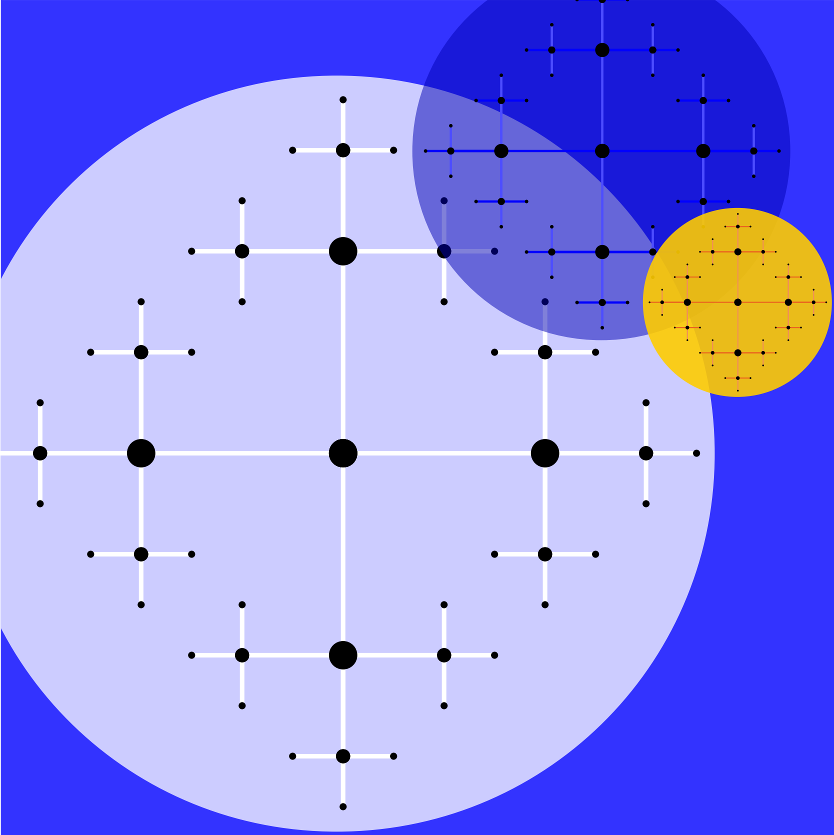

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Group Cohomology & Bounded Cohomology

An introduction for topologists

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Preliminary version

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1 Group cohomology

1.1 Introduction

What is group cohomology? Group cohomology is a contravariant functor turning groups and modules over groups into graded Abelian groups. I.e., on objects group cohomology looks like

$$H^n(G; A),$$

where

- the number $n \in \mathbb{N}$ is the degree in the grading of the graded Abelian group $H^*(G; A)$,
- the first parameter is a (discrete) group G,
- and the second ("Abelian") parameter is a $\mathbb{Z}G$-module A, the so-called coefficients.

Similarly, group homology is a covariant functor turning groups and modules over groups into graded Abelian groups.

How can we construct group cohomology? There are three main (equivalent) descriptions of group (co)homology:

- Topologically (via classifying spaces)
- Combinatorially (via the bar resolution)
- Algebraically (via derived functors).

Why is group cohomology interesting? First, group cohomology is an interesting theory in its own right providing a beautiful link between algebra and topology. Second, group (co)homology helps to solve the following problems:

- Given two groups, what extension groups with the given "kernel" and the given "quotient" group do there exist? (Section 1.4.4)
- How do cyclic Galois field extensions look like? (Hilbert 90) (Section 1.4.3)
- How surjective is the Hurewicz homomorphism in degree 2? (Section 1.6.7)
- Which (finite) groups can act freely on spheres? (Section 1.6 and 1.9)

Overview

In the *first part* of the semester we will study the following topics:

- Understand and compare the three basic descriptions of group (co)homology
- First applications of group (co)homology
- Transfer
- Product structures
- Cohomology of finite groups and periodic cohomology
- The Hochschild-Serre spectral sequence

In the *second part* of the semester we will look at a functional analytic variant of group cohomology, called bounded cohomology, and its application to the simplicial volume.

1.2 The domain categories for group (co)homology

The basic algebraic objects in the world of group (co)homology are group rings and modules over group rings.

Definition 1.2.1 (Group ring)**.** Let G be a group. The (integral) *group ring* of G is the ring $\mathbb{Z}G$ (sometimes also denoted $\mathbb{Z}[G]$ to avoid misunderstandings)

- whose underlying Abelian group is the free $\mathbb{Z}$-module $\bigoplus_{g\in G} \mathbb{Z} \cdot g$,
- and whose multiplication is the $\mathbb{Z}$-linear extension of composition in G, i.e.:

$$\begin{aligned} \cdot : \mathbb{Z}G \times \mathbb{Z}G &\longrightarrow \mathbb{Z}G \\ \Bigl(\sum_{g\in G} a_g \cdot g, \sum_{g \in G} b_g \cdot g\Bigr) &\longmapsto \sum_{g\in G}\sum_{h\in G} a_g \cdot b_{g^{-1}h} \cdot g \end{aligned}$$

  (where all sums are “finite”).

Example 1.2.2 (Group rings)**.**

- We have $\mathbb{Z}[1] \cong \mathbb{Z}$.
- The group ring $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ of the integers is nothing but the ring of Laurent polynomials.
- For all $n \in \mathbb{N}_{>0}$ we have $\mathbb{Z}[\mathbb{Z}/n] \cong \mathbb{Z}[t]/(t^n - 1)$.

Group rings and modules over group rings occur, for example, naturally in topology:

Example 1.2.3. Let X be a topological space and let G be a discrete group that acts continuously on X.

- Let $n \in \mathbb{N}$. Then the G-action on X induces a $\mathbb{Z}G$-module structure

$$\mathbb{Z}G \times C_n(X;\mathbb{Z}) \longrightarrow C_n(X;\mathbb{Z})$$

on the chain group $C_n(X;\mathbb{Z})$, given by

$$\begin{aligned} G \times \operatorname{map}(\Delta^n, X) &\longrightarrow \operatorname{map}(\Delta^n, X) \\ (g,\sigma) &\longmapsto \bigl(t \mapsto g \cdot \sigma(t)\bigr). \end{aligned}$$

– It is not difficult to see that the differential of the singular chain complex $C_*(X;\mathbb{Z})$ is a $\mathbb{Z}G$-morphism; hence, $C_*(X;\mathbb{Z})$ is naturally a $\mathbb{Z}G$-chain complex.

Notice: if X is a CW-complex and if G acts cellularly on X, then also the cellular chain complex $C_*^{\text{cell}}(X;\mathbb{Z})$ naturally is a $\mathbb{Z}G$-chain complex.

A fundamental special case is the following: If X is a connected CW-complex (with a chosen base point in the universal covering of X), then the singular/cellular chain complex of the universal covering of X naturally is a free $\mathbb{Z}\pi_1(X)$-chain complex.

Definition 1.2.4 (Invariants and coinvariants)**.** Let G be a discrete group and let A be a (left) $\mathbb{Z}G$-module. We call the submodule

$$A^G := \{a \in A \mid \forall_{g\in G} \;\; g \cdot a = a\}$$

the *invariants of* A. We call the quotient

$$A_G := A/\operatorname{span}_{\mathbb{Z}}\{g \cdot a - a \mid a \in A,\; g \in G\}$$

the *coinvariants of* A.

If G is a non-Abelian group, then the ring $\mathbb{Z}G$ is not commutative! Therefore, we need to distinguish between left and right modules over $\mathbb{Z}G$. In view of the following convention, we can restrict ourselves to left modules over $\mathbb{Z}G$, though.

Convention 1.2.5 (Tensor products and homomorphism modules over group rings)**.** Let G be a discrete group. We follow the convention that (if not explicitly specified otherwise) *all* $\mathbb{Z}G$-modules are left modules; this is possible, because taking inverses in G leads to an involution on the group ring $\mathbb{Z}G$ and hence we can canonically turn right $\mathbb{Z}G$-modules into left $\mathbb{Z}G$-modules and vice versa.

More explicitly, we use the following conventions for the tensor product and the group of $\mathbb{Z}G$-linear homomorphisms: Let A and B be two left $\mathbb{Z}G$-modules, and let $\bar{A}$ be the right $\mathbb{Z}G$-module obtained from A via the canonical involution on $\mathbb{Z}G$.

– Then we write
$$A \otimes_G B := \bar{A} \otimes_{\mathbb{Z}G} B.$$
Hence, $A \otimes_G B = (A \otimes_{\mathbb{Z}} B)_G$, where G acts diagonally on $A \otimes_{\mathbb{Z}} B$.
– Dually, we write
$$\operatorname{Hom}_G(A, B) := \operatorname{Hom}_{\mathbb{Z}G}(A, B).$$
Therefore, $\operatorname{Hom}_G(A, B) = \operatorname{Hom}_{\mathbb{Z}}(A, B)^G$, where G acts "diagonally" on $\operatorname{Hom}_{\mathbb{Z}}(\bar{A}, B)$.

Example 1.2.6. If G is a discrete group and A is a $\mathbb{Z}G$-module, then (where G acts trivially on $\mathbb{Z}$)
$$A^G = \operatorname{Hom}_G(\mathbb{Z}, A) \qquad \text{and} \qquad A_G = A \otimes_G \mathbb{Z}.$$

Convention 1.2.7 (Differentials on compound $\mathbb{Z}G$-chain complexes)**.** Let G be a discrete group, let (C_*, ∂_*) be a $\mathbb{Z}G$-chain complex, and let A be a $\mathbb{Z}G$-module. Then
$$C_* \otimes_G A := (C_n \otimes_G A)_{n\in\mathbb{N}}$$
is a $\mathbb{Z}$-chain complex with the differential $(\partial_n \otimes_G \mathrm{id}_A)_{n\in\mathbb{N}}$. Dually, we write
$$\operatorname{Hom}_G(C_*, A) := \bigl(\operatorname{Hom}_G(C_n, A)\bigr)_{n\in\mathbb{N}}$$
for the $\mathbb{Z}$-cochain complex equipped with the differential
$$\begin{aligned}\operatorname{Hom}_G(C_n, A) &\longrightarrow \operatorname{Hom}_G(C_{n+1}, A)\\ f &\longmapsto (-1)^{n+1} \cdot \bigl(c \mapsto f(\partial_{n+1}(c))\bigr).\end{aligned}$$

As indicated in the introduction, the domain categories for group homology and group cohomology incorporate both a group parameter and an Abelian parameter, the coefficients:

Definition 1.2.8 (GrpMod, GrpMod^{-})**.** The categories GrpMod, GrpMod^{-} are defined as follows:

1. GrpMod: The objects of the category GrpMod are pairs (G, A), where G is a discrete group and A is a (left) $\mathbb{Z}G$-module.
   The set of morphisms in GrpMod between two objects (G, A) and (H, B) is the set of pairs (φ, Φ), where

- $\varphi\colon G \longrightarrow H$ is a group homomorphism, and
- $\Phi\colon A \longrightarrow \varphi^* B$ is a $\mathbb{Z}G$-module homomorphism; here, $\varphi^* B$ is the $\mathbb{Z}G$-module whose underlying additive group coincides with B and whose $\mathbb{Z}G$-structure is given by

$$\begin{aligned} G \times B &\longrightarrow B \\ (g, b) &\longmapsto \varphi(g) \cdot b. \end{aligned}$$

The composition of morphisms is defined by composing both components (notice that this is well-defined in the second component).

2. GrpMod^-: The category GrpMod^- has the same objects as the category GrpMod, i.e., pairs of groups and modules over the corresponding group rings.
   The set of morphisms in GrpMod^- between two objects (G, A) and (H, B) is the set of pairs (φ, Φ), where
   - $\varphi\colon G \longrightarrow H$ is a group homomorphism, and
   - $\Phi\colon \varphi^* B \longrightarrow A$ is a $\mathbb{Z}G$-module homomorphism.

   The composition of morphisms is defined by covariant composition in the first component and contravariant composition in the second component.

The following, simple, example lies at the heart of group cohomology:

Example 1.2.9 (Invariants and coinvariants, functorially)**.** It is not difficult to see that we can extend the definition of coinvariants to a functor GrpMod $\longrightarrow$ Ab and the definition of invariants to a (contravariant) functor $\mathrm{GrpMod}^- \longrightarrow$ Ab.

1.3 Group cohomology, topologically

Topologically, group (co)homology can be defined by applying (co)homology with twisted coefficients to classifying spaces of groups; schematically, we can depict this as follows:

$$\begin{array}{llcccc}
\textit{Group homology} & \mathrm{GrpMod} & \longrightarrow & \mathrm{Top} & \longrightarrow & \mathrm{Ab}_* \\
 & (G,A) & \longmapsto & BG & \longmapsto & H_*(BG;A) \\
\textit{Group cohomology} & \mathrm{GrpMod}^{-} & \longrightarrow & \mathrm{Top} & \longrightarrow & \mathrm{Ab}_* \\
 & (G,A) & \longmapsto & BG & \longmapsto & H^*(BG;A)
\end{array}$$

1.3.1 Classifying spaces

The key to the topological definition of group (co)homology is the homotopy theoretical picture of group theory provided by classifying spaces.

Definition 1.3.1 (Model of BG)**.** Let G be a discrete group. A pointed connected CW-complex (X, x) together with an isomorphism $\pi_1(X, x) \cong G$ (this identification is part of the structure!) is a *model of (the classifying space) BG* if the universal covering of X is contractible (equivalently, the universal covering of X has the integral homology of a point, or, the homotopy groups $\pi_n(X, x)$ are trivial for all $n \in \mathbb{N}_{>1}$).

In particular, a model of BG is nothing but a polarised Eilenberg-Mac Lane space of type $(G, 1)$.

Theorem 1.3.2 (Existence and uniqueness of models of BG)**.**

1. *For every discrete group G there exists a model of BG.*
2. *Let G and H be discrete groups, and let $((X, x), \varphi_X)$ and $((Y, y), \varphi_Y)$ be models of BG and BH respectively (φ_X and φ_Y are the identifications of the fundamental groups of X and Y with G and H*

respectively). Then

$$\begin{aligned} \big[(X,x),(Y,y)\big]_\bullet &\longrightarrow \operatorname{Hom}(G,H) \\ [f] &\longmapsto \varphi_Y \circ \pi_1(f,x) \circ \varphi_X^{-1} \end{aligned}$$

is a natural bijection; here, $[\,\cdot\,,\,\cdot\,]_\bullet$ *denotes the set of pointed homotopy classes of pointed maps.*

3. *In particular, for every discrete group* G *there is up to canonical homotopy equivalence exactly one model of* BG.

Proof. The third part is a direct consequence of the first two parts. The first part can be proved by successively killing higher homotopy groups, and the second part can be proved by inductively constructing maps and homotopies on the skeleta [12, Section 8.8]. □

In view of this uniqueness result, we will sometimes abuse notation and write just BG to denote some model of BG; moreover, the chosen base point and the identification of the fundamental group with G are usually omitted in the notation. If $\varphi\colon G \longrightarrow H$ is a group homomorphism, we lazily write $B\varphi\colon BG \longrightarrow BH$ for some representative of the homotopy class of maps $BG \longrightarrow BH$ that induce φ on the level of fundamental groups; as long as we are only interested in notions up to homotopy or homology, this will not cause any problems.

Example 1.3.3 (Models of BG)**.**

- The one point space is a model of $B1$.
- The circle S^1 is a model of $B\mathbb{Z}$.
- Let $n \in \mathbb{N}_{>0}$, and let F_n be the free group of rank n. Then the n-fold wedge of circles is a model for BF_n (see Figure 1.1).
- If M is an oriented, closed, connected surface of genus at least 1, then M is a model of $B\pi_1(M)$.
- The infinite-dimensional projective space $\mathbb{R}\mathrm{P}^\infty$ is a model of $B\mathbb{Z}/2$.
- Let G be a Lie group with only finitely many connected components and let K be a maximal compact subgroup. Then G/K is homeomorphic to a Euclidean space. So, if $\Gamma \subset G$ is a discrete torsion-free subgroup, then $\Gamma \setminus G/K$ is a model for $B\Gamma$.
- Let G be a torsion-free discrete group and let Δ^G be the (infinite-dimensional) simplex spanned by G. Then the left translation action

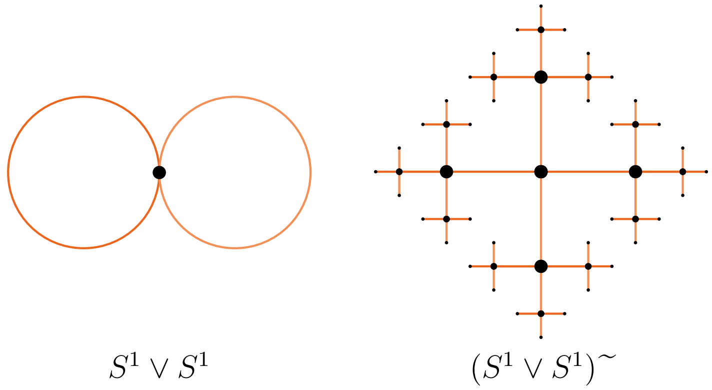


Figure 1.1: The space $S^1 \vee S^1$ together with its universal covering

of G on G induces a continuous action of G on Δ^G. Using the fact that G is torsion-free, one can show that the quotient $G \setminus \Delta^G$ is a model for BG. Notice however, that this model is quite "big."

Remark 1.3.4 (Classifying bundles)**.** Let G be a discrete group. Then the space BG classifies principal G-bundles: For all CW-complexes X the pull-back of bundles provides a natural bijection [12, Chapter 14]

$$[X, BG] \longrightarrow \text{Isomorphism classes of principal } G\text{-bundles over } X.$$

In view of Theorem 1.3.2, we can define a first simple version of group (co)homology (with trivial coefficients) from the category Grp of groups to the category Ab_* of graded Abelian groups:

Group homology	Grp	$\longrightarrow$	Ab_*
on objects	G	$\longmapsto$	$H_*(BG;\mathbb{Z})$
on morphisms	$\varphi\colon G \to H$	$\longmapsto$	$H_*(B\varphi)\colon H_*(BG) \to H_*(BH)$
Group cohomology	Grp	$\longrightarrow$	Ab_*
on objects	G	$\longmapsto$	$H^*(BG;\mathbb{Z})$
on morphisms	$\varphi\colon G \to H$	$\longmapsto$	$H^*(B\varphi)\colon H^*(BH) \to H^*(BG)$

Notice that group homology is covariant while group cohomology is contravariant.

1.3.2 (Co)Homology with twisted coefficients

In order to define group (co)homology via (co)homology of the corresponding classifying space, we need to incorporate the coefficient modules into the process of forming (co)homology. Here, we do not follow the most elegant, but a pragmatic approach towards (co)homology with twisted coefficients; the following convention is not a mathematical necessity, but will prove to be quite convenient:

Convention 1.3.5 (Pointed CW-complexes)**.** By a *pointed CW-complex* we mean a CW-complex X together with a chosen base point in X as well as a chosen base point in the universal covering $\widetilde{X}$ lying over x. (In particular, there is a well-defined action of $\pi_1(X,x)$ on $\widetilde{X}$ and hence on the singular/cellular chain complex of $\widetilde{X}$).

Definition 1.3.6 ((Co)Homology with twisted coefficients)**.** Let X be a (pointed) connected CW-complex, let G be the fundamental group of X, let $\widetilde{X}$ be the universal covering of X, and let A be a (left) $\mathbb{Z}G$-module.

– *Homology with twisted coefficients.* We write

$$C_*(X;A) := C_*(\widetilde{X};\mathbb{Z}) \otimes_G A.$$

Tensoring the differential of $C_*(\widetilde{X};\mathbb{Z})$ with the identity on A, we obtain a differential on $C_*(X;A)$ turning $C_*(X;A)$ into chain complex (Convention 1.2.7). The homology

$$H_*(X;A) := H_*\bigl(C_*(X;A)\bigr)$$

of this complex is called *homology of X with twisted coefficients in A.*

– *Cohomology with twisted coefficients.* Dually,

$$C^*(X;A) := \operatorname{Hom}_G\bigl(C_*(\widetilde{X};\mathbb{Z}), A\bigr)$$

is a cochain complex with respect to the dual of the differential on $C_*(\widetilde{X};\mathbb{Z})$ (see Convention 1.2.7), and we call

$$H^*(X;A) := H^*\bigl(C^*(X;A)\bigr)$$

cohomology of X with twisted coefficients in A.

(Co)Homology with twisted coefficients is functorial in the following sense: Before explaining this functoriality, we first define the underlying domain category – similarly, to the domain category for group (co)homology.

Definition 1.3.7 (TopMod, TopMod^-)**.** The categories TopMod, TopMod^- are defined as follows:

1. TopMod: The objects of the category TopMod are pairs (X, A), where X is a pointed connected CW-complex and A is a (left) $\mathbb{Z}\pi_1(X)$-module.

   The set of morphisms in TopMod between two objects (X, A) and (Y, B) is the set of pairs (f, Φ), where
   - $f\colon X \longrightarrow Y$ is a pointed continuous map, and
   - $\Phi\colon A \longrightarrow \pi_1(f)^*B$ is a $\mathbb{Z}\pi_1(X)$-module homomorphism.

   The composition of morphisms is defined by composing both components (notice that this well-defined in the second component).
2. TopMod^-: The category TopMod^- has the same objects as the category TopMod, i.e., pairs of pointed connected CW-complexes and modules over the fundamental group.

   The set of morphisms in TopMod^- between two objects (X, A) and (Y, B) is the set of pairs (f, Φ), where
   - $f\colon X \longrightarrow Y$ is a pointed, continuous map, and
   - $\Phi\colon \pi_1(f)^*B \longrightarrow A$ is a $\mathbb{Z}\pi_1(X)$-module homomorphism.

   The composition of morphisms is defined by covariant composition in the first component and contravariant composition in the second component.

Remark 1.3.8 (Functoriality of (co)homology with twisted coefficients)**.**

- *Homology.* Let $(f, \Phi)\colon (X, A) \longrightarrow (Y, B)$ be a morphism in the category TopMod, and let $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ be the lift of $f\colon X \longrightarrow Y$ to the universal coverings mapping the base point of $\widetilde{X}$ to the one of $\widetilde{Y}$; such a lift exists and is unique by covering theory. We then define

  $$C_*(f; \Phi) := C_*(\widetilde{f}) \otimes_G \Phi\colon C_*(X; A) \longrightarrow C_*(Y; B),$$

  which is a chain map. Let

  $$H_*(f; \Phi) := H_*\bigl(C_*(f; \Phi)\bigr)\colon H_*(X; A) \longrightarrow H_*(Y; B).$$

  Clearly, this definition turns homology with twisted coefficients into a functor $\mathrm{TopMod} \longrightarrow \mathrm{Ab}_*$.

– *Cohomology.* Dually, let $(f, \Phi)\colon (X, A) \longrightarrow (Y, B)$ be a morphism in the category TopMod⁻, and let $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ be the base point preserving lift of f to the universal coverings. We then define

$$C^*(f; \Phi) := \operatorname{Hom}_G\bigl(C_*(\widetilde{f}), \Phi\bigr)\colon C^*(Y; B) \longrightarrow C^*(X; A),$$

which is a cochain map. Let

$$H^*(f; \Phi) := H^*\bigl(C^*(f; \Phi)\bigr)\colon H^*(Y; B) \longrightarrow H^*(X; A).$$

Clearly, this definition turns cohomology with twisted coefficients into a contravariant functor $\mathrm{TopMod}^- \longrightarrow \mathrm{Ab}_*$.

Because the singular chain complex and the cellular chain complex of (the universal covering of) a CW-complex are (equivariantly) homotopy equivalent, we can take either one of them in the above definitions without changing the resulting (co)homology theory.

Example 1.3.9.

– On the category of connected CW-complexes, (co)homology with twisted coefficients in the trivial module $\mathbb{Z}$ (i.e., the fundamental groups act trivially on $\mathbb{Z}$) coincides with ordinary (co)homology with integral coefficients.
– If X is a connected CW-complex with fundamental group G, then

$$H_*(X; \mathbb{Z}G) = H_*\bigl(C_*(\widetilde{X}; \mathbb{Z}) \otimes_G \mathbb{Z}G\bigr) = H_*\bigl(C_*(\widetilde{X}; \mathbb{Z})\bigr) = H_*\bigl(\widetilde{X}; \mathbb{Z}\bigr).$$

Notice however, that the analogous statement for cohomology is wrong in general (see the example below).
– We compute (co)homology of S^1 with twisted coefficients via the standard CW-structure of S^1 consisting of one 0-cell and one 1-cell. Then the cellular chain complex of the corresponding cell structure on the universal covering $S^1 \cong \mathbb{R}$ is

$$\begin{array}{ccc} \text{dimension } 1 & & \text{dimension } 0 \\ \mathbb{Z}[\mathbb{Z}] & \xrightarrow[\cdot(t-1)]{} & \mathbb{Z}[\mathbb{Z}] \end{array}$$

where t corresponds to the generator $+1$ in $\pi(S^1) \cong \mathbb{Z}$. Hence, if A is a $\mathbb{Z}[\mathbb{Z}]$-module, we see that the cellular twisted chain complex $C_*^{\text{cell}}(S^1; A)$ is given by

$$\begin{array}{ccc} \text{dimension } 1 & & \text{dimension } 0 \\ A & \xrightarrow[(t-1)\cdot]{} & A \end{array}$$

Hence,

$$H_k(S^1; A) = \begin{cases} A_{\mathbb{Z}} = A/\big((t-1)\cdot A\big) & \text{if } k = 0 \\ A^{\mathbb{Z}} = \ker((t-1)\cdot : A \to A) & \text{if } k = 1 \\ 0 & \text{for all } k \in \mathbb{N}_{>1}. \end{cases}$$

Similarly,

$$H^k(S^1; A) = \begin{cases} A^{\mathbb{Z}} = \ker((t-1)\cdot : A \to A) & \text{if } k = 0 \\ A_{\mathbb{Z}} = A/\big((t-1)\cdot A\big) & \text{if } k = 1 \\ 0 & \text{for all } k \in \mathbb{N}_{>1}. \end{cases}$$

In particular, notice that $H^1(S^1; \mathbb{Z}\pi_1(S^1)) \cong \mathbb{Z} \neq 0 = H^1(\widetilde{S^1}; \mathbb{Z})$.

Geometrically, (co)homology with twisted coefficients can also be obtained by considering so-called local coefficient systems [10, Chapter 5]; e.g., local coefficient systems occur naturally when considering fibrations – the higher homotopy groups of the fibre yield local coefficient systems over the base. The definition via local coefficient systems is independent of the choice of base points, but is unwieldy for concrete computations.

1.3.3 Group cohomology, topologically

Finally, we are able to give the topological definition of group cohomology:

Definition 1.3.10 (Group homology, topologically)**.** Group homology is the functor $\text{GrpMod} \longrightarrow \text{Ab}_*$ defined as follows: For every discrete group G we choose a model X_G of BG; moreover, for every homomorphism $\varphi \colon G \longrightarrow H$ of groups we choose a continuous map $f_\varphi \colon X_G \longrightarrow X_H$ realising φ on the level of fundamental groups (see Theorem 1.3.2 for the existence of such objects).

– On objects: Let (G, A) be an object in GrpMod, i.e., G is a discrete group and A is a $\mathbb{Z}G$-module. Then we define *group homology of G with coefficients in A* by

$$H_*(G;A) := H_*\big(X_G;A\big).$$

– On morphisms: Let $(\varphi,\Phi)\colon (G,A) \longrightarrow (H,B)$ be a morphism in the category GrpMod. Then we define $H_*(\varphi;\Phi)$ through the following diagram:

$$\begin{array}{ccc} H_*(G;A) & \overset{H_*(\varphi;\Phi)}{\dashrightarrow} & H_*(H;B) \\ \| & & \| \\ H_*(X_G;A) & \underset{H_*(f_\varphi;\Phi)}{\longrightarrow} & H_*(X_H;B) \end{array}$$

Definition 1.3.11 (Group cohomology, topologically)**.** Group homology is the contravariant functor $\text{GrpMod}^- \longrightarrow \text{Ab}_*$ defined as follows: For every discrete group G we choose a model X_G of BG; moreover, for every homomorphism $\varphi\colon G \longrightarrow H$ of groups we choose a continuous map $f_\varphi\colon X_G \longrightarrow X_H$ realising φ on the level of fundamental groups (see Theorem 1.3.2 for the existence of such objects).

– On objects: Let (G, A) be an object in GrpMod^-, i.e., G is a discrete group and A is a $\mathbb{Z}G$-module. Then we define *group cohomology of G with coefficients in A* by

$$H^*(G;A) := H^*\big(X_G;A\big).$$

– On morphisms: Let $(\varphi,\Phi)\colon (G,A) \longrightarrow (H,B)$ be a morphism in the category GrpMod^-. Then we define $H^*(\varphi;\Phi)$ through the following diagram:

$$\begin{array}{ccc} H^*(H;B) & \overset{H^*(\varphi;\Phi)}{\dashrightarrow} & H^*(G;A) \\ \| & & \| \\ H^*(X_H;B) & \underset{H^*(f_\varphi;\Phi)}{\longrightarrow} & H^*(X_G;A) \end{array}$$

Notice that group homology and group cohomology defined in this way indeed are functorial (homology covariantly, cohomology contravariantly). This definition of group homology and group cohomology is independent of the chosen models in the following sense: Any two choices of models of classifying spaces and of maps between them leads to naturally and canonically isomorphic functors.

1.3.4 Group (co)homology, first examples

Proposition 1.3.12 (Group (co)homology – low degrees)**.**

1. *In degree* 0*, the group homology functor coincides with (i.e., is naturally isomorphic to) the functor* $\mathrm{GrpMod} \longrightarrow \mathrm{Ab}$ *of taking coinvariants.*
2. *In degree* 0*, the group cohomology functor coincides with the contravariant functor* $\mathrm{GrpMod}^{-} \longrightarrow \mathrm{Ab}$ *of taking invariants.*
3. *In degree* 1*, the functor* $\mathrm{Grp} \longrightarrow \mathrm{Ab}$ *of taking group homology with trivial* $\mathbb{Z}$*-coefficients coincides with the Abelianisation functor.*
4. *In degree* 1*, the (contravariant) functor* $\mathrm{Grp} \longrightarrow \mathrm{Ab}$ *of taking group cohomology with trivial* $\mathbb{Z}$*-coefficients coincides with* $\mathrm{Hom}_{\mathrm{Grp}}(\,\cdot\,, \mathbb{Z})$*.*

Proof. Let (G, A) and (H, B) be two objects in GrpMod (or, equivalently in GrpMod^{-}), and let $\varphi\colon G \longrightarrow H$ be a group homomorphism.

We start by choosing convenient models X_G, X_H and $f_\varphi\colon X_G \longrightarrow X_H$ of BG, BH and $B\varphi$ respectively: Homotopy theory shows that we can assume without loss of generality that X_G and X_H have exactly one 0-cell each and that the 1-skeleton of X_G and X_H is just G and H respectively; moreover, we may assume that $f\colon X_G \longrightarrow X_H$ corresponds on the 1-skeleton to the map $\varphi\colon G \longrightarrow H$. Hence, we obtain the diagram in Figure 1.2 for the cellular chain complexes of the universal coverings (the actions on the terms in dimension 1 is given by the canonical action on the left factor). We now prove the first two parts:

1. For the claim in group homology, let $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ be a morphism in GrpMod. Then on the level of cellular chain complexes with coefficients, we obtain the diagram on the left hand side of Figure 1.3. Hence, the diagram

$$
\begin{array}{ccc}
\text{dimension 1} & & \text{dimension 0} \\
1 \otimes e_g & \longmapsto & g - 1 \\
\mathbb{Z}G \otimes_{\mathbb{Z}} \bigoplus_{g \in G} \mathbb{Z} \cdot e_g & \longrightarrow & \mathbb{Z}G \\
 & & \big\downarrow {\scriptstyle \mathbb{Z}\varphi} \\
\mathbb{Z}H \otimes_{\mathbb{Z}} \bigoplus_{h \in H} \mathbb{Z} \cdot e_h & \longrightarrow & \mathbb{Z}H \\
1 \otimes e_h & \longmapsto & h - 1
\end{array}
$$

Figure 1.2: A nice cellular model of a group homomorphism

$$
\begin{array}{ccc}
\text{dimension 1} & & \text{dimension 0} \\
a \cdot e_G & \longmapsto & g \cdot a - a \\
\bigoplus_{g \in G} A \cdot e_g & \longrightarrow & A \\
 & & \big\downarrow {\scriptstyle \Phi} \\
\bigoplus_{h \in H} B \cdot e_h & \longrightarrow & B \\
b \cdot e_h & \longmapsto & h \cdot b - b
\end{array}
\qquad
\begin{array}{ccc}
\text{dimension 1} & & \text{dimension 0} \\
(g \cdot a - a)_{g \in G} & \longleftarrow\!\!\shortmid & a \\
\prod_{g \in G} A \cdot e_g & \longleftarrow & A \\
 & & \big\uparrow {\scriptstyle \Phi} \\
\prod_{h \in H} B \cdot e_h & \longleftarrow & B \\
(h \cdot b - b)_{h \in H} & \longleftarrow\!\!\shortmid & b
\end{array}
$$

Figure 1.3: Computing group (co)homology in degree 0

$$\begin{array}{ccc} H_0(G;A) & \xrightarrow{H_0(\varphi;\Phi)} & H_0(H;B) \\ \| & & \| \\ H_0(X_G;A) & \xrightarrow{H_0(f_G;\Phi)} & H_0(X_H;B) \\ \| & & \| \\ A_G & \xrightarrow[\Phi_G]{} & B_H \end{array}$$

is commutative.

2. The argument for cohomology analogously uses the right hand side diagram of Figure 1.3 (for a morphism $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ in the category GrpMod⁻).

The third and fourth part are just a reformulation of a well-known fact about (co)homology of spaces (and the Hurewicz homomorphism). □

Example 1.3.13. Using the concrete models of classifying spaces given in Section 1.3.1, we obtain our first examples of group (co)homology:

– Trivial group: Let A be a $\mathbb{Z}$-module. Then clearly

$$H_k(1;A) = \begin{cases} A & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases} \quad \text{and} \quad H^k(1;A) = \begin{cases} A & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

for all $k \in \mathbb{N}$, because the one point space is a model of $B1$.

– The infinite cyclic group: Let t be a generator of $\mathbb{Z}$ and let A be a $\mathbb{Z}[\mathbb{Z}]$-module. Then looking at the standard cell decomposition of $S^1 \simeq B\mathbb{Z}$ we obtain (see Example 1.3.9)

$$H_k(\mathbb{Z};A) = \begin{cases} A_{\mathbb{Z}} & \text{if } k = 0 \\ A^{\mathbb{Z}} & \text{if } k = 1 \\ 0 & \text{for all } k \in \mathbb{N}_{>1} \end{cases} \quad \text{and} \quad H^k(\mathbb{Z};A) = \begin{cases} A^{\mathbb{Z}} & \text{if } k = 0 \\ A_{\mathbb{Z}} & \text{if } k = 1 \\ 0 & \text{for all } k \in \mathbb{N}_{>1}. \end{cases}$$

– Free groups: Let $n \in \mathbb{N}_{>0}$. Because the n-fold wedge of circles is a model of BF_n, we obtain (where F_n acts trivially on $\mathbb{Z}$)

$$H_k(F_n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^n & \text{if } k = 1 \\ 0 & \text{for all } k \in \mathbb{N}_{>1}. \end{cases}$$

– Surface groups: Let M_g be an oriented closed connected surface of genus $g > 0$. Then M_g is a model of $B\pi_1(\pi_1(M_g))$ and so (where $\pi_1(M_g)$ acts trivially on $\mathbb{Z}$)

$$H_k\bigl(\pi_1(M_g);\mathbb{Z}\bigr) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^{2\cdot g} & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{for all } k \in \mathbb{N}_{>2}. \end{cases}$$

– The cyclic group $\mathbb{Z}/2$ of order 2: Looking at the standard cell decomposition of infinite-dimensional projective space $\mathbb{R}\mathrm{P}^\infty$, which is a model of $B\mathbb{Z}/2$, we see that (where $\mathbb{Z}/2$ acts trivially on $\mathbb{Z}$)

$$H_k(\mathbb{Z}/2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } k \in \mathbb{N} \text{ is odd} \\ 0 & \text{if } k \in \mathbb{N}_{>0} \text{ is even} \end{cases} \quad\text{and}\quad H^k(\mathbb{Z}/2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{N} \text{ is odd} \\ \mathbb{Z}/2 & \text{if } k \in \mathbb{N}_{>0} \text{ is even.} \end{cases}$$

This might come as a surprise: There exist “small” groups with “large” homology. In particular, this computation shows that there is no finite-dimensional model of $B\mathbb{Z}/2$.

1.3.5 Products and free products

Several topological constructions preserve the property of being classifying spaces of discrete groups. For example, the product of two classifying spaces is a classifying space for the product group and glueings of classifying spaces lead to classifying spaces for amalgamated free products of groups.

Proposition 1.3.14 (Group cohomology of product groups)**.** *Let G_1 and G_2 be two discrete groups, and let X_1 and X_2 be models of the classifying space of G_1 and G_2 respectively.*

1. *Then $X_1 \times X_2$ is a model of $B(G_1 \times G_2)$.*
2. *Consequently, for all principal ideal rings A (on which the groups act trivially) we obtain natural short exact Künneth sequences*

$$0 \to \bigoplus_{p+q=k} H_p(G_1;A) \otimes_A H_q(G_2;A) \to H_k(G_1 \times G_2;A) \to \bigoplus_{p+q=k-1} \operatorname{Tor}_1^A\bigl(H_p(G_1;A), H_q(G_2;A)\bigr) \to 0$$

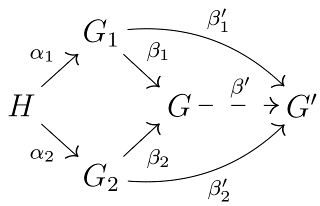
for all $k \in \mathbb{N}$ (induced by the homological cross product). These sequences split, but the splittings are not necessarily natural.
If all homology groups $H_k(G_1; A)$ are finitely generated over A, then there also are corresponding short exact Künneth sequences for cohomology of groups.

Proof. The first part follows from the observation that $X_1 \times X_2$ has fundamental group $G_1 \times G_2$ and that the product of the universal coverings of X_1 and X_2 is a universal covering for $X_1 \times X_2$ (in particular, $X_1 \times X_2$ has contractible universal covering).

For the second part, we apply the Künneth formula to the first part. □

Definition 1.3.15 (Amalgamated free products)**.** Let $\alpha_1 \colon H \longrightarrow G_2$ and $\alpha_2 \colon H \longrightarrow G_2$ be homomorphism of groups. An *amalgamated free product of* G_1 *and* G_2 *over* H is a group G together with homomorphisms $\beta_1 \colon G_1 \longrightarrow G$, and $\beta_2 \colon G_2 \longrightarrow G$ with $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ satisfying the following universal property:

For all groups G' and homomorphisms $\beta'_1 \colon G_1 \longrightarrow G'$ and $\beta'_2 \colon G_2 \longrightarrow G'$ with $\beta'_1 \circ \alpha_1 = \beta'_2 \circ \alpha_2$ there is exactly one homomorphism $\beta' \colon G \longrightarrow G'$ making the following diagram commutative:



Of course, amalgamated free products are determined uniquely (upt to canonical isomorphism) by this universal property. That all amalgamated free products indeed exist can be shown by giving explicit constructions of such groups [48]. On the other hand, amalgamated free products naturally occur in topology – for example, when computing the fundamental group of a glued space via the Seifert and van Kampen theorem [30, 31].

Proposition 1.3.16 (Group (co)homology of amalgamated free products)**.** *Let $\alpha_1 \colon H \longrightarrow G_1$ and $\alpha_2 \colon H \longrightarrow G_2$ be two injective group homomorphisms of discrete groups.*

1. *There exist models Y, X_1, X_2 of BH, BG_1 and BG_2 with the following properties: The complex Y is a subcomplex of X_1 and X_2 and the push-out $X_1 \cup_Y X_2$ is a model of $B(G_1 *_H G_2)$.*
2. *Consequently, there are long exact Mayer-Vietoris sequences*

$$\cdots \to H_k(H;A) \to H_k(G_1;A) \oplus H_k(G_2;A) \to H_k(G;A) \to H_{k-1}(H;A) \to \dots$$
$$\cdots \to H^{k-1}(H;A) \to H^k(G;A) \to H^k(G_1;A) \oplus H^k(G_2;A) \to H^k(H;A) \to \dots$$

for all $\mathbb{Z}$-modules A with trivial group action.

3. *In particular: For all discrete groups G_1 and G_2 we obtain*

$$H_k(G_1 * G_2;A) \cong H_k(G_1;A) \oplus H_k(G_2;A)$$
$$H^k(G_1 * G_2;A) \cong H^k(G^2;A) \oplus H^k(G_2;A)$$

for all $k \in \mathbb{N}_{>0}$ and all $\mathbb{Z}$-modules A with trivial group action.

Proof. We start by constructing nice models of the various classifying spaces: Let Y, X_1, and X_2 be models of BH, BG_1, and BG_2 respectively. Then we can realise the group homomorphisms $\alpha_1 \colon H \longrightarrow G_2$ and $\alpha_2 \colon H \longrightarrow G_2$ by continuous maps $f_1 \colon Y \longrightarrow X_1$ and $f_2 \colon Y \longrightarrow X_2$; using cellular approximation and taking mapping cylinders, we may assume that the maps f_1 and f_2 actually are inclusions of subcomplexes. In the following, we write

$$X := X_1 \cup_Y X_2.$$

In view of the Seifert and van Kampen theorem, the complex X has fundamental group $G_1 *_H G_2$; moreover, the inclusions of the subcomplexes Y, X_1, and X_2 into X induce the structure homomorphisms of the groups H, G_1, and G_2 into $G_1 *_H G_2$. These three homomorphisms are injective by the structure theory of amalgamated free products (and the assumption on injectivity of α_1 and α_2).

Therefore, it remains to show that the universal covering $\widetilde{X}$ of X is contractible: Because $\widetilde{X}$ is a simply connected CW-complex it suffices to establish that $H_k(\widetilde{X};\mathbb{Z}) = 0$ for all $k \in \mathbb{N}_{>1}$. To this end we decompose $\widetilde{X}$ into smaller pieces and apply the Mayer-Vietoris sequence to this decomposition: Let $\pi \colon \widetilde{X} \longrightarrow X$ be the universal covering map and let

$$\bar{Y} := \pi^{-1}(Y), \qquad \bar{X}_1 := \pi^{-1}(X_1), \qquad \bar{X}_2 := \pi^{-1}(X_2)$$

be the inverse images of the subcomplexes Y, X_1, X_2 in $\widetilde{X}$. In view of the Mayer-Vietoris sequence for the glueing

$$\widetilde{X} = \bar{X}_1 \cup_{\bar{Y}} \bar{X}_2$$

it suffices to show that $\bar{Y}$, $\bar{X}_1$, $\bar{X}_2$ have trivial homology.

In the following, we give the argument only for $\bar{Y}$ – the arguments for $\bar{X}_1$ and $\bar{X}_2$ being similar. Because Y is a model of BH, the universal covering of Y is contractible; hence, it suffices to show that the connected components of $\bar{Y}$ are simply connected. For any base point y in $\bar{Y}$ we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\bar{Y}, y) & \longrightarrow & \pi_1(\widetilde{X}) = 1 \\ {\scriptstyle \pi_1(\pi)}\downarrow & & \downarrow{\scriptstyle \pi_1(\pi)} \\ \pi_1(Y, \pi(y)) & \longrightarrow & \pi_1(X) \end{array}$$

where the horizontal arrows are induced by the inclusions of complexes. In particular, the lower horizontal arrow corresponds to the inclusion of H into G. As the left vertical arrow is injective by covering theory, we conclude that $\pi_1(\bar{Y}, y)$ is trivial, as desired. This finishes the proof of the first part.

The second part and the third part now follow by applying the long exact Mayer-Vietoris sequences to the models constructed in the first part. □

Remark 1.3.17. There are also versions of the previous propositions with twisted coefficients; however, for the sake of simplicity, we only treated trivial coefficients.

Using the above propositions one can for example compute the group (co)homology of the infinite dihedral group. Moreover – as soon as we know the group (co)homology of all finite cyclic groups (see Section 1.6.2) – we can compute the group (co)homology of all finitely generated Abelian groups and of the matrix group $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$.

Recall that singular and cellular (co)homology on the category of CW-complexes can be characterised uniquely by the Eilenberg-Steenrod axioms. Are there analogous axioms for group (co)homology? Indeed, there are such axioms; however, in contrast to the geometric situation it is imperative to incorporate *all* (twisted) coefficients into the discussion. We will explain this approach in more detail in Section 1.5.6.

1.4 Group cohomology, combinatorially

It might seem unsatisfactory that in order to obtain an invariant of groups we had to take the detour via classifying spaces – it is only natural to suspect that there is a purely algebraic description of group (co)homology. Translating the functorial simplicial models of classifying spaces into algebra, we obtain the bar resolution of a discrete group. Twisting this equivariant chain complex with modules over the group in question yields functorial (co)chain complexes that in turn give rise to functors $\mathrm{GrpMod} \longrightarrow \mathrm{Ab}_*$ and $\mathrm{GrpMod}^- \longrightarrow \mathrm{Ab}_*$. In fact – as we will see in Section 1.5.6 – these functors are naturally isomorphic to the group (co)homology functors constructed topologically in Section 1.3.

An interesting aspect of the approach via the bar resolution is that no choices (such as choosing models of classifying spaces) are involved, and that the construction is functorial even on the level of (co)chain complexes. Moreover, this approach is visibly linked to algebraic properties of groups, leading to applications in the classification of group extensions and in Galois theory (Sections 1.4.4 and 1.4.3).

A disadvantage of the approach via the bar resolution is that we loose some flexibility (in the topological setting, we can choose a model of the classifying space suitable for the problem at hand); concrete calculations in higher degrees are close to impossible via the bar resolution.

1.4.1 The bar resolution

Basically, the bar resolution is an algebraic counterpart of the functorial simplicial models of classifying spaces of discrete groups:

Definition 1.4.1 (The bar resolution)**.** Let G be a discrete group. The *bar resolution* of G is the $\mathbb{Z}G$-chain complex $C_*(G)$ defined as follows:

– For $n \in \mathbb{N}$ let

$$C_n(G) := \bigoplus_{g \in G^{n+1}} \mathbb{Z} \cdot g_0 \cdot [g_1|\cdots|g_n]$$

with the G-action characterised by

$$h \cdot \bigl(g_0 \cdot [g_1|\cdots|g_n]\bigr) := (h \cdot g_0) \cdot [g_1|\cdots|g_n]$$

for all $h \in G$ and all $g \in G^{n+1}$.

– The differential is the $\mathbb{Z}G$-homomorphism $\partial\colon C_*(G) \longrightarrow C_{*-1}(G)$ uniquely determined by

$$\begin{aligned} C_n(G) &\longrightarrow C_{n-1}(G) \\ g_0 \cdot [g_1|\cdots|g_n] &\longmapsto g_0 \cdot g_1 \cdot [g_2|\cdots|g_n] \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \cdot g_0 \cdot [g_1|\cdots|g_{j-1}|g_j \cdot g_{j+1}|g_{j+2}|\cdots|g_n] \\ &\quad + (-1)^n \cdot g_0 \cdot [g_1|\cdots|g_{n-1}]. \end{aligned}$$

A straightforward computation shows that $\partial \circ \partial = 0$ and hence that $C_*(G)$ is a $\mathbb{Z}G$-chain complex.

Using tensor products and the Hom-functor, we also obtain versions with twisted coefficients (see Convention 1.2.7 for our conventions on tensor products and Hom-complexes of $\mathbb{Z}G$-chain complexes):

Definition 1.4.2 (The bar construction with coefficients)**.** Let G be a discrete group and let A be a $\mathbb{Z}G$-module. Let $C_*(G;A)$ be the $\mathbb{Z}$-chain complex given by

$$C_*(G;A) := C_*(G) \otimes_G A,$$

and let $C^*(G;A)$ be the $\mathbb{Z}$-cochain complex given by

$$C^*(G;A) := \operatorname{Hom}_G\bigl(C_*(G), A\bigr).$$

Of course, these constructions are functorial with respect to morphisms in the categories GrpMod and GrpMod⁻:

Definition 1.4.3 (The bar construction on morphisms)**.** Let G and H be discrete groups, let A be a G-module, and let B be an H-module. Furthermore, let $\varphi\colon G \longrightarrow H$ be a group homomorphism.

– We let $C_*(\varphi)\colon C_*(G) \longrightarrow C_*(H)$ be the $\mathbb{Z}G$-chain map uniquely determined by

$$\begin{aligned} C_n(G) &\longrightarrow C_n(H) \\ g_0 \cdot [g_1|\cdots|g_n] &\longmapsto \varphi(g_0) \cdot \bigl[\varphi(g_1) \bigm| \cdots \bigm| \varphi(g_n)\bigr]. \end{aligned}$$

– If $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ is a morphism in GrpMod, then we write

$$C_*(\varphi; \Phi) := C_*(\varphi) \otimes_G \Phi\colon C_*(G; A) \longrightarrow C_*(H; B);$$

notice that $C_*(\varphi; \Phi)$ is a well-defined $\mathbb{Z}$-chain map.

– Dually, if $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ is a morphism in GrpMod^-, then we write

$$C^*(\varphi; \Phi) := \mathrm{Hom}_G(C_*(\varphi), \Phi)\colon C^*(H; B) \longrightarrow C^*(G; A);$$

this is a well-defined $\mathbb{Z}$-cochain map.

1.4.2 Group cohomology, combinatorially

Definition 1.4.4 (Group homology, combinatorially)**.** Group homology is the functor GrpMod $\longrightarrow \mathrm{Ab}_*$ defined as follows:

– On objects: Let (G, A) be an object in GrpMod, i.e., G is a discrete group and A is a $\mathbb{Z}G$-module. Then we define *group homology of G with coefficients in A* by

$$H_*(G; A) := H_*\bigl(C_*(G; A)\bigr).$$

– On morphisms: Let $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ be a morphism in the category GrpMod. Then we define

$$H_*(\varphi; \Phi) := H_*\bigl(C_*(\varphi; \Phi)\bigr)\colon H_*(G; A) \longrightarrow H_*(H; B).$$

Definition 1.4.5 (Group cohomology, combinatorially)**.** Group cohomology is the functor $\mathrm{GrpMod}^- \longrightarrow \mathrm{Ab}_*$ defined as follows:

– On objects: Let (G, A) be an object in GrpMod^-, i.e., G is a discrete group and A is a $\mathbb{Z}G$-module. Then we define *group cohomology of G with coefficients in A* by

$$H^*(G; A) := H^*\bigl(C^*(G; A)\bigr).$$

– On morphisms: Let $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ be a morphism in the category GrpMod^{-}. Then we define

$$H^*(\varphi; \Phi) := H^*\bigl(C^*(\varphi; \Phi)\bigr)\colon H^*(H; B) \longrightarrow H^*(G; A).$$

Notice that $H_*\colon \mathrm{GrpMod} \longrightarrow \mathrm{Ab}_*$ indeed is a (covariant) functor and that $H^*\colon \mathrm{GrpMod}^{-} \longrightarrow \mathrm{Ab}_*$ is a (contravariant) functor.

Remark 1.4.6. We will prove in Section 1.5.6 that indeed the combinatorial description and the topological description give rise to naturally isomorphic group (co)homology functors. In the remainder of the present section, we always refer to the combinatorial description of group (co)homology, when writing expressions like $H_*(G; A)$ or $H^*(G; A)$.

1.4.3 Application: Cyclic Galois extensions (Hilbert 90)

The key tool in the classification of cyclic Galois extensions is *Hilberts Satz 90*. Recall that a Galois extension is called *cyclic* if its Galois group is a cyclic group.

Theorem 1.4.7 (Hilbert 90 – cohomological version)**.** *Let L/K be a finite Galois extension of fields with Galois group G. Then*

$$H^1(G; L^\times) = \{1\},$$

where the G-action on the coefficients $L^\times$ is the Galois action (and we think of the coefficients $L^\times$ as well as the cohomology group as multiplicative groups).

Proof. We follow the general strategy of averaging – as the group G is finite, we can sum up expressions over all elements of G.

Let $f \in C^1(G; L^\times) = \mathrm{Hom}_G(C_1(G), L^\times)$ be a cocycle. Because characters $L^\times \longrightarrow L^\times$ are linearly independent over L [26, Theorem VI.4.1], there exists an element $x \in L$ such that

$$\bar{x} := \sum_{\tau \in G} f\bigl(1 \cdot [\tau]\bigr) \cdot \tau(x) \in L$$

is non-zero, and hence lies in $L^\times$. We now claim that the $\mathbb{Z}G$-homomorphism $\bar{f}$ given by

$$\begin{aligned} \bar{f}\colon C_0(G) &\longrightarrow L^\times \\ \sigma_0 &\longmapsto \sigma_0(\bar{x}) \end{aligned}$$

witnesses that f is a coboundary: Indeed, for all $\sigma_1 \in G$ we have

$$\begin{aligned} (\delta\bar{f})\bigl(1\cdot[\sigma_1]\bigr) &= \bar{f}((-1)\cdot(\sigma_1 - 1)) \\ &= \frac{\bar{x}}{\sigma_1(\bar{x})} \\ &= \frac{\bar{x}}{\sum_{\tau\in G}\sigma_1\bigl(f(1\cdot[\tau])\bigr)\cdot\sigma_1\circ\tau(x)} \\ &= \frac{\bar{x}}{\sum_{\tau\in G} f(\sigma_1\cdot[\tau])\cdot\sigma_1\circ\tau(x)} \\ &= \frac{\bar{x}}{\sum_{\tau\in G}\frac{1}{f(1\cdot[\sigma_1])}\cdot f(1\cdot[\sigma_1\circ\tau])\cdot\sigma_1\circ\tau(x)} \\ &= \bar{x}\cdot f(1\cdot[\sigma_1])\cdot\frac{1}{\bar{x}}, \end{aligned}$$

because f is a cocycle. Using G-equivariance of $\bar{f}$, $\delta\bar{f}$, and f, we obtain $\delta\bar{f} = f$, as desired. □

Corollary 1.4.8 (Hilbert 90)**.** *Let L/K be a finite cyclic Galois extension, let σ be a generator of the Galois group G, and let $x \in L$. Then the following are equivalent:*

1. *The norm $N_{L/K}(x)$ equals* 1.
2. *There exists an $a \in L^\times$ satisfying $x = a/\sigma(a)$.*

Proof. Recall that the norm $N_{L/K}(x)$ is the determinant of the K-homomorphism $L \longrightarrow L$ given by multiplication with x; we use the following well-known properties of the norm:

– The norm is invariant under the action of the Galois group.
– More precisely, $N_{L/K}(x) = \prod_{\tau\in G}\tau(x)$.

So if $x = a/\sigma(a)$ for some $a \in L^\times$, then

$$N_{L/K}(x) = \frac{N_{L/K}(a)}{N_{L/K}(\sigma(a))} = \frac{N_{L/K}(a)}{N_{L/K}(a)} = 1.$$

Conversely, suppose that $N_{L/K}(x) = 1$, and that the Galois group (and hence σ) has order n. Then the $\mathbb{Z}G$-homomorphism $f\colon C_1(G) \longrightarrow L^\times$ determined uniquely by

$$\begin{aligned} 1 \cdot [1] &\longmapsto 1 \\ 1 \cdot [\sigma] &\longmapsto x \\ &\vdots \\ 1 \cdot [\sigma^{n-1}] &\longmapsto x \cdot \sigma(x) \cdot \dots \cdot \sigma^{n-2}(x) \end{aligned}$$

is easily seen to be a cocyle (because $\prod_{j=0}^{n-1} \sigma^j(x) = N_{L/K}(x) = 1$). Hence, by the theorem, it is a coboundary, say $f = \delta \bar{f}$ for some $\bar{f} \in C^0(G; L^\times)$. In particular,

$$x = f(1 \cdot [\sigma]) = \delta \bar{f}(1 \cdot [\sigma]) = \frac{\bar{f}(1)}{\bar{f}(\sigma)} = \frac{f(1)}{\sigma(f(1))}. \qquad \square$$

In order to understand cyclic Galois extensions, one now applies the corollary to roots of unity in the base field: If L/K is a cyclic Galois extension of degree n with $\operatorname{char} K \nmid n$, if $\zeta \in K$ is a primitive n-th root of unity and if σ is a generator of the Galois group of L/K, then the corollary provides an element $a \in L$ with $\zeta = a/\sigma(a)$. In particular, $a^n \in L^\sigma = K$, so $X^n - a^n$ is a polynomial over K and one can show in this situation, that in fact L is the splitting field of $X^n - a^n$ over K.

Remark 1.4.9. There is also an additive version of the above results, involving the trace and cohomology with coefficients in the extension field instead of the norm and cohomology with coefficients in the units of the extension field.

1.4.4 Application: Group extensions with Abelian kernel

In the following, we study the question of how to classify extension groups of a given Abelian kernel and a given quotient group.

Definition 1.4.10 ((Equivalence of) extensions)**.** Let Q be a group and let A be an Abelian group.

– An *extension of Q by A* is an exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of groups.

– Two extensions $0 \to A \to G \to Q \to 1$ and $0 \to A \to G' \to Q \to 1$ are *equivalent* if there is a homomorphism $\varphi\colon G \longrightarrow G'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & G' & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

commutative; notice that in this case φ necessarily is an isomorphism.

Remark 1.4.11 (The conjugation action of the quotient on the kernel)**.** An extension $0 \to A \to G \to Q \to 1$ of a group Q by an Abelian group A induces a $\mathbb{Z}Q$-module structure on A as follows: The group G acts by conjugation on the normal subgroup A. Because A is Abelian, the conjugation action of A on itself is trivial. Therefore, the quotient $Q = A \setminus G$ acts by "conjugation" on A, and so we obtain a $\mathbb{Z}Q$-module structure on A.

Example 1.4.12 (Actions on the kernel)**.**

– Let A be an Abelian group and let Q be some group. Then the action of Q on A induced by the product extension

$$0 \longrightarrow A \longrightarrow A \times Q \longrightarrow Q \longrightarrow 1$$

(where $A \longrightarrow A \times Q$ is the inclusion of the first factor and $A \times Q \longrightarrow Q$ is the projection onto the second factor) is the trivial action.

– In the extension

$$0 \longrightarrow A_3 \longrightarrow S_3 \longrightarrow S_3/A_3 \longrightarrow 1$$

the non-trivial element of the quotient group $S_3/A_3 \cong \mathbb{Z}/2$ acts by taking inverses on $A_3 \cong \mathbb{Z}/3$.

Definition 1.4.13. Let Q be a group and let A be a $\mathbb{Z}Q$-module. We write $E(Q, A)$ for the set of all equivalence classes of extensions of Q by A that induce the given $\mathbb{Z}Q$-module structure on A. (It is not difficult to see that equivalent extensions induce the same action on the kernel.)

Theorem 1.4.14 (Classification of group extensions with Abelian kernel)**.** *Let Q be a group and let A be a $\mathbb{Z}Q$-module. Then the maps*

$$\begin{aligned} H^2(Q; A) &\longleftrightarrow E(Q, A) \\ [f] &\longmapsto [0 \to A \to G_f \to Q \to 1] \\ \eta_E &\longleftarrow E \end{aligned}$$

are mutually inverse bijections (the definition of the extensions G_f and the classes η_E is provided in the course of the proof below).

Clearly, this theorem is a great help in classifying groups (see Section 1.6.4 for examples). Conversely, by exhibiting non-trivial extensions, we can construct non-trivial cohomology classes.

As we will see later (Section 1.5), group cohomology is related to the Ext-functors, which owe their name to the classification of extensions as described in Theorem 1.4.14.

Proof. We start with the map from the right hand side to the left hand side. More precisely, we show how to obtain a 2-cocycle from a given extension and that the corresponding cohomology class does not change when replacing the extension by an equivalent one:

Let $E \in E(Q, A)$, and let

$$0 \longrightarrow A \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } Q \longrightarrow 0$$

be an extension of Q by A inducing the given Q-action on A that represents E. The idea is to measure the failure of π to be a split group homomorphism by a 2-cocycle:

- *Choosing a section.* Let $s\colon Q \longrightarrow G$ be a set-theoretic section of $\pi\colon G \longrightarrow Q$. Of course, the map s in general will *not* be a group homomorphism. Measuring the failure of s being a group homomorphism leads to the map

$$\begin{aligned} \bar{f}\colon Q \times Q &\longrightarrow A \\ (q_1, q_2) &\longmapsto s(q_1) \cdot s(q_2) \cdot s(q_1 \cdot q_2)^{-1}; \end{aligned}$$

here, $\bar f$ maps to A because s is a section of π. (This map $\bar f$ is the key to obtaining a cocycle out of the given extension.)

– *Rewriting the group structure on G.* Using the map $\bar f$ we can recover the group structure on G from the group structure on Q and the Q-action on A as follows: First of all,

$$\begin{aligned} G &\longrightarrow A \times Q \\ g &\longmapsto \bigl(g \cdot s(\pi(g))^{-1}, \pi(g)\bigr) \\ a \cdot s(q) &\longleftarrow\!\shortmid (a, q) \end{aligned}$$

are mutually inverse bijections. Using these bijections, the composition on G translates into the composition (where $\bullet$ denotes the Q-action on A induced by the given extension)

$$\begin{aligned} (A \times Q) \times (A \times Q) &\longrightarrow A \times Q \\ \bigl((a, q), (a', q')\bigr) &\longmapsto \bigl(a + q \bullet a' + \bar f(q, q'), q \cdot q'\bigr). \end{aligned}$$

Indeed, for all (a, q), $(a', q') \in A \times Q$ we have (in G)

$$\begin{aligned} a \cdot s(q) \cdot a' \cdot s(q') &= a \cdot q \bullet a' \cdot s(q) \cdot s(q') \\ &= a \cdot q \bullet a' \cdot \bar f(q_1, q_2) \cdot s(q \cdot q'), \end{aligned}$$

because the action of Q on A is given by conjugation in G.

– *Constructing a cocycle.* Because the composition on G is associative, a short calculation shows that

$$\bar f(q_1, q_2) + \bar f(q_1 \cdot q_2, q_3) - q_1 \bullet \bar f(q_2, q_3) - \bar f(q_1, q_2 \cdot q_3) = 0$$

for all $q_1, q_2, q_3 \in Q$. We now define $f \in C^2(Q; A) = \operatorname{Hom}_Q(C_2(Q), A)$ to be the homomorphism given by

$$\begin{aligned} C_2(Q) &\longrightarrow A \\ q_0 \cdot [q_1|q_2] &\longmapsto q_0 \bullet \bar f(q_1, q_2). \end{aligned}$$

The relation above for $\bar f$ following from the associativity of G shows that f is a cocycle.

- *How does the choice of section affect the cocycle?* Let $s'\colon Q \longrightarrow G$ be another section of π and let $f' \in C^2(Q;A)$ be the corresponding cocycle. Because s and s' are sections of π, there is a function $\bar{c}\colon Q \longrightarrow A$ such that
$$s'(q) = \bar{c}(q) \cdot s(q)$$
for all $q \in Q$. A straightforward computation shows that $c \in C^1(Q;A)$ given by
$$\begin{aligned} c\colon C_1(Q) &\longrightarrow A \\ q_0 \cdot [q_1] &\longmapsto q_0 \bullet \bar{c}(q_1) \end{aligned}$$
satisfies $\delta(c) = f' - f$. In fact,
$$\begin{aligned} f'(1 \bullet [q_1|q_2]) &= s'(q_1) \cdot s'(q_2) \cdot s'(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= \bar{c}(q_1) \cdot s(q_1) \cdot \bar{c}(q_2) \cdot s(q_2) \cdot s(q_1 \cdot q_2)^{-1} \cdot \bar{c}(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= \bar{c}(q_1) \cdot \bigl(q_1 \bullet \bar{c}(q_2)\bigr) \cdot s(q_1) \cdot s(q_2) \cdot s(q_1 \cdot q_2)^{-1} \cdot \bar{c}(q_1 \cdot q_2)^{-1} && \text{in } G \\ &= \bar{c}(q_1) + q_1 \bullet \bar{c}(q_2) + f(1 \bullet [q_1|q_2]) - \bar{c}(q_1 \cdot q_2) && \text{in } A \\ &= \delta(c)(1 \bullet [q_1|q_2]) + f(1 \bullet [q_1|q_2]) \end{aligned}$$
for all $q_1, q_2 \in Q$; here, we used the fact that the Q-action on A is induced from conjugation in G. Now using Q-equivariance we obtain $\delta(c) = f' - f$.
- *How does the choice of extension affect the cocycle?* Let
$$0 \longrightarrow A \xrightarrow{\ i\ } G' \xrightarrow{\ \pi'\ } Q \longrightarrow 0$$
be an extension equivalent to the one involving G. Furthermore, let $\varphi\colon G' \longrightarrow G$ be an isomorphism witnessing that these extensions are equivalent. If $s'\colon Q \longrightarrow G'$ is a section of π', then $\varphi \circ s'\colon Q \longrightarrow G$ is a section of π and the cocycles corresponding to s' and to $\varphi \circ s'$ coincide. Therefore, the previous paragraph shows that the extension G' leads to the same cohomology class as the extension G.

Using the cocycles corresponding to sections of π we therefore obtain a well-defined cohomology class $\eta_E \in H^2(Q;A)$.

Conversely, suppose we are given a cohomology class $\eta \in H^2(Q;A)$. We show how to construct an equivalence class of extensions of Q by A out of this cohomology class. More precisely, we construct extensions out of 2-cocycles and show that cohomologous cocycles lead to equivalent extensions:

– *A group structure out of a cocycle.* Let $f \in C^2(Q;A)$ be a cocycle, and let

$$\begin{aligned} \bar{f} \colon Q \times Q &\longrightarrow A \\ (q_1, q_2) &\longmapsto f(1 \cdot [q_1|q_2]). \end{aligned}$$

Inspired by the first part of the proof, on the set $A \times Q$ we define the composition

$$\begin{aligned} (A \times Q) \times (A \times Q) &\longrightarrow A \times Q \\ \bigl((a,q),(a',q')\bigr) &\longmapsto \bigl(a + q \bullet a' + \bar{f}(q,q'), q \cdot q'\bigr). \end{aligned}$$

The same calculation as above shows that f being a cocycle implies that this composition is associative.
Moreover, using the cocycle property of f once more, we see that $(e,1)$ is a neutral element for this composition, where

$$e := -\bar{f}(1,1) = -f(1 \bullet [1|1]).$$

An easy computation shows that every element of $A \times Q$ has an inverse element with respect to this composition and the neutral element e. So $G_f := A \times Q$ is a group with respect to this composition.

– *An extension out of a cocycle.* Via the homomorphisms

$$\begin{aligned} i_f \colon A &\longrightarrow G_f = A \times Q \\ a &\longmapsto (a + e, 1) \end{aligned}$$

and

$$\begin{aligned} \pi_f \colon G_f = A \times Q &\longrightarrow Q \\ (a,q) &\longmapsto q. \end{aligned}$$

the group G_f can be viewed as an extension of Q by A:

$$0 \longrightarrow A \xrightarrow{i_f} G_f \xrightarrow{\pi_f} Q \longrightarrow 1$$

– *The induced action on the kernel.* The map

$$\begin{aligned} s_f \colon Q &\longrightarrow G_f = A \times Q \\ q &\longmapsto (0, q) \end{aligned}$$

is a section of π_f. Hence, the Q-action $*$ of the above extension on A is given by

$$\begin{aligned} q * a &= s_f(q) \cdot (a + e, 1) \cdot s_f(q)^{-1} \\ &= (0, q) \cdot (a + e, 1) \cdot (0, q)^{-1} \\ &= \bigl(0 + q \bullet (a + e) + \bar f(q, 1), q\bigr) \cdot (0', q^{-1}) \\ &= \bigl(q \bullet (a + e) + \bar f(q, 1) + q \bullet 0' + \bar f(q, q^{-1}), 1\bigr) \\ &= \bigl(q \bullet (a + e) + \bar f(q, 1) + e, 1\bigr) \\ &= (q \bullet a + e, 1) \\ &= i_f(q \bullet a) \end{aligned}$$

for all $q \in Q$ and all $a \in A$; here, we wrote $(0', q^{-1}) = (0, q)^{-1}$, and (in the penultimate step) we used the cocycle property of f on $1 \cdot [q|1|1]$. So this extension $0 \longrightarrow A \longrightarrow G_f \longrightarrow Q \longrightarrow 1$ induces the given Q-action on A and thus represents a class in $E(Q, A)$.

– *What happens if we change the cocycle?* Similarly as above we see that changing the cocycle f by a coboundary leads to an equivalent extension.

Therefore, we obtain a well-defined map $H^2(Q; A) \longrightarrow E(Q, A)$.

That these two maps are mutually inverse to each other follows from the concrete constructions, and is left as an exercise to the reader. □

Example 1.4.15 (Extensions of free groups)**.** Of course, any extension of a free group by an Abelian group (more generally, by any group) splits. This is consistent with the classification result above: Taking the equivalence of group cohomology defined topologically and of group cohomology defined via the bar resolution for granted, we see that the cohomology groups occurring in Theorem 1.4.14 vanish (recall that there are one-dimensional models for the classifying spaces of free groups (Example 1.3.3)).

Remark 1.4.16 (The extension corresponding to the trivial cohomology class)**.** Let Q be a group and let A be a $\mathbb{Z}Q$-module. Then the proof of the theorem above shows that the zero class in $H^2(Q; A)$ corresponds to the semi-direct product

$$0 \longrightarrow A \longrightarrow A \rtimes Q \longrightarrow Q \longrightarrow 1,$$

where the action used to construct the semi-direct product is nothing but the given action of Q on A.

Conversely, non-trivial extensions lead to non-trivial cohomology classes in degree 2; for example, we can use this to show that $H^2(\mathbb{Z}/3; \mathbb{Z})$ is non-trivial (where $\mathbb{Z}/3$ acts trivially on $\mathbb{Z}$).

Theorem 1.4.17 (Functoriality of the classification of extensions)**.** *The classification of extensions with Abelian kernel is functorial in the following sense: Let* $(\varphi, \Phi)\colon (Q, A) \longrightarrow (Q', A')$ *be a morphism in the category* GrpMod.[1] *Moreover, let* $E \in E(Q, A)$ *and* $E' \in E(Q', A')$ *be represented by* $0 \to A \to G \to Q \to 1$ *and* $0 \to A' \to G' \to Q' \to 1$ *respectively. Then there is a group homomorphism* $\widetilde{\varphi}\colon G \longrightarrow G'$ *making the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & {\scriptstyle \Phi}\downarrow & & {\scriptstyle \widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & A' & \longrightarrow & G' & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

commutative if and only if (in $H^2(Q, \varphi^* A')$*)*

$$H^2(\mathrm{id}_Q; \Phi)(\eta_E) = H^2(\varphi; \mathrm{id}_{A'})(\eta_{E'}).$$

Proof. Exercise (this can be seen by studying the explicit constructions in the proof of Theorem 1.4.14). ☐

Corollary 1.4.18. *Let* Q *be a discrete group, let* A *be a* $\mathbb{Z}Q$*-module, let*

$$0 \longrightarrow A \longrightarrow G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

be an extension inducing the given Q*-action on* A*, and let* E *be the corresponding equivalence class in* $E(Q, A)$*. Then* $H^2(\pi; \mathrm{id}_{\pi^* A})(\eta_E) = 0$.

---

[1] i.e., Q and Q' are discrete groups, A is a $\mathbb{Z}Q$-module, A' is a $\mathbb{Z}Q'$-module, $\varphi\colon Q \longrightarrow Q'$ is a group homomorphism, and $\Phi\colon A \longrightarrow \varphi^* A'$ is a $\mathbb{Z}Q$-homomorphism.

Proof. Using π, we obtain the G-module π^*A out of A, and hence the semi-direct product extension $0 \to \pi^*A \to \pi^*A \rtimes G \to G \to 1$. The group homomorphism

$$\begin{aligned} \widetilde{\pi}\colon \pi^*A \rtimes G &\longrightarrow G \\ (a, g) &\longmapsto a \cdot g \end{aligned}$$

leads to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*A & \longrightarrow & \pi^*A \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & A & \longrightarrow & G & \underset{\pi}{\longrightarrow} & Q & \longrightarrow & 1. \end{array}$$

Recalling that the cohomology class corresponding to the semi-direct product $\pi^*A \rtimes G$ is trivial (see Remark 1.4.16), we obtain

$$H^2(\pi; \mathrm{id}_A)(\eta_E) = H^2(\mathrm{id}_Q; \mathrm{id}_A)(\eta_{\text{semi-direct product}}) = 0$$

from Theorem 1.4.17, as claimed. □

More generally, also extensions of quotient groups by non-Abelian kernels can be classified by means of group cohomology [4, Section IV.6]; however, this classification is much more delicate – for example, given a quotient and a kernel group with an outer action by the quotient, there does not necessarily exist any extension inducing the given action (i.e., there is no non-Abelian analogue of the semi-direct product).

A related question is to determine the conjugacy classes of all splittings of a split extension with Abelian kernel; this is related to the first cohomology group of the quotient group with coefficients in the kernel [4, Section IV.2].

1.5 Group cohomology via derived functors

Our aim is now to find an *algebraic* description of group (co)homology that has some built-in flexibility (like the definition of group (co)homology via classifying spaces), and to show that both the topological definition and the combinatorial definition of group (co)homology are instances of this more general framework.

The solution is to interpret group (co)homology in terms of derived functors; natural questions now are:

– What is a derived functor?
– Wich functors are group homology and group cohomology derived from?

In the present section, we will start with a review of homological algebra, then we will define/characterise derived functors, and finally, we will show how group (co)homology fits into this setting and why all three approaches to group (co)homology lead to the same theory. During this section it might prove useful to keep the following slogan in mind:

> *Homological algebra measures non-exactness, both on the level of objects (homology) and on the level of morphisms (derived functors).*

For convenience, we will do homological algebra only in module categories; in a way this is also the most general case – like all manifolds can be assumed to be submanifolds of Euclidean space; of course, this embedding point of view also has its drawbacks, but for the applications we have in mind it is appropriate.

Convention 1.5.1 (Rings)**.** In the following, by a *ring* we always mean an associative, not necessarily commutative, ring with unit. If R is a ring, then R-Mod denotes the category of left R-modules and Mod-R denotes the category of right R-modules.

We assume that the reader has some basic familiarity with homological algebra, especially with the notions of chain complexes, homology, chain homotopies, and the snake lemma as provided by a basic course on algebraic topology [13, 53] (see also Appendix **??**).

Large parts of this section are inspired by the excellent book of Weibel [53, Chapter 2].

1.5.1 Right/left exact functors

Recall that a sequence of morphisms of modules over a ring is called *exact* if for every morphism in the sequence, the image equals the kernel of the next morphism. I.e., a sequence

$$\cdots \longrightarrow A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \longrightarrow \cdots$$

of modules over a given ring is exact if and only if $\operatorname{im} f_n = \ker f_{n-1}$ for all n.

Definition 1.5.2 (Right/left exact functors)**.** Let R and S be two rings.

- A (covariant or contravariant) functor R-Mod $\longrightarrow S$-Mod is *additive* if it preserves the sum operation on the homomorphism groups.
- An additive functor $F\colon R\text{-Mod} \longrightarrow S\text{-Mod}$ is *right exact* if for every short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in R-Mod the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

  is exact (in S-Mod); analogously, *left exact* functors are defined.
- An additive contravariant functor $F\colon R\text{-Mod} \longrightarrow S\text{-Mod}$ is *left exact* if for every short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in R-Mod the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$$

  is exact (in S-Mod); analogously, *right exact* contravariant functors are defined.

– (Contravariant) Functors R-Mod $\longrightarrow$ S-Mod that are both right exact and left exact (i.e., that map short exact sequences to short exact sequences) are called *exact*.

The same terminology is used when R-Mod or S-Mod is replaced by the corresponding category of right modules.

Remark 1.5.3 (Exact functors and (long) exact sequences)**.** Using splicing of short exact sequences it follows that exact functors also map long exact sequences to long exact sequences.

The fundamental example of left exact functors are Hom-functors:

Proposition 1.5.4 (Left exactness of Hom)**.** *Let R be a ring, and let A be a left R-module.*

1. *Then* $\operatorname{Hom}_R(A, \cdot\,)\colon R\text{-Mod} \longrightarrow \text{Ab}$ *is a left exact functor.*
2. *The functor* $\operatorname{Hom}_R(\,\cdot\,, A)\colon R\text{-Mod} \longrightarrow \text{Ab}$ *is left exact.*

(Analogously, the Hom*-functors on right R-modules are left exact).*

Proof. Exercise (this is a straightforward calculation). □

A rich source of semi-exact functors is provided by adjoint functors:

Definition 1.5.5 (Adjoint functors)**.** Let $F\colon \text{Mod-}R \longrightarrow \text{Mod-}S$ and $G\colon \text{Mod-}S \longrightarrow \text{Mod-}R$ be additive functors. The functor G is *right adjoint to F* (the functor F is *left adjoint to G*) if there is a natural isomorphism

$$\operatorname{Hom}_S\bigl(F(\,\cdot\,),\,\cdot\,\bigr) \cong \operatorname{Hom}_R\bigl(\,\cdot\,, G(\,\cdot\,)\bigr)$$

of Abelian groups, i.e., for all $X \in \operatorname{Ob}(\text{Mod-}R)$ and all $Y \in \operatorname{Ob}(\text{Mod-}S)$ there is an isomorphism $\varphi_{X,Y}\colon \operatorname{Hom}_S(F(X), Y) \longrightarrow \operatorname{Hom}_R(X, G(Y))$ such that for all morphisms $f \in \operatorname{Hom}_R(X', X)$ and all $g \in \operatorname{Hom}_S(Y, Y')$ the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_S\bigl(F(X), Y\bigr) & \xrightarrow{\ \varphi_{X,Y}\ } & \operatorname{Hom}_R\bigl(X, G(Y)\bigr) \\
{\scriptstyle \operatorname{Hom}_S(F(f),g)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(f,G(g))} \\
\operatorname{Hom}_S\bigl(F(X'), Y'\bigr) & \xrightarrow[\ \varphi_{X',Y'}\ ]{} & \operatorname{Hom}_R\bigl(X', G(Y')\bigr)
\end{array}$$

is commutative.

Proposition 1.5.6 (Adjointness implies exactness)**.** *Let R and S be two rings, and suppose that the additive functor $F\colon \text{Mod-}R \longrightarrow \text{Mod-}S$ admits a right adjoint functor $G\colon \text{Mod-}S \longrightarrow \text{Mod-}R$. Then F is right exact and G is left exact.*

Proof. Exercise (use the exactness properties of the Hom-functors and apply a Yoneda-ish trick). □

Example 1.5.7 (Tensor products and homomorphism modules)**.** Let R be a ring, and let A be a left R-module. Then $\cdot \otimes_R A\colon \text{Mod-}R \longrightarrow \text{Ab}$ and $\text{Hom}_\mathbb{Z}(A, \cdot)\colon \text{Ab} \longrightarrow \text{Mod-}R$ are adjoint functors, because

$$\text{Hom}_\mathbb{Z}(\cdot \otimes_R A, \cdot) \cong \text{Hom}_R\bigl(\cdot, \text{Hom}_\mathbb{Z}(A, \cdot)\bigr).$$

Therefore, $\cdot \otimes_R A$ is right exact.

In particular: If G is a group, then the coinvariants functor $\cdot_G = \cdot \otimes_G \mathbb{Z}$ is right exact and the invariants functor $\cdot^G = \text{Hom}_G(\mathbb{Z}, \cdot)$ is left exact.

Example 1.5.8 (Pulling back module structures)**.** Let $\varphi\colon G \longrightarrow H$ be a group homomorphism. Then the functor $\varphi^*\colon \mathbb{Z}H\text{-Mod} \longrightarrow \mathbb{Z}G\text{-Mod}$ that is given by pulling back the module structures via φ is exact – a sequence of modules over a ring is exact if and only if the underlying sequence of Abelian groups is exact.

Remark 1.5.9 (A more general setup)**.** The notion of exactness cannot only be formulated in module categories over a ring but more general in the setting of so-called Abelian or triangulated categories.

However, by the Freyd-Mitchell embedding theorem, every small (part of an) Abelian category is isomorphic via an exact functor to a full subcategory of some module category [53, 16]. So we do not loose too much in generality by considering only module categories, while keeping the comfort of proofs using "elements."

1.5.2 Derived functors, schematically

The derived functor of a left or right exact functor is an "exact extension" of the functor in question, which is universal in a certain sense.

Definition 1.5.10 (Homological ∂-functor, cohomological δ-functor)**.** Let R and S be two rings.

– A *homological* ∂*-functor* is an additive functor $T_*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ together with S-morphisms $(\partial_n\colon T_n(A'') \longrightarrow T_{n-1}(A'))_{n\in\mathbb{N}}$ defined for every short exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules such that T_* and ∂_* fit together in natural long exact sequences. I.e., for every commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows, the associated diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & T_n(A') & \longrightarrow & T_n(A) & \longrightarrow & T_n(A'') & \xrightarrow{\partial_n} & T_{n-1}(A') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & T_n(B') & \longrightarrow & T_n(B) & \longrightarrow & T_n(B'') & \xrightarrow[\partial_n]{} & T_{n-1}(B') & \longrightarrow & \cdots \end{array}$$

of S-modules is commutative and has exact rows (here, the upper ∂_n is the morphism associated with the upper short exact sequence of R-modules, and the lower ∂_n is the morphism associated with the lower short exact sequence of R-modules).

– Similarly, a (covariant) *cohomological* δ*-functor* is an additive functor $T^*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ together with a sequence of S-morphisms $(\delta^n\colon T^n(A'') \longrightarrow T^{n+1}(A'))_{n\in\mathbb{N}}$ defined for every short exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules such that T^* and δ^* fit together in natural long exact sequences.

Definition 1.5.11 (Left derived functor)**.** Let R and S be two rings, and let $F\colon R\text{-Mod} \longrightarrow S\text{-Mod}$ be a right exact functor. A homological ∂-functor $L_*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ is a *left derived functor of* F if it satisfies the following properties:

– *Extension.* The functor L_* extends F in the sense that L_0 and F are naturally isomorphic functors $R\text{-Mod} \longrightarrow S\text{-Mod}$.

– *Universality.* If $T_*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ is a homological ∂-functor and if $\tau_0\colon T_0 \longrightarrow F$ is a natural transformation, then there is a unique natural transformation $\tau_* \colon T_* \longrightarrow L_*$ of homological ∂-functors extending τ_0.

Definition 1.5.12 (Right derived functor)**.** Let R and S be two rings, and let $F \colon R\text{-Mod} \longrightarrow S\text{-Mod}$ be a left exact functor. A cohomological δ-functor $R^* \colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ is a *right derived functor of* F if it satisfies the following properties:

- *Extension.* The functor R^* extends F in the sense that R^0 and F are naturally isomorphic functors $R\text{-Mod} \longrightarrow S\text{-Mod}$.
- *Universality.* If $T^* \colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ is a cohomological δ-functor and if $\tau^0 \colon F \longrightarrow T^0$ is a natural transformation, then there is a unique natural transformation $\tau^* \colon R^* \longrightarrow T^*$ of homological δ-functors extending τ^0.

In view of universality, derived functors of a given functor are unique (up to canonical natural isomorphism).

Example 1.5.13 (Derived functors of exact functors)**.** Let R and S be rings and let $F \colon R\text{-Mod} \longrightarrow S\text{-Mod}$ be an exact functor. Then the functor $L_* \colon R\text{-Mod} \longrightarrow S\text{-Mod}$ given by $L_0 := F$ and $L_n := 0$ for all $n \in \mathbb{N}_{>0}$ is easily seen to be a left derived functor of F (analogously we obtain a right derived functor).

Moreover, derived functors always exist and can be constructed by the following recipe (as explained in detail in the subsequent sections):

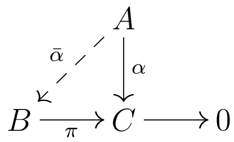
- We replace the objects by a decomposition of the objects into "simpler" objects (projective and injective resolutions).
- Apply the functor in question to these decompositions and measure the failure of exactness via (co)homology.

In the case of group (co)homology, for a fixed group, we will see that group (co)homology is obtained by deriving the coinvariants and the invariants functors.

1.5.3 Projective and injective resolutions

One of the fundamental ideas in homological algebra is to replace objects by sequences of objects that are easier to understand. Tractable objects in the sense of homological algebra are projectives (a generalisation of free modules) and injectives (a generalisation of divisible Abelian groups).

Definition 1.5.14 (Projective modules)**.** Let R be a ring. An R-module A is *projective* if it has the following lifting property: For every surjective R-homomorphism $\pi\colon B \longrightarrow C$ and every R-homomorphism $\alpha\colon A \longrightarrow C$ there is an R-homomorphism $\bar{\alpha}\colon A \longrightarrow B$ such that $\pi \circ \bar{\alpha} = \alpha$. Schematically:



Proposition 1.5.15 (Characterisations of projectivity)**.** *Let R be a ring, and let A be an R-module. Then the following are equivalent:*

1. *The module A is projective.*
2. *The module A is a direct summand in a free R-module.*
3. *The functor* $\operatorname{Hom}_R(A, \cdot\,)\colon R\text{-Mod} \longrightarrow \text{Ab}$ *is exact.*
4. *Every short exact sequence* $0 \to B' \to B \to A \to 0$ *of R-modules splits.*

Proof. Using the definition of projectivity it is not difficult to see that 1 and 2 are equivalent, and that 1 and 3 are equivalent. Moreover, it is easily seen that 2 and 4 are equivalent. □

Example 1.5.16 (Projective modules)**.** Of course, all free modules are projective. The converse is not true in general: For example, the module $\mathbb{Z} \times \{0\}$ is a projective $\mathbb{Z} \times \mathbb{Z}$-module, but not a free $\mathbb{Z} \times \mathbb{Z}$-module.

Other prominent examples of projective modules that are not free occur naturally in the context of topological K-theory.

Remark 1.5.17 (Flat modules)**.** All projective modules are flat: Let R be a ring, and let A be a projective R-module, i.e., A is a direct summand in a free R-module. Hence, the functor $\cdot \otimes_R A\colon \text{Mod-}R \longrightarrow \text{Ab}$ is exact; in other words, the R-module A is *flat*.

Notice that not all flat modules are projective; for example, the $\mathbb{Z}$-module $\mathbb{Q}$ is flat, but not a direct summand in a free $\mathbb{Z}$-module and therefore not projective.

The dual concept to projectivity is injectivity – on a formal level, it can be obtained by reversing arrows:

Definition 1.5.18 (Injective modules)**.** Let R be a ring. An R-module A is called *injective* if it satisfies the following extension property: For every injective R-module homomorphism $i \colon B \longrightarrow C$ and every R-homomorphism $\alpha \colon B \longrightarrow A$ there is an extension $\bar{\alpha} \colon C \longrightarrow A$ such that $\bar{\alpha} \circ i = \alpha$. Schematically:

$$\begin{array}{ccccc} & & A & & \\ & & \uparrow{\scriptstyle\alpha} & \nwarrow{\scriptstyle\bar{\alpha}} & \\ 0 & \longrightarrow & B & \xrightarrow[i]{} & C \end{array}$$

Remark 1.5.19 (Injective modules and homomorphism modules)**.** Let R be a ring and let A be a left R-module. Then the homomorphism functor $\operatorname{Hom}_R(\,\cdot\,, A) \colon R\text{-Mod} \longrightarrow \mathrm{Ab}$ is exact if and only if the R-module A is injective.

Example 1.5.20 (Injective modules)**.**

- The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective (apply Zorn's Lemma to the partially ordered set of extensions).
- Let R be a ring. Then left the R-module

  $$R' := \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$$

  (where we consider the first argument R as right R-module) is injective: Apply Remark 1.5.19 to

  $$\begin{aligned} \operatorname{Hom}_R(\,\cdot\,, R') &\cong \operatorname{Hom}_R\bigl(\,\cdot\,, \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})\bigr) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

  and the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. (Notice that a sequence of R-modules is exact if and only if the underlying sequence of $\mathbb{Z}$-modules is exact).

The replacement of objects by simpler objects mentioned above is formalised by the notions of projective and injective resolutions:

Definition 1.5.21 (Projective/injective resolutions)**.** Let R be a ring, and let A be an R-module.

– A *projective resolution* of A is a chain complex (P_*, ∂_*) of projective R-modules together with an R-homomorphism $\varepsilon\colon P_0 \longrightarrow A$ such that the concatenated sequence

$$\cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

is exact; we denote the concatenated sequence by $P_* \mathbin{\square} \varepsilon$.

– Dually, an *injective resolution* of A is a cochain complex (I^*, δ^*) of injective R-modules together with an R-homomorphism $\eta\colon A \longrightarrow I_0$ such that the concatenated sequence

$$0 \longrightarrow A \xrightarrow{\eta} I^0 \xrightarrow{\delta^0} I^1 \longrightarrow \cdots$$

is exact; we denote the concatenated sequence by $\eta \mathbin{\square} I^*$.

Example 1.5.22 (Bar resolution)**.** The bar resolution $C_*(G)$ is a projective $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$: The complex

$$\begin{array}{ccccccc} \cdots \xrightarrow{\partial_2} & C_1(G) & \xrightarrow{\partial_1} & C_0(G) & \xrightarrow{\varepsilon} & \mathbb{Z} \\ & & & g & \longmapsto & 1 \end{array}$$

is exact, as can be seen via an explicit $\mathbb{Z}$-chain contraction. Moreover, for all $n \in \mathbb{N}$ the n-th chain module

$$C_n(G) = \bigoplus_{g \in G^{n+1}} \mathbb{Z} \cdot g_0 \cdot [g_1|\cdots|g_n] \cong \bigoplus_{g \in G^n} \mathbb{Z}G \cdot [g_1|\cdots|g_n]$$

is a free (thus projective) $\mathbb{Z}G$-module (recall the definition of the G-action on $C_n(G)$ given in Definition 1.4.1).

However, in general, $C_*(G) \otimes_{\mathbb{Z}} A$ is not a projective $\mathbb{Z}G$-resolution of the $\mathbb{Z}G$-module A.

Example 1.5.23 (Classifying spaces)**.** Let G be a discrete group and let X_G be a model of the classifying space BG. Because the universal covering $\widetilde{X_G}$ is contractible, the (singular or cellular) chain complex $C_*(\widetilde{X_G}; \mathbb{Z})$ can be canonically extended by a surjective augmentation $\varepsilon\colon C_0(\widetilde{X_G}; \mathbb{Z}) \longrightarrow H_0(\widetilde{X_G}; \mathbb{Z}) = \mathbb{Z}$ such that $C_*(\widetilde{X_G}; \mathbb{Z}) \mathbin{\square} \varepsilon$ is exact.

Moreover, by the definition of the singular/cellular chain complex and the fact that G acts freely on $\widetilde{X_G}$, the chain modules $C_n(\widetilde{X_G}; \mathbb{Z})$ are free $\mathbb{Z}G$-modules.

Of course, an important step in our agenda to replace objects by sequences of simpler objects is to prove that such replacements indeed exist for every object:

Proposition 1.5.24 (Enough projectives/enough injectives)**.** *Let R be a ring.*

1. *The category of R-modules has enough projective modules, i.e., every R-module is a quotient of a projective R-module.*
2. *The category of R-modules has enough injective modules, i.e., every R-module is a submodule of an injective R-module.*

Proof. Clearly every R-module is a quotient of a free R-module (e.g., for an R-module A we could consider the free module generated by the set A and map every generator to the corresponding element in A).

Moreover, every R-module A is a submodule of an injective R-module: It is not difficult to see that products of injective modules are injective; so $\prod_{\operatorname{Hom}_R(A,R')} R'$, where $R' := \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, is an injective module by Example 1.5.20. The evaluation homomorphism

$$\begin{aligned} A &\longrightarrow \prod_{\operatorname{Hom}_R(A,R')} R' \\ a &\longmapsto \bigl(f(a)\bigr)_{f\in \operatorname{Hom}_R(A,R')} \end{aligned}$$

is injective: If $a \in A \setminus \{0\}$, then using the injectivity of $\mathbb{Q}/\mathbb{Z}$ over $\mathbb{Z}$ we find a $\mathbb{Z}$-homomorphism $\bar{f}\colon A \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $\bar{f}(a) \neq 0$; then

$$\begin{aligned} f\colon A &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) = R' \\ b &\longmapsto \bigl(r \mapsto \bar{f}(r \cdot b)\bigr) \end{aligned}$$

is an R-homomorphism with $f(a) \neq 0$. □

Caveat 1.5.25 (Enough projectives/injectives in Abelian categories)**.** By the Freyd-Mitchell embedding theorem every small Abelian category can be viewed as a full subcategory of a module category; however, even though module categories contain enough projectives and enough injectives, not all subcategories of module categories do so. Therefore, there exist Abelian categories that do *not* have enough projectives/injectives.

Corollary 1.5.26 (Existence of resolutions)**.** *Let R be a ring.*

1. *Every R-module has a projective resolution.*
2. *Every R-module has an injective resolution.*

Proof. The proofs of both cases are similar, so we only prove the claim about the projective resolutions. Let A be an R-module. We inductively construct a projective resolution of A:

Because the category of R-modules has enough projectives (Proposition 1.5.24), there is a projective R-module P_0 admitting an epimorphism

$$\pi_0\colon P_0 \longrightarrow A.$$

Now let $n \in \mathbb{N}$ and suppose inductively, that we already constructed a partial projective resolution

$$P_n \xrightarrow{\pi_n} P_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A.$$

Because the category of R-modules has enough projectives, there is a projective R-module P_{n+1} admitting an epimorphism $\pi_{n+1}\colon P_{n+1} \longrightarrow \ker \pi_n$. Then the sequence

$$P_{n+1} \xrightarrow{\pi_{n+1}} P_n \xrightarrow{\pi_n} P_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A$$

is exact, which completes the induction step. □

1.5.4 The fundamental lemma of homological algebra

As next step, we will prove that the replacements of objects (i.e., projective and injective resolutions) are essentially unique:

Proposition 1.5.27 (Fundamental lemma of homological algebra)**.** *Let R be a ring, let A and B be two R-modules, and let $f\colon A \longrightarrow B$ be an R-module homomorphism.*

1. *Let $P_* \square (\varepsilon\colon P_0 \to A)$ be an R-chain complex where all P_n are projective, and let $C_* \square (\gamma\colon C_0 \to B)$ be an exact sequence of R-modules. Then f can be extended to a chain map $f_* \square f\colon P_* \square \varepsilon \longrightarrow C_* \square \gamma$; moreover, the extension $f_*\colon P_* \longrightarrow C_*$ is unique up to R-chain homotopy.*

2. *Let* $(\eta\colon B \to I^0) \square I^*$ *be an* R*-cochain complex where all modules* I^n *are injective, and let* $(\gamma\colon A \to C^0) \square C^*$ *be an exact cochain complex of* R*-modules.*
   Then f *can be extended to a cochain map* $f \square f^*\colon \gamma \square C^* \longrightarrow \eta \square I^*$*; moreover, the extension* $f^*\colon C^* \longrightarrow I^*$ *is unique up to* R*-cochain homotopy.*

Proof. The statement about injective resolutions can be proved in the same way as the statement about projective resolutions. Moreover, we only sketch the proof of the existence part (the uniqueness part follows by similar arguments [53, Theorem 2.2.6]); to this end, we inductively construct an extension

$$f_* \square f\colon P_* \square \varepsilon \longrightarrow C_* \square \gamma$$

of f: Because P_0 is projective and γ is surjective, we find an R-homomorphism $f_0\colon P_0 \longrightarrow C_0$ making the diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{\varepsilon} & A \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ C_0 & \xrightarrow[\gamma]{} & B \end{array}$$

commutative.

Let $n \in \mathbb{N}$ and suppose inductively that a chain map $f_*\colon P_* \square \varepsilon \longrightarrow C_* \square \gamma$ extending f is constructed up to degree n. In order to construct f_{n+1} we proceed as follows: Because $C_* \square \gamma$ is exact, $\operatorname{im} f_n \circ \partial^P_{n+1} \subset \ker \partial^C_n = \operatorname{im} \partial^C_{n+1}$. Using projectivity of P_{n+1} we obtain an R-homomorphism f_{n+1} fitting into the commutative diagram

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\partial^P_{n+1}} & P_n & \xrightarrow{\partial^P_n} & P_{n-1} \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C_{n+1} & \xrightarrow[\partial^C_{n+1}]{} & C_n & \xrightarrow[\partial^C_n]{} & C_{n-1} \end{array}$$

as desired. $\square$

Corollary 1.5.28 (Uniqueness of resolutions)**.** *Let* R *be a ring and let* A *an* R*-module.*

1. *Then up to canonical R-chain homotopy equivalence there is exactly one projective resolution of A. I.e., if* $P_* \,\square\, \varepsilon$ *and* $P'_* \,\square\, \varepsilon'$ *are two projective R-resolutions of A, then there is a canonical R-chain homotopy equivalence* $P_* \simeq P'_*$.
2. *Dually, up to canonical R-chain homotopy equivalence there is exactly one injective resolution of A.*

Proof. The existence of projective and injective resolutions is provided by Proposition 1.5.26, uniqueness follows from the fundamental lemma (applied to the identity map $\mathrm{id}_A\colon A \longrightarrow A$). □

Another invaluable tool in the context of derived functors is the following observation, which roughly says that projective resolutions can be manifactured in a natural way for short exact sequences of modules:

Proposition 1.5.29 (Horseshoe lemma)**.** *Let R be a ring, let*

$$0 \longrightarrow A' \xrightarrow{f'} A \xrightarrow{f''} A'' \longrightarrow 0$$

be a short exact sequence of R-modules, and suppose that $P'_* \,\square\, \varepsilon'$ *and* $P''_* \,\square\, \varepsilon''$ *are projective resolutions of* A' *and* A'' *respectively. Then there exists a projective resolution* $P_* \,\square\, \varepsilon$ *of A and R-chain maps* $f'_*\colon P'_* \,\square\, \varepsilon' \longrightarrow P_* \,\square\, \varepsilon$ *and* $f''_*\colon P_* \,\square\, \varepsilon \longrightarrow P''_* \,\square\, \varepsilon''$ *extending* f' *and* f'' *respectively such that*

$$0 \longrightarrow P'_n \xrightarrow{f'_n} P_n \xrightarrow{f''_n} P''_n \longrightarrow 0$$

is an exact sequence in every degree $n \in \mathbb{N}$.

Proof. We set

$$P_n := P'_n \oplus P''_n$$

for all $n \in \mathbb{N}$, and we define $f'_n\colon P'_n \longrightarrow P_n$ and $f''_n\colon P_n \longrightarrow P''_n$ to be the canonical injection and the canonical projection respectively; of course, this is basically the only way to construct a resolution of A inducing short exact sequences in each degree, because short exact sequences over projective modules split.

Similar to the arguments in the proof of the fundamental lemma of homological algebra, using projectivity we inductively construct a surjective R-homomorphism $\varepsilon\colon P_0 \longrightarrow A$ and R-homomorphisms $P_n \longrightarrow P_{n-1}$ fitting

$$\begin{array}{cccccccccccc}
 & 0 & & 0 & & \cdots & & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\longrightarrow & P'_n & \longrightarrow & P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & P'_0 & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \\
 & {\scriptstyle f'_n}\downarrow & & {\scriptstyle f'_{n-1}}\downarrow & & & & \downarrow{\scriptstyle f'_0} & & \downarrow{\scriptstyle f'} & & \\
 & P_n & \dashrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
 & {\scriptstyle f''_n}\downarrow & & {\scriptstyle f''_{n-1}}\downarrow & & & & \downarrow{\scriptstyle f''_0} & & \downarrow{\scriptstyle f''} & & \\
\longrightarrow & P''_n & \longrightarrow & P''_{n-1} & \longrightarrow & \cdots & \longrightarrow & P''_0 & \xrightarrow{\varepsilon''} & A'' & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & \cdots & & 0 & & 0 & &
\end{array}$$

Figure 1.4: Proof of the horseshoe lemma

into the commutative "horseshoe" diagrams in Figure 1.4 (in such a way that the middle row is a complex).

That $P_* \square \varepsilon$ indeed is a resolution of A follows from the snake lemma and the five lemma. □

Proposition 1.5.30 (Horseshoe lemma, naturality)**.** *Let R be a ring and let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of R-modules with exact rows. Then there exist projective resolutions of these six modules together with chain maps between them extending this ladder diagram such that in every degree we obtain a corresponding commutative diagram with exact rows.

Proof. We choose any projective resolutions of the outer modules A', B', A'', and B'' and lift the outer vertical maps to chain maps between these resolutions. As next step, we choose projective resolutions of A and B as in the horseshoe lemma. To finish the proof we need a lift of the middle vertical morphism to the projective resolutions of A and B such that the corresponding ladder diagrams in each degree are commutative; this again is an exercise in juggling with projectives and inductive constructions of chain maps [53, Proof of Theorem 2.4.6]. □

Similarly, horseshoes of injective resolutions can (naturally) be filled with injective resolutions.

1.5.5 Derived functors, construction

Using the techniques developed in the previous sections, we finally can prove the existence of derived functors:

Theorem 1.5.31 (Left derived functors)**.** *Let R and S be two rings, and let $F\colon R\text{-Mod} \longrightarrow S\text{-Mod}$ be a right exact functor. Then there exists a left derived functor of F (and by definition of left derived functors, it is essentially unqiue).*

Proof (of Theorem 1.5.31).

- *Construction.* Following the recipe given in Section 1.5.2, we construct a left derived functor $L_*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$: For every R-module A we choose a projective R-resolution $P^A_* \,\square\, (\varepsilon_A\colon P^A_0 \to A)$ of A; then we define
  $$L_n(A) := H_n\bigl(F(P^A_*)\bigr)$$
  for all $n \in \mathbb{N}$. If $f\colon A \longrightarrow B$ is an R-homomorphism, we choose a lift $f_*\colon P^A_* \,\square\, \varepsilon_A \longrightarrow P^B_* \,\square\, \varepsilon_B$ of f, and define
  $$L_n(f) := H_n\bigl(F(f_*)\bigr)\colon L_n(A) \longrightarrow L_n(B)$$
  for all $n \in \mathbb{N}$. Because projective resolutions and lifts of homomorphisms to projective resolutions are essentially unique by the fundamental lemma of homological algebra (Proposition 1.5.27), the definition of L_* is (up to canonical natural isomorphism) independent of these choices and L_* indeed is a functor.
  Notice that $L_n(P) = 0$ for all projective R-modules and all degrees $n \in \mathbb{N}_{>0}$, because the projective module P admits a projective resolution concentrated in degree 0:
  $$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\mathrm{id}_P} P \longrightarrow 0.$$

– *The functor $L_* \colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ extends F.* Let $f\colon A \longrightarrow B$ be an R-homomorphism and let $f_* \square f\colon P_*^A \square \varepsilon_A \longrightarrow P_*^B \square \varepsilon_B$ be as above; in particular, there is a commutative diagram

$$\begin{array}{ccccccc}
P_1^A & \xrightarrow{\partial_1^A} & P_0^A & \xrightarrow{\varepsilon_A} & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
P_1^B & \xrightarrow[\partial_1^B]{} & P_0^B & \xrightarrow[\varepsilon_B]{} & B & \longrightarrow & 0
\end{array}$$

with exact rows. Applying F and homology, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
L_0(A) & = & H_0(F(P_*^A)) & = & F(P_0^A)/F(\operatorname{im}\partial_1^A) & = & F(P_0^A)/F(\ker\varepsilon_A) & = & F(A) \\
\downarrow{\scriptstyle L_0(f)} & & \downarrow{\scriptstyle H_0(F(f_*))} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle F(f)} \\
L_0(B) & = & H_0(F(P_*^B)) & = & F(P_0^B)/F(\operatorname{im}\partial_1^B) & = & F(P_0^B)/F(\ker\varepsilon_B) & = & F(B)
\end{array}$$

because F is right exact. Therefore, L_0 coincides with F.

– *The functor $L_* \colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ is a homological ∂-functor.* We now construct the natural transformations $(\partial_n)_{n\in\mathbb{N}}$: Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of R-modules with exact rows. According to the extended horseshoe lemma (Proposition 1.5.30), there are projective resolutions $Q_*^{A'} \square \varepsilon'$, $Q_*^{A} \square \varepsilon$, $Q_*^{A''} \square \varepsilon''$, $Q_*^{B'} \square \eta'$, $Q_*^{B} \square \eta$, $Q_*^{B''} \square \eta''$ of these six modules such that the corresponding ladder diagrams in each degree are commutative and have exact rows; because of projectivity, these exact rows are even split exact sequences. Therefore, applying the additive functor F, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(Q_*^{A'}) & \longrightarrow & F(Q_*^{A}) & \longrightarrow & F(Q_*^{A''}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(Q_*^{B'}) & \longrightarrow & F(Q_*^{B}) & \longrightarrow & F(Q_*^{B''}) & \longrightarrow & 0
\end{array}$$

of R-chain complexes whose rows are exact in every degree. Now the snake lemma provides us with natural exact sequences in homology. Because the chain complexes $F(Q^?_*)$ and $F(P^?_*)$ are canonically chain homotopic by the fundamental lemma of homological algebra, we also obtain natural long exact sequences in homology when we replace the resolutions $Q^?_*$ by the resolutions $P^?_*$ we chose at the begin of the proof.
Translated into L_* this is nothing but saying that we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_n(A') & \longrightarrow & L_n(A) & \longrightarrow & L_n(A'') & \xrightarrow{\partial_n} & L_{n-1}(A') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & L_n(B') & \longrightarrow & L_n(B) & \longrightarrow & L_n(B'') & \xrightarrow[\partial_n]{} & L_{n-1}(B') & \longrightarrow & \cdots \end{array}$$

with exact rows. Furthermore, by construction, L_* is additive. So, L_* is a homological ∂-functor.

– *The functor L_* is universal.* Let $T_*\colon R\text{-Mod} \longrightarrow S\text{-Mod}_*$ be a homological ∂-functor and suppose there is a natural transformation $\tau_0\colon T_0 \longrightarrow F = L_0$. We inductively extend τ_0 to a natural transformation $\tau_*\colon T_* \longrightarrow L_*$ by *dimension shifting*: Let $n \in \mathbb{N}$ and suppose that $\tau_n\colon T_n \longrightarrow L_n$ is already constructed; we now construct τ_{n+1}: Let A be an R-module. Then there is a short exact sequence

$$0 \longrightarrow A' \xrightarrow{i_A} P \xrightarrow{\pi_A} A \longrightarrow 0$$

of R-modules, where P is projective. Because T_* and L_* are homological ∂-functors, because $L_{n+1}(P) = 0$, and because τ_n is natural, we obtain the following commutative diagram (the solid arrows) with exact rows:

$$\begin{array}{ccccccc} & & T_{n+1}(A) & \xrightarrow{\partial_{n+1}} & T_n(A') & \xrightarrow{T_n(i_A)} & T_n(P) \\ & & \Big\downarrow{\scriptstyle \tau_{n+1}(A)} & & \Big\downarrow{\scriptstyle \tau_n(A')} & & \Big\downarrow{\scriptstyle \tau_n(P)} \\ 0 = L_{n+1}(P) & \longrightarrow & L_{n+1}(A) & \xrightarrow[\partial_{n+1}]{} & L_n(A') & \xrightarrow[L_n(i_A)]{} & L_n(P) \end{array}$$

A simple diagram chase shows that there is a unique S-homomorphism $\tau_{n+1}(A)$ making the left hand square commutative. (The lower line of this diagram explains the term "dimension shifting": The module $L_{n+1}(A)$ can be recovered from L_n applied to a different module A'.)

We now have to show that $\tau_{n+1}(A)$ indeed is compatible with the connecting homomorphisms ∂_{n+1} of *all* short exact sequences with A as quotient and that τ_{n+1} is compatible with homomorphisms of modules.

1. *The homomorphism $\tau_{n+1}(A)$ is compatible with ∂_{n+1}.* Let

$$0 \longrightarrow B' \longrightarrow B \longrightarrow A \longrightarrow 0$$

be a short exact sequence of R-modules. As P is projective, a diagram chase reveals that there are R-homomorphisms f and f' fitting into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & P_A & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We now consider the diagram in Figure 1.5; The small outer four quadrangles commute because T_* and L_* are homological ∂-functors and because τ_n is natural by induction. Moreover, the large outer square commutes by construction of $\tau_{n+1}(A)$. Hence, also the inner square is commutative, as was to be shown.

2. *Moreover, τ_{n+1} is compatible with homomorphisms of modules.* Let $f\colon A \longrightarrow B$ be an R-homomorphism. Let

$$0 \longrightarrow A' \longrightarrow P_A \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow B' \longrightarrow P_B \longrightarrow B \longrightarrow 0$$

be the short exact sequences with projective R-modules P_A and P_B used in the definition of $\tau_{n+1}(A)$ and $\tau_{n+1}(B)$, respectively. Similarly as above, we can find R-homomorphisms p and f' fitting into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & P_A & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle f}\downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & P_B & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

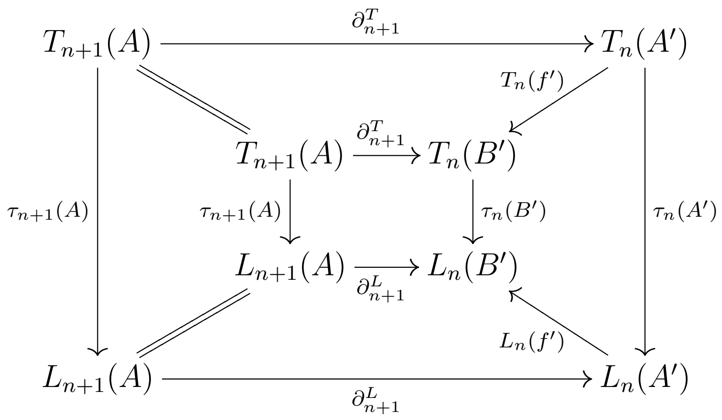


Figure 1.5: Compatibility of $\tau_{n+1}(A)$ with ∂_{n+1}

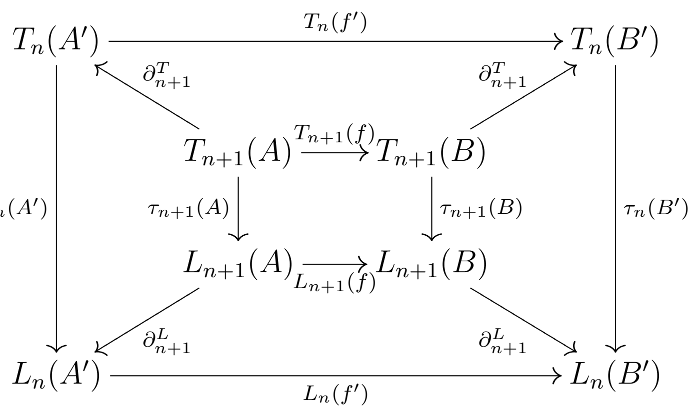


Figure 1.6: Naturality of τ_{n+1}

Considering the diagram in Figure 1.6 and arguing as above, we find that

$$\partial_{n+1}^L \circ \tau_{n+1}(B) \circ T_{n+1}(f) = \partial_{n+1}^L \circ L_{n+1}(f) \circ \tau_{n+1}(A).$$

Because $L_{n+1}(P_B) = 0$, the morphism $\partial_{n+1}^L \colon L_{n+1}(B) \longrightarrow L_n(B')$ is injective, and thus we obtain

$$\tau_{n+1}(B) \circ T_{n+1}(f) = L_{n+1}(f) \circ \tau_{n+1}(A).$$

as desired.

This finishes the proof that L_* indeed is a left derived functor of F. □

Replacing projectives by injectives, we can show with the same type of arguments that left exact functors admit right derived functors:

Theorem 1.5.32 (Right derived functors)**.** *Let R and S be two rings, and let $F\colon R$-Mod $\longrightarrow S$-Mod be a left exact functor. Then there exists a right derived functor of F (and by definition of right derived functors, it is essentially unique).*

Prominent examples of derived functors are Tor and Ext, which play a crucial rôle in the Künneth formula and the universal coefficient theorem, as well as in group (co)homology (see below):

Definition 1.5.33 (Tor and Ext)**.** Let R be a ring, and let A be a left R-module.

- Then we define

$$\operatorname{Tor}_*^R(\,\cdot\,, A)\colon \text{Mod-}R \longrightarrow \text{Ab}_*$$

as the left derived functor of $\,\cdot \otimes_R A\colon \text{Mod-}R \longrightarrow \text{Ab}$.

- And we define

$$\operatorname{Ext}_R^*(A, \,\cdot\,)\colon R\text{-Mod} \longrightarrow \text{Ab}_*$$

as the right derived functor of $\operatorname{Hom}_R(A, \,\cdot\,)\colon R\text{-Mod} \longrightarrow \text{Ab}$.

Remark 1.5.34 (Etymology of Tor and Ext)**.** The functor Tor is related to torsion in modules (e.g., over the integers); the functor Ext is related to extension problems (such as in Section 1.4.4).

Further examples of derived functors are the higher lim-terms (derived functor of inverse limits), sheaf cohomology (derived functor of the sections functor), and higher direct images in sheaf theory (derived functor of direct images).

1.5.6 Group cohomology, via derived functors

Using the fact that the coinvariants functor is right exact and that the invariants functor is left exact (Example 1.5.7), we can define group (co)homology in terms of derived functors:

Definition 1.5.35 (Group homology, as derived functor)**.** *Group homology* is the functor $H_*\colon \text{GrpMod} \longrightarrow \text{Ab}_*$ defined as follows:

- If G is a discrete group, then

$$H_*(G;\,\cdot\,) := \operatorname{Tor}_*^{\mathbb{Z}G}(\bar{\cdot},\mathbb{Z})\colon \mathbb{Z}G\text{-Mod} \longrightarrow \text{Ab}_*$$

  is the derived functor of $\cdot_G = \cdot \otimes_G \mathbb{Z}\colon \mathbb{Z}G\text{-Mod} \longrightarrow \text{Ab}$ of G; here, G acts trivially on $\mathbb{Z}$, and $\bar{\cdot}$ denotes the conversion of left $\mathbb{Z}G$-modules into right $\mathbb{Z}G$-modules using the canonical involution on $\mathbb{Z}G$.
- If $\varphi\colon G \longrightarrow H$ is a group homomorphism, then by universality of the derived functor $H_*(H;\,\cdot\,)$, there is a unique natural transformation

$$\tau_*^\varphi\colon H_*(G;\varphi^*\cdot\,) \longrightarrow H_*(H;\,\cdot\,)$$

  of homological ∂-functors $\mathbb{Z}H\text{-Mod} \longrightarrow \text{Ab}_*$ extending the natural transformation $(\varphi^*\cdot\,)_G \longrightarrow \cdot_H$ induced by φ; notice that $H_*(G;\varphi^*\cdot\,)$ indeed is a homological ∂-functor, because $\varphi^*\colon \mathbb{Z}H\text{-Mod} \longrightarrow \mathbb{Z}G\text{-Mod}$ is exact (Example 1.5.8).
- If $(\varphi,\Phi)\colon (G,A) \longrightarrow (H,B)$ is a morphism in GrpMod, then we write

$$H_*(\varphi;\Phi) := \tau_*^\varphi(B) \circ \operatorname{Tor}_*^{\mathbb{Z}G}(\Phi,\mathbb{Z}).$$

  which is a homomorphism $H_*(G;A) \longrightarrow H_*(H;B)$ in Ab_*.

Definition 1.5.36 (Group cohomology, as derived functor)**.** *Group cohomology* is the (contravariant) functor $H^*\colon \text{GrpMod}^- \longrightarrow \text{Ab}_*$ defined as follows:

- If G is a discrete group, then

$$H^*(G;\,\cdot\,) := \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},\,\cdot\,)\colon \mathbb{Z}G\text{-Mod} \longrightarrow \text{Ab}_*$$

  is the derived functor of $\cdot^G = \operatorname{Hom}_G(\mathbb{Z},\,\cdot\,)\colon \mathbb{Z}G\text{-Mod} \longrightarrow \text{Ab}_*$; here, G acts trivially on $\mathbb{Z}$.

– If $\varphi\colon G \longrightarrow H$ is a group homomorphism, then by universality of the derived functor $H^*(H;\,\cdot\,)$ there is a unique natural transformation
$$\tau^*_\varphi\colon H^*(H;\,\cdot\,) \longrightarrow H^*(G;\varphi^*\,\cdot\,)$$
of cohomological δ-functors $\mathbb{Z}H$-Mod $\longrightarrow \mathrm{Ab}_*$ extending the natural transformation $\,\cdot\,^H \longrightarrow (\varphi^*\,\cdot\,)^G$ induced by φ; notice that $H^*(G;\varphi^*\,\cdot\,)$ indeed is a cohomological δ-functor, because φ^* is exact.

– If $(\varphi,\Phi)\colon (G,A) \longrightarrow (H,B)$ is a morhpism in GrpMod⁻, then we write
$$H^*(\varphi;\Phi) := \mathrm{Ext}^*_{\mathbb{Z}G}(\Phi,\mathbb{Z}) \circ \tau^*_\varphi(B),$$
which is a homomorphism $(\varphi,\Phi)\colon (G,A) \longrightarrow (H,B)$.

Notice that group homology and group cohomology defined in this way indeed are functors of the given types; for compatibility with composition, we need the uniqueness of the natural transformations extending (co)invariants (exercise).

Recalling the construction of derived functors via projective/injective resolutions, we see that we can use *any* projective/injective resolution of the coefficient module to compute group (co)homology, which provides significant flexibility. In general, it is quite difficult to find nice “small” resolutions by algebraic means, though. In Section 1.6, we will use geometric input to construct accessible resolutions.

1.5.7 Group cohomology, axiomatically

Imitating the characterisation of derived functors, we are led to the following axiomatic description of group homology and group cohomology.

Theorem 1.5.37 (Group homology, axiomatically)**.** *There exists, up to natural isomorphism, exactly one functor $H_*\colon$ GrpMod $\longrightarrow \mathrm{Ab}_*$ together with connecting homomorphisms $\partial_*\colon H_*(G;A'') \longrightarrow H_{*-1}(G;A')$ for every discrete group G and every short exact sequence $0 \to A' \to A \to A'' \to 0$ of $\mathbb{Z}G$-modules such that the following properties are satisfied:*

– Extension of coinvariants. *The functor H_* extends the coinvariants functor* GrpMod $\longrightarrow$ Ab *(Example 1.2.9), i.e., H_0 and the coinvariants functor are naturally isomorphic.*

– Long exact sequences. *Let G be a discrete group, and let*

$$0 \longrightarrow A' \xrightarrow{\Phi'} A \xrightarrow{\Phi''} A'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{Z}G$*-modules. Then there is a natural (see explanations below) long exact sequence*

$$\cdots \longrightarrow H_n(G;A') \xrightarrow{H_n(\mathrm{id}_G;\Phi')} H_n(G;A) \xrightarrow{H_n(\mathrm{id}_G;\Phi'')} H_n(G;A'') \xrightarrow{\partial_n} H_{n-1}(G;A') \longrightarrow \cdots$$

– Vanishing. *For all discrete groups G, all projective* $\mathbb{Z}G$*-modules P, and all* $n \in \mathbb{N}_{>0}$ *we have*

$$H_n(G;P) = 0.$$

As in the case of derived functors, these axioms are a formal way of saying that all information of the group homology functor is stored in degree 0 (and the coefficient modules); the natural long exact sequences and the vanishing axiom allow to reconstruct the rest of the functor.

Remark 1.5.38. It is natural to ask whether we can interpret the whole group homology functor GrpMod $\longrightarrow$ Ab_* as derived functor of some functor GrpMod $\longrightarrow$ Ab. However, as the category GrpMod is not a nice category (it is not Abelian or exact), this is not possible.

What does "naturality of the long exact sequence" mean in this context? Let $\varphi\colon G \longrightarrow H$ be a group homomorphism, and let

$$\begin{gathered} 0 \longrightarrow A' \xrightarrow{\Phi'} A \xrightarrow{\Phi''} A'' \longrightarrow 0 \\ 0 \longrightarrow B' \xrightarrow[\Psi']{} B \xrightarrow[\Psi'']{} B'' \longrightarrow 0 \end{gathered}$$

be short exact sequences of $\mathbb{Z}G$-modules and $\mathbb{Z}H$-modules respectively. Moreover, suppose there are morphisms

$$\begin{aligned} (\varphi,\Xi')\colon (G,A') &\longrightarrow (H,B') \\ (\varphi,\Xi)\colon (G,A) &\longrightarrow (H,B) \\ (\varphi,\Xi'')\colon (G,A'') &\longrightarrow (H,B'') \end{aligned}$$

in GrpMod making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\Phi'} & A & \xrightarrow{\Phi''} & A'' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Xi'} & & \Big\downarrow{\scriptstyle \Xi} & & \Big\downarrow{\scriptstyle \Xi''} & & \\
0 & \longrightarrow & \varphi^* B' & \xrightarrow[\varphi^* \Psi']{} & \varphi^* B & \xrightarrow[\varphi^* \Psi'']{} & \varphi^* B'' & \longrightarrow & 0
\end{array}$$

of $\mathbb{Z}G$-modules commutative (notice that the lower line is exact as well, because φ^* is an exact functor). Then the corresponding diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(G;A') & \xrightarrow{H_n(\mathrm{id}_G;\Phi')} & H_n(G;A) & \xrightarrow{H_n(\mathrm{id}_G;\Phi'')} & H_n(G;A'') & \xrightarrow{\partial_n} & H_{n-1}(G;A') & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle H_n(\varphi;\Xi')} & & \Big\downarrow{\scriptstyle H_n(\varphi;\Xi)} & & \Big\downarrow{\scriptstyle H_n(\varphi;\Xi'')} & & \Big\downarrow{\scriptstyle H_{n-1}(\varphi;\Xi')} & & \\
\cdots & \longrightarrow & H_n(H;B') & \xrightarrow[H_n(\mathrm{id}_H;\Psi')]{} & H_n(H;B') & \xrightarrow[H_n(\mathrm{id}_H;\Psi'')]{} & H_n(H;B'') & \xrightarrow[\partial_n]{} & H_{n-1}(H;B') & \longrightarrow & \cdots
\end{array}$$

should be commutative as well.

Proof. In Theorems 1.5.40, 1.5.41, 1.5.42 we will establish that such a functor exists.

The rest of this proof is concerned with the uniqueness of such a functor: Let H_* and K_* be two functors GrpMod $\longrightarrow$ Ab_* satisfying the axioms above. Similarly, as in the proof of universality in Theorem 1.5.31, we proceed inductively by *dimension shifting*:

By the first axiom, there is a natural isomorphism $\tau_0 \colon K_0 \longrightarrow H_0$. Suppose $n \in \mathbb{N}$ and that we constructed a natural isomorphism $\tau_n \colon K_n \longrightarrow H_n$; we now construct τ_{n+1}:

Let (G, A) be an object in GrpMod. Then there is a short exact sequence

$$0 \longrightarrow A' \xrightarrow{i_A} P_A \xrightarrow{\pi_A} A \longrightarrow 0$$

of $\mathbb{Z}G$-modules, where P_A is projective. In view of the vanishing axiom, $H_{n+1}(G; P_A) = 0$. Therefore, there is a unique homomorphism

$$\tau_{n+1}(A) \colon K_{n+1}(G; A) \longrightarrow H_{n+1}(G; A)$$

fitting into the commutative diagram

$$\begin{array}{ccccccc}
K_{n+1}(G;P_A) & \longrightarrow & K_{n+1}(G;A) & \xrightarrow{\partial^K_{n+1}} & K_n(G;A') & \longrightarrow & K_n(G;P_A) \\
 & & \Big\downarrow{\scriptstyle \tau_{n+1}(A)} & & \Big\downarrow{\scriptstyle \tau_n(A')} & & \Big\downarrow{\scriptstyle \tau_n(P_A)} \\
0 & \longrightarrow & H_{n+1}(G;A) & \xrightarrow[\partial^H_{n+1}]{} & H_n(G;A') & \longrightarrow & H_n(G;P_A)
\end{array}$$

with exact rows (the rows are exact because of the long exact sequence axiom). The same arguments as in the proof of universality in Theorem 1.5.31 show that τ_{n+1} is compatible with the morphisms ∂_{n+1} and that it is compatible with morphisms in GrpMod; so τ_* is a natural transformation between K_* and H_*.

Reversing the rôles of H_* and K_* und using the uniqueness of the resulting natural transformations we conclude, that H_* and K_* are naturally isomorphic. □

Theorem 1.5.39 (Group cohomology, axiomatically)**.** *There is up to natural isomorphism exactly one contravariant functor* $H^*\colon \text{GrpMod}^- \longrightarrow \text{Ab}_*$ *(together with connecting homomorphisms* $\delta^*\colon H^*(G;A'') \longrightarrow H^{*+1}(G;A')$ *for every discrete group* G *and every short exact sequence* $0 \to A' \to A \to A'' \to 0$ *of* $\mathbb{Z}G$*-modules such that the following properties are satisfied:*

- Extension of invariants. *The functor* H^* *extends the invariants functor* $\text{GrpMod}^- \longrightarrow \text{Ab}$ *(Example 1.2.9), i.e.,* H^0 *and the invariants functor are naturally isomorphic.*
- Long exact sequences. *Let* G *be a discrete group, and let*

  $$0 \longrightarrow A' \xrightarrow{\Phi'} A \xrightarrow{\Phi''} A'' \longrightarrow 0$$

  be a short exact sequence of $\mathbb{Z}G$*-modules. Then there is a natural long exact sequence*

  $$\cdots \longrightarrow H^n(G;A') \xrightarrow{H^n(\mathrm{id}_G;\Phi')} H^n(G;A) \xrightarrow{H^n(\mathrm{id}_G;\Phi'')} H^n(G;A'') \xrightarrow{\delta^n} H^{n+1}(G;A') \longrightarrow \cdots$$

- Vanishing. *For all discrete groups* G*, all injective* $\mathbb{Z}G$*-modules* I*, and all* $n \in \mathbb{N}_{>0}$ *we have*

  $$H^n(G;I) = 0.$$

Proof. Existence is contained in Theorems 1.5.40, 1.5.41, 1.5.42. Uniqueness follows in the same way as in the group homology case. □

Theorem 1.5.40 (Topological definition and the axioms)**.**
- *The topological definition of group homology (Definition 1.3.10) satisfies the axioms of Theorem 1.5.37.*
- *The topological definition of group cohomology (Definition 1.3.11) satisfies the axioms of Theorem 1.5.39.*

Proof. We only give the proof for homology; the proof for cohomology can then be obtained by the usual modifications.
- *Extension of coinvariants.* That the topologically defined functor H_0 coincides with the coinvariants functor is the content of Proposition 1.3.12.
- *Long exact sequences.* Let G be a group and let X_G be a model of the classifying space BG. Then the (singular) chain complex $C_*(\widetilde{X_G};\mathbb{Z})$ of the universal covering is a chain complex consisting of *free* $\mathbb{Z}G$-modules.
  So for any short exact sequence $0 \to A' \to A \to A'' \to 0$ of $\mathbb{Z}G$-modules, the induced sequences
  $$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(X_G;A') & \longrightarrow & C_n(X_G;A) & \longrightarrow & C_n(X_G;A'') & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & C_n(\widetilde{X_G};\mathbb{Z}) \otimes_G A' & \to & C_n(\widetilde{X_G};\mathbb{Z}) \otimes_G A & \to & C_n(\widetilde{X_G};\mathbb{Z}) \otimes_G A'' & \to & 0 \end{array}$$
  are exact in every degree $n \in \mathbb{N}$. Therefore, the snake lemma provides us with a long exact sequence in homology.
  Moreover, this long exact sequence is natural as the construction on the level of chain complexes is natural enough and the snake lemma yields a natural long exact sequence in homology.
- *Vanishing.* Let G be a group, let P be a projective $\mathbb{Z}G$-module, and let X_G be a model of BG. Because $\widetilde{X_G}$ is contractible, we have
  $$H_n\bigl(C_*(\widetilde{X_G};\mathbb{Z})\bigr) = H_n(\widetilde{X_G};\mathbb{Z}) = 0$$
  for all $n \in \mathbb{N}_{>0}$; as P is projective, and hence flat, also the sequence $C_*(X_G;P) = C_*(\widetilde{X_G};\mathbb{Z}) \otimes_G P$ is exact in all degrees greater than 0. In particular,
  $$H_n(G;P) = H_n\bigl(C_*(X_G;P)\bigr) = 0$$

for all $n \in \mathbb{N}_{>0}$. □

Theorem 1.5.41 (Combinatorial definition and the axioms)**.**

- *The combinatorial definition of group homology via the bar resolution (Definition 1.4.4) satisfies the axioms of Theorem 1.5.37.*
- *The combinatorial definition of group cohomology via the bar resolution (Definition 1.4.5) satisfies the axioms of Theorem 1.5.39.*

Proof. This is essentially the same proof as in the topological case. We only give the proof for homology; the proof for cohomology can then be obtained by the usual manipulations.

- *Extension of coinvariants.* This is the same computation as in Proposition 1.3.12 (there the models have been chosen in such a way that the cellular complexes look like the start of the bar resolution).
- *Long exact sequences.* This is the same argument as in the topological case above – the bar construction consists of free modules and the bar construction is functorial on the chain level.
- *Vanishing.* Because $H_n\big(C_*(G)\big) = 0$ for all $n \in \mathbb{N}_{>0}$ and the chain modules are projective, the same reasoning as in the topological case applies. □

Theorem 1.5.42 (Derived functor definition and the axioms)**.**

- *The definition of group homology via deriving coinvariants functors (Definition 1.5.35) satisfies the axioms of Theorem 1.5.37.*
- *The definition of group cohomology via deriving invariants functors (Definition 1.5.36) satisfies the axioms of Theorem 1.5.39.*

Proof. We only give the proof for homology; the proof for cohomology can then be obtained by the usual manipulations.

- *Extension of coinvariants.* This follows from the definition of group homology via derived functors: Let $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ be a morphism in GrpMod. Then the diagram in Figure 1.7 is commutative; i.e., H_0 coincides with the coinvariants functor.
- *Long exact sequences.* By definition of group homology via derived functors we obtain long exact sequences in group homology, and these long exact sequences are natural in the module parameter if the group is fixed.

$$
\begin{array}{ccccccccc}
H_0(G;A) & = & \operatorname{Tor}_0^{\mathbb{Z}G}(A,\mathbb{Z}) & = & A\otimes_G\mathbb{Z} & = & A_G & & [a] \\
\Big\downarrow{\scriptstyle H_0(\varphi;\Phi)} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_0^{\mathbb{Z}G}(\Phi,\mathbb{Z})} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \operatorname{Tor}_0^{\mathbb{Z}G}(\varphi^*B,\mathbb{Z}) & = & \varphi^*B\otimes_G\mathbb{Z} & = & (\varphi^*B)_G & & \\
 & & \Big\downarrow{\scriptstyle \tau_0^{\varphi}(B)} & & \Big\downarrow & & \Big\downarrow & & \\
H_0(H;B) & = & \operatorname{Tor}_0^{\mathbb{Z}H}(B,\mathbb{Z}) & = & B\otimes_H\mathbb{Z} & = & B_H & & [\Phi(a)]
\end{array}
$$

Figure 1.7: The derived functor version of group homology in degree 0

$$
\begin{array}{ccccccccccc}
\cdots \to & H_n(G;A') & \xrightarrow{H_n(\mathrm{id}_G;\Phi')} & H_n(G;A) & \xrightarrow{H_n(\mathrm{id}_G;\Phi'')} & H_n(G;A'') & \xrightarrow{\partial_n} & H_{n-1}(G;A') & \to \cdots \\
 & \Big\downarrow{\scriptstyle \operatorname{Tor}_n^{\mathbb{Z}G}(\Phi';\mathbb{Z})} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_n^{\mathbb{Z}G}(\Phi;\mathbb{Z})} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_n^{\mathbb{Z}G}(\Phi'';\mathbb{Z})} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_{n-1}^{\mathbb{Z}G}(\Phi';\mathbb{Z})} & \\
\cdots \to & H_n(G;\varphi^*B') & \xrightarrow[H_n(\mathrm{id}_G;\varphi^*\Psi')]{} & H_n(G;\varphi^*B') & \xrightarrow[H_n(\mathrm{id}_G;\varphi^*\Psi'')]{} & H_n(G;\varphi^*B'') & \xrightarrow[\partial_n]{} & H_{n-1}(H;\varphi^*B') & \to \cdots \\
 & \Big\downarrow{\scriptstyle \tau_n^{\varphi}(B')} & & \Big\downarrow{\scriptstyle \tau_n^{\varphi}(B)} & & \Big\downarrow{\scriptstyle \tau_n^{\varphi}(B'')} & & \Big\downarrow{\scriptstyle \tau_{n-1}^{\varphi}(B')} & \\
\cdots \to & H_n(H;B') & \xrightarrow[H_n(\mathrm{id}_H;\Psi')]{} & H_n(H;B') & \xrightarrow[H_n(\mathrm{id}_H;\Psi'')]{} & H_n(H;B'') & \xrightarrow[\partial_n]{} & H_{n-1}(H;B') & \to \cdots
\end{array}
$$

Figure 1.8: Naturality in GrpMod of the long exact sequences of the derived functor version of group homology

Why do we have naturality in GrpMod as described above? For every commutative ladder

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\Phi'} & A & \xrightarrow{\Phi''} & A'' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Xi'} & & \Big\downarrow{\scriptstyle \Xi} & & \Big\downarrow{\scriptstyle \Xi''} & & \\
0 & \longrightarrow & \varphi^*B' & \xrightarrow[\varphi*\Psi']{} & \varphi^*B & \xrightarrow[\varphi*\Psi'']{} & \varphi^*B'' & \longrightarrow & 0
\end{array}
$$

with exact rows as in the explanation of the axioms of group homology above (following Theorem 1.5.37), we obtain a diagram as in Figure 1.8.

Because group homology in the derived functor sense is a homological ∂-functor in the second variable and because φ^* is an exact functor, all three rows are exact. The top ladder commutes because $\operatorname{Tor}_*^{\mathbb{Z}G}(\,\bar{\cdot}\,,\mathbb{Z})$ is a homological ∂-functor; the lower ladder commutes because τ_*^{φ} is a natural transformation of homological ∂-functors. Hence, also the outer ladder has to be commutative. By definition, the compositions of the vertical arrows are nothing but $H_*(\varphi;\Xi')$, $H_*(\varphi;\Xi)$, and $H_*(\varphi;\Xi'')$, which concludes the proof of naturality.

– *Vanishing.* This vanishing property was established in the construction of left derived functors (see proof of Theorem 1.5.31). □

In view of the axiomatic characterisation of group (co)homology, we finally see that our three descriptions of group (co)homology give rise to the same functors; in particular, for problems concerning group (co)homology we are free to choose an appropriate description or method of computation of group (co)homology.

Corollary 1.5.43 (Group (co)homology, equivalence of the three descriptions)**.**

1. *The topological definition of group homology, the combinatorial definition of group homology, and the definition of via deriving coinvariants functors are naturally isomorphic functors* $\mathrm{GrpMod} \longrightarrow \mathrm{Ab}_*$.
2. *The topological definition of group cohomology, the combinatorial definition of group cohomology, and the definition of group cohomology via deriving the invariants functors are naturally isomorphic contravariant functors* $\mathrm{GrpMod}^{-} \longrightarrow \mathrm{Ab}_*$.

1.5.8 Computing group (co)homology – summary

Often one is interested in a (co)homology group only up to isomorphism of a given group and given coefficients and not in the whole group (co)homology functor. In the following, we briefly summarise the different methods for computing group (co)homology established so far (see Corollary 1.5.43). Let G be a discrete group and let A be a $\mathbb{Z}G$-module.

- *Group homology.*
  - Pick a projective $\mathbb{Z}G$-resolution $P_* \square (\varepsilon \colon P_0 \to A)$ of A and compute
  $$H_*(G;A) = \operatorname{Tor}_*^{\mathbb{Z}G}(\bar{\cdot}, \mathbb{Z}) \cong H_*\bigl((P_*)_G\bigr).$$
  - Pick a projective $\mathbb{Z}G$-resolution $P_* \square (\varepsilon \colon P_0 \to \mathbb{Z})$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$; by the fundamental lemma of homological algebra, P_* is $\mathbb{Z}G$-homotopy equivalent to the bar complex $C_*(G)$ of G. Therefore, $P_* \otimes_G A$ and $C_*(G) \otimes_G A$ are $\mathbb{Z}$-homotopy equivalent and so
  $$H_*(G;A) = H_*\bigl(C_*(G) \otimes_G A\bigr) \cong H_*(P_* \otimes_G A).$$
- *Group cohomology.*
  - Pick an injective $\mathbb{Z}G$-resolution $(\eta \colon A \to I^0) \square I^*$ of A and compute
  $$H^*(G;A) = \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \,\cdot\,) \cong H^*\bigl((I^*)^G\bigr).$$
  - Pick a projective $\mathbb{Z}G$-resolution $P_* \square (\varepsilon \colon P_0 \to \mathbb{Z})$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$; by the fundamental lemma of homological algebra, P_* is $\mathbb{Z}G$-homotopy equivalent to the bar complex $C_*(G)$ of G. Therefore, $\operatorname{Hom}_G(P_*, A)$ and $\operatorname{Hom}_G(C_*(G), A)$ are $\mathbb{Z}$-homotopy equivalent and so
  $$H^*(G;A) = H^*\bigl(\operatorname{Hom}_G(C^*(G), A)\bigr) \cong H^*\bigl(\operatorname{Hom}_G(P_*, A)\bigr).$$

For example, nice such projective resolutions $P_* \square \varepsilon$ can be obtained by looking at a suitable model of the classifying space BG or more generally by studying appropriate actions of G.

1.6 Group cohomology and group actions

The full power of the theory of group (co)homology lies in its tight connection with group actions: Nice group actions give rise to nice projective resolutions – and hence to computations of group (co)homology; consequently, group (co)homology also can be viewed as an obstruction for the existence of certain group actions.

In the present section, we will enjoy the interplay between group (co)homology and group actions while studying the question of which finite groups can act freely on spheres. In particular, we will compute the (co)homology of finite cyclic groups (Section 1.6.2), we will classify certain p-groups using group cohomology (Section 1.6.4), and we will deduce an algebraic obstruction for the existence of free actions on spheres in terms of Sylow subgroups (Sections 1.6.1, 1.6.3, 1.6.5).

Furthermore, we explain how to measure the surjectivity of the Hurewicz homomorphism using group homology (Sections 1.6.6 and 1.6.7) and, more generally, we give a brief overview of spectral sequences linking group homology and homology associated with certain group actions (Section 1.6.8) – generalising the description of group homology via classifying spaces in several directions.

1.6.1 Application: Groups acting on spheres I

Finite groups acting freely on spheres admit particularly nice resolutions thanks to the homological structure of spheres:

Theorem 1.6.1 (Groups acting on spheres)**.** *Let G be a finite group acting freely on an n-dimensional sphere S^n.*

1. *If n is even, then G is trivial or $G \cong \mathbb{Z}/2$.*

2. *If n is odd, then there exists a periodic projective $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ of period $n+1$.*

This theorem is the first step towards a classification of all finite groups admitting free actions on a sphere; we will continue this point of view in Section 1.9 by investigating groups with periodic cohomology. A thorough treatment of this classical problem is provided in the lecture notes by Davis and Milgram [11].

The basic idea behind the proof is to consider a cellular $\mathbb{Z}G$-chain complex of S^n and to use the fact that $H_0(S^n;\mathbb{Z}) \cong \mathbb{Z} \cong H_n(S^n;\mathbb{Z})$ in splicing together infinitely many copies of the cellular chain complex to obtain a periodic resolution. However, for this we need to know which $\mathbb{Z}G$-module structure the G-action on S^n induces on $\mathbb{Z} \cong H_n(S^n;\mathbb{Z})$. The key to succesfully studying this $\mathbb{Z}G$-module structure is the Lefschetz fixed point theorem:

Definition 1.6.2 (Lefschetz number)**.** Let X be a finite CW-complex and let $f\colon X \longrightarrow X$ be a continuous map. Then the *Lefschetz number of f* is the integer defined by

$$\Lambda(f) := \sum_{n\in\mathbb{N}} (-1)^n \cdot \operatorname{tr}_{\mathbb{Z}}\bigl(H_n(f;\mathbb{Z})\colon H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z})\bigr);$$

here, $\operatorname{tr}_{\mathbb{Z}}$ denotes the trace given by the usual integral trace of $\mathbb{Z}$-linear endomorphisms on the free part of the corresponding finitely generated Abelian group.

Example 1.6.3 (Lefschetz number and Euler characteristic)**.** If X is a finite CW-complex, then the Lefschetz number of the identitiy map id_X coincides with the Euler characteristic of X:

$$\Lambda(\mathrm{id}_X) = \chi(X)$$

Theorem 1.6.4 (Lefschetz fixed point theorem [21, Theorem 2C.3])**.** *Let X be a finite CW-complex and let $f\colon X \longrightarrow X$ be a continuous map. If f has no fixed points, then $\Lambda(f) = 0$.*

The converse of the Lefschetz fixed point theorem obviously does not hold in general; for example, the identity map on S^1 has Lefschetz number equal to 0 despite of having lots of fixed points.

Corollary 1.6.5 (Groups acting on spheres and top homology)**.** *Let G be a group that acts freely on an n-sphere S^n.*

1. *If n is even, then G is the trivial group or $G \cong \mathbb{Z}/2$.*
2. *If n is odd, then G acts trivially on $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$.*

Proof. As the 0-dimensional sphere S^0 is nothing but a disjoint union of two points (on which only the trivial group and groups isomorphic to $\mathbb{Z}/2$ can act freely) we assume $n > 0$ in the following.

Let $f\colon S^n \longrightarrow S^n$ be the continuous map induced by the action of a given non-trivial element of G on S^n; this map f does not have a fixed point because the G-action on S^n is free. As the homology of S^n is concentrated in the degrees 0 and n (which are distinct!), we obtain by the Lefschetz fixed point theorem

$$\begin{aligned} 0 &= \Lambda(f) \\ &= \operatorname{tr}_{\mathbb{Z}} H_0(f;\mathbb{Z}) + (-1)^n \cdot \operatorname{tr}_{\mathbb{Z}} H_n(f;\mathbb{Z}) \\ &= 1 + (-1)^n \cdot \deg f. \end{aligned}$$

So, if n is odd, then $\deg f = 1$ and hence $H_n(f;\mathbb{Z}) = \mathrm{id}$. Thus G acts trivially on $H_n(S^n;\mathbb{Z})$.

If n is even, then $\deg f = -1$; thus, all non-trivial elements of G act by multiplication by -1 on $H_n(S^n;\mathbb{Z})$. Consequently, if g, $g' \in G \setminus \{1\}$, then $g \cdot g'$ acts by multiplication by 1 on $H_n(S^n;\mathbb{Z})$; therefore, $g \cdot g' = 1$. In particular, $G = 1$ or $G \cong \mathbb{Z}/2$. □

Proof (of Theorem 1.6.1). If n is even, then G is trivial or isomorphic to $\mathbb{Z}/2$ by Corollary 1.6.5. Notice that the cellular chain complex of $\mathbb{RP}^\infty$, which is a model of $B\mathbb{Z}/2$, associated with the standard cell structure gives rise to a projective $\mathbb{Z}[\mathbb{Z}/2]$-resolution

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathbb{Z}$ of period 2 (where t denotes the generator of $\mathbb{Z}/2$ and $\varepsilon\colon \mathbb{Z}[\mathbb{Z}/2] \longrightarrow \mathbb{Z}$ is the homomorphism sending 1 and t to 1).

If n is odd, we argue as follows: We choose a G-equivariant CW-structure on S^n such that the G-action freely permutes the cells (such a cell structure always exists [**?**]). Let C_* be the associated cellular chain complex; because G acts freely on the cells, the complex C_* in fact is a $\mathbb{Z}G$-complex consisting of free $\mathbb{Z}G$-modules. Notice that $C_k = 0$ for all $k \in \mathbb{N}_{>n}$ because S^n is n-dimensional. Let

$$\eta\colon \mathbb{Z} \cong H_n(S^n;\mathbb{Z}) = \ker \partial_n \longrightarrow C_n,$$
$$\varepsilon\colon C_0 \longrightarrow C_0/\operatorname{im}\partial_1 = H_0(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

be the inclusion and projection, respectively; hence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is exact. Because the G-action on $\mathbb{Z} \cong H_0(S^n;\mathbb{Z})$ induced by the G-action is trivial (continuous maps between connected spaces induce the identity on zero-th homology), and because the G-action on $\mathbb{Z} \cong H_n(S^n;\mathbb{Z})$ induced by the G-action on S^n is trivial by Corollary 1.6.5, this is an exact sequence in $\mathbb{Z}G$-Mod.

We can now splice together infinitely many copies of these exact sequences and thus obtain a free $\mathbb{Z}G$-resolution

$$\cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\eta\circ\varepsilon} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ of period $n+1$. □

1.6.2 (Co)Homology of finite cyclic groups

A prominent class of groups admitting nice actions on spheres is the class of finite cyclic groups; therefore, we can apply Theorem 1.6.1 to obtain a straightforward computation of (co)homology of finite cyclic groups. In a way, this is a generalisation of the computation of (co)homology of $\mathbb{Z}/2$ via $\mathbb{R}\mathrm{P}^\infty \cong B\mathbb{Z}/2$ – the geometric analogue of the algebraic resolutions produced by Theorem 1.6.1 are increasing unions of lens spaces.

Corollary 1.6.6 ((Co)Homology of finite cyclic groups)**.** *Let* $n \in \mathbb{N}_{>0}$, *let*

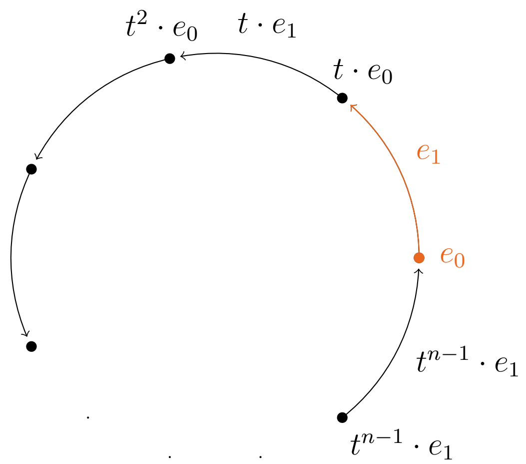


Figure 1.9: A free $\mathbb{Z}/n$-equivariant CW-structure on S^1

G be a cyclic group of order n, and let A be a $\mathbb{Z}G$-module. Then

$$H_k(G;A) \cong \begin{cases} A_G & \text{if } k = 0 \\ A^G/N \cdot A & \text{if } k \text{ is odd} \\ \ker(N\colon A \to A)/(t-1) \cdot A & \text{if } k > 0 \text{ is even} \end{cases}$$

$$H^k(G;A) \cong \begin{cases} A^G & \text{if } k = 0 \\ \ker(N\colon A \to A)/(t-1) \cdot A & \text{if } k \text{ is odd} \\ A^G/N \cdot A & \text{if } k > 0 \text{ is even,} \end{cases}$$

where $t \in G$ is a generator and $N := 1 + t + \cdots + t^{n-1}$ is the norm element. *In particular,*

$$H_k(G;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/n & \text{if } k \text{ is odd} \\ 0 & \text{if } k > 0 \text{ is even.} \end{cases}$$

Proof. The cyclic group $G \cong \mathbb{Z}/n$ acts freely on the circle by rotation around $2\pi/n$. A corresponding free G-equivariant CW-structure on S^1 is depicted in Figure 1.9. In view of the construction in the proof of

Theorem 1.6.1, this CW-structure leads to the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \longrightarrow H_1(S^1;\mathbb{Z}) = \mathbb{Z} \xrightarrow{\ \eta\ } \mathbb{Z}G \xrightarrow{\ t-1\ } \mathbb{Z}G \xrightarrow{\ \varepsilon\ } \mathbb{Z} = H_0(S^1;\mathbb{Z}) \longrightarrow 0$$

What is the map ε? Every 0-cell corresponds to the generator 1 in $\mathbb{Z}$; hence, $\varepsilon(g) = 1$ for all $g \in G$.

What is the map η? The sum of all 1-cells, i.e., the *norm element* (compare also the proof of Corollary 1.4.8)

$$N := 1 + t + t^2 + \cdots + t^{n-1} \in \mathbb{Z}G,$$

is a generator of $H_1(S^1;\mathbb{Z})$; hence, $\eta(1) = N$.

Therefore, as in the proof of Theorem 1.6.1, splicing these sequences together yields the following free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of period 2:

$$\cdots \xrightarrow{\ N\ } \mathbb{Z}G \xrightarrow{\ t-1\ } \mathbb{Z}G \xrightarrow{\ N\ } \mathbb{Z}G \xrightarrow{\ t-1\ } \mathbb{Z}G \xrightarrow{\ \varepsilon\ } \mathbb{Z}$$

Applying $\,\cdot\, \otimes_G A$ and $\operatorname{Hom}_G(\,\cdot\,, A)$ to this resolution and taking (co)homology afterwards, we obtain the (co)homology of $G = \mathbb{Z}/n$ with coefficients in A (Section 1.5.8), and thus the stated results. □

Corollary 1.6.7 (Classifying spaces of non-torsion-free groups)**.** *Let G be a group that is not torsion-free. Then there is no finite dimensional model of the classifying space BG.*

Proof. Exercise. □

Example 1.6.8 ((Co)Homology of finitely generated Abelian groups)**.** Using the computation of (co)homology of cyclic groups (Corollary 1.6.6 and Example 1.3.13), the classification of finitely generated Abelian groups, and the Künneth theorem (Proposition 1.3.14), we can compute (co)homology of finitely generated Abelian groups. For example, for all $n \in \mathbb{N}_{>0}$ and all even $k \in \mathbb{N}_{>0}$ we have

$$H_k(\mathbb{Z}/n \times \mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}/n^{\oplus k/2+1},$$

which is not periodic.

The homology of Abelian groups carries an additional structure, the *Pontrjagin product*. In the case of finitely generated Abelian groups, the ring structure of group homology can also be determined [4, Section V.6].

1.6.3 Application: Groups acting on spheres II

In view of the computation of (co)homology of cyclic groups, we obtain a first group-theoretic obstruction for finite groups to act freely on spheres:

Corollary 1.6.9 (Finite higher rank Abelian groups cannot act freely on spheres)**.**

1. *For $n \in \mathbb{N}_{>1}$ the group $\mathbb{Z}/n \times \mathbb{Z}/n$ cannot act freely on a sphere.*
2. *Consequently, all Abelian subgroups of a finite group acting freely on a sphere must be cyclic.*

Proof. Using the Künneth theorem (Proposition 1.3.14) and the computation of the homology of cyclic groups (Corollary 1.6.6) shows that the homology of $\mathbb{Z}/n \times \mathbb{Z}/n$ is not periodic; hence, there is no periodic projective $\mathbb{Z}/n \times \mathbb{Z}/n$-resolution of $\mathbb{Z}$. So, by Theorem 1.6.1, $\mathbb{Z}/n \times \mathbb{Z}/n$ cannot act freely on a sphere.

The second part follows from the first part with help of the classification of finite Abelian groups. □

Corollary 1.6.10 (p-Groups acting freely on spheres)**.** *Let $p \in \mathbb{N}$ be a prime and let G be a non-trivial p-group acting freely on a sphere. Then G contains a unique subgroup of order p.*

Proof. As G is a non-trivial p-group, its centre is non-trivial [26, Theorem I.6.5], and thus a non-trivial Abelian p-group; in particular, G contains a central subgroup C of order p.

Assume for a contradiction that G contained another subgroup C' of order p. Then the subgroup of G generated by C and C' is Abelian and is easily seen to be isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$, contradicting Corollary 1.6.9. □

In the next section, we will classify all finite p-groups with a unique subgroup of order p.

1.6.4 Application: Classifying p-groups with a unique subgroup of order p

In the following, we will classify all finite p-groups with a unique subgroup of order p (Corollary 1.6.13), which is a result due to Burnside [**?**].

The proof consists of an induction over the order of the p-groups in question. For the induction step we will need the classification of all p-groups that contain a cyclic subgroup of index p:

Theorem 1.6.11 (Classification of p-groups with a cyclic subgroup of index p)**.** *Let $p \in \mathbb{N}$ be a prime. Any p-group with a cyclic subgroup of index p is isomorphic to one of the groups in the following list:*

1. *Let $p \in \mathbb{N}$ be a prime.*
   - A. *$\mathbb{Z}/p^n$ for all $n \in \mathbb{N}_{>0}$.*
   - B. *$\mathbb{Z}/p^n \times \mathbb{Z}/p$ for all $n \in \mathbb{N}_{>0}$.*
   - C. *$\mathbb{Z}/p^n \rtimes \mathbb{Z}/p$ for all $n \in \mathbb{N}_{>1}$, where the generator $[1]$ of $\mathbb{Z}/p$ acts on $\mathbb{Z}/p^n$ by multiplication with $1 + p^{n-1}$. (Because $(1 + p^{n-1})^p$ is congruent to 1 modulo p^n this indeed defines a $\mathbb{Z}/p$-action on $\mathbb{Z}/p^n$.)*

   Note that no two of these groups are isomorphic.
2. *For the prime 2 there are three more families of groups to consider:*
   - A. *Dihedral 2-groups. Let $n \in \mathbb{N}_{>2}$. The dihedral group D_{2^n} is the semi-direct product group $\mathbb{Z}/2^n \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts on $\mathbb{Z}/2^n$ by multiplication by -1.*
   - B. *Generalised quaternion 2-groups. Let $\mathbb{H}$ be the quaternion algebra and let $n \in \mathbb{N}_{>0}$. Then the generalised quaternion group Q_{2^n} is the subgroup of the units of $\mathbb{H}$ generated by $e^{\pi \cdot i/2^n}$ and j; alternatively, we can describe Q_{2^n} by the presentation*

     $$Q_{2^n} = \langle x,\ y \mid y^4 = 1,\ y^2 = x^{2^n},\ y \cdot x \cdot y^{-1} = x^{-1} \rangle.$$

     Notice that the extension $0 \longrightarrow \mathbb{Z}/2^{n+1} \longrightarrow Q_{2^n} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ (given by the inclusion of the subgroup generated by x) does not split and that the quotient acts by multiplication by -1 on the kernel. For example, Q_2 is the ordinary quaternion group (of order 8).
   - C. *$\mathbb{Z}/2^n \rtimes \mathbb{Z}/2$ for all $n \in \mathbb{N}_{>2}$, where $\mathbb{Z}/2$ acts on $\mathbb{Z}/2^n$ by multiplication by $-1 + 2^{n-1}$.*

Again, no two of these groups are isomorphic, and none of these groups is isomorphic to a group of type 1.

Clearly, all of these p-groups contain a cyclic subgroup of index p. Conversely, every such p-group is of this type:

The idea of the proof is to view the p-group in question as an extension of $\mathbb{Z}/p$ by the cyclic subgroup of index p and to use the classification of extensions with Abelian kernel. Therefore, as a preparation, we first have to understand all actions of $\mathbb{Z}/p$ on a cyclic group $\mathbb{Z}/p^n$:

Lemma 1.6.12 (Classifying $\mathbb{Z}/p$-actions on $\mathbb{Z}/p^n$)**.** *Let $p \in \mathbb{N}$ be a prime, let $n \in \mathbb{N}_{>1}$, and let $a \in \mathbb{Z}$ with $a^p \equiv 1 \bmod p^n$.*

1. *If p is odd, then $a \equiv 1 \bmod p^{n-1}$.*
2. *If $p = 2$, then $a \equiv \pm 1 \bmod 2^{n-1}$.*

Proof. Exercise (by Fermat's little theorem, we can write a in the form $a = 1 + k \cdot p^d$ where $d > 0$ and where p does not divide k). □

Proof (of Theorem 1.6.11). Let G be a p-group containing a cyclic subgroup C of index p; then G is finite and C is a normal subgroup of G (by a classical result in the theory of p-groups [26, Lemma I.6.7]). Hence, we have an extension

$$0 \longrightarrow C \longrightarrow G \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

of groups; here, $C \cong \mathbb{Z}/p^n$ for some $n \in \mathbb{N}$, and we may assume without loss of generality that $n > 0$.

This extension induces an action of $\mathbb{Z}/p$ on C (via conjugation in G). If this action is trivial, then G is Abelian; so, G is of type 1A or 1B, by the classification of finitely generated Abelian groups.

In the following, we assume that the action of $\mathbb{Z}/p$ on C is non-trivial (in particular, $n > 1$).

1. If p is odd, then by Lemma 1.6.12 the $\mathbb{Z}/p$-action on $C \cong \mathbb{Z}/p^n$ is given by multiplication by a number $a \in \mathbb{Z}$ with $a \equiv 1 \bmod p^{n-1}$.
   We show now that the only extensions inducing this action are the ones isomorphic to the semi-direct product extension corresponding to groups of type 1C: In view of the classification of group extensions with Abelian kernel (Theorem 1.4.14), it suffices to show that one of

the generators of $\mathbb{Z}/p$ acts by multiplication with $1 + p^{n-1}$ on C and that $H^2(\mathbb{Z}/p; C) = 0$.
Because $[a] \in \mathbb{Z}/p^n$ and $[1+p^{n-1}]$ are elements of order p in $C = \mathbb{Z}/p^n$, we have

$$\begin{aligned}\bigl\{[a^0], \dots, [a^{p-1}]\bigr\} &= \bigl\{[1 + j \cdot p^{n-1}] \bigm| j \in \{0, \dots, p-1\}\bigr\} \\ &= \bigl\{[1 + p^{n-1}]^j \bigm| j \in \{0, \dots p-1\}\bigr\},\end{aligned}$$

and thus we can assume without loss of generality that $a = 1 + p^{n-1}$. Recall that (Corollary 1.6.6)

$$H^2(\mathbb{Z}/p; C) = C^{\mathbb{Z}/p}/N \cdot C,$$

where $N := \sum_{j=0}^{p-1}[a^j] \in C$ is the norm element. Then for the invariants we obtain

$$\begin{aligned}C^{\mathbb{Z}/p} &= C^{[1+p^{n-1}]} \\ &= \bigl\{[x] \in \mathbb{Z}/p^n \bigm| p^{n-1} \cdot x \equiv 0 \bmod p^n\bigr\} \\ &= p \cdot \mathbb{Z}/p^n;\end{aligned}$$

on the other hand,

$$\begin{aligned}N &= \sum_{j=0}^{p-1}[a^j] \\ &= \sum_{j=0}^{p-1}[1 + j \cdot p^{n-1}] \\ &= [p] + \Bigl[\frac{p \cdot (p-1)}{2} \cdot p^{n-1}\Bigr] \\ &= [p],\end{aligned}$$

and hence $N \cdot C = p \cdot \mathbb{Z}/p^n$. So

$$H^2(\mathbb{Z}/p; C) = C^{\mathbb{Z}/p}/N \cdot C = 0.$$

2. Similar arguments as in the first part show that the only possible 2-groups with cyclic subgroup of index 2 are the ones in the list above [4, proof of Theorem IV.4.1]. □

Corollary 1.6.13 (Classification of p-groups with a unique subgroup of order p)**.** *Let $p \in \mathbb{N}$ be a prime and let G be a finite p-group. If G contains a unique group of order p, then G is cyclic or a generalised quaternion group.*

Proof. We prove the assertion by induction over the order of G. If $|G| = p$, then G must be cyclic.

For the induction step we may assume that every proper subgroup of G is cyclic or a generalised quaternion group. By the classification result Theorem 1.6.11, it suffices to show that G contains a cyclic subgroup of index p: the only groups in the list above with a unique subgroup of order p are the groups of type 1A (cyclic groups) and the groups of type 2B (generalised quaternion groups). As G is a p-group there is a normal subgroup H of G of index p [26, Corollary I.6.6].

- If H is cyclic, then we obviously are in that situation.
- If H is a generalised quaternion group, then we argue as follows: Looking at the conjugation action of G on the set of cyclic subgroups of H of index 2 in H (a set with an odd number of elements; this can be shown by inspecting the orders of elements in generalised quaternion groups), we see that H contains a cyclic subgroup C of index 2 that is normal in G. Using the conjugation action of G/C on C, we obtain an epimorphism

  $$G/C \longrightarrow (\mathbb{Z}/|C|)^\times \longrightarrow (\mathbb{Z}/4)^\times = \{-1, +1\};$$

  let K/C be the kernel of this epimorphism. By construction, the generator of K/C acts trivially on C; therefore, K is not generalised quaternion and thus cyclic by the induction hypothesis. On the other hand, K has index 2 in G, which finishes the proof. □

1.6.5 Application: Groups acting on spheres III

Combining the obstruction in terms of periodic resolutions (Corollary 1.6.10) with Burnside's classification result (Corollary 1.6.13), we obtain a nice algebraic obstruction for finite groups to be able to act freely on spheres:

Corollary 1.6.14 (Sylow groups of finite groups acting on spheres)**.** *Let G be a finite group acting freely on a sphere. Then every Sylow subgroup of G is cyclic or generalised quaternion.*

Proof. Let $p \in \mathbb{N}$ be a prime, and let S be a p-Sylow subgroup of G; then S contains a unique subgroup of order p (Corollary 1.6.10).

Hence, Burnside's classification result (Corollary 1.6.13) tells us that S is a cyclic or a generalised quaternion group. □

While all finite cyclic groups can act freely on S^1 and all generalised quaternion groups can act freely on S^3 (the unit sphere in $\mathbb{H}$), not every finite group all of whose Sylow subgroups are cyclic or generalised quaternion can act freely on a sphere (Example 1.6.17). The latter result is a consequence of Milnor's study of elements of order 2 in groups acting freely on spheres:

Theorem 1.6.15 (Milnor's generalisation of the Borsuk-Ulam theorem [33, Theorem 1])**.** *Let $n \in \mathbb{N}_{>0}$ and let $T \colon S^n \longrightarrow S^n$ be a continuous map without fixed points that satisfies $T \circ T = \mathrm{id}_{S^n}$. Then for every continuous map $f \colon S^n \longrightarrow S^n$ of odd degree there exists an $x \in S^n$ with*

$$T \circ f(x) = f \circ T(x).$$

When $T \colon S^n \longrightarrow S^n$ is the antipodal map, then the statement of this theorem is the ordinary Borsuk-Ulam theorem.

Corollary 1.6.16 (Elements of order 2 in groups acting on spheres [33, Remark on p. 624])**.** *Any finite group acting freely on a sphere can contain at most one element of order 2.*

Proof. Let $n \in \mathbb{N}_{>0}$ and let G be a finite group acting freely on S^n. Every homeomorphism of S^n has odd degree (namely 1 or -1); in particular, by the theorem, every element in G of order 2 commutes with every homeomorphism of S^n. Therefore, all elements of G of order 2 are central.

Because the centre of G cannot contain a copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$ (Corollary 1.6.9), we conclude that G can contain at most one element of order 2 (and this is central). □

Example 1.6.17 (Symmetric groups and actions on spheres [33, Corollary 2]). In view of the previous corollary, the symmetric group S_n on $n \in \mathbb{N}$ letters cannot act freely on a sphere if $n \geq 3$ (because the transpositions are elements of order 2).

Notice however that the group S_3 does have periodic cohomology; in fact all finite groups satisfying the Sylow subgroup condition of Corollary 1.6.14 have periodic cohomology (Corollary 1.9.30).

1.6.6 Group actions on highly connected spaces

Of course, in general a free action of a group on a highly connected space will not give rise to a periodic projective resolution; however, we still obtain a partial projective resolution. For simplicity, we consider only the case of trivial coefficients.

Proposition 1.6.18 (Group (co)homology and partial resolutions). *Let G be a group and let*

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a partial projective $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ of length n. Then $H_k((P_*)_G) \cong H_k(G;\mathbb{Z})$ for all $k \in \{0,\dots,n-1\}$ and there is an exact sequence

$$0 \longrightarrow H_{n+1}(G;\mathbb{Z}) \longrightarrow \big(H_n(P_*)\big)_G \longrightarrow H_n\big((P_*)_G\big) \longrightarrow H_n(G;\mathbb{Z}) \longrightarrow 0.$$

Proof. Exercise. □

Corollary 1.6.19 (Partial resolutions via group actions). *Let G be a group, let $n \in \mathbb{N}$ and suppose that there is an $(n-1)$-connected free G-CW-complex X. Then the cellular chain complex of X gives a partial projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of length n. Hence, $H_k(X/G;\mathbb{Z}) \cong H_k(G;\mathbb{Z})$ for all $k \in \{0,\dots,n-1\}$ and there is an exact sequence*

$$\big(H_n(X;\mathbb{Z})\big)_G \longrightarrow H_n(X/G;\mathbb{Z}) \longrightarrow H_n(G;\mathbb{Z}) \longrightarrow 0.$$

Proof. Using the Hurewicz theorem, this follows from the above proposition. □

1.6.7 Application: The Hurewicz homomorphism in degree 2

For example, using partial resolutions provided by free actions on highly connected spaces, we can analyse the Hurewicz homomorphism; it turns out that group homology is a measure for the surjectivity of the Hurewicz homormophism (Proposition 1.6.21).

Definition 1.6.20 (Hurewicz homomorphism)**.** Let X be a pointed connected CW-complex, and let $n \in \mathbb{N}$. Then the *Hurewicz homomorphism in degree n* is the homomorphism

$$\begin{aligned} h_n^X \colon \pi_n(X) = [S^n, X]_\bullet &\longrightarrow H_n(X;\mathbb{Z}) \\ [f] &\longmapsto H_n(f;\mathbb{Z})([S^n]_\mathbb{Z}), \end{aligned}$$

where $[S^n]_\mathbb{Z} \in H_n(S^n;\mathbb{Z})$ denotes the fundamental class of S^n.

Proposition 1.6.21 (Measuring surjectivity of the Hurewicz homomorphism)**.** *Let $n \in \mathbb{N}_{\geq 2}$, and let X be a connected, pointed CW-complex whose universal covering $\widetilde{X}$ is $(n-1)$-connected (i.e., $\pi_j(\widetilde{X}) = 0$ for all $j \in \{0, \dots, n-1\}$). Then there is an exact sequence*

$$\pi_n(X) \xrightarrow{h_n^X} H_n(X;\mathbb{Z}) \longrightarrow H_n\bigl(\pi_1(X);\mathbb{Z}\bigr) \longrightarrow 0$$

(The map $H_n(X;\mathbb{Z}) \longrightarrow H_n(\pi_1(X);\mathbb{Z})$ in this sequence can be shown to be induced by the classifying map $X \longrightarrow B\pi_1(X)$.)

Proof. Exercise (use Proposition 1.6.18 and the Hurewicz theorem). □

Corollary 1.6.22 (Measuring surjectivity of the Hurewicz homomorphism in degree 2)**.** *Let X be a pointed connected CW-complex. Then there is an exact sequence*

$$\pi_2(X) \xrightarrow{h_2(X)} H_2(X;\mathbb{Z}) \longrightarrow H_2\bigl(\pi_1(X);\mathbb{Z}\bigr) \longrightarrow 0.$$

Example 1.6.23 (Free fundamental groups and Hurewicz homomorphism in degree 2)**.** If X is a pointed connected CW-complex with free fundamental group, then the Hurewicz homomorphism $h_2^X \colon \pi_2(X) \longrightarrow H_2(X;\mathbb{Z})$ in degree 2 is surjective (Example 1.3.13).

1.6.8 Preview: Group actions and spectral sequences

If G is a discrete group and A is a $\mathbb{Z}G$-module, then

$$H_*(BG; A) \cong H_*(G; A)$$

by the topological description of group homology. This identity can be read in two ways:

- Group homology is a means to compute homology of certain topological spaces.
- Certain topological spaces compute group homology.

Both of these interpretations allow for generalisations in terms of spectral sequences [4, Chapter VII] (spectral sequences will be the topic of Section 1.10):

Theorem 1.6.24 (Homology of free quotients via group homology)**.** *Let G be a discrete group, let X be a free G-CW-complex, and let A be a $\mathbb{Z}G$-module. Then there is a converging spectral sequence of the following type:*

$$E^2_{pq} = H_p\bigl(G; H_q(X; A)\bigr) \Longrightarrow H_{p+q}(X/G; A).$$

Theorem 1.6.25 (Group homology via actions on acyclic spaces)**.** *Let G be a discrete group, let X be a connected (not necessarily free) G-CW-complex, and let A be a $\mathbb{Z}G$-module. If X is acyclic (i.e., X has the integral homology of a point), then there is a converging spectral sequence of the following type:*

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma; A_\sigma) \Longrightarrow H_{p+q}(G; A).$$

Here, we use the following notation:

- *For $p \in \mathbb{N}$ we choose a set Σ_p of representatives of the G-orbits of p-cells of X.*
- *For $p \in \mathbb{N}$ and $\sigma \in \Sigma_p$, the group G_σ is the subgroup of G consisting of all elements mapping the cell σ to itself.*
- *For $p \in \mathbb{N}$ and $\sigma \in \Sigma_p$, the $\mathbb{Z}G_\sigma$-module A is obtained from the $\mathbb{Z}G$-module A by twisting the action according to the orientation character $G_\sigma \longrightarrow \mathbb{Z}/2$ of the G_σ-action on the cell σ.*

1.7 Cohomology of subgroups

In this section, we will study the relation between the (co)homology of subgroups and the (co)homology of ambient groups:

- How can we compute (co)homology of a subgroup in terms of the (co)homology of the ambient group? More precisely, let H be a subgroup of a discrete group G and let B be a $\mathbb{Z}H$-module. How can we express $H_*(H; B)$ in terms of $H_*(G; \,\cdot\,)$?
- How can we assemble (co)homology of a group out of the (co)homology of its proper subgroups? More precisely, let G be a group and let A be a $\mathbb{Z}G$-module. How can we find a family S of proper subgroups of G that is as small as possible such that we can recover $H_*(G; A)$ from $(H_*(H; \,\cdot\,))_{H \in S}$?

The first question is answered by Shapiro's lemma (Section 1.7.2), the second question is addressed by the transfer technique (Section 1.7.3). In particular, we will derive the decomposition of (co)homology of finite groups into a direct sum of the (co)homology of its Sylow subgroups (Section 1.7.5). We start with some algebraic preparations in Section 1.7.1.

1.7.1 Induction, coinduction, and restriction

How can we compute (co)homology of a subgroup in terms of the (co)homology of the ambient group? As first step we need a way to turn modules over the subgroup into modules over the ambient group:

Definition 1.7.1 (Induction, coinduction)**.** Let G be a group and let H be a subgroup.

- *Induction.* We write

$$\operatorname{Ind}_H^G := \mathbb{Z}G \otimes_{\mathbb{Z}H} \,\cdot\, : \mathbb{Z}H\text{-Mod} \longrightarrow \mathbb{Z}G\text{-Mod}$$

for the *induction* functor. Here, for a $\mathbb{Z}H$-module B, the $\mathbb{Z}G$-module structure on $\operatorname{Ind}_H^G(B) = \mathbb{Z}G \otimes_{\mathbb{Z}H} B$ is given by

$$\begin{aligned} G \times \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B \\ (g, g' \otimes b) &\longmapsto g \cdot g' \otimes b; \end{aligned}$$

Notice that we use $\cdot \otimes_{\mathbb{Z}H} \cdot$ instead of $\cdot \otimes_H \cdot$.

– *Coinduction.* We write

$$\operatorname{Coind}_H^G := \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, \,\cdot\,) \colon \mathbb{Z}H\text{-Mod} \longrightarrow \mathbb{Z}G\text{-Mod}$$

for the *coinduction* functor. For a $\mathbb{Z}H$-module B, the $\mathbb{Z}G$-module structure on $\operatorname{Coind}_H^G(B) = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B)$ is given by

$$\begin{aligned} G \times \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) &\longrightarrow \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) \\ (g, f) &\longmapsto \bigl(g' \mapsto f(g' \cdot g)\bigr) \end{aligned}$$

Example 1.7.2 (Induction modules)**.** Let H be a subgroup of a group G.

– *Induction of the group ring* $\mathbb{Z}H$. Of course,

$$\operatorname{Ind}_H^G(\mathbb{Z}H) = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong \mathbb{Z}G.$$

Notice however that $\operatorname{Coind}_H^G(\mathbb{Z}H) \ncong \mathbb{Z}G$ in general.

– *Induction of the trivial module* $\mathbb{Z}$. Then we have an isomorphism

$$\operatorname{Ind}_H^G(\mathbb{Z}) = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} \cong \mathbb{Z}[G/H]$$

of $\mathbb{Z}G$-modules; here, the G-action on $\mathbb{Z}[G/H] := \bigoplus_{G/H} \mathbb{Z}$ is the one induced by the left translation action of G on the coset space G/H.

Proposition 1.7.3 ((Co)induction for finite index subgroups)**.** *Let G be a group and let H be a subgroup of finite index. Then there is a natural isomorphism*

$$\operatorname{Ind}_H^G(B) \cong \operatorname{Coind}_H^G(B)$$

for all $\mathbb{Z}H$-modules B.

Proof. A straightforward computation shows that the two homomorphisms

$$\begin{aligned} \varphi \colon \operatorname{Ind}_H^G(B) = \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\longrightarrow \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) = \operatorname{Coind}_H^G(B) \\ g \otimes b &\longmapsto \bigl(g' \mapsto \chi_H(g' \cdot g) \cdot (g' \cdot g) \cdot b\bigr) \end{aligned}$$

and

$$\begin{aligned}\psi\colon \operatorname{Coind}_H^G = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B)\\ f &\longmapsto \sum_{gH \in G/H} g \otimes f(g^{-1})\end{aligned}$$

are well-defined and $\mathbb{Z}G$-linear. Here, $\chi_H\colon G \longrightarrow \{0,1\}$ denotes the characteristic function of H in G; notice that the term following $\chi_H(g'\cdot g)$ only makes sense if $g' \cdot g \in H$.

Moreover, φ and ψ are mutually inverse: It is clear that $\psi\circ\varphi = \operatorname{id}_{\operatorname{Ind}_H^G(B)}$. Conversely, let $f \in \operatorname{Coind}_H^G(B)$. Then

$$\begin{aligned}\varphi \circ \psi(f) &= \varphi\Bigl(\sum_{gH\in G/H} g \otimes f(g^{-1})\Bigr)\\ &= \sum_{gH\in G/H} \bigl(g' \mapsto \chi_H(g'\cdot g)\cdot (g'\cdot g)\cdot f(g^{-1})\bigr)\\ &= \sum_{gH\in G/H} \bigl(g' \mapsto \chi_H(g'\cdot g)\cdot f(g'\cdot g\cdot g^{-1})\bigr)\\ &= \sum_{gH\in G/H} \bigl(g' \mapsto \chi_{Hg^{-1}}(g')\cdot f(g')\bigr)\\ &= f,\end{aligned}$$

and thus $\varphi\circ\psi = \operatorname{id}_{\operatorname{Coind}_H^G(B)}$. □

Conversely, we can just forget about parts of the action of the ambient group, thereby turning modules over the ambient group into modules over subgroups:

Definition 1.7.4 (Restriction)**.** Let G be a group, let H be a subgroup of G, and let $i\colon H \longrightarrow G$ be the inclusion. Then we write

$$\operatorname{Res}_H^G := i^* \cdot : \mathbb{Z}G\text{-Mod} \longrightarrow \mathbb{Z}H\text{-Mod}$$

for the *restriction* functor.

Lemma 1.7.5 (Restrictions of projectives)**.** *Restrictions of projectives are projective. More precisely: Let G be a group, let H be a subgroup, and let P be a projective $\mathbb{Z}G$-module. Then $\operatorname{Res}_H^G P$ is a projective $\mathbb{Z}H$-module.*

Proof. As a module is projective if and only if it is a direct summand in a free module and as the functor Res_H^G is compatible with direct sums, it suffices to show that $\operatorname{Res}_H^G \mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Indeed, as $\mathbb{Z}H$-module, $\mathbb{Z}G$ is nothing but a direct sum of $|G/H|$ copies of the $\mathbb{Z}H$-module $\mathbb{Z}H$; in particular, $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. □

Corollary 1.7.6 (Restrictions of resolutions)**.** *Let G be a group, let H be a subgroup, and let $P_* \,\square\, \varepsilon$ be a projective $\mathbb{Z}G$-resolution of a $\mathbb{Z}G$-module A. Then $\operatorname{Res}_H^G P_* \,\square\, \varepsilon$ is a projective $\mathbb{Z}H$-resolution of the $\mathbb{Z}H$-module $\operatorname{Res}_H^G A$.*

Proposition 1.7.7 (Mixing induction and restriction)**.** *Let G be a group, let H be a discrete subgroup, and let A be a $\mathbb{Z}G$-module. Then there is a natural isomorphism*

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G A \cong \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} A$$

of $\mathbb{Z}G$-modules, where G acts diagonally on $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} A$.

Proof. A straightforward computation shows that

$$\begin{aligned} \operatorname{Ind}_H^G \operatorname{Res}_H^G A = \mathbb{Z}G \otimes_{\mathbb{Z}H} A &\longrightarrow \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} A \\ g \otimes a &\longmapsto gH \otimes g \cdot a \\ g \otimes g^{-1} \cdot a &\longleftarrow\!\shortmid gH \otimes a \end{aligned}$$

are well-defined $\mathbb{Z}G$-homomorphisms that are mutually inverse. □

The converse composition $\operatorname{Res}_H^G \operatorname{Ind}_H^G$ can be described using double cosets [4, Proposition III.5.6].

1.7.2 Shapiro's lemma

Shapiro's lemma shows that (co)homology of a subgroup can indeed be computed in terms of (co)homology of the ambient group with suitably (co)induced coefficients:

Proposition 1.7.8 (Shapiro's lemma)**.** *Let G be a discrete group, let H be a subgroup, and let B be a $\mathbb{Z}H$-module. We write $i\colon H \longrightarrow G$ for the inclusion. Moreover, we define $\mathbb{Z}H$-homomorphisms I and C by*

$$\begin{aligned} I\colon B &\longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} B = \operatorname{Ind}_H^G(B) \\ b &\longmapsto 1 \otimes b, \end{aligned}$$

$$\begin{aligned} C\colon \operatorname{Coind}_H^G(B) = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B) &\longrightarrow B \\ f &\longmapsto f(1); \end{aligned}$$

notice that I is split injective while C is split surjective (as $\mathbb{Z}H$-homomorphism).

1. *Then $(i, I)\colon (H, B) \longrightarrow (G, \operatorname{Ind}_H^G(B))$ is a morphism in* GrpMod *and $(i, C)\colon (H, B) \longrightarrow (G, \operatorname{Coind}_H^G(B))$ is a morphism in* GrpMod^-.
2. *The induced homomorphisms*

$$\begin{aligned} H_*(i; I)\colon H_*(H; B) &\longrightarrow H_*\bigl(G; \operatorname{Ind}_H^G(B)\bigr), \\ H^*(i; C)\colon H^*\bigl(G; \operatorname{Coind}_H^G(B)\bigr) &\longrightarrow H^*(H; B) \end{aligned}$$

   are natural isomorphisms.

The heart of the proof of Shapiro's lemma is the algebraic fact that for all $\mathbb{Z}G$-modules A and all $\mathbb{Z}H$-modules B there are natural isomorphisms (of Abelian groups)

$$\begin{aligned} A \otimes_G \mathbb{Z}G \otimes_{\mathbb{Z}H} B &\cong A \otimes_{\mathbb{Z}H} B, \\ \operatorname{Hom}_G\bigl(A, \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B)\bigr) &\cong \operatorname{Hom}_H(A, B) \end{aligned}$$

and that projective $\mathbb{Z}G$-resolutions can be viewed as projective $\mathbb{Z}H$-resolutions.

Proof (of Shapiro's lemma). The first part follows directly from the definition. For the second part we argue as follows:

We consider the following diagram:

$$\begin{array}{ccccc} H_*(H;B) & = & H_*\bigl(C_*(H)\otimes_H B\bigr) & = & H_*\bigl(C_*(H)\otimes_H B\bigr) \\ \downarrow{\scriptstyle H_*(i;I)} & & \downarrow{\scriptstyle H_*(C_*(i)\otimes_H I)} & & \downarrow{\scriptstyle H_*(C_*(i)\otimes_H \mathrm{id}_B)} \\ H_*\bigl(G;\operatorname{Ind}_H^G(B)\bigr) & = & H_*\bigl(C_*(G)\otimes_G \operatorname{Ind}_H^G(B)\bigr) & \xleftarrow[H_*(\varphi_*)]{} & H_*\bigl(\operatorname{Res}_H^G C_*(G)\otimes_H B\bigr) \end{array}$$

The left square is commutative by the description of group homology in terms of the bar resolution.

What about the right square? The map φ_* is the chain map given by

$$\begin{aligned}\varphi_n\colon \operatorname{Res}_H^G C_n(G) \otimes_H B &\longrightarrow C_n(G) \otimes_G \operatorname{Ind}_H^G(B)\\ c \otimes b &\longmapsto c \otimes 1 \otimes b,\end{aligned}$$

which is an isomorphism (see the remark preceding the proof). Therefore, $H_*(\varphi_*)$ is also an isomorphism. It follows directly from the definition of I and φ_* that the right square is commutative as well.

In order to prove the Shapiro lemma for group homology, it therefore remains to show that als the right vertical arrow $H_*(C_*(i) \otimes_H \mathrm{id}_B)$ is an isomorphism; thus it suffices to show that

$$C_*(i)\colon C_*(H) \longrightarrow \operatorname{Res}_H^G C_*(G)$$

is a chain homotopy equivalence of $\mathbb{Z}H$-chain complexes.

By Corollary 1.7.6, the complex $\operatorname{Res}_H^G C_*(G)$ (together with the canonical augmentation $C_0(G) \longrightarrow \mathbb{Z}$) is a projective $\mathbb{Z}H$-resolution of $\mathbb{Z}$. As the $\mathbb{Z}H$-chain map $C_*(i)$ induces an isomorphism on the resolved module $\mathbb{Z}$ (namely the identity homomorphism), the fundamental lemma of homological algebra implies that $C_*(i)$ is a $\mathbb{Z}H$-chain homotopy equivalence.

Similar arguments prove the statement about cohomology. □

Corollary 1.7.9. *Let H be a subgroup of a group G. Then*

$$H_*(H;\mathbb{Z}) \cong H_*(G;\mathbb{Z}[G/H])$$

by Shapiro's lemma and Example 1.7.2. □

Corollary 1.7.10. *Let G be a discrete group satisfying $H_*(G;A) \cong H_*(1;A)$ for all $\mathbb{Z}G$-modules A. Then G is the trivial group.*

Proof. Assume for a contradiction that G is non-trivial. Then G contains a non-trivial cyclic group. Using Shapiro's lemma, the computation of homology of cyclic groups (or the computation of $H_1(\,\cdot\,;\mathbb{Z})$), and the assumption on G, we obtain

$$\begin{aligned}0 \ncong C &\cong H_1(C;\mathbb{Z})\\ &\cong H_1\bigl(G;\operatorname{Ind}_C^G(\mathbb{Z})\bigr)\\ &\cong H_1\bigl(1;\operatorname{Ind}_C^G(\mathbb{Z})\bigr)\\ &= 0,\end{aligned}$$

which is absurd. Hence G must be trivial. □

Corollary 1.7.11. *Let G be a group and let H be a subgroup of finite index. Then there is a canonical isomorphism*

$$\begin{aligned} H^*(H;\mathbb{Z}H) &\cong H^*(G;\operatorname{Coind}_H^G \mathbb{Z}H) \cong H^*(G;\operatorname{Ind}_H^G \mathbb{Z}H) \\ &\cong H^*(G;\mathbb{Z}G) \end{aligned}$$

by Proposition 1.7.3 and Shapiro's lemma. □

This corollary is interesting in view of the topological/geometric interpretations of *co*homology of a group with group ring coefficients via ends of groups [49, 17].

1.7.3 The transfer

How can we assemble (co)homology of a group out of the (co)homology of its proper subgroups? The answer to this question lies in the study of the relation between (co)homology of a group and (co)homology of its finite index subgroups.

Definition 1.7.12 (Restriction and corestriction)**.** Let G be a group, let H be a subgroup, and let $i\colon H \longrightarrow G$ be the inclusion. Furthermore, let A be a $\mathbb{Z}G$-module.

- The map

$$\operatorname{res}_H^G := H^*(i;\operatorname{id}_A)\colon H^*(G;A) \longrightarrow H^*(H;\operatorname{Res}_H^G A)$$

  is called *restriction*.
- The map

$$\operatorname{cor}_H^G := H_*(i;\operatorname{id}_A)\colon H_*(H;\operatorname{Res}_H^G A) \longrightarrow H_*(G;A)$$

  is called *corestriction*.

If H is a finite index subgroup of G, then there are non-trivial homomorphisms going in the other direction, the so-called *transfer* maps; in

$$\begin{array}{ccc} H_*(H; \operatorname{Res}^G_H A) & \overset{\cong}{\longleftrightarrow} & H_*(G; \operatorname{Ind}^G_H \operatorname{Res}^G_H A) \\ {\scriptstyle \operatorname{tr}^G_H} \uparrow & & \updownarrow {\scriptstyle \cong} \\ H_*(G; A) & \xrightarrow[H_*(\operatorname{id}_G;\Phi)]{} & H_*(G; \operatorname{Coind}^G_H \operatorname{Res}^G_H A) \end{array} \qquad \begin{array}{ccc} H^*(H; \operatorname{Res}^G_H A) & \overset{\cong}{\longleftrightarrow} & H^*(G; \operatorname{Coind}^G_H \operatorname{Res}^G_H A) \\ {\scriptstyle \operatorname{tr}^G_H} \downarrow & & \updownarrow {\scriptstyle \cong} \\ H^*(G; A) & \xleftarrow[H^*(\operatorname{id}_G;\Psi)]{} & H^*(G; \operatorname{Ind}^G_H \operatorname{Res}^G_H A) \end{array}$$

Figure 1.10: Transfer, via Shapiro's lemma

the following, we will construct these maps. In order to answer the question above we will investigate the relation between the transfer and the (co)restriction maps; moreover, we will have to consider the "action" of the ambient group G on the (co)homology of its subgroups (Section 1.7.4).

We start with the description of the transfer via Shapiro's lemma:

Definition 1.7.13 (Transfer map, via Shapiro's lemma)**.** Let G be a group, let H be a subgroup of finite index, and let A be a $\mathbb{Z}G$-module. The transfer maps

$$\begin{aligned} \operatorname{tr}^G_H = \operatorname{res}^G_H \colon H_*(G; A) &\longrightarrow H_*(H; \operatorname{Res}^G_H A) \\ \operatorname{tr}^G_H = \operatorname{cor}^G_H \colon H^*(H; \operatorname{Res}^G_H A) &\longrightarrow H^*(G; A) \end{aligned}$$

are defined by the commutative diagrams in Figure 1.10:

Here, in the homological transfer, the lower horizontal arrow is the map induced by the (injective) $\mathbb{Z}G$-homomorphism

$$\begin{aligned} \Phi \colon A &\longrightarrow \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) = \operatorname{Coind}^G_H \operatorname{Res}^G_H A \\ a &\longmapsto (g \mapsto g \cdot a), \end{aligned}$$

the right vertical isomorphism is the isomorphism induced by the isomorphism $\operatorname{Coind}^G_H \operatorname{Res}^G_H A \cong \operatorname{Ind}^G_H \operatorname{Res}^G_H A$ from Proposition 1.7.3, and the upper horizontal arrow is the isomorphism from Shapiro's lemma (Proposition 1.7.8).

In the cohomological transfer, the lower horizontal arrow is the map induced by the (surjective) $\mathbb{Z}G$-homomorphism

$$\begin{aligned} \Psi \colon \operatorname{Ind}^G_H \operatorname{Res}^G_H A = \mathbb{Z}G \otimes_{\mathbb{Z}H} A &\longrightarrow A \\ g \otimes a &\longmapsto g \cdot a, \end{aligned}$$

the right vertical arrow is the isomorphism induced by the corresponding isomorphism on the coefficient modules (Proposition 1.7.3), and the upper horizontal arrow is the isomorphism from Shapiro's lemma.

The slight abuse of notation for res^G_H and cor^G_H might be confusing in the beginning; however, this notation will pay off later, because it allows to treat homology and cohomology uniformly.

Remark 1.7.14 (The nature of transfer)**.**

- As the isomorphisms of Proposition 1.7.3 and of Shapiro's lemma are natural in the module variable, also the transfer maps are natural in the module variable.
- One can show that the transfer map itself in general is *not* induced by a morphism in GrpMod or GrpMod⁻ respectively. (Exercise).

While the above definition is a concise description of the transfer it is not very illuminating. In the following, we derive a more explicit description on the level of (co)chain complexes and give a topological interpretation:

Remark 1.7.15 (Transfer map, explicit description)**.** Let H be a finite index subgroup of a group G and let A be a $\mathbb{Z}G$-module. Moreover, let $P_* \mathbin{\square} \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$; in particular, $P_* \mathbin{\square} \varepsilon$ can also be viewed as a projective $\mathbb{Z}H$-resolution of $\mathbb{Z}$ (Corollary 1.7.6). We now give an explicit description of the transfer maps

$$\operatorname{tr}^G_H \colon H_*(G;A) = H_*(P_* \otimes_G A) \longrightarrow H_*(P_* \otimes_H A) = H_*(H; \operatorname{Res}^G_H A)$$
$$\operatorname{tr}^G_H \colon H^*(H;\operatorname{Res}^G_H A) = H^*\bigl(\operatorname{Hom}_H(P_*,A)\bigr) \longrightarrow H^*\bigl(\operatorname{Hom}_G(P_*,A)\bigr) = H^*(G;A)$$

in terms of the resolution $P_* \mathbin{\square} \varepsilon$: We start with the homological transfer. Recalling the proof of Proposition 1.7.3 and Shapiro's lemma, we see that the diagram in Figure 1.11 is commutative, where Φ is as in the definition of the transfer and where ① and ② are the following chain maps:

$$\text{①}\colon P_* \otimes_G \operatorname{Coind}^G_H \operatorname{Res}^G_H A \longrightarrow P_* \otimes_G \operatorname{Ind}^G_H \operatorname{Res}^G_H A$$
$$p \otimes f \longmapsto \sum_{gH \in G/H} g \otimes f(g^{-1})$$

$$\text{②}\colon P_* \otimes_G \operatorname{Ind}^G_H \operatorname{Res}^G_H A \longrightarrow P_* \otimes_H A$$
$$p \otimes g \otimes a \longmapsto g^{-1}p \otimes a.$$

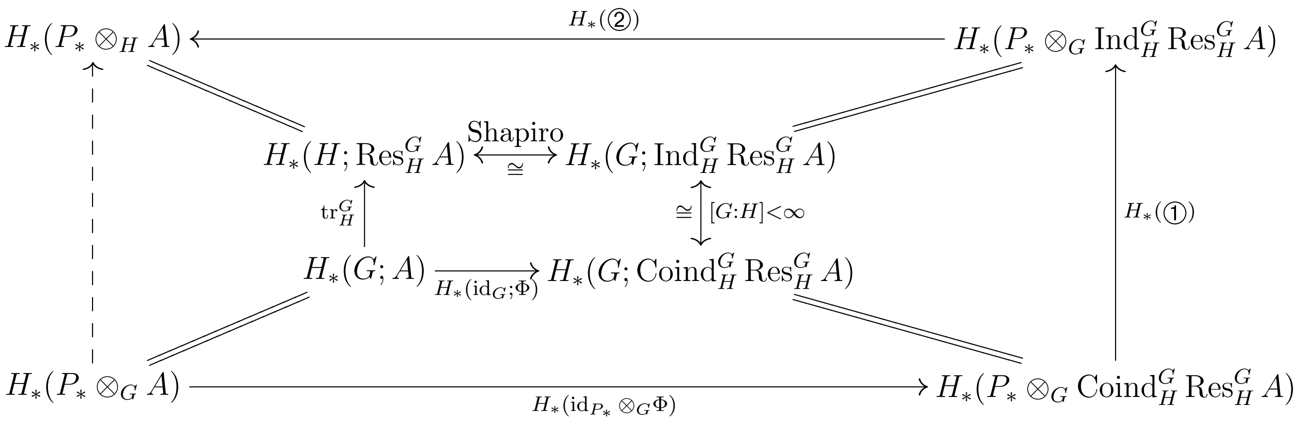


Figure 1.11: Transfer, via resolutions of the ambient group

Therefore, the dashed left vertical arrow (which corresponds to the homological transfer) is induced from the chain map

$$\begin{aligned} P_* \otimes_G A &\longrightarrow P_* \otimes_H A \\ p \otimes a &\longmapsto \sum_{gH \in G/H} g^{-1} p \otimes g^{-1} a = \sum_{Hg \in H \backslash G} gp \otimes ga. \end{aligned}$$

Similarly, one shows that the cohomological transfer is induced by the cochain map

$$\begin{aligned} \operatorname{Hom}_H(P_*, A) &\longrightarrow \operatorname{Hom}_G(P_*, A) \\ f &\longrightarrow \Bigl(p \mapsto \sum_{gH \in G/H} f(g \cdot p)\Bigr). \end{aligned}$$

In other words: the transfer map can be viewed as some sort of averaging process. In the context of bounded cohomology (Section 2.6), we will extensively use a similar averaging process (over amenable objects).

Remark 1.7.16 (Transfer map, topologically)**.** Let G be a group and let H be a subgroup of G of finite index. Then the transfer

$$\operatorname{tr}_H^G \colon H_*(G; \mathbb{Z}) \longrightarrow H_*(H; \mathbb{Z})$$

can be described topologically as follows: Let X_G be a model of the classifying space of G, and let $\pi_H \colon X_H \longrightarrow X_G$ be the covering associated with the subgroup H of $G = \pi_1(X_G)$; by covering theory, we have a commutative diagram

$$\begin{array}{ccc} & \widetilde{X_G} = \widetilde{X_H} & \\ & \downarrow {\scriptstyle \widetilde{\pi_H}} & \\ {\scriptstyle \widetilde{\pi_G}} & X_H & \\ & \downarrow {\scriptstyle \pi_H} & \\ & X_G & \end{array}$$

of covering maps, and X_H is a model of BH.

Taking $C_*(\widetilde{X_G};\mathbb{Z})$ as projective $\mathbb{Z}G$-resolution (Example 1.5.23), we obtain (Remark 1.7.15) that the homological transfer is induced by the chain map

$$\begin{aligned} \textcircled{1} \colon C_*(\widetilde{X_G};\mathbb{Z}) \otimes_G \mathbb{Z} &\longrightarrow C_*(\widetilde{X_G};\mathbb{Z}) \otimes_H \mathbb{Z} \\ \sigma \otimes 1 &\longmapsto \sum_{Hg \in H\backslash G} g\sigma \otimes 1. \end{aligned}$$

We now translate this chain map into a chain map between the chain complexes of X_G and X_H: To this end we look for a nice description of the right vertical arrow in the commutative diagram

$$\begin{array}{ccc} C_*(\widetilde{X_G};\mathbb{Z}) \otimes_H \mathbb{Z} & = & C_*(X_H;\mathbb{Z}); \\ \textcircled{1} \uparrow & & \uparrow \textcircled{2} \\ C_*(\widetilde{X_G};\mathbb{Z}) \otimes_G \mathbb{Z} & = & C_*(X_G;\mathbb{Z}) \end{array}$$

here, the upper horizontal identification maps $\sigma \otimes 1$ to $\widetilde{\pi_H} \circ \sigma$. For a singular simplex $\sigma \in \operatorname{map}(\Delta^k, X_G)$ we denote by $\pi_H^{-1}(\sigma)$ the set of all π_H-lifts of σ to X_H (see Figure 1.12). Then covering theory shows that the chain map ② is given by

$$\begin{aligned} C_*(X_G;\mathbb{Z}) &\longrightarrow C_*(X_H;\mathbb{Z}) \\ \sigma &\longmapsto \sum_{\widetilde{\sigma} \in \pi_H^{-1}(\sigma)} \sigma. \end{aligned}$$

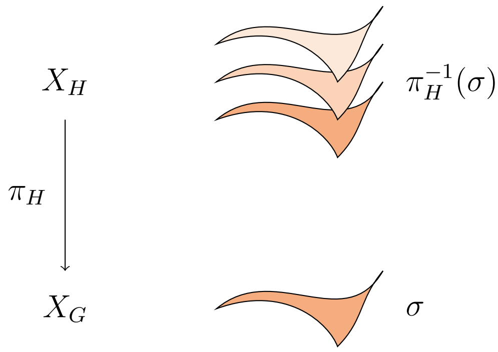


Figure 1.12: Transfer, topologically: lifting simplices

Similarly, the topological transfer can also be described using cellular (co)homology.

Remark 1.7.17 (Transfer and homological ∂-functors)**.** Using (co)effaceability, the transfer maps in homology and cohomology can also be described in terms of homological ∂-functors and cohomological δ-functors respectively [4, Section III.9].

One of the key features of the transfer is its tight connection with the index of the subgroup in question; for example, this property of the transfer allows to deduce many torsion results:

Proposition 1.7.18 (Transfer and index)**.** *Let G be a group, let $H \subset G$ be a subgroup of finite index, and let A be a $\mathbb{Z}G$-module. For all $\alpha \in H_*(G;A)$ and all $\alpha \in H^*(G;A)$ the following relation holds:*

$$\operatorname{cor}_H^G \circ \operatorname{res}_H^G(\alpha) = [G:H] \cdot \alpha$$

Notice that in the homological case, res_H^G denotes the transfer and that in the cohomological case, cor_H^G denotes the transfer; this slight abuse of notation allows us to state properties of the transfer in a uniform way.

Proof. We prove only the statement in homology; the cohomological case is similar. For the proof we use the description of the transfer given in Remark 1.7.15:

Let $P_* \twoheadrightarrow \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Then the composition

$$H_*(G;A) \xrightarrow{\operatorname{tr}_H^G = \operatorname{res}_H^G} H_*(H;\operatorname{Res}_H^G A) \xrightarrow{\operatorname{cor}_H^G} H_*(G;A)$$

is modeled on the level of chain complexes by the composition of the two chain maps

$$\begin{aligned} P_* \otimes_G A &\longrightarrow P_* \otimes_H A \\ p \otimes a &\longmapsto \sum_{Hg \in H \backslash G} gp \otimes ga \end{aligned}$$

$$\begin{aligned} P_* \otimes_H A &\longrightarrow P_* \otimes_G A \\ p \otimes a &\longmapsto p \otimes a; \end{aligned}$$

this composition is nothing but multiplication by $[G : H]$, because by definition of $\otimes_G$ we have

$$gp \otimes ga = p \otimes a$$

in $P_* \otimes_G A$ for all $p \in P_*$, all $a \in A$, and all $g \in G$. □

The converse composition of (co)restriction maps will be studied in Section 1.7.4.

Corollary 1.7.19 (Torsion in group (co)homology)**.** *Let G be a group, let H be a subgroup of finite index, let A be a $\mathbb{Z}G$-module, and let $k \in \mathbb{N}$.*

1. *If $H_k(H;\operatorname{Res}_H^G A) = 0$, then $[G : H] \cdot H_k(G;A) = 0$. Similarly, if $H^k(H;\operatorname{Res}_H^G A) = 0$, then $[G : H] \cdot H^k(G;A) = 0$.*
2. *In particular: If $[G : H]$ is invertible in A and $H_k(H;\operatorname{Res}_H^G A) = 0$, then $H_k(G;A) = 0$. Similarly, if $[G : H]$ is invertible in A and if $H^k(H;\operatorname{Res}_H^G A) = 0$, then $H^k(G;A) = 0$.*

Proof. This immediately follows from the properties of the transfer established in Proposition 1.7.18. □

Example 1.7.20 (Infinite groups whose higher (co)homology is torsion)**.**

1. The group $\operatorname{SL}(2,\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ contains a free group of index 12. Because there are one-dimensional models of classifying spaces of free groups, it follows that

$$12 \cdot H_k\bigl(\operatorname{SL}(2,\mathbb{Z});A\bigr) = 0 \quad \text{and} \quad 12 \cdot H^k\bigl(\operatorname{SL}(2,\mathbb{Z});A\bigr) = 0$$

   for all $k \in \mathbb{N}_{>1}$ and all coefficient modules A.

2. The infinite dihedral group D_∞ contains an infinite cyclic group of index 2. Therefore,

$$2 \cdot H_k(D_\infty; A) = 0 \quad \text{and} \quad 2 \cdot H^k(D_\infty; A) = 0$$

for all $k \in \mathbb{N}_{>1}$ and all coefficient modules A.

Corollary 1.7.21 ((Co)homology of finite groups is torsion)**.** *Let G be a finite group.*

1. *If A is a $\mathbb{Z}G$-module and $k \in \mathbb{N}_{>0}$, then*

$$|G| \cdot H_k(G; A) = 0 \quad \textit{and} \quad |G| \cdot H^k(G; A) = 0.$$

2. *In particular,*

$$H_*(G; \mathbb{Q}) = H_*(1; \mathbb{Q}) \quad \textit{and} \quad H^*(G; \mathbb{Q}) = H^*(1; \mathbb{Q}).$$

Proof. This corollary follows from the previous corollary applied to the trivial group as subgroup. □

Hence, if we want to compute the (co)homology of a finite group G, we only need to compute the p-primary parts for all primes p dividing $|G|$. These p-primary parts in turn are related to the (co)homology of p-Sylow subgroups (Section 1.7.5); in order to describe the precise relation, we need to study the "action" of G on the (co)homology of its Sylow subgroups (Section 1.7.4).

1.7.4 Action on (co)homology of subgroups

When trying to assemble (co)homology of the ambient group out of the (co)homology of certain subgroups, it is natural to take the "action" of the ambient group on these (co)homology groups into account.

Definition 1.7.22 (Action of the ambient group on (co)homology of subgroups)**.** Let G be a group, let H be a subgroup, and let A be a $\mathbb{Z}G$-module. Moreover, let $g \in G$.

– We write

$$\begin{aligned} c(g) &:= (h \mapsto g \cdot h \cdot g^{-1}, a \mapsto g \cdot a) \\ &: (H, \operatorname{Res}_H^G A) \longrightarrow (gHg^{-1}, \operatorname{Res}_{gHg^{-1}}^G A) \end{aligned}$$

for the isomorphism in GrpMod given by conjugation with g. Then for $\alpha \in H_*(H; A)$ we set

$$g \bullet \alpha := H_*\big(c(g)\big)(\alpha) \in H_*(gHg^{-1}; A).$$

– Similarly, we write

$$\begin{aligned} c(g)^- &:= (g \cdot h \cdot g^{-1} \mapsto h, a \mapsto g \cdot a) \\ &: (gHg^{-1}, \operatorname{Res}_{gHg^{-1}}^G A) \longrightarrow (H, \operatorname{Res}_H^G A) \end{aligned}$$

for the isomorphism in GrpMod^- given by conjugation with g. For $\alpha \in H^*(H; A)$ we set

$$g \bullet \alpha := H^*\big(c(g)^-\big)(\alpha) \in H^*(gHg^{-1}; \operatorname{Res}_{gHg^{-1}}^G A).$$

Notice that in the definition above

$$(g' \cdot g) \bullet \alpha = g' \bullet (g \bullet \alpha)$$

holds for all g, $g' \in G$ and all α in the (co)homology of H with coefficients in $\operatorname{Res}_H^G A$; clearly, if H is not normal in G, this is not an honest action on (co)homology of H with coefficients in $\operatorname{Res}_H^G A$, but close to such an action.

Proposition 1.7.23 (Action of the ambient group, via resolutions). *Let G be a group, let H be a subgroup, and let A be a $\mathbb{Z}G$-module. If $P_* \square \varepsilon$ is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then the action of an element $g \in G$ can be described by the commutative diagram*

$$\begin{array}{ccc} H_*(H; \operatorname{Res}_H^G A) & = & H_*(P_* \otimes_H \operatorname{Res}_H^G A) \\ {\scriptstyle g\bullet\cdot}\downarrow & & \downarrow \\ H_*(gHg^{-1}; \operatorname{Res}_{gHg^{-1}}^G A) & = & H_*(P_* \otimes_{gHg^{-1}} \operatorname{Res}_{gHg^{-1}}^G A), \end{array}$$

where the right vertical arrow is the map induced by the chain map

$$\begin{aligned} P_* \otimes_H \operatorname{Res}_H^G A &\longrightarrow P_* \otimes_{gHg^{-1}} \operatorname{Res}_{gHg^{-1}}^G A \\ p \otimes a &\longmapsto gp \otimes ga. \end{aligned}$$

Similarly, the action of g on cohomology is induced by the cochain map

$$\begin{aligned} \operatorname{Hom}_H(P_*, \operatorname{Res}_H^G A) &\longrightarrow \operatorname{Hom}_{gHg^{-1}}(P_*, \operatorname{Res}_{gHg^{-1}}^G A) \\ f &\longmapsto \bigl(a \mapsto g \cdot f(g^{-1} \cdot a)\bigr). \end{aligned}$$

Proof. We give only the proof for homology; the cohomological case is similar. By Corollary 1.7.6, we can view $P_* \square \varepsilon$ also as projective $\mathbb{Z}H$-resolution and as projective $\mathbb{Z}gHg^{-1}$-resolution of $\mathbb{Z}$. The description of group homology in terms of projective resolutions of the trivial module $\mathbb{Z}$ shows that the homomorphism $g \bullet \cdot$ is induced from the chain map $f_* \otimes_H (a \mapsto g \cdot a)$, where $f_* \colon \operatorname{Res}_H^G P_* \longrightarrow \operatorname{Res}_{gHg^{-1}}^G P_*$ is any chain map extending the homomorphism $\mathrm{id}_{\mathbb{Z}} \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ on the resolved module and that is compatible with the $\mathbb{Z}H$-action on the domain and the $\mathbb{Z}gHg^{-1}$-action on the target with respect to the group homomorphism

$$\begin{aligned} H &\longrightarrow gHg^{-1} \\ h &\longmapsto g \cdot h \cdot g^{-1}; \end{aligned}$$

such a chain map is for instance given by

$$\begin{aligned} f_* \colon \operatorname{Res}_H^G P_* &\longrightarrow \operatorname{Res}_{gHg^{-1}}^G P_* \\ p &\longmapsto g \cdot p. \end{aligned}$$

Therefore, the claim follows. □

Proposition 1.7.24 (The action of subgroups on their own (co)homology)**.** *Let G be a group, let H be a subgroup, and let A be a $\mathbb{Z}G$-module.*

1. *For all $h \in H$ and all $\alpha \in H_*(H; \operatorname{Res}_H^G A)$ and all $\alpha \in H^*(H; \operatorname{Res}_H^G A)$, we have*
$$h \bullet \alpha = \alpha.$$
2. *In particular, if H is a normal subgroup of G, then the conjugation action of G on (H, A) induces an action (in the ordinary sense) of the quotient group G/H on $H_*(H; \operatorname{Res}_H^G A)$ and on $H^*(H; \operatorname{Res}_H^G A)$.*

Proof. The second part directly follows from the first part. For the first part we use the description of the "action" in terms of projective resolutions (Proposition 1.7.23): Let $P_* \twoheadrightarrow \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$; because $hHh^{-1} = H$ the diagram

$$\begin{array}{ccc} H_*(H; \operatorname{Res}^G_H A) & = & H_*(P_* \otimes_H \operatorname{Res}^G_H A) \\ {\scriptstyle h\bullet\cdot}\downarrow & & \downarrow{\scriptstyle H_*(f_* \otimes_H (a \mapsto h\cdot a))} \\ H_*(H; \operatorname{Res}^G_H A) & = & H_*(P_* \otimes_H \operatorname{Res}^G_H A) \end{array}$$

is commutative, where $f_* \colon \operatorname{Res}^G_H P_* \longrightarrow \operatorname{Res}^G_H P_*$ is the chain map

$$\begin{aligned} f_* \colon \operatorname{Res}^G_H P_* &\longrightarrow \operatorname{Res}^G_H P_* \\ p &\longmapsto h \cdot p. \end{aligned}$$

Because the chain map $f_* \otimes_H (a \mapsto h \cdot a)$ maps $p \otimes a \in \operatorname{Res}^G_H P_* \otimes_H \operatorname{Res}^G_H A$ to

$$hp \otimes ha = p \otimes a \in \operatorname{Res}^G_H P_* \otimes_H \operatorname{Res}^G_H A,$$

the right vertical arrow is the identity, i.e., the element h acts trivially on $H_*(H; \operatorname{Res}^G_H A)$.

Similarly, the statement about cohomology can be obtained. □

The "action" of the ambient group provides a convenient means to describe the behaviour of the transfer:

Definition 1.7.25 (Invariant classes in (co)homology)**.** Let G be a group, let H be a subgroup of finite index, and let A be a $\mathbb{Z}G$-module. A class $\alpha \in H_*(H; \operatorname{Res}^G_H A)$ (or in $H^*(H; \operatorname{Res}^G_H A)$) is *$G$-invariant*, if

$$\operatorname{res}^{gHg^{-1}}_{H \cap gHg^{-1}} g \bullet \alpha = \operatorname{res}^{H}_{H \cap gHg^{-1}} \alpha$$

holds for all $g \in G$.

Proposition 1.7.26 (Transfer and the action of the ambient group)**.** *Let G be a group, let H be a subgroup of finite index, and let A be a $\mathbb{Z}G$-module.*

1. *Then*

$$\operatorname{res}^G_H \circ \operatorname{cor}^G_H \alpha = \sum_{HgH \in H\backslash G/H} \operatorname{cor}^H_{H \cap gHg^{-1}} \circ \operatorname{res}^{gHg^{-1}}_{H \cap gHg^{-1}} g \bullet \alpha$$

for all $\alpha \in H_*(H; \operatorname{Res}_H^G A)$ *and all* $\alpha \in H^*(H; \operatorname{Res}_H^G A)$ *respectively. In particular: If* H *is normal in* G*, then*

$$\operatorname{res}_H^G \circ \operatorname{cor}_H^G \alpha = \sum_{gH \in G/H} g \bullet \alpha$$

for all $\alpha \in H_*(H; \operatorname{Res}_H^G A)$ *and all* $\alpha \in H^*(H; \operatorname{Res}_H^G A)$ *respectively.*

2. *The restriction maps are* G*-invariant in the following sense: For all classes* $\alpha \in H_*(G; A)$ *and all* $\alpha \in H^*(G; A)$ *we have*

$$g \bullet \operatorname{res}_H^G \alpha = \operatorname{res}_{gHg^{-1}}^G \alpha$$

for all $g \in G$*. In particular, all elements in the image of* res_H^G *are* G*-invariant in the sense of Definition 1.7.25.*

Proof. We prove only the statement about homology; the proof of the statement about cohomology is analogous.

Again, we use the description of the "action" of G on homology and of the transfer in terms of projective resolutions (Remark 1.7.15 and Proposition 1.7.23); let $P_* \,\square\, \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. In the following, we also denote the chain map

$$\begin{aligned} P_* \otimes_H \operatorname{Res}_H^G A &\longrightarrow P_* \otimes_{gHg^{-1}} \operatorname{Res}_{gHg^{-1}}^G A \\ p \otimes a &\longmapsto gp \otimes ga. \end{aligned}$$

by $g \bullet \cdot$ as well, and analogously, we also use the notation res and cor for the corresponding chain level descriptions; moreover, we decorate all tensor products explicitly with the group in question to keep track of the different actions, and we use the abbreviation $H_g := H \cap gHg^{-1}$.

For the first part, we start by computing the right hand side on the chain level: Notice that the (co)restriction maps on the right hand side indeed are well-defined as the group/subgroup pairs occurring in this expression all are of finite index. For all $p \otimes_H a \in P \otimes_H A$ the chain level expression for the right hand side equals

$$\begin{aligned}
&\sum_{HgH\in H\backslash G/H} \operatorname{cor}^{H}_{gHg^{-1}} \circ \operatorname{res}^{gHg^{-1}}_{H_g} g \bullet (p \otimes_H a) \\
&= \sum_{HgH\in H\backslash G/H} \operatorname{cor}^{H}_{gHg^{-1}} \circ \operatorname{res}^{gHg^{-1}}_{H_g} (gp \otimes_{gHg^{-1}} ga) \\
&= \sum_{HgH\in H\backslash G/H} \operatorname{cor}^{H}_{gHg^{-1}} \sum_{H_g ghg^{-1}\in H_g\backslash gHg^{-1}} ghg^{-1}\cdot gp \otimes_{H_g} ghg^{-1}\cdot ga \\
&= \sum_{HgH\in H\backslash G/H} \operatorname{cor}^{H}_{gHg^{-1}} \sum_{H_g ghg^{-1}\in H_g\backslash gHg^{-1}} ghp \otimes_{H_g} gha \\
&= \sum_{HgH\in H\backslash G/H} \sum_{H_g ghg^{-1}\in H_g\backslash gHg^{-1}} ghp \otimes_{H} gha \\
&= \sum_{Hg\in H\backslash G} gp \otimes_H ga;
\end{aligned}$$

by the description of the transfer in terms of projective resolution (Remark 1.7.15) we know that this term is nothing but the chain level expression for the left hand side. This proves the first part.

We now come to the second part: Again, we evaluate the corresponding expressions on the chain level. Notice that in the homology case, res^G_H is the transfer map. For all $p \otimes_G a \in P_* \otimes_G A$ we have

$$\begin{aligned}
g \bullet \operatorname{res}^G_H(p \otimes_G a) &= g \bullet \sum_{Hg'\in H\backslash G} g'p \otimes_H g'a \\
&= \sum_{Hg'\in H\backslash G} gg'p \otimes_{gHg^{-1}} gg'a \\
&= \sum_{gHg^{-1}g'\in gHg^{-1}\backslash G} g'p \otimes_{gHg^{-1}} g'a \\
&= \operatorname{res}^G_{gHg^{-1}}(p \otimes_G a).
\end{aligned}$$

In particular, the corresponding relation in homology holds as well.

Furthermore, for any $\alpha \in H_*(G; A)$ we obtain that $\beta := \operatorname{res}^G_H \alpha$ is G-invariant in the sense of Definition 1.7.25 because

$$\begin{aligned}
\operatorname{res}^{gHg^{-1}}_{H\cap gHg^{-1}} g \bullet \beta &= \operatorname{res}^{gHg^{-1}}_{H\cap gHg^{-1}} (g \bullet \operatorname{res}^G_H \alpha) \\
&= \operatorname{res}^{gHg^{-1}}_{H\cap gHg^{-1}} \circ \operatorname{res}^G_{gHg^{-1}} \alpha \\
&= \operatorname{res}^{G}_{H\cap gHg^{-1}} \alpha \\
&= \operatorname{res}^{H}_{H\cap gHg^{-1}} \circ \operatorname{res}^G_{H} \alpha \\
&= \operatorname{res}^{H}_{H\cap gHg^{-1}} \beta;
\end{aligned}$$

in the third and in the fifth step we used "associativity" of res, which is easily established by looking at the description via projective resolutions. (Exercise). □

1.7.5 Decomposing group (co)homology into primary parts

Using the transfer and the "action" of a finite group on the (co)homology of its Sylow subgroups, we obtain a nice decomposition of the (co)homology of finite groups into pieces related to the Sylow subgroups. We start by decomposing group (co)homology into its primary parts:

Definition 1.7.27 (Primary parts)**.** Let A be an Abelian group, and let $p \in \mathbb{N}$ be a prime. The *p-primary part* of A is the subgroup

$$A_{[p]} := \{a \in A \mid \exists_{n\in\mathbb{N}}\ p^n \cdot a = 0\}.$$

Remark 1.7.28 (Decomposing torsion groups into primary parts)**.** Let A be an Abelian group that consists only of torsion elements. Then

$$A = \bigoplus_{p\in\mathbb{N}\text{ prime}} A_{[p]}$$

because: As all elements of A are torsion, all cyclic subgroups of A are torsion groups. Using the classification of finite Abelian groups, we see that any such cyclic subgroup is generated by elements of the primary parts of A. Therefore, $A = \sum_{p\in\mathbb{N}\text{ prime}} A_{[p]}$. A straightforward computation shows that this sum is direct.

Proposition 1.7.29 (Primary decomposition of group (co)homology I). *Let G be a finite group, let A be a $\mathbb{Z}G$-module, and let $k \in \mathbb{N}_{>0}$. Then*

$$H_k(G;A) = \bigoplus_{p \in P(|G|)} H_k(G;A)_{[p]},$$
$$H^k(G;A) = \bigoplus_{p \in P(|G|)} H^k(G;A)_{[p]},$$

where $P(|G|)$ denotes the set of all positive prime numbers dividing $|G|$.

Proof. By the remark above, any Abelian group that consists only of torsion elements is the direct sum of its primary parts. Now the claim follows with help of Corollary 1.7.21. □

Proposition 1.7.30 (Primary decomposition of group (co)homology II). *Let G be a finite group. For each prime $p \in P(|G|)$ let G_p be a p-Sylow subgroup of G. Moreover, let $k \in \mathbb{N}_{>0}$.*

1. *For $p \in P(|G|)$, the map $\operatorname{res}^G_{G_p} \colon H_k(G;A)_{[p]} \longrightarrow H_k(G_p; \operatorname{Res}^G_{G_p} A)$ is injective and the image consists exactly of the G-invariant elements of $H_k(G_p; \operatorname{Res}^G_{G_p} A)$ in the sense of Definition 1.7.25. Literally the same statement also holds in cohomology.*
2. *In particular: If $p \in P(|G|)$, and if G_p is normal in G, then*

$$H_k(G;A)_{[p]} \cong H_k(G_p; \operatorname{Res}^G_{G_p} A)^{G/G_p},$$
$$H^k(G;A)_{[p]} \cong H^k(G_p; \operatorname{Res}^G_{G_p} A)^{G/G_p}.$$

3. *Putting it all together, we obtain the primary decompositions*

$$H_k(G;A) \cong \bigoplus_{p \in P(|G|)} H_k(G_p; \operatorname{Res}^G_{G_p} A)^G,$$
$$H^k(G;A) \cong \bigoplus_{p \in P(|G|)} H^k(G_p; \operatorname{Res}^G_{G_p} A)^G$$

(here, we used $\cdot^G$ to denote the G-invariants in the sense of Definition 1.7.25).

Proof. We prove only the homological statements; the results in cohomology follow from similar arguments.

The second part follows directly from the first part, and the third part can be deduced from the first part with help of Proposition 1.7.29. Therefore, it suffices to prove the first part:

The map $\operatorname{res}^G_{G_p}$ is injective on $H_*(G;A)_{[p]}$: As G_p is a p-Sylow subgroup of G, the index $[G : G_p]$ is not divisible by p and hence invertible in the p-primary part $H_*(G;A)_{[p]}$. Because

$$\operatorname{cor}^G_{G_p} \circ \operatorname{res}^G_{G_p} = [G : G_p]$$

it follows that $\operatorname{res}^G_{G_p}$ indeed is injective on $H_*(G;A)_{[p]}$.

What is the image of $\operatorname{res}^G_{G_p}$? We have already seen in Proposition 1.7.26 that the image of $\operatorname{res}^G_{G_p}$ consists of G-invariant elements. Hence, it remains to prove that any G-invariant element of $H_*(G_p; \operatorname{Res}^G_{G_p} A)$ lies in the image $\operatorname{res}^G_{G_p}(H_*(G;A)_{[p]})$: Let $\beta \in H_*(G_p; \operatorname{Res}^G_{G_p} A)$ be G-invariant. We then consider the class

$$\alpha := \frac{1}{[G : G_p]} \cdot \operatorname{cor}^G_{G_p} \beta \in H_*(G;A)_{[p]};$$

notice that $\operatorname{cor}^G_{G_p} \beta$ lies in the p-primary part of $H_*(G;A)$ because the order of β is a prime power (as G_p is a p-group). Moreover, $[G : G_p]$ is invertible in the p-primary part $H_*(G;A)_{[p]}$. Therefore, α indeed is a well-defined element of $H_*(G;A)_{[p]}$.

Furthermore, $\beta = \operatorname{res}^G_H \alpha$ because by G-invariance of β and Proposition 1.7.18

$$\begin{aligned}
\operatorname{res}^G_{G_p} \circ \operatorname{cor}^G_{G_p} \beta &= \sum_{G_p g G_p \in G_p \backslash G / G_p} \operatorname{cor}^{G_p}_{G_p \cap g G_p g^{-1}} \circ \operatorname{res}^{g G_p g^{-1}}_{G_p \cap g G_p g^{-1}} g \bullet \beta \\
&= \sum_{G_p g G_p \in G_p \backslash G / G_p} \operatorname{cor}^{G_p}_{G_p \cap g G_p g^{-1}} \circ \operatorname{res}^{G_p}_{G_p \cap g G_p g^{-1}} \beta \\
&= \sum_{G_p g G_p \in G_p \backslash G / G_p} [G_p : G_p \cap g G_p g^{-1}] \cdot \beta \\
&= [G : G_p] \cdot \beta.
\end{aligned}$$

The last equality follows by decomposing G/G_p into left G_p-orbits. □

Example 1.7.31 (The symmetric group S_3)**.** For example, using the primary decomposition in Proposition 1.7.30, it is possible to compute the (co)homology of the symmetric group S_3; for all $k \in \mathbb{N}$ we have (Exercise):

$$H_k(S_3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \equiv 0 \bmod 4 \text{ and } k > 0 \\ \mathbb{Z}/2 & \text{if } k \equiv 1 \bmod 4 \\ 0 & \text{if } k \equiv 2 \bmod 4 \\ \mathbb{Z}/6 & \text{if } k \equiv 3 \bmod 4, \end{cases}$$

$$H^k(S_3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/6 & \text{if } k \equiv 0 \bmod 4 \text{ and } k > 0 \\ 0 & \text{if } k \equiv 1 \bmod 4 \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \bmod 4 \\ 0 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Later we will see that there are several possibilities to simplify this slightly cumbersome computation and we will also explain the obvious patterns in these (co)homology groups (using product structures, periodicity, or spectral sequences (Sections 1.8, 1.9, 1.10).

Using more sophisticated transfer maps it is possible to obtain more fine-grained information about the (co)homology of finite groups; an example of such a refined transfer map is the Evens transfer [**?**].

1.7.6 Application: Generalising the group-theoretic transfer

The transfer in group (co)homology discussed above actually is a generalisation of the classical transfer in group theory [14]:

Definition 1.7.32 (Classical transfer)**.** Let G be a group and let H be a subgroup of finite index n. Then the *classical transfer* is the homomorphism

$$\begin{aligned} G_{\mathrm{ab}} &\longrightarrow H_{\mathrm{ab}} \\ [g] &\longmapsto \Biggl[\prod_{k=1}^{n} g_k \cdot g \cdot R(g_k \cdot g)^{-1}\Biggr]; \end{aligned}$$

here, $g_1, \dots, g_n$ is a set of representatives of $H\backslash G$, and $R\colon G \longrightarrow \{g_1, \dots, g_n\}$ is the map associating to each group element $g \in G$ the representative of the corresponding coset $H \cdot g$.

One can show that this indeed is a well-defined group homomorphism that is independent of the chosen set of representatives.

Proposition 1.7.33 (Classical transfer and group homology)**.** *Let G be a group and let H be a subgroup of finite index. Then the transfer map*

$$\mathrm{tr}_H^G\colon H_1(G;\mathbb{Z}) \longrightarrow H_1(H;\mathbb{Z})$$

coincides with the classical transfer of Definition 1.7.32, when we identify the first group homology with $\mathbb{Z}$-coefficients with the Abelianisation of the group in question (Proposition 1.3.12).

Proof. This can be proved by carefully analysing the homological transfer in terms of the topological description or via bar resolutions. (Exercise). □

For instance, the classical transfer is related to quadratic residues in number theory:

Example 1.7.34 (Legendre symbol via transfer)**.** Let $p \in \mathbb{N}_{>2}$ be a prime. Then the transfer map corresponding to the subgroup $\{+1, -1\}$ of the units $\mathbb{Z}/p^\times$ coincides with the Legendre symbol associated with p; this is a consequence of the Gauß lemma on quadratic residues. (Exercise).

1.8 Product structures

Until now, we viewed group homology and group cohomology only as graded Abelian groups; however, group (co)homology carries an additional, multiplicative, structure.

Like products in (co)homology of topological spaces, these products provide a finer structure on group (co)homology that allows us to understand certain phenomena in group (co)homology better and that also simplifies calculations substantially.

Before starting with the details of the constructions, we first give a brief overview of the products to come: In the following list, we only indicate the basic shape of the products. The exact types (including coefficients) are given during the detailed discussions and constructions below:

– *(Co)Homological cross-product.* The homological cross-product is of the shape

$$H_p \otimes_{\mathbb{Z}} H_{p'} \longrightarrow H_{p+p'},$$

the cohomological cross-product is of the shape

$$H^p \otimes_{\mathbb{Z}} H^{p'} \longrightarrow H^{p+p'}.$$

These products both are external products.

– *Cup-product.* The cup-product is an internal product of the shape

$$H^p \otimes_{\mathbb{Z}} H^{p'} \longrightarrow H^{p+p'}.$$

– *Cap-product.* The cap-product is a product of the shape

$$H^p \otimes_{\mathbb{Z}} H_{p'} \longrightarrow H_{p'-p}.$$

In the rest of this section, we will focus mainly on the cup-product as it is the most important of these products. We will start with an axiomatic description for the cup-product, and we then will construct all of the products listed above by tensor product constructions on the (co)chain level, and derive further properties.

For the sake of simplicity, we will pursue a rather elementary approach to products – a more conceptual point of view would be based on the notion of differential graded algebras [1].

1.8.1 A multiplicative structure for group cohomology

Theorem 1.8.1 (Cup-product in group cohomology, axiomatically)**.** *There is exactly one multiplicative structure* $\cdot \cup \cdot$*, the so-called* cup-product*, on group cohomology satisfying the following axioms: For every group* G*, all* $\mathbb{Z}G$*-modules* A *and* A'*, and all degrees* p*,* $p' \in \mathbb{N}$ *the cup-product structure provides a* $\mathbb{Z}$*-linear map*

$$\cdot \cup \cdot : H^p(G; A) \otimes_{\mathbb{Z}} H^{p'}(G; A') \longrightarrow H^{p+p'}(G; A \otimes_{\mathbb{Z}} A');$$

here, G *acts diagonally on the coefficients* $A \otimes_{\mathbb{Z}} A'$*.*

- Degree 0. *For every group* G *and all* $\mathbb{Z}G$*-modules* A *and* A'*, the cup-product*

  $$A^G \otimes_{\mathbb{Z}} A'^G = H^0(G; A) \otimes_{\mathbb{Z}} H^0(G; A') \longrightarrow H^0(G; A \otimes_{\mathbb{Z}} A') = (A \otimes_{\mathbb{Z}} A')^G$$

  coincides with the $\mathbb{Z}$*-homomorphism induced from the canonical inclusions* $A^G \longrightarrow A$ *and* $A'^G \longrightarrow A'$*.*
- Naturality with respect to connecting homomorphisms. *Let* G *be a group, let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be a short exact sequence of* $\mathbb{Z}G$*-modules, and let* B *be a* $\mathbb{Z}G$*-module such that the induced sequence* $0 \longrightarrow A' \otimes_{\mathbb{Z}} B \longrightarrow A \otimes_{\mathbb{Z}} B \longrightarrow A'' \otimes_{\mathbb{Z}} B \longrightarrow 0$ *is exact. Then*

  $$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta$$

  for all $\alpha \in H^*(G; A'')$ *and all* $\beta \in H^*(G; B)$*, where the* δ*s are the connecting homomorphisms of the long exact cohomology sequences corresponding to the above short exact sequence of* $\mathbb{Z}G$*-modules.*

  (An analogous statement holds when the factors are swapped; however, a sign has to be introduced in that case.)

Proof. Uniqueness. Uniqueness is a consequence of a dimension shifting argument: Let G be a group. We prove uniqueness of the cup-product on the cohomology of G by double induction over the degrees of the domain of the product:

The cup-product $H^0 \otimes_{\mathbb{Z}} H^0 \longrightarrow H^0$ is determined uniquely by the first axiom.

Let B be a $\mathbb{Z}G$-module and let $\beta \in H^{p'}(G; B)$. We show now inductively that the cup-product $\cdot \cup \beta \colon H^* \longrightarrow H^{*+p'}$ is determined uniquely by the cup-product $\cdot \cup \beta \colon H^0 \longrightarrow H^{p'}$ in degree 0: For the induction step, let $p \in \mathbb{N}$ and assume that we proved the claim for $\cdot \cup \beta \colon H^p \longrightarrow H^{p+p'}$. Let A be a $\mathbb{Z}G$-module. Notice that there is a $\mathbb{Z}G$-embedding

$$\begin{aligned} A &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A) = \operatorname{Coind}_1^G A \\ a &\longmapsto (g \mapsto g \cdot a), \end{aligned}$$

which is split injective over $\mathbb{Z}$; in particular, the associated short exact sequence $0 \longrightarrow A \longrightarrow \operatorname{Coind}_1^G A \longrightarrow (\operatorname{Coind}_1^G A)/A \longrightarrow A$ stays exact when tensored with B. Furthermore, $H^k(G; \operatorname{Coind}_1^G A) = 0$ for all $k \in \mathbb{N}_{>0}$ by Shapiro's lemma (Proposition 1.7.8). Hence, the second axiom provides us with a commutative diagram

$$\begin{array}{ccccc} H^p\big(G; (\operatorname{Coind}_1^G A)/A\big) & \xrightarrow{\delta} & H^{p+1}(G; A) & \longrightarrow & H^{p+1}(G; \operatorname{Coind}_1^G A) = 0 \\ \downarrow{\scriptstyle \cdot\cup\beta} & & \downarrow{\scriptstyle \cdot\cup\beta} & & \downarrow \\ H^{p+p'}\big(G; (\operatorname{Coind}_1^G A)/A \otimes_{\mathbb{Z}} B\big) & \xrightarrow[\delta]{} & H^{p+1+p'}(G; A \otimes_{\mathbb{Z}} B) & \longrightarrow & H^{p+1+p'}(G; \operatorname{Coind}_1^G A \otimes_{\mathbb{Z}} B) \end{array}$$

with exact rows; therefore, $\delta \colon H^p(G; \operatorname{Coind}_1^G A/A) \longrightarrow H^{p+1}(G; A)$ is surjective and so $\cdot \cup \beta \colon H^{p+1}(G; A) \longrightarrow H^{p+1+p'}(G; A \otimes_{\mathbb{Z}} B)$ is determined uniquely by $\cdot \cup \beta H^p \longrightarrow H^{p+p'}$.

Using the swapped version of the second axiom, in the same way we can prove inductively that $\alpha \cup \cdot : H^* \longrightarrow H^{p+*}$ is determined uniquely by the cup-product $\alpha \cup \cdot : H^0 \longrightarrow H^p$.

Putting both inductions together, we see that the cup-product indeed is determined uniquely by the cup-product $H^0 \otimes_{\mathbb{Z}} H^0 \longrightarrow H^0$, which in turn is covered by the first axiom.

Existence. We will prove existence by giving an explicit construction in terms of projective resolutions (Section 1.8.2–1.8.4). $\square$

When proving existence below, we will also see that the cup-product enjoys several other properties as well that we would expect of a nice product, such as associativity, graded commutativity, existence of a unit, naturality with respect to morphisms in GrpMod⁻ etc. (see Proposition 1.8.18).

Caveat 1.8.2 (Homological products)**.** In general, there is no sensible map of type $A_G \otimes_{\mathbb{Z}} A'_G \longrightarrow (A \otimes_{\mathbb{Z}} A')_G$. Therefore, there are no obvious analogous axioms for a product on group homology.

For Abelian groups, there is the so-called Pontryagin product [4, Chapter V.5] on group homology, and in the general case one can define a coproduct instead of a product.

1.8.2 Algebraic preliminaries: tensor products of resolutions

The construction of the cup-product will be achieved by the same algebraic means as the construction of the cup-product for singular or cellular (co)homology of topological spaces. In particular, the construction relies on an understanding of tensor products of chain complexes:

Convention 1.8.3 (Tensor products of chain complexes)**.** For $\mathbb{Z}$-chain complexes (C_*, ∂) and (C'_*, ∂'_*), the *tensor product* $C_* \otimes_{\mathbb{Z}} C'_*$ is the chain complex given by

$$(C_* \otimes_{\mathbb{Z}} C'_*)_n := \bigoplus_{p \in \mathbb{N}} C_p \otimes_{\mathbb{Z}} C'_{n-p}$$

for all $n \in \mathbb{N}$ equipped with the boundary operator given by

$$\begin{aligned} (C_* \otimes_{\mathbb{Z}} C'_*)_n &\longrightarrow (C_* \otimes_{\mathbb{Z}} C'_*)_{n-1} \\ x \otimes x' &\longmapsto \partial x \otimes x' + (-1)^{|x|} x \otimes \partial' x' \end{aligned}$$

for all $n \in \mathbb{N}$; here, $|\cdot|$ stands for the degree of an element in a chain complex. Notice that the square of this boundary operator indeed is zero.

The sign is chosen according to the following convention: whenever an operator of degree d is moved past an element of degree p, then the sign $(-1)^{d \cdot p}$ is introduced. Other choices of signs are possible and appropriate in certain circumstances; however, in the following, the above choice of signs proves to be convenient.

Proposition 1.8.4 (Tensor products of resolutions)**.** *Let G and G' be two groups, and let $P_* \mathbin{\square} \varepsilon$ and $P'_* \mathbin{\square} \varepsilon'$ be a $\mathbb{Z}G$-resolution and a $\mathbb{Z}G'$-resolution of $\mathbb{Z}$ respectively. Then the tensor product $(P_* \otimes_{\mathbb{Z}} P'_*) \mathbin{\square} (\varepsilon \otimes_{\mathbb{Z}} \varepsilon')$ is a projective $\mathbb{Z}[G \times G']$-resolution of $\mathbb{Z}$; here, the group $G \times G'$ acts componentwise on the tensor products.*

Proof. We have to show that all the chain modules of $P_* \otimes_{\mathbb{Z}} P'_*$ are projective $\mathbb{Z}[G \times G']$-modules and that the complex $(P_* \otimes_{\mathbb{Z}} P'_*) \mathbin{\square} (\varepsilon \otimes_{\mathbb{Z}} \varepsilon')$ is acyclic.

The latter easily follows from the Künneth theorem and the right exactnes of the tensor product, because the two factors $P_* \mathbin{\square} \varepsilon$ and $P'_* \mathbin{\square} \varepsilon'$ are acyclic.

As projective modules can be characterised by being direct summands of free modules, and as tensor products are compatible with direct sums, the first statement follows from the fact that

$$\begin{aligned} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G'] &\longrightarrow \mathbb{Z}[G \times G'] \\ g \otimes g' &\longmapsto (g, g') \end{aligned}$$

is a $\mathbb{Z}[G \times G']$-isomorphism. □

Corollary 1.8.5 (Diagonal approximation)**.** *Let G be a group, and let $P_* \mathbin{\square} \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Then up to $\mathbb{Z}G$-homotopy equivalence there is exactly one chain map*

$$\Delta_* \colon P_* \longrightarrow P_* \otimes_{\mathbb{Z}} P_*$$

compatible with ε and $\varepsilon \otimes_{\mathbb{Z}} \varepsilon$ and extending $\mathrm{id}_{\mathbb{Z}} \colon \mathbb{Z} \longrightarrow \mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$*; here, G acts diagonally on $P_* \otimes_{\mathbb{Z}} P_*$.*

Any such chain map Δ_* is called a diagonal approximation *of the resolution $P_* \mathbin{\square} \varepsilon$.*

Proof. By Proposition 1.8.4, the tensor product $(P_* \otimes_{\mathbb{Z}} P'_*) \mathbin{\square} (\varepsilon \otimes_{\mathbb{Z}} \varepsilon)$ is a projective $\mathbb{Z}[G \times G]$-resolution of $\mathbb{Z}$. Moreover, the restriction functor $\mathbb{Z}[G \times G]$-Mod $\longrightarrow \mathbb{Z}G$-Mod induced by the diagonal homomorphism

$$\begin{aligned} G &\longrightarrow G \times G \\ g &\longmapsto (g, g) \end{aligned}$$

is exact (Example 1.5.8). Therefore, the claim follows from the fundamental lemma of homological algebra (Proposition 1.5.27). □

In general, it is not easy to find a nicely expressible diagonal approximation of a given resolution. Fortunately, for the bar resolution and the standard periodic resolutions of finite cyclic groups, convenient diagonal approximations are known:

Example 1.8.6 (The Alexander-Whitney map)**.** Let G be a group. Then the *Alexander-Whitney map* $\Delta_*\colon C_*(G) \longrightarrow C_*(G) \otimes_{\mathbb{Z}} C_*(G)$ given by

$$\begin{aligned} C_n(G) &\longrightarrow \bigl(C_*(G) \otimes_{\mathbb{Z}} C_*(G)\bigr)_n \\ g_0 \cdot [g_1|\cdots|g_n] &\longmapsto \sum_{p=0}^{n} g_0 \cdot [g_1|\cdots|g_p] \otimes g_0 \cdot g_1 \cdot \dots \cdot g_p \cdot [g_{p+1}|\cdots|g_n] \end{aligned}$$

for all $n \in \mathbb{N}$ is a diagonal approximation over $\mathbb{Z}G$ (as is readily verified by a small computation).

The topological counterpart of this map maps a simplex to the sum of tensor products of its front and back faces, i.e., it gives an approximation of a bigger simplex by sums of products of smaller simplices.

Example 1.8.7 (A diagonal approximation for finite cyclic groups)**.** Let $n \in \mathbb{N}_{>0}$; we write $G := \mathbb{Z}/n$ for the cyclic group of order n. Let $P_* \mathbin{\square} \varepsilon$ be the standard projective $\mathbb{Z}G$-resolution

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

of $\mathbb{Z}$ (see Corollary 1.6.6); here, $t = [1] \in \mathbb{Z}/n$ is a generator of the cyclic group $G = \mathbb{Z}/n$, and N denotes the norm element.

Then the $\mathbb{Z}G$-map $\Delta_*\colon P_* \longrightarrow P_* \otimes_{\mathbb{Z}} P_*$ given by

$$\begin{aligned} P_{p+p'} = \mathbb{Z}[G] &\longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] = P_p \otimes_{\mathbb{Z}} P_{p'} \\ 1 &\longmapsto \begin{cases} 1 \otimes 1 & \text{if } p \text{ is even} \\ 1 \otimes t & \text{if } p \text{ is odd and } p' \text{ is even} \\ \sum_{j,k \in \{0,\dots,n-1\}, j<k} t^j \otimes t^k & \text{if } p \text{ is odd and } p' \text{ is odd} \end{cases} \end{aligned}$$

for all p, $p' \in \mathbb{N}$ is a diagonal approximation for the resolution $P_* \mathbin{\square} \varepsilon$; indeed, a straightforward but rather tedious computation shows that Δ_*

is compatible with the boundary operators and the augmentations ε and $\varepsilon \otimes_{\mathbb{Z}} \varepsilon$ respectively.

Notice that when computing with this diagonal approximation, some care has to be taken to keep all the parts in the different degrees apart from each other!

1.8.3 The cross-product

The first step towards the construction of the cup-product is the construction of an external product, the so-called cross-product; on the (co)chain level, the cross-products are nothing but tensor products:

Definition 1.8.8 (Cross-product on the (co)chain level)**.** Let G and G' be two groups, let A be a $\mathbb{Z}G$-module, let A' be a $\mathbb{Z}G'$-module, let $P_* \twoheadrightarrow \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and let $P'_* \twoheadrightarrow \varepsilon'$ be a projective $\mathbb{Z}G'$-resolution of $\mathbb{Z}$. Then the *cross-product* on the chain level is defined by

$$\begin{aligned} \times\colon (P_* \otimes_G A) \otimes_{\mathbb{Z}} (P'_* \otimes_{G'} A') &\longrightarrow (P_* \otimes_{\mathbb{Z}} P'_*) \otimes_{G \times G'} (A \otimes_{\mathbb{Z}} A') \\ (x \otimes a) \otimes (x' \otimes a') &\longmapsto (x \otimes x') \otimes (a \otimes a'); \end{aligned}$$

dually, the *cross-product* on the cochain level is defined by

$$\begin{aligned} \times\colon \operatorname{Hom}_G(P_*, A) \otimes_{\mathbb{Z}} \operatorname{Hom}_{G'}(P'_*, A') &\longrightarrow \operatorname{Hom}_{G \times G'}(P_* \otimes_{\mathbb{Z}} P'_*, A \otimes_{\mathbb{Z}} A') \\ f \otimes f' &\longmapsto \bigl(x \otimes x' \mapsto (-1)^{|f'|\cdot|x|} f(x) \otimes f'(x')\bigr). \end{aligned}$$

Using the description of group (co)homology via projective resolutions of $\mathbb{Z}$ (Section 1.5.8), we can thus define cross-products in group (co)homology:

Definition 1.8.9 ((Co)Homological cross-product)**.** Let G and G' be two groups. Moreover, let A and A' be a $\mathbb{Z}G$-module and a $\mathbb{Z}G'$-module respectively.

– *Homological cross-product.* The *homological cross-product* is defined by

$$\begin{array}{ccc} H_p(G;A) \otimes_{\mathbb{Z}} H_{p'}(G';A') & \xrightarrow{\ \cdot \times \cdot\ } & H_{p+p'}(G \times G'; A \otimes_{\mathbb{Z}} A') \\ \| & & \| \\ H_p(P_* \otimes_G A) \otimes_{\mathbb{Z}} H_{p'}(P'_* \otimes_{G'} A') & \longrightarrow & H_{p+p'}\bigl((P_* \otimes_{\mathbb{Z}} P'_*) \otimes_{G \times G'} (A \otimes_{\mathbb{Z}} A')\bigr) \\ [z] \otimes [z'] & \longmapsto & [z \times z'] \end{array}$$

for all $p, p' \in \mathbb{N}$, where $P_* \Box \varepsilon$ and $P'_* \Box \varepsilon'$ are some projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. (The isomorphism on the right is provided by Proposition 1.8.4).

– *Cohomological cross-product.* The *cohomological cross-product* is defined by

$$\begin{array}{ccc} H^p(G;A) \otimes_{\mathbb{Z}} H^{p'}(G';A') & \xrightarrow{\;\cdot \times \cdot\;} & H^{p+p'}(G \times G'; A \otimes_{\mathbb{Z}} A') \\ \| & & \| \\ H_p\big(\mathrm{Hom}_G(P_*, A)\big) \otimes_{\mathbb{Z}} H_{p'}\big(\mathrm{Hom}_{G'}(P'_*, A')\big) & \longrightarrow & H_{p+p'}\big(\mathrm{Hom}_{G\times G'}(P_* \otimes_{\mathbb{Z}} P'_*, A \otimes_{\mathbb{Z}} A')\big) \\ [f] \otimes [f'] & \longmapsto & [f \times f'] \end{array}$$

for all $p, p' \in \mathbb{N}$, where $P_* \Box \varepsilon$ and $P'_* \Box \varepsilon'$ are some projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. (The isomorphism on the right is provided by Proposition 1.8.4).

Remark 1.8.10 (Well-definedness of the (co)homological cross-product)**.** The (co)homological cross product in group (co)homology is well-defined in the following sense:

1. The cross-product of two classes does not depend on the choice of the representing (co)cycles.
2. The cross-product does not depend on the choice of projective resolution.

For the first part, we observe that we have in the situation of Definition 1.8.8 the relations

$$\partial(z \times z') = \partial z \times z' + (-1)^{|z|} \cdot z \times \partial z'$$
$$\delta(f \times f') = \delta f \times f' + (-1)^{|f|} \cdot f \times \delta f'$$

for all chains $z \in P_* \otimes_G A$, $z' \in P'_* \otimes_{G'} A'$ and all cochains $f \in \mathrm{Hom}_G(P_*, A)$, $f' \in \mathrm{Hom}_{G'}(P'_*, A')$.

For the second part, we consider the following (with the notation from Definition 1.8.8): Let $Q_* \Box \eta$ be another $\mathbb{Z}G$-resolution of $\mathbb{Z}$. By the fundamental lemma of homological algebra, there is a $\mathbb{Z}G$-chain homotopy equivalence $f \colon P_* \longrightarrow Q_*$ extending the identity on $\mathbb{Z}$. A simple computation shows that then

$$
\begin{gathered}
f \otimes_G \mathrm{id}_A \\
(f \otimes_{\mathbb{Z}} \mathrm{id}_{P'_*}) \otimes_{G\times G'} \mathrm{id}_{A\otimes_{\mathbb{Z}} A'} \\
\mathrm{Hom}_G(f, \mathrm{id}_A) \\
\mathrm{Hom}_{G\times G'}(f \otimes_{\mathbb{Z}} \mathrm{id}_{P'_*}, \mathrm{id}_{A\otimes_{\mathbb{Z}} A'})
\end{gathered}
$$

are $\mathbb{Z}$-chain homotopy equivalences of the complexes involved in the definition of the cross-products extending the identity on $\mathbb{Z}$, which are compatible with taking cross-products. As these chain homotopy equivalences induce the identity on group (co)homology, this shows that the (co)homological cross-product in group (co)homology is independent of the chosen resolution of G. Similarly, we can argue for the resolutions for the right hand factor.

1.8.4 The cup-product

As mentioned above, the cup-product is an internal product on group cohomology, derived from the cohomological cross-product:

Definition 1.8.11 (Cup-product)**.** Let G be a group, and let A and A' be two $\mathbb{Z}G$-modules. Then the map

$$
\cdot \cup \cdot \colon H^*(G;A) \otimes_{\mathbb{Z}} H^*(G;A') \longrightarrow H^*(G; A \otimes_{\mathbb{Z}} A')
$$

given by

$$
\begin{aligned}
H^p(G;A) \otimes_{\mathbb{Z}} H^{p'}(G;A') &\longrightarrow H^{p+p'}(G; A \otimes_{\mathbb{Z}} A') \\
\alpha \otimes \alpha' &\longmapsto H^{p+p'}(d; \mathrm{id}_{A\otimes_{\mathbb{Z}} A'})(\alpha \times \alpha')
\end{aligned}
$$

for all p, $p' \in \mathbb{N}$ is the *cup-product* on cohomology of G. Here, G acts diagonally on the coefficients $A \otimes_{\mathbb{Z}} A'$ and $d\colon G \longrightarrow G \times G$ is the diagonal homomorphism; hence, $(d, \mathrm{id}_{A\otimes_{\mathbb{Z}} A'})\colon (G, A \otimes_{\mathbb{Z}} A') \longrightarrow (G \times G, A \otimes_{\mathbb{Z}} A')$ indeed is a morphism in GrpMod^-.

Remark 1.8.12 (Topological cup-product)**.** If G is a group and if X_G is a model for BG, then the group cohomological cup-product and the topological cup-product on $H^*(G;\mathbb{Z}) \cong H^*(X_G;\mathbb{Z})$ coincide. (Exercise).

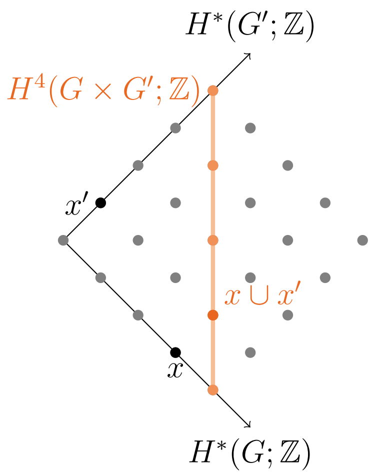


Figure 1.13: Cohomology ring of a product group, schematically

Example 1.8.13 (Cohomology of $\mathbb{Z}^2$)**.** Because the 2-torus is a model for $B(\mathbb{Z}\times\mathbb{Z})$, we obtain from the preceding remark that there is an isomorphism

$$H^*(\mathbb{Z}\times\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}[x]/(x^2)\otimes_{\mathbb{Z}}\mathbb{Z}[y]/(y^2)$$

of graded rings, where x and y have degree 1.

Remark 1.8.14 (The cohomology of product groups)**.** More generally, using the cohomological Künneth theorem, we can compute the cup-product structure on cohomology with $\mathbb{Z}$-coefficients of a product of two groups (as long as the group homology with $\mathbb{Z}$-coefficients of one of the two factors is finitely generated in each degree) in terms of the cup-product structure on the two factors (Figure 1.13).

For concrete computations it is often helpful to be able to express the cup-product in terms of the same projective resolution in domain and target; such a description can be given via diagonal approximations:

Remark 1.8.15 (Cup-product via diagonal approximations)**.** Let G be a group, let A and A' be two $\mathbb{Z}G$-modules, and let $P_* \,\square\, \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$; furthermore, let $\Delta_*\colon P_* \longrightarrow P_*\otimes_{\mathbb{Z}} P_*$ be a diagonal approximation. Then for all p, $p'\in\mathbb{N}$ the diagram in Figure 1.14 is

$$\begin{array}{ccccc}
 & & \overset{\cdot\cup\cdot}{\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}} & & \\
H^p(G;A)\otimes_{\mathbb{Z}} H^{p'}(G;A') & \xrightarrow[\cdot\times\cdot]{} & H^{p+p'}(G\times G;A\otimes_{\mathbb{Z}} A') & \xrightarrow[H^*(d;\mathrm{id})]{} & H^{p+p'}(G;A\otimes_{\mathbb{Z}} A') \\
\| & & \| & & \| \\
H^p\big(\mathrm{Hom}_G(P_*,A)\big)\otimes_{\mathbb{Z}} H^{p'}\big(\mathrm{Hom}_G(P_*,A')\big) & \xrightarrow{\cdot\times\cdot} & H^{p+p'}\big(\mathrm{Hom}_G(P_*\otimes_{\mathbb{Z}} P_*,A\otimes_{\mathbb{Z}} A')\big) & \xrightarrow{H^*(\mathrm{Hom}_G(\Delta_*,\mathrm{id}))} & H^{p+p'}\big(\mathrm{Hom}_G(P_*,A\otimes_{\mathbb{Z}} A')\big) \\
 & & \underset{H^*(\mathrm{Hom}_G(\Delta_*,\mathrm{id}_{A\otimes_{\mathbb{Z}}A'})\circ\,\cdot\,\times\,\cdot\,)}{\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}} & &
\end{array}$$

Figure 1.14: Computing the cup-product via diagonal approximations

commutative: the left square commutes by definition of the cohomological cross-product, the right square commutes by the fundamental lemma of homological algebra (the chain map Δ_* is compatible with the diagonal embedding $G \longrightarrow G \times G$).

Example 1.8.16 (Cohomology ring of finite cyclic groups)**.** Let $n \in \mathbb{N}_{>0}$. Then there is an isomorphism

$$H^*(\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}[x]/(n \cdot x)$$

of graded rings, where x is a generator of degree 2. In order to prove this, we compute cup-products in $H^*(\mathbb{Z}/n;\mathbb{Z})$ via the standard projective resolution of $\mathbb{Z}$: For brevity, we write $G := \mathbb{Z}/n = \langle t | t^n = 1\rangle$ and we write $P_* \,\square\, \varepsilon$ for the standard projective $\mathbb{Z}G$-resolution

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

of $\mathbb{Z}$ (see Corollary 1.6.6). For $p \in \mathbb{N}$ we let $f_{2p} \in \mathrm{Hom}_G(P_{2p}, \mathbb{Z})$ be the cocycle given by

$$\begin{aligned} f_{2p} \colon P_{2p} = \mathbb{Z}G &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto 1. \end{aligned}$$

We show now that we can take x to be the cohomology class $[f_2]$: Looking at the computation of $H^*(G;\mathbb{Z})$ via the resolution P_* (see Corollary 1.6.6), we see that f_{2p} corresponds to the generator $[1]$ of the cohomology group $H^{2p}(G;\mathbb{Z}) = \mathbb{Z}/n$.

Using the diagonal approximation $\Delta_*\colon P_* \longrightarrow P_* \otimes_{\mathbb{Z}} P_*$ from Example 1.8.7, we obtain

$$f_{2p} \cup f_{2p'} = f_{2(p+p')}$$

for all p, $p' \in \mathbb{N}$, because

$$\begin{aligned}(f_{2p} \cup f_{2p'})(1) &= (f_{2p} \times f_{2p'})\big(\Delta_*(1)\big)\\ &= (f_{2p} \times f_{2p'})(1 \otimes 1)\\ &= f_{2p}(1) \otimes f_{2p'}(1)\\ &= 1;\end{aligned}$$

in the second step we took advantage of the fact that the only term of $\Delta_*(1)$ (where $1 \in \mathbb{Z}G = P_{2(p+p')}$) that survives under the evaluation of $f_{2p} \times f_{2p'}$ is the one where the degrees of the components are $2p$ and $2p'$ respectively.

Therefore, $H^*(\mathbb{Z}/n;\mathbb{Z})$ is generated by the element $[f_2] \in H^2(\mathbb{Z}/n;\mathbb{Z})$, which has infinite multiplicative order and additive order n, as claimed.

Remark 1.8.17 (The cup-product in terms of bar resolutions)**.** Let G be a group, and let A and A' be two $\mathbb{Z}G$-modules. Then the cup-product on cohomology is induced by the following map on the bar resolution:

$$\begin{aligned}\operatorname{Hom}_G(C_p(G),A) \otimes_{\mathbb{Z}} \operatorname{Hom}_G(C_{p'}(G),A') &\longrightarrow \operatorname{Hom}_G(C_{p+p'}(G), A \otimes_{\mathbb{Z}} A')\\ f \otimes f' &\longmapsto \big(g_0 \cdot [g_1|\cdots|g_{p+p'}]\\ &\qquad \mapsto (-1)^{p \cdot p'} \cdot f(g_0 \cdot [g_1|\dots|g_p])\\ &\qquad \otimes f(g_0 \cdots g_p \cdot [g_{p+1}|\cdots|g_{p+p'}])\big);\end{aligned}$$

indeed, this follows from the description of the cup-product via diagonal approximations (Remark 1.8.15) and the definition of the Alexander-Whitney map (Example 1.8.6).

Using this description via the bar resolution, we can finally complete the proof of Theorem 1.8.1:

Proposition 1.8.18 (Cup-product, construction satisfies the axioms)**.** *The cup-product defined above satisfies the axioms of Theorem 1.8.1: Let G be a group.*

1. Degree 0. *For all* $\mathbb{Z}G$*-modules* A *and* A'*, the cup-product*

$$A^G \otimes_{\mathbb{Z}} A'^G = H^0(G;A) \otimes_{\mathbb{Z}} H^0(G;A') \longrightarrow H^0(G;A \otimes_{\mathbb{Z}} A') = (A \otimes_{\mathbb{Z}} A')^G$$

coincides with the $\mathbb{Z}$*-homomorphism induced from the canonical inclusions* $A^G \longrightarrow A$ *and* $A'^G \longrightarrow A'$.

2. Naturality with respect to connecting homomorphisms. *Let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be a short exact sequence of* $\mathbb{Z}G$*-modules and let* B *be a* $\mathbb{Z}G$*-module such that the induced sequence*

$$0 \longrightarrow A' \otimes_{\mathbb{Z}} B \longrightarrow A \otimes_{\mathbb{Z}} B \longrightarrow A'' \otimes_{\mathbb{Z}} B \longrightarrow 0$$

is exact. Then

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta$$

for all $\alpha \in H^*(G;A'')$ *and all* $\beta \in H^*(G;B)$*, where the* δ*s are the connecting homomorphisms of the long exact cohomology sequences corresponding to the above short exact sequence of* $\mathbb{Z}G$*-modules.*
(An analogous statement holds when the factors are swapped; however, a sign has to be introduced in that case.)

Proof. The first part can easily be verified using the description of the cup-product in terms of the bar resolution and the Alexander-Whitney map (Remark 1.8.17).

For the second part, we also use the description of the cup-product in terms of the bar resolutions: Let $p := |\beta|$, and let $f \in \operatorname{Hom}_G(C_p(G), B)$ be a cocycle representing the class $\beta \in H^p(G;B)$. Then we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_G(C_*(G), A') & \longrightarrow & \operatorname{Hom}_G(C_*(G), A) & \longrightarrow & \operatorname{Hom}_G(C_*(G), A'') & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \cdot \cup f} & & \downarrow {\scriptstyle \cdot \cup f} & & \downarrow {\scriptstyle \cdot \cup f} & & \\
0 & \to & \operatorname{Hom}_G(C_{*+p}(G), A' \otimes_{\mathbb{Z}} B) & \to & \operatorname{Hom}_G(C_{*+p}(G), A \otimes_{\mathbb{Z}} B) & \to & \operatorname{Hom}_G(C_{*+p}(G), A'' \otimes_{\mathbb{Z}} B) & \to & 0
\end{array}$$

induced by the given two short exact sequences of $\mathbb{Z}G$-modules and the cup-product via the Alexander-Whitney map on the bar-resolution. That this diagram is commutative can be read off the description of the cup-product via the Alexander-Whitney map. Moreover, the rows are exact because the chain modules of $C_*(G)$ all are projective $\mathbb{Z}G$-modules. Because f is a cocycle, we obtain

$$\begin{aligned}\delta(\,\cdot\,\cup f) &= (\delta\,\cdot\,)\cup f + (-1)^{|\,\cdot\,|}\cdot\,\cdot \cup \delta f\\ &= (\delta\,\cdot\,)\cup f,\end{aligned}$$

where δ here is the coboundary operator of the cochain complexes built out of $C_*(G)$ via Hom_G; hence, the vertical arrows are cochain maps as well.

The long exact sequence in cohomology is derived from this diagram via the snake lemma; the naturality part of the snake lemma therefore proves the desired cup-product relation on the level of group cohomology. □

We now come to the algebraic properties of the cup-product [4, Chapter V.III]:

Proposition 1.8.19 (Cup-product, algebraic properties)**.** *Let G be a group.*

1. Unit element. *The element $1 \in \mathbb{Z} = H^0(G;\mathbb{Z})$ acts as a unit element for the cup-product in the following sense: Let A be a $\mathbb{Z}G$-module. Then for all $\alpha \in H^*(G;A)$ we have the relation*

$$1 \cup \alpha = \alpha = \alpha \cup 1$$

in $H^*(G;A)$ (where we use the canonical isomorphisms $\mathbb{Z}\otimes_{\mathbb{Z}} A \cong A$ and $A \cong A \otimes_{\mathbb{Z}} \mathbb{Z}$).

2. Graded commutativity. *The cup-product is graded commutative in the following sense: Let A and A' be two $\mathbb{Z}G$-modules. Then for all $\alpha \in H^*(G;A)$ and all $\alpha' \in H^*(G;A')$ we have*

$$\alpha' \cup \alpha = (-1)^{|\alpha|\cdot|\alpha'|}\cdot H^*(\mathrm{id}_G;t)(\alpha\cup\alpha'),$$

where $t\colon A\otimes_{\mathbb{Z}} A' \longrightarrow A'\otimes_{\mathbb{Z}} A$ is the $\mathbb{Z}G$-isomorphism given by swapping the two factors.

3. Associativity. *The cup-product is associative in the following sense: Let A, A', and A'' be $\mathbb{Z}G$-modules. Then*

$$(\alpha\cup\alpha')\cup\alpha'' = \alpha\cup(\alpha'\cup\alpha'')$$

holds in $H^*(A\otimes_{\mathbb{Z}} A'\otimes_{\mathbb{Z}} A'')$ for all $\alpha\in H^*(G;A)$, $\alpha'\in H^*(G;A')$, and $\alpha''\in H^*(G;A'')$.

4. Naturality with respect to morphisms in GrpMod^{-}. *Let H be another group, let $\varphi\colon G \longrightarrow H$ be a group homomorphism, and suppose that $(\varphi, \Phi)\colon (G, A) \longrightarrow (H, B)$ and $(\varphi, \Phi')\colon (G, A') \longrightarrow (H, B')$ are morphisms in* GrpMod^{-}*. Then*

$$H^*(\varphi; \Phi \otimes_{\mathbb{Z}} \Phi')(\alpha \cup \alpha') = \bigl(H^*(\varphi; \Phi)(\alpha)\bigr) \cup \bigl(H^*(\varphi; \Phi')(\alpha')\bigr)$$

holds for all $\alpha \in H^*(H; B)$ and all $\alpha' \in H^*(H; B')$.

Proof. The first three parts can either be proved by looking at the description of the cup-product in terms of projective resolutions or by verifying the properties in degree 0 and then applying dimension shifting. (Exercise).

The last part follows directly from the definitions (or the description in terms of the bar resolution). □

Corollary 1.8.20 (Cohomology ring with $\mathbb{Z}$-coefficients)**.** *Let G be a group. Then the cohomology $H^*(G; \mathbb{Z})$ is a graded ring with respect to the multiplication given by the cup-product, which is graded commutative.*

Moreover, any group homomorphism $G \longrightarrow G'$ induces a unital ring homomorphism $H_*(G'; \mathbb{Z}) \longrightarrow H_*(G; \mathbb{Z})$) □

Of course, similar observations apply to more general coefficients (e.g., whenever the coefficients are an algebra on which the group acts trivially).

This ring structure demonstrates that group cohomology is actually quite rigid and so simplifies many calculations.

Example 1.8.21 (Endomorphisms of finite cyclic groups in cohomology)**.** Let n, $m \in \mathbb{N}_{>0}$ and let $\varphi\colon \mathbb{Z}/n \longrightarrow \mathbb{Z}/m$ be a group homomorphism. What is the induced homomorphism in group cohomology with integer coefficients?

We know that $H_*(\varphi; \mathrm{id}_{\mathbb{Z}})$ is a unital ring automorphism by the previous corollary; moreover, we know that the cohomology ring of a finite cyclic group is a polynomial algebra generated by a single element of degree 2 with infinite multiplicative order and having the group order as additive order (Example 1.8.16). Therefore, it suffices to compute $H^2(\varphi; \mathrm{id}_{\mathbb{Z}})\colon H^2(\mathbb{Z}/m; \mathbb{Z}) \longrightarrow H^2(\mathbb{Z}/n; \mathbb{Z})$. This in turn can be done via the universal coefficient theorem, which relates this homomorphism to the homomorphism that φ induces on H_1.

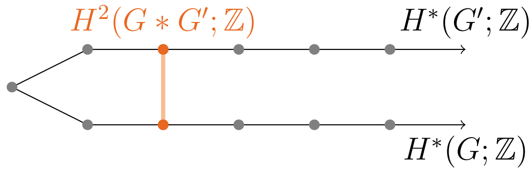


Figure 1.15: Cohomology ring of free products of groups, schematically

For example, if $\varphi\colon \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3$ is the group homomorphism given by $[1] \longmapsto [-1]$, then

$$\begin{aligned} H^{2p}(\varphi; \mathrm{id}_{\mathbb{Z}})\colon H^{2p}(\mathbb{Z}/3; \mathbb{Z}) &\longrightarrow H^{2p}(\mathbb{Z}/3; \mathbb{Z}) \\ [1] &\longmapsto [2^p] \end{aligned}$$

for all $p \in \mathbb{N}$.

Example 1.8.22 (Cohomology ring of free products of groups)**.** Let G and G' be two groups. Then there is an isomorphism

$$H^*(G * G'; \mathbb{Z}) \cong \mathbb{Z} \oplus \bigoplus_{k \in \mathbb{N}} \bigl(H^k(G; \mathbb{Z}) \times H^k(G'; \mathbb{Z})\bigr)$$

of graded rings; the grading on the right hand side is as follows: the first $\mathbb{Z}$-summand is the part in degree 0, and for $k \in \mathbb{N}_{>0}$ the degree k part is $H^k(G; \mathbb{Z}) \times H^k(G'; \mathbb{Z})$. Moreover, the product of two classes in non-zero degree that lie in different factors is zero (see also Figure 1.15).

This is a consequence of the fact that we know from the Mayer-Vietoris sequence (Proposition 1.3.16) that the homomorphism

$$H^*(G * G'; \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \bigoplus_{k \in \mathbb{N}} \bigl(H^k(G; \mathbb{Z}) \times H^k(G'; \mathbb{Z})\bigr)$$

of graded Abelian groups given by restriction to the subgroups G and G' respectively is an isomorphism of graded Abelian groups. By naturality, this isomorphism is compatible with the cup-products; thus, this homomorphism is an isomorphism of graded rings, as claimed.

Analogously to the additive primary decomposition of cohomology of finite groups, there is also a multiplicative decomposition. For simplicity, we only treat the case of trivial coefficients:

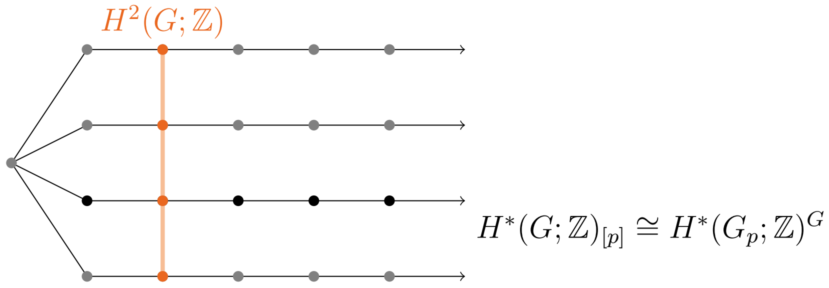


Figure 1.16: Cohomology ring of a finite group, schematically

Corollary 1.8.23 (Primary decomposition, multiplicatively). *Let G be a finite group, and let $P(|G|)$ be the set of positive primes dividing $|G|$.*

1. *Then there is a canonical isomorphism*

$$H^*(G;\mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p \in P(|G|)} H^*(G;\mathbb{Z})_{[p]}$$

of graded rings; here, $\mathbb{Z}$ is the part in degree 0 and for $k \in \mathbb{N}_{>0}$ the degree k part of the right hand side is formed by $\prod_{p \in P(|G|)} H^k(G\mathbb{Z})_{[p]}$ (see also Figure 1.16).

2. *Moreover, if for $p \in P(|G|)$ we write G_p for a p-Sylow subgroup of G, then there is also a canonical isomorphism*

$$H^*(G;\mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p \in P(|G|)} H^*(G_p;\mathbb{Z})^G$$

of graded rings (where the G-fixed points are to be understood as in Definition 1.7.25).

Proof. Because $P(|G|)$ is finite, we know already that there is an isomorphism

$$H^*(G;\mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \bigoplus_{p \in P(|G|)} H^*(G;\mathbb{Z})_{[p]} = \mathbb{Z} \oplus \prod_{p \in P(|G|)} H^*(G;\mathbb{Z})_{[p]}$$

of graded Abelian groups, given by projecting onto the primary parts (Proposition 1.7.29).

It is not difficult to see that these projections are compatible with the graded ring structure: If $p \in P(|G|)$, then the p-primary part $H^*(G;\mathbb{Z})_{[p]}$ clearly is a (graded) ideal in $H^*(G;\mathbb{Z})$. Hence also

$$\bigoplus_{q \in P(|G|)\setminus\{p\}} H^*(G;\mathbb{Z})_{[q]}$$

is a (graded) ideal in $H^*(G;\mathbb{Z})$. So the projection $H^*(G;\mathbb{Z}) \longrightarrow H^*(G;\mathbb{Z})_{[p]}$ is a ring homomorphism.

The second part follows directly from the first part: Let $p \in P(|G|)$. The isomorphism $H^*(G;\mathbb{Z})_{[p]} \longrightarrow H^*(G_p;\mathbb{Z})^G$ of graded Abelian groups in Proposition 1.7.30 is induced by $\operatorname{res}^G_{G_p}$, which is a morphism in GrpMod^-, and thus is a ring homomorphism; as the underlying homomorphism of Abelian groups is an isomorphism, this must be a ring isomorphism. □

Example 1.8.24 (Cohomology ring of the symmetric group S_3)**.** Using the multiplicative primary decomposition one can, for instance, compute the cohomology ring $H^*(S_3;\mathbb{Z})$ of S_3. (Exercise).

Proposition 1.8.25 (Cup-product and transfer)**.** *Let G be a group and let H be a subgroup of finite index; moreover, let A and A' be two $\mathbb{Z}G$-modules. Then*

$$\operatorname{cor}^G_H\big((\operatorname{res}^G_H \alpha) \cup \alpha'\big) = \alpha \cup \operatorname{cor}^G_H \alpha'$$

for all $\alpha \in H^*(G;A)$ and all $\alpha' \in H^*(H;\operatorname{Res}^G_H A')$.

Proof. We prove the corresponding relation on the cochain level. Let $P_* \,\square\, \varepsilon$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. For all cochains $f \in \operatorname{Hom}_G(P_*, A)$ and $f' \in \operatorname{Hom}_H(\operatorname{Res}^G_H P_*, \operatorname{Res}^G_H A')$ we have (in $\operatorname{Hom}_G(P_* \otimes_{\mathbb{Z}} P_*, A \otimes_{\mathbb{Z}} A')$)

$$\begin{aligned}
\operatorname{cor}^G_H\big((\operatorname{res}^G_H f) \cup f'\big) &= \sum_{gH \in G/H} g \cdot (f \times f') \\
&= \sum_{gH \in G/H} g \cdot f \times g \cdot f' \\
&= \sum_{gH \in G/H} f \times g \cdot f' \\
&= f \times \sum_{gH \in G/H} g \cdot f' \\
&= f \times \operatorname{res}^G_H f'.
\end{aligned}$$

□

1.8.5 The cap-product

Like for (co)homology of topological spaces there is also a cap-product in group cohomology, which is kind of a dual of the cup-product:

Definition 1.8.26 (Cap-product in group (co)homology)**.** Let G be a group, and let A and A' be two $\mathbb{Z}G$-modules. Then the *cap-product* is defined as follows: For all $p, p' \in \mathbb{N}$ with $p \leq p'$ it is the homomorphism

$$\cdot \cap \cdot : H^p(G; A) \otimes_{\mathbb{Z}} H_{p'}(G; A') \longrightarrow H_{p'-p}(G; A \otimes_{\mathbb{Z}} A')$$

that is induced by the following map

$$\begin{aligned} \operatorname{Hom}_G(P_p, A) \otimes_{\mathbb{Z}} \bigl((P_* \otimes_{\mathbb{Z}} P_*)_{p'} \otimes_G A'\bigr) &\longrightarrow P_{p'-p} \otimes_G (A \otimes_{\mathbb{Z}} A') \\ f \otimes (x \otimes x' \otimes a) &\longmapsto (-1)^{p \cdot p'} \cdot x \otimes f(x') \otimes a \end{aligned}$$

where $P_* \twoheadrightarrow \varepsilon$ is some projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Similarly to the case of the cross-product, one can show that this definition does not depend on the choice of (co)cycles or the projective resolution. Furthermore, as in the case of the cup-product one can get rid of the tensor product complex $P_* \otimes_{\mathbb{Z}} P_*$ by using a diagonal approximation.

The cap-product is dual to the cup-product in the following sense:

Proposition 1.8.27 (Evaluation and cup-/cap-products)**.** *Let G be a group, and let A, A', and B be $\mathbb{Z}G$-modules.*

1. *For all $\alpha \in H^p(G; A)$, all $\alpha' \in H^{p'}(G; A')$ and all $\beta \in H_{p+p'}(G; B)$ we have*
$$\langle \alpha \cup \alpha', \beta \rangle = \langle \alpha, \alpha' \cap \beta \rangle$$
in $(A \otimes_{\mathbb{Z}} A' \otimes_{\mathbb{Z}} B)_G$.
2. *In particular, for all $\alpha \in H^p(G; A)$ and all $\beta \in H_p(G; A')$, we have*
$$\alpha \cap \beta = \langle \alpha, \beta \rangle$$
in $A \otimes_G B$.

Here, the evaluation $\langle \cdot\,, \cdot \rangle$ is induced from the following map on the level of resolutions:

$$\begin{aligned} \langle \cdot\,, \cdot \rangle\colon \operatorname{Hom}_G(P_p, A) \otimes_{\mathbb{Z}} P_p \otimes_G A' &\longrightarrow A \otimes_G A' \\ f \otimes (x \otimes a) &\longmapsto f(x) \otimes a. \end{aligned}$$

Proof. The first part is a straightforward computation on the (co)chain level. The second part follows from the first part by taking the first factor to be the unit element in $H^0(G; \mathbb{Z})$. □

In the context of Tate cohomology of finite groups, the cap-product will provide us with striking duality phenomena.

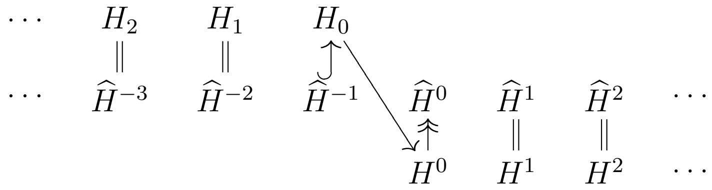
Example 1.8.28 (Poincaré duality groups)**.** An interesting class of groups that can be described in terms of the cap-product is the class of Poincaré duality groups, which play an important rôle in geometric topology.

1.9 Tate cohomology and periodic cohomology

If G is a finite group, then group homology $H_*(G;\,\cdot\,)$ and group cohomology $H^*(G;\,\cdot\,)$ behave similarly in certain situations; for example, if G is a finite group, then induced and coinduced modules coincide.

Tate's idea was to pack group homology and group cohomology of finite groups into a common $\mathbb{Z}$-graded functor $\widehat{H}^*$. Technically, this is achieved by extending $\mathbb{N}$-graded projective resolutions of $\mathbb{Z}$ to $\mathbb{Z}$-graded "projective resolutions" of $\mathbb{Z}$, so-called complete projective resolutions. A convenient theoretical framework for this type of homological algebra is relative homological algebra (which also will play a key rôle in the context of bounded cohomology).

Schematically, Tate cohomology looks as follows:



Example applications of Tate cohomology are:

- The theory of cohomologically trivial modules, which is interesting from an algebraic point of view [4, Chapter VI.8].
- The theory of periodic cohomology, which is related to the problem of which finite groups admit free actions on spheres (Section 1.9.5).

We will first describe Tate cohomology axiomatically; in a second step we sketch the construction of Tate cohomology via complete projective resolutions, and then give a brief overview of the relative homological algebra needed in the context of Tate cohomology. After discussing the relation between Tate cohomology and ordinary group (co)homology, we will have a look at the product structure on Tate cohomology. Finally, we will study periodic cohomology.

1.9.1 Tate cohomology – definition

We start with the definition of the domain category of Tate cohomology, we will then give an axiomatic description of Tate cohomology (additively), and then we will indicate how Tate cohomology can be constructed in terms of complete projective resolutions:

Definition 1.9.1 (The category $\mathrm{GrpMod}\widehat{\ }$)**.** The category $\mathrm{GrpMod}\widehat{\ }$ is defined as follows:

- The objects of $\mathrm{GrpMod}\widehat{\ }$ are pairs (G, A), where G is a finite group and A is a $\mathbb{Z}G$-module.
- A morphism $(i, \Phi)\colon (H, B) \longrightarrow (G, A)$ in $\mathrm{GrpMod}\widehat{\ }$ consists of the inclusion $i\colon H \longrightarrow G$ of a subgroup H of a finite group G and a $\mathbb{Z}H$-morphism $\Phi\colon i^* A = \mathrm{Res}^G_H A \longrightarrow B$. The composition of morphisms is defined by covariant composition in the first component and by contravariant composition in the second component.

Theorem 1.9.2 (Tate cohomology, axiomatically)**.** *Tate cohomology is the (up to natural isomorphism) unique contravariant functor*

$$\widehat{H}^*(\,\cdot\,;\,\cdot\,)\colon \mathrm{GrpMod}\widehat{\ } \longrightarrow \mathrm{Ab}_{\mathbb{Z}*}$$

(from the category $\mathrm{GrpMod}\widehat{\ }$ *to the category of* $\mathbb{Z}$*-graded Abelian groups) together with connecting homomorphisms* $\delta^*\colon \widehat{H}^*(G; A'') \longrightarrow \widehat{H}^{*+1}(G; A')$ *for all finite groups* G *and all short exact sequences* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *of* $\mathbb{Z}G$*-modules such that the following properties are satisfied:*

- Middle degrees. *For every finite group* G *and every* $\mathbb{Z}G$*-module* A *there is a natural (in both variables) exact sequence*

$$0 \longrightarrow \widehat{H}^{-1}(G; A) \longrightarrow H_0(G; A) = A_G \longrightarrow A^G = H^0(G; A) \longrightarrow \widehat{H}^0(G; A) \longrightarrow 0$$

  where the homomorphism $A_G \longrightarrow A^G$ *coincides with the homomorphism given by multiplication with the norm element* $\sum_{g \in G} g \in \mathbb{Z}[G]$.
- Long exact sequences. *For every finite group* G *and every short exact sequence* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *of* $\mathbb{Z}G$*-modules, there is a natural (in both variables) long exact sequence*

$$\cdots \longrightarrow \widehat{H}^k(G; A') \longrightarrow \widehat{H}^k(G; A) \longrightarrow \widehat{H}^k(G; A'') \xrightarrow{\delta^k} \widehat{H}^{k+1}(G; A') \longrightarrow \cdots$$

in Tate cohomology.
- Vanishing on induced modules. *For all finite groups G, all $\mathbb{Z}$-modules A, and all $k \in \mathbb{Z}$ we have $\widehat{H}^k(G; \mathbb{Z}G \otimes_G A) = 0$.*

With help of a dimension shifting argument we see that there is at most one such theory. The existence of such a theory relies on $\mathbb{Z}$-graded projective resolutions, so-called complete projective resolutions:

Definition 1.9.3 (Complete projective resolutions)**.** Let G be a group. A *complete projective $\mathbb{Z}G$-resolution* of $\mathbb{Z}$ is a $\mathbb{Z}$-graded $\mathbb{Z}G$-chain complex P_* consisting of projectives $\mathbb{Z}G$-modules together with a surjective $\mathbb{Z}G$-homomorphism $\varepsilon\colon P_0 \longrightarrow \mathbb{Z}$, the *augmentation*, such that the boundary operator $\partial_0\colon P_0 \longrightarrow P_{-1}$ factors over ε and an inclusion $\mathbb{Z} \hookrightarrow P_{-1}$:

$$\cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} P_{-1} \xrightarrow{\partial_{-1}} P_{-2} \longrightarrow \cdots, \qquad \partial_0 = \big(P_0 \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \hookrightarrow P_{-1}\big)$$

The complex P_* together with the augmentation and factorisation is denoted by $P_* \triangledown \varepsilon$.

Definition 1.9.4 (Tate cohomology)**.**
- Let G be a finite group, and let A be a $\mathbb{Z}G$-module. Then *Tate cohomology* of G with coefficients in A is defined by

$$\widehat{H}^k(G; A) := H^k\big(\operatorname{Hom}_G(P_*, A)\big)$$

  for all $k \in \mathbb{Z}$, where $P_* \triangledown \varepsilon$ is some complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.
- If $(i, \Phi)\colon (H, B) \longrightarrow (G, A)$ is a morphism in $\mathrm{GrpMod}\widehat{\ }$, then we define $\widehat{H}^*(i; \Phi)\colon \widehat{H}^*(G; A) \longrightarrow \widehat{H}^*(H; B)$ by
  1. choosing a complete projective $\mathbb{Z}G$-resolution $P_* \triangledown \varepsilon$ of $\mathbb{Z}$,
  2. taking a restriction type cochain map

$$\operatorname{Hom}_G(P_*, A) \longrightarrow \operatorname{Hom}_H(\operatorname{Res}^G_H P_*, \operatorname{Res}^G_H A),$$

  3. combining this with the cochain map induced by Φ,
  4. and finally taking cohomology.

In order for Tate cohomology defined in this way to be well-defined, we need the usual ingredients: we need that complete projective resolutions do exist and that they (and their morphisms) are essentially unique.

Proposition 1.9.5 (Existence and uniqueness of complete projective resolutions)**.** *Let G be a finite group.*

1. *Then there exists a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.*
2. *Moreover, between any two complete projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$ there exists up to $\mathbb{Z}G$-homotopy precisely one $\mathbb{Z}G$-chain map that preserves the augmentations.*

The existence part is covered by Example 1.9.7 below. In order to prove the uniqueness part, we need a new version of homological algebra, so-called relative homological algebra.

Caveat 1.9.6 (Extending maps between complete projective resolutions)**.** Let G be a finite group, and let $P_* \triangledown \varepsilon$ and $P'_* \triangledown \varepsilon'$ be two complete projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$. By projectivity, we can extend the identity on $\mathbb{Z}$ to a chain map between the positive parts of P_* and P'_* respectively. However, projectivity does not help us with extending such a map to the negative part (the arrows in the extension problems point in the wrong direction!).

If the modules in the negative parts were injective, we could just use the extension property provided by injectivity; of course, projective modules are not injective in general. However, for finite groups G, any projective $\mathbb{Z}G$-module is almost injective, namely relatively injective (see Definition 1.9.10 below).

We will later encounter a similar version of homological algebra again, namely, when developping the algebraic approach to bounded cohomology (Section 2.7).

Example 1.9.7 (Complete projective resolutions of finite type)**.** Let G be a finite group. Then we can construct a complete projective resolution of $\mathbb{Z}$ as follows (Exercise):

1. Because G is finite, the bar resolution $P_*^+ \square \varepsilon$ is a projective $\mathbb{Z}G$-resolution of finite type; a $\mathbb{Z}G$-resolution is of *finite type* if all its chain modules are finitely generated $\mathbb{Z}G$-modules.

2. Then the dual complex $\mathrm{moH}_G(\varepsilon, \mathrm{id}_{\mathbb{Z}}) \square \mathrm{moH}_G(P_*^+, \mathbb{Z}G)$ is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules.
   Here, for a $\mathbb{Z}G$-module A, we write $\mathrm{moH}_G(A, \mathbb{Z}G)$ for the $\mathbb{Z}G$-module whose underlying Abelian group is $\mathrm{Hom}_G(A, \mathbb{Z}G)$ together with the G-action

$$\begin{aligned} G \times \mathrm{moH}_G(A, \mathbb{Z}G) &\longrightarrow \mathrm{moH}_G(A, \mathbb{Z}G) \\ (g, f) &\longmapsto \bigl(x \mapsto f(x) \cdot g^{-1}\bigr). \end{aligned}$$

3. Splicing these two resolutions together, we obtain a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, which is of finite type.

Example 1.9.8 (Complete resolutions for finite groups acting freely on spheres)**.** For finite groups that act freely on spheres, we can just take the pieces constructed in Theorem 1.6.1 and splice them together to obtain a periodic complete projective resolution of $\mathbb{Z}$ of finite type.

Example 1.9.9 (Tate cohomology of finite cyclic groups)**.** Let $n \in \mathbb{N}_{>0}$, and let $G = \mathbb{Z}/n = \langle t \mid t^n = 1 \rangle$. Then (where $N \in \mathbb{Z}G$ is the norm element)

$$\begin{array}{c} \cdots \longrightarrow \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \cdots \\ \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}G \end{array}$$

is a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. In particular, we obtain

$$\widehat{H}^k(\mathbb{Z}/n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

for all $k \in \mathbb{Z}$.

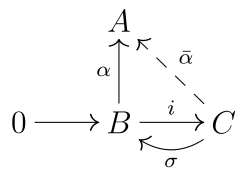
It is not difficult to check that the construction of Tate cohomology via complete projective resolutions indeed satisfies the axioms, as soon as uniqueness of complete projective resolutions and maps between such resolutions is established.

1.9.2 Relative homological algebra

We give a very brief introduction to the basic notions of relative homological algebra; the central concept is that of a relatively injective module – a variant of injective modules. When studying bounded cohomology, we will use a similar theory in a functional analytic context (Section 2.7).

Definition 1.9.10 (Relatively injective modules)**.** Let G be a finite group.

- An injective $\mathbb{Z}G$-morphism $i\colon B \longrightarrow C$ is said to be *relatively injective* (or *admissible*), if it splits as a $\mathbb{Z}$-homomorphism (there does not necessarily have to exist a G-equivariant split!).
- A $\mathbb{Z}G$-module A is *relatively injective*, if for every relatively injective $\mathbb{Z}G$-morphism $i\colon B \longrightarrow C$ and every $\mathbb{Z}G$-homomorphism $\alpha\colon B \longrightarrow A$ there exists a $\mathbb{Z}G$-homomorphism $\bar{\alpha}\colon C \longrightarrow A$ extending i:



Example 1.9.11 (Relatively injective modules)**.** Let G be a finite group.

1. *Coinduced modules are relatively injective:* more precisely, if A is a $\mathbb{Z}$-module, then $\operatorname{Coind}_1^G A = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$ is a relatively injective $\mathbb{Z}G$-module: Let $i\colon B \longrightarrow C$ be a relatively injective $\mathbb{Z}G$-homomorphism and let $\sigma\colon C \longrightarrow B$ be a $\mathbb{Z}$-split of i. If $\alpha\colon B \longrightarrow \operatorname{Coind}_1^G A$ is a $\mathbb{Z}G$-homomorphism, then
$$\begin{aligned} \bar{\alpha}\colon C &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A) = \operatorname{Coind}_1^G A \\ c &\longmapsto \bigl(g \mapsto \alpha\bigl(\sigma(g \cdot c)\bigr)(1)\bigr) \end{aligned}$$
is a $\mathbb{Z}G$-homomorphism satisfying $\bar{\alpha} \circ i = \alpha$.
2. *All projective $\mathbb{Z}G$-modules are relatively injective $\mathbb{Z}G$-modules:* Again, we use the fact that projective modules are the same as direct summands of free modules.
It is not difficult to see that direct summands of relatively injective modules are relatively injective again. Therefore, it suffices to show

that free $\mathbb{Z}G$-modules are relatively injective: Let S be a set. Because G is finite, there is an isomorphism

$$\bigoplus_S \mathbb{Z}G = \operatorname{Ind}_1^G \bigoplus_S \mathbb{Z} \cong \operatorname{Coind}_1^G \bigoplus_S \mathbb{Z}$$

of $\mathbb{Z}G$-modules. As the latter one is relatively injective by the first part, we obtain that $\bigoplus_S \mathbb{Z}G$ is relatively injective.
In particular, not every relatively injective $\mathbb{Z}G$-module is an injective $\mathbb{Z}G$-module.

Relative homological algebra is concerned with resolving modules by relatively injective modules. However, as relatively injective modules only solve certain extension problems, we can only hope for a corresponding version of the fundamental lemma of homological algebra, if we require resolutions not only to be acyclic but also to give rise to relatively injective mapping problems.

Definition 1.9.12 (Relatively injective resolutions)**.** Let G be a finite group. A *strong relatively injective resolution* of a $\mathbb{Z}G$-module A is a cochain complex I^* consisting of relatively injective $\mathbb{Z}G$-modules together with an augmentation $\eta\colon \mathbb{Z} \longrightarrow I^0$ such that the concatenated cochain complex $\eta \mathbin{\square} I^*$ is $\mathbb{Z}$-contractible.

In the context of Tate cohomology, the following is the key example of strong relatively injective resolutions:

Example 1.9.13 (Relatively injective resolutions out of projective resolutions)**.** Let G be a finite group. If $P_* \triangledown \varepsilon$ is a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then the negative part $(\mathbb{Z} \hookrightarrow P_{-1}) \mathbin{\square} (P_k)_{k\in\mathbb{Z}_{<0}}$ is a strong relatively injective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Indeed, all cochain modules are relatively injective by Example 1.9.11. Moreover, the concatenated complex $(\mathbb{Z} \hookrightarrow P_{-1}) \mathbin{\square} (P_k)_{k\in\mathbb{Z}_{<0}}$ is $\mathbb{Z}$-contractible because it consists of free $\mathbb{Z}$-modules and it is acyclic (Exercise).

Proposition 1.9.14 (Fundamental lemma of relative homological algebra)**.** *Let G be a finite group.*

1. *From any $\mathbb{Z}$-contractible $\mathbb{Z}G$-cochain complex that resolves $\mathbb{Z}$ to any $\mathbb{Z}G$-cochain complex of relatively injective modules together with an augmentation from $\mathbb{Z}$, there exists up to $\mathbb{Z}G$-homotopy exactly one $\mathbb{Z}G$-cochain map that is compatible with the augmentations.*

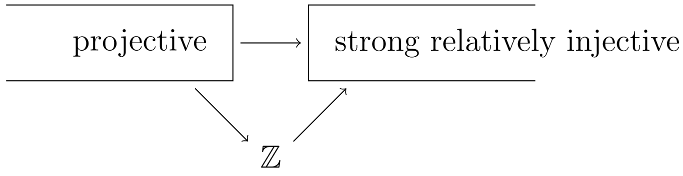


Figure 1.17: A fundamental lemma for complete projective resolutions: the classical fundamental lemma for projective resolutions in the positive part, combined with the fundamental lemma for strong relatively injective resolutions for the negative part

2. *In particular: Between any two strong relatively injective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$ there exists up to $\mathbb{Z}G$-chain homotopy exactly one $\mathbb{Z}G$-chain map that is compatible with the augmentations.*

Proof. This is proved by the same inductive arguments as the fundamental lemma of classical homological algebra. □

Combining this fundamental lemma with the classical one, we obtain also a fundamental lemma for complete projective resolutions of $\mathbb{Z}$ (see Figure 1.17).

1.9.3 Tate cohomology and ordinary group (co)homology

We now make the statement from the introduction more precise that Tate cohomology is a wrapper for both cohomology and homology of finite groups:

Proposition 1.9.15 (Tate cohomology and group cohomology)**.**

1. *In positive degrees, Tate cohomology coincides with ordinary group cohomology (and restriction homomorphisms), i.e., there is a natural (in* $\mathrm{GrpMod}\hat{\ }$*) isomorphism*

$$\widehat{H}^k(\,\cdot\,;\,\cdot\,) \cong H^k(\,\cdot\,;\,\cdot\,)$$

for all $k \in \mathbb{N}_{>0}$, and these natural isomorphism are compatible with the connecting homomorphisms and with the description of Tate cohomology in the middle degrees.

2. *In negative degrees, Tate cohomology coincides with ordinary group homology (and transfer homomorphisms), i.e., there is a natural (with respect to morphisms in* $\mathrm{GrpMod}\widehat{\ }$*) isomorphism*

$$\widehat{H}^k(\,\cdot\,;\,\cdot\,) \cong H_{-k-1}(\,\cdot\,;\,\cdot\,)$$

for all $k \in \mathbb{Z}_{<0}$, and these natural isomorphisms are compatible with the connecting homomorphisms and with the description of Tate cohomology in the middle degrees.

Proof. Let G be a finite group, and let A be a $\mathbb{Z}G$-module. We choose a complete projective $\mathbb{Z}G$-resolution $P_* \triangledown \varepsilon$ as constructed in Example 1.9.7.

That Tate cohomology of G with coefficients in A in positive degrees coincides with ordinary group cohomology of G with coefficients in A follows directly from the construction of Tate cohomology.

Using the fact that finitely generated projective $\mathbb{Z}G$-modules behave nicely with respect to taking Hom_G and $\otimes_G$, we deduce that Tate cohomology of G with coefficients in A in negative degrees coincides with ordinary group homology of G with coefficients in A: Indeed, we have

$$\begin{aligned}\mathrm{Hom}_G(P_*, A)^k &= \mathrm{Hom}_G\bigl(\text{ɯoH}_G(P^+_{-(*-1)}, \mathbb{Z}G), A\bigr)^k \\ &= P^+_{-k-1} \otimes_G A\end{aligned}$$

for all $k \in \mathbb{Z}_{<0}$, where P^+_* is the part of P_* in positive degree.

Similarly, one proves the assertions about morphisms by direct inspection on the (co)chain level. □

In particular, this proposition justifies the schematic picture for Tate cohomology in Figure 1.18, and we can deduce that

$$\widehat{H}^0(G;\mathbb{Z}) \cong \mathbb{Z}/|G| \qquad \text{and} \qquad \widehat{H}^{-1}(G;\mathbb{Z}) = 0$$

holds for all finite groups G.

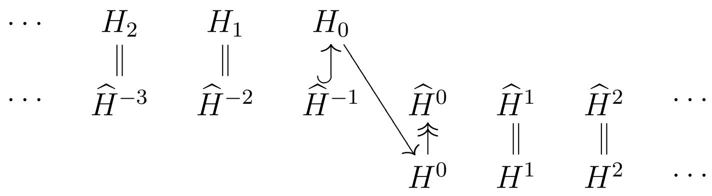


Figure 1.18: Tate cohomology, schematically

1.9.4 The product structure on Tate cohomology

The key feature of Tate cohomology is the cup-product structure, giving rise to a striking duality between ordinary group homology and cohomology (Theorem 1.9.18).

Theorem 1.9.16 (Tate cohomology, product structure)**.** *There is exactly one multiplicative structure $\cdot \cup \cdot$, the so-called* cup-product*, on Tate cohomology of finite groups satisfying the following axioms: For every finite group G, all $\mathbb{Z}G$-modules A and A', and all degrees p, $p' \in \mathbb{Z}$ the cup-product structure provides a $\mathbb{Z}$-linear map*

$$\cdot \cup \cdot : \widehat{H}^p(G; A) \otimes_{\mathbb{Z}} \widehat{H}^{p'}(G; A') \longrightarrow \widehat{H}^{p+p'}(G; A \otimes_{\mathbb{Z}} A');$$

here, G acts diagonally on the coefficients $A \otimes_{\mathbb{Z}} A'$.

– Degree 0. *For every finite group G and all $\mathbb{Z}G$-modules A and A', the cup-product*

$$\cdot \cup \cdot : \widehat{H}^0(G; A) \otimes_{\mathbb{Z}} \widehat{H}^0(G; A') \longrightarrow \widehat{H}^0(G; A \otimes_{\mathbb{Z}} A')$$

is induced from $\cdot \cup \cdot : H^0(G; A) \otimes_{\mathbb{Z}} H^0(G; A') \longrightarrow H^0(G; A \otimes_{\mathbb{Z}} A')$ (which is nothing but the canonical map $A^G \otimes_{\mathbb{Z}} A'^G \longrightarrow (A \otimes_{\mathbb{Z}} A')^G$) via the canonical projection $H^0 \longrightarrow \widehat{H}^0$.

– Naturality with respect to connecting homomorphisms. *Let G be a finite group, let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be a short exact sequence of $\mathbb{Z}G$-modules, and let B be a $\mathbb{Z}G$-module such that the* *induced sequence* $0 \longrightarrow A' \otimes_{\mathbb{Z}} B \longrightarrow A \otimes_{\mathbb{Z}} B \longrightarrow A'' \otimes_{\mathbb{Z}} B \longrightarrow 0$ *is exact. Then*

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta$$

for all $\alpha \in \widehat{H}^*(G; A'')$ *and all* $\beta \in \widehat{H}^*(G; B)$*, where the* δ*s are the connecting homomorphisms of the long exact Tate cohomology sequences corresponding to the above short exact sequence of* $\mathbb{Z}G$*-modules.*

(An analogous statement holds when the factors are swapped; however, a sign has to be introduced in that case.)

Proof. Uniqueness follows similarly as in the classical case (Theorem 1.8.1) by dimension shifting (notice however that we need to induct both up and down from 0).

Existence is much harder to establish than in the classical case, though: We will not provide a full proof here, but only sketch some of the ideas and problems (a complete proof is given in Brown's book [4, Chapter VI.5]).

The fundamental idea in the construction of the cup-product on ordinary group cohomology was the observation that tensor products of projective resolutions are projective resolutions again. However, when dealing with complete resolutions, two issues arise:

- The usual tensor products of chain complexes is not "big" enough – every element can only contain components of finitely many different degrees. For complete resolutions, we will need to use the *completed tensor product* $\widehat{\otimes}_{\mathbb{Z}}$, defined by

$$(C \,\widehat{\otimes}_{\mathbb{Z}}\, C')_n := \prod_{p \in \mathbb{Z}} C_p \otimes_{\mathbb{Z}} C'_{n-p}$$

  for all $\mathbb{Z}$-graded chain complexes C_* and C'_*, and all degrees $n \in \mathbb{Z}$.
- Moreover, if $P_* \triangledown \varepsilon$ is a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then the completed tensor product $(P_* \,\widehat{\otimes}_{\mathbb{Z}}\, P_*) \triangledown (\varepsilon \,\widehat{\otimes}_{\mathbb{Z}}\, \varepsilon)$ in general will not be a complete projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

However, the completed tensor product complex $P_* \,\widehat{\otimes}_{\mathbb{Z}}\, P_*$ is still nice enough such that by means of relative homological algebra one can construct a diagonal approximation $P_* \longrightarrow P_* \,\widehat{\otimes}_{\mathbb{Z}}\, P_*$ and that one can show that such diagonal approximations are essentially unique. Then the cup-product in Tate cohomology can be defined via a completed version of the cross-product followed by such a diagonal approximation. □

Corollary 1.9.17 (The cup-product on Tate cohomology and ordinary group cohomology). *Let G be a group and let A and A' be two $\mathbb{Z}G$-modules. Then for all p, $p' \in \mathbb{N}_{>0}$, the two cup-products*

$$\begin{aligned} \cdot \cup \cdot : \widehat{H}^p(G; A) \otimes_\mathbb{Z} \widehat{H}^{p'}(G; A') &\longrightarrow \widehat{H}^{p+p'}(G; A \otimes_\mathbb{Z} A') \\ \cdot \cup \cdot : H^p(G; A) \otimes_\mathbb{Z} H^{p'}(G; A') &\longrightarrow H^{p+p'}(G; A \otimes_\mathbb{Z} A') \end{aligned}$$

coincide.

Proof. This follows from Theorem 1.9.16, Theorem 1.8.1 and Proposition 1.9.15 via a standard dimension shifting argument. □

Theorem 1.9.18 (Tate cohomology, duality). *Let G be a finite group, and let $k \in \mathbb{Z}$. Then the cup-product*

$$\begin{aligned} \cdot \cup \cdot : \widehat{H}^k(G; \mathbb{Z}) \otimes_\mathbb{Z} \widehat{H}^{-k}(G; \mathbb{Z}) \longrightarrow \widehat{H}^0(G; \mathbb{Z}) = \mathbb{Z}/|G| &\hookrightarrow \mathbb{Q}/\mathbb{Z} \\ [1] &\mapsto [1/|G|] \end{aligned}$$

is a duality pairing. I.e., the induced homomorphisms

$$\begin{aligned} \widehat{H}^k(G; \mathbb{Z}) &\longrightarrow \operatorname{Hom}_\mathbb{Z}\bigl(\widehat{H}^{-k}(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}\bigr) \\ \alpha &\longmapsto \bigl(\beta \mapsto \alpha \cup \beta\bigr) \\ \widehat{H}^{-k}(G; \mathbb{Z}) &\longrightarrow \operatorname{Hom}_\mathbb{Z}\bigl(\widehat{H}^{k}(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}\bigr) \\ \beta &\longmapsto \bigl(\alpha \mapsto \alpha \cup \beta\bigr) \end{aligned}$$

are isomorphisms.

Proof. Using the universal coefficient theorem and the identifications

$$\begin{aligned} \widehat{H}^k(G; \mathbb{Z}) &\cong H^k(G; \mathbb{Z}) \\ \widehat{H}^{-k-1}(G; \mathbb{Z}) &\cong H_k(G; \mathbb{Z}) \end{aligned}$$

for all $k \in \mathbb{N}_{>0}$ (Proposition 1.9.15) it is not difficult to see that the groups $\widehat{H}^k(G; \mathbb{Z})$ and $\operatorname{Hom}_\mathbb{Z}(\widehat{H}^{-k}(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ are isomorphic Abelian groups: Using a complete projective $\mathbb{Z}G$-resolution of finite type and the fact that all (co)homology groups of G are $|G|$-torsion, we see that all (co)homology groups of G with $\mathbb{Z}$-coefficients are finite Abelian groups. Furthermore,

$\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ for all finite Abelian groups A. Notice how the degree shift in the Ext-terms in the universal coefficient theorem coincides with the degree shift in lower Tate cohomology.

However, it requires some work to establish that the cup-product provides such an isomorphism: This can, for example, be achieved as follows [9, 4, Theorem XII.6.5, Theorem VI.7.4]:

1. Show (for instance, by dimension shifting) that there exists an element $\zeta \in \widehat{H}_{-1}(G; \mathbb{Z})$ (the Tate homology in degree -1) such that the cap-product
$$\cdot \cap \zeta \colon \widehat{H}^k(G; A) \longrightarrow \widehat{H}_{-1-k}(G; A)$$
is an isomorphism for any $\mathbb{Z}G$-module A.
2. Use the duality between the cap-product and the cup-product to show with help of the first step that the cup-product
$$\cdot \cup \cdot \colon \widehat{H}^k(G; \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{H}^{-1-k}(G; \mathbb{Z}) \longrightarrow \widehat{H}^{-1}(G; \mathbb{Q}/\mathbb{Z})$$
is a duality pairing.
3. Use the long exact sequence associated with the short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ and the naturality of the cup-product to deduce that the cup-product
$$\cdot \cup \cdot \colon \widehat{H}^k(G; \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{H}^{-k}(G; \mathbb{Z}) \longrightarrow \widehat{H}^0(G; \mathbb{Z}) = \mathbb{Z}/|G|$$
is a duality pairing. □

Using dimension shifting, we can furthermore also derive the following properties of the cup-product on Tate cohomology:

– *Existence of a unit.* The element $[1] \in \mathbb{Z}/|G| = \widehat{H}^0(G; \mathbb{Z})$ is a left and right neutral element for the cup-product on Tate cohomology.
– *Associativity.* The cup-product on Tate cohomology is associative.
– *Graded commutativity.* The cup-product on Tate cohomology is graded commutative in the sense of Proposition 1.8.19.
– *Naturality.* The cup-product on Tate cohomology is natural with respect to morphisms in $\operatorname{GrpMod}^{\widehat{\ }}$.

In particular: If G is a finite group, then $\widehat{H}^*(G; \mathbb{Z})$ is a graded commutative, unital $\mathbb{Z}$-graded ring, and inclusions of subgroups of finite groups induce unital homomorphisms of $\mathbb{Z}$-graded rings.

Like in the case of ordinary group (co)homology, Tate cohomology of a given group G can be assembled out of the Tate cohomology of its Sylow groups and the "action" of G on the Tate cohomology of the Sylow subgroups (in the sense of Section 1.7.4):

Proposition 1.9.19 (Tate cohomology, primary decomposition)**.** *Let G be a finite group and let $P(|G|)$ be the set of positive primes dividing $|G|$.*

1. *There is an isomorphism*
$$\widehat{H}^*(G;\mathbb{Z}) \cong \prod_{p\in P(|G|)} \widehat{H}^*(G;\mathbb{Z})_{[p]}$$
of $\mathbb{Z}$-graded rings; for $k \in \mathbb{Z}$, the degree k part of the right hand side is $\prod_{p\in P(|G|)} \widehat{H}^k(G;\mathbb{Z})_{[p]}$.
2. *For $p \in P(|G|)$ let G_p be a Sylow subgroup of G. Then we have an isomorphism*
$$\widehat{H}^*(G;\mathbb{Z}) \cong \prod_{p\in P(|G|)} \widehat{H}^*(G_p;\mathbb{Z})^G$$
of $\mathbb{Z}$-graded rings.

Proof. Basically the same arguments as in the case of ordinary group cohomology (Corollary 1.8.23) let us derive this decomposition from the naturality properties of the cup-product in Tate cohomology. □

1.9.5 Periodic cohomology

Notice that the ordinary group cohomology ring cannot contain any invertible classes of positive degree (because there are no negative degrees to compensate . . .); the Tate cohomology ring, however, can contain also invertible elements in non-zero degrees. Any such element gives rise to a very strong periodicity of the Tate cohomology groups (see also Figure 1.19). The theory of such groups is one of the algebraic counterparts of the groups that admit free actions on spheres.

Definition 1.9.20 (Periodic cohomology)**.** A finite group G is said to have *periodic cohomology* if there is a degree $d \in \mathbb{N}_{>0}$ such that $\widehat{H}^d(G;\mathbb{Z})$ contains an element that is invertible in the Tate cohomology ring $\widehat{H}^*(G;\mathbb{Z})$. The smallest such d is the *period* of G.

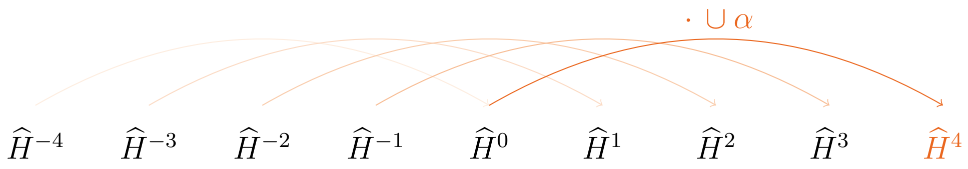


Figure 1.19: Periodic cohomology, schematically (for $d = 4$)

In view of the duality theorem (Theorem 1.9.18), periodicity in the sense of the previous definition can be reformulated in several ways:

Proposition 1.9.21 (Characterisations of periodic cohomology)**.** *Let G be a finite group. Then the following are equivalent:*

1. *The group G has periodic cohomology.*
2. *There exist $n \in \mathbb{Z}$ and $d \in \mathbb{N}_{>0}$ such that for all $\mathbb{Z}G$-modules A there is an isomorphism $\widehat{H}^n(G;A) \cong \widehat{H}^{n+d}(G;A)$.*
3. *For some degree $d \in \mathbb{N}_{>0}$ we have $\widehat{H}^d(G;\mathbb{Z}) \cong \mathbb{Z}/|G|$.*
4. *For some degree $d \in \mathbb{N}_{>0}$ the Tate cohomology group $\widehat{H}^d(G;\mathbb{Z})$ contains an element of (additive) order $|G|$.*

Proof. The implication "1 $\Rightarrow$ 2" is trivial.

For the implication "2 $\Rightarrow$ 3" we observe that $\widehat{H}^0(G;\mathbb{Z}) \cong \mathbb{Z}/|G|$, and that – by dimension shifting – we can assume without loss of generality that $n = 0$.

The implication "3 $\Rightarrow$ 4" is trivial.

For the implication "4 $\Rightarrow$ 1" we fall back on the duality theorem (Theorem 1.9.18): Let $\alpha \in \widehat{H}^d(G;\mathbb{Z})$ be an element of additive order $|G|$; because $\mathbb{Q}/\mathbb{Z}$ is injective, there is a homomorphism $\widehat{H}^d(G;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ that maps α to $[1/|G|]$. By the duality theorem, there is a class $\beta \in \widehat{H}^{-d}(G;\mathbb{Z})$ such that this homomorphism coincides with

$$\begin{aligned} \cdot \cup \beta \colon \widehat{H}^d(G;\mathbb{Z}) \longrightarrow \widehat{H}^0(G;\mathbb{Z}) \cong \mathbb{Z}/|G| &\hookrightarrow \mathbb{Q}/\mathbb{Z} \\ [1] &\mapsto [1/n]; \end{aligned}$$

in particular, $\alpha \cup \beta = [1]$, that is, α is invertible in $\widehat{H}^*(G;\mathbb{Z})$. In other words, G has periodic cohomology. □

Example 1.9.22 (The groups $\mathbb{Z}/n \times \mathbb{Z}/n$)**.** Let $n \in \mathbb{N}_{>0}$. Then the group $\mathbb{Z}/n \times \mathbb{Z}/n$ does not have periodic cohomology; this follows from the Künneth formula (Example 1.6.8), Proposition 1.9.15, and the characterisation 2 of the above proposition.

Example 1.9.23 (Finite groups acting freely on spheres)**.** Finite groups that act freely on a sphere have periodic cohomology in the sense of Definition 1.9.20: For example, we can deduce this from the fact that such groups admit periodic (complete) resolutions (Example 1.9.8) and the characterisation of periodicity (Proposition 1.9.21).

Alternatively, it is possible to prove by a direct argument that there is a cohomology class such that cup-product with this class induces an isomorphism in (Tate) cohomology. (Exercise).

Caveat 1.9.24. The symmetric group S_3 has periodic cohomology (in view of the computations of Example 1.7.31 or Corollary 1.9.30 below); however, S_3 cannot act freely on a sphere (Example 1.6.17).

As we know, the periods of finite groups that act freely on spheres are even (by Theorem 1.6.1), and all subgroups of groups that act freely on spheres also act freely on spheres. The analogous properties hold also for groups with periodic cohomology:

Proposition 1.9.25 (Basic properties of groups with periodic cohomology)**.** *Let G be a finite group with periodic cohomology.*

1. *If G is non-trivial, then the period of the cohomology of G is even.*
2. *All subgroups of G have periodic cohomology as well.*

Proof. By definition, periodicity of the cohomology of the group G entails the existence of an invertible class $\alpha \in \widehat{H}^d(G;\mathbb{Z})$, where $d \in \mathbb{N}_{>0}$ is the period.

Why is the period d even if G is non-trivial? *Assume* for a contradiction that d is odd. Because the product structure on Tate cohomology is graded commutative, we deduce

$$\alpha \cup \alpha = (-1)^{|\alpha|\cdot|\alpha|} \cdot \alpha \cup \alpha = (-1)^{d\cdot d} \cdot \alpha \cup \alpha = -\alpha \cup \alpha.$$

As α is invertible in $\widehat{H}^*(G;\mathbb{Z})$, it follows that α has additive order at most 2. On the other hand, looking at the isomorphism $\cdot \cup \alpha \colon \widehat{H}^0(G;\mathbb{Z}) \longrightarrow$

$\widehat{H}^d(G;\mathbb{Z})$ shows that α has additive order $|G|$. Therefore $G \cong \mathbb{Z}/2$ or G is trivial. In the first case, G has period 2 (Corollary 1.6.6 and Example 1.9.23), contradicting our assumption. Hence, d has to be odd.

We now come to the proof of the second part: Let $H \subset G$ be a subgroup. Because the restriction $\operatorname{res}_H^G \colon \widehat{H}^*(G;\mathbb{Z}) \longrightarrow \widehat{H}^*(H;\mathbb{Z})$ is a unital homomorphism of $\mathbb{Z}$-graded rings, the image $\operatorname{res}_H^G(\alpha) \in \widehat{H}^d(H;\mathbb{Z})$ is invertible in $\widehat{H}^d(G;\mathbb{Z})$; that is, H has periodic cohomology. □

Example 1.9.26 (Finite groups with periodic cohomology of period 2)**.** A finite group has periodic cohomology with period 2 if and only if it is cyclic (and non-trivial): Clearly, any finite cyclic (non-trivial) group has periodic cohomology with period 2 (Corollary 1.6.6 and Example 1.9.23).

Conversely, suppose that G is a finite group with periodic cohomology of period 2; of course, G is then non-trivial. Moreover,

$$\begin{aligned} G_{\mathrm{ab}} &\cong H_1(G;\mathbb{Z}) \\ &\cong \widehat{H}^{-2}(G;\mathbb{Z}) \\ &\cong \widehat{H}^0(G;\mathbb{Z}) \\ &\cong \mathbb{Z}/|G|. \end{aligned}$$

Because the Abelianisation G_{ab} is a quotient group of G, the cardinalities force $G \cong G_{\mathrm{ab}} \cong \mathbb{Z}/|G|$; in particular, G is cyclic.

Example 1.9.27 (Finite groups with periodic cohomology of period 4)**.** Let G be a finite group with periodic cohomology of period 4. Then

$$\widehat{H}^k(G;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/|G| & \text{if } k \equiv 0 \bmod 4 \\ 0 & \text{if } k \equiv 1 \bmod 4 \\ G_{\mathrm{ab}} & \text{if } k \equiv 2 \bmod 4 \\ 0 & \text{if } k \equiv 3 \bmod 4; \end{cases}$$

(in particular, we can also read off the homology and cohomology groups of G with $\mathbb{Z}$-coefficients).

Example 1.9.28 (Generalised quaternion groups)**.** In particular, this allows to compute (co)homology of the generalised quaternion groups. (Exercise).

1.9.6 Characterising groups with periodic cohomology

In Section 1.6 we considered the problem of which finite groups admit free actions on spheres; in particular, we derived an algebraic necessary conditions for such groups: namely, all Sylow subgroups of a finite group acting freely on a sphere have to be cyclic or generalised quaternion (Corollary 1.6.14). However, the converse is not true in general, and the classification of all finite groups acting freely on spheres is quite sophisticated.

In the previous section we saw that all finite groups acting freely on spheres have periodic cohomology. In the present section, we will show that indeed all finite groups all of whose Sylow subgroups are cyclic or generalised quaternion do have periodic cohomology.

Theorem 1.9.29 (Characterising p-groups with periodic cohomology)**.** *Let $p \in \mathbb{N}$ be a prime and let G be a finite p-group. Then the following are equivalent:*

1. *The group G has periodic cohomology.*
2. *All Abelian subgroups of G are cyclic.*
3. *The group G is cyclic or a generalised quaternion group.*
4. *The group G acts freely on a sphere.*

Proof. If G has periodic cohomology, then so do all of its (Abelian) subgroups (Proposition 1.9.25). In view of Example 1.9.22 and the classification of finite Abelian groups, we obtain that all Abelian subgroups of G have to be cyclic (this is the same argument as in Corollary 1.6.9).

If all Abelian subgroups of G are cyclic, then G contains a unique group of order p (arguing as in the proof of Corollary 1.6.10), and hence G is cyclic or generalised quaternion by Burnside's classification result (Corollary 1.6.13).

If the group G is cyclic or a generalised quaternion group, then G acts freely on a sphere and hence has periodic cohomology (Example 1.9.23). □

While free actions of all of the Sylow subgroups in general cannot be assembled into a free action of the whole group (e.g., the symmetric group S_3

has cyclic Sylow subgroups but cannot act freely on a sphere), the analogous statement for periodic cohomology is true:

Corollary 1.9.30 (Characterising groups with periodic cohomology)**.**

1. *A finite group has periodic cohomology if and only if all of its Sylow subgroups have periodic cohomology.*
2. *A finite group has periodic cohomology if and only if all of its Sylow subgroups are cyclic or generalised quaternion groups.*

Proof. In view of Theorem 1.9.29 it suffices to prove the first part: Let G be a finite group all of whose Sylow subgroups have periodic cohomology. In order to prove that G has periodic cohomology as well, we use the primary decomposition of the Tate cohomology of G:

For every $p \in P(|G|)$ let G_p be a Sylow subgroup of G; because G_p has periodic cohomology by assumption, there is a $d_p \in \mathbb{N}_{>0}$ such that $\widehat{H}^{d_p}(G_p;\mathbb{Z})$ contains an invertible element α_p.

However, not all of $\widehat{H}^*(G_p;\mathbb{Z})$ is contained in the Tate cohomology of G, but only the G-invariant part. So we first show that some power of α_p is G-invariant (with G-invariant inverse): To this end, let $e_p \in \mathbb{N}_{>0}$ with

$$\forall_{x \in (\mathbb{Z}/|G_p|)^\times} \quad x^{e_p} = 1;$$

for example, we could take $e_p = \varphi(|G_p|)$, where φ is the Euler φ-function. For $g \in G$, we now consider the classes

$$\begin{aligned} \rho_p(g) &:= \operatorname{res}^{G_p}_{(G_p)_g} \alpha_p \in \widehat{H}^{d_p}\big((G_p)_g;\mathbb{Z}\big), \\ \sigma_p(g) &:= \operatorname{res}^{gG_pg^{-1}}_{(G_p)_g} g \bullet \alpha_p \in \widehat{H}^{d_p}\big((G_p)_g;\mathbb{Z}\big), \end{aligned}$$

where we used the abbreviation $(G_p)_g := G_p \cap gG_pg^{-1}$. Because α_p is invertible, so are $\rho_p(g)$ and $\sigma_p(g)$ (the restrictions and the action $g \bullet \cdot$ are compatible with cup-products); therefore, we obtain

$$\widehat{H}^{d_p}\big((G_p)_g;\mathbb{Z}\big) \cong \widehat{H}^0\big((G_p)_g;\mathbb{Z}\big) \cong \mathbb{Z}/|(G_p)_g|,$$

and in particular, there is a $\iota_p(g) \in (\mathbb{Z}/|(G_p)_g|)^\times$ satisfying the relation $\sigma_p(g) = \iota_p(g) \cdot \rho_p(g)$. By construction of e_p it follows that $\iota_p^{e_p} = 1$ (there is a surjective homomorphism $(\mathbb{Z}/|G_p|)^\times \longrightarrow (\mathbb{Z}/|(G_p)_g|)^\times)$ and thus

$$\begin{aligned}
\operatorname{res}^{G_p}_{(G_p)_g} g \bullet \alpha_p^{e_p} &= \bigl(\operatorname{res}^{G_p}_{(G_p)_g} g \bullet \alpha_p\bigr)^{e_p} \\
&= \sigma_p(g)^{e_p} \\
&= \iota_p(g)^{e_p} \cdot \rho_p(g)^{e_p} \\
&= \rho_p(g)^{e_p} \\
&= \bigl(\operatorname{res}^{G_p}_{(G_p)_g} \alpha_p\bigr)^{e_p} \\
&= \operatorname{res}^{G_p}_{(G_p)_g} \alpha_p^{e_p},
\end{aligned}$$

i.e., the class $\alpha_p^{e_p}$ is G-invariant; analogously, we see that the inverse $\alpha_p^{-e_p}$ is G-invariant.

Taking $d \in \mathbb{N}_{>0}$ as the least common multiple of all the products $d_p \cdot e_p$, we deduce with help of the primary decomposition of Tate cohomology (Proposition 1.9.19) that $\widehat{H}^d(G;\mathbb{Z})$ contains a class that is invertible in $\widehat{H}^*(G;\mathbb{Z})$. This finishes the proof that G has periodic cohomology. □

Remark 1.9.31 (Estimating the period of a group)**.** The proof of the previous theorem also provides an explicit estimate for the period of a group all of whose Sylow subgroups are cyclic or generalised quaternion: The period is at most as big as the least common multiple of all $\varphi(|G_p|) \cdot d_p$, where d_p is the period of the Sylow subgroup G_p.

This can be further simplified by taking into account that cyclic groups have period 2 and generalised quaternion groups have period 4 (they admit a free action on S^3).

Example 1.9.32 (The symmetric group S_3)**.** For example, looking at the Sylow subgroups of S_3, we can deduce that S_3 has periodic cohomology and that the period is at most 4.

Using Proposition 1.9.25 and Example 1.9.26 we can rule out the periods 0, 1, 2 and 3. So S_3 has periodic cohomology with period 4. Therefore, we can easily read off all the (co)homology groups of S_3 (Example 1.7.31).

Recall that S_3 cannot act freely on a sphere, though (Example 1.6.17).

Corollary 1.9.33 (Groups with periodic cohomology and odd cohomology)**.** *Let G be a finite group with periodic cohomology. If $k \in \mathbb{Z}$ is odd, then $\widehat{H}^k(G;\mathbb{Z}) = 0$.*

Proof. In view of the primary decomposition of Tate cohomology (Proposition 1.9.19), it suffices to prove the corresponding statement for all p-groups with periodic cohomology. (Exercise). □

1.10 The Hochschild-Serre spectral sequence

How can we compute the (co)homology of an extension group if we know the (co)homology of the quotient and the kernel? The methods we studied so far do not give a satisfying solution to this problem – sometimes we can obtain partial results via the Shapiro lemma or the transfer, but these methods are not powerful enough to treat the general case.

In the present section, we will have a brief look at an algebraic tool that allows us to attack this problem, so-called *spectral sequences*. Spectral sequences can be thought of as a sophisticated version of long exact homology sequences approximating the homology of a graded chain complex in terms of smaller pieces – in the case of a grading of a chain complex of length 1 the corresponding spectral sequence is nothing but the long exact homology sequence.

In the first section, we will explain what a spectral sequence is and some of the terminology that goes with it. We will then state some examples of spectral sequences and derive some basic applications. In Section 1.10.3 and Section 1.10.4, we explain how the Hochschild-Serre spectral sequence

$$E^2_{pq} = H_p\big(Q; H_q(N; \operatorname{Res}^G_N A)\big) \Longrightarrow H_{p+q}(G; A),$$
$$E_2^{pq} = H^p\big(Q; H^q(N; \operatorname{Res}^G_N A)\big) \Longrightarrow H^{p+q}(G; A)$$

of a group extension $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ computes the (co)homology of an extension group in terms of the kernel and the quotient.

Mainly, there will be no proofs in this section, but only explanations of the terminology and example computations; furthermore, for simplicity we will restrict our discussion to the setting that all spectral sequences reside in the first quadrant. Extensive treatments of spectral sequences (including proofs) can be found in the books by Weibel [53], Hatcher [22], McCleary [32], and in the lecture notes of Bauer [1].

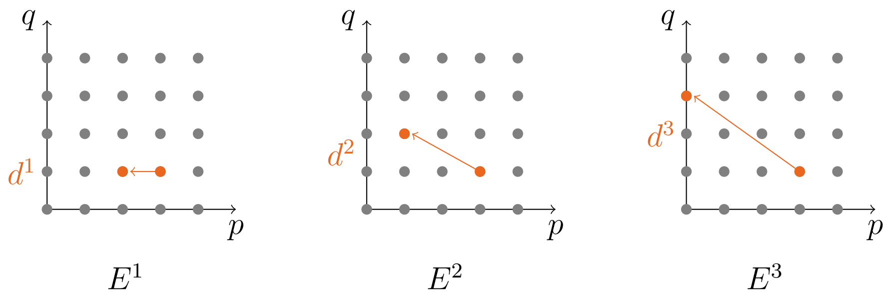


Figure 1.20: Pages of a homological spectral sequence

1.10.1 Spectral sequences in a nutshell

We will now explain the principle of spectral sequences. In the beginning, spectral sequences and all the notation might seem frightening and very technical; however, with a little bit of practice one will sooner or later appreciate their power and fall for the challenge of tricking spectral sequences into revealing all their secrets.

A spectral sequence is a sequence of bigraded modules, where the next bigraded module is obtained from the previous one by taking homology (see also Figure 1.20 for an illustration):

Definition 1.10.1 (Homological spectral sequence)**.** A *(bigraded, homological) spectral sequence* over a ring R is a sequence $(E^r, d^r)_{r\in\mathbb{N}_{>0}}$ of bigraded R-modules (i.e., every E^r is family $(E^r_{pq})_{p,q\in\mathbb{N}}$ of R-modules) and R-homomorphisms $d^r\colon E^r \longrightarrow E^r$ with the following properties:

- For every $r \in \mathbb{N}_{>0}$ the map d^r has degree $(-r, r-1)$, and $d^r \circ d^r = 0$.
- For every $r \in \mathbb{N}_{>0}$ there is an isomorphism

$$E^{r+1} \cong H_*(E^r, d^r) = \frac{\ker d^r}{\operatorname{im} d^r},$$

  and this isomorphism is a part of the data of the spectral sequence.

The term E^r is also called the *r-th page* of (E^*, d^*).

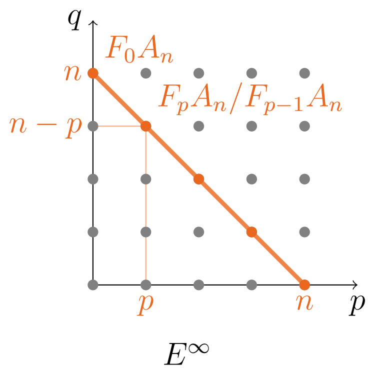


Figure 1.21: Convergence of a homological spectral sequence, schematically

Definition 1.10.2 (The ∞-page of a homological spectral sequence)**.** Let $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ be a homological spectral sequence (in the sense of the previous definition). Because all the (E^r, d^r) reside in the first quadrant, for every p, $q \in \mathbb{N}$ there exists an $s \in \mathbb{N}_{>0}$ such that

$$E^s_{pq} = E^{s+1}_{pq} = E^{s+2}_{pq} = \dots;$$

we then define $E^\infty_{pq} := E^s_{pq}$.

Definition 1.10.3 (Collapsing of a spectral sequence)**.** Let $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ be a homological spectral sequence and let $s \in \mathbb{N}_{>0}$. We say that this spectral sequence *collapses at stage s* if

$$E^s = E^{s+1} = E^{s+2} = \dots$$

(In this case, in particular $E^s = E^\infty$.)

Until now, nothing really happened yet – we just introduced some notation. The next definition is crucial for the applications of spectral sequences; it allows us to relate a spectral sequence to something we want to compute:

Definition 1.10.4 (Convergence of a spectral sequence)**.** Let R be a ring, let A be an $\mathbb{N}$-graded R-module, and let $(F_n A)_{n \in \mathbb{N}}$ be an increasing filtration of A that is compatible with the grading of A. We say that a spectral sequence $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ over R *converges to* A if the following conditions are satisfied (see also Figure 1.21):

– For all $p, q \in \mathbb{N}$ we have (with $F_{-1}A := 0$)

$$E^\infty_{pq} \cong \frac{F_p A_{p+q}}{F_{p-1} A_{p+q}}.$$

– The spectral sequence is exhaustive, i.e., $F_n A_n = A_n$ for all $n \in \mathbb{N}$.

In this case one writes

$$E^2_{pq} \Longrightarrow A_{p+q}.$$

Remark 1.10.5 (Stepping through a spectral sequence)**.** What is the typical "usage" of a spectral sequence? We might be interested in some graded object A (in most cases: homology of something) for which there happens to exist a (homological) spectral sequence $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ converging to A, where the E^2-term is something accessible:

$$E^2_{p+q} \Longrightarrow A_{p+q}.$$

Usually, one then proceeds as follows:

1. Try to compute as many of the modules of the E^2-term as possible; in general, the more zeroes, the better!
2. Try to prove that many of the differentials d^2_{pq} in the E^2-term are zero – e.g., using the degree, torsion phenomena, product structures, ...
3. Using the results of the first two steps, try to compute as much of the E^3-term and the differential d^3 as possible.
   Note. Many spectral sequences collapse (at least to a large extent) at the E^2-stage or the E^3-stage!
4. Carry on like that and try to compute as much of the E^∞-term as possible.
5. Try to solve the extension problems arising when reconstructing A out of E^∞.

Caveat 1.10.6. If a spectral sequence $(E^r, d^r)_{r \in \mathbb{N}_{>0}}$ converges to a graded filtred module A, and if we know this spectral sequence, then this does not necessarily mean that we can actually compute A – we only obtain the quotients $F_*A/F_{*-1}A$ of the associated filtration (as depicted in Figure 1.21)! I.e., we still have to solve a sequence of extension problems (Figure 1.22).

In most cases, one is not able to determine the differentials $(d^r)_{r \in \mathbb{N}_{>0}}$ explicitly; however, the degrees of these differentials already reveal a lot

$$
\begin{array}{c}
0 \longrightarrow F_0A_n \longrightarrow F_1A_n \longrightarrow \dfrac{F_1A_n}{F_0A_n} \longrightarrow 0 \\
0 \longrightarrow F_1A_n \longrightarrow F_2A_n \longrightarrow \dfrac{F_2A_n}{F_1A_n} \longrightarrow 0 \\
\vdots \\
0 \longrightarrow F_{n-1}A_n \longrightarrow F_nA_n = A_n \longrightarrow \dfrac{F_nA_n}{F_{n-1}A_n} \longrightarrow 0
\end{array}
$$

Figure 1.22: Convergence of a homological spectral sequence, extensions

about the spectral sequence and its long-term development, and additional external input might provide enough information to extract non-trivial conclusions out of a spectral sequence.

In a way, spectral sequences behave more like puzzles than like deterministic processes. We will explain some of the basic techniques for handling spectral sequences below (Section 1.10.4).

Dually to the concept of homological spectral sequences there is also a notion of cohomological spectral sequences:

Definition 1.10.7 (Cohomological spectral sequence)**.** A *(bigraded) cohomological spectral sequence* over a ring R is a sequence $(E_r, d_r)_{r\in\mathbb{N}_{>0}}$ of bigraded R-modules E_r and R-homomorphisms $d_r \colon E_r \longrightarrow E_r$ with the following properties:

- For every $r \in \mathbb{N}_{>0}$ the map d_r has degree $(r, -r+1)$, and $d_r \circ d_r = 0$.
- For every $r \in \mathbb{N}_{>0}$ there is an isomorphism

$$E_{r+1} \cong H_*(E_r, d_r) = \frac{\ker d_r}{\operatorname{im} d_r},$$

and this isomorphism is a part of the data of the spectral sequence.

Similar to the homological case, the ∞-page and collapsing are defined for cohomological spectral sequences.

Definition 1.10.8 (Convergence of a cohomological spectral sequence)**.** Let R be a ring, let A be an $\mathbb{N}$-graded R-module, let A be an $\mathbb{N}$-graded

R-module, and let $(F_n A)_{n\in\mathbb{N}}$ be an decreasing filtration of A that is compatible with the grading of A. We say that a cohomological spectral sequence $(E_r, d_r)_{r\in\mathbb{N}_{>0}}$ *converges to* A if the following conditions are satisfied:

- For all $p, q \in \mathbb{N}$ we have
$$E_\infty^{pq} \cong \frac{F_p A_{p+q}}{F_{p+1} A_{p+q}}$$
- The filtration $F_* A$ is exhaustive and Hausdorff, i.e., $F_0 A = A$ and $F_{n+1} A = 0$.

1.10.2 Some classic spectral sequences

In the following, we list some classic spectral sequences that converge to interesting objects; of course, this list is by no means complete.

Where do spectral sequences come from? Two of the main sources are the spectral sequences associated with double complexes (or filtred complexes), and the Grothendieck spectral sequence:

- *Double complexes.* For every $\mathbb{N} \times \mathbb{N}$-graded double complex, there are two spectral sequences [53, Chapter 5.6]: one relates the vertical homology of the horizontal homology to the homology of the total complex, and the other one relates the horizontal homology of the vertical homology to the homology of the total complex.
- *Grothendieck spectral sequence.* The Grothendieck spectral sequence allows to compute derived functors of compositions of functors in terms of the derived functors of the factors [53, Chapter 5.8].

These two spectral sequences are the foundation for many classic spectral sequences in algebraic topology:

- *Künneth theorem.* For instance, the Künneth theorem can be viewed as a special case of the double complex spectral sequences [53, Theorem 5.6.4].
- *Group actions.* The spectral sequences for group actions (see Section 1.6.8) can be derived from double complexes [4, Chapter VII.7].
- *Leray spectral sequence.* The Leray spectral sequence allows to compute homology of a space in terms of nerves of coverings with low multiplicity and acyclic intersections [4, Theorem VII.4.4].

- *Leray-Serre spectral sequence.* The Leray-Serre spectral sequence allows to express the (co)homology of the total space of a fibration in terms of the (co)homology of the base and fibre [53, Chapter 5.3].
- *Atiyah-Hirzebruch spectral sequence.* The Atiyah-Hirzebruch spectral sequence relates values of generalised cohomology theories to values of singular homology with coefficients in the so-called coefficients of the generalised homology theory [51, Chapter 15].

In the context of group cohomology, the central spectral sequence is the Hochschild-Serre spectral sequence, which describes the (co)homology of an extension group in terms of the (co)homology of the quotient and the kernel (Section 1.10.3).

What are typical results that can be proved via spectral sequences?

- *Long exact homology sequences.* Spectral sequences whose E^2-terms are concentrated in two adjacent rows give rise to long exact sequences [53, Exercise 5.2.2].
  A more careful analysis shows that every first quadrant spectral sequence gives rise to a five term exact sequence in low degrees [53, Exercise 5.1.3].
- *Vanishing/torsion results.* Vanishing and torsion properties are preserved under taking homology and thus survive until the E^∞-page. Usually, these vanishing and torsion properties can then also be transferred to the graded target object.
- *Dimension results/Euler characteristic.* Similarly, also dimension properties and Euler characteristics survive the travel through the pages until the E^∞-page and can then be transferred to the graded target object.

1.10.3 The Hochschild-Serre spectral sequence

In group (co)homology, one of the central spectral sequences is the Hochschild-Serre spectral sequence (see Figure 1.23):

Theorem 1.10.9 (The Hochschild-Serre spectral sequence)**.** *Let G be a group fitting into a short exact sequence $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ of groups, and let A be a $\mathbb{Z}G$-module.*

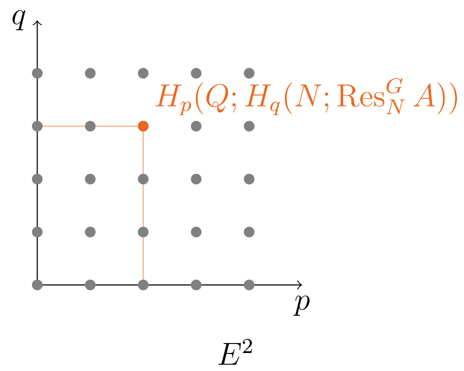


Figure 1.23: The Hochschild-Serre spectral sequence, schematically

1. *Then there is a homological spectral sequence*

$$E^2_{pq} = H_p\bigl(Q; H_q(N; \mathrm{Res}_N^G A)\bigr) \Longrightarrow H_{p+q}(G; A);$$

here, $Q \cong G/N$ acts on the coefficients $H_*(N; \mathrm{Res}_N^G A)$ as described in Proposition 1.7.24.

2. *Similarly, there is a cohomological spectral sequence*

$$E_2^{pq} = H^p\bigl(Q; H^q(N; \mathrm{Res}_N^G A)\bigr) \Longrightarrow H^{p+q}(G; A).$$

If A is a $\mathbb{Z}G$-algebra, then this spectral sequence is multiplicative in the following sense: All terms $(E_r)_{r \in \mathbb{N}_{\geq 2}}$ carry the structure of a differential graded algebra (i.e., they are equipped with a graded commutative product such that the differentials satisfy the Leibniz rule) such that
   - *the product on the E_2-term coincides with the cup-product on the cohomology $H^*(Q; H^*(N; \mathrm{Res}_N^G A))$, and*
   - *such that the filtration on $H^*(G; A)$ induced by this spectral sequence is also compatible with the cup-product on $H^*(G; A)$.*

Proof. Each of the descriptions of group (co)homology provides a proof of the Hochschild-Serre spectral sequence (and its convergence):

– *Topologically.* Associated with the group extension in the statement of the theorem there is a fibration $BN \longrightarrow BG \longrightarrow BQ$ of (models of) the classifying spaces (i.e., the short exact sequence on π_1 induced by this fibration coincides with the given group extension). Then for every $\mathbb{Z}G$-module A, the Serre spectral sequence

$$E^2_{pq} = H_p\bigl(BQ; H_q(BN; \operatorname{Res}^G_N A)\bigr) \Longrightarrow H_{p+q}(G; A)$$

of the fibration $BN \longrightarrow BG \longrightarrow BQ$ coincides with the Hochschild-Serre spectral sequence of this group extension [**?**]; of course, the same argument applies to cohomology.

– *Via projective resolutions of* $\mathbb{Z}$. When expressing group (co)homology in terms of projective resolutions of the trivial module $\mathbb{Z}$, the Hochschild-Serre spectral sequence can be obtained from the spectral sequences associated to double complexes [4, Chapter VII.6].
– *Via derived functors.* Decomposing the G-(co)invariants functor into a composition of the N-(co)invariants functor followed by the Q-(co)-invariants functor, we can deduce the Hochschild-Serre spectral sequence from the Grothendieck spectral sequence [**?**]. □

Remark 1.10.10 (Naturality of the Hochschild-Serre spectral sequence)**.** The Hochschild-Serre spectral sequence is natural in the following sense: For simplicity, let A be a $\mathbb{Z}$-module on which all of the following groups act trivially. If

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N' & \longrightarrow & G' & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

is a commutative diagram of groups with exact rows, then the corresponding induced homomorphisms on homology fit together to form a morphism

$$\begin{array}{ccc} E^2_{pq} = H_p\bigl(Q; H_q(N; \operatorname{Res}^G_N A)\bigr) & \Longrightarrow & H_{p+q}(G; A) \\ \downarrow & & \downarrow \\ E^2_{pq} = H_p\bigl(Q'; H_q(N'; \operatorname{Res}^{G'}_{N'} A)\bigr) & \Longrightarrow & H_{p+q}(G'; A) \end{array}$$

of spectral sequences (i.e., homomorphisms between the corresponding pages of the spectral sequences that are compatible with the differentials, and such that the map between the $(r+1)$-st pages is the map induced on homology by the map between the r-th pages).

Similarly, the cohomological Hochschild-Serre spectral sequence is natural with respect to such morphisms (and if the coefficients are an algebra, then the morphism of cohomological spectral sequences is also compatible with the product structure).

Example 1.10.11 (Inheritance through spectral sequences)**.** As indicated above, certain properties survive the passage to the E^∞-page and allow to deduce inheritance properties: Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be an extension of groups. Then the cohomological dimension satisfies

$$\operatorname{cd} G \leq \operatorname{cd} N + \operatorname{cd} Q$$

(Exercise), and (whenever these Euler characteristics are defined)

$$\chi(BG) = \chi(BN) \cdot \chi(BQ).$$

Moreover, also torsion results and duality properties can be derived.

1.10.4 Sample computations for group extensions

In the following, we give some sample computations to illustrate basic techniques in spectral sequence computations.

Collapsing at the E^2-stage, trivial extension problems. We start illustrating the use of the Hochschild-Serre spectral sequence by computing, again, the homology of the symmetric group S_3:

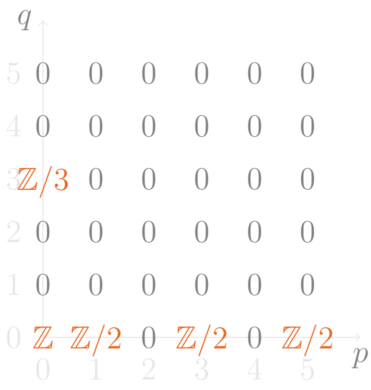
Example 1.10.12 (The symmetric group S_3)**.** The symmetric group S_3 fits into a group extension

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/2$ acts on the kernel $\mathbb{Z}/3$ by taking inverses. The Hochschild-Serre spectral sequence then gives us:

$$E^2_{pq} = H_p\bigl(\mathbb{Z}/2; H_q(\mathbb{Z}/3;\mathbb{Z})\bigr) \Longrightarrow H_{p+q}(S_3;\mathbb{Z}),$$

where $\mathbb{Z}/2$ acts on the coefficients $H_*(\mathbb{Z}/3;\mathbb{Z})$ by the maps induced by taking inverses on $\mathbb{Z}/3$; i.e., for $k \in \mathbb{N}$, the group $\mathbb{Z}/2$ acts by multiplication by $(-1)^k$ on $H_{2k+1}(\mathbb{Z}/3;\mathbb{Z}) \cong \mathbb{Z}/3$ (see Example 1.4.12).



$E^2 = E^\infty$

Figure 1.24: The Hochschild-Serre spectral sequence for S_3

1. How does the E^2-term look like? The description of the $\mathbb{Z}/2$-action on the homology of $\mathbb{Z}/3$ gives the vertical axis of the E^2-term of the Hochschild-Serre spectral sequence (recall that zero-th homology is given by taking coinvariants).
   Of course, the horizontal axis is nothing but $H_*(\mathbb{Z}/2;\mathbb{Z})$. In view of the torsion results provided by the transfer (Corollary 1.7.19), we obtain
   $$E^2_{pq} = H_p\bigl(\mathbb{Z}/2; H_q(\mathbb{Z}/3;\mathbb{Z})\bigr) = 0$$
   for all p, $q \in \mathbb{N}_{>0}$. Therefore, the E^2-term looks as depicted in Figure 1.24.
2. Are there non-trivial differentials? For any $r \in \mathbb{N}_{\geq 2}$, the differential d^r of the Hochschild-Serre spectral sequence has degree $(-r, r-1)$; in particular, the horizontal and the vertical component of the bidegree have different parity. Hence, all differentials $(d^r)_{\geq 2}$ have to be trivial in this example. In other words, the spectral sequence corresponding to the above extension collapses at the E^2-stage, and therefore $E^\infty = E^2$.
3. What about the extension problems? From the E^∞-page of the spectral sequence, for $k \in \mathbb{N}_{>0}$ we obtain short exact sequences of Abelian groups of the following types:

$$\begin{array}{ll} 0 \to H_k(S_3;\mathbb{Z}) \to \mathbb{Z}/2 \to 0 & \text{if } k \equiv 1 \bmod 4 \\ 0 \to H_k(S_3;\mathbb{Z}) \to 0 & \text{if } k \equiv 2 \bmod 4 \\ 0 \to \mathbb{Z}/3 \to H_k(S_3;\mathbb{Z}) \to \mathbb{Z}/2 \to 0 & \text{if } k \equiv 3 \bmod 4 \\ 0 \to H_k(S_3;\mathbb{Z}) \to 0 & \text{if } k \equiv 0 \bmod 4. \end{array}$$

The classification of finitely generated Abelian groups tells us that all these extensions have to be trivial. Therefore, we obtain

$$H_k(S_3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } k \equiv 1 \bmod 4 \\ 0 & \text{if } k \equiv 2 \bmod 4 \\ \mathbb{Z}/6 & \text{if } k \equiv 3 \bmod 4 \\ 0 & \text{if } k \equiv 4 \bmod 4 \text{ and } k > 0. \end{cases}$$

Collapsing at the E^2-stage, non-trivial extension problems. We now give an example of an instance of the Hochschild-Serre spectral sequence that still collapses at the E^2-term, but where the resulting extension problems are non-trivial:

Example 1.10.13 (The infinite dihedral group)**.** We consider the infinite dihedral group $D_\infty = \langle s, t \mid s^2 = 1, sts = t^{-1}\rangle$; it is not difficult to see that the group D_∞ fits into an extension

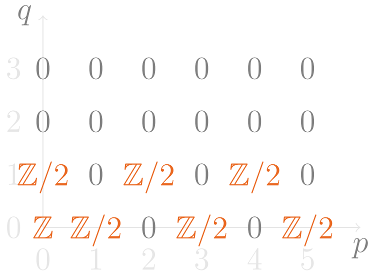
$$1 \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/2$ acts on the kernel $\mathbb{Z}$ by taking inverses. The Hochschild-Serre spectral sequence then gives us:

$$E^2_{pq} = H_p\bigl(\mathbb{Z}/2; H_q(\mathbb{Z};\mathbb{Z})\bigr) \Longrightarrow H_{p+q}(D_\infty;\mathbb{Z}),$$

where $\mathbb{Z}/2$ acts on the coefficients $H_*(\mathbb{Z};\mathbb{Z})$ by the maps induced by taking inverses in $\mathbb{Z}$; i.e., the group $\mathbb{Z}/2$ acts trivially on $H_0(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$ and by multiplication by -1 on $H_1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$.

1. How does the E^2-term look like? With help of the standard periodic $\mathbb{Z}/2$-resolution of $\mathbb{Z}$ (see the proof of Corollary 1.6.6) we see that the E^2-term of this spectral sequence has the shape depicted in Figure 1.25.



$$E^2 = E^\infty$$

Figure 1.25: The Hochschild-Serre spectral sequence for D_∞

2. Are there non-trivial differentials? Looking at the degrees of the differentials, we see that the differentials all start or end in 0; so, there are no non-trivial differentials. In other words, the spectral sequence collapses at the E^2-stage, and thus $E^\infty = E^2$.
3. What about the extension problems? We obtain $H_0(D_\infty; \mathbb{Z}) \cong \mathbb{Z}$, and $H_k(D_\infty; \mathbb{Z}) = 0$ for all even $k \in \mathbb{N}_{>0}$ from the spectral sequence. For all odd $k \in \mathbb{N}$, the E^∞-term of the Hochschild-Serre spectral sequence gives us short exact sequences of the following type:

$$0 \to \mathbb{Z}/2 \to H_k(D_\infty; \mathbb{Z}) \to \mathbb{Z}/2 \to 0.$$

Now the classification of finitely generated Abelian groups shows that $H_k(D_\infty; \mathbb{Z}) \cong \mathbb{Z}/4$ or $H_k(D_\infty; \mathbb{Z}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
In order to find out which of these alternatives actually happens, we take advantage of the fact that the group extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

we started with splits. In particular, we see that the identity homomorphism $\mathbb{Z}/2 \cong H_k(\mathbb{Z}/2; \mathbb{Z}) \longrightarrow H_k(\mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/2$ factors through $H_k(D_\infty; \mathbb{Z})$. Hence, $H_k(D_\infty; \mathbb{Z})$ cannot be isomorphic to $\mathbb{Z}/4$. Therefore,

$$H_k(D_\infty; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

for all odd $k \in \mathbb{N}$, which completes the computation of $H_*(D_\infty; \mathbb{Z})$ via the Hochschild-Serre spectral sequence.

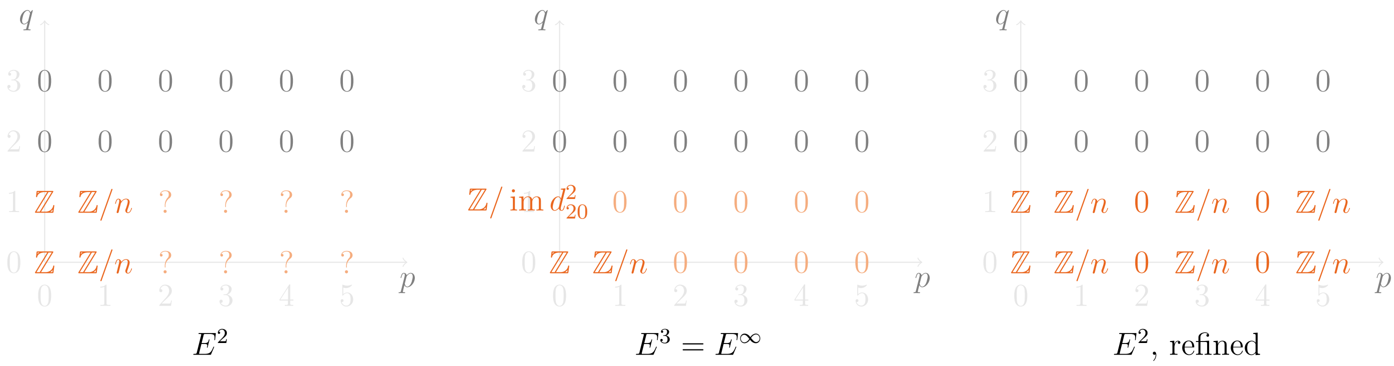


Figure 1.26: The Hochschild-Serre spectral sequence for $\mathbb{Z}/n$

Reverse engineering. Sometimes, the Hochschild-Serre spectral sequence also allows to compute the (co)homology of the quotient or the kernel if the (co)homology of the extension group is known. As a toy example of this principle, we demonstrate how to compute the homology of finite cyclic groups out of the homology of $\mathbb{Z}$:

Example 1.10.14 (Finite cyclic groups)**.** Let $n \in \mathbb{N}_{>0}$. We consider the group extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

given by multiplication by n on $\mathbb{Z}$; here, $\mathbb{Z}/n$ acts trivially on the kernel $\mathbb{Z}$. The Hochschild-Serre spectral sequence then gives us:

$$E^2_{pq} = H_p\bigl(\mathbb{Z}/n; H_q(\mathbb{Z};\mathbb{Z})\bigr) \Longrightarrow H_{p+q}(\mathbb{Z};\mathbb{Z}),$$

where $\mathbb{Z}/n$ acts trivially on the coefficients $H_*(\mathbb{Z};\mathbb{Z})$.

1. How does the E^2-term look like? In this case, we do not have the complete information needed to compute the E^2-term – because $H_*(\mathbb{Z}/n;\mathbb{Z})$ is what we *want* to compute. However, we know enough to describe the basic shape of the E^2-term: Because

$$H_0(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}, \qquad \forall_{k \in \mathbb{N}_{>1}} \; H_k(\mathbb{Z};\mathbb{Z}) = 0$$

all entries above height 1 in the E^2-term have to be zero.

As $H_0(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z} \cong H_1(\mathbb{Z};\mathbb{Z})$ is an isomorphism of $\mathbb{Z}[\mathbb{Z}/n]$-modules, the zero-th and the first row have to contain isomorphic entries; i.e., $E^2_{p0} \cong E^2_{p1}$ for all $p \in \mathbb{N}$.
Clearly, $E^2_{00} = H_0(\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}$. Using the fact that the first homology group with integral coefficients coincides with the Abelianisation, we obtain

$$E^2_{10} = H_1(\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}/n.$$

All this information is illustrated in Figure 1.26.
However, at this point, we do not know what the other entries in the zero-th and first row are. In particular, we cannot say anything about the differential d^2, yet.

2. How does the E^3-term look like? Looking at the degrees of the differentials, we see that the spectral sequence collapses at the E^3-term; hence, $E^\infty = E^3$.
Because $H_k(\mathbb{Z};\mathbb{Z}) = 0$ for all $k \in \mathbb{N}_{>1}$ this implies that the only non-trivial entries in the zeroth and the first row can be in the positions $(0,0)$, $(1,0)$ or $(0,1)$.
Furthermore, on the one hand, $H_1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$, and on the other hand, $H_1(\mathbb{Z};\mathbb{Z})$ fits into the short exact sequence

$$0 \longrightarrow \mathbb{Z}/\operatorname{im} d^2_{20} \longrightarrow H_1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

derived from the E^∞-term. Therefore, $\operatorname{im} d^2_{20} = 0$.

3. Refining the information on the E^2-term. Using the information on the E^3-term we want to derive information on the E^2-term. By definition,

$$E^3_{pq} = \frac{\ker d^2_{pq}}{\operatorname{im} d^2_{p+2,q-1}}$$

for all p, $q \in \mathbb{N}$. In particular, we obtain:
   - for all $p \in \mathbb{N}_{>1}$ the differential d^2_{p0} is injective, and
   - for all $p \in \mathbb{N}_{>2}$ the differential d^2_{p0} is surjective;

i.e., d^2_{p0} is an isomorphism for all $p \in \mathbb{N}_{>2}$. Together with the tiny part of E^2 that we already computed, we deduce inductively that

$$E^2_{p0} = E^2_{p1} \cong \begin{cases} \mathbb{Z}/n & \text{if } p \in \mathbb{N} \text{ is odd} \\ 0 & \text{if } p \in \mathbb{N}_{>0} \text{ is even.} \end{cases}$$

Therefore, $H_k(\mathbb{Z}/n;\mathbb{Z}) = 0$ for even $k \in \mathbb{N}_{>0}$, and $H_k(\mathbb{Z}/n) \cong \mathbb{Z}/n$ for all odd $k \in \mathbb{N}$.

Non-trivial differentials in E^2. Another nice example of the Hochschild-Serre spectral sequence is the computation of the Heisenberg group via the "obvious" central extension:

Example 1.10.15 (The Heisenberg group)**.** Using the Hochschild-Serre spectral sequence, one can also determine the (co)homology of the 3-dimensional discrete Heisenberg group. (Exercise.)

In this example, the necessary information about the differentials in the E^2-term can be obtained from the fact that the first homology with integral coefficients coincides with the Abelianisation.

Naturality, multiplicativity. Of course, the multiplicative structure and naturality conveniently add rigidity to the spectral sequences:

Example 1.10.16 (The dihedral group D_4)**.** The dihedral group D_4 fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}/4 \longrightarrow D_4 \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the quotient $\mathbb{Z}/2$ acts on the kernel $\mathbb{Z}/4$ by taking inverses.

We now wish to compute cohomology of $H^*(D_4;\mathbb{Z}/2)$ (an example that we would not be able to treat without the technique of spectral sequences!). Again, it is not clear a priori how the differentials on the E_2-term look like.

Comparing the cohomological Hochschild-Serre spectral sequence for the extension for D_4 above with the cohomological Hochschild-Serre spectral sequence of the trivial extension

$$1 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \longrightarrow 1 \longrightarrow 1,$$

and using the fact that the cohomological Hochschild-Serre spectral sequence is multiplicative, one can show that

$$H^k(D_4;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{k+1}$$

for all $k \in \mathbb{N}_{>0}$; moreover, it is also possible to determine the product structure [1, Beispiel 7.3.1].

1.11 Exercises

Most of the exercises are grouped into collections of four exercises, covering the material of one week of lectures. The exercises vary in difficulty; some of them are straightforward applications of the material presented in the text, while others require additional knowledge (e.g., from algebraic topology).

Exercise sheet #1

Exercise 1.1 (Group rings and subrings of $\mathbb{C}$)**.**

1. Let $\zeta_5 \in \mathbb{C}$ be a primitive fifth root of unity. Are the rings $\mathbb{Z}[\mathbb{Z}/5]$ and $\mathbb{Z}[\zeta_5] \subset \mathbb{C}$ isomorphic?
2. Are there non-trivial groups whose integral group ring is isomorphic to a subring of $\mathbb{C}$?

Exercise 1.2 (A functorial model of classifying spaces)**.** Let G be a torsion-free group, and let Δ^G be the (infinite-dimensional) simplex (with the weak topology) spanned by G. The left translation action of G on G induces a continuous G-action on Δ^G. Show that the quotient space $G \setminus \Delta^G$ is a model of BG.

Exercise 1.3 (Homology of the projective plane with twisted coefficients)**.**

1. Let $\mathbb{Z}_w$ be the $\mathbb{Z}[\mathbb{Z}/2]$-module whose underlying $\mathbb{Z}$-module is $\mathbb{Z}$ with the $\mathbb{Z}/2$-action given by the generator of $\mathbb{Z}/2$ acting by multiplication by -1. Compute $H_*(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_w)$ and $H^*(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_w)$. Compare the results with (co)-homology of $\mathbb{R}\mathrm{P}^2$ with $\mathbb{Z}$-coefficients.
2. Formulate generalised Poincaré duality with twisted coefficients for closed connected (but not necessarily orientable) manifolds.

Exercise 1.4 ((Co)Homology of $\mathbb{Z}/2$)**.**

1. Compute group homology and group cohomology of $\mathbb{Z}/2$ with arbitrary coefficients (using the topological description).
2. Generalise the result for $H_*(\mathbb{Z}/2; \mathbb{Q})$ and $H^*(\mathbb{Z}/2; \mathbb{Q})$ to arbitrary finite groups!

Exercise sheet $\#1\frac{1}{2}$

Exercise 1.5 (The infinite dihedral group)**.** The *infinite dihedral group* is the isometry group of $\mathbb{Z}$ (equipped with the metric induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ into the Euclidean line). We denote the infinite dihedral group by D_∞.

1. Find a nice presentation of D_∞ by generators and relations (for instance, analogous to finite dihedral groups).
2. How can D_∞ be written as a free product of two small groups?
3. Compute the (co)homology of D_∞ with constant coefficients via the topological description.
4. Deduce that there is no finite-dimensional model of the classifying space of D_∞.
   [There is however a finite-dimensional model of the classifying space for proper actions of D_∞.]
5. Prove that D_∞ is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}$; what is the induced action of $\mathbb{Z}/2$ on $\mathbb{Z}$? Is the cohomology class corresponding to this extension trivial?
   In particular, D_∞ is virtually $\mathbb{Z}$; compare $H_*(D_\infty;\mathbb{Q})$ with $H_*(\mathbb{Z};\mathbb{Q})$.
6. Compute $H_1(\mathbb{Z}/2;\mathbb{Z})$ for all $\mathbb{Z}[\mathbb{Z}/2]$-module structures on $\mathbb{Z}$ via the combinatorial description.
7. Compute group (co)homology of the Klein four group D_2 with constant coefficients via the topological description and compare the result with group (co)homology of $\mathbb{Z}/2$ and D_∞ respectively.

Exercise sheet #2

Exercise 1.6 (Bar resolution)**.** Let G be a discrete group.

1. Show that $C_*(G)$ together with the map $\partial\colon C_*(G) \longrightarrow C_*(G)$ (see Definition **??**) indeed forms a $\mathbb{Z}G$-chain complex.
2. Show that the complex $C_*(G) \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}$ is contractible by giving an explicit $\mathbb{Z}$-chain contraction; here, $\varepsilon\colon C_0(G) \longrightarrow \mathbb{Z}$ is the $\mathbb{Z}$-homomorphism mapping all elements of G to 1.

Exercise 1.7 (Extensions of $\mathbb{Z}/3$ by $\mathbb{Z}/2$)**.**

1. Show that $H^2(\mathbb{Z}/3;\mathbb{Z})$ contains at least three elements by giving three non-equivalent extensions of $\mathbb{Z}/3$ by $\mathbb{Z}$ (here, $\mathbb{Z}$ carries the only possible $\mathbb{Z}/3$-action, the trivial one).
2. Conclude that there is no $\mathbb{Z}[\mathbb{Z}/3]$-chain contraction of $C_*(G) \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}$.

Exercise 1.8 (Classification of extensions)**.** Let Q be a discrete group and let A be a $\mathbb{Z}Q$-module. Show that the maps

$$\begin{aligned} H^2(Q;A) &\longleftrightarrow E(Q,A) \\ [f] &\longmapsto [0 \to A \to G_f \to Q \to 0] \\ \eta_E &\longleftarrow\!\shortmid E \end{aligned}$$

constructed in **??** are mutually inverse.

Exercise 1.9 (Naturality of the classification of extensions)**.** Show that the classification of extensions with Abelian kernel is natural in the following sense: Let $(\varphi,\Phi)\colon (Q,A) \longrightarrow (Q',A')$ be a morphism in GrpMod, and let $E \in E(Q,Q)$ and $E' \in E(Q',A')$ be represented by the extensions $0 \to A \to G \to Q \to 1$ and $0 \to A' \to G' \to Q' \to 1$ respectively. Show that there exists a group homomoprhism $\widetilde{\varphi}\colon G \longrightarrow G'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \widetilde{\varphi}} & & \Big\downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & A' & \longrightarrow & G' & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

commutative if and only if (in $H^2(Q;\varphi^*A')$)

$$H^2(\mathrm{id}_Q;\Phi)(\eta_E) = H^2(\varphi;\mathrm{id}_{A'})(\eta_{E'}).$$

Exercise sheet #3

Exercise 1.10 (Projective modules over group rings)**.**

1. For which groups G is the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ projective?
2. For which groups G is the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ projective?

Exercise 1.11 (Cohomological dimension)**.** Let G be a group. Then

$$\operatorname{cd} G := \inf\bigl\{n \in \mathbb{N} \cup \{\infty\} \bigm| \mathbb{Z} \text{ admits a projective } \mathbb{Z}G\text{-resolution of length } n\bigr\}$$

is the *cohomological dimension of* G; analgously,

$$\operatorname{gdim} G := \inf\bigl\{n \in \mathbb{N} \cup \{\infty\} \bigm| \text{there is a model of } BG \text{ of dimension } n\bigr\}$$

is the *geometric dimension of* G.

1. Show that the cohomological dimension of a group is at moast as big as the geometric dimension.
2. Which groups have cohomological dimension equal to 0?
3. Let $n \in \mathbb{N}$. Determine the cohomological and the geometrical dimension of $\mathbb{Z}^n$.

Exercise 1.12 (Exactness properties of Hom)**.** Let R be a ring.

1. Show that a seqeuence $B' \xrightarrow{f'} B \xrightarrow{f''} B'' \longrightarrow 0$ of R-modules is exact if and only if for all R-modules A the corresponding sequence

$$0 \longrightarrow \operatorname{Hom}_R(B'', A) \xrightarrow{\operatorname{Hom}_R(f'', \operatorname{id}_A)} \operatorname{Hom}_R(B, A) \xrightarrow{\operatorname{Hom}_R(f', \operatorname{id}_A)} \operatorname{Hom}_R(B', A)$$

is exact.

2. Give an example for a $\mathbb{Z}$-module A and a short exact sequence of $\mathbb{Z}$-modules witnessing that the functor $\operatorname{Hom}_{\mathbb{Z}}(\,\cdot\,, A)$ is not exact.

Exercise 1.13 (Adjointness and exactness)**.** Let R and S be rings.

1. Prove that an additive functor $F\colon \text{Mod-}R \longrightarrow \text{Mod-}S$ having a right adjoint functor is right exact.
2. Show that the tensor product functor $\,\cdot\, \otimes_R A\colon \text{Mod-}R \longrightarrow \text{Ab}$ is right exact for every left R-module A.

Exercise sheet #4

Exercise 1.14 (Tor, examples)**.**

1. Compute the groups $\operatorname{Tor}_*^{\mathbb{Q}[X]}\bigl(\mathbb{Q}[X]/X^{2009}, \mathbb{Q}[X]/X^{2010}\bigr)$ via a suitable projective resolution.
2. Does $\operatorname{Tor}_2^{\mathbb{Z}[X]}(A, B) = 0$ hold for all $\mathbb{Z}[X]$-modules A and B ?

Exercise 1.15 (Partial projective resolutions)**.** Let G be a discrete group, let $n \in \mathbb{N}$, and let

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a partial projective $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ of length n (i.e., the $\mathbb{Z}G$-modules $P_0, \dots, P_n$ are projective, the sequence above is exact, but the $\mathbb{Z}G$-homomorphism ∂_n is not necessarily injective).

1. Show that the partial projective resolution above can be extended to a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.
2. Show that there is an exact sequence of the following type:

$$0 \longrightarrow H_{n+1}(G;\mathbb{Z}) \longrightarrow \bigl(H_n(P_*)\bigr)_G \longrightarrow H_n\bigl((P_*)_G\bigr) \longrightarrow H_n(G;\mathbb{Z}) \longrightarrow 0$$

Exercise 1.16 (Hurewicz homomorphism and group cohomology)**.**

1. Let $n \in \mathbb{N}_{\geq 2}$, and let X be a (pointed) CW-complex with fundamental group G whose universeal covering is $(n-1)$-connected. Show that there is an exact sequence

$$\pi_n(X) \xrightarrow{h_n^X} H_n(X;\mathbb{Z}) \longrightarrow H_n(G;\mathbb{Z}) \longrightarrow 0,$$

   where $h_n^X \colon \pi_n(X) \longrightarrow H_n(X;\mathbb{Z})$ is the Hurewicz homomorphism in degree n.

   Hints. Use Exercise 1.15.
2. Conclude that the Hurewicz homomorphism $\pi_2(X) \longrightarrow H_2(X;\mathbb{Z})$ is surjective for all connected (pointed) CW-complexes X with free fundamental group.

Exercise 1.17 (Grp and GrpMod and pre-additive categories)**.** A category C is called *pre-additive* if for all objects X and Y in C the sets $\operatorname{Mor}_C(X,Y)$ can be endowed with the structure of Abelian groups in such a way that composition of morphisms is bilinear.

1. Is there a pre-additive structure on the category Grp of groups?
2. Is there a pre-additive structure on the category GrpMod ?

Exercise sheet #5

Exercise 1.18 (Geometric dimension and torsion)**.**

1. Let G be a discrete group that is not torsion-free. Show that
$$\operatorname{gdim} G = \infty = \operatorname{cd} G.$$
(I.e., there is no finite dimensional model of BG and there is no projective resolution of finite length of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.)
2. Does there exist a torsion-free group with infinite geometric/cohomological dimension?

Exercise 1.19 ($\mathbb{Z}/p$-actions on $\mathbb{Z}/p^n$)**.** Let $p \in \mathbb{N}$ prim, let $n \in \mathbb{N}_{>1}$, and let $a \in \mathbb{Z}$ with $a^p \equiv 1 \bmod p^n$. Prove the following:

1. If $p \neq 2$, then $a \equiv 1 \bmod p^{n-1}$.
2. If $p = 2$, then $a \equiv \pm 1 \bmod 2^{n-1}$.

Exercise 1.20 (Generalised quaternion groups)**.** For $m \in \mathbb{N}_{\geq 2}$ let Q_m denote the *m-th generalised quaternion group*, i.e., Q_m is the subgroup of the quaternions $\mathbb{H}$ generated by the elements $e^{\pi i/m}$ and j. Determine $H_{2009}(Q_m; \mathbb{Z})$, where Q_m acts trivially on $\mathbb{Z}$.

Exercise 1.21 (Homology of $\mathrm{SL}(2,\mathbb{Z})$)**.**

1. Let m, $n \in \mathbb{Z}$, and let $\varphi \colon \mathbb{Z}/m \longrightarrow \mathbb{Z}/n$ be a group homomorphism. Compute the induced homomorphism
$$H_*(\varphi; \mathrm{id}_\mathbb{Z}) \colon H_*(\mathbb{Z}/m; \mathbb{Z}) \longrightarrow H_*(\mathbb{Z}/n; \mathbb{Z})$$
(where the groups in question act trivially on the coefficients $\mathbb{Z}$).
2. Compute $H_*(\mathrm{SL}(2,\mathbb{Z}); \mathbb{Z})$, where $\mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathbb{Z}$.
Hints. Use the fact that $\mathrm{SL}(2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$, where the amalgamated free product is taken with respect to the homomorphisms $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/4$ and $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/6$ given by multiplication by 2 and 3 respectively.

Exercise sheet #6

Choose one of the following topics (or any other topic related to group (co)homology), and answer the questions (partially) with help of the literature – try to find suitable literature on http://www.ams.org/mathscinet, http://books.google.com, http://scholar.google.com, ...; further hints are given below. You do not have to understand these works in detail – it suffices to understand the important ideas and concepts and the overall structure.

Write a summary of you results (1–4 pages), preferably in such a way that you are able to give a short talk in class.

Exercise 1.22 (More homological algebra).

1. What are Abelian and derived categories?
2. What are derived functors in this context?

Exercise 1.23 (Groups with small homology).

1. Are the non-trivial discrete groups G with $H_*(G;\mathbb{Z}) \cong H_*(1;\mathbb{Z})$?
2. What consequences does this have for the realisability of given (co)homology groups as group (co)homology?

Exercise 1.24 (Geometric meaning of group cohomology in degree 1). Let G be a discrete group.

1. What is the geometric meaning of $H^1(G;\mathbb{Z}G)$?
2. What can be said about the shape of $H^1(G;\mathbb{Z}G)$?

Exercise sheet #7

Exercise 1.25 (Transfer and chains of subgroups)**.** Let G be a group and let $H \subset G$ and $K \subset G$ be subgroups of finite index with $K \subset H$. Show that

$$\begin{aligned} \operatorname{res}_K^H \circ \operatorname{res}_H^G &= \operatorname{res}_K^G, \\ \operatorname{cor}_H^G \circ \operatorname{cor}_K^H &= \operatorname{cor}_K^G \end{aligned}$$

holds in (co)homology with arbitrary $\mathbb{Z}G$-coefficients.

Exercise 1.26 ((Co)Homology of S_3)**.** Compute the (co)homology of S_3 with $\mathbb{Z}$-coefficients (with trivial S_3-action) with help of the primary decomposition.

Exercise 1.27 (Classical transfer)**.**

1. Prove (e.g., using the topological description of transfer) that homological transfer with $\mathbb{Z}$-coefficients coincides with the classical transfer from group theory. The group theoretical transfer for a subgroup $H \subset G$ of finite index is given by

$$\begin{aligned} G_{\mathrm{ab}} &\longrightarrow H_{\mathrm{ab}} \\ [g] &\longmapsto \Biggl[\prod_{k=1}^{[G:H]} g_k \cdot g \cdot R(g_k \cdot g)^{-1}\Biggr], \end{aligned}$$

   where $\{g_1, \dots, g_{[G:H]}\} \subset G$ is a system of representatives of $H \setminus G$ and $R \colon G \longrightarrow \{g_1, \dots, g_{[G:H]}\}$ is the map associating with every group element $g \in G$ the representative of the corresponding coset $H \cdot g$.
2. Let $p \in \mathbb{N}_{>2}$ be prime. Show that the transfer corresponding to the (multiplicative) subgroup $\{-1, 1\}$ in $\mathbb{Z}/p^\times$ coincides with the Legendre symbol with respect to p.
   Hints. Use the Gauß lemma for quadratic residues.

Exercise 1.28 (Transfer is invisible in GrpMod and GrpMod$^-$)**.** Show that the transfer maps in group (co)homology in general are not induced by morphisms in GrpMod or GrpMod$^-$.

Exercise sheet #8

Exercise 1.29 (Cohomology ring of S_3)**.** Calculate the cohomology ring of the symmetric group S_3.

Exercise 1.30 (Cohomology ring of D_∞)**.** Calculate the cohomology ring of the infinite dihedral group $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

Exercise 1.31 (Comparison of the product structure with the topological cup-product)**.** Let G be a group and let X_G be a model of BG. Show that the canonical isomorphism $H^*(G;\mathbb{Z}) \cong H^*(X_G;\mathbb{Z})$ is compatible with the cup-product in group cohomology and the cup-product on singular (or cellular) cohomology.

Exercise 1.32 (Algebraic properties of the cup-product)**.** Show that the cup-product in group cohomology has the following properties: Let G be a discrete group.

1. *Multiplicative unit.* Show that the class $1 \in \mathbb{Z} \cong H^0(G;\mathbb{Z})$ is a unit for the cup-product, i.e., show that
$$1 \cup \alpha = \alpha = \alpha \cup 1$$
holds for all $\mathbb{Z}G$-modules A and all classes $\alpha \in H^*(G;A)$, where we use the canonical identifications $\mathbb{Z} \otimes_\mathbb{Z} A \cong A \cong A \otimes_\mathbb{Z} \mathbb{Z}$.
2. *Graded commutativity.* Let A and A' be two $\mathbb{Z}G$-modules. Prove that
$$\alpha' \cup \alpha = (-1)^{|\alpha|\cdot|\alpha'|} \cdot H^*(\mathrm{id}_G;t)(\alpha \cup \alpha')$$
holds for all cohomology classes $\alpha \in H^*(G;A)$ and $\alpha' \in H^*(G;A')$, where $t\colon A \otimes_\mathbb{Z} A' \longrightarrow A' \otimes_\mathbb{Z} A$ is the $\mathbb{Z}G$-homomorphism interchanging the two factors.

Exercise sheet #9

Exercise 1.33 (Groups with periodic cohomology)**.**

1. Describe the group (co)homology of groups having periodic cohomology with period 4 in as simple terms as possible.
2. Compute the group (co)homology of gerenalised quaternion groups.
3. Deduce the following: If G is a group with periodic cohomology and if $k \in \mathbb{Z}$ is odd, then $\widehat{H}^k(G; \mathbb{Z}) = 0$.

Exercise 1.34 (Finite groups acting freely on spheres)**.** Let G be a finite group that acts freely on a sphere. Prove that G has periodic cohomology without using the duality theorem for Tate cohomology.
Hints. A strategy of proof is outlined in Brown's book [4, Exercise V.3.3].

Exercise 1.35 (Some projective resolutions)**.** Let G be a finite group.

1. Show that if $0 \to \mathbb{Z} \xrightarrow{\eta} P_0 \to P_1 \to \cdots$ is an exact $\mathbb{Z}G$-chain complex with projective modules $(P_n)_{n \in \mathbb{N}}$, then $\eta \square P_*$ is a strong relatively injective $\mathbb{Z}G$-resolution of $\mathbb{Z}$.
2. Prove the following: If P is a finitely generated projective $\mathbb{Z}G$-module, then also $\mathrm{moH}_G(P, \mathbb{Z}G)$ is a finitely generated projective $\mathbb{Z}G$-module and for all $\mathbb{Z}G$-modules A there is a natural isomorphism

$$\mathrm{Hom}_G\bigl(\mathrm{moH}_G(P, \mathbb{Z}G), A\bigr) \cong P \otimes_G A$$

of Abelian groups. (This was used when comparing Tate cohomology in negative degrees with ordinary group homology.)
Hints. For a $\mathbb{Z}G$-module P we denote by $\mathrm{moH}_G(P, \mathbb{Z}G)$ the $\mathbb{Z}G$-module whose underlying Abelian group is $\mathrm{Hom}_G(P, \mathbb{Z}G)$ endowed with the following G-action:

$$\begin{aligned} G \times \mathrm{moH}_G(P, \mathbb{Z}G) &\longrightarrow \mathrm{moH}_G(P, \mathbb{Z}G) \\ (g, f) &\longmapsto \bigl(x \mapsto f(x) \cdot g^{-1}\bigr). \end{aligned}$$

Exercise 1.36 (Periodic cohomology with large period)**.** Are there groups with periodic cohomology of arbitrarily large period?

Exercise sheet #10

The *three-dimensional Heisenberg group* is the group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,\ y,\ z \in \mathbb{Z} \right\} \subset \mathrm{SL}(3, \mathbb{Z}).$$

Exercise 1.37 (Algebraic properties of the Heisenberg group)**.**

1. Show that the Heisenberg group fits into a central extension of the type

$$1 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 1.$$

2. Compute the Abelianisation H_{ab} of the Heisenberg group.

Exercise 1.38 (Homology of the Heisenberg group)**.** Calculate the homology $H_*(H; \mathbb{Z})$ of the Heisenberg group with trivial coefficients via the Hochschild-Serre spectral sequence.

Exercise 1.39 (Cohomological dimension of the Heisenberg group)**.**

1. Let G be a group. Show that the cohomological dimension (see exercise sheet #3) can also be described via

$$\operatorname{cd} G = \inf\bigl\{n \in \mathbb{N} \bigm| \text{for all } j \in \mathbb{N}_{>n} \text{ we have } H^j(G; \,\cdot\,) = 0\bigr\}.$$

2. Using the Hochschild-Serre spectral sequence prove that

$$\operatorname{cd} G \leq \operatorname{cd} Q + \operatorname{cd} N$$

   holds for all group extensions $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$.
3. Conclude that the Heisenberg group has cohomological dimension 3.

Exercise 1.40 (A model of the classifying space for the Heisenberg group)**.**

1. Show that there exists an oriented closed connected three-dimensional manifold that is a model of the classifying space of the Heisenberg group H.
2. Use this model and Poincaré duality to compute the homology of H with $\mathbb{Z}$-coefficients.

2 Bounded cohomology

2.1 Introduction

There are many examples of successfully enriching algebraic structures and algebraic invariants related to group theory with metric data. One instance of this paradigm is bounded cohomology.

What is bounded cohomology? Bounded cohomology is a functional analytic variant of group cohomology turning groups and Banach modules over groups into graded semi-normed vector spaces. I.e., on objects bounded cohomology looks like

$$H^n_{\mathrm{b}}(G; V),$$

where

- the number $n \in \mathbb{N}$ is the degree in the grading of the graded semi-normed vector space $H^*_{\mathrm{b}}(G; V)$,
- the first parameter is a (discrete) group G,
- and the second parameter is a Banach G-module V, the so-called coefficients.

The homological sibling of bounded cohomology is so-called ℓ^1-homology.

How can we construct bounded cohomology? The main theme in constructing bounded cohomology is to pass to a functional analytic setting by replacing the ring $\mathbb{Z}G$ by $\ell^1(G)$ and by replacing ordinary cocycles by bounded cocycles; there are three main (equivalent) descriptions of bounded cohomology:

- Topologically (via classifying spaces)
- Combinatoriallly (via the Banach bar resolution)
- Algebraically (via relative homological algebra).

Why is bounded cohomology interesting? First, bounded cohomology is a fascinating theory in its own right linking topology, group theory, functional analysis and measure theory; while it is similar to group cohomology and singular cohomology in several respects, there are also striking differences – e.g., the mapping theorem for bounded cohomology of spaces (Theorem 2.8.2).

Second, bounded cohomology helps to solve the following problems:

- Which mapping degrees can occur for maps between two given manifolds? (Section 2.9)
- Can we measure the size of certain characteristic classes? [19, 5]
- Which groups can act interestingly on the circle? [18]
- Does a given group admit quasi-morphisms to $\mathbb{R}$ that are not a perturbation of an actual group homomorphism? (A *quasi-morphism* from a discrete group G to $\mathbb{R}$ is a map $G \longrightarrow \mathbb{R}$ that satisfies multiplicativity up to a uniformly bounded error.) (Section 2.5.4)

Moreover, there are many applications of bounded cohomology to rigidity theory; unfortunately these are beyond the scope of these lectures [8, 38].

Overview

In the *second part* of the semester we will study the following topics:

- Understand and compare the three basic descriptions of bounded cohomology
- Application of bounded cohomology to the study of quasi-morphisms (Section 2.5.4)
- Relation of bounded cohomology and amenability; amenable groups will play a rôle similar to the one of finite groups in ordinary group cohomology (Section 2.6)
- The mapping theorem in bounded cohomology (Section 2.8)
- Applications to the simplicial volume (Section 2.9)

For simplicity, we will only treat the case of bounded cohomology of discrete groups; an extensive treatment of the general case is provided in Monod's book [37].

2.2 The domain category for bounded cohomology

The basic algebraic objects in the world of bounded cohomology are Banach modules with isometric group actions and the ℓ^1-completion of the real group ring.

Definition 2.2.1 (Banach G-modules)**.** Let G be a (discrete) group.
- A *(left) normed G-module* is a normed $\mathbb{R}$-vector space V together with an isometric G-action $G \times V \longrightarrow V$. (Analogously, right normed G-modules are defined.)
- A *Banach G-module* is a Banach space together with an isometric G-action.
- A *morphism* of normed G-modules is a G-equivariant bounded linear map between normed G-modules.

Recall that a *Banach space* is nothing but a normed $\mathbb{R}$-vector space that is complete with respect to the given norm; a linear map $\Phi\colon V \longrightarrow W$ between normed vector spaces is *bounded* if

$$\|\Phi\| := \sup_{x \in V \setminus \{0\}} \frac{\|\Phi(x)\|}{\|x\|}$$

is finite.

A fundamental example of a Banach G-module is the ℓ^1-completion of the real group ring:

Definition 2.2.2 (ℓ^1-group algebra)**.** Let G be a discrete group. The *ℓ^1-group algebra* $\ell^1(G)$ is the completion of the real group ring $\mathbb{R}G := \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}G$ with respect to the *ℓ^1-norm*

$$\begin{aligned} \|\cdot\|_1 \colon \mathbb{R}G &\longrightarrow \mathbb{R} \\ \sum_{g \in G} a_g \cdot g &\longmapsto \sum_{g \in G} |a_g|. \end{aligned}$$

I.e., the underlying Banach space of $\ell^1(G)$ is the set of all ℓ^1-summable series over G with coefficients in $\mathbb{R}$, and the multiplication is given by the usual multiplication of series.

Notice that every Banach G-module is a module over $\ell^1(G)$ (because the action is isometric).

Example 2.2.3 (ℓ^1-group algebras)**.**
- If G is a finite group, then $\ell^1(G) = \mathbb{R}G$.
- For the group $\mathbb{Z}$ of the integers, $\ell^1(\mathbb{Z})$ is the space of absolutely convergent formal power series in one variable.

Moreover, Banach G-modules arise naturally in the ℓ^1-completion of the bar resolution complex or in the ℓ^1-completion of the singular chain complex of the universal covering of a space with fundamental group G.

As indicated in the introduction, the domain categories for bounded cohomology and ℓ^1-homology incorporate both a group parameter and a module parameter, the coefficients. The exact definition is just a straightforward translation of the the definition of the domain categories GrpMod and GrpMod⁻ for ordinary group (co)homology into our Banach setting:

Definition 2.2.4 (GrpBan, GrpBan⁻)**.** The categories GrpBan, GrpBan⁻ are defined as follows:

1. GrpBan : The objects of the category GrpBan are pairs (G, V), where G is a discrete group and V is a (left) Banach G-module.
   The set of morphisms in GrpBan between two objects (G, V) and (H, W) is the set of paris (φ, Φ), where
   - $\varphi\colon G \longrightarrow H$ is a group homomorphism, and
   - $\Phi\colon V \longrightarrow \varphi^* W$ is a morphism of Banach G-modules; here, $\varphi^* W$ is the Banach G-module whose underlying Banach space is W and whose (isometric) G-action is given by

$$\begin{aligned} G \times W &\longrightarrow W \\ (g, w) &\longmapsto \varphi(g) \cdot w. \end{aligned}$$

   The composition of morphisms is defined by composing both components (notice that this is well-defined in the second component).

2. GrpBan^- : The category GrpBan^- has the same objects as the category GrpBan, i.e., pairs of groups and Banach modules over this group.
   The set of morphisms in GrpBan^- between two objects (G, V) and (H, W) is the set of pairs (φ, Φ), where
   - $\varphi\colon G \longrightarrow H$ is a group homomorphism, and
   - $\Phi\colon \varphi^* W \longrightarrow V$ is a morphism of Banach G-modules.

   The composition of morphisms is defined by covariant composition in the first component and contravariant composition in the second component.

As in the case of ordinary group (co)homology, we need viable notions of homomorphism spaces, tensor products, invariants and coinvariants.

Definition 2.2.5 (Bounded operators, projective tensor products)**.** Let U and V be two normed $\mathbb{R}$-vector spaces.

- We write $B(U, V)$ for the normed $\mathbb{R}$ vector space of all bounded linear operators of type $U \longrightarrow V$ with respect to the operator norm. (If V is a Banach space, then so is $B(U, V)$).
- We call $U^{\#} := B(U, \mathbb{R})$ the *dual space* of U; the operator norm on $U^{\#}$ is denoted by $\|\cdot\|_\infty$.
- We write $U \overline{\otimes} V$ for the *projective tensor product* of U and V; the projective tensor product is the completion of the tensor product $U \otimes_{\mathbb{R}} V$ of $\mathbb{R}$-vector spaces with respect to the *projective norm* given by

$$\|x\| := \inf\biggl\{\sum_j \|u_j\| \cdot \|v_j\| \biggm| \sum_j u_j \otimes v_j \text{ represents } x \in U \otimes_{\mathbb{R}} V\biggr\}$$

  for all $x \in U \otimes_{\mathbb{R}} V$.

The norm on the tensor product described above is the biggest sensible norm on the tensor product; the name projective tensor product derives from the fact that the projective tensor product is well-behaved with respect to quotient maps of Banach spaces [46, Proposition 2.5].

Remark 2.2.6 (Adjunction properties)**.** The operations B and $\overline{\otimes}$ are adjoint on the category of Banach spaces in the following sense: For all

Banach spaces U, V and W there is a natural isometric isomorphism

$$\begin{aligned} B(U \mathbin{\overline{\otimes}} V, W) &\longleftrightarrow B\bigl(U, B(V,W)\bigr) \\ f &\longmapsto \bigl(u \mapsto (v \mapsto f(u \otimes v))\bigr) \\ \bigl(u \otimes v \mapsto f(u)(v)\bigr) &\longleftarrow f \end{aligned}$$

of Banach spaces.

In a way, bounded cohomology and ℓ^1-homology can be viewed as being "derived functors" of the invariants and coinvariants functors in this Banach context:

Definition 2.2.7 ((Co)invariants of Banach G-modules)**.** Let G be a discrete group, and let V be a Banach G-module.

– The *invariants* of V are the Banach space

$$V^G := \{x \in V \mid \forall_{g \in G} \;\; g \cdot x = x\}.$$

– The *coinvariants* of V are given by

$$V_G := V/\overline{W},$$

where $W \subset V$ is the subspace generated by $\{g \cdot v - v \mid g \in G,\; v \in V\}$. *Caveat.* The subspace W is *not* closed in V in general; therefore, the algebraically defined coinvariants V/W in general do not form a Banach space with respect to the norm on V. On the other hand, the norm on V turns $V_G = V/\overline{W}$ into a Banach space.

Convention 2.2.8 (Morphism spaces and projective tensor products of Banach G-modules)**.** Let G be a discrete group. We follow the convention that (if not explicitly stated otherwise) all Banach G-modules are left Banach G-modules; taking inverses in G leads to an involution on $\ell^1(G)$ that enables us to convert left Banach G-modules into right Banach G-modules and vice versa.

More explicitly, we use the following conventions for tensor products and morphism spaces: Let U and V be two left Banach G-modules.

– Then $B(U,V)$ is a Banach G-module with respect to the diagonal action

$$\begin{aligned} G \times B(U,V) &\longrightarrow B(U,V) \\ (g,f) &\longmapsto \bigl(u \mapsto g \cdot f(g^{-1} \cdot u)\bigr), \end{aligned}$$

and we write $B_G(U,V) := \big(B(U,V)\big)^G$.
- Moreover, $U \mathbin{\overline{\otimes}} V$ is a Banach G-module with respect to the diagonal action (given by the extension to the completion of)

$$\begin{aligned} G \times U \mathbin{\overline{\otimes}} V &\longrightarrow U \mathbin{\overline{\otimes}} V \\ (g, u \otimes v) &\longmapsto (g \cdot u) \otimes (g \cdot v), \end{aligned}$$

and we write $U \mathbin{\overline{\otimes}}_G V := (U \mathbin{\overline{\otimes}} V)_G$.

Example 2.2.9. If G is a discrete group and V is a Banach G-module, then by construction (where G acts trivially on $\mathbb{R}$)

$$V^G = B_G(\mathbb{R}, V) \qquad \text{and} \qquad V_G = V \mathbin{\overline{\otimes}}_G \mathbb{R}.$$

Example 2.2.10 (Invariants and coinvariants as functors)**.** It is not difficult to see that we can extend the definition of invariants to a (contravariant) functor $\mathrm{GrpBan}^- \longrightarrow \mathrm{Ban}$ and that we can extend the definition of coinvariants to a functor $\mathrm{GrpBan} \longrightarrow \mathrm{Ban}$.

Here, Ban denotes the category of Banach spaces (with bounded operators as morphisms).

Clearly, the adjunction in Remark 2.2.6 is G-equivariant.

2.3 Homology of normed chain complexes

In the following, we introduce the basic homological framework – normed chain complexes and their homology. A normed chain complex is a chain complex equipped with a norm such that the boundary operators all are bounded operators. In particular, the homology of a normed chain complex inherits a semi-norm. For example, in the case of the singular chain complex equipped with the ℓ^1-norm this semi-norm contains valuable geometric information such as the simplicial volume.

In order to understand this semi-norm in homology it suffices to understand the semi-norm in homology of the corresponding completed chain complex or the dual cochain complex; in the case of singular homology, this corresponds to the investigation of ℓ^1-homology and bounded cohomology respectively.

Also in the case of bounded cohomology of groups, completions and dual cochain complexes lie at the heart of the constructions; there we start out with the $\mathbb{R}$-valued bar complex.

After introducing the basic definitions for normed chain complexes, we discuss the fundamental properties of semi-norms in homology. The last section is concerned with the equivariant setting.

The discussion of the main examples of these concepts follows in the subsequent sections.

2.3.1 Normed chain complexes

In the following, we use the convention that Banach spaces are Banach spaces over $\mathbb{R}$ and that all (co)chain complexes are indexed over the set $\mathbb{N}$ of natural numbers.

Definition 2.3.1 (Normed chain complexes).

- A *normed chain complex* is a chain complex of normed vector spaces all of whose boundary morphisms are bounded linear operators. Analogously, *normed cochain complexes* are defined.
- A *Banach (co)chain complex* is a normed (co)chain complex consisting of Banach spaces.
- A *morphism of normed (co)chain complexes* is a (co)chain map between normed (co)chain complexes consisting of bounded linear operators.

In our context, the fundamental examples of the concept of normed chain complexes are given by the singular chain complex and the bar resolution equipped with the obvious ℓ^1-norms:

Example 2.3.2 (ℓ^1-Norms)**.** Both the singular chain complex and the $\mathbb{R}$-valued bar complex carry natural ℓ^1-norms:

- Let X be a topological space. We define the *ℓ^1-norm* $\|\cdot\|_1$ on the singular chain complex $C_*(X;\mathbb{R})$ as the ℓ^1-norm with respect to the basis given by the set of singular simplices. In other words: For $n \in \mathbb{N}$ let (where the chains are supposed to be in *reduced form*, that is no singular simplex occurs more than once in the sum representation)

$$\begin{aligned} C_n(X;\mathbb{R}) &\longrightarrow \mathbb{R} \\ \sum_{j=0}^{k} a_j \cdot \sigma_j &\longmapsto \sum_{j=0}^{k} |a_j|. \end{aligned}$$

  This norm turns the singular chain complex into a normed chain complex: the boundary operator in degree n has operator norm at most $n+1$, as follows easily from the definitions.
  If $f\colon X \longrightarrow Y$ is a continuous map, then the corresponding chain map $C_*(f;\mathrm{id}_\mathbb{R})\colon C_*(X;\mathbb{R}) \longrightarrow C_*(Y;\mathbb{R})$ consists of bounded linear maps (of operator norm at most 1), and so is a morphism of normed chain complexes.
- Let G be a discrete group. Then the $\mathbb{R}$-*valued bar complex*

$$C^{\mathbb{R}}_*(G) := C_*(G) \otimes_{\mathbb{Z}} \mathbb{R}$$

  is a normed chain complex with respect to the *ℓ^1-norm* $\|\cdot\|_1$ defined by

$$\begin{aligned} C_n^{\mathbb{R}}(G) &\longrightarrow \mathbb{R} \\ \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1|\cdots|g_n] &\longmapsto \sum_{g \in G^{n+1}} |a_g|; \end{aligned}$$

again, the definition of the boundary operator in degree n shows that it is a bounded operator of operator norm at most $n+1$.
If $\varphi\colon G \longrightarrow H$ is a group homomorphism, then the corresponding chain map $C_*^{\mathbb{R}}(\varphi)\colon C_*^{\mathbb{R}}(G) \longrightarrow C_*^{\mathbb{R}}(H)$ consists of bounded linear operators of operator norm at most 1; therefore, $C_*^{\mathbb{R}}(\varphi)$ is a morphism of normed chain complexes.

Definition 2.3.3 (Dual cochain complex)**.** Let (C_*, ∂_*) be a normed chain complex, and let V be a Banach space. Then the *dual cochain complex* $((C^\#)^*, (\partial^\#)^*)$ is the Banach cochain complex defined by

$$(C^\#)^n := (C_n)^\#$$

for all $n \in \mathbb{N}$, together with the coboundary operators

$$\begin{aligned} (\partial^\#)^n \colon (C^\#)^n &\longrightarrow (C^\#)^{n+1} \\ f &\longmapsto (-1)^{n+1} \cdot \bigl(c \mapsto f(\partial_{n+1}(c))\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$; recall that $\cdot^\#$ stands for the (topological) dual normed vector space and that dual vector spaces are complete.

Definition 2.3.4 (Completions of normed chain complexes)**.** Let (C_*, ∂_*) be a normed chain complex. Then the boundary operators ∂_n can be extended to boundary operators $\overline{\partial}_n\colon \overline{C}_n \longrightarrow \overline{C}_{n-1}$ of the completions, which are bounded operators as well and which satsify $\overline{\partial}_{n+1} \circ \overline{\partial}_n = 0$. The Banach chain complex $(\overline{C}_*, \overline{\partial}_*)$ is the *completion* of (C_*, ∂_*).
(Similarly, completions of normed cochain complexes are defined.)

For example, completing the bar complex or the singular chain complex with respect to the ℓ^1-semi-norm leads to ℓ^1-homology.

Clearly, for all normed chain complexes C_* we have $(C^\#)^* = (\overline{C}^\#)^*$.

2.3.2 Semi-norms in homology

The presence of chain complexes calls for the investigation of the corresponding homology. In the case of normed chain complexes, the homology groups carry an additional piece of information – the semi-norm.

Definition 2.3.5 (Semi-norm on homology)**.**

- Let (C_*, ∂_*) be a normed chain complex and let $n \in \mathbb{N}$. The *n-th homology* of C_* is the quotient
$$H_n(C_*) := \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$
- Let (C^*, δ^*) be a normed cochain complex and let $n \in \mathbb{N}$. The *n-th cohomology* of C^* is the quotient
$$H^n(C^*) := \frac{\ker(\delta^n \colon C^n \to C^{n-1})}{\operatorname{im}(\delta^{n-1} \colon C^{n-1} \to C^n)}.$$
- Let (C_*, ∂_*) be a normed chain complex. The norm $\|\cdot\|$ on C_* induces a semi-norm, also denoted by $\|\cdot\|$, on the homology $H_*(C_*)$ as follows: If $\alpha \in H_n(C_*)$, then
$$\|\alpha\| := \inf\bigl\{\|c\| \bigm| c \in C_n,\ \partial(c) = 0,\ [c] = \alpha\bigr\}.$$
Similarly, we define a semi-norm on the cohomology of normed cochain complexes.

Caveat 2.3.6. In general, the semi-norm on the homology of a normed cochain complex is *not* a norm because the images of the boundary operators are not necessarily closed (even if the complex in question is a Banach chain complex). Therefore, it is sometimes convenient to look at the reduced (co)homology (defined as the quotient of the kernel by the *closure* of the image).

Example 2.3.7 (ℓ^1-Semi-norm in homology)**.** For example, the ℓ^1-norms from Example 2.3.2 induce ℓ^1-semi-norms on singular homology with real coefficients and on group homology with real coefficients.

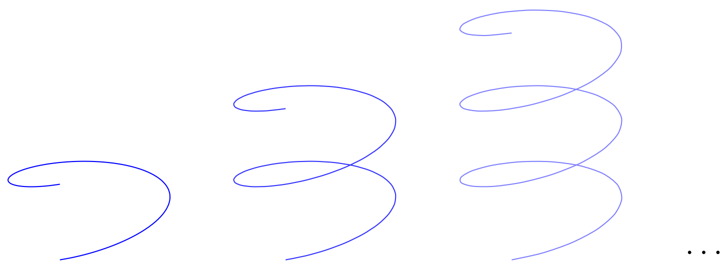


Figure 2.1: Simplices wrapping around the circle

An example of an interesting invariant for oriented closed connected manifolds emerging from the ℓ^1-norm in singular homology is the simplicial volume, which is nothing but the ℓ^1-semi-norm of the fundamental class (see Section 2.9).

Definition 2.3.8 (Simplicial volume)**.** Let M be an oriented closed connected manifold of dimension n. The *simplicial volume* of M is defined by

$$\|M\| := \|[M]\|_1 = \inf\bigl\{\|c\|_1 \bigm| c \in C_n(M;\mathbb{R}) \text{ is a fundamental cycle of } M\bigr\}$$

where $[M] \in H_n(M;\mathbb{R})$ is the fundamental class of M with real coefficients.

Example 2.3.9 (Simplicial volume of the circle)**.** The simplicial volume of the circle is zero: For $d \in \mathbb{N}_{>0}$ let $\sigma_d \colon [0,1] \longrightarrow S^1$ be the singular 1-simplex given by wrapping the unit interval d times around S^1 (Figure 2.1). A straightforward calculation shows that $1/d \cdot \sigma_d$ is an $\mathbb{R}$-fundamental cycle of S^1; hence,

$$0 \leq \|S^1\| \leq \inf_{d \in \mathbb{N}_{>0}} \Bigl\|\frac{1}{d} \cdot \sigma_d\Bigr\|_1 = \inf_{d \in \mathbb{N}_{>0}} \frac{1}{d} = 0,$$

as claimed.

We will discuss the simplicial volume and its fascinating relationship with Riemannian geometry and bounded cohomology in more detail in Section 2.9.

How can we compute such semi-norms in homology? It turns out that the semi-norm in homology of a normed chain complex can be described in terms of the homology of the completion or the dual of the chain complex in question. This means, for example, that the simplicial volume can be described in terms of ℓ^1-homology and bounded cohomology; moreover, these theories seem to be more appropriate for the study of simplicial volume than singular (co)homology.

Proposition 2.3.10 (Semi-norm on homology via completions [47, Lemma 2.9])**.**

1. *Let D_* be a normed chain complex and let C_* be a dense subcomplex. Then the induced map $H_*(C_*) \longrightarrow H_*(D_*)$ is isometric with respect to the induced semi-norms in homology.*
2. *If C_* is a normed chain complex, then the map $H_*(C_*) \longrightarrow H_*(\overline{C}_*)$ induced by the inclusion $C_* \subset \overline{C}_*$ into the completion is isometric. In particular: If $k \in \mathbb{N}$ satisfies $H_k(\overline{C}_*) = 0$, then the induced semi-norm on $H_k(C_*)$ is zero.*

Proof. The second part is a special case of the first part. So it suffices to prove the first part: The idea of the proof is to approximate boundaries in D_* by boundaries in C_*. In the following, we write $i\colon C_* \hookrightarrow D_*$ for the inclusion and $\|\cdot\|$ for the norm on D_*.

Because C_* is a subcomplex, $\|H_*(i)\| \leq 1$. Conversely, let $z \in C_n$ be a cycle and let $\overline{z} \in D_n$ be a cycle with $[\overline{z}] = H_n(i)(z) \in H_n(D_*)$; furthermore, let $\varepsilon \in \mathbb{R}_{>0}$. To prove the first part, it suffices to find a cycle $z' \in C_n$ satisfying

$$[z'] = [z] \in H_n(C_*) \qquad \text{and} \qquad \|z'\| \leq \|\overline{z}\| + \varepsilon.$$

By definition of $\overline{z}$, there must be a chain $\overline{w} \in D_{n+1}$ with $\partial_{n+1}(\overline{w}) = i(z) - \overline{z}$. Because C_{n+1} is dense in D_{n+1} and because $\|\partial_{n+1}\|$ is finite, there is a chain $w \in C_{n+1}$ such that

$$\big\|\overline{w} - i(w)\big\| \leq \frac{\varepsilon}{\|\partial_{n+1}\|}.$$

Then $z' := z + \partial_{n+1}(w) \in C_n$ is a cycle with $[z'] = [z]$ in $H_n(C_*)$, and

$$\big\|\overline{z} - i(z')\big\| = \big\|\partial_{n+1}(\overline{w} - i(w))\big\| \leq \varepsilon.$$

In particular, $\|z'\| \leq \|\overline{z}\| + \varepsilon$. Hence, $H_n(i)$ is an isometry. □

Caveat 2.3.11. The previous proposition is surprising in the respect that usually the processes of completing and taking homology do not harmonise (Exercise).

Definition 2.3.12 (Kronecker products)**.** Let C_* be a normed chain complex. The evaluation maps $\langle\,\cdot\,,\,\cdot\,\rangle\colon (C^\#)^n \times C_n \longrightarrow \mathbb{R}$ induce a well-defined $\mathbb{R}$-linear map

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon H^*((C^\#)^*) \otimes_\mathbb{R} H_*(C_*) \longrightarrow \mathbb{R},$$

the so-called *Kronecker product.*

Proposition 2.3.13 (Duality principle for semi-norms [19, 2, p. 17, Proposition F.2.2])**.** *Let C_* be a normed chain complex and let $n \in \mathbb{N}$. Then*

$$\|\alpha\| = \sup\Bigl\{\frac{1}{\|\varphi\|_\infty}\Bigm|\varphi \in H^n((C^\#_*)^*) \text{ and } \langle\varphi,\alpha\rangle = 1\Bigr\}$$

holds for all $\alpha \in H_n(C_*)$; here, $\sup\emptyset := 0$.

In particular: If $k \in \mathbb{N}$ satisfies $H^k((C^\#)^*) = 0$, then the induced semi-norm on $H_k(C_*)$ is zero.

Proof. If $\alpha \in H_n(C_*)$ and $\varphi \in H^n((C^\#)^*)$, then

$$\bigl|\langle\varphi,\alpha\rangle\bigr| \leq \|\alpha\| \cdot \|\varphi\|_\infty.$$

This shows that $\|\alpha\|$ is at least as large as the supremum. Now suppose that $\|\alpha\| \neq 0$; in particular, if $c \in C_n$ is a cycle representing α, then $c \notin \overline{\operatorname{im}\partial_{n+1}}$. Thus, by the Hahn-Banach theorem there exists a linear functional $f\colon C_n \longrightarrow \mathbb{R}$ satisfying

$$f|_{\operatorname{im}\partial_{n+1}} = 0, \qquad f(c) = 1, \qquad \|f\|_\infty \leq 1/\|\alpha\|.$$

So $f \in (C^\#)^n$ is a cocycle; let $\varphi := [f] \in H^n((C^\#)^*)$ be the associated cohomology class. By construction, $\langle\varphi,\alpha\rangle = 1$ and $\|\varphi\|_\infty \leq \|f\|_\infty \leq 1/\|\alpha\|$. Hence, $\|\alpha\|$ is at most as large as the supremum. □

Caveat 2.3.14. In general it is *not* possible to compute the semi-norm on the cohomology of the dual cochain complex in terms of the semi-norm on the homology [27, p. 38].

Caveat 2.3.15. There is no analogue of the universal coefficient theorem for topological duals of Banach chain complexes [27, Remark 3.4].

Furthermore, the induced semi-norm in homology is compatible with the mechanism of producing long exact sequence via the snake lemma:

Proposition 2.3.16 (Snake lemma)**.** *Let* $0 \longrightarrow C_* \xrightarrow{i} D_* \xrightarrow{p} E_* \longrightarrow 0$ *be a short exact sequence of Banach chain complexes. Then there is a natural long exact sequence*

$$\cdots \longrightarrow H_n(C_*) \xrightarrow{H_n(i)} H_n(D_*) \xrightarrow{H_n(p)} H_n(E_*) \xrightarrow{\partial_n} H_{n-1}(C_*) \longrightarrow \cdots$$

in homology, and the connecting homomorphism ∂_* *is continuous (with respect to the induced semi-norms in homology).*

In the same way, short exact sequences of Banach cochain complexes give rise to natural long exact sequences in cohomology with continuous connecting homomorphisms.

Proof. That the mentioned sequence in homology is exact is a purely algebraic fact following from the snake lemma for $\mathbb{R}$-chain complexes.

The continuity of the connecting homomorphisms can, for example, be derived from its concrete construction [37, proof of Proposition 8.2.1]. □

2.3.3 The equivariant setting

Of course, we will also need complexes of Banach G-modules and some basic constructions on them:

Definition 2.3.17 (Banach G-chain complexes)**.** Let G be a discrete group.

- A *normed G-(co)chain complex* is a (co)chain complex consisting of normed G-modules all of whose (co)boundary operators are bounded linear G-equivariant maps.
- A *Banach G-(co)chain complex* is a normed G-(co)chain complex consisting of Banach G-modules.
- A *morphism of normed/Banach G-(co)chain complexes* is a (co)chain map of normed/Banach (co)chain complexes that consists of G-morphisms.

– Two morphisms of normed/Banach G-(co)chain complexes are *G-homotopic* if there exists a (co)chain homotopy between them consisting of G-morphisms.

Example 2.3.18 (The singular chain complex)**.** Let X be a topological space and let G be a discrete group that acts continuously on X (for example the action of the fundamental group of a pointed CW-complex on the universal covering).

– Let $n \in \mathbb{N}$. Then the G-action on X induces a G-action

$$\begin{aligned} G \times C_n(X;\mathbb{R}) &\longrightarrow C_n(X;\mathbb{R}) \\ (g,\sigma) &\longmapsto g \cdot \sigma := \bigl(t \mapsto g \cdot \sigma(t)\bigr) \end{aligned}$$

that is isometric with respect to the ℓ^1-norm. So $C_n(X;\mathbb{R})$ is a normed G-module.

– It is not difficult to see that the boundary operator of the singular chain complex $C_*(X;\mathbb{R})$ is compatible with this G-action. In Example 2.3.2 we have already seen that this boundary operator is a bounded operator.

Hence, $C_*(X;\mathbb{R})$ is a normed G-chain complex. In general, this chain complex is *not* complete; so in general $C_*(X;\mathbb{R})$ is not a Banach G-chain complex.

Example 2.3.19 (The bar complex)**.** Let G be a discrete group. Then the G-action given by the $\mathbb{R}G$-module structure on the $\mathbb{R}$-valued bar complex $C^{\mathbb{R}}_*(G)$ (see Definition 1.4.1 and Example 2.3.2) is isometric with respect to the ℓ^1-norm on $C^{\mathbb{R}}_*(G)$. Therefore, $C_*(G;\mathbb{R})$ is a normed G-chain complex. Again, in general, $C^{\mathbb{R}}_*(G)$ is not a Banach G-chain complex.

Convention 2.3.20 (Complexes of morphisms, tensor products)**.** Let G be a discrete group, let (C_*, ∂_*) be a normed G-chain complex, and let V be a Banach G-module.

– The cochain complex $B(C_*, V)$ consists of the chain modules $B(C_n, V)$ together with the operator norm and the boundary operator

$$\begin{aligned} B(\partial_n, \mathrm{id}_V)\colon B(C_n, V) &\longrightarrow B(C_{n+1}, V) \\ f &\longmapsto (-1)^{n+1}\bigl(c \mapsto f(\partial_{n+1} c)\bigr); \end{aligned}$$

as ∂_* is a bounded operator, so is $B(\partial_*, \mathrm{id}_V)$. Hence, $B(C_*, V)$ is a Banach cochain complex.
Moreover, the diagonal G-action on $B(C_*, V)$ given by

$$\begin{aligned} G \times B(C_n, V) &\longrightarrow B(C_n, V) \\ (g, f) &\longmapsto \bigl(c \mapsto g \cdot f(g^{-1} \cdot c)\bigr) \end{aligned}$$

is isometric and compatible with the coboundary operator, and therefore turns $B(C_*, V)$ into a Banach G-cochain complex.

- The chain complex $C_* \mathbin{\overline{\otimes}} V$ consists of the chain modules $C_n \mathbin{\overline{\otimes}} V$ together with the projective tensor product norm and the boundary operator $\partial_* \mathbin{\overline{\otimes}} \mathrm{id}_V$; as ∂_* is a bounded operator, so is $\partial_* \mathbin{\overline{\otimes}} \mathrm{id}_V$.
  Moreover, the diagonal G-action on $C_* \mathbin{\overline{\otimes}} V$ is isometric and compatible with the boundary operator, and so $C_* \mathbin{\overline{\otimes}} V$ is a Banach G-chain complex.

2.4 Bounded cohomology, topologically

The singular chain complex (with real coefficients) of a topological space is a normed chain complex with respect to the ℓ^1-norm. Taking the completion and the topological dual of the singular chain complex gives rise to ℓ^1-homology and bounded cohomology of spaces respectively.

$$
\begin{array}{ccc}
 & & C_{\mathrm{b}}^*(X;\mathbb{R}) \rightsquigarrow H_{\mathrm{b}}^*(X;\mathbb{R}) \\
 & \overset{\text{topological dual}}{\nearrow} & \\
C_*(X;\mathbb{R}) & & \\
 & \underset{\text{completion}}{\searrow} & \\
 & & C_*^{\ell^1}(X;\mathbb{R}) \rightsquigarrow H_*^{\ell^1}(X;\mathbb{R})
\end{array}
$$

Topologically, bounded cohomology can be defined by applying bounded cohomology with twisted coefficients to classifying spaces of groups; schematically, we can depict this as follows:

ℓ^1-Homology	GrpBan	$\longrightarrow$	Top	$\longrightarrow$	$\mathrm{Vec}_*^{\|\cdot\|}$
	(G,V)	$\longmapsto$	BG	$\longmapsto$	$H_*^{\ell^1}(BG;V)$
Bounded cohomology	GrpBan^-	$\longrightarrow$	Top	$\longrightarrow$	$\mathrm{Vec}_*^{\|\cdot\|}$
	(G,V)	$\longmapsto$	BG	$\longmapsto$	$H_{\mathrm{b}}^*(BG;V)$

Here, $\mathrm{Vec}_*^{\|\cdot\|}$ denotes the category of semi-normed graded vector spaces with bounded linear operators as morphisms.

In the following, we give the precise definitions for bounded cohomology and ℓ^1-homology of spaces with twisted coefficients and study some basic properties of these theories.

2.4.1 Bounded cohomology of spaces

The singular chain complex with real coefficients is a normed chain complex with respect to the ℓ^1-norm introduced in Example 2.3.18. More generally, the singular chain complex of the universal covering of a space is a normed equivariant chain complex (Example 2.3.18). Therefore, we can translate the definition of (co)homology with twisted coefficients in a straightforward way to our Banach setting:

Definition 2.4.1 (Bounded cohomology with twisted coefficients)**.** Let X be a pointed connected CW-complex (in the sense of Convention 1.3.5), let G be the fundamental group of X, let $\widetilde{X}$ be the universal covering of X, and let V be a (left) Banach G-module. Then $C_*(\widetilde{X};\mathbb{R})$ is a normed G-chain complex with respect to the ℓ^1-norm (Example 2.3.18).

– *Bounded cohomology with twisted coefficients.* We write

$$C^*_{\mathrm{b}}(X;V) := B_G\bigl(C_*(\widetilde{X};\mathbb{R}), V\bigr)$$

(which is a Banach cochain complex with respect to the coboundary operator introduced in Convention 2.3.20). We call

$$H^*_{\mathrm{b}}(X;V) := H^*\bigl(C^*_{\mathrm{b}}(X;V)\bigr)$$

bounded cohomology of X with twisted coefficients in V.

– *ℓ^1-Homology with twisted coefficients.* We write

$$C^{\ell^1}_*(X;V) := C_*(\widetilde{X};\mathbb{R}) \mathbin{\overline{\otimes}_G} V$$

(which is a Banach chain complex with respect to the boundary operator introduced in Convention 2.3.20). We call

$$H^{\ell^1}_*(X;V) := H_*\bigl(C^{\ell^1}_*(X;V)\bigr)$$

ℓ^1-homology of X with twisted coefficients in V.

The norms on $C^*_{\mathrm{b}}(X;V)$ and $C^{\ell^1}_*(X;V)$ induce semi-norms in bounded cohomology and ℓ^1-homology respectively; we always consider these semi-norms on $H^*_{\mathrm{b}}(X;V)$ and $H^{\ell^1}_*(X;V)$.

Let TopBan and TopBan^- be the categories defined in the same way as TopMod and TopMod^- but using equivariant Banach modules instead of modules over the group ring (Definition 1.3.7). Then we can extend bounded cohomology and ℓ^1-homology to functors

$$H_b^*\colon \mathrm{TopBan}^- \longrightarrow \mathrm{Vec}_*^{\|\cdot\|}$$
$$H_*^{\ell^1}\colon \mathrm{TopBan} \longrightarrow \mathrm{Vec}_*^{\|\cdot\|}.$$

Remark 2.4.2 (Functoriality of bounded cohomology and ℓ^1-homology with twisted coefficients)**.**

- *Bounded cohomology.* Let $(f,\Phi)\colon (X,V) \longrightarrow (Y,W)$ be a morphism in the category TopBan^-, and let $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ be the unique lift of $f\colon X \longrightarrow Y$ to the universal coverings mapping the base-point of $\widetilde{X}$ to the one of $\widetilde{Y}$; such a lift exists and is unique by covering theory. We then define
$$C_b^*(f;\Phi) := B_G\bigl(C_*(\widetilde{f};\mathrm{id}_{\mathbb{R}}),\Phi\bigr)\colon C_b^*(Y;W) \longrightarrow C_b^*(X;V),$$
which is a cochain map consisting of bounded linear operators of norm at most $\|\Phi\|$. Let
$$H_b^*(f;\Phi) := H^*\bigl(C_b^*(f;\Phi)\bigr)\colon H_b^*(Y;W) \longrightarrow H_b^*(X;V);$$
clearly, this map has operator norm at most $\|\Phi\|$ with respect to the induced semi-norms on bounded cohomology.
It is not difficult to see that this definition turns bounded cohomology into a contravariant functor $\mathrm{TopBan}^- \longrightarrow \mathrm{Vec}_*^{\|\cdot\|}$.
- *ℓ^1-Homology.* Similarly, the induced morphisms in ℓ^1-homology are defined by taking tensor products of maps.

Proposition 2.4.3 (ℓ^1-Semi-norm via bounded cohomology and ℓ^1-homology)**.** *Let X be a pointed connected CW-complex.*

1. *The homomorphism $H_*(X;\mathbb{R}) \longrightarrow H_*^{\ell^1}(X;\mathbb{R})$ induced by the inclusion of the corresponding chain complexes is isometric with respect to the ℓ^1-semi-norms on homology.*
2. *For all $n \in \mathbb{N}$ and all $\alpha \in H_n(X;\mathbb{R})$ we have*
$$\|\alpha\|_1 = \sup\Bigl\{\frac{1}{\|\varphi\|_\infty} \Bigm| \varphi \in H_b^n(X;\mathbb{R}),\ \langle\varphi,\alpha\rangle = 1\Bigr\}.$$

Proof. This follows directly from the definitions and the corresponding statements about completions and duals of normed chain complexes (Proposition 2.3.10 and Proposition 2.3.13). □

In particular, ℓ^1-homology and bounded cohomology can be used to study the simplicial volume; in fact, this was one of the motivations for Gromov to investigate bounded cohomology in his seminal article *Volume and Bounded Cohomology* [19].

2.4.2 Elementary properties of bounded cohomology

In the following, we will discuss some of the elementary properties of bounded cohomology. We will mainly focus on bounded cohomology; similar properties hold for ℓ^1-homology [27, 28, Chapter 2, Section 3].

Proposition 2.4.4 (Bounded cohomology of a point)**.** *Let $\bullet$ denote the one-point space, and let V be a Banach space. Then $H^0_{\mathrm{b}}(\bullet; V) \cong V$ and $H^k_{\mathrm{b}}(\bullet; V) = 0$ for all $k \in \mathbb{N}_{>0}$.*

Proof. Because there is only one singular simplex in every dimension on $\bullet$, we have $C^*_{\mathrm{b}}(\bullet; V) = C^*(\bullet; V)$, and hence

$$H^*_{\mathrm{b}}(\bullet; V) = H^*(\bullet; V).$$ □

Proposition 2.4.5 (Homotopy invariance)**.** *Let X and Y be two connected pointed CW-complexes with fundamental groups G and H respectively, let V be a Banach G-module, and let W be a Banach H-module. If (f_0, Φ) and $(f_1, \Phi)\colon (X, V) \longrightarrow (Y, W)$ are morphisms in* TopBan^- *with homotopic (base-point preserving) maps f_0 and f_1, then*

$$H^*_{\mathrm{b}}(f_0, \Phi) = H^*_{\mathrm{b}}(f_1, \Phi)\colon H^*_{\mathrm{b}}(Y; W) \longrightarrow H^*_{\mathrm{b}}(X; V).$$

Proof. For simplicity, we give the proof only in the case that V and W are the constant coefficients $\mathbb{R}$ and that $\Phi = \mathrm{id}_{\mathbb{R}}$.

The classic construction [13, Proposition III.5.7] of subdividing a homotopy between f_0 and f_1 in an appropriate way gives rise to a chain homotopy

$$h_*\colon C_*(X; \mathbb{R}) \longrightarrow C_*(Y; \mathbb{R})$$

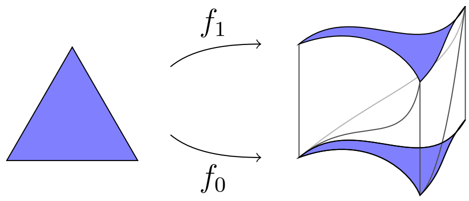


Figure 2.2: Homotopy invariance of bounded cohomology

between $C_*(f_0; \mathrm{id}_\mathbb{R})$ and $C_*(f_1; \mathrm{id}_\mathbb{R})$ that is bounded in every degree (see also Figure 2.2). Therefore, the dual of h_* leads to a cochain homotopy between $C^*_\mathrm{b}(f_0; \mathrm{id}_\mathbb{R})$ and $C^*_\mathrm{b}(f_1; \mathrm{id}_\mathbb{R})$ that is bounded in every degree. Now the claim follows. □

Proposition 2.4.6 (Bounded cohomology in degree 0)**.** *The functor*

$$H^0_\mathrm{b} \colon \mathrm{TopBan}^- \longrightarrow \mathrm{Vec}^{\|\cdot\|}$$

coincides with the invariants functor (and the induced semi-norm on the zero-th bounded cohomology coincides with the (restricted) norm on the coefficient module).

Proof. This is a straightforward calculation on the bounded cochain complex. (Exercise). □

Caveat 2.4.7 (ℓ^1-Homology in degree 0)**.** Notice that ℓ^1-homology in degree 0 in general neither gives the algebraic coinvariants nor the reduced coinvariants (in the sense of Definition 2.2.7), but something between these two notions of coinvariants [28, Section 3.2.1].

The first surprise might be the following observation – bounded cohomology in degree 1 with trivial coefficients is zero:

Proposition 2.4.8 (First bounded cohomology)**.** *Let X be a pointed (connected) CW-complex. Then $H^1_\mathrm{b}(X; \mathbb{R}) = 0$.*

Proof. For simplicity, we give the proof only in the case that X is connected. (The general case requires more complicated notation [50, Corollary 2.14]).

– *The ℓ^1-semi-norm on $H_1(X;\mathbb{R})$ is trivial:* By the universal coefficient theorem

$$H_1(X;\mathbb{R}) = H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

and in view of the Hurewicz theorem, every class in $H_1(X;\mathbb{Z})$ can be represented by a single loop. Therefore, every class α in $H_1(X;\mathbb{R})$ can be written in the form

$$\alpha = \sum_{j=1}^{k} a_j \cdot H_1(f_j;\mathrm{id}_{\mathbb{R}})([S^1]_{\mathbb{R}}),$$

where $a_1,\dots,a_k \in \mathbb{R}$ and $f_1,\dots,f_k \colon S^1 \longrightarrow X$ are continuous maps; i.e., we decompose α into "loops." Using Example 2.3.9, we obtain

$$\begin{aligned} \|\alpha\|_1 &\leq \sum_{j=1}^{k} |a_j| \cdot \big\|H_1(f_j;\mathrm{id}_{\mathbb{R}})([S^1]_{\mathbb{R}})\big\|_1 \\ &\leq \sum_{j=1}^{k} |a_j| \cdot \|S^1\| \\ &\leq 0. \end{aligned}$$

– *The first bounded cohomology group is trivial:* Let $\varphi \in H^1_{\mathrm{b}}(X;\mathbb{R})$, and let $f \in C^1_{\mathrm{b}}(X;\mathbb{R}) \subset C^1(X;\mathbb{R})$ be a bounded cocycle representing φ. In view of the first part and continuity of the evaluation map, we obtain

$$|\langle \varphi, \alpha \rangle| \leq \|\varphi\|_\infty \cdot \|\alpha\|_1 = 0$$

for all $\alpha \in H_1(X;\mathbb{R})$. By the universal coefficient theorem for singular cohomology, there exists a cochain $u \in C^0(X;\mathbb{R})$ with

$$\delta^0(u) = f;$$

however, in general, this cochain will not be bounded. In the following, we will modify u in order to obtain a bounded cochain witnessing that $\varphi = [f] = 0$ in bounded cohomology: Let $x \in X$ be the basepoint of X. We then define $\overline{u} \in C^0(X;\mathbb{R})$ to be the linear extension of

$$\begin{aligned} X = \mathrm{map}(\Delta^0, X) &\longrightarrow \mathbb{R} \\ y &\longmapsto u(y) - u(x). \end{aligned}$$

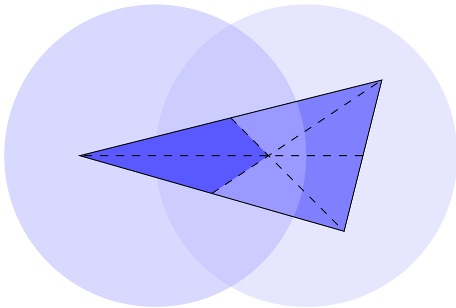


Figure 2.3: Excision and barycentric subdivision, schematically

The cochain $\overline{u}$ is bounded: Because X is path-connected, for every $y \in X$ there is a path $\sigma_y\colon [0,1] \longrightarrow X$ with $\sigma_y(0) = y$ and $\sigma_y(1) = x$. Therefore, we obtain

$$\begin{aligned} \bigl|\overline{u}(y)\bigr| &= \bigl|u(y) - u(x)\bigr| \\ &= \bigl|u(\sigma_y(0)) - u(\sigma_y(1))\bigr| \\ &= \bigl|u(\partial_1 \sigma_y)\bigr| \\ &= \bigl|(\delta^0 u)(\sigma_y)\bigr| \\ &= \bigl|f(\sigma_y)\bigr| \\ &\leq \|f\|_\infty \end{aligned}$$

for all $y \in X$. In particular, $\overline{u} \in C^0_{\mathrm{b}}(X;\mathbb{R})$.
Moreover, a straightforward computation shows that $\delta^0 \overline{u} = \delta^0 u = f$, and hence $\varphi = [f] = 0$ in $H^1_{\mathrm{b}}(X;\mathbb{R})$, as claimed. □

Caveat 2.4.9 (Bounded cohomology and excision/Mayer-Vietoris property)**.** Bounded cohomology (and ℓ^1-homology) in general do *not* satisfy excision. For example, one can show that $H^k_{\mathrm{b}}(S^1;\mathbb{R}) = 0$ for all $k \in \mathbb{N}_{>0}$, but $H^2_{\mathrm{b}}(S^1 \vee S^1;\mathbb{R}) \neq 0$ (Theorem 2.5.17, Theorem 2.6.14, and Section 2.6.2). In particular, there are no cellular versions of bounded cohomology or ℓ^1-homology.

The geometric reason behind this phenomenon is the following: Singular homology and cohomology satisfy excision, because any singular homology class can be represented by a singular cycle consisting of "small" singular simplices. This is achieved by applying barycentric subdivision (see Figure 2.3) sufficiently often. However, in an (infinite) ℓ^1-chain $\sum_{n\in\mathbb{N}} a_n \cdot \sigma_n$, the number of barycentric subdivisions needed for the simplices $(\sigma_n)_{n\in\mathbb{N}}$ might be unbounded.

This failure of excision is both a curse and a blessing. On the one hand, the lack of excision makes concrete computations via the usual divide and conquer approach almost impossible; on the other hand, it turns out that bounded cohomology and ℓ^1-homology depend only on the fundamental group (Theorem 2.8.2) and hence can be computed in terms of certain nice resolutions (Section 2.7).

Caveat 2.4.10 (The comparison map)**.** Let X be a pointed CW-complex. Then the inclusion $C^*_{\mathrm{b}}(X;\mathbb{R}) \longrightarrow \operatorname{Hom}_{\mathbb{R}}(C_*(X;\mathbb{R}),\mathbb{R}) = C^*(X;\mathbb{R})$ of the bounded cochain complex into the singular cochain complex induces a map on cohomology, the so-called *comparison map*

$$H^*_{\mathrm{b}}(X;\mathbb{R}) \longrightarrow H^*(X;\mathbb{R}).$$

In general, this map is neither injective (for example, for the space $S^1 \vee S^1$ in degree 2, nor surjective (for example, for S^1 in degree 1 (Caveat 2.4.9)).

2.4.3 Bounded cohomology of groups

Like in the case of ordinary group (co)homology, the key to the topological definition of bounded cohomology is the homotopy theoretical picture of group theory provided by classifying spaces (see Section 1.3.1 for a short introduction to classifying spaces of discrete groups).

Definition 2.4.11 (Bounded cohomology, topologically)**.** Bounded cohomology of groups is the contravariant functor $\mathsf{GrpBan}^- \longrightarrow \mathsf{Vec}_*^{\|\cdot\|}$ defined as follows: For every discrete group G we choose a model X_G of BG; moreover, for every homomorphism $\varphi\colon G \longrightarrow H$ of groups we choose a continuous map $f_\varphi\colon G \longrightarrow H$ realising φ on the level of fundamental groups (see Theorem 1.3.2).

- On objects: Let (G,V) be an object in GrpBan^-, i.e., G is a discrete group and V is a Banach G-module. Then we define *bounded* *cohomology of G with coefficients in V* by

$$H_b^*(G;V) := H_b^*(X_G;V).$$

– On morphisms: Let $(\varphi,\Phi)\colon (G,V) \longrightarrow (H,W)$ be a morphism in the category GrpBan⁻. Then we define $H_b^*(\varphi;\Phi)$ through the following commutative diagram:

$$\begin{array}{ccc} H_b^*(H;W) & \overset{H_b^*(\varphi;\Phi)}{\dashrightarrow} & H_b^*(G;V) \\ \| & & \| \\ H_b^*(X_H;W) & \underset{H_b^*(f_\varphi;\Phi)}{\longrightarrow} & H_b^*(X_G;V) \end{array}$$

In the same way, ℓ^1-homology can be defined.

Notice that – because of homotopy invariance – bounded cohomology and ℓ^1-homology indeed are functorial and that the definition is independent of the chosen models in the following sense: Any two choices of models of classifying spaces and of maps between them leads to naturally and canonically isomorphic functors.

Example 2.4.12 (Bounded cohomology of the trivial group)**.** For all Banach spaces V we have $H_b^*(1;V) = H^*(1;V)$ (Proposition 2.4.4).

Example 2.4.13 (Bounded cohomology in degree 0)**.** Bounded cohomology of groups in degree 0 (defined topologically) coincides with the invariants functor GrpBan⁻ $\longrightarrow \mathrm{Vec}^{\|\cdot\|}$. The case of ℓ^1-homology is slightly more involved [28, Section 3.2].

Example 2.4.14 (First bounded cohomology)**.** For all discrete groups G we have $H_b^1(G;\mathbb{R}) = 0$ by Proposition 2.4.8.

Caveat 2.4.15 (Comparison map)**.** The comparison map between bounded cohomology and singular cohomology for spaces leads to a comparison map

$$H_b^*(G;\mathbb{R}) \longrightarrow H^*(G;\mathbb{R})$$

for every discrete group G. However, this comparison map in general is neither injective (for example, for $\mathbb{Z} * \mathbb{Z}$ in degree 2 (Theorem 2.6.14)), nor surjective (for example, in degree 1 (Example 2.4.14)).

Surprisingly, bounded cohomology of groups cannot only be computed via bounded cohomology of classifying spaces but much more generally via bounded cohomology of *any* topological space with the given fundamental group (Theorem 2.8.2). This is a major difference between bounded cohomology and ordinary group cohomology!

2.5 Bounded cohomology, combinatorially

As in the case of ordinary group cohomology we might wish for a description of bounded cohomology and ℓ^1-homology that does not involve the choice of a classifying space.

There is a straightforward translation of the bar construction into the setting of bounded cohomology, which provides a description of bounded cohomology that is functorial even on the cochain level.

In this section, we introduce the relevant cochain complexes and derive basic properties of this combinatorial description of bounded cohomology. Moreover, we give an example of a group with non-trivial second bounded cohomology (the free group on two generators) and we explain how bounded cohomology can help to understand quasi-morphisms. In the last section, we show that the combinatorial description and the topological description of bounded cohomology of groups indeed coincide.

2.5.1 The Banach bar resolution

The basic building block for the combinatorial description of bounded cohomology of groups is the Banach bar complex:

Definition 2.5.1 (The Banach bar complex)**.** Let G be a discrete group. The *Banach bar complex* of G is the completion

$$C_*^{\ell^1}(G) := \overline{C_*^{\mathbb{R}}(G)}$$

of the $\mathbb{R}$-valued bar complex $C_*^{\mathbb{R}}(G) = C_*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ of G with respect to the ℓ^1-norm (Example 2.3.2).

More explicitly, for a discrete group G, the Banach bar complex $C_*^{\ell^1}(G)$ in degree n consists of all (infinite) sums $\sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1 | \dots | g_n]$ with real coefficients satisfying $\sum_{g \in G^{n+1}} |a_g| < \infty$. As in the case of the bar complex, the G-action is given by the left action on the first component of the tuples in G^{n+1}, and the boundary operator is given as in Definition 1.4.1.

In Section 2.7, we will study a version of homological algebra suitable for bounded cohomology and ℓ^1-homology. It will turn out that the complexes $C^{\ell^1}_*(G;V)$ and $C^*_{\mathrm{b}}(G;V)$ (together with the obvious augmentation maps) are resolutions in the appropriate sense.

Definition 2.5.2 (The Banach bar complexes with coefficients)**.** Let G be a discrete group, and let V be a Banach G-module. Then we write (where the projective tensor product and the space of bounded linear functionals are taken with respect to the ℓ^1-norm on $\mathbb{C}^{\mathbb{R}}_*(G)$)

$$\begin{aligned} C^{\ell^1}_*(G;V) &:= C^{\mathbb{R}}_*(G) \mathbin{\overline{\otimes}}_G V \cong C^{\ell^1}_*(G) \mathbin{\overline{\otimes}}_G V, \\ C^*_{\mathrm{b}}(G;V) &:= B\bigl(C^{\mathbb{R}}_*(G), V\bigr) \cong B\bigl(C^{\ell^1}_*(G), V\bigr). \end{aligned}$$

The Banach bar construction is functorial on the categories GrpBan and GrpBan⁻, respectively:

Definition 2.5.3 (The Banach bar construction on morphisms)**.** Let G and H be discrete groups, let V be a Banach G-Module, let W be a Banach H-module, and let $\varphi\colon G \longrightarrow H$ be a group homomorphism.

- We write $C^{\mathbb{R}}_*(\varphi) := C_*(\varphi) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}}$; i.e., $C^{\mathbb{R}}_*(\varphi)$ is the map translating bar elements over G into bar elements over H, using the homomorphism φ (see also Definition 1.4.3).
- If $(\varphi, \Phi)\colon (G,V) \longrightarrow (H,W)$ is a morphism in GrpBan, then we write

$$C^{\ell^1}_*(\varphi;\Phi) := C^{\mathbb{R}}_*(\varphi) \mathbin{\overline{\otimes}}_G \Phi\colon C^{\ell^1}_*(G;V) \longrightarrow C^{\ell^1}_*(H;W).$$

- Dually, if $(\varphi, \Phi)\colon (G,V) \longrightarrow (H,W)$ is a morphism in GrpBan⁻, then we write

$$C^*_{\mathrm{b}}(\varphi;\Phi) := B\bigl(C^{\mathbb{R}}_*(\varphi), \Phi\bigr)\colon C^*_{\mathrm{b}}(H;W) \longrightarrow C^*_{\mathrm{b}}(G;V).$$

2.5.2 Bounded cohomology, combinatorially

Using the Banach bar construction, we obtain a combinatorial version of bounded cohomology:

Definition 2.5.4 (Bounded cohomology, combinatorially)**.** Bounded cohomology is the functor $\mathrm{GrpBan}^{-} \longrightarrow \mathrm{Vec}^{\|\cdot\|}{}_{*}$ defined as follows:

- On objects: Let (G, V) be an object in GrpBan, i.e., G is a discrete group and V is a Banach G-module. Then we define *bounded cohomology of G with coefficients in V* by

$$H^*_{\mathrm{b}}(G;V) := H^*\big(C^*_{\mathrm{b}}(G;V)\big).$$

- On morphisms: Let $(\varphi, \Phi)\colon (G, V) \longrightarrow (H, W)$ be a morphism in the category GrpBan^{-}. Then we define

$$H^*_{\mathrm{b}}(\varphi;\Phi) := H^*\big(C^*_{\mathrm{b}}(\varphi;\Phi)\big)\colon H^*_{\mathrm{b}}(H;W) \longrightarrow H^*_{\mathrm{b}}(G;V).$$

Notice that the ℓ^1-norm on $C^{\mathbb{R}}_*(G)$ induces a semi-norm on $H^*_{\mathrm{b}}(G;V)$. Similarly, ℓ^1-homology $H^{\ell^1}_*(\,\cdot\,;\,\cdot\,)\colon \mathrm{GrpBan} \longrightarrow \mathrm{Vec}^{\|\cdot\|}_*$ can be defined combinatorially, using $C^{\ell^1}_*(\,\cdot\,;\,\cdot\,)$.

We will establish the equivalence of the topological and the combinatorial description of bounded cohomology (and ℓ^1-homology) in Section 2.5.5.

In order to get used to this setting, we look at some basic properties:

Proposition 2.5.5 (Bounded cohomology in degree 0)**.** *The (contravariant) functor $H^0_{\mathrm{b}}\colon \mathrm{GrpBan}^{-} \longrightarrow \mathrm{Vec}^{\|\cdot\|}$ given by the combinatorial description of bounded cohomology of groups coincides with the invariants functor.*

Proof. This is a straightforward computation similar to the corresponding argument in ordinary group cohomology (Proposition 1.3.12). □

Caveat 2.5.6 (ℓ^1-Homology in degree 0)**.** ℓ^1-Homology of groups in degree 0 (as given by the combinatorial description) does neither give the algebraic coinvariants nor the reduced coinvariants, but something between these two notions of coinvariants. (Exercise)

Proposition 2.5.7 (Bounded cohomology in degree 1)**.** *Let G be a discrete group. Then $H^1_{\mathrm{b}}(G;\mathbb{R}) = 0$, where G acts trivially on $\mathbb{R}$.*

Proof. This is a straightforward computation, relying on the fact that there are no non-trivial group homomorphisms $G \longrightarrow \mathbb{R}$ with bounded image. (Exercise) □

Just as in the topological setting, we can also describe the comparison map on the level of bar complexes:

Definition 2.5.8 (Comparison map, combinatorially)**.** Let G be a discrete group and let V be a Banach G-module.

– Then the inclusion $C^*(G;V) \longrightarrow C^*_{\mathrm{b}}(G;V)$ induces a homomorphism on the level of cohomology, the so-called *comparison map*

$$H^*_{\mathrm{b}}(G;V) \longrightarrow H^*(G;V).$$

– We denote the kernel of the comparison map by $EH^*_{\mathrm{b}}(G;V)$.

2.5.3 The second bounded cohomology of free groups

Finally, we arrive at our first example of a non-trivial bounded cohomology group (in non-zero degree):

Theorem 2.5.9 (Bounded cohomology of free groups)**.** *Let F be a free group of rank at least 2. Then $H^2_{\mathrm{b}}(F;\mathbb{R})$ is infinite-dimensional.*

Proof (of Theorem 2.5.9). We follow Mitsumatsu's [36] streamlined version of Brooks's argument [3]. The basic strategy is to find a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded cocycles and a sequence $(c_n)_{n\in\mathbb{N}}$ of ℓ^1-cycles in degree 2 such that

$$\langle f_k, c_n \rangle = -\delta_{kn}$$

for all k, $n \in \mathbb{N}$. Clearly, the existence of such (co)cycles proves that the bounded cohomology $H^2_{\mathrm{b}}(F;\mathbb{R})$ is infinite-dimensional.

In the following, we work with a free generating set S of F. Let a and b be two distinct elements of S.

– *Construction of the bounded cocycles.* For an element $w \in F$, we define

$$\begin{aligned} \psi_w \colon G &\longrightarrow \mathbb{R} \\ g &\longmapsto \#(\text{occurences of } w \text{ in } g) - \#(\text{occurences of } w^{-1} \text{ in } g), \end{aligned}$$

where we view all elements of F as (reduced) words in S; we now consider the corresponding element $\overline{\psi}_w \in C^1(F;\mathbb{R}) = \operatorname{Hom}_F(C_1(F),\mathbb{R})$ given by

$$\begin{aligned} \overline{\psi}_w \colon C_1(F) &\longrightarrow \mathbb{R} \\ g_0 \cdot [g_1] &\longmapsto \psi_w(g_1). \end{aligned}$$

Of course, $\overline{\psi}_w$ is not bounded in general; however, the coboundary $\delta\overline{\psi}_w$ *is* bounded (as can be shown by an easy computation, see Lemma 2.5.11 below), and hence yields a cocycle in $C^2_{\mathrm{b}}(F;\mathbb{R})$. We then set for all $k \in \mathbb{N}$

$$f_k := \delta\overline{\psi}_{[a^k,b^k]} \in C^2_{\mathrm{b}}(F;\mathbb{R}).$$

– *Construction of the ℓ^1-cycles.* For $n \in \mathbb{N}$ we let

$$c_n := \sum_{j\in\mathbb{N}} 2^{-j-1} \cdot \bigl[[a^n,b^n]^{2^j} \bigm| [a^n,b^n]^{2^j}\bigr] - b_n \in C_2^{\ell^1}(F;\mathbb{R}),$$

where

$$\begin{aligned} b_n &:= [a^n \mid a^{-n}b^{-n}] + [b^{-n} \mid b^n a^{-n} b^{-n}] - [a^n \mid b^n a^{-n} b^{-n}] \\ &\in C_2(F;\mathbb{R}) \subset C_2^{\ell^1}(F;\mathbb{R}). \end{aligned}$$

Clearly, c_n is a well-defined ℓ^1-chain. Because the boundary of the infinite sum in c_n equals $[a^n, b^n]$ (telescope!), and the boundary of b_n equals $[a^n, b^n]$ as well, the chain c_n is indeed a cycle.

By construction, for all k, $n \in \mathbb{N}$ we have

$$\langle f_k, c_n\rangle = \Bigl\langle f_k, \sum_{j\in\mathbb{N}} 2^{-j-1} \cdot \bigl[[a^n,b^n]^{2^j} \bigm| [a^n,b^n]^{2^j}\bigr] \Bigr\rangle - \langle f_k, b_n\rangle.$$

Looking at the definition of f_k and $\overline{\psi}_{[a^k,b^k]}$, one can show that

$$\begin{aligned} &\Bigl\langle f_k, \sum_{j\in\mathbb{N}} 2^{-j-1} \cdot \bigl[[a^n,b^n]^{2^j} \bigm| [a^n,b^n]^{2^j}\bigr] \Bigr\rangle \\ &= \sum_{j\in\mathbb{N}} 2^{-j-1} \cdot \bigl\langle \delta\overline{\psi}_{[a^n,b^n]}, \bigl[[a^n,b^n]^{2^j} \bigm| [a^n,b^n]^{2^j}\bigr] \bigr\rangle \\ &= 0. \end{aligned}$$

Furthermore, the second summand reduces to

$$\begin{aligned} \langle f_k, b_n\rangle &= \langle \delta\overline{\psi}_{[a^k,b^k]}, b_n\rangle \\ &= \langle \overline{\psi}_{[a^k,b^k]}, \partial b_n\rangle \\ &= \langle \overline{\psi}_{[a^k,b^k]}, [a^n,b^n]\rangle \\ &= \delta_{kn}, \end{aligned}$$

as desired. □

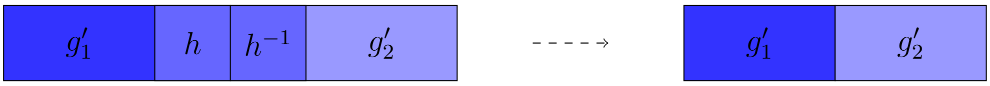


Figure 2.4: Rewriting composed words

Caveat 2.5.10. In the last few steps of the proof, we could not have reduced $\langle f_k, c_n\rangle = \langle \delta\overline{\psi}_{[a^k,b^k]}, c_n\rangle$ to $\langle \overline{\psi}_{[a^k,b^k]}, \partial c_n\rangle = \langle \overline{\psi}_{[a^k,b^k]}, 0\rangle = 0$, because the cochain $\overline{\psi}_{[a^k,b^k]}$ is in general unbounded (and so cannot be evaluated on general ℓ^1-chains).

Lemma 2.5.11 (Counting words gives a bounded cochain)**.** *In the situation of the proof of the above theorem, the cochain $\delta\overline{\psi}_w \colon C_2(F) \longrightarrow \mathbb{R}$ is bounded.*

Proof. Let $g_0, g_1, g_2 \in F$. By definition, we have

$$\begin{aligned}
\delta\overline{\psi}_w(g_0 \cdot [g_1|g_2]) &= \psi_w(g_1) + \psi_w(g_2) - \psi_w(g_1 \cdot g_2)\\
&= \#(\text{occurences of } w \text{ in } g_1)\\
&\quad - \#(\text{occurences of } w^{-1} \text{ in } g_1)\\
&\quad + \#(\text{occurences of } w \text{ in } g_2)\\
&\quad - \#(\text{occurences of } w^{-1} \text{ in } g_2)\\
&\quad - \#(\text{occurences of } w \text{ in } g_1 \cdot g_2)\\
&\quad + \#(\text{occurences of } w^{-1} \text{ in } g_1 \cdot g_2).
\end{aligned}$$

We now write $g_1 = g_1' \cdot h$ and $g_2 = h^{-1} \cdot g_2'$ in such a way that the word $g_1' \cdot g_2'$ is in reduced form (and represents $g_1 \cdot g_2$ by construction) and such that the word h is in reduced form (see Figure 2.4). Then

$$\begin{aligned}
\delta\overline{\psi}_w(g_0 \cdot [g_1|g_2]) &= \#(\text{occurences of } w \text{ in } g_1' \cdot h)\\
&\quad - \#(\text{occurences of } w^{-1} \text{ in } g_1' \cdot h)\\
&\quad + \#(\text{occurences of } w \text{ in } h^{-1} \cdot g_2')\\
&\quad - \#(\text{occurences of } w^{-1} \text{ in } h^{-1} \cdot g_2')\\
&\quad - \#(\text{occurences of } w \text{ in } g_1' \cdot g_2')\\
&\quad + \#(\text{occurences of } w^{-1} \text{ in } g_1' \cdot g_2').
\end{aligned}$$

Using the fact that for all words g, h such that the concatenation $g \cdot h$ is in reduced form the number

$$\big|\#(\text{occurences of } w \text{ in } g \cdot h) - \#(\text{occurences of } w \text{ in } g) - \#(\text{occurences of } w \text{ in } h)\big|$$

is bounded by the length of w with respect to S, we obtain that $\delta\overline{\psi}_w$ is bounded, as claimed. □

More generally, Mineyev [34, 35] proved that a finitely presented discrete group G is word hyperbolic if and only if for all Banach G-modules V and all $k \in \mathbb{N}_{\geq 2}$ the comparison map $H^k_b(G; V) \longrightarrow H^k(G; V)$ is surjective. So word hyperbolic groups of real cohomological dimension bigger than 1 give rise to non-trivial bounded cohomology classes.

2.5.4 Application: Quasi-morphisms

Allowing for a uniformly bounded additive error in the definition of a group homomorphism leads to the notion of quasi-morphisms; in this section, we give a brief introduction into quasi-morphisms and their relation to bounded cohomology.

Definition 2.5.12 (Quasi-morphisms)**.** Let G be a discrete group. A *quasi-morphism* on G is a map $f\colon G \longrightarrow \mathbb{R}$ such that

$$\sup_{g,h\in G} \big|f(g) + f(h) - f(g \cdot h)\big| < \infty.$$

We denote the $\mathbb{R}$-vector space of quasi-morphisms on G by $\operatorname{QM}(G)$.

Example 2.5.13 (Trivial quasi-morphisms)**.** Let G be a discrete group, let $f\colon G \longrightarrow \mathbb{R}$ be a group homomorphism, and let $b\colon G \longrightarrow \mathbb{R}$ be a bounded function. Then $f + b\colon G \longrightarrow \mathbb{R}$ is a quasi-morphism.

Definition 2.5.14 (Trivial quasi-morphisms)**.** Let G be a discrete group. A quasi-morphism $f\colon G \longrightarrow \mathbb{R}$ is called *trivial* if there exists a group homomorphism $f'\colon G \longrightarrow \mathbb{R}$ such that

$$\sup_{g\in G}\big|f(g) - f'(g)\big| < \infty.$$

The subspace of $\operatorname{QM}(G)$ of all trivial quasi-morphisms on G is denoted by $\operatorname{QM}_0(G)$.

These notions lead to the following natural question: Is every quasi-morphism trivial? How can we get access to the space $\operatorname{QM}(G)/\operatorname{QM}_0(G)$?

Starting from the observation that the "coboundary" of a quasi-morphism can be viewed as a bounded cocycle, we obtain that quasi-morphisms can be studied via bounded cohomology [**?**, 45]:

Theorem 2.5.15 (Quasi-morphisms and bounded cohomology)**.** *Let G be a discrete group.*

1. *There is a canonical isomorphism*

$$\operatorname{QM}(G)/\operatorname{QM}_0(G) \cong EH^2_{\mathrm{b}}(G;\mathbb{R});$$

recall that $EH^2_{\mathrm{b}}(G;\mathbb{R})$ is, by definition, the kernel of the comparison map $H^2_{\mathrm{b}}(G;\mathbb{R}) \longrightarrow H^2(G;\mathbb{R})$.

2. *In particular: If $H^2(G;\mathbb{R}) = 0$, then $\operatorname{QM}(G)/\operatorname{QM}_0(G) \cong H^2_{\mathrm{b}}(G;\mathbb{R})$.*

Proof. It suffices to prove the first part: To this end we consider the following diagram:

$$\begin{array}{ccc} C^1(G;\mathbb{R}) & \xrightarrow{\ \delta\ } & C^2(G;\mathbb{R}) \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \operatorname{QM}(G) & \xrightarrow[\ \psi\ ]{} & C^2_{\mathrm{b}}(G;\mathbb{R}) \end{array}$$

Here, the maps φ and ψ are the cochain maps defined by

$$\begin{aligned} \varphi\colon \operatorname{QM}(G) &\longrightarrow C^1(G;\mathbb{R}) \\ f &\longmapsto \bigl(g_0\cdot[g_1] \mapsto f(g_1)\bigr) \\ \psi\colon \operatorname{QM}(G) &\longrightarrow C^2_{\mathrm{b}}(G;\mathbb{R}) \\ f &\longmapsto \bigl(g_0\cdot[g_1|g_2] \mapsto f(g_1)+f(g_2)-f(g_1\cdot g_2)\bigr). \end{aligned}$$

Clearly, the images of both maps contain only G-equivariant maps, and the image of ψ consists of bounded cochains in view of the defining property of quasi-morphisms; so φ and ψ are well-defined. Moreover, a straightforward calculation shows that this diagram is commutative, and hence that the image of ψ consists of bounded *cocycles*.

In particular, the top row of the diagram witnesses that we obtain an induced homomorphism

$$\Psi\colon \operatorname{QM}(G) \longrightarrow EH^2_{\mathrm{b}}(G;\mathbb{R}).$$

In order to prove the theorem, we will identify the image and the kernel of the homomorphism Ψ:

– *The image of* Ψ*.* We show that Ψ is surjective: Let $f \in C^2_{\mathrm{b}}(G;\mathbb{R})$ be a bounded cocycle representing an element of the kernel $EH^2_{\mathrm{b}}(G;\mathbb{R})$ of the comparison map; i.e., there is a cochain $b \in C^1(G;\mathbb{R})$ satisfying $\delta b = f$. Then the map

$$\begin{aligned} \widetilde{f}\colon G &\longrightarrow \mathbb{R} \\ g &\longmapsto b(1 \cdot [g]) \end{aligned}$$

is a quasi-morphism (because $\delta b = f$ is bounded) and, by construction, $\Psi(\widetilde{f}) = [f]$.

– *The kernel of* Ψ*.* We show that the kernel of Ψ coincides with the subspace $\operatorname{QM}_0(G)$ of trivial quasi-morphisms: Let $f \in \operatorname{QM}(G)$ such that $\Psi(f) = 0$ in $EH^2_{\mathrm{b}}(G;\mathbb{R})$; i.e., there exists a bounded cochain $b \in C^2_{\mathrm{b}}(G;\mathbb{R})$ such that $\psi(f) = \delta b$. Viewing b as a (trivial) quasi-morphism (as we may), we obtain that $f - b \in \ker \psi$.
On the other hand, the kernel of ψ obviously consists of all homomorphisms $G \longrightarrow \mathbb{R}$. Therefore, f is a trivial quasi-homomorphism. □

Corollary 2.5.16 (Quasi-morphisms of free groups)**.** *Let F be a non-Abelian free group. Then the $\mathbb{R}$-vector space* $\operatorname{QM}(F)/\operatorname{QM}_0(F)$ *is infinite-dimensional.*

Proof. On the one hand, we have $H^2(F;\mathbb{R}) = 0$ (Example 1.3.13), and so $\operatorname{QM}(F)/\operatorname{QM}_0(F) \cong H^2_{\mathrm{b}}(F;\mathbb{R})$ (Theorem 2.5.15); on the other hand, we know that $H^2_{\mathrm{b}}(F;\mathbb{R})$ is infinite-dimensional (Theorem 2.5.9).

(Actually, the proof of Theorem 2.5.9 relied on the construction of certain quasi-morphisms, namely the ψ_w.) □

2.5.5 Comparing the topological and the combinatorial definition of bounded cohomology

As was to be expected, the topological description and the combinatorial description of bounded cohomology of groups give rise to the same theory:

Theorem 2.5.17 (Bounded cohomology, topologically vs. combinatorially)**.** *Let G be a discrete group and let X be a model of BG.*

1. *There are $\mathbb{R}G$-chain maps*

$$\varphi_* \colon C_*(\widetilde{X};\mathbb{R}) \longrightarrow C^{\mathbb{R}}_*(G)$$
$$\psi_* \colon C^{\mathbb{R}}_*(G) \longrightarrow C_*(\widetilde{X};\mathbb{R})$$

   with the following properties:
   - *With respect to the corresponding ℓ^1-norms we have*

   $$\|\varphi_*\| \leq 1 \qquad \textit{and} \qquad \|\psi_*\| \leq 1.$$

   - *Moreover, the compositions $\varphi_* \circ \psi_*$ and $\psi_* \circ \varphi_*$ are homotopic to the identity on $C^{\mathbb{R}}_*(G)$ and $C_*(\widetilde{X};\mathbb{R})$ respectively, and there exist corresponding $\mathbb{R}G$-chain homotopies that consist of* bounded *linear maps in every degree.*
2. *In particular: If V is a Banach G-module, then there is an isomorphism*

$$H^*_{\mathrm{b}}(X;V) \cong H^*\bigl(C^*_{\mathrm{b}}(G;V)\bigr),$$

   and this isomorphism is isometric with respect to the induced seminorms in cohomology. Moreover, this isomorphism is natural in the second variable.
   Similarly, $H^{\ell^1}_*(X;V) \cong H_*\bigl(C^{\ell^1}_*(G;V)\bigr)$.

Proof. We first show how the *second part* can be derived from the first part: Because the chain maps φ_* and ψ_* are morphisms of normed chain complexes they induce cochain maps

$$B(\varphi, \mathrm{id}_V) \colon C^*_{\mathrm{b}}(G;V) = B\bigl(C^{\mathbb{R}}_*(G), V\bigr) \longrightarrow B\bigl(C_*(\widetilde{X};\mathbb{R}), V\bigr) = C^*_{\mathrm{b}}(G;V),$$
$$B(\psi, \mathrm{id}_V) \colon C^*_{\mathrm{b}}(X;V) = B\bigl(C_*(\widetilde{X};\mathbb{R}), V\bigr) \longrightarrow B\bigl(C^{\mathbb{R}}_*(G), V\bigr) = C^*_{\mathrm{b}}(G;V)$$

of norm at most 1. Moreover, the bounded(!) chain homotopies provided by the first part extend to cochain homotopies of the compositions $B(\varphi, \mathrm{id}_V) \circ B(\psi, \mathrm{id}_V)$ and $B(\psi, \mathrm{id}_V) \circ B(\varphi, \mathrm{id}_V)$ to the corresponding identity maps.

In particular, we obtain the claimed natural isometric isomorphism

$$H^*_{\mathrm{b}}(X; V) \cong H^*\bigl(C^*_{\mathrm{b}}(G; V)\bigr).$$

Similarly, the statement about ℓ^1-homology can be proved.

It remains to prove the *first part:* To this end, we proceed in two steps:

- We first replace the complex $C^{\mathbb{R}}_*(G)$ by a complex $D^{\mathbb{R}}_*(G)$ (see below), which is more appropriate in this simplicial context.
- We then construct chain maps

$$\overline{\varphi}_* \colon C_*(\widetilde{X}; \mathbb{R}) \longrightarrow D^{\mathbb{R}}_*(G)$$
$$\overline{\psi}_* \colon D^{\mathbb{R}}_*(G) \longrightarrow C_*(\widetilde{X}; \mathbb{R})$$

  by induction over the dimension of simplices/chains with properties analogous to those of φ_* and ψ_* stated in the theorem.

How does the chain complex $D^{\mathbb{R}}_*(G)$ look like? For $n \in \mathbb{N}$ we let

$$D^{\mathbb{R}}_n(G) := \bigoplus_{g \in G^{n+1}} \mathbb{R} \cdot (g_0, \dots, g_n);$$

the chain module $D^{\mathbb{R}}_n(G)$ is a normed G-module with respect to the ℓ^1-norm given by the basis G^{n+1} and the diagonal G-action. The maps

$$\begin{aligned} D^{\mathbb{R}}_n(G) &\longrightarrow D^{\mathbb{R}}_{n-1}(G) \\ (g_0, \dots, g_n) &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot (g_0, \dots, \widehat{g_j}, \dots, g_n) \end{aligned}$$

clearly form a G-equivariant boundary operator on $D^{\mathbb{R}}_*(G)$ that is bounded in every degree. Moreover, a straightforward computation shows that

$$\begin{aligned} C^{\mathbb{R}}_*(G) &\longleftrightarrow D^{\mathbb{R}}_*(G) \\ g_0 \cdot [g_1 | \cdots | g_n] &\longmapsto (g_0, g_0 \cdot g_1, g_0 \cdot g_1 \cdot g_2, \dots, g_0 \cdot \dots \cdot g_n) \\ g_0 \cdot [g_0^{-1} g_1 | \dots | g_{n-1}^{-1} \cdot g_n] &\longleftarrow (g_0, \dots, g_n) \end{aligned}$$

are mutually inverse isometric G-chain maps. Hence, in the following discussion we can replace $C^{\mathbb{R}}_*(G)$ by $D^{\mathbb{R}}_*(G)$.

We now come to the construction of the chain maps $\overline{\varphi}_*$ and $\overline{\psi}_*$: We start with a few preparations. Let $F \subset \widetilde{X}$ be a (set-theoretic, strict) fundamental domain for the G-action on the the universal covering $\widetilde{X}$ of X; without loss of generality we may assume that F contains the base point x_0 of the universal covering $\widetilde{X}$. For $x \in \widetilde{X}$ let $g_x \in G$ be the group element uniquely determined by the property

$$x \in g_x \cdot F.$$

Moreover, in the following for $n \in \mathbb{N}$ we denote the vertices of the standard simplex Δ^n by $v_0, \dots, v_n$.

- *Construction of* $\overline{\varphi}_* \colon C_*(\widetilde{X};\mathbb{R}) \longrightarrow D^{\mathbb{R}}_*(G)$*:* For $n \in \mathbb{N}$ we define $\overline{\varphi}_n$ to be the linear extension of the map

$$\begin{aligned} \operatorname{map}(\Delta^n, \widetilde{X}) &\longrightarrow D^{\mathbb{R}}_n(G) \\ \sigma &\longmapsto (g_{\sigma_{v_0}}, \dots, g_{\sigma(v_n)}). \end{aligned}$$

  Then clearly $\overline{\varphi}_* \colon C_*(\widetilde{X};\mathbb{R}) \longrightarrow D^{\mathbb{R}}_*(G)$ is an $\mathbb{R}G$-chain map that has norm at most 1.
- *Construction of* $\overline{\psi}_* \colon C^{\mathbb{R}}_*(G) \longrightarrow C_*(\widetilde{X};\mathbb{R})$*:* We proceed by induction over the dimension of simplices. For the induction start we define

$$\begin{aligned} D^{\mathbb{R}}_0(G) &\longrightarrow C_0(\widetilde{X};\mathbb{R}) \\ g_0 &\longmapsto g_0 \cdot x_0. \end{aligned}$$

  For the induction step let $n \in \mathbb{N}$ and suppose that $\overline{\psi}_*$ is already constructed up to dimension n in such a way that any tuple $(g_0, \dots, g_k)$ over G with $k \leq n$ is mapped to a single singular simplex on $\widetilde{X}$ with vertices $g_0 \cdot x_0, \dots, g_k \cdot x_0$.
  We now extend the definition of $\overline{\psi}_*$ to degree/dimension $n+1$: Let $(g_0, \dots, g_{n+1}) \in G^{n+1}$. Because $\pi_n(\widetilde{X}, x_0) = 0$, we can find a singular $(n+1)$-simplex $\sigma \colon \Delta^{n+1} \longrightarrow \widetilde{X}$ with the following property: For all $j \in \{0, \dots, n+1\}$ the restriction of σ to the j-th face of Δ^{n+1} coincides with the simplex $\overline{\psi}_n(g_0, \dots, \widehat{g_j}, \dots, g_{n+1})$ (see Figure 2.5). We then define

$$\overline{\psi}_{n+1}(g_0, \dots, g_{n+1}) := \sigma \in C_{n+1}(\widetilde{X};\mathbb{R});$$

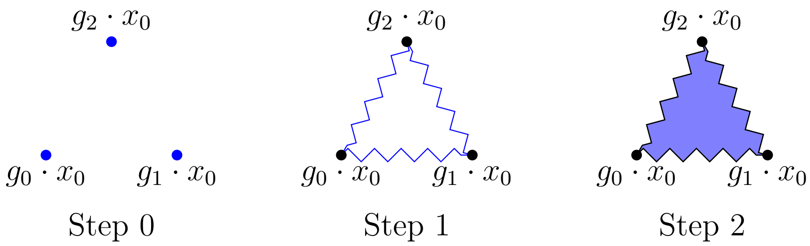


Figure 2.5: Construction of $\overline{\psi}_2$

notice that $\sigma(v_j) = g_j \cdot x_0$ for all $j \in \{0, \dots, n+1\}$, that $\overline{\psi}_{n+1}$ has norm at most 1, and that $\partial_{n+1} \circ \overline{\psi}_{n+1} = \overline{\psi}_n \circ \partial_n$.
Because the diagonal action of G on G^{n+1} is free, and because $\overline{\psi}_n$ is G-equivariant by induction, we can arrange that $\overline{\psi}_{n+1}$ is G-equivariant as well.

– *The composition* $\overline{\varphi}_* \circ \overline{\psi}_*$*:* By construction, we have

$$\overline{\varphi}_* \circ \overline{\psi}_* = \mathrm{id}_{D^{\mathbb{R}}_*(G)} .$$

– *The composition* $\overline{\psi}_* \circ \overline{\varphi}_*$*:* In order to construct an $\mathbb{R}G$-chain homotopy $h_* \colon \overline{\psi}_* \circ \overline{\varphi}_* \simeq \mathrm{id}_{C_*(\widetilde{X};\mathbb{R})}$ that is bounded in every degree we inductively construct compatible homotopies between singular simplices and their image under $\overline{\psi}_* \circ \overline{\varphi}_*$:
Because all homotopy groups of $\widetilde{X}$ are trivial and G acts freely on $\widetilde{X}$, we can inductively construct for every singular simplex $\sigma \colon \Delta^n \longrightarrow \widetilde{X}$ a continuous map $\tau_\sigma \colon \Delta^n \times [0,1] \longrightarrow \widetilde{X}$ in such a way that (see Figure 2.6):
  – On $\Delta^{n+1} \times \{0\}$ the map τ_σ coincides with the singular simplex $\overline{\psi}_{n+1} \circ \overline{\varphi}_{n+1}(\sigma)$.
  – On $\Delta^{n+1} \times \{1\}$ the map τ_σ coincides with the singular simplex σ.
  – The restriction $\tau_\sigma|_{\partial\Delta^n \times [0,1]}$ coincides with the maps given by the τ's corresponding to the faces of σ.
  – The construction is G-equivariant in the sense that $\tau_{g \cdot \sigma} = g \cdot \tau_\sigma$ for all $g \in G$.

Then we define $h_* \colon C_*(\widetilde{X};\mathbb{R}) \longrightarrow C_{*+1}(\widetilde{X};\mathbb{R})$ by applying the canonical subdivision of prisms $\Delta^n \times [0,1]$ into $(n+1)$-simplices to the family $\{\tau_\sigma \mid \sigma \colon \Delta^* \longrightarrow \widetilde{X}\}$ constructed above.
A standard computation shows that h_* indeed is an $\mathbb{R}G$-chain ho-

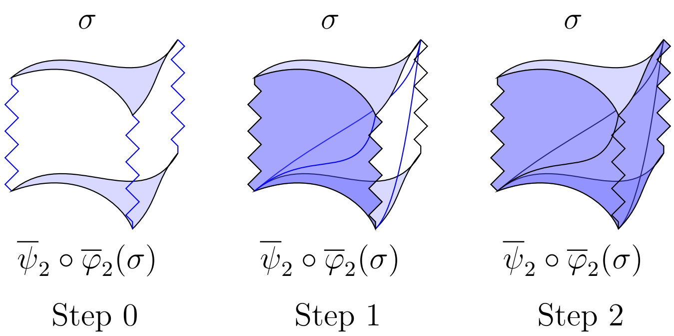


Figure 2.6: Construction of h_2

motopy between $\overline{\psi}_* \circ \overline{\varphi}_*$ and the identity that is bounded in every degree. □

Remark 2.5.18 (Naturality)**.** It is not too difficult (but technically slightly cumbersome) to show that the isomorphisms constructed above are also natural with respect to group homomorphisms. Therefore, we obtain an (isometric) isomorphism of functors identifying bounded cohomology via the topological description with bounded cohomology via the combinatorial description (and similarly for ℓ^1-homology).

Notice that the same arguments also allow to prove the corresponding statement about ordinary group (co)homology.

However, much more than the theorem above is true – by Gromov's mapping theorem (see Section 2.8), for *any* countable connected CW-complex with fundamental group G there is a canonical isometric isomorphism

$$H_b^*(X;\mathbb{R}) \cong H_b^*(G;\mathbb{R}).$$

The key to the mapping theorem is the relationship of bounded cohomology with amenability and the description of bounded cohomology in terms of suitable resolutions; these topics are the content of the following sections.

2.6 Amenable groups

The rôle of finite groups in the setting of bounded cohomology is played by the so-called amenable groups. These groups can be characterised as those groups admitting an invariant mean on the set of bounded functions from the group in question to $\mathbb{R}$; hence, such a mean allows for a nice averaging.

Using this averaging on bounded cochains, transfer maps in bounded cohomology can be defined; this important fact will eventually lead to the mapping theorem in bounded cohomology.

Moreover, the relation of bounded cohomology with amenability provides a link of bounded cohomology with ergodic theory and geometric/measurable group theory; while ordinary group cohomology can be computed nicely by looking at complexes with small skeleta and proper group actions, bounded cohomology asks for amenable actions and ergodic theory. (However, we will not have the time to discuss these aspects during these lectures).

In this section, we give a brief introduction to amenable groups and show that amenable groups can be characterised in terms of bounded cohomology. The deeper connections of amenability with bounded cohomology will be investigated in later chapters. For simplicity, we will only consider the case of discrete groups.

2.6.1 Amenable groups via means

Definition 2.6.1 (Amenable groups)**.** A discrete group G is *amenable* if there exists a (left) G-invariant mean on the set $B(G,\mathbb{R})$ of bounded functions of type $G \longrightarrow \mathbb{R}$.

A *G-invariant mean* on $B(G,\mathbb{R})$ is an $\mathbb{R}$-linear map $m\colon B(G,\mathbb{R}) \longrightarrow \mathbb{R}$ satisfying the following properties:

– For all $f \in B(G, \mathbb{R})$ we have

$$\inf_{g \in G} f(g) \leq m(f) \leq \sup_{g \in G} f(g);$$

in particular, $m(1) = 1$.
– For all $f \in B(G, \mathbb{R})$ and all $g \in G$ we have

$$m\bigl(h \mapsto f(g^{-1} \cdot h)\bigr) = m(f).$$

2.6.2 Examples of amenable groups

Example 2.6.2 (Amenability: finite groups)**.** Finite groups are amenable; for example, averaging functions on the finite group in question gives rise to an equivariant mean as required in the definition of amenability.

Moreover, all Abelian groups are amenable: The proof that every Abelian group admits an invariant mean is highly non-constructive; a basic ingedrient of the proof is the following fixed point theorem [43, Proposition 0.14]:

Theorem 2.6.3 (Markov-Kakutani fixed point theorem)**.** *Let K be a non-empty compact convex subset of a locally convex $\mathbb{R}$-vector space. Let G be an Abelian group of continuous affine transformations $K \longrightarrow K$ (i.e., continuous maps that are compatible with convex combinations). Then the set K contains a G-fixed point.*

Example 2.6.4 (Amenability: Abelian groups)**.** Every discrete Abelian group is amenable: Let G be an Abelian group. We consider the action

$$\begin{aligned} G \times B(G, \mathbb{R})^{\#} &\longrightarrow B(G, \mathbb{R})^{\#} \\ (g, m) &\longmapsto \bigl(h \mapsto f(g^{-1} \cdot h\bigr) \end{aligned}$$

of G on the dual vector space $B(G, \mathbb{R})^{\#}$ endowed with the weak* topology. The subset $M(G) \subset B(G, \mathbb{R})^{\#}$ of non-negative functionals on $B(G, R)$ mapping the constant function 1 to 1 is a convex closed subset of $B(G, R)^{\#}$; the set $M(G)$ is non-empty (it contains, for instance, evaluation on the neutral element of G) and it is compact by the Banach-Alaoglu theorem. Moroever, $G \cdot M(G) \subset M(G)$ by definition of the G-action.

Because G is Abelian, the Markov-Kakutani fixed point theorem applies and provides us with a fixed point $m \in M(G)$ of this G-action. Clearly, such a fixed point m is nothing but a G-invariant mean on G. In particular, G is amenable.

In contrast, free groups are not amenable – in a sense they are the prototypical examples of non-amenable groups (Caveat 2.6.10):

Example 2.6.5 (Amenability: free groups)**.** The free group F on two generators (say a and b) is *not* amenable: *Assume* for a contradiction that F is amenable; hence, there is an F-invariant mean $m\colon B(F,\mathbb{R}) \longrightarrow \mathbb{R}$. We write μ for the associated (finitely additive) F-invariant probability measure

$$\begin{aligned} \mu\colon P(F) &\longrightarrow \mathbb{R}_{\geq 0} \\ A &\longmapsto \chi(A); \end{aligned}$$

here, $P(F)$ denotes the power set of F.

For a letter $g \in \{a, b, a^{-1}, b^{-1}\}$ let W_g be the set of (reduced) words in a, b, a^{-1}, b^{-1} that start with g. Because F is the free group freely generated by a and b, we obtain

$$1 = \mu(F) = \mu(W_a) + \mu(W_{a^{-1}}) + \mu(\{1\}) + \mu(W_b) + \mu(W_{b^{-1}}).$$

Moreover, $W_a = a \cdot (W_b \sqcup W_{b^{-1}} \sqcup W_a \sqcup \{1\})$, and so (by F-invariance and finite additivity of μ)

$$\mu(W_a) = \mu(W_b) + \mu(W_{b^{-1}}) + \mu(W_a) + \mu(\{1\}),$$

which implies $\mu(W_b) = 0 = \mu(W_{b^{-1}})$ and $\mu(\{1\}) = 0$. Similarly, we obtain $\mu(W_a) = 0 = \mu(W_{a^{-1}})$. Therefore, $1 = \mu(F) = 0$, a contradiction. Thus, F is not amenable.

Caveat 2.6.6. Notice that the finitely additive invariant probability measure on an amenable group in general is *not* σ-additive! So some care has to been taken when arguing using these finitely additive measures.

Remark 2.6.7 (Banach-Tarski paradoxon)**.** The decomposition of the free group on two generators used in the proof above is a so-called *paradoxical* *decomposition*. Non-amenable groups can be characterised via paradoxical decompositions.

Such decompositions play a decisive rôle in the proof of the *Banach-Tarski paradoxon*: The unit 2-sphere can be decomposed into finitely many pieces that in turn can be put together to form *two* unit 2-spheres. (Of course, these pieces cannot be Lebesgue-measurable.) The source of such decompositions of the 2-sphere are paradoxical decompositions of $\mathbb{Z} * \mathbb{Z}$, which is a subgroup of $\mathrm{SO}(3, \mathbb{R})$.

2.6.3 Inheritance properties of amenable groups

The class of (discrete) amenable groups behaves nicely with respect to the basic operations on groups, such as taking subgroups, quotients and extensions:

Proposition 2.6.8 (Amenable groups, inheritance properties)**.**

1. *Subgroups of amenable groups are amenable*
2. *Quotients groups of amenable groups are amenable.*
3. *Extensions of amenable groups by amenable groups are amenable.*

Proof. These properties can be deduced from the definition in terms of invariant means in a fairly straightforward manner (Exercise) [43, Proposition 0.16]. For the first part it is convenient to choose a set of representatives of the action of the given subgroup on the ambient group; for the second part, one can push forward an invariant mean on the given amenable group to yield an invariant mean on the quotient; for the third part, one can use averaging over the amenable kernel to turn an invariant mean on the quotient into one of the extension group. □

Example 2.6.9 (Amenability: solvable groups)**.** As all discrete Abelian groups are amenable (Example 2.6.4), and as the class of discrete amenable groups is closed with respect to taking extensions, it follows inductively that all solvable groups are amenable.

Caveat 2.6.10 (Von Neumann problem)**.** As the free group on two generators is not amenable (Example 2.6.5), and as amenability passes down to subgroups, we obtain: every discrete group that contains a free group on two generators as a subgroup is not amenable.

Conversely, von Neumann asked whether a discrete group that is not amenable has to contain a free group on two generators as a subgroup; Olshanskii was the first to prove that there are non-amenable groups that do not contain the free group on two generators – more precisely, he produced torsion(!) groups that are non-amenable [41].

Astonishingly, analogous questions in the context of geometric and measurable group theory do have a positive answer [54, 15].

2.6.4 Geometric characterisations of amenable groups

A geometric and quite concrete approach to amenability is provided by the notion of a Følner sequence, which is a precise way of saying that a group is amenable if it contains subsets of finite non-zero measure that are almost invariant under translation.

Definition 2.6.11 (Følner sequence)**.** Let G be a discrete group. A *Følner sequence* for G is a sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of G such that for every $g \in G$ we have

$$\lim_{n\to\infty} \frac{\#\bigl(S_n \bigtriangleup g \cdot S_n\bigr)}{\#S_n} = 0.$$

(Here, $\bigtriangleup$ stands for the symmetric difference of sets.)

Example 2.6.12 (A Følner sequence for $\mathbb{Z}^d$)**.** Let $d \in \mathbb{N}_{>0}$. Then a straightforward computation shows that $(\{-n, \dots, n\}^d)_{n\in\mathbb{N}}$ is a Følner sequence for $\mathbb{Z}^d$; roughly speaking, the symmetric differences occuring in the Følner condition grow polynomially with exponent $d - 1$, but the given sequence grows polynomially with exponent d, see Figure 2.7.

Theorem 2.6.13 (Characterising amenable groups through Følner sequences)**.** *A countable discrete group is amenable if and only if it admits a Følner sequence.*

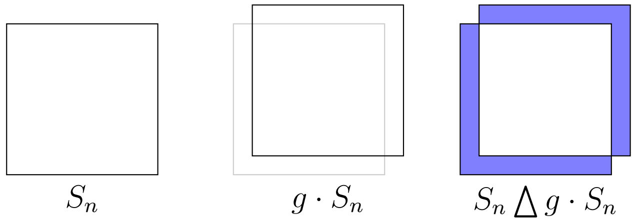


Figure 2.7: The Følner condition in $\mathbb{Z}^2$, schematically

Sketch proof. The proof makes use of the axiom of choice in the form of non-principal ultrafilters on $\mathbb{N}$; ultra-limits are used to convert Følner sequences into invariant means.

Conversely, in a suitable topology, invariant means can be approximated by ℓ^1-step functions on G; these step functions give rise to almost invariant sets [55, Chapter 10]. □

The theme of amenability being a strong invariance property can also be translated into further fields [55, Chapter 10]:

- *Ergodic theory.* A discrete group G is amenable if and only if for every continuous G-action on a non-empty compact metrisable space X there exists a G-invariant probability measure on X.
- *Representation theory.* A discrete group G is amenable if and only if the regular representation of G on $L^2(G, \mathbb{C})$ has almost invariant vectors.

2.6.5 Application: Characterising amenable groups via bounded cohomology

In the following we present Noskov's characterisation of amenable groups in terms of bounded cohomology [40, 24]:

Theorem 2.6.14 (Characterisation of amenability by bounded cohomology)**.** *Let G be a discrete group. Then the following are equivalent:*

1. *The group G is amenable.*

2. *For all Banach G-modules V and all $k \in \mathbb{N}_{>0}$ we have*

$$H^k_{\mathrm{b}}(G; V^{\#}) = 0.$$

3. *For all Banach G-modules V we have $H^1_{\mathrm{b}}(G; V^{\#}) = 0$.*

Proof. We start with the proof of the implication "1 $\Longrightarrow$ 2:" Suppose the group G is amenable; i.e., there is a G-invariant mean $m\colon B(G, \mathbb{R}) \longrightarrow \mathbb{R}$. Let V be a Banach G-module. As first step, using m, we construct a G-equivariant mean $m_V\colon B(G, V^{\#}) \longrightarrow V^{\#}$ via

$$\begin{aligned} m_V\colon B(G, V^{\#}) &\longrightarrow V^{\#} \\ f &\longmapsto \Bigl(v \mapsto m\bigl(g \mapsto (f(g))(v)\bigr)\Bigr); \end{aligned}$$

clearly, m_V is a bounded linear functional of norm at most 1, and using the invariance property of m it is not difficult to see that m_V indeed is G-equivariant.

Using this mean m_V we can now derive triviality of the higher bounded cohomology with coefficients in $V^{\#}$ via a suitable transfer map, which is one of the key arguments in the theory of bounded cohomology: We define the transfer $t^*\colon B\bigl(C^{\mathbb{R}}_*(G), V^{\#}\bigr) \longrightarrow C^*_{\mathrm{b}}(G; V^{\#})$ by

$$\begin{aligned} B\bigl(C^{\mathbb{R}}_n(G), V^{\#}\bigr) &\longrightarrow B_G\bigl(C^{\mathbb{R}}_n(G), V^{\#}\bigr) = C^n_{\mathrm{b}}(G; V^{\#}) \\ f &\longmapsto \Bigl(g_0 \cdot [g_1|\cdots|g_n] \mapsto m_V\bigl(g \mapsto f(g^{-1} \cdot g_0 \cdot [g_1|\cdots|g_n])\bigr)\Bigr); \end{aligned}$$

because m_V is G-equivariant, this map is well-defined and t^* indeed is a cochain map. Because m_V acts as the identity on constant maps (a property inherited from m), we obtain

$$t^* \circ i^* = \mathrm{id}_{C^*_{\mathrm{b}}(G; V^{\#})},$$

where $i^*\colon C^*_{\mathrm{b}}(G; V^{\#}) = B_G\bigl(C^{\mathbb{R}}_n(G), V^{\#}\bigr) \longrightarrow B\bigl(C^{\mathbb{R}}_*(G), V^{\#}\bigr)$ denotes the inclusion. Therefore, on the level of cohomology we have

$$H^*(t^*) \circ H^*(i^*) = \mathrm{id}_{H^*_{\mathrm{b}}(G; V^{\#})},$$

which proves that $H^*(i^*)$ is injective. On the other hand, it is not difficult to see that the complex $B(C^{\mathbb{R}}_*(G), V^{\#})$ has trivial cohomology in all degrees bigger than 1 (because $C^{\mathbb{R}}_*(G) \longrightarrow \mathbb{R}$ admits a chain contraction that is bounded in every degree (Proposition 2.7.7) and this chain contraction gives rise to a cochain contraction of the complex of bounded maps to $V^\#$). Hence, $H^k_{\mathrm{b}}(G; V^\#) = 0$ for all $k \in \mathbb{N}_{>0}$.

Obviously, statement “2” implies “3”.

It remains to prove “3 $\Longrightarrow$ 1:” Suppose that $H^1_{\mathrm{b}}(G; V^\#) = 0$ for all Banach G-modules V. In the following, we consider the Banach G-module

$$V := B(G, \mathbb{R})/\mathbb{R} \cdot 1$$

of bounded functions on G modulo the constant functions; hence, we can identify $V^\#$ with the space of bounded functions on $B(G, \mathbb{R})$ that vanish on the constant functions. The idea is now to construct a 1-bounded cocycle with coefficients in $V^\#$ and to use a 0-cochain hitting this cocycle via the coboundary operator in order to find an invariant mean on $B(G, \mathbb{R})$:

Let $\mu \in B(G; \mathbb{R})^\#$ with $\mu(1) = 1$; for example, let μ be evaluation at the neutral element. Then

$$\begin{aligned} f \colon C^{\mathbb{R}}_1(G) &\longrightarrow V^\# \\ g_0 \cdot [g_1] &\longmapsto g_0 \cdot g_1 \cdot \mu - g_0 \cdot \mu \end{aligned}$$

is well-defined, bounded and G-equivariant and so an element of $C^1_{\mathrm{b}}(G; V^\#)$; moreover, a simple calculation shows that f is a cocycle.

Because $H^1_{\mathrm{b}}(G; V^\#) = 0$, there exists a cochain $b \in C^0_{\mathrm{b}}(G; V^\#)$ satisfying

$$f = \delta b.$$

Let $\nu := b(1) \in V^\#$. Then, by definition of the coboundary operator,

$$(g - 1) \cdot \mu = f(g \cdot [1]) = \delta b(g \cdot [1]) = (g - 1) \cdot \nu.$$

for all $g \in G$. In other words, for all $g \in G$ we obtain

$$g \cdot (\mu - \nu) = \mu - \nu,$$

and thus $\mu - \nu$ is a G-invariant bounded linear functional on $B(G, \mathbb{R})$ satisfying $(\mu - \nu)(1) = \mu(1) - \nu(1) = 1$.

So $\mu - \nu$ almost is a G-invariant mean on $B(G, \mathbb{R})$; however, in general, the difference $\mu - \nu$ will not be non-negative. Similarly to the Hahn decomposition of signed measures of finite total variation there is a "minimal" decomposition

$$\mu - \nu = \varphi_+ - \varphi_-$$

into non-negative functionals $\varphi_+, \varphi_- \in B(G, \mathbb{R})^\#$. In addition, this decomposition is unique in a certain sense [**?**]; this uniqueness is strong enough to show that for all $g \in G$ the corresponding decomposition of $g \cdot (\mu - \nu)$ is $g \cdot \varphi_+ - g \cdot \varphi_-$. Because $\mu - \nu$ is G-invariant, so is the decomposition. In particular, φ_+ and φ_- are G-invariant.

Because $(\mu - \nu)(1) = 1$, the functional φ_+ cannot be trivial. Hence, a suitable normalisation of φ_+ is a G-invariant mean on G, showing that G is amenable. □

In particular, we obtain that amenable groups do not admit any non-trivial quasi-morphisms:

Corollary 2.6.15 (Quasi-morphisms on amenable groups)**.** *Any quasi-morphism on a discrete amenable group is trivial (in the sense of Definition 2.5.14).*

Proof. Let A be a discrete amenable group. In view of Theorem 2.5.15, the space $\operatorname{QM}(A)/\operatorname{QM}_0(A)$ is isomorphic to the kernel of the comparison map $H^2_{\mathrm{b}}(A;\mathbb{R}) \longrightarrow H^2(A;\mathbb{R})$.

However, by the previous theorem, $H^2_{\mathrm{b}}(A;\mathbb{R}) = H^2_{\mathrm{b}}(A;\mathbb{R}^\#) = 0$. Hence, all quasi-morphisms $A \longrightarrow \mathbb{R}$ are trivial. □

Caveat 2.6.16. In general, bounded cohomology in degree 1 of an amenable group does not vanish for *all* (even those that admit no pre-dual) coefficients; for example, there exist Banach $\mathbb{Z}$-modules V such that $H^1_{\mathrm{b}}(\mathbb{Z}; V)$ is infinite dimensional [40].

Using a suitable duality on the level of (co)homology, one can deduce from Theorem 2.6.14 an analogous characterisation of amenable groups in terms of ℓ^1-homology [28, Corollary 5.5].

2.7 Bounded cohomology, algebraically

Our aim is now to find an *algebraic* description of bounded cohomology that has some built-in flexibility (like the definition of bounded cohomology via classifying spaces) and to understand how much flexibility we have. In classical group cohomology, the solution to this problem is to interpret group cohomology as a derived functor. Unfortunately, in the Banach world, homological algebra is obstructed by the fact that the category of Banach modules is not a nice category in the sense of classical homological algebra (i.e., it is not an Abelian category).

While not actually deriving a functor in the literal, axiomatic, sense, we can model the construction of derived functors in this Banach setting, thereby obtaining a description of bounded cohomology in terms of certain resolutions. Technically, these resolutions are part of a Banach version of the relative homological algebra that was also used in the context of Tate cohomology.

Bühler developed a version of ℓ^1-homology that *is* a derived functor of the reduced coinvariants [7]; however, this approach requires a more abstract background and is not suitable for the type of applications we have in mind.

The full strength of the approach to bounded cohomology via (relative) homological algebra becomes visible when combined with amenability – see the discussion of the mapping theorem in the next chapter.

2.7.1 Relative homological algebra, Banach version

The basic idea of homological algebra is to approximate objects by simpler objects – i.e., to replace objects by injective or projective resolutions. The right version of homological algebra for bounded cohomology is relative homological algebra. Recall that in relative homological algebra we restrict the class of mapping problems to injective/surjective maps that split in a weak sense (cf. Section 1.9.2). In order to obtain enough control over the norms (and hence over the semi-norm in bounded cohomology) we need an additional condition on the splittings in terms of operator norms:

Definition 2.7.1 (Relatively injective/projective morphisms of Banach modules)**.** Let G be a discrete group, and let U and W be two Banach G-modules.

- A G-morphism $\pi \colon U \longrightarrow W$ is called *relatively projective* if there is a (not necessarily equivariant) linear map $\sigma \colon W \longrightarrow U$ satisfying
$$\pi \circ \sigma = \mathrm{id}_W \qquad \text{and} \qquad \|\sigma\| \leq 1.$$
- A G-morphism $i \colon U \longrightarrow W$ is called *relatively injective* if there is a (not necessarily equivariant) linear map $\sigma \colon W \longrightarrow U$ satisfying
$$\sigma \circ i = \mathrm{id}_U \qquad \text{and} \qquad \|\sigma\| \leq 1.$$

Definition 2.7.2 (Relatively injective/projective Banach modules)**.** Let G be a discrete group and let V be a Banach G-module.

- The module V is called *relatively projective* if for each relatively projective G-morphism $\pi \colon U \longrightarrow W$ and each G-morphism $\alpha \colon V \longrightarrow W$ there is a G-morphism $\beta \colon V \longrightarrow U$ such that
$$\pi \circ \beta = \alpha \qquad \text{and} \qquad \|\beta\| \leq \|\alpha\|.$$
- The module V is called *relatively injective* if for each relatively injective G-morphism $i \colon U \longrightarrow W$ and for each G-morphism $\alpha \colon U \longrightarrow V$ there is a G-morphism $\beta \colon W \longrightarrow V$ such that
$$\beta \circ i = \alpha \qquad \text{and} \qquad \|\beta\| \leq \|\alpha\|.$$

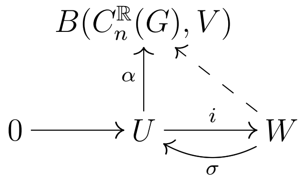
The mapping problems arising in the definition of relatively projective and relatively injective Banach G-modules are depicted in Figure 2.8. Sometimes, "relatively injective" and "relatively projective" morphisms are also called "admissible monomorphisms" and "admissible epimorphisms" respectively.

Similarly, to the situation in the context of Tate cohomology, we see that induced and coinduced modules are relatively projective and relatively injective respectively:

Figure 2.8: Mapping problems for relatively projective and relatively injective Banach G-modules respectively

Example 2.7.3 (Relatively injective/projective modules)**.** Let G be a discrete group, let V be a Banach G-module, and let $n \in \mathbb{N}$.

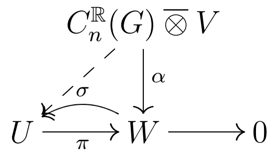
- The Banach G-module $B(C_n^{\mathbb{R}}(G), V)$ (with the diagonal G-action) is relatively injective: Any mapping problem in the sense of Definition 2.7.2 of the form



  can be solved by the G-morphism

$$\begin{aligned} W &\longrightarrow B\bigl(C_n^{\mathbb{R}}(G), V\bigr) \\ w &\longmapsto \Bigl(g_0 \cdot [g_1|\cdots|g_n] \mapsto \bigl(\alpha(g_0 \cdot \sigma(g_0^{-1} \cdot w))\bigr)(g_0 \cdot [g_1|\cdots|g_n])\Bigr). \end{aligned}$$

- The Banach G-module $C_n^{\mathbb{R}}(G) \mathbin{\overline{\otimes}} V$ (with the diagonal G-action) is relatively projective: Any mapping problem in the sense of Definition 2.7.2 of the form



  is solved by the G-morphism

$$\begin{aligned} C_n^{\mathbb{R}}(G) \mathbin{\overline{\otimes}} V &\longrightarrow U \\ g_0 \cdot [g_1|\cdots|g_n] \otimes v &\longmapsto g_0 \cdot \sigma \circ \alpha\bigl(1 \cdot [g_1|\cdots|g_n] \otimes (g_0^{-1} \cdot v)\bigr). \end{aligned}$$

In particular, the category of Banach G-modules contains enough relatively projective objects and enough relatively injective objects; as in the case of classical group cohomology this observation is interesting in the context of dimension shifting and proving uniqueness of derived functors.

Remark 2.7.4 (Enough relatively projectives/enough relatively injectives)**.** Let G be a discrete group and let V be a Banach G-module.

- Then there exists a relatively injective Banach G-module W together with a relatively injective G-morphism $V \longrightarrow W$; for example, we can take the relatively injective Banach G-module $W := B(\ell^1(G), V)$ (see Example 2.7.3) together with the canonical embedding

$$\begin{aligned} V &\longrightarrow B(\ell^1(G), V) \\ v &\longmapsto (g \mapsto g \cdot v). \end{aligned}$$

- Similarly, there exists a relatively projective Banach G-module W together with a relatively projective G-morphism $W \longrightarrow V$; for example, the relatively projective Banach G-module $W := \ell^1(G) \mathbin{\overline{\otimes}} V$ together with the canonical projection

$$\begin{aligned} \ell^1(G) \mathbin{\overline{\otimes}} V &\longrightarrow V \\ g \mathbin{\overline{\otimes}} v &\longmapsto g \cdot v. \end{aligned}$$

Example 2.7.5 (Relative projectivity/injectivity of the trivial module $\mathbb{R}$)**.** Let G be a discrete group.

- The trivial Banach G-module $\mathbb{R}$ is relatively injective if and only if G is amenable (Exercise).
- The trivial BAnach G-module $\mathbb{R}$ is relatively projective if and only if G is finite (Exercise).

Taking duals transforms relatively projective modules into relatively injective modules [27, Proposition A.4].

As in the classical case, resolutions provide the means to describe general objects (equivariant Banach modules) in terms of simpler ones (relatively injective/relatively projective Banach modules); the restricted power of *relatively* injective/projective Banach modules is reflected by the fact that we have to use *strong* resolutions.

Definition 2.7.6 (Strong relatively injective/projective resolutions)**.** Let G be a discrete group and let V be a Banach G-module.

- A *(homological) resolution* of V is a Banach G-chain chain complex C_* together with a G-morphism $\varepsilon\colon C_0 \longrightarrow V$, the *augmentation*, satisfying $H_*(C_* \,\square\, \varepsilon) = 0$; recall that "$\square$" denotes the concatenation of (co)chain complexes.
  Similarly, *cohomological resolutions* are defined.
- A resolution of V by Banach G-modules is called *strong* if the concatenated Banach G-(co)chain complex admits a (not necessarily equivariant) (co)chain contraction of norm at most 1.
- A resolution of V by Banach G-modules is called *relatively injective* (or *relatively projective*) if it consists of relatively projective Banach G-modules (or relatively injective Banach G-modules respectively).

Crucial examples for strong relatively injective/projective resolutions are provided by the Banach bar constructions; later, we will see that also topological spaces give rise to a large number of nice resolutions.

Proposition 2.7.7 (The Banach bar resolution)**.** *Let G be a discrete group, and let V be a Banach G-module.*

1. *Then $C^{\mathbb{R}}_*(G)\,\overline{\otimes}\,V$ together with $\varepsilon\,\overline{\otimes}\,\mathrm{id}_V$ is a strong relatively projective G-resolution of V.*
2. *Dually, $B(C^{\mathbb{R}}_*(G), V)$ together with $B(\varepsilon, \mathrm{id}_V)$ is a strong relative injective G-resolution of V.*

Here, ε denotes the canonical augmentation map

$$\begin{aligned} \varepsilon\colon C^{\mathbb{R}}_0(G) &\longrightarrow \mathbb{R} \\ G \ni g_0 &\longmapsto 1. \end{aligned}$$

Proof. As preparation, we exhibit a contracting chain homotopy s_* of the concatenated chain complex $C^{\mathbb{R}}_*(G) \,\square\, \varepsilon$ of norm at most 1: We define $s_{-1}\colon \mathbb{R} \longrightarrow C^{\mathbb{R}}_0(G)$ by $s_{-1}(1) := 1 \cdot []$, and for $n \in \mathbb{N}$ we set

$$\begin{aligned} s_n\colon C^{\mathbb{R}}_n(G) &\longrightarrow C^{\mathbb{R}}_{n+1}(G) \\ g_0 \cdot [g_1|\cdots|g_n] &\longrightarrow (-1)^{n+1} \cdot 1 \cdot [g_0|g_1|\cdots|g_n]; \end{aligned}$$

a straightforward computation shows that s_* has the stated properties.

Then $s_* \mathbin{\overline{\otimes}} \mathrm{id}_V$ is a contracting chain homotopy of $(C^{\mathbb{R}}_*(G) \mathbin{\overline{\otimes}} V) \square (\varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V)$ that also has norm as most 1; dually, $B(s_*, \mathrm{id}_V)$ is a contracting cochain homotopy of $B(\varepsilon, \mathrm{id}_V) \square B(C^{\mathbb{R}}_*(G), V)$ of norm at most 1.

Moreover, the complexes $C^{\mathbb{R}}_*(G) \mathbin{\overline{\otimes}} V$ and $B(C^{\mathbb{R}}_*(G), V)$ consist of relatively projective and relatively injective Banach G-modules respectively (Example 2.7.3), which finishes the proof of the proposition. □

Again, as in the classical case, strong relatively injective and strong relatively projective resolutions are essentially unique:

Proposition 2.7.8 (Fundamental lemma of relative homological algebra, Banach version)**.** *Let G be a discrete group and let $f\colon V \longrightarrow W$ be a morphism of Banach G-modules.*

1. *Let $P_* \square (\varepsilon\colon P_0 \to V)$ be a Banach G-chain complex where all P_n are relatively projective, and let $C_* \square (\gamma\colon C_0 \to W)$ be a strong homological resolution of W by Banach G-modules.*
   Then f can be extended to a morphism $f_* \square f\colon P_* \square \varepsilon \longrightarrow C_* \square \gamma$ of Banach G-chain complexes; moreover, the extension $f_* \square P_* \longrightarrow C_*$ is unique up to bounded (in every degree) G-chain homotopy.
2. *Let $(\eta\colon W \to I^0) \square I^*$ be a Banach G-cochain complex where all I^n are relatively injective Banach G-modules, and let $(\gamma\colon V \to C^0) \square C^*$ be a strong cohomological resolution of V by Banach G-modules.*
   Then f can be extended to a morphism $f \square f^*\colon \gamma \square C^* \longrightarrow \eta \square I^*$ of Banach G-cochain complexes; moreover, the extension f^* is unique up to bounded (in every degree) G-cochain homotopy.

Proof. This version of the fundamental lemma of homological algebra can be proved by the same inductive arguments as the classical version. (Exercise) [37, Chapter 7.2]. □

Corollary 2.7.9 (Uniqueness of strong resolutions)**.** *Let G be a discrete group and let V be a Banach G-module.*

1. *Then up to canonical (in every degree) bounded G-chain homotopy equivalence there is exactly one strong relatively projective G-resolution of V; i.e., between any two strong relatively projective G-resolutions of V there exists a canonical bounded G-chain homotopy equivalence.*

2. *Dually, up to canonical (in every degree) bounded G-cochain homotopy equivalence there is exactly one strong relatively injective G-resolution of V.*

Proof. Existence is covered by Proposition 2.7.7; uniqueness follows form the fundamental lemma (applied to the identity map $V \longrightarrow V$). □

2.7.2 Bounded cohomology, algebraically

Using strong resolutions, we see that bounded cohomology and ℓ^1-homology can be computed in the same way as derived functors of $\cdot^G$ and $\cdot_G$ can be described in the classical case. Namely, for a discrete group G and a Banach G-module V bounded cohomology can be obtained as follows:

- Replace V by an approximation through simpler objects; i.e., replace V by a strong relatively injective resolution I^*.
- Apply the invariants functor $\cdot^G$ to I^*.
- Take cohomology; the result is $H_b^*(G; V)$.

This is justified by the following theorem:

Theorem 2.7.10 (Bounded cohomology, algebraically)**.** *Let G be a discrete group and let V be a Banach G-module.*

1. *If I^* together with the augmentation $\eta\colon V \to I^0$ is a strong relatively injective G-resolution of V, then there is a canonical isomorphism*

$$H^*(I^{*G}) \longrightarrow H_b^*(G; V)$$

having norm at most 1 in every degree (with respect to the induced semi-norms).

2. *If P_* together with the augmentation $\varepsilon\colon P_0 \to V$ is a strong relatively projective G-resolution of V, then there is a canonical isomorphism*

$$H_*\big(B_G(P_*, V)\big) \longrightarrow H_b^*(G; V)$$

that is functorial in the second variable and that has norm at most 1 in every degree.

In addition, these isomorphisms are functorial in the following sense: If $(\varphi, \Phi)\colon (G, V) \longrightarrow (H, W)$ is a morphism in GrpBan$^-$*, and if $\eta \mathbin{\square} I^*$ and*

$\vartheta \,\square\, J^*$ are strong relatively injective resolutions of V (over G) and of W (over H), then the following diagram is commutative:

$$\begin{array}{ccc} H^*_{\mathrm{b}}(H;W) & \longleftarrow & H^*(J^{*H}) \\ {\scriptstyle H^*_{\mathrm{b}}(\varphi;\Phi)}\downarrow & & \downarrow \\ H^*_{\mathrm{b}}(G;V) & \longleftarrow & H^*(I^{*G}) \end{array}$$

Here, the horizontal arrows are the canonical isomorphisms provided by the first part; the left vertical map is induced by $C^*_{\mathrm{b}}(\varphi;\Phi)$; the right vertical map is induced by the unique (up to homotopy) morphism $\varphi^* J^* \longrightarrow I^*$ of Banach G-cochain complexes extending $\Phi\colon \varphi^* W \longrightarrow V$.

Similar statements also apply to ℓ^1-homology [28, Theorem 3.7].

In view of the first two parts of the theorem, the semi-norms on $H^*_{\mathrm{b}}(G;\,\cdot\,)$ and $H^{\ell^1}_*(G;\,\cdot\,)$ that are induced by the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ respectively on the bar constructions are called the *canonical semi-norms*. Notice that by Theorem 2.5.17 the canonical semi-norm coincides with the one given by the topological description of bounded cohomology.

Caveat 2.7.11. It is not difficult to construct (e.g., via scaling) examples of strong relatively projective/injective resolutions that do *not* induce the canonical semi-norms on (co)homology.

Proof (of Theorem 2.7.10). For the first part, recall that

$$H^*_{\mathrm{b}}(G;V) = H^*\big(C^*_{\mathrm{b}}(G;V)\big) = H^*\big(B(C^{\mathbb{R}}_*(G),V)^G\big)$$

and that $B(C^{\mathbb{R}}_*(G),V)$ is a strong relatively injective G-resolution of V (see Proposition 2.7.7). Thus, in view of the fundamental lemma (Proposition 2.7.8), there is a canonical isomorphism $H^*(I^{*G}) \cong H^*_{\mathrm{b}}(G;V)$ of graded semi-normed vector spaces coming from the identity morphism on V.

Using a small calculation one can show that this isomorphism indeed has norm at most 1 [23, Proof of Theorem (3.6)]: Namely, if s^* is a cochain contraction of $\eta\square I^*$ of norm at most 1, then the maps $f^n\colon I^n \longrightarrow B(C^{\mathbb{R}}_n(G),V)$

defined inductively by

$$\begin{aligned} f^0 \colon I^0 &\longrightarrow B\bigl(C_0^{\mathbb{R}}(G), V\bigr) \\ x &\longmapsto \bigl(g_0 \mapsto g_0 \cdot s^0(g_0^{-1} \cdot x)\bigr), \end{aligned}$$

$$\begin{aligned} f^{n+1} \colon I^{n+1} &\longrightarrow B\bigl(C_{n+1}^{\mathbb{R}}(G), V\bigr) \\ x &\longmapsto \bigl(g_0 \cdot [g_1|\cdots|g_{n+1}] \mapsto f^n(g_0 \cdot s^{n+1}(g_0^{-1} \cdot x))(g_0 \cdot g_1 \cdot [g_2|\cdots|g_{n+1}])\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$ form a G-equivariant cochain map of norm at most 1 extending the identity id_V on the coefficients V.

Similarly, the second part follows because $C_*^{\mathbb{R}}(G) \mathbin{\overline{\otimes}} \mathbb{R}$ is a strong relatively projective G-resolution of $\mathbb{R}$ (Proposition 2.7.7); the fact about the semi-norms in this case can be established using results by Park [42, Theorem 2.2].

The third part is also a consequence of the fundamental lemma (Proposition 2.7.8): the commutativity of the diagram follows because also

$$B(\varphi, \Phi) \colon \varphi^* B\bigl(C_*^{\mathbb{R}}(H), W\bigr) \longrightarrow B\bigl(C_*^{\mathbb{R}}(G), V\bigr)$$

is a morphism of Banach G-cochain complexes extending Φ. □

Moreover, in the context of bounded cohomology there is also a dimension shifting mechanism available:

Theorem 2.7.12 (Bounded cohomology, dimension shifting)**.** *Let G be a discrete group.*

1. *If $0 \longrightarrow U \xrightarrow{\ i\ } V \xrightarrow{\ p\ } W \longrightarrow 0$ is a short exact sequence of Banach G-modules where the projection $p \colon V \longrightarrow W$ is relatively projective, then there is a corresponding natural long exact sequence in bounded cohomology:*

$$\cdots \longrightarrow H_b^k(G;U) \xrightarrow{H_b^k(\mathrm{id}_G;i)} H_b^k(G;V) \xrightarrow{H_b^k(\mathrm{id}_G;p)} H_b^k(G;W) \xrightarrow{\ \delta^k\ } H_b^{k+1}(G;U) \longrightarrow \cdots$$

   The connecting homomorphism $\delta^k \colon H_b^k(G;W) \longrightarrow H_b^{k+1}(G;U)$ is continuous with respect to the canonical semi-norm.
2. *For all relatively injective Banach G-modules I we have $H_b^k(G;I) = 0$ for all $k \in \mathbb{N}_{>0}$.*

Proof. For the proof of the first part we use the description of bounded cohomology via the Banach bar complex, which is a strong relatively projective G-resolution of $\mathbb{R}$. It is not difficult to show that taking G-morphisms from a relatively projective G-module to a short exact sequence of Banach G-modules with a relatively projective projection preserves exactness [37, Chapter 8.2]. Therefore, we obtain the natural long exact sequences in bounded cohomology from the normed version of the snake lemma (Proposition 2.3.16).

The vanishing of bounded cohomology in higher degree on relatively injective modules is a consequence of the description in terms of strong relatively injective resolutions (Theorem 2.7.10). □

Caveat 2.7.13 (Bounded cohomology, axiomatically)**.** It is tempting to combine the dimension shifting mechanism with the fact that there are enough injective modules in this context (Remark 2.7.4) and the fact that bounded cohomology in degree 0 coincides with the invariants functor $\mathrm{GrpBan}^{-} \longrightarrow \mathrm{Vec}^{\|\cdot\|}$ in order to obtain an axiomatic description of bounded cohomology along the lines of derived functors. Notice however that the connecting homomorphism, even if it is bijective, in general will only be continuous but not necessarily bicontinuous or even isometric. So the naïve axiomatic approach kills the information on the semi-norm, which is central for the applications.

2.8 The mapping theorem

The characterisation of amenable groups in terms of the vanishing of higher bounded cohomology (Theorem 2.6.14) gave already a taste of what to expect from bounded cohomology in the context of amenability. In the following, we will look at further results in this direction:

– Algebraically, averaging over amenable groups shows that bounded cohomology cannot see amenable normal subgroups.
– The negligence of bounded cohomology towards amenable objects can be exploited in the context of bounded cohomology of topological spaces: looking at the homotopy groups of a space, we see that almost all homotopy theoretic information is Abelian (and hence amenable) – except for the fundamental group. This observation is the key to Gromov's mapping theorem [19, 23]: Continuous maps between spaces that on the level of fundamental groups are surjective and have amenable kernel induce isometric isomorphisms in bounded cohomology; in particular, all spaces with a given fundamental group compute bounded cohomology of this particular group.

In this section, we will first sketch a proof of the algebraic mapping theorem on amenable kernels. After that we will sketch a proof of the topological mapping theorem – following the strategy of Ivanov [23]. Applications of the mapping theorem to geometry and topology are given in the next chapter.

2.8.1 The mapping theorem, algebraically

Recall that a group is amenable if it possesses an invariant mean on the set of bounded functions (Definition 2.6.1). Using this mean, averaging and splitting maps on the level of bounded cochains can be produced. An example of such a result is the following generalisation of the vanishing result for bounded cohomology of amenable groups [19, 23, 40, 37]:

Theorem 2.8.1 (Mapping theorem, algebraically)**.** *Let G be a discrete group, let A be an amenable normal subgroup, and let V be a Banach G-module. We write $\pi\colon G \longrightarrow G/A$ for the canonical projection and $I\colon V^{\#A} \longrightarrow V^{\#}$ for the inclusion of the A-fixed points into the dual Banach G-module of V; notice that $V^{\#A}$ inherits the structure of a Banach G/A-module from the Banach G-module structure on $V^{\#}$ and that $(\pi, I)\colon (G, V^{\#}) \longrightarrow (G/A, V^{\#A})$ is a morphism in the category* GrpBan⁻*.*

Then the induced map (the so-called inflation homomorphism*)*

$$H_{\mathrm{b}}^*(\pi; I)\colon H_{\mathrm{b}}^*(G/A; V^{\#A}) \longrightarrow H_{\mathrm{b}}^*(G; V^{\#})$$

is an isometric isomorphism of graded semi-normed vector spaces (with respect to the canonical semi-norm).

Sketch of proof. Using amenability of A, i.e., averaging over A, one can construct a morphism

$$t^*\colon B\bigl(C_*^{\mathbb{R}}(G), V^{\#}\bigr) \longrightarrow B\bigl(\pi^* C_*^{\mathbb{R}}(G/A), V^{\#}\bigr)$$

of Banach G-cochain complexes extending $\mathrm{id}_{V^{\#}}$ and satisfying

$$t^* \circ B\bigl(C_*^{\mathbb{R}}(\pi), \mathrm{id}_{V^{\#}}\bigr) = \mathrm{id}_{B(\pi^* C_*^{\mathbb{R}}(G/A), V^{\#})}\,.$$

For example, in degree 0 we can take (where $m_V\colon B(A, \mathrm{res}_A^G V^{\#}) \longrightarrow V^{\#}$ is an A-equivariant mean as in the proof of Theorem 2.6.14)

$$\begin{aligned} B\bigl(C_0^{\mathbb{R}}(G), V^{\#}\bigr) &\longrightarrow B\bigl(\pi^* C_0^{\mathbb{R}}(G/A), V^{\#}\bigr) \\ f &\longmapsto \bigl(g_0 \cdot A \mapsto g_0 \cdot m_V(a \mapsto g_0^{-1} \cdot f(g_0 \cdot a))\bigr). \end{aligned}$$

This has several consequences:

- For all $n \in \mathbb{N}$ the Banach G-module $B(\pi^* C_n^{\mathbb{R}}(G/A), V^{\#})$ is relatively injective.
  Hence, $B(\pi^* C_*^{\mathbb{R}}(G/A), V^{\#})$ can be viewed as a strong relatively injective G-resolution of $V^{\#}$. In particular, we obtain an isomorphism

$$\begin{aligned} H_{\mathrm{b}}^*(G/A; V^{\#A}) &= H^*\bigl(B(C_*^{\mathbb{R}}(G/A), V^{\#A})^{G/A}\bigr) \\ &= H^*\bigl(B(C_*^{\mathbb{R}}(G/A), V^{\#})^{G}\bigr) \\ &\cong H^*\bigl(B(C_*^{\mathbb{R}}(G), V^{\#})^{G}\bigr) \\ &= H_{\mathrm{b}}^*(G; V^{\#}), \end{aligned}$$

which has norm at most 1 (Theorem 2.7.10); the uniqueness part of the fundamental lemma of relative homological algebra allows us to conclude that this isomorphism is $H_{\mathrm{b}}^*(\pi; I) = H^*(B(C_*^{\mathbb{R}}(\pi), I)^G)$.
– On the other hand, the one-sided inverse t^* of $B(C_*^{\mathbb{R}}(\pi), I)$ has also norm at most 1 and therefore witnesses that the canonical semi-norms on $H_{\mathrm{b}}^*(G/A; V^{\#A})$ and $H_{\mathrm{b}}^*(G; V^{\#})$ are the same. □

More results in the same spirit can be obtained by similar arguments [37, Chapter 8.5].

2.8.2 The mapping theorem, topologically

One of the most startling results in the theory of bounded cohomology is the topological mapping theorem – in a cunning way it uses the interaction between topology, homotopy theory, functional analysis and homological algebra.

The original proof by Gromov [19] relies on a variation of simplicial sets and simplicial topology; in the following, we will sketch a proof [27] in terms of homological algebra along the lines of Brooks [3], Ivanov [23], Noskov [40] and Monod [37].

Theorem 2.8.2 (Mapping theorem, topologically)**.** *Let X be a countable connected pointed* CW*-complex with fundamental group G, let V be a Banach G-module, and let X_G be a model of the classifying space BG. Then the classifying map $c\colon X \longrightarrow X_G$ induces an isometric isomorphism*

$$H_{\mathrm{b}}^*(c; \mathrm{id}_{V^{\#}})\colon H_{\mathrm{b}}^*(G; V^{\#}) = H_{\mathrm{b}}^*(X_G; V^{\#}) \longrightarrow H_{\mathrm{b}}^*(X; V^{\#})$$

of graded semi-normed vector spaces (which is natural in the coefficients).

Recall that the *classifying map* of a connected pointed CW-complex X with fundamental group G is the unique (up to homotopy) continuous map $X \longrightarrow BG$ that induces the identity on the level of fundamental groups. The classifying map of a space can be viewed as a geometric incarnation of the $\mathbb{Z}G$-chain map $C_*(\widetilde{X}; \mathbb{Z}) \longrightarrow C_*(\widetilde{BG}; \mathbb{Z})$ provided by the fundamental lemma of homological algebra.

Sketch of proof. Intuitively, this theorem relies on the following facts:

– The homotopy type of a CW-complex is determined by its homotopy groups.
– All higher homotopy groups are Abelian and hence amenable (thus invisible in bounded cohomology).

More concretely, Ivanov's proof of this theorem (with trivial coefficients) consists of the following two steps:

1. Show that $B(C_*(\widetilde{X};\mathbb{R}), V^\#)$ together with the canonical augmentation map is an approximate strong relatively injective G-resolution of $V^\#$.

   Definition 2.8.3 (Approximate strong resolutions)**.** Let G be a discrete group.
   - If C^* is a Banach G-cochain complex and $n \in \mathbb{N}$, we define the *truncated cochain complex* $C^*|_n$ to be the Banach G-cochain complex derived from C^* by keeping only the modules (and the corresponding coboundary operators) in degree $0, \dots, n$ and defining all modules in higher degrees to be 0.
   - An augmented Banach G-cochain complex $(C^*, \varepsilon\colon V \to C^0)$ is an *approximate strong resolution* of the Banach G-module V if for every $n \in \mathbb{N}$, the truncated complex $C^*|_n$ admits a *partial contracting cochain homotopy*, i.e., linear maps $(s_j\colon C^j \to C^{j-1})_{j\in\{1,\dots,n\}}$ and $s_0\colon C^0 \to V$ of norm at most 1 satisfying

   $$\forall_{j\in\{1,\dots,n-1\}} \;\; \delta^{j-1} \circ s_j + s_{j+1} \circ \delta^j = \mathrm{id}_{C^j}$$

   as well as $s_0 \circ \varepsilon = \mathrm{id}_V$.
2. Deduce that $H^*_\mathrm{b}(c;\mathrm{id}_{V^\#})$ is a continuous isomorphism (of norm at most 1).
3. Show that the isomorphism $H^*_\mathrm{b}(c;\mathrm{id}_{V^\#})$ is isometric.

Ad 1. We start by decomposing the universal covering space $\widetilde{X}$ according to its homotopy groups in highly connected pieces: Because $\widetilde{X}$ is a simply connected countable CW-complex, there is a sequence

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 := \widetilde{X}$$

of principal bundles $(p_n)_{n\in\mathbb{N}_{>0}}$ with Abelian structure groups such that

$$\forall_{j\in\{0,\dots,n\}} \;\; \pi_j(X_n) = 0 \qquad \text{and} \qquad \forall_{j\in\mathbb{N}_{>n}} \;\; \pi_j(X_n) = \pi_j(\widetilde{X})$$

holds for all $n \in \mathbb{N}_{>0}$ [23, p. 1096]. In particular, all the X_n are simply connected. Here, the structure groups have the homotopy type of Eilenberg-Mac Lane spaces associated with higher homotopy groups of X.

Let $n \in \mathbb{N}$. Since X_n is n-connected, one can explicitly construct a partial chain contraction

$$\mathbb{R} \xrightarrow{r_0} C_0(X_n;\mathbb{R}) \xrightarrow{r_1} \cdots \xrightarrow{r_n} C_n(X_n;\mathbb{R})$$

with $\|r_j\| \leq 1$ for all $j \in \{0,\dots,n\}$ [23, p. 1097]; this is similar to the construction in the proof of Theorem 2.5.17. Because r_* is bounded, we obtain a partial cochain contraction

$$V^\# = B(\mathbb{R},V^\#) \xleftarrow{B(r_0,\mathrm{id}_{V^\#})} B\bigl(C_0(X_n;\mathbb{R}),V^\#\bigr) \xleftarrow{B(r_1,\mathrm{id}_{V^\#})} \cdots \xleftarrow{B(r_n,\mathrm{id}_{V^\#})} B\bigl(C_n(X_n;\mathbb{R}),V^\#\bigr)$$

with norm at most 1. We now wish to translate this partial contracting cochain map of X_n into one of $\widetilde{X}$: At this point, we make use of the fact that the fibre bundles p_j have Abelian (and hence amenable) structure groups; more precisely, similar to the algebraic mapping theorem we have [27, Lemma B.4]:

Lemma 2.8.4. *Let X and Y be simply connected spaces, let $p\colon X \longrightarrow Y$ be a principal bundle whose structure group is an Abelian topological group G, and let V be a Banach space. Then for each $n \in \mathbb{N}$ there is a partial split of $C^*_{\mathrm{b}}(p;V^\#)|_n$, i.e., a cochain map*

$$t|^*_n\colon C^*_{\mathrm{b}}(X;V^\#)|_n \longrightarrow C^*_{\mathrm{b}}(Y;V^\#)|_n$$

of truncated complexes satisfying for all $j \in \{0,\dots,n\}$

$$t|_n{}^j \circ C^j_{\mathrm{b}}(p;\mathrm{id}_{V^\#}) = \mathrm{id} \qquad \textit{and} \qquad \|t|_n{}^j\| \leq 1.$$

By the above Lemma 2.8.4, for all $j \in \{1,\dots,n\}$ we find partial splits

$$t(j)|^*_n\colon C^*_{\mathrm{b}}(X_{j+1};V^\#)|_n \longrightarrow C^*_{\mathrm{b}}(X_j;V^\#)|_n$$

of $C^*_{\mathrm{b}}(p_j;\mathrm{id}_{V^\#})|_n$. As indicated in Figure 2.9, we then consider the maps $V^\# \xleftarrow{s_0} C^0_{\mathrm{b}}(\widetilde{X};V^\#) \xleftarrow{s_1} \cdots \xleftarrow{s_n} C^n_{\mathrm{b}}(\widetilde{X};V^\#)$ defined by

$$s_j := t(1)|^{j-1}_n \circ \cdots \circ t(n-1)|^{j-1}_n \circ B(r_j,\mathrm{id}_{V^\#}) \circ C^j_{\mathrm{b}}(p_{n-1};V^\#) \circ \cdots \circ C^j_{\mathrm{b}}(p_1;V^\#)$$

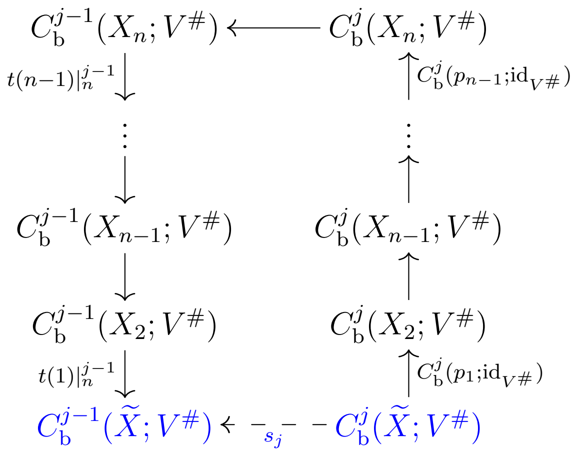


Figure 2.9: Constructing a partial cochain contraction for the bounded cochains of the universal covering space

for all $j \in \{0, \dots, n\}$. By construction, $\|s_j\| \leq 1$ and $s_0, \dots, s_n$ form a partial cochain contraction [23, p. 1096].

It remains to show that the Banach G-modules $B(C_n(\widetilde{X}; \mathbb{R}), V^\#)$ are relatively injective: Let $F \subset \widetilde{X}$ be a fundamental domain for the G-action on $\widetilde{X}$. For $n \in \mathbb{N}$, we write $F_n \subset C_n(\widetilde{X}; \mathbb{R})$ for the subspace generated by all singular simplices mapping the zeroth vertex of Δ^n into F. Then

$$C_n(\widetilde{X}; \mathbb{R}) = \mathbb{R}G \otimes_{\mathbb{R}} F_n$$

(as $\mathbb{R}G$-modules). In particular, we obtain

$$B\big(C_n(\widetilde{X}; \mathbb{R}), V^\#\big) = B\big(\mathbb{R}G \otimes F_n, V^\#\big) = B\big(\ell^1(G), B(F_n, V^\#)\big).$$

Because $B(\ell^1(G), B(F_n, V^\#))$ is a relatively injective Banach G-module (Example 2.7.3), it follows that $B(C_*(\widetilde{X}; \mathbb{R}); V^\#)$ is a relatively injective Banach G-module.

Hence, the cochain complex $B(C_*(\widetilde{X}; \mathbb{R}), V^\#)$ together with the obvious augmentation map is an approximate strong relatively injective G-resolution of $V^\#$.

Ad 2. Let $\widetilde{c}\colon \widetilde{X} \longrightarrow \widetilde{X}_G$ be the base-point preserving lift of c to the universal coverings. Because the inductive proof of the fundamental lemma of homological algebra depends only on finite initial parts of the resolutions in question, it follows that the morphism

$$B\big(C_*(\widetilde{c};\mathbb{R}), \mathrm{id}_{V^\#}\big)\colon B\big(C_*(\widetilde{X}_G,\mathbb{R}), V^\#\big) \longrightarrow B\big(C_*(\widetilde{X};\mathbb{R}), V^\#\big)$$

of approximate strong relatively injective G-resolutions of $V^\#$ (that extends the identity on $V^\#$) induces an isomorphism $H^*_{\mathrm{b}}(X_G; V^\#) \longrightarrow H^*_{\mathrm{b}}(X; V^\#)$. Moreover, this isomorphism has norm at most 1 by construction.

Ad 3. Similarly to the proof of Theorem 2.5.17 we can construct a morphism $C_*(\widetilde{X};\mathbb{R}) \longrightarrow C^{\mathbb{R}}_*(G) \longrightarrow C_*(\widetilde{X}_G;\mathbb{R})$ of normed G-chain complexes of norm at most 1. This morphism induces a morphism

$$B\big(C_*(\widetilde{X}_G;\mathbb{R}), V^\#\big) \longrightarrow B\big(C_*(\widetilde{X};\mathbb{R}), V^\#\big)$$

of Banach G-cochain complexes of norm at most 1 extending the identity on $V^\#$.

Because the inductive proof of the fundamental lemma of homological algebra depends only on finite initial parts of the resolutions in question it follows that this morphism is G-homotopy inverse to $B(C_*(\widetilde{c};\mathbb{R}), \mathrm{id}_{V^\#})$. Hence, $H^*_{\mathrm{b}}(c; \mathrm{id}_{V^\#})$ must be isometric.

Alternatively, for the third step, one can argue using the second step in combination with the properties of the canonical semi-norm on the bounded cohomology $H^*_{\mathrm{b}}(G; V^\#)$ (see Theorem 2.7.10). □

In particular, combining this result with the algebraic version of the mapping theorem (Theorem 2.8.1), we obtain the following astonishing consequences:

Corollary 2.8.5 (Mapping theorem with amenable kernels, topologically)**.** *Let $f\colon X \longrightarrow Y$ be a (base-point preserving) continuous map between countable connected pointed* CW*-complexes such that the induced homomorphism $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and has amenable kernel A. Then for all Banach $\pi_1(X)$-modules V, the induced map*

$$H^*_{\mathrm{b}}(f; I)\colon H^*_{\mathrm{b}}(Y; V^{\#A}) \longrightarrow H^*_{\mathrm{b}}(X; V^\#),$$

where $I\colon V^{\#A} \longrightarrow V^\#$ denotes the inclusion of the A-fixed points, is an isometric isomorphism of semi-normed vector spaces. □

Corollary 2.8.6 (The ℓ^1-semi-norm on singular homology and amenable kernels)**.** *Let $f\colon X \longrightarrow Y$ be a (base-point preserving) continuous map between countable connected pointed* CW*-complexes such that the induced homomorphism $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and has amenable kernel. Then the induced map*

$$H_*(f;\mathrm{id}_\mathbb{R})\colon H_*(X;\mathbb{R}) \longrightarrow H_*(Y;\mathbb{R})$$

is isometric with respect to the ℓ^1-semi-norm.

Proof. This is a consequence of the version of the topological mapping theorem in Corollary 2.8.5 and the duality principle for semi-norms (Proposition 2.3.13). □

Corollary 2.8.7 (The ℓ^1-semi-norm and amenable fundamental group)**.** *Let X be a countable connected* CW*-complex with amenable fundamental group, and let $k \in \mathbb{N}_{>0}$. Then the ℓ^1-semi-norm on $H_k(X;\mathbb{R})$ is zero.* □

In Section 2.9, we will have a closer look at the effect of the mapping theorem on the behaviour of the simplicial volume.

2.8.3 Duality and mapping theorems in ℓ^1-homology

The proofs given for the mapping theorems in bounded cohomology cannot be transferred literally to ℓ^1-homology – the problem being that in general the modules $C_n^\mathbb{R}(G/A)\,\overline{\otimes}\,V$ are not relatively projective Banach G-modules if A is an amenable normal subgroup of G [28, Caveat 5.7].

However, using a suitable duality on the level of (co)homology, the mapping theorems in bounded cohomology *can* be translated into mapping theorems in ℓ^1-homology [28, 7]:

Theorem 2.8.8 (Mapping theorem in ℓ^1-homology)**.** *Let $f\colon X \longrightarrow Y$ be a continuous map between countable connected pointed* CW*-complexes such that the induced map $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and has amenable kernel A; moreover, let V be a Banach $\pi_1(X)$-module. Then*

$$H_*^{\ell^1}(f;P)\colon H_*^{\ell^1}(X;V) \longrightarrow H_*^{\ell^1}(Y;V_A)$$

is an isometric isomorphism.

2.9 Application: Simplicial volume

The simplicial volume is a homotopy invariant of oriented closed connected manifolds, introduced by Gromov in his proof of Mostow rigidity [39, 19]. In a way, the simplicial volume measures how difficult it is to build a manifold out of simplices (allowing non-integral coefficients):

Definition 2.9.1 (Simplicial volume)**.** Let M be an oriented closed connected n-manifold. Then the *simplicial volume* of M is defined as

$$\|M\| := \bigl\{\|c\|_1 \bigm| c \in C_n(M;\mathbb{R}) \text{ is an } \mathbb{R}\text{-fundamental cycle of } M\bigr\} \in \mathbb{R}_{\geq 0},$$

where $\|\cdot\|_1$ denotes the ℓ^1-norm on the singular chain complex with respect to the basis given by all singular simplices (Example 2.3.7).

This harmless looking homotopy invariant is surprisingly closely related to the Riemannian volume (which is not a homotopy invariant in general!); for example, it gives non-trivial lower bounds for the minimal volume and it allows to prove that the volume of hyperbolic closed manifolds is a homotopy invariant(!). In particular, one can obtain degree theorems via the simplicial volume.

In view of these geometric aspects of the simplicial volume, it is attractive to find algebraic tools that help analysing its behaviour; one such tool is bounded cohomology. Indeed, via the duality principle for semi-norms, we see that the simplicial volume can be expressed in terms of bounded cohomology.

In the following, we will give a brief introduction to the simplicial volume and its beautiful properties. We will start with a general discussion of functorial semi-norms and degree theorems; in Sections 2.9.1 through 2.9.3 we have a closer look at the connection between simplicial and Riemannian volume. Then we show how the interplay of simplicial volume with bounded cohomology can be used to prove non-trivial vanishing and inheritance results. Finally, we will present a short list of open problems.

2.9.1 Functorial semi-norms and degree theorems

A degree theorem is a statement of the following form:

Theorem 2.9.2 (Metatheorem – degree theorem)**.** *Let D and T be certain suitable classes of Riemannian manifolds of the same dimension – the domain manifolds and the target manifolds. Then there is a constant $c \in \mathbb{R}$ with the following property: For all $M \in D$, all $N \in T$, and all continuous maps $f \colon M \longrightarrow N$ we have*

$$|\deg f| \leq c \cdot \frac{\operatorname{vol} M}{\operatorname{vol} N}.$$

Of course, the classes of domain and target manifolds have to be chosen very carefully; for example (see Corollary 2.9.10 below):

Theorem 2.9.3 (Degree theorem for hyperbolic manifolds)**.** *Let $n \in \mathbb{N}_{>1}$, and let M and N be oriented closed connected hyperbolic n-manifolds. Then*

$$|\deg f| \leq \frac{\operatorname{vol} M}{\operatorname{vol} N}$$

for all continuous maps $f \colon M \longrightarrow N$.

How can such a degree theorem be proved? A basic strategy is the following: Find a real-valued invariant v on the class of target and domain manifolds D and T respectively that has the following properties:

1. For all continuous maps $f \colon M \longrightarrow N$ with $M \in D$ and $N \in T$ we have $|\deg f| \cdot v(N) \leq v(M)$.
2. There is a $c_D \in \mathbb{R}$ such that for all $M \in D$ we have $v(M) \leq c_D \cdot \operatorname{vol} M$.
3. There is a $c_T \in \mathbb{R}$ such that for all $N \in T$ we have $v(N) \geq c_T \cdot \operatorname{vol} N$.

Putting all three properties together we obtain a degree theorem. The art is now to find appropriate invariants v and good estimates from above and below.

Using the definition of the mapping degree in terms of singular homology in the top degree, we see that the simplicial volume is functorial in the following sense:

Remark 2.9.4 (Functoriality of the simplicial volume)**.** For all continuous maps $f\colon M \longrightarrow N$ of oriented closed connected manifolds of the same dimension we have

$$|\deg f| \cdot \|N\| \leq \|M\|.$$

As homotopy equivalences of oriented closed connected manifolds have degree 1 or -1 it follows that the simplicial volume indeed is a homotopy invariant.

Example 2.9.5 (Simplicial volume of spheres and tori)**.** If $n \in \mathbb{N}_{>0}$, then the sphere S^n and the torus $(S^1)^n$ have self-maps of degree 2; therefore, $\|S^n\| = 0$ and $\|(S^1)^n\| = 0$.

Remark 2.9.6 (Simplicial volume and covering maps)**.** If $f\colon M \longrightarrow N$ is a covering of oriented closed connected manifolds, then

$$|\deg f| \cdot \|N\| = \|M\|.$$

In view of Remark 2.9.4 it suffices to prove "$\geq$;" this can be shown with a suitable transfer map on the level of singular chains. (Exercise)

More generally, the ℓ^1-semi-norm is an example of a so-called functorial semi-norm on singular homology with real coefficients [20, 5.34]. These functorial semi-norms play an important rôle in the study of degree theorems and questions about mapping degrees in general.

2.9.2 Simplicial volume and negative curvature

All examples we discussed so far had vanishing simplicial volume. In the following, we will briefly explain one source of non-vanishing simplicial volume: negative curvature.

Theorem 2.9.7 (Simplicial volume of hyperbolic manifolds)**.** *Let $n \in \mathbb{N}_{>1}$, and let M be an oriented closed connected hyperbolic n-manifold. Then*

$$\|M\| = \frac{\operatorname{vol} M}{v_n},$$

where v_n is the supremal volume of all geodesic n-simplices in hyperbolic n-space.

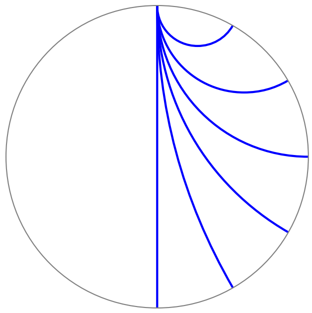


Figure 2.10: Geodesics in hyperbolic n-space $\mathbb{H}^n$

What is a hyperbolic manifold? A *hyperbolic manifold* is a Riemannian manifold whose Riemannian universal covering is isometric to hyperbolic space; this is equivalent to the property that the sectional curvature of the Riemannian metric is everywhere equal to -1.

For $n \in \mathbb{N}_{>1}$ *hyperbolic n-space* $\mathbb{H}^n$ is a Riemannian n-manifold that is homeomorphic to the open n-ball; the Riemannian metric has among others the following features [2, 44]:

- Between any two points in $\mathbb{H}^n$ there exists a unique (up to parametrisation) geodesic.
- In particular, using (ordered) convex combinations via geodesics, for any finite sequence of points in $\mathbb{H}^n$ there is a corresponding geodesic simplex with the given vertices.
- The maximal geodesics in $\mathbb{H}^n$ are precisely those circles in the n-ball that intersect the boundary of the closed n-ball orthogonally (Figure 2.10).
- It is a curious property of hyperbolic n-space that the supremum over the volumes of all geodesic n-simplices in $\mathbb{H}^n$ is *finite*. The reason for this behaviour is the following crucial property of negative curvature: geodesic triangles are "thin" (Figure 2.11). In fact, it is possible to define negative curvature in terms of this property.

Proof (of Theorem 2.9.7). We only sketch Thurston's proof for the estimate

$$\|M\| \geq \frac{\operatorname{vol} M}{v_n};$$

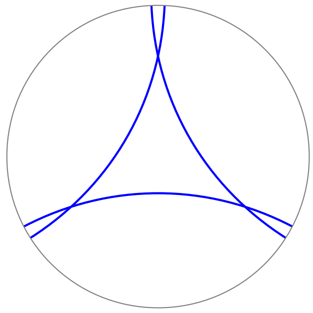


Figure 2.11: Geodesic triangles in negatively curved spaces are thin

the proof of the converse inequality requires different means (Thurston's smearing [52] or discrete versions of it [2]).

Because M is hyperbolic, we can "straighten" any singular simplex on M by lifting it to the universal covering $\mathbb{H}^n$, replacing the lifted simplex by the geodesic simplex on $\mathbb{H}^n$ with the same vertices, and projecting the resulting simplex back to M (Figure 2.12).

This process leads to a chain map $\operatorname{str}_*\colon C_*(M;\mathbb{R}) \longrightarrow C_*(M;\mathbb{R})$ chain homotopic to the identity; this can be proved with arguments similar to those in the proof of Theorem 2.5.17.

Because integrating (smooth) fundamental cycles yields the volume of the manifold in question and because the universal covering map $\mathbb{H}^n \longrightarrow M$ is a local isometry, we obtain

$$\begin{aligned}
\operatorname{vol} M &= \int_{\operatorname{str}_n c} \operatorname{vol}_M \\
&= \sum_{j=0}^{r} a_j \cdot \int (\operatorname{str}_n \sigma_j)^* \operatorname{vol}_M \\
&= \sum_{j=0}^{r} a_j \cdot \int (\operatorname{str}_n \sigma_j)^{\sim} \operatorname{vol}_{\mathbb{H}^n} \\
&\leq \sum_{j=0}^{r} |a_j| \cdot v_n \\
&\leq \|c\|_1 \cdot v_n
\end{aligned}$$

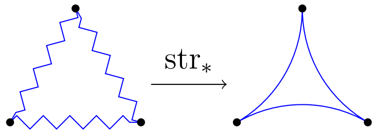


Figure 2.12: The straightening map, schematically

for all fundamental cycles $c = \sum_{j=0}^{r} a_j \cdot \sigma_j$ (in reduced form) of M; taking the infimum over all fundamental cycles gives $\|M\| \geq \operatorname{vol} M / v_n$. □

Corollary 2.9.8 (Simplicial volume of surfaces)**.** *Let M be an oriented closed connected surface of genus $g \geq 2$. Then*

$$\|M\| = 4 \cdot g - 4.$$

Proof. Under the given assumptions, the surface M is hyperbolic [2]. Therefore, we obtain from the Gauß-Bonnet theorem that

$$\|M\| = \frac{\operatorname{vol} M}{v_2} = 4 \cdot g - 4.$$

In fact, in this case, it is possible to give an ad-hoc argument showing that $\|M\| \leq 4 \cdot g - 4$ via finite coverings and explicit triangulations (Exercise). □

Recalling that the simplicial volume and the dimension of oriented closed connected manifolds is a homotopy invariant, we obtain the following astonishing consequence of Theorem 2.9.7:

Corollary 2.9.9 (Homotopy invariance of hyperbolic volume)**.** *The volume of oriented closed connected hyperbolic manifolds is a homotopy invariant.* □

In fact, the simplicial volume originated in Gromov's proof of Mostow rigidity [39] where this result is heavily used.

Moreover, we obtain the following degree theorem:

Corollary 2.9.10 (Degree theorem for hyperbolic manifolds)**.** *Let $n \in \mathbb{N}_{>1}$, and let M and N be oriented closed connected hyperbolic n-manifolds. Then*

$$|\deg f| \leq \frac{\operatorname{vol} M}{\operatorname{vol} N}$$

for all continuous maps $f \colon M \longrightarrow N$. □

Remark 2.9.11 (Generalisations of non-vanishing results)**.**

- Using Mineyev's result on non-vanishing of higher bounded cohomology of hyperbolic groups together with the mapping theorem, one obtains: The simplicial volume of oriented closed connected rationally essential manifolds of dimension at least 2 with word-hyperbolic fundamental group is non-zero.
- Lafont and Schmidt showed that also the simplicial volume of compact locally symmetric spaces of non-compact type is non-zero [25].
- Bucher-Karlsson computed the exact value of the simplicial volume of the product of two oriented closed connected hyperbolic surfaces [6].

Notice that Theorem 2.9.7 also has non-trivial consequences for the fundamental groups of closed hyperbolic manifolds (see the vanishing results of Section 2.9.4) [23].

2.9.3 Simplicial volume and Riemannian volume

In the previous section we saw that the simplicial volume and the Riemannian volume of hyperbolic manifolds coincide up to a constant factor depending only on the dimension of the manifolds in question. Of course, in general, the simplicial volume does not coincide with the Riemannian volume – however, astonishingly, there still are non-trivial connections. We now present two such connections: the relation with the minimal volume and the proportionality principle.

Definition 2.9.12 (Minimal volume)**.** Let M be an oriented closed connected smooth manifold. Then the *minimal volume* of M is defined by

$$\operatorname{minvol} M := \inf\bigl\{\operatorname{vol}(M,g) \bigm| g \text{ is a Riemannian metric on } M \text{ with } |\sec g| \leq 1\bigr\}.$$

A priori it is not clear that there exist smooth manifolds with non-zero minimal volume. However, Gromov discovered the following estimate [19]:

Theorem 2.9.13 (Simplicial volume and minimal volume)**.** *Let M be an oriented closed connected smooth n-manifold. Then*

$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol} M.$$

Example 2.9.14. So in particular, oriented closed connected hyperbolic manifolds have non-zero minimal volume (Theorem 2.9.7).

Another generalisation of the computation of the simplicial volume of hyperbolic manifolds is the following proportionality principle of Gromov and Thurston [20, 52, 50]:

Theorem 2.9.15 (Proportionality principle for simplicial volume)**.** *Let M and N be two oriented closed connected Riemannian manifolds with isometric universal Riemannian coverings. Then*

$$\frac{\|M\|}{\operatorname{vol} N} = \frac{\|N\|}{\operatorname{vol} N}.$$

All known proofs of this theorem make use of the description of the simplicial volume in terms of bounded cohomology.

For example, the non-vanishing result of Lafont and Schmidt for locally symmetric spaces of non-compact type (Remark 2.9.11) relies on the proportionality principle in order to reduce the problem to the case of irreducible locally symmetric spaces of non-compact type.

2.9.4 Simplicial volume and the mapping theorem in bounded cohomology

The duality principle for semi-norms (Proposition 2.3.13) and the mapping theorem in bounded cohomology (Theorem 2.8.2) open the door to many vanishing results and inheritance properties of the simplicial volume. Recall that the mapping theorem roughly says that bounded cohomology of a space depends only on its classifying map (which encodes the fundamental group) and that bounded cohomology cannot see amenable groups.

Example 2.9.16 (Simplicial volume and amenable fundamental group)**.** Let M be an oriented closed connected manifold of non-zero dimension whose fundamental group is amenable; then

$$\|M\| = 0.$$

In order to appreciate the power of the mapping theorem it is a good exercise to sit down and try to prove by geometric means that the simplicial volume of all simply connected manifolds is zero (as far as I know, no such proof is known to date ...).

Example 2.9.17 (Simplicial volume and free fundamental group)**.** Let M be an oriented closed connected manifold of non-zero dimension whose fundamental group is a free group; then

$$\|M\| = 0.$$

Caveat 2.9.18 (Simplicial volume and fundamental group)**.** The simplicial volume does *not* depend only on the fundamental group, but on the classifying map of the manifold in question; namely, if M is an oriented closed connected manifold with non-zero simplicial volume (such manifolds exist: Theorem 2.9.7), then $\pi_1(M) \cong \pi_1(S^2 \times M)$, but $\|S^2 \times M\| = 0$ because $\|S^2\| = 0$ (Theorem 2.9.19).

2.9.5 Simplicial volume, inheritance properties

As indicated above, we can use bounded cohomology (and its nice invariance properties) to derive some inheritance properties of the simplicial volume:

Theorem 2.9.19 (Simplicial volume and products)**.** *Let M and N be oriented closed connected manifolds. Then*

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\| \cdot \|N\|.$$

Proof. For the left hand inequality, use the cohomological cross-product and the duality principle for semi-norms; for the right hand inequality, use the homological cross-product and its explicit description through the shuffle product (Exercise) [19, 2]. □

Theorem 2.9.20 (Simplicial volume and connected sums)**.** *Let M and N be oriented closed connected manifolds of the same dimension ≥ 3. Then*

$$\|M \# N\| = \|M\| + \|N\|.$$

Proof. We prove only the weaker statement that

$$\frac{\|M\| + \|N\|}{2} \leq \|M \# N\| \leq \|M\| + \|N\|.$$

(The proof of the equality as stated in the theorem requires a careful combinatorial analysis of the universal covering of the wedge of two classifying spaces [19].)

Using the continuous maps $M \# N \longrightarrow M$ and $M \# N \longrightarrow N$ of degree 1 collapsing N and M respectively, we obtain

$$\|M \# N\| \geq \|M\| \qquad \text{and} \qquad \|M \# N\| \geq \|N\|,$$

and therefore $\|M \# N\| \geq 1/2 \cdot (\|M\| + \|N\|)$.

To prove the estimate $\|M \# N\| \leq \|M\| + \|N\|$ we look at the pinching map

$$p \colon M \# N \longrightarrow M \vee N$$

that collapses the sphere along which M and N are glued together; because $\dim M = \dim N \geq 3$, the theorem of Seifert and van Kampen shows that the induced homomorphism $\pi_1(p) \colon \pi_1(M \# N) \longrightarrow \pi_1(M \vee N)$ is an isomorphism.

Therefore, the the mapping theorem and the duality principle for semi-norms imply that $H_n(p; \mathrm{id}_\mathbb{R}) \colon H_n(M \# N; \mathbb{R}) \longrightarrow H_n(M \vee N; \mathbb{R})$ is isometric with respect to the ℓ^1-semi-norm (Corollary 2.8.6). In particular,

$$\|M \# N\| = \big\|H_n(p; \mathbb{R})([M \# N]_\mathbb{R})\big\|_1;$$

on the other hand, in view of the Mayer-Vietoris sequence, we know that the class $H_n(p; \mathbb{R})([M \# N]_\mathbb{R})$ is nothing but the sum of the two classes $[M]_\mathbb{R}$ and $[N]_\mathbb{R}$ in

$$H_n(M \# N; \mathbb{R}) \cong H_n(M; \mathbb{R}) \oplus H_n(N; \mathbb{R}).$$

This shows that

$$\|M \# N\| = \big\|H_n(p; \mathbb{R})([M \# N]_\mathbb{R})\big\|_1 \leq \|M\| + \|N\|,$$

as desired. □

The above theorem does *not* hold in dimension 2: The simplicial volume of the torus is 0, whereas the simplicial volume of oriented closed connected surfaces of higher genus is non-zero (Corollary 2.9.8).

Theorem 2.9.21 (Simplicial volume and fibrations)**.** *Let $p\colon M \longrightarrow B$ be a fibration of oriented closed connected manifolds whose fibre F also is an oriented closed connected manifold of non-zero dimension, and suppose that the fundamental group of F is amenable. Then*

$$\|M\| = 0.$$

Proof. This is a straightforward application of the mapping theorem [29, Exercise 14.15 and p. 556]. □

2.9.6 Open problems

1. *Infinite amenable normal subgroups (W. Lück).* For which oriented closed connected manifolds M whose fundamental group contains an infinite amenable normal subgroup does $\|M\| = 0$ hold?
2. *Simplicial volume and Euler characteristic (M. Gromov).* Does for all oriented closed connected aspherical manifolds M with $\|M\| = 0$ follow that $\chi(M) = 0$?
3. *Integral simplicial volume.* What is the exact relation between the simplicial volume of oriented closed connected aspherical manifolds and the integral simplicial volume (defined via fundamental cycles with *integral* coefficients) of all finite coverings of the manifold in question?
4. *Simplicial volume and ℓ^1-homology.* Do there exist oriented closed connected manifolds M with $\|M\| = 0$ such that the image of the fundamental class $[M]_{\mathbb{R}}$ in ℓ^1-homology is non-zero?

2.10 Exercises

Most of the exercises are grouped into collections of four exercises, covering the material of one week of lectures. The exercises vary in difficulty; some of them are straightforward applications of the material presented in the text, while others require additional knowledge (e.g., from algebraic topology).

Exercise sheet #11

Exercise 2.1 (Homology of normed chain complexes)**.** Give an example of a normed chain complex C_* and a $k \in \mathbb{N}$ with the property that the k-th reduced homology of C_* vanishes but the k-th cohomology of the dual complex $C^{\#*}$ does *not* vanish. In particular, the induced semi-norm on cohomology of the dual complex can in general not be expressed in terms of the induced semi-norm on homology.

Exercise 2.2 (Simplicial volume and mapping degrees)**.**

1. Let $f\colon M \longrightarrow N$ be a continuous map of oriented closed connected manifolds of the same dimension with non-zero degree $\deg f$. Show that
$$\|N\| \leq \frac{1}{|\deg f|} \cdot \|M\|$$
and determine the simplicial volume of spheres and tori.
2. Let $f\colon M \longrightarrow N$ be a d-sheeted covering of oriented closed connected manifolds. Prove that
$$\|M\| = d \cdot \|N\|.$$

Exercise 2.3 (Simplicial volume and products)**.** Let M and N be oriented closed connected manifolds. Show that

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\| \cdot \|N\|.$$

Hints. For the right hand inequality use the homological cross-product, for the left hand inequality use the cohomological cross-product.

Exercise 2.4 (ℓ^p-norms on the singular chain complex)**.** Let X be a topological space. For $p \in [1, \infty]$ the *ℓ^p-norm* $\|\cdot\|_p$ on the singular chain complex $C_*(X; \mathbb{R})$ is the p-norm with respect to the (unordered) basis given by the set of all singular simplices; i.e., for a chain $c = \sum_{j=1}^{k} a_j \cdot \sigma_j \in C_*(X; \mathbb{R})$ in reduced form,

$$\|c\|_p := \left(\sum_{j=1}^{k} |a_j|^p \right)^{1/p}.$$

In the following, let $p \in (1, \infty]$ and let X be a path-connected space that contains at least two points.

1. Show that $C_*(X; \mathbb{R})$ is *not* a normed chain complex with respect to the ℓ^p-norm.
2. The norm $\|\cdot\|_p$ induces a semi-norm on homology $H_*(X; \mathbb{R})$. What can be said about this semi-norm?

Exercise sheet #12

Exercise 2.5 (Bounded cohomology in degree 0)**.**

1. Show that $H_b^0(\,\cdot\,;\,\cdot\,)\colon \mathrm{GrpBan}^- \longrightarrow \mathrm{Vec}^{\|\cdot\|}$ coincides with the invariants functor.
2. What happens in ℓ^1-homology in degree 0 ?

Exercise 2.6 (Bounded cohomology in degree 1)**.**

1. Let G be a discrete group. Using the description of bounded cohomology in terms of the bar resolution, show that $H_b^1(G;\mathbb{R}) = 0$.
2. What about $H_1^{\ell^1}(\,\cdot\,;\mathbb{R})$?

Exercise 2.7 (Quasi-morphisms)**.** Let G be a discrete group. A quasi-morphism $f\colon G \longrightarrow \mathbb{R}$ is called *homogeneous* if

$$f(g^n) = n \cdot f(g)$$

for all $g \in G$ and all $n \in \mathbb{Z}$.

1. Is every quasi-morphism homogeneous (up to a uniformly bounded additive error)?
2. Are there any non-trivial quasi-morphisms on Abelian groups?

Exercise 2.8 (Simplicial volume of surfaces)**.**

1. Let $d \in \mathbb{N}_{>0}$. Using the fundamental group, show that every oriented closed connected surface of genus at least 1 has a d-sheeted covering by a connected surface.
2. Give a graphical representaion of a double covering of an oriented closed connected surface of genus 2.
   Hints. As first step, draw a double covering of $S^1 \vee S^1$ by $S^1 \vee S^1 \vee S^1$.
3. Let F be an oriented closed connected surface of genus $g \in \mathbb{N}_{>0}$. Show that $\|F\| \leq 4 \cdot g - 4$.
   Hints. In fact, even $\|F\| = 4 \cdot g - 4$ holds; the proof of this fact requires some background in Riemannian geometry.

Exercise sheet #13

Exercise 2.9 (Inheritance properties of amenable groups)**.** In the following, all groups are assumed to be discrete.

1. Show that subgroups of amenable groups are amenable.
2. Show that quotients of amenable groups are amenable.
3. Show that extensions of amenable groups by amenable groups are amenable.

Exercise 2.10 (Amenable groups and Følner sequences)**.**

1. Prove that a discrete countable group is amenable if and only if it admits a Følner sequence.
   Hints. Use (non-principal) ultrafilters on $\mathbb{N}$ to obtain an invariant mean out of a Følner sequence.
   For the (more difficult) converse direction it is helpful to have a closer look at the duality of B and ℓ^1.
2. Using the characterisation via Følner sequences show that the free group on two generators is not amenable.

Exercise 2.11 (Relatively injective/projective modules)**.**

1. Let G be a discrete group. Show that the trivial Banach G-module $\mathbb{R}$ is relatively injective if and only if G is amenable.
2. For which discrete groups G is the trivial Banach G-module $\mathbb{R}$ relatively projective?

Exercise 2.12 (Fundamental theorem of homological algebra, Banach version)**.** Let G be a discrete group and let $f\colon U \longrightarrow V$ be a morphism of Banach G-modules; moreover, let $(C^*, \vartheta\colon U \hookrightarrow C^0)$ be a strong (cohomological) G-resolution of U and let $(I^*, \eta\colon V \hookrightarrow I^0)$ be an augmented Banach G-cochain complex consisting of relatively injective Banach G-modules. Prove the following:

1. The morphism f can be extended to a morphism $f \mathbin{\square} f^*\colon \vartheta \mathbin{\square} C^* \longrightarrow \eta \mathbin{\square} I^*$ of Banach G-cochain complexes.
2. The extension $f^*\colon C^* \longrightarrow I^*$ is unique up to G-homotopy (that is bounded in every degree).

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