ℓ^1-Homology and Simplicial Volume

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Introduction

A pervasive theme of contemporary mathematics is to explore rigidity phenomena caused by the symbiosis of algebraic topology and Riemannian geometry on manifolds. In this context, the term "rigidity" refers to the astounding fact that certain topological invariants provide obstructions for geometric structures. Consequently, topological invariants of this type serve as interfaces between topology and geometry, thereby generating a rewarding exchange between both fields.

Over time many such interfaces evolved, the forefather being the Gauß-Bonnet theorem [30; Chapter 9], which reveals the Euler characteristic of compact surfaces as an obstruction for specific types of curvature.

Another way to think of the Gauß-Bonnet theorem is to view it as a topological bound for the minimal volume of compact surfaces [18; p. 5], where the minimal volume of a smooth manifold M is defined as

$$\begin{aligned}\operatorname{minvol} M := \inf \bigl\{ \operatorname{vol}(M, g) \bigm| & \ g \text{ a complete Riemannian metric on } M \\ & \text{with } |\sec(g)| \leq 1 \bigr\}.\end{aligned}$$

By the Gauß-Bonnet theorem, the minimal volume of an oriented, closed, connected surface M can be estimated from below by $|2\pi \cdot \chi(M)|$, which is a topological invariant.

Similarly, the Euler characteristic of higher dimensional oriented, closed, connected, smooth manifolds yields lower bounds for the minimal volume [18; p. 6]. However, the vanishing of the Euler characteristic of oriented, closed, connected, odd-dimensional manifolds suggests to strive for other topological invariants that encode information on the minimal volume.

One example of such an invariant is the so-called simplicial volume: The simplicial volume is a proper homotopy invariant of oriented manifolds measuring the complexity of the fundamental class with real coefficients with respect to the ℓ^1-norm: If M is an oriented n-manifold, then the simplicial volume of M is defined as

$$\|M\| := \inf\bigl\{\|c\|_1 \bigm| c \text{ is an } \mathbf{R}\text{-fundamental cycle of } M\bigr\},$$

where $\bigl\|\sum_j a_j \cdot \sigma_j\bigr\|_1 := \sum_j |a_j|$.

For example, the simplicial volume of spheres and tori is zero, whereas the simplicial volume of oriented, closed, connected, negatively curved manifolds is non-zero [57, 18, 24].

Originally, the simplicial volume was designed by Gromov to give an alternative proof of the Mostow rigidity theorem [44, 1]. In his pioneering article *Volume and bounded cohomology* [18], Gromov establishes a vast number of links between the simplicial volume and Riemannian geometric quantities, one of them being the volume estimate [18; p. 12, p. 73]: If M is an oriented, connected, smooth n-manifold without boundary, then

$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol} M.$$

Together with the explicit computation of the simplicial volume of closed hyperbolic manifolds in terms of the hyperbolic volume [57, 18, 1, 50; Chapter 6, Section 2.2, Theorem C.4.2, Theorem 11.4.3] and the proportionality principle [18, 57, 56; Section 2.3, p. 6.9, Chapter 5], the volume estimate indicates that the simplicial volume indeed can be understood as a "topological approximation" of the Riemannian volume.

In view of these geometric consequences, it is desirable to find topological and algebraic tools that compute the simplicial volume. In the present thesis, we investigate such a tool – a functional analytic variant of singular homology, called ℓ^1-homology.

ℓ^1-Homology and bounded cohomology

Taking the completion and the topological dual of the singular chain complex with $\mathbf{R}$-coefficients of a space X with respect to the ℓ^1-norm gives rise to the ℓ^1-homology $H_*^{\ell^1}(X)$ and bounded cohomology $H_{\mathrm{b}}^*(X)$ of X respectively.

Both the ℓ^1-norm on the ℓ^1-chain complex $C_*^{\ell^1}(X)$ and the supremum norm on the bounded cochain complex $C_{\mathrm{b}}^*(X)$ induce semi-norms on the level of (co)homology. Gromov observed that the simplicial volume of oriented, closed, connected algebra — functional analysis

singular cohomology — bounded cohomology

$$\begin{tikzcd}
H^*(X) \arrow[d, leftrightarrow, "\text{universal coefficient theorem}"'] & C^*(X) \arrow[l, squiggly] & C^*_{\mathrm{b}}(X) \arrow[r, squiggly] & H^*_{\mathrm{b}}(X) \arrow[d, phantom, "?"] \\
H_*(X) & C_*(X) \arrow[l, squiggly] \arrow[u, squiggly, "\text{alg. dual}"] \arrow[ur, squiggly, "\text{top. dual}"'] \arrow[r, squiggly, "\text{completion}"'] & C^{\ell^1}_*(X) \arrow[u, squiggly, "\text{top. dual}"'] \arrow[r, squiggly] & H^{\ell^1}_*(X)
\end{tikzcd}$$

singular homology — *ℓ^1-homology*

Linking various (co)homology theories related to singular homology

manifolds can be expressed in terms of the semi-norm on bounded cohomology [18; p. 17]. Similarly, we show that the simplicial volume can be computed by the ℓ^1-semi-norm on ℓ^1-homology, both in the compact and in the non-compact case (Section 5.3). For example, the simplicial volume of a closed manifold with vanishing ℓ^1-homology or vanishing bounded cohomology is zero.

By passing from singular cohomology to bounded cohomology deep properties of the simplicial volume become visible; in fact, the work of Gromov [18] and Ivanov [25] shows that the seemingly small difference in the definition of bounded cochains and singular cochains has drastic consequences for the behaviour of the corresponding cohomology theories:

- Bounded cohomology of spaces with amenable fundamental group vanishes (in non-zero degree) [18, 25].
- Bounded cohomology depends only on the fundamental group [18, 25].
- Bounded cohomology admits a description in terms of certain injective resolutions [25, 42, 45].

Singular homology and cohomology are intrinsically tied together by the universal coefficient theorem. Therefore, it is natural to ask whether ℓ^1-homology and bounded cohomology admit a similar link; more specifically:

- Does ℓ^1-homology of spaces with amenable fundamental group vanish?
- Does ℓ^1-homology depend only on the fundamental group?
- Does ℓ^1-homology admit a description in terms of certain projective resolutions?

- More generally: What is the relation between bounded cohomology and ℓ^1-homology? Is there some kind of duality?

It is the purpose of this thesis to give a both uniform and lightweight approach to answer these questions.

A first step towards results of this type is the insight of Matsumoto and Morita that ℓ^1-homology of a space is trivial if and only if bounded cohomology of the space in question is trivial [38; Corollary 2.4]. In particular, ℓ^1-homology of a space with amenable fundamental group is zero in non-zero degree. This led Matsumoto and Morita to suspect that – like bounded cohomology – also ℓ^1-homology depends only on the fundamental group [38; Remark 2.6].

Bouarich was the first to prove that ℓ^1-homology indeed depends only on the fundamental group [5; Corollaire 6]; his proof is based on the result of Matsumoto and Morita that ℓ^1-homology of simply connected spaces is trivial and on an ℓ^1-version of Brown's theorem.

The translation mechanism

Although there is no real duality between bounded cohomology and ℓ^1-homology (in the spirit of the universal coefficient theorem, cf. Remark (3.4)), ℓ^1-homology and bounded cohomology are strongly linked by the translation mechanism below.

The ℓ^1-chain complex of a space is an example of a so-called Banach chain complex, i.e., it is a chain complex of Banach spaces whose boundary operators are bounded operators. Other examples of Banach chain complexes occurring naturally in our context are ℓ^1-chain complexes of discrete groups (i.e., the ℓ^1-completion of the bar resolution) as well as ℓ^1-chain complexes of spaces and discrete groups with twisted coefficients.

Matsumoto and Morita's vanishing result can be generalised to an approximation of the universal coefficient theorem: The homology of a Banach chain complex vanishes if and only if the cohomology of the topological dual of the complex vanishes [26; Proposition 1.2]. Applying this duality principle to mapping cones of morphisms of Banach chain complexes, we derive the following relative version (Theorem (3.1)):

Theorem (Translation mechanism for isomorphisms). *Let $f\colon C \longrightarrow D$ be a morphism of Banach chain complexes and let $f'\colon D' \longrightarrow C'$ be its topological dual.*

1. *Then the induced homomorphism $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism*

of vector spaces if and only if $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ *is an isomorphism of vector spaces.*

2. *Furthermore, if* $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ *is an isometric isomorphism, then also* $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ *is an isometric isomorphism.*

By applying this translation mechanism to suitable morphisms of Banach chain complexes, we can transfer results on bounded cohomology to the realm of ℓ^1-homology (see Chapter 4). The main results that we deduce using this technique are the following:

Corollary (Mapping theorem for ℓ^1-homology). *Let* $f\colon X \longrightarrow Y$ *be a continuous map between countable, connected CW-complexes such that the induced map* $\pi_1(f)$ *is surjective and has amenable kernel. Then the induced homomorphism*

$$H_*^{\ell^1}(f)\colon H_*^{\ell^1}(X) \longrightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism.

Corollary (ℓ^1-Homology via projective resolutions). *Let X be a countable, connected CW-complex with fundamental group G and let V be a Banach G-module.*

1. *Then there is a canonical isometric isomorphism*

$$H_*^{\ell^1}(X;V) \cong H_*^{\ell^1}(G;V).$$

2. *If C is a strong relatively projective G-resolution of V, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X;V) \cong H_*(C_G).$$

3. *If C is a strong relatively projective G-resolution of the trivial Banach G-module* $\mathbf{R}$, *then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X;V) \cong H_*\big((C \mathbin{\overline{\otimes}} V)_G\big).$$

Park tried to use an approach similar to Ivanov's work [25] on bounded cohomology to prove these results (with trivial coefficients) [47]. However, not all of Ivanov's arguments can be carried over to ℓ^1-homology and her proofs contain a significant gap – this issue is addressed in Caveats (4.13) and (4.15).

As an example application of the description of ℓ^1-homology via projective resolutions, we obtain a straightening on ℓ^1-homology generalising Thurston's straightening on non-positively curved Riemannian manifolds (Section 4.4). The ℓ^1-straightening in turn gives a new, homological, proof of the fact that measure homology and singular homology are isometrically isomorphic (Appendix D).

A finiteness criterion for the simplicial volume of non-compact manifolds

The results presented so far might give the impression that ℓ^1-homology is merely a shadow of bounded cohomology, its only advantage being that the stored information is more accessible than in bounded cohomology – due to the fact that homology usually is more intuitive than cohomology. But there are also genuine applications of ℓ^1-homology, as the following example shows:

The simplicial volume of non-compact manifolds, defined via locally finite fundamental cycles, is not finite in general. It might even then be infinite if the non-compact manifold in question is the interior of a compact manifold with boundary. According to Gromov, the vanishing of the simplicial volume of the boundary of the compactification is a necessary condition for the simplicial volume of the interior to be finite [18; p. 17]. More generally, we show that ℓ^1-homology allows to give a necessary and sufficient condition for the finiteness of the simplicial volume of the interior (Theorem (6.1)):

Theorem (Finiteness criterion). *Let $(W, \partial W)$ be an oriented, compact, connected n-manifold with boundary and let $M := W^\circ$. Then the following are equivalent:*

1. *The simplicial volume $\|M\|$ is finite.*
2. *The fundamental class of the boundary ∂W vanishes in ℓ^1-homology, i.e.,*

$$H_{n-1}(i_{\partial W})\bigl([\partial W]\bigr) = 0 \in H^{\ell^1}_{n-1}(\partial W),$$

where $i_{\partial W} \colon C_*(\partial W) \longrightarrow C^{\ell^1}_*(\partial W)$ is the natural inclusion of the singular chain complex of ∂W into the ℓ^1-chain complex.

Bounded cohomology, on the other hand, in general cannot detect whether a given class in ℓ^1-homology is zero; therefore, the finiteness criterion cannot be formulated in terms of bounded cohomology.

The finiteness criterion gives rise to a number of computations, or at least estimates, of the simplicial volume in the non-compact case (Section 6.4). These examples might be the starting point for a more detailed analysis of simplicial volume of non-compact manifolds via ℓ^1-homology.

Organisation of this work

Scriptum est omne divisum in partes tres: The first part, Chapters 1 through 4, deals with ℓ^1-homology and bounded cohomology as well as with the relation between these theories – the main goal being the link given by the translation mechanism.

Chapters 5 and 6 constitute the second part, which is more geometric in nature; this part deals with the simplicial volume and applications of ℓ^1-homology to the simplicial volume.

The Appendices A to D form the third part. The appendices contain material used in the rest of the thesis, but leading too far astray to be included in the main text.

We now describe the chapters in more detail: In Chapter 1, we introduce the basic objects of study – normed chain complexes and their homology. The main examples of these concepts are ℓ^1-homology and bounded cohomology, which enter the scene in Chapter 2. In addition to basic properties of ℓ^1-homology and bounded cohomology, a survey of the distinguishing features of bounded cohomology – such as the mapping theorem – is given in Section 2.4.

In Chapter 3, we return to the more abstract setting, investigating the duality between homology and cohomology of Banach chain complexes. In particular, we give a full proof of the translation mechanism. In Chapter 4, with help of the translation mechanism, we transfer results from bounded cohomology to ℓ^1-homology.

The simplicial volume and its relation to both ℓ^1-homology and bounded cohomology is studied in Chapter 5; in Section 5.4, we give a survey of known properties of simplicial volume. Chapter 6 is devoted to the finiteness criterion and its consequences.

The sequence of appendices starts with a review of the version of homological algebra suitable for our Banach-flavoured setting (Appendix A). In the second appendix (Appendix B), we generalise Ivanov's proof that bounded cohomology of spaces coincides with bounded cohomology of the fundamental group to bounded cohomology with twisted coefficients. Appendix C contains a proof of Gromov's description of simplicial volume of non-compact manifolds in terms of bounded cohomology. Finally, Appendix D gives an introduction to measure homology and its ℓ^1-version. Using the techniques established in Section 4.4, we give a new, homological, proof of the fact that measure homology and singular homology are isometrically isomorphic.

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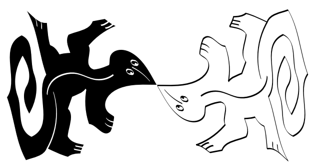

Contents

1 Homology of normed chain complexes

In this chapter, we introduce the basic objects of study – normed chain complexes and their homology. A normed chain complex is a chain complex equipped with a norm such that the boundary operators all are bounded operators. In particular, the homology of a normed chain complex inherits a semi-norm. For example, in the case of the singular chain complex equipped with the ℓ^1-norm this semi-norm contains valuable geometric information such as the simplicial volume.

In order to understand this semi-norm on homology it suffices to understand the semi-norm on homology of the corresponding completed normed chain complex (Proposition (1.7)). Therefore, in many situations we can restrict ourselves to the investigation of Banach chain complexes; in the case of singular homology this corresponds to the investigation of ℓ^1-homology.

In the first section we give concise definitions of the categories of normed and Banach chain complexes and introduce basic constructions on them. Section 1.2 deals with the homology of normed chain complexes, emphasising the induced semi-norms. Finally, in Section 1.3 we present the corresponding equivariant setting, because many normed chain complexes in fact naturally occur as (co)invariants of equivariant ones.

The discussion of examples of these concepts follows in Chapter 2.

1.1 Normed chain complexes

In this text, we use the convention that Banach spaces are Banach spaces over $\mathbf{R}$ and that all (co)chain complexes are indexed over the set $\mathbf{N}$ of natural numbers.

Definition (1.1). 1. A **normed chain complex** is a chain complex of normed vector spaces, where all boundary morphisms are bounded linear operators. Analogously, a **normed cochain complex** is a cochain complex of normed vector spaces, where all coboundary morphisms are bounded linear operators.

2. A **Banach (co)chain complex** is a normed (co)chain complex consisting of Banach spaces.
3. A **morphism of normed (co)chain complexes** is a (co)chain map between normed (co)chain complexes consisting of bounded operators. ◇

In our context, the fundamental examples of the concept of normed chain complexes are given by the singular chain complex and the bar resolution equipped with the obvious ℓ^1-norm (Chapter 2).

The direct sum of two normed (Banach) chain complexes again is a normed (Banach) chain complex, where the norm on the direct sum complex is the sum of the norms on the summands. Moreover, if C is a normed (Banach) chain complex and $D \subset C$ is a *closed* subcomplex, then the quotient C/D is a normed (Banach) chain complex with respect to the quotient norm.

Definition (1.2). Let (C, ∂) be a normed chain complex. Then the **dual cochain complex** (C', ∂') is the normed cochain complex defined by

$$\forall_{n \in \mathbf{N}} \quad (C')^n := (C_n)',$$

where $\cdot'$ stands for taking the (topological) dual normed vector space, together with the coboundary operators

$$\begin{aligned}(\partial')^n := (\partial_{n+1})' \colon (C')^n &\longrightarrow (C')^{n+1} \\ f &\longmapsto \bigl(c \mapsto f(\partial_{n+1}(c))\bigr)\end{aligned}$$

and the norm given by $\|f\|_\infty := \sup\bigl\{|f(c)| \bigm| c \in C_n, \|c\| = 1\bigr\}$ for $f \in (C')^n$. ◇

Remark (1.3). 1. If C is a normed (co)chain complex, then the (co)boundary operator can be extended to a (co)boundary operator on the completion $\overline{C}$ that is bounded in each degree. Hence, the completion $\overline{C}$ of C is a Banach (co)chain complex.

2. If C is a Banach chain complex, then its dual C' is also complete and thus a Banach cochain complex. Moreover, if C is a normed chain complex, then we have $C' = (\overline{C})'$. □

Examples of Banach (co)chain complexes include the ℓ^1-completion of the singular chain complex of topological spaces (Sections 2.1 and 2.3) and the ℓ^1-completion of the bar resolution of discrete groups (Section 2.2), which give rise to the different types of ℓ^1-homology. The corresponding dual complexes are the source for the various incarnations of bounded cohomology.

1.2 (Semi-)norms in homology

Clearly, the presence of chain complexes calls for the investigation of the corresponding homology. In the case of normed chain complexes, the homology groups carry an additional piece of information – the semi-norm.

Definition (1.4). 1. Let (C, ∂) be a normed chain complex and let $n \in \mathbf{N}$. The **n-th homology** of C is the quotient

$$H_n(C) := \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$

2. Dually, if (C, δ) is a normed cochain complex, then its **n-th cohomology** is the quotient

$$H^n(C) := \frac{\ker(\delta^n \colon C^n \to C^{n+1})}{\operatorname{im}(\delta^{n-1} \colon C^{n-1} \to C^n)}.$$

3. Let C be a normed chain complex. The norm $\|\cdot\|$ on C induces a semi-norm, also denoted by $\|\cdot\|$, on the homology $H_*(C)$ as follows: If $\alpha \in H_n(C)$, then

$$\|\alpha\| := \inf\bigl\{\|c\| \bigm| c \in C_n,\, \partial(c) = 0,\, [c] = \alpha\bigr\}.$$

Similarly, we define a semi-norm on the cohomology of normed cochain complexes. ◇

Because the images of the (co)boundary operators of Banach (co)chain complexes are not necessarily closed, the induced semi-norms on (co)homology need not be norms. Therefore, it is sometimes convenient to look at the corresponding reduced versions instead:

Definition (1.5). 1. Let (C, ∂) be a normed chain complex and let $n \in \mathbf{N}$. Then the n**-th reduced homology** of C is given by

$$\overline{H}_n(C) := \ker \partial_n / \overline{\operatorname{im} \partial_{n+1}},$$

where $\overline{\cdot}$ denotes the closure in C.

2. Analogously, if (C, δ) is a normed cochain complex and $n \in \mathbf{N}$, then the n**-th reduced cohomology** of C is given by

$$\overline{H}^n(C) := \ker \delta^n / \overline{\operatorname{im} \delta^{n-1}}. \qquad \diamond$$

Remark (1.6). Any morphism $f \colon C \longrightarrow D$ of normed chain complexes induces linear maps $H_n(f) \colon H_n(C) \longrightarrow H_n(D)$ for each $n \in \mathbf{N}$. Since f is continuous in each degree, these maps descend to linear maps $\overline{H}_n(f) \colon \overline{H}_n(C) \longrightarrow \overline{H}_n(D)$. Moreover, the maps $H_n(f)$ and $\overline{H}_n(f)$ are bounded linear operators. $\square$

In order to understand semi-norms on the homology of normed chain complexes it suffices to consider the case of Banach chain complexes [52; Lemma 2.9]; in the case of singular homology this amounts to restrict attention to ℓ^1-homology (cf. Proposition (2.5)).

Proposition (1.7). *Let D be a normed chain complex and let C be a dense subcomplex. Then the induced map $H_*(C) \longrightarrow H_*(D)$ is isometric. In particular, the induced map $\overline{H}_*(C) \longrightarrow \overline{H}_*(D)$ must be injective.*

Proof. In the following, we write $i \colon C \hookrightarrow D$ for the inclusion and $\|\cdot\|$ for the norm on D.

Because C is a subcomplex, $\|H_*(i)\| \leq 1$. Conversely, let $z \in C_n$ be a cycle and let $\overline{z} \in D_n$ be a cycle such that $[\overline{z}] = H_n(i)([z]) \in H_n(D)$. Furthermore, let $\varepsilon \in \mathbf{R}_{>0}$. To prove the proposition, it suffices to find a cycle $z' \in C_n$ satisfying

$$[z'] = [z] \in H_n(C) \qquad \text{and} \qquad \|z'\| \leq \|\overline{z}\| + \varepsilon.$$

By definition of $\overline{z}$, there must be a chain $\overline{w} \in D_{n+1}$ with $\partial_{n+1}(\overline{w}) = i(z) - \overline{z}$. Since C_{n+1} lies densely in D_{n+1} and since $\|\partial_{n+1}\|$ is finite, there is a chain $w \in C_{n+1}$ such that

$$\big\|\overline{w} - i(w)\big\| \leq \frac{\varepsilon}{\|\partial_{n+1}\|}.$$

Then $z' := z + \partial(w) \in C_n$ is a cycle with $[z'] = [z] \in H_n(C)$ and

$$\big\|\overline{z} - i(z')\big\| = \big\|\partial_{n+1}(\overline{w} - i(w))\big\| \leq \varepsilon.$$

In particular, $\|z'\| \leq \|\overline{z}\| + \varepsilon$. Hence, $H_n(i)$ is an isometry. $\square$

The previous proposition is surprising in the respect that usually the processes of completing and taking homology do not harmonise:

Example (1.8). *There exist normed chain complexes C with $H_*(C) = 0$ and $\overline{H}_*(\overline{C}) \neq 0$, and vice versa:*

1. Let $f\colon C_1 \longrightarrow C_0$ be a continuous operator and let $\overline{f}\colon \overline{C}_1 \longrightarrow \overline{C}_0$ be its extension. Suppose that f is an isomorphism, but $\ker \overline{f} \neq 0$. For example, for $x := \sum_{j\in\mathbf{N}} 1/2^{j+1} \cdot e_j \in \ell^1(\mathbf{N}, \mathbf{R})$, we consider $f\colon \bigoplus_{\mathbf{N}} \mathbf{R} \longrightarrow \operatorname{im} f$ given by $f(e_n) := e_n - x$; here, $\bigoplus_{\mathbf{N}} \mathbf{R}$ and $\ell^1(\mathbf{N}, \mathbf{R})$ are endowed with the ℓ^1-norm.
   We now view the operator $f\colon C_1 \longrightarrow C_0$ as a normed chain complex, concentrated in degrees 0 and 1. By construction, $H_*(C) = 0$, but
   $$\overline{H}_1(\overline{C}) = \ker \overline{f}/\overline{0} \neq 0.$$
2. Let V be a normed vector space that is not complete. Then we view the inclusion $i\colon V \hookrightarrow \overline{V}$ as a normed chain complex C, which is concentrated in degree 0 and 1. Since V is not complete,
   $$H_0(C) = \overline{V} / \operatorname{im} i = \overline{V}/V \neq 0.$$
   But the completion $\overline{C}$ of C is the Banach chain complex given by $\mathrm{id}\colon \overline{V} \longrightarrow \overline{V}$ and hence $H_*(\overline{C}) = 0 = \overline{H}_*(\overline{C})$.

In particular, if a morphism $f\colon C \longrightarrow D$ of normed chain complexes induces an isometric isomorphism on homology, then $\overline{H}_*(\overline{f})$ as well as $H_*(\overline{f})$ in general are neither injective, nor surjective, nor isometric. $\diamond$

An ubiquitous tool in any decent (co)homology theory are long exact sequences provided by the snake lemma. In the world of Banach (co)chain complexes, the snake lemma takes the following form:

Proposition (1.9) (Snake lemma). *Let* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ *be a short exact sequence of Banach chain complexes. Then there is a natural long exact sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(p)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

in homology, and the connecting homomorphism ∂ *is continuous.*

In the same way, short exact sequences of Banach cochain complexes give rise to long exact sequences in cohomology with continuous connecting homomorphisms.

Proof. That the mentioned sequence in homology is exact is a purely algebraic fact following from the snake lemma for **R**-chain complexes [7; Proposition 0.4].

In order to convince ourselves of the continuity of the connecting homomorphism ∂, we first recall the definition of ∂: Let $\gamma \in H_n(C)$, and let $c \in C_n$ be a cycle representing γ. Because $p_n \colon B_n \longrightarrow C_n$ is surjective, there is a chain $b \in B_n$ with $p_n(b) = c$. By construction, $p_{n-1}(\partial_n^B(b)) = \partial_n^C(p_n(b)) = \partial_n^C(c) = 0$, and hence there is an $a \in A_{n-1}$ satisfying $i_{n-1}(a) = \partial_n^B(b)$; moreover, a is a cycle and the class $[a] \in H_{n-1}(A)$ is independent of the choices made. The connecting homomorphism is then defined by

$$\partial(\gamma) := [a] \in H_{n-1}(A).$$

We now proceed using an argument by Monod [42; proof of Proposition 8.2.1]: Let $\varepsilon \in \mathbf{R}_{>0}$. By definition of the semi-norm on homology, the cycle c can be chosen in such a way that $\|c\| \leq \|\gamma\| + \varepsilon$. The open mapping theorem applied to the operators $\widetilde{p}_n \colon B/\ker p_n \longrightarrow C$ and $\widetilde{i}_{n-1} \colon A_{n-1} \longrightarrow \operatorname{im} i_{n-1} = \ker p_{n-1}$ induced by p_n and i_{n-1} respectively shows that these operators have bounded inverses. Therefore, we can choose the chains $b \in B_n$ and $a \in A_{n-1}$ in a such a way that

$$\begin{aligned} \|b\| &\leq \|\widetilde{p}_n^{-1}\| \cdot \big(\|c\| + \varepsilon\big) \\ &\leq \|\widetilde{p}_n^{-1}\| \cdot \big(\|\gamma\| + 2 \cdot \varepsilon\big), \\ \|a\| &\leq \|\widetilde{i}_{n-1}^{-1}\| \cdot \|\partial_n^B(b)\|. \end{aligned}$$

This shows that

$$\|\partial\| \leq \|\widetilde{i}_{n-1}^{-1}\| \cdot \|\partial_n^B\| \cdot \|\widetilde{p}_n^{-1}\| < \infty,$$

i.e., the connecting homomorphism ∂ is continuous. □

1.3 The equivariant setting

We now turn our attention towards equivariant normed chain complexes, i.e., normed chain complexes that additionally carry an isometric group action. Many normed chain complexes in fact are naturally derived from equivariant normed chain complexes by taking (co)invariants.

1.3.1 Banach G-modules

As first step, we recall the definitions of the categories of normed G-modules and of Banach G-modules respectively.

Definition (1.10). Let G be a discrete group.

1. A normed vector space together with an isometric (left) G-action is called a **normed G-module**. A **Banach G-module** is a normed G-module that is complete.
2. A **G-morphism** between normed G-modules is a G-equivariant bounded linear operator. $\diamond$

The most basic example of a Banach G-module with non-trivial group action is $\ell^1(G)$, the set of all ℓ^1-functions $G \longrightarrow \mathbf{R}$ with the G-action given by shifting the argument. Obviously, any Banach G-module is a module over $\ell^1(G)$.

Geometrically, normed G-modules arise in the following way: Let X be a topological space with a continuous G-action and let $n \in \mathbf{N}$. Then the singular chain group $C_n(X)$ inherits a G-action, which is isometric with respect to the ℓ^1-norm.

Definition (1.11). Let G be a group and let V be a normed G-module. The set of **invariants** of V is defined by

$$V^G := \{v \in V \mid \forall_{g \in G} \quad g \cdot v = v\}.$$

The set of **coinvariants** of V is the quotient

$$V_G := V/\overline{W},$$

where $W \subset V$ is the subspace generated by the set $\{g \cdot v - v \mid v \in V, g \in G\}$. $\diamond$

Clearly, if V is a normed (Banach) G-module, then V^G is a normed vector (Banach) space with respect to the restricted norm and V_G is a normed vector (Banach) space with respect to the quotient norm [49; Proposition 2.1.5] – because a *closed* subspace is quotiened out. However, notice that the space W itself used in the previous definition in general need not be closed in V.

Any G-morphism $f\colon V \longrightarrow W$ between normed G-modules induces a bounded linear operator $f_G\colon V_G \longrightarrow W_G$ satisfying $f_G \circ (V \twoheadrightarrow V_G) = (W \twoheadrightarrow W_G) \circ f$ [49; Proposition 2.1.7]. Similarly, any G-morphism $f\colon V \longrightarrow W$ restricts to a bounded linear operator $f^G\colon V^G \longrightarrow W^G$. Clearly, both $\cdot_G$ and $\cdot^G$ are functorial.

Like in the (algebraic) category of G-modules the most basic functors are tensor products and taking homomorphisms:

Definition (1.12). Let G be a discrete group and let U and V be two Banach G-modules.

1. The **projective tensor product** $U \mathbin{\overline{\otimes}} V$ is the Banach G-module whose underlying Banach space is the projective tensor product $U \mathbin{\overline{\otimes}} V$ of Banach spaces, i.e., the completion of the tensor product $U \otimes V$ of $\mathbf{R}$-vector spaces with respect to the norm

$$\forall_{c \in U \otimes V} \quad \|c\| := \inf\biggl\{ \sum_j \|u_j\| \cdot \|v_j\| \biggm| \sum_j u_j \otimes v_j \text{ represents } c \in U \otimes V \biggr\}.$$

   The (diagonal) G-action on $U \mathbin{\overline{\otimes}} V$ is the G-action uniquely determined by

$$\forall_{g \in G} \quad \forall_{u \in U} \quad \forall_{v \in V} \quad g \cdot (u \otimes v) := (g \cdot u) \otimes (g \cdot v).$$

2. The Banach space $B(U, V)$ of all bounded linear functions from U to V (with the operator norm) is a Banach G-module with respect to the G-action

$$\begin{aligned} G \times B(U,V) &\longrightarrow B(U,V) \\ (g, f) &\longmapsto \bigl(u \mapsto g \cdot (f(g^{-1} \cdot u))\bigr). \end{aligned}$$

   In particular, $U' = B(U, \mathbf{R})$ is a Banach G-module (where $\mathbf{R}$ is regarded as the trivial Banach G-module). ◇

If U and V are two Banach G-modules, then $(U \mathbin{\overline{\otimes}} V)_G$ can be viewed as an equivariant tensor product "$U^{\mathrm{op}} \mathbin{\overline{\otimes}}_G V$." We prefer the notation $(U \mathbin{\overline{\otimes}} V)_G$ because it allows us to speak only about left G-modules.

As expected, the functors $\overline{\otimes}$ and B are adjoint in the following sense:

Remark (1.13). Let G be a discrete group and let U, V, and W be Banach G-modules. Then

$$\begin{aligned} B(U \mathbin{\overline{\otimes}} V, W) &\longrightarrow B\bigl(U, B(V, W)\bigr) \\ f &\longmapsto \bigl(u \mapsto (v \mapsto f(u \otimes v))\bigr) \\ \bigl(u \otimes v \mapsto f(u)(v)\bigr) &\longleftarrow f \end{aligned}$$

is an isometric isomorphism of Banach G-modules. □

Furthermore, taking duals converts coinvariants to invariants:

Proposition (1.14). *For all Banach G-modules V the map*

$$\begin{aligned} \varphi\colon (V_G)' &\longrightarrow (V')^G \\ f &\longmapsto f \circ \pi \end{aligned}$$

is a natural isometric isomorphism, where $\pi\colon V \longrightarrow V_G$ *is the canonical projection.*

Proof. It is not hard to see that φ is well-defined and $\|\varphi\| \leq 1$. Conversely, we consider the map

$$\begin{aligned} \psi\colon (V')^G &\longrightarrow (V_G)' \\ f &\longmapsto \overline{f}, \end{aligned}$$

where $\overline{f}\colon V_G \longrightarrow \mathbf{R}$ is the unique continuous functional satisfying $\overline{f} \circ \pi = f$. Moreover, $\|\overline{f}\|_\infty \leq \|f\|_\infty$. Again, it is not difficult to check that ψ is well-defined and that $\|\psi\| \leq 1$.

By construction, $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$, which implies that φ must be an isometric isomorphism. □

1.3.2 Banach G-chain complexes

Assembling normed G-modules and G-morphisms into chain complexes yields the category of normed G-chain complexes:

Definition (1.15). Let G be a discrete group.

1. A **normed G-(co)chain complex** is a normed (co)chain complex consisting of normed G-modules whose (co)boundary operators are G-morphisms.
2. A **Banach G-(co)chain complex** is a normed G-(co)chain complex consisting of Banach G-modules.
3. A **morphism of normed G-(co)chain complexes** is a (co)chain map of Banach G-(co)chain complexes that consists of G-morphisms.
4. Two morphisms of normed G-(co)chain complexes are **G-homotopic** if there exists a (co)chain homotopy between them consisting of G-morphisms. ◇

Fundamental examples of normed G-chain complexes are the bar resolution of G as well as singular chain complexes of G-spaces (Sections 2.2 and 2.3).

The functors of taking invariants and coinvariants respectively extend to functors on the category of Banach G-chain complexes:

Definition (1.16). Let G be a discrete group.

1. If (C, δ) is a Banach G-cochain complex, then C^G is the Banach cochain complex given by $(C^G)^n := (C^n)^G$ with the coboundary operator δ^G induced by δ.
2. If (C, ∂) is Banach G-chain complex, then C_G is the Banach chain complex given by $(C_G)_n := (C_n)_G$ and the boundary operator ∂_G induced by ∂. $\diamond$

Every G-morphism $f\colon C \longrightarrow D$ of normed/Banach G-(co)chain complexes induces morphisms $f^G\colon C^G \longrightarrow D^G$ and $f_G\colon C_G \longrightarrow D_G$ of normed/Banach (co)chain complexes, and these constructions are functorial.

Of course, the operations $\overline{\otimes}$ and B also have a pendant on the level of G-chain complexes:

Definition (1.17). Let G be a discrete group, let (C, ∂) be a Banach G-chain complex and let V be a Banach G-module.

1. The **projective tensor product** $C \mathbin{\overline{\otimes}} V$ is the Banach G-chain complex with
$$(C \mathbin{\overline{\otimes}} V)_n := C_n \mathbin{\overline{\otimes}} V$$
and the boundary operator $\partial \mathbin{\overline{\otimes}} \mathrm{id}_V$.
2. The Banach G-cochain complex $B(C, V)$ is defined by $B(C, V)^n := B(C_n, V)$, equipped with the coboundary operator
$$\begin{aligned} B(\partial_{n+1}, \mathrm{id}_V)\colon B(C, V)^n &\longrightarrow B(C, V)^{n+1} \\ f &\longmapsto \bigl(c \mapsto f(\partial_{n+1}(c))\bigr). \end{aligned}$$
$\diamond$

A straightforward computation (using Remark (1.13)) shows that the functors B and $\overline{\otimes}$ are adjoint. I.e., for all Banach G-chain complexes C and all Banach G-modules U, V, there is a natural isometric isomorphism
$$B(C \mathbin{\overline{\otimes}} U, V) \cong B\bigl(C, B(U, V)\bigr).$$
of Banach G-cochain complexes.

The tools introduced so far allow us to imitate standard constructions of homological algebra related to group (co)homology. For example, if C is a Banach G-chain complex and V is a Banach G-module, then we can build the Banach (co)chain complexes $(C \mathbin{\overline{\otimes}} V)_G$ and $B(C, V)^G$, and, in a second step, take their homology. By choosing suitable complexes C, these (co)chain complexes give rise to ℓ^1-homology and bounded cohomology with twisted coefficients (Sections 2.2 and 2.3). More generally, one can develop a theory of projective and injective resolutions in the framework of Banach G-(co)chain complexes (Appendix A).

2 ℓ^1-Homology and bounded cohomology

The singular chain complex (with real coefficients) of a topological space is a normed chain complex with respect to the ℓ^1-norm. Taking the completion and the topological dual of the singular chain complex gives rise to ℓ^1-homology and bounded cohomology of spaces respectively (Section 2.1).

Similarly, also the bar resolution of a discrete group can be turned into a normed chain complex by introducing a suitable ℓ^1-norm. The completion and the topological dual of this normed chain complex give rise to ℓ^1-homology and bounded cohomology of discrete groups respectively (Section 2.2).

Both constructions can be decorated with equivariant Banach modules, yielding the corresponding theories with (twisted) coefficients (Sections 2.2 and 2.3).

In this chapter, we give an introduction to ℓ^1-homology and bounded cohomology for topological spaces and discrete groups and study their basic properties (Sections 2.1.2 and 2.2.4).

The last section (Section 2.4) contains a survey of more sophisticated properties of bounded cohomology, mainly concerning amenable groups. In Chapter 4, we apply the methods developed in Chapter 3 to transfer these properties to ℓ^1-homology.

2.1 ℓ^1-Homology of spaces

In the next paragraphs, we introduce ℓ^1-homology of topological spaces (with trivial coefficients). Other flavours of ℓ^1-homology like ℓ^1-homology of discrete groups or ℓ^1-homology with twisted coefficients are presented in Section 2.2 and Section 2.3 respectively.

Definition (2.1). Let (X, A) be a pair of topological spaces. The **ℓ^1-norm** on the singular chain complex $C_*(X)$ with real coefficients is defined as follows: For a chain $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_*(X)$ in reduced form we set

$$\|c\|_1 := \sum_{j=0}^{k} |a_j|.$$

The semi-norm on the quotient $C_*(X, A) = C_*(X)/C_*(A)$ induced by $\|\cdot\|_1$ is a norm because $C_*(A)$ is ℓ^1-closed in $C_*(X)$. This norm on $C_*(X, A)$ is also denoted by $\|\cdot\|_1$. $\diamond$

The boundary operator $\partial_n \colon C_n(X, A) \longrightarrow C_{n-1}(X, A)$ is a bounded operator with respect to the ℓ^1-norm of operator norm at most $(n+1)$. Hence, $C_*(X, A)$ is a normed chain complex. Clearly, $C_*(X)$ and $C_*(X, A)$ are in general not complete and thus these complexes are no Banach chain complexes.

Definition (2.2). Let (X, A) be a pair of topological spaces. The **ℓ^1-chain complex** of (X, A) is the ℓ^1-completion

$$C_*^{\ell^1}(X, A) := \overline{C_*(X, A)}^{\ell^1}$$

of the normed chain complex $C_*(X, A)$. We abbreviate $C_*^{\ell^1}(X, \emptyset)$ by $C_*^{\ell^1}(X)$. $\diamond$

By Remark (1.3), the completion $C_*^{\ell^1}(X, A)$ is a Banach chain complex. Furthermore, because $C_*(A)$ is ℓ^1-closed in $C_*(X)$, there is an isometric isomorphism

$$C_*^{\ell^1}(X, A) \cong C_*^{\ell^1}(X)/C_*^{\ell^1}(A)$$

of Banach chain complexes.

Definition (2.3). If (X, A) is a pair of topological spaces, then the Banach cochain complex

$$C_{\mathrm{b}}^*(X, A) := \big(C_*^{\ell^1}(X, A)\big)' = \big(C_*(X, A)\big)'$$

is the **bounded cochain complex** of (X, A). We write $C_{\mathrm{b}}^*(X) := C_{\mathrm{b}}^*(X, \emptyset)$. $\diamond$

Using the isomorphism $C_*^{\ell^1}(X, A) \cong C_*^{\ell^1}(X)/C_*^{\ell^1}(A)$, it is not difficult to see that there is for all $n \in \mathbf{N}$ an isometric isomorphism [49; Proposition 2.1.7]

$$C_{\mathrm{b}}^n(X, A) \cong \big\{f \in C_{\mathrm{b}}^n(X) \bigm| f|_{C_n^{\ell^1}(A)} = 0\big\}.$$

If $f\colon (X, A) \longrightarrow (Y, B)$ is a continuous map of pairs of topological spaces, then the induced map $C_*(f)\colon C_*(X, A) \longrightarrow C_*(Y, B)$ is a chain map that is bounded in each degree (with operator norm equal to 1), i.e., it is a morphism of normed chain complexes. Its extension $C_*^{\ell^1}(f)\colon C_*^{\ell^1}(X, A) \longrightarrow C_*^{\ell^1}(Y, B)$ is a morphism of Banach chain complexes and its dual $C_{\mathrm{b}}^*(f)\colon C_{\mathrm{b}}^*(Y, B) \longrightarrow C_{\mathrm{b}}^*(X, A)$ is a morphism of Banach cochain complexes.

Definition (2.4). Let (X, A) be a pair of topological spaces.

1. Then ℓ^1**-homology** of (X, A) is defined as

   $$H_*^{\ell^1}(X, A) := H_*\big(C_*^{\ell^1}(X, A)\big).$$

   Dually, **bounded cohomology** of (X, A) is given by

   $$H_{\mathrm{b}}^*(X, A) := H^*\big(C_{\mathrm{b}}^*(X, A)\big).$$

   We write $H_*^{\ell^1}(X) := H_*^{\ell^1}(X, \emptyset)$ and $H_{\mathrm{b}}^*(X) := H_{\mathrm{b}}^*(X, \emptyset)$ for short.
2. The semi-norms on $H_*^{\ell^1}(X, A)$ and $H_{\mathrm{b}}^*(X, A)$ are the ones induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively and are also denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$.
3. If $f\colon (X, A) \longrightarrow (Y, B)$ is a continuous map of pairs of topological spaces, then the maps on ℓ^1-homology and bounded cohomology induced by $C_*^{\ell^1}(f)$ and $C_{\mathrm{b}}^*(f)$ are denoted by $H_*^{\ell^1}(f)$ and $H_{\mathrm{b}}^*(f)$ respectively. $\diamond$

By construction, both ℓ^1-homology and bounded cohomology are functorial with respect to composition of continuous maps of (pairs of) topological spaces.

2.1.1 The ℓ^1-semi-norm on ℓ^1-homology of spaces

Despite of the relation between ℓ^1-homology or bounded cohomology and singular (co)homology being rather obscure, both ℓ^1-homology and bounded cohomology compute the ℓ^1-semi-norm on singular homology (Proposition (2.5) and Theorem (3.8)). In fact, it turns out that these theories are much better adapted to the behaviour of the ℓ^1-semi-norm than singular homology itself.

Proposition (2.5). *Let (X, A) be a pair of topological spaces. Then the homomorphism $H_*(X, A) \longrightarrow H_*^{\ell^1}(X, A)$ induced by the inclusion $i_{X,A} \colon C_*(X, A) \hookrightarrow C_*^{\ell^1}(X, A)$ is isometric with respect to the semi-norms on $H_*(X, A)$ and $H_*^{\ell^1}(X, A)$ induced by the ℓ^1-norm.*

In particular, if $H_n^{\ell^1}(X, A) = 0$, then $\|\alpha\|_1 = 0$ for all $\alpha \in H_n(X, A)$.

Proof. This follows from Proposition (1.7) because $C_*(X, A)$ is, by definition, a dense subcomplex of $C_*^{\ell^1}(X, A)$ with respect to $\|\cdot\|_1$. □

An interesting invariant defined in terms of the ℓ^1-semi-norm on singular homology is the simplicial volume (Chapter 5). Proposition (2.5) is the key to study the simplicial volume via ℓ^1-homology and the associated tools.

Moreover, from Proposition (2.5) we can deduce that ℓ^1-homology of topological spaces is not always zero:

Example (2.6). *For any $n \in \mathbf{N} \setminus \{1\}$ there is a topological space with $H_n^{\ell^1}(X) \neq 0$:*

For example, if M is an oriented, closed, connected, hyperbolic manifold of dimension $n \geq 2$, it is well-known that all non-zero classes in $H_2(M), \dots, H_n(M)$ have non-zero ℓ^1-semi-norm [57; p. 6.6]. By Proposition (2.5), the image of these classes in ℓ^1-homology cannot be zero.

It is not difficult to see that $H_0^{\ell^1}(X) \neq 0$ for all topological spaces. On the other hand, $H_1^{\ell^1}(X) = 0$ (Proposition (2.7)). ◇

2.1.2 Basic properties of ℓ^1-homology of spaces

In the following, we study the basic properties of ℓ^1-homology, i.e., we analyse to which extent ℓ^1-homology satisfies the Eilenberg-Steenrod axioms.

Proposition (2.7) (Basic properties of ℓ^1-homology of spaces).

1. Homotopy invariance. *The functor ℓ^1-homology is homotopy invariant, i.e., if $f, g \colon (X, A) \longrightarrow (Y, B)$ are homotopic maps of pairs, then*
$$H_*^{\ell^1}(f) = H_*^{\ell^1}(g) \colon H_*^{\ell^1}(X, A) \longrightarrow H_*^{\ell^1}(Y, B).$$

2. ℓ^1-Homology of the point. *For the one-point-space $\bullet$ we obtain $H_0^{\ell^1}(\bullet) = \mathbf{R}$ and $H_k^{\ell^1}(\bullet) = 0$ for all $k \in \mathbf{N}_{>0}$.*
3. The first ℓ^1-homology group. *For any space X we have $H_1^{\ell^1}(X) = 0$.*
4. Long exact sequence of the pair. *Let (X, A) be a pair of topological spaces. Then there is a natural long exact sequence*

$$\cdots \xrightarrow{\partial} H_k^{\ell^1}(A) \longrightarrow H_k^{\ell^1}(X) \longrightarrow H_k^{\ell^1}(X,A) \xrightarrow{\partial} H_{k-1}^{\ell^1}(A) \longrightarrow \cdots,$$

where ∂ is induced by the map $\partial\colon C_k^{\ell^1}(X,A) \longrightarrow C_k^{\ell^1}(A)$, and the other maps are the homomorphisms induced by the natural inclusions $A \hookrightarrow X \hookrightarrow (X,A)$.
5. Finite disjoint unions. *If I is a finite set and $(X_i)_{i\in I}$ are topological spaces, then $H_*^{\ell^1}(\coprod_{i\in I} X_i)$ is isometrically isomorphic to $\bigoplus_{i\in I} H_*^{\ell^1}(X_i)$. Here, the direct sum is equipped with the sum of the norms.*

Proof. Apart from part 3, all properties can be shown by the same arguments as in singular homology:

Homotopy invariance. If f and g are homotopic, the classic construction [16; Proposition III.5.7] of subdividing a homotopy between f and g in an appropriate way gives rise to a chain homotopy $h\colon C_*(X,A) \longrightarrow C_*(Y,B)$ between $C_*(f)$ and $C_*(g)$ that is bounded in each degree. Therefore, h can be extended to the ℓ^1-chain complexes, and a straightforward calculation shows that this extension is a chain homotopy between $C_*^{\ell^1}(f)$ and $C_*^{\ell^1}(g)$. In particular, $H_*^{\ell^1}(f) = H_*^{\ell^1}(g)$.

ℓ^1-Homology of the point. By definition of the ℓ^1-chain complex, $C_*^{\ell^1}(\bullet) = C_*(\bullet)$. In particular, $H_k^{\ell^1}(\bullet) = H_k(\bullet)$, which equals $\mathbf{R}$ if $k = 0$ and which is 0 otherwise.

The first ℓ^1-homology group. As first step, we show that each class in $H_1^{\ell^1}(X)$ can be represented by an ℓ^1-cycle consisting of *loops* in X:

In order to keep the notational overhead limited, we only consider the case that X is path-connected (the general case can be treated by the same arguments, but requires an additional layer of indices). We choose a base point $x \in X$ and for each point $y \in X$ we choose a path $p_y\colon [0,1] \longrightarrow X$ joining y and x. Then for each 1-simplex $\sigma\colon [0,1] \longrightarrow X$ there is a homotopy $h_\sigma\colon [0,1]\times[0,1] \longrightarrow X$ satisfying

$$h_\sigma(\,\cdot\,,0) = \sigma, \qquad h_\sigma(0,\,\cdot\,) = p_{\sigma(0)}, \qquad h_\sigma(1,\,\cdot\,) = p_{\sigma(1)}.$$

We now define

$$\begin{aligned} \ell\colon C_1(X) &\longrightarrow C_1(X) \\ \sigma &\longmapsto h_\sigma(\,\cdot\,,1). \end{aligned}$$

By construction $\|\ell\| \leq 1$ and all chains in the image of ℓ are sums of loops (based at x) and hence are cycles. Moreover, subdividing $[0,1] \times [0,1]$ into the obvious two triangles gives rise to a homomorphism $H\colon C_1(X) \longrightarrow C_2(X)$ with

$$\partial\big(H(\sigma)\big) + p_{\sigma(1)} - p_{\sigma(0)} = \sigma - \ell(\sigma)$$

for all $\sigma \in \operatorname{map}(\Delta^1, X)$ and $\|H\| \leq 2$. Therefore, we obtain extensions

$$\begin{aligned} \overline{\ell}\colon C_1^{\ell^1}(X) &\longrightarrow C_1^{\ell^1}(X), \\ \overline{H}\colon C_1^{\ell^1}(X) &\longrightarrow C_2^{\ell^1}(X) \end{aligned}$$

of ℓ and H respectively such that $\overline{H}$ witnesses the fact that for all cycles $c \in C_1^{\ell^1}(X)$ the image $\overline{\ell}(c)$ is a cycle and

$$[\overline{\ell}(c)] = [c] \in H_1^{\ell^1}(X).$$

Thus, it suffices to show $[\overline{\ell}(c)] = 0 \in H_1^{\ell^1}(X)$ for all cycles $c \in C_1^{\ell^1}(X)$: By construction, $\overline{\ell}(c)$ is an ℓ^1-sum $\sum_{n\in\mathbf{N}} a_n \cdot \sigma_n$ of loops $\sigma_n\colon [0,1] \longrightarrow X$. In particular, we have the decomposition $\sigma_n = \sigma_n^\circ \circ \tau$, where $\tau\colon [0,1] \longrightarrow S^1$ wraps exactly once around S^1 and where $\sigma_n^\circ \in \operatorname{map}(S^1, X)$.

Let $d\colon S^1 \longrightarrow S^1$ be an orientation preserving double covering. Then there is a chain $b \in C_2(S^1)$ with

$$\partial b = \tau - \frac{1}{2} \cdot C_1(d)(\tau).$$

We consider the formal sum

$$b\big(\overline{\ell}(c)\big) := \sum_{n\in\mathbf{N}} a_n \cdot \sum_{j\in\mathbf{N}} \frac{1}{2^j} \cdot C_2(\sigma_n^\circ) \circ C_2(d^j)(b),$$

which clearly is ℓ^1-summable. Hence, $b(\overline{\ell}(c)) \in C_2^{\ell^1}(X)$, and one easily checks that $\partial b(\overline{\ell}(c)) = \overline{\ell}(c) \in C_1^{\ell^1}(X)$. In particular, $[c] = [\overline{\ell}(c)] = 0 \in H_1^{\ell^1}(X)$.

Matsumoto and Morita show how $H_1^{\ell^1}(X) = 0$ can be derived from knowledge on bounded cohomology [38; Corollary 2.7]. On the other hand, Bouarich's proof [5; p. 164] is not entirely correct [37] – his argument shows only that the reduced (in the normed sense) first ℓ^1-homology vanishes.

Long exact sequence. By the snake lemma (Proposition (1.9)), the short exact sequence $0 \longrightarrow C_*^{\ell^1}(A) \longrightarrow C_*^{\ell^1}(X) \longrightarrow C_*^{\ell^1}(X, A) \longrightarrow 0$ induces a long exact sequence in homology; moreover, the construction of the connecting homomorphism shows that it indeed is of the mentioned form.

Finite disjoint unions. Because the standard simplices Δ^k are connected, it follows that there is an isometric isomorphism $C_*(X) \cong \bigoplus_{i\in I} C_*(X_i)$ of normed chain complexes. The finiteness of I ensures that $C^{\ell^1}_*(X)$ and $\bigoplus_{i\in I} C^{\ell^1}_*(X_i)$ are isometrically isomorphic. Therefore, we obtain an isometric isomorphism

$$H^{\ell^1}_*(X) \cong \bigoplus_{i\in I} H^{\ell^1}_*(X_i). \qquad \square$$

Similar arguments as in the previous proposition show that bounded cohomology satisfies the corresponding properties [56; Sections 2.1.1 and 2.2.2].

However, ℓ^1-homology (as well as bounded cohomology) does *not* satisfy excision. The geometric reason behind this phenomenon is the following: Singular homology satisfies excision, because any singular homology class can also be represented by a singular cycle consisting of "small" simplices. This is achieved by applying barycentric subdivision sufficiently often. However, in an (infinite) ℓ^1-chain $\sum_{n\in\mathbf{N}} a_n \cdot \sigma_n$, the number of barycentric subdivisions needed for the simplices $(\sigma_n)_{n\in\mathbf{N}}$ might be unbounded. Therefore, an ℓ^1-homology class can in general not be represented by cycles consisting only of "small" simplices.

Example (2.8). We will see that $H^{\ell^1}_*(S^1) \cong H^{\ell^1}_*(\bullet)$ (Corollary (4.2)). But the second ℓ^1-homology $H^{\ell^1}_2(S^1 \vee S^1)$ is not even finite dimensional because there is an isomorphism $H^{\ell^1}_2(S^1 \vee S^1) \cong H^{\ell^1}_2(\mathbf{Z} * \mathbf{Z})$ (Corollary (4.14)), and the latter term is not finite dimensional [6, 41]. In particular, there can be no cellular version of ℓ^1-homology. $\diamond$

This failure of excision is both a curse and a blessing. On the one hand, the lack of excision makes concrete computations via the usual divide and conquer approach significantly harder; on the other hand, it turns out that both bounded cohomology and ℓ^1-homology depend only on the fundamental group (Theorem (2.26) and Corollary (4.3)) and hence can be computed in terms of certain nice resolutions (Theorem (2.28) and Corollary (4.14)).

Remark (2.9). It is not clear, whether ℓ^1-homology of an *infinite* disjoint union of spaces coincides with the ℓ^1-sum of the ℓ^1-homology of the pieces. Namely, let $(X_n)_{n\in\mathbf{N}}$ be a family of spaces and let $X := \coprod_{n\in\mathbf{N}} X_n$. Clearly, we have an isometric isomorphism $C^{\ell^1}_*(X) \cong \bigoplus^{\ell^1}_{n\in\mathbf{N}} C^{\ell^1}_*(X_n)$. But there might exist a cycle $c \in C^{\ell^1}_*(X)$ such that its restrictions to all subspaces X_n are null-homologous in $C^{\ell^1}_*(X_n)$, but such that the sum of all such null-homologies is not ℓ^1-summable. $\diamond$

In Section 2.4, we list some of the more sophisticated properties of bounded cohomology; their analogues in ℓ^1-homology are derived in Chapter 4.

2.1.3 Where is ℓ^p-homology?

It is tempting to consider not only the ℓ^1-norm on the singular chain complex, but also ℓ^p-norms with $p > 1$. However, the singular chain complex is not a normed chain complex with respect to these norms (Proposition (2.11)) – hence, the boundary operator cannot be extended to the completion of the singular chain complex with respect to the ℓ^p-norm.

Definition (2.10). Let X be a topological space and let $p \in [1, \infty]$. The **ℓ^p-norm** on the singular chain complex $C_*(X)$ is defined by

$$\|c\|_p := \begin{cases} \sqrt[p]{\sum_{j=0}^{k} |a_j|^p} & \text{if } p \neq \infty \\ \sup\{|a_j| \mid j \in \{0, \dots, k\}\} & \text{if } p = \infty \end{cases}$$

for all $c := \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(X)$. ◇

Proposition (2.11). *Let X be a path-connected topological space with at least two points and let $p \in (1, \infty]$. Then the singular chain complex $C_*(X)$ of X is* not *a normed chain complex with respect to the ℓ^p-norm.*

Proof. In the following, let $n \in \mathbf{N}_{>0}$. Let $x, y \in X$ be two points with $x \neq y$, and let $\sigma \in \operatorname{map}(\Delta^n, X)$ be a singular n-simplex of X with $\partial\sigma \neq 0$; for example, any singular n-simplex that has one vertex in x and n vertices in y enjoys this property.

For each $d \in \mathbf{N}$ we can find d different n-simplices of X having the same boundary as σ; for example, using a cofibration $\partial\Delta^n \cup \{x_1, \dots, x_d\} \hookrightarrow \Delta^n$, where $x_1, \dots, x_d \in (\Delta^n)^\circ$, we can find singular n-simplices $\sigma_1, \dots, \sigma_d \colon \Delta^n \longrightarrow X$ such that

$$\sigma_j(x_k) = \begin{cases} x & \text{if } k \neq j \\ y & \text{if } k = j \end{cases}$$

and $\sigma_j|_{\partial\Delta^n} = \sigma|_{\partial\Delta^n}$ holds for all $j, k \in \{1, \dots, d\}$. In particular, the $\sigma_1, \dots, \sigma_d$ are d different singular simplices with $\partial\sigma_j = \partial\sigma$.

We then consider the chain

$$c_d := \frac{1}{d} \cdot \sum_{j=1}^{d} \sigma_j \in C_n(X).$$

By construction, $\partial_n c_d = \partial\sigma \neq 0$. On the other hand, $\lim_{d\to\infty} \|c_d\|_p = 0$ because $p > 1$. Therefore, the boundary operator ∂_n is not a bounded operator. □

Furthermore, a refined version of this argument can be applied to show that the semi-norms on singular homology induced by $\|\cdot\|_p$ with $p > 1$ are trivial.

2.2 ℓ^1-Homology of discrete groups

Like in ordinary group (co)homology, there are several ways to introduce their functional analytic analogues – bounded cohomology and ℓ^1-homology of discrete groups:

- Firstly, ℓ^1-homology and bounded cohomology of discrete groups can be explicitly defined in terms of the Banach bar resolution, which is the ℓ^1-completion of the bar resolution (Sections 2.2.1 and 2.2.2).
- Secondly, ℓ^1-homology and bounded cohomology of discrete groups can also be described in the language of certain projective and injective resolutions (Section 2.2.3).

In Section 2.2.4, we examine basic properties of ℓ^1-homology of groups.

2.2.1 The Banach bar resolution

The bar resolution with **R**-coefficients of a discrete group can be equipped with an ℓ^1-norm turning this resolution into a normed chain complex. The corresponding completion is the so-called Banach bar resolution of the group. Standard operations like projective tensor products and spaces of bounded operators then create the Banach (co)chain complexes that underlie the definitions of ℓ^1-homology and bounded cohomology of discrete groups with coefficients.

Definition (2.12). Let G be a discrete group. The **Banach bar resolution** of G is the ℓ^1-completion of the bar resolution of G, i.e., the Banach G-chain complex defined as follows:

1. For each $n \in \mathbf{N}$ let
$$C_n^{\ell^1}(G) := \left\{ \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1|\dots|g_n] \;\middle|\; \forall_{g \in G^{n+1}} \;\; a_g \in \mathbf{R} \text{ and } \sum_{g \in G^{n+1}} |a_g| < \infty \right\}$$
together with the norm $\big\| \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1|\dots|g_n] \big\|_1 := \sum_{g \in G^{n+1}} |a_g|$ and the G-action characterised by
$$h \cdot \big(g_0 \cdot [g_1|\dots|g_n]\big) := (h \cdot g_0) \cdot [g_1|\dots|g_n]$$
for all $g \in G^{n+1}$ and all $h \in G$.

2. The boundary operator is the G-morphism uniquely determined by

$$\begin{aligned} C_n^{\ell^1}(G) &\longrightarrow C_{n-1}^{\ell^1}(G) \\ g_0 \cdot [g_1 | \dots | g_n] &\longmapsto g_0 \cdot g_1 \cdot [g_2 | \dots | g_n] \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \cdot g_0 \cdot [g_1 | \dots | g_{j-1} | g_j \cdot g_{j+1} | g_{j+2} | \dots | g_n] \\ &\quad + (-1)^n \cdot g_0 \cdot [g_1 | \dots | g_{n-1}]. \end{aligned}$$

3. Moreover, we define the augmentation $\varepsilon \colon C_0^{\ell^1}(G) \longrightarrow \mathbf{R}$ by adding up the coefficients. $\diamond$

Definition (2.13). Let G be a discrete group and let V be a Banach G-module.

1. Let $C_*^{\ell^1}(G; V)$ be the Banach G-chain complex given by

$$C_*^{\ell^1}(G; V) := C_*^{\ell^1}(G) \mathbin{\overline{\otimes}} V.$$

2. Dually, we define the Banach G-cochain complex $C_{\mathrm{b}}^*(G; V)$ by

$$C_{\mathrm{b}}^*(G; V) := B\bigl(C_*^{\ell^1}(G), V\bigr).$$

(Details on the corresponding norms, G-actions and (co)boundary operators can be found in Definitions (1.12) and (1.17)). $\diamond$

Remark (2.14). Let G be a discrete group. Then, $C_*^{\ell^1}(G; \mathbf{R}) = C_*^{\ell^1}(G)$. If V is a Banach G-module, the adjointness relation between $\overline{\otimes}$ and $\cdot'$ (see Remark (1.13)) shows that

$$C_{\mathrm{b}}^*(G; V') = \bigl(C_*^{\ell^1}(G; V)\bigr)'. \qquad \square$$

Remark (2.15). For any discrete group G, the bijections

$$\begin{aligned} G^{n+1} &\longrightarrow G^{n+1} \\ g_0 \cdot [g_1 | \dots | g_n] &\longmapsto (g_n^{-1}, \dots, g_0^{-1}) \end{aligned}$$

induce an isometric isomorphism $(C_*^{\ell^1}(G))' \longrightarrow C_{\mathrm{b}}^*(G)$ of Banach G-cochain complexes, where $C_{\mathrm{b}}^*(G)$ is the cochain complex defined by Ivanov [25; Section 3.4]. $\square$

Of course, these constructions are functorial with respect to group homomorphisms and change of coefficients:

Remark (2.16). Let $\varphi\colon G \longrightarrow H$ be a homomorphism of discrete groups, let V be a Banach G-module and let W be a Banach H-module. Then

$$\begin{aligned} C_n^{\ell^1}(\varphi)\colon C_n^{\ell^1}(G) &\longrightarrow \varphi^*\bigl(C_n^{\ell^1}(H)\bigr) \\ g_0 \cdot [g_1|\dots|g_n] &\longmapsto \varphi(g_0)\cdot\bigl[\varphi(g_1)|\dots|\varphi(g_n)\bigr] \end{aligned}$$

defines a morphism $C_*^{\ell^1}(\varphi)\colon C_*^{\ell^1}(G) \longrightarrow \varphi^* C_*^{\ell^1}(H)$ of Banach G-chain complexes of norm 1; here, $\varphi^*(\,\cdot\,)$ stands for the Banach G-module structure on the Banach H-module in question that is induced by φ. In particular, for any morphism $f\colon V \longrightarrow \varphi^* W$ of Banach G-modules, the map

$$C_*^{\ell^1}(\varphi; f) := C_*^{\ell^1}(\varphi)\,\overline{\otimes}\, f\colon C_*^{\ell^1}(G;V) \longrightarrow \varphi^*\bigl(C_*^{\ell^1}(H;W)\bigr)$$

is a morphism of Banach G-chain complexes (of norm at most $\|f\|$). Analogously, for any morphism $f\colon \varphi^*W \longrightarrow V$ of Banach G-modules,

$$C_b^*(\varphi; f) := B\bigl(C_*^{\ell^1}(\varphi), f\bigr)\colon \varphi^*\bigl(C_b^*(H;W)\bigr) \longrightarrow C_b^*(G;V)$$

is a morphism of Banach G-cochain complexes (of norm at most $\|f\|$). $\square$

2.2.2 Definition of ℓ^1-homology of discrete groups

The definition of ℓ^1-homology and bounded cohomology of discrete groups is now a straightforward adaption of the classic definition of group (co)homology in terms of the bar resolution:

Definition (2.17). Let G be a discrete group and let V be a Banach G-module.

1. The **ℓ^1-homology of G with coefficients in V**, denoted by $H_*^{\ell^1}(G;V)$, is the homology of the Banach chain complex $C_*^{\ell^1}(G;V)_G$.
   We write $H_*^{\ell^1}(G) := H_*^{\ell^1}(G;\mathbf{R})$, where $\mathbf{R}$ is the trivial Banach G-module.
2. The **bounded cohomology of G with coefficients in V** is defined as
   $$H_b^*(G;V) := H^*\bigl(C_b^*(G;V)^G\bigr),$$
   and we write $H_b^*(G) := H_b^*(G;\mathbf{R})$, where $\mathbf{R}$ is the trivial Banach G-module.
3. If $\varphi\colon G \longrightarrow H$ is a homomorphism of discrete groups, W is a Banach H-module, and $f\colon V \longrightarrow \varphi^*W$ is a morphism of Banach G-modules, we write
   $$H_*^{\ell^1}(\varphi; f)\colon H_*^{\ell^1}(G;V) \longrightarrow H_*^{\ell^1}(H;W)$$
   for the homomorphism induced by the composition
   $$p \circ C_*^{\ell^1}(\varphi; f)_G\colon C_*^{\ell^1}(G;V)_G \longrightarrow \varphi^*(C_*^{\ell^1}(H;W))_G \longrightarrow C_*^{\ell^1}(H;W)_H.$$

4. Dually, if $\varphi\colon G \longrightarrow H$ is a homomorphism of discrete groups, W is a Banach H-module and $f\colon \varphi^* W \longrightarrow V$ is a morphism of Banach G-modules, then we write
$$H_b^*(\varphi; f)\colon H_b^*(H; W) \longrightarrow H_b^*(G; V)$$
for the homomorphism induced by the composition
$$C_b^*(\varphi; f)^G \circ i\colon C_b^*(H; W)^H \hookrightarrow \varphi^*(C_b^*(H; W))^G \longrightarrow C_b^*(G; V)^G. \qquad \diamond$$

Notice that the ℓ^1-norm on the Banach bar resolution $C_*^{\ell^1}(G)$ induces semi-norms on $H_*^{\ell^1}(G; V)$ and on $H_b^*(G; V)$.

2.2.3 ℓ^1-Homology of discrete groups via projective resolutions

In this section, we show that ℓ^1-homology (and bounded cohomology) of discrete groups enjoy the same flexibility as ordinary group (co)homology: namely, both ℓ^1-homology and bounded cohomology can be computed by means of relative homological algebra, as studied by Brooks, Ivanov, Monod, and Park [6, 25, 42, 47]. An introduction to this version of homological algebra is given in Appendix A.

Theorem (2.18). *Let G be a discrete group and let V be a Banach G-module.*

1. *For any strong relatively projective G-resolution $(C, \eta\colon C_0 \to V)$ of V there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*
$$H_*^{\ell^1}(G; V) \cong H_*(C_G).$$
2. *For any strong relatively injective G-resolution $(C, \eta\colon V \to C^0)$ of V there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*
$$H_b^*(G; V) \cong H^*(C^G).$$
3. *If $(C, \eta\colon C_0 \to \mathbf{R})$ is a strong relatively projective G-resolution of the trivial Banach G-module $\mathbf{R}$, then there are canonical isomorphisms (degreewise isomorphisms of semi-normed vector spaces)*
$$H_*^{\ell^1}(G; V) \cong H_*\big((C \mathbin{\overline{\otimes}} V)_G\big),$$
$$H_b^*(G; V) \cong H^*\big(B(C, V)^G\big).$$

The semi-norms on $H^{\ell^1}_*(\,\cdot\,;\cdot\,)$ and $H^*_{\mathrm{b}}(\,\cdot\,;\cdot\,)$ induced by the standard resolutions $C^{\ell^1}_*(G;V)$ and $C^*_{\mathrm{b}}(G;V)$ coincide with the canonical semi-norms in the sense of Ivanov [25, 47, 42; Corollary 3.6.1, Corollary 2.3, Corollary 7.4.7]. However, it is clear that not every strong relatively projective/injective resolution induces the same semi-norm in (co)homology.

Of course, not only the groups $H^{\ell^1}_*(\,\cdot\,;\cdot\,)$ and $H^*_{\mathrm{b}}(\,\cdot\,;\cdot\,)$ can be described via resolutions, but also the morphisms $H^{\ell^1}_*(\varphi;f)$ and $H^*_{\mathrm{b}}(\varphi;f)$ [42, 33; Section 8, Section 5.1].

Bühler developed a description of ℓ^1-homology and bounded cohomology as derived functors via exact categories [11], thereby providing an even more conceptual approach, which paves the way to applying standard methods from homological algebra directly to ℓ^1-homology and bounded cohomology.

Proof (of Theorem (2.18)). The fundamental lemma of homological algebra in the context of Banach G-modules (Propositions (A.7) and (A.9)) shows that the two terms $H_*(C_G)$ and $H^*(C^G)$ do not depend (up to canonical isomorphisms) on the chosen strong relatively projective resolutions and strong relatively injective resolutions respectively.

Therefore, for the first and the second part it suffices to show that $C^{\ell^1}_*(G;V)$ and $C^*_{\mathrm{b}}(G;V)$ are strong relatively projective/injective resolutions of V. This is done in Proposition (2.19).

It remains to prove the third part: If $(C,\eta\colon C_0 \to R)$ is a strong relatively projective resolution of $\mathbf{R}$, there exist by the fundamental lemma of homological algebra (Proposition (A.9)) mutually G-homotopy inverse G-chain homotopy equivalences $\varphi : C \mathbin{\square} \eta \simeq C^{\ell^1}_*(G) \mathbin{\square} \varepsilon : \psi$; here, the symbol "$\square$" denotes concatenation of chain complexes. Hence, $\varphi \mathbin{\overline{\otimes}} \mathrm{id}_V$ and $\psi \mathbin{\overline{\otimes}} \mathrm{id}_V$ clearly are (mutually G-homotopy inverse) G-chain homotopy equivalences

$$(C \mathbin{\overline{\otimes}} V) \mathbin{\square} (\eta \mathbin{\overline{\otimes}} \mathrm{id}_V) \simeq \bigl(C^{\ell^1}_*(G) \mathbin{\overline{\otimes}} V\bigr) \mathbin{\square} (\varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V) = C^{\ell^1}_*(G;V) \mathbin{\square} (\varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V).$$

In particular, we obtain an isomorphism

$$H_*\bigl((C \mathbin{\overline{\otimes}} V)_G\bigr) \cong H_*\bigl(C^{\ell^1}_*(G;V)_G\bigr),$$

which is in each degree an isomorphism of semi-normed vector spaces. Therefore, it follows that

$$H^{\ell^1}_*(G;V) \cong H_*\bigl((C \mathbin{\overline{\otimes}} V)_G\bigr).$$

Analogously, $B(\varphi,\mathrm{id}_V)$ and $B(\psi,\mathrm{id}_V)$ are (mutually G-homotopy inverse) G-cochain equivalences between $C^*_{\mathrm{b}}(G;V) = B(C^{\ell^1}_*(G),V)$ and $B(C,V)$. Thus, we see that there is a canonical isomorphism $H_b^*(G;V) \cong H^*(B(C,V)^G)$, which is in each degree an isomorphism of semi-normed vector spaces. $\square$

Proposition (2.19). *Let G be a discrete group and let V be a Banach G-module.*

1. *Then $C_*^{\ell^1}(G;V)$ together with the augmentation $\varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V$ is a strong relatively projective G-resolution of V.*
2. *Dually, $C_b^*(G;V)$ together with the augmentation $B(\varepsilon, \mathrm{id}_V)$ is a strong relatively injective G-resolution of V.*

Here, $\varepsilon\colon C_0^{\ell^1}(G) \longrightarrow \mathbf{R}$ is the augmentation introduced in Definition (2.12).

Proof. Ad 1. Park showed that $(C_*^{\ell^1}(G), \varepsilon)$ is a strong relatively projective resolution of the trivial Banach G-module $\mathbf{R}$ [47; p. 596f]: A contracting chain homotopy s of the concatenated chain complex $C_*^{\ell^1}(G) \mathbin{\square} \varepsilon$ of norm at most 1 can, for example, be defined by $s_{-1}(1) := 1 \cdot [\,]$ and

$$\begin{aligned} s_n\colon C_n^{\ell^1}(G) &\longrightarrow C_{n+1}^{\ell^1}(G) \\ g_0 \cdot [\,g_1 | \dots | g_n\,] &\longmapsto (-1)^{n+1} \cdot [\,g_0 \,|\, g_1 | \dots | g_n\,]. \end{aligned}$$

Therefore, $s \mathbin{\overline{\otimes}} \mathrm{id}_V$ is a contracting chain homotopy of $C_*^{\ell^1}(G;V) \mathbin{\square} (\varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V)$, which also has norm at most 1. Hence, $(C_*^{\ell^1}(G;V), \varepsilon \mathbin{\overline{\otimes}} \mathrm{id}_V)$ is a strong resolution of the Banach G-module $\mathbf{R} \mathbin{\overline{\otimes}} V = \overline{V} = V$.

For each $n \in \mathbf{N}$, the Banach G-module $C_n^{\ell^1}(G;V) = C_n^{\ell^1}(G) \mathbin{\overline{\otimes}} V$ is relatively projective because any mapping problem (in the sense of Definition (A.1)) of the form

$$\begin{tikzcd} & C_n^{\ell^1}(G;V) \arrow[d, "\alpha"] \arrow[dl, dashed] & \\ U \arrow[r, "\pi"'] & W \arrow[r] \arrow[l, bend right, "\sigma"'] & 0 \end{tikzcd}$$

is solved by the G-morphism given by

$$\begin{aligned} C_n^{\ell^1}(G;V) &\longrightarrow U \\ g_0 \cdot [g_1 | \dots | g_n] \otimes v &\longmapsto g_0 \cdot \sigma \circ \alpha\bigl(1 \cdot [g_1 | \dots | g_n] \otimes (g_0^{-1} \cdot v)\bigr). \end{aligned}$$

Ad 2. Dually, it is not difficult to see that $B(s, \mathrm{id}_V)$ is a contracting cochain homotopy of $B(\varepsilon, \mathrm{id}_V) \mathbin{\square} B(C_*^{\ell^1}(G), V)$. Furthermore, for each $n \in \mathbf{N}$ the Banach

G-module $C_b^n(G;V) = B(C_n^{\ell^1}(G), V)$ is relatively injective because any mapping problem (in the sense of Definition (A.1)) of the form

$$\begin{tikzcd} & C_b^n(G;V) & \\ 0 \arrow[r] & U \arrow[u, "\alpha"] \arrow[r, "i"] & W \arrow[ul, dashed] \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

can be solved by the G-morphism

$$\begin{aligned} W &\longrightarrow C_b^n(G;V) = B\big(C_n^{\ell^1}(G), V\big) \\ w &\longmapsto \Big(g_0 \cdot [g_1|\dots|g_n] \mapsto \big(\alpha(g_0 \cdot \sigma(g_0^{-1} \cdot w))\big)(g_0 \cdot [g_1|\dots|g_n])\Big). \end{aligned}$$

If we were only interested in the case of $V = W'$ for some Banach G-module W, then we could just apply Proposition (A.8) to the first part. $\square$

2.2.4 Basic properties of ℓ^1-homology of discrete groups

As expected, in degree zero ℓ^1-homology and bounded cohomology of a discrete group coincide with the (co)invariants of the coefficients. Like the corresponding theories for topological spaces, ℓ^1-homology and bounded cohomology of discrete groups with $\mathbf{R}$-coefficients vanish in degree 1.

Proposition (2.20) (Low dimensions). *Let G be a discrete group and let V be a Banach G-module. Then*

$$\begin{aligned} H_0^{\ell^1}(G;V) &\cong V_G, \\ H_b^0(G;V) &\cong V^G, \\ H_1^{\ell^1}(G) = 0 &= H_b^1(G). \end{aligned}$$

Proof. (Co)homology in degree 0*:* Almost the same calculations as in ordinary group (co)homology prove the statements on ℓ^1-homology and bounded cohomology in degree 0. The only noteworthy difference is that V_G is obtained from V by dividing out the *closure* of $\operatorname{span}\{g \cdot v - v \mid v \in V,\ g \in G\}$.

(Co)homology in degree 1*:* For each $g \in G$ the chain

$$s(g) := \sum_{n \in \mathbf{N}} \frac{1}{2^{n+1}} \cdot \big[g^{2^n} \mid g^{2^n}\big] \in C_2^{\ell^1}(G)$$

constructed by Mitsumatsu [41] satisfies $\partial_G s(g) = 1 \cdot [g]$ in the quotient $C_1^{\ell^1}(G)_G$. Hence, the map $\partial_G \colon C_2^{\ell^1}(G)_G \longrightarrow C_1^{\ell^1}(G)_G$ induced by the boundary operator on $C_*^{\ell^1}(G)$ is surjective. This implies that the dual map

$$\begin{tikzcd}
\big(C_1^{\ell^1}(G)_G\big)' \arrow[r] \arrow[d, equal, "(1.14)"'] & \big(C_2^{\ell^1}(G)_G\big)' \arrow[d, equal, "(1.14)"] \\
C_{\mathrm{b}}^1(G)^G \arrow[r] & C_{\mathrm{b}}^2(G)^G
\end{tikzcd}$$

is injective. Therefore, we obtain $H_1^{\ell^1}(G) = 0$ and $H_{\mathrm{b}}^1(G) = 0$. □

Moreover, like in ordinary group cohomology, short exact sequences on the level of coefficients give rise to long exact sequences in (co)homology [26, 42, 11]:

Proposition (2.21) (Long exact sequence). *Let G be a discrete group, and let*

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{p} W \longrightarrow 0$$

be a short exact sequence of Banach G-modules with relatively injective i. Then there are natural long exact sequences

$$\cdots \longrightarrow H_n^{\ell^1}(G;U) \longrightarrow H_n^{\ell^1}(G;V) \longrightarrow H_n^{\ell^1}(G;W) \longrightarrow H_{n-1}^{\ell^1}(G;U) \longrightarrow \cdots,$$
$$\cdots \longrightarrow H_{\mathrm{b}}^n(G;W') \longrightarrow H_{\mathrm{b}}^n(G;V') \longrightarrow H_{\mathrm{b}}^n(G;U') \longrightarrow H_{\mathrm{b}}^{n+1}(G;W') \longrightarrow \cdots$$

with continuous connecting homomorphisms.

Proof. For every $n \in \mathbf{N}$, the Banach G-module $C_n^{\ell^1}(G)$ is relatively projective (Proposition (2.19)). Therefore, we obtain short exact sequences

$$0 \longrightarrow C_*^{\ell^1}(G,U)_G \longrightarrow C_*^{\ell^1}(G;V)_G \longrightarrow C_*^{\ell^1}(G;W)_G \longrightarrow 0,$$
$$0 \longrightarrow C_{\mathrm{b}}^*(G;W')^G \longrightarrow C_{\mathrm{b}}^*(G;V')^G \longrightarrow C_{\mathrm{b}}^*(G;U')^G \longrightarrow 0$$

of Banach G-(co)chain complexes (Proposition (A.3)).

Now the snake lemma (Proposition (1.9)) gives the two natural long exact sequences on (co)homology. □

A closer look at the proof of Proposition (A.3) reveals that in the situation of the above proposition it is not necessary to assume that i is relatively injective [26, 42; Proposition 2.10, Section 8.2].

2.3 ℓ^1-Homology of spaces with twisted coefficients

Analogously to singular (co)homology, there are also versions of ℓ^1-homology and bounded cohomology of spaces with twisted coefficients.

Definition (2.22). Let X be a connected topological space with universal covering $\widetilde{X}$ and fundamental group G, and let V be a Banach G-module.

1. The **ℓ^1-chain complex of X with twisted coefficients in V** is defined as the Banach chain complex of coinvariants
$$C_*^{\ell^1}(X;V) := \big(C_*^{\ell^1}(\widetilde{X}) \mathbin{\overline{\otimes}} V\big)_G.$$
Here, $C_*^{\ell^1}(\widetilde{X})$ inherits the G-action from the action of the fundamental group on the universal covering $\widetilde{X}$.
2. The **ℓ^1-homology of X with twisted coefficients in V**, denoted by $H_*^{\ell^1}(X;V)$, is the homology of the Banach chain complex $C_*^{\ell^1}(X;V)$.
3. The **bounded cochain complex of X with twisted coefficients in V** is defined as the Banach cochain complex of invariants
$$C_b^*(X;V) := B\big(C_*^{\ell^1}(\widetilde{X}), V\big)^G.$$
4. **Bounded cohomology of X with twisted coefficients in V** is the cohomology of the Banach cochain complex $C_b^*(X;V)$ and is denoted by $H_b^*(X;V)$.

(Details on the definition of the Banach G-(co)chain complexes $C_*^{\ell^1}(\widetilde{X}) \mathbin{\overline{\otimes}} V$ and $B(C_*^{\ell^1}(\widetilde{X}), V)$ can be found in Definitions (1.12) and (1.17)). ◇

The ℓ^1-chain complex and the bounded cochain complex of X (as defined in Section 2.1) can be recovered from this definition by taking trivial coefficients; namely, as Park stated [47; proof of Theorem 4.1], the ℓ^1-chain complex of X can be viewed as the coinvariants of the ℓ^1-chain complex of $\widetilde{X}$:

Proposition (2.23). *If X is a connected topological space admitting a universal covering $\pi\colon \widetilde{X} \longrightarrow X$, then the morphism $C_*^{\ell^1}(\pi)\colon C_*^{\ell^1}(\widetilde{X}) \longrightarrow C_*^{\ell^1}(X)$ induces an isometric isomorphism*
$$\varphi\colon C_*^{\ell^1}(\widetilde{X})_{\pi_1(X)} \longrightarrow C_*^{\ell^1}(X)$$
of Banach chain complexes.

Therefore, $C_*^{\ell^1}(X;\mathbf{R}) = C_*^{\ell^1}(X)$, and we obtain from Proposition (1.14) that

$$C_b^*(X;\mathbf{R}) \cong \bigl(C_*^{\ell^1}(\widetilde{X})'\bigr)^{\pi_1(X)} \cong \bigl(C_*^{\ell^1}(\widetilde{X})_{\pi_1(X)}\bigr)' \cong \bigl(C_*^{\ell^1}(X)\bigr)' \cong C_b^*(X).$$

Proof (of Proposition (2.23)). For brevity, we write G for $\pi_1(X)$ and W for the subcomplex $\operatorname{span}\{g \cdot c - c \mid c \in C_*^{\ell^1}(\widetilde{X}), g \in G\}$.

Since $C_*^{\ell^1}(\pi)$ is continuous (with norm 1) and since $C_*^{\ell^1}(\pi)$ clearly vanishes on W, it also vanishes on the closure $\overline{W}$. In particular, $C_*^{\ell^1}(\pi)$ induces a morphism

$$\varphi\colon C_*^{\ell^1}(\widetilde{X})_G \longrightarrow C_*^{\ell^1}(X).$$

of Banach chain complexes with norm equal to 1 [49; Proposition 2.1.7].

We now construct an inverse to φ: To this end, for each $\tau \in \operatorname{map}(\Delta^*, X)$ we choose a π-lift $\widetilde{\tau} \in \operatorname{map}(\Delta^*, \widetilde{X})$. Then

$$\begin{aligned} \psi\colon C_*^{\ell^1}(X) &\longrightarrow C_*^{\ell^1}(\widetilde{X})_G \\ \sum_{j\in\mathbf{N}} a_j \cdot \tau_j &\longmapsto \sum_{j\in\mathbf{N}} a_j \cdot \widetilde{\tau}_j \ + \overline{W} \end{aligned}$$

is a linear map, which satisfies $\|\psi\| \leq 1$. (As we will see in the following paragraph, ψ is the inverse of φ and thus is also compatible with the boundary operators).

Clearly, $\varphi \circ \psi = \mathrm{id}$. Conversely, let $c = \sum_{j\in\mathbf{N}} a_j \cdot \sigma_j + \overline{W} \in C_*^{\ell^1}(\widetilde{X})_G$. For every index $j \in \mathbf{N}$ there exists a $g_j \in G$ such that $(\pi \circ \sigma_j)\tilde{} = g_j \cdot \sigma_j$. Therefore, we obtain

$$\begin{aligned} (\psi \circ \varphi)(c) - c &= \Bigl(\sum_{j\in\mathbf{N}} a_j \cdot (\pi \circ \sigma_j)\tilde{} - \sum_{j\in\mathbf{N}} a_j \cdot \sigma_j\Bigr) \ + \overline{W} \\ &= \sum_{j\in\mathbf{N}} a_j \cdot (g_j \cdot \sigma_j - \sigma_j) \ + \overline{W}. \end{aligned}$$

Because the series $\sum_{j\in\mathbf{N}} |a_j|$ converges, the sum $\sum_{j\in\mathbf{N}} a_j \cdot (g_j \cdot \sigma_j - \sigma_j)$ lies in the ℓ^1-closure of W, i.e., in $\overline{W}$. This implies $(\psi \circ \varphi)(c) - c = 0$ and hence $\psi \circ \varphi = \mathrm{id}$. This proves the proposition. □

Basic properties of ℓ^1-homology and bounded cohomology with twisted coefficients can be derived by similar means as in Sections 2.1.2 and 2.2.4. We refrain from doing so because ℓ^1-homology as well as bounded cohomology of spaces actually coincide with the corresponding theories of discrete groups (Theorem (2.28) and Corollary (4.14)).

2.4 Peeking into the mirror universe

The seemingly small difference in the definition of bounded cochains and singular cochains has drastic consequences for the behaviour of the corresponding cohomology theories. In this section, we give a short survey of the astonishing properties of bounded cohomology. Unfortunately, the beautiful geometric applications of bounded cohomology to rigidity theory, as for example developed by Burger, Monod and Shalom [12, 43], are beyond the scope of this thesis. Instead we focus on the classic results concerning amenable and hyperbolic groups.

The material presented below (especially Sections 2.4.3 and 2.4.4) does not only serve as a guideline for the properties of ℓ^1-homology we can expect to hold, but also enables us to derive similar results for ℓ^1-homology by means of the translation mechanism (see Chapters 3 and 4).

2.4.1 Amenable groups

Before studying the relation between amenable groups and bounded cohomology, we first recall the definition of amenable groups:

Definition (2.24). A discrete group A is called **amenable** if there is a left-invariant mean on the set $B(A, \mathbf{R})$ of bounded functions from A to $\mathbf{R}$, i.e., if there is a linear map $m\colon B(A, \mathbf{R}) \longrightarrow \mathbf{R}$ satisfying

$$\forall_{f \in B(A,\mathbf{R})} \quad \forall_{a \in A} \quad m(f) = m\bigl(b \mapsto f(a^{-1} \cdot b)\bigr)$$

and

$$\forall_{f \in B(A,\mathbf{R})} \quad \inf\bigl\{f(a) \bigm| a \in A\bigr\} \leq m(f) \leq \sup\bigl\{f(a) \bigm| a \in A\bigr\}. \qquad \diamond$$

Every finite, every Abelian, and every solvable group is amenable. The class of amenable groups is closed under taking subgroups and quotients. An example of a non-amenable group is the free group $\mathbf{Z} * \mathbf{Z}$. A detailed discussion of amenable groups can be found in Paterson's book [48].

2.4.2 Amenable groups and bounded cohomology

The proof of the following proposition contains the prototypic argument in the setting of bounded cohomology and shows how amenability can be exploited in the theory of bounded cohomology [18; p. 39]:

Proposition (2.25). *Let X be a connected, aspherical CW-complex with amenable fundamental group. Then $H_b^k(X) = 0$ for all $k \in \mathbf{N}_{>0}$.*

Proof. For brevity, we write $G := \pi_1(X)$ for the fundamental group of X and $\pi\colon \widetilde{X} \longrightarrow X$ for the universal covering. The space $\widetilde{X}$ is contractible because X is aspherical. In particular, $H_b^k(\widetilde{X}) = 0$ for all $k \in \mathbf{N}_{>0}$.

Because G is amenable, there is a G-invariant mean $m\colon B(G, \mathbf{R}) \longrightarrow \mathbf{R}$. This mean allows to construct a split of the homomorphism $H_b^*(\pi)\colon H_b^*(X) \longrightarrow H_b^*(\widetilde{X})$ as follows: Covering theory shows that

$$\begin{aligned} s\colon C_b^*(\widetilde{X}) &\longrightarrow C_b^*(X) \\ f &\longmapsto \bigl(\sigma \mapsto m(g \mapsto f(g \cdot \widetilde{\sigma}))\bigr) \end{aligned}$$

(where $\widetilde{\sigma}$ denotes any π-lift of the singular simplex σ) is a well-defined chain map satisfying

$$s \circ C_b^*(\pi) = \mathrm{id}_{C_b^*(X)}\,.$$

Therefore, we obtain the relation $H^*(s) \circ H_b^*(\pi) = \mathrm{id}_{H_b^*(X)}$ on the level of bounded cohomology. In particular, $H^*(s)$ is surjective.

Now the claim follows because the bounded cohomology of $\widetilde{X}$ is trivial in non-zero degree. □

Of course, the construction of s in the proof would not be possible on the singular cochain complex because the mean m can only be applied to *bounded* functions.

Moreover, the argument in the proof of the previous proposition admits no analogue on the level of ℓ^1-chain complexes (cf. Caveats (4.13) and (4.13)). Because the construction in the proof of Proposition (2.25) lies at the heart of most results on bounded cohomology related to amenability, the corresponding statements for ℓ^1-homology cannot be proved by imitating the cohomological proofs.

2.4.3 The mapping theorem in bounded cohomology

Gromov [18; p. 40] and Ivanov [25; Theorem 4.3] established that bounded cohomology of spaces depends only on the fundamental group. More generally, they showed the following:

Theorem (2.26) (Mapping theorem for bounded cohomology). *If $f\colon X \longrightarrow Y$ is a continuous map of connected, countable CW-complexes such that the induced map $\pi_1(f)$ is surjective and has amenable kernel, then the induced homomorphism*

$$H^*_{\mathrm{b}}(f)\colon H^*_{\mathrm{b}}(Y) \longrightarrow H^*_{\mathrm{b}}(X)$$

is an isometric isomorphism.

In particular, the bounded cohomology of spaces with amenable fundamental group vanishes in non-zero degree.

Similarly, bounded cohomology of groups cannot see amenable, normal subgroups [25, 45, 42; Section 3.8, Theorem 1, Corollary 8.5.2]:

Theorem (2.27). *Let G be a discrete group, let $A \subset G$ be an amenable normal subgroup and let V be a Banach G-module. Then the projection $G \longrightarrow G/A$ induces an isometric isomorphism*

$$H^*_{\mathrm{b}}(G/A; V'^A) \cong H^*_{\mathrm{b}}(G; V').$$

Conversely, the vanishing of bounded cohomology (with respect to *all* twisted coefficients) even characterises amenable groups [26].

2.4.4 Bounded cohomology of spaces via injective resolutions

Singular cohomology of an aspherical space coincides with group cohomology of the fundamental group, and hence singular cohomology of aspherical spaces can be computed by injective resolutions and vice versa. The same is true for bounded cohomology. But since bounded cohomology depends only on the fundamental group (Theorem (2.26)), the corresponding statement is much stronger:

Theorem (2.28). *Let X be a countable, connected CW-complex, let $G := \pi_1(X)$, and let V be a Banach G-module. Then there is a natural isometric isomorphism*

$$H^*_{\mathrm{b}}(X; V') \cong H^*_{\mathrm{b}}(G; V').$$

As a consequence, bounded cohomology of spaces can also be computed via strong relatively injective resolutions (Theorem (2.18)).

In the case of trivial coefficients, the theorem was proved by Ivanov [25; Theorem 4.1] (based on work of Brooks [6]). A proof of the generalised version is given in Appendix B.

2.4.5 Bounded cohomology and hyperbolic groups

Using straightening (see Section 4.4), Thurston shows that for all oriented, closed, connected, hyperbolic manifolds M the comparison map $H_b^k(M) \longrightarrow H^k(M)$ is surjective for all $k \in \mathbf{N}_{\geq 2}$. In particular, bounded cohomology is not always zero.

An interesting class of groups, containing all fundamental groups of oriented, closed, connected, hyperbolic manifolds, is the class of hyperbolic groups introduced by Gromov [19]. Mineyev [39, 40] extended Thurston's result to all hyperbolic groups and discovered that this property characterises hyperbolicity:

Theorem (2.29) (Characterisation of hyperbolic groups by bounded cohomology). *Let G be a finitely presented group. Then G is hyperbolic if and only if for all Banach G-modules V and all $k \in \mathbf{N}_{\geq 2}$ the comparison map $H_b^k(G;V) \longrightarrow H^k(G;V)$ is surjective.*

In particular, the mapping theorem (Theorem (2.26)) yields: If X is an aspherical, countable, connected CW-complex with hyperbolic fundamental group, then the comparison map $H_b^k(X) \longrightarrow H^k(X)$ is surjective for all $k \in \mathbf{N}_{\geq 2}$.

3 Duality

The universal coefficient theorem shows that the algebraic dual of homology of a chain complex of vector spaces coincides with the cohomology of the algebraic dual cochain complex. In this chapter, we investigate the effect of replacing algebraic duals by topological duals.

While the naïve analogue of the universal coefficient theorem fails in this Banach setting, we present the following translation mechanism (Theorem (3.1)): A morphism of Banach chain complexes induces an isomorphism on homology if and only if its dual induces an isomorphism on cohomology of the corresponding dual Banach cochain complexes. Additionally, if the isomorphism on cohomology is isometric, then so is the isomorphism on homology.

A first step towards a proof is the observation that taking topological duals is compatible with acyclicity of Banach chain complexes. Using mapping cones, we can transform this compatibility into the translation mechanism.

We first give a precise statement of the translation principle and discuss duality in the Banach setting (Section 3.1). In the second step, mapping cones are studied (Section 3.2). The proof of the translation mechanism is given in Section 3.3. In the last section, we have a closer look at the relation between cohomological and homological comparison maps.

Applications of the translation mechanism to ℓ^1-homology and bounded cohomology are given in Chapter 4.

3.1 Linking homology and cohomology

3.1.1 Statement of the main theorem

Unlike taking algebraic duals of chain complexes of vector spaces, taking topological duals of normed chain complexes does not commute with homology (Remark (3.4)). However, it is still possible to transfer certain information from homology of a Banach chain complex to cohomology of the dual Banach cochain complex and vice versa:

Theorem (3.1) (Translation mechanism for isomorphisms). *Let $f\colon C \longrightarrow D$ be a morphism of Banach chain complexes and let $f'\colon D' \longrightarrow C'$ be its dual.*

1. *Then the induced homomorphism $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism of vector spaces if and only if $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ is an isomorphism of vector spaces.*
2. *Furthermore, if $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ is an isometric isomorphism, then also $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isometric isomorphism.*

The proof of the first part relies on the following duality principle: A Banach chain complex is acyclic if and only if the corresponding dual Banach cochain complex is acyclic (Theorem (3.5)). The key to lifting this duality to morphisms is to apply the duality principle to mapping cones of morphisms of Banach chain complexes.

The second part can be derived from the first part because the semi-norm on homology of a normed chain complex can be computed in terms of the semi-norm on cohomology of its dual complex (Theorem (3.8)). On the other hand, the semi-norm on cohomology of the dual in general cannot be computed in terms of the semi-norm on homology. Therefore, we cannot expect that the converse of the second part holds.

Before delving into the details of the proof of the translation mechanism, we first introduce the Kronecker product and shed some light on the relation it induces between homology of Banach chain complexes and cohomology of the corresponding dual cochain complexes.

3.1.2 The Kronecker product in the normed setting

Analogous to the algebraic setting, evaluation links homology of a normed chain complex and cohomology of its dual complex.

Definition (3.2). Let C be a normed chain complex. Evaluation $C'^n \otimes C_n \longrightarrow \mathbf{R}$ induces linear maps, the so-called **Kronecker products**,

$$\begin{aligned}
\langle \cdot\,, \cdot \rangle \colon H^*(C') \otimes H_*(C) &\longrightarrow \mathbf{R},\\
\langle \cdot\,, \cdot \rangle \colon H^*(C') \otimes \overline{H}_*(C) &\longrightarrow \mathbf{R},\\
\langle \cdot\,, \cdot \rangle \colon \overline{H}^*(C') \otimes \overline{H}_*(C) &\longrightarrow \mathbf{R}.
\end{aligned}$$
◇

These Kronecker products are well-defined because all elements in C' are, by definition, continuous.

Remark (3.3). Let $f \colon C \longrightarrow D$ be a morphism of normed chain complexes and let $n \in \mathbf{N}$. Then the induced homomorphisms $H_n(f)$ and $H^n(f')$ are adjoint in the sense that

$$\big\langle \varphi, H_n(f)(\alpha)\big\rangle = \big\langle H^n(f')(\varphi), \alpha\big\rangle$$

for all $\alpha \in H_n(C)$ and all $\varphi \in H^n(D')$. Analogously, $\overline{H}_n(f)$ and $\overline{H}^n(f')$ are adjoint with respect to $\langle \cdot\,, \cdot \rangle$. □

By the universal coefficient theorem, the algebraic dual of homology of a chain complex of vector spaces coincides with the cohomology of the algebraic dual complex. However, taking topological duals (even of complete normed chain complexes) fails to commute with homology:

Remark (3.4). *There is no obvious duality isomorphism between homology and cohomology of Banach chain complexes*:

Let C be a Banach chain complex. Then we have the following commutative diagram

$$\begin{tikzcd}
H^*(C') \arrow[r] \arrow[d] \arrow[dr] & \hom_{\mathbf{R}}(H_*(C), \mathbf{R}) \\
\overline{H}^*(C') \arrow[r] & \big(\overline{H}_*(C)\big)', \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are the homomorphisms induced by the Kronecker products (i.e., they are induced by evaluation of elements in C' on elements in C), the left vertical arrow is the canonical projection and the right vertical arrow is the composition $(\overline{H}_*(C))' \hookrightarrow \hom_{\mathbf{R}}(\overline{H}_*(C), \mathbf{R}) \hookrightarrow \hom_{\mathbf{R}}(H_*(C), \mathbf{R})$ of inclusions.

The lower horizontal morphism, and hence also the diagonal morphism, is surjective by the Hahn-Banach theorem. Moreover, Matsumoto and Morita showed that the diagonal morphism is injective if and only if $H^*(C') = \overline{H}^*(C')$ holds [38; Theorem 2.3].

Obviously, this is not the case in general. It is even wrong if $C = C_*^{\ell^1}(X)$ for certain topological spaces X [54, 53]. Hence, there is no obvious duality between ℓ^1-homology and bounded cohomology.

Even the lower horizontal arrow is in general not injective: The kernel of the evaluation map

$$\ker \partial'^{n+1} \longrightarrow \big(\ker \partial_n / \overline{\operatorname{im} \partial_{n+1}}\big)' = \big(\overline{H}_n(C)\big)'$$

equals $({}^{\perp}\operatorname{im}(\partial'^n))^{\perp}$, which is the weak*-closure of $\operatorname{im} \partial'^n$ [51; Theorem 4.7]. Furthermore, the norm-closure $\overline{\operatorname{im} \partial'^n}$ and the weak*-closure $({}^{\perp}\operatorname{im}(\partial'^n))^{\perp}$ coincide if and only if $\operatorname{im} \partial_{n+1}$ is closed [51; Theorem 4.14]. Thus there is also no obvious duality isomorphism between reduced ℓ^1-homology and reduced bounded cohomology. $\diamond$

Nevertheless, the Kronecker product is strong enough to give sufficient conditions for (co)homology classes to be non-trivial. For example, if $\alpha \in H_*(C)$ and $\varphi \in H^*(C')$ with $\langle \varphi, \alpha \rangle = 1$, then neither α, nor φ can be zero. This effect can be used to show that ℓ^1-homology and bounded cohomology of certain surface groups are non-trivial [41].

3.1.3 Duality tools for the proof of the translation mechanism

Surprisingly, there is the following relation between homology of Banach chain complexes and cohomology of their duals, which has been discovered by Johnson [26; Proposition 1.2] as well as by Matsumoto and Morita [38; Corollary 2.4].

Theorem (3.5) (Duality principle). *Let C be a Banach chain complex. Then $H_*(C)$ vanishes if and only if $H^*(C')$ vanishes.*

Here, the "$*$" carries the meaning "*All* of the $H_n(C)$ are zero if and only if *all* the $H^n(C')$ are zero."

For the sake of completeness we provide a proof of this theorem. The proof is based on the following fact, stating that taking dual Banach spaces pretends to be an exact functor; it is not a genuine exact functor because the categories involved are not Abelian.

Lemma (3.6). *Let $f\colon U \longrightarrow V$ and $g\colon V \longrightarrow W$ be two bounded operators of Banach spaces satisfying $g \circ f = 0$. Then the following two statements are equivalent:*

1. *The image of g is closed and* $\operatorname{im} f = \ker g$.
2. *The image of f' is closed and* $\operatorname{im}(g') = \ker(f')$.

Proof. The various kernels and images are related as follows [51; Theorem 4.7 and Theorem 4.12], where $\overline{\operatorname{im}(g')}^{*}$ denotes the weak*-closure of $\operatorname{im}(g')$:

$$(\operatorname{im} f)^{\perp} = \ker(f'),$$
$$(\ker g)^{\perp} = \bigl({}^{\perp}\operatorname{im}(g')\bigr)^{\perp} = \overline{\operatorname{im}(g')}^{*}.$$

Suppose the image of g is closed and $\operatorname{im} f = \ker g$. Then also $\operatorname{im} f$ is closed. Hence, $\operatorname{im}(f')$ and $\operatorname{im}(g')$ are (weak*-)closed by the closed range theorem [51; Theorem 4.14]. Therefore, we obtain $\ker(f') = \overline{\operatorname{im}(g')}^{*} = \operatorname{im}(g')$.

Conversely, suppose the image of f' is closed and $\operatorname{im}(g') = \ker(f')$. Thus, also $\operatorname{im}(g')$ is closed. By the closed range theorem, $\operatorname{im}(g')$ is even weak*-closed and both $\operatorname{im} f$ and $\operatorname{im} g$ are closed. In particular,

$$(\operatorname{im} f)^{\perp} = (\ker g)^{\perp}.$$

Because the image $\operatorname{im} f$ is closed and $\operatorname{im} f \subset \ker g$, the Hahn-Banach theorem shows that $\operatorname{im} f = \ker g$. □

Proof (of Theorem (3.5)). If the (co)homology of a Banach (co)chain complex vanishes, then the images of all (co)boundary operators are kernels of bounded operators and hence closed. Therefore, the theorem follows from Lemma (3.6). □

Remark (3.7). Lemma (3.6) can also be used to give stronger versions of Theorem (3.5); for example, one can loosen the restriction on the degrees [38; Theorem 2.3]. However, we do not need these generalisations for the applications we have in mind and therefore stick to the more streamlined formulation of Theorem (3.5). ◇

Moreover, Gromov realised that the semi-norms on homology and cohomology are intertwined in the following way [18, 1; p. 17, Proposition F.2.2]:

Theorem (3.8) (Duality principle for semi-norms). *Let C be a normed chain complex and let $n \in \mathbf{N}$. Then*

$$\|\alpha\| = \sup\Bigl\{ \frac{1}{\|\varphi\|_\infty} \Bigm| \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \Bigr\}$$

holds for each $\alpha \in H_n(C)$. Here, $\sup \emptyset := 0$.

Proof. If $\alpha \in H_n(C)$ and $\varphi \in H^n(C')$, then

$$\big|\langle \varphi, \alpha \rangle\big| \leq \|\alpha\| \cdot \|\varphi\|_\infty .$$

This shows that $\|\alpha\|$ is at least as large as the supremum. Now suppose $\|\alpha\| \neq 0$, i.e., if c is a cycle representing α, then $c \notin \overline{\operatorname{im} \partial_{n+1}}$. Thus, by the Hahn-Banach theorem there exists a functional $f \colon C_n \longrightarrow \mathbf{R}$ satisfying

$$f|_{\operatorname{im} \partial_{n+1}} = 0, \qquad f(c) = 1, \qquad \|f\|_\infty \leq 1/\|\alpha\|.$$

In particular, $f \in C'^n$ is a cocycle. Let $\varphi := [f] \in H^n(C')$ be the corresponding cohomology class. Then, by construction, $\langle \varphi, \alpha \rangle = 1$ and $\|\varphi\|_\infty \leq \|f\|_\infty \leq 1/\|\alpha\|$. Hence, $\|\alpha\|$ is at most as large as the supremum. ☐

The discussion in Remark (3.4) shows however that the semi-norm on $H^*(C')$ can in general not be computed by the semi-norm on $H_*(C)$. (It might happen that the reduced homology $\overline{H}_*(C)$ is zero, but $\overline{H}^*(C')$ is non-zero).

3.2 Mapping cones

Mapping cones of chain maps are a device translating questions about isomorphisms on homology into questions about the vanishing of homology groups (Lemma (3.10)). Like many concepts in homological algebra, the mapping cone is modeled on its topological counterpart – the mapping cone of continuous maps.

Definition (3.9). 1. Let $f \colon (C, \partial^C) \longrightarrow (D, \partial^D)$ be a morphism of normed chain complexes. Then the **mapping cone** of f, denoted by $\operatorname{Cone}(f)$, is the normed chain complex defined by

$$\operatorname{Cone}(f)_n := C_{n-1} \oplus D_n,$$

linked by the boundary operator that is given by the matrix

$$\begin{tikzcd}
\operatorname{Cone}(f)_n \arrow[d, "{\begin{pmatrix} -\partial^C & 0 \\ f & \partial^D \end{pmatrix}}"'] & = C_{n-1} \arrow[d, "-\partial^C"'] \arrow[drr, "f"] & \oplus & D_n \arrow[d, "\partial^D"] \\
\operatorname{Cone}(f)_{n-1} & = C_{n-2} & \oplus & D_{n-1}.
\end{tikzcd}$$

2. Dually, if $f\colon (D,\delta_D) \longrightarrow (C,\delta_C)$ is a morphism of normed cochain complexes, then the **mapping cone** of f, also denoted by $\mathrm{Cone}(f)$, is the normed cochain complex defined by

$$\mathrm{Cone}(f)^n := D^{n+1} \oplus C^n$$

with the coboundary operator determined by the matrix

$$\begin{tikzcd}
\mathrm{Cone}(f)^n \arrow[d, "\begin{pmatrix} -\delta_D & 0 \\ f & \delta_C \end{pmatrix}"'] & = D^{n+1} \arrow[d, "-\delta_D"'] \arrow[dr, "f"] & \oplus & C^n \arrow[d, "\delta_C"] \\
\mathrm{Cone}(f)^{n+1} & = D^{n+2} & \oplus & C^{n+1}.
\end{tikzcd}$$

In the first case, we equip the mapping cone with the direct sum of the norms, in the second case, we use the maximum norm. ◇

Clearly, if f is a morphism of Banach (co)chain complexes, then the mapping cone $\mathrm{Cone}(f)$ is also a Banach (co)chain complex.

3.2.1 Mapping cones and homology isomorphisms

The main feature of mapping cones is being able to detect isomorphisms on homology:

Lemma (3.10). 1. *Let $f\colon C \longrightarrow D$ be a morphism of normed chain complexes. Then the induced map $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism (of vector spaces) if and only if all homology groups $H_*(\mathrm{Cone}(f))$ are zero.*

2. *Dually, let $f\colon D \longrightarrow C$ be a morphism of normed cochain complexes. Then the induced map $H^*(f)\colon H^*(D) \longrightarrow H^*(C)$ is an isomorphism if and only if all cohomology groups $H^*(\mathrm{Cone}(f))$ are zero.*

In the proof of the lemma, we use the following notation:

Definition (3.11). If C is a normed chain complex, the normed chain complex ΣC that is derived from C via $(\Sigma C)_n := C_{n-1}$ is called the **suspension of** C. For a normed cochain complex C, the suspension ΣC is defined by $(\Sigma C)^n := C^{n-1}$. ◇

Proof. The sequence (where the morphisms are given by the obvious inclusion and the negative of the projection) $0 \longrightarrow D \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma C \longrightarrow 0$ of normed chain complexes is exact. Hence, we obtain a long exact sequence in homology (Proposition (1.9)) whose connecting morphism is easily seen to coincide with $H_*(f)$ [7; Proposition 0.6]:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n\big(\mathrm{Cone}(f)\big) \arrow[r] & H_n(\Sigma C) \arrow[r] \arrow[d, equal] & H_{n-1}(D) \arrow[r] & H_{n-1}\big(\mathrm{Cone}(f)\big) \arrow[r] & \cdots \\
& & H_{n-1}(C) \arrow[ur, "H_{n-1}(f)"'] & & &
\end{tikzcd}$$

This proves the first part. The second part can be shown in the same way, making use of the long exact cohomology sequence corresponding to the short exact sequence $0 \longrightarrow C \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma^{-1}D \longrightarrow 0$ of normed cochain complexes. $\square$

Notice that both the lemma and its proof are completely algebraic in nature – we used only the underlying **R**-chain complexes.

3.2.2 Mapping cones of dual morphisms

In order to understand the relation between the induced maps $H_*(f)$ and $H^*(f')$ it remains to relate the mapping cone of f to the one of f'.

Lemma (3.12). *Let $f\colon C \longrightarrow D$ be a morphism of normed chain complexes and let $f'\colon D' \longrightarrow C'$ the induced morphism between the dual complexes. Then there is a natural isomorphism*

$$\mathrm{Cone}(f)' \cong \Sigma\mathrm{Cone}(-f')$$

of normed cochain complexes, relating the mapping cones of f and $-f'$. In particular,

$$H^*\big(\mathrm{Cone}(f)'\big) \cong H^*\big(\Sigma\mathrm{Cone}(-f')\big).$$

Proof. For each $n \in \mathbf{N}$, there is an isomorphism

$$\begin{aligned}
\big(\mathrm{Cone}(f)'\big)^n = (C_{n-1} \oplus D_n)' &\longrightarrow (D_n)' \oplus (C_{n-1})' = \mathrm{Cone}(-f')^{n-1} \\
\varphi &\longmapsto (-1)^n \cdot \big(d \mapsto \varphi(0,d), c \mapsto \varphi(c,0)\big) \\
(-1)^n \cdot \big((c,d) \mapsto \varphi(c) + \psi(d)\big) &\longleftarrow\!\shortmid (\psi, \varphi)
\end{aligned}$$

of normed vector spaces. By definition, the coboundary operator of $\mathrm{Cone}(f)'$ is given by

$$\begin{aligned}
(C_{n-1} \oplus D_n)' &\longrightarrow (C_n \oplus D_{n+1})' \\
\varphi &\longmapsto \Big((c,d) \mapsto \varphi\big(-\partial^C(c), f(c) + \partial^D(d)\big)\Big),
\end{aligned}$$

which corresponds under the isomorphisms given above to the coboundary operator on $\Sigma\mathrm{Cone}(-f')$. Hence, we obtain an isomorphism $\mathrm{Cone}(f)' \cong \Sigma\mathrm{Cone}(-f')$ of normed cochain complexes. $\square$

3.3 Transferring isomorphisms

In this section, we put all the pieces together and complete the proof of the translation mechanism (Theorem (3.1)).

3.3.1 Transferring algebraic isomorphisms

Fusing the properties of mapping cones with the duality principle (Theorem (3.5)) yields a proof of the first part of Theorem (3.1):

Theorem (3.13). *Let $f\colon C \longrightarrow D$ be a morphism of Banach chain complexes. Then the induced homomorphism $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism of vector spaces if and only if the induced homomorphism $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ is an isomorphism of vector spaces.*

Proof. By Lemma (3.10), the induced homomorphism $H_*(f)$ is an isomorphism if and only if $H_*(\mathrm{Cone}(f)) = 0$. In view of the duality principle (Theorem (3.5)) and Lemma (3.12), this is equivalent to

$$0 = H^*\big(\mathrm{Cone}(f)'\big) \cong H^*\big(\Sigma\mathrm{Cone}(-f')\big) = H^{*-1}\big(\mathrm{Cone}(-f')\big).$$

(The duality principle is applicable because the cone of a morphism of Banach chain complexes is a Banach chain complex.) On the other hand, the cohomology groups $H^{*-1}(\mathrm{Cone}(-f'))$ are all zero if and only if $H^*(-f')\colon H^*(D') \longrightarrow H^*(C')$ is an isomorphism (Lemma (3.10)). Moreover, $H^*(f') = -H^*(-f')$, and therefore the claim follows. $\square$

3.3.2 Transferring isometric isomorphisms

Similarly, combining the properties of mapping cones with the duality principle for semi-norms (Theorem (3.8)) proves the second part of Theorem (3.1):

Theorem (3.14). *Let $f\colon C \longrightarrow D$ be a morphism of normed chain complexes that induces an isometric isomorphism $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ between the cohomology groups of the topological duals. Then also $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is isometric.*

Proof. That the homomorphism $H_*(f)$ is isometric is a consequence of the duality principle for semi-norms (Theorem (3.8)), namely:

Let $n \in \mathbf{N}$ and let $\alpha \in H_n(C)$. Using the duality principle for semi-norms twice and the fact that $H^*(f')$ is an isometric isomorphism, we obtain

$$\begin{aligned}
\big\|H_n(f)(\alpha)\big\| &= \sup\Big\{\frac{1}{\|\psi\|_\infty}\ \Big|\ \psi \in H^n(D') \text{ and } \big\langle \psi, H_n(f)(\alpha)\big\rangle = 1\Big\}\\
&= \sup\Big\{\frac{1}{\|\psi\|_\infty}\ \Big|\ \psi \in H^n(D') \text{ and } \big\langle H^n(f')(\psi), \alpha\big\rangle = 1\Big\}\\
&= \sup\Big\{\frac{1}{\|H^n(f')(\psi)\|_\infty}\ \Big|\ \psi \in H^n(D') \text{ and } \big\langle H^n(f')(\psi), \alpha\big\rangle = 1\Big\}\\
&= \sup\Big\{\frac{1}{\|\varphi\|_\infty}\ \Big|\ \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha\rangle = 1\Big\}\\
&= \|\alpha\|,
\end{aligned}$$

as desired. □

3.4 Comparing comparison maps

In addition to the translation mechanism (Theorem (3.1)), there also exists a link between the cohomological and homological comparison maps, i.e., between the maps measuring the difference between the algebraic and the functional analytic settings.

Proposition (3.15). *Let C be a normed chain complex, let $n \in \mathbf{N}$, and let $i\colon C \hookrightarrow \overline{C}$ and $j\colon C' \hookrightarrow \hom_{\mathbf{R}}(C, \mathbf{R})$ be the canonical inclusions. If the cohomological comparison map $H^n(j)\colon H^n(C') \longrightarrow H^n(\hom_{\mathbf{R}}(C, \mathbf{R}))$ is surjective, then the homological comparison map $H_n(i)\colon H_n(C) \longrightarrow H_n(\overline{C})$ is injective.*

Proof. Let $\alpha \in H_n(C) \setminus \{0\}$. Clearly, it suffices to show $\|H_n(i)(\alpha)\| \neq 0$.

By the universal coefficient theorem, the Kronecker product induces an isomorphism $H^n(\hom_{\mathbf{R}}(C, \mathbf{R})) \cong \hom_{\mathbf{R}}(H_n(C), \mathbf{R})$. Hence, there is a cohomology class $\psi \in H^n(\hom_{\mathbf{R}}(C, \mathbf{R}))$ with $\langle \psi, \alpha \rangle = 1$. By assumption, the cohomological comparison map $H^n(j)$ is surjective, i.e., there is a $\varphi \in H^n(C')$ with $H^n(j)(\varphi) = \psi$. In particular,

$$\bigl\langle \varphi, H_n(i)(\alpha) \bigr\rangle = \bigl\langle H^n(j)(\varphi), \alpha \bigr\rangle = \langle \psi, \alpha \rangle = 1.$$

Therefore, $\varphi \neq 0$ and we obtain $\|H_n(i)(\alpha)\| \neq 0$ from the duality principle for semi-norms (Theorem (3.8)). □

However, the converse of this proposition is not true without imposing substantial finiteness conditions as the following example shows.

Example (3.16). *There exist normed chain complexes C and $n \in \mathbf{N}$ such that the homological comparison map $H_n(C) \longrightarrow H_n(\overline{C})$ is injective, but the cohomological comparison map $H^n(C') \longrightarrow H^n(\hom_{\mathbf{R}}(C, \mathbf{R}))$ is not surjective:*

Let C be the Banach chain complex concentrated in degree n with $C_n = \ell^1(\mathbf{Z})$. Then $\overline{C} = C$ and hence the homological comparison map $H_n(C) \longrightarrow H_n(\overline{C})$ is injective.

By construction,

$$\begin{aligned} H^n\bigl(C'\bigr) &= B\bigl(\ell^1(\mathbf{Z}), \mathbf{R}\bigr) \\ H^n\bigl(\hom_{\mathbf{R}}(C, \mathbf{R})\bigr) &= \hom_{\mathbf{R}}\bigl(\ell^1(\mathbf{Z}), \mathbf{R}\bigr). \end{aligned}$$

Therefore, the cohomological comparison map $H^n(C') \longrightarrow H^n(\hom_{\mathbf{R}}(C, \mathbf{R}))$ cannot be surjective – the space $\ell^1(\mathbf{Z})$ is infinite dimensional. ◇

While it is easy to see (for example, using the Kronecker product) that the image of the cohomological comparison map is always a Banach space [6, 17; p. 60, Corollary 1.12], it is unknown for which normed chain complexes C the semi-norm on the image of the homological comparison map $H_*(C) \longrightarrow H_*(\overline{C})$ is a norm; a discussion of this issue in the case that C is the singular chain complex of a manifold can be found in Section 6.3.1.

4 Isomorphisms in ℓ^1-homology

In this chapter, we apply the translation mechanism established in the previous chapter (Theorem (3.1)) to ℓ^1-homology, thereby gaining a uniform, lightweight approach to the following results:

- Like bounded cohomology ℓ^1-homology of a space depends only on the fundamental group (Corollary (4.3)).
- More generally, ℓ^1-homology of spaces as well as of discrete groups cannot see amenable, normal subgroups (Corollaries (4.2) and (4.12)).
- ℓ^1-Homology of spaces can be computed via certain projective resolutions (Corollary (4.14)).

In Section 4.4, we present an example application of these results – a "straightening" of chains on the level of ℓ^1-homology. The motivation for this application is the geometric straightening in the non-positively curved setting. One of the consequences of the ℓ^1-straightening is a homological proof of the fact that measure homology and singular homology are isometrically isomorphic (Appendix D).

4.1 Isomorphisms in ℓ^1-homology of spaces

The translation mechanism (Theorem (3.1)) allows to transfer certain results from bounded cohomology to ℓ^1-homology. In this section, we present the simplest applications of this type, concerning ℓ^1-homology of spaces with $\mathbf{R}$-coefficients.

4.1.1 The translation mechanism for ℓ^1-homology of spaces

In the language of ℓ^1-homology, the translation mechanism reads as follows:

Corollary (4.1). *Let $f\colon (X,A) \longrightarrow (Y,B)$ be a continuous map of pairs of topological spaces.*

1. *The induced homomorphism $H_*^{\ell^1}(f)\colon H_*^{\ell^1}(X,A) \longrightarrow H_*^{\ell^1}(Y,B)$ is an isomorphism if and only if $H_\mathrm{b}^*(f)\colon H_\mathrm{b}^*(Y,B) \longrightarrow H_\mathrm{b}^*(X,A)$ is an isomorphism.*
2. *If $H_\mathrm{b}^*(f)\colon H_\mathrm{b}^*(Y,B) \longrightarrow H_\mathrm{b}^*(X,A)$ is an isometric isomorphism, then $H_*^{\ell^1}(f)$ is also an isometric isomorphism.*
3. *In particular, $H_*^{\ell^1}(X,A)$ vanishes if and only if $H_\mathrm{b}^*(X,A)$ vanishes.*

Proof. By definition, $C_\mathrm{b}^*(X,A) = (C_*^{\ell^1}(X,A))'$ and $C_\mathrm{b}^*(Y,B) = (C_*^{\ell^1}(Y,B))'$. The cochain map $C_\mathrm{b}^*(f)\colon C_\mathrm{b}^*(Y,B) \longrightarrow C_\mathrm{b}^*(X,A)$ coincides with $(C_*^{\ell^1}(f))'$. Applying the translation mechanism Theorem (3.1) to $C_*^{\ell^1}(f)$ proves the Corollary. □

4.1.2 The mapping theorem in ℓ^1-homology

For example, Corollary (4.1) yields a new, lightweight proof of the fact that ℓ^1-homology depends only on the fundamental group (Corollary (4.3)) and that amenable groups are a blind spot of ℓ^1-homology (Corollary (4.2)). A short introduction to amenable groups is given in Section 2.4.1.

Corollary (4.2) (Mapping theorem for ℓ^1-homology). *Let $f\colon X \longrightarrow Y$ be a continuous map of connected, countable CW-complexes such that $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and has amenable kernel. Then the induced homomorphism*

$$H_*^{\ell^1}(f)\colon H_*^{\ell^1}(X) \longrightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism.

Proof. It is a classic result in the theory of bounded cohomology that in this situation $H^*_b(f)\colon H^*_b(Y) \longrightarrow H^*_b(X)$ is an isometric isomorphism (Theorem (2.26)). Applying Corollary (4.1) completes the proof. $\square$

Corollary (4.3). *The ℓ^1-homology of connected, countable CW-complexes depends only on the fundamental group.*

Proof. Let X be a connected, countable CW-complex. Its fundamental group is countable, and hence there is a model of the classifying space $B\pi_1(X)$ that is a countable, connected CW-complex. Therefore, we can apply the previous corollary. $\square$

Bouarich gave the first proof that ℓ^1-homology depends only on the fundamental group [5; Corollaire 6]. His proof is based on Theorem (3.5), the fact that bounded cohomology of simply connected spaces vanishes, and an ℓ^1-version of Brown's theorem. Moreover, Park [47; Corollary 4.2] already claimed that Corollary (4.2) holds. However, due to a gap in her argument, her proof is not complete. This issue is addressed in Caveat (4.13) and Caveat (4.15).

Corollary (4.2) also gives a new proof of the following result of Bouarich [5; Corollaire 5]:

Corollary (4.4). *Let $p\colon E \longrightarrow B$ be a fibration of connected, countable CW-complexes with path-connected fibre F. If the fundamental group $\pi_1(F)$ is amenable, then the induced map $H^{\ell^1}_*(p)\colon H^{\ell^1}_*(E) \longrightarrow H^{\ell^1}_*(B)$ is an isometric isomorphism.*

Proof. From the portion

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \xrightarrow{\pi_1(p)} \pi_1(B) \longrightarrow \pi_0(F) = 0$$

of the long exact sequence associated to the fibration p, we obtain that $\pi_1(p)$ is surjective and that its kernel $\ker \pi_1(p)$, as homomorphic image of the amenable group $\pi_1(F)$, must be amenable [48; Proposition 1.12 and 1.13]. Now the result follows from Corollary (4.2). $\square$

4.1.3 Amenable subsets and ℓ^1-homology

Gromov introduced the notion of amenable subsets of spaces [18; p. 40]; amenable subsets serve as a generalisation of sets that are contractible in the ambient space.

Definition (4.5). Let X be a topological space. A subset $A \subset X$ is called **amenable** if the subgroups $\operatorname{im} \pi_1(B \hookrightarrow X) \subset \pi_1(X)$ are amenable for all path-connected components B of A. $\diamond$

Corollary (4.6) (Amenable subsets and ℓ^1-homology). *Let X be a connected, countable CW-complex and suppose that $A \subset X$ is an amenable subcomplex with finitely many connected components. Then*

$$\operatorname{im} H_k^{\ell^1}(A \hookrightarrow X) = 0$$

for all $k \in \mathbf{N}_{>0}$.

Proof. Keeping in mind that ℓ^1-homology is additive with respect to finite disjoint unions (Proposition (2.7)), we can restrict ourselves to the case where A is connected.

Killing the kernel of $\pi_1(A \hookrightarrow X)$ by gluing in disks shows that we can find a connected, countable CW-complex $\overline{X}$ containing X and a subcomplex $\overline{A}$ of $\overline{X}$ containing A with the following properties [25; p. 1110]: The group $\pi_1(\overline{A})$ is amenable, the inclusion $X \hookrightarrow \overline{X}$ induces an isomorphism on the level of fundamental groups, and the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \overline{A} & \longrightarrow & \overline{X} \end{array}$$

of inclusions is commutative. By the mapping theorem (Corollary (4.2)), the right vertical arrow induces an isometric isomorphism on the level of ℓ^1-homology and $H_k^{\ell^1}(\overline{A}) = 0$ for all $k \in \mathbf{N}_{>0}$. Therefore, the corresponding diagram in ℓ^1-homology shows that $\operatorname{im} H_k^{\ell^1}(A \hookrightarrow X) = 0$. $\square$

For bounded cohomology, the comparison map in high degree is well understood in the case that the space in question admits an open, amenable covering with controlled multiplicity [18, 25; p. 40, Corollary 6.3]. Probably this is also true for ℓ^1-homology – for example, one could try to transfer Ivanov's argument to ℓ^1-homology.

4.1.4 Hyperbolic groups and ℓ^1-homology

Mineyev showed that hyperbolicity gives rise to large bounded cohomology (Section 2.4.5). A small part of this result can also be formulated in terms of ℓ^1-homology:

Corollary (4.7). *Let X be an aspherical, countable, connected CW-complex with hyperbolic fundamental group. Then the comparison map $H_k(X) \longrightarrow H_k^{\ell^1}(X)$ is injective for all $k \in \mathbf{N}_{\geq 2}$.*

Proof. Let $k \in \mathbf{N}_{\geq 2}$. By Mineyev's result (Section 2.4.5), the cohomological comparison map $H_{\mathrm{b}}^k(X) \longrightarrow H^k(X)$ is surjective. Therefore, Proposition (3.15) applied to the normed chain complex $C_*(X)$ shows that the homological comparison map $H_k(X) \longrightarrow H_k^{\ell^1}(X)$ is injective. □

4.2 Isomorphisms in ℓ^1-homology of discrete groups

Analogously to the previous section, the translation mechanism can also be applied to ℓ^1-homology of discrete groups (Corollary (4.8)). For example, this transforms the characterisation of amenable groups via bounded cohomology into a characterisation in terms of ℓ^1-homology (Corollary (4.11)). Moreover, we deduce that like bounded cohomology, ℓ^1-homology ignores amenable normal subgroups (Corollary (4.12)).

4.2.1 The translation mechanism for ℓ^1-homology of discrete groups

Corollary (4.8). *Let $\varphi\colon G \longrightarrow H$ be a homomorphism of discrete groups, let V be a Banach G-module, let W be a Banach H-module and suppose $f\colon V \longrightarrow \varphi^* W$ is a morphism of Banach G-modules.*

1. *Then the homomorphism $H_*^{\ell^1}(\varphi; f)\colon H_*^{\ell^1}(G; V) \longrightarrow H_*^{\ell^1}(H; W)$ is an isomorphism if and only if $H_{\mathrm{b}}^*(\varphi; f')\colon H_{\mathrm{b}}^*(H; W') \longrightarrow H_{\mathrm{b}}^*(G; V')$ is an isomorphism.*
2. *If $H_{\mathrm{b}}^*(\varphi; f')$ is an isometric isomorphism, then so is $H_*^{\ell^1}(\varphi; f)$.*

Proof. By definition (Definition (2.17)), we have

$$H_*^{\ell^1}(\varphi; f) = H_*\big(p \circ C_*^{\ell^1}(\varphi; f)_G\big),$$
$$C_{\mathrm{b}}^*(\varphi; f') = H^*\big(C_{\mathrm{b}}^*(\varphi; f')^G \circ i\big),$$

where $p\colon (\varphi^* C_*^{\ell^1}(H; W))_G \longrightarrow C_*^{\ell^1}(H; W)_H$ denotes the canonical projection and $i\colon C_{\mathrm{b}}^*(H; W')^H \longrightarrow (\varphi^* C_{\mathrm{b}}^*(H; W'))^G$ is the inclusion.

$$
\begin{tikzcd}
\big(C^{\ell^1}_*(H;W)_H\big)' \arrow[r, equal, "(1.14)"] \arrow[d, "p'"] \arrow[dd, bend right=60, "(p\circ C^{\ell^1}_*(\varphi;f)_G)'"'] & \big(C^{\ell^1}_*(H;W)'\big)^H \arrow[r, equal, "(2.14)"] & C^*_b(H;W')^H \arrow[d, "i"'] \\
\big((\varphi^*C^{\ell^1}_*(H;W))_G\big)' \arrow[r, equal, "(1.14)"] \arrow[d, "(C^{\ell^1}_*(\varphi;f)_G)'"] & \big(\varphi^*C^{\ell^1}_*(H;W)'\big)^G \arrow[r, equal, "(2.14)"] & \big(\varphi^*C^*_b(H;W')\big)^G \arrow[d, "C^*_b(\varphi;f')^G"'] \\
\big(C^{\ell^1}_*(G;V)_G\big)' \arrow[r, equal, "(1.14)"'] & \big(C^{\ell^1}_*(G;V)'\big)^G \arrow[r, equal, "(2.14)"'] & C^*_b(G;V')^G
\end{tikzcd}
$$

Figure (4.9). Linking ℓ^1-homology and bounded cohomology of discrete groups (proof of Corollary (4.8))

A straightforward calculation shows that the diagram in Figure (4.9) is a commutative diagram of morphisms of Banach chain complexes, where all horizontal morphisms are isometric isomorphisms.

Therefore, applying the translation mechanism (Theorem (3.1)) to the morphism $p \circ C^{\ell^1}_*(\varphi; f)_G$ of Banach chain complexes proves the corollary. □

Corollary (4.10). *Let G be a discrete group and let V be a Banach G-module. Then $H^{\ell^1}_*(G;V) \cong H^{\ell^1}_*(1;V)$ if and only if $H^*_b(G;V') \cong H^*_b(1;V')$.* □

4.2.2 Amenable groups and ℓ^1-homology of discrete groups

Corollary (4.8) enables us to carry over many results on bounded cohomology of discrete groups to ℓ^1-homology. In the following, we present two examples of this kind:

Corollary (4.11) (Characterisation of amenable groups by ℓ^1-homology). *For a discrete group G the following are equivalent:*

1. *The group G is amenable.*
2. *For all Banach G-modules V, the ℓ^1-homology $H^{\ell^1}_*(G;V)$ of G with coefficients in V is trivial, i.e., $H^{\ell^1}_*(G;V) \cong H^{\ell^1}_*(1;V)$.*

Proof. Amenable groups can be characterised by the vanishing of bounded cohomology with arbitrary (dual) coefficients in non-zero degree [26, 45]. Therefore the claim follows with help of Corollary (4.10). □

Corollary (4.12). *Let G be a discrete group, let $A \subset G$ be an amenable, normal subgroup and let V be a Banach G-module. Then the projection $G \longrightarrow G/A$ induces an isometric isomorphism*

$$H_*^{\ell^1}(G;V) \cong H_*^{\ell^1}(G/A;V_A).$$

Proof. The corresponding homomorphism

$$H_{\mathrm{b}}^*(G \twoheadrightarrow G/A; V'^A \hookrightarrow V') \colon H_{\mathrm{b}}^*(G/A; V'^A) \longrightarrow H_{\mathrm{b}}^*(G;V')$$

is an isometric isomorphism [45, 42; Theorem 1, Corollary 8.5.2] (the case with $\mathbf{R}$-coefficients was already treated by Ivanov [25; Section 3.8]). Because the inclusion $V'^A \hookrightarrow V'$ is the dual of the projection $V \longrightarrow V_A$ (which follows from Proposition (1.14)), we can apply Corollary (4.8). ☐

Caveat (4.13). Let G be a discrete group and let $A \subset G$ be an amenable normal subgroup.

Ivanov proved that the cochain complex $C_{\mathrm{b}}^*(G/A)$ is a strong relatively injective G-resolution of the trivial G-module $\mathbf{R}$ [25; Theorem 3.8.4] by showing that the G-morphisms $C_{\mathrm{b}}^*(G/A) \longrightarrow C_{\mathrm{b}}^*(G)$ induced by the projection $G \longrightarrow G/A$ are split injective [25; Lemma 3.8.1 and Corollary 3.8.2].

Analogously, Park claimed that the G-morphisms $C_n^{\ell^1}(G) \longrightarrow C_n^{\ell^1}(G/A)$ are split surjective [47; Lemma 2.4 and Lemma 2.5] and concluded that the $C_n^{\ell^1}(G/A)$ are relatively projective G-modules. Unfortunately, Park's proof [47; proof of Lemma 2.4] contains an error: the A-invariant mean on $B(A,\mathbf{R})$ provided by amenability of A in general is *not* σ-additive.

In fact, $C_n^{\ell^1}(G/A)$ in general is *not* a relatively projective G-module as the following example shows: Let G be an infinite amenable group (e.g., $G = \mathbf{Z}$) and let $A := G$. Then the G-action on $G/A = 1$ is trivial. However, since G is infinite, the G-modules $C_n^{\ell^1}(G)$ do not contain any non-zero G-invariant elements. Therefore, any G-morphism of type $C_n^{\ell^1}(G/A) \longrightarrow C_n^{\ell^1}(G)$ must be trivial. We now consider the mapping problem

$$\begin{tikzcd}
 & C_n^{\ell^1}(G/A) = \mathbf{R} \arrow[dl, dotted, "?"'] \arrow[d, "\mathrm{id}"] & \\
C_n^{\ell^1}(G) \arrow[r, "\pi"'] & \mathbf{R} \arrow[r] & 0
\end{tikzcd}$$

with the G-morphism π given by $g_0 \cdot [g_1|\dots|g_n] \longmapsto 1$, which obviously admits a (non-equivariant) split of norm 1. The argument above shows that this mapping problem cannot have a solution, and hence that $C_n^{\ell^1}(G/A)$ cannot be a relatively projective G-module.

This problem also affects several other results of Park, e.g., her proof of the fact that ℓ^1-homology depends only on the fundamental group [47; Theorem 4.1] and of the equivalence theorem [47; Theorem 3.7 and 4.4]. $\diamond$

4.3 ℓ^1-Homology of spaces via projective resolutions

Finally, we are able to identify ℓ^1-homology of topological spaces with ℓ^1-homology of the associated fundamental groups:

Corollary (4.14). *Suppose X is a countable, connected CW-complex with fundamental group G, and let V be a Banach G-module.*

1. *There is a canonical isometric isomorphism*
$$H_*^{\ell^1}(X;V) \cong H_*^{\ell^1}(G;V).$$
2. *In particular: If C is a strong relatively projective G-resolution of V, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*
$$H_*^{\ell^1}(X;V) \cong H_*(C_G).$$
3. *If C is a strong relatively projective G-resolution of the trivial Banach G-module $\mathbf{R}$, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*
$$H_*^{\ell^1}(X;V) \cong H_*\big((C \mathbin{\overline{\otimes}} V)_G\big).$$

Therefore, the results of Section 4.2 are also valid for ℓ^1-homology with twisted coefficients and provide generalisations of the results presented in Section 4.1.

Caveat (4.15). Ivanov proved the corresponding theorem for bounded cohomology with $\mathbf{R}$-coefficients [25; Theorem 4.1] by verifying that $C_b^*(\widetilde{X})$ is a strong relatively injective resolution of the trivial Banach G-module $\mathbf{R}$ [25; Theorem 2.4].

The proof that the resolution $C_b^*(\widetilde{X})$ is strong relies heavily on the fact that certain chain maps are split injective (see Lemma (B.4)). However, for the same reasons as explained in Caveat (4.13), it is not possible to translate these arguments into the language of ℓ^1-chain complexes. Hence, it seems impossible to prove that the chain complex $C_*^{\ell^1}(\widetilde{X})$ is a *strong* resolution. In particular, Park's proof [47; proof of Theorem 4.1] of Corollary (4.14) (with **R**-coefficients) is not complete. $\diamond$

Using the techniques developed in Chapter 3, we can derive Corollary (4.14) from the corresponding result in bounded cohomology (cf. Theorem (2.28), which is proved in Appendix B).

Proof (of Corollary (4.14)). Ad 1. In order to prove the first part of Corollary (4.14), we proceed as follows:

1. We establish a connection between $C_*^{\ell^1}(\widetilde{X}; V)$ and the strong relatively projective resolution $C_*^{\ell^1}(G; V)$.
2. The dual of this morphism, when restricted to the invariants, induces an isometric isomorphism on the level of cohomology of the invariants (Theorem (2.28)).
3. Finally, we apply the translation mechanism (Theorem (3.1)) to transfer this isometric isomorphism back to ℓ^1-homology.

First step. Park [47; proof of Theorem 4.1] constructed the following map ("pre-dually" to Ivanov's construction [25; proof of Theorem 4.1]):

Let $F \subset \widetilde{X}$ be a (set-theoretic) fundamental domain of the G-action on $\widetilde{X}$. In the following, the vertices of the standard n-simplex Δ^n are denoted by $v_0, \dots, v_n$. For a singular simplex $\sigma \in \operatorname{map}(\Delta^n, \widetilde{X})$ let $g_0(\sigma), \dots, g_n(\sigma) \in G$ be the group elements characterised uniquely by

$$\begin{aligned} g_0(\sigma)^{-1} \cdot \sigma(v_0) &\in F \\ g_1(\sigma)^{-1} \cdot g_0(\sigma)^{-1} \cdot \sigma(v_1) &\in F \\ &\vdots \\ g_n(\sigma)^{-1} \cdot \dots \cdot g_1(\sigma)^{-1} \cdot g_0(\sigma)^{-1} \cdot \sigma(v_n) &\in F. \end{aligned}$$

Then the map $\eta \colon C_*^{\ell^1}(\widetilde{X}) \longrightarrow C_*^{\ell^1}(G)$ given by

$$\begin{aligned} C_n^{\ell^1}(\widetilde{X}) &\longrightarrow C_n^{\ell^1}(G) \\ \sigma &\longmapsto g_0(\sigma) \cdot \bigl[g_1(\sigma)\bigm| \dots \bigm| g_n(\sigma)\bigr] \end{aligned}$$

is a morphism of Banach G-chain complexes. Hence,

$$\eta_V := \eta \mathbin{\overline{\otimes}} \mathrm{id}_V \colon C_*^{\ell^1}(\widetilde{X}; V) \longrightarrow C_*^{\ell^1}(G; V)$$

is also a morphism of Banach G-chain complexes.

Let $(\eta_V)_G \colon C_*^{\ell^1}(\widetilde{X};V)_G \longrightarrow C_*^{\ell^1}(G;V)_G$ denote the morphism of Banach chain complexes induced by η_V. We show now that a different choice of fundamental domain $F^* \subset \widetilde{X}$ leads to a map chain homotopic to $(\eta_V)_G$:

Homological algebra shows that there is up to G-homotopy only one G-morphism $C_*^{\ell^1}(\widetilde{X}) \longrightarrow C_*^{\ell^1}(G)$ (Proposition (A.7)); in fact, $C_*^{\ell^1}(\widetilde{X})$ is a Banach G-chain complex consisting of relatively projective G-modules [47; p. 611] and $C_*^{\ell^1}(G)$ is a strong relatively projective resolution of $\mathbf{R}$ (Proposition (2.19)). But η and η^*, the map obtained via F^*, are such G-morphisms and hence are G-homotopic. Therefore, also $\eta \mathbin{\overline{\otimes}} \mathrm{id}_V$ and $\eta_V^* := \eta^* \mathbin{\overline{\otimes}} \mathrm{id}_V$ must be G-homotopic, which implies that the induced maps $(\eta_V)_G$ and $(\eta_V^*)_G$ are homotopic. In particular,

$$H_*\big((\eta_V)_G\big) \colon H_*\big(C_*^{\ell^1}(\widetilde{X};V)_G\big) \longrightarrow H_*\big(C_*^{\ell^1}(G;V)_G\big)$$

does not depend on the choice of fundamental domain.

Second step. The dual of the G-morphism η_V coincides under the natural isometric isomorphisms $(C_*^{\ell^1}(\widetilde{X};V))' \cong C_b^*(\widetilde{X};V')$ and $(C_*^{\ell^1}(G;V))' \cong C_b^*(G;V')$ of Banach G-cochain complexes (Remarks (1.13) and (2.14)) with $\vartheta_{V'} \colon C_b^*(G;V') \longrightarrow C_b^*(\widetilde{X};V')$, the morphism of Banach G-cochain complexes given by

$$\begin{aligned} C_b^n(G;V') &\longrightarrow C_b^n(\widetilde{X};V') \\ f &\longmapsto \big(\sigma \mapsto f(g_0(\sigma),\dots,g_n(\sigma))\big). \end{aligned} \tag{4.16}$$

In other words, the diagram

$$\begin{tikzcd} \big(C_*^{\ell^1}(G;V)\big)' \arrow[r, "(\eta_V)'"] \arrow[d, equal, "(2.14)"'] & \big(C_*^{\ell^1}(\widetilde{X};V)\big)' \arrow[d, equal, "(1.13)"] \\ C_b^*(G;V') \arrow[r, "\vartheta_{V'}"'] & C_b^*(\widetilde{X};V') \end{tikzcd}$$

is commutative. Taking G-invariants of this diagram yields the commutative diagram of morphisms of Banach cochain complexes in Figure (4.17).

The restriction $(\vartheta_{V'})^G$ to the subcomplexes of G-invariants induces an isometric isomorphism on the level of cohomology (Theorem (2.28)/Theorem (B.1)). Hence, also the top row of the diagram (i.e, $(\eta_V)_G{}'$) must induce an isometric isomorphism on the level of cohomology.

Third step. Therefore, we can derive from the translation mechanism (Theorem (3.1)) that

$$(\eta_V)_G \colon C_*^{\ell^1}(X;V) = C_*^{\ell^1}(\widetilde{X};V)_G \longrightarrow C_*^{\ell^1}(G;V)_G$$

$$\begin{array}{ccc}
\bigl(C^{\ell^1}_*(G;V)_G\bigr)' & \xrightarrow{(\eta_V)_G{}'} & \bigl(C^{\ell^1}_*(\widetilde{X};V)_G\bigr)' \\
{\scriptstyle (1.14)}\Big\| & & \Big\|{\scriptstyle (1.14)} \\
\bigl(C^{\ell^1}_*(G;V)'\bigr)^G & \xrightarrow{(\eta_V)'^G} & \bigl(C^{\ell^1}_*(\widetilde{X};V)'\bigr)^G \\
{\scriptstyle (1.13)}\Big\| & & \Big\|{\scriptstyle (1.13)} \\
C^*_{\mathrm{b}}(G;V')^G & \xrightarrow[(\vartheta_{V'})^G]{} & C^*_{\mathrm{b}}(\widetilde{X};V')^G
\end{array}$$

Figure (4.17). Relating η_V and $\vartheta_{V'}$

induces a (canonical) isometric isomorphism on the level of homology. This finishes the proof of the first part.

Ad 2. and 3. These statements follow from the first part combined with the corresponding results on ℓ^1-homology of discrete groups (Theorem (2.18)). $\square$

4.4 Example application – a generalised straightening

For manifolds of non-positive sectional curvature, the existence of unique geodesics on the universal covering shows that the singular chain complex can be replaced by the chain complex of so-called straight simplices (Proposition (4.20)). Straight simplices in this context are defined as projections of geodesic simplices on the universal covering.

Based on ℓ^1-homology, we exhibit a generalised straightening that is valid for all countable, connected CW-complexes (Theorem (4.21)).

As first step, we define straight simplices for general spaces, starting with the notationally more transparent case of universal coverings. Geodesic simplices on the universal covering of a manifold with non-positive sectional curvature depend only the set of vertices, leading to the following definition:

Definition (4.18). Let X be a connected space with fundamental group G that admits a universal covering $\widetilde{X} \longrightarrow X$.

1. The real vector space $\mathbf{R}[\widetilde{X}^{n+1}]$, i.e., the $\mathbf{R}$-vector space with basis $\widetilde{X}^{n+1}$, with the obvious ℓ^1-norm carries an isometric G-action given by the diagonal action of G on $\widetilde{X}^{n+1}$.
2. The **straight chain complex** of $\widetilde{X}$ is the normed G-chain complex $S_*(\widetilde{X})$ defined by
$$S_n(X) := \mathbf{R}[\widetilde{X}^{n+1}]$$
together with the ℓ^1-norm and the boundary operator given by
$$\begin{aligned} S_n(\widetilde{X}) &\longrightarrow S_{n-1}(\widetilde{X}) \\ (x_0, \dots, x_n) &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot (x_0, \dots, \widehat{x_j}, \dots, x_n). \end{aligned}$$
3. Moreover, we define the **straightening map** $s_{\widetilde{X}} \colon C_*(\widetilde{X}) \longrightarrow S_*(\widetilde{X})$ via
$$\begin{aligned} C_n(\widetilde{X}) &\longrightarrow S_n(\widetilde{X}) \\ \sigma &\longmapsto \big(\sigma(v_0), \dots, \sigma(v_n)\big), \end{aligned}$$
where $v_0, \dots, v_n$ are the vertices of the standard n-simplex Δ^n. $\diamond$

Clearly, the straightening $s_{\widetilde{X}}$ is a well-defined chain map, which is bounded in each degree and which is compatible with the respective G-actions. I.e., $s_{\widetilde{X}}$ is a morphism of normed G-chain complexes. In order to obtain a straightening not only for the universal covering $\widetilde{X}$ but also for the space X itself, we pass to the coinvariants:

Definition (4.19). Let X be a connected, topological space that admits a universal covering $\widetilde{X} \longrightarrow X$. Let G be the fundamental group of X.

1. The **straight chain complex** of X is the normed chain complex defined by
$$S_*(X) := S_*(\widetilde{X})_G.$$
2. The **straight ℓ^1-chain complex** of X, denoted by $S_*^{\ell^1}(X)$, is the completion of the normed chain complex $S_*(X)$.
3. The **straightening** and **ℓ^1-straightening** of X respectively are given by
$$\begin{aligned} s_X &:= (s_{\widetilde{X}})_G \colon C_*(X) = C_*(\widetilde{X})_G \longrightarrow S_*(X), \\ \overline{s}_X &:= (\overline{s_{\widetilde{X}}})_G \colon C_*^{\ell^1}(X) = C_*^{\ell^1}(\widetilde{X})_G \longrightarrow S_*^{\ell^1}(X), \end{aligned}$$
where $\overline{s_{\widetilde{X}}}$ denotes the extension of the morphism $s_{\widetilde{X}} \colon C_*(\widetilde{X}) \longrightarrow S_*(\widetilde{X})$ of normed chain complexes to the respective completions. $\diamond$

For the universal covering $\widetilde{X}$ this "new" definition of the straight chain complex and the straightening map coincides with the first definition.

Proposition (4.20) (Straightening in the non-positively curved case). *Let M be a connected, Riemannian manifold with non-positive sectional curvature. Then straightening $s_M \colon C_*(M) \longrightarrow S_*(M)$ induces an isometric isomorphism*

$$H_*(s_M) \colon H_*(M) \longrightarrow H_*(S_*(M)).$$

If M is a Riemannian manifold with non-positive sectional curvature, any two points in the Riemannian universal covering $\widetilde{M}$ of M are connected by precisely one geodesic (up to parametrisation). In particular, it makes sense to speak of convex combinations of points in $\widetilde{M}$. A singular n-simplex σ of $\widetilde{M}$ is called *geodesic* if there exist $(m_0, \dots, m_n) \in \widetilde{M}^{n+1}$ such that σ is of the form

$$\begin{aligned} \Delta^n &\longrightarrow \widetilde{M} \\ (t_0, \dots, t_n) &\longmapsto \sum_{j=0}^{n} t_j \cdot m_j. \end{aligned}$$

Proof (of Proposition (4.20)). For brevity, we write $G := \pi_1(M)$. Because M is non-positively curved, for each $(n+1)$-tuple $(m_0, \dots, m_n)$ of points in the universal covering $\widetilde{M}$, there exists exactly one geodesic n-simplex $\Delta^n \longrightarrow \widetilde{M}$ whose vertices are $m_0, \dots, m_n$. Therefore, the chain complex $S_*(\widetilde{M})$ is isometrically G-isomorphic to the subcomplex $C^{\mathrm{g}}_*(\widetilde{M})$ of $C_*(\widetilde{M})$ generated by all geodesic simplices.

Furthermore, the geometry of $\widetilde{M}$ allows to find an explicit G-homotopy between the identity and the morphism

$$C_*(\widetilde{M}) \xrightarrow{s_{\widetilde{M}}} S_*(\widetilde{M}) \longrightarrow C^{\mathrm{g}}_*(\widetilde{M}) \longrightarrow C_*(\widetilde{M})$$

of Banach G-chain complexes [50; Lemma 2 on p. 531]. Clearly, this equivariant chain homotopy descends to a chain homotopy on M and thus $H_*(s_M)$ is an isomorphism.

The isomorphism $H_*(s_M)$ is even isometric because both $s_{\widetilde{M}}$ and the composition $S_*(\widetilde{M}) \longrightarrow C^{\mathrm{g}}_*(\widetilde{M}) \longrightarrow C_*(\widetilde{M})$ are norm-decreasing. □

Of course, in general, the geometry and topology of the universal covering is much more complicated and the corresponding statement would be false. However, if we look at the completion of the chain complexes, i.e., if we step into the ℓ^1-world, then straightening induces an isometric isomorphism on the level of homology:

Theorem (4.21) (ℓ^1-Straightening). *Let X be a countable, connected CW-complex. Then straightening $\overline{s}_X\colon C^{\ell^1}_*(X) \longrightarrow S^{\ell^1}_*(X)$ induces an isometric isomorphism*

$$H_*(\overline{s}_X)\colon H^{\ell^1}_*(X) \longrightarrow H_*\bigl(S^{\ell^1}_*(X)\bigr).$$

Proof. One can show that $S^{\ell^1}_*(\widetilde{X})$, with the augmentation $S^{\ell^1}_0(\widetilde{X}) \longrightarrow \mathbf{R}$ given by adding up the coefficients, is a strong relatively projective G-resolution of the trivial G-module $\mathbf{R}$. This is similar to the proof that $C^{\ell^1}_*(G)$ is a strong relatively projective resolution of $\mathbf{R}$ (Proposition (2.19)):

Namely, the Banach G-modules $S^{\ell^1}_n(\widetilde{X})$ are relatively projective because they are ℓ^1-completions of free $\mathbf{R}G$-modules [47; Lemma 2.1]. For the chain contraction, we choose a point $p \in \widetilde{X}$ and define

$$\begin{aligned} k_n\colon S^{\ell^1}_n(\widetilde{X}) &\longrightarrow S^{\ell^1}_{n+1}(\widetilde{X}) \\ (x_0,\dots,x_n) &\longmapsto (-1)^{n+1}\cdot(p,x_0,\dots,x_n), \end{aligned}$$

as well as

$$\begin{aligned} k_{-1}\colon \mathbf{R} &\longrightarrow S^{\ell^1}_0(\widetilde{X}) \\ 1 &\longmapsto 1\cdot p. \end{aligned}$$

It is not difficult to see that $(k_n)_{n\in\mathbf{Z}_{\geq -1}}$ indeed is a chain contraction of $S^{\ell^1}_*(\widetilde{X})$ (concatenated with the augmentation) of norm at most 1. Hence, $S^{\ell^1}_*(\widetilde{X})$ is a strong relatively projective G-resolution of $\mathbf{R}$.

Therefore, the fundamental lemma of homological algebra (Proposition (A.7)) implies that $\overline{s}_{\widetilde{X}}$ is a G-chain homotopy equivalence and hence that $\overline{s}_X = (\overline{s}_{\widetilde{X}})_G$ is a chain homotopy equivalence. In particular, $H_*(\overline{s}_X)$ is an isomorphism.

It remains to show that this isomorphism is isometric: By construction, the canonical morphism $\eta\colon C^{\ell^1}_*(\widetilde{X}) \longrightarrow C^{\ell^1}_*(G)$ of Banach G-chain complexes (introduced in the proof of Corollary (4.14)) factorises over the straight ℓ^1-chain complex:

$$\begin{tikzcd} C^{\ell^1}_*(\widetilde{X}) \arrow[rr, "\eta"] \arrow[dr, "\overline{s}_{\widetilde{X}}"'] & & C^{\ell^1}_*(G) \\ & S^{\ell^1}_*(\widetilde{X}) \arrow[ur] & \end{tikzcd}$$

Moreover, both diagonal arrows are of norm at most 1. Because $H_*(\eta_G)$ is an isometric isomorphism (Corollary (4.14)), we deduce that the isomorphism $H_*(\overline{s}_X) = H_*((\overline{s}_{\widetilde{X}})_G)$ is isometric. □

An important aspect of the abstract straightening as described above is that it allows to get control of the semi-norm in measure homology (see Appendix D). Measure homology in turn is the foundation for Thurston's smearing technique, which is a useful tool in the study of simplicial volume.

Remark (4.22). If M is a Riemannian manifold with non-positive sectional curvature and if $N \subset M$ is a convex subset, then also the corresponding relative version of Proposition (4.20) holds. Here, the subset N is said to be convex if all connected components of its preimage in $\widetilde{M}$ under the universal covering map are convex.

It is not entirely clear whether the same holds for the generalised straightening in Theorem (4.21); namely, it is difficult to check whether the cochain contractions of $C_b^n(\widetilde{X})$ can be chosen to be natural with respect to inclusion maps. $\diamond$

5 Simplicial volume and ℓ^1-homology

The simplicial volume of oriented (not necessarily compact) manifolds is a proper homotopy invariant measuring the complexity of (generalised) triangulations, i.e., the complexity of the fundamental class in locally finite homology with respect to the ℓ^1-semi-norm.

Gromov introduced the simplicial volume in order to give an alternative proof of Mostow rigidity. Subsequently, in his seminal paper *Volume and bounded cohomology* even more relations, such as the volume estimate, between simplicial volume and Riemannian geometry are uncovered.

On the other hand, the simplicial volume is accessible by powerful algebraic tools – both ℓ^1-homology and bounded cohomology compute the simplicial volume.

Before giving the precise definition of simplicial volume in Section 5.2, we first recapitulate locally finite homology and the local characterisation of fundamental cycles (Section 5.1). Section 5.3 contains the description of simplicial volume in terms of ℓ^1-homology and bounded cohomology. Finally, in Section 5.4 properties of the simplicial volume are collected.

A closer investigation of the simplicial volume of non-compact manifolds is performed in Chapter 6.

5.1 Fundamental cycles of manifolds

In this section, we recall the definition of fundamental cycles of oriented manifolds, which serve as an easier to handle replacement of triangulations, as well as their local characterisation (Theorem (5.4)). Since fundamental cycles of non-compact manifolds live in locally finite homology, we first give a short introduction into locally finite homology and its contravariant companion – cohomology with compact supports.

5.1.1 Locally finite homology

By definition, the singular chain complex contains only finite chains. On the other hand, triangulations of non-compact manifolds need not be finite. Therefore, some questions on non-compact manifolds force to allow certain infinite chains, leading to locally finite homology. Similarly, singular cohomology needs to be replaced by cohomology with compact supports:

Definition (5.1). Let X be a topological space and let $k \in \mathbf{N}$.

1. We write $C(X)$ for the set of all compact, connected, non-empty subsets of X.
2. A set $A \subset \operatorname{map}(\Delta^k, X)$ is called **locally finite** if any compact subset of X intersects the image of only finitely many elements of A. The set of all locally finite subsets of $\operatorname{map}(\Delta^k, X)$ is denoted by $S_k^{\mathrm{lf}}(X)$.
3. The **locally finite chain complex** of X is the chain complex $C_*^{\mathrm{lf}}(X)$ consisting of the vector spaces
$$C_k^{\mathrm{lf}}(X) := \left\{ \sum_{\sigma \in A} a_\sigma \cdot \sigma \;\middle|\; A \in S_k^{\mathrm{lf}}(X) \text{ and } (a_\sigma)_{\sigma \in A} \subset \mathbf{R} \right\}$$
of (formal, possibly infinite) sums equipped with the boundary operator given by the alternating sums of the $(k-1)$-faces.
4. The homology $H_*^{\mathrm{lf}}(X)$ of the locally finite chain complex is called **locally finite homology** of X.

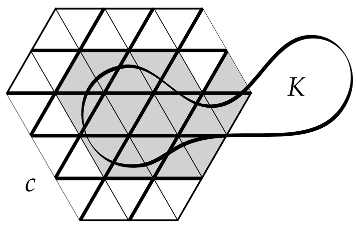

Figure (5.2). Restriction of chains; the shaded part is the restriction $c|_K$ of c (Definition (5.3))

5. A cochain $f \in C^k(X)$ has **compact support** if there exists a compact subset K in X such that $f(\sigma) = 0$ holds whenever $\sigma \in \operatorname{map}(\Delta^k, X \setminus K)$. The **cochain complex with compact supports** of X is the subcomplex $C^*_{\mathrm{cs}}(X)$ of $C^*(X)$ of cochains with compact support.
6. The **cohomology with compact supports** of X, denoted by $H^*_{\mathrm{cs}}(X)$, is the cohomology of $C^*_{\mathrm{cs}}(X)$. ◇

By definition, for compact spaces singular homology and locally finite homology coincide. Dually, in this case also singular cohomology and cohomology with compact supports are equal.

Notice that we can evaluate cochains with compact support on locally finite chains, and this evaluation descends to (co)homology.

Algebraically, the locally finite chain complex of X can also be expressed as the inverse limit

$$C^{\mathrm{lf}}_*(X) = \varprojlim_{K \in C(X)} C_*(X, X \setminus K),$$

where the set $C(X)$ is directed by inclusion. For K, $L \in C(X)$ with $L \subset K$ the structure map $C_*(X, X \setminus K) \longrightarrow C_*(X, X \setminus L)$ is the one induced by the inclusion $(X, X \setminus K) \longrightarrow (X, X \setminus L)$. Dually,

$$C^*_{\mathrm{cs}}(X) = \operatorname*{colim}_{K \in C(X)} C^*(X, X \setminus K).$$

Definition (5.3). Let X be a topological space, let $k \in \mathbf{N}$, and let $K \in C(X)$. The **restriction** of a chain $c = \sum_{\sigma \in A} a_\sigma \cdot \sigma \in C^{\mathrm{lf}}_k(X)$ to the subspace K is defined as

$$c|_K := \sum_{\substack{\sigma \in A, \\ \sigma(\Delta^k) \cap K \neq \emptyset}} a_\sigma \cdot \sigma \in C_k(X). \qquad \diamond$$

Restriction of chains is illustrated in Figure (5.2). Clearly, $\cdot|_K$ gives rise to a chain map $\cdot|_K \colon C^{\mathrm{lf}}_*(X) \longrightarrow C_*(X, X \setminus K)$ and hence to a homomorphism

$$\cdot|_K \colon H^{\mathrm{lf}}_*(X) \longrightarrow H_*(X, X \setminus K).$$

If $L \in C(X)$ with $L \subset K$, then the restrictions $\cdot|_K$ and $\cdot|_L$ are compatible with the inclusion $i^K_L \colon C_*(X, X \setminus K) \longrightarrow C_*(X, X \setminus L)$, i.e., $\cdot|_L = i^K_L \circ \cdot|_K$. For example, the natural map $C^{\mathrm{lf}}_*(X) \longrightarrow \varprojlim_{K \in C(M)} C_*(X, X \setminus K)$ is given by the restriction maps.

5.1.2 Homology of manifolds in the top dimension

The top-dimensional (locally finite) homology with $\mathbf{R}$-coefficients of oriented, connected manifolds is one-dimensional and contains a distinguished generator, the so-called fundamental class. Moreover, this generator can be described "locally," i.e., by restrictions to small sets:

Theorem (5.4) (Fundamental classes of manifolds).

1. *Let M be an oriented, connected n-manifold without boundary and let $K \in C(M)$. Then*
$$H_n(M, M \setminus K) \cong \mathbf{R},$$
and there is a unique generator $[M, M \setminus K]$ with the following property: For all points $x \in K$, the restriction $[M, M \setminus K]|_{\{x\}} \in H_n(M, M \setminus \{x\})$ coincides (under change of coefficients) with the image of the generator of $H_n(M, M \setminus \{x\}; \mathbf{Z})$ given by the (homological) orientation of M. In particular, if $L \in C(M)$ with $L \subset K$, then the restriction homomorphism
$$\cdot|_L \colon H_n(M, M \setminus K) \longrightarrow H_n(M, M \setminus L)$$
is an isomorphism mapping $[M, M \setminus K]$ to $[M, M \setminus L]$.
2. *Let M be an oriented, connected n-manifold without boundary. Then $H^{\mathrm{lf}}_n(M) \cong \mathbf{R}$ and there is a unique class $[M] \in H^{\mathrm{lf}}_n(M)$ such that*
$$[M]|_K = [M, M \setminus K] \in H_n(M, M \setminus K)$$
holds for all $K \in C(M)$.
3. *Let $(M, \partial M)$ be an oriented, compact, connected n-manifold with boundary ∂M. Then $H_n(M, \partial M) \cong \mathbf{R}$ and there is a unique class $[M, \partial M] \in H_n(M, \partial M)$ such that*
$$[M, \partial M]|_K = [M^\circ, M^\circ \setminus K] \in H_n(M^\circ, M^\circ \setminus K) \cong H_n(M, M \setminus K)$$

holds for all $K \in C(M^\circ)$. Furthermore,

$$\partial[M, \partial M] = \sum_{N \in \pi_0(\partial M)} [N] \in H_{n-1}(\partial M).$$

Proof. Almost any textbook on algebraic topology contains a proof of the first and the third part [36; Chapter XIV].

For the second part, we use the description of $C^{\mathrm{lf}}_*(M)$ as inverse limit. Clearly, the directed system $(C_*(M, M \setminus K))_{K \in C(M)}$ satisfies the Mittag-Leffler condition. Moreover, $H_{n+1}(M, M \setminus K) = 0$ for all $K \in C(M)$ [36; Lemma XIV.2.3]. Therefore, the $\lim^1$-term vanishes and we obtain [59; Theorem 3.5.8]

$$H^{\mathrm{lf}}_n(M) \cong \varprojlim_{K \in C(M)} H_n(M, M \setminus K).$$

In view of the first part, it follows that $H^{\mathrm{lf}}_n(M) \cong \mathbf{R}$ and that there exists a unique class $[M] \in H^{\mathrm{lf}}_n(M)$ with the desired properties. □

Definition (5.5). The classes $[M, M \setminus K]$, $[M]$, and $[M, \partial M]$ in Theorem (5.4) are called **fundamental classes** of the respective objects. Cycles representing such a class are called **fundamental cycles**. ◇

For example, any triangulation of a manifold gives rise to a fundamental cycle – i.e., fundamental cycles can be viewed as generalised triangulations. But the concept of fundamental cycles is much more flexible, especially when considering homology with $\mathbf{R}$-coefficients:

Example (5.6). For each $d \in \mathbf{N}_{>0}$, the chain $1/d \cdot \sigma_d$ is a fundamental cycle of the circle S^1, where $\sigma_d \colon [0,1] \longrightarrow S^1$ is given by $\sigma_d(t) := e^{2\pi i \cdot d \cdot t}$. ◇

Dually, there are also cohomological versions of the fundamental class:

Corollary (5.7) (Cohomological fundamental classes of manifolds). *In the corresponding situations of Theorem (5.4) there are cohomology classes $[M, M \setminus K]^* \in H^n(M, M \setminus K) \cong \mathbf{R}$, $[M]^* \in H^n_{\mathrm{cs}}(M) \cong \mathbf{R}$, and $[M, \partial M]^* \in H^n(M, \partial M) \cong \mathbf{R}$ uniquely determined by the relations*

$$\begin{aligned} \big\langle [M, M \setminus K]^*, [M, M \setminus K] \big\rangle &= 1, \\ \big\langle [M]^*, [M] \big\rangle &= 1, \\ \big\langle [M, \partial M]^*, [M, \partial M] \big\rangle &= 1. \end{aligned}$$

Proof. The compact case follows from Theorem (5.4) by means of the universal coefficient theorem. For the non-compact case, it suffices to note that (because the set $C(M)$ is directed by inclusion [55; p. 162])

$$H^n_{\mathrm{cs}}(M) = H^n\Big(\operatorname*{colim}_{K\in C(M)} C^*(M, M\setminus K)\Big) \cong \operatorname*{colim}_{K\in C(M)} H^n(M, M\setminus K)$$

and $H^{\mathrm{lf}}_n(M) \cong \varprojlim_{K\in C(M)} H_n(M, M\setminus K)$ (proof of Theorem (5.4)). $\square$

Definition (5.8). The classes $[M, M\setminus K]^*$, $[M]^*$, and $[M, \partial M]^*$ in Corollary (5.7) are called **cohomological fundamental classes** of the respective objects and all cocycles representing such a class are called **fundamental cocycles**. $\diamond$

Theorem (5.4) can easily be generalised to cover also the case of non-connected manifolds (which naturally occur as boundaries of compact manifolds). Hence, we can also speak of fundamental (co)cycles and (cohomological) fundamental classes of oriented, non-connected manifolds.

5.2 Definition of simplicial volume

The simplicial volume of oriented manifolds is a proper homotopy invariant measuring the complexity of (generalised) triangulations, i.e., the complexity of the fundamental class with respect to the ℓ^1-semi-norm.

We start with the compact case (Section 5.2.1) and then consider one possible generalisation to the non-compact case (Section 5.2.2).

5.2.1 The compact case

As already indicated, the simplicial volume is defined as the evaluation of the ℓ^1-semi-norm on the fundamental class [18; p. 8]:

Definition (5.9). Let $(M, \partial M)$ be an oriented, closed, connected n-manifold with boundary ∂M. The **simplicial volume** of $(M, \partial M)$ is given by

$$\begin{aligned}\|M, \partial M\| &:= \big\|[M, \partial M]\big\|_1 \\ &= \inf\big\{\|c\|_1 \bigm| c \in C_n(M) \text{ is a relative fundamental cycle of } (M, \partial M)\big\} \\ &\in [0, \infty).\end{aligned}$$

$\diamond$

The term "relative fundamental cycle in $C_n(M)$" refers to a chain that under the projection $C_n(M) \longrightarrow C_n(M, \partial M)$ is mapped to a fundamental cycle of $(M, \partial M)$.

The definition of simplicial volume can easily be generalised to cover also non-connected manifolds (by taking the sum of the simplicial volumes of the connected components) and non-orientable manifolds (by dividing the simplicial volume of the orientation covering by 2).

Example (5.10). A simple, yet instructive, example is to compute the simplicial volume of S^1: For each $d \in \mathbf{N}_{>0}$, there is a fundamental cycle $1/d \cdot \sigma_d \in C_1(S^1)$, where σ_d is a singular simplex on the circle (cf. Example (5.6)). Therefore, we obtain $\|S^1\| \leq 1/d$ for all $d \in \mathbf{N}_{>0}$ and hence $\|S^1\| = 0$. The same type of argument shows that the simplicial volume of all spheres and tori (of non-zero dimension) equals zero. $\diamond$

A more extensive collection of examples and properties of simplicial volume is given in Section 5.4.

5.2.2 The non-compact case

Clearly, also the chain complex $C^{\mathrm{lf}}_*(X)$ of locally finite singular chains of a topological space becomes a "normed" chain complex with respect to the ℓ^1-norm; the reason we put quotation marks here is that the ℓ^1-norm of locally finite chain complexes is not necessarily finite.

In particular, we obtain a notion of simplicial volume for non-compact manifolds [18; p. 8], which in the compact case coincides with the one defined in Section 5.2.1:

Definition (5.11). Let M be a connected n-manifold without boundary. The **simplicial volume** of M is defined as

$$\begin{aligned} \|M\| &:= \big\|[M]\big\|_1 \\ &= \inf\big\{\|c\|_1 \bigm| c \in C^{\mathrm{lf}}_n(M) \text{ is a fundamental cycle of } M\big\} \\ &\in [0, \infty]. \end{aligned}$$ $\diamond$

The tamest examples of non-compact manifolds are manifolds that are the interior of a compact manifold with boundary. The simplicial volume of such manifolds is related to the simplicial volume of the ambient compact manifold as follows:

Proposition (5.12). *Let $(W, \partial W)$ be an oriented, compact, connected manifold with boundary and let $M := W^\circ$. Then*

$$\|M\| \geq \|W, \partial W\|.$$

Proof. Because M is homeomorphic to $M' := W \cup_{\partial W} \partial W \times [0, \infty)$ [14, 8], it suffices to show that $\|M'\| \geq \|W, \partial W\|$.

Let $c' \in C_n^{\mathrm{lf}}(M')$ be a locally finite fundamental cycle of M'. Pushing the restriction $c'|_W$ via the obvious projection $M' \longrightarrow W$ to W yields a chain $c \in C_n(W)$ with $\partial c \in C_{n-1}(\partial W)$. Because c and c' coincide on W°, it follows from the local characterisation of fundamental cycles (Theorem (5.4)) that c is a relative fundamental cycle of $(W, \partial W)$.

By construction, $\|c\|_1 \leq \|c'\|_1$, which implies $\|M\| = \|M'\| \geq \|W, \partial W\|$. □

In general, this inequality is a strict inequality – namely, the simplicial volume of the interior might be infinite, whereas the relative simplicial volume of the compactification is always finite. In Chapter 6, we present a necessary and sufficient finiteness criterion for such interiors and investigate some examples of simplicial volumes of such manifolds (Section 6.4).

5.3 Computing simplicial volume

Both ℓ^1-homology and bounded cohomology provide a convenient setting for computing the simplicial volume in a systematic way. We first look at the compact case and, in a second step, discuss the corresponding generalisations for the non-compact case. Because ℓ^1-homology and bounded cohomology are homotopy invariants, but the simplicial volume of non-compact manifolds is only invariant under proper homotopy equivalences, the latter case is more involved.

In both cases, we first present the homological approach and then the cohomological one because – especially in the non-compact case – the connection with ℓ^1-homology is much more transparent.

5.3.1 The compact case – homological approach

Using the comparison map from singular homology to ℓ^1-homology, we can express the simplicial volume in terms of ℓ^1-homology:

Proposition (5.13). *Let $(M, \partial M)$ be an oriented, compact, connected n-manifold with boundary. Then*

$$\|M, \partial M\| = \big\|H_n(i_{M,\partial M})\big([M, \partial M]\big)\big\|_1,$$

where $i_{M,\partial M}\colon C_*(M, \partial M) \longrightarrow C^{\ell^1}_*(M, \partial M)$ is the canonical inclusion.

Proof. This is a special case of Proposition (2.5). □

Hence, if M is closed and connected, then information on ℓ^1-homology of the fundamental group $\pi_1(M)$ transforms into computations of the simplicial volume of M, which is particularly suited for vanishing results:

Corollary (5.14). *Let M be an oriented, closed, connected n-manifold with classifying map $f\colon M \longrightarrow B\pi_1(M)$. Then*

$$\begin{aligned}\|M\| &= \big\|H^{\ell^1}_n(f)\big(H_n(i_M)([M])\big)\big\|_1 \\ &= \big\|H_n(f)([M])\big\|_1.\end{aligned}$$

In particular, if $H^{\ell^1}_n(\pi_1(M)) = 0$, then $\|M\| = 0$.

Proof. By the mapping theorem for ℓ^1-homology (Corollary (4.2)), the induced homomorphism $H^{\ell^1}_*(f)\colon H^{\ell^1}_*(M) \longrightarrow H^{\ell^1}_*(B\pi_1(M))$ is an isometric isomorphism. Therefore, the first equality follows from Proposition (5.13). The second one can be derived from the first one by applying Proposition (2.5) to the classifying space $B\pi_1(M)$.

Furthermore, $H^{\ell^1}_*(\pi_1(M)) \cong H^{\ell^1}_*(B\pi_1(M))$ (Corollary (4.14)), which implies the last statement. □

The first proof of the second equality of Corollary (5.14) was originally given by Gromov [18; Corollary B on p. 40], using bounded cohomology.

5.3.2 The compact case – cohomological approach

The duality principle for semi-norms translates the ℓ^1-homological description of simplicial volume (Proposition (5.13)) into one in terms of bounded cohomology (Proposition (5.15)). Historically, this cohomological interpretation was the first algebraic approach to simplicial volume [18; p. 17].

Proposition (5.15) (Duality principle for compact manifolds). *Let $(M, \partial M)$ be an oriented, compact, connected n-manifold with boundary. Then*

$$\|M, \partial M\| = \sup\Bigl\{ \frac{1}{\|\varphi\|_\infty} \Bigm| \varphi \in H^n_{\mathrm{b}}(M, \partial M),\ \langle \varphi, [M, \partial M]\rangle = 1 \Bigr\} = \frac{1}{\bigl\|[M, \partial M]^*\bigr\|_\infty}.$$

Here, $\sup \emptyset := 0$.

Because not all fundamental cocycles are bounded, the value $\|[M, \partial M]^*\|_\infty$ can be infinite; in this case, we use the convention $1/\infty = 0$.

Proof. The first equality follows from the duality principle for semi-norms (Theorem (3.8)). The second equality is easily derived from the fact that

$$\|[M, \partial M]^*\|_\infty = \inf\bigl\{ \|\varphi\|_\infty \bigm| \varphi \in H^n_{\mathrm{b}}(M),\ H^n(j_M)(\varphi) = [M, \partial M]^* \bigr\},$$

where $j_M \colon C^*_{\mathrm{b}}(M, \partial M) \longrightarrow C^*\,(M, \partial M)$ is the inclusion. □

5.3.3 The non-compact case – homological approach

Analogously to the compact case (Proposition (5.13)), also the simplicial volume of non-compact manifolds can be expressed with help of ℓ^1-homology.

Definition (5.16). Let M be an oriented, connected n-manifold without boundary. We write $[M]^{\ell^1} \subset H^{\ell^1}_n(M)$ for the set of all homology classes that are represented by at least one locally finite fundamental cycle (with finite ℓ^1-norm). ◇

If M is compact, then the set $[M]^{\ell^1}$ contains exactly one element, namely the class $H_n(i_M)([M])$. However, if M is non-compact the set $[M]^{\ell^1}$ may be empty (this happens if and only if $\|M\| = \infty$) or consist of more than one element.

Proposition (5.17). *If M is an oriented, connected n-manifold without boundary, then*

$$\|M\| = \inf\bigl\{ \|\alpha\|_1 \bigm| \alpha \in [M]^{\ell^1} \subset H^{\ell^1}_n(M) \bigr\}.$$

Here, $\inf \emptyset := \infty$.

Proof. Let $i \colon C^{\mathrm{lf}}_*(M) \cap C^{\ell^1}_*(M) \hookrightarrow C^{\mathrm{lf}}_*(M)$ and $j \colon C^{\mathrm{lf}}_*(M) \cap C^{\ell^1}_*(M) \hookrightarrow C^{\ell^1}_*(M)$ denote the inclusions. By definition,

$$\begin{aligned} \|M\| &= \inf\bigl\{ \|\alpha\|_1 \bigm| \alpha \in H_*(i)^{-1}([M]) \bigr\} \\ &= \inf\bigl\{ \|\alpha\|_1 \bigm| \alpha \in H_*(j)^{-1}([M]^{\ell^1}) \bigr\}. \end{aligned}$$

The sequence

$$C_*(M) \hookrightarrow C_*^{\mathrm{lf}}(M) \cap C_*^{\ell^1}(M) \hookrightarrow C_*^{\ell^1}(M)$$

of inclusions of normed chain complexes shows that the middle complex is a dense subcomplex of the ℓ^1-chain complex $C_*^{\ell^1}(M)$. Thus, the induced map

$$H_*(j)\colon H_*\big(C_*^{\mathrm{lf}}(M) \cap C_*^{\ell^1}(M)\big) \longrightarrow H_*^{\ell^1}(M)$$

on homology is isometric (Proposition (1.7)). This yields the desired description of $\|M\|$. □

For example, in combination with Corollary (4.14) the previous proposition gives rise to the following vanishing result:

Corollary (5.18). *Let M be an oriented, connected n-manifold with $H_n^{\ell^1}(\pi_1(M)) = 0$. Then $\|M\| \in \{0, \infty\}$.* □

5.3.4 The non-compact case – cohomological approach

There are two possible generalisations of Proposition (5.15) to the non-compact case. One can either dualise the computation via ℓ^1-homology (leading to Proposition (5.19)), or, as indicated by Gromov [18; p. 17], one can try to find a suitable semi-norm on cohomology with compact supports and evaluate it on the dual fundamental class (Theorem (5.20)).

Proposition (5.19). *Let M be an oriented, connected n-manifold without boundary. Then*

$$\|M\| = \inf_{\alpha \in [M]^{\ell^1}} \sup\Big\{ \frac{1}{\|\varphi\|_\infty} \;\Big|\; \varphi \in H_{\mathrm{b}}^n(M),\, \langle \varphi, \alpha \rangle = 1 \Big\}.$$

Proof. Combining the computation of $\|M\|$ via ℓ^1-homology (Proposition (5.17)) with the duality principle of semi-norms (Theorem (3.8)) proves the proposition. □

Theorem (5.20) (Duality principle for non-compact manifolds). *Let M be an oriented, connected manifold without boundary. Then*

$$\|M\| = \frac{1}{\big\|[M]^*\big\|_\infty^{\mathrm{lf}}}.$$

This theorem is proved in Appendix C, where also the exact definition of the semi-norm $\|\cdot\|_{\infty}^{\mathrm{lf}}$ on $H^n_{\mathrm{cs}}(M)$ is given (Definition (C.1)).

The advantage of the first version is being closely related to bounded cohomology; but in general the semi-norm of more than one cohomology class has to be computed. The second version needs only knowledge about the dual fundamental class, but the semi-norm involved is quite difficult to control. Hence, Proposition (5.19) is more suitable for vanishing results and Theorem (5.20) is to be preferred for calculations that involve concrete constructions on the dual fundamental class (e.g., product formulae – see Theorem (C.7)).

5.4 A collection of properties of simplicial volume

In this section, we present a collection of topological as well as geometric properties of simplicial volume. The geometric ones, such as proportionality, the minimal volume estimate and the computation for hyperbolic manifolds, demonstrate that the simplicial volume also can be viewed as a topological approximation of the Riemannian volume.

Degree estimate. Let $f\colon M \longrightarrow N$ be a proper, continuous map of oriented, connected manifolds (of the same dimension) of non-zero degree. Then

$$\|N\| \leq \frac{1}{|\deg f|} \cdot \|M\| \,.$$

This also holds for compact manifolds with boundary and maps relative to the boundary.

Self-maps. Let $f\colon M \longrightarrow M$ be a continuous self-map of the oriented, closed, connected manifold M with $|\deg f| \geq 2$. Then $\|M\| = 0$. Clearly, this also holds for compact manifolds with boundary and self-maps respecting the boundary.

Example (5.21). In particular, the simplicial volume of spheres and tori of non-zero dimension is zero. $\diamond$

Proper homotopy invariance. If the two oriented, connected manifolds M and N are properly homotopy equivalent, then $\|M\| = \|N\|$. This is also true for compact manifolds with boundary and homotopy equivalences relative to the boundary.

This follows from the degree estimate because properly homotopic manifolds have the same dimension, any proper homotopy equivalence has degree 1.

Finite coverings. Let $p\colon M \longrightarrow N$ be a finite covering of oriented, connected manifolds. Then
$$\|M\| = |\deg p| \cdot \|N\| .$$

This also holds for compact manifolds with boundary and finite covering maps respecting the boundaries.

The estimate "$\leq$" follows from the degree estimate. The reverse inequality can be shown by summing up all p-lifts of the simplices in a fundamental cycle of M (which gives a fundamental cycle of N).

Connected sums. Let M and N be oriented, closed, connected manifolds of the same dimension > 2. Then
$$\|M \# N\| = \|M\| + \|N\| .$$

The pinching map $M \# N \longrightarrow M \vee N$ induces an isomorphism on the level of fundamental groups. Therefore, the mapping theorem of ℓ^1-homology (or bounded cohomology) can be used to show that "$\leq$" holds. Gromov proves the (more complicated) estimate "$\geq$" by looking at a concrete description of the universal covering of $M \vee N$ as a so-called tree-like complex [18; Section 3.5].

Similar arguments apply not only to connected sums, but also to "amenable gluings," i.e., to manifolds that are glued along a common submanifold of codimension 1 that is amenable [18, 28; p. 55, Chapter 3].

Products. Let M and N be oriented, closed, connected manifolds. Then
$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\| \cdot \|N\| .$$

The first inequality remains true if the compactness condition on one of the two factors is dropped, the second estimate even holds if both manifolds are non-compact (Theorem (C.7)). However, the first inequality fails in general for non-compact manifolds, and for compact manifolds with boundary (cf. Sections C.4 and 6.4).

Because the cross-product of two fundamental cycles is a fundamental cycle of the product manifold, the second inequality follows. Gromov's proof of the first inequality takes advantage of the cohomological description of simplicial volume; a detailed proof is given in Section C.4.

Example (5.22). Bucher-Karlsson computes the first concrete value of a non-trivial product [10]: If M_1 and M_2 are two oriented, closed, connected, hyperbolic surfaces, then $\|M_1 \times M_2\| = 3/2 \cdot \|M_1\| \cdot \|M_2\|$. Moreover, in this case the simplicial volumes of the factors M_1 and M_2 can be computed explicitly (Example (5.23)). $\diamond$

Fibrations. Let $F \longrightarrow M \longrightarrow B$ be a fibration of oriented, closed, connected manifolds with $\dim F > 0$. If $\pi_1(F)$ is amenable, then $\|M\| = 0$.

A spectral sequence argument shows that $\dim M > \dim B$; therefore, the mapping theorem of ℓ^1-homology/bounded cohomology yields $\|M\| = 0$ [35; Exercise 14.15 and p. 556].

Circle actions. Let M be an oriented, closed, connected, smooth manifold with non-trivial, smooth S^1-action. Then $\|M\| = 0$.

Yano describes a quite concrete, geometric construction to reduce general S^1-actions to the product case [60]. Gromov proves the vanishing via the corresponding statement on the minimal volume [18; p. 93].

Proportionality principle. Let M and N be oriented, closed, connected, Riemannian manifolds with isometric universal covering. Then

$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N}.$$

Thurston sketches a homological proof using measure homology [57, 56; p. 6.9, Chapter 5] (see also Appendix D, especially Section D.3). A skeleton for a cohomological proof is given by Gromov [18; Section 2.3]. Both proofs depend on normalised Haar measures on the compact quotients of the group of orientation preserving isometries on the common universal covering divided by the respective fundamental groups.

The proportionality principle in general does not hold in the non-compact or in the relative cases [18; p. 59].

Volume estimate. Let M be an oriented, smooth n-manifold without boundary. Then the simplicial volume is bounded from above by the minimal volume $\operatorname{minvol} M$ in the sense that

$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol} M.$$

Both the original proof of Gromov, as well as the version of Besson, Courtois and Gallot rely on the description of simplicial volume via bounded cohomology and comass estimates [18, 2].

Negative curvature. Let M be an oriented, closed, connected Riemannian n-manifold of negative curvature. Then $\|M\| > 0$. If M is hyperbolic, then

$$\|M\| = \frac{\operatorname{vol} M}{v_n},$$

where $v_n \in (0, \infty)$ is the maximal volume of an ideal n-simplex in hyperbolic n-space.

The curvature condition ensures that there is an upper bound on the volumes of geodesic simplices in the universal covering of $\widetilde{M}$. Thurston's straightening (Proposition (4.20)) shows that the simplicial volume of a negatively curved manifold can be computed by looking only at fundamental cycles consisting of (projections of) geodesic simplices. Therefore, integration over the volume form of M shows that the simplicial volume is bounded from below [57, 24].

The upper bound can be obtained by Thurston's smearing construction [57; Chapter 6] or the corresponding discrete version [1, 50; Theorem C.4.2, Theorem 11.4.3].

Example (5.23). If F_g is the oriented, closed, connected surface of genus g at least 2, then $\|F_g\| = 4 \cdot g - 4$. $\diamond$

Locally symmetric spaces of non-compact type. Oriented, closed, connected, locally symmetric spaces of non-compact type have non-zero simplicial volume.

Lafont and Schmidt apply a refined straightening procedure to show positivity [29], modulo two special cases. These special cases are covered by work of Thurston [57; Chapter 6] and Bucher-Karlsson [9] respectively.

Hyperbolic fundamental group. Oriented, closed, connected, aspherical manifolds with hyperbolic fundamental group have non-zero simplicial volume.

This follows from Mineyev's work on bounded cohomology of hyperbolic groups (Section 2.4.5) together with Proposition (5.15).

Amenable fundamental group. Let M be an oriented, connected n-manifold (without boundary) of dimension at least 1 with amenable fundamental group. Then $\|M\| \in \{0, \infty\}$. If M is compact, then $\|M\| = 0$.

By Corollary (4.12) we have $H_n^{\ell^1}(\pi_1(M)) = 0$, and hence Proposition (5.18) applies.

Amenable coverings. More generally, let M be an oriented, closed, connected manifold of dimension n that possesses a covering by open, amenable subsets of multiplicity at most n. Then $\|M\| = 0$.

In this situation, the comparison map $H^*_b(M) \longrightarrow H^*(M)$ factors over the cohomology of the nerve of the open covering [18, 25]. Therefore, the class $[M]^*$ does not lie in the image of the comparison map $H^n_b(M) \longrightarrow H^n(M)$. In other words, $\|[M]^*\|_\infty = \infty$. This implies that $\|M\| = 0$ (Proposition (5.15)).

Free fundamental group. Let M be an oriented, closed, connected manifold of dimension at least 1 with free fundamental group. Then $\|M\| = 0$.

The homology of free groups vanishes in degrees bigger than 1, and ℓ^1-homology vanishes in degree 1 (Proposition (2.7)); thus, the image of $[M]$ under the classifying map $M \longrightarrow B\pi_1(M)$ is zero in ℓ^1-homology. Therefore, we obtain $\|M\| = 0$ (Corollary (5.14)).

The corresponding statement for the relative simplicial volume of compact manifolds with boundary and for the simplicial volume of non-compact manifolds is not true (Section 6.4.3).

The definition of the simplicial volume of non-compact manifolds given in Section 5.2.2 is of more topological than geometric nature – it does not reflect (Riemannian) geometric properties as well as the simplicial volume in the category of compact manifolds. By imposing geometric conditions on the locally finite fundamental cycles that appear in the infimum of Definition (5.11), one obtains versions of the simplicial volume that carry more geometric information. Of course, these geometric versions are in general only invariant under proper homotopy equivalences that are compatible with the geometric structures involved.

One example of such a variant of simplicial volume is the so-called *Lipschitz simplicial volume* [18, 34], where the infimum of ℓ^1-norms is taken over the set of those locally finite fundamental cycles whose simplices satisfy a uniform Lipschitz condition. The Lipschitz simplicial volume can be used to establish proportionality principles and product formulae for certain classes of non-compact manifolds [34]. The price for this gain is that we lose the strong connection with homological tools such as ℓ^1-homology and bounded cohomology.

In the next chapter, we study the question under which conditions the simplicial volume of non-compact manifolds is finite in a special case – namely for manifolds that are the interior of a compact manifold with boundary.

6 A finiteness criterion for simplicial volume

The simplicial volume of non-compact manifolds is not finite in general. It might even then be infinite if the non-compact manifold M in question is the interior of a compact manifold $(W, \partial W)$ with boundary. Gromov showed that the vanishing of the simplicial volume of the boundary ∂W is a necessary condition for $\|M\|$ to be finite.

It turns out that ℓ^1-homology allows to give a necessary and sufficient condition for the finiteness of $\|M\|$ if M has such a nice compactification (Theorem (6.1)). More precisely: the simplicial volume of M is finite if and only if the fundamental class of ∂W is mapped to zero in ℓ^1-homology under the comparison map, i.e., if ∂W is "ℓ^1-invisible." Clearly, this is a purely topological condition.

Since bounded cohomology cannot see whether a given class in ℓ^1-homology is zero, this finiteness criterion cannot be formulated in terms of bounded cohomology.

The finiteness criterion is stated in Section 6.1 and proved in Section 6.2. We then investigate the class of ℓ^1-invisible manifolds (Section 6.3). In the last section, we show how the finiteness criterion can help getting a better understanding of the behaviour of simplicial volume of non-compact manifolds.

6.1 Stating the finiteness criterion

One of the virtues of ℓ^1-homology is its ability to characterise the finiteness of simplicial volume of certain non-compact manifolds:

Theorem (6.1) (Finiteness criterion). *Let $(W, \partial W)$ be an oriented, compact, connected n-manifold with boundary and let $M := W^\circ$. Then the following are equivalent:*

1. *The simplicial volume $\|M\|$ is finite.*
2. *The fundamental class of the boundary ∂W vanishes in ℓ^1-homology, i.e.,*

$$H_{n-1}(i_{\partial W})\big([\partial W]\big) = 0 \in H_{n-1}^{\ell^1}(\partial W),$$

where $i_{\partial W} \colon C_*(\partial W) \longrightarrow C_*^{\ell^1}(\partial W)$ is the natural inclusion.

The Kronecker product of bounded cohomology and ℓ^1-homology factorises over reduced ℓ^1-homology. Therefore, bounded cohomology in general cannot detect the vanishing of a given class in ℓ^1-homology – bounded cohomology can only tell whether the corresponding class in reduced ℓ^1-homology is zero, i.e., whether the ℓ^1-semi-norm of the considered class is zero.

Combining the finiteness criterion with the fact that the comparison map between singular homology and ℓ^1-homology is isometric (Proposition (2.5)), we obtain the necessary condition formulated by Gromov [18; p. 17]:

Corollary (6.2). *Let $(W, \partial W)$ be an oriented, connected, compact manifold with boundary and let $M := W^\circ$. If $\|M\|$ is finite, then $\|\partial W\| = 0$.* □

A more thorough discussion of the relation between the second item of the finiteness criterion and the vanishing of simplicial volume of the boundary is given in Section 6.3.1.

The first sufficient condition for finiteness of simplicial volume of non-compact manifolds needs self-maps on the boundary of non-trivial degree [18; p. 8]. In addition, Gromov states also sufficient conditions for finiteness using amenable coverings and minimal volume [18; p. 58, p. 12/p. 73]. However, his techniques are much more sophisticated and less transparent. Furthermore, the estimate via the minimal volume is not a topological condition, but depends on a smooth structure.

Of course, it would be interesting to know whether the finiteness criterion (Theorem (6.1)) can be generalised to cover all non-compact manifolds. A possible strategy might be to find a suitable definition of "ℓ^1-homology at infinity" and to study its relation with the fundamental group at infinity.

6.2 ℓ^1-Invisibility and the proof of the finiteness criterion

Before starting with the proof of the finiteness criterion, we introduce the notion of ℓ^1-invisibility, which is a shorthand for the second item in the statement of the finiteness criterion.

Definition (6.3). An oriented, closed n-manifold M is called **ℓ^1-invisible** if its fundamental class vanishes in ℓ^1-homology, i.e., if

$$H_n(i_M)\big([M]\big) = 0 \in H_n^{\ell^1}(M). \qquad \diamond$$

If M is ℓ^1-invisible, then $\|M\| = 0$. More generally, we can reformulate the ℓ^1-invisibility condition as follows – which is also a step towards the proof of the finiteness criterion:

Proposition (6.4). *Let M be an oriented, closed n-manifold. Then the following are equivalent:*

1. *The manifold M is ℓ^1-invisible.*
2. *There are fundamental cycles $(z_k)_{k\in\mathbf{N}} \subset C_n\,(M)$ and chains $(b_k)_{k\in\mathbf{N}} \subset C_{n+1}\,(M)$ satisfying*

$$\begin{aligned} \forall_{k\in\mathbf{N}} \quad \partial(b_k) &= z_{k+1} - z_k, \\ \sum_{k\in\mathbf{N}} \|b_k\|_1 &< \infty, \\ \sum_{k\in\mathbf{N}} \|z_k\|_1 &< \infty. \end{aligned}$$

Proof. The proof is nothing but a rearrangement of absolutely convergent series in the ℓ^1-chain complex.

1 $\Rightarrow$ 2 Suppose that M is ℓ^1-invisible. Let $z \in C_n(M)$ be a fundamental cycle of M. Since M is ℓ^1-invisible, there exists an ℓ^1-chain $b \in C^{\ell^1}_{n+1}(M)$ with

$$\partial b = -z.$$

The chain b can be written in the form $b = \sum_{k\in\mathbf{N}} a_k \cdot \sigma_k \in C^{\ell^1}_{n+1}(M)$ with $a_k \in \mathbf{R}$ and different $\sigma_k \in \operatorname{map}(\Delta^{n+1}, M)$. We now set $b_k := a_k \cdot \sigma_k$ and

$$z_k := z + \sum_{j=0}^{k-1} \partial(b_k) \in C_n(M)$$

for all $k \in \mathbf{N}$. This implies $\sum_{k\in\mathbf{N}} \|b_k\|_1 < \infty$ and $\partial(b_k) = z_{k+1} - z_k$ for all $k \in \mathbf{N}$. Moreover, the definition of ∂ in the ℓ^1-chain complex shows that

$$\sum_{k\in\mathbf{N}} \partial(b_k) = \partial b = -z = -z_0.$$

In order to satisfy the additional summability condition on the $\|z_k\|_1$, we construct a suitable subsequence of $(z_k)_{k\in\mathbf{N}}$ and then modify the $(b_k)_{k\in\mathbf{N}}$ accordingly: From the considerations above we deduce that

$$\begin{aligned}
\lim_{k\to\infty} z_k &= \lim_{k\to\infty} \left(z_0 + \sum_{j=0}^{k-1} \partial(b_j) \right) \\
&= z_0 + \lim_{k\to\infty} \sum_{j=0}^{k-1} \partial(b_j) \\
&= z_0 - z_0 \\
&= 0
\end{aligned}$$

(in particular, the limit $\lim_{k\to\infty} z_k$ indeed exists). We set $s(0) := 0$ and choose inductively $s(k) \in \mathbf{N}$ large enough so that $s(k) > s(k-1)$ and

$$\|z_{s(k)}\|_1 \leq \frac{1}{2^k} \cdot \|z_0\|_1.$$

Then the resulting sequences $(z'_k)_{k\in\mathbf{N}} \subset C_{n-1}(\partial W)$ and $(b'_k)_{k\in\mathbf{N}} \subset C_n(\partial W)$ defined by

$$\begin{aligned}
z'_k &:= z_{s(k)}, \\
b'_k &:= \sum_{j=s(k)}^{s(k+1)-1} b_j
\end{aligned}$$

for all $k \in \mathbf{N}$ satisfy condition 2.

$2 \Rightarrow 1$ Conversely, suppose that part 2 is satisfied. Then the infinite sum

$$b := \sum_{k\in\mathbf{N}} b_k$$

is a well-defined chain in $C^{\ell^1}_{n+1}(M)$. Since $\sum_{k\in\mathbf{N}} \|b_k\|_1 < \infty$ as well as $\sum_{k\in\mathbf{N}} \|z_k\|_1 < \infty$, we can compute the boundary of b via

$$\partial b = \partial\biggl(\sum_{k\in\mathbf{N}} b_k\biggr) = \sum_{k\in\mathbf{N}} \partial(b_k) = \sum_{k\in\mathbf{N}} (z_{k+1} - z_k) = z_0.$$

Because z_0 is a fundamental cycle of M, this proves that M is ℓ^1-invisible. $\square$

We now turn to the geometric part of the proof of the finiteness criterion:

Proof (of Theorem (6.1)). The theorem trivially holds if the boundary ∂W is empty; therefore, we assume for the rest of the proof that $\partial W \neq \emptyset$. The homeomorphism [8, 14]

$$M \cong W \sqcup_{\partial W} \partial W \times [0,\infty) =: M'$$

shows that we can also look at the notationally more convenient manifold M' instead of M.

$1 \Rightarrow 2$ Suppose the simplicial volume $\|M\| = \|M'\|$ is finite. In other words, there is a locally finite fundamental cycle $c \in C^{\mathrm{lf}}_n(M')$ with $\|c\|_1 < \infty$. We now restrict c to a cylinder lying in $\partial W \times [0,\infty) \subset M'$. The boundary of this restriction is a fundamental cycle of ∂W and the restriction itself gives rise to the desired boundary in the ℓ^1-chain complex. In the following, we explain this procedure in more detail (the notation is illustrated in Figure (6.5)):

For $t \in (0,\infty)$, we consider the cylinder

$$Z_t := \partial W \times [t,\infty) \subset M'.$$

Because c is locally finite, there exists a $t \in (0,\infty)$ such that the restriction $c|_{Z_t} \in C^{\mathrm{lf}}_n(M')$ does not meet W.

Let

$$\begin{aligned} p_t &\colon \partial W \times [0,\infty) \longrightarrow Z_t,\\ q_t &\colon \partial W \times [0,\infty) \longrightarrow \partial W \times [0,t) \end{aligned}$$

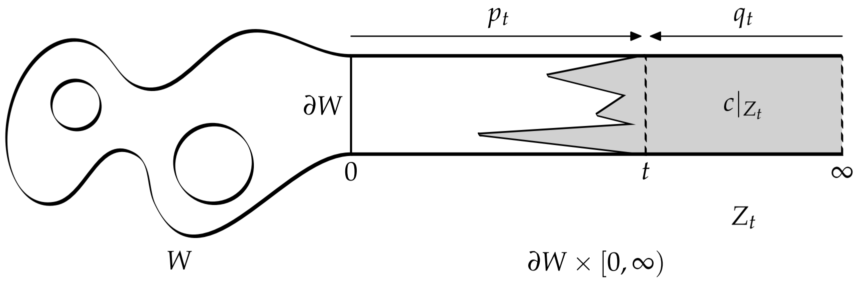

Figure (6.5). The proof of "$1 \Rightarrow 2$" of the finiteness criterion

denote the canonical projections. Clearly,

$$c_t := C_n^{\mathrm{lf}}(p_t)(c|_{Z_t})$$

is a locally finite chain on Z_t whose boundary lies in $\partial W \times \{t\} = \partial Z_t$. In the following paragraph, we show that ∂c_t is a fundamental cycle of $\partial W \times \{t\}$:

The boundary of the restriction

$$c_t^{t+1} := C_n(q_{t+1})(c_t|_{\partial W \times [t,t+1]}) \in C_n(\partial W \times [t, t+1])$$

lies in $\partial W \times \{t, t+1\}$. By construction, we have $c_t^{t+1}|_{\{x\}} = c|_{\{x\}}$ for all points $x \in \partial W \times (t, t+1)$. Therefore, the local characterisation of fundamental cycles (Theorem (5.4)) implies that c_t^{t+1} is a relative fundamental cycle of the manifold $\partial W \times [t, t+1]$ with boundary. In particular,

$$z_t := \partial c_t = (\partial c_t^{t+1})|_{\partial W \times \{t\}} \in C_{n-1}(\partial W \times \{t\})$$

is a fundamental cycle of $\partial W \times \{t\}$ (Theorem (5.4)).

The finiteness of $\|c\|_1$ yields $c \in C_n^{\ell^1}(M')$ and $c_t \in C_n^{\ell^1}(Z_t)$. Therefore,

$$b_t := C_n^{\ell^1}(q_t)(c_t) \in C_n^{\ell^1}(\partial W \times \{t\})$$

and

$$\partial(b_t) = C_{n-1}^{\ell^1}(q_t)\big(\partial(c_t)\big) = C_{n-1}^{\ell^1}(q_t)(z_t) = z_t.$$

I.e., $H_{n-1}(i_{\partial W \times \{t\}})([\partial W \times \{t\}]) = 0 \in H_{n-1}^{\ell^1}(\partial W \times \{t\})$. Using the identification $\partial W \cong \partial W \times \{t\}$, we see that ∂W is ℓ^1-invisible and hence that part 2 is satisfied.

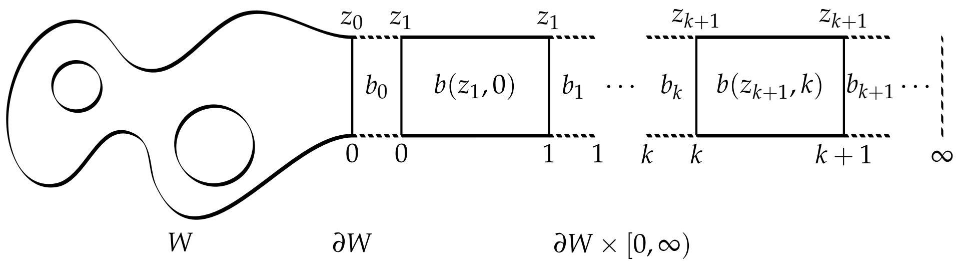

Figure (6.6). The proof of "$2 \Rightarrow 1$" of the finiteness criterion

$2 \Rightarrow 1$ Conversely, suppose that part 2 holds, i.e., that ∂W is ℓ^1-invisible. By Proposition (6.4), we find fundamental cycles $(z_k)_{k\in\mathbf{N}} \subset C_{n-1}(\partial W)$ and chains $(b_k)_{k\in\mathbf{N}} \subset C_n(\partial W)$ with

$$\forall_{k\in\mathbf{N}} \quad \partial(b_k) = z_{k+1} - z_k,$$
$$\sum_{k\in\mathbf{N}} \|b_k\|_1 < \infty,$$
$$\sum_{k\in\mathbf{N}} \|z_k\|_1 < \infty.$$

The idea is – similarly to Gromov's argument in a special case [18; p. 8] – to take a relative fundamental cycle of $(W, \partial W)$ and to glue the $(b_k)_{k\in\mathbf{N}}$ to its boundary. To ensure that the resulting chain is locally finite, we spread out the chain $\sum_{k\in\mathbf{N}} b_k$ over the cylinder $\partial W \times [0, \infty)$.

More precisely, let $c \in C_n(W)$ be a relative fundamental cycle of the manifold $(W, \partial W)$ with boundary. Then $\partial c \in C_{n-1}(\partial W)$ is a fundamental cycle of the oriented, compact manifold ∂W. Of course, we may assume that $\partial c = z_0$.

The spreading out of $(b_k)_{k\in\mathbf{N}}$ is achieved by using the following chains: For any cycle $z \in C_{n-1}(\partial W)$ and any $k \in \mathbf{N}$ we can find a chain $b(z,k) \in C_n(\partial W \times [0, \infty))$ such that

$$\partial\big(b(z,k)\big) = C_{n-1}(j_{k+1})(z) - C_{n-1}(j_k)(z),$$
$$\big\|b(z,k)\big\|_1 \leq n \cdot \|z\|_1\,;$$

here, $j_k \colon \partial W \hookrightarrow \partial W \times \{k\} \hookrightarrow \partial W \times [0, \infty)$ denotes the inclusion. For example, such a chain $b(z,k)$ can be constructed by looking at the canonical triangulation of $\Delta^{n-1} \times [0,1]$ into n-simplices.

We now define (see also Figure (6.6))

$$b := \sum_{k \in \mathbf{N}} \bigl(C_n\,(j_k)(b_k) + b(z_{k+1}, k) \bigr)$$

and $\overline{c} := c + b$. Because all b_k and all $b(z_{k+1}, k)$ are finite, the stretched chain b is a well-defined locally finite n-chain of M'. Therefore, also $\overline{c} \in C_n^{\mathrm{lf}}\,(M')$.

In the chain complex $C_*^{\mathrm{lf}}\,(M')$ of locally finite chains we can compute

$$\begin{aligned} \partial(\overline{c}) &= \partial(c) + \partial(b) \\ &= z_0 + \sum_{k \in \mathbf{N}} \quad \bigl(C_{n-1}\,(j_k)(z_{k+1}) - C_{n-1}\,(j_k)(z_k) \\ &\qquad\qquad + C_{n-1}\,(j_{k+1})(z_{k+1}) - C_{n-1}\,(j_k)(z_{k+1}) \bigr) \\ &= z_0 + \sum_{k \in \mathbf{N}} \quad \bigl(C_{n-1}\,(j_{k+1})(z_{k+1}) - C_{n-1}\,(j_k)(z_k) \bigr) \\ &= z_0 - z_0 \\ &= 0. \end{aligned}$$

In other words, $\overline{c}$ is a cycle. The construction shows that $\overline{c}|_M = c|_M$ and hence that $\overline{c}$ is a locally finite fundamental cycle of M' (Theorem (5.4)). Furthermore, we obtain

$$\begin{aligned} \|\overline{c}\|_1 &\leq \|c\|_1 + \|b\|_1 \\ &\leq \|c\|_1 + \sum_{k \in \mathbf{N}} \bigl(\|b_k\|_1 + \bigl\|b(z_{k+1}, k)\bigr\|_1 \bigr) \\ &\leq \|c\|_1 + \sum_{k \in \mathbf{N}} \|b_k\|_1 + \sum_{k \in \mathbf{N}} n \cdot \|z_{k+1}\|_1 \\ &< \infty, \end{aligned}$$

which shows that $\|M'\| < \infty$.

This finishes the proof of the finiteness criterion. □

6.3 A closer look at ℓ^1-invisible manifolds

In view of the finiteness criterion (Theorem (6.1)), it is interesting to analyse ℓ^1-invisibility in more detail.

In this section, we study the relation of ℓ^1-invisibility to the vanishing of simplicial volume (Section 6.3.1), the behaviour with respect to standard topological constructions (Section 6.3.2), the influence of the fundamental group (Section 6.3.3), the relation to curvature (Section 6.3.4), as well as a possible connection with L^2-Betti numbers (Section 6.3.5).

For applications, we refer to Section 6.4.

6.3.1 ℓ^1-Invisibility and simplicial volume

If a manifold is ℓ^1-invisible, then its simplicial volume is zero because the comparison map to ℓ^1-homology is isometric (Proposition (2.5)).

Matsumoto and Morita [38] introduce the so-called uniform boundary condition for normed chain complexes. This framework allows in certain cases to derive ℓ^1-invisibility from the vanishing of simplicial volume (Proposition (6.8)).

Definition (6.7). A normed chain complex $(C, \|\cdot\|)$ is said to satisfy the **uniform boundary condition in degree** q if there is a constant $K \in \mathbf{R}_{>0}$ such that for any null-homologous cycle $z \in C_q$ there exists a chain $b \in C_{q+1}$ with

$$\partial(b) = z \qquad \text{and} \qquad \|b\| \leq K \cdot \|z\|. \qquad \diamond$$

Proposition (6.8). *Let M be an oriented, closed, connected n-manifold with $\|M\| = 0$.*

1. *If the chain complex $(C_*(M), \|\cdot\|_1)$ satisfies the uniform boundary condition in degree n, then M is ℓ^1-invisible.*
2. *If $H_b^{n+1}(M) = 0$, then M is ℓ^1-invisible.*

Proof. In the first case, condition 2 of the technical characterisation of ℓ^1-invisibility, Proposition (6.4), is satisfied. The second case can be reduced to the first case by a result of Matsumoto and Morita [38; Theorem 2.8]. $\square$

On the other hand, if M is an oriented, closed, connected, ℓ^1-invisible n-manifold, then $(C_*(M), \|\cdot\|_1)$ does not necessarily satisfy the uniform boundary condition in degree n, as the following example shows:

Example (6.9). Let $n \in \mathbf{N}_{\geq 5}$. Then there exists a finitely presented group G such that $H_b^{n+1}(G)$ is non-zero (for example, we could take G to be the fundamental group of an oriented, closed, connected hyperbolic $(n+1)$-manifold, cf. Theorem (2.29)). Furthermore, there is also an oriented, closed, connected manifold N of dimension $(n-1)$ with fundamental group G [36; p. 114f].

We now consider the oriented, closed, connected n-manifold

$$M := N \times S^2.$$

By construction, $\pi_1(M) \cong G$ and hence $H_b^{n+1}(M) \cong H_b^{n+1}(G) \neq 0$ by the mapping theorem in bounded cohomology (Theorem (2.28)). In particular, the comparison map $H_b^{n+1}(M) \longrightarrow H^{n+1}(M) = 0$ is *not* injective. By a result of Matsumoto and Morita [38; Theorem 2.8], this means that $C_*(M)$ does not satisfy the uniform boundary condition in degree n. On the other hand, one can show that M is ℓ^1-invisible (Proposition (6.10).2/5). $\diamond$

It could be true that the vanishing of the simplicial volume of an oriented, closed, connected manifold already implies ℓ^1-invisibility. However, this seems to be rather unlikely:

If a manifold is ℓ^1-invisible, there must exist fundamental cycles and boundaries between them of small ℓ^1-norm (Proposition (6.4)). If the simplicial volume of an oriented, closed, connected n-manifold M is zero, a priori we can only deduce that there is a sequence $(z_n)_{n\in\mathbf{N}} \subset C_n(M)$ of fundamental cycles with $\lim_{n\to\infty}\|z_n\|_1 = 0$. But this does not give any control over the ℓ^1-norms of the set of all chains $b_k \in C_{n+1}(M)$ with $\partial b_k = z_{k+1} - z_k$. It is conceivable that there are manifolds such that $\|b_k\|_1$ must be large, whenever $\|z_k\|_1$ and $\|z_{k+1}\|_1$ are small.

Unfortunately, it also seems to be very difficult to prove existence of a counterexample, let alone exhibit a concrete counterexample. For example, it is not possible to use the work on non-Banach bounded cohomology because bounded cohomology cannot see the difference between ℓ^1-homology and reduced ℓ^1-homology. Furthermore, in almost all cases where it is known that the simplicial volume is zero, the underlying reason is strong enough to also give ℓ^1-invisibility, as we will see in the following subsections.

6.3.2 ℓ^1-Invisibility and standard constructions

Proposition (6.10). *Let M and N be two oriented, closed, connected manifolds of dimension at least* 1.

1. Non-trivial maps. *Suppose M and N are of the same dimension and there exists a continuous map $f\colon M \longrightarrow N$ of non-zero degree. If M is ℓ^1-invisible, then N is also ℓ^1-invisible.*
2. Non-trivial self-maps. *If M admits a self-map $f\colon M \longrightarrow M$ with $|\deg f| \geq 2$, then M is ℓ^1-invisible.*

3. Connected sums. *Suppose M and N are of the same dimension at least* 3. *If both M and N are ℓ^1-invisible, then also the connected sum $M \# N$ is ℓ^1-invisible.*
4. Amenable gluings. *Let (W_1, A) and (W_2, A) be oriented, compact, connected m-manifolds with boundary A and let $M := W_1 \cup_A W_2$. Suppose that*
$$H_m(i_{(W_j,A)})([W_j, A]) = 0 \in H_m^{\ell^1}(W_j, A)$$
for $j \in \{1,2\}$ and that $\operatorname{im} H_m^{\ell^1}(A \hookrightarrow M) = 0$; *the second condition is for example satisfied if A is an amenable subset of M or of one of the W_j (Definition (4.5) and Corollary (4.6)). Then the oriented, closed, connected m-manifold M is ℓ^1-invisible.*
5. Products. *If M is ℓ^1-invisible, then also the product $M \times N$ is ℓ^1-invisible.*
6. Fibrations. *Let $p\colon M \longrightarrow B$ be a fibration of oriented, closed, connected manifolds whose fibre F is also an oriented, closed, connected manifold of non-zero dimension. If $\pi_1(F)$ is amenable, then M is ℓ^1-invisible.*
7. Circle actions. *If M is smooth and admits a smooth S^1-action that is either free or has at least one fixed point, then M is ℓ^1-invisible.*
8. Proportionality. *Suppose M and N are equipped with a Riemannian metric such that the Riemannian universal coverings of M and N are isometric. Then M is ℓ^1-invisible if and only if N is ℓ^1-invisible.*

Comparing this list with the properties of simplicial volume (Section 5.4) raises the question whether all oriented, closed, connected manifolds with vanishing minimal volume are ℓ^1-invisible.

The proofs for most of the parts of the proposition are modeled on the proofs for the corresponding properties of simplicial volume.

Proof. In the following, we write $m := \dim M$ and $n := \dim N$.

Non-trivial maps. By definition of the mapping degree, we have
$$\begin{aligned} H_m(i_N)\big([N]\big) &= \frac{1}{\deg f} \cdot H_m(i_N)\big(H_m(f)\big([M]\big)\big) \\ &= \frac{1}{\deg f} \cdot H_m^{\ell^1}(f)\big(H_m(i_M)\big([M]\big)\big) \\ &= 0. \end{aligned}$$

Hence, N is ℓ^1-invisible.

Non-trivial self-maps. Let $z \in C_m(M)$ be a fundamental cycle of M. For brevity, we write $d := \deg f$. The definition of the mapping degree of the mapping degree shows that there is a chain $b \in C_{m+1}(M)$ satisfying

$$\partial b = z - \frac{1}{d} \cdot C_m(f)(z).$$

The chains $(z_k)_{k\in\mathbf{N}} \subset C_m(M)$ and $(b_k)_{k\in\mathbf{N}} \subset C_{m+1}(M)$ defined by

$$z_k := \frac{1}{d^k} \cdot C_m(f^k)(z),$$
$$b_k := \frac{1}{d^k} \cdot C_{m+1}(f^k)(b)$$

for all $k \in \mathbf{N}$ satisfy condition 2 of Proposition (6.4). Therefore, M is ℓ^1-invisible.

Connected sums. The pinching map $f\colon M \# N \longrightarrow M \vee N$ induces an isomorphism on the level of fundamental groups because $\dim M = \dim N \geq 3$ (by the Seifert-van Kampen theorem). By the mapping theorem for ℓ^1-homology (Corollary (4.2)), the induced homomorphism $H_m^{\ell^1}(f)\colon H_m^{\ell^1}(M \# N) \longrightarrow H_m^{\ell^1}(M \vee N)$ therefore is an (isometric) isomorphism.

Let $j_M\colon M \longrightarrow M \vee N$ and $j_N\colon N \longrightarrow M \vee N$ be the canonical inclusions. The Mayer-Vietoris sequence for singular homology shows that

$$H_m(j_M) \oplus H_m(j_N)\colon H_m(M) \oplus H_m(N) \longrightarrow H_m(M \vee N)$$

is an isomorphism satisfying

$$H_m(j_M) \oplus H_m(j_N)\bigl([M],[N]\bigr) = H_m(f)\bigl([M \# N]\bigr).$$

We now consider the commutative diagram

$$\begin{tikzcd}
H_m(M \# N) \arrow[r] \arrow[d, "H_m(f)"'] & H_m^{\ell^1}(M \# N) \arrow[d, "H_m^{\ell^1}(f)"] \\
H_m(M \vee N) \arrow[r] & H_m^{\ell^1}(M \vee N) \\
H_m(M) \oplus H_m(N) \arrow[r] \arrow[u, "H_m(j_M)\oplus H_m(j_N)"] & H_m^{\ell^1}(M) \oplus H_m^{\ell^1}(N) \arrow[u, "H_m^{\ell^1}(j_M)\oplus H_m^{\ell^1}(j_N)"']
\end{tikzcd}$$

where the horizontal maps are given by the respective comparison maps between singular homology and ℓ^1-homology.

Both M and N are ℓ^1-invisible, and thus the lower horizontal arrow maps $([M],[N])$ to 0. Therefore, the comparison map of $M \vee N$ maps $H_m(f)([M \# N])$

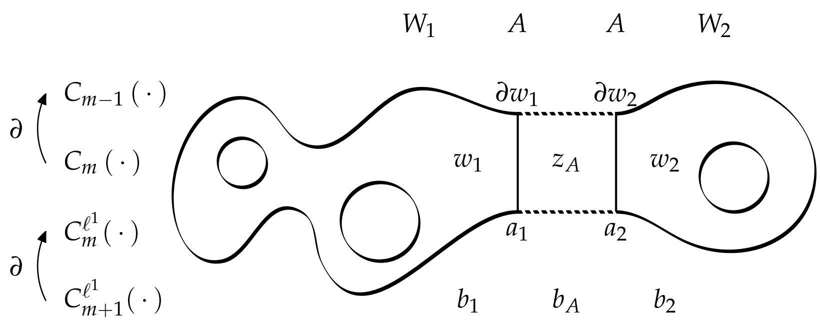

Figure (6.11). Amenable gluings

to 0. Now the fact that $H_m^{\ell^1}(f)$ is injective shows that the comparison map of $M \# N$ maps $[M \# N]$ to 0; thus, the connected sum $M \# N$ is ℓ^1-invisible.

Amenable gluings. Let $w_1 \in C_m(W_1)$ and $w_2 \in C_m(W_2)$ be relative fundamental cycles of (W_1, A) and (W_2, A) respectively. Since the corresponding relative fundamental classes are zero in ℓ^1-homology, there exist chains $b_j \in C_{m+1}^{\ell^1}(W_j)$ as well as $a_j \in C_m^{\ell^1}(A)$ satisfying

$$\partial b_1 = w_1 + a_1 \qquad \text{and} \qquad \partial b_2 = w_2 + a_2.$$

(All the notation is illustrated in Figure (6.11)). Because w_j is a relative fundamental cycle, both ∂w_1 and ∂w_2 are fundamental cycles of (the not necessarily connected) closed $(m-1)$-manifold A. In particular, there is a chain $z_A \in C_m(A)$ with

$$\partial z_A = \partial w_2 - \partial w_1.$$

We now consider the chain

$$z := w_1 - w_2 + z_A \in C_m(M).$$

By construction, z is a cycle and because z coincides on the open subset $W_1^\circ \subset M$ with the (relative) fundamental cycle w_1, the cycle z must be a fundamental cycle of M (Theorem (5.4)).

On the other hand, $\partial(z_A - a_1 + a_2) = 0$ shows that $z - a_1 + a_2 \in C_m^{\ell^1}(A)$ is a cycle. Because $\operatorname{im} H_m^{\ell^1}(A \hookrightarrow M) = 0$, there is a $b_A \in C_{m+1}^{\ell^1}(M)$ with

$$\partial b_A = z_A - a_1 + a_2.$$

Putting it all together yields

$$\partial(b_1 - b_2 + b_A) = w_1 - w_2 + z_A = z.$$

I.e., the ℓ^1-chain $b_1 - b_2 + b_A \in C^{\ell^1}_{m+1}(M)$ is a witness for the ℓ^1-invisibility of M.

(We also could have derived the previous item on connected sums from this one – however, the direct proof for connected sums looks more straightforward).

Products. We choose fundamental cycles $z_M \in C_m(M)$ and $z_N \in C_n(N)$ of M and N respectively. Therefore, the cross product $z := z_M \times z_N$ is a fundamental cycle of $M \times N$ by the Künneth theorem.

It is not difficult to see that the homological cross product

$$\times\colon C_p(M) \otimes C_q(N) \longrightarrow C_{p+q}(M \times N)$$

can be extended to a cross product

$$\times\colon C^{\ell^1}_p(M) \otimes C^{\ell^1}_q(N) \longrightarrow C^{\ell^1}_{p+q}(M \times N)$$

on ℓ^1-chains. Moreover, the relation $\partial(c_M \times c_N) = \partial c_M \times c_N + (-1)^p \cdot c_M \times \partial c_N$ for all $c_M \in C_p(M)$ and all $c_N \in C_q(N)$ also carries over to ℓ^1-chains.

Because M is ℓ^1-invisible, there is a chain $b_M \in C^{\ell^1}_{m+1}(M)$ with $\partial b_M = z_M$. We now consider the chain

$$b := b_M \times z_N \in C^{\ell^1}_{m+n}(M \times N).$$

By construction, in the chain complex $C^{\ell^1}_*(M \times N)$ we have the relation

$$\begin{aligned}\partial b &= \partial b_M \times z_N + (-1)^m \cdot b_M \times \partial z_N\\ &= z_M \times z_N + (-1)^m \cdot b_M \times 0\\ &= z_M \times z_N.\end{aligned}$$

In other words, the product $M \times N$ is ℓ^1-invisible.

Fibrations. In this situation, the dimension of B is at most $m-1$ [35; Exercise 14.15 and p. 556]. In particular, $H_m(B) = 0$. Since $\pi_1(F)$ is amenable, the induced map $H^{\ell^1}_m(p)\colon H^{\ell^1}_m(M) \longrightarrow H^{\ell^1}_m(B)$ is an (isometric) isomorphism (Corollary (4.4)). Therefore, the commutative diagram

$$\begin{array}{ccc} H_m(M) & \xrightarrow{H_m(i_M)} & H^{\ell^1}_m(M) \\ {\scriptstyle H_m(p)}\big\downarrow & & \big\downarrow{\scriptstyle H^{\ell^1}_m(p)} \\ 0 = H_m(B) & \xrightarrow[H_m(i_B)]{} & H^{\ell^1}_m(B) \end{array}$$

shows that M is ℓ^1-invisible.

Circle actions. We first look at the case that the S^1-action on M has at least one fixed point: Then

$$H_m(f)\big([M]\big) = 0 \in H_m\big(B\pi_1(M)\big),$$

where $f\colon M \longrightarrow B\pi_1(M)$ is the classifying map [18, 35; p. 95, Lemma 1.42]. On the other hand, the induced map $\pi_1(f)$ on the level of fundamental groups is an isomorphism, and hence $H_m^{\ell^1}(f)\colon H_m^{\ell^1}(M) \longrightarrow H_m^{\ell^1}(B\pi_1(M))$ is an (isometric) isomorphism (Corollary (4.2)). Now the commutative diagram

$$\begin{tikzcd}
H_m(M) \arrow[r, "H_m(i_M)"] \arrow[d, "H_m(f)"'] & H_m^{\ell^1}(M) \arrow[d, "H_m^{\ell^1}(f)"] \\
H_m\big(B\pi_1(M)\big) \arrow[r, "H_m(i_{B\pi_1(M)})"'] & H_m^{\ell^1}\big(B\pi_1(M)\big)
\end{tikzcd}$$

allows us to deduce that M must be ℓ^1-invisible if the action has fixed points.

What happens if the S^1-action on M is free? Because M is compact, the S^1-operation in question is proper. Therefore, the quotient M/S^1 is a compact manifold of dimension $m-1$. Hence, $H_m(M/S^1) = 0$. Furthermore, it follows that the projection $p\colon M \longrightarrow M/S^1$ is an S^1-principal bundle and thus in particular a fibration with fibre S^1. Because the fundamental group $\pi_1(S^1) \cong \mathbf{Z}$ is amenable, the induced map $H_m^{\ell^1}(p)\colon H_m^{\ell^1}(M) \longrightarrow H_m^{\ell^1}(M/S^1)$ is an (isometric) isomorphism (Corollary (4.4)). Therefore, we deduce (like in the previous item) from the commutative diagram

$$\begin{tikzcd}
H_m(M) \arrow[r, "H_m(i_M)"] \arrow[d, "H_m(p)"'] & H_m^{\ell^1}(M) \arrow[d, "H_m^{\ell^1}(p)"] \\
0 = H_m\big(M/S^1\big) \arrow[r, "H_m(i_{M/S^1})"'] & H_m^{\ell^1}\big(M/S^1\big)
\end{tikzcd}$$

that $H_m(i_M)([M]) = 0$. I.e., M is ℓ^1-invisible.

Proportionality. Thanks to symmetry, it suffices to show that if M is ℓ^1-invisible, then also N is ℓ^1-invisible. Moreover, since the universal coverings of M and N coincide, the dimensions of M and N must be equal.

Thurston's proof of the proportionality principle for simplicial volume relies on the so-called "smearing" of (smooth) singular chains. Smearing is a chain map

$$\mathrm{smear}_{M,N}\colon C_*^{\mathrm{s}}(M) \longrightarrow C_*^{\mathrm{s}}(N),$$

$$
\begin{tikzcd}
H_n\,(M) \arrow[r, "H_n(i_M)"] & H_n^{\ell^1}(M) \\
H_n^{\mathrm{s}}\,(M) \arrow[u, "\cong"] \arrow[r] \arrow[d, "H_n(\mathrm{smear}_{M,N})"'] \arrow[dd, bend right=60, "\cdot\frac{\operatorname{vol} M}{\operatorname{vol} N}"'] & H_n^{\mathrm{s},\ell^1}(M) \arrow[u, "\cong"'] \arrow[d, "H_n(\mathrm{smear}^{\ell^1}_{M,N})"] \\
\mathcal{H}_n^{\mathrm{s}}\,(N) \arrow[r] & \mathcal{H}_n^{\mathrm{s},\ell^1}(N) \\
H_n^{\mathrm{s}}\,(N) \arrow[u, "\cong"] \arrow[r] \arrow[d, "\cong"'] & H_n^{\mathrm{s},\ell^1}(N) \arrow[u, "\subset"'] \arrow[d, "\cong"] \\
H_n\,(N) \arrow[r, "H_n(i_N)"'] & H_n^{\ell^1}(N)
\end{tikzcd}
$$

Figure (6.12). Proof of part 8 of Proposition (6.10) – proportionality

where $C_*^{\mathrm{s}}\,(M) \subset C_*\,(M)$ is the subcomplex generated by smooth singular chains and $\mathcal{C}_*^{\mathrm{s}}\,(N)$ is the smooth version of the measure chain complex of N. The two distinguishing features of this chain map are that it is bounded with norm at most 1 and that it maps (smooth) fundamental cycles of M to (smooth) measure cycles representing $\operatorname{vol} M/\operatorname{vol} N$ times the fundamental class of N; more details on measure homology and smearing are provided in Appendix D, especially in Theorem (D.13).

Because $\mathrm{smear}_{M,N}$ is bounded, it extends to a morphism

$$\mathrm{smear}^{\ell^1}_{M,N} \colon C_*^{\mathrm{s},\ell^1}(M) \longrightarrow \mathcal{C}_*^{\mathrm{s},\ell^1}(N)$$

of the corresponding completed chain complexes (Theorem (D.13)). We obtain the commutative ladder in Figure (6.12), where

- except for the maps $H_n(\mathrm{smear}_{M,N})$ and $H_n(\mathrm{smear}^{\ell^1}_{M,N})$, all homomorphisms are the ones induced by the canonical inclusions of the underlying chain complexes;
- all arrows labeled with "$\cong$" are (isometric) isomorphisms (Proposition (D.5) and Corollary (D.12));
- the arrow labeled with "$\subset$" is an (isometric) injection (Theorem (D.9)).

Hence, chasing through the diagram shows that if $H_n(i_M)([M]) = 0 \in H_n^{\ell^1}(M)$, then also $H_n(i_N)([N]) = 0 \in H_n^{\ell^1}(N)$ holds. This finishes the proof of proportionality for ℓ^1-invisibility. $\square$

6.3.3 ℓ^1-Invisibility and the fundamental group

Proposition (6.13). *Let M be an oriented, closed, connected m-manifold.*

1. Group homology. *If $H_m(\pi_1(M)) = 0$, then M is ℓ^1-inviable.*
2. Amenable fundamental group. *If the fundamental group of M is amenable and if $m > 0$, then M is ℓ^1-invisible.*
3. Free fundamental group. *If the fundamental group of M is free and $m > 0$, then M is ℓ^1-invisible.*

Proof. Group homology. Let $f\colon M \longrightarrow B\pi_1(M)$ be the classifying map. We consider the commutative diagram

$$\begin{tikzcd} H_m(M) \arrow[r, "H_m(i_M)"] \arrow[d, "H_m(f)"'] & H_m^{\ell^1}(M) \arrow[d, "H_m^{\ell^1}(f)"] \\ H_m\bigl(B\pi_1(M)\bigr) \arrow[r, "H_m(i_{B\pi_1(M)})"'] & H_m^{\ell^1}\bigl(B\pi_1(M)\bigr). \end{tikzcd}$$

Because $H_m^{\ell^1}(f)$ is an (isometric) isomorphism by the mapping theorem (Corollary (4.2)) and $H_m(B\pi_1(M)) = H_m(\pi_1(M)) = 0$, it follows that $H_m(i_M)([M]) = 0$.

Amenable fundamental group. If $\pi_1(M)$ is amenable, then $H_m^{\ell^1}(M) = 0$ (Corollary (4.2)). In particular, $H_m(i_M)([M]) = 0 \in H_m^{\ell^1}(M)$, i.e., M is ℓ^1-invisible.

Free fundamental group. If $\pi_1(M)$ is a free group, then we have $H_k(\pi_1(M)) = 0$ for all $k \in \mathbf{N}_{>1}$. Therefore, we can apply part 1 whenever $m > 1$. If $m = 1$, then $M = S^1$, which is ℓ^1-invisible (for example, by part 2). $\square$

6.3.4 ℓ^1-Invisibility and curvature

Corollary (6.14). *Let M be an oriented, closed, connected, Riemannian manifold.*

1. Positive curvature. *If M has positive sectional curvature, then M is ℓ^1-invisible.*
2. Flat manifolds. *If M is flat, then M is ℓ^1-invisible.*
3. Negative curvature. *If M has negative sectional curvature, then M is* not *ℓ^1-invisible.*

Proof. Positive curvature. By Myers's theorem, complete, connected manifolds of positive sectional curvature have finite – and hence amenable – fundamental group [30; Theorem 11.8]. Therefore, these manifolds are ℓ^1-invisible (Proposition (6.13).2)

Flat manifolds. The classification of compact, flat manifolds shows that they all are finitely covered by tori and hence have amenable fundamental group [13; Theorem II.5.3].

Moreover, ℓ^1-invisibility of flat manifolds can also be derived from the proportionality property of ℓ^1-invisibility (Proposition (6.10).8): Namely, the Riemannian universal covering of any flat manifold is isometric to the Riemannian universal covering of the torus of the same dimension. Because tori are ℓ^1-invisible, all flat manifolds must be ℓ^1-invisible.

Negative curvature. Oriented, closed, connected manifolds of negative sectional curvature have non-zero simplicial volume (Section 5.4). Therefore, these manifolds cannot be ℓ^1-invisible. □

6.3.5 ℓ^1-Invisibility and L^2-Betti numbers

Gromov asked if the L^2-Betti numbers of an oriented, closed, connected, aspherical manifold with vanishing simplicial volume are zero [20; Section 8A]. For an introduction to L^2-Betti numbers we refer to Lück's extensive textbook [35].

The notion of ℓ^1-invisibility gives rise to the following approach to Gromov's question: Let M be an oriented, closed, connected n-manifold with fundamental group G, and let $\mathcal{N}G$ be the group von Neumann algebra of G. Then there is a commutative diagram relating L^2-(co)homology to bounded cohomology and ℓ^1-homology:

$$\begin{array}{ccc}
H^k(M;\mathcal{N}G) & \xrightarrow{\cdot\cap[M]} & H_{n-k}(M;\mathcal{N}G) \\
\uparrow & & \downarrow \\
H^k_{\mathrm{b}}(M;\mathcal{N}G) & \xrightarrow[\cdot\cap H_n(i_M)([M])]{} & H^{\ell^1}_{n-k}(M;\mathcal{N}G)
\end{array}$$

Here, the vertical arrows are the respective comparison maps. The upper horizontal arrow is an isomorphism (Poincaré duality with twisted coefficients – for cellular L^2-(co)homology this follows from a theorem of Wall [58; Theorem 2.1, p. 23]).

The lower cap-product is based on literally the same definition as the usual cap-product in singular (co)homology with twisted coefficients: In view of Proposition (2.23), we can identify the two Banach chain complexes $C_*^{\ell^1}(M)$ and $C_*^{\ell^1}(\widetilde{M})_G$. Then the cap-product is defined on generators by

$$\begin{array}{ccc} C_b^k(M;\mathcal{N}G) \otimes C_n^{\ell^1}(M) & \xrightarrow{\ \cdot \cap \cdot\ } & C_{n-k}^{\ell^1}(M;\mathcal{N}G) \\ \| & & \| \\ B\big(C_k^{\ell^1}(\widetilde{M}),\mathcal{N}G\big)^G \otimes C_n^{\ell^1}(\widetilde{M})_G & \longrightarrow & \big(C_{n-k}^{\ell^1}(\widetilde{M}) \mathbin{\overline{\otimes}} \mathcal{N}G\big)_G \\ \cup & & \cup \\ f \otimes [\sigma] & \longmapsto & \big[{}_{n-k}\lfloor\sigma \otimes f(\sigma\rfloor_k)\big] \end{array}$$

where the brackets $[\,\cdot\,]$ denote the corresponding equivalence classes in the coinvariants, and ${}_{n-k}\lfloor\sigma$ and $\sigma\rfloor_k$ are the $(n-k)$-back face and the k-front face of σ respectively.

The same calculations as in singular (co)homology show that this definition is independent of the chosen representatives in the coinvariants. Moreover, the norm $\|f \cap c\|_1 \leq \|f\|_\infty \cdot \|c\|_1$ is finite for all $f \in C_b^k(M;V)$ and all $c \in C_n^{\ell^1}(M)$. Therefore, this cap-product is well-defined on the (co)chain level. In addition, this cap-product descends to a cap-product on the level of (co)homology.

Suppose that the comparison map $H_b^*(M;\mathcal{N}G) \longrightarrow H^*(M;\mathcal{N}G)$ is surjective and that the comparison map $H_*(M;\mathcal{N}G) \longrightarrow H_*^{\ell^1}(M;\mathcal{N}G)$ is injective (up to $\mathcal{N}G$-dimension). Then the commutative diagram above shows: If M is ℓ^1-invisible, then the upper horizontal arrow is the zero map up to $\mathcal{N}G$-dimension and thus all L^2-Betti numbers of M must be zero. This leads to the following question:

Question (6.15). Let M be an oriented, closed, connected, aspherical manifold with fundamental group G. Is the comparison map $H_b^*(M;\mathcal{N}G) \longrightarrow H^*(M;\mathcal{N}G)$ surjective and is the comparison map $H_*(M;\mathcal{N}G) \longrightarrow H_*^{\ell^1}(M;\mathcal{N}G)$ injective (at least up to $\mathcal{N}G$-dimension)? $\diamond$

6.4 Applying the finiteness criterion

Combining the finiteness criterion (Theorem (6.1)) with the properties of ℓ^1-invisibility established in Section 6.3 gives access to a collection of simple examples demonstrating the odd behaviour of simplicial volume of non-compact manifolds.

In Section 6.4.1, we give a basic vanishing result, which can be applied to compute the simplicial volume of Euclidean spaces (Section 6.4.2). Section 6.4.3 contains an elementary example of a non-compact manifold whose simplicial volume is neither zero nor infinite. The lack of homotopy invariance of the simplicial volume in the category of non-compact manifolds is discussed in Section 6.4.4. A geometrically interesting class of non-compact manifolds with nice boundaries are locally symmetric spaces of finite volume (Section 6.4.5). Finally, the last section deals with products of non-compact manifolds.

6.4.1 Vanishing results

The computation of simplicial volume of non-compact manifolds in terms of ℓ^1-homology leads to the following vanishing result:

Corollary (6.16). *Let $(W, \partial W)$ be an oriented, compact, connected n-manifold with boundary and let $M := W^\circ$. If $H_n^{\ell^1}(W) = 0$ and ∂W is ℓ^1-invisible, then $\|M\| = 0$.*

Proof. Because M is homotopy equivalent to W, we know that $H_n^{\ell^1}(M) = 0$. In particular, $\|M\| \in \{0, \infty\}$ (Corollary (5.18)). According to the finiteness criterion (Theorem (6.1)) the simplicial volume of M cannot be infinite because ∂W is ℓ^1-invisible. Therefore, $\|M\| = 0$. □

In particular, if $(W, \partial W)$ is an oriented, compact, connected n-manifold with boundary with $H_n^{\ell^1}(W) = 0$ and $H_{n-1}^{\ell^1}(\partial W) = 0$, then $\|W^\circ\| = 0$.

6.4.2 Euclidean spaces

For example, Corollary (6.16) allows to compute the simplicial volume of Euclidean spaces:

Example (6.17). Since $\|S^0\| = 2$, the finiteness criterion (Theorem (6.1)) shows that $\|\mathbf{R}\| = \|[0,1]^\circ\| = \infty$.

On the other hand, for all $n \in \mathbf{N}_{>1}$, the sphere S^{n-1} is ℓ^1-invisible (Proposition (6.10)) and hence $\|\mathbf{R}^n\| = \|(D^n)^\circ\| < \infty$ by the finiteness criterion. Moreover, $H_n^{\ell^1}(\mathbf{R}^n) = 0$. Using Corollary (6.16), we deduce that $\|\mathbf{R}^n\| = 0$.

The second part can also be shown via self-maps of (D^n, S^{n-1}) of non-trivial degree [18; p. 8f]. ◇

It is one of the confusing aspects of the simplicial volume of non-compact manifolds that (by the example above) the simplicial volume of hyperbolic n-space is zero if $n > 1$. This contrasts the classic result that the simplicial volume of oriented, closed, connected hyperbolic manifolds is positive.

6.4.3 Non-compact manifolds with finite, non-zero simplicial volume

The finiteness criterion provides concrete examples of non-compact manifolds whose simplicial volume is non-trivial, i.e., it is neither zero nor infinite:

Example (6.18). Let M be an oriented, closed, connected manifold with $\|M\| \neq 0$ of dimension $n \geq 2$ (for example a hyperbolic manifold of dimension at least 2), and let N be a non-compact manifold obtained from M by removing a finite number of points. Then

$$0 < \|N\| < \infty.$$

This can be seen as follows: Let $(N', \partial N')$ be the compact manifold with boundary obtained from M by removing the same (finite) number of open n-balls. By construction, N is homeomorphic to the interior of N' and the boundary of N' consists of a finite disjoint union of $(n-1)$-spheres.

In particular, the boundary of N' is ℓ^1-invisible (Proposition (6.10).2). Hence, the finiteness criterion (Theorem (6.1)) shows that $\|N\| < \infty$.

Why is $\|N\|$ non-zero? By Proposition (5.12) it suffices to show $\|N', \partial N'\| > 0$. By construction, $N' = M \setminus (D_1^\circ \sqcup \dots D_k^\circ)$, where $D_1, \dots, D_k \subset M$ are small n-disks. Because $H_{\mathrm{b}}^n(D_1 \sqcup \dots \sqcup D_k) = 0$, the singular chain complex $C_*\,(D_1 \sqcup \dots \sqcup D_k)$ satisfies the uniform boundary condition in degree $n-1$ [38; Theorem 2.8]. Let $K \in \mathbf{R}_{>0}$ be the corresponding constant as in the Definition (6.7).

If $c' \in C_n\,(N')$ is a relative fundamental cycle of $(N', \partial N')$, then $\partial c' \in C_{n-1}\,(\partial N')$ is a cycle; because $H_{n-1}\,(D_1 \sqcup \dots \sqcup D_k) = 0$, this cycle is null-homologous in the chain complex $C_*\,(D_1 \sqcup \dots \sqcup D_k)$.

Therefore, the uniform boundary condition guarantees the existence of a chain $b \in C_n\,(D_1 \sqcup \dots \sqcup D_k)$ with

$$\partial b = -\partial c' \qquad \text{and} \qquad \|b\|_1 \leq K \cdot \|\partial c'\|_1 \leq (n+1) \cdot K \cdot \|c'\|_1.$$

In particular, the chain $c := c' + b \in C_n\,(M)$ is a cycle. Because c and c' coincide on N'°, the cycle c is a fundamental cycle of M (Theorem (5.4)).

This implies

$$\|M\| \leq \|c\|_1 \leq \|c'\|_1 + (n+1) \cdot K \cdot \|c'\|_1$$

and hence $\|N', \partial N'\| \geq 1/(1 + (n+1) \cdot K) \cdot \|M\| > 0$, as desired. $\diamond$

For example, many of the oriented, hyperbolic surfaces of finite volume can be obtained in this way. In fact, all oriented, hyperbolic surfaces have finite, non-zero simplicial volume and the value can be computed explicitly (in terms of the volume) [57, 34]. However, these computations are beyond the scope of this text.

6.4.4 Lack of homotopy invariance in the non-compact case

We have already seen that the simplicial volume of non-compact manifolds is not homotopy invariant in general (Example (6.17)), but only invariant under proper homotopy equivalences. With little more effort, we can also show that the simplicial volume of non-compact manifolds is not even homotopy invariant in the class of non-compact manifolds with *finite* simplicial volume:

Example (6.19). Let $(W, \partial W)$ be an oriented, compact, connected surface of genus at least 2 with non-empty boundary. Clearly, $W^\circ \simeq W^\circ \times \mathbf{R}^2$, but

$$\begin{aligned} 0 &< \|W^\circ\| < \infty, \\ 0 &= \|W^\circ \times \mathbf{R}^2\|. \end{aligned}$$

The first line is an instance of Example (6.18), the second line follows in view of $\|W^\circ\| < \infty$ and $\|\mathbf{R}^2\| = 0$ from the product formula (Theorem (C.7)). $\diamond$

6.4.5 Locally symmetric spaces of non-compact type of finite volume

A class of non-compact manifolds with nice compactification arises naturally in differential geometry, namely in the study of locally symmetric spaces of non-compact type with finite volume.

We begin with hyperbolic manifolds of finite volume (Example (6.20)). In Examples (6.21) and (6.22), we study locally symmetric spaces of non-compact type given by arithmetic lattices; an introduction to arithmetic lattices and the corresponding locally symmetric spaces can be found in the book of Borel and Ji [3; Section III.2], where also a number of examples of such spaces is discussed.

Example (6.20). Let M be an oriented, connected, complete hyperbolic n-manifold. Then a classic theorem [1; Corollary D.3.14] asserts that there is an oriented, compact, connected n-manifold $(W, \partial W)$ with boundary such that $M \cong W^\circ$ and all connected components of the boundary ∂W are manifolds admitting a flat Riemannian structure.

By Corollary (6.14), oriented, closed, connected, flat manifolds are ℓ^1-invisible. Therefore, the finiteness criterion (Theorem (6.1)) yields that the simplicial volume of M is finite.

In this situation, one can even show that $\|M\| = \|W, \partial W\|$: The chain complex $C_*(\partial W)$ satisfies the uniform boundary condition in degree $n-1$ because all connected components of ∂W are amenable (cf. proof of Corollary (6.14)) and hence $H^n_{\mathrm{b}}(\partial W) = 0$ [38; Theorem 2.8]. Let $K \in \mathbf{R}_{>0}$ be the corresponding constant as in Definition (6.7).

The proof of the finiteness criterion shows that if $c \in C_n(W)$ is a relative fundamental cycle of $(W, \partial W)$, then we can find a locally finite fundamental cycle c' of M satisfying $\|c'\|_1 \leq \|c\|_1 + 2 \cdot K \cdot \|\partial c\|_1$. Therefore, Gromov's "equivalence theorem" [18; p. 57] lets us deduce that $\|M\| \leq \|W, \partial W\|$. On the other hand, we also have the reverse inequality by Proposition (5.12).

Notice however that there does not seem to be a complete proof of the equivalence theorem in the literature. Gromov's proof is rather sketchy, and Park's approach [47, 46] is incomplete (in the ℓ^1-case) and does deal with a different semi-norm on the relative homology groups (both in the ℓ^1-case as well as in the bounded cohomology case). $\diamond$

Example (6.21). Let $M = \Gamma \backslash G / K$ be an oriented, connected, locally symmetric space (of non-compact type) with finite volume, where Γ is an arithmetic lattice of $\mathbf{Q}$-rank equal to 1. Then the simplicial volume of M is finite:

In this situation, it is known that M is homeomorphic to the interior of an oriented, compact manifold $(W, \partial W)$ with boundary, the Borel-Serre compactification [4, 3].

Moreover, each connected component of ∂W is a fibre bundle, whose fibre is an oriented, closed, connected nil-manifold and whose base is an oriented, closed, connected manifold [3; Proposition III.9.20]; the stratification of the boundary in this case is especially simple because in the $\mathbf{Q}$-rank 1 case all proper, $\mathbf{Q}$-parabolic subgroups are minimal [3; III.1.8]. The fundamental groups of nil-manifolds are nilpotent and hence amenable [48; p. 13]; thus, the fibres have amenable fundamental group.

Therefore, ∂W is ℓ^1-invisible (Proposition (6.10).6) and hence $\|M\|$ is finite by the finiteness criterion (Theorem (6.1)). $\diamond$

Example (6.22). Let $M = \Gamma \backslash G / K$ be an oriented, connected, locally symmetric space (of non-compact type) with finite volume, where Γ is an arithmetic lattice of $\mathbf{Q}$-rank at least 3. Then $\|M\|$ is finite:

Let $n := \dim M$. It is known that under the assumptions above M is the interior of an oriented, compact, connected n-manifold $(W, \partial W)$, where the boundary ∂W is connected and the map $\pi_1(\partial W) \longrightarrow \pi_1(W)$ induced by the inclusion is an isomorphism [4; Section 11.2].

By the mapping theorem (Corollary (4.2)), the homomorphism $H_*^{\ell^1}(\partial W) \longrightarrow H_*^{\ell^1}(W)$ induced by the inclusion is an isometric isomorphism. Therefore, the long exact sequence in ℓ^1-homology (Proposition (2.7)) of the pair $(W, \partial W)$ shows that $H_n^{\ell^1}(W, \partial W) = 0$. Because $\partial[W, \partial W] = [\partial W]$, the commutative diagram

$$\begin{tikzcd} H_n(W, \partial W) \arrow[r] \arrow[d] & H_n^{\ell^1}(W, \partial W) \arrow[d] \\ H_{n-1}(W) \arrow[r] & H_n^{\ell^1}(W) \end{tikzcd}$$

(where the horizontal arrows are the comparison maps and the vertical arrows are the boundary operators of the respective long exact sequences in homology) yields that ∂W is ℓ^1-invisible. In particular, $\|M\|$ is finite (Theorem (6.1)).

Notice that $H_n^{\ell^1}(W, \partial W) = 0$ implies $\|W, \partial W\| = 0$ (Proposition (5.13)) – in contrast to the closed case (Section 5.4). It might be even true that $\|M\| = 0$. $\diamond$

Because all the examples of locally symmetric spaces considered in this section have complete Riemannian metrics of finite volume and bounded sectional curvature, their minimal volume is finite. Therefore, also Gromov's estimate of the simplicial volume by the minimal volume (Section 5.4) shows that the simplicial volume of these examples must be finite.

The advantage of the approach via the finiteness criterion is that it depends only on topological information and that it is more concrete – one can actually see where the finiteness comes from.

6.4.6 Products of manifolds

The simplicial volume of products of manifolds can be estimated from above by the product of the simplicial volumes of the factors (cf. Section 5.4 and Theorem (C.7)). But this estimate cannot deal with the tricky case that one of the factors has zero simplicial volume and the other one has infinite simplicial volume [18; p. 10].

In a very special case, ℓ^1-invisibility completely determines the simplicial volume of such products:

Proposition (6.23). *Let M be an oriented, closed, connected n-manifold. Then*

$$\|M \times \mathbf{R}\| = \begin{cases} 0 & \text{if } M \text{ is } \ell^1\text{-invisible,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Because $M \times \mathbf{R}$ is the interior of the compact manifold $M \times [0,1]$ with boundary $M \sqcup M$, the finiteness criterion (Theorem (6.1)) yields that $\|M \times \mathbf{R}\|$ is finite if and only if M is ℓ^1-invisible.

Therefore, it remains to show that $\|M \times \mathbf{R}\| = 0$ provided M is ℓ^1-invisible: In the case that M is ℓ^1-invisible, the proof of the finiteness criterion (applied to $M \times [0,1]$) shows that there is a locally finite chain $c \in C^{\mathrm{lf}}_{n+1}(M \times [0,\infty))$ such that the sequence $(c_k)_{k \in \mathbf{N}}$ of chains given by

$$c_k := c|_{M \times (k,\infty)}$$

satisfies the following properties:

1. For each $k \in \mathbf{N}$ we have $c_k \in C^{\mathrm{lf}}_{n+1}(M \times [k,\infty))$ and $\partial c_k \in C_n(M \times \{k\})$.
2. For each $k \in \mathbf{N}$ and each $x \in M \times (k,\infty)$ the restriction $c_k|_{\{x\}}$ is a relative fundamental cycle of $(M \times [k,\infty), M \times [k,\infty) \setminus \{x\})$.
3. Moreover, $\lim_{k\to\infty} \|c_k\|_1 = 0$.

For each $k \in \mathbf{N}$, we now consider the mirror images

$$\overline{c_k} := C^{\mathrm{lf}}_{n+1}(\mathrm{id}_M \times r_k)(c_k) \in C^{\mathrm{lf}}_{n+1}(M \times (-\infty,k]),$$

where $r_k \colon \mathbf{R} \longrightarrow \mathbf{R}$ denotes reflection at k. Then $c_k - \overline{c_k} \in C^{\mathrm{lf}}_{n+1}(M \times \mathbf{R})$ is a cycle, which is a fundamental cycle of $M \times \mathbf{R}$ (this follows from 2. and Theorem (5.4)). Therefore, we obtain

$$\|M \times \mathbf{R}\| \le \inf_{k \in \mathbf{N}} \|c_k - \overline{c_k}\|_1 \le \inf_{k \in \mathbf{N}} 2 \cdot \|c_k\|_1 = 0. \qquad \square$$

Hence, any oriented, closed, connected manifold with vanishing simplicial volume that is not ℓ^1-invisible would produce the first example of two manifolds M and N with $\|M\| = 0$, $\|N\| = \infty$ and $\|M \times N\| \neq 0$.

A related problem is to find an example of two open manifolds whose product has non-zero simplicial volume. Again, the finiteness criterion helps us to catch such an example red-handed:

Example (6.24). Let $(M, \partial M)$ be an oriented, compact, connected surface of genus at least 1 with non-empty boundary. Then

$$\|M^\circ \times \mathbf{R}\| = \infty.$$

This is a consequence of the finiteness criterion: Clearly, $M^\circ \times \mathbf{R}$ is the interior of the compact manifold $M \times [0,1]$ with boundary

$$\partial\bigl(M \times [0,1]\bigr) = M \times \{0,1\} \cup_{\partial M \times \{0,1\}} \partial M \times [0,1],$$

which is nothing but an oriented, closed, connected surface of genus at least 2. Because hyperbolic surfaces are not ℓ^1-invisible (Corollary (6.14)), the finiteness criterion (Theorem (6.1)) shows that $\|M^\circ \times \mathbf{R}\|$ is infinite. $\diamond$

Surprisingly, all information on the factors is lost when considering the simplicial volume of threefold products of open manifolds (Example (C.9)).

A Homological algebra of Banach G-modules

Brooks [6] and Ivanov [25] adapted (relative) homological algebra in the sense of Hochschild [23] to fit the needs of bounded cohomology of discrete groups. In this chapter, we introduce the basic objects of this theory and investigate their compatibility with taking (topological) duals. The key concepts are strong relatively projective and injective resolutions, which lead to the desirable fundamental lemma (Proposition (A.7)). Concrete examples of these concepts are studied in Section 2.2.

A more detailed account of the material collected in this chapter is, for example, presented in the work of Ivanov [25] and Monod [42], as well as (for the non-Banach case) in the book of Guichardet [21]. An alternative approach to homological algebra of Banach G-modules was created by Bühler [11].

A.1 Relatively injective and relatively projective Banach G-modules

The atoms of the variant of (relative) homological algebra presented in the following are Banach G-modules with a suitable notion of projectivity and injectivity.

$$\begin{tikzcd} & V \arrow[d, "\alpha"] \arrow[dl, dashed, "\beta"'] & \\ U \arrow[r, "\pi"'] & W \arrow[r] \arrow[l, bend right, "\sigma"'] & 0 \end{tikzcd} \qquad \begin{tikzcd} & V & \\ 0 \arrow[r] & U \arrow[u, "\alpha"] \arrow[r, "i"] & W \arrow[ul, dashed, "\beta"'] \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

Figure (A.2). Mapping problems for relatively projective and relatively injective Banach G-modules respectively

Definition (A.1). Let G be a discrete group.

1. A **Banach G-module** is a Banach space V with a G-action $G \times V \longrightarrow V$ such that for each $g \in G$ the linear map $v \mapsto g \cdot v$ is an isometry.
2. A **G-morphism** is a bounded linear map between Banach G-modules that is G-equivariant.
3. A G-morphism $\pi\colon U \longrightarrow W$ is called **relatively projective** if there is a (not necessarily equivariant) linear map $\sigma\colon W \longrightarrow U$ satisfying
$$\pi \circ \sigma = \mathrm{id}_W \qquad \text{and} \qquad \|\sigma\| \leq 1.$$
4. A G-morphism $i\colon U \longrightarrow W$ is called **relatively injective** if there is a (not necessarily equivariant) linear map $\sigma\colon W \longrightarrow U$ satisfying
$$\sigma \circ i = \mathrm{id}_U \qquad \text{and} \qquad \|\sigma\| \leq 1.$$
5. A Banach G-module V is called **relatively projective** if for each relatively projective G-morphism $\pi\colon U \longrightarrow W$ and each G-morphism $\alpha\colon V \longrightarrow W$ there is a G-morphism $\beta\colon V \longrightarrow U$ such that
$$\pi \circ \beta = \alpha \qquad \text{and} \qquad \|\beta\| \leq \|\alpha\|.$$
6. A Banach G-module V is called **relatively injective** if for each relatively injective G-morphism $i\colon U \longrightarrow W$ and for each G-morphism $\alpha\colon U \longrightarrow V$ there is a G-morphism $\beta\colon W \longrightarrow V$ such that
$$\beta \circ i = \alpha \qquad \text{and} \qquad \|\beta\| \leq \|\alpha\|. \qquad \diamond$$

The mapping problems arising in the definition of relatively projective and relatively injective Banach G-modules are depicted in Figure (A.2). Sometimes, "relatively injective" and "relatively projective" morphisms are also called "admissible monomorphisms" and "admissible epimorphisms" respectively.

The most basic example of a Banach G-module with non-trivial group action is $\ell^1(G)$, the set of all ℓ^1-functions $G \longrightarrow \mathbf{R}$ with the G-action given by shifting the argument. Obviously, any Banach G-module is a module over $\ell^1(G)$. However, the homological algebra we use does not coincide with the homological algebra in the category of $\ell^1(G)$-modules. Even worse, the category of Banach G-modules is (like the category of Banach spaces) not Abelian.

Relatively projective Banach G-modules are flat in the following sense [11]:

Proposition (A.3). *Let G be a discrete group, let $0 \longrightarrow U \xrightarrow{i} V \xrightarrow{p} W \longrightarrow 0$, be a short exact sequence of G-morphisms of Banach G-modules with relatively injective i, and let A be a relatively projective G-module. Then the induced sequences*

$$0 \longrightarrow B(A,W')^G \longrightarrow B(A,V')^G \longrightarrow B(A,U')^G \longrightarrow 0,$$
$$0 \longrightarrow (A \mathbin{\overline{\otimes}} U)_G \longrightarrow (A \mathbin{\overline{\otimes}} V)_G \longrightarrow (A \mathbin{\overline{\otimes}} W)_G \longrightarrow 0$$

are exact.

Proof. We obtain from the duality principle (Theorem (3.5)) that the induced sequence

$$0 \longrightarrow W' \xrightarrow{p'} V' \xrightarrow{i'} U' \longrightarrow 0$$

is exact. By assumption, i is relatively injective and therefore its dual i' is relatively projective. Because the Banach G-module A is relatively projective, it follows that $B(\mathrm{id}_A, i')^G \colon B(A,V')^G \longrightarrow B(A,U')^G$ is surjective.

A straightforward calculation yields that also the truncated sequence

$$0 \longrightarrow B(A,W')^G \longrightarrow B(A,V')^G \longrightarrow B(A,U')^G$$

is exact. Therefore, the first part of the proposition is proved.

For the second part, we observe that there is a commutative ladder

$$\begin{array}{ccccc}
\big((A \mathbin{\overline{\otimes}} W)_G\big)' & \longrightarrow & \big((A \mathbin{\overline{\otimes}} V)_G\big)' & \longrightarrow & \big((A \mathbin{\overline{\otimes}} U)_G\big)' \\
\| & & \| & & \| \\
B(A,W')^G & \longrightarrow & B(A,V')^G & \longrightarrow & B(A,U')^G,
\end{array}$$

whose vertical morphisms are isometric isomorphisms of Banach G-modules (Remark (1.13) and Proposition (1.14)). The duality principle (Theorem (3.5)) shows that also the sequence $0 \longrightarrow (A \mathbin{\overline{\otimes}} U)_G \longrightarrow (A \mathbin{\overline{\otimes}} V)_G \longrightarrow (A \mathbin{\overline{\otimes}} W)_G \longrightarrow 0$ is exact. $\square$

$$
\begin{tikzcd}
 & V' & \\
0 \arrow[r] & U \arrow[u, "\alpha"] \arrow[r, "i"] & W \arrow[ul, dashed, "\beta"'] \arrow[l, bend left, "\sigma"]
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
 & V \arrow[d, "j_V"] & \\
 & V'' \arrow[d, "\alpha'"] & \\
W' \arrow[r, "i'"'] & U' \arrow[r] \arrow[l, bend right, "\sigma'"'] & 0
\end{tikzcd}
$$

(b)

$$
\begin{tikzcd}
 & V' & \\
 & U'' \arrow[u, "(\alpha' \circ j_V)'"] \arrow[r, "i''"] & W'' \arrow[ul, "\gamma'"'] \\
0 \arrow[r] & U \arrow[u, "j_U"] \arrow[r, "i"] & W \arrow[u, "j_W"']
\end{tikzcd}
$$

(c)

Figure (A.5). Diagrams occurring in the proof of Proposition (A.4)

Taking duals transforms relatively projective modules into relatively injective modules; because not all Banach spaces are reflexive, it seems unlikely that the converse of this proposition holds.

Proposition (A.4). *Let V be a relatively projective Banach G-module. Then its dual V' is a relatively injective Banach G-module.*

Proof. In order to show that V' is a relatively injective Banach G-module we have to find a G-morphism $\beta\colon W \longrightarrow V'$ fitting into the diagram Figure (A.5)(a) whenever $\alpha\colon U \longrightarrow V'$ is a G-morphism and $i\colon U \longrightarrow W$ is a G-morphism admitting a (not necessarily equivariant) split $\sigma\colon W \longrightarrow U$ satisfying $\sigma \circ i = \mathrm{id}_U$ and $\|\sigma\| \leq 1$.

There is an isometric embedding [49; 2.3.7]

$$
\begin{aligned}
j_V\colon V &\longrightarrow V'' \\
v &\longmapsto \big(f \mapsto f(v)\big),
\end{aligned}
$$

which is G-equivariant, of V into its double dual V''. (However, this embedding is not surjective in general). Taking the dual of the solid part of diagram Figure (A.5)(a) thus gives rise to Figure (A.5)(b). Clearly, we have $i' \circ \sigma' = \mathrm{id}_{W'}$ and $\|\sigma'\| \leq \|\sigma\| \leq 1$. Because V is relatively projective, we there exits a G-morphism $\gamma\colon V \longrightarrow W'$ such that $i' \circ \gamma = \alpha' \circ j_V$ and $\|\gamma\| \leq \|\alpha \circ j_V\| \leq \|\alpha\|$.

Dualising a second time yields the commutative diagram Figure (A.5)(c). Unfolding the various definitions shows that $(\alpha' \circ j_V)' \circ j_U = \alpha$. Hence, $\beta := \gamma' \circ j_W$ is a G-morphism with $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\gamma'\| \cdot 1 \leq \|\alpha\|$. $\square$

A.2 Relatively injective and relatively projective resolutions

The key concept of homological algebra is the adequate notion of projective and injective resolutions leading to the fundamental lemma of homological algebra (Proposition (A.7)). In our case, the special form of the mapping problems occurring in the definition of relatively projective/injective G-modules forces us to consider so-called "strong" resolutions.

Definition (A.6). Let G be a discrete group and let V be a Banach G-module.

1. Let (C, ∂) be a Banach G-chain complex (cf. Definition (1.15)). An **augmentation** of C with respect to V is a G-morphism $\varepsilon \colon C_0 \longrightarrow V$ satisfying $\varepsilon \circ \partial_1 = 0$. If ε is an augmentation of C, then the concatenation of C and $\varepsilon \colon C_0 \longrightarrow V$ is a Banach G-chain complex, which we denote by $C \,\square\, \varepsilon$.
2. Dually, an **augmentation** of a Banach G-cochain complex (C, δ) is a G-morphism $\varepsilon \colon C_0 \longrightarrow V$ satisfying $\delta_0 \circ \varepsilon = 0$. The concatenation of $\varepsilon \colon V \longrightarrow C_0$ and C is then a Banach G-cochain complex, which is denoted by $\varepsilon \,\square\, C$.
3. A **(left) resolution** of V is a Banach G-chain complex C together with an augmentation $\varepsilon \colon C_0 \longrightarrow V$ such that $H_*(C \,\square\, \varepsilon) = 0$.
4. A **(right) resolution** of V is Banach G-cochain complex C together with an augmentation $\varepsilon \colon V \longrightarrow C_0$ such that $H^*(\varepsilon \,\square\, C) = 0$.
5. A resolution of V by Banach G-modules is called **strong** if the concatenated Banach G-(co)chain complex admits a (not necessarily equivariant) chain contraction of norm at most 1.
6. A resolution of V is called **relatively projective** (or **relatively injective**) if it consists of relatively projective Banach G-modules (or relatively injective Banach G-modules respectively). $\diamond$

Now the fundamental lemma reads as follows:

Proposition (A.7) (Fundamental lemma). *Let G be a discrete group, let V and W be two Banach G-modules, and let $f \colon V \longrightarrow W$ be a G-morphism.*

1. *If $(C, \varepsilon \colon C_0 \to V)$ is an augmented Banach G-chain complex consisting of relatively projective G-modules and $(D, \eta \colon D_0 \to W)$ is a strong resolution of W,*

then f can be extended to a morphism $C \mathbin{\square} \varepsilon \longrightarrow D \mathbin{\square} \varepsilon$ of Banach G-chain complexes. Moreover, this morphism is unique up to G-homotopy.

2. *Dually, if $(D, \eta \colon W \to D^0)$ is an augmented Banach G-cochain complex consisting of relatively injective G-modules and if $(C, \varepsilon \colon V \to C^0)$ is a strong resolution of V, then f can be extended to a morphism $\varepsilon \mathbin{\square} C \longrightarrow \eta \mathbin{\square} D$ of Banach G-cochain complexes and this morphism is unique up to G-homotopy.*

Proof. The fundamental lemma can be proved using standard techniques from homological algebra [21, 42; Proposition 2.2, Lemma 7.2.4]. For example, in order to find an extension of f in the first part, we inductively solve mapping problems of the form

$$\begin{tikzcd} & C_{n+1} \arrow[dl, dashed, "f_{n+1}"'] \arrow[d, "f_n \circ \partial^C_{n+1}"] \\ D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & \operatorname{im} \partial^D_{n+1} \end{tikzcd}$$

(where $f_{-1} := f$). This is a mapping problem in the sense of Definition (A.1) because $\operatorname{im} \partial^D_{n+1} = \ker \partial^D_n$ is closed – and hence indeed a Banach G-module – and any contracting homotopy of D provides a (non-equivariant) split of the G-morphism $\partial^D_{n+1} \colon D_{n+1} \longrightarrow \operatorname{im} \partial^D_{n+1}$ of norm at most 1. Therefore, the relative projectivity of C_{n+1} ensures the existence of a solution f_{n+1}. □

Proposition (A.4) extends to resolutions and thus dualising transforms (strong) relatively projective resolutions into (strong) relatively injective ones:

Proposition (A.8). *Let G be a discrete group and let $(C, \varepsilon \colon C_0 \to V)$ be a relatively projective resolution of the Banach G-module V. Then its dual $(C', \varepsilon' \colon V' \to C'^0)$ is a relatively injective resolution of the Banach G-module V'.*

If the resolution (C, ε) is strong, then so is (C', ε').

Proof. By Proposition (A.4) the Banach G-cochain complex C' consists of relatively injective Banach G-modules. Since (C, ε) is a resolution, $H_*(C \mathbin{\square} \varepsilon) = 0$. Because the Banach G-cochain complexes $(C \mathbin{\square} \varepsilon)'$ and $\varepsilon' \mathbin{\square} C'$ are isomorphic, we obtain $H_*(\varepsilon' \mathbin{\square} C') = 0$ from the duality principle (Theorem (3.5)). Hence, (C', ε') is a resolution of V'.

If (C, ε) is strong, then the dual of a chain contraction of $C \mathbin{\square} \varepsilon$ with norm at most 1 is a cochain contraction of the dual $\varepsilon' \mathbin{\square} C'$ with norm at most 1, i.e., (C', ε') is a strong resolution of V'. □

The following consequence of the fundamental lemma (Proposition (A.7)) lies at the heart of the definition of group (co)homology in this Banach-flavoured setting (see Definition (2.17) and Theorem (2.18)).

Proposition (A.9). *Let G be a discrete group and let V be a Banach G-module.*

1. *If $(C, \varepsilon\colon C_0 \to V)$ and $(D, \eta\colon D_0 \to V)$ are two strong relatively projective (left) resolutions of V, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*(C_G) \cong H_*(D_G).$$

2. *Dually, if $(C, \varepsilon\colon V \to C^0)$ and $(D, \eta\colon V \to D^0)$ are two strong relatively injective (right) resolutions of V, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H^*(C^G) \cong H^*(D^G).$$

However, the canonical isomorphisms mentioned in the proposition need not be isometric.

Proof. Clearly, any morphism $\varphi\colon C \mathbin{\square} \varepsilon \longrightarrow D \mathbin{\square} \varepsilon$ of Banach G-chain complexes induces a morphism $C_G \longrightarrow D_G$ of Banach chain complexes. Similarly, G-homotopies descend to (bounded) homotopies on the coinvariants. Hence, Proposition (A.7) applied to the G-morphism $\mathrm{id}\colon V \longrightarrow V$ proves the first part.

In the same way the second part can be derived from Proposition (A.7). □

B Bounded cohomology with twisted coefficients

Ivanov proved that bounded cohomology of topological spaces (with $\mathbf{R}$-coefficients) can be computed in terms of strong relatively injective resolutions of $\mathbf{R}$ [25; Theorem 4.1]. This appendix is devoted to the generalisation of Ivanov's result to bounded cohomology with twisted coefficients:

Theorem (B.1). *Let X be a countable, connected CW-complex with fundamental group G and let V be a Banach G-module.*

1. *The morphism $\vartheta_{V'} \colon C_b^*(G; V') \longrightarrow C_b^*(\widetilde{X}; V')$ of Banach G-cochain complexes (defined in (4.16)) induces an isometric isomorphism*

$$H_b^*(X; V') \cong H_b^*(G; V').$$

   Moreover, this isometric isomorphism does not depend on the choice of fundamental domain used in the definition of the $\vartheta_{V'}$.

2. *In particular: If C is a strong relatively injective G-resolution of V', then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_b^*(X; V') \cong H^*(C^G).$$

3. *If C is a strong relatively projective G-resolution of $\mathbf{R}$, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_b^*(X; V') \cong H^*\big(B(C, V')^G\big).$$

The proof of the first part relies on the following observation:

Lemma (B.2). *Let X be a countable, connected CW-complex with fundamental group G and let V be a Banach G-module. The cochain complex $C^*_b(\widetilde{X}; V') = B(C^{\ell^1}_*(\widetilde{X}), V')$ together with the augmentation $\varepsilon_X \colon V' \longrightarrow C^0_b(\widetilde{X}; V')$ given by the obvious inclusion is an approximate strong relatively injective G-resolution of V'.*

Definition (B.3). Let G be a discrete group.

1. If C is a Banach G-cochain complex and $n \in \mathbf{N}$, we define the **truncated cochain complex** $C|_n$ to be the Banach G-cochain complex derived from C by keeping only the modules (and the corresponding coboundary operators) in degree $0, \dots, n$ and defining all modules in higher degrees to be 0.
2. An augmented Banach G-cochain complex $(C, \varepsilon \colon V \to C^0)$ is an **approximate strong resolution** of the Banach G-module V if for every $n \in \mathbf{N}$, the truncated complex $C|_n$ admits a **partial contracting cochain homotopy**, i.e., linear maps $(K_j \colon C^j \to C^{j-1})_{j \in \{1,\dots,n\}}$ and $K_0 \colon C^0 \to V$ of norm at most 1 satisfying
$$\forall_{j \in \{1,\dots,n-1\}} \quad \delta^{j-1} \circ K_j + K_{j+1} \circ \delta^j = \mathrm{id}_{C^j}$$
as well as $K_0 \circ \varepsilon = \mathrm{id}_V$. ◇

The proof of Lemma (B.2) is a straightforward generalisation of Ivanov's proof of the fact that $C^*_b(\widetilde{X})$ is a strong relatively injective $\pi_1(X)$-resolution of $\mathbf{R}$, one of the main steps being the following splitting:

Lemma (B.4). *Let X and Y be simply connected spaces, let $p \colon X \longrightarrow Y$ be a principal bundle whose structure group is an Abelian topological group G, and let V be a Banach space. Then for each $n \in \mathbf{N}$ there is a partial split of $C^*_b(p; V')|_n$, i.e., a cochain map*

$$A|_n \colon C^*_b(X; V')|_n \longrightarrow C^*_b(Y; V')|_n$$

of truncated complexes satisfying for all $j \in \{0, \dots, n\}$

$$A|_n{}^j \circ C^j_b(p; V') = \mathrm{id} \qquad \textit{and} \qquad \|A|_n{}^j\| \leq 1.$$

Proof (of Lemma (B.4)). This can be shown in exactly the same way as the corresponding statement for $\mathbf{R}$-coefficients [25; Theorem 2.2]:

Let $n \in \mathbf{N}$. Then the group $G_n := \operatorname{map}(\Delta^n, G)$ is Abelian and hence amenable (when regarded as discrete group). Therefore, there exists a G_n-equivariant mean

$$m \colon B(G_n, V') \longrightarrow V',$$

where V' is endowed with the trivial G_n-action. Such a mean can, for example, be constructed via

$$\begin{aligned} m\colon B(G_n, V') &\longrightarrow V' \\ f &\longmapsto \Bigl(v \mapsto m_{\mathbf{R}}\bigl(g \mapsto (f(g))(v)\bigr)\Bigr), \end{aligned}$$

where $m_{\mathbf{R}}\colon B(G_n, \mathbf{R}) \longrightarrow \mathbf{R}$ is a G_n-invariant mean provided by amenability of G_n.

Now the same construction as in Ivanov's proof [25; proof of Theorem 2.2] gives rise to the partial split $A|_n$. (Perhaps there is no total split $C^*_{\mathrm{b}}(X; V') \longrightarrow C^*_{\mathrm{b}}(Y; V')$ because – unlike in the case with $\mathbf{R}$-coefficients [25; p. 1094] – the theorem of Banach-Alaoglu cannot be applied directly to the space $B(B(G_n, V'), V')$. But for our applications the partial splits suffice.) □

Using Lemma (B.4), we can construct the required partial contracting homotopies of the bounded chain complex with twisted coefficients as in Ivanov's proof for $\mathbf{R}$-coefficients:

Proof (of Lemma (B.2)). Because $\widetilde{X}$ is a simply connected countable CW-complex, there is a sequence

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 := \widetilde{X}$$

of principal bundles $(p_n)_{n \in \mathbf{N}_{>0}}$ with Abelian structure groups such that

$$\forall_{j \in \{0,\dots,n\}} \quad \pi_j(X_n) = 0 \qquad \text{and} \qquad \forall_{j \in \mathbf{N}_{>n}} \quad \pi_j(X_n) = \pi_j(\widetilde{X})$$

holds for all $n \in \mathbf{N}_{>0}$ [25; p. 1096]. In particular, all the X_n are simply connected.

Let $n \in \mathbf{N}$. Since X_n is n-connected, one can explicitly construct a partial chain contraction

$$\mathbf{R} \xrightarrow{L_0} C_0(X_n) \xrightarrow{L_1} \cdots \xrightarrow{L_n} C_n(X_n)$$

with $\|L_j\| \leq 1$ for all $j \in \{0, \dots, n\}$ [25; p. 1097]. Because L is bounded, it can be extended to a partial cochain contraction

$$\mathbf{R} \xrightarrow{\overline{L_0}} C_0^{\ell^1}(X_n) \xrightarrow{\overline{L_1}} \cdots \xrightarrow{\overline{L_n}} C_n^{\ell^1}(X_n),$$

which also satisfies $\|\overline{L_j}\| \leq 1$. Therefore, the induced maps

$$V' = B(\mathbf{R}, V') \xleftarrow{B(\overline{L_0}, \mathrm{id}_{V'})} C^0_{\mathrm{b}}(X_n; V') \xleftarrow{B(\overline{L_1}, \mathrm{id}_{V'})} \cdots \xleftarrow{B(\overline{L_n}, \mathrm{id}_{V'})} C^n_{\mathrm{b}}(X_n; V')$$

form a partial cochain contraction with norm at most 1. Using the splits from Lemma (B.4), we can transfer this partial contracting cochain map of X_n to one of X: By Lemma (B.4), for $j \in \{1, \dots, n\}$ we find partial splits

$$A(j)|_n \colon C_b^*(X_{j+1}; V')|_n \longrightarrow C_b^*(X_j; V')|_n$$

of $C_b^*(p_j)|_n$. We consider the maps

$$V' \xleftarrow{K_0} C_b^0(\widetilde{X}; V') \xleftarrow{K_1} \cdots \xleftarrow{K_n} C_b^n(\widetilde{X}; V')$$

defined by

$$K_j := A(1)|_n \circ \cdots \circ A(n-1)|_n \circ B(\overline{L_j}, \mathrm{id}_{V'}) \circ C_b^j(p_{n-1}; V') \circ \cdots \circ C_b^j(p_1; V')$$

for all $j \in \{0, \dots, n\}$. By construction, $\|K_j\| \leq 1$ and $K_0, \dots, K_n$ form a partial cochain contraction [25; p. 1096].

It remains to show that the Banach G-modules $C_b^n(\widetilde{X}; V')$ are relatively injective: Let $F \subset \widetilde{X}$ be a fundamental domain for the G-action on $\widetilde{X}$. For $n \in \mathbf{N}$, we write $F_n \subset C_n^{\ell^1}(\widetilde{X})$ for the Banach subspace generated by all singular simplices mapping the zeroth vertex of Δ^n into F. Then

$$C_n^{\ell^1}(\widetilde{X}) = \ell^1(G) \mathbin{\overline{\otimes}} F_n$$

(as Banach G-modules). In particular, we obtain (cf. Remark (1.13))

$$C_b^n(\widetilde{X}; V') = B\big(C_n^{\ell^1}(\widetilde{X}), V'\big) = B\big(\ell^1(G) \mathbin{\overline{\otimes}} F_n, V'\big) = B\big(\ell^1(G), B(F_n, V')\big).$$

Because $B(\ell^1(G), B(F_n, V'))$ is a relatively injective Banach G-module [42; Proposition 4.4.1], it follows that $C_b^n(\widetilde{X}; V')$ is relatively injective.

Hence, the cochain complex $(C_b^*(\widetilde{X}; V'), \varepsilon_X)$ is an approximate strong relatively injective resolution of V'. □

Theorem (B.1) can now be deduced from Lemma (B.2) by means of homological algebra:

Proof (of Theorem (2.28)). Ad 1. The pair $(C_b^*(\widetilde{X}; V'), \varepsilon_X \colon V' \to C_b^0(\widetilde{X}; V'))$ is an approximate strong relatively injective resolution of V' by Lemma (B.2).

The morphism $\vartheta_{V'} \colon C_b^*(G; V') \longrightarrow C_b^*(\widetilde{X}; V')$ of Banach G-cochain complexes constructed in the proof of Corollary (4.14) satisfies (where $\varepsilon \colon C_0^{\ell^1}(G) \longrightarrow \mathbf{R}$ is the augmentation of Definition (2.12))

$$\varepsilon_X \circ \vartheta_{V'}^0 = B(\varepsilon, \mathrm{id}_V) \circ \mathrm{id}_{V'}.$$

I.e., $\mathrm{id}_{V'} \mathbin{\square} \vartheta_{V'} \colon B(\varepsilon, \mathrm{id}_V) \mathbin{\square} C^*_{\mathrm{b}}(G;V') \longrightarrow \varepsilon_X \mathbin{\square} C^*_{\mathrm{b}}(\widetilde{X};V')$ is a morphism of Banach G-cochain complexes.

The inductive proof of Proposition (A.7) depends only on finite initial parts of the resolutions in question. Because $(C^*_{\mathrm{b}}(G;V'), B(\varepsilon, \mathrm{id}_V))$ is a strong relatively injective resolution (Proposition (2.19)), it follows that $\vartheta_{V'}$ is the (up to G-homotopy) unique morphism of Banach G-cochain complexes from $C^*_{\mathrm{b}}(G;V')$ to $C^*_{\mathrm{b}}(\widetilde{X};V')$ and that $\vartheta_{V'}$ admits a G-homotopy inverse.

In particular, the restriction of $\vartheta_{V'}$ to the G-invariants induces an isomorphism

$$H^*\big(C^*_{\mathrm{b}}(\widetilde{X};V')\big) \cong H^*\big(C^*_{\mathrm{b}}(G;V')^G\big) = H^*_{\mathrm{b}}(G;V'),$$

which is independent of the choice of fundamental domain used in the definition of $\vartheta_{V'}$.

Furthermore, this isomorphism is even isometric: By construction, $\|\vartheta_{V'}\| \leq 1$. Conversely, it is known that the semi-norm on $H^*_{\mathrm{b}}(G;V')$ induced by the norm on the standard resolution $C^*_{\mathrm{b}}(G;V')$ is "minimal" [42; Corollary 7.4.7, Theorem 7.3.1]. Therefore, the isomorphism on cohomology induced by $\vartheta_{V'}$ must be isometric.

Ad 2. and 3. Combining the first part with Theorem (2.18) shows that we can also compute $H^*_{\mathrm{b}}(X;V')$ in the stated form via strong relatively injective resolutions. $\square$

C Gromov's duality principle for non-compact manifolds

It is natural to ask whether the simplicial volume of a non-compact manifold can also be computed in terms of a suitable semi-norm evaluated on the dual fundamental class in cohomology with compact supports. Although the naïve duality $\|M\| = 1/\,\|[M]^*\|_\infty$ fails in general (Remark (C.4)), it is still possible to explicitly describe a semi-norm $\|\cdot\|_\infty^{\mathrm{lf}}$ on cohomology with compact supports such that

$$\|M\| = \frac{1}{\big\|[M]^*\big\|_\infty^{\mathrm{lf}}}$$

holds (Theorem (C.2)). In this appendix, we give a complete proof of this duality and apply it to derive a generalised product formula (Theorem (C.7)) as indicated by Gromov [18; p. 17f].

C.1 Statement of the non-compact duality principle

As first step, we recall Gromov's definition [18; p. 17] of the semi-norm $\|\cdot\|_\infty^{\mathrm{lf}}$ on cohomology with compact supports:

Definition (C.1). Let M be a topological space, let $N \subset M$ be a subspace, let $k \in \mathbf{N}$, and let $A \in S_k^{\mathrm{lf}}(M)$; the definition of $S_k^{\mathrm{lf}}(M)$ is contained in Definition (5.1).

1. If $f \in C^k(M)$, then
$$\|f\|_\infty^A := \sup_{\sigma \in A} \bigl|f(\sigma)\bigr| \in [0, \infty].$$
The corresponding semi-norms $\|\cdot\|_\infty^A$ on the cohomology groups $H^k(M,N)$ and $H^k_{\mathrm{cs}}(M)$ are defined as the infimum of the above norm of all representatives of the cohomology class in question.
2. Moreover, for $\varphi \in H^k(M,N)$ or $\varphi \in H^k_{\mathrm{cs}}(M)$, we write
$$\|\varphi\|_\infty^{\mathrm{lf}} := \sup_{A \in S_k^{\mathrm{lf}}(M)} \|\varphi\|_\infty^A \in [0, \infty].$$
Notice that the semi-norm $\|\cdot\|_\infty^{\mathrm{lf}}$ only exists on the level of cohomology, but not on the level of cochains.
3. Similarly, if $c = \sum_{\sigma \in B} a_\sigma \cdot \sigma$ is a chain in $C_k^{\mathrm{lf}}(M)$ or $C_k(M,N)$ that is in reduced form, then
$$\|c\|_1^A := \sum_{\sigma \in B \cap A} |a_\sigma| + \sum_{\sigma \in B \setminus A} \infty \in [0, \infty]$$
(where $\sum_\emptyset \infty := 0$). The corresponding semi-norms $\|\cdot\|_1^A$ on the homology groups $H_k^{\mathrm{lf}}(M)$ and $H_k(M,N)$ are defined as the infimum of the above norm of all representatives.
4. If M is an oriented, connected manifold without boundary and if K is a non-empty, compact, connected subset of M, then
$$\|M, M \setminus K\|^A := \bigl\|[M, M \setminus K]\bigr\|_1^A,$$
$$\|M\|^A := \bigl\|[M]\bigr\|_1^A \qquad \diamond$$

Using this terminology, Gromov [18; p. 17] stated the following generalisation of Proposition (5.15) to non-compact manifolds:

Theorem (C.2) (Duality principle for non-compact manifolds). *Let M be an oriented, connected manifold without boundary. Then the simplicial volume of M can be computed by*
$$\|M\| = \frac{1}{\bigl\|[M]^*\bigr\|_\infty^{\mathrm{lf}}}.$$

Because the cochain complex of cochains with compact support is not large enough, we cannot directly apply the duality principle for semi-norms (Theorem (3.8)) to obtain this non-compact version. Rather we have to take a little detour where we show that $\|M\|$ can be expressed nicely in terms of relative simplicial volumes (Proposition (C.3)). We then apply duality to these relative terms (Proposition (C.6)). Because there is no proof of Theorem (C.2) in the literature and because the whole construction is not as straightforward as one might suspect, a full proof is presented in the following sections.

C.2 The homological version

Before proving the duality principle for non-compact manifolds, we first establish an approximation result for the simplicial volume in terms of (relative) simplicial volumes of compact pieces of the manifold in question:

Proposition (C.3) (Simplicial volume in terms of relative simplicial volumes). *If M is an oriented, connected n-manifold without boundary, then*

$$\|M\| = \inf_{A \in S_n^{\mathrm{lf}}(M)} \|M\|^A,$$

and for all $A \in S_n^{\mathrm{lf}}(M)$ we have

$$\|M\|^A = \sup_{K \in C(M)} \|M, M \setminus K\|^A.$$

Recall that $C(M)$ is the set of all compact, connected, non-empty subsets of M.

Remark (C.4). In the second part of the proposition it is essential that we restrict ourselves to locally finite (in the sense of Definition (5.1)) subsets of $\operatorname{map}(\Delta^n, M)$; for example, the naïve approximation $\|M\| = \sup_{K \in C(M)} \|M, M \setminus K\|$ does not hold in general: Restriction of fundamental cycles shows that the estimate "$\geq$" always holds. However, the reverse inequality may fail badly:

If $M = W^\circ$, where $(W, \partial W)$ is an oriented, compact, connected manifold with boundary, then

$$\sup_{K \in C(M)} \|M, M \setminus K\| = \|W, \partial W\|,$$

which is always finite. On the other hand, $\|M\|$ is not finite in general (see Chapter 6).

This example also shows why the naïve duality $\|M\| = 1/\,\|[M]^*\|_\infty$ cannot be correct in general [18; p. 17]: By definition of $[M]^*$ via cocycles with compact support, we have $\|[M]^*\|_\infty = \inf_{K\in C(M)} \|[M, M\setminus K]^*\|_\infty$. Therefore, in the case $M = W^\circ$ as above, it follows that

$$\frac{1}{\|[M]^*\|_\infty} = \frac{1}{\inf_{K\in C(M)} \|[M, M\setminus K]^*\|_\infty} = \sup_{K\in C(M)} \|M, M\setminus K\| = \|W, \partial W\|\,,$$

which in general does not coincide with $\|M\|$. $\diamond$

Proof (of Proposition (C.3)). The equality $\|M\| = \inf_{A\in S_n^{\mathrm{lf}}(M)} \|M\|^A$ follows directly from the definition of $\|M\|$ by locally finite fundamental cycles.

Therefore, it remains to show that for each $A \in S_n^{\mathrm{lf}}(M)$ we can compute $\|M\|^A$ as the supremum $\sup_{K\in C(M)} \|[M, M\setminus K]\|^A$:

We first show that "$\geq$" holds: If $c \in C_n^{\mathrm{lf}}(M)$ is a locally finite fundamental cycle and if $K \in C(M)$, then the restriction $c|_K \in C_n(M, M\setminus K)$ is a relative fundamental cycle of $(M, M\setminus K)$ and

$$\|c\|_1^A \geq \big\|c|_K\big\|_1^A.$$

Hence, taking the infimum over all fundamental cycles c yields

$$\|M\|_1^A \geq \sup_{K\in C(M)} \|M, M\setminus K\|^A\,.$$

It remains to show "$\leq$": Let $(K_m)_{m\in\mathbf{N}} \subset C(M)$ be an increasing sequence satisfying $\bigcup_{m\in\mathbf{N}} K_m = M$. It suffices to prove that

$$\|M\|^A \leq \sup_{m\in\mathbf{N}} \|M, M\setminus K_m\|^A\,.$$

If the supremum $\sup_{m\in\mathbf{N}} \|M, M\setminus K_m\|^A$ is infinite, there is nothing to show; so we assume in the following that this supremum is finite.

The idea is now to choose $\|\cdot\|_1^A$-small relative fundamental cycles c_m of the pairs $(M, M\setminus K_m)$ supported on A and to construct – via diagonalisation – a locally finite fundamental cycle c of M out of the sequence $(c_m)_{m\in\mathbf{N}}$. The fact that *all* c_m are supported on A and the local finiteness of A ensure that this limit cycle exists, is locally finite with support in A, and indeed represents the fundamental class.

More precisely: Let $\varepsilon \in \mathbf{R}_{>0}$ and for each $m \in \mathbf{N}$ let $c_m \in C_n(M)$ be a relative fundamental cycle of $(M, M \setminus K_m)$ with

$$\|c_m\|_1^A < \|M, M \setminus K_m\|^A + \varepsilon.$$

The $\|M, M \setminus K_m\|^A$ all being finite, we can write

$$c_m = \sum_{\sigma \in A} a_m^\sigma \cdot \sigma$$

for certain real coefficients a_m^σ. Furthermore because $\sup_{m \in \mathbf{N}} \|c_m\|_1^A$ is finite, for each $\sigma \in A$ the sequence $(a_m^\sigma)_{m \in \mathbf{N}} \subset \mathbf{R}$ is bounded and thus possesses at least one limit point.

We choose an enumeration $(\sigma_k)_{k \in \mathbf{N}}$ of A; this is possible because A is locally finite and hence countable. By induction on $k \in \mathbf{N}$, we can find subsequences

$$(m_r^{(k)})_{r \in \mathbf{N}} \subset (m_r^{(k-1)})_{r \in \mathbf{N}}$$

(where $m_r^{(-1)} := r$ for all $r \in \mathbf{N}$) such that the limit

$$a_k := \lim_{r \to \infty} a^{\sigma_k}_{m_r^{(k)}}$$

exists. Thus

$$c := \sum_{k \in \mathbf{N}} a_k \cdot \sigma_k$$

is a locally finite chain on M with support in A.

Lemma (C.5). *For the chain $c \in C_n^{\mathrm{lf}}(M)$ constructed as above, the following holds:*

1. *The chain $c \in C_n^{\mathrm{lf}}(M)$ is a cycle.*
2. *The cycle c represents the fundamental class of M.*
3. *Moreover,*
$$\|c\|_1^A \leq \sup_{m \in \mathbf{N}} \|M, M \setminus K_m\|^A + \varepsilon.$$

Before proving the lemma, we first complete the proof of the theorem. Thanks to the lemma we obtain

$$\|M\|^A \leq \|c\|_1^A \leq \sup_{m \in \mathbf{N}} \|M, M \setminus K_m\|^A + \varepsilon,$$

which implies (by letting $\varepsilon \to 0$) that $\|M\|^A \leq \sup_{m \in \mathbf{N}} \|M, M \setminus K_m\|^A$, as desired. □

Proof (of Lemma (C.5)). We use the same notation as established in the proof of the theorem. To prove the lemma, we make use of the following fact: For any $m \in \mathbf{N}$ there is a $k \in \mathbf{N}$ such that: If $\sigma \in A$ and $\sigma(\Delta^n) \cap K_m \neq \emptyset$, then $\sigma \in \{\sigma_1, \dots, \sigma_k\}$. (This is a direct consequence of the local finiteness of A).

1. *The chain c is a cycle:* It suffices to show that

$$\forall_{m\in\mathbf{N}} \quad \partial\big(c|_{K_m}\big) \in C_{n-1}\,(M \setminus K_m)$$

because $(K_m)_{m\in\mathbf{N}}$ is an increasing, exhausting sequence of M, the chain c is locally finite, and $C^{\mathrm{lf}}_*\,(M) = \varprojlim_{K\in C(M)} C_*\,(M, M\setminus K)$. Let $m \in \mathbf{N}$, and let $k \in \mathbf{N}$ be chosen as above. Furthermore, we assume that k is the smallest such number. Thus,

$$c|_{K_m} = \sum_{j=0}^{k} a_j \cdot \sigma_j.$$

Therefore, we obtain

$$\begin{aligned}
\partial\big(c|_{K_m}\big) &= \partial\Bigg(\sum_{j=0}^{k}\Big(\lim_{r\to\infty} a^{\sigma_j}_{m_r^{(k)}}\Big)\cdot\sigma_j\Bigg)\\
&= \partial\Bigg(\lim_{r\to\infty}\sum_{j=0}^{k} a^{\sigma_j}_{m_r^{(k)}}\cdot\sigma_j\Bigg),
\end{aligned}$$

where the last limit is taken with respect to $\|\cdot\|_1$. The boundary operator ∂ is $\|\cdot\|_1$-continuous. This implies

$$\begin{aligned}
\partial\big(c|_{K_m}\big) &= \lim_{r\to\infty}\partial\Bigg(\sum_{j=0}^{k} a^{\sigma_j}_{m_r^{(k)}}\cdot\sigma_j\Bigg)\\
&= \lim_{r\to\infty}\partial\big(c_{m_r^{(k)}}|_{K_m}\big).
\end{aligned}$$

If $r \in \mathbf{N}$ is large enough, then $m_r^{(k)} \geq m$, and therefore $\partial(c_{m_r^{(k)}})$ lies in the chain complex $C_*(M \setminus K_{m_r^{(k)}}) \subset C_*\,(M \setminus K_m)$. But then also

$$\partial\big(c_{m_r^{(k)}}|_{K_m}\big) \in C_*\,(M \setminus K_m).$$

In other words, $\partial(c|_{K_m})$ is an ℓ^1-limit of chains in $C_*\,(M \setminus K_m)$. Because the subcomplex $C_*\,(M \setminus K_m)$ is ℓ^1-closed in $C_*\,(M)$, we see that $\partial(c|_{K_m}) \in C_*\,(M \setminus K_m)$.

Therefore, we deduce that $c \in C^{\mathrm{lf}}_n\,(M) = \varprojlim_{m\in\mathbf{N}} C_n\,(M, M \setminus K_m)$ is a cycle.

2. *The cycle c is a fundamental cycle of V:* By the local characterisation of fundamental cycles (Theorem (5.4)), it suffices to show that for some $m \in \mathbf{N}$

$$\big\langle [M, M \setminus K_m]^*, [c|_{K_m}] \big\rangle = 1$$

holds. Let $m \in \mathbf{N}$, and let $k \in \mathbf{N}$ be as above. Moreover, let $f_m \in C^n(M, M \setminus K_m)$ be a cocycle representing $[M, M \setminus K_m]^*$. Then

$$\begin{aligned}
\big\langle [M, M \setminus K_m]^*, [c|_{K_m}] \big\rangle &= f_m\Big(\sum_{j=0}^{k} a_j \cdot \sigma_j\Big) \\
&= f_m\Big(\sum_{j=0}^{k} \Big(\lim_{r\to\infty} a^{\sigma_j}_{m_r^{(k)}}\Big) \cdot \sigma_j\Big) \\
&= f_m\Big(\lim_{r\to\infty} \sum_{j=0}^{k} a^{\sigma_j}_{m_r^{(k)}} \cdot \sigma_j\Big).
\end{aligned}$$

The restriction of f_m to the ($\|\cdot\|_1$-closed) finite dimensional subspace $\bigoplus_{j=0}^{k} \mathbf{R} \cdot \sigma_j$ is continuous with respect to the norm $\|\cdot\|_1$. Therefore, we obtain

$$\begin{aligned}
\big\langle \varphi_m, [c|_{K_m}] \big\rangle &= \lim_{r\to\infty} f_m\Big(\sum_{j=0}^{k} a^{\sigma_j}_{m_r^{(k)}} \cdot \sigma_j\Big) \\
&= \lim_{r\to\infty} f_m\big(c_{m_r^{(k)}}|_{K_m}\big).
\end{aligned}$$

For all large enough $r \in \mathbf{N}$, we have $m_r^{(k)} \geq m$ and hence the restriction $c_{m_r^{(k)}}|_{K_m}$ is a relative fundamental cycle of $(M, M \setminus K_m)$. This implies

$$\big\langle [M, M \setminus K_m]^*, [c|_{K_m}] \big\rangle = \lim_{r\to\infty} \big\langle [M, M \setminus K_m]^*, [M, M \setminus K_m] \big\rangle = 1,$$

as was to be shown.

3. *The norm of c is small enough:* Because the support of c lies in A, it follows that $\|c\|_1^A = \sum_{k\in\mathbf{N}} |a_k| = \lim_{k\to\infty} \sum_{j=0}^{k} |a_j|$. For any $k \in \mathbf{N}$, we have

$$\begin{aligned}
\sum_{j=0}^{k} |a_j| = \sum_{j=0}^{k} \lim_{r\to\infty} \big|a^{\sigma_j}_{m_r^{(k)}}\big| &= \lim_{r\to\infty} \sum_{j=0}^{k} \big|a^{\sigma_j}_{m_r^{(k)}}\big| \\
&\leq \sup_{r\to\infty} \big\|c_{m_r^{(k)}}\big\|_1^A \leq \sup_{m\in\mathbf{N}} \|c_m\|_1^A \\
&\leq \sup_{m\in\mathbf{N}} \|M, M \setminus K_m\|^A + \varepsilon,
\end{aligned}$$

and hence $\|c\|_1^A \leq \sup_{m\in\mathbf{N}} \|M, M \setminus K_m\|^A + \varepsilon$. $\square$

C.3 The dual point of view

The proof of the duality principle for non-compact manifolds (Theorem (C.2)) is based on the homological approximation (Proposition (C.3)) combined with a duality expressing $\|M, M \setminus K\|^A$ in terms of the corresponding dual fundamental class:

Proposition (C.6) (Duality principle for the locally finite semi-norms). *Let M be an oriented, connected n-manifold without boundary. Then*

$$\|M, M \setminus K\|^A = \frac{1}{\|[M, M \setminus K]^*\|_\infty^A}$$

for all $A \in S_n^{\mathrm{lf}}(M)$ and all $K \in C(M)$.

Proof. The norm $\|\cdot\|_1^A$ is only defined on the chain group in degree n because A is a set of n-simplices. Furthermore, this norm also may take the value ∞. Therefore, we cannot directly apply the duality principle for semi-norms (Theorem (3.8)).

However, it is not difficult to see that the semi-norm $\|\cdot\|_\infty^A$ is dual to $\|\cdot\|_1^A$ and that the Hahn-Banach theorem also applies in the case that the norm is infinite. Literally the same arguments as in the proof of Theorem (3.8) show that the equality $\|M, M \setminus K\|^A = 1/\,\|[M, M \setminus K]^*\|_\infty^A$ holds. □

Proof (of Theorem (C.2)). By definition of the dual fundamental class $[M]^*$ via cocycles with compact support, we have

$$\|[M]^*\|_\infty^A = \inf_{K \in C(M)} \|[M, M \setminus K]^*\|_\infty^A$$

for each $A \in S_n^{\mathrm{lf}}(M)$. Therefore, the duality principle for the locally finite semi-norms (Proposition (C.6)) and the approximation of simplicial volume in terms of relative simplicial volumes (Proposition (C.3)) allow us to deduce that

$$\begin{aligned}
\|M\| &= \inf_{A \in S_n^{\mathrm{lf}}(M)} \sup_{K \in C(M)} \|M, M \setminus K\|^A \\
&= \inf_{A \in S_n^{\mathrm{lf}}(M)} \sup_{K \in C(M)} \frac{1}{\|[M, M \setminus K]^*\|_\infty^A} \\
&= \inf_{A \in S_n^{\mathrm{lf}}(M)} \frac{1}{\|[M]^*\|_\infty^A} \\
&= \frac{1}{\|M\|_\infty^{\mathrm{lf}}}.
\end{aligned}$$

This finishes the proof of the duality principle. □

C.4 A generalised product formula

With help of the duality provided by Theorem (C.2), we can prove the following version of the product formula for simplicial volume [18; p. 17f]:

Theorem (C.7) (Product formula for simplicial volume). *Let M and N be oriented, connected, manifolds without boundary of dimension m and n respectively.*

1. *Then*
$$\|M \times N\| \leq \binom{m+n}{m} \cdot \|M\| \cdot \|N\|.$$

2. *If N is compact, then*
$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{m+n}{m} \cdot \|M\| \cdot \|N\|.$$

Here, $x \cdot \infty := \infty$ for all $x \in (0, \infty]$. However, in the case that one of the factors has zero simplicial volume and the other one has infinite simplicial volume, the formula does not tell anything about the simplicial volume of the product (see also Section 6.4.6).

Proof. Ad 1. The first part follows, like in the compact case [18, 1; p. 10, Theorem F.2.5], by taking the homological cross-product of fundamental cycles: The cross-product of two locally finite cycles again is a locally finite cycle and restriction to a point shows that the cross-product of two fundamental cycles indeed is a fundamental cycle of the product (by the Künneth theorem and the local characterisation in Theorem (5.4)). Therefore, the explicit form of the homological cross-product [16; Exercise 12.26.2] shows that

$$\|M \times N\| \leq \binom{m+n}{m} \cdot \|M\| \cdot \|N\|.$$

Ad 2. The cohomological cross-product of two cochains with compact support is not necessarily a cochain with compact support. However, if one of the two cochains is a cochain on a compact space, then the explicit form of the cohomological cross-product [15; p. 65] shows that the cross-product again has compact support. Furthermore, evaluating the cohomological cross-product on the homological cross-product of fundamental cycles of both factors (which is a fundamental cycle of the product) shows that the cross-product of a fundamental cocycle (with compact support) of M and a fundamental cocycle of N is a fundamental cocycle (with compact support) of the product $M \times N$. I.e., on the level of cohomology we obtain the relation

$$[M]^* \times [N]^* = [M \times N]^*.$$

The duality principle for non-compact manifolds (Theorem (C.2)) yields

$$\begin{aligned}\|M\| &= \frac{1}{\|[M]^*\|_\infty^{\mathrm{lf}}},\\ \|N\| &= \frac{1}{\|[N]^*\|_\infty^{\mathrm{lf}}} = \frac{1}{\|[N]^*\|_\infty},\\ \|M \times N\| &= \frac{1}{\|[M \times N]^*\|_\infty^{\mathrm{lf}}}.\end{aligned}$$

Therefore, it remains to prove that

$$\|[M \times N]^*\|_\infty^{\mathrm{lf}} \leq \|[M]^*\|_\infty^{\mathrm{lf}} \cdot \|[N]^*\|_\infty.$$

That is, given a locally finite set $B \in S_{m+n}^{\mathrm{lf}}(M \times N)$ and $\varepsilon \in \mathbf{R}_{>0}$ it suffices to find a cocycle $h_B^\varepsilon \in C_{\mathrm{cs}}^{m+n}(M \times N)$ representing $[M \times N]^*$ such that

$$\|h_B^\varepsilon\|_\infty^B \leq \big(\|[M]^*\|_\infty^{\mathrm{lf}} + \varepsilon\big) \cdot \big(\|[N]^*\|_\infty + \varepsilon\big).$$

By definition, for any $\varepsilon \in \mathbf{R}_{>0}$ and any $A \in S^{\mathrm{lf}}_m(M)$ there exist fundamental cocycles $f^\varepsilon_A \in C^n_{\mathrm{cs}}(M)$ and $g^\varepsilon \in C^m(N)$ such that

$$\|f^\varepsilon_A\|^A_\infty \leq \|[M]^*\|^{\mathrm{lf}}_\infty + \varepsilon \qquad \text{and} \qquad \|g^\varepsilon\|_\infty \leq \|[N]^*\|_\infty + \varepsilon.$$

For $B \in S^{\mathrm{lf}}_{n+m}(M \times N)$ and $\varepsilon \in \mathbf{R}_{>0}$ we now consider the cochain

$$h^\varepsilon_B := f^\varepsilon_A \times g^\varepsilon,$$

with $A := \{p_M \circ \sigma\rfloor_m \mid \sigma \in B\}$, where $p_M \colon M \times N \longrightarrow M$ is the projection.

Why is A locally finite? Let $K \subset M$ be compact. Then $L := K \times N \subset M \times N$ is compact. Therefore, the set $\{\sigma \in B \mid \sigma(\Delta^{m+n}) \cap L \neq \emptyset\}$ is finite. But then also

$$\begin{aligned}\big\{\tau \in A \bigm| \tau(\Delta^m) \cap K \neq \emptyset\big\} &\subset \big\{p_M \circ \sigma\rfloor_m \bigm| \sigma \in B,\ p_M \circ \sigma\rfloor_m(\Delta^m) \cap K \neq \emptyset\big\} \\ &\subset \big\{p_M \circ \sigma\rfloor_m \bigm| \sigma \in B,\ \sigma(\Delta^{m+n}) \cap L \neq \emptyset\big\}\end{aligned}$$

must be finite. Hence, A is locally finite and thus h^ε_B is well-defined. Moreover, h^ε_B is a cocycle with compact support representing $[M]^* \times [N]^* = [M \times N]^*$. By construction,

$$\begin{aligned}\|h^\varepsilon_B\|^B_\infty &\leq \sup_{\sigma \in B} \big|f^\varepsilon_A(p_N \circ {}_n\lfloor\sigma)\big| \cdot \big|g^\varepsilon(p_M \circ \sigma\rfloor_m)\big| \\ &\leq \|f^\varepsilon_A\|^A_\infty \cdot \|g^\varepsilon\|_\infty \\ &\leq \big(\|[M]^*\|^{\mathrm{lf}}_\infty + \varepsilon\big) \cdot \big(\|[N]^*\|_\infty + \varepsilon\big).\end{aligned}$$

This proves the product formula. □

In the second part of the product formula (Theorem (C.7)), the restriction that one of the factors has to be compact might seem artificial. However, the following example [18; p. 10] shows that some condition on the factors is needed:

Example (C.8). The simplicial volume of $\mathbf{R}$ is infinite, but $\|\mathbf{R} \times \mathbf{R}\| = 0$ (see Example (6.17)). ◇

Gromov mentions that the same phenomenon can also occur if the simplicial volumes of the factors are finite:

Example (C.9). Gromov shows that if M_1, M_2, M_3 are oriented, connected, non-compact manifolds (without boundary) of dimension at least 3, then the simplicial volume $\|M_1 \times M_2 \times M_3\|$ is zero [18; p. 59]. On the other hand, the M_j can be chosen in such a way that $\|M_j\|$ is finite and non-zero (Example (6.18)). ◇

However, compactness of one of the factors clearly is not a necessary condition for the second part of the product formula, as the example

$$\|\mathbf{R}^4\| = 0 = \|\mathbf{R}^2\| \cdot \|\mathbf{R}^2\|$$

shows (Example (6.17)).

It is natural to ask whether there is at least a product formula for the simplicial volume of manifolds with boundary. Again, it is easy to show that there is an estimate of the form $\|(W_1, \partial W_1) \times (W_2, \partial W_2)\| \leq \text{const} \cdot \|W_1, \partial W_1\| \cdot \|W_2, \partial W_2\|$ by looking at the homological cross-product of relative fundamental cycles. In order to get an estimate from below one would have to consider the cohomological cross product of two relative fundamental cocycles. However, to construct the cohomological cross-product

$$H^*(W_1, \partial W_1) \otimes H^*(W_2, \partial W_2) \longrightarrow H^*\big((W_1, \partial W_1) \times (W_2, \partial W_2)\big)$$

one has to use the inverse of the isomorphism

$$\begin{array}{c} H^*\big((W_1, \partial W_1) \times (W_2, \partial W_2)\big) \\ \Big\downarrow \\ H^*\Big(\hom_{\mathbf{R}}\big(C_*(W_1 \times W_2)/(C_*(\partial W_1 \times W_2) + C_*(W_1 \times \partial W_2)), \mathbf{R}\big)\Big) \end{array}$$

given by excision. Therefore, control over the norm of the cohomological cross-product is lost in this step.

D Measure homology and measure ℓ^1-homology

In his study of simplicial volume of hyperbolic manifolds, Thurston introduced a new homology theory, called measure homology [57; p. 6.6–6.7]. Measure homology is a cunning variation of singular homology: Let X be a topological space and let $n \in \mathbf{N}$. The idea is to think of a singular chain $\sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(X)$ with real coefficients as a signed measure on $\operatorname{map}(\Delta^n, X)$ carrying mass a_j on the set $\{\sigma_j\}$. The measure chain complex of X consists of all signed measures on $\operatorname{map}(\Delta^n, X)$ satisfying some finiteness conditions. Measure homology is then defined to be the homology of this chain complex.

We first give an introduction into measure homology and its counterpart in the smooth category – smooth measure homology – as well as their ℓ^1-versions (Section D.1). In Section D.2, we investigate the relation between ℓ^1-homology and measure ℓ^1-homology. We conclude with a short discussion of the smearing map (Section D.3), which is the key step in Thurston's proof of the proportionality principle of simplicial volume, and which we use in the proof of the corresponding statement about ℓ^1-invisibility (Proposition (6.10)).

D.1 Measure homology and measure ℓ^1-homology

In this section, we recall the definition of (smooth) measure homology and introduce the corresponding ℓ^1-versions.

D.1.1 Measure homology – the topological version

As indicated above, the measure chain complex consists of signed measures on the space of singular simplices that satisfy certain finiteness conditions. More detailed accounts of measure homology are given in the articles of Hansen [22] and Zastrow [61].

Definition (D.1). Let X be a topological space and let $n \in \mathbf{N}$.

1. The **n-th measure chain group**, denoted by $\mathcal{C}_n(X)$, is the $\mathbf{R}$-vector space of signed measures on $\operatorname{map}(\Delta^n, X)$ possessing a compact determination set and finite total variation, where $\operatorname{map}(\Delta^n, X)$ is equipped with the compact-open topology. The elements of $\mathcal{C}_n(X)$ are called **measure n-chains**.
2. For each $j \in \{0, \dots, n+1\}$ the inclusion $\partial_j \colon \Delta^n \longrightarrow \Delta^{n+1}$ of the j-th face induces a continuous map $\operatorname{map}(\Delta^{n+1}, X) \longrightarrow \operatorname{map}(\Delta^n, X)$ and hence a homomorphism (which we also denote by ∂_j)
$$\begin{aligned} \partial_j \colon \mathcal{C}_{n+1}(X) &\longrightarrow \mathcal{C}_n(X) \\ \mu &\longmapsto \mu^{(\sigma \mapsto \sigma \circ \partial_j)}. \end{aligned}$$
The **boundary operator** of measure chains is then defined by
$$\partial := \sum_{j=0}^{n+1} (-1)^j \cdot \partial_j \colon \mathcal{C}_{n+1}(X) \longrightarrow \mathcal{C}_n(X).$$
3. The $\mathbf{R}$-vector space $\mathcal{H}_n(X) := H_n\big(\mathcal{C}_*(X), \partial\big)$ is called the **n-th measure homology group** of X. $\diamond$

Zastrow showed that $(\mathcal{C}_* (X), \partial)$ indeed is a chain complex [61; Corollary 2.9]. In particular, measure homology is well-defined. Furthermore, each continuous map $f \colon X \longrightarrow Y$ induces a chain map [61; Lemma-Definition 2.10(iv)]

$$\begin{aligned} \mathcal{C}_* (f) \colon \mathcal{C}_* (X) &\longrightarrow \mathcal{C}_* (Y) \\ \mu &\longmapsto \mu^f, \end{aligned}$$

which does not increase the total variation. Therefore, we obtain a homomorphism $\mathcal{H}_* (f) \colon \mathcal{H}_* (X) \longrightarrow \mathcal{H}_* (Y)$ with $\|\mathcal{H}_* (f)(\mu)\| \leq \|\mu\|$ for all $\mu \in \mathcal{H}_* (X)$. Clearly, this turns $\mathcal{H}_*$ into a functor.

By verifying the Eilenberg-Steenrod axioms for measure homology, Hansen [22] and Zastrow [61] deduced that measure homology and singular homology coincide on the category of CW-complexes. More precisely:

Theorem (D.2). *For all CW-complexes the inclusion of the singular chain complex into the measure chain complex induces an isomorphism on homology.*

Measure homology therefore combines in a beautiful way the rigidity of singular homology with the flexibility of ℓ^1-chains.

Like the singular chain complex the measure chain complex comes with a natural norm, the total variation, and it is not difficult to see that total variation turns the measure chain complex into a normed chain complex.

Definition (D.3). Let X be a topological space.

1. The **measure ℓ^1-chain complex** $\mathcal{C}^{\ell^1}_* (X)$ of X is the completion (in the sense of Remark (1.3)) of the measure chain complex $\mathcal{C}_* (X)$ with respect to total variation.
2. The homology $\mathcal{H}^{\ell^1}_* (X)$ of the measure ℓ^1-chain complex of X is called the **measure ℓ^1-homology** of X. $\diamond$

Because the space of all signed measures on a given measurable space is a Banach space with respect to total variation, the n-chains of the ℓ^1-measure chain complex $\mathcal{C}^{\ell^1}_* (X)$ also can be viewed as measures on $\operatorname{map}(\Delta^n, X)$; more precisely, $\mathcal{C}^{\ell^1}_n (X)$ consists of the signed measures on $\operatorname{map}(\Delta^n, X)$ of finite total variation that have a σ-compact determination set.

D.1.2 Smooth singular homology

For the main application of measure homology – Thurston's smearing construction – it is necessary to replace measure homology by a corresponding theory based on smooth singular simplices. Before modifying the definition of measure homology to support Thurston's construction, we first introduce the smooth version of singular homology for smooth manifolds, which links smooth measure homology and singular homology, as well as the corresponding ℓ^1-versions.

Definition (D.4). Let M be a smooth manifold.

1. The **smooth singular chain complex** of M is the normed subcomplex $C^{\mathrm{s}}_*(M)$ of the singular chain complex $C_*(M)$, equipped with the ℓ^1-norm, that is generated by all smooth singular simplices; a singular simplex $\sigma\colon \Delta^n \longrightarrow M$ is called **smooth** if it can be extended to a smooth map on an open neighbourhood of Δ^n in $\mathbf{R}^{n+1}$.
2. The homology $H^{\mathrm{s}}_*(M)$ of $C^{\mathrm{s}}_*(M)$ is the **smooth singular homology** of M.
3. The completion of $C^{\mathrm{s}}_*(M)$ with respect to the ℓ^1-norm is called **smooth ℓ^1-chain complex** of M and is denoted by $C^{\mathrm{s},\ell^1}_*(M)$.
4. The homology $H^{\mathrm{s},\ell^1}_*(M)$ of $C^{\mathrm{s},\ell^1}_*(M)$ is the **smooth ℓ^1-homology** of M. $\diamond$

Proposition (D.5). *Let M be a smooth manifold.*

1. *The inclusion $C^{\mathrm{s}}_*(M) \hookrightarrow C_*(M)$ induces an isometric isomorphism*
$$H^{\mathrm{s}}_*(M) \cong H_*(M).$$
2. *The inclusion $C^{\mathrm{s},\ell^1}_*(M) \hookrightarrow C^{\ell^1}_*(M)$ induces an isometric isomorphism*
$$H^{\mathrm{s},\ell^1}_*(M) \cong H^{\ell^1}_*(M).$$

Proof. We write $j\colon C^{\mathrm{s}}_*(M) \hookrightarrow C_*(M)$ and $\bar{j}\colon C^{\mathrm{s},\ell^1}_*(M) \hookrightarrow C^{\ell^1}_*(M)$ for the inclusions.

With help of the Whitney approximation theorem one can construct a smoothing operator $s\colon C_*(M) \longrightarrow C^{\mathrm{s}}_*(M)$ with the following properties: the map s is a chain map of norm 1 and the composition $s \circ j$ is homotopic to the identity on $C^{\mathrm{s}}_*(M)$ via a chain homotopy that is bounded in each degree [31; p. 417ff]. In particular, $H_*(j)$ is an isomorphism with inverse $H_*(s)$. Because both $H_*(j)$ and $H_*(s)$ do not increase the norm, $H_*(j)$ must be isometric.

Furthermore, the boundedness properties ensure that both s and the mentioned chain homotopy can be extended to the completions of the chain complexes involved. Therefore, the same argument shows that $H_*(\bar{j})$ is an isometric isomorphism. $\square$

D.1.3 Measure homology – the smooth version

Similarly to measure homology for general spaces we can define a version of measure homology tailored for smooth manifolds [61, 56]:

Definition (D.6). Let M be a smooth manifold.

1. The **smooth measure chain complex** $\mathcal{C}^{\mathrm{s}}_*(M)$ of M is defined like the measure chain complex $\mathcal{C}_*(M)$, but using measures on the space $\mathrm{map}_\infty(\Delta^*, M)$ of all smooth singular simplices instead.
2. The homology $\mathcal{H}^{\mathrm{s}}_*(M)$ of the smooth measure chain complex of M is called **smooth measure homology** of M. ◇

As topology on $\mathrm{map}_\infty(\Delta^n, M)$ we choose the $\mathcal{C}^1$-topology, i.e., the unique topology that turns the differential $\mathrm{map}_\infty(\Delta^n, M) \longrightarrow \mathrm{map}(T\Delta^n, TM)$ into a homeomorphism onto its image, where $\mathrm{map}(T\Delta^n, TM)$ is given the compact-open topology.

The advantage of the $\mathcal{C}^1$-topology is being fine enough to ensure that integration of smooth measure chains on a Riemannian manifold over the volume form is well-defined [50, 56; Lemma 3 in Section 11.5, Section 4.4].

Classical tools of algebraic topology show that smooth measure homology and (smooth) singular homology coincide on the category of smooth manifolds [61, 56; Theorem 3.4, Theorem 4.10]:

Theorem (D.7). *For any smooth manifold the inclusion of the smooth singular chain complex into the smooth measure chain complex induces an isomorphism in homology.*

As in the the topological case, total variation turns the smooth measure chain complex into a normed chain complex.

Definition (D.8). Let M be a smooth manifold.

1. The **smooth measure ℓ^1-chain complex** of M is defined as the completion of the smooth measure chain complex of M with respect to total variation. This chain complex is denoted by $\mathcal{C}^{\mathrm{s},\ell^1}_*(M)$.
2. The homology $\mathcal{H}^{\mathrm{s},\ell^1}_*(M)$ of $\mathcal{C}^{\mathrm{s},\ell^1}_*(M)$ is the so-called **smooth measure ℓ^1-homology** of M. ◇

D.2 Relating ℓ^1-homology and measure ℓ^1-homology

The straightening technique (Section 4.4) allows to deduce that (smooth) measure ℓ^1-homology contains a copy of ordinary ℓ^1-homology and thus that (smooth) measure homology can be used to compute the ℓ^1-semi-norm on singular homology.

Theorem (D.9). 1. *Let X be a countable, connected CW-complex. Then the homomorphism $H_*(\bar{j}_X)\colon H_*^{\ell^1}(X) \longrightarrow \mathcal{H}_*^{\ell^1}(X)$ induced by the inclusion $\bar{j}_X\colon C_*^{\ell^1}(X) \hookrightarrow \mathcal{C}_*^{\ell^1}(X)$ is injective and isometric.*

2. *Let M be a connected, smooth manifold. Then $H_*(\bar{j}_M^s)\colon H_*^{s,\ell^1}(M) \longrightarrow \mathcal{H}_*^{s,\ell^1}(M)$, the homomorphism induced by the natural inclusion $\bar{j}_M^s\colon C_*^{s,\ell^1}(M) \hookrightarrow \mathcal{C}_*^{s,\ell^1}(M)$, is injective and isometric.*

During the proof of this theorem we have to compare (smooth) measure chain complexes of spaces with the (smooth) measure chain complexes of their universal covering. At this point, the following observation is helpful:

Lemma (D.10). 1. *Let X be a countable, connected CW-complex with universal covering $\pi\colon \widetilde{X} \longrightarrow X$ and let $n \in \mathbf{N}$. Then there is a measurable section*

$$\operatorname{map}(\Delta^n, X) \longrightarrow \operatorname{map}(\Delta^n, \widetilde{X})$$

of the map $\operatorname{map}(\Delta^n, \widetilde{X}) \longrightarrow \operatorname{map}(\Delta^n, X)$ *induced by π.*

2. *Let M be a connected, smooth manifold, let $\pi\colon \widetilde{M} \longrightarrow M$ be the (smooth) universal covering, and let $n \in \mathbf{N}$. Then there is a measurable section*

$$\operatorname{map}_\infty(\Delta^n, M) \longrightarrow \operatorname{map}_\infty(\Delta^n, \widetilde{M})$$

of the map $\operatorname{map}_\infty(\Delta^n, \widetilde{M}) \longrightarrow \operatorname{map}(\Delta^n, M)$ *induced by π.*

Proof. For both parts, there exist proofs based on geometric arguments [32, 56; Appendix, Section 4.3.3].

Moreover, the first part can also be treated by the following reasoning based on descriptive set theory: The simplex Δ^n is compact and metrisable, and the manifolds M and $\widetilde{M}$ are Polish; therefore, $\operatorname{map}(\Delta^n, M)$ and $\operatorname{map}(\Delta^n, \widetilde{M})$ are standard Borel spaces [27; Theorem 4.19]. The map $\operatorname{map}(\Delta^n, \widetilde{M}) \longrightarrow \operatorname{map}(\Delta^n, M)$ induced by π is Borel (even continuous) and countable-to-one (because $\pi_1(M)$ is countable). Now descriptive set theory shows that this map admits a measurable section [27; Exercise 18.14/Theorem 18.10]. $\square$

Proof (of Theorem (D.9)). Ad 1. The basic idea is to look for a factorisation of the ℓ^1-straightening map $\overline{s}_X$ of the following type:

$$\begin{tikzcd} \mathcal{C}^{\ell^1}_*(X) \arrow[rr, "\overline{s}_X"] \arrow[dr, "\overline{j}_X"'] & & S^{\ell^1}_*(X) \\ & \mathcal{C}^{\ell^1}_*(X) \arrow[ur, "\mathcal{S}_X"'] & \end{tikzcd}$$

The measure straightening $\mathcal{S}_X$ is constructed as follows: For each $n \in \mathbf{N}$ let $s_n\colon \operatorname{map}(\Delta^n, X) \longrightarrow \operatorname{map}(\Delta^n, \widetilde{M})$ be a measurable section of the map induced by π, as provided by Lemma (D.10). For $(x_0, \dots, x_n) \in \widetilde{X}^{n+1}$ the set

$$A_{(x_0,\dots,x_n)} := \bigl\{\tau \in \operatorname{map}(\Delta^n, \widetilde{X}) \bigm| \forall_{j \in \{0,\dots,n\}} \quad \tau(v_j) = x_j\bigr\} \subset \operatorname{map}(\Delta^n, \widetilde{X})$$

is closed an hence Borel; here, $v_0, \dots, v_n$ denote the vertices of the standard simplex Δ^n. By definition, the $A_{(x_0,\dots,x_n)}$ are pairwise disjoint. Then for all $\mu \in \mathcal{C}^{\ell^1}_n(M)$, at most countably many of the values $\mu^{s_n}(A_x)$ are non-zero and

$$\sum_{x \in \widetilde{X}^{n+1}} \bigl|\mu^{s_n}(A_x)\bigr| \leq \|\mu^{s_n}\| \leq \|\mu\| < \infty,$$

where μ^{s_n} is the push-forward measure on $\operatorname{map}(\Delta^n, \widetilde{M})$ of μ via s_n (Lemma (D.11) below). We define the straightening $\mathcal{S}_X$ in degree n by

$$\begin{aligned} \mathcal{C}^{\ell^1}_n(X) &\longrightarrow S^{\ell^1}_n(X) = S^{\ell^1}_n(\widetilde{X})_{\pi_1(X)} \\ \mu &\longmapsto \Bigl[\sum_{x \in \widetilde{X}^{n+1}} \mu^{s_n}(A_x)\Bigr]. \end{aligned}$$

It is not difficult to see that this definition does not depend on the actual choice of the measurable section s_n and that the corresponding map $\mathcal{S}_X\colon \mathcal{C}^{\ell^1}_*(X) \longrightarrow S^{\ell^1}_*(X)$ is a chain map of norm at most 1. Furthermore, the triangle above is commutative.

In particular, $H_*(s_X) \circ H_*(\overline{j}_X) = H_*(\overline{s}_X)$. Because $H_*(\overline{s}_X)$ is an isometric isomorphism (Theorem (4.21)), it follows that $H_*(\overline{j}_X)$ is injective. Moreover, both $H_*(\overline{j}_X)$ and $H_*(\mathcal{S}_X)$ do not increase the norm. Thus, $H_*(\overline{j}_X)$ is isometric.

Ad 2. Similarly, using a measurable section $\operatorname{map}_\infty(\Delta^n, M) \longrightarrow \operatorname{map}_\infty(\Delta^n, \widetilde{M})$ (Lemma (D.10)) we can construct a straightening map $\mathcal{S}^{\mathrm{s}}_M \colon \mathcal{C}^{\mathrm{s},\ell^1}_*(M) \longrightarrow S_*(M)$ fitting into the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & C^{\ell^1}_*(M) \arrow[dr, "\bar{s}_M"] & \\
C^{\mathrm{s},\ell^1}_*(M) \arrow[ur] \arrow[dr, "\bar{j}^{\mathrm{s}}_M"'] & & S^{\ell^1}_*(M). \\
 & \mathcal{C}^{\mathrm{s},\ell^1}_*(M) \arrow[ur, "\mathcal{S}^{\mathrm{s}}_M"'] &
\end{tikzcd}$$

Because the top left arrow induces an isometric isomorphism on the level of homology (Proposition (D.5)), the same arguments as in the first part show that also $H_*(\bar{j}^{\mathrm{s}}_M)$ is an isometric injection. $\square$

In the course of the proof, we made use of the following measure theoretic fact:

Lemma (D.11). *Let μ be a signed measure with finite total variation on a measurable space A and let $(A_i)_{i\in I}$ be a family of pairwise disjoint, measurable subsets. Then at most countably many of the numbers $\mu(A_i)$ are non-zero, and*

$$\sum_{i\in I} |\mu(A_i)| \leq \|\mu\|.$$

Proof. Because the $(A_i)_{i\in I}$ are pairwise disjoint and μ has finite total variation, for each $m \in \mathbf{N}$ the set $\{i \in I \mid |\mu(A_i)| > 1/m\}$ is finite. In particular, the set of indices $i \in I$ with $\mu(A_i) \neq 0$ is at most countable.

Therefore, σ-additivity of the variation $|\mu| = \mu^+ + \mu^-$ shows that

$$\sum_{i\in I} |\mu(A_i)| \leq \sum_{i\in I} |\mu|(A_i) \leq |\mu|(A) = \|\mu\|. \qquad \square$$

In particular, we obtain homological (and hence a bit more transparent) versions of the original proofs [56, 32; Section 4.3, Theorem 1.1 and 1.2] that measure homology and singular homology are *isometrically* isomorphic.

Corollary (D.12).
1. *For countable, connected CW-complexes measure homology is isometrically isomorphic to singular homology.*
2. *For connected, smooth manifolds smooth measure homology and singular homology are isometrically isomorphic.*

Proof. Ad 1. Let X be a countable, connected CW-complex. In view of Theorem (D.2), it suffices to show that the homomorphism on homology induced by the inclusion $C_*(X) \hookrightarrow \mathcal{C}_*(X)$ is isometric. To this end we consider the commutative diagram

$$\begin{tikzcd} C_*(X) \arrow[r] \arrow[d] & \mathcal{C}_*(X) \arrow[d] \\ C^{\ell^1}_*(X) \arrow[r] & \mathcal{C}^{\ell^1}_*(X) \end{tikzcd}$$

of inclusions of normed chain complexes. Theorem (D.9) and Proposition (1.7) show that the lower horizontal arrow and the vertical arrows induce isometries on the level of homology. Therefore, also the top horizontal arrow induces an isometry on the level of homology.

Ad 2. Let M be a smooth manifold. Literally the same argument as in the first part shows that the inclusion $C^{s}_*(M) \hookrightarrow \mathcal{C}^{s}_*(M)$ induces an isometric isomorphism on the level of homology. On the other hand, smooth singular homology and singular homology are isometrically isomorphic (Proposition (D.5)). Therefore, also smooth measure homology and singular homology of M are isometrically isomorphic. $\square$

However, Theorem (D.9) cannot be derived from Corollary (D.12) by general arguments, as Example (1.8) shows.

D.3 Smearing

The main feature of (smooth) measure homology is Thurston's construction of the smearing map [57; p. 6.8–6.9], which constitutes the lion share of his proof of the proportionality principle of simplicial volume (see also Remark (D.14) below). When deriving proportionality for ℓ^1-invisibility (Proposition (6.10)), we use the following version of smearing on the level of ℓ^1-homology:

Theorem (D.13) (Smearing). *Let M and N be oriented, closed, connected, Riemannian n-manifolds whose Riemannian universal covering spaces are isometric. Then there are* *chain maps*

$$\begin{aligned}\operatorname{smear}_{M,N}\colon C^s_*(M) &\longrightarrow \mathcal{C}^s_*(N),\\ \operatorname{smear}^{\ell^1}_{M,N}\colon C^{s,\ell^1}_*(M) &\longrightarrow \mathcal{C}^{s,\ell^1}_*(N)\end{aligned}$$

of norm 1 *making the diagram*

$$\begin{tikzcd} H_n(M) \arrow[d, "\cdot\frac{\operatorname{vol} M}{\operatorname{vol} N}"'] & H^s_n(M) \arrow[l] \arrow[r, equal] \arrow[d, "\cdot\frac{\operatorname{vol} M}{\operatorname{vol} N}"'] & H^s_n(M) \arrow[r] \arrow[d, "H_n(\operatorname{smear}_{M,N})"] & H^{s,\ell^1}_n(M) \arrow[d, "H_n(\operatorname{smear}^{\ell^1}_{M,N})"] \\ H_n(N) & H^s_n(N) \arrow[l] \arrow[r] & \mathcal{H}^s_n(N) \arrow[r] & \mathcal{H}^{s,\ell^1}_n(N) \end{tikzcd}$$

commutative. Here, the horizontal arrows are the maps induced by the canonical inclusions on the level of chain complexes.

Proof. The leftmost square is clearly commutative and all four vector spaces involved are isometric to $\mathbf{R}$, generated by the corresponding fundamental classes. Therefore, it makes sense to speak of "multiplication by $\operatorname{vol} M/\operatorname{vol} N$."

The smearing $\operatorname{smear}_{M,N}\colon C^s_*(M) \longrightarrow \mathcal{C}^s_*(N)$ can be constructed by averaging over the Haar measure of the compact quotient $\pi_1(N)\setminus \operatorname{Isom}^+(\widetilde{M})$ [57, 56; p. 6.8, Section 5.4].

If $c \in C^s_n(M)$ is a smooth fundamental cycle of M, then integration of the measure cycle $\operatorname{smear}_{M,N}(c)$ over the volume form of N shows that $\operatorname{smear}_{M,N}(c)$ represents $\operatorname{vol} M/\operatorname{vol} N$ times the fundamental class of N [57, 56; p. 6.9, Theorem 5.23]. Furthermore, the smearing is constructed in such a way that it does not increase the norm [56; Lemma 5.22].

This shows that the middle square of the diagram is commutative. Because $\operatorname{smear}_{M,N}$ is a morphism of normed chain complexes, it extends to the completions and hence we obtain the desired chain map $\operatorname{smear}^{\ell^1}_{M,N}$. By construction, then also the rightmost square of the diagram is commutative. □

Remark (D.14). The proportionality principle of simplicial volume directly follows from this theorem [57, 56; p. 6.9, Section 5.5]: If M and N are oriented, closed, connected, Riemannian manifolds with isometric Riemannian universal covering, then the previous theorem shows that

$$\frac{\operatorname{vol} M}{\operatorname{vol} N}\cdot \|N\| \le \|M\|.$$

Symmetry yields the converse inequality and hence we obtain the proportionality $\|M\|/\operatorname{vol} M = \|N\|/\operatorname{vol} N$. □

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Table of Notation

Symbols

B

C

L

M

N

P

Q

R

S

T

V

Z

Index

I

K

L

M

N

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