited on page: 178

[31] R.C. Kirby, L.C. Siebenmann. On the triangulation of manifolds and the Hauptvermutung, *Bull. Amer. Math. Soc.*, 75, pp. 742–749, 1969. Cited on page: 181

[32] R.C. Kirby and L.C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, 1977. Cited on page: 181

[33] S. MacLane. *Categories for the Working Mathematician*, second edition, Springer, 1998. Cited on page: 17

[34] J. Milnor. On spaces having the homotopy type of CW-complex, *Trans. Amer. Math. Soc.*, 90, pp. 272–280, 1959. Cited on page: 181

[35] B. Mohar, C. Thomassen. *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 2001. Cited on page: 161

[36] J.M. Lee. *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, 218, 2002. Cited on page: 148

[37] C. Löh. Measure homology and singular homology are isometrically isomorphic, *Math. Z.*, 253(1), pp. 197–218, 2006. Cited on page: 131

[38] C. Löh. Simplicial Volume, *Bull. Man. Atl.*, pp. 7–18, 2011. Cited on page: 170, 171

[39] C. Löh. *Geometric Group Theory. An Introduction*, Universitext, Springer, 2018. Cited on page: 67, 100, 103, A.19, A.20, A.22

[40] C. Löh. Rank gradient vs. stable integral simplicial volume, *Period. Math. Hung.*, 76(1), pp. 88–94, 2018. Cited on page: 171

[41] C. Löh. Cost vs. integral foliated simplicial volume, *Groups Geom. Dyn.*, 14(3), pp. 899–916, 2020. Cited on page: 171

[42] C. Löh. Simplicial volume with $\mathbb{F}_p$-coefficients, *Period. Math. Hung.*, 80(1), pp. 38–58, 2020. Cited on page: 171

[43] C. Löh. *Group Cohomology*, lecture notes, Universität Regensburg, 2019.
http://www.mathematik.uni-regensburg.de/loeh/teaching/grouphom_ss19/lecture_notes.pdf Cited on page: 102, 112

[44] C. Löh. *Ergodic theoretic methods in group homology. A minicourse on L^2-Betti numbers in group theory*, SpringerBriefs in Mathematics, Springer, 2020. Cited on page: 101

[45] C. Löh, S. Krauss, N. Kilbertus (eds.). *Quod erat knobelandum. Themen, Aufgaben und Lösungen des Schülerzirkels Mathematik der Universität Regensburg.*, Springer Spektrum, 2016. Cited on page: 165

[46] C. Löh, M. Moraschini, R. Sauer. Amenable covers and integral foliated simplicial volume, preprint, available at arXiv:2112.12223 [math.GT], 2021. Cited on page: 171

[47] C. Löh, C. Pagliantini. Integral foliated simplicial volume of hyperbolic 3-manifolds, *Groups Geom. Dyn.*, 10(3), pp. 825–865, 2016. Cited on page: 170, 171

[48] A.T. Lundell, S. Weingram. *Topology of CW-complexes.* Van Nostrand, New York, 1969. Cited on page: 183

[49] C. Malkiewich. The stable homotopy category, preprint, 2014.
https://faculty.math.illinois.edu/~cmalkiew/stable.pdf Cited on page: 116

[50] A.A. Markov. Insolubility of the problem of homeomorphy, (Russian) *Proc. Internat. Congress Math. 1958*, pp. 300–306, Cambridge University Press, 1960. Cited on page: 68

[51] W.S. Massey. *Algebraic Topology: an Introduction*, Graduate Texts in Mathematics, 56, seventh edition, Springer, 1989. Cited on page: 55, 82, 103, 208

[52] W.S. Massey. *A Basic Course in Algebraic Topology*, Graduate Texts in Mathematics, 127, third edition, Springer, 1997. Cited on page: 138, 161

[53] H.J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979*, Lecture Notes in Mathematics, 788, pp. 109–124. Springer, 1980. Cited on page: 171

[54] J.R. Munkres. *Elements of algebraic topology*, Addison-Wesley, 1984. Cited on page: 67, 131

[55] J.R. Munkres. *Topology*, second edition, Pearson, 2003. Cited on page: A.2

[56] J. Nash. Non-cooperative games, *Ann. of Math.* (2), 54(2), pp. 286–295, 1951. Cited on page: 41

[57] V.P. Orevkov. A constructive map of the square into itself, which moves every constructive point, *Dokl. Akad. Nauk SSSR*, 152, pp. 55–58, 1963. Cited on page: 41

[58] B. Pierce. *Basic Category Theory for Computer Scientists*, Foundations of Computing, MIT University Press, 1991. Cited on page: 17, 27

[59] P.H. Potgieter. Computable counter-examples to the Brouwer fixed-point theorem, preprint, available at arXiv:0804.3199 [math.GM], 2008. Cited on page: 41

[60] E. Riehl. *Category Theory in Context*, Aurora: Dover Modern Math Originals, 2016. Cited on page: 17

[61] J.J. Rotman. *An Introduction to the Theory of Groups*, fourth edition, Graduate Texts in Mathematics, 148, Springer, 1999. Cited on page: 67, A.21

[62] M. Staley. Understanding quaternions and the Dirac belt trick, *European J. Phys.*, 31(3), pp. 467–478, 2010. Cited on page: 87

[63] L.A. Steen. *Counterexamples in Topology*, Dover, 1995. Cited on page: A.2, A.8

[64] N.E. Steenrod. A convenient category of topological spaces, *Michigan Math. J.*, 14(2), pp. 133–152, 1967. Cited on page: 185

[65] J. Strom. *Modern Classical Homotopy Theory*, American Mathematical Society, 2012. Cited on page: 62

[66] R.M. Switzer. *Algebraic Topology – Homotopy and Homology*, Springer, 1975. Cited on page: 131, 132

[67] W.P. Thurston. *Geometry and Topology of 3-Manifolds.* Lecture notes, Princeton, 1978.
Available at http://www.msri.org/publications/books/gt3m. Cited on page: 171

[68] T. tom Dieck. *Algebraic Topology*, European Mathematical Society, 2008. Cited on page: 62, 87, 98, 131, 148, 170, 174, 198, A.43, A.45

[69] C. Weibel. *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, 1995. Cited on page: 112, 200

[70] G.W. Whitehead. *Elements of Homotopy Theory*, Graduate Texts in Mathematics, 61, Springer, 1978. Cited on page: 199, A.43, A.44

[71] A. Zastrow. On the (non)-coincidence of Milnor-Thurston homology theory with singular homology theory, *Pacific Journal of Mathematics*, 186(2), pp. 369–396, 1998. Cited on page: 131, 199

Deutsch → English

A

Abbildungsgrad	mapping degree	41
Abbildungskegel	mapping cone	127, B.5
Abbildungszylinder	mapping cylinder	A.46
Abelianisierung, Abelisierung	abelianisation	172
abgeschlossene Menge	closed set	A.2
Abschluss	closure	A.3
absolute Homologie	absolute homology	108
Additivitätsaxiom	additivity axiom	107
algebraische Topologie	algebraic topology	1
Antipodenabbildung	antipodal map	43
Ausschneidung	excision	106
Automorphismengruppe	automorphism group	A.23

B

Bahn	orbit	A.24
Bahnenraum	orbit space	A.24
Ball	ball	6
baryzentrische Unterteilung	barycentric subdivision	150
Basispunkt	basepoint	32
Basisraum	base space	71
Bettizahl	Betti number	205
Blatt	sheet	72
Bordismus	bordism	131
Bündel	bundle	71

C

CW-Komplex	CW-complex	180
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darstellbarer Funktor	representable functor	27
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Erzeuger	generator	67
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