Interpreting Quantum Chromodynamics from Spin Glass- and Polymer Analogies

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Abstract

In an informal way some kind of Ising Lattice QCD is introduced which allows to interpret and discuss parts of the well-known theory of Quantum Chromodynamics (confinement, quarks and gluons etc.) from simple phenomena of magnetism and polymer physics. Also the viewpoint of duality is stressed. Moreover, the non-abelian character of the Lie Group SU(3) is important. Essential arguments for confinement and asymptotic freedom are given at the very end of the manuscript, after an appendix.

1 Introduction

This is an informal paper, not intended for publication\(^1\). The QCD (Quantum Chromodynamics, i.e. the theory of strong interaction) is perhaps the most-important theory of present-day high energy physics, dealing with quarks, gluons, confinement, and perhaps the quark-gluon plasma transition predicted to happen at ultra-high energies around 200 MeV. In contrast, ferromagnetism has characteristic temperatures around 1000 K or below, corresponding to 0.1 eV, and spin glass magnetism happens even below 30 K. Thus, since the phenomena differ by nine or more energy decades, one would tend to believe that they are unrelated. However, one should be warned by the facts that (i) the seminal paper of Franz Wegner on lattice-gauge theories, \cite{1}, is based on Ising spins, and (ii) that a strongly pedagogical and interdisciplinary paper of John Kogut, \cite{2}, also stresses the relation between QCD and spins, and (iii) that according to a recent colloquium talk of Fritjof Karsch, \cite{3}, a 2nd-order phase transition to a quark-gluon plasma, which seems to exist at temperatures around \( T_c \approx 200 \text{ MeV} / k_B \), belongs apparently to the universality class of the 3d Ising model.

2 Spin glasses and ”frustration”

In fact, in the theory of Ising spin glasses one is working with the Hamiltonian

\[
\mathcal{H} = - \sum_{i,k} s_i J_{i,k} s_k ,
\]

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Here the spin variables are Ising degrees of freedom, \( s_i = \pm 1 \), and the \( J_{i,k} \) are real numbers, e.g. quenched Gaussian random numbers, or they are also binary random numbers.

In the last-mentioned case, where one is dealing with a so-called ±1 spin glass, the notion of "frustration" plays an important role: One considers a closed loop \( \mathcal{W} \) on the lattice containing the sites \( i \) and \( k \), and on this loop one considers the product \( P_{\mathcal{W}} := J_{i,k} \cdot J_{k,l} \cdot \ldots \cdot J_{n,m} \cdot J_{m,i} \).

The loop \( \mathcal{W} \) is called "frustrated", \cite{4}, if the product is negative.

3 Relation to QCD

The relation to the Quantum Chromodynamics is essentially a matter of three steps:

- At first one interpretes the spin variables \( s_i \) as "quarks" and the \( J_{i,k} \) as "gluons". (This is of course only semantics).
- Secondly, both quantities are now allowed to fluctuate according to the thermal laws. (This is an essential physical step).
- Finally one adds to \( \mathcal{H} \) the so-called Wilson loop contributions \( J_{i,k}J_{k,l}J_{l,m}J_{m,i} \), i.e. the so-called plaquette variables, defined on the boundary of the side faces of the four-dimensional elementary hypercubes defining the four-dimensional euclidean lattice corresponding to the (3 plus 1)-dimensional Minkowski space underlying at \( T=0 \) the quantum chromodynamics. In this way one obtains some kind of "Ising Lattice QCD".

Besides: This contribution corresponds to the above-mentioned "frustration", but one works no longer with a "quenched" situation, but with "annealed" degrees of freedom.

4 Gauge invariance

The Hamiltonian is now explicitly gauge invariant, namely against the simultaneous transformations

\[
\begin{align*}
    s_i &\rightarrow s_i \cdot \epsilon_i, \\
    s_k &\rightarrow s_k \epsilon_k, \\
    J_{i,k} &\rightarrow \epsilon_k J_{i,k} \epsilon_k.
\end{align*}
\]

(2)

Here \( \epsilon_i = \pm 1 \) and \( \epsilon_k = \pm 1 \) are independent random numbers.

Of course this is gross simplification. In the QCD, the \( \epsilon \) parameters are replaced by SU(3) matrices. But the essential philosophy is recovered by the Ising approximation.

This corresponds to a very large extra degeneracy, which already plagues the spin glass physics at low temperatures and also comes into play in the QCD (see below) and means that one is dealing with a very large entropy.

5 Analogies and Consequences

The gauge invariance leads to the notion of a Mattis spin glass, \cite{5}, i.e. a configuration, looking perfectly random, i.e. \( s_i \equiv \epsilon_i \), but which is actually a "ferromagnet in disguise".

This example means that, in any case, one should be very careful with observations.

We start with parallel one-dimensional ferromagnetic Ising chains, e.g. all in the direction of the time axis. If the "gluons" are quenched, as in the spin-glass analogy, while the "quark"
degrees of freedom are allowed to fluctuate thermodynamically, then one has of course weak interaction at large distances and strong interaction at neighbouring lattice points. This is just the opposite situation as in Quantum Chromodynamics: there one has asymptotic freedom at low distances and confinement at large ones, [6].

In fact, the kink excitations in the one-dimensional Ising model, see [2], in particular those on vertically parallel Ising chains, immediately destroy the long-range order, which in the one-dimensional Ising model exists only at T=0. As a consequence, right from the beginning one should not work with quenched gluon degrees of freedom, as in spin glasses but better with the dual situation of quenched quark degrees of freedom. In fact this was the preferred approximation in the early years of QCD, as sketched e.g. in the standard book of Becher, Böhm and Joos, [7].

6 Duality

The importance of duality has already been stressed in the seminal paper of F. Wegner, [1], where it has been stated among other points that the low temperature results of the original model corresponds to the high temperature results of a dual one. To be specific, the frozen gluon approximation of the spin-glass theory and the frozen quark approximation of the early years of the QCD are somehow dual to each other, and the asymptotic freedom as a short-distance behaviour of the QCD is somehow dual to the usual behaviour of magnetic phases at long distances.

Moreover, also the usual 1/r-behaviour of V(r) in the non-confined phases and the strange r-behaviour of the stretched loops under confinement can be considered as somehow dual to each other.

Here the caveat ”somehow” should not be overlooked. In fact, the original models considered by [1] are Ising models with a gauge invariant 2n-point interaction in d dimensions, which is relevant in our case for n=2 and d=4 (respectively for d=3 at high temperatures, where in statistical mechanics a phase-space integral along the time axis is replaced by a sum of finite Matsubara frequencies, see e.g. Abrikosov et al, [9]). The Hamiltonian corresponds to the isolated gluon Hamiltonian

\[ H = K \sum_{\partial \text{plaquettes}} J_{i,k} J_{k,l} J_{l,m} J_{m,i} , \]

with positive K, taken around the edges of a ”plaquette”, i.e. over the boundary of all faces of the d-dimensional simple cubes, defining our lattice. For d≤ 3 this creates a simple d-dimensional hypercubic lattice itself, the ”dual” one, since e.g. \([\binom{3}{2}] = 3\) again. However for d=4, \([\binom{4}{2}] = 6\), i.e. there are now too many faces for simple duality.

However, in this case one may distinguish the three faces going out from ”the lower left” of a hypercube from those coming in towards the ”upper right”, i.e. one has the possibility to get different parity with respect to the center of the hypercube, and may in this way distinguish ”electric” and ”magnetic” field quantities (see below.)

7 Rubber elasticity

On the other hand the above-mentioned crucial behaviour characterizing the confinement, namely the linear increase of V(r) with increasing distance r, which is usually compared on
pedagogical reasons with the behaviour of a stretched rubber band, has also a direct physical meaning: it is well known that rubber elasticity means entropy elasticity and is physically related with the entangling, disentangling and deforming of the many loops in polymers, [8]. Thus one should not be astonished that here Wilson loops, i.e. the dominance of gluon disorder ("annealed disorder", to be precise!), leads to the phenomenon of confinement. Essentially it is the huge degeneracy mentioned above, i.e. essentially a consequence of gauge invariance, which is finally responsible for the rubber-like confinement.

8 Random fields as dual quantities to spin glass terms

If the quarks are frozen and the gluons fluctuating, one can describe the first part of our Hamiltonian, the term usually called the ”spin glass Hamiltonian”, i.e. (1), as a ”random field term”, where the random fields for the gluons, i.e. for the link variables $J_{i,k}$, are given by the momentaneous values of the variables $s_i$ and $s_k$, i.e. by the ”quarks”. However, once again, one has not ”quenched” randomness, but annealed one, and the gauge invariance should be important again.

9 Polymer states, confinement and asymptotic freedom

There is another virtue of the polymer analogy: the fact that in these systems there is apparently everywhere a finite probability to have entanglement between different loops makes it plausible that in these systems one would have an area law behaviour, and not a circumferential one, of the Wilson loop variables. As is known, this is the prerequisite for confinement.

Here some remarks about the difference between QED (Quantum Electrodynamics) and the QCD (Quantum Chromodynamics) are in order: In the QED, with an abelian gauge group, the U(1), the loop segments are uncharged, i.e. unstructured, whereas in the QCD (non-abelian gauge group SU(3)) the loop segments carry colour-charge, i.e. the loop segments themselves generate a colour-electric field $\mathbf{E}$ or a colour-magnetic field $\mathbf{B}$ (quantities with 8 complex 3x3-matrices); i.e. the segments themselves generate the surface-densities $\text{rot} \mathbf{E}$ or $\text{rot} \mathbf{B}$ giving rise to the area-law behaviour. These topological consequences of the non-abelian character of the gauge group, which creates some kind of entangling property of the loops, corresponds in polymer physics to the presence of chemical units, by which loop segments are structured, such that different loops are glued together.

Concerning the non-abelian character of the Wilson products $P_W = J_1 \cdot \ldots \cdot J_4$ (using a symbolical writing) in the groups SU(3), one should note that one is working with a closed surface (i.e. the sum of the edges along the faces of a simple hypercube) and that the sum of the contributions from the faces can be transformed into a volume integral by a kind of Gaussian theorem. However, using general statements similar to ”$\text{div} \text{rot} \equiv 0$” one gets zero for the abelian case. In contrast, if the J-quantities are taken from a non-abelian Lie group, then the above-mentioned identity must be replaced by a nontrivial one involving the structure constants of the group.

Also the property of ”asymptotic freedom” finds a natural explanation through the polymer-physics analogy, namely by the presence of the network of entangled loops: below a characteristic length of that network, e.g. the ”reptation length” of the loops, the influence on the correlations is felt. This characteristic length, $\sim 1$ fm, corresponds directly to the ”bag radius” of the old-fashioned bag-models of the nucleon.
10 The quark-gluon plasma

Ultimately, e.g. if temperatures or pressure begin to correspond roughly to the masses of the heavier quarks, roughly 1 GeV, one should definitely give up any quenched approximation and should treat quarks and gluons on the same footing, namely both as equally fluctuating quantities. The corresponding behaviour leads to the prediction of a 2nd order quark-gluon plasma phase transition, which corresponds to the universality class of the 3d Ising model. That one has a three-dimensional lattice, and no longer a four-dimensional one, is natural, since in the numerical simulation the time-axis is no longer critical, [3]. Moreover, already Franz Wegner, with his generalized Ising ansatz, had a precise duality relation leading to the 3d-Ising model for this case.

11 Analogy between confinement and Abrikosov’s type-II superconductivity

The analogies between QCD and solid-state phenomena are further supported by a correspondence between the confinement of the colour-field into hadrons and the basic phenomenon of the theory of type-II superconductivity - a theory that is essentially two-dimensional, with a natural continuation to E. Artin’s ”braid group”, [10], which is nonabelian. In this theory the U(1)-magnetic field is confined to the center of Abrikosov’s vortex lattice, [11]. I.e., the London penetration depth $\lambda$ of that theory corresponds to the confinement radius $R_c$ of QCD.

It can be shown that this correspondence is based on firm mathematical grounds, namely on a re-interpretation of a term $\propto g_\mathbf{A}_q$ in the Lagrangian of the QCD.

12 Conclusion

The phenomena of lattice QCD, i.e. Quantum Chromodynamics, quarks, gluons, confinement, asymptotic freedom, gauge-invariance, the quark-gluon plasma transition, and so on, have been given certain pedagogical plausibility interpretations by referring to well-known phenomena in solid-state physics, e.g. spin-glass and polymer behaviour, in particular to a ”Mattis spin glass”, i.e. a ”ferromagnet in disguise”, [5], and to random-field physics, entanglement-probability, and ”frustation”.

Moreover, the property of duality is stressed: i.e. the high-temperature phase of the quarks (or at least dominated by them) seems to correspond by duality to the low-temperature phase of the gluons (or dominated by them). At the same time, by this duality, there is apparently a $1/r \rightarrow r$ correspondence, i.e. between asymptotic freedom (Coulomb-like behaviour) and confinement.

The duality is a direct one, since the centers of the simple hypercubes of the original lattice correspond to the sites of the dual lattice. Also the splitting of the colour-fields into electric and magnetic parts can be understood on topological reasonings, and the importance of the non-abelian group character is seen.

Then, finally, at temperatures of roughly $200 \text{ MeV}/k_B$ the quark-gluon plasma transition happens, where both particles should be treated on equal footing and where, on thermal averaging, the non-commutativity is lost, and also the criticality along the time-axis, such that the transition is of the simple universality class of the 3d-Ising model.
Appendix and some arguments

The term corresponding to "div rot $P_W$" for a non-abelian Lie group is well known to be

$$\text{"div rot } P_W\text{" } \to f_{a,b}^c (P_W)_b^\nu \mu (P_W)_\mu ^\nu ,$$

where as usual one has to sum with respect to the indexes appearing two times. The $f$-quantities are the so-called structure constants of the Lie group, antisymmetric with respect to $b$ and $c$. The indexes $a,b,$ and $c$ are shifted from upwards to downwards and vice versa with the trivial signature $(+,\ldots,+)$ from $a=1$ to $a=8$. Also the $\mu$ and $\nu$ indexes are transformed with the trivial signature, but a four-dimensional one, since one is working in Euclidean four-space. This $a$-dependent quantity ($a=1,\ldots,8$) has to be discussed on the "dual" lattice, i.e. on the the centers of the hypercubes.

Now the important qualitative arguments:

- At high energies, on the one hand, this "pure gluon" term, which apparently leads to confinement, is most important, $\propto E^8$, compared with that one where the quark-gluon coupling appears, i.e. compared with the "spin glass" one, $\propto E^3$.

- On the other hand, the structure constants $f_{a,b}^c$ keep their $O(1)$ values, i.e. the non-abelian character becomes more and more negligible with increasing energy even at relatively low temperatures. This is the asymptotic freedom.\(^2\)

- At very high temperatures the non-abelian character is negligible also at smaller energies, and - as stressed above - there is the possibility of a continuous phase transition to a quark-gluon plasma, with the universality class of the 3d-Ising model.

More mathematically, we have the following argument:

$F \equiv D \wedge A := d \wedge A - ig \cdot A \wedge A$ is the Lie-algebra-valued field strength of the system, where $d$ corresponds to the partial derivative, and $D$ to the covariant one, and where $g$ corresponds to the quark-gluon coupling, i.e. to the non-Abelian character of the group, while the final "wedge product" compensates a factor $-i$ and leads to the structure constants. $F$ corresponds to a two-form, and $A$ to a one-form.

Now, because of $d \wedge d \equiv 0$, we have only a nonvanishing volume-form through the non-abelian character of the group, i.e. through the gluons. In fact, $D \wedge F$ is the gauge-invariant three-form $-ig \cdot d \wedge \{A \wedge A\} - ig \cdot A \wedge (d \wedge A - ig \cdot A \wedge A) = -g^2 A \wedge (A \wedge A)$, where the second and third terms on the l.h.s. come from the non-gauge-invariant difference between $d$ and $D$.

To the gauge-invariant three-form $D \wedge F$ a corresponding volume can be defined, [12], the "characteristic volume" $\frac{4\pi}{3} R^3_c$, where $R$ is the so-called "bag radius" of the system. On the other hand, very high energies of the system mean very low distances $r$, and in the limit $0 \leftarrow r \ll R_c$ the system is "asymptotically free", as if the polymer had crystallized.

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\(^2\)I.e., the system shows the following behaviour: in the limit $0 \leftarrow r \ll R_c$, where $R_c$ is a correlation length of the glued loops, which corresponds to the critical radius ($\sim 1$ fm) of the old-fashioned 'bag'-models of the QCD, all correlations vanish.
References


[10] The essential point of E. Artin’s braid groups is the fact that one braid can wind arbitrarily often around the other. The braid groups are nonabelian.

[11] Abrikosov’s theory of type-II superconductivity is described in all textbooks on superconductivity.

[12] The detailed prescription remains somewhat vague at present.